The Coordinate Bethe Ansatz for the Heisenberg Spin Chain

Tobias Kästli

May 16, 2018

Abstract

The Heisenberg spin chain is a one dimensional model for ferro and antiferro magnetism. It has very nice properties, which allow us to solve it exactly using the coordinate Bethe ansatz first proposed by Hans Bethe in 1931. We sketch the problem in detail and start by introducing a basic solution. We interpret it as a pseudo particle and make an ansatz inspired by that. In the end we are left with equations describing the wave function, energy dispersion and momenta of these particles.

Contents

1 Introduction **2**

2 The Heisenberg Spin Chain **2**
2.1 The Hamiltonian 2
2.2 Symmetries of the Spin Chain 3

3 The One particle Sector **4**
3.1 The Vacuum Energy 4
3.2 Solving the One Particle Sector 4

4 The Bethe Ansatz for the Two Particle Sector **6**
4.1 The Coordinate Bethe Ansatz 6
4.2 Solving the Two Particle Sector 6

5 The General Bethe Ansatz **8**

6 Conclusion **11**

1 Introduction

The Heisenberg spin chain is a model for ferro and anti ferro magnetism. We imagine a one dimensional array of spins sitting on a fixed evenly spaced lattice. We are considering the case for spin 1/2 and only take into account nearest neighbour interaction. In addition we want the chain to be periodic, meaning we bring its edges together thus forming a ring.

This model was proposed by Werner Heisenberg [1], and Hans Bethe [2] in 1931 found a way of calculating its eigenstates and eigenvalues exactly using a method now called coordinate Bethe ansatz. It was found out later that it works because the model is integrable. As the name suggests the method consists of making an educated guess about the form of the wave function. This guess can then be plugged in to the equations to find its parameters. In this report we are going to formulate the coordinate Bethe ansatz and use it on a spin chain.

In Section 2 we describe the model, its Hamiltonian and its symmetries in detail. These symmetries require that the Hamiltonian takes block diagonal form. We use this fact to construct the eigenstates step by step. In order to understand the problem better Section 3 shows the solution of the two smallest and therefore simplest blocks. This is done without using the Bethe ansatz. Section 4 then introduces the coordinate Bethe ansatz for a more difficult case. After that we present the general form of the ansatz in Section 5.

2 The Heisenberg Spin Chain

2.1 The Hamiltonian

As sketched in the introduction we consider the Heisenberg XXZ-chain. As can be seen in Figure 1, this is a cyclic chain of L spins, which sit on an equally spaced one dimensional lattice. We choose the spacing to be equal to one.

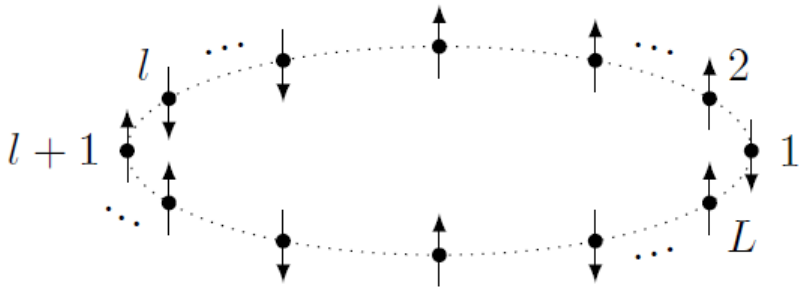

Figure 1: Schematic of the Heisenberg spin chain. From Lamers [6].

All those spins live in a two dimensional Hilbert space $V_l \simeq \mathbb{C}^2$. The state of the whole chain is therefore an element of the tensor products of all those spin spaces. This is a 2^L dimensional complex vector space.

$$\mathcal{H} = \bigotimes_{l \in \mathbb{Z}_L} V_l \simeq \bigotimes_{l \in \mathbb{Z}_L} \mathbb{C}^2 \simeq \mathbb{C}^{2^L} \tag{1}$$

Here $\mathbb{Z}_L := \mathbb{Z}/L\mathbb{Z}$ is the cyclic group of length L. We call the spin operator of these spaces $\mathbf{S} = (S^x, S^y, S^z)$. It consists of Pauli matrices. We generalise those for a use on $\mathcal{H}$ by tensoring them with identities $\mathbb{I}_2$.

$$\mathbf{S}_l := \underbrace{\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{l-1 \text{ times}} \otimes \mathbf{S} \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{L-l-1 \text{ times}} \tag{2}$$

