İşaret İşleme
Zaman Domaini Analizi

(Birim darbe cevabı

&

Sıfır durum cevabı)

H5CD1

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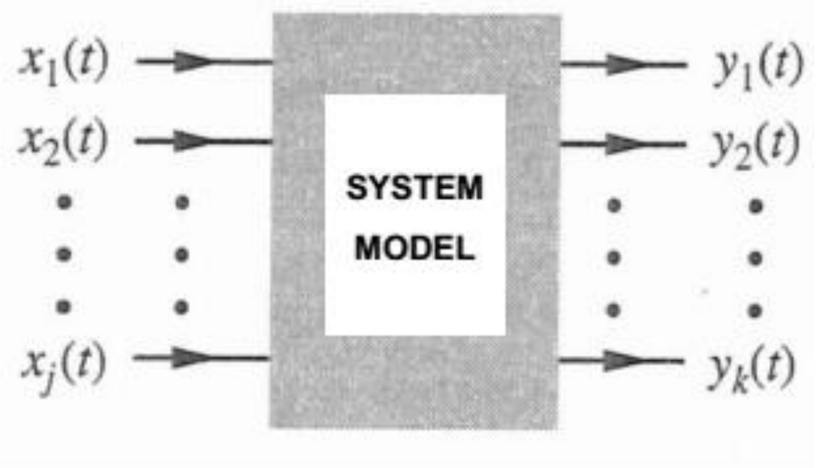
versiyon111020

Bir sistemin toplam tepkisi - (Total response)



Remember that for a Linear System

Total response = zero-input response + zero-state response

Önceki derslerimizde sistemin sıfır giriş cevabının nasıl hesaplanacağını öğrenmiştik

Sıfır Giriş Cevabı (SGC)-(Zero input response)

In this lecture, we will focus on a linear system's **zero-input response**, $y_0(t)$, which is the solution of the system equation when input $x(t)$ = 0.

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

$$\Rightarrow \quad Q(D)y(t) = P(D)x(t)$$

$$\Rightarrow \quad Q(D)y_0(t) = 0$$

$$\Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_0(t) = 0$$

Characteristic Polynomial of a system

$Q(\lambda)$ is called the **characteristic polynomial** of the system

$Q(\lambda) = 0$ is the **characteristic equation** of the system

The **roots** to the characteristic equation $Q(\lambda) = 0$, i.e. $\lambda_1, \lambda_2, \ldots, \lambda_N$, are extremely important.

They are called by different names:

- Characteristic values
- **Eigenvalues**
- **Natural frequencies**

The exponentials $e^{\lambda_i t} (i = 1, 2, \ldots, n)$ are the **characteristic modes** (also known as **natural modes**) of the system

Characteristics modes determine the system's behaviour

Q(λ) is called the **characteristic polynomial** of the system

The **roots** to the characteristic equation $Q(\lambda) = 0$, i.e. $\lambda_1, \lambda_2, \ldots, \lambda_N$, are extremely important.

- **NOT:**
  - **Sistemin karakteristik polinomuna ait kökler (λ) 3 farklı durumda olabilir.**
    - **Katsız (tekrar etmeyen) kök**
    - **Katlı (tekrar eden) kök**
    - **Kompleks kök**
  - **Her 3 durum için sistemin sıfır giriş cevabı farklı genel çözümlerle bulunur.**

- Sistemin karakteristik polinomuna ait kökler (λ_i) Katsız (tekrar etmeyen) kökler olduğunda sistemin sıfır giriş cevabına ait genel çözüm;

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}$$

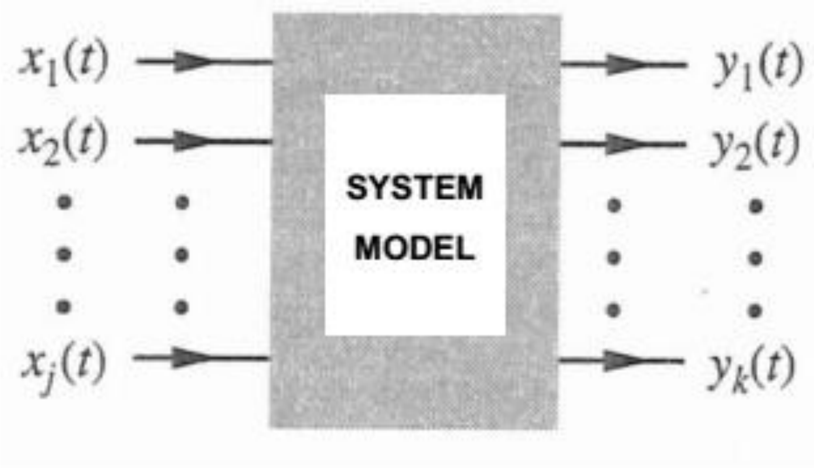
- Sistemin karakteristik polinomuna ait kökler (λ_i) Katlı (tekrar eden) kökler olduğunda sistemin sıfır giriş cevabına ait genel çözüm;

$$y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1}) e^{\lambda t}$$

- Sistemin karakteristik polinomuna ait kökler (λ_i) Kompleks kökler olduğunda sistemin sıfır giriş cevabına ait genel çözüm;

$$\begin{aligned} y_0(t) &= \frac{c}{2} e^{j\theta} e^{(\alpha + j\beta)t} + \frac{c}{2} e^{-j\theta} e^{(\alpha - j\beta)t} \\ &= \frac{c}{2} e^{\alpha t} \left[e^{j(\beta t + \theta)} + e^{-j(\beta t + \theta)} \right] \\ &= c e^{\alpha t} \cos(\beta t + \theta) \end{aligned}$$

