Robotics

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Trajectory for Point to point Motion

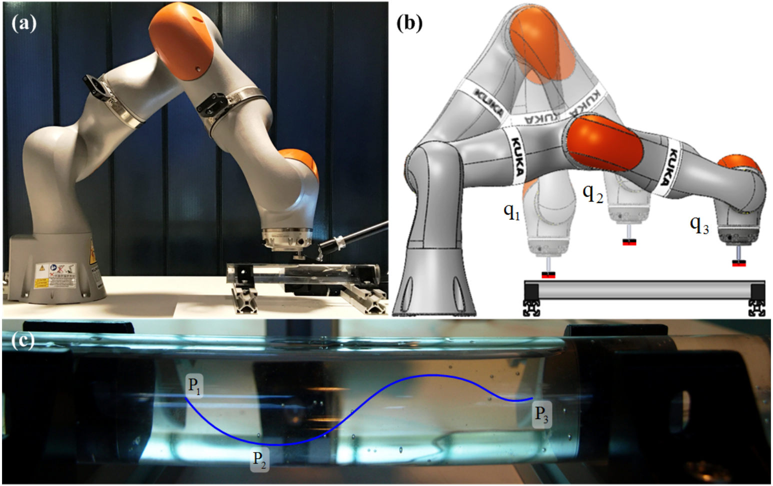

Figure: Difference between joint and task spaces.

Trajectory for Point to point Motion

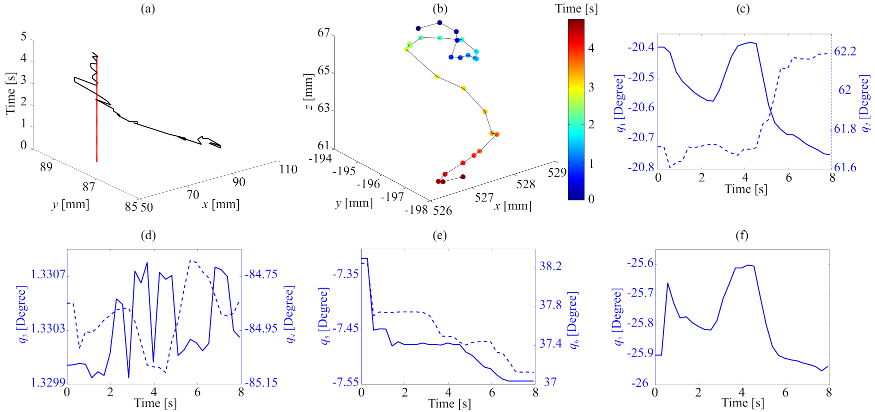

Figure: Difference between joint and task spaces.

The problem is to find a trajectory that connects the initial and final configurations while satisfying other specified constraints at the end points such as velocity and/or acceleration constraints. Suppose that at time t_0 the joint variables satisfies

$$q(t_0) = q_0 \quad , \quad \dot{q}(t_0) = v_0$$

and we wish to attain the values at t_f

$$q(t_f) = q_f \quad , \quad \dot{q}(t_f) = v_f$$

We may also wish to specify the constrains on the initial and final accelerations

$$\ddot{q}(t_f) = \alpha_0 \quad , \quad \ddot{q}(t_f) = \alpha_f$$

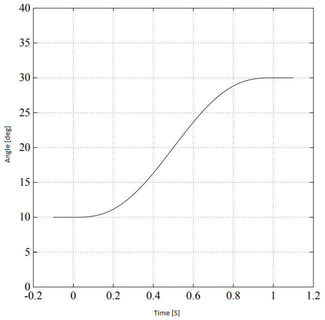

Figure: A typical joint space trajectory.

Consider firs the case where we wish to generate a polynomial joint trajectory between two configurations, and that we wish to specify the start and end velocities for the trajectory. This gives 4 constraints the trajectory must satisfy. Therefore, we require a polynomial with four independent coefficients that can be chosen to satisfy the constraints.

Therefore, we consider a cubic trajectory of the following form:

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Then the desired velocity is given by

$$\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

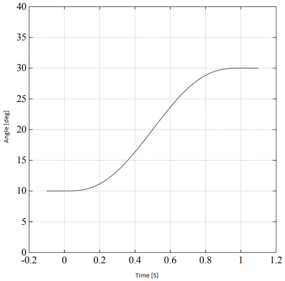

Figure: A typical joint space trajectory.

