Trigonometric tables

Θ°	Θ^R	sin(Θ)	cos(Θ)	tan(Θ)	cot(Θ)	sec(Θ)	csc(Θ)
0°	0	0	+1	0	±inf	+1	±inf
30°	π/6	+1/2	+√3/2	+√3/3	+√3	+2√3/3	+2
45°	π/4	+√2/2	+√2/2	+1	+1	+√2	+√2
60°	π/3	+√3/2	+1/2	+√3	+√3/3	+2	+2√3/3
90°	π/2	+1	0	±inf	0	±inf	+1
120°	2π/3	+√3/2	−1/2	−√3	−√3/3	−2	+2√3/3
135°	3π/4	+√2/2	−√2/2	−1	−1	−√2	+√2
150°	5π/6	+1/2	−√3/2	−√3/3	−√3	−2√3/3	+2
180°	π	0	−1	0	±inf	−1	±inf
210°	7π/6	−1/2	−√3/2	+√3/3	+√3	−2√3/3	−2
225°	5π/4	−√2/2	−√2/2	+1	+1	−√2	−√2
240°	4π/3	−√3/2	−1/2	+√3	+√3/3	−2	−2√3/3
270°	3π/2	−1	0	±inf	0	±inf	−1
300°	5π/3	−√3/2	+1/2	−√3	−√3/3	+2	−2√3/3
315°	7π/4	−√2/2	+√2/2	−1	−1	+√2	−√2
330°	11π/6	−1/2	+√3/2	−√3/3	−√3	+2√3/3	−2
360°	2π	0	+1	0	±inf	+1	±inf

Geometry

Circles: Circ. $C=2\pi r$ Area $A=\pi r^2$

Cylinders: Volume $V=\pi r^2 h$ Area $S=2\pi r^2+2\pi rh$

Cones: Volume $V=1/3\pi r^2 h$ Area $S=\pi r^2+\pi r\sqrt{(r=r^2+h^2)}$

Spheres: Volume $V=4/3\pi r^3$ Area $S=4\pi r^2$

Triangles:

For any Triangle with sides <a, b, c> and angles <A, B, C> (with sides opposite the angle with the matching letter):

sum of angles<A, B, C>=180° area = 1/2base*hight

Semiperimiter: $s=(a+b+c)/2$

Herons Formula: $area=\sqrt{s(s-a)(s-b)(s-c)}$

The law of Sines: $\frac{\sin(A)}{a}=\frac{\sin(B)}{b}=\frac{\sin(C)}{c}$

The Law of Cosines: $a^2=b^2+c^2-2bc\cos(A)$

$b^2=a^2+c^2-2ac\cos(B)$ $c^2=a^2+b^2-2ab\cos(C)$

The Law of Tangents: $\frac{\tan[(A-B)/2]}{\tan[(A+B)/2]}=\frac{a-b}{a+b}$

Right Triangles:

For a Right Triangle with sides <abc> and angles <ABC> and C being the 90° angle:

Pythagorean Theorem: $a^2+b^2=c^2$

$A=\tan^{-1}(a/b)$ $\sin(A)=a/c$ $\cos(A)=b/c$ $\tan(A)=a/b$

$B=\tan^{-1}(b/a)$ $\sin(B)=b/c$ $\cos(B)=a/c$ $\tan(B)=b/a$

$C=90°$ $\sin(C)=1$ $\cos(C)=0$ $\tan(C)=+\text{inf}$

Trigonometric Identities

$\sin$=oposite/hypotenuse cos=adjacent/hypotenuse

$\tan(x)=\sin(x)/\cos(x)$ tan=oposite/adjacent

$\cot(x)=\cos(x)/\sin(x)$ $\cot(x)=1/\tan(x)$

$\sec(x)=1/\cos(x)$ $\csc(x)=1/\sin(x)$

$1+\tan^2(x)=\sec^2(x)$ $1+\cot^2(x)=\csc^2(x)$

$\sin(x+/-\pi/4)=+/-\cos(x)$ $\cos(x+/-\pi/4)=-/+\sin(x)$

$\sin(x+/-\pi/2)=-\sin(x)$ $\cos(x+/-\pi/2)=-\cos(x)$

Odd-even: **Co-function**:

$\sin(-x)=-\sin(x)$ $\sin(\pi/2-x)=\cos(x)$

$\cos(-x)=\cos(x)$ $\cos(\pi/2-x)=\sin(x)$

$\tan(-x)=-\tan(x)$ $\tan(\pi/2-x)=\cot(x)$

Pythagorean:

$\sin^2(x)+\cos^2(x)=1$ $\sinh^2(x)-\cosh^2(x)=1$

Eulers Formulas:

$e^{j\Theta}=\cos(\Theta)+j*\sin(\Theta)$ $e^{j\pi/2}=j$ $e^{j\pi}=-1$ $e^{j\pi}+1=0$

$\sin(\Theta)=\frac{e^{j\Theta}-e^{-j\Theta}}{2j}$ $\cos(\Theta)=\frac{e^{j\Theta}+e^{-j\Theta}}{2}$

$\sin(\Theta)=\Theta-\Theta^3/3!+\Theta^5/5!-\Theta^7/7!+\ldots$ $\cos(\Theta)=1-\Theta^2/2!+\Theta^4/4!-\Theta^6/6!+\ldots$

Addition/Subtraction:

$\sin(x+/-y)=\sin(x)\cos(y)+/-\cos(x)\sin(y)$

$\cos(x+/-y)=\cos(x)\cos(y)-/+\sin(x)\sin(y)$

$\tan(x+/-y)=\frac{\tan(x)+/-\tan(y)}{1-/+\tan(x)\tan(y)}$

Double-angles:

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

$\sin(2x)=2\sin(x)\cos(x)$ $\tan(2x)=\frac{2*\tan(x)}{1-\tan^2(x)}$

Half-angles:

$\sin(x/2)=\pm\sqrt{((1-\cos(x))/2)}$ $\cos(x/2)=\pm\sqrt{((1+\cos(x))/2)}$

Sums:

$\sin(x)+\sin(y)=2\sin((x+y)/2)\cos((x-y)/2)$

$\cos(x)+\cos(y)=2\cos((x+y)/2)\cos((x-y)/2)$

Products:

$\sin(x)\sin(y)={}^1/_2[\cos(x-y)-\cos(x+y)]$ $\sin^2(x)={}^1/_2(1-\cos(2x))$

$\cos(x)\cos(y)={}^1/_2[\cos(x+y)+\cos(x-y)]$ $\cos^2(x)={}^1/_2(1+\cos(2x))$

$\sin(x)\cos(y)={}^1/_2[\sin(x+y)-\sin(y-x)]$

Hyperbolic Functions

$\sinh(x)={}^1/_2(e^x-e^{-x})$ $\cosh(x)={}^1/_2(e^x+e^{-x})$

$\tanh(x)=\sinh(x)/\cosh(x)$ $\coth(x)=\cosh(x)/\sinh(x)$

$\text{sech}(x)=1/\cosh(x)$ $\text{csch}(x)=1/\sinh(x)$

Exponential Functions

$\ln(1)=0$ $\ln(a*b)=\ln(a)+\ln(b)$

$\ln(e)=1$ $\ln(a/b)=\ln(a)-\ln(b)$

$\log_a(x)=\ln(x)/\ln(a)$ $\ln(1/x)=-\ln(x) \Leftrightarrow \ln(1/x)+\ln(x)=0$

$\log_b a*\log_a c=\log_b c$ $\ln(1/e^x)=-\ln(e^x)=-x$

$\ln(x^n)=x*\ln(x)$ $\ln(a^x)=x*\ln(a)$

$\ln(e^x)=x$ $e^{\ln(x)}=x$ for x>0

$a=b^{\log b(a)}$ $a^{\log b(n)}=n^{\log b}(a)$

$\ln(x)=1+\sum[n=1..\infty](-1^n(x-1)^n/n)$ $n!=\text{round}(\sqrt{[\pi(2n+1/3)]}n^n e^{-n})$

$e^x=\sum n=0..\infty[x^n/x!]$ $(1+x/n)^n\to e^x$ as $n\to\infty$

$[e^x]^y=e^{(x*y)}$ $e^x*e^y=e^{(x+y)}$

$[e^x]^{-y}=e^{(x/y)}$ $e^x/e^y=e^{(x-y)}$

$x^n=e^{n*\ln(x)}$ $x^n*x^m=e^{(n+m)*\ln(x)}$

$x^x=e^{x*\ln(x)}$ $x=e^y \Leftrightarrow y=\ln(x)$

$a^x=e^{x*\ln(a)}$ $y=\lambda e^{-\lambda x} \Leftrightarrow x=\ln(y/\lambda)/\lambda$

$e^x=1+x+x^2/2!+x^3/3!+x^4/4!+x^5/5!+\ldots$

$\int_{-\infty}^{\infty}\lambda x^n e^{-\lambda x}=\frac{n!}{x^n}$ $[e^a]\wedge(e^b)=e^{(ae\wedge b)}$

Complex Numbers

$z=[x+jy]=r\angle\Theta=re^{j\Theta}$ $\cos(\Theta)=\text{Re}(e^{j\Theta})$

$z=r[\cos(\Theta)+j\sin(\Theta)]$ $\sin(\Theta)=\text{Im}(e^{j\Theta})$

$z^*=[x-jy]=r\angle-\Theta=re^{-j\Theta}$ $\text{Re}(e^{(\sigma+j\omega)t})=\text{Re}(e^{\sigma t}e^{j\omega t})=e^{\sigma t}\cos(\omega t)$

$\Theta=\tan^{-1}(y/x)$ $\text{Im}(e^{(\sigma+j\omega)t})=\text{Im}(e^{\sigma t}e^{j\omega t})=e^{\sigma t}\sin(\omega t)$

$r=|z|=\sqrt{x^2+y^2}$ $zz^*=\sqrt{x+jy}=\sqrt{r}\angle^{\Theta}/_2$

$e^{j\Theta}=\cos(\Theta)+j\sin(\Theta)$ $|e^{j\Theta}|=\sqrt{\cos^2\Theta+\sin^2\Theta}=1$

$z^n=(x+jy)^n=r^n\angle n\Theta=r^n e^{jn\Theta}=$ $z^n=r^n[\cos(n\Theta)+j\sin(n\Theta)]$

$z^{1/n}=(x+jy)^{1/n}=r^{1/n}\angle[{}^{\Theta}/_n+{}^{2\pi k}/_n]_{k=0,1,\ldots n-1}$ $z^{1/n}=r^{1/n}[\cos(\emptyset)+j\sin(\emptyset)]$, $\emptyset={}^{\Theta}/_n+{}^{2\pi k}/_n$ $_{k=0,1,\ldots n-1}$

$\ln(re^{j\Theta})=\ln(r)+j\Theta+j2k\pi$ $_{k=\text{integer}}$ $e^{j\Theta}=1+j\Theta-\Theta^2/2!-\Theta^3/3!+\Theta^4/4!+\Theta^5/5!+\ldots$

Miscelaneous

Quadratic Formula: If $ax^2+bx+c=0$ then,

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

lines:

Slope from P_1 to P_2: $m=\frac{(y_2-y_1)}{(x_2-x_1)}$

$P_1=(x_1,y_1)$ $P_2=(x_2,y_2)$

Point-Slope equation: $y-y_1=m(x-x_1)$

Slope-Intercept equation: $y=mx+b$ slope=m, y intercept=b

Degrees <-> Radians: 180° = π Radians

Factoring Polynomials:

x^2+y^2 = **(x-jy)(x+jy) or (y-jx)(y+jx)** $x^2-y^2 = (x+y)(x-y)$

$x^3+y^3 = (x+y)(x^2-xy+y^2)$ $x^3-y^3 = (x-y)(x^2+xy+y^2)$

$(x+y)^2 = x^2+2xy+y^2$ $(x-y)^2 = x^2-2xy+y^2$

$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$ $(x-y)^3 = x^3-3x^2y+3xy^2-y^3$

$(x+y)^4 = x^4+4x^3y+6x^2y^2+3xy^3+y^4$ $(x-y)^4 = x^4-3x^3y+6x^2y^2-3xy^3+y^4$

$x^n+x^{n-1}+x^{n-2}\ldots x^2+x+1 = \left(\frac{x^{n+1}-1}{x-1}\right)$

Use Pascals Triangle for higner powers of $(x+y)^n$ or $(x-y)^n$

Equations of Common Shapes

Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=K$, K > 0

Hyperbolic Cylinder of 2 sheets $\frac{x^2}{a^2}-\frac{y^2}{b^2}=K$, K > 0

Hyperboloid of 1 sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=K$, K > 0

Elliptic Cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$

Hyperboloid of 2 sheets $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=K$, K > 0

Elliptic Paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$

Hyperboloc Paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$

Vectors in 2D

For any points $p_1=(x_1,y_1)$ and $p_2=(x_2,y_2)$

Distance from p_1 to p_2: $|p_1\,p_2|=\sqrt{((x_2-x_1)^2+(y_2-y_1)^2)}$

Vector from p_1 to p_2: $<p_1\,p_2>=<(x_2-x_1),\ (y_2-y_1)>$

Midpoint of p_1 and p_2: $m_{12}=\left(\frac{(x_2+x_1)}{2},\ \frac{(y_2+y_1)}{2}\right)$

For any vectors **u**, **v**, and **w** and any scalars a, b and c, the following relationships hold: **i**=unit vector in x direction **j**=unit vector in y direction **k**=unit vector in z direction

i·j=0, j·k=0, k·i=0 **i·i=i, j·j=j, k·k=k**

u+v=v+u **(u+v)+w=u+(v+w)**

u+0=0+u=u **u+(-u)=0**

0=0u=<0,0> **-u=**$<-u_1,\ -u_2>$

uc=cu=$<cu_1,\ cu_2>$ |c**u**|=|c| |**u**|

a(b**u**)=(ab)**u**=**u**(ab) a(**u+v**)=a**u**+a**v**

(a+b)**u**=a**u**+b**u** 1**u**=**u**

u+v$=<u_x+v_x,u_y+v_y>$ (vect) **u·v**$=u_xv_x+u_yv_y$ (scalar)

u·v=v·u **u·v**=0 Iff **u** ⊥ **v**

u·(v+w)=u·v+u·w c(**u·v**)=(c**u**)·**v**=**u**·(c**v**)

$|\mathbf{u}|=\sqrt{((u_x)^2+(u_y)^2)}$ (scalar) **0·u**=0

$|\mathbf{u}|^2=(u_x)^2+(u_y)^2$ (scalar) $\mathbf{u\cdot u}=|\mathbf{u}|^2$

Unit Vector (length=1) **U=u/|u|**

$\mathbf{u\cdot v}=|\mathbf{u}||\mathbf{v}|\cos(\Theta)$ $|\mathbf{u-v}|^2=|\mathbf{u}|^2+|\mathbf{v}|^2-2|\mathbf{u}||\mathbf{v}|\cos(\Theta)$

Vectors in 3D

all 2D vector ops apply to 3D

i×j=k, j×k=i, k×i=j **i×i=0, j×j=0, k×k=0**

$\mathbf{u}=<u_x,u_y,u_z>$ $\mathbf{v}=<v_x,v_y,v_z>$ $\mathbf{u}=u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k}$ $\mathbf{v}=v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k}$

$|\mathbf{u}|=\sqrt{((u_x)^2+(u_y)^2+(u_z)^2)}$ (scalar)

$\mathbf{u\cdot v}=u_xv_x+u_yv_y+u_zv_z$ (scalar)

$\mathbf{u+v}=<u_x+v_x,u_y+v_y,u_z+v_z>$ (vector)

$\mathbf{u\times v}=<(u_yv_z-u_zv_y)\mathbf{i},(u_zv_x-u_xv_z)\mathbf{j},(u_xv_y-u_yv_x)\mathbf{k}>$ (vector)

u×v=0 Iff **u** is parallel to **v** **u×v** is ⊥ to both **u** and **v**

u×(u×v) Forms a right triangle **u**,**v**,and **u×v** form a ⊥ triple

u×u=0 **u·u×v**=0

u×v=-v×u **u×(v+w)=u×v+u×w**

c(**u×v**)=(c**u**)×**v**=**u**×(c**v**) **u×0=0×u=0**

u×(v·w)=u·(v×w) **u×(v×w)=(u·w)v-(u·v)w**

$|\mathbf{u\times v}|=|\mathbf{u}|\ |\mathbf{v}|\sin(\Theta)$ $|\mathbf{u\times v}|^2=|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u\cdot v})^2$

$\Theta=\cos^{-1}\left(\frac{|\mathbf{u}|}{|\mathbf{v}|}\right)$ where **v** is the longer vector $\Theta=\cos^{-1}\left(\frac{\mathbf{u\cdot v}}{|\mathbf{u}||\mathbf{v}|}\right)$

Vector Geometry in 3D

Area of a **parallelogram**: **u×v**

Volume of a **parallelopiped**: |**u·v×w**|=|**v·u×w**|=|**w·u×v**|

Standard Equation of a **Plane:** $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$
A, B, and C are the factors of the Normal Vector.
Point $P_0(x_1, y_1, z_1)$ is a point on the plane.

Standard Equation of a **Sphere:** $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=R^2$
Point $P_0(x_1, y_1, z_1)$ is the location of the center of the sphere.
R is the radius of the sphere.

Standard Equation of a **Line:** $Ax+By+Cz+D=0$

Parametric Equation of a **Line:** $x=x_0+at,\ y=y_0+bt,\ z=z_0+ct$
Point $P_0(x_0, y_0, z_0)$ is a point on the line. a, b, and c are the factors of a Vector parallel to the line.

Symetric Equation of a **Line:** $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$

Vector Valued Functions in 3D

Vector Valued Function: $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$

Velocity: $\mathbf{v}(t)=\mathbf{r}'(t)$

Acceleration: $\mathbf{a}(t)=\mathbf{v}'(t)=\mathbf{r}''(t)$

Speed: $v(t)=|\mathbf{v}(t)|$

Unit Tangent Vector: $\mathbf{T}(t)=\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$

Curvature: $k(t)=\frac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|^3}=\frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{|\mathbf{v}(t)|^3}$

Radius: $1/k$

Length of a Curve: $s=\int\sqrt{f'(t)^2+g(t)^2+h(t)^2}$

Normal Vector: $\mathbf{N}(t)=\left(\frac{|\mathbf{v}(t)|^2}{|\mathbf{v}(t)\times\mathbf{a}(t)|}\right)\mathbf{T}'(t)$

Acceleration: $\mathbf{a}(t)=\mathbf{a}_t(t)+\mathbf{a}_n(t)$

$\mathbf{a}_{\text{TANGENT}}$ $\mathbf{a}_t(t)=\mathbf{T}(t)\cdot\mathbf{a}(t)$ $=\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$

$\mathbf{a}_{\text{NORMAL}}$ $\mathbf{a}_n(t)=|\mathbf{T}(t)\times\mathbf{a}(t)|$ $=\frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$

Coordinate Transformations

PolarCoordinates: **r, Θ**

0 ≤ **r** ≤ +inf from origin 0 ≤ **Θ** ≤ 2π ↺ from +X axis

xy->Polar

$\mathbf{r}=\sqrt{(x^2+y^2)}$ $\Theta=\tan^{-1}(y/x)$ x>0 $\pi+\tan^{-1}(y/x)$ x<0

$\tan(\Theta)=y/x$ $\Theta=\pi/2$ x=0, y>0 $3\pi/2$ x=0, y<0

$\Theta=0$ x=0, y=0

polar->xy

$x=\mathbf{r}\cos(\Theta)$ $y=\mathbf{r}\sin(\Theta)$

Cylindrical Coordinates: **r, Θ, z**

0 ≤ **r** ≤ +inf from origin 0 ≤ **Θ** ≤ 2π from +X axis −inf ≤ **z** ≤ +inf from XY plane

cyl->xyz

$x=\mathbf{r}\cos(\Theta)$ $y=\mathbf{r}\sin(\Theta)$ $z=\mathbf{z}$

xyz->cyl

$\mathbf{z}=z$ $\Theta=\tan^{-1}(y/x)$ x>0 $\pi+\tan^{-1}(y/x)$ x<0

$\mathbf{r}=\sqrt{x^2+y^2}$ $\Theta=\pi/2$ x=0, y>0 $3\pi/2$ x=0, y<0

$\Theta=0$ x=0, y=0

Spherical Coordinates: **ρ, Θ, Φ**

0 ≤ **ρ** ≤ +inf from origin 0 ≤ **Θ** ≤ 2π from +X axis −π ≤ **Φ** ≤ π from XY plane

sph->xyz

$x=\rho\sin(\Phi)\cos(\Theta)$ $y=\rho\sin(\Phi)\sin(\Theta)$ $z=\rho\cos(\Phi)$

xyz->sph

$\rho=\sqrt{(x^2+y^2+z^2)}$ $\Theta=\tan^{-1}(y/x)$ x>0 $\pi+\tan^{-1}(y/x)$ x<0

$\Phi=\cos^{-1}(z/\rho)$ $\Theta=\pi/2$ x=0, y>0 $3\pi/2$ x=0, y<0

$\Phi=\tan^{-1}(z/\sqrt{(x^2+y^2)})$ $\Theta=0$ x=0, y=0

sph->cyl

$\mathbf{r}=\rho\sin(\Phi)$ $\Theta=\Theta$ $\mathbf{z}=\rho\cos(\Phi)$

cyl->sph

$\rho=\sqrt{(\mathbf{r}^2+\mathbf{z}^2)}$ $\Theta=\Theta$ $\Phi=\tan^{-1}(\mathbf{z}/\mathbf{r})$

Derivatives

$D_x\ x^n = nx^{n-1}$	$D_x\ \lvert x\rvert = \frac{\lvert x\rvert}{x}$
$D_x\ \sin(x)=\cos(x)$	$D_x\ \cos(x)=-\sin(x)$
$D_x\ \tan(x)=\sec^2(x)$	$D_x\ \cot(x)=-\csc^2(x)$
$D_x\ \sec(x)=\sec(x)\tan(x)$	$D_x\ \csc(x)=-\csc(x)\cot(x)$
$D_x\ \sinh(x)=\cosh(x)$	$D_x\ \cosh(x)=\sinh(x)$
$D_x\ \tanh(x)=\text{sech}^2(x)$	$D_x\ \coth(x)=-\text{csch}^2(x)$
$D_x\ \text{sech}(x)=-\text{sech}(x)\tanh(x)$	$D_x\ \text{csch}(x)=-\text{csch}(x)\coth(x)$
$D_x\ e^x=e^x$	$D_x\ a^x=a^x \ln(a)$
$D_x\ \ln(a*x)=1/x$	$D_x\ \ln(x)=1/x$
$D_x\ \log_a(x)= \frac{1}{x \ln(a)}$	$D_x\ \log_x(a)= \frac{1}{a \ln(x)}$
$D_x\ \sin^{-1}(x)= +\frac{1}{\sqrt{(-x^2+1)}}$	$D_x\ \cos^{-1(x)}= -\frac{1}{\sqrt{(-x^2+1)}}$
$D_x\ \tan^{-1}(x)= +\frac{1}{(+x^2+1)}$	$D_x\ \sec^{-1}(x)= +\frac{1}{x\sqrt{(+x^2-1)}}$
$D_x\ \cot^{-1}(x)= -\frac{1}{(+x^2+1)}$	$D_x\ \csc^{-1}(x)= -\frac{1}{x\sqrt{(+x^2-1)}}$
$D_x\ \sinh^{-1}(x)= +\frac{1}{\sqrt{(+x^2+1)}}$	$D_x\ \cosh^{-1}(x)= +\frac{1}{\sqrt{(+x^2-1)}}$
$D_x\ \tanh^{-1}(x)= +\frac{1}{(-x^2+1)}$	$D_x\ \text{sech}^{-1}(x)= -\frac{1}{x\sqrt{(-x^2+1)}}$
$D_x\ \coth^{-1}(x)= +\frac{1}{(-x^2+1)}$	$D_x\ \text{csch}^{-1}(x)= -\frac{1}{\lvert x\rvert\sqrt{(+x^2+1)}}$

