The Geometric Series.

$$\sum_{i=1}^{n} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$S = \sum_{i=1}^{n} ar^{n-1} = \frac{a}{1-r}, \quad \underbrace{|r|<1}$$

if $|r| \geq 1$, the geometric series is <u>divergent</u>.

<u>Ex:</u> <u>Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?</u>

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 3^{1-n} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

notice this is a geometric series with $a = 4$, $r = \frac{4}{3}$,

since $r = \frac{4}{3}$, the series <u>diverges</u>.

<u>Ex:</u> Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

$$\sum_{n=0}^{\infty} x^n = \underbrace{1 + x + x^2 + \cdots}$$

geometric series with $a = 1$, and $r = x$, since $|x| < 1$ we know this series <u>converges</u>.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Ex: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

$$1 = (A+B)n + A$$

$$A = 1$$
$$B = -1$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) + \cdots$$

$$S_n = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cdots + \cdots \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} 1 - \frac{1}{n+1} = 1 - (0) = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

- Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Solution: for this particular series, it's convenient to consider the partial sums $\underbrace{S_2, S_4, S_8, S_{16}, S_{32},}_{S_{2^n}}$ and show that they become large.

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) =$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{3}{2}$$

$$S_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{4}{2}$$

$$S_{32} > 1 + \frac{5}{2}$$

$$S_{64} > 1 + \frac{6}{2}$$

in general $\quad S_{2^n} > 1 + \frac{n}{2}$

notice $\lim\limits_{n \to \infty} S_{2^n} = \infty$ so $\{S_n\}$ is divergent. Therefore the harmonic series diverges.

Theorem: If the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent, then $\lim\limits_{n \to \infty} a_n = 0$

The converse of this theorem is $\boxed{\text{not true}}$ in general.

'n^{th} term test' for divergence

- If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then series $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex: Show that $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5}$$

by the n^{th} term test the series diverges.