MTH 532: Differential Topology Problem Sets

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Textbook: *Differential Topology* by Guillermin and Pollack

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Contents

Chapter 1

Homework 1

1.1 §1 Definitions: 2, 3, 12

1.1.1 2: Suppose that X is a subset of $\mathbb{R}^N$ and Z is a subset of X. Show that the restriction to Z of any smooth map on X is a smooth map on Z.

Let $X \subset \mathbb{R}^N, Z \subset X$. Let $f : X \to \mathbb{R}^M$ smooth.
Then for all $x \in X$, there is an open $x \in U_x \subset \mathbb{R}^N$ with a map $F_x : U_x \to \mathbb{R}^M$ with continuous partial derivatives and satisfying $F_x|_{U_x \cap X} = f|_{U_x \cap X}$.
Let

$$\begin{aligned} g_f : Z &\to \mathbb{R}^N \\ z &\mapsto f(z) \end{aligned}$$

For all $z \in Z$, of course $z \in X$. Then there are smooth maps $F_Z : U_z \subset \mathbb{R}^N \to \mathbb{R}^M$ as above.
Then define the corresponding restriction

$$\begin{aligned} G_Z : U_z \cap Z &\to \mathbb{R}^M \\ z &\mapsto G_Z(z) := F_z(z) \end{aligned}$$

This is smooth since F_z is smooth and $U_z \cap Z$ is open. Hence we have a smooth map satisfying, for all $z \in Z$,

$$G_z|_{U_z \cup Z} = f|_{U_z \cup Z}$$

1.1.2 3: Let $X \subset \mathbf{R}^N, Y \subset \mathbf{R}^M, Z \subset \mathbf{R}^L$ be arbitrary subsets, and let $f : X \to Y$, $g : Y \to Z$ be smooth maps. Then the composite $g \circ f : X \to Z$ is smooth. If f and g are diffeomorphisms, so is $g \circ f$.

Let $X \subset \mathbb{R}^N, Y \subset \mathbb{R}^M, Z \subset \mathbb{R}^L$ and given two smooth maps $f : X \to Y$, $g : Y \to Z$,
WTS: $g \circ f : X \to Z$ **is smooth.**
Needed is for all $x \in X$, an open $x \in U_x \subset \mathbb{R}^N$ with a smooth map $GF : U_x \to Z$ satisfying

$$GF|_{U_x \cup Z} = g \circ f|_{U_x \cup Z}$$

By smoothness we are guaranteed an F_x on $U_x \subset \mathbb{R}^N$ and a G_y on $U_y \subset \mathbb{R}^M$ that agree respectively with f and g. Consider now the composition, for all $x \in X$,

$$\begin{aligned} G_y \circ F_x =: GF : & U_x \to Z \\ & x \mapsto GF(x) = G(F(x)) \end{aligned}$$

Consequently by smoothness, the restriction satisfies $G|_{F(U_x)\cap Y}(F|_{U_x \cap X}(x)) = G|_{F(U_x)\cap Y}(f(x)) = g(f(x))$ and smoothness of the composition is guaranteed by real analysis.
WTS: $g \circ f : X \to Z$ **is a diffeomorphism given** f **and** g **are.**
The composition is clearly a bijection, hence an inverse exists. Needed is the smoothness of the inverse.
Since $g(f)$ is a diffeomorphism, g^{-1} (f^{-1}) is smooth. Then by the first part the composition $f^{-1} \circ g^{-1} : Z \to X$ is smooth, and is the inverse to the composition $g \circ f$:

$$f^{-1}(g^{-1}(g(f(x)))) = x$$

1.1.3 12: Stereographic projection is a map π from the punctured sphere $S^2 - \{N\}$ onto $\mathbf{R}^2$, where N is the north pole $(0, 0, 1)$. For any $p \in S^2 - \{N\}$, $\pi(p)$ is defined to be the point at which the line through N and p intersects the xy plane. Prove that $\pi : S^2 - \{N\} \to \mathbf{R}^2$ is a diffeomorphism. (To do so, write π explicitly in coordinates and solve for π^{-1}.)

