Numerical Modelling

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the anatomy of an ocean model

Outline

- **Lesson 1 :** [D109]
  - Introduction
  - Equations of motions
  - *Activity 1 [run an ocean model]*
- **Lesson 2 :** [D109]
  - Subgrid-scale parameterization
  - Dynamics of the ocean gyre
  - *Activity 2 [Dynamics of an ocean gyre]*
- **Lesson 3 :** [D109]
  - Horizontal Discretization
  - Vertical coordinates
  - *Activity 2 [Dynamics of an ocean gyre]*
  - *Activity 3 [Impacts of numerics / topography]*
- **Lesson 4 :** [D109]
  - Numerical schemes
  - Presentation of the model CROCO
  - *Activity 3 [Impacts of numerics / topography]*
- **Lesson 5 :** [D109]
  - Boundary Forcings
  - *Activity 5 [Design a realistic simulation]*
- **Lesson 6 :** [D109]
  - Diagnostics and validation
  - *Activity 6 [Analyze a realistic simulation]*
- **Lesson 7 :** [D109]
  - *Project*

Presentations and material will be available at :

jgula.fr/ModNum/

Useful references

Extensive courses:

- MIT: https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-950-atmospheric-and-oceanic-modeling-spring-2004/lecture-notes/
- Princeton: https://stephengriffies.github.io/assets/pdfs/GFM_lectures.pdf

Overview on ocean modelling and current challenges:

- Griffies et al., 2000, Developments in ocean climate modelling, Ocean Modelling.http://jgula.fr/ModNum/Griffiesetal00.pdf
- Griffies, 2006, "Some Ocean Model Fundamentals", In "Ocean Weather Forecasting: An Integrated View of Oceanography", 2006, Springer Netherlands. http://jgula.fr/ModNum/Griffies_Chapter.pdf
- Fox-Kemper et al, 19, “Challenges and Prospects in Ocean Circulation Models" http://jgula.fr/ModNum/FoxKemperetal19.pdf

CROCO/CROCO:

- https://www.myCROCO.org/wiki/
- Shchepetkin, A., and J. McWilliams, 2005: The Regional Oceanic Modeling System (CROCO): A split-explicit, free-surface, topography-following- coordinate ocean model. Ocean Modell. http://jgula.fr/ModNum/ShchepetkinMcWilliams05.pdf

#4 Vertical Discretization

Master's degree 2nd year Marine Physics

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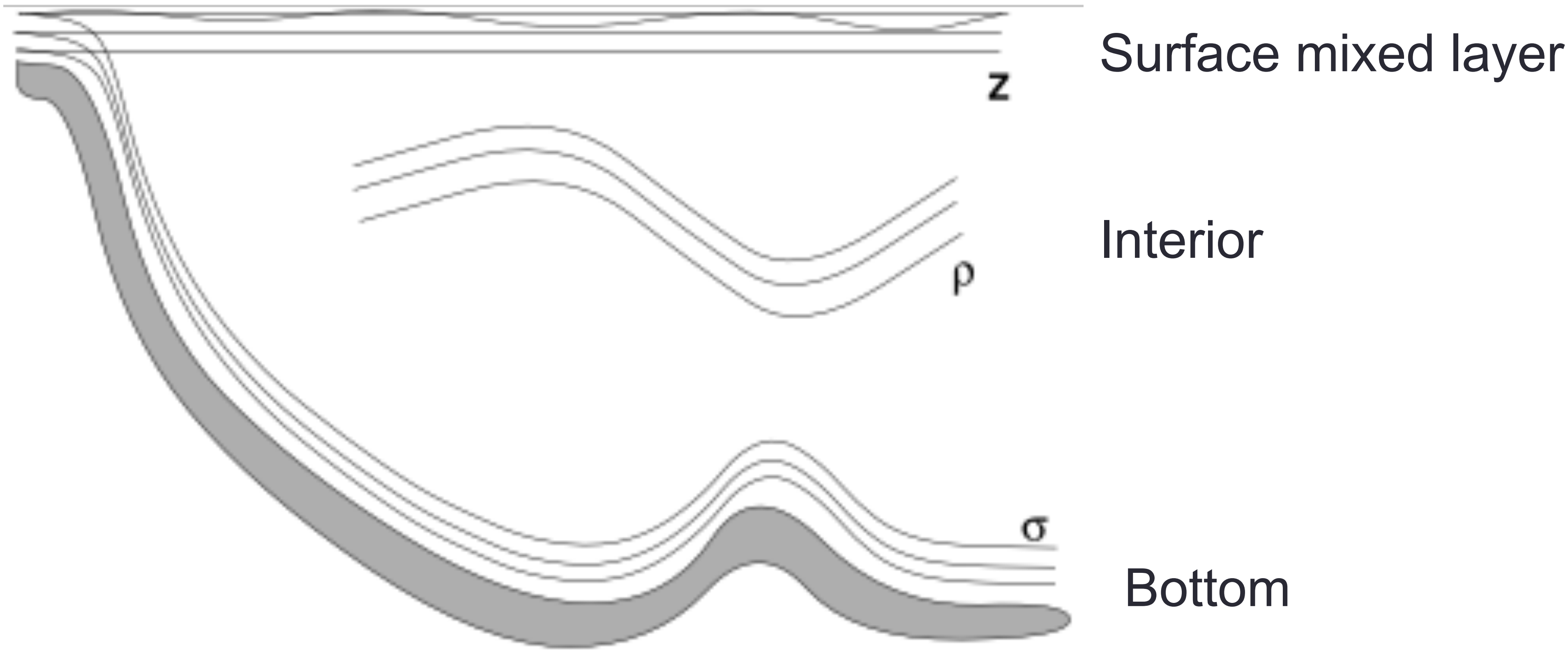
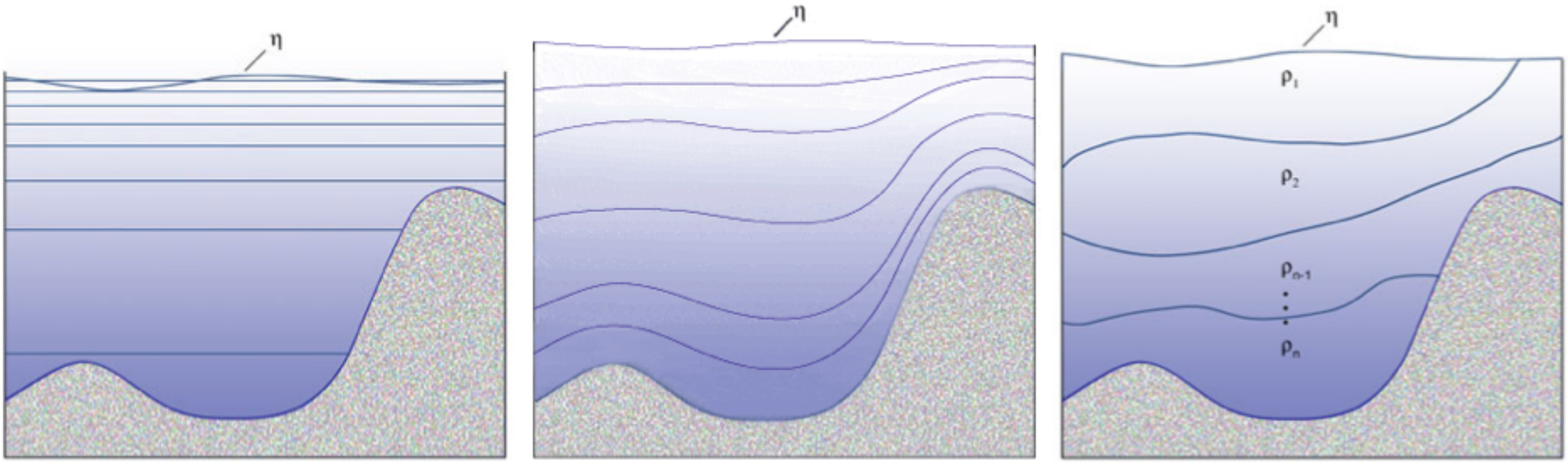
Vertical discretization