Similarly we define the spin flip operators $S_l^{\pm} := S_l^x \pm iS_l^y$, which are ladder operators for the S^z eigenbasis of V_l. We can write them as matrices in this basis:

$$S_l^+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_l^- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad S_l^z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

With these notations we can formulate the Hamiltonian of the Heisenberg XXZ-chain and rewrite it in a convenient way using ladder operators.

$$H_{XXZ} = -J \sum_{l \in \mathbb{Z}_L} (S_l^x S_{l+1}^x + S_l^y S_{l+1}^y + \Delta S_l^z S_{l+1}^z) \tag{4}$$

$$= -\frac{J}{2} \sum_{l \in \mathbb{Z}_L} (S_l^+ S_{l+1}^- + S_l^- S_{l+1}^+ + 2\Delta S_l^z S_{l+1}^z) \tag{5}$$

Here Δ is an anisotropy factor. If it is set to one, we get the so called XXX-chain. We will however keep it for the sake of generality. The overall factor J is a coupling parameter. It is sometimes referred to as exchange integral [3]. The case where $J > 0$ is called ferromagnetic, if $J < 0$ anti ferromagnetic. For now we stick with the ferromagnetic case. It is however possible to treat the anti ferromagnetic Hamiltonian in a similar way (see [5]).

2.2 Symmetries of the Spin Chain

The most obvious symmetry of our system is its translational invariance. We can apply the shift operator L times and get the same system back. This fact will help us to construct some simple solutions of our model (see Section 3).

Our Hamiltonian is also invariant under rotations around the z-axis. This can easily be seen by looking at the total spin operator in z-direction: $S^z := \sum_{l \in \mathbb{Z}_L} S_l^z$. One can check that S^z commutes with the Hamiltonian. Since S^z is the generator of rotations around the z-axis, such rotations are also symmetries of the spin chain. We can construct eigenstates of S^z by starting from the state, in which all spins point up, and then flip single spins using the spin flip operators S_l^- from Section 2.1.

$$|\Omega\rangle := |\uparrow \cdots \uparrow\rangle \tag{6}$$

$$|l_1 \cdots l_M\rangle := \hbar^{-M} S_{l_1}^- \cdots S_{l_M}^- |\Omega\rangle \qquad 1 \leq l_1 < \cdots < l_M \leq L \tag{7}$$

The $|\Omega\rangle$ in (6) is called pseudo vacuum and the basis we get from (7) is called coordinate basis. As said these basis elements are eigenstates of S^z and since $[S^z, H_{XXZ}] = 0$ the Hamiltonian takes block diagonal form in this basis. This is shown in Figure 2. The M-th of these blocks is called M-particle sector and it has a size of $\binom{L}{M} \times \binom{L}{M}$. We can now solve our Schrödinger equation for every sector separately.

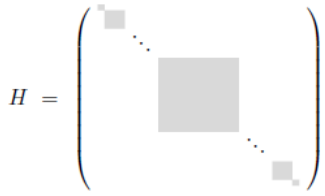

Figure 2: If we choose the coordinate basis from (7) and sort the basis kets by there total spin in z-direction, our Hamiltonian takes this block diagonal form. From Lamers [6].

3 The One particle Sector

3.1 The Vacuum Energy

From now on we choose our units such that $J = \hbar = 1$. We start solving the smaller sectors since they are easier. The uppermost one has only size 1×1 and has our pseudo vacuum $|\Omega\rangle$ as eigenstate. As warm up we calculate its energy. The action of the S_l^+ and S_l^z operators on $|\Omega\rangle$ are simple:

$$S_l^+ |\Omega\rangle = 0 \qquad S_l^z |\Omega\rangle = \frac{\hbar}{2} |\Omega\rangle \qquad \forall l \in \mathbb{Z}_L \tag{8}$$

The two spin flipping terms in our Hamiltonian vanish therefore and we are left with the L S_l^z terms.

$$H_{XXZ} |\Omega\rangle = -\frac{1}{2} \sum_{l \in \mathbb{Z}_L} \left(S_l^+ S_{l+1}^- + S_l^- S_{l+1}^+ + 2\Delta S_l^z S_{l+1}^z \right) |\Omega\rangle = - \sum_{l \in \mathbb{Z}_L} \Delta S_l^z S_{l+1}^z |\Omega\rangle \tag{9}$$

$$= - \sum_{l \in \mathbb{Z}_L} \Delta \frac{1}{4} |\Omega\rangle = \frac{\Delta L}{4} |\Omega\rangle = E_0 |\Omega\rangle \tag{10}$$

So our vacuum energy is $E_0 = \frac{\Delta L}{4}$. In the later chapters we are going to set it to zero. This will not change our physics since there is no fixed zero point for the energy.