Note that if p is near N, then $|\pi(p)|$ is large. Thus π allows us to think of S^2 a copy of $\mathbf{R}^2$ compactified by the addition of one point "at infinity." Since we can define stereographic projection by using the south pole instead of the north, S^2 may be covered by two local parametrizations.
The stereographic projecton is the pair of maps

$$\begin{aligned}
f :& S^2 \setminus (0,0,1) \to \mathbb{R}^2 \\
&(x,y,z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z}) \\
f^{-1} :& \mathbb{R}^2 \to S^2 \setminus (0,0,1) \\
&(x,y) \mapsto (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2})
\end{aligned}$$

which can be derived using high school geometry (perhaps the inverse is not that trivial).

1. f is injective.
   This follows from the fact that there is only one line(up to diffeomorphism) passing through two points.

2. f is surjective.
   For a given z, the x coordinate is a polynomial αx, same for the y entry. Then suppose some point $p = (p_x, p_y)$ is not in image $f(S^2 \setminus (0,0,1))$. Clearly, $f(\frac{p_x}{\alpha}, \frac{p_y}{\alpha}, 1 - \frac{1}{\alpha})$ maps to p and since α can be made arbitrarily large or small, by z approaching 1 or zero, this point is on S^2 for the appropriate z.

3. f is smooth.
   This follows since the north pole is missing and $z \neq 1$, the composition of elementary functions is smooth.

4. f^{-1} is smooth.
   This time given by the fact that $1+x^2+y^2$ is bounded below by 1 and hence never vanishes. Then $\frac{1}{1+x^2+y^2}$ is bounded above by 1. Hence the components are each, again, composition of polynomials with division, which is smooth given the condition of the denominator.

1.2 §2 Derivatives and Tangents: 2, 4, 8

1.2.1 2: If U is an open subset of the manifold X, check that $T_x(U) = T_x(X)$ for $x \in U$

Consider a manifold $X \subset \mathbb{R}^N$. For every $x \in X$, there is a local parametrization $\phi_x : U_x \to X$ with the constraint $\phi^x(0) = x$ by definition of a (smooth) manifold. The tangent space of X at x is defined as the image

$$T_x(X) = d\phi_0^x(U_x) \tag{1.1}$$

Since U_x is open, it itself is a manifold and we have local parametrixations as above. Then the tangent space at x is

$$T_x(U_x) = d\psi_0^x(V_x) \tag{1.2}$$

where $\psi_0^x : V_x \subset \mathbb{R}^k \to U$. But since the parametrization does not matter, and $U_x \subset X$ one can use the ψ_0^x for the derivative in (1.1).

1.2.2 4: Suppose that $f : X \to Y$ is a diffeomorphism, and prove that at each x its derivative df_x is an isomorphism of tangent spaces.

Given a diffeomorphism $f : X \to Y$, first we note that for each $x \in X$ and $y \in Y$

$$\dim T_x(X) = \dim X = \dim Y = \dim T_y(Y) \tag{1.3}$$

where the bijection property is used.
Since the inverse is again a diffeomorphism, we have decomposition of the identity on X:

$$\mathrm{id}_X = f^{-1} \circ f$$

The order of the composition can be swapped to work in Y and to again show the proof of equality of dimension. By the chain rule obtained is the derivative

$$d(\mathrm{id}_X)_x = df^{-1}_{f(x)} \circ df_x$$

The important step here is to notice the Jacobian of the identity map is the identity matrix. This holds since the entries in the Euclidean basis satisfy

$$(d(\mathrm{id}_X)_x)_{ij} = \frac{\partial f_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

where δ is the Kronecker delta or the identity matrix of dimension $\dim X$. Then df_x is an isomorphism as we have shown the existence of an inverse.

1.2.3 8: What is the tangent space to the paraboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$, where $(a > 0)$?