Depending on the depth, motions in the ocean will be mostly aligned along **geopotentials, isopycnal or topography.**



Vertical discretization

Several choices are possible to define the vertical coordinates system:

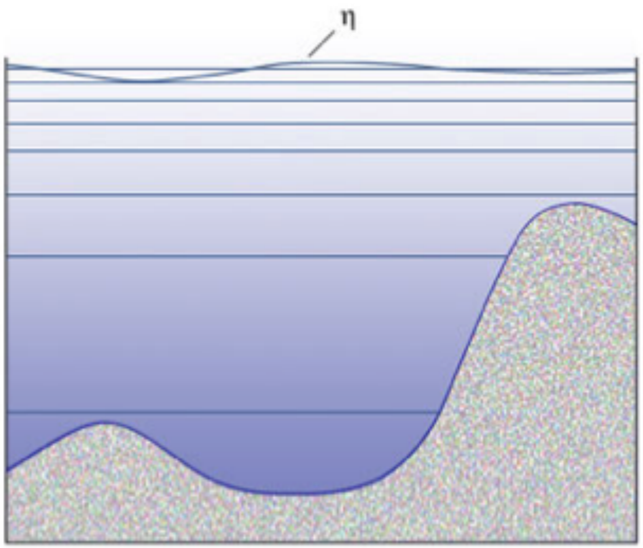


z-coordinates	**sigma-coordinates**	**isopycnal-coordinates**
Vertical coordinate is height (or depth)	The vertical coordinate follows the bathymetry	The vertical coordinate is the potential density

The vertical coordinate is the major difference between models.

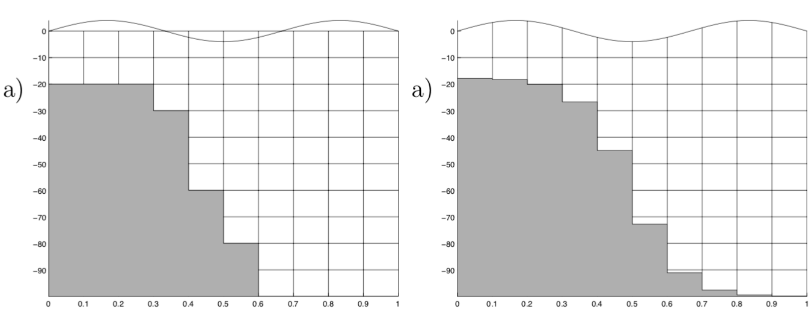
Vertical discretization

z-coordinates



PROS

- Natural in the upper ocean and for mixed-layer processes
- Ideal to compute horizontal (pressure) gradients
- Easier to implement and use

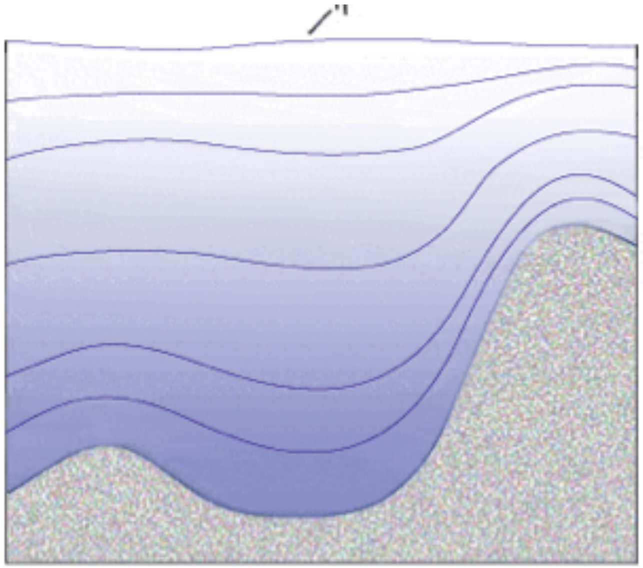


CONS

- The representation of tracer advection and diffusion along inclined density surfaces in the ocean interior is cumbersome.
- Representation of bottom topography is difficult.
- Need for bottom and lateral conditions
- Representation and parameterization of the BBL is unnatural.