3.2 Solving the One Particle Sector

The second smallest sector is the one particle sector. It has size $L \times L$ and its eigenstates can be built out of basis elements with only one down spin. As seen in (7) we write these basis states as $|l\rangle$ for a $l \in \mathbb{Z}_L$. In order to solve the one particle sector we can make use of the translational symmetry of our system (see Section 2.2).

The unitary translation operator U shifts all lattice points of our chain one step to the right i.e. the state $|l\rangle$ goes to $|l+1\rangle$ under U. The generator of U is called momentum. Since this is a symmetry transformation the eigenstates of the Hamiltonian $|\Psi_1; p\rangle$ also need to be eigenstates of U for a momentum $-p$, where p is real.

$$U |\Psi_1; p\rangle = e^{-ip} |\Psi_1; p\rangle \tag{11}$$

We now write our Hamiltonian eigenstate $|\Psi_1, p\rangle$ in terms of our coordinate basis elements $|l\rangle$. We want to find the wave functions $\psi_p(l)$.

$$|\Psi_1; p\rangle = \sum_{l \in \mathbb{Z}_L} \psi_p(l) |l\rangle \tag{12}$$

This can be done using (11). The $(l+1)$-th coefficient $\psi_p(l+1)$ is simply given by the projection of $|\Psi_1; p\rangle$ onto $|l+1\rangle$. We can then express $|l+1\rangle$ as U acting on $|l\rangle$. Using the fact that the eigenstates of H_{XXZ} are also eigenstates of U, this gives us a recursion relation for ψ_p.

$$\psi_p(l+1) = \langle l+1 | \Psi_1; p\rangle = \langle l | U^\dagger |\Psi_1; p\rangle = e^{ip} \langle l | \Psi_1; p\rangle = e^{ip} \psi_p(l) \tag{13}$$

This recursion is solved by a plane wave:

$$|\Psi_1; p\rangle = \frac{1}{\sqrt{L}} \sum_{l \in \mathbb{Z}_L} e^{ipl} |l\rangle \tag{14}$$

These are the eigenstates of H_{XXZ} in the one particle sector. We call these solutions magnons and treat them as pseudo particles. We will later see that this approach is useful. It also explains why we call this matrix block one particle sector.

As next step we need to take care of our boundary conditions. We can see in (14) that there are in principle infinitely many solutions, since we can choose an infinite amount of different momenta. Our periodic boundary conditions will confine the solution to a finite set of allowed momenta.

Our boundary conditions tell us, that the eigenstate shifted by L around the chain should again be the same state. Formally this means:

$$U^L = e^{-ipL} = id \tag{15}$$

$$\Rightarrow p = \frac{2\pi}{L} \qquad \text{for } n \in \mathbb{Z}_L \tag{16}$$

This quantisation of p leaves us with L allowed solutions, which is exactly enough for our one particle sector of size $L \times L$.

We also want to know the energy eigenvalues. We can now compute them by acting with H_{XXZ} on our solution (14). To this end we first look at its action on a general basis state $|l'\rangle$.

$$H_{XXZ}\left|l'\right\rangle = -\frac{1}{2}\sum_{l\in\mathbb{Z}_L}\left(S_l^+S_{l+1}^- + S_l^-S_{l+1}^+ + 2\Delta S_l^zS_{l+1}^z\right)\left|l'\right\rangle \tag{17}$$

$$= -\frac{1}{2}\left(\left|l'+1\right\rangle + \left|l'-1\right\rangle + \frac{\Delta}{2}\left((L-2)-2\right)\left|l'\right\rangle\right) \tag{18}$$

$$= -\frac{1}{2}\left(\left|l'+1\right\rangle + \left|l'-1\right\rangle + \frac{\Delta}{2}\left(L-4\right)\left|l'\right\rangle\right) \tag{19}$$

