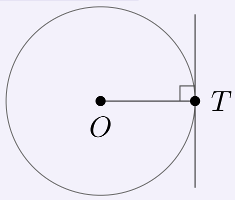
Circle Properties and Theorems

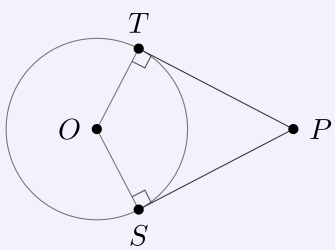
Math 10 · Mr. Merrick · February 13, 2026

1) Radius ⊥ Tangent



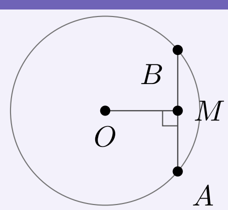
$OT \perp$ tangent at T.

2) Equal Tangents



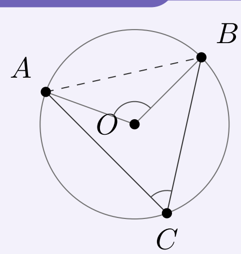
$PT = PS$.

3) ⊥ from Centre Bisects Chord



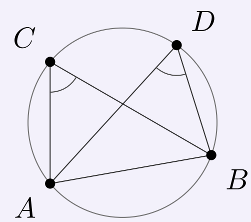
$OM \perp AB \;\Rightarrow\; AM = MB$.

4) Star Trek Theorem



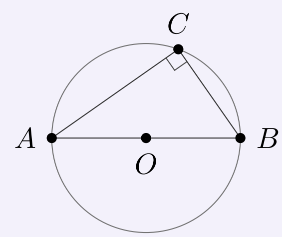
$\angle AOB = 2\angle ACB$.

5) Angles in the Same Segment



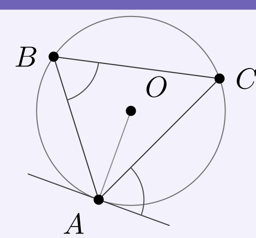
$\angle ACB = \angle ADB$.

6) Angle in a Semicircle



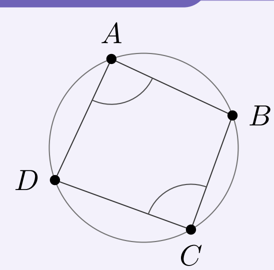
AB diameter $\Rightarrow \angle ACB = 90^\circ$.

7) Alternate Segment Theorem



$\angle(\text{tangent at } A,\, AC) = \angle ABC$.

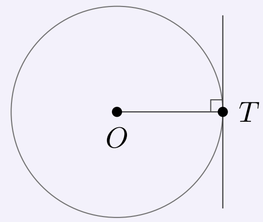
8) Cyclic Quadrilateral



$\angle A + \angle C = 180^\circ \quad (\text{and } \angle B + \angle D = 180^\circ)$.

Proof: 1) Radius $\perp$ Tangent

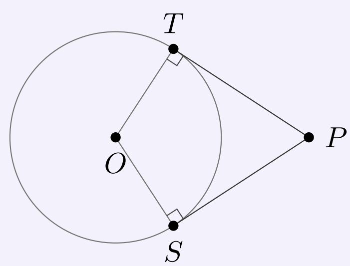
Claim. A radius is perpendicular to a tangent at the point of contact.



Proof: A tangent line intersects the circle at exactly one point (definition of tangent). Among all segments from a point to a line, the perpendicular segment has the shortest length. If OT were not perpendicular to the tangent at T, then the perpendicular from O to that tangent would meet it at a point $X \neq T$ with $OX < OT$. But then X would be closer to O than the radius length, so X would lie inside the circle. That contradicts the fact that a tangent touches the circle at exactly one point. Hence $OT \perp$ tangent at T.

Proof: 2) Equal Tangents

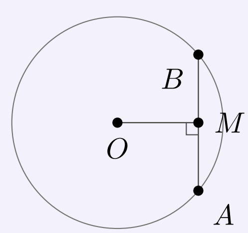
Claim. Tangents from the same external point are equal in length.



Proof. $OT \perp PT$ and $OS \perp PS$ (Theorem 1). Also $OT = OS$ (radii) and OP is common. Thus right triangles $\triangle OTP$ and $\triangle OSP$ are congruent (RHS/HL), so $PT = PS$.

Proof: 3) $\perp$ from Centre Bisects Chord

Claim. The perpendicular from the centre to a chord bisects the chord.