Integrals

$\int x\ dv$	=	$\int v\ dx = xv$	
$\int x^n\ dx$	=	$\frac{1}{(n+1)} x^{n+1}+C$	$n \ne -1$
$\int (1/x)\ dx$	=	$\ln(\lvert x\rvert)+C$	
$\int e^x\ dx$	=	e^x+C	
$\int a^x\ dx$	=	$\frac{a^x}{\ln(a)}+C$	
$\int xe^x\ dx$	=	$(x-1)e^x+C$	
$\int x^n e^x\ dx$	=	$x^n e^x-n(\int x^{n-1}e^x\ dx)$	
	=	$x^n e^x-nx^{n-1}e^x+n(n-1)x^{n-2}e^x\ldots n!e^x+C$	
$\int xe^{ax}\ dx$	=	$(e^{ax}/a^2)(ax-1)+C$	
$\int x^2 e^{ax}\ dx$	=	$(e^{ax}/a^3)(a^2x^2+2ax+2)+C$	
$\int \ln(x)dx$	=	$x(\ln(x)-1)+C$	
$\int \sin(x)\ dx$	=	$-\cos(x)+C$	
$\int \cos(x)\ dx$	=	$\sin(x)+C$	
$\int \sec^2(x)\ dx$	=	$\tan(x)+C$	
$\int \csc^2(x)\ dx$	=	$-\cot(x)+C$	
$\int \sec(x)\tan(x)\ dx$	=	$\sec(x)+C$	
$\int \csc(x)\cot(x)\ dx$	=	$-\csc(x)+C$	
$\int \tan(x)\ dx$	=	$\ln(\lvert\sec(x)\rvert)+C$	
$\int \cot(x)\ dx$	=	$\ln(\lvert\sin(x)\rvert)+C$	
$\int \sec(x)\ dx$	=	$\ln(\lvert\sec(x)+\tan(x)\rvert)+C$	
$\int \csc(x)\ dx$	=	$\ln(\lvert\csc(x)+\cot(x)\rvert)+C$	
$\int \frac{+1}{(+x^2+a^2)}\ dx$	=	$\frac{1}{a}\tan^{-1}(\frac{x}{a})+C$	$a \ne 1$
$\int \frac{+1}{(+x^2+1)}\ dx$	=	$\tan^{-1}(x)+C$	
$\int \frac{+1}{\sqrt{(+x^2+1)}}\ dx$	=	$\sinh^{-1}(x)+C$	
$\int \frac{+1}{\sqrt{(+x^2-1)}}\ dx$	=	$\cosh^{-1}(x)+C$	
$\int \frac{+1}{(-x^2+a^2)}\ dx$	=	$\frac{1}{2a}\ln(\lvert\frac{x+a}{x-a}\rvert)+C$	
$\int \frac{+1}{(-x^2+1)}\ dx$	=	$\tanh^{-1}(x)+C$ << seperate domain >>	
$\int \frac{+1}{(-x^2+1)}\ dx$	=	$\coth^{-1}(x)+C$	
$\int \frac{+1}{(-x^2-1)}\ dx$	=	$\cot^{-1}(x)+C$	
$\int \frac{+1}{\sqrt{(-x^2+a^2)}}\ dx$	=	$\sin^{-1}(\frac{x}{a})+C$	$a \ne 1$
$\int \frac{+1}{\sqrt{(-x^2+1)}}\ dx$	=	$\sin^{-1}(x)+C$ << seperate domain >>	
$\int \frac{+1}{\sqrt{(-x^2+1)}}\ dx$	=	$-\cos^{-1}(x)+C$	
$\int \frac{+1}{\sqrt{(+x^2 \pm a^2)}}\ dx$	=	$\ln(x+\sqrt{(x^2 \pm a^2)})+C$	
$\int \frac{+1}{x\sqrt{(-x^2+a^2)}}\ dx$	=	$\frac{1}{a}\sec^{-1}(\lvert\frac{x}{a}\rvert)+C$	
$\int \frac{-1}{x\sqrt{(-x^2+1)}}\ dx$	=	$\text{sech}^{-1}(x)+C$	
$\int \frac{+1}{x\sqrt{(+x^2-1)}}\ dx$	=	$\sec^{-1}(x)+C$	
$\int \frac{-1}{x\sqrt{(+x^2-1)}}\ dx$	=	$\csc^{-1}(x)+C$	
		<< seperate domain >>	
$\int \frac{-1}{\lvert x\rvert\sqrt{(+x^2+1)}}\ dx$	=	$\text{csch}^{-1}(x)+C$	
$\int \frac{1}{(x \ln(a))}\ dx$	=	$\log_a(x)+C$	
$\int \frac{1}{(a \ln(x))}\ dx$	=	$\log_x(a)+C$	
$\int \sin^2(x)\ dx$	=	$\frac{1}{2}x - \frac{1}{4}\sin(2x)+C$	
$\int \cos^2(x)\ dx$	=	$\frac{1}{2}x + \frac{1}{4}\sin(2x)+C$	
$\int \tan^2(x)\ dx$	=	$\tan(x)-x+C$	
$\int \cot^2(x)\ dx$	=	$-\cot(x)-x+C$	
$\int \sin^3(x)\ dx$	=	$-{}^1/_3(2+\sin^2(x))\cos(x)+C$	
$\int \cos^3(x)\ dx$	=	$-{}^1/_3(2+\cos^2(x))\sin(x)+C$	
$\int \tan^3(x)\ dx$	=	$+{}^1/_2\tan^2(x)+\ln(\lvert\cos(x)\rvert)+C$	
$\int x \sin(x)\ dx$	=	$\sin(x)-x*\cos(x)+C$	
$\int x \cos(x)\ dx$	=	$\cos(x)-x*\sin(x)+C$	
$\int x^n\sin(x)\ dx$	=	$-x^n\cos(x)+n(\int x^{n-1}\cos(x)\ dx)+C$	
$\int x^n\cos(x)\ dx$	=	$-x^n\sin(x)+n(\int x^{n-1}\sin(x)\ dx)+C$	
$\int \sin^{-1}(x)\ dx$	=	$x*\sin^{-1}(x)+\sqrt{(1-x^2)}+C$	
$\int \tan^{-1}(x)\ dx$	=	$x*\tan^{-1}(x)-{}^1/_2\ln(1+x^2)$	
$\int \sinh(x)\ dx$	=	$\cosh(x)+C$	
$\int \cosh(x)\ dx$	=	$\sinh(x)+C$	
$\int \tanh(x)\ dx$	=	$\ln(\cosh(x))+C$	
$\int \text{sech}^2(x)\ dx$	=	$\tanh(x)+C$	
$\int \text{csch}^2(x)\ dx$	=	$\coth(x)+C$	
$\int \text{sech}(x)\tanh(x)\ dx$	=	$-\text{sech}(x)+C$	
$\int \text{csch}(x)\coth(x)\ dx$	=	$-\text{csch}(x)+C$	

Laplace Transforms

$\mathcal{L}[f(t)]$	$= F(s)$
$\mathcal{L}^{-1}[F(s)]$	$= f(t)$
$\mathcal{L}[u(t)]$	$= \frac{1}{s}$
$\mathcal{L}[u(t-a)]$	$= \frac{e^{-as}}{s}$
$\mathcal{L}[tu(t)]$	$= \frac{1}{s^2}$
$\mathcal{L}[e^{at}u(t)]$	$= \frac{1}{s-a}$
$\mathcal{L}[te^{-at}u(t)]$	$= \frac{1}{(s+a)^2}$
$\mathcal{L}[\cos(\omega t)u(t)]$	$= \frac{s}{s^2+\omega^2}$
$\mathcal{L}[\sin(\omega t)u(t)]$	$= \frac{\omega}{s^2+\omega^2}$
$\mathcal{L}[\cos(\omega t+\Theta)u(t)]$	$= \frac{s*\sin(\Theta)+\omega*\cos(\Theta)}{s^2+\omega^2}$
$\mathcal{L}[\sin(\omega t+\Theta)u(t)]$	$= \frac{s*\cos(\Theta)-\omega*\sin(\Theta)}{s^2+\omega^2}$
$\mathcal{L}[\delta(t)]$	$= 1$
$\mathcal{L}[\delta(t-a)]$	$= e^{-as}$
$\mathcal{L}[1]$	$= \frac{1}{s}$
$\mathcal{L}[t]$	$= \frac{1}{s^2}$
$\mathcal{L}[t^n]$	$= \frac{n!}{s^{n+1}}$
$\mathcal{L}[\frac{1}{\sqrt{\pi t}}]$	$= \frac{1}{\sqrt{s}}$
$\mathcal{L}[t^a]$	$= \frac{\Gamma(a+1)}{s^{a+1}}$
$\mathcal{L}[t^n e^{at}]$	$= \frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}[\cos(\omega t)]$	$= \frac{s}{s^2+\omega^2}$
$\mathcal{L}[e^{at}\cos(\omega t)]$	$= \frac{s-a}{(s-a)^2+\omega^2}$
$\mathcal{L}[e^{at}\sin(\omega t)]$	$= \frac{k}{(s-a)^2+\omega^2}$
$\mathcal{L}[\cosh(\omega t)]$	$= \frac{s}{s^2-\omega^2}$
$\mathcal{L}[\sinh(\omega t)]$	$= \frac{\omega}{s^2-\omega^2}$
$\mathcal{L}[\frac{1}{2u(t)^3}(\sin(\omega t)-kt\cos(\omega t))]$	$= \frac{1}{(s^2+\omega^2)^2}$
$\mathcal{L}[\frac{1}{2u(t)}(\sin(\omega t))]$	$= \frac{s}{(s^2+\omega^2)^2}$
$\mathcal{L}[\frac{1}{2k}(\sin(\omega t)+kt\cos(\omega t))]$	$= \frac{s^2}{(s^2+\omega^2)^2}$
$\mathcal{L}[\int_{[0\ldots 1]} f(\tau)g(t-\tau)d\tau]$	$= F(s)G(s)$

Linearity:
$\mathcal{L}[a*f(t)+b*g(t)] = a*\mathcal{L}[f(t)]+b*\mathcal{L}[g(t)]$

Scaling:
$\mathcal{L}[f(at)] = \frac{1}{a}F(\frac{s}{a})$

Time Shift:
$\mathcal{L}[u(t-a)f(t-a)] = e^{as}F(s)$ $\qquad$ $\mathcal{L}[u(t+a)f(t+a)] = e^{-as}F(s)$

Time Differientiation:
$\mathcal{L}[f'(t)] = sF(s)-f(0)$ $\qquad$ $\mathcal{L}[f''(t)] = s^2F(s)-sf(0)-f'(0)$

$\mathcal{L}[f^{(n)}(t)] = s^nF(s)-s^{n-1}f(0)-\ldots-f^{(n-1)}(0)$

Time Integration:
$\mathcal{L}[\int_{[0\ldots t]} f(t)dt] = \frac{F(s)}{s}$

Frequency Shift:
$\mathcal{L}[e^{at}f(t)u(t)] = F(s-a)$ $\qquad$ $\mathcal{L}[e^{-at}f(t)u(t)] = F(s+a)$

Frequency Differientiation:
$\mathcal{L}[tf(t)] = -F'(s)$ $\qquad$ $\mathcal{L}[t^nf(t)] = (-1)^nF^{(n)}(s)$

Frequency Integration:
$\mathcal{L}[\frac{f(t)}{t}] = \int_{[s\ldots\infty]}F(s)ds$ $\qquad$ $\mathcal{L}[\frac{f(t)}{t^n}] = \int^{(n)}_{[s\ldots\infty]}F(s)ds$

Periodic Function: $\quad f(t) = f(t + nT)$, for all $n \ne 0$
$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_{[0\ldots T]}e^{st}f(t)dt$

Square Wave:
$\mathcal{L}[(-1)^{[[at]]}] = \frac{1}{s}\tanh(\frac{as}{2})$

Step Wave:
$\mathcal{L}[[[\frac{t}{a}]]] = \frac{e^{-as}}{s(1-e^{-as})}$

Initial and Final Value of f(t):
$f(0^+) = \lim_{[s\to\infty]}sF(s)$ $\qquad$ $f(\infty) = \lim_{[s\to 0]}sF(s)$

Convolution:
$f(t)\otimes g(t) = \mathcal{L}[f(t)]*\mathcal{L}[g(t)]$

Fourier Transforms

$f(t)=1$	$F(\omega)=2\pi\delta(\omega)$	$\lvert F(\omega)\rvert=2\pi$ at 0
$f(t)=\delta(t)$	$F(\omega)=1$	$\lvert F(\omega)\rvert=1$
$f(t)=\delta(t-a)$	$F(\omega)=e^{-j\omega a}$	$\lvert F(\omega)\rvert=1$
$f(t)=u(t)$	$F(\omega)=\pi\delta(\omega)+1/j\omega$	$\lvert F(\omega)\rvert=\pi$ at 0
$f(t)=\text{sign}(t)$	$F(\omega)=\frac{2}{j\omega}$	
$f(t)=\lvert t\rvert$	$F(\omega)=\frac{-2}{\omega^2}$	
$f(t)=e^{-j\omega 0t}$	$F(\omega)=2\pi\delta(\omega-\omega_0)$	$\lvert F(\omega)\rvert=2\pi$ at ω_0
$f(t)=\cos(\omega_0 t)$	$F(\omega)=\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$	$\lvert F(\omega)\rvert=\pi$ at $\pm\omega_0$
$f(t)=\sin(\omega_0 t)$	$F(\omega)=j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	$\lvert F(\omega)\rvert=j\pi@-\omega_0,-j\pi@\omega_0$
$f(t)=e^{-\alpha t}u(t)$	$F(\omega)=\frac{1}{(\alpha-j\omega)}$	
$f(t)=t^n e^{-\alpha t}u(t)$	$F(\omega)=\frac{n!}{(\alpha-j\omega)^{n+1}}$	
$f(t)=e^{-\alpha t}\cos(\omega_0 t)u(t)$	$F(\omega)=\frac{\alpha+j\omega}{(\alpha+j\omega)^2+\omega_0^2}$	
$f(t)=e^{-\alpha t}\sin(\omega_0 t)u(t)$	$F(\omega)=\frac{\omega_0}{(\alpha+j\omega)^2+\omega_0^2}$	

Linearity:
$f(t)=a*g(t)+b*h(t)$ $\qquad$ $F(\omega)=a*G(\omega)+b*H(\omega)$

Scaling:
$f(t)=g(a*t)$ $\qquad$ $F(\omega)=\frac{1}{\lvert a\rvert}G(\frac{\omega}{a})$

Time Shift:
$f(t)=g(t-a)u(t-a)$ $\qquad$ $F(\omega)=e^{-j\omega a}G(\omega)$

Time Differientiation:
$g(t)=f'(t)$ $\qquad$ $G(\omega)=j\omega F(\omega)$

$g(t)=f^{(n)}(t)$ $\qquad$ $G(\omega)=(j\omega)^nF(\omega)$

Time Integration:
$g(t)=\int_{[-\infty\ldots t]}f(t)dt$ $\qquad$ $G(\omega)=\frac{F(\omega)}{j\omega}+\pi F(0)\delta(\omega)$

Frequency Shift:
$f(t)=e^{-j\omega 0t}g(t)$ $\qquad$ $F(\omega)=G(\omega-\omega_0)$

Frequency Differientiation:
$g(t)=f'(t)$ $\qquad$ $G(\omega)=j\omega F(\omega)$

Unit Pulse:
$f(t)=u(t+\frac{T}{2})-u(t-\frac{T}{2})$ $\qquad$ $F(\omega)=\frac{2}{\omega}\sin(\omega t)$ $\qquad$ period=T

Time Reversal:
$g(t)=f(-t)$ $\qquad$ $G(\omega)=F(-\omega)$ or $F^*(\omega)$

Duality:
$g(t)=F(t)$ $\qquad$ $G(\omega)=2\pi f(-\omega)$

Convolution:
$f(t)=g(t)\otimes h(t)$ $\qquad$ $F(\omega)=G(\omega)*H(\omega)$

$f(t)=g(t)*h(t)$ $\qquad$ $F(\omega)=\frac{1}{2\pi}G(\omega)\otimes H(\omega)$

Amplitude Modulation:
$f(t)=\cos(\omega_0 t)g(t)$ $\qquad$ $F(\omega)=\frac{1}{2}[G(\omega+\omega_0)+G(\omega-\omega_0)]$

Differentiation

Constant Function: $f(x)=k$ $D_x[k]=0$

Identity Function: $f(x)=x$ $D_x(x)=1$

Power Rule: $f(x)=x^n$ $D_x[x^n]=nx^{n-1}$

Constant Multiples: $g(x)=kf(x)$ $D_x[k\cdot f(x)]=k\cdot D_x[f(x)]$

Sums: $h(x)=f(x)+g(x)$
$D_x[f(x)+g(x)]=D_x[f(x)]+D_x[g(x)]$

Differences: $h(x)=f(x)-g(x)$
$D_x[f(x)-g(x)]=D_x[f(x)]-D_x[g(x)]$

Product & Quotients: The derivative of a product of functions is **NOT** equal to the product of the derivatives of the functions.

Products: $h(x)=f(x)g(x)$
$D_x[f(x)g(x)]=f(x)D_x[g(x)]+g(x)D_x[f(x)]$

Quotients: $h(x)=f(x)/g(x)$
$$D_x[f(x)/g(x)]=\frac{g(x)D_x[f(x)]-f(x)D_x[g(x)]}{[g(x)]^2}$$

Chain Rule: $y=f(u)$ and $u=g(x)$
$D_x[f(g(x))]=D_x[f(g(x))]D_x[g(x)]$

L'Hopitals Rule: If limit[$\mathbf{f(x)=0}$] and limit[$\mathbf{g(x)=0}$],
$$\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right)=\lim_{x\to 0}\left(\frac{f'(x)}{g'(x)}\right)$$

Partial Derivatives

Partial Derivative of $\mathbf{f(x,y)}$ with respect to x:
Treat y as a constant and take the x derivative of $\mathbf{f(x,y)}$.

$\partial f/\partial x$ and $f_x(x,y)$ denotes the partial derivative of f with respect to x. $\partial^2 f/\partial y\partial x$ and $f_{xy}(x,y)$ denote the partial derivative of f with respect to x and then to y. $\partial f/\partial x(P_o)$ denotes $\partial f/\partial x$ evaluated at point P_o

For any continuous function $f(x,y)$: $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$

Harmonic functions:
For a function $f(x,y)$, if $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ the function is said to be "Harmonic".

Gradient Vector:
$\nabla f(x,y,x)$ is the vector that points in the direction of maximum increase of $f(x,y,z)$ from the point $P_o=(x_0, y_0, z_0)$. The max rate of increase of f is $|\nabla f(x,y,x)|$. The gradient at point P_o is perpindicular to the level curve of $f(x,y)$ and the level surface of $f(x,y,z)$ that goes through P_o.
$\nabla f=\partial f/\partial x(P_o)\mathbf{i}+\partial f/\partial y(P_o)\mathbf{j}+\partial f/\partial z(P_o)\mathbf{k}$

The gradient vector evaluated at point P_o.
$\nabla f(P_o)=\partial f/\partial x(x_0)\mathbf{i}+\partial f/\partial y(y_0)\mathbf{j}+\partial f/\partial z(z_0)\mathbf{k}$

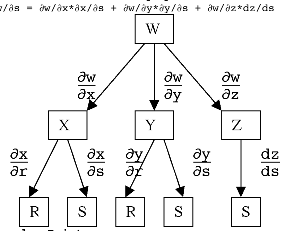
dir of max decrease$=-\nabla[f(P_o)]$ $\nabla[f(P_o)+g(P_o)]=\nabla f(P_o)+\nabla g(P_o)$
$\nabla[kf(P_o)]=k\nabla[f(P_o)]$ $\nabla[f(P_o)-g(P_o)]=\nabla f(P_o)-\nabla g(P_o)$
$\nabla[f(P_o)g(P_o)]=g(P_o)\nabla f(P_o)+f(P_o)\nabla g(P_o)$

Directional Derivative:
Given a point P_o, a gradient vector $\nabla f(P_o)$ and a unit vector in some direction **u**:
$D_u f(P_o)=\mathbf{u}\cdot\nabla f(P_o)$ $D_u f(x,y)=u_1 f_x(x,y)+u_2 f_y(x,y)$

Chain Rule:
Let $z=f(x,y)$ be differientiable at $(x(t), y(t))$:
$D_x f(x,y) = dx/dt[f(x)] = \partial f/\partial x*dx/dt + \partial f/\partial y*dy/dt$

chain rule with partial derivatives:
$f_t(x,y) = \partial f/\partial t = \partial f/\partial x*\partial x/\partial t + \partial f/\partial y*\partial y/\partial t$
$f_t(x,y,z) = \partial f/\partial t = \partial f/\partial x*\partial x/\partial t + \partial f/\partial y*\partial y/\partial t + \partial f/\partial z*\partial z/\partial t$

Let $w=f(x,y,z)$; $x=g(r,s)$, $y=h(r,s)$, $z=l(s)$
$\partial w/\partial r = \partial w/\partial x*\partial x/\partial r + \partial w/\partial y*\partial y/\partial r$
$\partial w/\partial s = \partial w/\partial x*\partial x/\partial s + \partial w/\partial y*\partial y/\partial s + \partial w/\partial z*dz/ds$



Boundry Points:
The set S of points within a defined range of x and y coordinates or a distance from a point P_o.

Stationary Points:
The set of points where $f(x,y)$ is differentiable and $|\nabla f(x,y)|=0$ (the tangent plane is horozontal).

Singular Points:
The set of points where $f(x,y)$ is not differentiable.

Second Partials Test:
If $f(x,y)$ has continuous second partials and $\nabla f(x,y) = 0$:
$D=D(x_0, y_0)=f_{xx}(x,y)f_{yy}(x,y)-(f_{xy}(x,y))^2$
if $D>0\&f_{xx}(x,y)<0\&f_{yy}(x,y)<0$, then $f(x,y)$ is a local maximum
if $D>0\&f_{xx}(x,y)>0\&f_{yy}(x,y)>0$, then $f(x,y)$ is a local minimum
if $D<0$, then $f(x,y)$ is a saddle point
if $D=0$, then the test is inconclusive

Vector Fields

Gradient of a Scalar Field
Given a scalar function $f(x,y,z)$, the gradient of f is:
$\mathbf{F}(x,y,z)=\nabla f=\partial f/\partial x\mathbf{i}+\partial f/\partial y\mathbf{j}+\partial f/\partial z\mathbf{k}$
F is a Conservative Vector Field and $f(x,y,z)$ is the Potential Function of **F**.

Given a vector Function $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$,
i.e. **curl(F)** $=\partial M/\partial y= \partial N/\partial x$, $\partial M/\partial z= \partial P/\partial x$, and $\partial N/\partial z= \partial P/\partial y$
or if **F** is a 2D function then: **curl(F)** $=\partial M/\partial y=\partial N/\partial x$
F is Conservative if and only if **curl(F)** = **0** (**zero** vector). if $\mathbf{F}(x,y,z)$ is Conservative then a Potential Function f exists, such that $\mathbf{F}=\nabla f$.

Partials Vector $\nabla=\partial/\partial x\mathbf{i}+\partial/\partial y\mathbf{j}+\partial/\partial z\mathbf{k}$
Divergence $\mathrm{div}(\mathbf{F}) =\nabla\cdot\mathbf{F}$
Curl $\mathrm{curl}(\mathbf{F})=\nabla\times\mathbf{F}$
$\mathrm{div}(\mathbf{F})=\partial M/\partial x+\partial N/\partial y+\partial P/\partial z$
$\mathrm{curl}(\mathbf{F})=(\partial P/\partial y-\partial N/\partial z)\mathbf{i}+(\partial M/\partial z-\partial P/\partial x)\mathbf{j}+(\partial N/\partial x-\partial M/\partial y)\mathbf{k}$
Instead of multiplying, apply the matching partial derivative to the **i**, **j** and **k** terms of the function **F** and then add as normal.

Divergence is the degree that the vector field **F** diverges away from a point P (div > 0) or converges (div < 0).
Curl is the direction of the axis around which the vector field **F** rotates most rapidly. |curl(**F**)| is the speed of rotation.

Curves and Surfaces in 3D

Level Curve:
Given a function $z=f(x,y)$, hold z constant and solve $f(x,y)=c$ for x and y.

Level Surface:
Given a function $w=f(x,y,z)$ hold w constant and solve $w=f(x,y,z)$ for x, y, and z.

Tangent Plane:
For the surface $\mathbf{f(x,y,z)=k}$ at any point (x_K, y_K, z_K) where $f(x,y,z)=k$ the standard equation of the Tangent Plane is:
$f_x(x_K, y_K, z_K)(x-x_K)+f_y(x_K, y_K, z_K)(y-y_K)+f_z(x_K, y_K, z_K)(z-z_K) = 0$

For the surface $\mathbf{z = f(x,y)}$ the standard equation of the Tangent Plane at
$P_0 = (x_0, y_0, f(x_0, y_0))$ is:
$z-z_0 = f_x(x_0, y_0)(x-x_0) + f_y(x_1, y_1)(y-y_0)$

The Gradient Vector is the Normal to the Tangent Plane and to the Level Curve or Level Surface at point P_0. Convert a formula into the form $f(x,y,z)=k$, then the factors of the Gradient Vector of that formula can be plugged into the Standard Formula for a Plane to get the formula for the Tangent Plane.

Methods of Integration

Power Rule
$$\int x^n\,dx = \frac{1}{(n+1)}x^{n+1}+C \quad n \neq 1$$

Linearity
$\int k\,f(x)\,dx = k\int f(x)\,dx$
$\int [\,f(x) + g(x)\,]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int [\,f(x) - g(x)\,]\,dx = \int f(x)\,dx - \int g(x)\,dx$

Additive property
$$\int_{x=a}^{x=c}[f(x)]\,dx = \int_{x=a}^{x=b}[f(x)\,dx + \int_{x=b}^{x=c}[f(x)\,dx$$
regardless of the order of a, b and c.

Generalized Power Rule
$$\int [g(x)]^r g'(x)\,dx = \frac{[g(x)]^{r+1}}{r+1} + C$$

Subsitiution Rule
To simplify an Integral, find any function g(x) in the table of standard integrals such that g(x) and g'(x) are both present in the expression f(x) dx. If $u=g(x)$ and $du/dx=g'(x)\,dx$ and $\int f(x)\,dx=\int g(x)g'(x)\,dx$ then $\int f(x)\,dx=\int h(u)\,du$

Integration by Parts
To simplify an Integral, F(x), find any 2 functions U(x) and V(x) (with V(x) in the table of standard integrals) such that U(x) and V'(x) are both present in the expression F(x). Then solve the Integral for V(x)U'(x) dx. If the new integral is more complex then you have choosen the wrong substitution.
$\int F(x)\,dx = \int u(x)v'(x)\,dx = U(x)V(x) - \int V(x)U'(x)\,dx$

Order of Substitutions ILATE
1 **I**nverse trig functions
2 **L**ogarithm functions
3 **A**lgebraic functions
4 **T**rig functions
5 **E**xponential functions

Double Integrals

Compute a Double integral with respect to a rectangular area by integrating with respect to y, holding x constant and and then evaluating the definite integral for the range of y, then take the resulting formula and integrate with respect to x, holding y constant and evaluate the definite integral for the range of x.

Cartesian Coordinates:
$dA = \frac{df}{dy}\frac{dy}{dx}$ $f(x) = \mathrm{lower}(y)$ $g(x) = \mathrm{upper}(y)$
$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b}\left[\int_{y=f(x)}^{y=g(x)}[\,f(x,y)\,]\,dy\,\right]dx$$
Note: The bounds of the outer intergral are always constants.
Graph the formula(s) of the given bounds of the Integral in Cartesian coords. and examine the graph to determine the correct bounds for the Iterated Integral.