Bir sistemin toplam tepkisi - (Total response)



Remember that for a Linear System

Total response = zero-input response + zero-state response

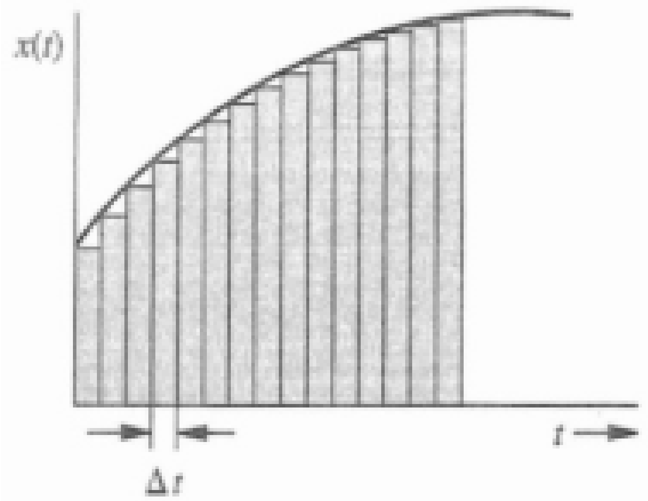
Sistemin sıfır durum cevabını bulmak için önce birim darbe cevabı hesaplanmalıdır.

Birim Darbe Cevabı
(Unit Impulse Response)

The importance of Impulse Response h(t)

- **Zero-state response** assumes that the system is in “rest” state, i.e. all internal system variables are zero.
- Deriving and understanding zero-state response depends on knowing the **impulse response** $h(t)$ to a system.
- Any input $x(t)$ can be broken into many **narrow rectangular pulses**. Each pulse produces a system response.

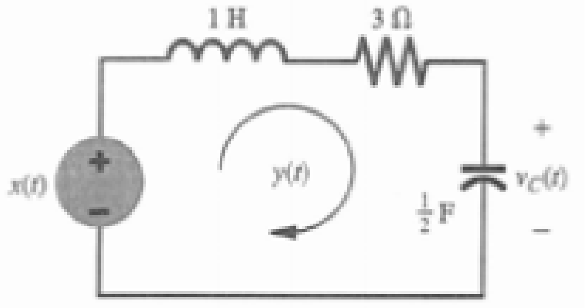
- Since the system is linear and time invariant, the system response to $x(t)$ is the sum of its responses to all the impulse components.
- $h(t)$ is the system response to the rectangular pulse at $t=0$ as the pulse **width approaches zero**.



How to determine the unit impulse response h(t)? (1)

Given that a system is specified by the following differential equation, determine its unit impulse response h(t).

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y(t)=\frac{dx}{dt}$$



Remember the general system equation:

$$Q(D)y(t)=P(D)x(t)$$

It can be shown that the impulse response *h(t)* is given by:

$$h(t)=[P(D)y_n(t)]\,u(t) \quad \text{..... (4.3.1)}$$

where *u(t)* is the unit step function, and **$y_n(t)$ is a linear combination of the characteristic modes of the system.**

$$y_n(t)=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}+........+c_Ne^{\lambda_N t}$$

How to determine the unit impulse response h(t)? (2)

The constants c_i are determined by the following initial conditions:

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = ... = y_n^{(N-2)}(0) = 0, \quad y_n^{(N-1)}(0) = 1.$$

Note $y_n^{(k)}(0)$ is the kth derivative of $y_n(t)$ at t = 0.

The above is true if M, the order of P(D), is less than N, the order of Q(D) (which is generally the case for most stable systems).

The Example (1)

Determine the impulse response for the system: $(D^2 + 3D + 2)\, y(t) = Dx(t)$

This is a second-order system (i.e. N=2, M=1) and the characteristic polynomial is:

$$(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2)$$

The characteristic roots are λ = -1 and λ = -2.

Therefore : $y_n(t) = c_1 e^{-t} + c_2 e^{-2t}$

Differentiating this equation yields: $\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$

The initial conditions are

$$\dot{y}_n(0) = 1 \qquad \text{and} \qquad y_n(0) = 0$$

The Example (2)

Setting $t = 0$ and substituting the initial conditions yield:

$$0 = c_1 + c_2$$
$$1 = -c_1 - 2c_2$$

The solution of these equations are:

$$c_1 = 1 \qquad \text{and} \qquad c_2 = -1$$

Therefore we obtain

$$y_n(t) = e^{-t} - e^{-2t}$$

Remember that $h(t)$ is given by:

$$h(t) = [P(D)y_n(t)]\,u(t)$$

and $P(D) = D$ in this case.

Therefore

$$h(t) = [P(D)y_n(t)]u(t) = (-e^{-t} + 2e^{-2t})u(t)$$

$$P(D)y_n(t) = Dy_n(t) = \dot{y}_n(t) = -e^{-t} + 2e^{-2t}$$

Sıfır Durum Cevabı
(Zero State Response)

Zero-state Response (1)

We now consider how to determine the **system response** $y(t)$ to an input $x(t)$ when system is in zero state.

