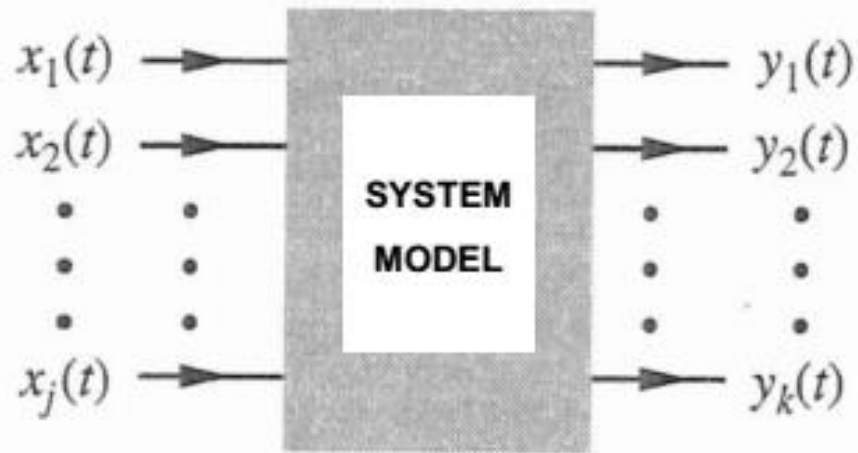


# İşaret İşleme

## Zaman Domaini Analizi-H3CD2

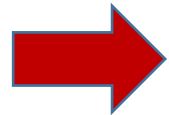
Dr. Meriç Çetin  
versiyon081022

Sürekli zamanlı sistemleri  
tekrar hatırlayalım



In general, relationship between  $x(t)$  and  $y(t)$  in a **linear time-invariant (LTI)** differential system is given by (where all coefficients  $a_i$  and  $b_i$  are constants):

$$\begin{aligned} \frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy}{dt} + a_N y(t) \\ = b_{N-M} \frac{d^M x}{dt^M} + b_{N-M+1} \frac{d^{M-1} x}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx}{dt} + b_N x(t) \end{aligned}$$



Use compact notation **D** for **operator  $d/dt$** ,

$$\frac{dy}{dt} \equiv Dy(t) \text{ and } \frac{d^2 y}{dt^2} \equiv D^2 y(t)$$

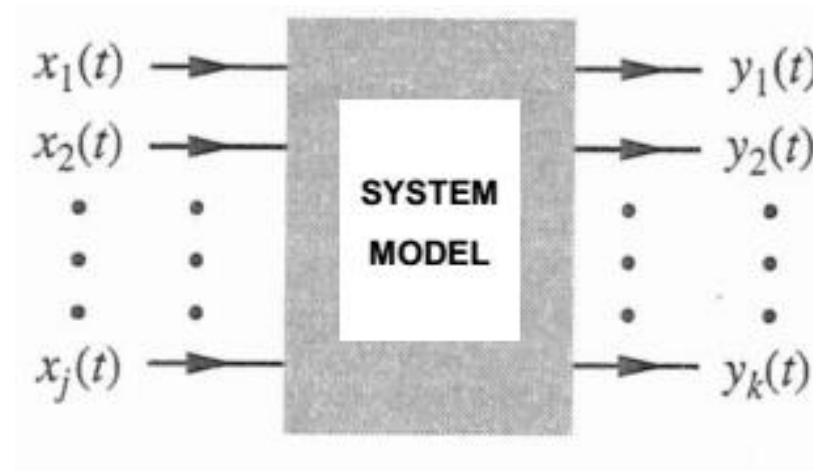
We get:  $(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t)$   
 $= (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$

or  $Q(D)y(t) = P(D)x(t)$

$$Q(D) = D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N$$

$$P(D) = b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N$$

## Bir sistemin toplam tepkisi - (Total response)



Remember that for a Linear System

**Total response = zero-input response + zero-state response**

# Characteristic Polynomial of a system

---

$Q(\lambda)$  is called the **characteristic polynomial** of the system

$Q(\lambda) = 0$  is the **characteristic equation** of the system

The **roots** to the characteristic equation  $Q(\lambda) = 0$ , i.e.  $\lambda_1, \lambda_2, \dots, \lambda_N$ , are extremely important.

