d. $$\begin{aligned} x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 &= 0 \\ 2x_1 + 2x_2 - 4x_3 - 4x_4 + x_5 &= 0 \\ x_1 - x_2 + 2x_3 + 4x_4 + x_5 &= 0 \\ -2x_1 - 4x_2 + 8x_3 + 10x_4 + x_5 &= 0 \end{aligned}$$

Exercise 1.3.6

a. Does Theorem 1.3.1 imply that the system $\left\{\begin{array}{l} -z+3y=0 \\ 2x-6y=0 \end{array}\right.$ has nontrivial solutions? Explain.

b. Show that the converse to Theorem 1.3.1 is not true. That is, show that the existence of nontrivial solutions does *not* imply that there are more variables than equations.

Exercise 1.3.7 In each case determine how many solutions (and how many parameters) are possible for a homogeneous system of four linear equations in six variables with augmented matrix A. Assume that A has nonzero entries. Give all possibilities.

a. Rank $A = 2$.

b. Rank $A = 1$.

c. A has a row of zeros.

d. The row-echelon form of A has a row of zeros.

Exercise 1.3.8 The graph of an equation $ax+by+cz=0$ is a plane through the origin (provided that not all of a, b, and c are zero). Use Theorem 1.3.1 to show that two planes through the origin have a point in common other than the origin $(0, 0, 0)$.

Exercise 1.3.9

a. Show that there is a line through any pair of points in the plane. [*Hint*: Every line has equation $ax+by+c=0$, where a, b, and c are not all zero.]

b. Generalize and show that there is a plane $ax+by+cz+d=0$ through any three points in space.

Exercise 1.3.10 The graph of

$$a(x^2+y^2)+bx+cy+d=0$$

is a circle if $a \neq 0$. Show that there is a circle through any three points in the plane that are not all on a line.

Exercise 1.3.11 Consider a homogeneous system of linear equations in n variables, and suppose that the augmented matrix has rank r. Show that the system has nontrivial solutions if and only if $n > r$.

Exercise 1.3.12 If a consistent (possibly nonhomogeneous) system of linear equations has more variables than equations, prove that it has more than one solution.

1.4 An Application to Network Flow

There are many types of problems that concern a network of conductors along which some sort of flow is observed. Examples of these include an irrigation network and a network of streets or freeways. There are often points in the system at which a net flow either enters or leaves the system. The basic principle behind the analysis of such systems is that the total flow into the system must equal the total flow out. In fact, we apply this principle at every junction in the system.

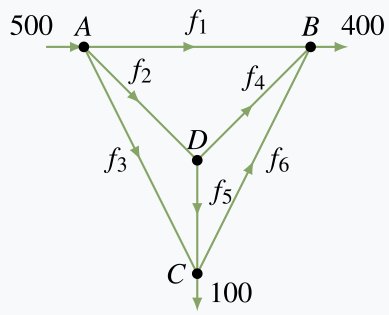
Junction Rule

At each of the junctions in the network, the total flow into that junction must equal the total flow out.

This requirement gives a linear equation relating the flows in conductors emanating from the junction.

Example 1.4.1

A network of one-way streets is shown in the accompanying diagram. The rate of flow of cars into intersection A is 500 cars per hour, and 400 and 100 cars per hour emerge from B and C, respectively. Find the possible flows along each street.



Solution. Suppose the flows along the streets are f_1, f_2, f_3, f_4, f_5, and f_6 cars per hour in the directions shown.
Then, equating the flow in with the flow out at each intersection, we get

$$\begin{array}{ll} \text{Intersection } A & 500 = f_1 + f_2 + f_3 \\ \text{Intersection } B & f_1 + f_4 + f_6 = 400 \\ \text{Intersection } C & f_3 + f_5 = f_6 + 100 \\ \text{Intersection } D & f_2 = f_4 + f_5 \end{array}$$

These give four equations in the six variables $f_1, f_2, \dots, f_6$.

$$\begin{array}{rcl} f_1 + f_2 + f_3 &=& 500 \\ f_1 + f_4 + f_6 &=& 400 \\ f_3 + f_5 - f_6 &=& 100 \\ f_2 - f_4 - f_5 &=& 0 \end{array}$$

The reduction of the augmented matrix is

$$\left[\begin{array}{rrrrrr|r} 1 & 1 & 1 & 0 & 0 & 0 & 500 \\ 1 & 0 & 0 & 1 & 0 & 1 & 400 \\ 0 & 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrrrr|r} 1 & 0 & 0 & 1 & 0 & 1 & 400 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, when we use f_4, f_5, and f_6 as parameters, the general solution is

$$f_1 = 400 - f_4 - f_6 \qquad f_2 = f_4 + f_5 \qquad f_3 = 100 - f_5 + f_6$$

This gives all solutions to the system of equations and hence all the possible flows.
Of course, not all these solutions may be acceptable in the real situation. For example, the flows $f_1, f_2, \dots, f_6$ are all *positive* in the present context (if one came out negative, it would mean traffic flowed in the opposite direction). This imposes constraints on the flows: $f_1 \geq 0$ and $f_3 \geq 0$ become

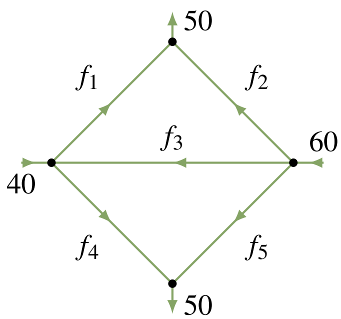
$$f_4 + f_6 \leq 400 \qquad f_5 - f_6 \leq 100$$

Further constraints might be imposed by insisting on maximum values on the flow in each street.

