- Applied Mathematics -
Introduction to perturbation methods

Practical exercises : Perturbation methods for differential equations
WKB method

Exercice 1 : Projectile with friction

Consider the following problem of a projectile launched vertically with a fluid friction ($x^*(t^*)$ is the position of the projectile at time t^*):

$$m\frac{d^2x^*}{dt^{*2}} + k\frac{dx^*}{dt^*} = -mg, \quad x^*(0) = 0 \quad \frac{dx^*}{dt^*}(0) = V.$$

1. Use the following scaling to non-dimensionalize this problem :

$$x = \frac{x^*}{V^2 g^{-1}}, \qquad t = \frac{t^*}{V g^{-1}}, \qquad \beta = \frac{kV}{mg}$$

2. With an expansion in powers of β (approximation of low friction) determine at order 2 in β the solution of the problem.

3. Supplementary question: find at order 2 in β the time t_m at which the projectile reaches its highest position x_m.

Exercice 2 : WKB on an initial value problem

Using the WKB method at order 1 (you want S_0 and S_1), find an approximate solution of the following ordinary differential equation (ODE)

$$\epsilon^2 y''(x) = \frac{1}{(1+x^2)^2}\, y(x) \quad y(0) = 0, \quad y'(0) = 1$$

Exercice 3 : WKB method for a boundary layer problem

We consider the following boundary value problem (BVP)

$$\epsilon y'' + 2y' + y = 0 \quad y(0) = 0, \quad y(1) = 1$$

The WKB method consists in seeking approximate solutions of the form

$$y(x) = \exp\left(\sum_{n=0}^{+\infty} \delta^{n-1} S_n(x)\right)$$

1. Adapt the WKB method to the present case (presence of y) and determine the differential equations for $S_0(x)$ et $S_1(x)$.

2. Solve these equations without taking into account the boundary conditions so that the solution reads
$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$
where c_1 and c_2 are two constants of integration.

3. Use the boundary conditions to find those two constants.

Exercice 4 :

Determine the approximated solutions of the following equation using the WKB method for $x \longrightarrow \infty$:
$$y'' = (\frac{x^2}{4} - \nu - \frac{1}{2})y$$
where ν is a fixed positive parameter.

Hint: To make appearing ϵ explicitly use the change of variable $x = t/\epsilon$ then consider $\epsilon \longrightarrow 0$. Since no initial conditions nor boundary conditions are given, you need to find two independant functions with their constant of integration. At the end, the integral that will apear will be approximated by using the property $x \longrightarrow \infty$.