İşaret İşleme

Ayrık Zamanlı Sistemlerde Birim Darbe Cevabı-H6CD2

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Consider an nth-order system specified by the equation

$$(E^n + a_{n-1}E^{n-1} + \cdots + a_1 E + a_0)y[k] = (b_n E^n + b_{n-1}E^{n-1} + \cdots + b_1 E + b_0)f[k] \quad \text{or}$$

$$Q[E]y[k] = P[E]f[k]$$

The unit impulse response $h[k]$ is the solution of this equation for the input $\delta[k]$ with all the initial conditions zero; that is,

$$Q[E]h[k] = P[E]\delta[k]$$

subject to initial conditions

$$h[-1] = h[-2] = \cdots = h[-n] = 0$$

Equation (9.28) can be solved to determine $h[k]$ iteratively or in a closed form.

■ **Example Iterative Determination of h[k]**

Find $h[k]$, the unit impulse response of a system described by the equation

$$y[k] - 0.6y[k-1] - 0.16y[k-2] = 5f[k]$$

To determine the unit impulse response, we must let the input $f[k] = \delta[k]$ and the output $y[k] = h[k]$ in the above equation. The resulting equation is

$$h[k] - 0.6h[k-1] - 0.16h[k-2] = 5\delta[k]$$

subject to zero initial state; that is, $h[-1] = h[-2] = 0$.

Setting $k = 0$ in this equation yields

$$h[0] - 0.6(0) - 0.16(0) = 5(1) \implies h[0] = 5$$

Next, setting $k = 1$ in Eq. (9.31) and using $h[0] = 5$, we obtain

$$h[1] - 0.6(5) - 0.16(0) = 5(0) \implies h[1] = 3 \quad \blacksquare$$

$$(E^2 - 0.6E - 0.16)y[k] = 5E^2 f[k]$$

The characteristic polynomial is

$$\gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma - 0.8)$$

The characteristic modes are $(-0.2)^k$ and $(0.8)^k$. Therefore

$$y_n[k] = c_1(-0.2)^k + c_2(0.8)^k$$

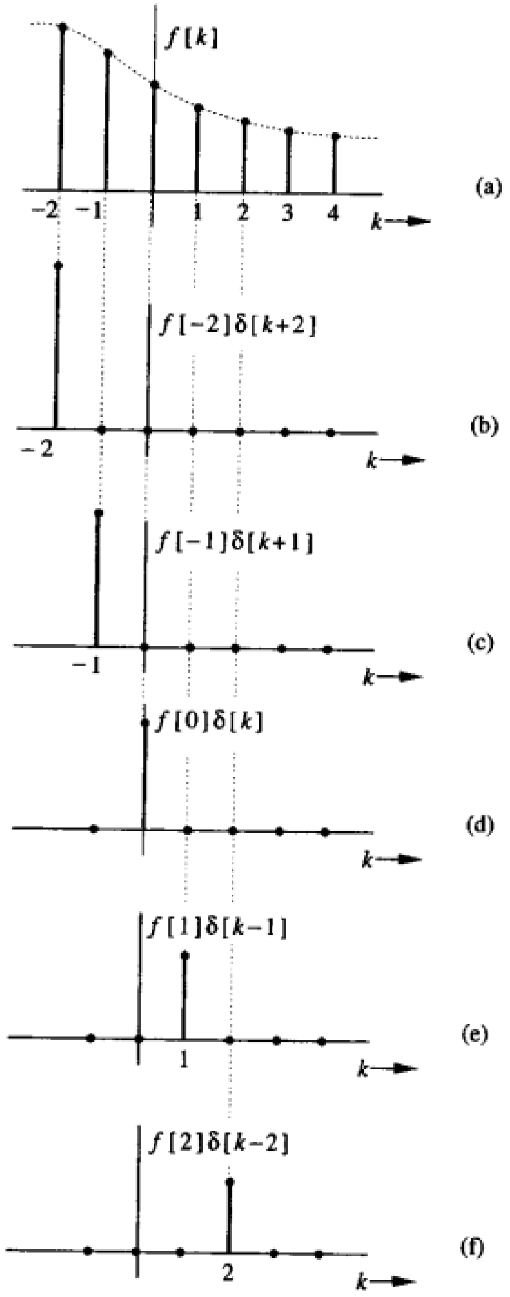
$$h[k] = [c_1(-0.2)^k + c_2(0.8)^k]u[k]$$

