

MTH 532: Differential Topology Problem Sets

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Chapter 1

Homework 1

1.1 §1 Definitions: 2, 3, 12

1.1.1 2

Let $X \subset \mathbb{R}^N, Z \subset X$. Let $f : X \rightarrow \mathbb{R}^M$ smooth.

Then for all $x \in X$, there is an open $x \in U_x \subset \mathbb{R}^N$ with a map $F_x : U_x \rightarrow \mathbb{R}^M$ with continuous partial derivatives and satisfying $F_x|_{U_x \cap X} = f|_{U_x \cap X}$.

Let

$$\begin{aligned} g_f : Z &\rightarrow \mathbb{R}^M \\ z &\mapsto f(z) \end{aligned}$$

For all $z \in Z$, of course $z \in X$. Then there are smooth maps $F_z : U_z \subset \mathbb{R}^N \rightarrow \mathbb{R}^M$ as above.

Then define the corresponding restriction

$$\begin{aligned} G_z : U_z \cap Z &\rightarrow \mathbb{R}^M \\ z &\mapsto G_z(z) := F_z(z) \end{aligned}$$

This is smooth since F_z is smooth and $U_z \cap Z$ is open. Hence we have a smooth map satisfying, for all $z \in Z$,

$$G_z|_{U_z \cap Z} = f|_{U_z \cap Z}$$

1.1.2 3

Let $X \subset \mathbb{R}^N, Y \subset \mathbb{R}^M, Z \subset \mathbb{R}^L$ and given two smooth maps $f : X \rightarrow Y, g : Y \rightarrow Z$,

WTS: $g \circ f : X \rightarrow Z$ is smooth.

Needed is for all $x \in X$, an open $U_x \subset \mathbb{R}^N$ with a smooth map $GF : U_x \rightarrow Z$ satisfying

$$GF|_{U_x \cup Z} = g \circ f|_{U_x \cup Z}$$

By smoothness we are guaranteed an F_x on $U_x \subset \mathbb{R}^N$ and a G_y on $U_y \subset \mathbb{R}^M$ that agree respectively with f and g . Consider now the composition, for all $x \in X$,

$$\begin{aligned} G_y \circ F_x &=: GF : U_x \rightarrow Z \\ x &\mapsto GF(x) = G(F(x)) \end{aligned}$$

Consequently by smoothness, the restriction satisfies $G|_{F(U_x) \cap Y}(F|_{U_x \cap X}(x)) = G|_{F(U_x) \cap Y}(f(x)) = g(f(x))$ and smoothness of the composition is guaranteed by real analysis.

WTS: $g \circ f : X \rightarrow Z$ is a diffeomorphism given f and g are.

The composition is clearly a bijection, hence an inverse exists. Needed is the smoothness of the inverse.

Since $g(f)$ is a diffeomorphism, $g^{-1}(f^{-1})$ is smooth. Then by the first part the composition $f^{-1} \circ g^{-1} : Z \rightarrow X$ is smooth, and is the inverse to the composition $g \circ f$:

$$f^{-1}(g^{-1}(g(f(x)))) = x$$

1.1.3 12

The stereographic projection is the pair of maps

$$\begin{aligned} f : S^2 \setminus (0, 0, 1) &\rightarrow \mathbb{R}^2 \\ (x, y, z) &\mapsto \left(\frac{x}{1-z}, \frac{y}{1-z} \right) \\ f^{-1} : \mathbb{R}^2 &\rightarrow S^2 \setminus (0, 0, 1) \\ (x, y) &\mapsto \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right) \end{aligned}$$

which can be derived using high school geometry (perhaps the inverse is not that trivial).

1. f is injective.

This follows from the fact that there is only one line (up to diffeomorphism) passing through two points.

2. f is surjective.

For a given z , the x coordinate is a polynomial αx , same for the y entry. Then suppose some point $p = (p_x, p_y)$ is not in image $f(S^2 \setminus (0, 0, 1))$. Clearly, $f(\frac{p_x}{\alpha}, \frac{p_y}{\alpha}, 1 - \frac{1}{\alpha})$ maps to p and since α can be made arbitrarily large or small, by z approaching 1 or zero, this point is on S^2 for the appropriate z .