Define a pulse $p(t)$ of unit height and width $\Delta\tau$ at t=0:

Input $x(t)$ can be represented as sum of narrow rectangular pulses.

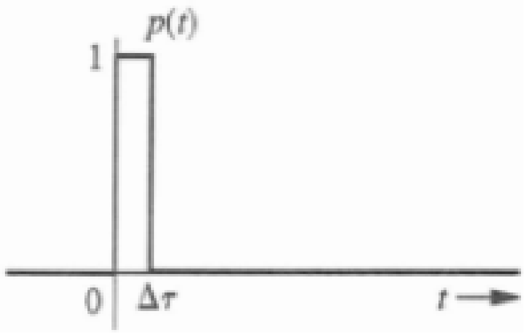
The pulse at $t = n\Delta\tau$ has a height $x(t) = x(n\Delta\tau)$.

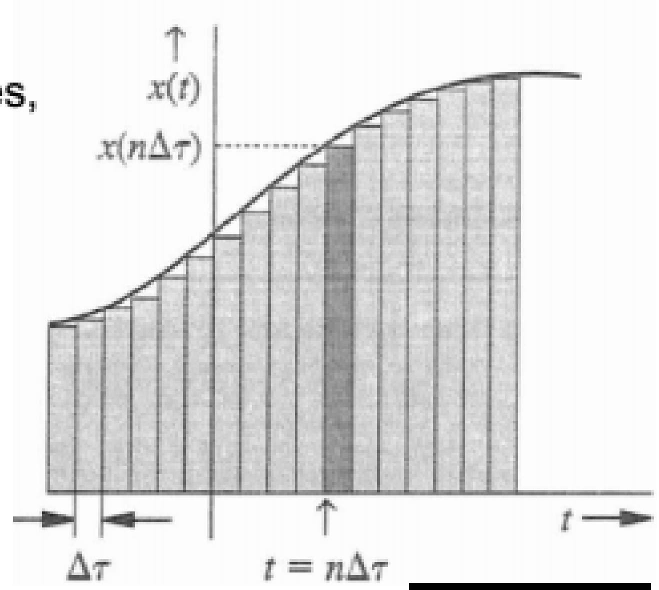
This can be expressed as $x(n\Delta\tau)\, p(t - n\Delta\tau)$.

Therefore $x(t)$ is the sum of all $[x(n\Delta\tau)/\Delta\tau]$. such pulses, i.e.

$$x(t) = \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)\, p(t - n\Delta\tau)$$

$$= \lim_{\Delta\tau \to 0} \sum_{\tau} \left[\frac{x(n\Delta\tau)}{\Delta\tau} \right] p(t - n\Delta\tau)\Delta\tau$$





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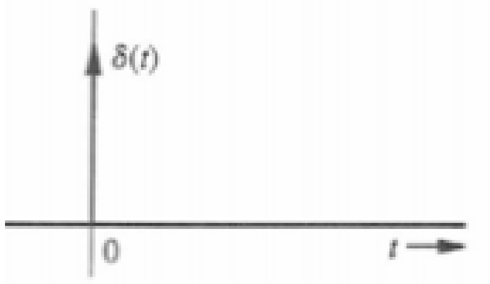
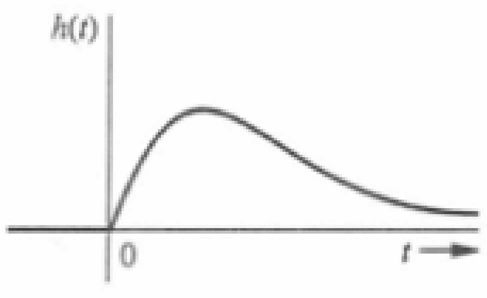
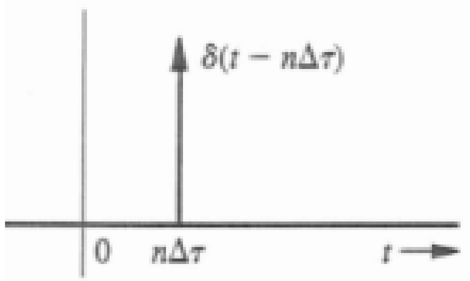
Zero-state Response (2)