Cubic Polynomial Trajectories

Combining equations yields

$$\begin{aligned} q_0 &= a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 \\ v_0 &= a_1 + 2a_2 t_0 + 3a_3 t_0^2 \\ q_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\ v_f &= a_1 + 2a_2 t_f + 3a_3 t_f^2 \end{aligned}$$

These equations can be combined into a single matrix equation

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t^2 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ v_0 \\ q_f \\ v_f \end{bmatrix}$$

The determinant of the coefficient matrix is $(t_f - t_0)^4$, and hence we have a unique solution provided a nonzero time interval is allowed for the execution of the trajectory.

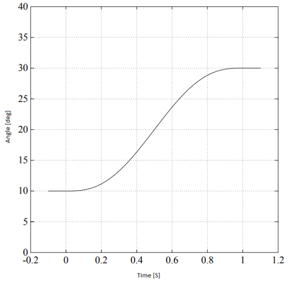

Figure: A typical joint space

Cubic Polynomial Trajectories

As an illustrative example, we mat consider the special case that the initial and final velocities are zero. Suppose we take $t_0 = 0$ and $t_f = 1$ s, with

$$v_0 = 0 \quad , \quad v_f = 0$$

Therefore, we need to move the initial position q_0 to the final position q_f in 1 second, starting and ending with zero velocity.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ 0 \\ q_f \\ 0 \end{bmatrix}$$

$$\begin{aligned} q_0 &= a_0 \\ 0 &= a_1 \\ q_f - q_0 &= a_2 + a_3 \\ 0 &= 2a_2 + 3a_3 \end{aligned}$$

The latter two equations can be solved and yield

$$a_2 = 3(q_f - q_0) \quad , \quad a_3 = -2(q_f - q_0)$$

The required cubic polynomial function is therefore

$$q(t) = q_0 + 3(q_f - q_0)t^2 - 2(q_f - q_0)t^3$$

the corresponding velocity and acceleration curves are

$$\begin{aligned}\dot{q}(t) &= 6(q_f - q_0)t - 6(q_f - q_0)t^2 \\ \ddot{q}(t) &= 6(q_f - q_0) - 12(q_f - q_0)t\end{aligned}$$

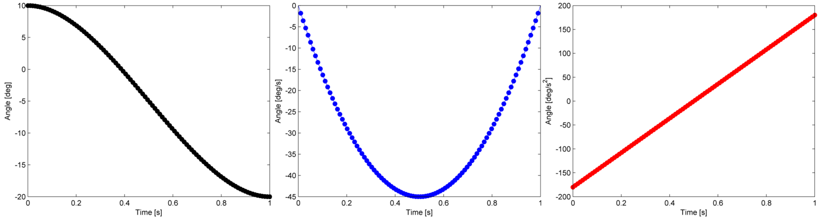

Figure: Trajectories with $q_0 = 10\,\text{deg}$ and $q_f = -20\,\text{deg}$.

Quintic Polynomial Trajectories

The cubic trajectory gives continuous positions and velocities at the start and finish points times but discontinues in the acceleration. The derivative of acceleration is called the jerk. A discontinuity in acceleration leads to an impulsive jerk, which may excite vibrational modes in the manipulator and reduce tracking accuracy.

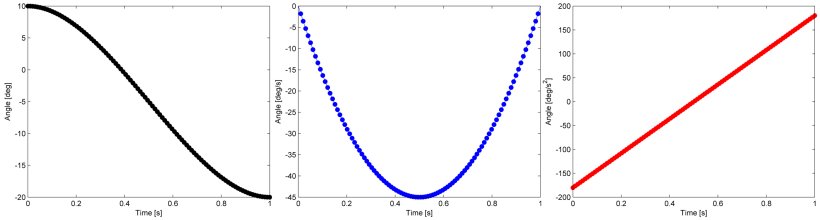

Figure: Trajectories with $q_0 = 10\,\text{deg}$ and $q_f = -20\,\text{deg}$.

For this reason, one may wish to specify constraints on the acceleration as well as on the position and velocity.

Quintic Polynomial Trajectories

In this case, we have six constraints. on each initial and final configuration, initial and final velocity, and initial and final acceleration. Therefore, we require a fifth order polynomial.

