TD N°2 Ex 1

1) $m x^{*''}(t^*) + k x^{*'} = -mg \qquad x^*(0) = 0 \qquad x^{*'}(0) = V$

$$x = \frac{x^*}{V^2 g^{-1}} \qquad t = \frac{t^*}{V g^{-1}} \qquad \beta = \frac{kV}{mg}$$

$$x^{*'}(t^*) = V^2 g^{-1} \frac{dx}{dt}\frac{dt}{dt^*} = V^2 g^{-1} \times \frac{1}{V g^{-1}} \frac{dx}{dt} = V x'(t)$$

$$x^{*''}(t^*) = \left(x^{*'}(t^*)\right)' = V\left(x'(t)\right)' \frac{dt}{dt^*} = \frac{V}{V g^{-1}} x''(t) = g\, x''(t)$$

the problem becomes then:

$$(1) \qquad x''(t) + \beta x'(t) = -1 \qquad x(0) = 0 \qquad x'(0) = 1$$

2) $x(t) = x_0(t) + \beta x_1(t) + \beta^2 x_2(t) + O(\beta^3)$ (at order 2)

Then (1) $\Rightarrow \quad x_0'' + \beta x_1'' + \beta^2 x_2'' + \beta x_0' + \beta^2 x_1' + O(\beta^3) = -1$

With iniatial conditions:

$$\begin{cases} x_0(0) + \beta x_1(0) + \beta^2 x_2(0) + O(\beta^3) = 0 \\ x_0'(0) + \beta x_1'(0) + \beta^2 x_2'(0) + O(\beta^3) = 1 \end{cases}$$

So

$$\begin{cases} x_0''(t) = -1 & x_0(0) = 0 \quad x_0'(0) = 1 \\ x_1''(t) + x_0'(t) = 0 & x_1(0) = 0 \quad x_1'(0) = 0 \\ x_2''(t) + x_1'(t) = 0 & x_2(0) = 0 \quad x_2'(0) = 0 \end{cases}$$

by solving these problems we get:

$\rightarrow \quad x_0(t) = -\frac{t^2}{2} + t$

$\rightarrow \quad x_1''(t) = t - 1 \quad x_1(0) = 0 \quad x_1'(0) = 0$

so $\quad x_1(t) = \frac{t^3}{6} - \frac{t^2}{2}$

$\rightarrow \quad x_2''(t) = -\frac{t^2}{2} + t \quad x_2(0) = 0 \quad x_2'(0) = 0$

So $\quad x_2(t) = -\frac{t^4}{24} + \frac{t^3}{6}$

And thus the approximate solution is at order 2:

$$x(t) \underset{\beta \ll 1}{\sim} -\frac{t^2}{2} + t + \beta\left(\frac{t^3}{6} - \frac{t^2}{2}\right) + \beta^2\left(-\frac{t^4}{24} + \frac{t^3}{6}\right)$$

3) We are searching for t_m as $x'(t_m) = 0$

$$x'(t_m) = 0 \Rightarrow (-t_m + 1) + \beta\left(\frac{t_m^2}{2} - t_m\right) + \beta^2\left(-\frac{t_m^3}{6} + \frac{t_m^2}{2}\right) + O(\beta^3) = 0$$

if $t_m = t_{m_0} + \beta t_{m_1} + \beta^2 t_{m_2} + O(\beta^3)$

we get:

$$t_{m_0} - \beta t_{m_1} - \beta^2 t_{m_2} + 1 + \beta\left(\frac{t_{m_0}^2}{2} + \beta t_{m_0} t_{m_1} - t_{m_0} - \beta t_{m_1}\right)$$
$$+ \beta^2\left(-\frac{t_{m_0}^3}{6} + \frac{t_{m_0}^2}{2}\right) + O(\beta^3) = 0$$