Using this calculating the energy of a magnon becomes quite simple if we make a change of summation:

$$H_{XXZ}\left|\Psi_1;p\right\rangle = \frac{1}{\sqrt{L}}\sum_{l\in\mathbb{Z}_L}e^{ipl}H_{XXZ}\left|l\right\rangle \tag{20}$$

$$= -\frac{1}{\sqrt{L}}\sum_{l\in\mathbb{Z}_L}e^{ipl}\frac{1}{2}\left(\left|l+1\right\rangle + \left|l-1\right\rangle + \frac{\Delta}{2}\left(L-4\right)\left|l\right\rangle\right) \tag{21}$$

$$= -\frac{1}{2\sqrt{L}}\left(\sum_{l\in\mathbb{Z}_L}e^{ipl}\left|l+1\right\rangle + \sum_{l\in\mathbb{Z}_L}e^{ipl}\left|l-1\right\rangle + \frac{\Delta}{2}(L-4)\sum_{l\in\mathbb{Z}_L}e^{ipl}\left|l\right\rangle\right) \tag{22}$$

$$= -\frac{1}{2}\left(e^{-ip} + e^{ip} + \frac{\Delta(L-4)}{2}\right)\frac{1}{\sqrt{L}}\sum_{l\in\mathbb{Z}_L}e^{ipl}\left|l\right\rangle \tag{23}$$

$$= -\cos(p) - \frac{\Delta(L-4)}{4}\left|\Psi_1;p\right\rangle. \tag{24}$$

As already said we are now setting the vacuum energy E_0 to zero, meaning we subtract it from our magnon energy E_1.

$$\epsilon_1 = E_1 - E_0 = -\cos(p) - \frac{\Delta(L-4)}{4} + \frac{\Delta L}{4} = \Delta - \cos(p) \tag{25}$$

With all these informations we know now everything about the one particle sector. From here we go on to the two particle sector. We are going to solve it using an ansatz inspired by the one particle solution

4 The Bethe Ansatz for the Two Particle Sector

The two particle sector cannot be solved simply by exploiting symmetries. This is why we are going to need the coordinate Bethe ansatz. It gives us a guess for the structure of our wave function. We are going to see later that it works. This is because our system is integrable.

First we generalise some things we have done in Section 3.2. Our goal is to find eigenstates of our Hamiltonian expressed in the coordinate basis. Since H_{XXZ} takes block diagonal form (see Section 2.2), we do not need all coordinate basis elements for this expansion. For a eigenstate of the M-particle sector it is enough to use those with M spins pointing down.

$$|\Psi_M; p_1 \cdots p_M\rangle = \sum_{1 \le l_1 < \cdots < l_M \le L} \psi_{p_1 \cdots p_M}(l_1 \cdots l_M)\, |l_1 \cdots l_M\rangle \tag{26}$$

The p_i are going to be related to our ansatz. Their sum is the momentum of the solution. Later we are going to write the l_i and p_i as elements of vectors of length M: $\mathbf{l}$ and $\mathbf{p}$. The $\psi_{\mathbf{p}}(\mathbf{l})$ are characterised by the projection of the Schrödinger equation onto $|\mathbf{l}\rangle$.

$$\langle \mathbf{l}|\, H_{XXZ}\, |\Psi_M; \mathbf{p}\rangle = E_M(\mathbf{p})\, \psi_{\mathbf{p}}(\mathbf{l}) \tag{27}$$

In this equation $E_M(\mathbf{p})$ is the energy of the state.

4.1 The Coordinate Bethe Ansatz

Now the coordinate Bethe ansatz comes into play. Similar to the one particle solution we anticipate our $\psi_{p_1,p_2}(l_1, l_2)$ to be the sum of two plane wave factors, the first can be seen as a superposition of two free magnons, the second as a scattering state, where they have exchanged momentum.

$$\psi_{p_1,p_2}(l_1, l_2) = A(p_1, p_2)\, e^{i(p_1 l_1 + p_2 l_2)} + A'(p_1, p_2)\, e^{i(p_1 l_2 + p_2 l_1)} \tag{28}$$

We now need to identify the functions A and A' as well as $E_M(\mathbf{p})$ using the Schrödinger equation (27). After that we need to enforce periodic boundary conditions, which will give us quantisation conditions on our momenta p_1 and p_2.