3. f is smooth.

This follows since the north pole is missing and $z \neq 1$, the composition of elementary functions is smooth.

4. f^{-1} is smooth.

This time given by the fact that $1 + x^2 + y^2$ is bounded below by 1 and hence never vanishes. Then $\frac{1}{1+x^2+y^2}$ is bounded above by 1. Hence the components are each, again, composition of polynomials with division, which is smooth given the condition of the denominator.

1.2 §2 Derivatives and Tangents: 2, 4, 8

1.2.1 2

Consider a manifold $X \subset \mathbb{R}^N$. For every $x \in X$, there is a local parametrization $\phi_x : U_x \rightarrow X$ with the constraint $\phi_x(0) = x$ by definition of a (smooth) manifold. The tangent space of X at x is defined as the image

$$T_x(X) = d\phi_0^x(U_x) \quad (1.1)$$

Since U_x is open, it itself is a manifold and we have local parametrizations as above. Then the tangent space at x is

$$T_x(U_x) = d\psi_0^x(V_x) \quad (1.2)$$

where $\psi_0^x : V_x \subset \mathbb{R}^k \rightarrow U$. But since the parametrization does not matter, and $U_x \subset X$ one can use the ψ_0^x for the derivative in (1.1).

1.2.2 4

Given a diffeomorphism $f : X \rightarrow Y$, first we note that for each $x \in X$ and $y \in Y$

$$\dim T_x(X) = \dim X = \dim Y = \dim T_y(Y) \quad (1.3)$$

where the bijection property is used.

Since the inverse is again a diffeomorphism, we have decomposition of the identity on X :

$$\text{id}_X = f^{-1} \circ f$$

The order of the composition can be swapped to work in Y and to again show the proof of equality of dimension. By the chain rule obtained is the derivative

$$d(\text{id}_X)_x = df_{f(x)}^{-1} \circ df_x$$

The important step here is to notice the Jacobian of the identity map is the identity matrix. This holds since the entries in the Euclidean basis satisfy

$$(d(\text{id}_X)_x)_{ij} = \frac{\partial f_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

where δ is the Kronecker delta or the identity matrix of dimension $\dim X$. Then df_x is an isomorphism as we have shown the existence of an inverse.

1.2.3 8

Given is a paraboloid in \mathbb{R}^3

$$x^2 + y^2 - z^2 = a$$

What is the tangent space at $(\sqrt{a}, 0, 0)$?

First some intuition through geometry. Clearly the system is symmetric with respect to the coordinate axes. Fix $z = 0$ and we have a circle, so the system looks like two cones merged at this circle of radius \sqrt{a} . The further symmetry of the paraboloid under the swap $x \leftrightarrow y$ which again reflects the circular nature. Hence at the point of interest we are at the throat, and the tangent plane has the normal \hat{x} , it is the y - z plane shifted by \sqrt{a} .

Now time for an explicit calculation. An atlas is the two functions

$$f_{\pm}(y, z) = (\pm\sqrt{z^2 - y^2 + a}, y, z)$$

which is smooth if $z^2 - y^2 < a$ which certainly is true for our point. We have chosen to omit x since that is the only way to define a nonsingular derivative. Since the object at hand is two dimensional the expected tangent space is a plane. The Jacobian is a 2 by 3 matrix

$$df_{+, \vec{x}} = \begin{pmatrix} \frac{-1}{\sqrt{z^2 - y^2 + a}} y & \frac{1}{\sqrt{z^2 - y^2 + a}} z \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where we observe that f acts trivially on the x subspace. Clearly, this maps \mathbb{R}^2 to \mathbb{R}^3 . At our point this is the trivial

$$df_{+, \vec{x}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The image of this operator is really in \mathbb{R}^2 , as the first row is 0 the nontrivial component is the really the 2×2 block. Then the image is spanned by its columns, which are unit vectors in y and z directions. Hence our earlier observation is confirmed, the tangent space is given by the plane

$$x = \sqrt{a}$$

1.3 §4 Submersions: 1

1.3.1 1: If $f : X \rightarrow Y$ is a submersion and U is an open set of X , show that $f(U)$ is open in Y .