So
$$\begin{cases} -t_{m_0} + 1 = 0 \\ -t_{m_1} + \dfrac{t_{m_0}^2}{2} - t_{m_0} = 0 \\ -t_{m_2} + t_{m_0} t_{m_1} - t_{m_1} - \dfrac{t_{m_0}^3}{6} + \dfrac{t_{m_0}^2}{2} = 0 \end{cases}$$

Which gives: $t_m \sim 1 - \dfrac{\beta}{2} + \dfrac{\beta^2}{3}$ at order 2

Exercise 2

$$\varepsilon^2 y''(x) = \frac{1}{(1+x^2)^2}\, y(x) \quad x \in]0,1[$$

with $y(0) = 0$ and $y'(0) = 1$

if we let $Q(x) = \dfrac{1}{(1+x^2)^2}$ then we have

the same form of equation as in the WKB course:

$$\varepsilon^2 y''(x) = Q(x)\, y(x) \quad \text{and} \quad Q(x) \neq 0$$

So we can apply the results:

With $y(x) \sim \exp\left(\dfrac{S_0(x)}{\delta} + S_1(x)\right)$

we obtain: $\delta = \varepsilon$

$$S_0(x) = \pm \int^x \sqrt{Q(x)}\, dt$$

$$S_1(x) = \ln\left(Q(x)^{-\frac{1}{4}}\right)$$

We obtain:

$$S_0(x) = \pm \int^x \frac{1}{1+t^2}\,dt = \pm \operatorname{Arctan} x$$

$$S_1(x) = \ln\left(\sqrt{1+x^2}\right)$$

And thus:

$$y(x) \sim C_+ \sqrt{1+x^2}\, e^{+\frac{\operatorname{Arctan} x}{\varepsilon}} + C_- \sqrt{1+x^2}\, e^{-\frac{\operatorname{Arctan} x}{\varepsilon}}$$

Determination of the constants C_+ and C_- with the initial conditions:

$$y(0) = 0 \implies C_+ + C_- = 0 \implies C_- = -C_+$$

$$y'(0) = 1 \implies ?$$

Let's calculate $y'(x)$:

$$y'(x) = C_\pm \left(\sqrt{1+x^2}\right)' e^{\pm\frac{\operatorname{Arctan} x}{\varepsilon}} + C_\pm \sqrt{1+x^2} \left(e^{\pm\frac{\operatorname{Arctan} x}{\varepsilon}}\right)'$$

$$y'(x) = C_\pm \frac{x}{\sqrt{1+x^2}} e^{\pm\frac{\operatorname{Arctan} x}{\varepsilon}} + C_\pm \sqrt{1+x^2}\, \frac{(\pm 1)}{\varepsilon(1+x^2)}\, e^{\pm\frac{\operatorname{Arctan} x}{\varepsilon}}$$

$$y'(0) = 0 + C_+ \frac{1}{\varepsilon} e^0 + 0 - C_- \frac{1}{\varepsilon} e^0 = \frac{C_+ - C_-}{\varepsilon}$$

Thus $y'(0) = 1 \Rightarrow C_+ - C_- = \varepsilon$

But $y'(0) = 0 \Rightarrow C_- = -C_+$

Thus $C_+ = \dfrac{\varepsilon}{2}$ and $C_- = \dfrac{-\varepsilon}{2}$

The WKB approximation of the problem is thus:

$$y(x) \sim \frac{\varepsilon}{2}\sqrt{1+x^2}\, e^{\frac{1}{\varepsilon}\operatorname{Arctan} x} - \frac{\varepsilon}{2}\sqrt{1+x^2}\, e^{-\frac{1}{\varepsilon}\operatorname{Arctan} x}$$

thus

$$\boxed{y(x) \underset{\varepsilon \to 0}{\sim} \varepsilon\sqrt{1+x^2}\, \sinh\left(\frac{\operatorname{Arctan} x}{\varepsilon}\right)}$$

