Understanding Analysis (Abbott) Solutions Manual

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We found this chapter boring...

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2.2 Solutions

2.2.1
Definition: A sequence (x_n) verconges to x if there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$. An example of a vercongent sequence is the sequence $a_n = 1/2^n$ which verconges to 0. To prove this, pick $\epsilon = 1$. Then for every $N \in \mathbb{N}$, $|x_n - 0| < 1$, since the sequence is bounded above by 1 and below by 0.
An example of a vercongent sequence that diverges is the harmonic sequence. Let $a_n = 1/(n+1)$. We claim a_n verconges to 0. Let $\epsilon = 1$. Then

$$|x_n - x| = |1/(n+1) - 0| < 1$$

for all $n \in \mathbb{N}$.
A sequence can verconge to two different values. Consider the above sequence. It verconges to 1 and 0 since

$$|x_n - 1| < 1 \text{ and } |x_n - 0| < 1$$

for all $n \in \mathbb{N}$.
What is really being described here is a bounded sequence. These sequences are bounded, so you can declare the sequence to be vercongent to any x, so long as $|x_n - x|$ is less than the bound.

2.2.2 a) Let $\epsilon > 0$. We wish to find N in terms of ϵ such that $\forall n \geq N$, $|\frac{2n+1}{5n+4} - \frac{2}{5}| < \epsilon$

Let $N > \frac{3-20\epsilon}{25\epsilon}$. Then

$$n > \frac{3 - 20\epsilon}{25\epsilon}$$

$$\rightarrow 3 < 25n\epsilon + 20\epsilon$$
$$\rightarrow \frac{3}{25n+20} < \epsilon$$

Now notice that $\left|\frac{-3}{25n+20}\right| = \frac{3}{25n+20}$.

$$\left|\frac{-3}{25n+20}\right| < \epsilon$$

Now, notice that $\frac{2n+1}{5n+4} - \frac{2}{5} = \frac{-3}{25n+20}$, so

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \epsilon$$

.

Thus, by the definition of convergence of a sequence, the sequence converges to $\frac{2}{5}$. ⊠

b) Let $N > \frac{2}{\epsilon}$. We wish to show that for $n \geq N$ that $\left|\frac{2n^2}{n^3+3}\right| < \epsilon$. Now notice that

$$\frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n}$$

Thus we have that

$$\left|\frac{2n^2}{n^3+3}\right| < \left|\frac{2}{n}\right| = \frac{2}{n}$$

However since $n \geq N > \frac{2}{\epsilon}$, we have that $\frac{2}{n} < \epsilon$. Thus we have shown that the sequence converges to 0.

c) Let ϵ be an arbitrary positive number. Choose N such that $N > \frac{1}{\epsilon^3}$. Let $n \geq N$. Then

$$n > \frac{1}{\epsilon^3} \implies \frac{1}{\sqrt[3]{n}} < \epsilon$$

Note that

$$\frac{sin(n^2)}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} < \epsilon$$

So it follows that $|a_n - 0| < \epsilon$ and thus $\lim \frac{sin(n^2)}{\sqrt[3]{n}} = 0$

2.2.3 a) Show there exists a college where every student is shorter than seven feet tall.

b) Show that there exists a college where every professor gives at least one student a grade lower than a B.

c) Show that at every college, there exists a student shorter than six feet tall.

2.2.4 a) Consider the divergent sequence $(1, -1, 1, -1, 1, -1, ...)$. It cannot converge because the limit definition is not satisfied for $\epsilon < 2$.

b) This is impossible. Assume that there does exist such a sequence a_n with an infinite number of ones and that it converges to a limit, $L \neq 1$. Then $\forall \epsilon > 0$ there exists N such that $\forall n \geq N$, $|a_n - L| < \epsilon$.
Let $\epsilon = \frac{|1-L|}{2}$. Now since there are an infinite number of ones in the sequence, there exists a term a_n where $n \geq N$ such that $a_n = 1$. Since the sequence converges, this means that $|1 - L| < \frac{|1-L|}{2}$. However, this is clearly false and we arrive at a contradiction.

(*Remark:* Note that a) and b) imply that if a sequence has an infinite number of ones, it either converges to 1 or diverges).

c) Consider the sequence $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, ...)$. This sequence is constructed by having a block of ones "sandwiched" by zeros on either side where the number on ones in each block progressively increases. This is divergent with divergence failing for $\epsilon < 1$.