Given $f : X \subset \mathbb{R}^n \rightarrow Y \subset \mathbb{R}^m$ a submersion, i.e. the derivative is a surjective linear map everywhere, and U open in X , wish to show that $f(U)$ is open in Y . Note that $n \geq m$.

First some notation.

Let $x = (x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_n) \in U$. Also, denote $x|_m$ the first m coordinates subspace of x . Naturally, if $U \subset \mathbb{R}^n$ and $n \geq m$, define the "restriction of U to m dimensional subspace $U|_m$ " as

$$U|_m = \{(x_1, x_2, \dots, x_m) | x = (x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_n) \in U\} \quad (1.4)$$

Since f is a submersion, there are diffeomorphisms i.e. charts ψ and ϕ such that the composition

$$(\psi^{-1} \circ f \circ \phi)(x) = (x_1, x_2, \dots, x_m) = x|_m$$

i.e. coordinate systems in which the subspaces of the first m coordinates are "parallel".

Since $U \subset X \subset \mathbb{R}^n$, the preimage $\phi^{-1}(U) = \mathcal{U} \subset \mathbb{R}^n$ is open by diffeomorphism. Now let $U = \phi(\mathcal{U})$ and so

$$(f \circ \phi)(\mathcal{U}) = f(U) = \text{Im } \psi(x|_m) = \psi(\mathcal{U}|_m \cap Y)$$

where the last term is the intersection of Y with restriction of \mathcal{U} to \mathbb{R}^m . Since \mathcal{U} is open, the intersection is open in Y by the subspace topology.

Then since ψ is a diffeomorphism, image of an open set is open, and hence is $f(U)$.

Chapter 2

Homework 2

2.1 §3 Inverse Function Theorem and Immersions: 1, 2

2.1.1 1: Let A be a linear map of \mathbf{R}^n , and $b \in \mathbf{R}^n$. Show that the mapping $x \rightarrow Ax + b$ is a diffeomorphism of \mathbf{R}^n if and only if A is nonsingular.

Want to prove both ways.

(\implies) Suppose A is singular. Call the mapping $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$. It is clearly a composition of (multiply by A from left, m) and (add b , n). Since f is a diffeomorphism, it has an inverse, namely the inverse of the composition $f^{-1} = (n \circ m)^{-1} = m^{-1} \circ n^{-1}$. The fact that m is invertible means there is an inverse matrix of A , hence A is nonsingular. Contradiction.

(\impliedby) Suppose A is nonsingular and $f(x) = Ax + b$, with domains as above. Define g

$$\begin{aligned} g : \mathbf{R}^n &\longrightarrow \mathbf{R}^n \\ x &\mapsto A^{-1}(x - b) \end{aligned}$$

which is clearly an inverse of f , and we have A^{-1} by nonsingularity. Needed is continuity and smoothness, which are given by the fact that m and n are compositions of continuous and smooth operations. Furthermore, the derivative of f is A , which is an isomorphism since it is invertible. Hence f is a local diffeomorphism (by Inverse Function Theorem) which is bijective (we have an inverse). Then f globally is a diffeomorphism.

2.1.2 2: *2. Suppose that Z is an l -dimensional submanifold of X and that $z \in Z$. Show that there exists a local coordinate system $\{x_1, \dots, x_k\}$ defined in a neighborhood U of z in X such that $Z \cap U$ is defined by the equations $x_{l+1} = 0, \dots, x_k = 0$.

We want to use the Local Immersion Theorem to show the coordinate functions take the form of the canonical immersion, i.e the injection $Z \cap U \xrightarrow{\iota} X$. Since Z is a submanifold there is the injection $\iota : Z \rightarrow X$, which has unit derivative, and hence is an immersion. Considering a neighbourhood $U \subset X$ of $z \in Z$, $Z \cap U \subset Z$ is an (open) neighbourhood of z . Then by the local immersion theorem, on $Z \cap U \subset Z$ and on a neighbourhood of $\iota(z) \in X$, there are charts such that the injection looks like the canonical immersion of Z into X .

2.2 §4 Submersions: 1, 2, 7

2.2.1 1: *1. If $f : X \rightarrow Y$ is a submersion and U is an open set of X , show that $f(U)$ is open in Y .

Refer to 1.3.1.