Given is a paraboloid in $\mathbb{R}^3$

$$x^2 + y^2 - z^2 = a$$

What is the tangent space at $(\sqrt{a}, 0, 0)$?
First some intuition through geometry. Clearly the system is symmetric with respect to the coordinte axes. Fix $z = 0$ and we have a circle, so the system looks like two cones merged at this circe of radius $\sqrt{a}$. The further symmetry of the paraboloid under the swap $x \leftrightarrow y$ which again reflects the circular nature. Hence at the point of interest we are at the throat, and the tangent plane has the normal $\hat{x}$, it is the y-z plane shifted by $\sqrt{a}$.
Now time for an explicit calculation. An atlas is the two functions

$$f_\pm(y, z) = (\pm\sqrt{z^2 - y^2 + a}, y, z)$$

which is smooth if $z^2 - y^2 < a$ which certainly is true for our point. We have chosen to omit x since that is the only way to define a nonsingular derivative. Since the object at hand is two dimensional the expected tangent space is a plane. The Jacobian is a 2 by 3 matrix

$$df_{+,\vec{x}} = \begin{pmatrix} \frac{-1}{\sqrt{z^2-y^2+a}} y & \frac{1}{\sqrt{z^2-y^2+a}} z \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where we observe that f acts trivially on the x subspace. Clearly, this maps $\mathbb{R}^2$ to $\mathbb{R}^3$. At our point this is the trivial

$$df_{+,\vec{x}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The image of this operator is really in $\mathbb{R}^2$, as the first row is 0 the nontrivial component is the really the 2×2 block. Then the image is spanned by its columns, which are unit vectors in y and z directions. Hence our earlier observation is confirmed , the tangent space is given by the plane

$$x = \sqrt{a}$$

1.3 §4 Submersions: 1

1.3.1 1: If $f : X \to Y$ is a submersion and U is an open set of X, show that $f(U)$ is open in Y.

Given $f : X \subset \mathbb{R}^n \to Y \subset \mathbb{R}^m$ a submersion, i.e. the derivative is a surjective linear map everywhere, and U open in X, wish to show that $f(U)$ is open in Y. Note that $n \geq m$.
First some notation.
Let $x = (x_1, x_2, \cdots, x_m, x_{m+1}, \cdots, x_n) \in U$. Also, denote $x|_m$ the first m coordinates subsapce of x. Naturally, if $U \subset \mathbb{R}^n$ and $n \geq m$, define the "restriction of U to m dimensional subspace $U|_m$" as

$$U|_m = \{(x_1, x_2, \cdots, x_m) |\, x = (x_1, x_2, \cdots, x_m, x_{m+1}, \cdots, x_n) \in U\} \quad (1.4)$$

Since f is a submersion, there are diffeomorphisms i.e. charts ψ and ϕ such that the composition

$$(\psi^{-1} \circ f \circ \phi)(x) = (x_1, x_2, \cdots, x_m) = x|_m$$

i.e. coordinate systems in which the subspaces of the first m coordinates are "parallel".
Since $U \subset X \subset \mathbb{R}^n$, the preimage $\phi^{-1}(U) = \mathcal{U} \subset \mathbb{R}^n$ is open by diffeomorphism. Now let $U = \phi(\mathcal{U})$ and so

$$(f \circ \phi)(\mathcal{U}) = f(U) = \operatorname{Im} \psi(x|_m) = \psi(\mathcal{U}|_m \cap Y)$$

where the last term is the intersection of Y with restriction of $\mathcal{U}$ to $\mathbb{R}^m$. Since $\mathcal{U}$ is open, the intersection is open in Y by the subspace topology.
Then since ψ is a diffeomorphism, image of an open set is open, and hence is $f(U)$.

Chapter 2

Homework 2

2.1 §3 Inverse Function Theorem and Immersions: 1, 2

2.1.1 1: Let A be a linear map of $\mathbf{R}^n$, and $b \in \mathbf{R}^n$. Show that the mapping $x \to Ax + b$ is a diffeomorphism of $\mathbf{R}^n$ if and only if A is nonsingular.

