CMPS 101, Spring 2016: HW 1

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Q1:

- bubbleSort[A]
  i = 1 to n-1
  for j = i to n-1
  if A[j] > A[j+1]
  swap([A[j] , A[j+1]])

- You do n - i iteration of the inner loop and n - 1 iterations of the outerloop, so your total time complexity is $\Theta(n^2)$.

$$T(n) = \sum_{i=1}^{n-1}(n-i) = n(n-1) - \frac{n(n-1)}{2}$$

$$T(n) = \sum_{i=1}^{n-1}(n-i) = \frac{1}{2}n^2$$

$$\frac{1}{2}n^2 = \Theta(n^2)$$

- loop invariant: After the ith iteration of the outer loop the sub array A[n-i,...,n] is in sorted order.
  Base Case: i=0, it is trivially sorted
  Inductive Step: Set i $\geq$ 1. Assume the loop invariant is true for i-1. At the end of the i-1 iteration, A[n-(i-1),...,n] is sorted. The inner loop bubbles up the largest element from A[1,...,n-i] and puts it into its sorted position in A[n-i,...,n]. A[n-i,...,n] is now sorted. When i = n-1 A[n-(n-i),...,n], or A[1,...,n], is sorted and the Algorithm terminates.

Q2: Base Case: Consider n=1. 1 has $2^{1-1} = 1$ subset.

Hypothesis: The number of subsets of $1, 2, ..., k+1$ having an odd number of elements is $2^{k+1-1} = 2^k$.

Inductive Step: A set with $k+1$ elements will have 2^{k+1} total subsets. Half of these subsets will be odd, and half of these subsets will be even, so there are $\frac{1}{2} \cdot 2^{k+1} = 2^k$ odd subsets.

Q3:

$$\lim_{n\to\infty} \frac{a_0 + a_1 n + a_2 n^2 + \ldots + a_k n^k}{n^k} = lim_{n\to\infty} \frac{\frac{a_0}{nk} + \frac{a_1}{nk} + \frac{a_2}{nk} \ldots + a_k}{n^k}$$

$$= a_k$$

By the limit definition if $k' < k$, then the limit $\Rightarrow \infty$, therefore $f(n) \notin O(n^{k'})$.

Q4:

$$log_2 n = O(\sqrt{n}) \Rightarrow \lim_{n\to\infty} \frac{log_2 n}{\sqrt{n}} = const$$

$$\lim_{n\to\infty} \frac{log_2 n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{ln(n)}{ln(2)}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{d}{dn}\frac{ln(n)}{ln(2)}}{\frac{d}{dn}\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{2nln(2)} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}(ln(2))} = 0$$

For $log_2 n$ to equal $\Theta(\sqrt{n})$, $log_2 n$ must equal $O(\sqrt{n})$ and $\Omega(\sqrt{n})$. Since the limit $= 0$, $log_2 n$ cannot equal $\Omega(\sqrt{n})$, therefore $log_2 n$ cannot equal $\Theta(\sqrt{n})$.

Q5:

$$T(n) = \sum_{i=n-k}^{n} (i-1)$$

$$= \sum_{i=2}^{n}(i-1) - \sum_{i=2}^{n-k}(i-1)$$

$$= \frac{(n-1)n}{2} - \frac{(n-k-1)(n-k)}{2}$$

$$= \frac{n^2 - n - n^2 + kn + n + kn - k^2 - k}{2}$$

$$= \frac{2kn - k^2 - k}{2}$$

$$= O(kn)$$