Internal Waves in the Ocean

Master 2 — Physique de l'Océan et du Climat

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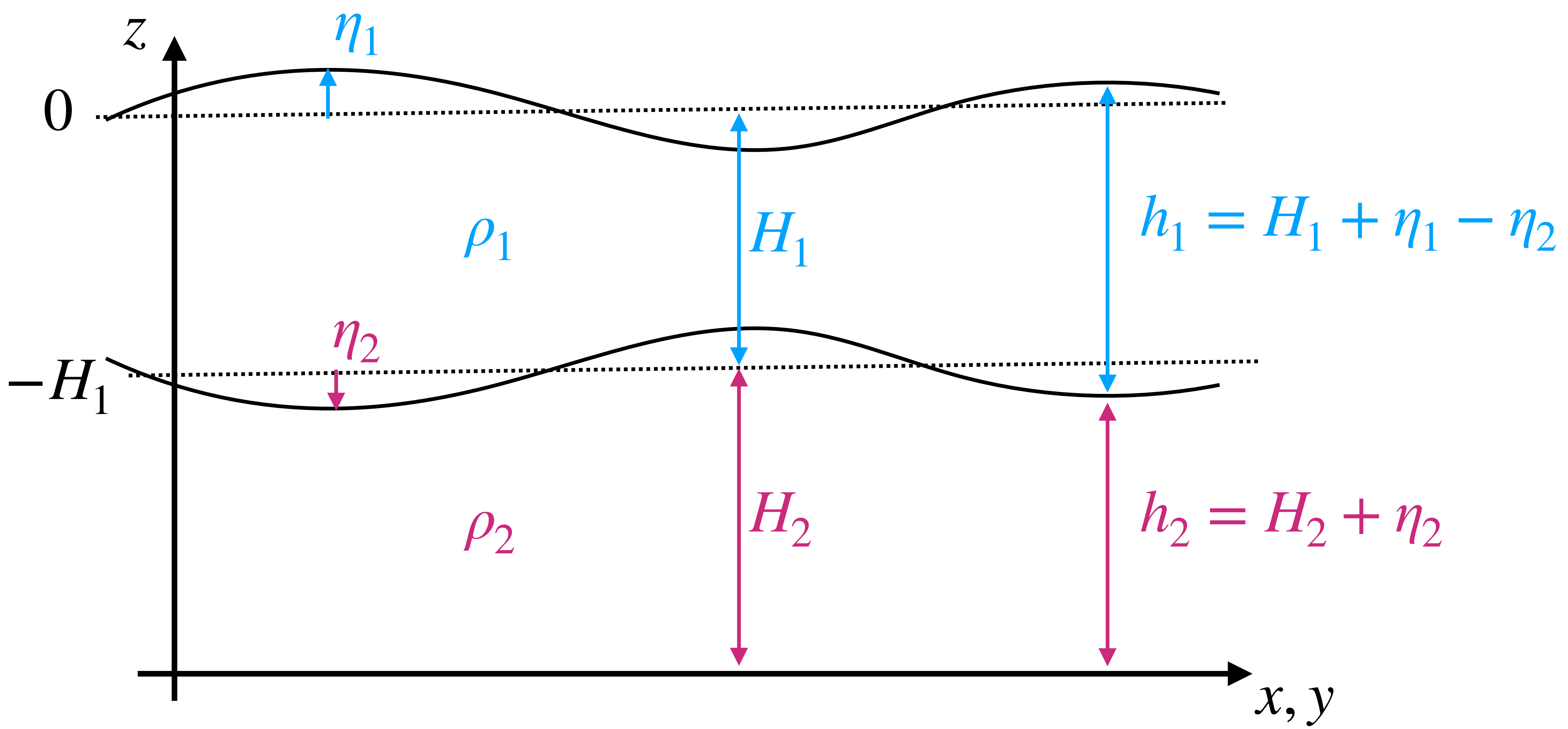
Outline

1. A general introduction to ocean waves
2. What are internal waves ? Why do we study internal waves ?
3. **Internal waves in the two-layer shallow-water model**
4. Internal waves in the continuously-stratified model
5. Generation of internal waves
6. Propagation of internal waves
7. Dissipation of internal waves and impacts