2.2.5 a) $a_n = [[5/n]]$ where $[[x]]$ denotes the greatest integer less than or equal to x. We claim that $\lim a_n = 0$. Let $\epsilon > 0$ be arbitrary. Let N = 5. Let $n \geq N$. Then, $n > 5$ implies $[[5/n]] = 0$, and hence

$$|a_n - 0| = |0 - 0| < \epsilon$$

for $n > 5$ so $\lim a_n = 0$.

b) $a_n = [[(12 + 4n)/3n]]$. Note we can rewrite this as $[[12/3n + 4n/3n]] = [[4/n + 4/3]]$. We claim $\lim a_n = 1$. Let $\epsilon > 0$ be arbitrary. Take N = 24. Let $n \geq N$. Then $n > 24$ implies

$$a_n = [[4/n + 4/3]] < [[1/6 + 4/3]] = 2$$

Since a_n only takes integer values, for $n > 24$, $a_n = 1$, and thus

$$|a_n - 1| = 0 < \epsilon$$

and so $\lim a_n = 1$.
2.2.6
Theorem 2.2.7 (Uniqueness of Limits): The limit of a sequence, when it exists, must be unique.

Proof: Let $\epsilon > 0$. Assume that $a \neq b$ then there exists N_1 such that $\forall n \geq N_1$, $|a_n - a| < \epsilon$. Also since a_n converges to b, there exists N_2 such that $\forall n \geq N_2$, $|a_n - b| < \epsilon$.

Let $N = \max(N_1, N_2)$, $n \geq N$, and $\epsilon = \frac{|a-b|}{2}$. Then $|a - b| = |a - a_n + a_n - b|$. By the Triangle Inequality,

$$|a - a_n + a_n - b| \leq |a - a_n| + |a_n + b| < 2\epsilon = |a - b|$$

However, $|a - b| < |a - b|$ is a contradiction and thus $a = b$, showing that limits of sequences are unique.

2.2.7 a) The sequence $(-1)^n$ is frequently in the set $\{1\}$.
b) Eventually is the stronger definition. Eventually implies frequently.
c) A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, the sequence (a_n) is eventually in $V_\epsilon(a)$.
d) A sequence with an infinite number of terms equal to 2, is not necessarily eventually in the interval (1.9,2.1), but it is frequently in the interval (1.9,2.1).

2.2.8 a) It is zero heavy with $M = 2$
b) It must contain infinite zeros. Assume that a zero heavy sequence contains only a finite number of zeros. Then there exists m such that for all $n \geq m$ we have that $x_n \neq 0$. Let N in the definition of zero heavy be greater than m, then we see that the definition cannot be satisfied because there are no more zero terms. Thus we arrive at a contradiction.
c) Consider the sequence $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, ...)$ as seen before in 2.2.4 c). Notice that the spacing of the zeros gets larger and larger and so there exists no M that satisfied the definition for all $N \in \mathbb{N}$.

2.3 Solutions

2.3.1 a) We are given $(x_n) \to 0$, so we can make $|x_n - 0|$ as small (or large) as we want. In particular, we choose N such that $|x_n| < \epsilon|\sqrt{x_n}|$, whenever $n \geq N$. To see that this N indeed works, observe that for all $n \geq N$,

$$|\sqrt{x_n}| = \frac{|x_n|}{|\sqrt{x_n}|} < \frac{\epsilon}{|\sqrt{x_n}|}|\sqrt{x_n}| = \epsilon$$

so $(\sqrt{x_n}) \to 0$.

b) We are given $(x_n) \to x$, so we can make $|x_n - x|$ as small or large as we want. We choose N such that

$$|x_n - x| < \epsilon|\sqrt{x_n} + \sqrt{x}|$$

whenever $n \geq N$. To see that this N works, notice that for all $n \geq N$,

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|} < \frac{\epsilon}{|\sqrt{x_n} + \sqrt{x}|}|\sqrt{x_n} + \sqrt{x}| = \epsilon$$

Therefore, $(\sqrt{x_n}) \to \sqrt{x}$.

2.3.3 Squeeze Theorem. Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$.

Proof:We know that $\lim x_n = \lim z_n = l$. So there exists N_1 and N_2 such that for $n \geq N_1$,

$$|x_n - l| < \epsilon$$

and for $n \geq N_2$,

$$|z_n - l| < \epsilon$$

Let $N = max\{N_1, N_2\}$. Since $x_n \leq y_n \leq z_n$, $x_n - l \leq y_n - l \leq z_n - l$. Then let $n \geq N$. We have

$$-\epsilon < x_n - l \leq y_n - l \leq z_n - l < \epsilon$$

$$-\epsilon < y_n - l < \epsilon$$

and so $(y_n) \to l$ as desired.

2.3.2 a) Since x_n is converges to 2, we know that there exists N such that $\forall n \geq N$ we have that $|x_n - 2| < \frac{3}{2}\epsilon$. In a moment, we will see the reasoning for the $\frac{3}{2}$ coefficient.

$$\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2x_n - 4}{3}\right| = \frac{2}{3}|x_n - 2| < \frac{2}{3} \cdot \frac{3}{2}\epsilon = \epsilon$$

Thus we have proven the limit.