Vertical discretization

sigma-coordinates



PROS

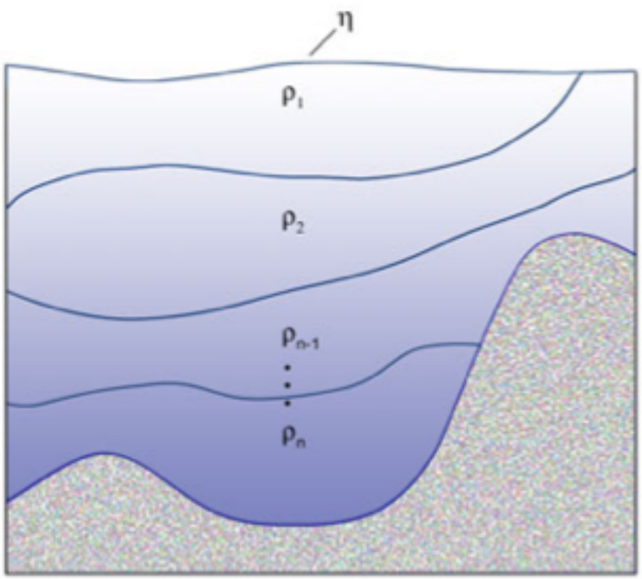
- Representation of bottom topography is natural (only bottom boundary condition)
- Representation and parameterization of the BBL is natural (more vertical res. in BBL)

CONS

- The representation of tracer advection and diffusion along inclined density surfaces in the ocean interior is cumbersome.
- Pressure gradient errors can be a problem

Vertical discretization

isopycnal-coordinates



PROS

- The representation of tracer advection and diffusion along inclined density surfaces in the ocean interior is natural.
- Water mass characteristics are preserved over long time scales

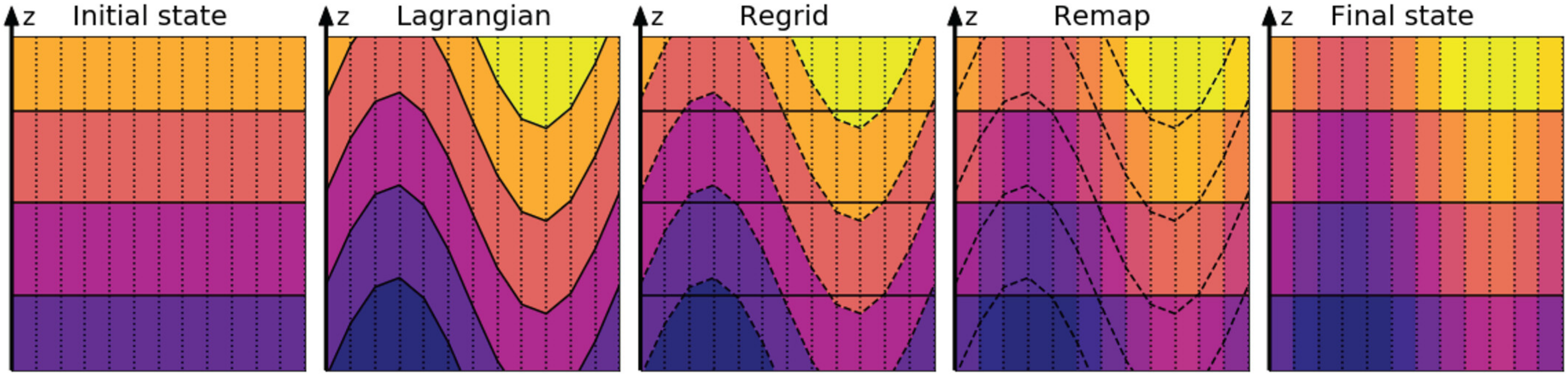
CONS

- Representing the effects of a realistic (non-linear) equation of state is cumbersome.
- Inappropriate for representing the surface mixed layer or BBL which are mostly unstratified.
- Non-hydrostatic effects/dynamics are not possible.
- Vertical and horizontal resolution are tightly connected in regions where isopycnals outcrop. This can lead to inadequate horizontal resolution in regions such as the ACC.

Vertical discretization

More recent developments use **Lagrangian vertical coordinates:**

- **Arbitrary Lagrangian-Eulerian (ALE) method**
- **Lagrangian-remap method**



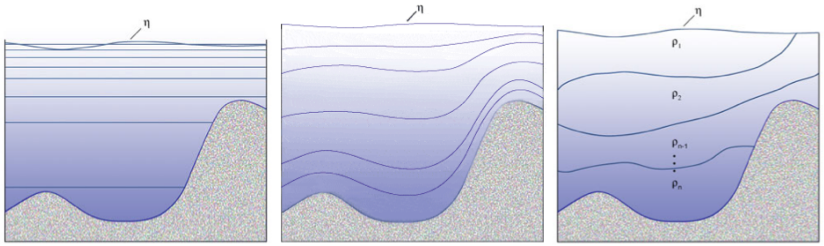
[Griffies et al, 20 https://agupubs.onlinelibrary.wiley.com/doi/full/10.1029/2019MS001954]

Vertical discretization

- Equations for a generalized coordinate system:

Consider a general vertical coordinate, r, which is assumed to be a monotonic function of height, z.

r can be for example: z, σ, or ρ



Any variable can then be written in the new coordinate system: $$A = A(x, y, z(x, y, r, t), t)$$

And vertical derivatives can be written

$$\frac{\partial A}{\partial r} = \frac{\partial z}{\partial r}\frac{\partial A}{\partial z} \qquad \frac{\partial A}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial A}{\partial r}$$

Vertical discretization

- Equations for a generalized coordinate system:

Other derivatives (for a horizontal coordinate s = x,y) can be written using the **chain rule:**

$$\left.\frac{\partial A}{\partial s}\right|_z = \left.\frac{\partial A}{\partial s}\right|_r - \left.\frac{\partial A}{\partial z}\,\frac{\partial z}{\partial s}\right|_r$$