Want to prove both ways.
($\Longrightarrow$) Suppose A is singular. Call the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. It is clearly a composition of (multiply by A from left, m) and (add b, n). Since f is a diffeomorphism, it is bijective, and so is n. Act on the right ith n^{-1}:

$$\begin{aligned} f &= n \circ m \\ n^{-1} \circ f &= m \end{aligned}$$

Then m is bijective. Contradiction.
($\Longleftarrow$) Suppose A is nonsingular and $f(x) = Ax + b$, with domains as above. Define g

$$\begin{aligned} g : &\mathbb{R}^n \longrightarrow \mathbb{R}^n \\ &x \mapsto A^{-1}(x - b) \end{aligned}$$

which is clearly an inverse of f, and we have A^{-1} by nonsingularity. Needed is continuity and smoothness, which are given by the fact that m and n are compositions of continuous and smooth operations. Furthermore, the derivative of f is A, which is an isomorphism since it is invertible. Hence f is a local diffeomorphism (by Inverse Function Theorem) which is bijective (we have an inverse). Then f globally is a diffeomorphism.

2.1.2 2: *2. Suppose that Z is an l-dimensional submanifold of X and that $z \in Z$. Show that there exists a local coordinate system $\{x_1, \dots, x_k\}$ defined in a neighborhood U of z in X such that $Z \cap U$ is defined by the equations $x_{l+1} = 0, \dots, x_k = 0$.

We want to use the Local Immersion Theorem to show the coordiate functions take the form of the canonical immersion, i.e the injection $Z \cap U \overset{\iota}{\hookrightarrow} X$. Since Z is a submanifold there is the injection $\iota : Z \to X$, which has unit derivative, and hence is an immersion. Considering a neighbourhood $U \subset X$ of $z \in Z$, $Z \cup U \subset Z$ is an (open) neighbourhood of z. Then by the local immersion theorem, on $Z \cup U \subset Z$ and on a neighbourhood of $\iota(z) \in X$, there are charts such that the injection looks like the canonical immersion of Z into X.

2.2 §4 Submersions: 1, 2, 7

2.2.1 1: *1. If $f : X \to Y$ is a submersion and U is an open set of X, show that $f(U)$ is open in Y.

Refer to 1.3.1.

2.2.2 2a: If X is compact and Y connected, show every submersion $f : X \to Y$ is surjective.

Trivially assume $f(X)$ nonempty. Since X is compact, so is $f(X)$ by smoothness. By the previous problem, $f(X)$ is open. But since Y is a manifold, hence is Hausdorff,, and then $f(X) \subset Y$ is closed as it is compact. Since Y is connected, the only nonempty clopen set is Y itself and as assumed $f(X)$ nonempty, it must be that $f(X) = Y$.

2.2.3 2b: Show that there exist no submersions of compact manifolds into Euclidean spaces.

Consider f as above but Y is some Euclidean $\mathbb{R}^n$. If there is such a submersion, $f(X) = \mathbb{R}^n$ is compact, contradiction with $\mathbb{R}^n$ not compact.

2.2.4 7: (Stack of Records Theorem.) Suppose that y is a regular value of $f : X \to Y$, where X is compact and has the same dimension as Y. Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$. Prove there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W_1 of x_i that are mapped diffeomorphically. Show that $f\left(X - \cup W_t\right)$ is compact and does not contain y.. See Figure 1-13.