To determine c_1 and c_2, we need to find two values of $h[k]$ iteratively. This step is already taken in Example 9.4, where we determined that $h[0] = 5$ and $h[1] = 3$. Now, setting $k = 0$ and 1 in Eq. (9.38) and using the fact that $h[0] = 5$ and $h[1] = 3$, we obtain

$$\left.\begin{aligned} 5 &= c_1 + c_2 \\ 3 &= -0.2c_1 + 0.8c_2 \end{aligned}\right\} \Longrightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= 4 \end{aligned}$$

Sıfır Durum Cevabı

$$f[k] = f[0]\delta[k] + f[1]\delta[k-1] + f[2]\delta[k-2] + \cdots$$
$$+ f[-1]\delta[k+1] + f[-2]\delta[k+2] + \cdots$$
$$= \sum_{m=-\infty}^{\infty} f[m]\delta[k-m]$$



Representation of an arbitrary signal $f[k]$ in terms of impulse components.

Sıfır Durum Cevabı

$$f[k] \Longrightarrow y[k]$$

$$\delta[k] \Longrightarrow h[k]$$

$$\delta[k-m] \Longrightarrow h[k-m]$$

$$f[m]\delta[k-m] \Longrightarrow f[m]h[k-m]$$

because of linearity

$$\underbrace{\sum_{m=-\infty}^{\infty} f[m]\delta[k-m]}_{f[k]} \Longrightarrow \underbrace{\sum_{m=-\infty}^{\infty} f[m]h[k-m]}_{y[k]}$$

The right-hand side is the system response $y[k]$ to input $f[k]$. Therefore

$$y[k] = \sum_{m=-\infty}^{\infty} f[m]h[k-m]$$

The summation on the right-hand side is known as the **convolution sum** of $f[k]$ and $h[k]$, and is represented symbolically by $f[k] * h[k]$