Which gives:

$$\left.\frac{\partial A}{\partial s}\right|_z = \left.\frac{\partial A}{\partial s}\right|_r - \left.\frac{\partial r}{\partial z}\frac{\partial A}{\partial r}\,\frac{\partial z}{\partial s}\right|_r$$

Vertical discretization

- Equations for a generalized coordinate system:

Using the chain rule, you can write the horizontal gradient: $\nabla_z A = \nabla_r A - \frac{\partial A}{\partial r}\frac{\partial r}{\partial z}\nabla_r z$

And the vertical velocity can be written:

$$
\begin{aligned}
w = D_t z &= \left.\frac{\partial z}{\partial t}\right|_r + \left.\frac{\partial z}{\partial x}\right|_r D_t x + \left.\frac{\partial z}{\partial y}\right|_r D_t y + \frac{\partial z}{\partial r} D_t r \\
&= \left.\frac{\partial z}{\partial t}\right|_r + \vec{v} \cdot \nabla_r z + \dot{r}\frac{\partial z}{\partial r}
\end{aligned}
$$

Vertical discretization

- Equations for a generalized coordinate system:

And the total derivative becomes:

$$
\begin{aligned}
D_t A &= \left.\frac{\partial A}{\partial t}\right|_z + \vec{v} \cdot \nabla_z A + w \frac{\partial A}{\partial z} \\
&= \left.\frac{\partial A}{\partial t}\right|_r + \vec{v} \cdot \nabla_r A + \left(w - \left.\frac{\partial z}{\partial t}\right|_r - \vec{v} \cdot \nabla_r z \right) \frac{\partial r}{\partial z} \frac{\partial A}{\partial r} \\
&= \left.\frac{\partial A}{\partial t}\right|_r + \vec{v} \cdot \nabla_r A + \dot{r} \frac{\partial A}{\partial r}
\end{aligned}
$$

And the horizontal pressure gradient:

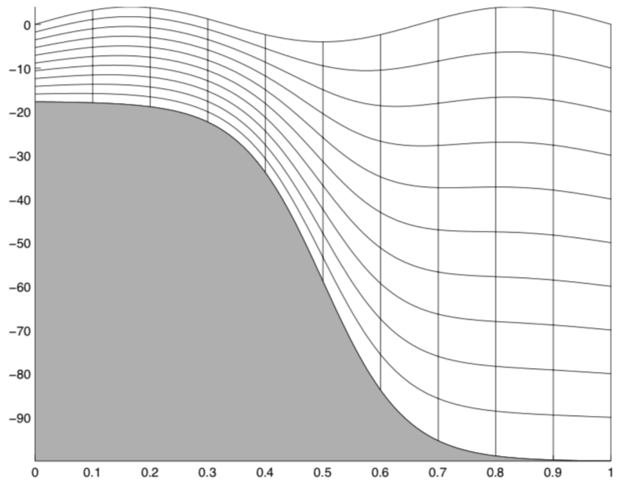
$$
\begin{aligned}
\nabla_z p &= \nabla_r p - \frac{\partial p}{\partial z} \nabla_r z \\
&= \nabla_r p + \rho \nabla_r g z
\end{aligned}
$$

Vertical grid : σ generalized coordinate

- Sigma-coordinate system:

Sigma-coordinates are terrain-following coordinates defined as:

$$r = \sigma = \frac{z}{H_z(x, y)}$$



$$\frac{\partial \sigma}{\partial z} = \frac{1}{H_z}$$

Vertical grid : σ generalized coordinate

- Horizontal and vertical derivatives can be written:

$$\left(\frac{\partial}{\partial x}\right)_z = \left(\frac{\partial}{\partial x}\right)_\sigma - \left(\frac{1}{H_z}\right)\left(\frac{\partial z}{\partial x}\right)_\sigma \frac{\partial}{\partial \sigma}$$

$$\left(\frac{\partial}{\partial y}\right)_z = \left(\frac{\partial}{\partial y}\right)_\sigma - \left(\frac{1}{H_z}\right)\left(\frac{\partial z}{\partial y}\right)_\sigma \frac{\partial}{\partial \sigma}$$

$$\frac{\partial}{\partial z} = \left(\frac{\partial s}{\partial z}\right)\frac{\partial}{\partial \sigma} = \frac{1}{H_z}\frac{\partial}{\partial \sigma}$$

Vertical grid : σ generalized coordinate

- Example: relative vorticity is

$$\zeta = \vec{\nabla} \times \vec{u} \cdot \vec{k} = \left.\frac{\partial v}{\partial x}\right|_z - \left.\frac{\partial u}{\partial y}\right|_z$$

- Beware: $$\left.\frac{\partial v}{\partial x}\right|_\sigma - \left.\frac{\partial u}{\partial y}\right|_\sigma \neq \left.\frac{\partial v}{\partial x}\right|_z - \left.\frac{\partial u}{\partial y}\right|_z$$

Vertical grid : σ generalized coordinate

- Example: (Hydrostatic) Potential vorticity (PV) is defined as

$$q = \left(f\vec{k} + \vec{\nabla} \times \vec{u} \right) . \vec{\nabla}\rho$$

$$q = \left[f + \left.\frac{\partial v}{\partial x}\right|_z - \left.\frac{\partial u}{\partial y}\right|_z \right] \frac{\partial \rho}{\partial z} - \frac{\partial v}{\partial z} \left.\frac{\partial \rho}{\partial x}\right|_z + \frac{\partial u}{\partial z} \left.\frac{\partial \rho}{\partial y}\right|_z$$

- Question: How do you compute PV in sigma-coordinates?
  - *Write the expression of q using only derivatives along sigma coordinates:*

$$\left.\frac{\partial}{\partial x}\right|_\sigma \qquad \left.\frac{\partial}{\partial y}\right|_\sigma \qquad \frac{\partial}{\partial \sigma}$$