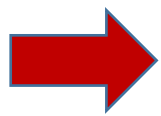
They are called by different names:

- Characteristic values
- **Eigenvalues**
- **Natural frequencies**

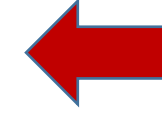
The exponentials  $e^{\lambda_i t}$  ( $i = 1, 2, \dots, n$ ) are the **characteristic modes** (also known as **natural modes**) of the system

Characteristics modes determine the system's behaviour

- Sistemin karakteristik polinomuna ait kökler ( $\lambda_i$ ) **Katsız (tekrar etmeyen) kökler** olduğunda sistemin sıfır giriş cevabına ait genel çözüm;

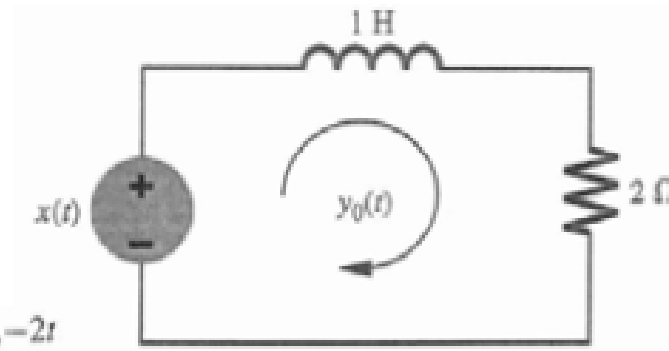


$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_N e^{\lambda_N t}$$



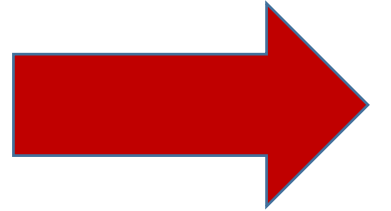
# Bir Örnek;

- ◆ This example demonstrates that any combination of characteristic modes can be sustained by the system with no external input.
- ◆ Consider this RL circuit:
- ◆ The loop equation is:  $(D + 2)y(t) = x(t)$
- ◆ It has a single characteristic root  $\lambda = -2$ , and the characteristic mode is  $e^{-2t}$
- ◆ Therefore, the loop current equation is  $y(t) = ce^{-2t}$



The loop current is sustained by the RL circuit on its own without any external input.

Sistemin karakteristik polinomuna ait kökler ( $\lambda$ ) **katlı** yani **tekrar eden** durumda ise sistemin sıfır giriş cevabı için genel çözüm formu değişecektir.





# Sıfır Giriş Cevabı Bulunurken Tekrar Eden Kökler Varsa;

- ◆ The discussions so far assume that all characteristic roots are **distinct**. If there are **repeated roots**, the form of the solution is modified.
- ◆ The solution of the equation:

$$(D - \lambda)^2 y_0(t) = 0$$

is given by:

$$y_0(t) = (c_1 + c_2 t) e^{\lambda t}$$

- ◆ In general, the characteristic modes for the differential equation:

$$(D - \lambda)^r y_0(t) = 0$$

are:

$$e^{\lambda t}, t e^{\lambda t}, t^2 e^{\lambda t}, \dots, t^{r-1} e^{\lambda t}$$

- ◆ The solution for  $y_0(t)$  is

$$y_0(t) = (c_1 + c_2 t + \dots + c_r t^{r-1}) e^{\lambda t}$$

# Tekrar Eden Kökler için Bir Örnek

Find  $y_0(t)$ , the zero-input component of the response for a LTI system described by the following differential equation:  $(D^2 + 6D + 9)y = (3D + 5)x(t)$

when the initial conditions are  $y_0(0) = 3$ ,  $\dot{y}_0(0) = -7$ .