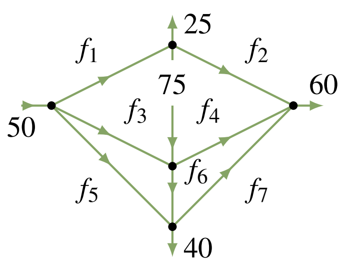
Exercises for 1.4

Exercise 1.4.1 Find the possible flows in each of the following networks of pipes.

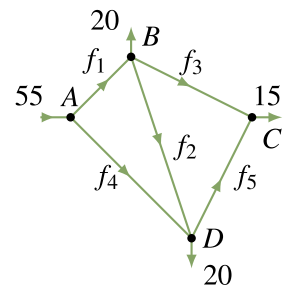
a.



b.



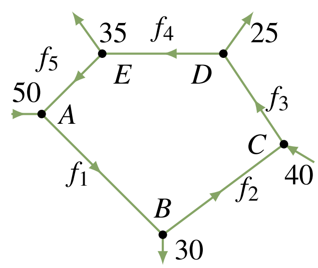
Exercise 1.4.2 A proposed network of irrigation canals is described in the accompanying diagram. At peak demand, the flows at interchanges A, B, C, and D are as shown.



a. Find the possible flows.

b. If canal BC is closed, what range of flow on AD must be maintained so that no canal carries a flow of more than 30?

Exercise 1.4.3 A traffic circle has five one-way streets, and vehicles enter and leave as shown in the accompanying diagram.



a. Compute the possible flows.

b. Which road has the heaviest flow?

1.5 An Application to Electrical Networks[7]

In an electrical network it is often necessary to find the current in amperes (A) flowing in various parts of the network. These networks usually contain resistors that retard the current. The resistors are indicated by a symbol (-\/\/\/-), and the resistance is measured in ohms (Ω). Also, the current is increased at various points by voltage sources (for example, a battery). The voltage of these sources is measured in volts (V),

[7]This section is independent of Section 1.4

1.2.9 b. Unique solution $x=-2a+b+5c$, $y=3a-b-6c$, $z=-2a+b+c$, for any a, b, c.

d. If $abc \neq -1$, unique solution $x=y=z=0$; if $abc=-1$ the solutions are $x=abt$, $y=-bt$, $z=t$.

f. If $a=1$, solutions $x=-t$, $y=t$, $z=-1$. If $a=0$, there is no solution. If $a\neq 1$ and $a\neq 0$, unique solution $x=\frac{a-1}{a}$, $y=0$, $z=\frac{-1}{a}$.

1.2.10 b. 1

d. 3

f. 1

1.2.11 b. 2

d. 3

f. 2 if $a=0$ or $a=2$; 3, otherwise.

1.2.12 b. False. $A=\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

d. False. $A=\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

f. False. $\begin{array}{r} 2x-y=0 \\ -4x+2y=0 \end{array}$ is consistent but $\begin{array}{r} 2x-y=1 \\ -4x+2y=1 \end{array}$ is not.

h. True, A has 3 rows, so there are at most 3 leading 1s.

1.2.14 b. Since one of $b-a$ and $c-a$ is nonzero, then

$$\left[\begin{array}{ccc} 1 & a & b+c \\ 1 & b & c+a \\ 1 & b & c+a \end{array}\right] \rightarrow \left[\begin{array}{ccc} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{array}\right] \rightarrow \left[\begin{array}{ccc} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc} 1 & 0 & b+c+a \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

1.2.16 b. $x^2+y^2-2x+6y-6=0$

1.2.18 $\frac{5}{20}$ in A, $\frac{7}{20}$ in B, $\frac{8}{20}$ in C.

Section 1.3

1.3.1 b. False. $A=\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

d. False. $A=\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$

f. False. $A=\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

h. False. $A=\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

1.3.2 b. $a=-3$, $x=9t$, $y=-5t$, $z=t$

d. $a=1$, $x=-t$, $y=t$, $z=0$; or $a=-1$, $x=t$, $y=0$, $z=t$

1.3.3 b. Not a linear combination.

d. $\mathbf{v}=\mathbf{x}+2\mathbf{y}-\mathbf{z}$

1.3.4 b. $\mathbf{y}=2\mathbf{a}_1-\mathbf{a}_2+4\mathbf{a}_3$.

1.3.5 b. $r\left[\begin{array}{c} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right]+s\left[\begin{array}{c} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{array}\right]+t\left[\begin{array}{c} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{array}\right]$

d. $s\left[\begin{array}{c} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{array}\right]+t\left[\begin{array}{c} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{array}\right]$

1.3.6 b. The system in (a) has nontrivial solutions.

1.3.7 b. By Theorem 1.2.2, there are $n-r=6-1=5$ parameters and thus infinitely many solutions.

d. If R is the row-echelon form of A, then R has a row of zeros and 4 rows in all. Hence R has $r=\text{rank } A=1$, 2, or 3. Thus there are $n-r=6-r=5$, 4, or 3 parameters and thus infinitely many solutions.

1.3.9 b. That the graph of $ax+by+cz=d$ contains three points leads to 3 linear equations homogeneous in variables a, b, c, and d. Apply Theorem 1.3.1.

1.3.11 There are $n-r$ parameters (Theorem 1.2.2), so there are nontrivial solutions if and only if $n-r>0$.

Section 1.4

1.4.1 b.
$$\begin{aligned} f_1 &= 85-f_4-f_7 \\ f_2 &= 60-f_4-f_7 \\ f_3 &= -75+f_4+f_6 \\ f_5 &= 40-f_6-f_7 \end{aligned}$$
f_4, f_6, f_7 parameters

1.4.2 b. $f_5=15$
$25 \leq f_4 \leq 30$

1.4.3 b. CD

Section 1.5