Vertical grid : σ generalized coordinate

$$\left(\frac{\partial}{\partial x}\right)_z = \left(\frac{\partial}{\partial x}\right)_\sigma - \left(\frac{1}{H_z}\right)\left(\frac{\partial z}{\partial x}\right)_\sigma \frac{\partial}{\partial \sigma}$$

$$\left(\frac{\partial}{\partial y}\right)_z = \left(\frac{\partial}{\partial y}\right)_\sigma - \left(\frac{1}{H_z}\right)\left(\frac{\partial z}{\partial y}\right)_\sigma \frac{\partial}{\partial \sigma}$$

$$\frac{\partial}{\partial z} = \left(\frac{\partial s}{\partial z}\right)\frac{\partial}{\partial \sigma} = \frac{1}{H_z}\frac{\partial}{\partial \sigma}$$

-

$$q = \left[f + \left.\frac{\partial v}{\partial x}\right|_z - \left.\frac{\partial u}{\partial y}\right|_z\right]\frac{\partial \rho}{\partial z} - \frac{\partial v}{\partial z}\left.\frac{\partial \rho}{\partial x}\right|_z + \frac{\partial u}{\partial z}\left.\frac{\partial \rho}{\partial y}\right|_z$$

Vertical grid : σ generalized coordinate

- Example: (Hydrostatic) Potential vorticity (PV) is defined as

$$q = \left[f + \left.\frac{\partial v}{\partial x}\right|_\sigma - \left.\frac{\partial u}{\partial y}\right|_\sigma \right] \frac{\partial \rho}{\partial z} - \frac{\partial v}{\partial z} \left.\frac{\partial \rho}{\partial x}\right|_\sigma + \frac{\partial u}{\partial z} \left.\frac{\partial \rho}{\partial y}\right|_\sigma$$

- Even if: $$\left.\frac{\partial v}{\partial x}\right|_\sigma - \left.\frac{\partial u}{\partial y}\right|_\sigma \neq \left.\frac{\partial v}{\partial x}\right|_z - \left.\frac{\partial u}{\partial y}\right|_z$$

Vertical grid : isopycnal coordinates

- For isopycnal coordinates, the vertical coordinate is: $r = \rho$

- Potential vorticity (PV) is easily defined as :

$$q = \frac{1}{h}\left[f + \left.\frac{\partial v}{\partial x}\right|_{\rho} - \left.\frac{\partial u}{\partial y}\right|_{\rho}\right]$$

with $$h = \frac{\partial z}{\partial \rho}$$

Vertical grid : σ generalized coordinate

- The vertical velocity in sigma-coordinates is:

$$\Omega(x, y, \sigma, t) = \frac{1}{H_z}\left[w - \left(\frac{z+h}{\zeta+h}\right)\frac{\partial \zeta}{\partial t} - u\frac{\partial z}{\partial x} - v\frac{\partial z}{\partial y}\right]$$

- And the “true” vertical velocity:

$$w = \frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} + \Omega H_z.$$

Vertical grid : σ generalized coordinate

- Equations for CROCO/CROCO become:

$$\frac{\partial u}{\partial t} - fv + \vec{v}\cdot\nabla u = -\frac{\partial \phi}{\partial x} - \left(\frac{g\rho}{\rho_o}\right)\frac{\partial z}{\partial x} - g\frac{\partial \zeta}{\partial x} + \frac{1}{H_z}\frac{\partial}{\partial \sigma}\left[\frac{K_m}{H_z}\frac{\partial u}{\partial \sigma}\right] + \mathcal{F}_u + \mathcal{D}_u$$

$$\frac{\partial v}{\partial t} + fu + \vec{v}\cdot\nabla v = -\frac{\partial \phi}{\partial y} - \left(\frac{g\rho}{\rho_o}\right)\frac{\partial z}{\partial y} - g\frac{\partial \zeta}{\partial y} + \frac{1}{H_z}\frac{\partial}{\partial \sigma}\left[\frac{K_m}{H_z}\frac{\partial v}{\partial \sigma}\right] + \mathcal{F}_v + \mathcal{D}_v$$

$$\frac{\partial C}{\partial t} + \vec{v}\cdot\nabla C = \frac{1}{H_z}\frac{\partial}{\partial \sigma}\left[\frac{K_C}{H_z}\frac{\partial C}{\partial \sigma}\right] + \mathcal{F}_T + \mathcal{D}_T$$

$$\rho = \rho(T, S, P)$$

$$\frac{\partial \phi}{\partial \sigma} = \left(\frac{-gH_z\rho}{\rho_o}\right)$$

$$\frac{\partial H_z}{\partial t} + \frac{\partial (H_z u)}{\partial x} + \frac{\partial (H_z v)}{\partial y} + \frac{\partial (H_z \Omega)}{\partial \sigma} = 0$$

Vertical grid + curvilinear grid

$$\frac{\partial}{\partial t}\left(\frac{H_z u}{mn}\right) + \frac{\partial}{\partial \xi}\left(\frac{H_z u^2}{n}\right) + \frac{\partial}{\partial \eta}\left(\frac{H_z uv}{m}\right) + \frac{\partial}{\partial \sigma}\left(\frac{H_z u\Omega}{mn}\right)$$

$$-\left\{\left(\frac{f}{mn}\right) + v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right\} H_z v =$$

$$-\left(\frac{H_z}{n}\right)\left(\frac{\partial \phi}{\partial \xi} + \frac{g\rho}{\rho_o}\frac{\partial z}{\partial \xi} + g\frac{\partial \zeta}{\partial \xi}\right) + \frac{1}{mn}\frac{\partial}{\partial \sigma}\left[\frac{K_m}{H_z}\frac{\partial u}{\partial \sigma}\right] + \frac{H_z}{mn}\left(\mathcal{F}_u + \mathcal{D}_u\right)$$

$$\frac{\partial}{\partial t}\left(\frac{H_z v}{mn}\right) + \frac{\partial}{\partial \xi}\left(\frac{H_z uv}{n}\right) + \frac{\partial}{\partial \eta}\left(\frac{H_z v^2}{m}\right) + \frac{\partial}{\partial \sigma}\left(\frac{H_z v\Omega}{mn}\right)$$