- ◆ The **characteristic polynomial** for this system is:

$$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$$

- ◆ The **repeated roots** are therefore  $\lambda_1 = -3$  and  $\lambda_2 = -3$ .

- ◆ The **zero-input response** is

$$y_0(t) = (c_1 + c_2 t)e^{-3t}$$

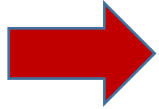
- ◆ Now, determine the constants using the initial conditions gives  $c_1 = 3$  and  $c_2 = 2$ .

- ◆ Therefore:

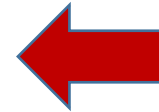
$$y_0(t) = (3 + 2t)e^{-3t} \quad t \geq 0$$

L2.2 p156

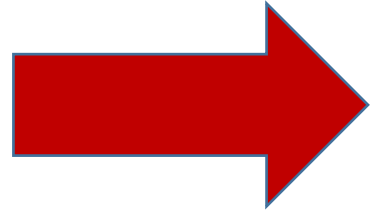
- Sistemin karakteristik polinomuna ait kökler ( $\lambda_i$ ) **Katlı (tekrar eden) kökler** olduğunda sistemin sıfır giriş cevabına ait genel çözüm;



$$y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1}) e^{\lambda t}$$



Sistemin karakteristik polinomuna ait kökler ( $\lambda$ ) **kompleks** ise sistemin sıfır giriş cevabı için genel çözüm formu değişecektir.



# Sıfır Giriş Cevabı Bulunurken Kompleks Kökler Varsa;

- ◆ Solutions of the characteristic equation may result in **complex roots**.
- ◆ For real (i.e. physically realizable) systems, all complex roots must occur in **conjugate pairs**. In other words, the coefficients of the characteristic polynomial  $Q(\lambda)$  are real.
- ◆ In other words, if  $\alpha + j\beta$  is a root, then there **must exists** the root  $\alpha - j\beta$ .
- ◆ The zero-input response corresponding to this pair of conjugate roots is:

$$y_0(t) = c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}$$

- ◆ For a real system, the response  $y_0(t)$  must also be real. This is possible only if  $c_1$  and  $c_2$  are conjugates too.
- ◆ Let
- ◆ This gives

$$\begin{aligned} c_1 &= \frac{C}{2} e^{j\theta} \quad \text{and} \quad c_2 = \frac{C}{2} e^{-j\theta} \\ y_0(t) &= \frac{C}{2} e^{j\theta} e^{(\alpha + j\beta)t} + \frac{C}{2} e^{-j\theta} e^{(\alpha - j\beta)t} \\ &= \frac{C}{2} e^{\alpha t} [e^{j(\beta t + \theta)} + e^{-j(\beta t + \theta)}] \\ &= C e^{\alpha t} \cos(\beta t + \theta) \end{aligned}$$

L2.2 p155

# Kompleks Kökler için Bir Örnek

Find  $y_0(t)$ , the zero-input component of the response for a LTI system described by the following differential equation:  $(D^2 + 4D + 40)x(t) = (D + 2)x(t)$

when the initial conditions are  $y_0(0) = 2$ ,  $\dot{y}_0(0) = 16.78$ .

- ◆ The characteristic polynomial for this system is:

$$\begin{aligned}\lambda^2 + 4\lambda + 40 &= (\lambda^2 + 4\lambda + 4) + 36 = (\lambda + 2)^2 + (6)^2 \\ &= (\lambda + 2 - j6)(\lambda + 2 + j6)\end{aligned}$$

- ◆ The complex roots are therefore  $\lambda_1 = -2 + j6$  and  $\lambda_2 = -2 - j6$

- ◆ The zero-input response in real form is ( $\alpha = -2$ ,  $\beta = 6$ )

$$y_0(t) = ce^{-2t} \cos(6t + \theta) \quad \dots (12.1)$$

# Kompleks Kökler için Bir Örnek-devam

- ◆ To find the constants  $c$  and  $\theta$ , we use the initial conditions  $y_0(0) = 2$ ,  $\dot{y}_0(0) = 16.78$ .
- ◆ Differentiating equation (12.1)  $y_0(t) = ce^{-2t} \cos(6t + \theta)$  gives:

$$\dot{y}_0(t) = -2ce^{-2t} \cos(6t + \theta) - 6ce^{-2t} \sin(6t + \theta)$$

- ◆ Using the initial conditions, we obtain:
$$2 = c \cos \theta$$
$$16.78 = -2c \cos \theta - 6c \sin \theta$$