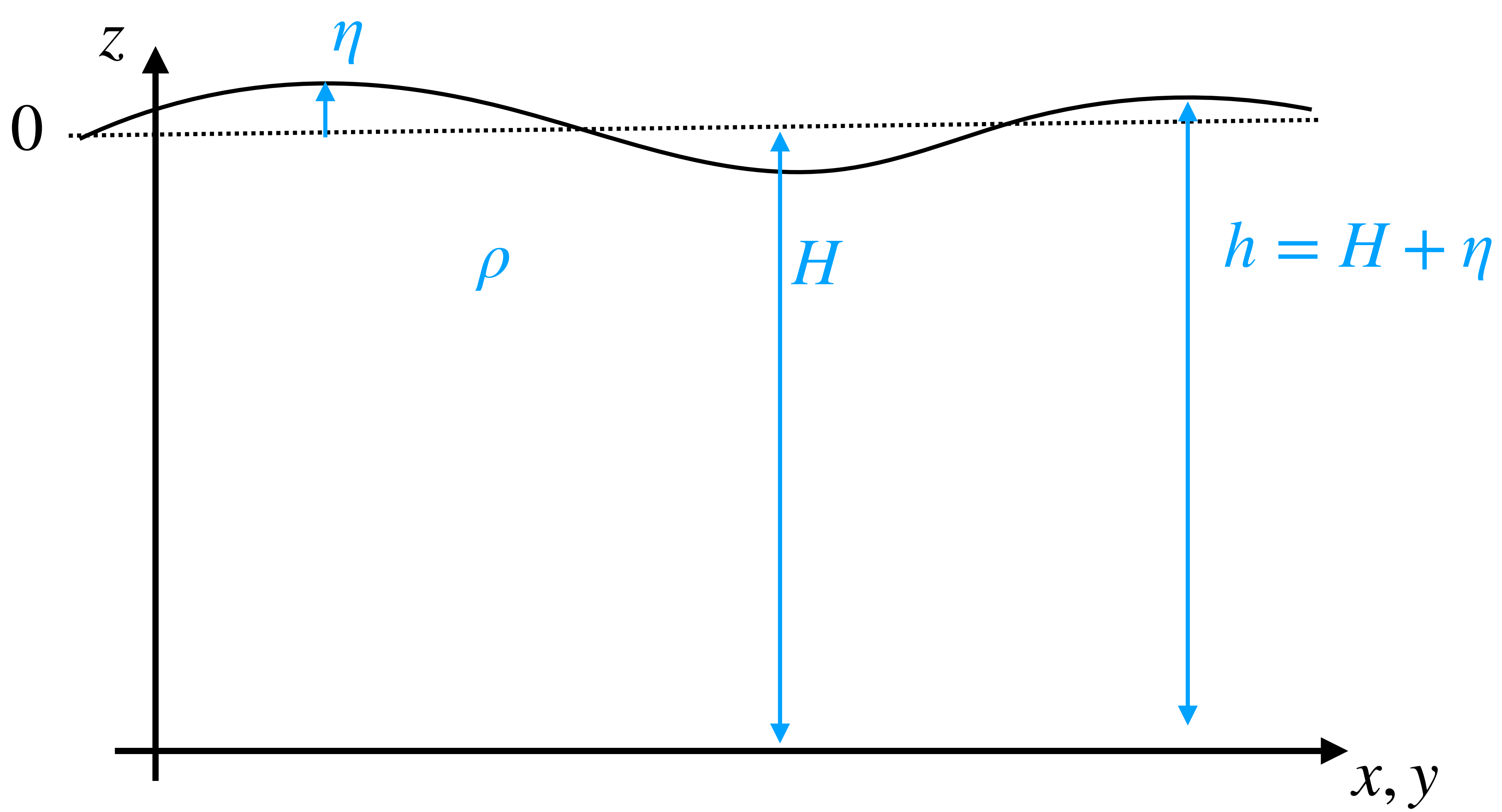
3. The two-layer shallow-water model



Reminder: Shallow-water equations

- No forcings/dissipation

- constant density / no vertical variations / hydrostatic



Reminder: Shallow-water equations

- No forcings/dissipation
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$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y},$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0,$$

