MTH 532: Differential Topology Problem Sets

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Chapter 1

Homework 1

1.1 §1 Definitions: 2, 3, 12

1.1.1 2: Suppose that X is a subset of $\mathbb{R}^N$ and Z is a subset of X. Show that the restriction to Z of any smooth map on X is a smooth map on Z.

Let $X \subset \mathbb{R}^N, Z \subset X$. Let $f : X \to \mathbb{R}^M$ smooth.
Then for all $x \in X$, there is an open $x \in U_x \subset \mathbb{R}^N$ with a map $F_x : U_x \to \mathbb{R}^M$ with continuous partial derivatives and satisfying $F_x|_{U_x \cap X} = f|_{U_x \cap X}$.
Let

$$\begin{aligned} g_f : Z &\to \mathbb{R}^N \\ z &\mapsto f(z) \end{aligned}$$

For all $z \in Z$, of course $z \in X$. Then there are smooth maps $F_Z : U_z \subset \mathbb{R}^N \to \mathbb{R}^M$ as above.
Then define the corresponding restriction

$$\begin{aligned} G_Z : U_z \cap Z &\to \mathbb{R}^M \\ z &\mapsto G_Z(z) := F_z(z) \end{aligned}$$

This is smooth since F_z is smooth and $U_z \cap Z$ is open. Hence we have a smooth map satisfying, for all $z \in Z$,

$$G_z|_{U_z \cup Z} = f|_{U_z \cup Z}$$

1.1.2 3: Let $X \subset \mathbf{R}^N, Y \subset \mathbf{R}^M, Z \subset \mathbf{R}^L$ be arbitrary subsets, and let $f : X \to Y$, $g : Y \to Z$ be smooth maps. Then the composite $g \circ f : X \to Z$ is smooth. If f and g are diffeomorphisms, so is $g \circ f$.

Let $X \subset \mathbb{R}^N, Y \subset \mathbb{R}^M, Z \subset \mathbb{R}^L$ and given two smooth maps $f : X \to Y$, $g : Y \to Z$,
WTS: $g \circ f : X \to Z$ **is smooth.**
Needed is for all $x \in X$, an open $x \in U_x \subset \mathbb{R}^N$ with a smooth map
$GF : U_x \to Z$ satisfying

$$GF|_{U_x \cup Z} = g \circ f|_{U_x \cup Z}$$

By smoothness we are guaranteed an F_x on $U_x \subset \mathbb{R}^N$ and a G_y on $U_y \subset \mathbb{R}^M$ that agree respectively with f and g. Consider now the composition, for all $x \in X$,

$$\begin{aligned} G_y \circ F_x =: GF :& U_x \to Z \\ & x \mapsto GF(x) = G(F(x)) \end{aligned}$$

Consequently by smoothness, the restriction satisfies $G|_{F(U_x)\cap Y}(F|_{U_x \cap X}(x)) = G|_{F(U_x)\cap Y}(f(x)) = g(f(x))$ and smoothness of the composition is guaranteed by real analysis.
WTS: $g \circ f : X \to Z$ **is a diffeomorphism given** f **and** g **are.**
The composition is clearly a bijection, hence an inverse exists. Needed is the smoothness of the inverse.
Since $g(f)$ is a diffeomorphism, g^{-1} (f^{-1}) is smooth. Then by the first part the composition $f^{-1} \circ g^{-1} : Z \to X$ is smooth, and is the inverse to the composition $g \circ f$:

$$f^{-1}(g^{-1}(g(f(x)))) = x$$

1.1.3 **12:** Stereographic projection is a map π from the punctured sphere $S^2 - \{N\}$ onto $\mathbf{R}^2$, where N is the north pole $(0,0,1)$. For any $p \in S^2 - \{N\}$, $\pi(p)$ is defined to be the point at which the line through N and p intersects the xy plane. Prove that $\pi : S^2 - \{N\} \to \mathbf{R}^2$ is a diffeomorphism. (To do so, write π explicitly in coordinates and solve for π^{-1}.)

Note that if p is near N, then $|\pi(p)|$ is large. Thus π allows us to think of S^2 a copy of $\mathbf{R}^2$ compactified by the addition of one point "at infinity." Since we can define stereographic projection by using the south pole instead of the north, S^2 may be covered by two local parametrizations.
The stereographic projecton is the pair of maps

$$\begin{aligned} f :& S^2 \setminus (0,0,1) \to \mathbb{R}^2 \\ &(x,y,z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z}) \\ f^{-1} :& \mathbb{R}^2 \to S^2 \setminus (0,0,1) \\ &(x,y) \mapsto (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}) \end{aligned}$$

which can be derived using high school geometry (perhaps the inverse is not that trivial).