input $\Longrightarrow$ output

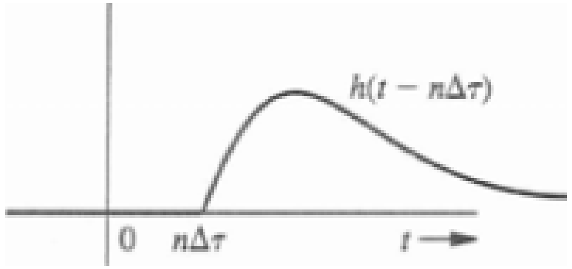


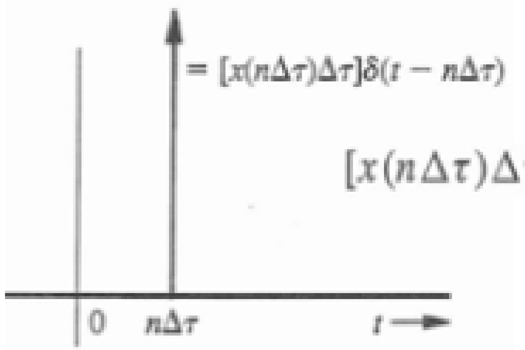
$$\delta(t) \Longrightarrow h(t)$$



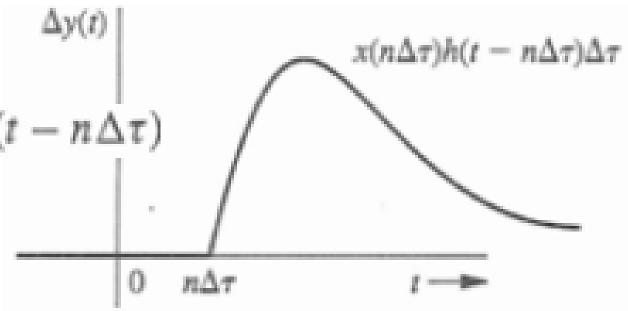


$$\delta(t-n\Delta\tau) \Longrightarrow h(t-n\Delta\tau)$$





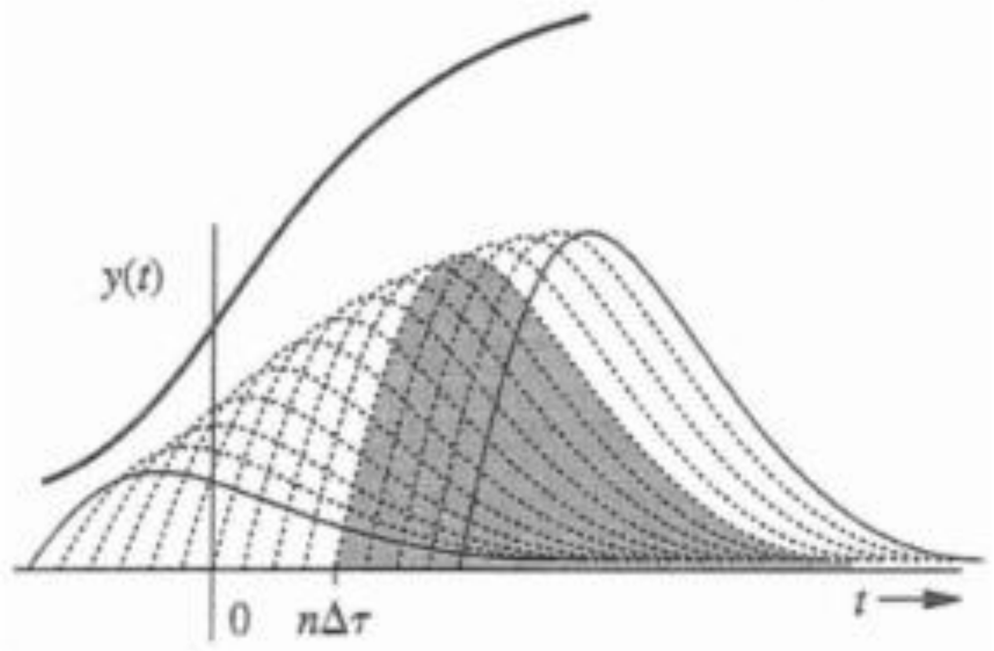
$$[x(n\Delta\tau)\Delta\tau]\delta(t-n\Delta\tau) \Longrightarrow [x(n\Delta\tau)\Delta\tau]h(t-n\Delta\tau)$$



Zero-state Response (3)

$$\text{input} \Longrightarrow \text{output}$$

$$\underbrace{\lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)\delta(t - n\Delta\tau)\,\Delta\tau}_{x(t)} \Longrightarrow \underbrace{\lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau}_{y(t)}$$



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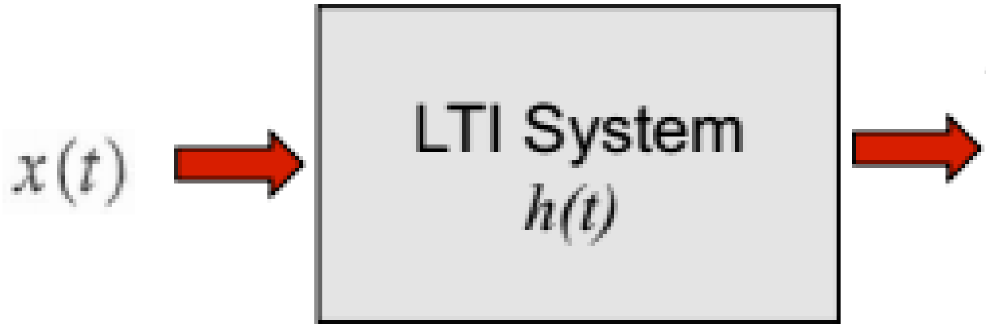
Zero-state Response (4)

- Therefore,

$$y(t) = \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

- Knowing h(t), we can determine the response y(t) to any input x(t).
- Observe the all-pervasive nature of the system's characteristic modes, which determines the impulse response of the system.