Reminder: linearized Shallow-water equations

- No forcings/dissipation
- constant density / no vertical variations / hydrostatic
- remove non-linear terms

$$\frac{\partial u}{\partial t} - f v = - g\frac{\partial \eta}{\partial x},$$

$$\frac{\partial v}{\partial t} + f u = - g\frac{\partial \eta}{\partial y},$$

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} + H\frac{\partial v}{\partial y} = 0,$$

Reminder: linearized Shallow-water equations

Linear shallow water equations on an f-plane

$$u_t - fv + g\eta_x = 0 \qquad (1)$$
$$v_t + fu + g\eta_y = 0 \qquad (2)$$
$$\eta_t + H(u_x + v_y) = 0 \qquad (3)$$

vorticity equation: $$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1) \rightarrow \quad \frac{\partial}{\partial t}(v_x - u_y) + f(u_x + v_y) = 0 \qquad (V)$$

divergence equation: $$\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) \rightarrow \quad \frac{\partial}{\partial t}(u_x + v_y) - f(v_x - u_y) + g\nabla^2\eta = 0 \qquad (D)$$

substitute (V) into (3) $$\eta_t - \frac{H}{f}\frac{\partial}{\partial t}(v_x - u_y) = 0$$

Reminder: linearized Shallow-water equations

substitute (D) into $\dfrac{\partial}{\partial t}$ (3)

$$\eta_{tt} + fH(v_x - u_y) - gH\nabla^2\eta = 0 \qquad \frac{\partial}{\partial t} \rightarrow$$

$$\eta_{ttt} + fH\frac{\partial}{\partial t}(v_x - u_y) - gH\nabla^2\eta_t = 0$$

substitute from $\quad \eta_t - \dfrac{H}{f}\dfrac{\partial}{\partial t}(v_x - u_y) = 0 \quad$ gives $\quad \eta_{ttt} + f^2\eta_t - gH\nabla^2\eta_t = 0$

With appropriate initial condition at *t = 0,* the departure from geostrophic disequilibrium follows:

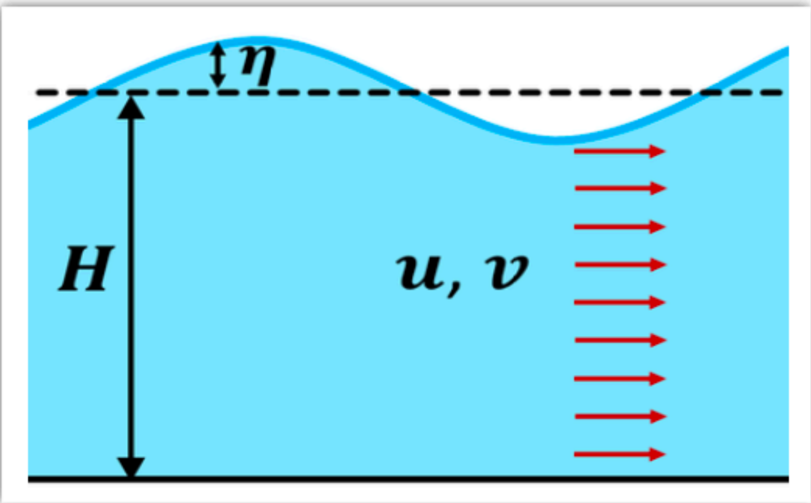
$\eta_{tt} - gH\nabla^2\eta + f^2\eta = 0 \qquad$ substitute solution $\qquad \eta = \tilde{\eta}e^{i(\mathbf{k}.\mathbf{x} - \omega t)}$

leads to dispersion relation

$$\boxed{\omega = \pm\sqrt{f^2 + gHk^2}}$$

Reminder: linearized Shallow-water equations

➯ *flat bottom, f-plane, linear perturbations u, v, η*



$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Wave solution: $(u, v, \eta) = (\tilde{u}, \tilde{v}, \tilde{\eta})e^{i(lx+my-\omega t)}$