1) $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$
Since y is regular, by the first version of the Preimage Theorem, the inverse image $f^{-1}(y)$ is a manifold of dimension $\dim X - \dim Y = 0$. Hence it is a set of points.
If it is infinite, it has a limit point in X and actually in itself, because the singleton sets are closed, hence by continuity the inverse image is closed and hence contains its limit points. Since y is regular, and dimensions are equal, f is a local diffeomorphism for all points in the preimage. But it cannot be injective at the limit point x since there is always anoter point from $f^{-1}(y)$ in every open set containing x. Then there is no limit point in $f^{-1}(y)$ and hence it is finite.
2) There exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W_i of x_i that are mapped diffeomorphically. Show that $f\left(X - \cup W_i\right)$ is compact and does not contain y.
As shown $f^{-1}(y) = \{x_1, \ldots, x_N\}$. Let W_i be disjoint neighbourhoods of x_i, respectively, where their existance is guaranteed by Dr Hausdorff. The complement of union of Ws in X is closed and hence compact, since X is

compact. By continuity, its image is also compact. y is not here as all points that map to y are contained in the W_i. The complement $U = Y - f(X - \cup W_i)$ is open and hence contains y. Finally, the preimage $f^{-1}(U) = f^{-1}(Y - f(X - \cup W_i)) = f^{-1}(Y) - (X - \cup W_i) = \cup W_i$.

2.3 §7 Sard's Theorem and Morse Functions: 1, 4

2.3.1 1: Show that $\mathbb{R}^k$ has measure zero in $\mathbb{R}^l$, $k < l$.

This follows from the fact $\mathbb{R}^k$ itself is covered by a single solid $S = (-\infty, \infty)$. When $\mathbb{R}^k$ is taken as a subset in $\mathbb{R}^l$, it is then covered by just the injection $\iota(S) = S \times \prod_k^l (0, 0)$, which trivially has volume zero.

2.3.2 4: Prove that rationals have measure zero in $\mathbb{R}$.

Cover $\mathbb{Q}$ by the solids (here just intervals) $(q_i - \frac{\varepsilon}{2^{i+1}}, q_i + \frac{\varepsilon}{2^{i+1}})$, where the subscript i is some arbitrary ordering of the rationals, starting from $i = 2$ for convenience, indexing allowed by countability. This has volume

$$\sum_{i=2}^{\infty} 2 * \frac{\varepsilon}{2^{i+1}} = \varepsilon \sum_{i=2}^{\infty} \frac{1}{2^i} = \varepsilon * \frac{1}{2} < \varepsilon$$

as desired.

Chapter 3

Homework 3

3.1 §5 Transversality: 2, 4, 7

3.1.1 2: Which of the following linear spaces intersect transversally?

1. The xy plane and the z axis in $\mathbb{R}^3$. Yes
2. The xy plane and the plane spanned by (3, 2, 0), (0, 4, -1) in $\mathbb{R}^3$ Yes
3. The plane spanned by (l, 0, 0), (2, I, 0) and the y axis in $\mathbb{R}^3$. No.
4. $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$. (Depends on k, I, n.) If $k + l < n$, no. else, yes.
5. $\mathbb{R}^k \times \{0\}$ and $\mathbb{R}^l \times \{0\}$ in $\mathbb{R}^n$. (Depends on k, I, n.) Only if k or l equals n.
6. $V \times \{0\}$ and the diagonal in $V \times V$. Yes.
7. The symmetric (A' = A) and skew symmetric (A' = -A) matrices in M(n). For any matrix

3.1.2 4: Let X and Z be transversal submanifolds of Y. Prove that if $y \in X \cap Z$, then the tangent space to the intersection is the intersection of the tangent spaces.

In notation, we want to show:

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z).$$

The first piece of knowledge is that for any $y \in X \cap Z$, one has

$$T_yX + T_yZ = T_yY$$

(just writing out transversality.) The intersection of the tangent spaces, naturally, is the point of intersection plus any common dimensions of the tangent spaces. What is meant by this is that if the sum of the dimensions of the tangent spaces are larger than of the ambient space, some of the vectors in T_yX lie in T_yZ.

3.2 §4 Submersions: 1, 2, 7

3.2.1 1: *1. If $f : X \to Y$ is a submersion and U is an open set of X, show that $f(U)$ is open in Y.

Refer to 1.3.1.

3.2.2 2a: If X is compact and Y connected, show every submersion $f : X \to Y$ is surjective.