$$+\left\{\left(\frac{f}{mn}\right) + v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right\} H_z u =$$

$$-\left(\frac{H_z}{m}\right)\left(\frac{\partial \phi}{\partial \eta} + \frac{g\rho}{\rho_o}\frac{\partial z}{\partial \eta} + g\frac{\partial \zeta}{\partial \eta}\right) + \frac{1}{mn}\frac{\partial}{\partial \sigma}\left[\frac{K_m}{H_z}\partial v \partial \sigma\right] + \frac{H_z}{mn}\left(\mathcal{F}_v + \mathcal{D}_v\right)$$

$$\frac{\partial}{\partial t}\left(\frac{H_z C}{mn}\right) + \frac{\partial}{\partial \xi}\left(\frac{H_z uC}{n}\right) + \frac{\partial}{\partial \eta}\left(\frac{H_z vC}{m}\right) + \frac{\partial}{\partial \sigma}\left(\frac{H_z \Omega C}{mn}\right) =$$

$$\frac{1}{mn}\frac{\partial}{\partial s}\left[\frac{K_C}{H_z}\frac{\partial C}{\partial \sigma}\right] + \frac{H_z}{mn}\left(\mathcal{F}_C + \mathcal{D}_C\right)$$

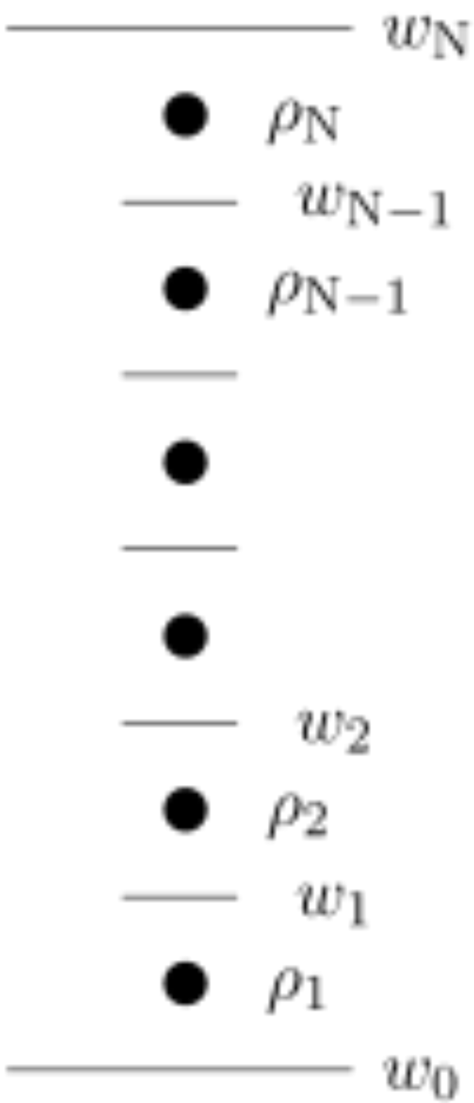
$$\rho = \rho(T, S, P)$$

$$\frac{\partial \phi}{\partial \sigma} = -\left(\frac{g H_z \rho}{\rho_o}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{H_z}{mn}\right) + \frac{\partial}{\partial \xi}\left(\frac{H_z u}{n}\right) + \frac{\partial}{\partial \eta}\left(\frac{H_z v}{m}\right) + \frac{\partial}{\partial \sigma}\left(\frac{H_z \Omega}{mn}\right) = 0.$$

Vertical discretization

CROCO: Staggered vertical grid



Vertical grid : σ generalized coordinate

CROCO: Generalized σ-Coordinate

Stretching & condensing of vertical resolution:

When activating the cpp key `NEW_S_COORD`, we have:

$$z(x,y,\sigma,t) = \zeta(x,y,\sigma) + [\zeta(x,y,t) + h(x,y)]z_0(x,y,\sigma)$$
$$z_0(x,y,\sigma) = \frac{h_c\sigma + h(x,y)Cs(\sigma)}{h_c + h(x,y)}$$

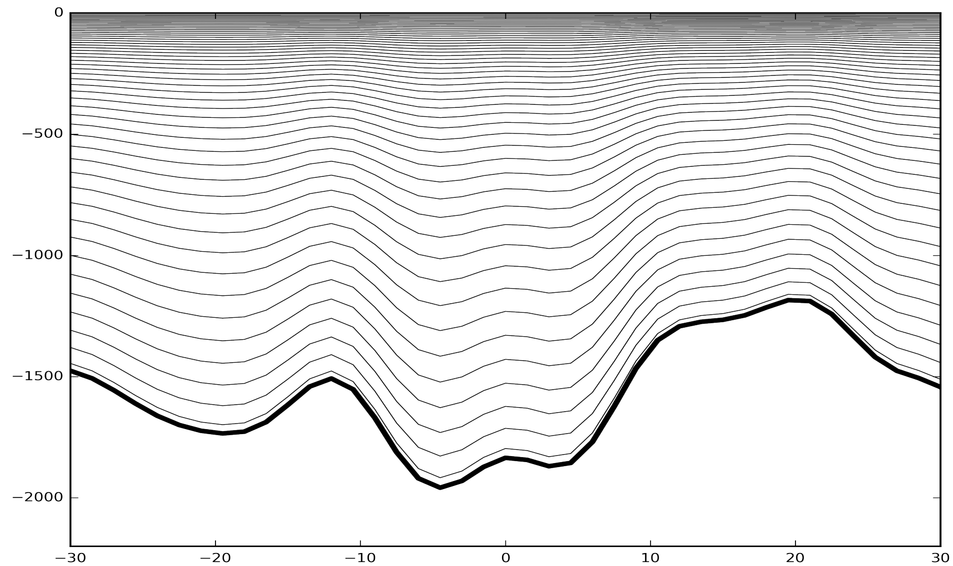
with :