Polar Coordinates:
$dA = r\,dr\,d\Theta$ $r = \sqrt{x^2 + y^2}$
$x = r\cos(\Theta)$ $y = r\sin(\Theta)$
Graph the bounds of the Integral in Cartesian coords. and examine the graph to determine the correct bounds for the function in Polar coordinates.

Triple Integrals

$dV = \frac{df}{dz}\frac{dz}{dy}\frac{dy}{dx}$ $f(x) = \mathrm{lower}(y)$ $g(x) = \mathrm{upper}(y)$
$h(x,y) = \mathrm{lower}(z)$ $j(x,y) = \mathrm{upper}(z)$
$$\iiint_V f(x,y)\,dV = \int_{x=a}^{x=b}\left[\int_{y=f(x)}^{y=g(x)}\left[\int_{z=h(x,y)}^{z=j(x,y)}[f(x,y,z)]dz\right]dy\right]dx$$
To find a Volume with a Triple Integral, use the given function(s) to find the bounds and integrate the constant function 1 over the bounds.

Cylindrical Coordinates:
$dV = r\,dz\,dr\,d\Theta$ $r = \sqrt{x^2 + y^2}$
$x = r\cos(\Theta)$ $y = r\sin(\Theta)$ $z = z$

Spherical Coordinates:
$dV=\rho^2\sin(\Phi)\,d\rho\,d\Phi\,d\Theta$ $\rho=\sqrt{(x^2+y^2+z^2)}$
$x=\rho\sin(\Phi)\cos(\Theta)$ $y=\rho\sin(\Phi)\sin(\Theta)$ $z=\rho\cos(\Phi)$
$\rho=\sqrt{(x^2+y^2+z^2)}$ $\Theta=\tan^{-1}(y/x)$ $x>0$ $\Theta=\pi+\tan^{-1}(y/x)$ $x<0$
$\Phi=\cos-1(z/\rho)$ $\Theta=\pi/2$ $x=0, y>0$ $\Theta=3\pi/2$ $x=0, y<0$
$\Phi=\tan-1(z/\sqrt{(x^2+y^2)})$ $\Theta=0$ $x=0, y=0$

Line Integrals

A Line Integral is the integral of some function $f(x,y,z)$, along a curve **c** such that the integral is the sum of $f(x,y,z)\Delta s$, where s is the length of the curve **c** and Δs is a small segment of the curve. To compute the Line integral, convert $f(x,y,z)$ into parametric form such that $x=X(t)$, $y=Y(t)$ and $z=Z(t)$ where t is the position along the curve **c**.

2D formula:
$$\oint_c f(x,y) = \int_a^b f(X(t),(Y(t))\sqrt{[X'(t)]^2 + [Y'(t)]^2}\,dt$$

3D formula:
$$\oint_c f(x,y,z) = \int_a^b f(X(t),(Y(t),Z(t))\sqrt{[X'(t)]^2+[Y'(t)]^2+[Z'(t)]^2}\,dt$$

Line Integrals of Vectors functions:
Where $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$
$$\oint_c \mathbf{F}(x,y,z)\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot d\mathbf{r} = \int_a^b M\,dx + N\,dy + P\,dz$$
$$\oint_c \nabla\mathbf{F}\cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$$
where f is the Potiential Function of **F** & **a**, **b** are the endpoints of the curve **c**

$\oint_c \nabla\mathbf{F}\cdot d\mathbf{r}$ is Independant of Path iff $\mathbf{F}(\mathbf{r})=\nabla f$ for some scalar function f. **F** is a Conservative Vector Field.

Green's Theorem
$$\oint_c M\,dx + N\,dy + P\,dz = \iint_s (\partial N/\partial x-\partial M/\partial y)\,dA$$
where **c** is a closed curve and **s** is the region enclosed by **c**.

Gauss' Divergence Theorem
Unit Tangent $\mathbf{T} = \partial x/\partial s\mathbf{i}+\partial y/\partial s\mathbf{j}$ Unit Normal $\mathbf{n} = \partial y/\partial s\mathbf{i}+\partial x/\partial s\mathbf{j}$
$$\oint_c \mathbf{F}(x,y,z)\cdot\mathbf{n}\,ds = \iint \mathrm{div}(\mathbf{F})\,dA = \iint \nabla\cdot\mathbf{F}\,dA$$

Gauss' Theorem in 3D
$$\iint_{\partial S} \mathbf{F}(x,y,z)\cdot\mathbf{n}\,dS = \iiint_V \mathrm{div}(\mathbf{F})\,dV = \iiint_S \nabla\cdot\mathbf{F}\,dV$$
$$\iint_{\partial S} \mathbf{F}(x,y,z)\cdot\mathbf{n}\,dS = \iiint_V [\,\partial M/\partial x + \partial N/\partial y + \partial P/\partial z\,]\,dV$$

Stokes' Theorem (around a closed curve)
$$\oint_c \mathbf{F}(x,y)\cdot\mathbf{T}\,ds = \iint_A \mathrm{curl}(\mathbf{F})\cdot\mathbf{k}\,dA$$

Stokes' Theorem in 3D (around a closed surface)
$$\oint_{\partial S} \mathbf{F}(x,y,z)\cdot\mathbf{T}\,dS = \iint_A \mathrm{curl}(\mathbf{F})\cdot\mathbf{n}\,dA$$

Surface Integrals

Surface Area
Given a function $z=f(x,y)$ and a region S in the x,y plane, find the surface area of the surface defined by $z=f(x,y)$.
$$\text{area} = \iint_S \sqrt{[\partial f/\partial x]^2+[\partial f/\partial y]^2+1}$$
Given a function $g(x,y,z)$ (scalar) and a region G (a surface), suppose G is the result of $z=f(x,y)$ for (x,y) in some region S in the x,y plane, then G sits above S.

$$dS = \sqrt{[\partial f/\partial x]^2+[\partial f/\partial y]^2+1}$$

Calculate a Surface Integral over G by replacing z by $f(x,y)$, wherever z occurrs in the function g. Then calculate the integral:
$$\iint_G g(x,y,z)\,dS= \iint_S g(x,y,f(x,y))\sqrt{[\partial f/\partial x]^2+[\partial f/\partial y]^2+1}\,dA$$

Surface Integrals

Special case of Surface Integral: given a vector function $\mathbf{g}(x,y,z)=\mathbf{F}(x,y,z)\cdot\mathbf{n}$ and a surface $z=f(x,y)$ over a region G. Take the gradient of the function $h(x,y,z)=[z-f(x,y)=0]$. ∇h is perpindicular to the surface $\boldsymbol{g}$. $\nabla h/|\nabla h|$ is a unit normal vector $\mathbf{n}$

$\mathbf{F}(x,y,z) = \mathbf{M}i + \mathbf{N}j + \mathbf{P}k$

$\mathbf{n} = [\partial f/\partial x\,\mathbf{i} + \partial f/\partial y\,\mathbf{j} + \mathbf{k}] \,/\, \sqrt{[\partial f/\partial x]^2+[\partial f/\partial y]^2+1}$

$\nabla\mathbf{F}\cdot\mathbf{n} = -M\partial f/\partial x + N\partial f/\partial y + P$

$$\iint_G g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\sqrt{[\partial f/\partial x]^2+[\partial f/\partial y]^2+1}\,dA$$

The Square Roots cancel leaving the formula:

$$\iint_G \mathbf{F}(x,y,z)\cdot\mathbf{n}\,ds = \iint_R [-\mathbf{M}\partial f/\partial x+\mathbf{N}\partial f/\partial y+\mathbf{P}]\,dA$$

Flux of F across C $= \int_C \mathbf{F}\cdot\mathbf{n}\,ds$

Mass Integrals

Mass m and Center of Mass $(\bar{x}\ \bar{y})$ of a Lamina

$\mathbf{m} = \iint \delta(x,y)\,dA$ where $\delta(x,y)$ is the density function.

$\bar{x} = \frac{1}{\mathbf{m}}\iint x\,\delta(x,y)\,dA$ $\qquad \bar{y} = \frac{1}{\mathbf{m}}\iint y\,\delta(x,y)\,dA$

Moment of Inertia

moment around x axis: $Ix = \iint x^2\,\delta(x,y)\,dA$

moment around y axis: $Iy = \iint y^2\,\delta(x,y)\,dA$

moment around z axis: $Iz = Ix + Iy$

Radius of Gyration: $\bar{r} = (\sqrt{I/m})$

Mass m and Center of Mass($\bar{x}\ \bar{y}\ \bar{z}$) of a Solid

Density function=$\delta(x,y,z)$

$\mathbf{m} = \iiint_V \delta(x,y)\,dV$ $\qquad \bar{z} = \frac{1}{\mathbf{m}}\iiint z\,\delta(x,y,z)dV$

$\bar{x} = \frac{1}{\mathbf{m}}\iiint x\,\delta(x,y,z)dV$ $\qquad \bar{y} = \frac{1}{\mathbf{m}}\iiint y\,\delta(x,y,z)dV$

Mass m and Center of Mass($\bar{r}\ \bar{\Theta}\ \bar{z}$)in cylindrical coords.

$\mathbf{m} = \iiint_V \delta(r\cos(\Theta),r\sin(\Theta),z)dV$

$\bar{z} = \frac{1}{\mathbf{m}}\iiint z\,\delta(r\cos(\Theta),r\sin(\Theta),z)dA$

$\bar{x} = \frac{1}{\mathbf{m}}\iiint r\cos(\Theta)\,\delta(r\cos(\Theta),r\sin(\Theta),z)dA$

$\bar{y} = \frac{1}{\mathbf{m}}\iiint r\sin(\Theta)\,\delta(r\cos(\Theta),r\sin(\Theta),z)dA$

$\mathbf{r} = \sqrt{\bar{x}^2+\bar{y}^2}$ $\qquad \boldsymbol{\Theta}=\tan^{-1}(\bar{y}\,/\,\bar{x})$

Mass m and Center of Mass $(\bar{\rho}\ \bar{\Theta}\ \bar{\Phi})$ in Spherical coords.

$\mathbf{m} = \iiint_V \delta(\rho\sin(\Phi)\cos(\Theta),\rho\sin(\Phi)\sin(\Theta),\rho\cos(\Phi))dV$

$\bar{z} = \frac{1}{\mathbf{m}}\iiint \rho\cos(\Phi)\delta(\rho\sin(\Phi)\cos(\Theta),\rho\sin(\Phi)\sin(\Theta),\rho\cos(\Phi))dA$

$\bar{x} = \frac{1}{\mathbf{m}}\iiint \rho\sin(\Phi)\cos(\Theta)\delta(\rho\sin(\Phi)\cos(\Theta),\rho\sin(\Phi)\sin(\Theta),\rho\cos(\Phi))dA$

$\bar{y} = \frac{1}{\mathbf{m}}\iiint \rho\sin(\Phi)\sin(\Theta)\delta(\rho\sin(\Phi)\cos(\Theta),\rho\sin(\Phi)\sin(\Theta),\rho\cos(\Phi))dA$

$\bar{\rho} = \sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}$ $\qquad \bar{\Theta}=\tan^{-1}(\bar{y}\,/\,\bar{x})$ $\qquad \bar{\Phi}=\cos^{-1}(\bar{z}\,/\,\bar{\rho})$

Differential Equations

Given any Differential Equation or word problem for which you want to set up a Differential Equation, you first must catagorize the type of problem:

Simple: $\frac{dy}{dx} = f(x)$ — Integrate both sides

Seperable: $\frac{dy}{dx}*f(x,y) = g(x,y)$ — Transform the equation

$f(x,y)=f1(x)*f2(y)$

$g(x,y)=g1(x)*g2(y)$

$\frac{f2(y)}{g2(y)}dy=\frac{g1(x)}{f1(x)}dx$ — Integrate both sides. Solve for Y

Linear: $\frac{dy}{dx}*f(x)+g(x,y)*Y=h(x)$ — Transform the equation

$\frac{dy}{dx}+\frac{g(x,y)}{f(x)}=\frac{h(x)}{f(x)}$ — IF $Y*P(x) = g(x,y)/f(x)$ and IF Y^{+1} is the only power of Y

$\frac{dy}{dx}+Y*P(x)=Q(x)$

$\rho(x)=e^{(\int P(x)dx)}$ — Compute the Integrating Factor

$\frac{d}{dx}[\rho(x)Y]=\rho(x)Q(x)$ — Integrate both sides

$\rho(x)Y=\int\rho(x)Q(x)dx+C$ — Solve for Y

$Y=\frac{(\int \rho(x)Q(x)dx)}{\rho(x)}+\frac{C}{\rho(x)}$

Homogenous: $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ — f,g polynomials, all x, y terms of same degree i.e. x^3, x^2y, xy^2. Transform the equation

$v=\frac{y}{x}$ $\qquad \frac{dy}{dx}=v+x\frac{dv}{dx}$ — Replace y by xv. The new D.E. is Seperable. Solve for v, replace v by y/x

Exact: $M(x,y)+N(x,y)\frac{dy}{dx}=0$ — If the D.E. can be written in this form the D.E. is Exact

$f(x,y)=\int M(x,y)\partial x$ — "constant" of integration is g(y)

$\frac{\partial f}{\partial y}[f(x,y)]=N(x,y)$ — Compute the partial derivative, set it equal to N(x,y), solve for g'(y) then integrate to get g(y)

$f(x,y)+g(y)=c$ — is the solution to the D.E.

Autonomous: $\frac{dx}{dt}=f(x)$ — f(x) does not depend on t

Critical: Roots of the RHS of the D.E. are the Critical Points

Stability: A Critical Point is stable if the sign of dx/dt is negative above the point and positive below it.

2nd Order: $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=F(x)$ — If P,Q are Constants and F=0 then the solution is as follows:

$ar^2+br+c=0$ — Find the roots r1 and r2 of the Characteristic Equation

Case 1: $y=C_1e^{R1X}+C_2e^{R2X}$ — 2 independant roots

Case 2: $y=C_1e^{R1X}+C_2Xe^{R1X}$ — 1 repeated root

Case 3: $ar^2+br+c=0$ $(b^2<4ac)$ Complex Roots $\alpha=\frac{-b}{2a}$ $\beta=\frac{(\sqrt{b^2-4ac})i}{2a}$

$y=e^{\alpha X}(c_1\cos(\beta X)+C_2\sin(\beta X))$

Exponential Decay: Radioactive Decay

$\frac{dN}{dt}=cN$ $\qquad c<1$

$N_t=N_0e^{-kt}$ — Halflife: $t=\ln(0.5)/-k$

Heating: Cooling $\frac{dT}{dt}=c(A-T)$ — Temperature approches room temp.

$T_t=A+(T-A)e^{-kt}$

Mixing: $\frac{dS}{dt}=RateIn-RateOut$

$RateOut=\frac{VolumeOut*Concentration}{Total\ Volume}$

Velocity: $F=F_G+F_R$ — F_G=gravity, F_R=Air Resistance

$m\frac{dv}{dt}=-kv-mg$ $\qquad \frac{dv}{dt}=-\rho v-g$ $\qquad \rho=\frac{k}{m}$

$v_t=\frac{g}{\rho}=\frac{mg}{k}$ — Terminal Velocity

$v(t)=[v_0+\frac{g}{\rho}]e^{-\rho t}-\frac{g}{\rho}$ — Velocity at time t

$y(t)=y_0+v_tt+\frac{1}{\rho}(v_0-v_t)(1-e^{-\rho t})$

Population Growth: Population Growth without Death

$\frac{dP}{dt}=cP$ $\qquad c>1$

$P_t=P_0e^{kt}$

Logistic Model: Population Growth with Death

$\frac{dP}{dt}=(\beta-\delta)P$ — β=Birth rate (const or αP); δ=Death rate (const or γP)

$\frac{dP}{dt}=kP(M-P)$ — Logistic Population $M=\frac{\beta}{k}$ Max Population

$\frac{dP}{dt}=kP(P-M)$ — Doomsday-Extinction $M=\frac{\delta}{k}$

$P(t)=\frac{MP_0}{P_0+(M-P_0)e^{-kMt}}$ — Logistic Equation. P->M as t->∝

$P(t)\ \frac{MP_0}{P_0+(M-P_0)e^{+kMt}}$ — Case 1: $P_0>M$ Doomsday. Case 2: $P_0<M$ Extinction

Differential Equations

Method of Undetermined Coefficients

If the differential equation has constant coefficients and the non-homogenous term **f(x)** is a polynmial, exponential, cosine or sine or sum of the above: $A\frac{d^2y}{dx^2}+B\frac{dy}{dx}+Cy=F(x)$

Guess y_p=a function of the same type and of the same order with all lower order terms. i.e. if $f(x)=x^2$ then guess $y_p=Dx^2+Ex+F$

Polynomial:
Guess a polunomial with all lower order terms.

Exponential:
Guess an exponential with the same exponent.

Sine or Cosine:
Guess a sum of sine and cosine with the same angular frequency.

Sum or Product of Polynomial, Exponential or Sine or Cosine:
Guess a sum or product with the same types of terms.
If any term of the guess for y_p overlaps with a term of y_c, then multiply that term of y_p by x. If that makes 2 parts of y_p overlap then multiply the overlapped term by x.

Method of Variation of parameters

If the differential equation has constant coefficients but the non-homogenous term f(x) is not a polynomial, exponential, sine or cosine or sum or product of them, then to find y_p, compute the complementary solution:

given D.E. $A\frac{d^2y}{dx^2}+B\frac{dy}{dx}+Cy=F(x)$

$y_c=c_1y_1(x)+c_2y_2(x)$

then $y_p=u_1(x)y_1(x)+u_2(x)y_2(x)$

where $u_1=\int\frac{-y_2(x)f(x)}{w(x)}dx$ $\qquad u_2=\int\frac{y_1(x)f(x)}{w(x)}dx$

$w(x)=y_1(x)*y_2'(x)-y_2(x)*y_1'(x)$

Mechanical Vibrations

Given Newtons Law, F=MA, the equation of a mass spring system:

$F=MA=M\frac{dV}{dt}=M\frac{d^2X}{dt^2}=mx''$ $\qquad mx''+cx'+kx=f(t)$

where m=mass, c=damping force proportional to velocity, k=spring constant, and f(t)=forcing function.

$\omega_0=\sqrt{(k/m)}$	$C=\sqrt{(A^2+B^2)}$	$x(t)=A\cos(\omega_0t)+B\sin(\omega_0t)$
$\alpha=\tan^{-1}(B/A)$	a>0, B>0	
$\alpha=\tan^{-1}(B/A)+\pi$	a<0	$x(t)=C\cos(\omega_0t-\alpha)$
$\alpha=\tan^{-1}(B/A)+2\pi$	a>0, B<0	
$T=2\pi/\omega_0$	$f=\omega_0/2\pi$	

Overdamped: $c^2>4km$ — 2 Real Roots r1, r2
The function damps down to an equilibrium value without any oscillations.
$x(t)=c_1e^{r1t}+c_2e^{r2t}$

Critically Damped: $c^2=4km$ — 1 Real Root r1
The function pases through the equilibrium value at most once and then damps down to an equilibrium value without any oscillations.
$x(t)=c_1e^{r1t}+c_2te^{r1t}$

Underdamped: $c^2<4km$ — ComplexRoots $\alpha=-c/2m$, $\beta=\sqrt{(4mk-c^2)}$
the function undergoes decreasing oscillations that damp down to zero amplitude.
$x(t)=e^{\alpha t}(A\cos(\beta t)+B\sin(\beta t))$

Forced Resonance: $mx''+cx'+kx=f_0\cos(\omega_0t)$ or $f_0\sin(\omega_0t)$

$A=\frac{(k-m\omega^2)f_0}{(k-m\omega^2)^2+(c\omega)^2}$ $\qquad B=\frac{c\omega f_0}{(k-m\omega^2)^2+(c\omega)^2}$

$C=\sqrt{(A^2+B^2)}$ $\qquad C=\frac{f_0}{\sqrt{[(k-m\omega^2)^2+(c\omega)^2]}}$

$\alpha=\tan^{-1}(B/A)$ $\qquad \alpha=\tan^{-1}(\frac{c\omega}{k-m\omega^2})$ $\qquad [+\pi$ if $k<m\omega^2]$

Practical Resonance occurs if $c<\sqrt{(2km)}$ at some frequency $\omega<\omega_0$.

Electrical Circuits: [RLC Circuits]

Mass m	≡ Inductance L
Damping c	≡ Resistance R
Spring constant k	≡ Reciprocal Capacitance 1/C
Position x	≡ Charge Q or Current I
Force F	≡ EMF E or it's derivative E'

$L\frac{dI}{dt}+RI+\frac{1}{C}Q=E(t)$ $\qquad LQ''+RQ'+\frac{1}{C}Q=E(t)$ $\qquad LI''+RI'+\frac{1}{C}I=E'(t)$

$I_{sp}(t)=\frac{E_0\cos(\omega t-\alpha)}{\sqrt{[R^2+(\omega L-\frac{1}{\omega C})^2]}}$

$\alpha=\tan^{-1}(\frac{\omega RC}{1-LC\omega^2})$

$Z=\sqrt{[R^2+(\omega L-\frac{1}{\omega C})^2]}$ — Z=Impedance

$S=\omega L-\frac{1}{\omega C}$ — S=Reactance

$Z=\sqrt{[R^2+S^2]}$

$\delta=\alpha-(1/2)\pi$ $\qquad \delta=\tan^{-1}(\frac{S}{R})$

$I_{sp}(t)=\frac{E_0}{Z}\sin(\omega t-\delta)$

Systems of First Order Differential; Equations:

Convert a higher order differential equation into a system of first order differential equations by substitution as follows:

Given a D.E. $Cx''+Dx'+Ex+F=G(x)$

Substitute: $y=x'$ $\qquad y'=x''$

Giving the set $X'=y$

$y'=-Dy-Ex-F-G(x)$

------------------------------Physics I Equations------------------------------

Physical Constants:

$c=1/\sqrt{(\varepsilon_0\mu_0)}=3\times10^8$m/sec		$e=1.60\times10^{-19}$C	Electron Charge
$M_p=1.67\times10^{-27}$kg	Proton Mass	$M_e=9.1\times10^{-31}$kg	Electron Mass
$K=8.99\times10^9$Nm2/C2	ElecStatConst	$K=8.99\times10^9$m/F	ElecStatConst
$\varepsilon_0=8.85\times10^{-12}C^2/Nm^2$	Permittivity	$\varepsilon_0=8.85\times10^{-12}$F/m	Permittivity
$\mu_0=4\pi\times10^{-7}$Vsec/Am	Permeability	$\mu_0=4\pi\times10^{-7}$H/m	Permeability
$h=6.63\times10^{-31}$Jsec	Planck Const	$h=4.14\times10^{-15}$eVsec	Planck Const
$eV=1.60\times10^{-19}$J	ElectronVolt	$N_a=6.02\times10^{23}$/mole	AvogadroNum
H=Vsec/A	Magnetic Field	N/C=1 V/m=1 J/Cm	Electric Field
C=1 Ampere*sec	Coulomb	V=1 Joul/Coulomb	Volt

Distance, Velocity, Acceleration, Time:

X=Displacement on X axis	Y=Displacement on Y axis
V=Linear Velocity	A=Linear Acceleration
$X_t-X_0=1/2(V_t+V_0)(t-t_0)$	$X_t=X_0+1/2(V_t+V_0)(t-t_0)$
$X_t-X_0=V_t(t-t_0)-1/2A(t-t_0)^2$	$X_t=X_0+V_t(t-t_0)-1/2A(t-t_0)^2$
$X_t-X_0=V_0(t-t_0)+1/2A(t-t_0)^2$	$X_t=X_0+V_0(t-t_0)+1/2A(t-t_0)^2$
$X_t-X_0=V_0\cos(\Theta)(t-t_0)$	$X_t=X_0+V_0\cos(\Theta)(t-t_0)$
$Y_t=Y_0+V_0\sin(\Theta)t-1/2gt^2$	$V_t^2=(V_0\sin(\Theta))^2-2g(Y_t-Y_0)$
$V_t-V_0=A(t-t_0)$	$V_t=V_0+A(t-t_0)$
$V_t^2-V_0^2=2A(X_t-X_0)$	$V_t^2=V_0^2+2A(X_t-X_0)$
$Vavg=(X_t-X_0)/(t-t_0)$	$Vavg=1/2(V_t+V_0)$
$Vavg=\Delta\lvert\mathbf{R}\rvert/\Delta t=dr/dt$	$Aavg=\Delta\lvert\mathbf{V}\rvert/\Delta t=dv/dt$
$Aavg=(V_t-V_0)/(t-t_0)$	$Aavg=1/2(A_t+A_0)$
free fall from $v_0=0,Y_0=h$	$t=\sqrt{(2(Y_t-Y_0)/g)}$
free rise ftom $V_0=v$ to $V_t=0$	$t=V_0/-2g$

Force:	F=force
F=MA	$F_g=Mg$
$F=F_x+F_y$	$F_{ab}=-F_{ba}$
$F=\sqrt{((MA_x)^2+(MA_y)^2)}$	$F_{friction}=\mu_kF_{normal}$

Energy, Work, Springs:

W=Work	E=Total Mechanical Energy
K=Kenetic Energy	U=Potential Energy
$K=(1/2)MV^2$	U=Mgh
$W=(1/2)(MV_f^2-MV_i^2)=\mathbf{F\cdot d}$	$W_S=(1/2)(kX_i^2-kX_f^2)$
$E=K_x+K_y+U_x+U_y$	$K_S=(1/2)kX^2$
$\Delta K=W_{cons}+W_{friction}$	$\Delta K=-\Delta U$
K=E-U	$\Delta U=-W$

