Sums and Differences for Two Random Variables

We are also interested in what happens if we take the sum of difference between two random variables X and Y. At the moment we will limit ourselves to random variables that are **independent**.

> ***Sums and Differences for Independent Random Variables***
> Let X and Y be two independent random variables.
>
> $$\begin{aligned} E(aX + bY) &= aE(Y) + bE(Y) \\ Var(aX + bY) &= a^2 Var(X) + b^2 Var(Y) \end{aligned}$$
>
> Although we know the mean and variance of $Z = aX + bY$, we don't necessarily know the distribution of Z. Proving these results requires a more thorough understanding of multivariate probability distributions. We will leave it for now.

Example 1: In a dungeons and dragons campaign you are first attacked by a swarm of 12 orcs, and then attacked by 4 mountain trolls. The chance of defeating an orc is $\frac{3}{4}$, and the chance of defeating a mountain troll is $\frac{1}{3}$. If you fail to defeat an opponent they escape into the enchanted forrest. You receive 5xp for each orc you defeat and 10 xp for each troll you defeat. In dungeons and dragons, battles are simulated by rolling dice so each battle independent.

(a) How many foes would you expect to defeat?

Here we may let O be the number of defeated Orcs, and T the number of defeated trolls. We have $O \sim \text{Binomial}\left(12, \frac{3}{4}\right)$ and $T \sim \text{Binomial}\left(4, \frac{1}{3}\right)$.

$$\begin{aligned} E(O + T) &= E(O) + E(T) \\ &= n_O p_O + n_T p_T \\ &= 12\left(\frac{3}{4}\right) + 4\left(\frac{1}{3}\right) \\ &= 10.3333 \end{aligned}$$

We would expect the defeat about then foes over the attacks.

(b) How much xp would you expect to gain?

We can now model the total xp as $5O + 10T$ and calculate expected xp for the two attacks.

$$\begin{aligned} E(5O + 10T) &= 5E(O) + 10E(T) \\ &= 5(12)\left(\frac{3}{4}\right) + 10(4)\left(\frac{1}{3}\right) \\ &= 58.3333 \end{aligned}$$

We would expect to gain about 59 xp over the two raids.

(c) Whats the probability of defeating two orcs **and** one troll in the two attacks?

Here we are looking for $P(O = 2 \cap T = 1)$.

$$\begin{aligned} P(O = 2 \cap T = 1) &= P(O = 2) \cdot P(T = 1) \\ &= \binom{12}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{10} \cdot \binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 \\ &= .00001398722 \end{aligned}$$

Pretty slim chance!

(d) If you continually faced the same two raids over and over again ad infintum, What would be the variance be for the total xp gained?

$$\begin{aligned} Var(5O + 10T) &= 25Var(O) + 100Var(T) \\ &= 25(12)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + 100(4)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \\ &= 145.1389 \end{aligned}$$

(e) What's the probability of defeating 2 foes?

This is a little bit more difficult. We must now consider the **total foes** defeated. We can create a table to show all possible values for the total defeated foes, $O + T$.

$O+T$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12	13
2	2	3	4	5	6	7	8	9	10	11	12	13	14
3	3	4	5	6	7	8	9	10	11	12	13	14	15
4	4	5	6	7	8	9	10	11	12	13	14	15	16

There are clearly three cases where exactly 2 foes are defeated. F can take on values from 0 to 16. Notice that we can also find all values for $P(O = o \cap T = t)$ using the same strategy as in part (c). We will use a more convenient notation in our table $P(O = o \cap T = t) = p(o, t)$

$P(O = o \cap T = t)$	**0**	**1**	$\cdots$	**11**	**12**	$P(O = o)$
0	$p(0,0)$	$p(1,0)$	$\cdots$	$p(11,0)$	$p(12,0)$	$p(0)$
1	$p(0,1)$	$p(1,1)$	$\cdots$	$p(11,1)$	$p(12,1)$	$p(1)$
2	$p(0,2)$	$p(1,2)$	$\cdots$	$p(11,2)$	$p(12,2)$	$p(2)$
3	$p(0,3)$	$p(1,3)$	$\cdots$	$p(11,3)$	$p(12,3)$	$p(3)$
4	$p(0,4)$	$p(1,4)$	$\cdots$	$p(11,4)$	$p(12,4)$	$p(4)$
$P(O = o)$	$p(0)$	$p(1)$	$\cdots$	$p(11)$	$p(12)$	1

We call this table the **joint distribution** of O, and T. It shows $p(o, t)$ for all values of o and t. We could also define a joint probability mass function.

$$p(o,t) = \binom{12}{o}\left(\frac{3}{4}\right)^{o}\left(\frac{1}{4}\right)^{12-o} \cdot \binom{4}{t}\left(\frac{1}{3}\right)^{t}\left(\frac{2}{3}\right)^{4-t}$$

Here the joint distribution is made up of two **marginal distributions**. The marginal distributions are the number of defeated orcs, $P(O = o)$, and defeated trolls, $P(T = t)$.

For this problem we are only interested in values where $O + T = 2$. This can happen three ways.

$$\begin{aligned} p(2,0) &= \binom{12}{2}\left(\frac{3}{4}\right)^{2}\left(\frac{1}{4}\right)^{10} \cdot \binom{4}{0}\left(\frac{1}{3}\right)^{0}\left(\frac{2}{3}\right)^{4} \\ &= 0.000006993612 \\ p(1,1) &= \binom{12}{1}\left(\frac{3}{4}\right)^{1}\left(\frac{1}{4}\right)^{11} \cdot \binom{4}{1}\left(\frac{1}{3}\right)^{1}\left(\frac{2}{3}\right)^{3} \\ &= 0.000000847711 \\ p(0,2) &= \binom{12}{0}\left(\frac{3}{4}\right)^{0}\left(\frac{1}{4}\right)^{12} \cdot \binom{4}{2}\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{2} \\ &= 0.00000001766 \end{aligned}$$

Now we can find the probability that the sum is equal to two.

$$\begin{aligned} P(\text{Defeat 2 Foes}) &= P(O = 2 \cap T = 0) + P(O = 1 \cap T = 1) + P(O = 0 \cap T = 2) \\ &= 0.000007858983 \end{aligned}$$

Your party is so legendary that there is a very small chance you only defeat 2 foes.