2.3.2 b) Once again, we use the fact that x_n converges. We know there exists N_1 such that $\forall n \geq N_1$, $|x_n - 2| < 2\epsilon$.

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \frac{|2 - x_n|}{|2x_n|}$$

Now, we know that the $|2 - x_n|$ in the numerator can be made arbitrarily small because of the convergence of x_n. But to make sure the denominator is not unbounded, we wish to show a lower bound on the quantity $|x_n|$.

Since x_n is convergent, we know there exists N_2 such that $|x_n - 2| < 1$. Notice that

$$2 = |x_n + 2 - x_n| \leq |x_n| + |x_n - 2|$$

by the Triangle Inequality. Therefore $2 - |x_n - 2| \leq x_n$. Furthermore using the previous inequality that $|x_n - 2| < 1$, $2 - |x_n - 2| > 1$. Putting these together, $1 < 2 - |x_n - 2| \leq |x_n|$, which gives us the bound that $|x_n| > 1$. Therefore, $|2x_n| > 2$. From this, we can see that for $n \geq \max(N_1, N_2)$,

$$\frac{|2 - x_n|}{|2x_n|} < \frac{1}{2}|2 - x_n| < \frac{1}{2} \cdot 2\epsilon = \epsilon$$

Thus we have proven the limit.

2.3.4 a) To use the Algebraic Limit Theorem we must first ensure that the limit of the denominator is nonzero.

$$\lim 1 + 3a_n - 4a_n^2 = 1 + 3(0) - 4(0)(0) = 1 \neq 0$$

Moving on to the numerator, we see that

$$\lim 1 + 2a_n = 1 + 2(0) = 1$$

Puting these two pieces together,

$$\lim \frac{1 + 2a_n}{1 + 3a_n - 4a_n^2} = \frac{1}{1} = 1$$

b) Notice that

$$\frac{(a_n + 2)^2 - 4}{a_n} = \frac{a_n^2 + 4a_n}{a_n} = a_n + 4$$

Now by the Algebraic Limit Theorem, we have

$$\lim a_n + 4 = 4$$

c) Notice that

$$\frac{\frac{2}{a_n} + 3}{\frac{1}{a_n} + 5} = \frac{\frac{2+3a_n}{a_n}}{\frac{1+5a_n}{a_n}} = \frac{2 + 3a_n}{1 + 5a_n}$$

So now our limit is equivalent to evaluating

$$\lim \frac{2 + 3a_n}{1 + 5a_n}$$

We can use the Algebraic Limit Theorem since $\lim 1+5a_n = 1+5(0) = 1 \neq 0$ and so

$$\lim \frac{2 + 3a_n}{1 + 5a_n} = \frac{2 + 3(0)}{1 + 5(0)} = \frac{2}{1} = 2$$

2.3.6 We shall first write the expression in a more convenient form by multiplying by the conjugate.

$$n - \sqrt{n^2 + 2n} = \frac{(n - \sqrt{n^2 + 2n})(n + \sqrt{n^2 + 2n})}{n + \sqrt{n^2 + 2n}} = \frac{-2n}{n + \sqrt{n^2 + 2n}} = \frac{-2n}{n\left(1 + \frac{\sqrt{n^2+2n}}{n}\right)}$$

$$= \frac{-2}{1 + \frac{\sqrt{n^2+2n}}{n}}$$

Now, we can invoke the Algebraic Limit Theorem, as long as the denominator is non-zero. We shall calculate that limit.

$$\lim 1 + \frac{\sqrt{n^2+2n}}{n} = \lim 1 + \lim \sqrt{\frac{n^2+2n}{n^2}} = 1 + \lim \sqrt{1 + \frac{2}{n}}$$

Now using exercise 2.3.1 and that $\left(\frac{1}{n}\right) \to 0$, we have that

$$\lim \sqrt{1 + \frac{2}{n}} = \sqrt{\lim \left(1 + \frac{2}{n}\right)} = \sqrt{1} = 1$$

and so $\lim 1 + \frac{\sqrt{n^2+2n}}{n} = 2$. Therefore,

$$\lim(n - \sqrt{n^2+2n}) = \lim \frac{-2}{1 + \frac{\sqrt{n^2+2n}}{n}} = \frac{-2}{2} = -1$$

.

2.3.8 a) We can write p as

$$p(t) = \sum_{k=0}^{l} a_k t^k$$

.

Thus,

$$\lim p(x_n) = \lim \sum_{k=0}^{l} a_k x_n^k$$

From here on, we will be invoking the Algebraic Limit Theorem without explicitly mentioning it

$$\lim \sum_{k=0}^{l} a_k x_n^k = \sum_{k=0}^{l} \lim a_k x_n^k = \sum_{k=0}^{l} a_k \lim x_n^k = \sum_{k=0}^{l} a_k x^k = p(x)$$

Thus we have shown the desired limit.

