Karakteristik polinomu:

$\det(A-\lambda I)=0$

$f_A(\lambda)=\det(A-\lambda I)$

$$A=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$f_A(\lambda)=-\lambda^3+7\lambda^2-14\lambda+8$

özdeğerler $\lambda_1=1,\ \lambda_2=2$ ve $\lambda_3=4$

özvektörler $v_1=\begin{bmatrix}1\\-1\\1\end{bmatrix}$ $v_2=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ ve $v_3=\begin{bmatrix}1\\2\\1\end{bmatrix}$

Lineer Denklem sistemlerinde iterasyon metodları

$AX=B$ sisteminde

Gauss-Jacobi metodu:

$$X_i=\frac{1}{a_{ii}}\left[b_i-\sum_{\substack{j=1\\ i\neq j}}^{n} a_{ij}x_j\right] \quad i=1,2,\ldots,n$$

iterasyonu

$$X_i^{(m+1)}=\frac{1}{a_{ii}}\left[b_i-\sum_{\substack{j=1\\ i\neq j}}^{n} a_{ij}x_j^{(m)}\right],\quad i=1,2,\ldots,n$$

kısaca $\underline{X}^{(m+1)}=b+B\underline{X}^{(m)}$ $(m\geq 0)$ olur.

örnek: $A=\begin{bmatrix} 10 & 3 & 1 \\ 2 & -10 & 3 \\ 1 & 3 & 10 \end{bmatrix}$, $b=\begin{bmatrix}14\\-5\\14\end{bmatrix}$

$\underline{X}^{(0)}=[0,0,0]^T$ başlangıç değerler için

$A\underline{X}=B$ sistemini Gauss-Jacobi ile $\underline{X}^{(6)}=?$

Çözüm:

$$10x_1 + 3x_2 + x_3 = 14$$
$$2x_1 - 10x_2 + 3x_3 = -5$$
$$x_1 + 3x_2 + 10x_3 = 14$$

$x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ başlangıç değeri için Gauss-Jacobi ile $\underline{x}^{(6)} = ?$

$$10x_1 = 14 - 3x_2 - x_3 \Rightarrow x_1 = \frac{1}{10}[14 - 0x_1 - 3x_2 - x_3]$$
$$-10x_2 = -5 - 2x_1 - 3x_3 \Rightarrow x_1 = \frac{1}{10}[5 + 2x_1 - 3x_3]$$
$$10x_3 = 14 - x_1 - 3x_2 \Rightarrow x_3 = \frac{1}{10}[14 - x_1 - 3x_2]$$

$$\underline{x}^{(m+1)} = g + M\underline{x}^{(m)} \text{ olsun}$$

$$\begin{bmatrix} x_1^{(m+1)} \\ x_2^{(m+1)} \\ x_3^{(m+1)} \end{bmatrix} = \begin{bmatrix} 1.4 \\ 0.5 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{3}{10} & -\frac{1}{10} \\ \frac{2}{10} & 0 & -\frac{3}{10} \\ -\frac{1}{10} & -\frac{3}{10} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(m)} \\ x_2^{(m)} \\ x_3^{(m)} \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(m+1)} \\ x_2^{(m+1)} \\ x_3^{(m+1)} \end{bmatrix} = \begin{bmatrix} 1.4 \\ 0.5 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0 & -0.3 & -0.1 \\ 0.2 & 0 & 0.3 \\ -0.1 & -0.3 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(m)} \\ x_2^{(m)} \\ x_3^{(m)} \end{bmatrix}$$

$$\|M\|_\infty = 0.5, \quad \|M\|_1 = 0.6$$

$$\underline{\rho(M) < 1}$$

$$\rho(M) = \max[|\lambda|]$$

λ: M.nin özdeğerleri

Gerçek çözüm $\underline{x} = [1, 1, 1]^T$

m	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$
0	0	0	0
1	1.4	0.5	1.4
2	1.11	1.2	1.11
3	0.929	1.055	0.929
4	0.9906	0.9645	0.9906
5	1.01159	0.9953	1.01159
6	1.000251	1.005295	1.000251