1. f is injective.
   This follows from the fact that there is only one line(up to diffeomorphism) passing through two points.

2. f is surjective.
   For a given z, the x coordinate is a polynomial αx, same for the y entry. Then suppose some point $p = (p_x, p_y)$ is not in image $f(S^2 \setminus (0,0,1))$. Clearly, $f(\frac{p_x}{\alpha}, \frac{p_y}{\alpha}, 1 - \frac{1}{\alpha})$ maps to p and since α can be made arbitrarily large or small, by z approaching 1 or zero, this point is on S^2 for the appropriate z.

3. f is smooth.
   This follows since the north pole is missing and $z \neq 1$, the composition of elementary functions is smooth.

4. f^{-1} is smooth.
   This time given by the fact that $1+x^2+y^2$ is bounded below by 1 and hence never vanishes. Then $\frac{1}{1+x^2+y^2}$ is bounded above by 1. Hence the components are each, again, composition of polynomials with division, which is smooth given the condition of the denominator.

1.2 §2 Derivatives and Tangents: 2, 4, 8

1.2.1 2: If U is an open subset of the manifold X, check that $T_x(U) = T_x(X)$ for $x \in U$

Consider a manifold $X \subset \mathbb{R}^N$. For every $x \in X$, there is a local parametrization $\phi_x : U_x \to X$ with the constraint $\phi^x(0) = x$ by definition of a (smooth) manifold. The tangent space of X at x is defined as the image

$$T_x(X) = d\phi_0^x(U_x) \tag{1.1}$$

Since U_x is open, it itself is a manifold and we have local parametrixations as above. Then the tangent space at x is

$$T_x(U_x) = d\psi_0^x(V_x) \tag{1.2}$$

where $\psi_0^x : V_x \subset \mathbb{R}^k \to U$. But since the parametrization does not matter, and $U_x \subset X$ one can use the ψ_0^x for the derivative in (1.1).

1.2.2 4: Suppose that $f : X \to Y$ is a diffeomorphism, and prove that at each x its derivative df_x is an isomorphism of tangent spaces.

Given a diffeomorphism $f : X \to Y$, first we note that for each $x \in X$ and $y \in Y$

$$\dim T_x(X) = \dim X = \dim Y = \dim T_y(Y) \tag{1.3}$$

where the bijection property is used.
Since the inverse is again a diffeomorphism, we have decomposition of the identity on X:

$$\mathrm{id}_X = f^{-1} \circ f$$

The order of the composition can be swapped to work in Y and to again show the proof of equality of dimension. By the chain rule obtained is the derivative

$$d(\mathrm{id}_X)_x = df^{-1}_{f(x)} \circ df_x$$

The important step here is to notice the Jacobian of the identity map is the identity matrix. This holds since the entries in the Euclidean basis satisfy

$$(d(\mathrm{id}_X)_x)_{ij} = \frac{\partial f_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

where δ is the Kronecker delta or the identity matrix of dimension $\dim X$. Then df_x is an isomorphism as we have shown the existence of an inverse.

1.2.3 8: What is the tangent space to the paraboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$, where $(a > 0)$?

Given is a paraboloid in $\mathbb{R}^3$

$$x^2 + y^2 - z^2 = a$$

What is the tangent space at $(\sqrt{a}, 0, 0)$?
First some intuition through geometry. Clearly the system is symmetric with respect to the coordinte axes. Fix $z = 0$ and we have a circle, so the system looks like two cones merged at this circe of radius $\sqrt{a}$. The further symmetry of the paraboloid under the swap $x \leftrightarrow y$ which again reflects the circular nature. Hence at the point of interest we are at the throat, and the tangent plane has the normal $\hat{x}$, it is the y-z plane shifted by $\sqrt{a}$.
Now time for an explicit calculation. An atlas is the two functions

$$f_{\pm}(y, z) = (\pm\sqrt{z^2 - y^2 + a}, y, z)$$

which is smooth if $z^2 - y^2 < a$ which certainly is true for our point. We have chosen to omit x since that is the only way to define a nonsingular derivative. Since the object at hand is two dimensional the expected tangent space is a plane. The Jacobian is a 2 by 3 matrix

$$df_{+,\vec{x}} = \begin{pmatrix} \frac{-1}{\sqrt{z^2-y^2+a}} y & \frac{1}{\sqrt{z^2-y^2+a}} z \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where we observe that f acts trivially on the x subspace. Clearly, this maps $\mathbb{R}^2$ to $\mathbb{R}^3$. At our point this is the trivial

$$df_{+,\vec{x}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The image of this operator is really in $\mathbb{R}^2$, as the first row is 0 the nontrivial component is the really the 2×2 block. Then the image is spanned by its columns, which are unit vectors in y and z directions. Hence our earlier observation is confirmed , the tangent space is given by the plane

$$x = \sqrt{a}$$

1.3 §4 Submersions: 1

1.3.1 1: If $f : X \to Y$ is a submersion and U is an open set of X, show that $f(U)$ is open in Y.