With: $\frac{\partial}{\partial x} \to il\,\times \quad \frac{\partial}{\partial y} \to im\,\times \quad \frac{\partial}{\partial t} \to -i\omega\times$

➯ *substitute wave solution: differential equations become linear algebraic equations*

$$-i\omega\tilde{u} - f\tilde{v} = -igl\tilde{\eta}$$
$$-i\omega\tilde{v} + f\tilde{u} = -igm\tilde{\eta}$$
$$-i\omega\tilde{\eta} + H(il\tilde{u} + im\tilde{v}) = 0$$

The unknowns are the wave amplitudes $\tilde{u}, \tilde{v}, \tilde{\eta}$

The parameters are the wave properties l, m, ω
and the geophysical constants f, g, H

Reminder: linearized Shallow-water equations

We need to solve algebraic system

$$\begin{pmatrix} -i\omega & -f & igl \\ f & -i\omega & igm \\ ilH & imH & -i\omega \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0$$

⇨ *trivial solution* $\tilde{u} = \tilde{v} = \tilde{\eta} = 0$ *(no flow)*

⇨ *The condition for having non-trivial solutions is hat the determinant of the matrix is zero.*

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \bullet \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \bullet \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \bullet \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

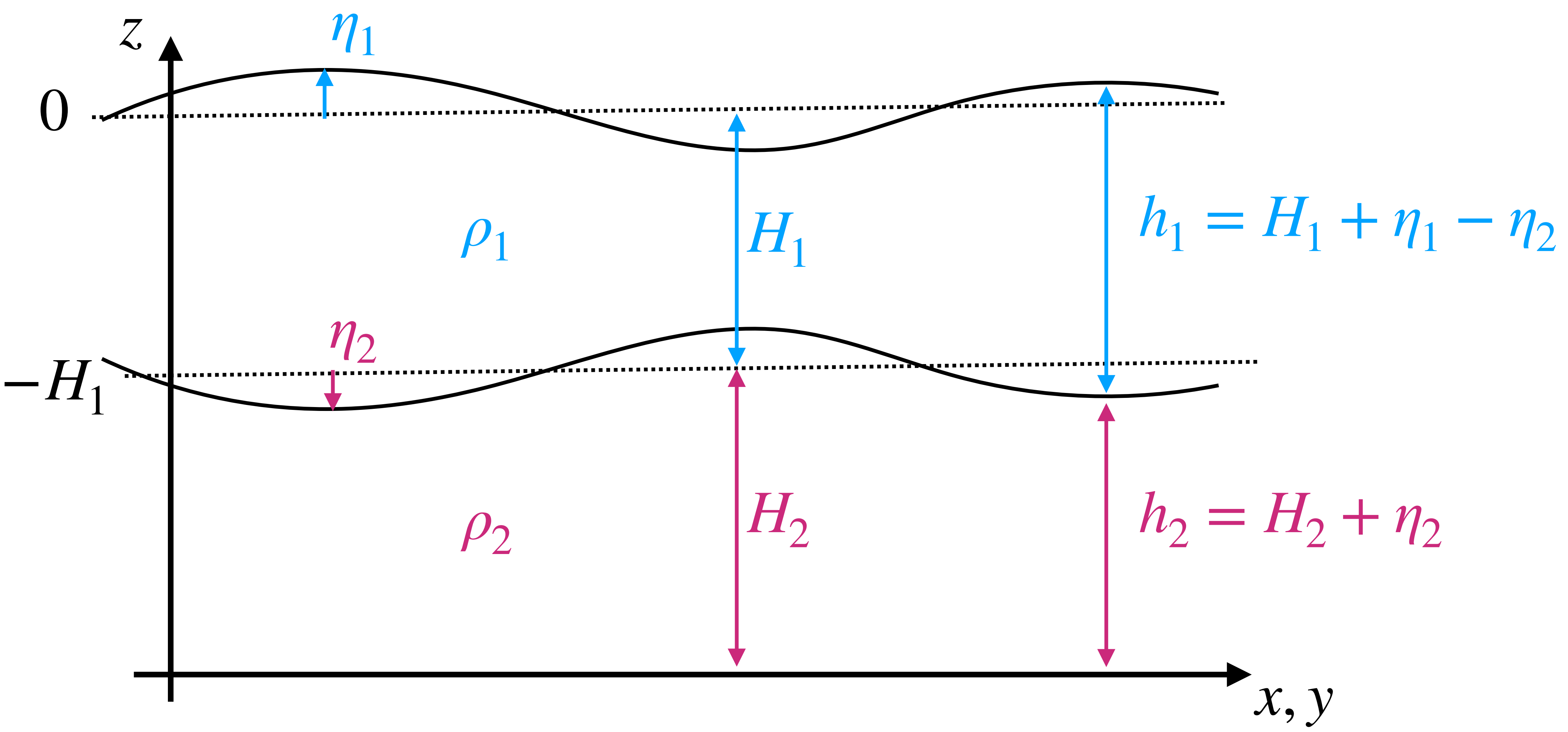
This leads to $\omega\left[\omega^2 - f^2 - gH(l^2 + m^2)\right] = 0$