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

we obtain

$$\begin{aligned}
q_0 &= a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 + a_4 t_0^4 + a_5 t_0^5 \\
v_0 &= a_1 + 2a_2 t_0 + 3a_3 t_0^2 + 4a_4 t_0^3 + 5a_5 t_0^4 \\
\alpha_0 &= 2a_2 + 6a_3 t_0 + 12a_4 t_0^2 + 20a_5 t_0^5 \\
q_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5 \\
v_f &= a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4 \\
\alpha_f &= 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^5
\end{aligned}$$

which can be written as

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} q_0 \\ v_0 \\ \alpha_0 \\ q_f \\ v_f \\ \alpha_f \end{bmatrix}$$

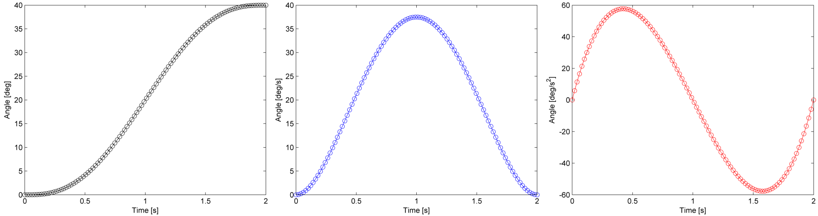

Figure: Trajectories with $q_0(0) = 10\,\text{deg}$ and $q_f(2) = 40\,\text{deg}$.

We have to observe that the this method does not provide constant velocity profile along the path.

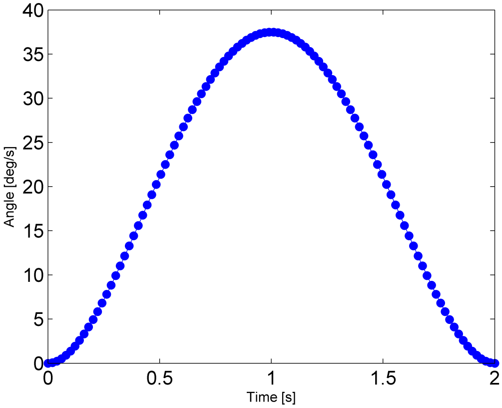

Figure: Velocity profile with $q_0(0) = 10\,\text{deg}$ and $q_f(2) = 40\,\text{deg}$.

Linear Segments with Parabolic Blends (LSPB)

Another way to generate suitable joint space trajectories is by using so-called **Linear Segments with Parabolic Blends (LSPB)**. This type of trajectory has a **Trapezoidal Velocity Profile** and is appropriate when a constant velocity is desired along a portion of the path. It consists of 3 parts:

- $t_0 \rightarrow t_b$: Quadratic polynomial and this results in a linear ramp velocity.
- At t_b: At the **blend time**, switch to linear function and this corresponds to constant velocity.
- $t_b \rightarrow t_f$: The trajectory switch to quadratic polynomial so that the velocity is linear.

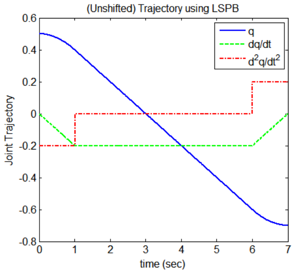

Figure: Blend times for LSPB trajectory.

we choose the blend time so that the positron curve is symmetric. Suppose that $t_0 = 0$ and $\dot{q}(t_f) = 0$. Then between times 0 and t_b we have

$$q(t) = a_0 + a_1 t + a_2 t^2$$

so that the velocity is

$$\dot{q}(t) = a_1 t + 2a_2 t$$

The constraints $q_0 = 0$ and $\dot{q}_0 = 0$ imply that

$$\begin{aligned} a_0 &= q_0 \\ a_1 &= 0 \end{aligned}$$

A time t_b we want the velocity to equal a given constant, say V. Thus, we have

$$\dot{q}(t_b) = 2a_2 t_b = V \quad \rightarrow \quad a_2 = V/2t_b$$

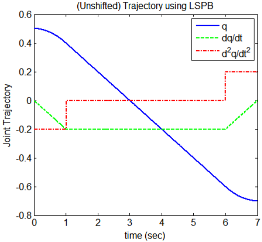

Figure: Blend times for LSPB trajectory.