$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

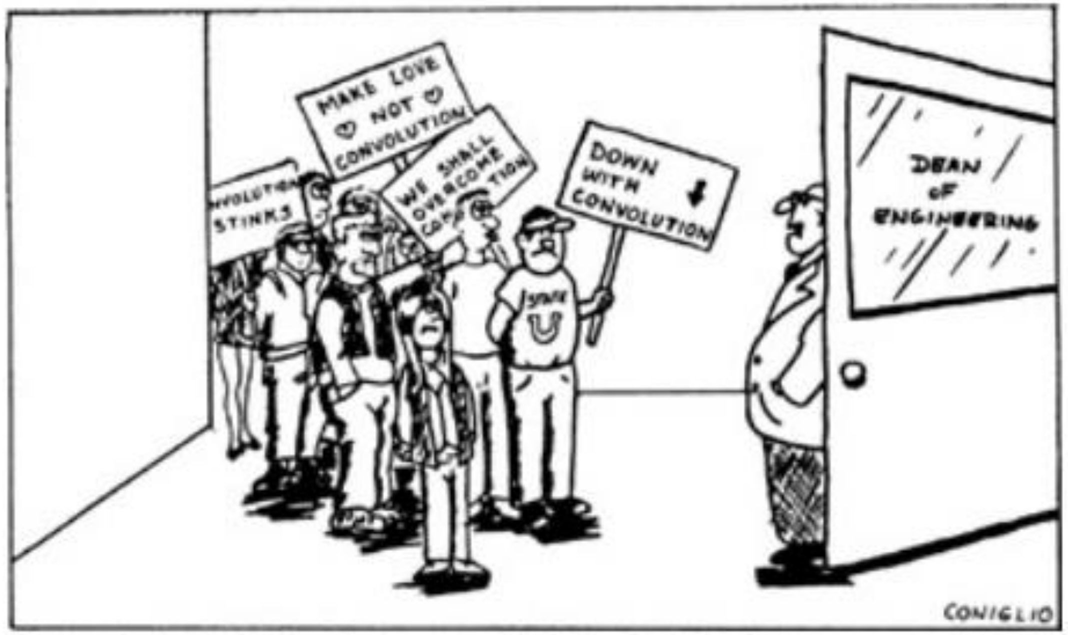
The Convolution Integral

- The derived integral equation occurs frequently in physical sciences, engineering and mathematics.
- It is given the name: the convolution integral.
- The convolution integral of two functions $x_1(t)$ and $x_2(t)$ is denoted symbolically as

$$x_1(t) * x_2(t) \equiv \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau)\, d\tau$$

- Note that the convolution operator is linear, i.e. it obeys the principle of superposition.

Time-domain analysis: Convolution



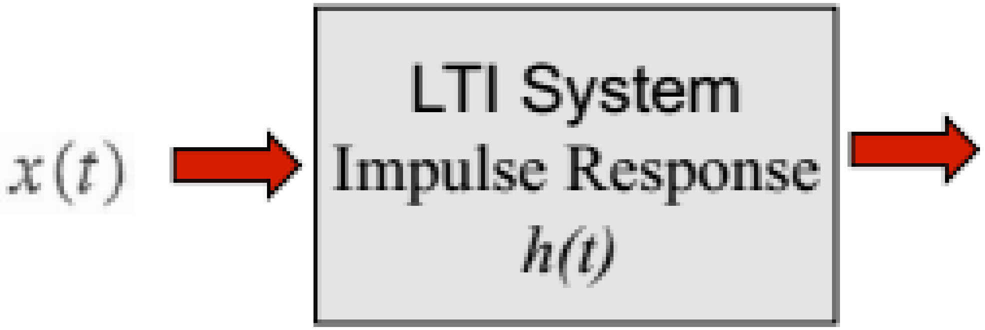
Convolution: its bark is worse than its bite!

Convolution Integral

- **Convolution Integral:**

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- System output (i.e. zero-state response) is found by convolving input x(t) with System's impulse response h(t).



$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Table 2.1: Convolution Table

Open table as spreadsheet

No.	$x_1(t)$	$x_2(t)$	$x_1(t)*x_2(t) = x_2(t)*x_1(t)$
1	$x(t)$	$\delta(t-T)$	$x(t-T)$
2	$e^{\lambda t}u(t)$	$u(t)$	$\frac{1-e^{\lambda t}}{-\lambda}u(t)$
3	$u(t)$	$u(t)$	$tu(t)$
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}u(t) \qquad \lambda_1 \neq \lambda_2$
5	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$
6	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$
7	$t^Nu(t)$	$e^{\lambda t}u(t)$	$\frac{N!\,e^{\lambda t}}{\lambda^{N+1}}u(t)-\sum_{k=0}^{N}\frac{N!\,t^{N-k}}{\lambda^{k+1}(N-k)!}u(t)$
8	$t^Mu(t)$	$t^Nu(t)$	$\frac{M!N!}{(M+N+1)!}t^{M+N+1}u(t)$