this is a more complicated dispersion relation !

three solutions: $\omega = 0$ *steady geostrophic flow*

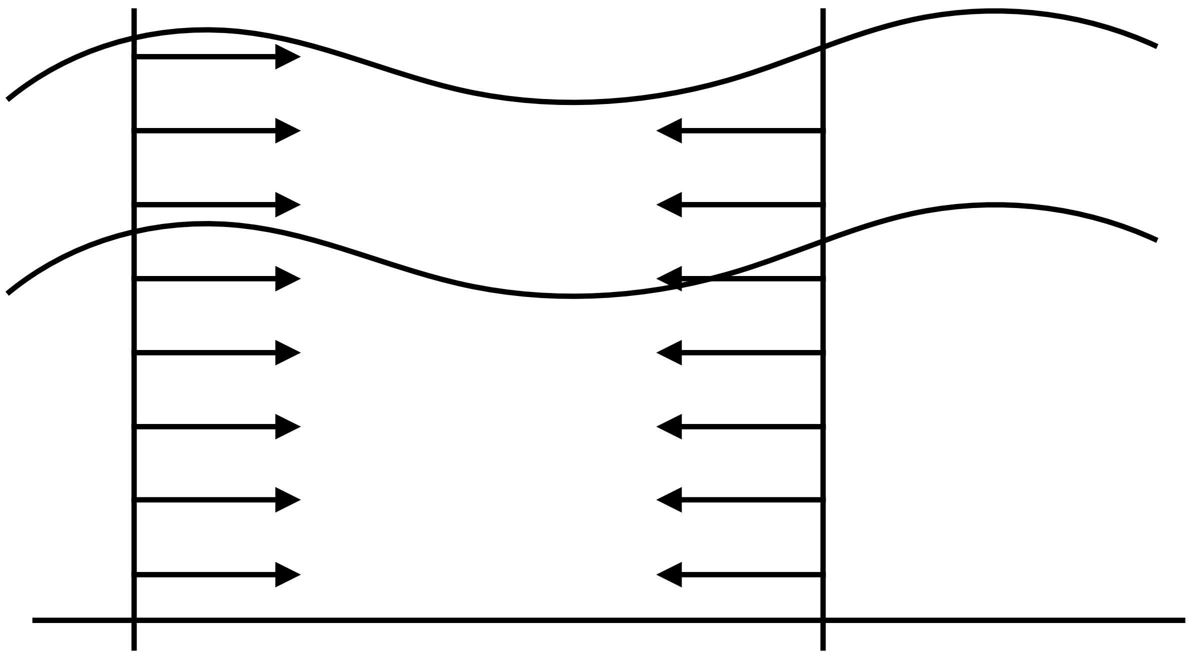
$\omega = \pm\sqrt{f^2 + gHk^2}$ *inertia-gravity waves*