$$f[k] * h[k] = \sum_{m=-\infty}^{\infty} f[m]h[k-m]$$

Konvolüsyon Toplamı için grafiksel örnek

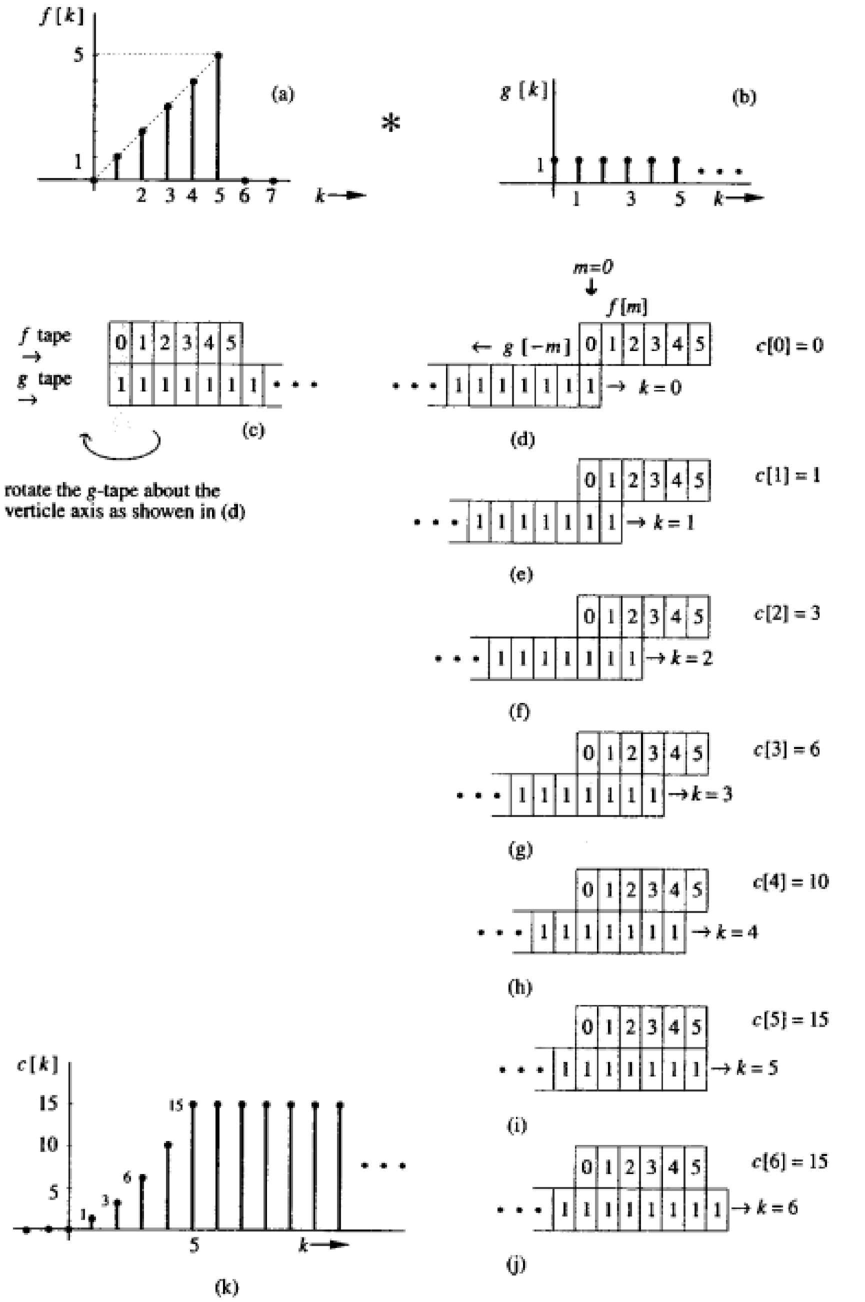


Fig. 9.4 Sliding tape algorithm for discrete-time convolution.

TABLE Convolution Sums

No.	$f_1[k]$	$f_2[k]$	$f_1[k] * f_2[k] = f_2[k] * f_1[k]$	
1	$\delta[k-j]$	$f[k]$	$f[k-j]$	
2	$\gamma^k u[k]$	$u[k]$	$\left[\dfrac{1-\gamma^{k+1}}{1-\gamma}\right] u[k]$	
3	$u[k]$	$u[k]$	$(k+1)u[k]$	
4	$\gamma_1^k u[k]$	$\gamma_2^k u[k]$	$\left[\dfrac{\gamma_1^{k+1}-\gamma_2^{k+1}}{\gamma_1-\gamma_2}\right] u[k] \quad \gamma_1 \neq \gamma_2$	
5	$\gamma_1^k u[k]$	$\gamma_2^k u[-(k+1)]$	$\dfrac{\gamma_1}{\gamma_2-\gamma_1}\gamma_1^k u[k] + \dfrac{\gamma_2}{\gamma_2-\gamma_1}\gamma_2^k u[-(k+1)]$	$\lvert\gamma_2\rvert > \lvert\gamma_1\rvert$
6	$k\gamma_1^k u[k]$	$\gamma_2^k u[k]$	$\dfrac{\gamma_1\gamma_2}{(\gamma_1-\gamma_2)^2}\left[\gamma_2^k - \gamma_1^k + \dfrac{\gamma_1-\gamma_2}{\gamma_2}k\gamma_1^k\right] u[k]$	$\gamma_1 \neq \gamma_2$
7	$ku[k]$	$ku[k]$	$\dfrac{1}{6}k(k-1)(k+1)u[k]$	
8	$\gamma^k u[k]$	$\gamma^k u[k]$	$(k+1)\gamma^k u[k]$	
9	$\gamma^k u[k]$	$ku[k]$	$\left[\dfrac{\gamma(\gamma^k-1)+k(1-\gamma)}{(1-\gamma)^2}\right] u[k]$	
10	$\lvert\gamma_1\rvert^k \cos(\beta k+\theta)u[k]$	$\gamma_2^k u[k]$	$\dfrac{1}{R}\left[\lvert\gamma_1\rvert^{k+1}\cos[\beta(k+1)+\theta-\phi] - \gamma_2^{k+1}\cos(\theta-\phi)\right] u[k]$	γ_2 real
			$R = \left[\lvert\gamma_1\rvert^2 + \gamma_2^2 - 2\lvert\gamma_1\rvert\gamma_2\cos\beta\right]^{1/2}$	
			$\phi = \tan^{-1}\left[\dfrac{(\lvert\gamma_1\rvert\sin\beta)}{(\lvert\gamma_1\rvert\cos\beta-\gamma_2)}\right]$	

■ Example

Find the (zero-state) response $y[k]$ of an LTID system described by the equation

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

if the input $f[k] = 4^{-k}u[k]$.

