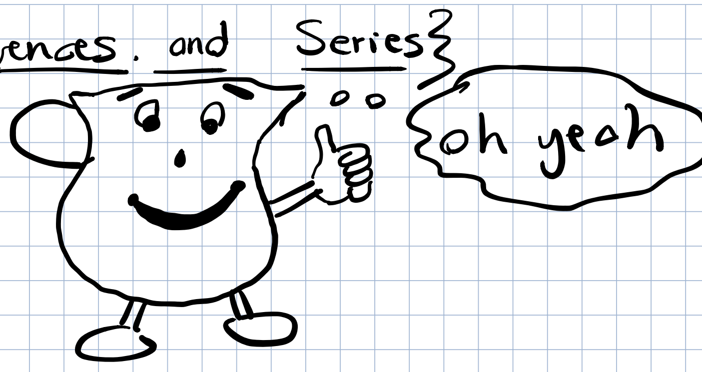
$\{$Sequences and Series$\}$



- A sequence can be thought of as a list of numbers written in a <u>definite</u> order:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

first term $\quad$ n^{th} term

- Notice that for all $n \in \mathbb{Z}^+$ there is a corresponding number a_n. We may define a sequence as a function whose domain is the set of positive integers.

- You might think $f(n)$ is a nice way to denote a sequence, however we denote a sequence a_n as so:

$\{a_1, a_2, a_3, \dots\}$ is denoted by

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

Σxample 1

* this is where Kevin said don

a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$, $a_n = \frac{n}{n+1}$, $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots\right\}$

b) $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$, $a_n = \frac{(-1)^n(n+1)}{3^n}$, $\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots\right\}$

c) $\{\sqrt{n-3}\}_{n=3}^{\infty}$, $\{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$

d) $\left\{\cos\left(\frac{n\pi}{6}\right)\right\}_{n=0}^{\infty}$, $\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\left(\frac{n\pi}{6}\right), \ldots\right\}$

Example 2: Find a formula for the general term a_n of the sequence:

$$\left\{\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

notice numerator starts at 3 and increases by 1 each term

$$(n+2)$$

notice that denominator is always a power of 5, denominator is 5^n

$$(-1)(-1)^n$$

so for ... $\left(\frac{n+2}{5^n}\right)$ $\quad (-1)^{n+1}$

we need negative even terms so we multiply by $(-1)^{n+1}$

$$a_n = \frac{(-1)^{n+1}(n+2)}{5^n}$$

Ex #3: Some sequences Do NOT have a simple defining equation.

a) Fibonacci sequence $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad n \geq 3$$

b) The sequence $\{p_n\}$ where p_n is the population of the world as of January 1^{st} in the year n.

c) Let a_n be the digit in the n^{th} decimal place of the number e, $\{a_n\}$ is a well-defined sequence whose first terms are

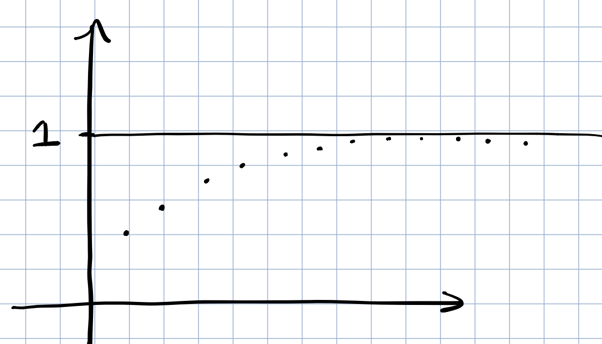
$$\{7, 1, 8, 2, 8, 1, \ldots\}$$

- Consider the sequence defined by:

$$a_n = \frac{n}{n+1}$$

- What happens to a_n as n gets large?

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$



- **Def**: A sequence has the limit L and we write:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

- If we can make the terms a_n as close to L as we like by taking n sufficiently large.
- If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence converges (or is conver nt). Otherwise, we say the sequence diverges (or is divergent).

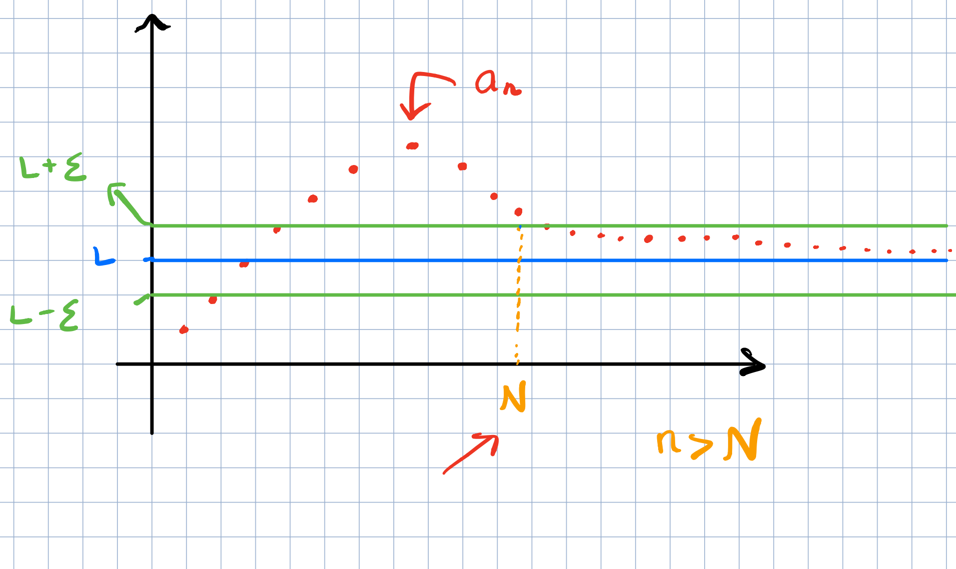
Definition (again...)