Trivially assume $f(X)$ nonempty. Since X is compact, so is $f(X)$ by smoothness. By the previous problem, $f(X)$ is open. But since Y is a manifold, it is embedded in some $\mathbb{R}^n$, and hence is Hausdorff. Then $f(X) \subset Y$ is closed as it is compact. Since Y is connected, and $f(X)$ nonempty, then it must be that $f(X) = Y$. Let f be a submersion. Then it has a surjective derivative everywhere on X.

3.2.3 2b: Show that there exist no submersions of compact manifolds into Euclidean spaces.

Consider f as above but Y is some Euclidean $\mathbb{R}^n$. If there is such a submersion, $f(X) = \mathbb{R}^n$ is compact, contradiction with $\mathbb{R}^n$ not compact.

3.2.4 7: (Stack of Records Theorem.) Suppose that y is a regular value of $f : X \to Y$, where X is compact and has the same dimension as Y. Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$. Prove there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W_1 of x_i that are mapped diffeomorphically. Show that $f\left(X - \cup W_t\right)$ is compact and does not contain y.. See Figure 1-13.

1) $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$
Since y is regular, by the first version of the Preimage Theorem, the inverse image $f^{-1}(y)$ is a manifold of dimension $\dim X - \dim Y = 0$. Hence it is a set of points.
If it is infinite, it has a limit point in X and actually in itself, since a set of points is closed and hence contains its limit points. Since y is regular, and dimensions are equal, f is a local diffeomorphism for all points in the preimage. But it cannot be injective at the limit point x since there is always anoter point from $f^{-1}(y)$ in every open set containing x. Then there is no limit point in $f^{-1}(y)$ and hence it is finite.
2) There exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W_i of x_i that are mapped diffeomorphically. Show that $f\left(X - \cup W_i\right)$ is compact and does not contain y.
As shown $f^{-1}(y) = \{x_1, \ldots, x_N\}$. Let W_i be disjoint neighbourhoods of x_i, respectively, where their existance is guaranteed by Dr Hausdorff. The complement of union of Ws in X is closed and hence compact, since X is compact. By continuity, its image is also compact. y is not here as all

points that map to y are contained in the W_i. The complement $U = Y - f(X - \cup W_i)$ is open and hence contains y. Finally, the preimage $f^{-1}(U) = f^{-1}(Y - f(X - \cup W_i)) = f^{-1}(Y) - (X - \cup W_i) = \cup W_i$.

3.3 §7 Sard's Theorem and Morse Functions: 1, 4

3.3.1 1: Show that $\mathbb{R}^k$ has measure zero in $\mathbb{R}^l$, $k < l$.

(I apologize for abuse of notation on the generalized product. I tried to make it clear by using $\times$ vs $*$.)
This follows from the fact the injection of any rectangular solid $S = \prod_i^k (a_i, b_i)$ with $a_i < b_i$ in $\mathbb{R}^k$ to $\mathbb{R}^l$ will be $\iota(S) = \prod_i^k (a_i, b_i) \times \prod_k^l (0,0)$ with $a_i < b_i$ with volume given $\prod_i^k |a_i - b_i| * \prod_k^l 0 = 0$. Hence $\mathbb{R}^k$ contains no rectangular solids and has measure 0.

3.3.2 4: Prove that rationals have measure zero in $\mathbb{R}$.

Cover $\mathbb{Q}$ by the solids (here just intervals) $(q_i - \frac{\varepsilon}{2^{i+1}}, q_i + \frac{\varepsilon}{2^{i+1}})$, where the subscript i is some arbitrary ordering of the rationals, starting from $i = 2$ for convenience, indexing allowed by countability. This has volume

$$\sum_{i=2}^{\infty} 2 * \frac{\varepsilon}{2^{i+1}} = \varepsilon \sum_{i=2}^{\infty} \frac{1}{2^i} = \varepsilon * \frac{1}{2} < \varepsilon$$

as desired.