- ◆ This reduces to:  $c \cos \theta = 2$

$$c \sin \theta = -3.463$$

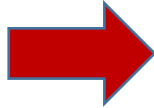
- ◆ Hence

$$c^2 = (2)^2 + (-3.464)^2 = 16 \implies c = 4 \qquad \theta = \tan^{-1} \left( \frac{-3.463}{2} \right) = -\frac{\pi}{3}$$

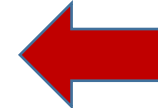
- ◆ Finally, the solution is

$$y_0(t) = 4e^{-2t} \cos \left( 6t - \frac{\pi}{3} \right)$$

- Sistemin karakteristik polinomuna ait kökler ( $\lambda_i$ ) **Kompleks** olduğunda sistemin sıfır giriş cevabına ait genel çözüm;



$$\begin{aligned}y_0(t) &= \frac{c}{2}e^{j\theta}e^{(\alpha+j\beta)t} + \frac{c}{2}e^{-j\theta}e^{(\alpha-j\beta)t} \\&= \frac{c}{2}e^{\alpha t} [e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}] \\&= ce^{\alpha t} \cos(\beta t + \theta)\end{aligned}$$





# The Resonance Behaviour

- ◆ Any signal consisting of a system's characteristic mode is **sustained by the system** on its own.
- ◆ In other words, the system offers NO obstacle to such signals.
- ◆ It is like asking an alcoholic to be a whisky taster.
- ◆ Driving a system with an input of the form of the characteristic mode will cause **resonance behaviour**.

Tacoma Bridge Disaster

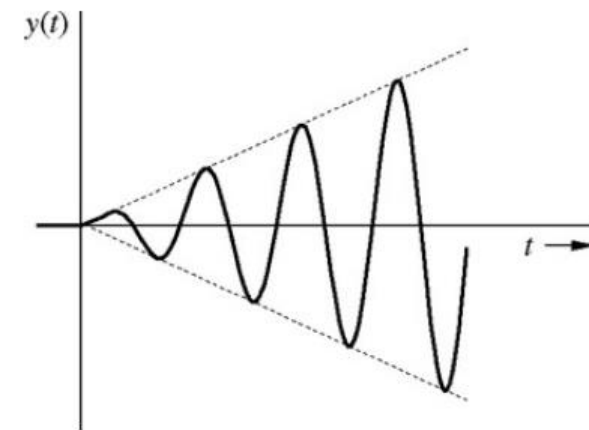
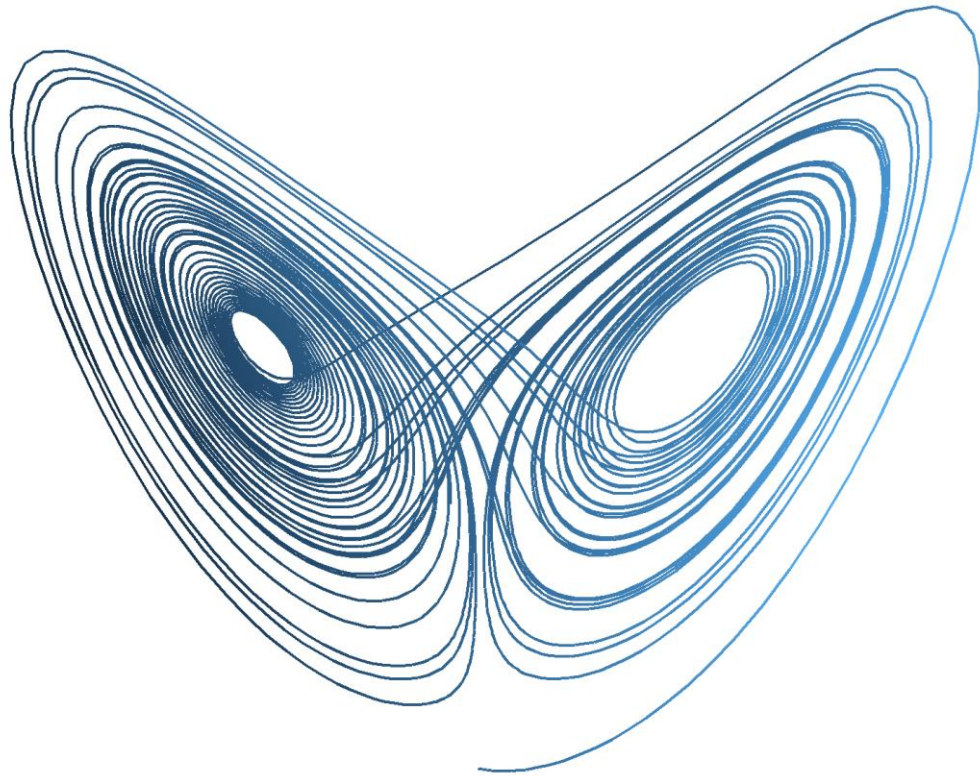