2.3.8 b) Consider the sequence $x_n = \frac{1}{n}$ that converges to 0 and the piecewise function

$$f(t) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

The sequence $\{f(x_n)\} = (0, 0, 0, ...)$ which converges to 0. However, $f(0) = 1$.

2.3.10 a) Consider the sequences $a_n = (-1)^n$ and $b_n = (-1)^(n+1)$. Notice that $\lim a_n - b_n = \lim 0 = 0$. However, neither a_n nor b_n converges.
b) Let $\epsilon > 0$. Since $b_n \to b$, there exists N such that $\forall n \geq N$, we have that $|b_n - b| < \epsilon$. By the Reverse Triangle inequality, we have that

$$||b_n| - |b|| \leq |b_n - b| < \epsilon$$

Thus by the definition of a limit, $|b_n| \to |b|$.
c) *It seems like textbooks meant to say $a_n \to a$ NOT $a_n \to 0$. The following solution will assume this correction.*
Let $\epsilon > 0$. Since $a_n \to a$, we know there exists N_1 such that $|a_n - a| < \frac{\epsilon}{2}$. Similarly, there exists N_2 such that $|b_n - a_n| < \frac{\epsilon}{2}$. Now, let $N = \max(N_1, N_2)$ and $n \geq N$, then

$$|b_n - a| = |b_n - a_n + a_n - a| \leq |b_n - a_n| + |a_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, we have proven the limit.
d) Let $\epsilon > 0$. Since $a_n \to 0$, there exists N such that $\forall n \geq N$, we have that $|a_n| < \epsilon$. Let $n \geq N$, then

$$|b_n - b| \leq a_n \leq |a_n| < \epsilon$$

Thus, we have proven the limit.

2.3.12 a) This problem seems to already assume that B is a nonempty subset of $\mathbb{R}$. Thus since B has an upper bound, $\sup B$ exists. Let $b = \sup B$.
Assume that a is not an upper bound of B, so $a < b$. Since $\{a_n\} \to a$, we know there exists N such that $\forall n \geq N$, we can have that $|a_n - a| < \frac{|b-a|}{2}$ by the definition of a limit. However, this implies that there is a_m where $m \geq N$ that is less than b. This contradicts that every a_n is an upper bound. Thus, our assumption that a is not an upper bound is incorrect and by contradiction we have that a must be an upper bound.
b) Assume that $a \in (0,1)$. Let $\gamma = \min(a, 1-a)$. By definition of a limit, there exists N such that $\forall n \geq N$, we have that $|a_n - a| < \frac{\gamma}{2}$. However, this implies there exists a_m with $m \geq N$ such that $a_m \in (0,1)$. This is a contradiction. Thus, $a \notin (0,1)$.
c) Let $a = \sqrt(2)$. We will proceed to construct a sequence implicitly that converges to a.
Let $\epsilon_n = \frac{1}{n}$. Because the rationals are dense over $\mathbb{R}$, there exists r such that $\sqrt{2} - \epsilon_n < r < \sqrt{2}$. For each n, choose a corresponding rational number a_n.
This process will produce a sequence $\{a_n\}$. We must now show that this sequence converges to $\sqrt{2}$. Let $\epsilon > 0$. First, notice that by construction,

$$|a_n - \sqrt{2}| < \epsilon_n = \frac{1}{n}$$

Let $n > \frac{1}{\epsilon}$, then $\frac{1}{n} < \epsilon$. Thus, we have that $|a_n - \sqrt(2)| < \epsilon$ and by definition, $\{a_n\}$ converges to $\sqrt(2)$. Notice that every a_n was rational but the limit is irrational.

2.4 Solutions

2.4.2 a) The problem is, we have yet to establish that the sequence even converges. Using y_1 to then recursively find further values in the sequence, we see that $y_1 = 1$, $y_2 = 2$, $y_3 = 1$. Notice that $y_3 = y_1$. Thus, the sequence will simply alternate between the values of 1 and 2.

Remark: For further information on when a recursive system such as the one above has cyclic behavior and so on, look into discrete dynamical systems.