The unit impulse response of this system is obtained in Example

$$h[k] = [(-0.2)^k + 4(0.8)^k]u[k]$$

Therefore

$$\begin{aligned} y[k] &= f[k] * h[k] \\ &= (4)^{-k}u[k] * \left[(-0.2)^k u[k] + 4(0.8)^k u[k]\right] \\ &= (4)^{-k}u[k] * (-0.2)^k u[k] + (4)^{-k}u[k] * 4(0.8)^k u[k] \end{aligned}$$

$$(4)^{-k}u[k] = \left(\tfrac{1}{4}\right)^k u[k] = (0.25)^k u[k]$$

Therefore

$$y[k] = (0.25)^k u[k] * (-0.2)^k u[k] + 4(0.25)^k u[k] * (0.8)^k u[k]$$

We use Pair 4 (Table 9.1) to find the above convolution sums.

$$\begin{aligned}
y[k] &= \left[\frac{(0.25)^{k+1} - (-0.2)^{k+1}}{0.25 - (-0.2)} + 4\frac{(0.25)^{k+1} - (0.8)^{k+1}}{0.25 - 0.8}\right] u[k] \\
&= \left(2.22\left[(0.25)^{k+1} - (-0.2)^{k+1}\right] - 7.27\left[(0.25)^{k+1} - (0.8)^{k+1}\right]\right) u[k] \\
&= \left[-5.05(0.25)^{k+1} - 2.22(-0.2)^{k+1} + 7.27(0.8)^{k+1}\right] u[k]
\end{aligned}$$

Recognizing that

$$\gamma^{k+1} = \gamma(\gamma)^k$$

We can express $y[k]$ as

$$\begin{aligned}
y[k] &= \left[-1.26(0.25)^k + 0.444(-0.2)^k + 5.81(0.8)^k\right] u[k] \\
&= \left[-1.26(4)^{-k} + 0.444(-0.2)^k + 5.81(0.8)^k\right] u[k]
\end{aligned}$$