Exercise 3

(1) $\varepsilon y''(x) + 2y'(x) + y(x) = 0 \qquad x \in]0,1[$

with $y(0) = 0$ and $y(1) = 1$

We are searching for a WKB approximation for $\varepsilon \ll 1$

$$y(x) \sim e^{\frac{1}{\delta} \sum_{n=0}^{+\infty} \delta^n S_n(x)}$$

Then $y'(x) \sim \left(\sum_{n \geq 0} \delta^{n-1} S_n'(x) \right) y(x)$

$$y''(x) \sim \left[\left(\sum_{n \geq 0} \delta^{n-1} S_n'(x) \right)^2 + \sum_{n \geq 0} \delta^{n-1} S_n''(x) \right] y(x)$$

Then (1) $\Rightarrow$

$$\left[\varepsilon \left(\sum_{n \geq 0} \delta^{n-1} S_n'(x) \right)^2 + \sum_{n \geq 0} \delta^{n-1} S_n''(x) \right] + 2 \left(\sum_{n \geq 0} \delta^{n-1} S_n'(x) \right) + 1 = 0$$

$$\Rightarrow \underbrace{\frac{\varepsilon}{\delta^2} S_0'^2}_{1} + \underbrace{2 \frac{\varepsilon}{\delta} S_0' S_1' + \frac{\varepsilon}{\delta} S_0''}_{2} + \underbrace{\frac{2}{\delta} S_0'}_{3} + \underbrace{2 S_1' + 1}_{4} + O(\varepsilon) = 0$$

order: $\frac{\varepsilon}{\delta^2} \qquad \frac{\varepsilon}{\delta} \qquad \frac{1}{\delta} \qquad 1$

if 1 $\sim$ 2 then $\delta \sim 1$ (we expect $\delta \ll 1$) NO

if 2 $\sim$ 3 then $\varepsilon \sim 1$ not possible ($\varepsilon \ll 1$) NO if 3 ~ 4 then $\delta \sim 1$ (idem) NO

if 2 ~ 4 then $\delta \sim \varepsilon$ then 2 ~ 4 $\sim 1 \ll$ 3 $\sim \frac{1}{\varepsilon}$ NO

(not a dominant balance)

if 1 ~ 3 then $\delta \sim \varepsilon$ then 1 ~ 3 $\sim \frac{1}{\varepsilon} \gg 1$

and 2 ~ 1 and 4 ~ 1 YES

thus 1 ~ 3 $\gg$ 2 ~ 4 this is a dominant balance

if 1 ~ 4 then $\delta \sim \sqrt{\varepsilon}$ then 1 ~ 4 ~ 1

but 3 $\sim \frac{1}{\delta} \sim \frac{1}{\sqrt{\varepsilon}} \gg 1$ thus 1 ~ 4 $\ll$ 3 NO

(not a dominant balance)

Conclusion: the only possible dominant balance is between 1 and 3 and conducts to the choice $\delta \sim \varepsilon$

With $\delta \sim \varepsilon$ the equation becomes:

$$\frac{1}{\delta}\left(S_0'^2 + 2S_0'\right) + \left(2S_0'S_1' + S_0'' + 2S_1' + 1\right) + O(\delta) = 0$$

Thus
$$\begin{cases} S_0'^2 + 2S_0' = 0 \\ 2S_0'S_1' + 2S_1' + S_0'' + 1 = 0 \end{cases}$$

- $S_0'^2 + 2S_0' = 0 \Rightarrow S_0'(S_0'+2) = 0 \Rightarrow \begin{cases} S_0' = 0 \\ \text{or} \\ S_0' = -2 \end{cases}$

$$\Rightarrow \begin{cases} S_0(x) = cst \\ \text{or} \\ S_0(x) = -2x + cst \end{cases}$$

- $2S_0'S_1' + 2S_1' + S_0'' + 1 = 0 \Rightarrow S_1' = \dfrac{-S_0'' - 1}{2S_0' + 2}$

if $S_0 = cst$: $(S_0' = 0, S_0'' = 0)$ then $S_1' = -\frac{1}{2}$

if $S_0 = -2x + cst$: $(S_0' = -2, S_0'' = 0)$ then $S_1' = \dfrac{-1}{-2} = \dfrac{1}{2}$

