Day 2 exercise (Medium)

Numerical Solution of BAE

The $su(2)$ Bethe equations

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \tag{0.1}$$

allow for complex solutions. The distribution of the Bethe roots on the complex plane could be quite complicated and curious. Mathematica allows understand them numerically.

It is convenient in some cases to work with the logarithmic version of the BAE equations

$$iL \log \frac{u_k + i/2}{u_k - i/2} + 2\pi n_k = \sum_{j\neq k}^{M} i \log \frac{u_k - u_j + i}{u_k - u_j - i} \ \ . \ \ k = 1, \ldots, M. \tag{0.2}$$

The integers n_k represent the $2\pi i$-ambiguity of the logarithm. The solution of (0.2) is uniquely specified by a set of integers n_k (at least for large L and small M). We will call them mode numbers. We will consider the case when all n_k are equal to n.

In order to find the solution numerically one has to specify starting points for the search. For generic parameters it is very difficult to give suitable starting points that lead to convergent iterations. However, it is pretty simple to find good starting points in the limit $L >> M$. In that limit we can expand

$$u_k = \frac{1}{2\pi n}\left(L + i z_k \sqrt{2L} + \mathcal{O}(1/L^0)\right) \tag{0.3}$$

1) Show that when L is large the equation (0.2) reduces to*

$$\sum_j \frac{1}{z_k - z_j} = z_k \tag{0.4}$$

2) Check that the equation is satisfied by zeroes of the Hermite polynomial $H_M(z_k) = 0$ (use $M = 12$).

Hint: Use `?*Hermit*` to find the necessary functions and `NSolve`.

3) Check that for $L = 1000$ the initial Bethe equations are almost satisfied with (0.3).

4) Use `FindRoot` and the the initial points generated ebove to solve the Bethe equations (0.1).

5) Plot the Bethe roots on the complex plane.

Hint: The option `AspectRation -> Automatic` of `ListPlot` could be useful

***6)** Make a loop with decreasing L, solving the Bethe equations on each step with the Bethe roots from the previous step as a starting point. $L = 30$ is your final goal. Try also to reach $L = 50,\ M = 20$.

Hint: For $L \sim 40$ you may need to increase the precision using options `WorkingPrecision` and `AccuracyGoal` of `FindRoot`. Use `??FindRoot` to see the complete list of options.

*We can think of this equations as the electrostatic equilibrium condition for $2D$ charges confined to live in a line.

Hint: You may want to use the function

```
Prn[a_] := (NotebookDelete[prntmp]; prntmp = PrintTemporary[a];)
```

to print the configuration of roots at each step.

***7)** In the previous task you should obtain a configuration called Bethe string where the roots are separated by i. Check how close to i the separation between the roots is.

****7)** What do you expect to get for different mode numbers n_j? Find configuration with two different mod numbers n_j.