It will later turn out that it is useful to make a slight change of notation in (28). Specifically we are dividing the ansatz by A and define the two body scattering matrix $S(p_1, p_2)$:

$$\psi_{p_1,p_2}(l_1, l_2) \propto e^{i(p_1 l_1 + p_2 l_2)} + S(p_1, p_2) e^{i(p_1 l_2 + p_2 l_1)}, \tag{29}$$

$$\text{with } S(p_1, p_2) = \frac{A'(p_1, p_2)}{A(p_1, p_2)}. \tag{30}$$

In (29) the $=$ has changed to a $\propto$ since we have divided by A. We can however fix this again using normalisation. Also notice that the two body scattering matrix S is despite its name simply a scalar. It can be thought of as the ratio between the scattered and the transmitted wave and assuming the only scattering process is that of simply interchanging momenta, which we implicitly assumed in our ansatz, it tells us everything there is to know about the scattering of two magnons.

4.2 Solving the Two Particle Sector

We now show how one can find the required parameters $E_2(p_1, p_2)$, $S(p_1, p_2)$ and p_1, p_2. If we find these, we can plug them into our ansatz and see if it works. We are going to start with $E_2(p_1, p_2)$.

The dispersion relation can be found considering $\psi_{p_1,p_2}(l_1,l_2)$ for $|l_1-l_2|>1$ and plugging it into our Schrödinger equation (27). This means that the excitations of our vacuum are not nearest neighbours. Since the Hamiltonian only describes interactions between such, it is not surprising that the result is more or less the same than for one excitation.

$$\sum_k S_k^{\pm} S_{k+1}^{\mp} |l_1,l_2\rangle = |l_1 \pm 1, l_2\rangle + |l_1, l_2 \pm 1\rangle \tag{31}$$

$$\sum_k S_k^z S_{k+1}^z |l_1,l_2\rangle = \frac{L-8}{4} |l_1,l_2\rangle \tag{32}$$

We now plug these into (27) and subtract the vacuum energy. This gives us a condition containing ψ and $\varepsilon_2 := E_2 - E_0$.

$$\begin{aligned} 2\,\varepsilon_2(\mathbf{p})\,\psi_{\mathbf{p}}(l_1,l_2) = 4\,\Delta\,\psi_{\mathbf{p}}(l_1,l_2) - \psi_{\mathbf{p}}(l_1-1,l_2) - \psi_{\mathbf{p}}(l_1+1,l_2) \\ -\psi_{\mathbf{p}}(l_1,l_2-1) - \psi_{\mathbf{p}}(l_1,l_2+1) \end{aligned} \tag{33}$$

Note that the terms containing $l_{1,2} \pm 1$ correspond to our change of summation in (25). We insert the coordinate Bethe ansatz (28) and find that the condition is satisfied if we choose

$$\varepsilon_2(p_1,p_2) = 2\,\Delta - \cos(p_1) - \cos(p_2) = \varepsilon_1(p_1) + \varepsilon_1(p_2). \tag{34}$$

This justifies our treatment of magnons as quasi particles since their energies add up. One has however to be careful to remember that we have said nothing about the allowed momenta. They are going to be different from the one particle case, which is why the energy is not simply the sum of two free magnons.

Next we are looking for the scattering matrix S. To this end we now consider $\psi_{p_1,p_2}(l_1,l_2)$ in the case, where $l_2 = l_1 + 1$. This now changes the action of our Hamiltonian considerably.

$$\sum_k S_k^+ S_{k+1}^- |l_1, l_1+1\rangle = |l_1, l_1+2\rangle \tag{35}$$

$$\sum_k S_k^- S_{k+1}^+ |l_1, l_1+1\rangle = |l_1-1, l_1+1\rangle \tag{36}$$

$$\sum_k S_k^z S_{k+1}^z |l_1, l_1+1\rangle = \frac{L-4}{4} |l_1, l_1+1\rangle \tag{37}$$

Again inserting this into (27) we get another condition for ψ:

$$2\,\varepsilon_2(\mathbf{p})\,\psi_{\mathbf{p}}(l_1,l_1+1) = 2\,\Delta\,\psi_{\mathbf{p}}(l_1,l_1+1) - \psi_{\mathbf{p}}(l_1-1,l_1+1) - \psi_{\mathbf{p}}(l_1,l_1+2). \tag{38}$$

Since we have already fixed ε_2, this constraint allows us to get the two body scattering matrix S. The easiest way to do this is using (33) again. One can easily check that with our choice of ε_2 it also holds for neighbouring spins. We can therefore subtract (33) from (38).

$$2\,\varepsilon_2(\mathbf{p})\,\psi_{\mathbf{p}}(l_1,l_1+1) = \psi_{\mathbf{p}}(l_1,l_1) + \psi_{\mathbf{p}}(l_1+1,l_1+1) \tag{39}$$