3. The two-layer shallow-water model

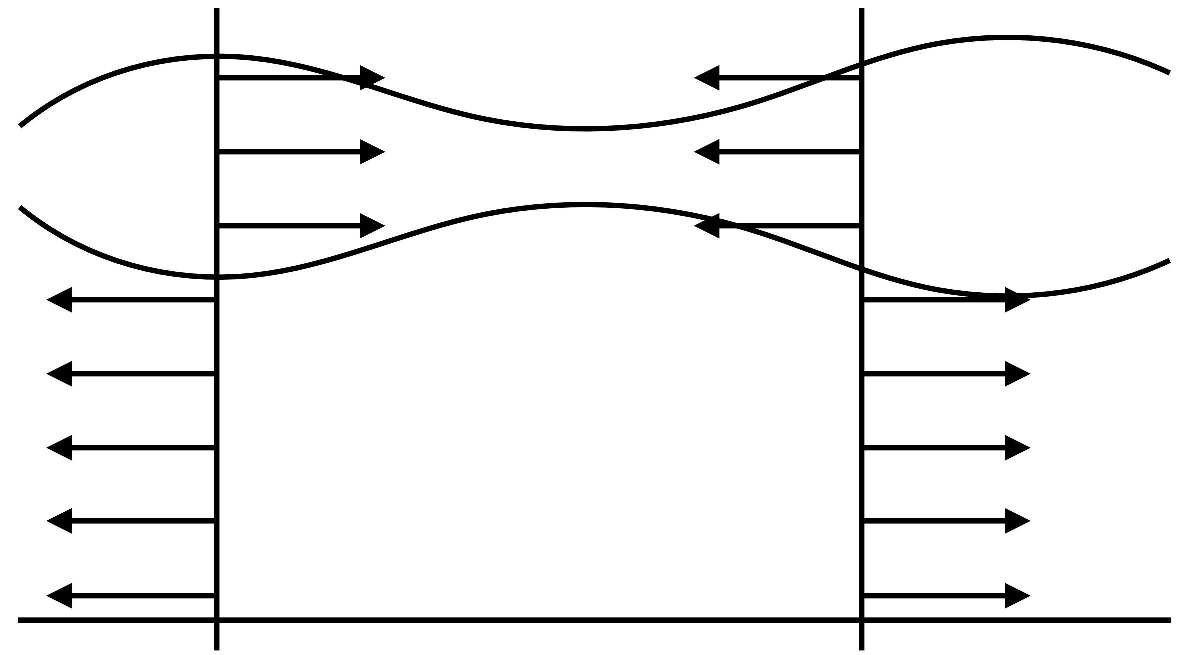


3. The two-layer shallow-water model

Structure of the waves:



Barotropic wave



Baroclinic wave

$$\omega_{bc}^2 = f^2 + g'\frac{H_1 H_2}{H}K^2$$ for the baroclinic mode, and

$$\omega_{bt}^2 = f^2 + gHK\left(1 - \frac{g' H_1 H_2}{gH^2}\right)$$ for the barotropic mode.

3. The two-layer shallow-water model

What is the surface signature of an internal wave ?

With typical values in the coastal ocean : H_1=30 m, H_2=70 m, g'=2x$10^{-3}g$, and an observed displacement of η_2=10 m ?

