PRACTICE VI

1. The length of a vine during a 12-hour period is given by a twice-differentiable function L, where $L(t)$ is measured in feet and t is measured in weeks for $0 \le t \le 12$. The graph of L is concave down on the interval $0 \le t \le 12$. Selected values of the derivative of L, $L'(t)$, are given in the table below. At time $t = 4$, the length of the vine is 5 feet.

t	2	4	5	8	10
$L'(t)$	1.0	0.8	0.7	0.4	0.2

(a) Use the tangent line approximation for L at time $t = 4$ to estimate $L(4.3)$, the length of the vine at time $t = 4.3$. Is the approximation an overestimate or an underestimate for $L(4.3)$? Give a reason for your answer.

$L_4(4.3) \approx L(4) + L'(4)(4.3-4) = 5 + 0.8(0.3) = \boxed{5.24 \text{ ft}}$

Since the graph of L is concave down on $4 \le t \le 4.3$, the tangent lies above L on this interval, $\therefore$ approximation is an overestimate.

(b) Use a left Riemann sum with four subintervals indicated by the data in the table to approximate $\int_2^{10} L'(t)dt$. Indicate the units of measure

$\int_2^{10} L'(t)dt \approx (4-2)L'(2) + (5-4)L'(4) + (8-5)L'(5) + (10-8)L'(8)$

$\boxed{= 5.7 \text{ ft}}$

(c) Is the approximation in part (b) an overestimate or an underestimate for $\int_2^{10} L'(t)dt$? Give a reason for your answer.

- Since the graph of L is concave down on $0 \le t \le 12$, L' is decreasing on $2 \le t \le 10$. Therefore riemann sum is an overestimate

Solve I.V.P. $y' + xy = 5$, $y(0) = 5$

2. Find two positive numbers whose sum is 300 and whose product is a maximum

$a, \quad b \qquad \boxed{a+b=300}$

$p'' = -2$

$p'' < 0$

for all values.

$p = a \cdot b = a(300-a)$

notice this is a parabola that opens downward. So we know it will have a maximum

$p = 300a - a^2$

$p' = 300 - 2a$

$0 = 300 - 2a$

$2a = 300$

$a = 150$

$b = 150$

3. Find two positive numbers whose product is 750 and for which the sum of one and 10 times the other is a minimum

$a \cdot b = 750$

$S = a + 10b$

$S = \frac{750}{b} + 10b$

$S' = -\frac{750}{b^2} + 10$

$0 = -\frac{750}{b^2} + 10$

$-10 = \frac{-750}{b^2}$

$10b^2 = 750$

$b = \pm\sqrt{\frac{750}{10}}$

$b = \sqrt{75}$

$a = \frac{750}{\sqrt{75}}$

$S'' = \frac{1500}{b^3}$

notice $S'' > 0$, provided $b > 0$, so S will be concave up, and point will give relative minimum.

4. Let x and y be two positive numbers such that $x + 2y = 50$ and $(x+1)(y+2)$ is a maximum. Find x and y

$x + 2y = 50 \Rightarrow x = 50 - 2y$

$P = (x+1)(y+2)$

notice this is a parabola that opens downward.

$P = (50 - 2y + 1)(y+2) = (51 - 2y)(y+2)$

$P' = -2(y+2) + (51 - 2y)$

$= -2y - 4 + 51 - 2y = -4y + 47$

$P'' = -4.$

$0 = -4y + 47$

$y = \frac{47}{4}$

$x = 50 - 2y = 50 - 2\left(\frac{47}{4}\right)$

$= \frac{5\sqrt{3}}{2}$

5. Find the linear approximation to $g(z) = \sqrt[4]{z}$ at $z = 2$. Use the linear approximation to approximate the value of $\sqrt[4]{3}$ and $\sqrt[4]{10}$. Compare the approximate values to the exact values.

$$L_2(x) = g(2) + g'(2)(x-2)$$

$$= \sqrt[4]{2} + \frac{1}{4(2)^{3/4}}(x-2)$$

$$L_2(3) = \sqrt[4]{2} + \frac{1}{4(2)^{3/4}}(1)$$

$$= \frac{\sqrt[4]{2} \cdot 4(\sqrt[4]{2})^3}{4(2)^{3/4}} + \frac{1}{4(2)^{3/4}} = \frac{(4 \cdot 2) + 1}{4(2)^{3/4}} = \frac{7}{4(2)^{3/4}}$$

6. Verify that $y = -t\cos t - t$ is a solution of the initial value problem

$$t\frac{dy}{dx} = y + t^2 \sin t, \quad y(\pi) = 0$$

$y(\pi) = -\pi\cos(\pi) - \pi$

$= \pi - \pi = 0$

$ty' = y + t^2\sin(t), \qquad y(\pi) = 0$

$y' = -\cos(t) + t\sin(t) - 1 \qquad y = -t\cos(t) - t.$

$t(-\cos(t) + t\sin(t) - 1) = -t\cos(t) - t + t^2\sin(t)$

$-t\cos(t) + t^2\sin(t) - t = -t\cos(t) - t + t^2\sin(t)$

$= -t\cos(t) + t^2\sin(t) - t$

$\square$

7. Find a solution to the initial-value problem

$$y' = -y^2, \;\; y(0) = \frac{1}{2}$$

$$-\frac{1}{y^2}\,dy = dx$$

$$\int -\frac{1}{y^2}\,dy = \int dx$$

$$\frac{1}{y} = x + c$$

$$y = \frac{1}{x+c}$$

$$y(0) = \frac{1}{2}$$

$$\therefore \quad \frac{1}{2} = \frac{1}{c}$$

$$c = 2$$

$$\boxed{y = \frac{1}{x+2}}$$

notice:

$$y' = -\frac{1}{(x+2)^2} = -y^2$$

8. Find a solution to the initial-value problem

$$y' = xy^3, \;\; y(0) = 2$$

$$\frac{dy}{dx} = xy^3$$

$$\int \frac{1}{y^3}\, dy = \int x\, dx$$

$$-\frac{1}{2y^2} = \frac{1}{2}x^2 + C$$

$$\frac{1}{y^2} = -x^2 + C$$

$$y = -\frac{1}{\sqrt{-x^2+C}} \quad \text{or} \quad y = \frac{1}{\sqrt{-x^2+C}}$$

$$y(0) = -\frac{1}{\sqrt{C}} \qquad\qquad y(0) = 2 = \frac{1}{\sqrt{C}}$$

$$\sqrt{C} = \frac{1}{2} \implies C = \frac{1}{4}$$

$$\underbrace{2 = -\frac{1}{\sqrt{C}}}_{\text{no } c \text{ satifies this equation}}$$

$$y = \frac{1}{\sqrt{-x^2+\frac{1}{4}}} = \boxed{\frac{2}{\sqrt{-4x^2+1}}}$$

9. **Without using technology** sketch the following:

(a) The solid formed when the region bound by $x = \sqrt{y}$, $x = \sqrt{-y}$, and $x = 4$, is revolved around the y-axis

(b) The solid formed when the region bound by $y = e^x$, $y = e^{-x} + 4$ and the y-axis is revolved around the line $x = 4$