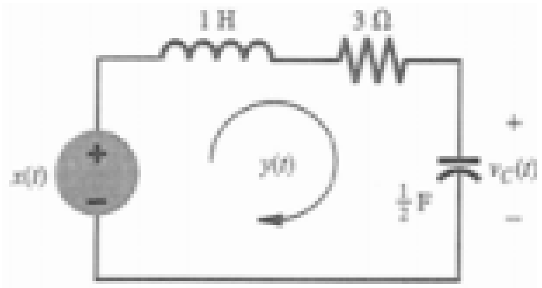
9	$te^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\dfrac{e^{\lambda_2 t} - e^{\lambda_1 t} + (\lambda_1 - \lambda_2)te^{\lambda_1 t}}{(\lambda_1 - \lambda_2)^2}u(t)$
10	$t^M e^{\lambda t}u(t)$	$t^N e^{\lambda t}u(t)$	$\dfrac{M!\,N!}{(N + M + 1)!}t^{M+N+1}e^{\lambda t}u(t)$
11	$t^M e^{\lambda_1 t}u(t)$ $\lambda_1 \neq \lambda_2$	$t^N e^{\lambda_2 t}u(t)$	$\displaystyle\sum_{k=0}^{M}\frac{(-1)^k M!(N+k)!\,t^{M-k}e^{\lambda_1 t}}{k!(M-k)!(\lambda_1-\lambda_2)^{N+k+1}}u(t) + \sum_{k=0}^{N}\frac{(-1)^k N!(M+k)!\,t^{N-k}e^{\lambda_2 t}}{k!(N-k)!(\lambda_2-\lambda_1)^{M+k+1}}u(t)$
12	$e^{-\alpha t}\cos(\beta t + \theta)u(t)$	$e^{\lambda_1 t}u(t)$	$\dfrac{\cos(\theta - \phi)e^{\lambda t} - e^{-\alpha t}\cos(\beta t + \theta - \phi)}{\sqrt{(\alpha + \lambda)^2 + \beta^2}}u(t)$
13	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(-t)$	$\dfrac{e^{\lambda_1 t}u(t) + e^{\lambda_2 t}u(-t)}{\lambda_2 - \lambda_1}$ $\quad \mathrm{Re}\,\lambda_2 > \mathrm{Re}\,\lambda_1$
14	$e^{\lambda_1 t}u(-t)$	$e^{\lambda_2 t}u(-t)$	$\dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_2 - \lambda_1}u(-t)$

Total Response

Total response = zero-input response + zero-state response

$$\text{total response} = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{\text{zero-input component}} + \underbrace{x(t) * h(t)}_{\text{zero-state component}}$$

- Let us put everything together, using our RLC circuit as an example.
- Let us assume $x(t) = 10e^{-3t}u(t), y(0) = 0, \quad \dot{y}(0) = -5.$
- In earlier slides, we have shown that



$$\text{total current} = \underbrace{(-5e^{-t} + 5e^{-2t})}_{\text{zero-input current}} + \underbrace{(-5e^{-t} + 20e^{-2t} - 15e^{-3t})}_{\text{zero-state current}} \qquad t \geq 0$$

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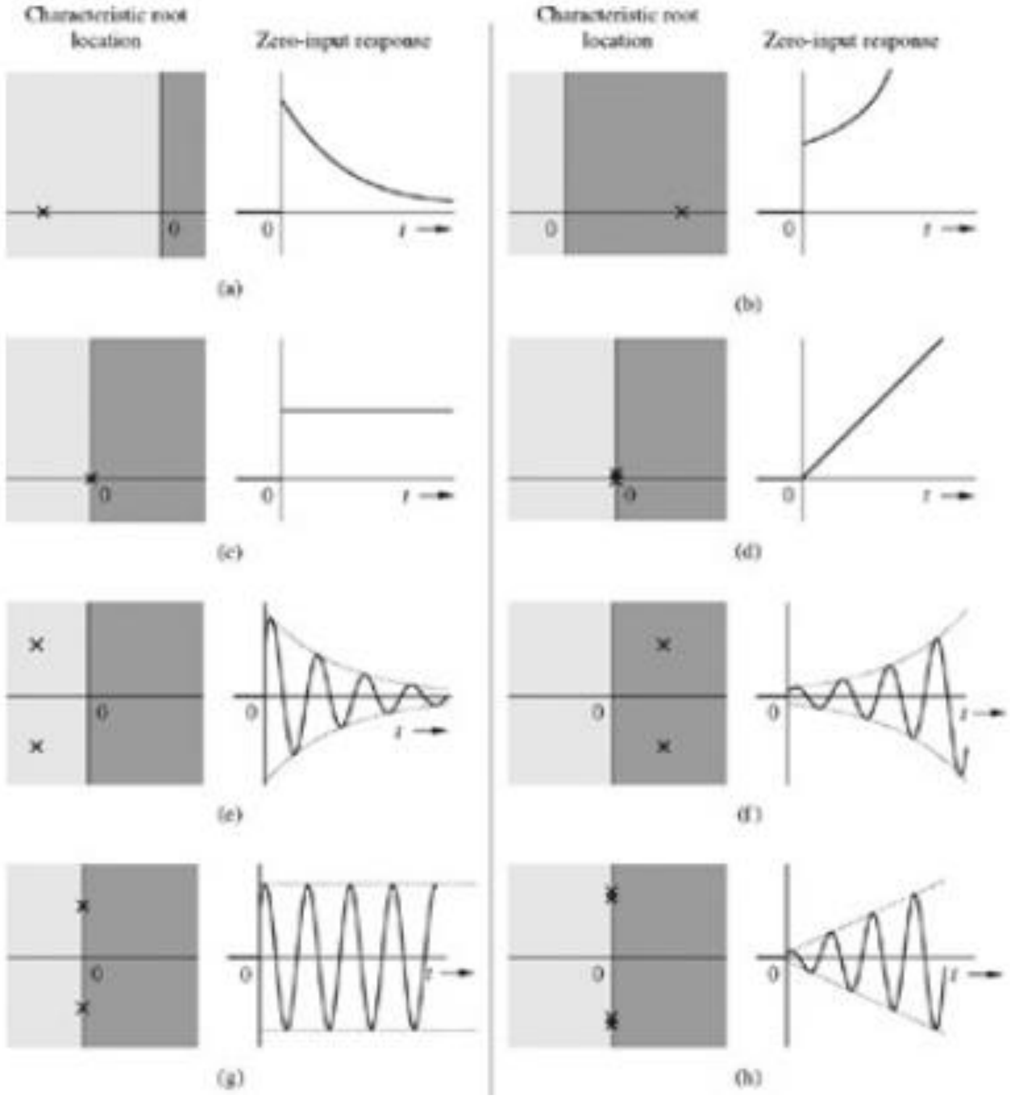
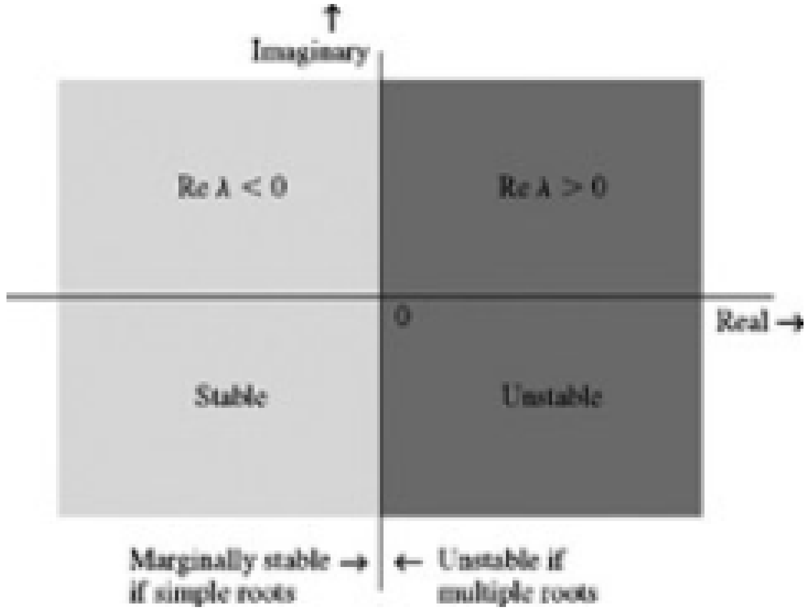
Internal (Asymptotic) Stability