We thus find two solutions:

$$\begin{cases} S_0 = 0 \quad , \quad S_1 = -\frac{1}{2}x \\ S_0 = -2x \quad , \quad S_1 = \frac{1}{2}x \end{cases} \qquad \text{(we have taken the cst to 0)}$$

We thus find a general WKB solution of the equation (1) under the form:

$$y(x) \sim C_1 \, e^{-\frac{x}{2}} + C_2 \, e^{-\frac{2x}{\delta} + \frac{x}{2}} \qquad \text{with } \delta = \varepsilon$$

We now need to apply boundary conditions to determine C_1 and C_2

$$y(0)=0 \Rightarrow c_1 + c_2 = 0$$
$$y(1)=1 \Rightarrow c_1 e^{-1/2} + c_2 e^{-\frac{2}{\varepsilon}+\frac{1}{2}} = 1$$

thus $c_2 = -c_1$

and $c_1\left(e^{-1/2} - e^{-\frac{2}{\varepsilon}+\frac{1}{2}}\right) = 1$

when $\varepsilon \to 0$, $e^{-\frac{2}{\varepsilon}} \to 0$ thus $c_1 \sim e^{\frac{1}{2}}$

Thus $c_1 = e^{\frac{1}{2}}$, $c_2 = e^{-\frac{1}{2}}$

And the WKB approximation at order 1 for the complete problem is

$$\boxed{y(x) \underset{\varepsilon \to 0}{\sim} e^{\frac{1}{2}}\left(e^{-\frac{x}{2}} - e^{-\frac{2x}{\varepsilon}+\frac{x}{2}}\right)}$$

Exercise 4

$$y''(x) = \left(\frac{x^2}{4} - \nu - \frac{1}{2}\right) y(x) \qquad x \to +\infty$$

We let $x = \frac{t}{\varepsilon}$ and $\varepsilon \ll 1$

$y(x) = Y(t)$ with $t = \varepsilon x$

$$\Rightarrow \frac{dy}{dx} = \frac{dY}{dt}\frac{dt}{dx} = \varepsilon Y'(t)$$

$$\Rightarrow \frac{d^2y}{dx^2} = y''(x) = \varepsilon^2 Y''(t)$$

$$\Rightarrow \varepsilon^2 Y''(t) = \underbrace{\left(\frac{t^2}{4\varepsilon^2} - \nu - \frac{1}{2}\right)}_{Q(t)} Y(t)$$

$$\frac{Q(t)}{\varepsilon^2} = \frac{t^2}{4\varepsilon^4} - \nu - \frac{1}{2} \gg 1 \quad \text{as } \varepsilon \to 0$$

$$\begin{cases} Y(t) \sim e^{\frac{1}{\delta}\sum_{n\geq 0} \delta^n S_n(t)} \\ Y'(t) \sim \left(\sum_{n\geq 0} \delta^{n-1} S_n'(t)\right) Y(t) \\ Y''(t) \sim \left(\left(\sum_{n\geq 0} \delta^{n-1} S_n'(t)\right)^2 + \sum_{n\geq 0} \delta^{n-1} S_n''(t)\right) Y(t) \end{cases}$$

$$\Rightarrow \varepsilon^2\Big(\underbrace{\frac{1}{\delta^2} S_0'^2}_{1} + \underbrace{\frac{2}{\delta} S_0' S_1' + \frac{1}{\delta} S_0''}_{2}\Big) Y(t) = \Big(\underbrace{\frac{t^2}{4\varepsilon^2}}_{3} \underbrace{- \nu - \frac{1}{2}}_{4}\Big) Y(t)$$

order: $\quad \frac{\varepsilon^2}{\delta^2} \qquad \frac{\varepsilon^2}{\delta} \qquad \frac{1}{\varepsilon^2} \qquad 1$