Figure 2.24: Buildup of system response in resonance.

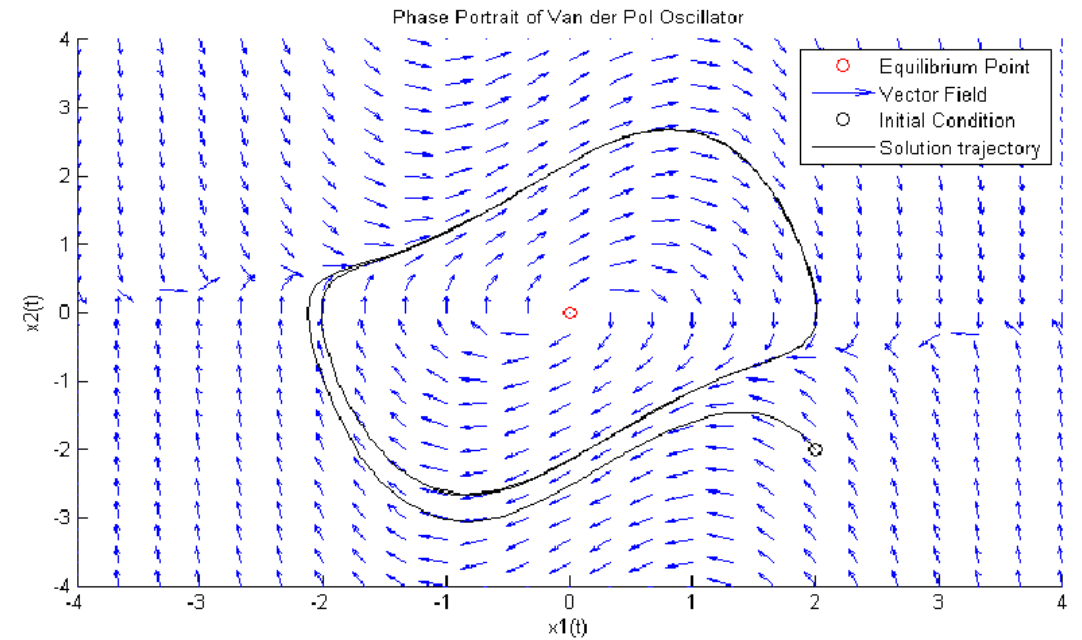
- <https://www.youtube.com/watch?v=3mclp9QmCGs>

← DEMO

- Strange Attractors
- <https://www.youtube.com/watch?v=dP3qAq9RNLg>



Van der Pol oscillator



# Bu ders notu için faydalanılan kaynaklar

## Lecture 3

### **Time-domain analysis: Zero-input Response** (Lathi 2.1-2.2)

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