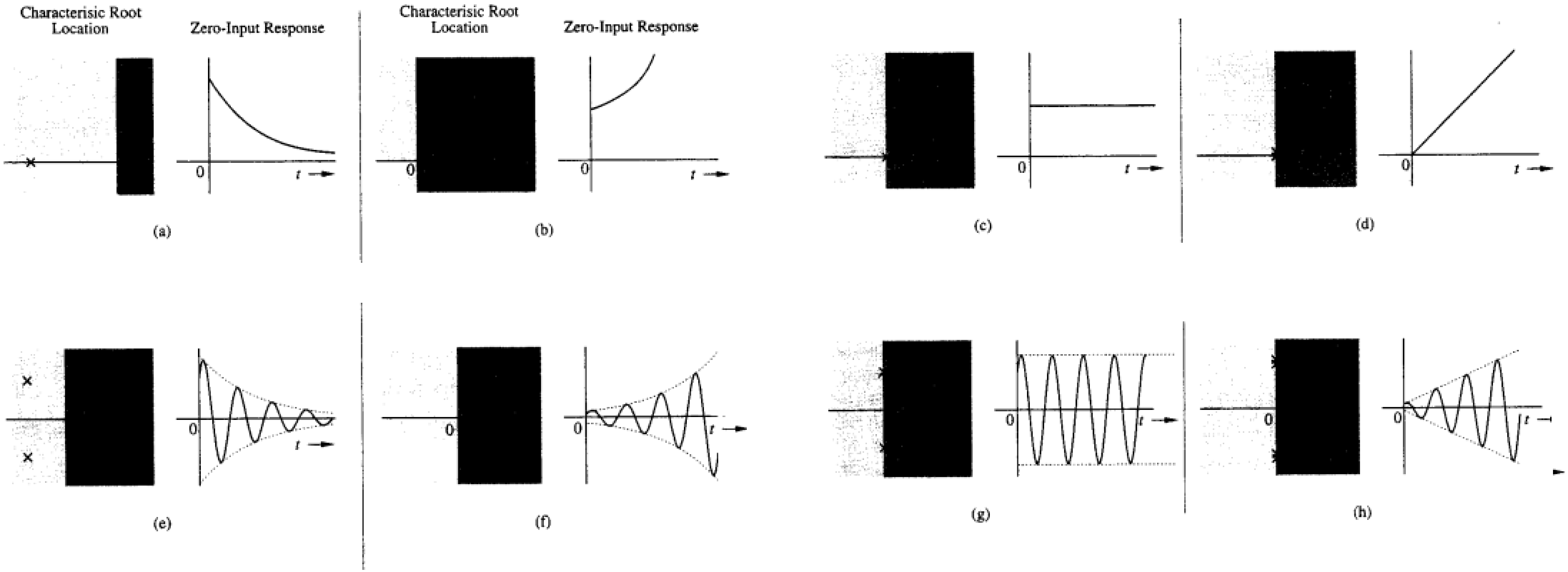


Fig. Location of characteristic roots and the corresponding characteristic modes.