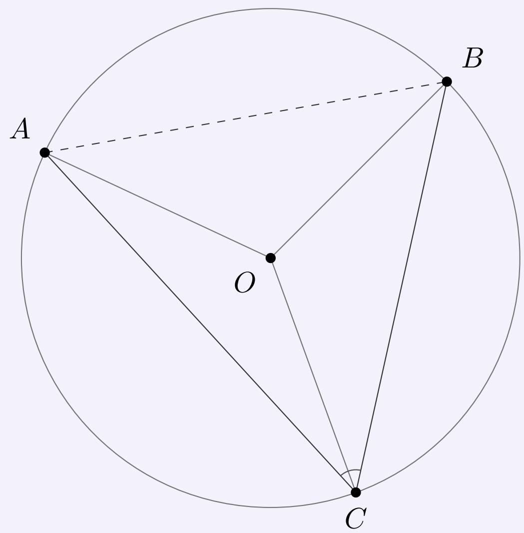


Proof. Let M be the point where the perpendicular from O meets chord AB (so $OM \perp AB$). Draw radii OA and OB. Then $OA = OB$ (radii), OM is common, and $\angle OMA = \angle OMB = 90°$. So right triangles $\triangle OMA$ and $\triangle OMB$ are congruent (RHS/HL), giving $AM = MB$.

Proof: 4) Star Trek Theorem (Angle at the Centre)

Claim. Let A, B, C be distinct points on a circle with centre O. Let $\widehat{AB}_C$ denote the arc AB *not containing* C, and let $\angle AOB_C$ denote the (possibly reflex) central angle that subtends this same arc $\widehat{AB}_C$. Then

$$\angle AOB_C = 2\angle ACB.$$



Proof

1. Draw the radii OA, OB, OC. Then $\triangle OAC$ and $\triangle OBC$ are isosceles because all radii are equal.

2. Let

$$x = \angle OCA \quad \text{and} \quad y = \angle BCO.$$

   Then the inscribed angle is $\angle ACB = x + y$.

3. In isosceles $\triangle OAC$, $\angle OAC = \angle OCA = x$, so

$$\angle AOC = 180^\circ - 2x.$$

4. In isosceles $\triangle OBC$, $\angle OBC = \angle BCO = y$, so

$$\angle COB = 180^\circ - 2y.$$

5. Consider the (non-reflex) angles $\angle AOC$ and $\angle COB$ obtained in steps 3–4. The rotation from ray OA to ray OB that *passes through* ray OC has measure $\angle AOC + \angle COB$. Therefore the rotation from ray OA to ray OB that *avoids* ray OC (i.e., the central angle subtending the arc $\widehat{AB}_C$) is

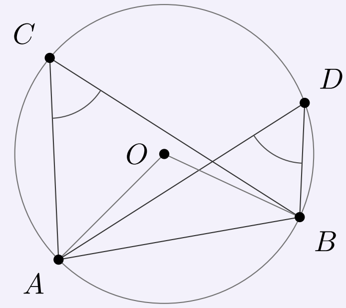
$$\angle AOB_C = 360^\circ - (\angle AOC + \angle COB).$$

6. Substitute from steps 3 and 4:

$$\angle AOB_C = 360^\circ - (180^\circ - 2x) - (180^\circ - 2y) = 2x + 2y = 2(x + y) = 2\angle ACB.$$

Proof: 5) Angles in the Same Segment

Claim. If A, B, C, D are distinct points on a circle and C and D lie on the same arc subtending chord AB (i.e., C and D are in the same segment determined by chord AB), then $\angle ACB = \angle ADB$.



Proof

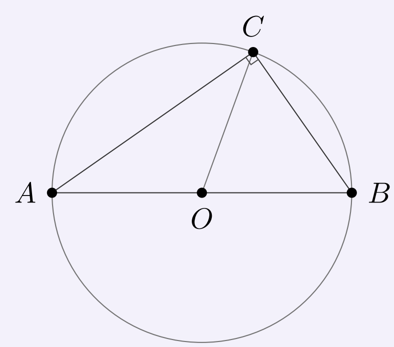
1. Because C and D are in the same segment with respect to chord AB, both angles $\angle ACB$ and $\angle ADB$ intercept the same arc $\widehat{AB}$ (the arc not containing C and also not containing D).
2. Let $\angle AOB_{\widehat{AB}}$ be the central angle subtending that same arc $\widehat{AB}$.
3. By the Star Trek theorem applied to that intercepted arc,

$$\angle AOB_{\widehat{AB}} = 2\angle ACB \quad \text{and} \quad \angle AOB_{\widehat{AB}} = 2\angle ADB.$$

4. Therefore $2\angle ACB = 2\angle ADB$, so $\angle ACB = \angle ADB$.

Proof: 6) Angle in a Semicircle

Claim. If AB is a diameter and C lies on the circle with $C \neq A, B$, then $\angle ACB = 90°$.