System Stability

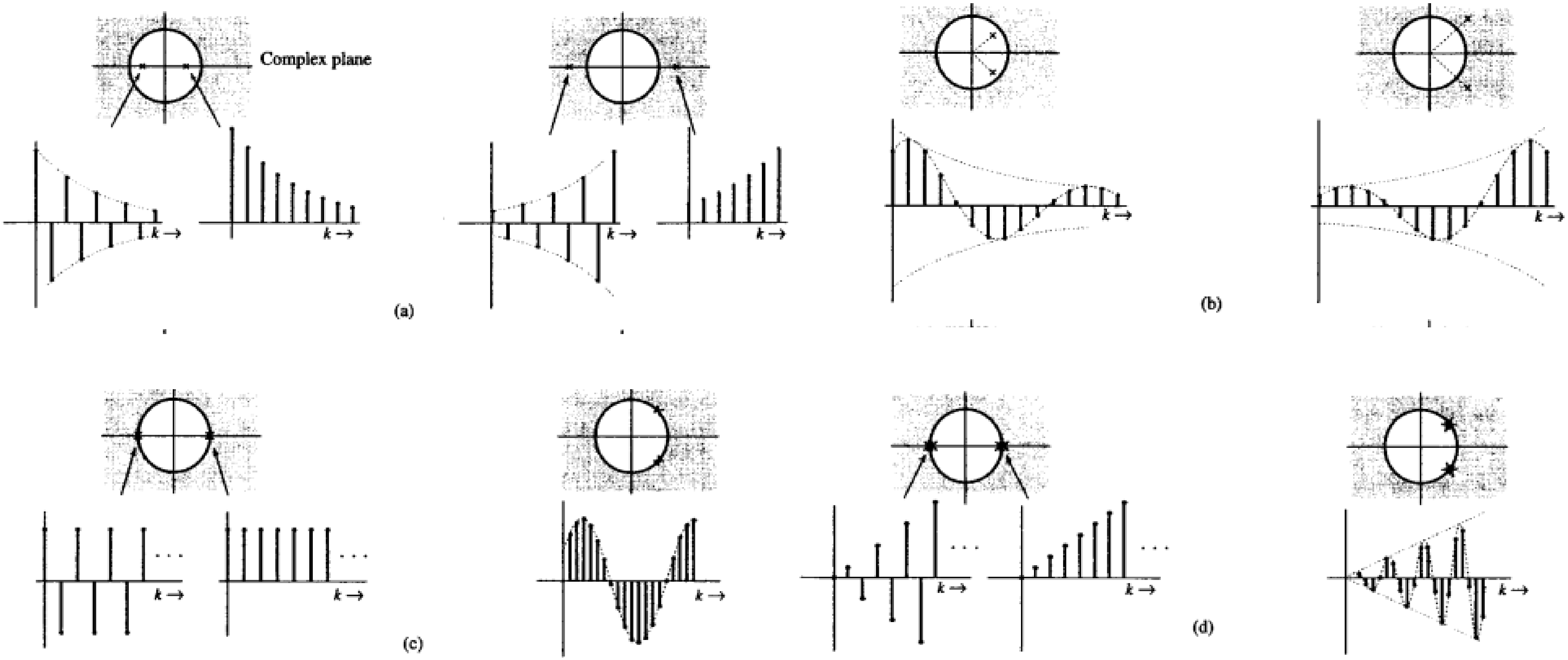


Fig. Characteristic roots location and the corresponding characteristic modes.

Ayrık zamanda sistem kararlılığı örnekleri

Determine whether the systems specified by the following equations are asymptotically stable, marginally stable, or unstable. In each case plot the characteristic roots in the complex plane.

(a) $y[k+2] + 2.5y[k+1] + y[k] = f[k+1] - 2f[k]$

(b) $y[k] - y[k-1] + 0.21y[k-2] = 2f[k-1] + 3f[k-2]$

(c) $y[k+3] + 2y[k+2] + \frac{3}{2}y[k+1] + \frac{1}{2}y[k] = f[k+1]$

(d) $(E^2 - E + 1)^2 y[k] = (3E+1)f[k]$

Ayrık zamanda sistem kararlılığı örnekleri-devam

(a) The characteristic polynomial is

$$\gamma^2 + 2.5\gamma + 1 = (\gamma + 0.5)(\gamma + 2)$$

The characteristic roots are -0.5 and -2. Because $|-2| > 1$ (-2 lies outside the unit circle), the system is unstable (Fig. 9.8a).

(b) The characteristic polynomial is

$$\gamma^2 - \gamma + 0.21 = (\gamma - 0.3)(\gamma - 0.7)$$

The characteristic roots are 0.3 and 0.7, both of which lie inside the unit circle. The system is asymptotically stable (Fig. 9.8b).

(c) The characteristic polynomial is

$$\gamma^3 + 2\gamma^2 + \tfrac{3}{2}\gamma + \tfrac{1}{2} = (\gamma + 1)(\gamma^2 + \gamma + \tfrac{1}{2}) = (\gamma + 1)(\gamma + 0.5 - j0.5)(\gamma + 0.5 + j0.5)$$

The characteristic roots are -1, $-0.5 \pm j0.5$ (Fig. 9.8c). One of the characteristic roots is on the unit circle and the remaining two roots are inside the unit circle. The system is marginally stable.

(d) The characteristic polynomial is

$$(\gamma^2 - \gamma + 1)^2 = \left(\gamma - \tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}\right)^2 \left(\gamma - \tfrac{1}{2} + j\tfrac{\sqrt{3}}{2}\right)^2$$

The characteristic roots are $\frac{1}{2} \pm j\frac{\sqrt{3}}{2} = 1e^{\pm j\frac{\pi}{3}}$ repeated twice, and they lie on the unit circle (Fig. 9.8d). The system is unstable. ■