It is important to realise that $\psi_{\mathbf{p}}(l_1,l_1)$ does not have any physical meaning because it is not possible for two down spins to lie at the same position. So the statement above is purely mathematical. Inserting the Bethe ansatz (28) we can solve it for $S = A'/A$.

$$S(p_1,p_2) = -\frac{1 - 2\,\Delta\,e^{ip_2} + e^{i(p_1+p_2)}}{1 - 2\,\Delta\,e^{ip_1} + e^{i(p_1+p_2)}} \tag{40}$$

Finally all this knowledge enables us to enforce periodic boundary conditions and thus get the allowed values for the momenta p_1 and p_2. Again these conditions require that moving the problem once around the chain should not change anything. Mathematically this means

$$\psi_{p_1,p_2}(l_2, l_1 + L) = \psi_{p_1,p_2}(l_1, l_2). \tag{41}$$

As before we plug in the Bethe ansatz (28). A comparison of coefficients in the resulting equation gives us a simple condition.

$$e^{ip_1 L} = S(p_1, p_2) \quad \text{and} \quad e^{ip_2 L} = S(p_1, p_2)^{-1} \tag{42}$$

Since the two body scattering matrix from (40) is unitary, it can be written as exponential of a real scattering phase Θ.

$$S(p_1, p_2) = e^{-i\Theta(p_1,p_2)} \tag{43}$$

We can therefore solve (42) formally by taking a logarithm. This gives us two new quantum numbers constraining the momenta. We call them Bethe quantum numbers λ_1 and λ_2. They are both integers between 1 and L.

$$Lp_1 = 2\pi\lambda_1 - \Theta(p_1, p_2) \quad \text{and} \quad Lp_2 = 2\pi\lambda_2 + \Theta(p_1, p_2) \qquad \lambda_1, \lambda_2 \in \mathbb{Z}_L \tag{44}$$

Theoretically we have now everything we need to solve this last equation, get the values for p_1 and p_2 and plug everything into our ansatz. However (44) is not easy to solve analytically. One usually uses computers. Karbach and Müller [4] did this for the case $\Delta = 1$. They also showed that we can find all our eigenstates using the coordinate Bethe ansatz.

5 The General Bethe Ansatz

We have now seen how the coordinate Bethe ansatz works for the two particle sector. The next and final thing we are going to do now is generalising our approach for an arbitrary number of magnons. To this end we first need a new ansatz.

$$\psi_{\mathbf{p}}(\mathbf{l}) = \sum_{\pi \in S_M} A_\pi(\mathbf{p})\, e^{i\mathbf{p}_\pi \cdot \mathbf{l}} \qquad \text{with } \mathbf{p}_\pi \cdot \mathbf{l} := \sum_{j=1}^{M} p_{\pi(j)} l_j. \tag{45}$$

Like in (28) we are implicitly assuming here that scattering can only exchange the momenta of two magnons. Under this assumption, ψ is the sum over all possible scattering outcomes. For $M = 1$ the ansatz reduces to (14) and for $M = 2$ to (28).

The things we need to find now are the same as before: The rescaled energy $\varepsilon_M(\mathbf{p})$, $A_\pi(\mathbf{p})$ and the allowed momenta $\mathbf{p}$. These need to satisfy the Schrödinger equation (27) and the periodic boundary conditions. Let us again start with the energy.

As in the two particle case we can find the energy considering the well separated case, where no two down spins lie next to each other. We plug this condition into (27) and get the following relation:

$$2\varepsilon_M(\mathbf{p})\psi_{\mathbf{p}}(\mathbf{l}) = 2M\Delta\psi_{\mathbf{p}}(\mathbf{l}) + \sum_{\mathbf{k}} \psi_{\mathbf{p}}(\mathbf{k}), \tag{46}$$

where $\mathbf{k}$ runs over all $2M$ configurations obtained by moving one spin in $\mathbf{l}$ to the left or to the right once. The condition is similar as for two particles in (33). This is not surprising, given that under our

Hamiltonian the separated spins do not see each other at all. The similarity also calls for a similar solution, which can be found to be

$$\varepsilon_M(\mathbf{p}) = M\Delta - \sum_{m=1}^{M} \cos(p_m) = \sum_{m=1}^{M} \varepsilon_1(p_m) \tag{47}$$

This again justifies our treatment of magnons as pseudo particles.

