The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.

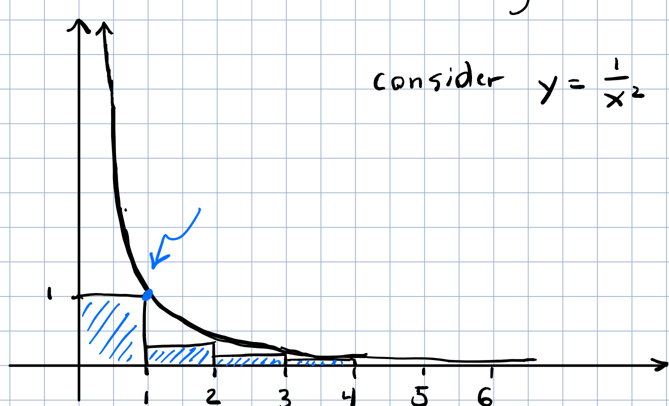
In other words:

* i) if $\int_1^{\infty} f(x)\,dx$ is convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) if $\int_1^{\infty} f(x)\,dx$ is divergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex:

Consider the series: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$



consider $y = \frac{1}{x^2}$

- we notice that our partial sums for the series must be <u>less than</u>:

$$1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 1 + (1) = 2$$

$$\lim_{t \to \infty} \int_1^{t} \frac{1}{x^2}\,dx$$

$$\lim_{t \to \infty} \left(-\frac{1}{x}\right)\Big|_1^{t} = \lim_{t \to \infty} \left(\cancel{-\frac{1}{t}}^{\,0} + 1\right)$$

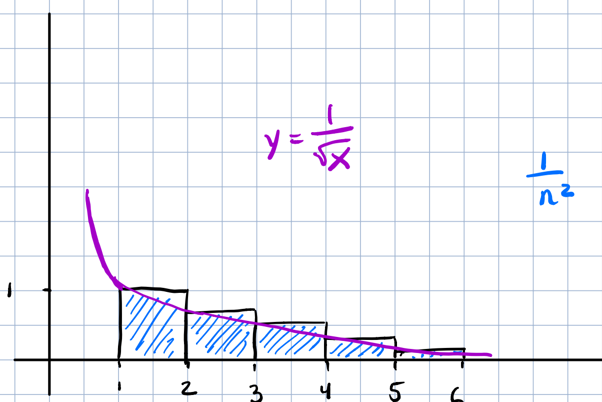
$$= 1$$

so we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots < 2$$

$\underline{Ex}$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots$

Show this is a divergent series.



notice that $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ will be less than $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, so if $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ is divergent, then we have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

$$\lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} 2\sqrt{x}\Big|_1^t$$

$$= \lim_{t\to\infty} 2\sqrt{t} - 2\sqrt{1} = \infty$$

$\underline{\text{Note}}$: When we use the integral test, it is not necessary to start the series or the integral at $n=1$.

$\underline{Ex}$:

$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$, we use $\int_4^{\infty} \frac{1}{(x-3)^2}\,dx$

Also, it is not necessary that f always be decreasing. We need f ultimately decreasing, which means decreasing for x larger then some N.

Ex: Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

first Notice $f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the integral test.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \arctan(x)\Big|_1^t$$

$$0 \le \frac{1}{n^2+1} \le \frac{1}{n^2}$$

$$= \lim_{t\to\infty} \underbrace{\arctan(t)}_{\pi/2} - \underbrace{\arctan(1)}_{\pi/4}$$

$$= \pi/2 - \pi/4 = \pi/4$$

- So, since this is a convergent integral, the series $\sum \frac{1}{(n^2+1)}$ is convergent.

- For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent.

  - if $p<0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$. If $p=0$ *, then $\boxed{\lim_{n\to\infty} \frac{1}{n^p} = 1.}$ In either case the series diverges by the n^{th} term test.

  - if $p>0$, the function is continuous, positive, and decreasing on $[1,\infty)$. From previous example

    - $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$, and diverges if $p \le 1$.

  - It follows from the <u>integral test</u> that the series $\sum \frac{1}{n^p}$ converges if $p>1$ and diverges if $0<p\le 1$.

$$\boxed{\text{– The p-series } \sum_{i=1}^{\infty} \frac{1}{n^p} \text{ is convergent if } p>1 \text{ and divergent if } p \le 1}$$

$$\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} \frac{1}{n^0} = \lim_{n\to\infty} \frac{1}{n^{1-1}}$$

$$= \lim_{n\to\infty} \frac{n}{n} = 1$$

Ex: • Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$ converges or diverges.

Solution: The function $f(x) = \frac{\ln(x)}{x}$ is positive and continuous for $x > 1$. However, it is not obvious whether or not f is decreasing... what do we do:

$$f'(x) = \frac{\frac{1}{x}x - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$$

$f'(x) < 0$ when $\ln(x) > 1$, or, $x > e$. So $f(x)$ is decreasing when $x > e$; so we may apply the integral test.

$$\int_1^{\infty} \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \left.\frac{(\ln(x))^2}{2}\right|_1^t$$

$$= \lim_{t\to\infty} \frac{(\ln(t))^2}{2} = \infty$$

• The series is divergent by the integral test.

Estimating the sum of a series.

- $\sum a_n$ is convergent, and now we want to find an approximation to the sum of the series, s.

- if we have S_n, S_n for any large n may be used as an approximation as $\lim_{n \to \infty} S_n = S$.

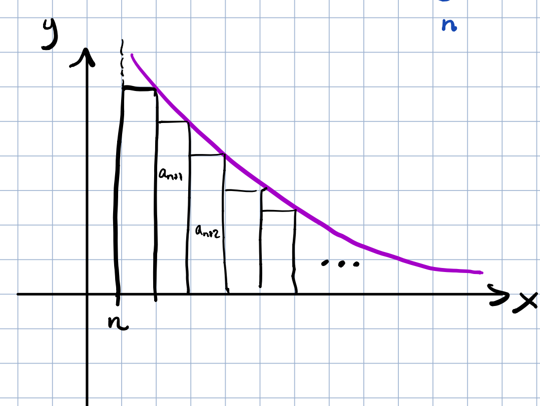
- We will now introduce remainder....

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

- The remainder R_n is the error made when s_n is used to approximate S.

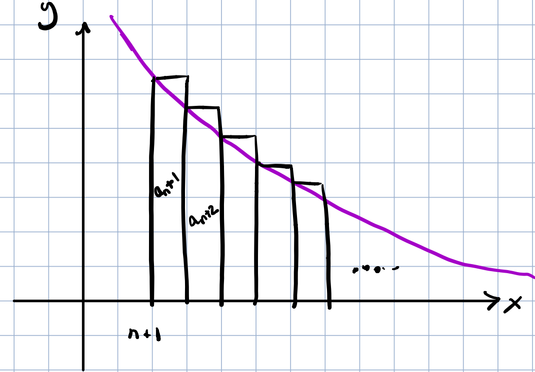
- We can use the same notation and ideas as in the integral test, assuming... f is decreasing on $[n, \infty)$.

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^\infty f(x)\,dx$$



also notice:

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx$$



Remainder Estimate for the Integral test

Suppose $f(k) = a_k$, where f is a continuous positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$ then:

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq s_n + \int_{n}^{\infty} f(x)\,dx$$