Given $f : X \subset \mathbb{R}^n \to Y \subset \mathbb{R}^m$ a submersion, i.e. the derivative is a surjective linear map everywhere, and U open in X, wish to show that $f(U)$ is open in Y. Note that $n \geq m$.
First some notation.
Let $x = (x_1, x_2, \cdots, x_m, x_{m+1}, \cdots, x_n) \in U$. Also, denote $x|_m$ the first m coordinates subsapce of x. Naturally, if $U \subset \mathbb{R}^n$ and $n \geq m$, define the "restriction of U to m dimensional subspace $U|_m$" as

$$U|_m = \{(x_1, x_2, \cdots, x_m)|\, x = (x_1, x_2, \cdots, x_m, x_{m+1}, \cdots, x_n) \in U\} \tag{1.4}$$

Since f is a submersion, there are diffeomorphisms i.e. charts ψ and ϕ such that the composition

$$(\psi^{-1} \circ f \circ \phi)(x) = (x_1, x_2, \cdots, x_m) = x|_m$$

i.e. coordinate systems in which the subspaces of the first m coordinates are "parallel".
Since $U \subset X \subset \mathbb{R}^n$, the preimage $\phi^{-1}(U) = \mathcal{U} \subset \mathbb{R}^n$ is open by diffeomorphism. Now let $U = \phi(\mathcal{U})$ and so

$$(f \circ \phi)(\mathcal{U}) = f(U) = \operatorname{Im}\psi(x|_m) = \psi(\mathcal{U}|_m \cap Y)$$

where the last term is the intersection of Y with restriction of $\mathcal{U}$ to $\mathbb{R}^m$. Since $\mathcal{U}$ is open, the intersection is open in Y by the subspace topology.
Then since ψ is a diffeomorphism, image of an open set is open, and hence is $f(U)$.

Chapter 2

Homework 2

2.1 §3 Inverse Function Theorem and Immersions: 1, 2

2.1.1 1: Let A be a linear map of $\mathbf{R}^n$, and $b \in \mathbf{R}^n$. Show that the mapping $x \to Ax + b$ is a diffeomorphism of $\mathbf{R}^n$ if and only if A is nonsingular.

Want to prove both ways.
($\Longrightarrow$) Suppose A is singular. Call the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. It is clearly a composition of (multiply by A from left, m) and (add b, n). Since f is a diffeomorphism, it has an inverse, namely the inverse of the composition $f^{-1} = (n \circ m)^{-1} = m^{-1} \circ n^{-1}$. The fact that m is invertible means there is an inverse matrix of A, hence A is nonsingular. Contradiction.
($\Longleftarrow$) Suppose A is nonsingular and $f(x) = Ax + b$, with domains as above. Define g

$$\begin{aligned} g :&\mathbb{R}^n \longrightarrow \mathbb{R}^n \\ &x \mapsto A^{-1}(x - b) \end{aligned}$$

which is clearly an inverse of f, and we have A^{-1} by nonsingularity. Needed is continuity and smoothness, which are given by the fact that m and n are compositions of continuous and smooth operations. Furthermore, the derivative of f is A, which is an isomorphism since it is invertible. Hence f is a local diffeomorphism (by Inverse Function Theorem) which is bijective (we have an inverse). Then f globally is a diffeomorphism.

2.1.2 2: *2. Suppose that Z is an l-dimensional submanifold of X and that $z \in Z$. Show that there exists a local coordinate system $\{x_1, \dots, x_k\}$ defined in a neighborhood U of z in X such that $Z \cap U$ is defined by the equations $x_{l+1} = 0, \dots, x_k = 0$.