A more challenging problem is to find A_π. We consider now a general $\mathbf{l}$ with N neighbouring down spins. In equation (46) the sum changes and now runs over only $2(M-N)$ possibilities of moving an excited spin to the left or right without hitting a neighbouring down spin.

$$2\varepsilon_M(\mathbf{p})\psi_{\mathbf{p}}(\mathbf{l}) = 2(M-N)\Delta\psi_{\mathbf{p}}(\mathbf{l}) + {\sum_{\mathbf{k}}}' \psi_{\mathbf{p}}(\mathbf{k}) \tag{48}$$

As for two particles in (38) the equation (46) formally also holds for neighbouring spins given our choice of ε_M. We can hence subtract the two and get the condition

$$2N\Delta\psi_{\mathbf{p}}(\mathbf{l}) = {\sum_{\mathbf{k}}}'' \psi_{\mathbf{p}}(\mathbf{k}), \tag{49}$$

where the sum now runs over all $2N$ possibilities of moving two neighbouring down spins on top of each other. If we for example consider only two neighbouring excitations ($N = 1$), the equation looks like:

$$2\Delta\psi_{\mathbf{p}}(1_1, \cdots, l_n, l_n+1, \cdots, l_M) = \psi_{\mathbf{p}}(l_1, \cdots, l_n, l_n, \cdots, l_M) + \psi_{\mathbf{p}}(l_1, \cdots, l_n+1, l_n+1, \cdots, l_M). \tag{50}$$

This reminds us of (39) and we can see, that (49) is just a sum of N of these equations. If we manage to solve all these equations separately, they will also satisfy (39). Therefore we are going to look at equation (50).

Plugging in the ansatz (45) and bringing everything to the left we get an equation for A_π. We use the notation $\mathbf{k} = (1_1, \cdots, l_n, l_n, \cdots, l_M)$.

$$\sum_{\pi \in S_M} s(p_{\pi(n)}, p_{\pi(n+1)})\, A_\pi(\mathbf{p})\, e^{i\mathbf{p}_\pi \cdot \mathbf{k}} = 0, \qquad \text{with } s(p_1, p_2) := 1 - 2\Delta e^{ip_1} + e^{i(p_1+p_2)} \tag{51}$$

We want to compare coefficients and notice, that the $e^{i\mathbf{p}_\pi \cdot \mathbf{k}}$ are not all independent. Since $k_n = k_{n+1}$, there are pairs of equal exponentials. Namely $e^{i\mathbf{p}_\pi \cdot \mathbf{k}} = e^{i\mathbf{p}_{\pi'} \cdot \mathbf{k}}$, if $\pi' = \pi \circ (n, n+1)$. All other exponentials should be independent so our equation links two of them.

$$\frac{A_{\pi'}(\mathbf{p})}{A_\pi(\mathbf{p})} = -\frac{s(p_{\pi(n)}, p_{\pi(n+1)})}{s(p_{\pi(n+1)}, p_{\pi(n)})} = S(p_{\pi(n)}, p_{\pi(n+1)}) \tag{52}$$

Here S is the two body scattering matrix from equation (40). We can now write every permutation $\pi \in S_M$ as product of transpositions of neighbours. By iteratively applying (52) the ratio between every A_π and A_{id} becomes a product of scattering matrices. A_{id} could be computed using normalisation.

$$\frac{A_\pi(\mathbf{p})}{A_{id}(\mathbf{p})} = \operatorname{sgn}(\pi) \prod_{1 \le m < m' \le M} \frac{s(p_{\pi(m')}, p_{\pi(m)})}{s(p_{m'}, p_m)} = \prod_{\substack{1 \le m < m' \le M \\ s.t.\ \pi(m) > \pi(m')}} S(p_m, p_{m'}) \tag{53}$$

This shows us that our problem is special. The A_π, which tell us about the probability of different scattering outcomes, consist of multiple application of S, which describes two body scattering. This means that all possible scattering processes consist of many two body scattering processes or that magnons only scatter in pairs. This property is why the model is called two body reducible.