Center of mass:	$M=\Sigma M_i$
$\mathbf{X}=(1/M)\Sigma M_iX_i$	$\mathbf{Y}=(1/M)\Sigma M_iY_i$
$\mathbf{Z}=(1/M)\Sigma M_iZ_i$	$\mathbf{R}_{cm}=i\mathbf{X}+j\mathbf{Y}+k\mathbf{Z}$
$V_{cm}=(1/M)\Sigma M_iV_i$	$A_{cm}=(1/M)\Sigma M_iA_i$
$P_{cm}=\Sigma M_iV_i$	$K_{cm}=\Sigma M_iV_i^2$
Momentum, collision:	$J=\Delta P=P_f-P_i=M(V_f-V_i)=0$
$P_i=P_f$ $P=\sqrt{(P_x^2+P_y^2)}$	$\Delta P_x=P_{xf}-P_{xi}$ $\Delta P_y=P_{yf}-P_{yi}$
$J=F_{avg}\Delta t$	$F_{avg}=\Delta P/\Delta t=J/\Delta t$ (inelastic)
$V_{cm}=P/(m_1+m_2)$	$V_{cm}=(P_{1i}+P_{2i})/(m_1+m_2)$
$P_{cm}=MV_{cm}=\Sigma M_iV_i$	If $E_f/E_i=1$ then E is conserved

Elastic: Energy & Momentum conserved

$P_{xi1}+P_{xi2}=P_{xf1}+P_{xf2}$	$P_{yi1}+P_{yi2}=P_{yf1}+P_{yf2}$
$K_{i1}+K_{i2}=K_{1f}+K_{2f}$	$M_{1i}V_{1i}+M_{2i}V_{2i}=M_{1f}V_{1f}+M_{2f}V_2$

$1/2M_1V_{1i}^2+1/2M_2V_{2i}^2=1/2M_1V_{1f}^2+1/2M_2V_{2f}^2$

$\mathbf{M_1}$--> $\mathbf{M_2}$ $V_{1i}>0$, $V_{2i}=0$	$\Delta P = 0$ (elastic)
$V_{1f}=V_{1i}((m_1-m_2)/(m_1+m_2))$	$V_{2f}=V_{1i}(2m_1/(m_1+m_2))$
$\mathbf{M_1}$--> <--$\mathbf{M_2}$ $V_{1i}>0$, $V_{2i}<0$	$\Delta P = 0$ (elastic)

$V_{1f}=V_{1i}((m_1-m_2)/(m_1+m_2)) + V_{2i}(2m_2/(m_1+m_2))$

$V_{2f}=V_{1i}(2m_1/(m_1+m_2)) + V_{2i}((m_2-m_1)/(m_1+m_2))$

Inelastic: Momentum conserved, Energy not conserved

$P_{xi1}+P_{xi2}=P_{xf1}+P_{xf2}$	$P_{yi1}+P_{yi2}=P_{yf1}+P_{yf2}$
$K_{1i}+K_{2i}\neq K_{1f}+K_{f2}$	$M_{1i}V_{1i}+M_{2i}V_{2i}=M_{1f}V_{1f}+M_{2f}V_{2f}$

$1/2M_1V_{1i}^2+1/2M_2V_{2i}^2\neq1/2(M_1+M_2)V_f^2$ (Energy not conserved)

Rocket:	$V_f=V_i+V_{exaust}Ln(M_i/M_f)$
Gravity: $G=6.67\times10^{-11}$ Nm^2kg^2	r_a=elliptic semi-major axis
$F_G=GM_1M_2/r^2$ $A_G=GM/r^2$	$V_{escape}=\sqrt{(2GM/r)}$
$K_G=GM_1M_2/2r_a$ $U_G=-GM_1M_2/r_a$	$E_G=-GM_1M_2/2r_a$
$T^2=(4\pi^2/GM)r_a^3$	dA/dt=constant r_a & ω_v vary
$dA/dt=1/2r^2\omega_v$	$A=1/2r^2\tan(\Theta)$
Mass of Sphere $M=4\rho\pi r^3/3$	Shell $M=4\rho\pi(r_2^3-r_1^3)/3$

Circular Motion:		R=radius	m
Θ=Angular Position	radian	S=Circumfral Position	m
T=Rorational Period	s	A_R=CentripitalAcceleration	m/s^2
V_T=TangentialVelocity	m/s	A_T=TangentialAcceleration	m/s^2
ω_v=Angular Velocity	Θ/s	α=AngularAcceleration	Θ/s^2
τ=Torque (Angular Force)		I=Moment of Inertia	kg
K_R=Angular Kinetic Energy		£=Angular Momentum	

$S=\Delta\Theta R$	$\Theta=(S_t-S_0)/R=\Delta S/R$
$V_T=(S_t-S_0)/(t-t_0)=\Delta S/\Delta t=R\omega$	$V_T^2=R^2\omega_v^2$
$T=2\pi R/V_T=2\pi/\omega_v$	$\omega_v=V_T/R=(\Theta_t-\Theta_0)/(t-t_0)=\Delta\Theta/\Delta t$
$A_R=V_T^2/R=R\omega_v^2$	$A_T=\alpha R=R\Delta\omega_v/\Delta t$
$\alpha=\Delta\omega_v/\Delta t=\Delta\Theta/\Delta t^2$	$\alpha=1/2(\omega_{vf}-\omega_{vi})=A_T/R$
$\Theta_t=\Theta_0+\omega_{v0}(t-t_0)+1/2\alpha(t-t_0)^2$	$\Theta_t=\Theta_0+\omega_{vt}(t-t_0)-1/2\alpha(t-t_0)^2$
$\Theta_t=\Theta_0+1/2(\omega_{vt}+\omega_{v0})(t-t_0)$	$\omega_{vavg}=(\Theta_t-\Theta_0)/(t-t_0)$
$\omega_{vt}=\omega_{v0}+\alpha(t-t_0)$	$\omega_{vt}^2=\omega_{v0}^2+2\alpha(\Theta_t-\Theta_0)^2$
$t=\sqrt{(2(\Theta_t-\Theta_0)/\alpha)}$ $[\omega_{v0}=0]$	$t=\omega_{v0}/-2\alpha$ $[\omega_{vt}=0]$
$K_r=1/2I\omega_v^2$	$£_r=I\omega_v=1/2I(\omega_{vf}-\omega_{vi})=RMV_T=R^2M\omega_v$
$\tau=I\alpha=F_\perp R=F_\angle R\sin(\angle_{FR})=\mathbf{R\times F}$	$\tau_{net}=d£/dt$
$W=\tau(\Theta_f-\Theta_i)=1/2I(\omega_{vf}^2-\omega_{vi}^2)$	$Power=dW/dt=\tau\omega_v$
Parallel Axis (d=axis<->cm)	$I=I_{cm}+Md^2$
Wheels: (Θ=angle of ramp)	$V_{cm}=R\omega_v$
$K=1/2I_{cm}\omega_v^2+1/2MV_{cm}^2$	$P=I_{cm}\omega_v+MV_{cm}=MV_T(I_{cm}+1)$
$A_{cm}=R\alpha=-g\sin\Theta/(1+I_{cm}/MR^2)$	$A_{cm,x}=-g\sin\Theta\cos\Theta/(1+I_{cm}/MR^2)$
$\tau_{friction}=R\mu_sMg\sin\Theta$	$A_{cm,y}=-g\sin^2\Theta/(1+I_{cm}/MR^2)$
Moment of Inertia:	I_{cm}=inertia (CenterOfMass)
thin hoop around axis:	$I_{cm}=MR^2$
thin hoop around diameter:	$I_{cm}=1/2MR^2$
thick wall tube:	$I_{cm}=1/2M(R_2^2+R_1^2)$
disk around axis:	$I_{cm}=1/2MR^2$
solid cyl. around axis:	$I_{cm}=1/2MR^2$
solid cyl. around cm:	$I_{cm}=1/4MR^2+1/12ML^2$
thin rod around cm:	$I_{cm}=1/12ML^2$
sphere:	$I_{cm}=2/5MR^2=1/10MD^2$
thin spherical shell:	$I_{cm}=2/3MR^2=1/6MD^2$
rect. slab around cm:	$I_{cm}=1/12M(L^2+W^2)$

Harmonic Motion, Waves:		f=Frequency	cycle/s
ω_F=AngularFrequency	Θ/s	T=Period sec/cycle	seconds
$[\omega_Ft+\Phi]$=AngularPhase	radians	Φ=Phase Constant	radians
A_T=Max Displ. Transverse		Y=Transverse Displacement	
A_L=Max Displ. Longitudinal		X=LongitudinalDisplacement	
A_Θ=Max Displ. Angular		Θ=Angular Displacement	
k=Linear Restoring Force N		κ=Angular Restoring Force N	
L=Length of Pendulum	meter	h=Distance Axis<->cm	meter
δ=Linear Density	kg/m	Ξ=String Tension	Newton
ρ=Volumetric Density	kg/m^3	B=Bulk Compressability	
λ=Wave Length	meter	n=Wave Number	1/meter
V_w=Wave Velocity	meter/s	V_P=Phase Velocity	meter/s

Simple Harmonic Motion:	E=K+U remains constant
$K=1/2MV^2$ $U=1/2kX^2$	$E=1/2M(\omega_FA)^2$
$K=1/2M\omega_F^2A^2\sin^2(\omega_Ft+\Phi)$	$U=1/2kA^2\cos^2(\omega_Ft+\Phi)$
$1/2M\omega_F^2A^2=1/2kA^2$	$X(x,t)=A_T\cos(\omega_Ft+\Phi)$
$v(x,t)=\omega_FA\sin(\omega_Ft+\Phi)$	$a(x,t)=-\omega_F^2A_L\cos(\omega_Ft+\Phi)$
$\lvert F_{max}\rvert=M\omega_F^2A$ $\lvert V_{max}\rvert=\omega_FA$	$\Phi=\tan^{-1}(-v_0/X_0\omega_F)$
Transverse Wave:	$T_T=1/f=2\pi\sqrt{(m/k_T)}$
$\omega_{FT}=2\pi/T=2\pi f=\sqrt{(k_T/m)}$	$f_T=\sqrt{(k_T/m)}/2\pi$
$F_Y=-k_TY$	$Y(x,t)=A_T\cos(nx\pm\omega_{FT}t+\Phi_T)$
Longitudinal Wave:	$T_L=1/f$
$\omega_{FL}=2\pi/T=2\pi f$	$f_L=1/T=\omega_{FL}/2\pi=V_w/\lambda$
$F_x=-k_LX$	$\Delta X(x,t)=A_T\cos(nx\pm\omega_{FL}t+\Phi_L)$
Wave Motion:	
$\lambda=2\pi/n=T\omega_F/n$	$n=2\pi/\lambda$
$T=1/f=\lambda n/\omega_F$	$f=1/T=\omega_F/2\pi=V_w/\lambda$
$V_w=\omega_F/n=\lambda/T=\lambda f$	$V_P=\omega_F/n$ +x dir, $-\omega_F/n$ -x dir
moving +x: $Phase=nx-\omega_Ft+\Phi$	moving -x: $Phase=nx+\omega_Ft+\Phi$
$v(x,t)=\omega_FA\sin(nx\pm\omega_Ft+\Phi)$	$a(x,t)=-\omega_F^2A\cos(nx\pm\omega_Ft+\Phi)$
$P=M\omega_FA\sin(nx\pm\omega_Ft+\Phi)=Mv$	$K=M\omega_F^2A^2\sin^2(nx\pm\omega_Ft+\Phi)=Mv^2$
$phase=2\pi(x/\lambda)\pm2\pi(t/T)$	
Torsional Motion:	$T_T=1/f=2\pi\sqrt{(I/\kappa)}$
$\omega_{F\Theta}=2\pi/T=2\pi f=\sqrt{(\kappa/I)}$	$f_T=1/T=\omega_F/2\pi=\sqrt{(\kappa/I)}/2\pi$
$\tau=-\kappa\Theta=I\alpha$	$\Theta=A_\Theta\cos(\omega_{F\Theta}t+\Phi_T)$
Pendulum Motion:	Physical $T_P=1/f=2\pi\sqrt{(I/Mgh)}$
Torsional $T_P=1/f=2\pi\sqrt{(I/\kappa)}$	Simple $T_P=1/f=2\pi\sqrt{(L/g)}$
$\omega_F=2\pi/T=2\pi f=\sqrt{(MgL/I)}\,\lvert\sqrt{(I/Mgh)}$	$f_P=1/T=\sqrt{(MgL/I)}/2\pi\,\lvert\sqrt{(Mgh/I)}/2\pi$
$\tau=-MgL\sin(\Theta)\,\lvert -Igh\sin(\Theta)$	$\Theta=A_\Theta\cos(\omega_{F\Theta}t+\Phi_\Theta)$
$\omega_v=v_\Theta=\omega_{F\Theta}A_\Theta\sin(\omega_{F\Theta}t+\Phi_\Theta)$	$\alpha=a_\Theta=-\omega_{F\Theta}^2A_\Theta\cos(\omega_{F\Theta}t+\Phi_\Theta)$
$P=I\omega_{F\Theta}A_\Theta\sin(\omega_{F\Theta}t+\Phi_\Theta)=I\omega_v$	$K=I\omega_{F\Theta}^2A_\Theta^2\sin^2(\omega_{F\Theta}t+\Phi_\Theta)=I\omega_v^2$
Vibrating String:	$f=\eta(V_w/2L)$ $[\eta=1,2,3,4\ldots]$
$V_w=\sqrt{(\Xi/\delta)}=\sqrt{tension/density}$	$Power=1/2\delta V_w\omega_F^2A_T^2$

Interference (same amplitude/phase/freq, same direction)

$Y(x,t)=2A\cos(\Delta\Phi/2)\sin(nx-\omega t+\Delta\Phi/2)$

Interference (same amplitude/phase/freq, opposite direction)

$Y(x,t)=2A\sin(nx)\cos(\omega_Ft)$

Vibrating Air:	V_{sound}=343m/s@20°C,331m/s@0°C
pipe open both ends:	$f=\eta(V_{sound}/2L)$ $[\eta=1,2,3,4\ldots]$
pipe open one end:	$f=\eta(V_{sound}/4L)$ $[\eta=1,3,5,7\ldots]$

------------------------------Physics II Equations------------------------------

Propagation of Light:		f=Frequency	cycle/s
ω_F=AngularFrequency	Θ/s	T=Period sec/cycle	seconds
$[\omega_Ft+\Phi]$=AngularPhase	radians	Φ=Angular Phase Diff.	radians
A_E=Max E Field	N/C	E=Electric Field	N/C
A_B=Max B Field	N/C	B=Magnetic Field	N/C
V_w=Wave Velocity	meter/s	V_P=Phase Velocity	meter/s
c=Speed of Light in a Vacuum		V_s=Speed of Light in Medium	
λ=Wave Length	meter	λ_i=Incident Wave Length	
λ_t=Transmitted Wave Length		λ_r=Reflected Wave Length	
Θ_i=Incident Wave Angle to ⊥		Θ_t=Transmitted Wave Angle to ⊥	
Θ_r=Reflected Wave Angle to ⊥			
N_m=Refraction Index of Medium		N_v=1 Refraction Index of Vacuum	
n=Wave Number	1/meter	m=Number of Waves	integer
a=Apeture Width of 1 Slit		d=Distance Between 2 Slits	
y=Distance to Point on Screen		D=Distance from Slits to Screen	

Speed, Wavelength, Frequency:

$c=f\lambda$ $f=c/\lambda$ $\lambda=c/f$ $V_m=f\lambda$ $f=V_m/\lambda$ $\lambda=V_m/f$

Index of Refraction:

$N_m=c/V_m$ $N_m=\lambda_i/\lambda_t$ $N_m=f_i/f_t$ $N_t/N_i=\sin(\Theta_t)/\sin(\Theta_i)$

$\Theta_i=\Theta_r$ Reflection $N_i\sin(\Theta_i)=N_t\sin(\Theta_t)$ Refraction

$\Theta_t=\sin^{-1}(N_i\sin(\Theta_i)/N_t)$ Total Internal Reflection if $N_i\sin(\Theta_i)>N_t$

$\Theta_B=\tan^{-1}(N_t/N_i)$ Brewster Angle $Wave_r$ Polarized /w **E** ⊥ to Surface

Spherical Mirrors:

p=Object Distance	i=Image Distance
f=Focal Length	m=Magnification
F_p=Focal Point	C_p=Center of Curvature
Upr=Image is Upright	Inv=Image is Inverted
f=r/2 Focal Length	m=-i/p=size(image)/size(object)
1/f=1/p+1/i	1/f=1/p+1/mp
Concave, Object btw F_p&Mirror	⇒ Upr Large Virt in Back
Concave, Object btw C_p&F_p	⇒ Inv Large Real in Front
Concave, Object before C_p	⇒ Inv Small Real in Front
Convex	⇒ Upr Small Virt in Back

Thin Spherical Lens:

r_1=Radius of Curvature of Lens Surface nearer to Object

r_2=Radius of Curvature of Lens Surface away from Object

$1/f=1/p+1/i=(N_m-1)(1/r_1-1/r_2)$

Constructive Interference: $\Phi=2m\pi$	$R_1=R_2+m\lambda$
Destructive Interference: $\Phi=(2m+1)\pi$	$R_1=R_2+(2m+1)\lambda/2$
2 Slit Interference:	
$d*\sin(\Theta)=m\lambda$ Maxima	$d*\sin(\Theta)=(2m+1)\lambda/2$ Minima
$\Theta=\sin^{-1}(m\lambda/d)$ Maxima	$\Theta=\sin^{-1}((2m+1)\lambda/2d)$ Minima
$d=m\lambda/\sin(\Theta)$ Maxima	$d=(2m+1)\lambda/2\sin(\Theta)$ Minima
$\tan(\Theta)=y/D$	$y=D\tan(\Theta)$ $D=y/\tan(\Theta)$
Grating Interference:	N_g=Number of Slits in Grating
$d*\sin(\Theta)=m\lambda$ Maxima	$d*\sin(\Theta)=2m\lambda/N_g$ Minima
Width of Central Maxima:	$\Theta=2\lambda/dN_g$
Dispersion of Grating:	D_Θ=Angle of Dispersion
Θ_m=Angle of m^{th} Maxima	$D_\Theta=derivative[\Theta_m/\lambda]=m/(d*\cos(\Theta_m))$
Resolving Power:	$R=\lambda/d\lambda=N_gm$
Thin Flm Interference:	τ=Thickness of Thin Film
$(2m-1)\lambda/2=2N\tau$ Maxima	$m\lambda=2N\tau$ Minima
$\tau=(2m-1)\lambda/4N$ Maxima	$\tau=m\lambda/2N$ Minima
1 Slit Diffraction:	$m\neq0$
$a*\sin(\Theta)=(2m+1)\lambda/2$ Maxima	$a*\sin(\Theta)=m\lambda$ Minima
$\Theta=\sin^{-1}((2m+1)\lambda/2a)$ Maxima	$\Theta=\sin^{-1}(m\lambda/a)$ Minima
$a=(2m+1)\lambda/2\sin(\Theta)$ Maxima	$a=m\lambda/\sin(\Theta)$ Minima
Special cases:	a=λ Central Max covers screen

Central Max gets sharper as a increases or λ decreases

Circular Apeture Diffraction:	Θ_R=smalest resolvable angle
$a*\sin(\Theta)=1.22\lambda$	$\sin(\Theta_R)=1.22\lambda/a$

Experement	ΔPhase	Maxima	Minima	Intensity
2Slit a≪λ	$\Phi=2\pi d\sin(\Theta)/\lambda$	$\Phi=2m\lambda$	$\Phi=(2m+1)\lambda$	$\cos^2(\Phi/2)$
1Slit	$\Phi=\pi a\sin(\Theta)/\lambda$	$\Phi=(2m+1)\lambda$	$\Phi=2m\lambda$	$\sin^2(\Theta)/\Theta$
Real 2Slit a>λ	$\Phi_I=2\pi d\sin(\Theta)/\lambda$ $\Phi_D=\pi a\sin(\Theta)/\lambda$	$\Phi_I=2m\lambda$	$\Phi_I=(2m+1)\lambda$	$\frac{\sin^2(\Phi_D)\cos^2(\Phi_I)}{2\Phi_D^2}$
nSlit a≤λ	$\Phi=2\pi d\sin(\Theta)/\lambda$	$\Phi=2m\lambda$	$\Phi=2m\lambda/N_g$	$\frac{\sin^2(\Theta N_g/2)}{N_g^2\sin^2(\Phi/2)}$
Real nSlit a>λ	$\Phi_I=2\pi d\sin(\Theta)/\lambda$ $\Phi_D=\pi a\sin(\Theta)/\lambda$	$\Phi_I=2m\lambda$	$\Phi_I=2m\lambda/N_g$	$\frac{\sin^2\Phi_D\sin^2(\Phi_IN_g/2)}{N_g^2\Phi_D\sin^2(\Phi_I/2)}$

Electromagnetic Waves:

Visible Spectrum=400nm-700nm	Peak Sensitivity=550nm
$E=E_{max}\sin(kx-\omega t)$	$B=B_{max}\sin(kx-\omega t)$
$c=E_{max}/B_{max}$	$c=E/B$
$c=1/\sqrt{(\mu_0\varepsilon_0)}$	$c=\omega/k$
$kE_{max}\cos(kx-\omega t)=\omega B_{max}\cos(kx-\omega t)$	$kE_{max}=\omega B_{max}$
$\mathbf{S}=(1/\mu_0)\mathbf{E\times B}$ Poynting Vector	**S**=instantaneous power/area W/m^2
$S=(1/\mu_0)\mathbf{EB}$	$S=(1/c\mu_0)E^2$
$E_{rms}=E_{max}/\sqrt{2}$	$E_{rms}=\sqrt{(P_sC\mu_0/4\pi r^2)}$
$I=P_s/4\pi r^2$ Intensity	$I=(1/c\mu_0)E_{rms}^2=(1/2c\mu_0)E_{max}^2$
$F=\Delta p/\Delta t$ Force	$\Delta U=IA\Delta t$ Energy
$\Delta p=\Delta U/c$ total absorption	$\Delta p=2\Delta U/c$ total reflection
F=IA/c total absorption	F=2IA/c total reflection
Pr=I/c Radiation Pressure	Pr=2I/c Radiation Pressure
Photons and Quantum Light:	$h=6.63x10^{-34}J\cdot s=4.14x10^{-15}eV\cdot s$
E=hf Photon Energy	$K_{max}=eV_{stop}$ Stopping Potential
$hf=K_{max}+\Phi$ Photoelectric Effect	$V_{stop}=(h/e)f-\Phi/e$
$p=hf/c=h/\lambda$ Photon Momentum	$\lambda=h/p$ de Broglie wavelength
$\Delta x\Delta p_x\geq h/2\pi$ $\Delta y\Delta p_y\geq h/2\pi$	$\Delta z\Delta p_z\geq h/2\pi$ Hiesenberg Uncertainty

$E_n=13.6eV/n^2$ Energy of Hydrogen

Electric Fields:

Q or q=Total Charge on Object	λs=Charge per Unit Length
σs=Charge per Unit Area	ρs=Charge per Unit Volume
E=Electric Field Vector	$\lvert\mathbf{E}\rvert$=Magnitude of Electric Field
F=Force due to Electric Field	$\lvert\mathbf{F}\rvert$=Magnitude of Electric Force
W=Work done by Electric Field	V=Voltage Potential of Field
$\varnothing_E$=Flux through a Surface	U=Potential Energy of Field
u_B=B field energy density	u_E=E field energy density
A=area	**p**=Dipole Moment
d=seperation between charges	τ=Torque due to Dipole Moment
κ=dielectric constant	ξ=emf
$\mathbf{F}=Kq_1q_2/r^2=(1/4\pi\varepsilon_0)q_1q_2/r^2$	$\mathbf{E}=\mathbf{F}/qt$ $F=E*q$
$\mathbf{E}=Kq_s/r^2=(1/4\pi\varepsilon_0)q_s/r^2$ Point	$\mathbf{E}=2K\lambda_s/r=(1/2\pi\varepsilon_0)\lambda_s/r$ Inf Line
$\mathbf{E}=(q_s/\varepsilon_0)(1-z/\sqrt{(z^2+r^2)})$ Disk	$\mathbf{E}=Kq_s/(z2+r2)^{3/2}$ Ring
$\mathbf{E}=(1/4\pi\varepsilon_0)q_s/r^2$ Outside Sphere	$\mathbf{E}=z\lambda_sr/2\varepsilon_0(z2+r2)^{3/2}$ Ring
E=0 Inside Sphere	**E**=0 Inside Ring
$\mathbf{E}=(1/4\pi\varepsilon_0r^2)(4\pi r^3\rho_s/3)$ In Volume	$\mathbf{p}=\lvert dq\rvert$dipole in dir of +Charge
$\mathbf{E}=\rho_sr/3\varepsilon_0$ In Spherical Volume	$\mathbf{E}=(1/2\pi\varepsilon_0)(\mathbf{p}/z^3)$ Dipole Field
$\mathbf{E}=\sigma_s/2\varepsilon_0$ Inf Sheet	$\tau=\mathbf{p\times E}=\mathbf{pE}\sin\Theta$ Torque on Dipole
$\mathbf{E}=\sigma_s/2\varepsilon_0$ Near Conducting Plate	$U=-\mathbf{p\cdot E}$ Potential of Dipole
$\mathbf{E}=(\sigma_+-\sigma_-)/2\varepsilon_0$ Btw 2 Cond Plates	$V=(1/4\pi\varepsilon_0)(\mathbf{p}\cos\Theta/r^2)$ Potential

$y=eEL^2/2m_ev^2$ bending of E beam

Electric Potential:

$E=-\Delta V/\Delta d$	$W_{applied}=-W$ Work on test charge
$U=-W_\infty$ Potential due to E Field	$W_{applied}=\Delta U=U_f-U_i$
V=U/q	$V=\Delta V=V_f-V_i$
$V=-W_\infty/q$ Potential from Inf	
$V=-\lvert E\rvert*d$ Constant E Field	$V=-\int_i^f \mathbf{E\cdot ds}$ Line Integral
$V=(1/4\pi\varepsilon_0)q_s/r$ Point Charge	
$A_e=E*e/m_e$ Electron in E Field	$v_e=\sqrt{(2Ev*1.6*10^{-19}/m_e)}$
$A_p=E*e/m_p$ Proton in E Field	$v_p=\sqrt{(2Ev*1.6*10^{-19}/m_p)}$

Gauss' Law:

$q_{enc}=\varepsilon_0\Phi_E$ Charge Inside Surface	$\Phi_E=\oint_s \mathbf{E\cdot dA}$ Surface Integral
$q_{enc}=\varepsilon_0E*A$ Uniform E Field	

Magnetic Fields:	$\mathbf{B}=\mathbf{F}_B/\lvert q\rvert v_q$
Ø=∠from **v** to **B**	Ø=∠from i**L** to **B**
$\mathbf{F}_B=q\mathbf{v\times B}$ B force on charge	$\mathbf{F}_B=ivB\sin$ Ø B force on charge
$\mathbf{F}_B=i\mathbf{L\times B}$ B force on wire	$\mathbf{F}_B=iLB\sin$ Ø B force on wire
$A_e=-e\mathbf{v\times B}/m_e$ Electron in B Field	$A_p=+e\mathbf{v\times B}/m_p$ Proton in B Field
v=E/B Crossed Fields	$m_e/q=B^2L^2/2yE$ Mass Spectrometer
$y=eBL^2/2m_ev$ bending of E beam	$E=v_dB$ Hall Effect
$r=m_ev/eB$ Circular E beam	$T=2\pi m_e/eB$ Orbital Period
$f=qB/2\pi m_e$ Orbital Frequency	$\omega=qB/m_e$ Angular Frequency
$v_\parallel=v*\cos$ Ø Helical paths	$v_\perp=v*\sin$ Ø Helical paths
$\tau_B=NiAB\sin\Theta$ Torque on N_{turn} coil	$\tau_B=\mu\times\mathbf{B}=\mu B\sin\Theta$ Torque on coil
Θ=∠ from ⊥ of coil to B field	μ=NiA Magnetic Moment
$\mathbf{B}=\mu_0i/2\pi r$ B field from wire	$U(\Theta)=-\mu\cdot\mathbf{B}$ Magnetic Potential
$\mathbf{B}=\mu_0iN/2r$ field @ center of coil	$\mathbf{B}=\mu_0i\Phi/2r$ field @ center of arc seg

$\mathbf{F}_{ba}=\mu_0Li_ai_b/2\pi d$ force btw 2 wires Parallel i attracts, anti repells

Ampere's Law: $\mu_0i_{enc}=\oint_s \mathbf{B\cdot ds}=\oint_s B\cos\Theta\, ds=B\oint_s ds=B(2\pi r)$ around wire

$\mathbf{B}=\mu_0in$ Solenoid	$\mu_0iN/2\pi r$ Torroid
$\Phi_B=BA$ Uniform B field	
$\xi=-d\Phi_B/dt$ Varying B Flux	$\Phi_B=\oint_s \mathbf{B\cdot dA}$ Surface Integral
$\xi=-Nd\Phi_B/dt$ N turn coil	$\Phi_B=0$ for closed surface

Lenz's Law: Changing Magnetic Flux Induces a Current in a coil such that the B Field produced by the Induced Current opposes the change in the Magnetic Flux.