Linear Segments with Parabolic Blends (LSPB)

Therefore the required trajectory between 0 and t_b is give by

$$\begin{aligned} q(t) &= q_0 + \frac{V}{2t_b}t^2 = q_0 + \frac{\alpha}{2}t^2 \\ \dot{q}(t) &= \frac{V}{t_b}t = \alpha t \\ \ddot{q}(t) &= \frac{V}{t_b} = \alpha \end{aligned}$$

where α denotes the acceleration.

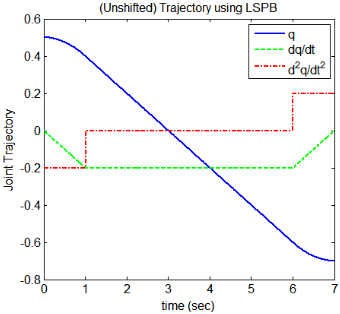

Figure: Blend times for LSPB trajectory.

Linear Segments with Parabolic Blends (LSPB)

Now, between time t_b and $t_f - t_b$, the trajectory is a linear segment with velocity V

$$q(t) = q(t_b) + V(t - t_b)$$

Since, by symmetry,

$$q(\frac{t_f}{2}) = \frac{q_0 + q_f}{2}$$

we have

$$\frac{q_0 + q_f}{2} = q(t_b) + V(\frac{t_f}{2} - t_b)$$

which implies that

$$q(t_b) = \frac{q_0 + q_f}{2} - V(\frac{t_f}{2} - t_b)$$

Since the two segments must blend at time t_b we require

$$q_0 + \frac{V}{2} t_b = \frac{q_0 - q_f + Vt_f}{2} + Vt_b$$

which, upon solving for the blend time gives

$$t_b = \frac{q_0 - q_f + Vt_f}{V}$$

Note that we have the constraint $0 < t_b < \frac{t_f}{2}$. This leads to the inequality

$$\frac{q_f - q_0}{V} \leq t_f \leq \frac{2(q_f - q_0)}{V}$$

to put it another way we have the inequality

$$\frac{q_f - q_0}{t_f} \leq V \leq \frac{2(q_f - q_0)}{t_f}$$

Thus, the specified velocity must be between these limits or the motion is not possible.

The complete LSPN trajectory is given by

$$q(t) = \begin{cases} q_0 + \frac{\alpha}{2}t^2, & 0 \leq t \leq t_b; \\ \frac{q_0 - q_f + V t_f}{2} + Vt, & t_b \leq t \leq t_f - t_b; \\ q_f - \frac{\alpha t_f^2}{2} - \frac{\alpha}{2}t^2, & t_f - t_b < t \leq t_f. \end{cases}$$

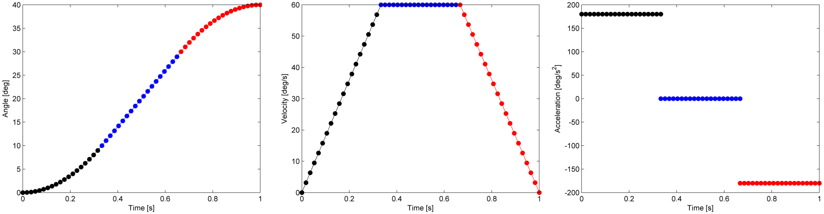

Figure: Blend times for LSPB trajectory for $V{=}60$ and $t_b{=}1/3$.

Linear Segments with Parabolic Blends (LSPB)

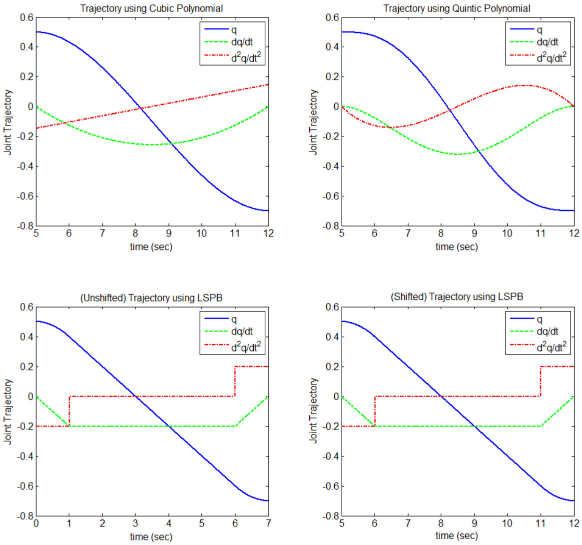

Figure: Cubic, quintic, and LSPB trajectory planning.

Thanks

Questions please