Proof

1. Since AB is a diameter, A, O, B are collinear, so the central angle subtending the semicircle is

$$\angle AOB = 180°.$$

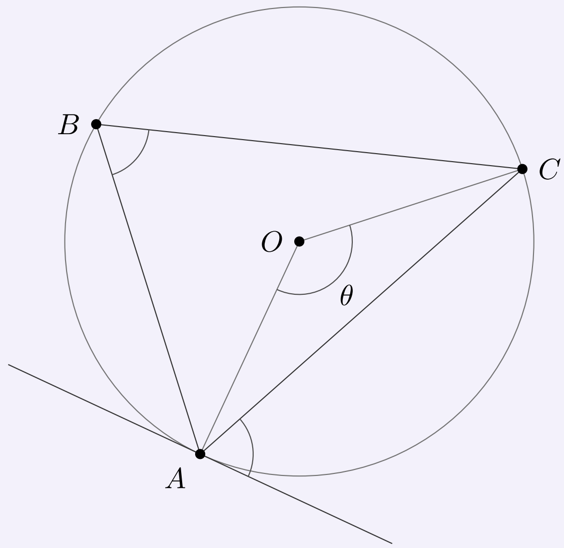
2. The inscribed angle $\angle ACB$ intercepts the semicircle arc $\widehat{AB}$, so by the Star Trek theorem,

$$\angle AOB = 2\angle ACB.$$

3. Thus $180° = 2\angle ACB$, so $\angle ACB = 90°$.

Proof: 7) Alternate Segment Theorem

Claim. The interior angle between a tangent and a chord equals the angle in the opposite segment.



Proof

1. By Theorem 1, the radius to the point of tangency is perpendicular to the tangent, so

$$OA \perp (\text{tangent at } A).$$

2. Let $\theta = \angle AOC$ be the *minor* central angle between rays OA and OC (so $0 < \theta < 180°$), subtending the minor arc $\overset{\frown}{AC}$.

3. In isosceles $\triangle AOC$ (since $OA = OC$), the base angles satisfy

$$\angle OAC = \angle ACO = \frac{180° - \theta}{2}.$$

4. The interior angle between the tangent at A and chord AC equals

$$\angle(\text{tangent at } A,\, AC) = 90° - \angle OAC = 90° - \frac{180° - \theta}{2} = \frac{\theta}{2}.$$

5. The inscribed angle $\angle ABC$ intercepts the same (minor) arc $\overset{\frown}{AC}$ as the central angle θ, so by the Star Trek theorem (Theorem 4),

$$\angle ABC = \frac{\theta}{2}.$$

6. Therefore

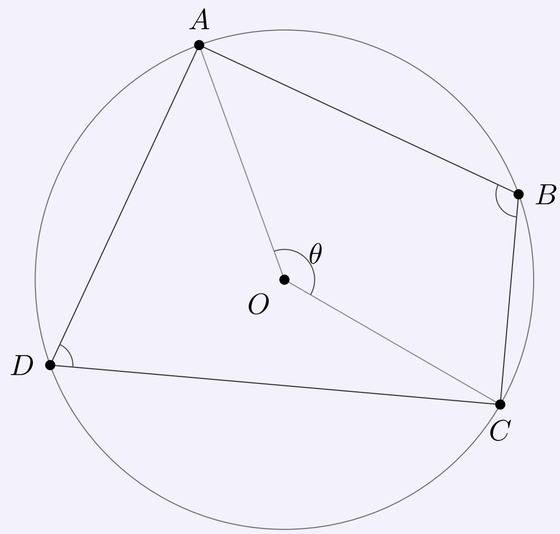
$$\angle(\text{tangent at } A,\, AC) = \angle ABC,$$

which is exactly the Alternate Segment Theorem.

Proof: 8) Cyclic Quadrilateral

Claim. If $ABCD$ is cyclic, then opposite angles are supplementary:

$$\angle A + \angle C = 180^\circ \quad \text{and} \quad \angle B + \angle D = 180^\circ.$$



Proof

1. Let $\theta = \angle AOC$, the (minor) central angle subtending the minor arc $\widehat{AC}$.
2. The inscribed angle $\angle ADC$ intercepts the minor arc $\widehat{AC}$, so by the Star Trek theorem (Theorem 4),
$$\angle ADC = \frac{\theta}{2}.$$
3. The other arc from A to C has measure $360^\circ - \theta$ (the major arc). The inscribed angle $\angle ABC$ intercepts the arc from A to C that does *not* contain B, which is the major arc in this configuration. Thus, by the Star Trek theorem,
$$\angle ABC = \frac{360^\circ - \theta}{2}.$$
4. Add the opposite angles:
$$\angle ABC + \angle ADC = \frac{360^\circ - \theta}{2} + \frac{\theta}{2} = 180^\circ.$$
5. Therefore $\angle B + \angle D = 180^\circ$.
6. Repeating the same argument with chord BD gives $\angle A + \angle C = 180^\circ$.