2.2.2 2a: If X is compact and Y connected, show every submersion $f : X \rightarrow Y$ is surjective.

Trivially assume $f(X)$ nonempty. Since X is compact, so is $f(X)$ by smoothness. By the previous problem, $f(X)$ is open. But since Y is a manifold, it is embedded in some \mathbb{R}^n , and hence is Hausdorff. Then $f(X) \subset Y$ is closed as it is compact. Since Y is connected, and $f(X)$ nonempty, then it must be that $f(X) = Y$. Let f be a submersion. Then it has a surjective derivative everywhere on X .

2.2.3 2b: Show that there exist no submersions of compact manifolds into Euclidean spaces.

Consider f as above but Y is some Euclidean \mathbb{R}^n . If there is such a submersion, $f(X) = \mathbb{R}^n$ is compact, contradiction with \mathbb{R}^n not compact.

2.2.4 7: (Stack of Records Theorem.) Suppose that y is a regular value of $f : X \rightarrow Y$, where X is compact and has the same dimension as Y . Show that $f^{-1}(y)$ is a finite set $\{x_1, \dots, x_N\}$. Prove there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \dots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U . [HINT: Pick disjoint neighborhoods W_i of x_i that are mapped diffeomorphically. Show that $f(X - \cup W_i)$ is compact and does not contain y . See Figure 1-13.

1) $f^{-1}(y)$ is a finite set $\{x_1, \dots, x_N\}$

Since y is regular, by the first version of the Preimage Theorem, the inverse image $f^{-1}(y)$ is a manifold of dimension $\dim X - \dim Y = 0$. Hence it is a set of points.

If it is infinite, it has a limit point in X and actually in itself, since a set of points is closed and hence contains its limit points. Since y is regular, and dimensions are equal, f is a local diffeomorphism for all points in the preimage. But it cannot be injective at the limit point x since there is always another point from $f^{-1}(y)$ in every open set containing x . Then there is no limit point in $f^{-1}(y)$ and hence it is finite.

2) There exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \dots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U . [HINT: Pick disjoint neighborhoods W_i of x_i that are mapped diffeomorphically. Show that $f(X - \cup W_i)$ is compact and does not contain y .

As shown $f^{-1}(y) = \{x_1, \dots, x_N\}$. Let W_i be disjoint neighbourhoods of x_i , respectively, where their existence is guaranteed by Dr Hausdorff. The complement of union of W s in X is closed and hence compact, since X is compact. By continuity, its image is also compact. y is not here as all

points that map to y are contained in the W_i . The complement $U = Y - f(X - \cup W_i)$ is open and hence contains y . Finally, the preimage $f^{-1}(U) = f^{-1}(Y - f(X - \cup W_i)) = f^{-1}(Y) - (X - \cup W_i) = \cup W_i$.

2.3 §7 Sard's Theorem and Morse Functions: 1, 4

2.3.1 1: Show that \mathbb{R}^k has measure zero in \mathbb{R}^l , $k < l$.

(I apologize for abuse of notation on the generalized product. I tried to make it clear by using \times vs $*$.)

This follows from the fact the injection of any rectangular solid $S = \prod_i^k (a_i, b_i)$ with $a_i < b_i$ in \mathbb{R}^k to \mathbb{R}^l will be $\iota(S) = \prod_i^k (a_i, b_i) \times \prod_k^l (0, 0)$ with $a_i < b_i$ with volume given $\prod_i^k |a_i - b_i| * \prod_k^l 0 = 0$. Hence \mathbb{R}^k contains no rectangular solids and has measure 0.

2.3.2 4: Prove that rationals have measure zero in \mathbb{R} .

Cover \mathbb{Q} by the solids (here just intervals) $(q_i - \frac{\varepsilon}{2^{i+1}}, q_i + \frac{\varepsilon}{2^{i+1}})$, where the subscript i is some arbitrary ordering of the rationals, starting from $i = 2$ for convenience, indexing allowed by countability. This has volume

$$\sum_{i=2}^{\infty} 2 * \frac{\varepsilon}{2^{i+1}} = \varepsilon \sum_{i=2}^{\infty} \frac{1}{2^i} = \varepsilon * \frac{1}{2} < \varepsilon$$

as desired.