2.4.2 b) To see if the strategy will even work, we must first establish that the sequence converges. Only then will the assumption that $\lim y_n = \lim y_{n+1}$ actually hold. To do this, we shall invoke the Monotone Convergence Theorem by first showing that the sequence is monotone increasing and bounded above. First, we shall show that $\{y_n\}$ is monotone increasing by induction. Note that $y_1 = 1$ and $y_2 = 3 - \frac{1}{1} = 2$. It is obvious that $y_1 \leq y_2$. This is our base case. For the induction step, we wish to show that $y_{n+1} \leq y_{n+2}$. We shall start from the induction hypothesis, $y_n \leq y_{n+1}$.

$$\begin{aligned}
y_n &\leq y_{n+1} \\
-y_n &\geq -y_{n+1} \\
-\frac{1}{y_n} &\leq -\frac{1}{y_{n+1}} \\
3 - \frac{1}{y_n} &\leq 3 - \frac{1}{y_{n+1}} \to y_{n+1} \leq y_{n+2}
\end{aligned}$$

Thus by induction, we see that the sequence is monotone increasing.
Now, we will show that $y_n \geq 1$ which will later be used to show that $y_n \leq 3$. Again, we shall use induction. First, we see that $y_1 = 1 \geq 1$. Next, notice that

$$\begin{aligned}
y_n &\geq 1 \\
\frac{1}{y_n} &\leq 1 \\
-\frac{1}{y_n} &\geq -1 \\
3 - \frac{1}{y_n} &\geq 2 \geq 1 \to y_{n+1} \geq 1
\end{aligned}$$

Thus, by induction, we have that $y_n \geq 1$. Now to show that $y_n \leq 3$. We have that $y_1 \leq 3$. Notice that

$$\begin{aligned} y_n &\geq 1 \\ -y_n &\geq -1 \\ -\frac{1}{y_n} &\leq -1 \\ 3-\frac{1}{y_n} &\leq 2 \leq 3 \rightarrow y_{n+1} \leq 3 \end{aligned}$$

By induction, we have that $y_n \leq 3$. Thus by the Monotone Convergence Theorem, we have that y_n converges. Since we have shown that the sequence converges to some limit, y, we can apply the strategy and we get the equation $y = 3 - \frac{1}{y}$, which when solved using the quadratic formula gives us $y = \frac{3 \pm \sqrt{5}}{2}$. Since all the terms are between 1 and 3, we see that the only reasonable value is $y = \frac{3+\sqrt{5}}{2}$.

2.4.4 a) *In the following paragraph, Archimadean Property will be used to refer to the first part of the Archimadean Property*

Assume the Archimedean Property does not hold. Let $x \in \mathbb{R}$ and $x > 0$. Consider the sequence $a_n = nx$. By our assumption that the Archimadean Property does not hold, we have that for $n \in \mathbb{B}$, $n \leq x$. This implies that $nx \leq x^2$ and therefore our sequence is bounded. Furthermore, $a_{n+1} - a_n = (n+1)x - nx = x > 0$ and so the sequence is monotone increasing. Therefore by the Monotone Convergence Theorem, we have that the sequence converges to some limit, L. Therefore, there exists N such that $\forall n \geq N$, we have that $|nx - L| < x$. Now since the sequence is monotone increasing, we must have that $L \geq nx$ $\forall n \in \mathbb{N}$. Therefore, $|nx - L| = L - nx < x \rightarrow L < (n+1)x$. However, this contradicts the previous statement that $L \geq nx$ $\forall n \in \mathbb{N}$. Therefore, our assumption that the Archimadean Property does not hold is false.

The second part of the Archimadean Property follows by letting $x \in \mathbb{R}$. We have just shown that the first part of the Archimadean Property holds, so there exists n such that $n > \frac{1}{x}$ which can then be rearranged to give $x > \frac{1}{n}$, as desired.
b) So we have that $I_n = [a_n, b_n]$ and $I_n \subseteq I_{n+1}$. Let us look at the sequence of a_n, the lower bounds of each interval. Now notice that each b_n serves as an upper bound and also that $a_n \leq a_{n+1}$, thus by the Monotone Convergence Theorem, we have that the sequence of a_n converges to some limit, a. Now notice that since the a_n are monotone increasing, we have that $a_n \leq a$ and also since each b_n is an upper bound, we have that $a_n \leq a \leq b_n$, showing that $a \in I_n$ for each $n \in \mathbb{N}$. Thus, we have proven the Nested Interval Property.

2.4.6 a) Notice that

$$\begin{aligned}(x-y)^2 = x^2+y^2-2xy = (x+y)^2 - 4xy &\geq 0 \\ \frac{(x+y)^2}{4} &\geq xy \\ \frac{x+y}{2} &\geq \sqrt{xy}\end{aligned}$$

Which is the desired result.

b) We already see from the part (a) that $x_{n+1} \leq y_{n+1}$ for all $n \in \mathbb{N}$. Thus,

$$x_n \leq y_n = \frac{x_n + y_n}{2} \leq \frac{2y_n}{2} = y_n$$

Also, we have that

$$x_{n+1} = \sqrt{x_n y_n} \geq \sqrt{x_n^2} = x_n$$

Thus from these inequalities, we have that $\{x_n\}$ is monotone increasing and $\{y_n\}$ is monotone decreasing. Also from the first inequality, we see that each x_n is a lower bound for each y_n and vice versa. Thus by the Monotone Convergence Theorem, both sequences must converge. Let $\lim x_n = x$ and $\lim y_n = y$. Then we can apply the strategy as seen in 2.4.2,

$$\begin{aligned}\lim x_{n+1} &= \lim \sqrt{x_n y_n} & \lim y_{n+1} &= \lim \frac{x_n + y_n}{2} \\ x &= \sqrt{xy} & y &= \frac{x+y}{2} \\ x^2 &= xy & 2y &= x+y \\ x &= y & x &= y\end{aligned}$$

Thus we have shown the desired result.