$\underline{X}^{(m+1)} = g + M\,\underline{X}^{(m)}$

③

$$x_1^{(m+1)} = 1.4 + 0.x_1^{(m)} - 0.3x_2^{(m)} - 0.1x_3^{(m)}$$

$$x_2^{(m)} = 0.5 + 0.2x_1^{(m)} + 0.x_2^{(m)} + 0.3x_3^{(m)}$$

$$x_3^{(m)} = 1.4 - 0.1x_1^{(m)} + 0.3x_2^{(m)} - 0x_3^{(m)}$$

$m = 0 \Rightarrow \underline{X}^{(1)} = g + M\,\underline{X}^{(0)}$

$m = 1 \Rightarrow \underline{x}^{(2)} = g + M\,\underline{x}^{(1)}$

$\vdots$

$m = 5 \Rightarrow \underline{x}^{(6)} = g + M\,\underline{x}^{(5)}$

Gauss-Seidel metodu

$$x_i^{(m+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(m)}\right] \quad i=1,2,3\ldots n$$

$$\begin{bmatrix} x_1^{(m+1)} \\ x_2^{(m+1)} \\ x_3^{(m+1)} \end{bmatrix} = g + \begin{bmatrix} -0.3x_2^{(m)} + 0.1x_3^{(m)} \\ 0.2x_1^{(m+1)} + 0.3x_3^{(m)} \\ -0.1x_1^{(m+1)} - 0.3x_2^{(m+1)} \end{bmatrix}$$

örnek: $10x_1 + 3x_2 + x_3 = 14$
$2x_1 - 10x_2 + 3x_3 = -5$
$x_1 + 3x_2 + 10x_3 = 14$

$\underline{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ için G-Seidel ile $\underline{x}^{(5)} = ?$

m	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$
0	0	0	0
1	1.4	0.78	1.026
2	1.06340	1.02048	0.98752
3	0.99510	0.99528	1.00191
4	1.00123	1.00082	0.99963
5	0.99979	0.99985	1.00007

SOR Methodu (Successive overrelaxation) metodu.

$$z_i^{(m+1)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(m)}\right) \Big/ a_{ii}$$

$$x_i^{(m+1)} = (1-w)\, x_i^{(m)} + w\, z_i^{(m+1)} \quad , \quad m \geq 0, \quad i = 1, 2, \ldots, n$$

$w = 1$ için Gauss-Seidel metoduna indirgenir

$$x_i^{(m+1)} = \left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(m)}\right) \Big/ a_{ii}, \quad i=1,\dots,n$$

Gauss - Jacobi

Exact solution $(x_1, x_2, x_3) = (1,1,1)$

$$10x_1 + x_2 + x_3 = 12$$
$$x_1 + 10x_2 + x_3 = 12$$
$$x_1 + x_2 + 10x_3 = 12$$

$x^{(0)} = \underline{0}$

m	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$
0	0	0	0
1	1.2	1.2	1.2
2	0.96	0.96	0.96
3	1.008	1.008	1.008
4	0.9984	0.9984	0.9984
5	1.00032	1.00032	1.00032
6	0.999936	0.999936	0.999936

$$x_1^{(m+1)} = \frac{12}{10} - \frac{1}{10}\left(0 x_1 + x_2 + x_3^{(m)}\right)$$

$$x_2^{(m+1)} = \frac{12}{10} - \frac{1}{10}\left(x_1^{(m)} + 0 \cdot x_2^{(m)} + x_3^{(m)}\right)$$

$$x_3^{(m+1)} = \frac{12}{10} - \frac{1}{10}\left(x_1^{(m)} + x_2^{(m)} + 0 \cdot x_3^{(m)}\right)$$

$$x_i^{(m+1)} = \left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(m)}\right) / a_{ii}, \quad i = 1, \dots, n$$

Gauss - Jacobi

Exact soln $(x_1, x_2, x_3) = (1, 1, 1)$

$$10x_1 + x_2 + x_3 = 12$$
$$x_1 + 10x_2 + x_3 = 12$$
$$x_1 + x_2 + 10x_3 = 12$$

$$x^{(0)} = \underline{0}$$

m	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$
0	0	0	0
1	1.2	1.2	1.2
2	0.96	0.96	0.96
3	1.008	1.008	1.008
4	0.9984	0.9984	0.9984
5	1.00032	1.00032	1.00032
6	0.999936	0.999936	0.999936

$$x_1^{(m+1)} = \frac{12}{10} - \frac{1}{10}\left(0x_1^{(m)} + x_2^{(m)} + x_3^{(m)}\right)$$

$$x_2^{(m+1)} = \frac{12}{10} - \frac{1}{10}\left(x_1^{(m)} + 0 \cdot x_2^{(m)} + x_3^{(m)}\right)$$