Faraday's Law: A Changing Magnetic Flux induces an emf in the space where the Flux changes, in a Loop, whose axis is tangent to the Magnetic Field lines.

$\xi_B=\oint_s \mathbf{E\cdot ds}=-d\Phi_B/dt$

A Changing B Field Induces an E Field at every point around the loop.

------------------------------Electronics Equations------------------------------

Definitions:

Node: Intersection of 2 or more circuit elements
Branch: A single element connected to 2 different nodes. 2 single elements in parallel between the same pair of nodes are 2 seperate branches
Loop: A Closed Path through a circuit that starts and ends at the same node
KVL: **Kirchhoff's Voltage Law:** The sum of the voltages across the elements of a Loop is ZERO
KCL: **Kirchhoff's Current Law:** The sum of all currents entering and leaving a node is ZERO
Peripheral: Current If a Current Source is present in only 1 mesh then the current of that mesh equals the Source
Voltage Sign: Voltage/Power across a circuit element is:
Power Sign: – if current enters the element at the – terminal; + if current enters the element at the + terminal

Voltages: Elements with a common Voltage are in Parallel
Series: $V_{eq}=V_1+V_2+V_3+...+V_n$
$V=i*(R_1+R_2+R_3+...+R_n)$
$V_{R1}=R_1/(R_1+R_2+R_3+...+R_n)$
Parallel: Independant voltage sources wired in Parallel must have the same voltage

Currents: Elements with a common Current are in Series
$i_{G1}=i_S G_1/(G_1+G_2+G_3+...+G_n)$
$i_{R1}=R_2/(R_1+R_2)$ $i_{R2}=R_1/(R_1+R_2)$
Series: Independant Current sources wired in Series must have the same current

Resistance: Ohms
Series: $R_{eq}=(R_1+R_2+R_3+...+R_n)$
Parallel: $R_{eq}=1/(1/R_1+1/R_2+1/R_3+...+1/R_n)$
$R_{eq}=(R_1*R_2)/(R_1+R_2)$

Conductance: Mhos or Siemens $G=1/R$
Series: $G_{eq}=1/(1/G_1+1/G_2+1/G_3+...+1/G_n)$
Parallel: $G_{eq}=G_1+G_2+G_3+...+G_n$

Real Sources: A Real Voltage or Current Source has an internal resistance. A Voltage Source V_S has a series resistance R_S. A Current Source I_S has a parallel resistance R_P. Both have a load R_L.

Voltage: Sources	$I_L=V_S/(R_S+R_L)$ $I_L=V_S/R_S-V_L/R_S$	$V_L=I_L R_L$ $I_{SC}=V_S/R_S$	$V_S=R_S I_L+V_L$ $V_{OC}=V_S$
Current: Sources	$I_L=I_S R_P/(R_P+R_L)$ $I_L=I_S-V_L/R_P$	$V_L=I_L R_L$ $I_{SC}=I_S$	$V_S=I_S R_P R_L/(R_P+R_L)$ $V_{OC}=I_S R_P$
Max Power:	Voltage Source Current Source	$R_S=R_L$ $R_P=R_L$	

$q(t)=\int_{t0}^{t} i\,dt + q(t_0)$ Charge transferred
$P=vi=i^2R=v^2/R$ Power Absorbed or Produced
$v=iR$ Ohms Law

Circuits:
$\xi=dWork/dq=iR$ $\xi-iR=0$ ideal battery circuit
$i=\xi/(R+r)$ $\xi-ir-iR=0$ real battery circuit
Parallel $i=i_1+i_2+...i_n$ $R_{eq}=1/(1/R_1+1/R_2+...1/R_n)$
$i=V(1/R_1+1/R_2+...1/R_n)$ $i_1=V/R_1$ $i_2=V/R_2$ … $i_n=V/R_n$
$\xi_{eq}=\xi_1=\xi_2=...\xi_n$ $P=i^2R_{eq}=V^2/R_{eq}$
Series $i=\xi/(R_1+R_2+...R_n)$ $R_{eq}=R_1+R_2+...R_n$
$\xi_{eq}=\xi_1+\xi_2+...\xi_n$ $P=i^2R_{eq}=\xi^2/R_{eq}$
$P=i^2R$ Resistive loss in Circuit $P=i(\xi-ir)=i\xi-i^2r$ Power in battery

Inductance:
$\xi_L=-Ldi/dt$ Self Induction $L=N\Phi_B/i$ Definition of Inductance
$\tau_L=L/R$ Inductive time constant $L/l=\mu_0 n^2 A=2\pi\mu_0 n^2 r$ Solenoid L/length
$i=(\xi/R)(1-e^{-tR/L})$ Charging $U_B=(1/2)Li^2$ Stored Magnetic Energy
$i=i_0 e^{-tR/L}$ Discharging $u_B=B^2/2\mu_0$ Magnetic Energy Density
$\Phi_{B21}=\pi\mu_0 i_1 N_1 r_1^2 r_2^2/2d^3$ B Flux in coil 2 from current i in coil 1
$M_{21}=N_2\Phi_{B21}/i_1$ Mutual Inductance in coil 2 from current i in coil
$1\xi_{M21}=-M_{21}di_1/dt$ Induced emf in coil 2 from current i in coil 1
$M=\pi\mu_0 N_1 r_1^2 N_2 r_2^2/2d^3$ Inductance btw 2 coils seperated by distance d
$\xi_2=-Mdi_1/dt$ Induced emf $\xi_1=-Mdi_2/dt$ Induced emf

Ampere-Maxwell Law:
$\oint_s \mathbf{B}\cdot d\mathbf{s} = \mu_0 i_{d,enc} + \mu_0 i_{enc}$
A Changing E Field Flux induces a Magnetic Field in the space where the Flux changes, in a Loop, whose axis is tangent to the E Field lines.
A Changing E Field Induces an B Field at every point around the loop of the same magnitude and direction as would have been produced by the displacement current. $i_d=\varepsilon_0 d\Phi_E/dt$ displacement current
$i_d=dq/dt$ charge on plates $i_d=\varepsilon_0 AdE/dt$ displacement current
$\mathbf{B}=(\mu_0 i_d/2\pi r)$ Outside Cap. r>R $\mathbf{B}=(\mu_0 i_d/2\pi R^2)r$ Inside Capacitor r<R

Capacitance: $C=q/V$
Parallel Plate Capacitor $C=\varepsilon_0 A/d$
Cylindrical Capacitor $C=2\pi\varepsilon_0(Length/\ln(r_{outer}/r_{inner}))$
Spherical Capacitor $C=4\pi\varepsilon_0(r_{outer}*r_{inner}/r_{outer}-r_{inner})$
Isolated Sphere $C=4\pi\varepsilon_0 r$
Parallel $V_1=V_2=...V_n$ $q=q_1+q_2+...q_n$ $C_{eq}=C_1+C_2+...C_n$
Series $V=V_1+V_2+...V_n$ $q_1=q_2=...q_n$ $C_{eq}=1/(1/C_1+1/C_2+...1/C_n)$
$U=q^2/2C=(1/2)CV^2$ Stored Energy $u_E=U/A*d=CV^2/2Ad$
$C=\kappa C_{air}$ Capacitor /w dielectric $u_E=(1/2)\varepsilon_0(V^2/d^2)=(1/2)\varepsilon_0 E^2$
$\varepsilon_0\rightarrow\kappa\varepsilon_0$ E field /w dielectric $V=\kappa E_0*d$

Current and Resistance:
$i=dq/dt$ $q=\int_0^t i\,dt$
$\mathbf{i}_0=\mathbf{i}_1+\mathbf{i}_2$ Wire Junctions
$\mathbf{J}=\mathbf{i}/A$ Current Density
$v_d=\mathbf{J}/(n*e)$ Drift Velocity $n=N_a$(density gm/cc)/(molar mass)
$R=V/\mathbf{i}$ Resistance $\rho=E/\mathbf{J}=(V/L)/(i/A)$ Resistivity
$R=\rho(Length/Area)$ $\sigma=1/\rho$ Conductivity
Ohms Law [V vs. i is linear] $\rho=m_e/e^2nt$ [t=mean time btw collision
$Power=iV=i^2R=V^2/R$ $\rho-\rho_0=\rho_0\alpha(t-t_0)$ Temp. Coefficient

Thevinin and Norton Equivalent Circuits:
Given any circuit as viewed from a single element (Z_{LOAD}), the rest of the circuit may be replaced with the Thevinin equivalent circuit consisting of a single voltage source V_{TH} in series with a single impedance Z_{TH}. The Norton equivalent circuit is obtained by a source transformation of V_{TH} in series with Z_{TH} to $I_N=V_{TH}/Z_{TH}$ in parallel with $Z_N=Z_{TH}$.
$V_{TH}=V_{OC}$ Voltage across the open circuited load terminals
I_{SC} Current through the short circuited load terminals
$Z_{TH}=Z_{OC}=V_{OC}/I_{SC}$ Impedance of circuit as viewed from load terminals

Source Transforms:
$\mathbf{V}_S$ in series with $\mathbf{Z}_S$ ==> $\mathbf{I}_S=\mathbf{V}_S/\mathbf{Z}_S$ in parallel with $\mathbf{Z}_S$
$\mathbf{I}_S$ in parallel with $\mathbf{Z}_S$ ==> $\mathbf{V}_S=\mathbf{I}_S*\mathbf{Z}_S$ in series with $\mathbf{Z}_S$

AC Power: P=Real Power
Q=Reactive Power S=Complex Power
Θ_V=Phase of Voltage at Source $\varnothing_I$=Phase of Current at Source
$V_{RMS}=V_{EFF}=V_{MAX}/\sqrt{2}$ $I_{RMS}=I_{EFF}=I_{MAX}/\sqrt{2}$
Power Factor: $P_F=R/Z=\cos(\Theta_V-\varnothing_I)$
Average Power: $P_{AVG}=V_{RMS}I_{RMS}\cos(\Theta_V-\varnothing_I)$
$P_{AVG}=V_{RMS}I_{RMS}R/Z$ $P_{AVG}=I_{RMS}^2R$
$P_{AVG}=(1/2)V_{MAX}I_{MAX}\cos(\Theta_V-\varnothing_I)$ $P_{AVG}=(1/2)Real(\mathbf{V}\bullet\mathbf{I})$
Complex Power: $\mathbf{S}=(1/2)(\mathbf{V}\bullet\mathbf{I})=P+jQ$
$P_{CIRCUIT}=\sum I_{EFF}^2R_N=\sum P_{SOURCES}$ $Q_{CIRCUIT}=\sum I_{EFF}^2X_N=\sum Q_{SOURCES}$
$\mathbf{S}_{SOURCE}=(1/2)\mathbf{V}_S\bullet conjugate(\mathbf{I})$ $\mathbf{S}_{CIRCUIT}=P_{CIRCUIT}+Q_{CIRCUIT}$
Phasors:
$\mathbf{V}=V_{MAX}\cos(\omega t-\Theta_V)$ $\mathbf{I}=I_{MAX}\cos(\omega t-\varnothing_I)$
$\mathbf{V}=V_{MAX\angle}\Theta_V$ $\mathbf{I}=I_{MAX\angle}\varnothing_I$
$Z_C=1/j\omega C=-j/\omega C$ $Z_L=j\omega L$
$Z_R=R$ $Z_{CIRCUIT}=\sum Z_N$

Complex Frequency: S Plane $[\sigma, j\omega]$
$v(t)=V_M e^{\sigma t}\cos(\omega t+\Theta)$ Sinusoid: $\sigma=0$, $v(t)=V_M\cos(\omega t+\Theta)$
$V_S=V_M\angle\Theta$ Exponential: $\omega=0$, $v(t)=V_M\cos(\Theta)e^{\sigma t}$
$S=(\sigma+j\omega)$ DC: $\sigma=0$, $\omega=0$, $v(t)=V_M\cos(\Theta)$

Driven RLC Curcuits:
ω_0=Resonant Angular Frequency ω_D=Driving Angular Frequency
Z=Circuit Impedance ξ_D=Driving emf
$f_0=\omega_0/2\pi$ Resonant Frequency $f_D=\omega_D/2\pi$ Driving Frequency
IFF $\omega_D=\omega_0\rightarrow$ circuit resonates $\omega_D=2\pi f_D$ Driving Angular Frequency
$V=V_{max}\sin(\omega_0 t)$ Driving Voltage $\xi_D=\xi_{max}\sin(\omega_D t)$ Driving emf
$Z=\sqrt{[R^2+(X_L-X_C)^2]}$ Impedance $\xi_D^2=V_R^2+(V_L-V_C)^2=(IR)^2+(IX_L-IX_C)^2$
$\varnothing=\tan^{-1}(Im(V/Z)/Re(V/Z))$ $\varnothing=\tan^{-1}((X_L-X_C)/R)$
$I_{max}=V_{max}/Z$ Current Amplitude $I_D=I_{max}\sin(\omega_D t-\varnothing)$ Driven Current
$V_D=I_D Z$ Driven Voltage $V_R=I_D R$ Voltage Across Resistor
$X_L=\omega_D L$ Inductive Reactance $V_L=I_D X_L$ Voltage Across Inductor
$X_C=1/\omega_D C$ Capacitive Reactance $V_C=I_D X_C$ Voltage Across Capacitor
$I_{rms}=I_{max}/\sqrt{2}$ Average Current $\xi_{rms}=\xi_{max}/\sqrt{2}$ Average emf
r_{loss}=DC resistance [Inductor] r_{leak}=DC resistance [Capacitor]
$R_{loss}=\rho^2/r_{loss}$ [Inductor] $R_{leak}=\rho^2/r_{leak}$ [Capacitor]

Resonance: $\omega_0 = 1/\sqrt{(L/C)}$
$V_{(\omega_0)} = V_{MAX} = IR$ $I_{(\omega_0)} = I_{MAX} = V/R$
$\mathbf{I} = \mathbf{V}\bullet\mathbf{Y} = V\bullet[1/R - j/\omega L + j\omega C]$ $\mathbf{V} = \mathbf{I}\bullet\mathbf{Z} = I\bullet[R + j\omega L - j/\omega C]$
$\rho = \sqrt{(L/C)}$

Parallel Circuits
$\frac{d^2v(t)}{dt^2}+\frac{1}{RC}\frac{dv(t)}{dt}+\frac{1}{LC}v(t) = 0$
$S^2 + \frac{1}{RC}S + \frac{1}{LC} = 0$
$S_{1,2}=-\frac{1}{RC}\pm\sqrt{\left(\frac{1}{4R^2C^2}\right)-\left(\frac{1}{LC}\right)}$
$\alpha = 1/(2RC)$
$\omega_0 = 1/\sqrt{(L/C)}$
$S_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$

Series Circuits
$\frac{d^2i(t)}{dt^2}+\frac{R}{L}\frac{di(t)}{dt}+\frac{1}{LC}i(t) = 0$
$S^2 + \frac{R}{L}S + \frac{1}{LC} = 0$
$S_{1,2}=-\frac{1}{RC}\pm\sqrt{\left(\frac{R^2}{4L^2}\right)-\left(\frac{1}{LC}\right)}$
$\alpha = R/(2L)$
$\omega_0 = 1/\sqrt{(L/C)}$
$S_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$

------------------------------Electronics Equations------------------------------

Frequency Response:
$\gamma = \omega_D/2\alpha$ $\beta = [1 - \omega_0^2/\omega_D^2]$
$y(\omega_D) = 1/\sqrt{(1+\gamma^2\beta^2)}$ $y(\omega_0) = 1$
$v(\omega_D) = V_{MAX}*y(\omega_D)$ $\Theta(\omega_D) = -\tan^{-1}(\gamma\beta)$
3dB Cutoff Frequency: $[\omega_1,\omega_2]$ $y(\omega_1)=y(\omega_2)=(1/\sqrt{2})y(\omega_0)$
$\omega_1 = -\alpha+\sqrt{(\alpha^2+\omega_0^2)}$ $\omega_2 = +\alpha+\sqrt{(\alpha^2+\omega_0^2)}$
$\omega_1 = \omega_0-\alpha$ $\omega_2 = \omega_0+\alpha$
$\Delta\omega = \omega_2-\omega_1$ $Q = \omega_0/\Delta\omega \approx \omega_0/2\alpha$
Over-Damped: $[S_1,S_2$ Real$]$ $v(t)=Ae^{-S1t} - Ae^{-S2t}$ $A=V_0$
$S_1 = -\alpha+\sqrt{(\alpha^2-\omega^2)}$ $S_2 = -\alpha-\sqrt{(\alpha^2-\omega^2)}$
Critical-Damped: $[S_1=S_2=-\alpha]$ $v(t)=Ate^{-St}$ $A=-V_0$
Under-Damped: $[S_1,S_2$ Complex$]$
$\omega_D = \sqrt{(\omega_0^2 - \alpha^2)}$ $v(t) = Ae^{-\alpha t}\cos(\omega_D t)$ $A=V_0$
$S_1 = -\alpha+j\omega_D$ $S_2 = -\alpha-j\omega_D$
$\tau = 1/\alpha$ $T_{zero} = 64\tau$
Computing [α] from measurements:

Over-Damped	Critically-Damped	Under-Damped
[V(t1)>0, V(t2)>0]	[V(t1)>0, V(t2)>0]	[V(t1)>0, V(t2)>0]
$S=\frac{\ln\left(\frac{V(t1)}{V(t2)}\right)}{(t2-t1)}$	$\alpha=\frac{\ln\left(\frac{V(t1)}{V(t2)}\right)-\ln\left(\frac{t1}{t2}\right)}{(t2-t1)}$	$\alpha=\frac{1}{(t2-t1)}\ln\left(\frac{V(t1)}{V(t2)}\right)$
$\alpha=\frac{-\left(s^2+(\omega_0)^2\right)}{2S}$		

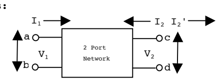
Undamped LC Circuits:
V_L=Voltage Across Inductor V_C=Voltage Across Capacitor
V_R=Voltage Across Resistor ω=Resonance Angular Frequency
$U_E=q^2/2C$ $U_B=Li^2/2$ $V_L=Ldi/dt$ $V_C=(1/C)q$ $V_R=iR$
$i=q/\sqrt{(LC)}$ $i=V\sqrt{(C/L)}$ $\omega=1/\sqrt{(LC)}$ $f=1/2\pi\sqrt{(LC)}$
$q_t=q_{max}\cos(\omega t+\varnothing)$ $i_t=-\omega q_{max}\sin(\omega t+\varnothing)$
$i_{max}=\omega q_{max}=V\sqrt{[C/L]}=\sqrt{[q_{max}^2/LC]}$ $i_t=-\omega i_{max}\sin(\omega t+\varnothing)$
$U_E=(q_{max}^2/2C)\cos^2(\omega t+\varnothing)$ $U_{Emax}=(q_{max}^2/2C)$
$U_B=(q_{max}^2/2C)\sin^2(\omega t+\varnothing)$ $U_{Bmax}=(q_{max}^2/2C)$
$U_E(t)+U_B(t)=(q_{max}^2/2C)$ $L(d^2q/dt^2)+(1/C)q=0$

Damped RC Circuits:
$\xi-iR-q/C=0$ Capacitor in circuit $\tau_c=RC$ Capacitive time constant
$\xi=R(dq/dt)+q/C$ ξ Charging $q=C\xi(1-e^{-t/RC})$ q Charging
$i=dq/dt=(\xi/R)e^{-t/RC}$ i Charging $V_c=q(t)/C=\xi(1-e^{-t/RC})$ V Charging
$iR+q/C=0$ Discharging $t=\ln(1/2)RC$ time to 1/2 charge
$R(dq/dt)+q/C=0$ Discharging $q=q_0e^{-t/RC}$ q Discharging
$i=-(q_0/RC)e^{-t/RC}$ i Discharging $V_c=V_0e^{-t/RC}$ V Discharging
$P=(\xi^2/R)e^{-2t/RC}$ P Charging $P=-(q_0^2/RC^2)e^{-2t/RC}$ P Discharging

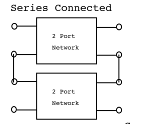
Damped RLC Circuits: ω'=Damped Angular Frequency
$\omega'=\sqrt{[(1/LC)-(R/2L)^2]}$ $q=q_{max}e^{-tR/2L}\cos(\omega't+\varnothing)$
$i=q_{max}e^{-tR/2L}[(R/2L)\cos(\omega't+\varnothing)-(\omega')\sin(\omega't+\varnothing)]$
$U=U_B+U_E=Li^2/2+q^2/2C$ Tot. Energy $L(d^2q/dt^2)+R(dq/dt)+(1/C)q=0$
$U_E=(q_{max}^2/2C)e^{-tR/L}\cos^2(\omega t+\varnothing)$ $U_{Emax}=(q_{max}^2/2C)e^{-tR/2L}$
$U_B=(q_{max}^2/2C)e^{-tR/L}\sin^2(\omega t+\varnothing)$ $U_{Bmax}=(q_{max}^2/2C)e^{-tR/2L}$
$U_{Emax}\Rightarrow U_B=0$ $U_{Bmax}\Rightarrow U_E=0$ $U_E(t)+U_B(t)=(q_{max}^2/2C)e^{-tR/2L}$

Laplace Transforms:
Resistor
$V(s)=R*\mathcal{L}[i(t)]$ $V(s)=R*I(s)$
Inductor
$V(s)=L*\mathcal{L}\left[\frac{di}{dt}\right]$ $V(s)=L*[sI(s)-i(0^-)]$ $I(s)=\frac{1}{sL}V(s)+\frac{i(0^-)}{s}$
Capacitor
$V(s)=R*\mathcal{L}\left[\frac{i(t)}{dt}\right]$ $V(s)=\frac{1}{sC}I(s)+\frac{v(0^-)}{s}$ $I(s)=C*[sV(s)-v(0^-)]$

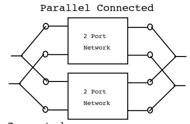
2 Port Networks:



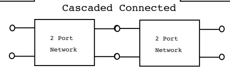
Series Connected



Parallel Connected



Cascaded Connected



Z-Parameters $[\dot{V}]=[\dot{Z}]\bullet[\dot{I}]$ $\begin{bmatrix}\mathbf{V_1}\\ \mathbf{V_2}\end{bmatrix}=\begin{bmatrix}z_{11} & z_{12}\\ z_{21} & z_{22}\end{bmatrix}\times\begin{bmatrix}I_1\\ I_2\end{bmatrix}$ $\Delta_Z=\begin{vmatrix}z_{11} & z_{12}\\ z_{21} & z_{22}\end{vmatrix}=z_{11}\times z_{22}-z_{21}\times z_{12}$	$\dot{Z}=\begin{bmatrix}\left[z_{11}=\frac{V_1}{I_1}\middle\vert\begin{matrix}OC[cd]\\ I_2=0\end{matrix}\right] & \left[z_{12}=\frac{V_1}{I_2}\middle\vert\begin{matrix}OC[ab]\\ I_1=0\end{matrix}\right]\\ \left[z_{21}=\frac{V_2}{I_1}\middle\vert\begin{matrix}OC[cd]\\ I_2=0\end{matrix}\right] & \left[z_{22}=\frac{V_2}{I_2}\middle\vert\begin{matrix}OC[ab]\\ I_1=0\end{matrix}\right]\end{bmatrix}$
Y-Parameters $[\dot{I}]=[\dot{Y}]\bullet[\dot{V}]$ $\begin{bmatrix}I_1\\ I_2\end{bmatrix}=\begin{bmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix}\times\begin{bmatrix}V_1\\ V_2\end{bmatrix}$ $\Delta_Y=\begin{vmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{vmatrix}=y_{11}\times y_{22}-y_{21}\times y_{12}$	$\dot{Y}=\begin{bmatrix}\left[y_{11}=\frac{I_1}{V_1}\middle\vert\begin{matrix}SC[cd]\\ V_2=0\end{matrix}\right] & \left[y_{12}=\frac{I_1}{V_2}\middle\vert\begin{matrix}SC[ab]\\ V_1=0\end{matrix}\right]\\ \left[y_{21}=\frac{I_2}{V_1}\middle\vert\begin{matrix}SC[cd]\\ V_2=0\end{matrix}\right] & \left[y_{22}=\frac{I_2}{V_2}\middle\vert\begin{matrix}SC[ab]\\ V_1=0\end{matrix}\right]\end{bmatrix}$
T-Parameters $[\dot{In}]=[\dot{T}]\bullet[\dot{Out}]$ $\begin{bmatrix}V_1\\ I_1\end{bmatrix}=\begin{bmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{bmatrix}\times\begin{bmatrix}V_2\\ I_2'\end{bmatrix}$ $\Delta_T=\begin{vmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{vmatrix}=t_{11}\times t_{22}-t_{21}\times t_{12}$	$\dot{T}=\begin{bmatrix}\left[t_{11}=\frac{V_1}{V_2}\middle\vert\begin{matrix}OC[cd]\\ I_2=0\end{matrix}\right] & \left[t_{12}=\frac{V_1}{I_2'}\middle\vert\begin{matrix}SC[cd]\\ V_2=0\end{matrix}\right]\\ \left[t_{21}=\frac{I_1}{V_2}\middle\vert\begin{matrix}OC[cd]\\ I_2'=0\end{matrix}\right] & \left[t_{22}=\frac{I_1}{I_2'}\middle\vert\begin{matrix}SC[cd]\\ V_2=0\end{matrix}\right]\end{bmatrix}$
Series Connection $[\dot{Z}]=[\dot{Z}_A]+[\dot{Z}_B]$	$\begin{bmatrix}\mathbf{V_1}\\ \mathbf{V_2}\end{bmatrix}=\begin{bmatrix}V_{1A}+V_{1B}\\ V_{2A}+V_{2B}\end{bmatrix}=\begin{bmatrix}z_{11B}+z_{11B} & z_{12A}+z_{12B}\\ z_{21A}+z_{21B} & z_{22A}+z_{22B}\end{bmatrix}\times\begin{bmatrix}I_1\\ I_2\end{bmatrix}$
Parallel Connection $[\dot{Y}]=[\dot{Y}_A]+[\dot{Y}_B]$	$\begin{bmatrix}I_1\\ I_2\end{bmatrix}=\begin{bmatrix}I_{1A}+I_{1B}\\ I_{2A}+I_{2B}\end{bmatrix}=\begin{bmatrix}y_{11A}+y_{11B} & y_{12A}+y_{12B}\\ y_{21A}+y_{21B} & y_{22A}+y_{22B}\end{bmatrix}\times\begin{bmatrix}V_1\\ V_2\end{bmatrix}$

Cascaded Connection

$[\dot{T}]=[\dot{T}_A]\times[\dot{T}_B]$ $\begin{bmatrix}V_1\\ I_1\end{bmatrix}=\begin{bmatrix}t_{11A} & t_{12A}\\ t_{21A} & t_{22A}\end{bmatrix}\times\begin{bmatrix}t_{11B} & t_{12B}\\ t_{21B} & t_{22B}\end{bmatrix}\times\begin{bmatrix}V_2\\ I_2'\end{bmatrix}=\begin{bmatrix}t_{11A}t_{11B}+t_{12A}t_{21B} & t_{11A}t_{12B}+t_{12A}t_{22B}\\ t_{21A}t_{11B}+t_{22A}t_{21B} & t_{21A}t_{12B}+t_{22A}t_{22B}\end{bmatrix}\times\begin{bmatrix}V_2\\ I_2'\end{bmatrix}$

\from to \	Z	Y	T
Z	$\begin{bmatrix}z_{11} & z_{12}\\ z_{21} & z_{22}\end{bmatrix}$	$\begin{bmatrix}\frac{y_{22}}{\Delta_Y} & \frac{-y_{12}}{\Delta_Y}\\ \frac{-y_{21}}{\Delta_Y} & \frac{y_{11}}{\Delta_Y}\end{bmatrix}$	$\begin{bmatrix}\frac{t_{11}}{t_{21}} & \frac{\Delta_T}{t_{21}}\\ \frac{1}{t_{21}} & \frac{t_{22}}{t_{21}}\end{bmatrix}$
Y	$\begin{bmatrix}\frac{z_{22}}{\Delta_Z} & \frac{z_{12}}{\Delta_Z}\\ \frac{z_{21}}{\Delta_Z} & \frac{z_{11}}{\Delta_Z}\end{bmatrix}$	$\begin{bmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix}$	$\begin{bmatrix}\frac{t_{22}}{t_{12}} & \frac{\Delta_T}{t_{12}}\\ \frac{-1}{t_{12}} & \frac{t_{11}}{t_{12}}\end{bmatrix}$
T	$\begin{bmatrix}\frac{z_{11}}{z_{21}} & \frac{\Delta_Z}{z_{21}}\\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}}\end{bmatrix}$	$\begin{bmatrix}\frac{-y_{22}}{y_{21}} & \frac{-1}{y_{21}}\\ \frac{-\Delta_Y}{y_{21}} & \frac{-y_{11}}{y_{21}}\end{bmatrix}$	$\begin{bmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{bmatrix}$

Discrete Structures Equations

Logic

Proposition / **P**=TRUE — The statement that proposition P is TRUE

¬P / **P**=FALSE, **¬P**=TRUE — The statement that proposition P is FALSE

Disjunction — Either P OR Q is TRUE

Conjunction — Both P AND Q are TRUE

Exclusive OR — Either P OR Q is TRUE, but NOT both P and Q

Implication → — If P then Q

BiConditional ↔ — P if and only if Q and Q if and only if P

Equivalent ⇔ — Two Propositions P and Q are Logically Equivalent IFF **P↔Q** is a Tautology. Identical Truth Tables

Contradiction / [P=TRUE]=FALSE — A statement that proposition P is TRUE, when P is FALSE

Tautology — A proposition that is ALWAYS TRUE

Propositional Function / P(x1,...xn)=TRUE/FALSE — A function that maps it's input to TRUE or FALSE

Identity Laws	P∨F⇔P	P∧T⇔P
Domination Laws	P∨T⇔T	P∧F⇔F
Idempotent Laws	P∨P⇔P	P∧P⇔P
Double Negation	¬(¬P)⇔P	
ContraPositive	(P→Q)⇔(¬Q→¬P)	
Commutative Laws	P∨Q⇔Q∨P	P∧Q⇔Q∧P
Associative Laws	(P∨Q)∨R⇔P∨(Q∨R) (P∧Q)∧R⇔P∧(Q∧R)	
Distributive Laws	P∨(Q∧R)⇔(P∨Q)∧(P∨R) P∧(Q∨R)⇔(P∧Q)∨(P∧R)	
Demorgans Laws	¬(P∨Q)⇔¬P∧¬Q	¬(P∧Q)⇔¬P∨¬Q
Logical Equivalances	P∨¬P⇔T (P→Q)⇔(¬P∨Q)	P∧¬P⇔F

Quantifiers

Universal Quantifier / **∀xP(x)** / [for all x∈U: P(x)=T] — The Universal Quantification of P(x) is the proposition that P(x) is true for ALL values of x in the Universal Set

Existential Quantifier / **∃xP(x)** / [for some x∈U: P(x)=T] — The Existential Quantification of P(x) is the proposition that there exists an element x in the Universal Set such that P(x) is true

Set / {x1,x2,x3...} — A collection of arbitrary elements with no duplicates

Z — The set of integers from {-∞ to +∞}

Z+ or **P** — The set of integers from {1 to +∞}

ℵ — The set of Natural Numbers {0 to +∞}

ℜ — The set of Real Numbers {-∞ to +∞}

Ø — The Empty Set

U — The Universal Set {The set of ALL values in the universe of discourse}

Set Builder / $\{x \mid P(x)\}$ — The set of all values x such that the proposition P(x)=TRUE

Cardinality / $\lvert S\rvert$=# of elements in S — The cardinality of a finite set is the number of distinct elements in the set

SubSet / {a1,a2...}∴∀a∈A:(a∈B) / $\lvert subset(S)\rvert$= 2^$\lvert S\rvert$ — Set A is a subset of set B if and only if every element of set A is also a element of set B

Infinite Set — A set is an Infinite Set if the number of distinct elements in the set is infinite

Power Set / $\lvert powerset(S)\rvert$= 2^2^$\lvert S\rvert$ — Given a Set S, the Power Set of S is the Set whose elements are ALL of the possible SubSets of the Set S

Ordered n-tuple — An ordered n-tuple is an ordered set of n elements where identical sets of elements in different order are different n-tuples

Cartesian Product / $A\times B=\{(a,b) \mid a\in A \wedge b\in B\}$ — Given 2 sets A and B, The Cartesian Product A×B is the set of all ordered pairs (a,b) where a∈A and b∈B

Functions — Let A and B be sets; a Function from A to B is an assignment of exactly 1 element of B to each element of A Given a Function F from A to B, A is the Domain of F, B is the Codomain of F. If f(a∈A)=b∈B then b is the Image of a, a is the pre-image of b

1 to 1 — A Function is said to be 1 to 1 if and only if f(x)=f(y)→x=y for all x,y in the domain of f

Onto — A Function from A to B is said to be Onto IF and only IF for every b∈B there is an a∈A such that f(a)=b

1 to 1 Correspondance — A Function from A to B is said to be a 1 to 1 Correspondance if it is both 1 to 1 and Onto

Counting

Sum Rule — If task#1 can be done n1 ways and task#2 can be done n2 ways and task#1, task#2 are mutually exclusive then there are n1+n2 ways to do either task#1 or task#2

Product Rule — If task#1 can be done n1 ways and task#2 can be done n2 ways and you do task#1 followed by task#2 then there are n1*n2 ways to do task#1 + task#2

Pigeonhole Principle — K+1 objects are placed into k boxes then at least 1 box must contain 2 or more of the objects. N objects are placed into k boxes then at least 1 box must contain ⌈N/k⌉ of the object

Permutations — A Permutation of a set of distinct objects is an ordered arraingement of the elements of the set

P(n) / P(n)=n!=n(n-1)(n-2)...1 — The number of Permutations of n elements

P(n,r) / P(n,r)=n!/r! =n(n-1)(n-2)...(n-r+1) — The number of Permutations of r elements taken from a set of n elements (0 ≤ r ≤ n)

Derangement — A Permutation where ALL elements are moved to a different position

Combinations — A Combination of a set is a unordered subset of the set

C(n,r) / C(n,r)=n!/(r!*(n-r)!) / C(n,r)=P(n,r)/P(r,r) / C(n,r)=C(n,n-r) — An r-combination is an unordered subset of r elements from a set of n elements (0 ≤ r ≤ n)

Sum of Combinations — $\sum$[k=0 to n](C(n,k))=2^n ; $\sum$[k=0 to n](-1^kC(n,k))=0

Binomial Theorem — $(X + Y)^n$=$\sum$[j=0 to n]C(n,j)$X^{n-j}Y^j$

Vandermonde's Identity — C(m+n,r)=$\sum$[k=0 to r](C(m,r-k)C(n,k))

Pascals Identity — C(n+1,k)=C(n,k-1)+C(n,k) i.e. **C(6,3)=C(5,2)+C(5,3)**

```
                        C(0,0)
                    C(1,0) C(1,1)
                C(2,0) C(2,1) C(2,2)
            C(3,0) C(3,1) C(3,2) C(3,3)
        C(4,0) C(4,1) C(4,2) C(4,3) C(4,4)
    C(5,0) C(5,1) C(5,2) C(5,3) C(5,4) C(5,5)
C(6,0) C(6,1) C(6,2) C(6,3) C(6,4) C(6,5) C(6,6)
```

Pascals Triangle

N	Coefficients of $(X + Y)^N$
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1
9	1 9 36 84 126 126 84 36 9 1
10	1 10 45 120 210 254 210 120 45 10 1

N	Coefficients of $(X + 2)^N$
0	1
1	1 2
2	1 4 4
3	1 6 12 8
4	1 8 24 32 16
5	1 10 40 80 80 32

Discrete Structures Equations

Modulo Arithmatic

Divides / $a \mid b$ IFF b=ac — **a divides b** if there exists an integer c such that b=ac

$a\mid b \wedge a\mid c \rightarrow a\mid (b+c)$	If $a\mid b$ and $a\mid c$ then $a\mid (b+c)$
$a\mid b \vee a\mid c \rightarrow a\mid bc$	If $a\mid b$ or $a\mid c$ then $a\mid bc$
$a\mid b \rightarrow a\mid bc$	If $a\mid b$ then $a\mid bc$ for all integer c
$a\mid b \wedge b\mid c \rightarrow a\mid c$	If $a\mid b$ and $b\mid c$ then $a\mid c$

Primes — A positive integer **P** is **Prime** IFF the only positive integer factors of **P** are **P** and **1**

Fundamental Theorem of Arithmatic — Every positive integer can be written uniquely as a product of primes.

Composite — An integer **n** is composite if it has factors other than **n** and **1**

Factors — If n is composite then it has a prime factor x <= √n

Division / **a**=dividend **d**=divisor / **q**=quotient **r**=remainder — If **a** is an integer and **d** is a positive integer then there are unique integers **q** and **r** with **0≤r≤d** such that **a=dq+r**

Greatest Common Divisor / **GCD(a,b)** — If **a** and **b** are integers, both ≠ 0 the largest integer **d** such that $\mathbf{d\mid a}$ and $\mathbf{d\mid b}$ = **GCD(a,b)**

Relative Primes — Integers **a** and **b** are relatively prime if **GCD(a,b)=1**

Pairwise Primes — The integers a_1, a_2,...a_n are Pairwise Relative Primes if GCD(a_i,a_j)=1 for all i≠j

Least Common Multiple / **LCM(a,b)** — If **a** and **b** are integers, both ≠ 0 the smallest positive integer **d** such that $\mathbf{a\mid d}$ and $\mathbf{b\mid d}$ = **LCM(a,b)**

Congruent Mod m / a≡b(mod m) / → $m\mid (a-b)$ / → a(mod m)=b(mod m) / → a=b+km — If a and b are integers and m is a positive integer then we say a is congruent to b mod m IFF $m\mid (a-b)$ or a(mod m)=b(mod m) or a=b+km for some integer k

Congruent Arithmatic / a≡b(mod m)∧c≡d(mod m) / →a+c≡b+d(mod m) / →ac≡bd(mod m) — If m is a positive integer and a, b, c and d are integers then if a≡b(mod m) and c≡d(mod m) then a+c≡b+d(mod m) and ac≡bd(mod m)

Linear Combination — If a and b are positive integers then there exist integers s and t such that GCD(a,b)=sa+tb

GCD Divides — If a, b, c are positive integers such that GCD(a,b)=1 and $a\mid bc$ then $a\mid c$

Prime Divides — If P is Prime and $P\mid a_1a_2a_3...a_n$ where each a_i is an integer then $p\mid a_i$ for some i

Congruent Cancellation — If m is a positive integer and if a, b and c are integers then if ac≡bc(mod m) and GCD(c,m)=1 then a≡b(mod m)

Linear Congruences / aX≡b(mod m) — If a, b are integers and m is a positive integer and X is an integer variable then aX≡b(mod m) is a Linear Congruence

Inverse mod m / sa+tm=1 / →sa+tm=1(mod m) and tm=0(mod m) / →sa=1(mod m) → a^{-1}=s — If a and m are relatively prime integers and m>1 then there is a unique multiplicative inverse a^{-1} mod m such that a*a^{-1}(mod m)=1

Chinese Remainder Thm — If X≡a_1(mod m_1) and X≡a_2(mod m_2) ...X≡a_n(mod m_n) and m_1,m_2...m_n are pairwise relative primes there is a unique solution X(mod m) where m=$m_1m_2...m_n$ and 0≤X≤m

Z_m — If N is a positive integer **Z_n** is {0,1...n-1} plus math mod m

Unit — A number a in **Z_m** is a Unit if a is relatively prime to m (GCD(a,m)=1, a has an inverse)

Eulers Number / ζ(m)=m-$2^{(\#\text{ of prime factors of }m)}$ / ζ(P)=P-1 if P is prime — The number of Units in **Z_m** ζ(m)=∏[1..k](1-P_i) where each P_i is one of the prime factors of m

Eulers Theorem — Suppose a is a unit mod m, then $a^{\zeta(m)}$≡1(mod m)

Little Fermats Theorem — If P is Prime, a^P≡a(mod P) for ALL a

a*b (mod m) = [(a mod m)*(b mod m)] mod m
a+b (mod m) = [(a mod m)+(b mod m)] mod m
a^b (mod m) = [$(a \bmod m)^b$] mod m
n^x≡n (mod x)
$a^{\zeta(x)}$≡1 (mod x)
a^P≡a (mod P)

a*10^n≡a*0	mod 2	[take 1's digit mod 2]
a*10^n≡a*1^n	mod 3	[876 mod 3 = 8+7 mod 3] discard 6 & 3
a*10^n≡a*2^n	mod 4	[876 mod 4 = 2 mod 4] discard 8 & 4 discard even powers of 10: 10^2,10^4,10^6...
a*10^n≡a*0	mod 5	[take 1's digit mod 5]
a*10^n≡a*4^n	mod 6	[765 mod 6 = 1*4^3+5 mod 5] discard 6
a*10^n≡a*3^n	mod 7	[765 mod 7 = 6*3^2+5 mod 7] discard 7
a*10^n≡a*2^n	mod 8	[987 mod 8 = 1*2^3+7 mod 8] discard 8
a*10^n≡a*1^n	mod 9	[98765 mod 9 = 8+7+6+5 mod 9] discard 9
a*10^n≡a*0^n	mod 10	[take 1's digit mod 10]
a*10^n≡a*-1^n	mod 11	[98765 mod 11=9-8+7-6+5 mod 9]

(a+b) mod m = 0 — b is the additive inverse of a mod m
(a*b) mod m = 1 — b is the multiplicative inverse of a mod m The mult. inverse does not always exist.
GCD(a,b)=a*s+b*t — If GCD(a,b)>1 then s ant t exist and are
GCD(a,b)*LCM(a,b)=ab — positive or negative integers.

RSA — Rivest, Shamir, Adelman

P,Q=factors	(1) Pick 2 large Prime Numbers P & Q each > 512 bits long
N=modulus	
n=ζ(N)	(2) Compute N=PQ, n=(P-1)(Q-1)
E=Encryption Key	(3) Pick a number E, > 512 bits long and relatively Prime to n
D=Decryption Key	
M=original message	(4) Compute D=E^{-1} (E inverse mod N)
M_E=encrypted message	(5) Publish N,E on Public key site
Note: ED≡1(mod N)	(6) Encrypt: compute $M_E=(M)^E$(mod N) send to recipient
Public N,E	
Secret P,Q,ζ(N),D	(7) Decrypt: compute $M=(M_E)^D$(mod N)

Truth Tables

P	Q	P∧Q	¬[P∧Q]	P∨Q	¬[P∨Q]	P⊕Q	P→Q	P↔Q
F/0	F/0	F	T	F	T	F	T	T
F/0	T/1	F	T	T	F	T	T	F
T/1	F/0	F	T	T	F	T	F	F
T/1	T/1	T	F	T	F	F	T	T

Quantifier	When true	When False
∀xP(x)	P(x)=T for all x	P(x)=F for any x
∃xP(x)	P(x)=T for any x	P(x)=F for all x
¬∀xP(x) ⇔ ∃x¬P(x)	P(x)=F for any x	P(x)=T for all x
¬∃xP(x) ⇔ ∀x¬P(x)	P(x)=F for all x	P(x)=T for any x

Common Summations

$\sum$i=1 to n[i]	n*(n+1)/2	$(n^2+n)/2$
$\sum$i=1 to n[i^2]	n*(n+1)*(2n+1)/6	$(2n^3+3n^2+n)/6$
$\sum$i=1 to n[i^3]	$n^2*(n+1)^2/4$	
$\sum$i=0 to n[a*X^i]	$\frac{a(X^{n+1}-1)}{(X-1)}$	X≠1
$\sum$i=0 to n[X^i]	$\left(\frac{X^{n+1}-1}{X-1}\right)$	

Order of Complexity

O(f(x)+g(x)) = max(O(f(x)),O(g(x))) O(f(x)) <= f(x)
O(f(x)*g(x)) = O(f(x))*O(g(x)) O(f(g(x)) <= f(g(x))
$(n/2)^{(n/2)}$<n!<n^n

1 O(1)	7 O(x^2)	13 O(e^x) or O(n^x)
2 O(log(log(x))	8 O(x^2*log(log(x))	14 O($e^{x*\log(x)}$)
3 O(log(x))	9 O(x^2*log(x))	15 O($e^{x\wedge 2}$)
4 O(x)	10 O(x^n)	16 O(x!)
5 O(x*log(log(x))	11 O(x^n*log(log(x))	17 O(x^x)
6 O(x*log(x))	12 O(x^n*log(x))	18 O($x^{x\wedge x\{x\text{ superexpt }n\text{ times}\}}$)

Discrete Structures Equations

Methods of Proof

Rule of Inference	Tautology	Method Name
P is TRUE $\therefore P \vee Q$	$P \rightarrow P \vee Q$	Given P is TRUE, by **Addition** $P \vee Q$ is TRUE
P∧Q is TRUE $\therefore P$	$P \wedge Q \rightarrow P$	Given $P \wedge Q$ is TRUE, by **Simplification** P is TRUE
$P \rightarrow Q$ $\therefore \neg Q \rightarrow \neg P$	$(P \rightarrow Q) \Leftrightarrow (\neg Q \rightarrow \neg P)$	Given $P \rightarrow Q$ is TRUE, by **ContraPositive** $\neg Q \rightarrow \neg P$ is TRUE
P Q $\therefore P \wedge Q$	$((P) \wedge (Q)) \rightarrow (P \wedge Q)$	Given P and given Q, by **Conjunction** $P \wedge Q$ is TRUE
P $P \rightarrow Q$ $\therefore Q$	$[P \wedge (P \rightarrow Q)] \rightarrow Q$	Given P and given $P \rightarrow Q$, by **Modes Ponens** Q is TRUE
$\neg Q$ $P \rightarrow Q$ $\therefore \neg P$	$[\neg Q \wedge (P \rightarrow Q)] \rightarrow \neg P$	Given $\neg Q$ and given $P \rightarrow Q$,by **Modes Tollens** P is FALSE
$P \rightarrow Q$ $Q \rightarrow R$ $\therefore P \rightarrow R$	$[P \rightarrow Q \wedge Q \rightarrow R] \rightarrow (P \rightarrow R)$	Given $P \rightarrow Q$ and given $Q \rightarrow R$, by **Hypothetical Syllogism** $P \rightarrow R$ is TRUE
$P \vee Q$ $\neg P$ $\therefore Q$	$[(P \vee Q) \wedge \neg P] \rightarrow Q$	Given $P \vee Q$ and given $\neg P$, by **Disjunctive Syllogism** Q is TRUE
Direct	$[P \rightarrow Q]$	Assume P is TRUE then use the Rules of Inference to show that Q MUST be TRUE
Indirect	$[\neg Q \rightarrow \neg P] \rightarrow [P \rightarrow Q]$	Assume Q is FALSE then use the Rules of Inference to show that $\neg Q \rightarrow \neg P$ is TRUE
Cases	$P \vee Q \vee R \ldots$=TRUE	Prove each case is TRUE
Vacuous	P=FALSE, $\therefore$ $[P \rightarrow Q]$=TRUE	Use Vacuous Proof for special cases (basis)
Trivial	Q=TRUE $\therefore$ $[P \rightarrow Q]$=TRUE	Use Trivial Proof for special cases (basis)
Contradiction	$[\neg P \rightarrow \neg Q]$=TRUE $\therefore$ $\neg P$=FALSE $\therefore$ P=TRUE	Assume P is FALSE and Q is TRUE then use the Rules of Inference to show that given $[\neg Q \rightarrow \neg P]$ that P MUST be FALSE for Q to be TRUE
Induction	Prove P(0) P(1) Prove $p(n) \rightarrow P(n+1)$	(1) Basis: Prove P(0),P(1) (2) Inductive Hypothesis: Assume P(n)=TRUE (3) Inductive Step: use P(n)=TRUE to prove P(n+1)=TRUE
Second Induction Principle	Prove P(0) P(1) Prove $p(1) \wedge \ldots \wedge P(n) \rightarrow P(n+1)$	(1) Basis: Prove P(0),P(1) (2) Inductive Hypothesis: Assume $P(1) \wedge \ldots \wedge P(n)$=TRUE (3) Inductive Step: use $P(1) \wedge \ldots \wedge P(n)$=TRUE to prove P(n+1)=TRUE