2.4.8 a) *Remark: s_k will denote the partial sum of the first k terms in this question*

First let's find the explicit formula for the partial sums.

$$\begin{aligned}s_k = \sum_{n=1}^{k} \frac{1}{2^n} &= \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^k} \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{k-1}}\right) \\ &= \frac{1}{2} + \frac{1}{2}\left(s_k + \frac{1}{2^k}\right)\end{aligned}$$

So from this, we have that $s_k = \frac{1}{2} + \frac{1}{2}\left(s_k + \frac{1}{2^k}\right)$. Solving for s_k, we find that $s_k = 1 + \frac{1}{2^k}$. Now notice that $\lim s_k = 1$, and thus the series converges.

(*Proof of the limit has been omitted but it boils down to showing that for every $\epsilon > 0$, there exists N such that $\forall k$, $\frac{1}{2^k} < \epsilon$. This is easily shown to be true.*)
b) Once again, we shall find an explicit formula for s_k. This time, notice that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. (This can be derived through partial fraction decomposition). Thus,

$$\begin{aligned}
\sum_{n=1}^{k} \frac{1}{n(n+1)} &= \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + ... + \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2} \right) + \left(-\frac{1}{3} + \frac{1}{3} \right) + ... + \left(-\frac{1}{k} + \frac{1}{k} \right) - \frac{1}{k+1} \\
&= 1 - \frac{1}{k+1}
\end{aligned}$$

We see that $\lim s_k = 1$ and thus the series converges to 1.
(*Proof of the limit has once again been omitted, but it comes down to showing that for every $\epsilon > 0$, there exists N such that $\forall k \geq N$, $\frac{1}{k+1} < \epsilon$. This is easily shown to be true.*)
c) To find an explicit formula for s_k, we will use the property of logarithms that $\log\left(\frac{a}{b}\right) = \log a - \log b$.

$$\begin{aligned}
s_k = \sum_{n=1}^{k} \log\left(\frac{n+1}{n}\right) &= \sum_{n=1}^{k} (\log n + 1 - \log n) \\
&= (\log 2 - \log 1) + (\log 3 - \log 2) + ... + (\log(k+1) - \log k) \\
&= -\log 1 + (\log 2 - \log 2) + (\log 3 - \log 3) + ... + (\log k - \log k) + \log(k+1) \\
&= \log(k+1) - \log 1 = \log(k+1)
\end{aligned}$$

So we have that $s_k = \log(k+1)$. However, this limit does not exist since $\log(k+1)$ is unbounded.

2.4.10 a) For a_n let's look at the partial products, p_k.

$$\begin{aligned}
p_k = \prod_{n=1}^{k} \left(1 + \frac{1}{n}\right) &= \prod_{n=1}^{k} \frac{n+1}{n} \\
&= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot ... \cdot \frac{k+1}{k} \\
&= k+1
\end{aligned}$$

Clearly, the partial products of these infinite product do not converge to any finite limit.
Now, let's discuss the $a_n = \frac{1}{n^2}$ case.

$$p_k = \prod_{n=1}^{k} \left(1 + \frac{1}{n^2}\right) = \prod_{n=1}^{k} \frac{n^2+1}{n^2}$$

Now let's look at specific partial products,

$$\begin{aligned}
p_1 &= 2 \\
p_2 &= \frac{5}{2} = 2.5 \\
p_3 &= \frac{25}{9} \approx 2.777... \\
p_4 &\approx 2.9513 \\
&. \\
&. \\
&. \\
p_{100} &\approx 3.6396 \\
&. \\
&. \\
&. \\
p_{900} &\approx 3.6719979143704307 \\
p_{901} &\approx 3.6720024376438865 \\
p_{902} &\approx 3.67200695089903 \\
p_{903} &\approx 3.6720114541691076
\end{aligned}$$

So basically it appears that the sequence of partial products at first seems to be growing rapidly, but much farther out in the sequence, it seems to be still increasing but not as rapidly and it appears to be approaching some value around 3.67. However, numerical evidence is not enough. In part (b) of this problem, we will properly prove that this infinite product has a finite value.

b) Note that in both the infinite product and the infinite sum, if any values of $a_n = 0$, then the result is not affected (because the $1 + a_n$ in the product will evaluate to 1, and the term will simply be 0 in the sum. In either case, the final value will not be affected). Thus, we can assume that our sequence of a_n is strictly positive.