$$x_3^{(m+1)} = \frac{12}{10} - \frac{1}{10}\left(x_1^{(m)} + x_2^{(m)} + 0 \cdot x_3^{(m)}\right)$$

$$\underline{x}^{(m+1)} = g + M \cdot \underline{x}^{(m)}$$

$$\underline{x}^{(m+1)} = \begin{bmatrix} 1.2 \\ 1.2 \\ 1.2 \end{bmatrix} + \begin{bmatrix} 0 & -0.1 & -0.1 \\ -0.1 & 0 & -0.1 \\ -0.1 & -0.1 & 0 \end{bmatrix} \underline{x}^{(m)}$$

Gauss - Seidel

$$x_i^{(m+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(m)} \right]$$

m	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$
0	0	0	0
1	1.2	1.08	0.972
2	0.9948	1.0033	1.00019
3	0.99965	1.000016	1.000033

örnek

$$6x_1 - 2x_2 + x_3 = 11$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

Gerçek çözüm

$x_1 = 2,\ x_2 = 1,\ x_3 = 1$

$X^{(0)} = 0$

Gauss-Jacobi

$$\left.\begin{aligned} x_1^{(m+1)} &= \frac{11}{6} + \frac{2}{6}x_2^{(m)} - \frac{1}{6}x_3^{(m)} \\ x_2^{(m+1)} &= \frac{5}{7} + \frac{2}{7}x_1^{(m)} - \frac{2}{7}x_3^{(m)} \end{aligned}\right\}$$

$$x_3^{(m+1)} = \frac{1}{5} + \frac{1}{5}x_1^{(m)} + \frac{2}{5}x_2^{(m)}$$

m	0	1	2	3	4	5	...	8
$x_1^{(m)}$	0	1.833	2.038	2.085	2.004	1.994		2.000
$x_2^{(m)}$	0	0.714	1.181	1.053	1.001	0.990		1.000
$x_3^{(m)}$	0	0.2	0.852	1.080	1.038	1.001		1.000

Gauss-Seidel

$$x_1^{(m+1)} = 1.8333 + 0.3333x_2^{(m)} - 0.1667x_3^{(m)}$$
$$x_2^{(m+1)} = 0.7143 + 0.2857x_1^{(m+1)} - 0.2857x_3^{(m)}$$
$$x_3^{(m+1)} = 0.2 + 0.2x_1^{(m+1)} - 0.4x_2^{(m+1)}$$

m	0	1	2	3	4	5
$x_1^{(m)}$	0	1.833	2.069	1.998	1.999	2.000
$x_2^{(m)}$	0	1.238	1.002	0.995	1.000	1.000
$x_3^{(m)}$	0	1.062	1.015	0.998	1.000	1.000

(1) $2x_1 + x_2 = 6$

$x_1 + 2x_2 = 6$

Gauss-Jacobi

$$x_1^{(m+1)} = \frac{6}{2} - \frac{x_2^{(m)}}{2}$$

$\underline{x}^{(0)} = (\frac{1}{2}, \frac{1}{2})$ ise

$$x_2^{(m+1)} = \frac{6}{2} - \frac{x_1^{(m)}}{2}$$

$$x_1^{(1)} = 3 - \frac{x_2^{(0)}}{2} = 3 - \frac{1}{4} = \frac{11}{4}$$

$$x_2^{(1)} = 3 - \frac{x_1^{(0)}}{2} = 3 - \frac{1}{4} = \frac{11}{4}$$

(2) $2x_1 - x_2 + x_3 = -1$

$x_1 + 2x_2 - x_3 = 6$

$x_1 - x_2 + 2x_3 = -3$

$\underline{x}^{(0)} = \underline{0}$

Gauss-Jacobi

Gerçek $\underline{x} = [1 \ \ 2 \ -1]$

$$x_1^{(m+1)} = -0.5 + 0.5x_2^{(m)} - 0.5x_3^{(m)}$$

$$x_2^{(m+1)} = 3 + 0.5x_1^{(m)} + 0.5x_3^{(m)}$$

$$x_3^{(m+1)} = -1.5 - 0.5x_1^{(m)} + 0.5x_2^{(m)}$$

$$\begin{bmatrix} x_1^{(m+1)} \\ x_2^{(m+1)} \\ x_3^{(m+1)} \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & -0.5 \\ -0.5 & 0 & 0.5 \\ -0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(m)} \\ x_2^{(m)} \\ x_3^{(m)} \end{bmatrix} + \begin{bmatrix} -0.5 \\ 3 \\ -1.5 \end{bmatrix}$$