Discrete Structures Algorithms

```
procedure FastMultiply(n1,a_{n1-1}...a_0, n2,B=b_{n2-1}...b_0)
  { F(2n)=3f(n)+8n+C }
  n:= log_2(max(n1,n2))/2
  A1:= a_{n1-1}...a_n; A0:=a_{n-1}...a_0; B1:=b_{n2-1}...b_n; B0:=b_{n-1}...b_0;
  if n1 < n2
    pad A1 on the left with n-n1 zeros;
  else
    pad B1 on the left with n-n2 zeros;
  AB1 := ShiftLeft(n, FastMultiply(n, A1, n, B1));
  AB2 := ShiftLeft(n, AB1);
  AB3 := ShiftLeft(n,
         FastMultiply(n, (A1-A0), n, (B1-B0)));
  AB4 := FastMultiply(n, A0, n, B0);
  AB5 := ShiftLeft(n, AB4);
{ AB  := (2^{2n}+2^n)*FastMultiply(n, A1, n, B1)+
         2^n*FastMultiply(n, (A1-A0), n, (B1-B0)) +
         (2^n+1)*FastMultiply(n, A0, n, B0) }
  AB  := AB1+AB2+AB3+AB4+AB5;
return AB;
```

```
procedure GCD(a,b)       {calc. greatest common divisor}
  while b > 0
  begin
      r    := a mod b; a := b; b  := r
  end
return a
```

```
procedure MultInverse(x,m) {calc. x^{-1}(mod m), 0≤x<m,m>1}
      a := x; b  := m; i  := 0
      s_0:= 1; t_0 := 0
      s_1:= 0; t_1 := 1
      while b > 0
      begin
        r   := a mod b; q   := (a-r)/b
        s_{i+2} := s_i-q*s_{i+1}; t_{i+2}  := t_i-q*t_{i+1}
      { Loop Invariant b_i=a_{i+1}=a_0*s_{i+1}+b_0*t_{i+1} }
        a := b; b  := r; i  := i+1
      end
      if a = 1
      then
        return s_{i-1}
      else
        return 0 {no inverse exists}
```

	a	b	q	r	s	t
0	x	m	x div m	x mod m	1	0
1	m	r_0	m div b_1	m mod b_1	0	1
2	b_1	r_1	$a_2$div b_2	$a_2$mod b_2	1	$-q_0$
3	b_2	r_2	$a_3$div b_3	$a_3$mod b_3	$-q_1$	$1+q_0q_1$
4	b_3	r_3	$a_4$div b_4	$a_4$mod b_4	$1+q_1q_2$	$-q_0-q_2(1+q_0q_1)$
5	b_4	r_4	$a_5$div b_5	$a_5$mod b_5	$s_3-q_3s_4$	$t_3-q_3t_4$

```
procedure ChineseRemainder(n, a_1,a_2...a_n, m_1,m_2...m_n)
  m=1; X=0
  for k = 1 to n
    m := m*m_k
  for k = 1 to n
    begin
      M_k := m/m_k; y_k := MultInverse(M_k,m_k); X:=X + a_k*M_k*y_k
    end
return X
```

Algorithms	An Algorithm has:
(1) Input	A finite set of inputs each from a specified set of valid values
(2) Output	A finite set of outputs each to a specified set of valid values
(3) Definiteness	All of the steps must be precisely and completely defined
(4) Correctness	Must produce correct output for every set of valid inputs
(5) Finiteness	Must terminate after a finite (perhaps large) number of steps
(5) Effectiveness	Must take finite number of steps each must be correct and finite
(6) Generality	Must be Correct and Effective for ALL values of the defined input set(s)
(7) Robustness	Must detect and report ALL invalid inputs and NOT attempt to process them

Discrete Structures Equations

Recurrance Relations

Linear, Homogenous, Constant Coefficients, Degree 1

$A_n=c*A_{n-1}$ $A_0=b$ Solution: $A_n=C^n*b$

Other forms:

$A_n=c*A_{n-1}+2$ Non-Homogeneous

$A_n=2n*A_{n-1}$ Non Constant Coefficients

$A_n=c*(A_{n-1})^2$ Non Linear

$A_n=c_1*A_{n-1}+c_2*A_{n-2}$ Degree 2

Non-Homogenous

$A_n=c*A_{n-1}+d$ $A_0=b$ Solution: $A_n=c^n*b+d\left(\frac{c^n-1}{c-1}\right)$

Degree 2

$A_n=c_1*A_{n-1}+c_2*A_{n-2}$ $A_0=b_0$ $A_1=b_1$ $c_2\neq0$ $c_1,c_2\in\Re$

Characteristic Equation $r^2-c_1r-c_2=0$

Use Quadratic Equation $a=1$ $b=-c_1$ $c=-c_2$

to solve for roots:

$\alpha_1=\frac{A_1-A_0r_2}{r_1-r_2}$ $r_1=\frac{c_1+\sqrt{(c_1)^2+4c_2}}{2}$

$\alpha_1=\frac{A_0r_1-A_1}{r_1-r_2}$ $r_2=\frac{c_1-\sqrt{(c_1)^2+4c_2}}{2}$

If $r_1\neq r_2$ then $A_n=\alpha_1r_1^n+\alpha_2r_2^n$

If $r_1=r_2$ then $A_n=\alpha_1r^n+\alpha_2nr^n$

Homogenous Degree k

$A_n=c_1*A_{n-1}+c_2*A_{n-2}+\ldots c_k*A_{n-k}$

$A_0=b_0$ $A_1=b_1$ … $A_{k-1}=b_{k-1}$

$c_k\neq0$ $c_1,c_2\ldots C_k\in\Re$

Characteristic Equation $r^k-c_1r^{k-1}-c_2r^{k-2}\ldots-c_k=0$

Solve the k^{th} order polynomial for the roots of the characteristic equation (try Pascals Triangle), then use the roots and the initial conditions to solve for the coefficients. Each r below is a root of the characteristic equation.

Case #1 $[r^k-c_1r^{k-1}-c_2r^{k-2}\ldots-c_k=0$ has only single roots]
then $H_n=[d_kr_k^n+d_{k-1}r_{k-1}^n\ldots+d_1r_1^n]$

Case #2 $[r^k-c_1r^{k-1}-c_2r^{k-2}\ldots-c_k=0$ multi-root m=multiplicity]
then $H_n=[d_{k-m}r_{k-m}^n+d_{k-m-1}r_{k-m-1}^n\ldots+d_1r_1^n]+[e_{m-1}n^{m-1}-e_{m-2}n^{t-2}\ldots-e_0]r_0^n$

H_n must satisfy the recurrance relation for n=k therefor
$H_k=[d_kr_k^n+d_{k-1}r_{k-1}^n\ldots+d_1r_1^n]=c_1*A_{k-1}+c_2*A_{k-2}+\ldots c_{k-1}*A_1+c_k*A_0$
substitute $[d_kr_k^{n-1}+d_{k-1}r_{k-1}^{n-1}\ldots+d_1r_1^{n-1}]$ for A_{k-1}
substitute $[d_kr_k^{n-2}+d_{k-1}r_{k-1}^{n-2}\ldots+d_1r_1^{n-2}]$ for A_{k-2}
contunue with substitutions for $A_{k-3}\ldots A_0$
Solve the resulting very messy equation for $d_k\ldots d_0$
Rearange the resulting messy equation to group equal roots on the left and all constants on the right then use the constant terms to solve for one of the d's. Use the solved d to solve for the next d and so on.

Non-Homogenous, Degree k

$A_n=c_1*A_{n-1}+c_2*A_{n-2}+\ldots c_k*A_{n-k}+f(n)$ $f(n)=[d_tn^t-d_{t-1}n^{t-1}\ldots-d_0]S^n$

$A_0=b_0$ $A_1=b_1$ … $A_{k-1}=b_{k-1}$ $c_k\neq0$ $c_1,c_2\ldots c_k\in\Re$

1) Find a ANY Particular Solution P_n that satisfies the recurrance relation (do NOT use the initial conditions) try functions that look like f(n)

2) Every solution to the Non-Homogenous recurrance relation has the form $A_n=P_n+H_n$ where H_n is the solution to the Homogenous recurrance relation

3) Solve the Homogenous recurrance relation for H_n

4) Use the initial conditions to define a set of simultaneous linear equations that can be used to find the values of the constants

How to find Particular Solution P_n
Let G(r) be the characteristic Equation of the

Homogenous system $A_n=C_1*A_{n-1}+C_2*A_{n-2}+\ldots C_k*A_{n-k}$
$G(r)=r^k-c_1r^{k-1}-c_2r^{k-2}\ldots-c_k=0$

Case #1 [S is NOT a root of G(r), $G(S)\neq0$]
then $P_n=[e_tn^t-e_{t-1}n^{t-1}\ldots-e_0]S^n$

Case #2 [S IS a root of G(r), G(S)=0, m=multiplicity]
then $P_n=n^m[e_tn^t-e_{t-1}n^{t-1}\ldots-e_0]S^n$

P_n must satisfy the recurrance relation for n=k therefor
$P_k=[e_tk^t-e_{t-1}k^{t-1}\ldots-e_0]S^k=c_1*A_{k-1}+c_2*A_{k-2}+\ldots c_{k-1}*A_1+c_k*A_0+f(k)$
substitute $[e_t(k-1)^t-e_{t-1}(k-1)^{t-1}\ldots^t-e_1(k-1)-e_0]S^{k-1}$ for A_{k-1}
substitute $[e_t(k-2)^t-e_{t-1}(k-2)^{t-1}\ldots^t-e_1(k-2)-e_0]S^{k-2}$ for A_{k-2}
contunue with substitutions for $A_{k-3}\ldots A_0$
$P_k=[e_tk^t-e_{t-1}k^{t-1}\ldots-e_0]S^k=c_1*A_{k-1}+c_2*A_{k-2}+\ldots c_{k-1}*A_1+c_k*A_0+f(k)$
Rearange the resulting messy equation to group equal exponents on the left and all constants on the right then divide both sides by the largest power of S that will go into all of the terms on the left.

Solve the resulting very messy equation for $e_t\ldots e_0$
using $P_k=c_1*b_{k-1}+c_2*b_{k-2}+\ldots c_{k-1}*b_1+c_k*b_0+f(k)$

Divide and Counquer

Linear

$f(n) = a*f(n/b)+c$ where a,b and c are constants

$f(n) = n^{logb(a)}[f(1) + \frac{c}{a-1}] - \frac{c}{a-1}$ $\{a\geq1,\ b\in\aleph,\ c\geq0\}$

Polynomial

$f(n) = a*f(n/b)+c*n^d$ where a,b,c and d are constants

$$f(n) = \begin{cases} O(n^d) & a<b^d \\ O(n^d\log(n)) & a=b^d \\ O(n^{logb(a)}) & a>b^d \end{cases} \qquad \{a\geq1,\ b\in\aleph,\ c\geq0,\ d\geq0\}$$

Logarithmic

$f(n) = a*f(n/b)+\log_c(n)$ where a,b and c are constants

$f(n) = O(n^{logb(a)})$ $\{a\geq1,\ b\in\aleph,\ c\geq0,\ d\geq0\}$

Inclusion-Exclusion

$|A\cup B| = |A|+|B|-|A\cap B|$
$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|B\cap C|-|A\cap C|+|A\cap B\cap C|$
$(A\cup B)^c = A^c\cap B^c$
Given a set $A\in X$
I_A is a function from $A\rightarrow X$ such that:
$I_A(x) = 1$ IFF $x\in A$; 0 otherwise
$I_A(X) = 1$ always
$I_A(\emptyset) = 0$ always
$I_{A\cup B} = I_A+I_B-I_{A\cap B}$
$I_{Ac} = 1-I_A$
$I_{A\cap B} = I_A*I_B$
$I_{B-A} = I_B-I_A$
$S(f) = \sum_{x\in X} [f(x)]$
$S(I_A) = |A|$
$S(f+g)= S(f)+S(g)$

------Statistics Equations------

Product Rule: If a series of independant operations can be performed n1, n2, n3 ... nk ways then the sequence of operations can be performed n1*n2*n2*...*nk number of ways

Permutations: A Permutation is an ordered arraingement of all or part of a set of objects

Combinations: A Combination is an unordered subset of all or part of a set of objects, also, a Combination is a partition of a set into 2 cells with r in cell#1 and n-r in cell#2

Number of Permutations

n distinct objects: $n!$

n taken r at a time: no repetition $\frac{n!}{(n-r)!}$ repetition n^r

n arrainged in a circle: $(n-1)!$

n objects of which n_1=type$_1$, ... n_k=type$_k$: $\frac{n!}{n_1!*n_2!*n_3!...n_k!}$

n yes/no experements: 2^n

Partitioning sets

partition a set of n objects into r cells with n_1 in cell$_1$, n_2 in cell$_2$... n_r in cell$_r$: $\frac{n!}{n_1!*n_2!...n_r!}$

Number of Combinations

n distinct objects taken r at a time: no repetition $\frac{n!}{r!(n-r)!}$ repetition $\frac{(n+r-1)!}{r!(n-1)!}$

Probability in Card Hands

#5 card hands..: $\frac{52!}{5!*(52-5)!} = \frac{52*51*50*49*48}{5*4*3*2} = 2,598,960$

P(Full House)..: $\frac{72}{4,165} = \frac{13*4*3*2*12*4*3}{2,598,960} = \frac{44,928}{2,598,960}$

P(3 of a kind).: $\frac{1}{21} = \frac{13*4*49*48}{2,598,960} = \frac{123,304}{2,598,960}$

P(4 of a kind).: $\frac{1}{4,165} = \frac{13*(52-4)}{2,598,960} = \frac{624}{2,598,960}$

P(Flush).......: $\frac{99}{4,165} = \frac{13*4*12*11*10*9}{2,598,960} = \frac{617,760}{2,598,960}$

P(Royal Flush).: $\frac{1}{649,740} = \frac{4}{2,598,960} = \frac{4}{2,598,960}$

Conditional Probability

$P(B|A) = P(A\cap B)/P(A)$ [P(A) > 0]

$P(A\cap B) = P(A)P(B|A)$

$P(B|A) = \frac{P(B)P(A|B)}{\sum[i=1 \text{ to } k][P(i)P(A|B)]}$

Probability Density Function

Discrete	Continuous
$f(x) \ge 0$	$f(x) \ge 0$
$\sum [f(x)] = 1$ [all $x \in$ domain(f)]	$\int[-\infty...\infty][f(x)]\,dx = 1$
$P(X=x) = f(x)$	$P(X=x) = \lim[dx\to 0]\, f(x)*dx$
	$P(a<X<b) = \int[a...b]f(x)\,dx$
$F(x)=P(X\le x) = \sum [f(t)]$ [all $t \in$ domain(f)|$t\le x$]	$F(x)=P(X\le x) = \int[-\infty...x]f(t)\,dt$

Cumulative Distribution

$F(x)=\sum[x\le X]f(x)$	$F(x)=\int[-\infty...x][f(x)]\,dx$
$P(a<X\le b) = F(b) - F(a)$	$P(a\le X\le b) = F(b) - F(a^-)$
$P(a<X<b) = F(b^-) - F(a)$	$P(a\le X<b) = F(b^-) - F(a^-)$

Joint Probability

$f(x,y) \ge 0$	$f(x,y) \ge 0$
$\sum[x\in X, y\in Y]f(x,y) = 1$	$\int[-\infty...\infty,-\infty...\infty]f(x,y)\,dy\,dx = 1$
$P((X,Y)\in A)$ $=\sum[(x,y)\in A]f(x,y)$	$P((X,Y)\in A)=\int[(x,y)\in A]f(x,y)\,dy\,dx$

Marginal Distribution

$g(x) = \sum[y\in Y]f(x,y)$	$g(x) = \int[-\infty...\infty][f(x,y)]\,dy$
$h(y) = \sum[x\in X]f(x,y)$	$h(y) = \int[-\infty...\infty][f(x,y)]\,dx$
$P(X=k)=g(k)=\sum[y\in Y]f(k,y)$	$P(X=k)=g(k)=\int[-\infty...\infty]f(k,y)\,dy$

Conditional Distribution

$P(y|x)=f(x,y)/g(x)$ [g(x)>0] $\quad P(x|y)=f(x,y)/h(y)$ [h(y)>0]

Statistical Independance

If X and Y are independant: $f(x|y)=g(x)$ $\quad f(y|x)=h(y)$ $\quad f(x,y)=g(x)h(y)$

Mean or Expected Value

$\mu_x=E(X)=\sum[x\in X]xf(x)$	$\mu_x=E(X)=\int[-\infty...\infty]x*f(x)\,dx$
$E[g(X)]=\sum[x\in X]g(x)f(x)$	$E[g(X)]=\int[-\infty...\infty]g(x)*f(x)\,dx$
$E[g(X,Y)]= \sum[x\in X,y\in Y]g(x,y)f(x,y)$	$E[g(X,Y)]= \int[-\infty...\infty,-\infty...\infty]g(x,y)f(x,y)\,dy\,dx$

Variance and Covariance

$\sigma^2_x=E[(X-\mu_x)^2]= E(X^2)-\mu_x^2$ $= \sum[x\in X](X-\mu_x)^2f(x)$	$\sigma^2_x=E[(X-\mu_x)^2]= E(X^2)-\mu_x^2$ $= \int[-\infty...\infty](X-\mu_x)^2f(x)\,dx$
$\sigma^2_{g(x)}=E[g(X)-\mu_{g(x)}]= \sum(g(X)-\mu_{g(x)})^2f(x)$	$\sigma^2_{g(x)}=E[g(X)-\mu_{g(x)}]= \int[-\infty...\infty](g(X)-\mu_{g(x)})^2f(x)\,dx$
$\sigma^2_{XY}=E[(X-\mu_x)(Y-\mu_y)]= \sum(X-\mu_x)(Y-\mu_y)f(x,y)$	$\sigma^2_x=E[(X-\mu_x)(Y-\mu_y)]= \int[-\infty...\infty,-\infty...\infty](X-\mu_x)(Y-\mu_y)f(x,y)\,dy\,dx$

$\sigma^2_{XY}=0$ if X and Y are independant random variables

Linearity of Expectation

$E[g(X)\pm h(X)]=E[g(X)]\pm E[h(X)]$

$E[g(X,Y)\pm h(X,Y)]=E[g(X,Y)]\pm E[h(X,Y)]$

$E[g(X)\pm h(Y)]=E[g(X)]\pm E[h(Y)]$

$E[X+Y]=E[X]+E[Y]$	$E[aX+b]=a^2\sigma^2_x=a^2E[(X-\mu_x)^2]$
$E[X-Y]=E[X]-E[Y]$	$E[aX]=a^2\sigma^2_x=a^2E[(X-\mu_x)^2]$
$E[X*Y]=E[X]*E[Y]$	$E[X+b]=\sigma^2_x=E[(X-\mu_x)^2]$
$E[X/Y]=E[X]/E[Y]$	$E[aX+bY]=a^2\sigma^2_x+b^2\sigma^2_Y$

Linearity of Variance and Covariance

$\sigma^2_x=E(X^2)-\mu_x^2$

$\sigma^2_{aX+b}=a^2\sigma^2_X$	$\sigma^2_{aXY}=a^2\sigma^2_{XY}$
$\sigma^2_{aX+bY}=a^2\sigma^2_X+b^2\sigma^2_Y$	$\sigma^2_{aX+bY}=a^2\sigma^2_X+b^2\sigma^2_Y$
$\sigma^2_{aX/bY}=a^2\sigma^2_X/b^2\sigma^2_Y$	$\sigma^2_{aX-bY}=a^2\sigma^2_X+b^2\sigma^2_Y$ << note: $(-b)^2=+b^2$

Standard Deviation and Correlation Coefficient

$\sigma_x=\mathrm{StdDev}(x)=+\sqrt{\sigma^2_x}$ $\quad \rho_{xy}=\mathrm{Corr}(x,y)=\sigma_{xy}/\sigma_x\sigma_y$

Chebychev's Theorem

$P(\mu-k\sigma<X<\mu+k\sigma)\ge 1-(1/k^2)$ $\quad P(|X-\mu|\ge e)= \sigma^2/e^2$

Functions of Random Variables

Given X is a continuous random variable with a distribution function X=f(x),where f(x) is a 1 to 1 function, and given Y=U(x) with inverse function X=W(y) then the distribution of Y is:

$$G(y) = f[W(y)]\,|D_y[w(y)]|$$

$$P(a<Y<b) = P[W(a)<X<W(b)]= \int_{W(a)}^{W(b)} f(x)\,dx = \int_a^b f[W(y)]\,|D_y[w(y)]|\,dy$$

Moments and Moment Generating Functions

$$\mu_x' = \sum_{x=-\infty}^{+\infty} x^r f(x) = E(x^r) \qquad \mu_x' = \int_{-\infty}^{+\infty} x^r f(x)\,dx \text{ or } \sum_{-\infty}^{+\infty} x^r f(x) = E(x^r)$$

$$M_x(t)=\sum_{x=-\infty}^{+\infty} e^{tx}f(x) = E(e^{tx}) \qquad \mu_x' = \int_{-\infty}^{+\infty} e^{tx}f(x)\,dx \text{ or } \sum_{-\infty}^{+\infty} x^r f(x) = E(e^{tx})$$

If 2 seperate distribution functions have the same moment generating function for all values of t over any interval that includes zero then the 2 functions have the same distribution

$M_{X+b}(t)=e^{bt}M_X(t)$	If X is normal distribution:
$M_{aX}(t)=M_X(at)$	$X = \sigma Z+\mu$
$M_{aX+b}(t)=e^{bt}M_X(at)$	$M_{\sigma Z+\mu}(t)=e^{\mu t}M_Z(\sigma t)=e^{\mu t}\, e^{1/2(\sigma^2 t^2)}$

If X_1, x_2...X_n are independant random variables with moment generating functions $M_{X1}(t)$, $M_{X2}(t)$...$M_{Xn}(t)$ then

$Y=X_1+x_2+...+X_n \Rightarrow M_Y(t)=M_{X1}(t)*M_{X2}(t)*...*M_{Xn}(t)$

$Y=a_1X_1+a_2x_2+...+a_nX_n \Rightarrow M_Y(t)=M_{X1}(a_1t)*M_{X2}(a_2t)*...*M_{Xn}(a_nt)$

Discrete Uniform Distribution

$f(x) = \frac{1}{B-A}$ $\quad$ {A≤x≤B} $\quad$ 0 elsewhere

$f(x;k)=\frac{1}{k}$ $\quad k=B-A$ $\quad$ {k=1,2,3...} 0 elsewhere

$\mu_x=E(X)=\frac{A+B}{2}$ $\quad \sigma^2_x=\frac{(B-A)^2}{12}$

$P(a<x<b) = (b-a)\,\frac{1}{B-A}$ $\quad$ {A<a<b<B}

$M_x(t)=\frac{e^t(1-e^{kt})}{k(1-e^t)} = \frac{e^t(1-e^{t(B-A)})}{(B-A)(1-e^t)}$

Continuous Uniform Distribution

$f(x;A,B)=\frac{1}{B-A}$ $\quad$ A≤x≤B, 0 elsewhere $\quad \mu_x = \frac{A+B}{2}$ $\quad \sigma^2_x = \frac{(B-A)^2}{12}$

------Statistics Equations------

Binomial Distribution

Given n Bernoulli trials with P(success)=P and P(failure)=Q=1-P

Each trial is independant and done With Replacement

The Binomial Distribution $b(x;n,p)=[n!/(x!(n-x)!)]p^xQ^{n-x}$

$\sum[x=0..n]b(x;n,p)=1$	$P(X<r)=\sum[x=0..r]b(x;n,p)$
$\mu_x=E(X)=nP$	$P(X\ge r)=1-P(x<r)=1-\sum[x=0..r]b(x;n,p)$
$\sigma^2_x=nPQ$ $\quad \sigma_x=+\sqrt{(nPQ)}$	$P(a<X<b)=\sum[x=a..b]b(x;n,p)$

$M_x(t)=(Pe^t+Q)^n$

Multinomial Distribution

n trials, k catagories with P(success)=P_i and P(failure)=Q_i

$\sum[i=1..k]x_i=k$ $\qquad \sum[i=1..k]P_i=1$

$f(n,x_1,x_2,...x_n;P_1,P_2,...P_n) = [n!/(x_1!*x_2!*...x_n!)]P_1^{x1}*P_2^{x2}*...P_n^{xn}$

Hypergeometric Distribution

Trials done Without Replacement, X=# of successes in a sample of size n selected from N items, k of N labeled success, N-k of N labeled failure; h(x;N,n,k)=P(X=# of successes in sample)

$$h(x;N,n,k)=\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} = \frac{k!}{x!(k-x)!}\;\frac{(N-k)!}{(n-x)!(N-k-n+x)!}\;\frac{n!(N-n)!}{N!}$$

$\mu_x=E(X)=\frac{nk}{N}$ $\qquad \sigma^2_x= \frac{N-n}{N-1} *n* \frac{k}{N}\left(1-\frac{k}{N}\right)$ $\quad M_x(t)=e^{tx}$

When N is large and n is much smaller than N

$(N-n)/(N-1) \to 1$ $\qquad P\approx k/N,\ Q\approx(1-k/N)$

$\mu_x=np=\frac{nK}{N}$ $\qquad \sigma^2_x=nPQ=n*\frac{K}{N}*(1- \frac{k}{N})$

Negative Binomial Distribution

Given Bernoulli trial with P(success)=P and P(failure)=Q=1-P

Each trial is independant and done With Replacement

X=# of trials on which the k^{th} success occurs

$$b^*(x;k,p)= \binom{x-1}{k-1}P^kQ^{x-k} = \frac{(x-1)!P^kQ^{x-k}}{(k-1)!(x-k)!}$$

$\mu_x=1/P$ $\qquad \sigma^2_x=(1-P)/P^2$

$P(X<r)=\sum[x=0..n]b^*(x;k,p)$

Geometric Distribution

Negative Binomial Distribution with k=1

$b^*(x;1,p)=PQ^{x-1}$ $\quad \mu_x=1/P$ $\quad \sigma^2_x=(1-P)/P^2$ $\quad M_x(t)=\frac{Pe^t}{1-Qe^t}$

Poisson Distribution

X=# of events occurring in a given time; the #of events in an interval is independant of other intervals; P(event) occurring in a short interval is proportional to the length of the interval; P(multiple events) occurring in a short interval is small

$X=p(x;\lambda t)=\frac{e^{-\lambda t}(\lambda t)^x}{x!}$

$\mu_x=\lambda t$ $\qquad \sigma^2_x=\lambda t$

$$F(a) = \sum_{x=0}^{a} \frac{e^{-\lambda t}(\lambda t)^x}{x!} = \frac{\Gamma(a+1,\lambda t)}{a!}$$

$$P(a\le x\le b) = \sum_{x=a}^{b} \frac{e^{-\lambda t}(\lambda t)^x}{x!} = \frac{\Gamma(b+1,\lambda t)}{b!} - \frac{\Gamma(a,\lambda t)}{(a-1)!}$$

$M_x(t)=e^{\mu(e^t-1)}$

NOTE: The Poisson Distribution is a Summation, not an Integral

Poisson Approximation to the Binomal

when $n\to\infty$, $p\to 0$, $\mu_x=np$ remains constant, $b(x;n,p)\to p(x;\mu)$

Normal Distribution

X=distribution of values about the mean.