Let s_k denote the partial sum in the infinite series and p_k be the partial product in the infinite product. We shall start with the backwards direction. So we shall assume that the infinite sum converges. Using the inequality given to us that $1 + x \leq 3^x$ for positive x, we have that

$$p_k = (1 + a_1)(1 + a_2)...(1 + a_k) \leq 3^{a_1} \cdot 3^{a_2} \cdot ... \cdot 3^{a_k} = 3^{a_1+a_2+...+a_k} = 3^{s_k}$$

Now since the partial sums converge, we have that the sequence of p_k must

converge. Thus, the infinite product converges.
Now for the forward direction. Notice that

$$a_k = a_1 + a_2 + ... + a_k \leq 1 + a_1 + a_2 + ... + a_k \leq (1 + a_1)(1 + a_2)...(1 + a_k) = p_k$$

By a similar argument as was used in the backwards direction, we have that since the sequence of p_k converge, the sequence of a_k must converge as well.

Section 2.5

2.5.2 a) Consider the subsequence $(x_2, x_3, x_4, ...)$. Since this converges, we can see that by simply adding on the first term x_1 to this subsequence will not affect the limit. Thus, the full sequence also converges to this same limit.
b) Assume the contrary: that (x_n) contains a divergent subsequence but x_n converges. This is contradiction because every subsequence must converge to the limit of x_n. Thus if there is a divergent subsequence, then x_n diverges.
c) Since the sequence is bounded, there exists a convergent subsequence that converges to some limit c. Now if any other subsequence also converged to c, this would imply that x_n would converge. However, this cannot be so there must be another subsequence that converges to a different limit.
d) Assume that the sequence is strictly monotone increasing. Let (x_{n_k}) be a convergent subsequence that converges to L. We will show that x_n also converges to L. To do this, we must show that for every $\epsilon > 0$ there exists N such that $\forall n \geq N$, we have that $|x_n - L| < \epsilon$.
Now since the subsequence converges to L, there exists N such that $\forall n_k \geq N$, we have that $|x_{n_k} - L| < \epsilon$. Now since the sequence is monotone increasing, there exists some $n \geq N$ such that

$$x_{n_k} \leq x_n \leq x_{n_{k+1}}$$

subtracting L and using that $|x_{n_k} - L| < \epsilon \to -\epsilon < x_{n_k} - L < 0$, we find that

$$-\epsilon < x_{n_k} - L \leq x_n - L \leq x_{n_{k+1}} - L < 0$$

From this, we see that $|x_n - L| < \epsilon$, as desired. A proof for strictly monotone decreasing functions follows the same lines except that we have

$$x_{n_k} \geq x_n \geq x_{n_{k+1}}$$

which in turn gives us

$$\epsilon > x_{n_k} - L \geq x_n - L \geq x_{n_{k+1}} - L > 0$$

From this, we also have that $|x_n - L| < \epsilon$. The last case is where all the terms are equal, which is trivially true.

2.5.4

2.5.6 We shall prove that the sequence converges to 1. We will split this into two cases: $b \geq 1$ and $b \leq 1$.
Case 1: If $b \geq 1$, then we see that $b^{\frac{1}{n}} \geq 1$. As a proof of this, assume that this were not so, then $b^{\frac{1}{n}} \leq 1$. Raising this to the n^{th} power, we see that $b \leq 1$. However this is a contradiction.
Next, we shall show that the sequence is monotone decreasing by induction. First notice that $b \leq b^2$ and after taking the square root of both sides, we see that $b^{\frac{1}{2}} \leq b$. This is our base case. Now, we wish to show that $b^{\frac{1}{n+1}} \leq b^{\frac{1}{n}}$. By our induction hypothesis, we have that $b^{\frac{1}{n}} \leq b^{\frac{1}{n-1}}$. From this, we see that

$$b \cdot b^{\frac{1}{n-1}} \geq b \cdot b^{\frac{1}{n}} = b^{1+\frac{1}{n}} \geq b$$

Thus we see that $b \leq b \cdot b^{\frac{1}{n}} = b^{\frac{n+1}{n}}$. Taking the $n+1$ root of both sides, we have that $b^{\frac{1}{n+1}} \leq b^{\frac{1}{n}}$. Thus by the Monotone Convergence Theorem, we have that the sequence converges for $b \geq 1$.
Case 2: If $0 \leq b \leq 1$, then by a similar argument for $b \geq 1$, we see that $b^{\frac{1}{n}} \leq 1$. Furthermore, we also see by induction that $b^{\frac{1}{n+1}} \geq b^{\frac{1}{n}}$ and so the sequence is increasing. Therefore the sequence also converges for $b \leq 1$ by the Monotone Convergence Theorem.
So we see that the sequence converges to some limit, l. And so we can see every subsequence must also converge to l. In particular, the subsequence $\left(b^{\frac{1}{2n}}\right)$ converges to l. So we have that $\lim b^{\frac{1}{n}} = \lim b^{\frac{1}{2n}}$ which implies that $l = \sqrt{l}$ (we have used exercise 2.3.1 in doing this) and thus we see that $l = 1$.