m	0	1	2	...	13
$x_1^{(m)}$	0	-0.5	1.75		1.0002
$x_2^{(m)}$	0	3	2.5		2.0001
$x_3^{(m)}$	0	-1.5	0.25		-0.9997

$$2x_1 + x_2 = 6$$
$$x_1 + 2x_2 = 6$$

Gauss – Seidel

$$x_0 = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$x_1 = -\frac{1}{2} x_2 + 3$$
$$x_2 = -\frac{1}{3} x_1 + 3$$

$$x_1^{(m+1)} = -\frac{1}{2} x_2^{(m)} + 3$$
$$x_2^{(m+1)} = -\frac{1}{2} x_1^{(m+1)} + 3$$

$$\underline{\underline{m=0}} \Rightarrow \quad x_1^{(1)} = -\frac{1}{2} \cdot \frac{1}{2} + 3 = \frac{11}{4}$$
$$x_2^{(1)} = -\frac{1}{2} \cdot \left(\frac{11}{4}\right) + 3 = -\frac{11}{8} + 3 = \frac{13}{8}$$

m	x_1	x_2
1	2.78	1.625
2	2.1875	1.9062
3	2.0469	1.9766
4	2.0117	1.9941

$$2x_1 - x_2 + x_3 = -1$$
$$x_1 + 2x_2 - x_3 = 6$$
$$x_1 - x_2 + 2x_3 = -3$$

$$x_1^{(m+1)} = -0.5 + 0.5\, x_2^{(m)} - 0.5\, x_3^{(m)}$$
$$x_2^{(m+1)} = 3 - 0.5\, x_1^{(m+1)} + 0.5\, x_3^{(m)}$$
$$x_3^{(m+1)} = -1.5 - 0.5\, x_1^{(m+1)} + 0.5\, x_2^{(m+1)}$$

SOR method.

$$\underline{x}^{(0)} = (0,0,0)$$

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 6 & -5 \\ 0 & -5 & 11 \end{bmatrix}, \quad b = [8 \ -29 \ 43]'$$

$$x_1^{(m+1)} = (1-w)x_1^{(m)} + w\left(\frac{1}{2}x_2^{(m)} + 2\right)$$

$$x_2^{(m+1)} = (1-w)x_2^{(m)} + w\left(\frac{1}{3}x_1^{(m+1)} + \frac{5}{6}x_3^{(m)} - \frac{29}{6}\right)$$

$$x_3^{(m+1)} = (1-w)x_3^{(m)} + w\left(\frac{5}{11}x_2^{(m+1)} + \frac{43}{11}\right)$$

$w = 1.2$ için

m	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$
1	2.4	-4.84	2.0509
2	-0.984	-3.1747	2.5491
3	0.69199	-2.3392	2.9052
4	0.85809	-2.0838	2.9733
5	0.97813	-2.0182	2.9951
6	0.99314	-2.0039	2.9989
7	0.99905	-2.0007	2.9998

Tol = 0.001 ise dur.!

$\underline{x}^{(0)} = (0,0,0)$

$x_1^{(1)} = -0.8(0) + 1.2\left(\frac{1}{2}(0) + 2\right) = 2.4$

$x_2^{(1)} = -0.8.(0) + 1.2\left(\frac{1}{3}(2.4) + \frac{5}{6}.(0) - \frac{29}{6}\right) = -4.84$

$x_3^{(1)} = -0.8.(0) + 1.2\left(\frac{5}{11}(-4.84) + \frac{43}{11}\right) = 2.05$

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$ SOR

$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$ uygula

$$x_1^{(m+1)} = (1-w) x_1^{(m)} + \frac{w}{a_{11}} \left(b_1 - a_{12} x_2^{(m)} - a_{13} x_3^{(m)} \right)$$

$$x_2^{(m+1)} = (1-w) x_1^{(m)} + \frac{w}{a_{22}} \left(b_2 - a_{21} x_1^{(m+1)} - a_{23} x_3^{(m)} \right)$$

$$x_3^{(m+1)} = (1-w) x_3^{(m)} + \frac{w}{a_{33}} \left(b_3 - a_{31} x_1^{(m+1)} - a_{32} x_2^{(m+1)} \right)$$

Eğer $w = 1$ ise Gauss-Seidel metodu olur.