We want to use the Local Immersion Theorem to show the coordiate functions take the form of the canonical immersion, i.e the injection $Z \cap U \overset{\iota}{\hookrightarrow} X$. Since Z is a submanifold there is the injection $\iota : Z \to X$, which has unit derivative, and hence is an immersion. Considering a neighbourhood $U \subset X$ of $z \in Z$, $Z \cup U \subset Z$ is an (open) neighbourhood of z. Then by the local immersion theorem, on $Z \cup U \subset Z$ and on a neighbourhood of $\iota(z) \in X$, there are charts such that the injection looks like the canonical immersion of Z into X.

2.2 §4 Submersions: 1, 2, 7

2.2.1 1: *1. If $f : X \to Y$ is a submersion and U is an open set of X, show that $f(U)$ is open in Y.

Refer to 1.3.1.

2.2.2 2a: If X is compact and Y connected, show every submersion $f : X \to Y$ is surjective.

Trivially assume $f(X)$ nonempty. Since X is compact, so is $f(X)$ by smoothness. By the previous problem, $f(X)$ is open. But since Y is a manifold, it is embedded in some $\mathbb{R}^n$, and hence is Hausdorff. Then $f(X) \subset Y$ is closed as it is compact. Since Y is connected, and $f(X)$ nonempty, then it must be that $f(X) = Y$. Let f be a submersion. Then it has a surjective derivative everywhere on X.

2.2.3 2b: Show that there exist no submersions of compact manifolds into Euclidean spaces.

Consider f as above but Y is some Euclidean $\mathbb{R}^n$. If there is such a submersion, $f(X) = \mathbb{R}^n$ is compact, contradiction with $\mathbb{R}^n$ not compact.

2.2.4 7: (Stack of Records Theorem.) Suppose that y is a regular value of $f : X \to Y$, where X is compact and has the same dimension as Y. Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$. Prove there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W_1 of x_i that are mapped diffeomorphically. Show that $f\,(X - \cup W_t)$ is compact and does not contain y.. See Figure 1-13.

1) $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$
Since y is regular, by the first version of the Preimage Theorem, the inverse image $f^{-1}(y)$ is a manifold of dimension $\dim X - \dim Y = 0$. Hence it is a set of points.
If it is infinite, it has a limit point in X and actually in itself, since a set of points is closed and hence contains its limit points. Since y is regular, and dimensions are equal, f is a local diffeomorphism for all points in the preimage. But it cannot be injective at the limit point x since there is always anoter point from $f^{-1}(y)$ in every open set containing x. Then there is no limit point in $f^{-1}(y)$ and hence it is finite.
2) There exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W_i of x_i that are mapped diffeomorphically. Show that $f\,(X - \cup W_i)$ is compact and does not contain y.
As shown $f^{-1}(y) = \{x_1, \ldots, x_N\}$. Let W_i be disjoint neighbourhoods of x_i, respectively, where their existance is guaranteed by Dr Hausdorff. The complement of union of Ws in X is closed and hence compact, since X is compact. By continuity, its image is also compact. y is not here as all

points that map to y are contained in the W_i. The complement $U = Y - f(X - \cup W_i)$ is open and hence contains y. Finally, the preimage $f^{-1}(U) = f^{-1}(Y - f(X - \cup W_i)) = f^{-1}(Y) - (X - \cup W_i) = \cup W_i$.

2.3 §7 Sard's Theorem and Morse Functions: 1, 4

2.3.1 1: Show that $\mathbb{R}^k$ has measure zero in $\mathbb{R}^l$, $k < l$.

(I apologize for abuse of notation on the generalized product. I tried to make it clear by using $\times$ vs $*$.)
This follows from the fact the injection of any rectangular solid $S = \prod_i^k (a_i, b_i)$ with $a_i < b_i$ in $\mathbb{R}^k$ to $\mathbb{R}^l$ will be $\iota(S) = \prod_i^k (a_i, b_i) \times \prod_k^l (0,0)$ with $a_i < b_i$ with volume given $\prod_i^k |a_i - b_i| * \prod_k^l 0 = 0$. Hence $\mathbb{R}^k$ contains no rectangular solids and has measure 0.

2.3.2 4: Prove that rationals have measure zero in $\mathbb{R}$.

Cover $\mathbb{Q}$ by the solids (here just intervals) $(q_i - \frac{\varepsilon}{2^{i+1}}, q_i + \frac{\varepsilon}{2^{i+1}})$, where the subscript i is some arbitrary ordering of the rationals, starting from $i = 2$ for convenience, indexing allowed by countability. This has volume

$$\sum_{i=2}^{\infty} 2 * \frac{\varepsilon}{2^{i+1}} = \varepsilon \sum_{i=2}^{\infty} \frac{1}{2^i} = \varepsilon * \frac{1}{2} < \varepsilon$$

as desired.