The last thing left to do is finding the allowed momenta. This is done using the periodic boundary conditions. Again we require $\psi_{\mathbf{p}}$ to be the same, when l_1 is shifted L times.

$$\psi_{\mathbf{p}}(l_1, \cdots, l_M) = \psi_{\mathbf{p}}(l_2, \cdots, l_M, l_1 + L) \qquad 1 \leq l_1 < \cdots < l_M \leq L \tag{54}$$

The Bethe wave function on the right hand side can be expressed using the cyclical permutation $\sigma = (12 \cdots M) \in S_M$:

$$\psi_{\mathbf{p}}(l_2, \cdots, l_M, l_1 + L) = \sum_{\pi'} A_{\pi'}(\mathbf{p}) e^{ip_{\pi'(1)}L} e^{i\mathbf{p}_{\pi'}\cdot\mathbf{l}} \qquad \text{with } \pi' = \pi \circ \sigma^{-1}. \tag{55}$$

We equate coefficients in (54) and obtain $M!$ Bethe ansatz equations.

$$e^{ip_{\pi(1)}L} = \frac{A_\pi(\mathbf{p})}{A_{\pi\sigma}(\mathbf{p})} \qquad \text{with } \pi \in S_M \tag{56}$$

We have already found an expression for the A_π in (53). We can now plug this in. There is going to be a massive cancelling out of the factors with $2 \leq m < m' \leq M$ in the nominator and the factors with $1 \leq m < m' \leq M-1$ in the denominator. We are left with the remaining product and the signum of the cyclic permutation, which can be rewritten as a product of two body scattering matrices.

$$e^{ip_{\pi(1)}L} = (-1)^{M-1} \prod_{j=2}^{M} \frac{s(p_{\pi(j)}, p_{\pi(1)})}{s(p_{\pi(1)}, p_{\pi(j)})} = (-1)^{M-1} \prod_{j=2}^{M} -S(p_{\pi(j)}, p_{\pi(1)}) \tag{57}$$

We are now interested in p_m so we choose a π with $\pi(1) = m$. Since S is a scalar, it does not matter in which order we multiply the different S. In other words, the result for π and π' will be the same as long as $m = \pi(1) = \pi'(1)$. We also notice that the product consists of $M-1$ factors and the -1 therefore vanishes. Changing notation to $\pi(j) = n$ and choosing to multiply the terms with smaller n first we get the final equations for our momentum.

$$e^{ip_m L} = \prod_{\substack{n=1 \\ n \neq m}}^{M} S(p_n, p_m) \tag{58}$$

This can be physically interpreted as the magnon acquiring a phase due to scattering with all the other magnons when it is moved once around the circle.

This is the last information we needed. We have now transformed the problem of diagonalising the M-particle sector of the Hamiltonian in to solving the equations (58) and after that (53). These are highly coupled systems of equations and hence very hard to solve. They are however much less sensitive to the size of the chain and the number of excitations than the brute force method. Therefore they can be more efficiently tackled using numerical approximations and computers. One can even consider the thermodynamic limit $L \to \infty$ analytically and study thermodynamic properties of the chain. This is not possible at all starting from the initial formulation or the problem.

6 Conclusion

We have seen that the coordinate Bethe ansatz transforms the problem of diagonalising the Hamiltonian of our spin chain to solving a system of coupled equations (53) and (58). Since we did not use any approximations, the solutions are exact. However the equations are still very hard to solve. This means we have to use computers (see for example [4] and [5]) or go to the thermodynamic limit (see for example [7]). We can also ask ourselves if the equations give us all the eigenstates there are. This was extensively discussed by Bethe in his original paper [2] and many more.

Furthermore the coordinate Bethe ansatz does not really tell us why it works. In some way the properties of the model we used have to be special. We have for example seen two body reducibility in (53). To understand this, one has to look deeper into quantum integrability.

References

[1] W. Heisenberg, "Zur Theorie des Ferromagnetismus," Z. Phys., vol. 49, pp. 619-636, 1928

[2] H. Bethe, "Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette", Z. Phys. 71, 205 (1931).

[3] H. Ibach and H. Lüth, "Festkrperphysik, Einfhrung in die Grundlagen", 7th Edition Springer-Lehrbuch, 2009, page 197

[4] M. Karbach and G. Muller, "Introduction to the Bethe ansatz I", Computers in Physics 11, 36 (1997), arXiv:cond-mat/9809162

[5] M. Karbach, K. Hu and G. Muller, "Introduction to the Bethe ansatz II" (1998), arXiv:cond-mat/9809163

[6] J. Lamers, "A pedagogical introduction to quantum integrability, with a view towards theoretical high-energy physics", arXiv:/1501.06805

[7] M. Gaudin, "The Bethe Wavefunction", Cambridge University Press, 2014, Translated by J.-S. Caux