The Normal Distribution is completely defined by μ_x and σ_x

$$X = f(x) = n(x;\mu,\sigma)= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$p(a<X<b)= \int_a^b n(x;\mu,\sigma)\,dx = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$$

Let $Z = \frac{X-\mu}{\sigma}$ $\quad z_1= \frac{a-\mu}{\sigma}$ $\quad z_2= \frac{b-\mu}{\sigma}$

$$Z = f_Z(z) = n(z;0,1)= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$p(a<X<b)= \int_{z1}^{z2} n(x;\mu,\sigma)\,dz = \frac{1}{\sqrt{2\pi}}\int_{z1}^{z1} e^{-\frac{z^2}{2}}dz = \int_{z1}^{z1} n(z;0,1) = P(z_1<Z<z_2)$$

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx = \frac{\left[\mathrm{Erfi}\left(\frac{z_2}{\sqrt{2}}\right)-\mathrm{Erfi}\left(\frac{z_1}{\sqrt{2}}\right)\right]}{2\sqrt{\pi}}$$

Use Normal Table to compute $p(a<X<b) = P(z_1<Z<z_2) = \Phi(z_2)-\Phi(z_1)$ for Z values from -3.49 to +3.49. For other values integrate n(z;0,1) to find $P(z_1<Z<z_2)$

$M_x(t) = \exp\left(\frac{\mu_x t+\sigma_x^2+t^2}{2}\right)$

If X_1, x_2...X_n are independant random variables with Normal distribution with means $\mu_{X1},\mu_{X2},...\mu_{Xn}$ and variances $\sigma^2_{X1},\sigma^2_{X2},...\sigma^2_{Xn}$ then the random variable Y has mean and variance:

$\bar{X}=X_1+x_2+...+X_n$	$M_{\left[\frac{X-\mu}{\sigma/\sqrt{n}}\right]}=1+\mu t+(\sigma^2+\mu^2)\,\frac{t^2}{2}+O(t^3)$
$Y=a_1X_1+a_2x_2+...+a_nX_n$	
$\mu_Y=a_1\mu_{X1}+a_2\mu_{X2}+...a_n\mu_{Xn}$	$\sigma^2_Y=a_1^2\sigma^2_{X1}+a_2^2\sigma^2_{X2}+...a_n^2\sigma^2_{Xn}$

Normal Approximation to the Binomial

If X is a binomial random variable, then the limit of the binomial with mean μ_x=nP and variance σ^2_x=nPQ as $n\to\infty$ is:

$Z = \frac{X-np}{\sqrt{npq}}$ $\quad b(x;n,p) \to n(z;0,1)$

The Normal Approximation is "Good Enough" when n > 30.

Gamma Function

$\Gamma(\alpha)= \int_0^{+\infty} x^{\alpha-1}e^{-x}\,dx = (\alpha-1)\Gamma(\alpha-1)$ $\quad \{\alpha>0\}$ $\quad$ 0 elsewhere

Γ(n integer) = (n-1)!		{n>0}			
$\Gamma(n+0.5) = (n-1)!\sqrt{\pi}$		{n>0}			
Γ(x real) = $(\lceil x\rceil-2)!*\Gamma(x-\lfloor x\rfloor)$		{x>1}			
$\Gamma(0)$ = undefined					
$\Gamma(0.5) = \sqrt{\pi}$	= 1.7725	$\Gamma(2)$	= 1!	= 1	
$\Gamma(1) = 0!$	= 1	$\Gamma(2.5)$	= $\sqrt{\pi}$	= 1.7725	
$\Gamma(1.5) = \sqrt{\pi}$	= 1.7725	$\Gamma(3)$	= 2!	= 2	

Incomplete Gamma Function

$\Gamma(\alpha,\beta)= \int_\beta^{+\infty} x^{\alpha-1}e^{-x}\,dx$ $\quad \{\alpha,\beta>0\}$ $\quad$ 0 elsewhere

Use the Incomplete Gamma table at the back of the book.

Gamma Distribution

Time to the n^{th} Poisson event occurring with arrival rate λ

$\alpha=n$ $\qquad \beta=1/\lambda$

$f(x) = \frac{1}{\beta^\alpha\Gamma(\alpha)}X^{\alpha-1}e^{-x/\beta}$ $\quad \{\alpha,x>0\}$ $\quad$ 0 elsewhere

$\mu = \alpha\beta$ $\qquad \sigma^2 = \alpha\beta^2$ $\quad \sigma = \beta\sqrt{\alpha}$

$M_x(t)=1+\mu t+\beta(1-t)^{-\alpha}-\beta$

$$P(a<X<b) = \int_a^b \frac{1}{\beta^\alpha\Gamma(\alpha)}X^{\alpha-1}e^{-x/\beta}\,dx = \frac{1}{\Gamma(\alpha)}\left(\Gamma\left(\alpha,\frac{a}{\beta}\right)-\Gamma\left(\alpha,\frac{b}{\beta}\right)\right) =$$

$$\frac{1}{2\,\mathrm{Gamma}\,\alpha}\Big(e^{-\mathrm{Log}\,\alpha\,\beta}\Big(2\,\mathrm{Gamma}\Big[0,\ a\Big(-\mathrm{Log}\ \alpha+\frac{1}{\beta}\Big)\Big] - 2\,\mathrm{Gamma}\Big[0,\ b\Big(-\mathrm{Log}\ \alpha+\frac{1}{\beta}\Big)\Big] + \mathrm{Log}[\beta/(a(-1+\mathrm{Log}\ \alpha\,\beta))] - \mathrm{Log}[\beta/(b(-1+\mathrm{Log}\ \alpha\,\beta))] + 2\,\mathrm{Log}\Big[-\frac{1}{\beta}(a(-1+\mathrm{Log}\ \alpha\,\beta))\Big] - \mathrm{Log}\Big[\frac{1}{\beta}(a(-1+\mathrm{Log}\,\alpha\,\beta))\Big] - 2\,\mathrm{Log}\Big[-\frac{1}{\beta}(b(-1+\mathrm{Log}\,\alpha\,\beta))\Big] + \mathrm{Log}\Big[\frac{1}{\beta}(b(-1+\mathrm{Log}\,\alpha\,\beta))\Big]\Big)\Big)$$

Exponential Distribution

Gamma Distribution with α=1 is an Exponential Distribution and β is the Mean Time Between Events or Mean Time to First Event

$X = f(x) = \frac{1}{\beta}e^{-x/\beta}$ $\quad$ {x>0} $\quad$ 0 elsewhere

$\mu = \beta$ $\qquad \sigma^2 = \beta^2$ $\quad \sigma = \beta$

$$P(a<X<b) = \int_a^b \frac{1}{\beta}e^{-x/\beta}\,dx = \frac{-\beta e^{\left(\frac{-a}{b}-\frac{a+b}{\beta}\right)}+\beta e^{\left(\frac{-a}{b}\right)}}{\beta} = e^{\left(\frac{-a}{\beta}\right)} - e^{\left(\frac{-b}{\beta}\right)}$$

$M_x(t)= e^{tx}$

Statistics Equations

Chi-Squared Distribution

Used in Statistical Inference, Sampling Distributions, Analysis of variance, and parametric Statistics

$\alpha=\nu/2$ {ν is a positive integer} $\beta=2$

$$X = \chi^2(\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} X^{\nu/2-1}e^{-x/2} \quad \{x>0\} \quad 0 \text{ elsewhere}$$

$$\mu = \nu \qquad \sigma^2 = 2\nu \qquad \sigma = \sqrt{2}\sqrt{\nu}$$

$$M_x(t) = (1-2t)^{-\nu/2}$$

$$P(a<x<b) = \frac{2(\Gamma(\nu/2,a/2)-\Gamma(\nu/2,b/2))}{\Gamma(\nu)} = \chi^2_\nu(b) - \chi^2_\nu(a)$$

If $X_1, x_2 \ldots X_n$ are independant random variables with Chi-Squared distributions with degrees of freedom $\nu_{X1}, \nu_{X2}, \ldots \nu_{Xn}$ then the random variable $Y=X_1+x_2+\ldots+X_n$ has a Chi-Squared distribution with degree of freedom $\nu_Y=\nu_{X1}+\nu_{X2}+\ldots\nu_{Xn}$

$Y=X_1+x_2+\ldots+X_n$ $\qquad \nu_Y=\nu_{X1}+\nu_{X2}+\ldots\nu_{Xn}$

$f(y)=\chi^2(\nu_Y)$ $\qquad \sigma^2=2\nu_Y$

$\mu_Y=\nu_Y$ $\qquad \sigma=\sqrt{2}\sqrt{\nu_Y}$

Lognormal Distribution

The randomm variable X has a Lognormal Distribution if the random variable Y=ln(X) has a normal distribution with mean μ and standard deviation σ

$$X = f(x) = \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} \quad \{\alpha>0, x\geq 0\} \ 0 \text{ elsewhere}$$

$$\mu_{ln} = e^{(\mu+1/2\sigma^2)} \qquad \sigma_{ln}^2 = e^{(2\mu+\sigma^2)}$$

Let $Z = \frac{\ln(X)-\mu}{\sigma}$ $\quad z_1 = \frac{\ln(a)-\mu}{\sigma}$ $\quad z_2 = \frac{\ln(b)-\mu}{\sigma}$

$P(X>a) = 1-P(X\leq a) = \Phi(z_1)$

$P(a<X<b) = P(z_1<Z<z_2) = \Phi(z_2)-\Phi(z_1)$

Use Table A3 to compute $\Phi(z)$ for Z values from -3.49 to +3.49.
For other values integrate $n(z;0,1)$ to find $P(z_1<Z<z_2)$

Weibull Distribution

X=Time to Failure or Life Length

$$X = f(x) = \alpha\beta x^{\beta-1}e^{-\sigma x^\beta} \quad \{\alpha, x>0\} \quad 0 \text{ elsewhere}$$

$$\mu = \alpha^{-1/\beta}\Gamma(1+1/\beta) \qquad \sigma^2 = \alpha^{-1/\beta}[\Gamma(1+2/\beta)-\Gamma(1+1/\beta)^2]$$

$$P(a<X<b) = \frac{e^{(\alpha b^\beta)} - e^{(\alpha a^\beta)}}{e^{(\alpha b^\beta)} * e^{(\alpha a^\beta)}}$$

$$F(t) = \int_0^t f(x)\,dx = \alpha t^\beta e^{-\alpha x^\beta}$$

$$R(t) = P(T>t) = 1-F(t) = 1 - \alpha t^\beta e^{-\alpha x^\beta}$$

The conditional probability that a component will fail in the interval from $T=t$ to $T=t+\Delta t$, given that it survived to time t

$$P(t<T<t+\Delta t) = \frac{F(t+\Delta t)-F(t)}{1-F(t)}$$

The Failure Rate of the component

$$Z(t) = \frac{f(t)}{R(t)} = \alpha\beta t^{\beta-1} \quad \{t>0\}$$

If and only if the time to failure has a Weibull Distribution

Beta Function

$\{\alpha,\beta\}\in\Re$ $\qquad \{\alpha,\beta\}\in$positive integers

$$\text{Beta}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha,\beta)} \qquad \text{Beta}(\alpha,\beta) = \frac{(\alpha-1)!(\beta-1!)}{\Gamma(\alpha,\beta)}$$

Incomplete Beta Function

$$\text{Beta}(t,\alpha,\beta) = \int_0^t x^{\alpha-1}(1-x)^{\beta-1}\,dx \quad \{0<t<1\} \ 0 \text{ elsewhere}$$

Beta Distribution

The probability that a component will fail in a specified time interval.

$$f(x) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)x^{\alpha-1}(1-x)^{\beta-1} \quad \{0<x<1\} \quad 0 \text{ elsewhere}$$

$\{0<a<1\}$

$\{0<b<1\}$

$$F(t) = \int_0^t f(x)\,dx = \frac{(\alpha+\beta)\text{Beta}(t,\alpha,\beta)}{\Gamma(\alpha,\beta)}$$

$$P(a<X<b) = F(b)-F(a) = \frac{(\alpha+\beta)[\text{Beta}(b,\alpha,\beta)-\text{Beta}(a,\alpha,\beta)]}{\Gamma(\alpha,\beta)}$$

Erlang Distribution

Given a Exponential Distribution where α is a positive integer n and β is the Mean Time Between Events, then the Erlang Distribution is the distribution of the time until the n^{th} exponentially distributed event occurs.

$$f(x) = \frac{x^{n-1}e^{-x/\beta}}{\beta^n(n-1)!} \quad \begin{Bmatrix} 0<x<1 \\ 0<n \\ 0<\beta \end{Bmatrix} \quad 0 \text{ elsewhere}$$

$$\mu = \frac{\beta\,[n!-\Gamma(n+1,\beta^{-1})]}{(n-1)!} \qquad \sigma^2 = \frac{\beta^2[(n+1)!-\Gamma(n+2,\beta^{-1})]}{(n-1)!}$$

$$P(a<X<b) = \int_a^b \frac{x^{n-1}e^{-x/\beta}}{\beta^n(n-1)!}\,dx = \frac{1}{(n-1)!}\Gamma\left(n,\frac{a}{\beta}\right)-\Gamma\left(n,\frac{b}{\beta}\right)$$

T Function

Given $X_1, x_2, \ldots, X_n$ where each X_i is a sample of an Independant Identical Distribution $f(x;\ \mu,\ \sigma)$; f=any distribution

$$T=\bar{X}=(1/n)\sum_{i=1}^{n}[X_i]$$

$$\mu_T=E(\bar{X})=\mu_X \qquad \sigma_T^2 = \frac{\sigma_X^2}{n} \qquad \sigma_T = \frac{\sigma_X}{\sqrt{n}}$$

$$Z=\frac{\bar{X}-\mu_x}{\sigma_X/\sqrt{n}} \qquad Z_1 = \frac{a-\mu_x}{\sigma_X/\sqrt{n}} \qquad Z_2 = \frac{b-\mu_x}{\sigma_X/\sqrt{n}}$$

$$P(a<\bar{X}<b) = P(Z_1<Z<Z_2) = \Phi(Z_2) - \Phi(Z_1)$$

S^2 Function

$$S^2 = \frac{n(\sum_{i=1}^{n}[X_i^2])-(\sum_{i=1}^{n}[X_i])^2}{(n-1)} = \frac{\sum_{i=1}^{n}[X_i-\bar{X}]^2}{(n-1)}$$

$$E(S^2) = \frac{n\sum_{i=1}^{n}[E(X_i^2)]-(\sum_{i=1}^{n}[n^2X_i^2])=\sigma_X^2}{n(n-1)}$$

$$\sum_{i=1}^{n}\frac{[X_i-\mu_x]^2}{\sigma_X^2} = \frac{(n-1)S^2}{\sigma_X^2} + \frac{(\bar{X}-\mu_x)^2}{\sigma_X^2/n} \qquad \sum_{i=1}^{n}\frac{[X_i-\bar{X}]^2}{\sigma_X^2} = \chi^2_{(n-1)}$$

$$\sum_{i=1}^{n}\frac{[X_i-\mu_x]^2}{\sigma_X^2} = \chi^2_n \qquad \frac{(n-1)S^2}{\sigma_X^2} = \chi^2_{(n-1)} \qquad \frac{(\bar{X}-\mu_x)^2}{\sigma_X^2/n} = \chi^2_1$$

100(1-α) confidence intervals

$$P(S^2>b) = P\left(\frac{(n-1)S^2}{\sigma^2}>\frac{(n-1)b}{\sigma^2}\right) = P\left(\frac{(n-1)S^2}{\sigma^2}>\left[\chi^2_{n-1}(\alpha)=\frac{(n-1)b}{\sigma^2}\right]\right)$$

$$P(a<S^2<b) = P\left(\frac{(n-1)a}{\sigma^2}<\frac{(n-1)S^2}{\sigma^2}<\frac{(n-1)b}{\sigma^2}\right) = P(\chi^2_{n-1}(\alpha_a)<\chi^2_{n-1}<\chi^2_{n-1}(\alpha_b))$$

$\chi^2_{n-1}(\alpha_a)=\frac{(n-1)a}{\sigma^2}$ $\quad \chi^2_{n-1}(\alpha_b)=\frac{(n-1)b}{\sigma^2}$ $\quad$ Lookup α_a and α_b in χ^2 table

Student T Distribution

If $\bar{X}$ is a is an estimated mean where $X_1, x_2, \ldots, X_n$ are known but μ and/or σ may be unknown then:

$$T = \frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{(\bar{X}-\mu)\sqrt{n}}{\sqrt{S^2}} \qquad Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \qquad V = \frac{(n-1)S^2}{\sigma^2}$$

If Z is a standard normal variable and V is a Chi-squared variable with ν degrees of freedom, then the random variable T where:

$$T = \frac{Z}{\sqrt{(V/\nu)}} = \frac{Z}{\sqrt{(V/(n-1))}} = \frac{(\bar{X}-\mu)/(\sigma/\sqrt{n})}{\sqrt{(S^2/\sigma^2)}} = \frac{(\bar{X}-\mu)\sqrt{n}}{\sqrt{S^2}}$$

has the T distribution with $\nu=n-1$ degrees of freedom:

$$T_\nu = h(t) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\pi}\sqrt{\nu}}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}$$

$$T_\nu(t) = h(t) = \frac{1}{\sqrt{\nu}}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \quad \{-\infty<t<+\infty\} \ \{\nu>0,\ \nu \text{ is an even integer}\}$$

$$T_\nu(t) = h(t) = \frac{(\nu-1)}{2\pi\sqrt{\nu}}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \quad \{-\infty<t<+\infty\} \ \{\nu>0,\ \nu \text{ is an odd integer}\}$$

100(1-α) confidence intervals

$$P(a<T<b) = P\left(a<\frac{(\bar{X}-\mu)\sqrt{n}}{S}<b\right) = P\left(-\bar{X}+a<\frac{-\mu\sqrt{n}}{S}<-\bar{X}+b\right) =$$

$$P\left(\bar{X}-b<\frac{\mu\sqrt{n}}{S}<\bar{X}-a\right) = P\left(\frac{(\bar{X}-b)S}{\sqrt{n}}<\mu<\frac{(\bar{X}-a)S}{\sqrt{n}}\right) = P(T_n(\alpha_b)<\mu<T_n(\alpha_a))$$

$T_n(\alpha_a)=\frac{(X-a)S}{\sqrt{n}}$ $\quad T_n(\alpha_b)=\frac{(X-b)S}{\sqrt{n}}$ $\quad$ Lookup α_a and α_b in T table

F Distribution

Given 2 independant random variables V_1 and V_2, each with Chi-Squared distributions with degrees of freedom ν_1 and ν_2, the Random variable:

$$F = \frac{V_1/\nu_1}{V_2/\nu_2}$$

has the F distribution with ν_1 and ν_2 degrees of freedom:

$$h(f) = \frac{\Gamma[(\nu_1+\nu_2)/2]\ (\nu_1/\nu_2)^{\nu 1/2}\ f^{(\nu 1/2-1)}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)(1+\nu_1 f/\nu_2)^{((\nu 1+\nu 2)/2)}} \quad \{f>0\}$$

The F Distribution depends on ν_1 and ν_2 and also on the order in which ν_1 and ν_2 are specified

$f_\alpha(\nu_1,\nu_2)$ = the F-value above which the F Distribution with degrees of freedom (ν_1,ν_2) has an area α

$f_{1-\alpha}(\nu_1,\nu_2) = 1/f_\alpha(\nu_2,\nu_1)$

Given 2 random samples with normal distributions with sample sizes n_1 and n_2 and variances σ^2_1 and σ^2_2

$$\chi^2_{n1} = \frac{(n_1-1)S^2_1}{\sigma^2_1} \qquad \chi^2_{n2} = \frac{(n_2-1)S^2_2}{\sigma^2_2}$$

$$F = \frac{S^2_1/\sigma^2_1}{S^2_2/\sigma^2_2} = \frac{\sigma^2_2 S^2_1}{\sigma^2_1 S^2_2}$$

Central Limit Theorem

Given n random Independant Identicaly Distributed samples with mean μ and variance σ^2 then Z is a good approximation for $n\geq 30$

$$Z=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \quad Z_a=\frac{a-\mu}{\sigma/\sqrt{n}} \quad Z_b=\frac{b-\mu}{\sigma/\sqrt{n}} \qquad \bar{X}=X_1+X_2+X_3\ldots X_n$$

$$P(a<\bar{X}<b) = P(Z_a<Z<Z_b) \xrightarrow[n\to\infty]{d} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^{\wedge}2}{2}}dz = \Phi(b)-\Phi(a)$$

$$M_{\left[\frac{X-\mu}{\sigma/\sqrt{n}}\right]} = e^{\frac{-\mu t\sqrt{n}}{\sigma}}\left[M_{X1}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n \qquad M_{X1}(t)=1+\mu t+(\mu^2+\sigma^2)t^2+O(t^3)$$

2 sample Central Limit Theorem

Given 2 random Independant Identicaly Distributed samples providing that both $n_1\geq 30$ and $n_2\geq 30$ or both X_1 and X_2 are approximately normal distributions:

$\text{Sample}_1=\bar{X}_1,\ \mu_1,\ \sigma_1,\ n_1$

$\text{Sample}_2=\bar{X}_2,\ \mu_2,\ \sigma_2,\ n_2$

$$Z\approx\frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{\sqrt{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)}}$$

Continuity Theorem

If Y_n is a discrete or continuous distribution such that the cumulative distribution F_{Yn} converges to F_Y for Y a continuous random variable then Y_n converges to Y in distribution

IF $Y_n \xrightarrow{d} Y$

$$P(a<Y_n<b) = F_{Yn}(b)-F_{Yn}(a) \Rightarrow P(a<Y<b) = F_Y(b)-F_Y(a)$$

1 Sided and 2 Sided Analysis

Given a sample and a stated claim about the μ or σ compute:

$$M_{\frac{(n-1)s^{\wedge}2}{\sigma^2}} = (1-2t)^{-1/2(n-1)} = \chi^2_{(n-1)}$$

Lookup the values for $\chi^2_{(n-1)}$, accept the claim if the computed mean or std-dev. is within the 95% confidence range (for the 1 Sided Test the value must be < table entry for column 0.05, for the 2 Sided Test the value must be between the entries for column 0.025 and 0.975)

Estimators

$$\hat{\Theta} = \frac{X_1+x_2+\ldots+X_n}{n} \qquad \{\text{minimum variance estimator}\}$$

Hypothesis about μ or σ

Given any test statistic such as $\bar{X}$ as an estimate of μ or S^2 as an estimate of σ^2, state a hypothesis H_0 about the value of the test statistic and an alternative hypothesis H_1. H_0 and H_1 are boolean expressions that relate the test statictic to μ, σ^2 or some value that is used to determine μ or σ^2 i.e. the success probability P for a binomial distribution

If the test for H_0 passes, then accept the hypothesis H_0 and reject H_1 else reject H_0 and accept the alternative hypothesis H_1

Type-1 Error

The probability of a Type-1 Error is P(Reject H_0, when H_0=true)

The Significance of a hypothesis is P(Type-1 Error)

Type-2 Error

The probability of a Type-2 Error is P(Accept H_0, when H_0=false)

P-Values

Given X is any estimation of the true mean μ, then the P-value for X is the minimum value of α such that the equations $(X-Z_{\alpha/2})\sigma/\sqrt{n}<\mu<(X+Z_{\alpha/2})\sigma/\sqrt{n}$ and $X-k\sigma<\mu<X+k\sigma$ are true where the values $-Z_{\alpha/2}=-\Phi(-k\sigma)$ and $+Z_{\alpha/2}=1-\Phi(+k\sigma)$

Minimum Sample Size

The minimum sample size n such that the probability that the difference between the sample mean and the true mean is within an error limit e is 100(1-α)

$$n=\left\lceil\left(\frac{Z_{\alpha/2}\sigma}{e}\right)^2\right\rceil$$