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n\to\infty$$

if for $\underset{\forall}{\text{every}}$ $\varepsilon > 0$, there is a corresponding integer N such that if

$$\underbrace{n > N} \quad \text{then} \quad \underbrace{|a_n - L| < \varepsilon}.$$



Theorem: If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ where n is an integer then: $\lim\limits_{n\to\infty} a_n = L$

Definition: $\lim\limits_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that if

$$n > N \quad \text{then} \quad a_n > M$$

Squeeze Theorem may be applied for sequences:

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \quad \lim_{n\to\infty} b_n = L$$

Theorem*: If $\lim\limits_{n\to\infty} |a_n| = 0$, then

$$\lim_{n\to\infty} a_n = 0$$

Ex: Evaluate: $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n}$ if it exists....

$(-1)^n$

Consider $\lim\limits_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

which means $\lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 0$

{Prove Theorem *}

$\lim\limits_{n\to\infty} a_n = 0$ then $\lim\limits_{n\to\infty} |a_n| = 0$

$\lim\limits_{n\to\infty} a_n = L$, then $\lim\limits_{n\to\infty} |a_n| = |L|$

<u>Theorem</u>: If $\lim\limits_{n\to\infty} a_n = L$, and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

<u>Ex</u>: Find $\lim\limits_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$

first notice the sine function is continuous at 0,

$$\lim_{n\to\infty} \sin(\pi/n) = \sin\Big(\underbrace{\lim_{n\to\infty} \frac{\pi}{n}}_{\to 0}\Big)$$

$$= \sin(0) = \boxed{0}$$

Ex: for what values of r is the sequence $\{r^n\}$ convergent.

we know that $\lim_{x\to\infty} a^x = \infty$ for all $a>1$ and $\lim_{x\to\infty} a^x = 0$ for $0<a<1$. From prior theorems we have:

$$\lim_{n\to\infty} r^n = \begin{cases} \infty, & \text{if } r>1 \\ 0, & 0<r<1 \end{cases}$$

It is clear that

$$\lim_{n\to\infty} 1^n = 1 \quad \text{and} \quad \lim_{n\to\infty} 0^n = 0$$

if $-1<r<0$, then $0<|r|<1$ so

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

Therefore $\lim_{n\to\infty} r^n = 0$ by prior theorem.

if $r \leq -1$ then $\{r^n\}$ diverges.

- The sequence $\{r^n\}$ is convergent if $-1<r\leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \end{cases}$$

Homework: Prove if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

- If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} -|a_n| = 0$, and since $-|a_n| \le a_n \le |a_n|$ we have $\lim_{n\to\infty} a_n = 0$, by squeeze theorem.

Definition: - A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \ge 1$, that is $a_1 < a_2 < a_3 < \cdots$.

- A sequence $\{a_n\}$ is called decreasing if $a_n > a_{n+1}$ for all $n \ge 1$.

- A sequence is monotonic if it is either increasing or decreasing.

Ex Show $\left\{ \frac{3}{n+5} \right\}$ increasing or $\boxed{\text{decreasing}}$.

we have

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

so $a_n > a_{n+1}$ for all $n \ge 1$

Ex: Show that $a_n = \frac{n}{n^2+1}$ is <u>decreasing</u>

we must show that $a_{n+1} < a_n$, or

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

$$(n+1)(n^2+1) < n\left[(n+1)^2+1\right]$$

$$n^3+n^2+n+1 < n^3+2n^2+2n$$

$$1 < n^2+n$$

Since $n \geq 1$, we know that $n^2+1>1$ is <u>true</u>. $\therefore$ $a_{n+1} < a_n$ and so $\{a_n\}$ is <u>decreasing</u>.

<u>Consider</u>: $f(x) = \frac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \underline{<0}$$

whenever $x^2>1$

Thus f is decreasing on $(1,\infty)$ and so $f(n) > f(n+1)$. Therefore $\{a_n\}$ is decreasing.

Definitions: • A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \le M$ for all $n \ge 1$

• It is **bounded below** if there is a number m such that $m \le a_n$ for all $n \ge 1$

• If it is bounded above, and below, then $\{a_n\}$ is a bounded sequence.

Monotonic Sequence Theorem: Every bounded monotonic sequence is convergent.

Series: If we add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form:

• $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

• which is an **infinite series** (or just series) and is denoted:

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

'Partial Sums' of a Series

in general the n^{th} partial sum of a series is denoted S_n;

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

We may treat each partial sum of a series as a sequence. $\{S_n\}_{n=1}^{\infty}$

Ex: Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is:

$$S_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{S_n\}$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}}$$

$$= \boxed{\frac{2}{3}}$$

Geometric Series

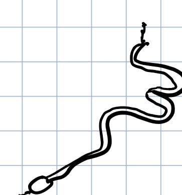
$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + ar^{n-1}$$

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + ar^n$$

$$S_n - rS_n = a - ar^n$$

3



$$S_n(1-r) = a(1-r^n)$$

$$\boxed{S_n = \frac{a(1-r^n)}{(1-r)}}$$

• for what values of r does $\{S_n\}$ converge?

- $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)}$ $\left\{ \text{left as exercize} \right\}$

The geometric series:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r|<1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r|<1$$

if $|r| \geq 1$, the series is <u>divergent</u>.