Dominant equilibrium:

$$① \sim ③ \Rightarrow \frac{\varepsilon^2}{\delta^2} \sim \frac{1}{\varepsilon^2} \Rightarrow \delta^2 \sim \varepsilon^4 \Rightarrow \delta \sim \varepsilon^2$$

$$\delta = \varepsilon^2 \Rightarrow ① \sim ③ \sim \frac{1}{\varepsilon^2} \gg ② \sim \frac{\varepsilon^2}{\delta} = \frac{\varepsilon^2}{\varepsilon^2} = 1 \sim ④ \quad \text{OK!}$$

and we can check that the other possible balances are not dominant.

$$\Rightarrow \frac{1}{\varepsilon^2} S_0'^2 + 2 S_0' S_1' + S_0'' = \frac{t^2}{4\varepsilon^2} - \nu - \frac{1}{2}$$

$$\Rightarrow \begin{cases} S_0'^2 = \frac{t^2}{4} \\ 2 S_0' S_1' + S_0'' = -\nu - \frac{1}{2} \end{cases} \Rightarrow \begin{cases} S_0' = \pm \frac{t}{2} \\ S_1' = \frac{-\nu - \frac{1}{2} - S_0''}{2 S_0'} \end{cases}$$

$$\Rightarrow \begin{cases} S_0 = \pm \frac{t^2}{4} \\ S_1' = \frac{-\nu - 1/2 - (\pm 1/2)}{\pm t} \end{cases} \quad \left(S_0' = \pm \frac{t}{2} \; , \; S_0'' = \pm \frac{1}{2} \right)$$

$$\Rightarrow \begin{cases} S_0 = + \frac{t^2}{4} \\ S_1' = \frac{-\nu - 1}{t} \end{cases} \quad \text{ou} \quad \begin{cases} S_0 = - \frac{t^2}{4} \\ S_1' = \frac{\nu}{t} \end{cases}$$

$$\Rightarrow \begin{cases} S_0 = \frac{t^2}{4} \\ S_1 = -(\nu+1) \ln t \end{cases} \quad \text{ou} \quad \begin{cases} S_0 = - \frac{t^2}{4} \\ S_1 = \nu \ln t \end{cases}$$

$$Y(t) \sim e^{\frac{S_0}{\delta} + S_1} \quad \text{with } \delta = \varepsilon^2$$

$$\sim e^{\frac{S_0}{\varepsilon^2} + S_1}$$

$$Y(t) \sim c_1\, e^{\frac{t^2}{4\varepsilon^2} - (\nu+1)\ln t} + c_2\, e^{-\frac{t^2}{4\varepsilon^2} + \nu \ln t}$$

$$Y(t) \sim c_1\, t^{-(\nu+1)} e^{\frac{t^2}{4\varepsilon^2}} + c_2\, t^{\nu} e^{-\frac{t^2}{4\varepsilon^2}}$$

We have $x = \frac{t}{\varepsilon} \Rightarrow t = \varepsilon x$

Thus $\boxed{y(x) \underset{x\to+\infty}{\sim} c_1 (\varepsilon x)^{-(\nu+1)} e^{\frac{x^2}{4}} + c_2 (\varepsilon x)^{\nu} e^{-\frac{x^2}{4}}}$

We can see that the approximate solution found depends on ε which is an artificial parameter introduced in the WKB resolution.
Now, we are interested in solutions for $x \to +\infty$. For a chosen ε value, as small as we like we can replace εx by x as $x \to +\infty$ in the WKB solution.

So: $\boxed{y(x) \sim c_1\, x^{-1-\nu} e^{\frac{x^2}{4}} + c_2\, x^{\nu} e^{-\frac{x^2}{4}}}$ $x \to +\infty$

(it is equivalent to say that the approximation is valid even if $\varepsilon = 1$ when $x \to +\infty$)