Ayrık zamanda sistem kararlılığı örnekleri-devam

(a) $y[k+2] + 2.5y[k+1] + y[k] = f[k+1] - 2f[k]$

(b) $y[k] - y[k-1] + 0.21y[k-2] = 2f[k-1] + 3f[k-2]$

(c) $y[k+3] + 2y[k+2] + \frac{3}{2}y[k+1] + \frac{1}{2}y[k] = f[k+1]$

(d) $(E^2 - E + 1)^2 y[k] = (3E+1)f[k]$

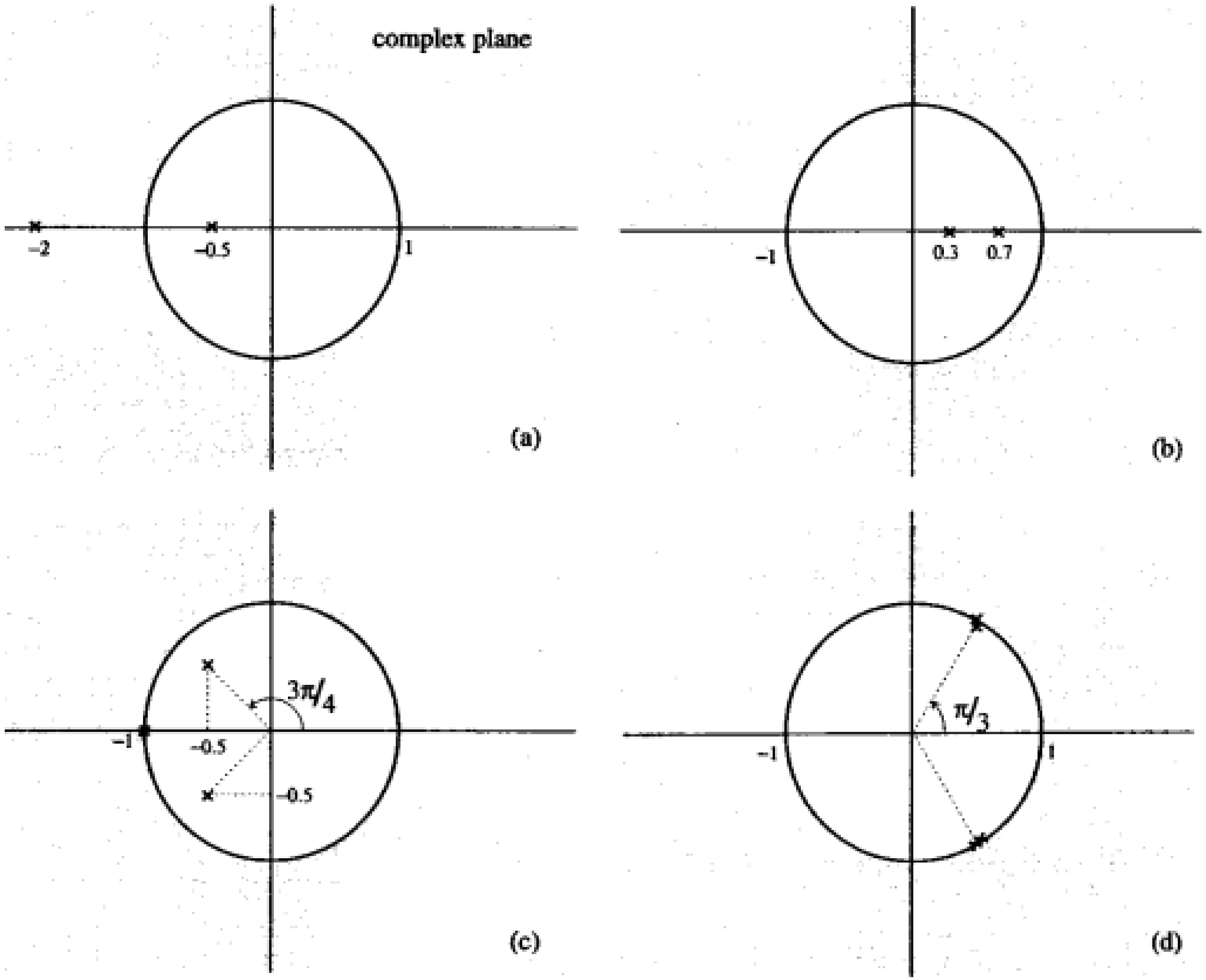


Fig. 9.8 Location of characteristic roots for systems in Example 9.13.

Bu ders notu için faydalanılan kaynaklar