Assignment #10

January 12, 2023

1. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}A_iP = D$ for each A_i.

$$A_1 = \begin{bmatrix} 2 & 6 & 0 \\ 0 & -1 & 0 \\ -1 & -2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 12 & 12 & 3 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

2. Find all distinct (real or complex) eigenvalues of A. Then find the basic eigenvectors of A corresponding to each eigenvalue. For each eigenvalue, specify the number of basic eigenvectors corresponding to that eigenvalue, then write the eigenvalue followed by the basic eigenvectors corresponding to that eigenvalue.

$$A = \begin{bmatrix} 3 & 13 \\ -2 & -7 \end{bmatrix}$$

3. Construct an example of a $2x2$ matrix, with one of its eigenvalues equal to -2, that is not diagonal or invertible, but is diagonalizable.

4. Find a formula in terms of k for the entries of A^k, where A is the diagonalizable matrix below and $A = PDP^{-1}$ for the matrices P and D below.

$$A = \begin{bmatrix} 10 & -8 \\ 12 & -10 \end{bmatrix}, \quad P = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

5. Find a formula in terms of k for the entries of A^k, where A is the diagonalizable matrix below and the eigenvalues of A are -2 and -3.

$$A = \begin{bmatrix} -6 & -6 \\ 2 & 1 \end{bmatrix}$$

6. Find a formula in terms of k for the entries of A^k, where A is the diagonalizable matrix below.

$$A = \begin{bmatrix} -18 & 20 \\ -15 & 17 \end{bmatrix}$$

7. The weather on any given day in a particular city can be sunny, cloudy or rainy. It has been observed to be predictable largely on the basis of the weather on the previous day. Specifically:

   - If it is sunny on one day, it will be sunny the next day $\frac{1}{5}$ of the time, and be cloudy the next day $\frac{2}{5}$ of the time.
   - If it is sunny on one day, it will be sunny the next day $\frac{3}{5}$ of the time, and be cloudy the next day $\frac{1}{5}$ of the time.
   - If it is sunny on one day, it will be sunny the next day $\frac{3}{5}$ of the time, and be cloudy the next day $\frac{1}{5}$ of the time.

   Using 'sunny', 'cloudy' and 'rainy' (in that order) as the states in a system, set up the transition matrix for a Markov chain to describe this system. Using your matrix to determine the probability that it will rain on Wednesday if it is sunny on Sunday.

8. Bob and Doug play a lot of Ping-Pong, but Doug is a much better player and wins 60% of their games.
   To make up for this, if Doug wins a game he will spot Bob five points in their next game. If Doug wins again he will spot Bob ten points the next game and if he still wins the next game he will spot him 15 points, and continue to spot him 15 points as he keeps winning.

   Whenever Bob wins a game he goes back to playing the next game with no advantage. It turns out that with a 5 point advantage Bob wins 60% of the time; He wins 90% of the time with a 10 point advantage and 90% of the time with a 15 point advantage.

   Model this situation as a Markov chain using the number of consecutive games won by Doug as the states.

   There should be 4 states representing 0, 1, 2 and 3 or more consecutive games won by Doug. Find the transition matrix of this system, the steady state vector for the system, and determine the proportion of games that Doug will win in the long run under these conditions.

9. Suppose that the sequence $x_0, x_1, x_2, \ldots$ is defined by $x_0 = 1$, $x_1 = 2$ and $x_{k+2} = -5x_{k+1} - 6x_k$, $k >= 0$. Find a general formula for X_k.