Sürekli zamanda sistem kararlılığı örnekleri

Investigate the asymptotic and the BIBO stability of LTIC system described by the following equations, assuming that the equations are internal system description.

a. $(D+1)(D^2+4D+8)y(t) = (D-3)x(t)$

b. $(D-1)(D^2+4D+8)y(t) = (D+2)x(t)$

c. $(D+2)(D^2+4)y(t) = (D^2+D+1)x(t)$

d. $(D+1)(D^2+4)^2y(t) = (D^2+2D+8)x(t)$

Sürekli zamanda sistem kararlılığı örnekleri-devam

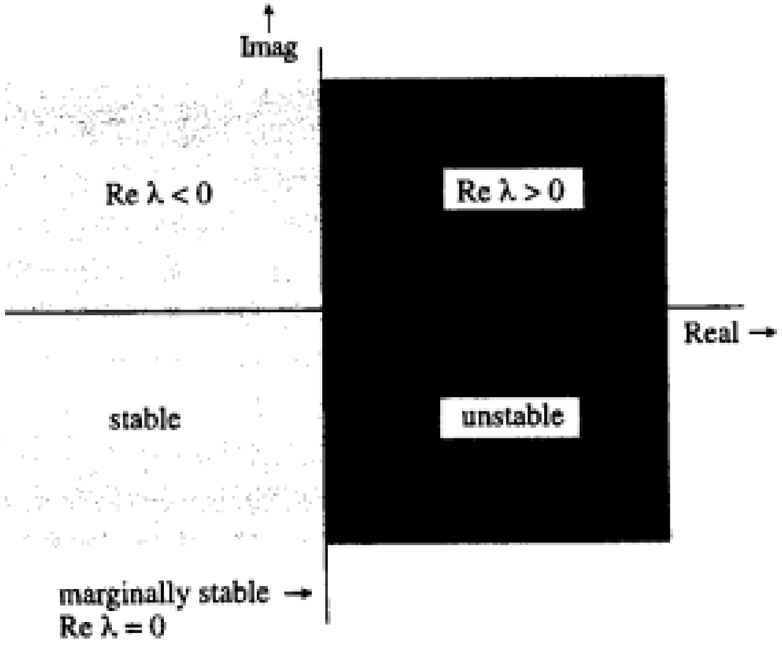
The characteristic polynomials of these systems are

a. $(\lambda + 1)(\lambda^2 + 4\lambda + 8) = (\lambda + 1)(\lambda + 2 - j2)(\lambda + 2 + j2)$

b. $(\lambda - 1)(\lambda^2 + 4\lambda + 8) = (\lambda - 1)(\lambda + 2 - j2)(\lambda + 2 + j2)$

c. $(\lambda + 2)(\lambda^2 + 4) = (\lambda + 2)(\lambda - j2)(\lambda + j2)$

d. $(\lambda + 1)(\lambda^2 + 4)^2 = (\lambda + 2)(\lambda - j2)^2(\lambda + j2)^2$



Characteristic roots location and system stability.

Sürekli zamanda sistem kararlılığı örnekleri-devam

a. $(\lambda + 1)(\lambda^2 + 4\lambda + 8) = (\lambda + 1)(\lambda + 2 - j2)(\lambda + 2 + j2)$

b. $(\lambda - 1)(\lambda^2 + 4\lambda + 8) = (\lambda - 1)(\lambda + 2 - j2)(\lambda + 2 + j2)$

c. $(\lambda + 2)(\lambda^2 + 4) = (\lambda + 2)(\lambda - j2)(\lambda + j2)$

d. $(\lambda + 1)(\lambda^2 + 4)^2 = (\lambda + 1)(\lambda - j2)^2(\lambda + j2)^2$

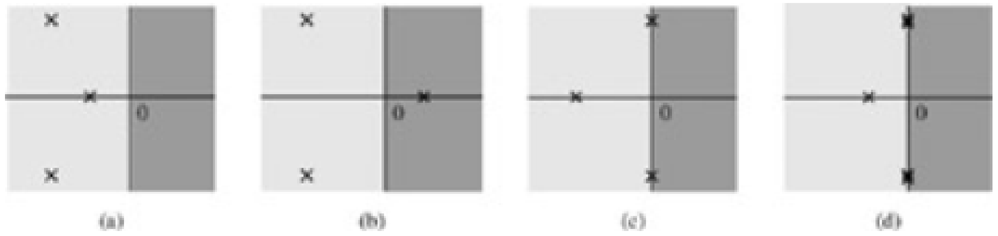
Consequently, the characteristic roots of the systems are (see Fig. 2.20):

a. $-1, -2 \pm j2$

b. $1, -2 \pm j2$

c. $-2 \pm j2$

d. $1, - \pm j2, \pm j2$



Bu ders notu için faydalanılan kaynaklar

Lecture 4

Time-domain analysis:
Zero-state Response
(Lathi 2.3-2.4.1)

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