2.5.8 a)
$(1, 2, 3, 4, 5, 6, ...)$ has 0 peaks.
$(1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, ...)$ has 1 peak.
$(2, 1, -\frac{1}{2}, -\frac{1}{3}, ...)$ has 2 peaks.
$(1, -1, 1, -1, ...)$ has infinitely many peaks.
b) A sequence has either finitely many peaks or infinitely many peaks.
Case 1: Assume a sequence x_n has infinitely many peaks. Then there exists $n_0 < n_1 < ...$ such that $a_{n_0} \geq a_{n_1} \geq a_{n_2} \geq ....$ This implies that the sequence of a_{n_k} is monotone decreasing.
Case 2: Assume the sequence has finitely many peaks. Let a_N be the last peak. Let $n_0 = N+1$. Since n_0 is not a peak (since a_n was the last peak), there exists $n_1 > n_0$ such that $a_{n_1} \geq a_{n_0}$ Similarly, there exists $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. Continuing this process, there exists $n_{k+1} > n_k$ such that $a_{n_{k+1}} \geq a_{n_k}$. Therefore, we see that we have $n_0 < n_1 < n_2 < ...$ and $a_{n_0} \leq a_{n_1} \leq a_{n_2} \leq ...$ which is a monotone increasing sequence.
Thus we have shown that every sequence has a monotone subsequence.

Now a proof of Bolzano-Weierstrass follows when x_n is a bounded sequence. Then we have that there is a monotone subsequence by the above proof. Since

x_n is bounded, the monotone subsequence of x_n is also bounded and thus by the Montone Convergence Theorem, we have that this subsequence must converge. Thus, we see that x_n contains a convergent subsequence.

2.6

2.6.2 a) The sequence $\{\frac{(-1)^n}{n}$ is convergent and therefore Cauchy and is clearly not monotone.
b) This cannot happen. Since Cauchy sequences converge, every subsequence must also converge and therefore every subsequence is bounded.
c) This cannot happen. Assume that this were possible. Let $\{x_n\}$ be a monotone increasing sequence and assume there exists a convergent (and therefore Cauchy) subsequence, $\{x_{n_k}\}$ that converges to a limit L. Since $\{x_n\}$ is monotone increasing and is divergent, there exists $x_\ell > L$. This means that all the $n_k < \ell$. However, this would imply that there are only a finite number of terms in the subsequence. This is contradiction.
A similar argument applies to monotone decreasing sequences by negating each term, resulting in a montone increasing sequence.
d) Consider the sequence $(0, 1, 0, 2, 0, 3, 0, 4, ...)$ then we can see that the sequence is clearly unbounded, however every odd term creates the subsequence $(0, 0, 0, ...)$ which is clearly Cauchy.

2.6.4 a) Let $\epsilon > 0$. Since a_n and b_n are Cauchy, there exists N_1 such that for $n, m \geq N_1$, we have that $|a_n - a_m| < \frac{\epsilon}{2}$ and similarly there exists N_2 such that for $n, m \geq N_2$, we have that $|b_n - b_m| < \frac{\epsilon}{2}$. Now, let $N = \max(N_1, N_2)$, then we see that

$$\begin{aligned}
|c_n - c_m| = ||a_n - a_m| - |a_m - b_m|| &\leq |a_n - a_m - b_n + b_m| \\
&\leq |a_n - a_m| + |b_n - b_m| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus we have shown that c_n is Cauchy.
b) It is not necessarily true that c_n is Cauchy. Let $a_n = 1$, which is clearly a Cauchy sequence. Then $c_n = (-1)^n$, which is clearly not Cauchy (consider $\epsilon < 2$).
c) It is not necessarily true that c_n is Cauchy. Let $a_n = \frac{(-1)^n}{n}$. Since a_n converges, it is Cauchy. However, we then have that

$$c_n = [[a_n]] = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Clearly, c_n is not Cauchy.

2.6.6 a) Let $a_n = \frac{(-1)^n}{n}$. Let $\epsilon > 0$ and $N > \frac{2}{\epsilon}$ then

$$|a_n - a_m| \leq |a_n| + |a_m| = \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

And so we see that $-\epsilon < a_n - a_m < \epsilon$ which gives us $a_m > a_n - \epsilon$. So this sequence is quasi-increasing.

b)

c)

2.7