3. The two-layer shallow-water model

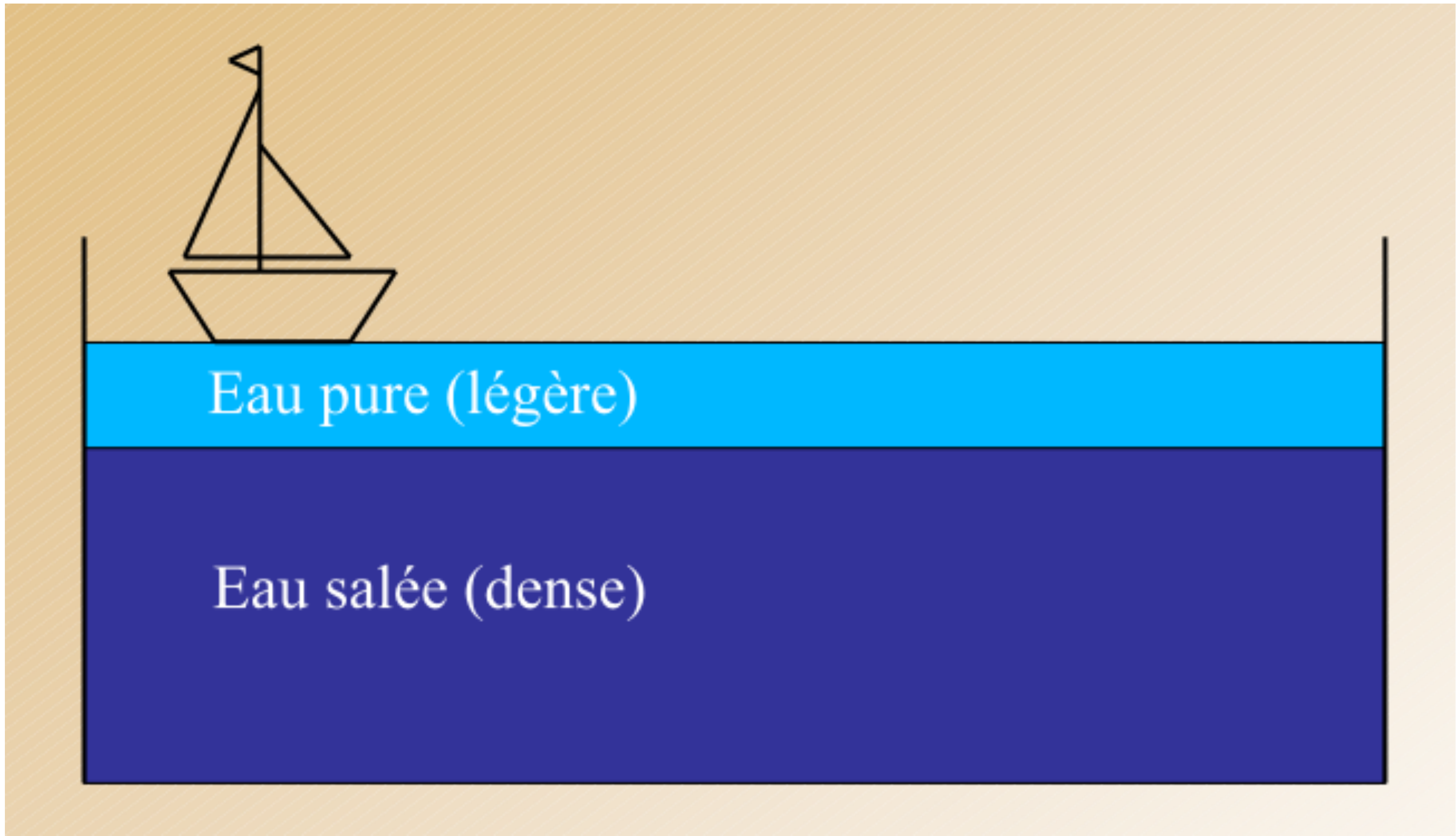
What is the surface signature of an internal wave ?

With typical values in the coastal ocean : H_1=30 m, H_2=70 m, g'=2x$10^{-3}g$, and an observed displacement of η_2=10 m ?

Using the η_1, η_2 relationship (slide #8), $\eta_1 = \dfrac{\eta_2}{1 + \dfrac{gH_1K^2}{f^2 - \omega^2}}$,

and with the dispersion relation, $K^2 = \dfrac{\omega^2 - f^2}{g'\dfrac{H_1H_2}{H}}$, so $\eta_1 = \dfrac{\eta_2}{1 - \dfrac{gH}{g'H_2}} = 1.4$ cm.

3. The two-layer shallow-water model



"When caught in dead water Fram appeared to be held back, as if by some mysterious force, and she did not always answer the helm. In calm weather, with a light cargo, Fram was capable of 6 to 7 knots. When in dead water she was unable to make 1.5 knots. We made loops in our course, turned sometimes right around, tried all sorts of antics to get clear of it, but to very little purpose."

Fridtjof Nansen (Norwegian Arctic explorer in 1893)

Example of use for the two-layer model: “dead water”