- $z_0(x,y,\sigma)$ a nonlinear vertical transformation
- $\zeta(x,y,\sigma)$ the free-surface
- $h(x,y)$ the ocean bottom
- σ a fractional vertical stretching coordinate, $-1 \leq \sigma \leq 0$
- h_c a positive thickness controlling the stretching
- $Cs(\sigma)$ a nondimensional, monotonic, vertical stretching, $-1 \leq (C\sigma) \leq 0$

Vertical grid : σ generalized coordinate

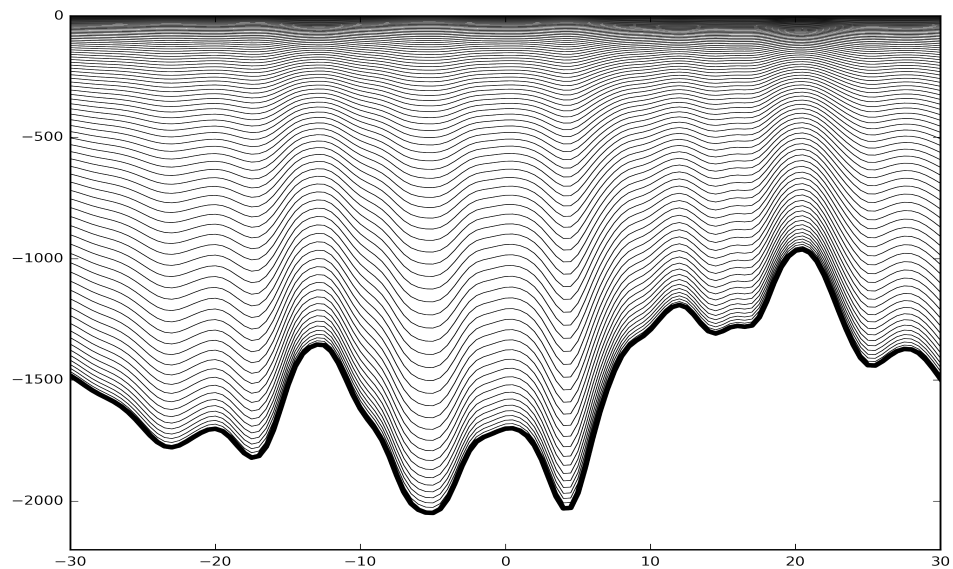
- 50 vertical levels

  $\theta=7$, $b=2$, $h_c=300$ m



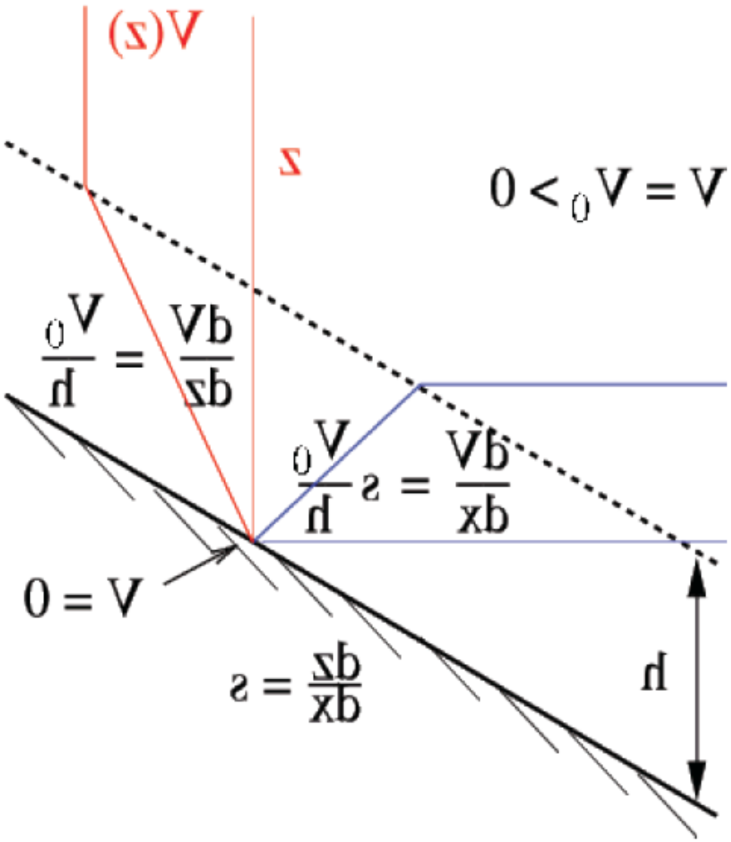
- 80 vertical levels

  $\theta=6$, $b=4$, $h_c=300$ m



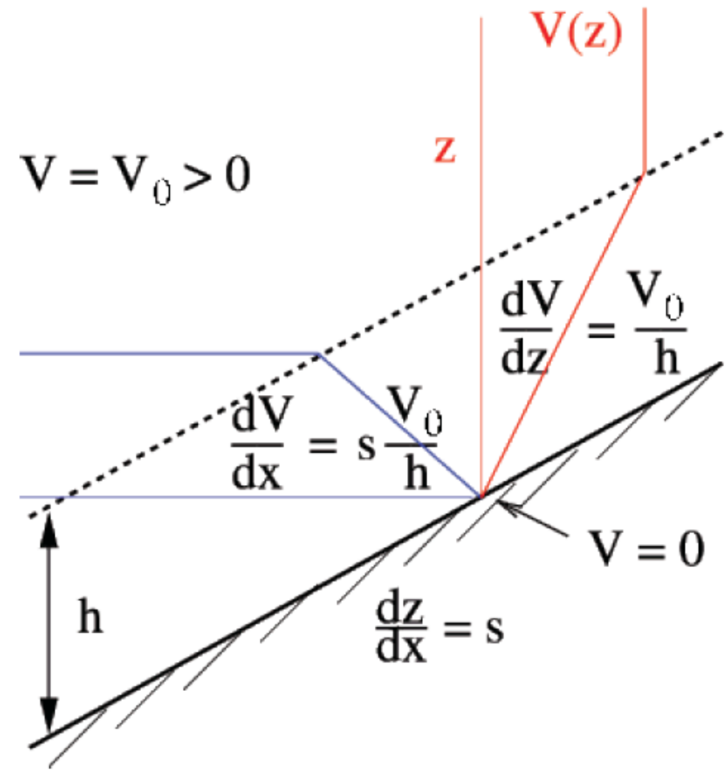
Ex: Topographic vorticity generation

Generation of vertical vorticity within the bottom boundary layer:



Current flowing with the coast on its left in the Northern hemisphere (opposite to Kelvin wave propagation)

= Positive vorticity generation

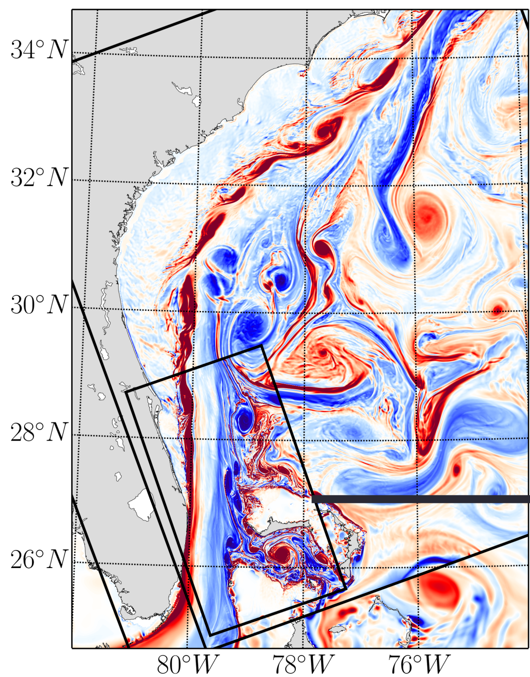


Current flowing with the coast on its right in the Northern hemisphere (same than Kelvin wave propagation)

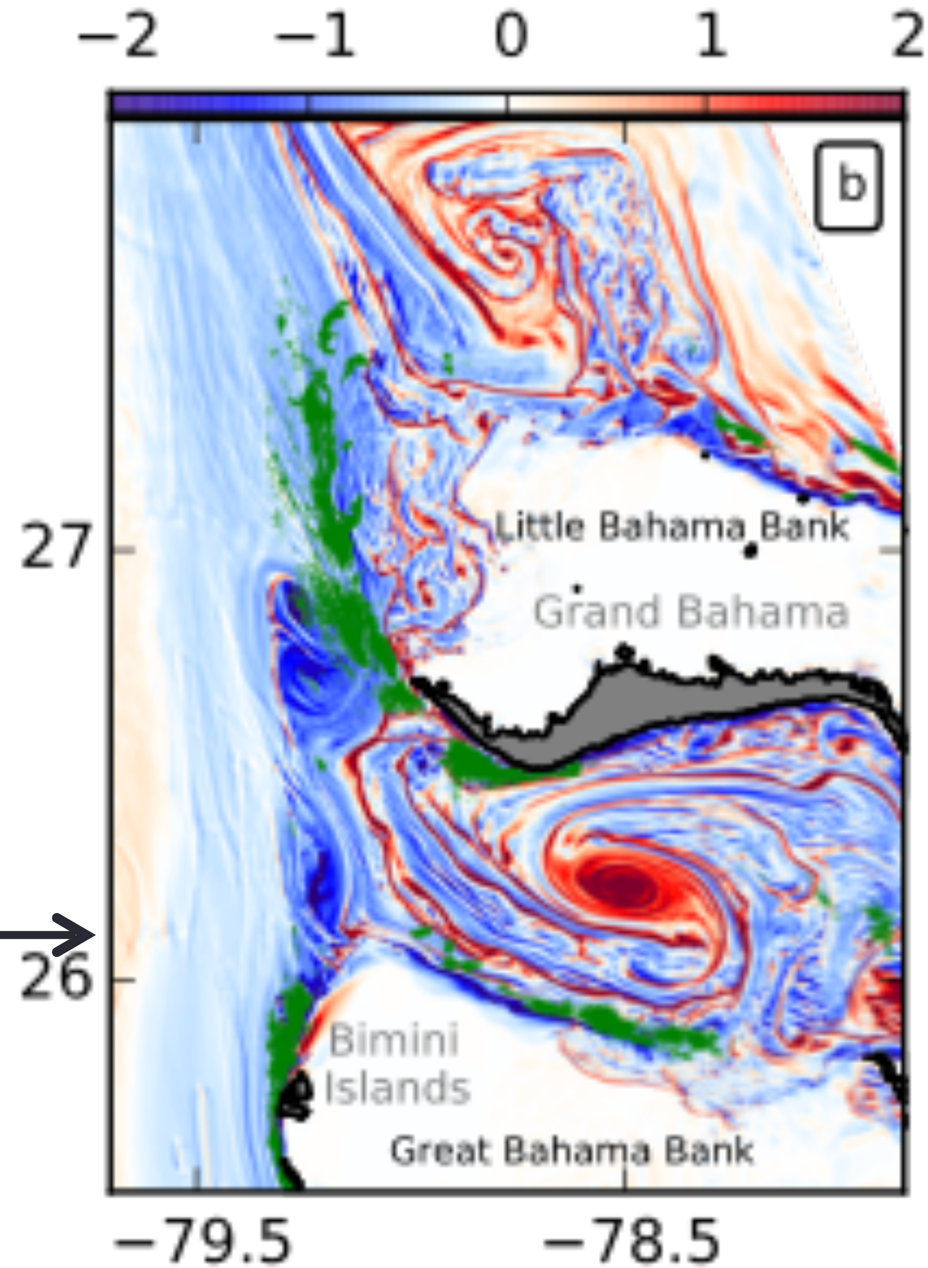
= Negative vorticity generation

Ex: Topographic vorticity generation

Anticyclonic vorticity generation by bottom drag on the slope on Bahamas slope

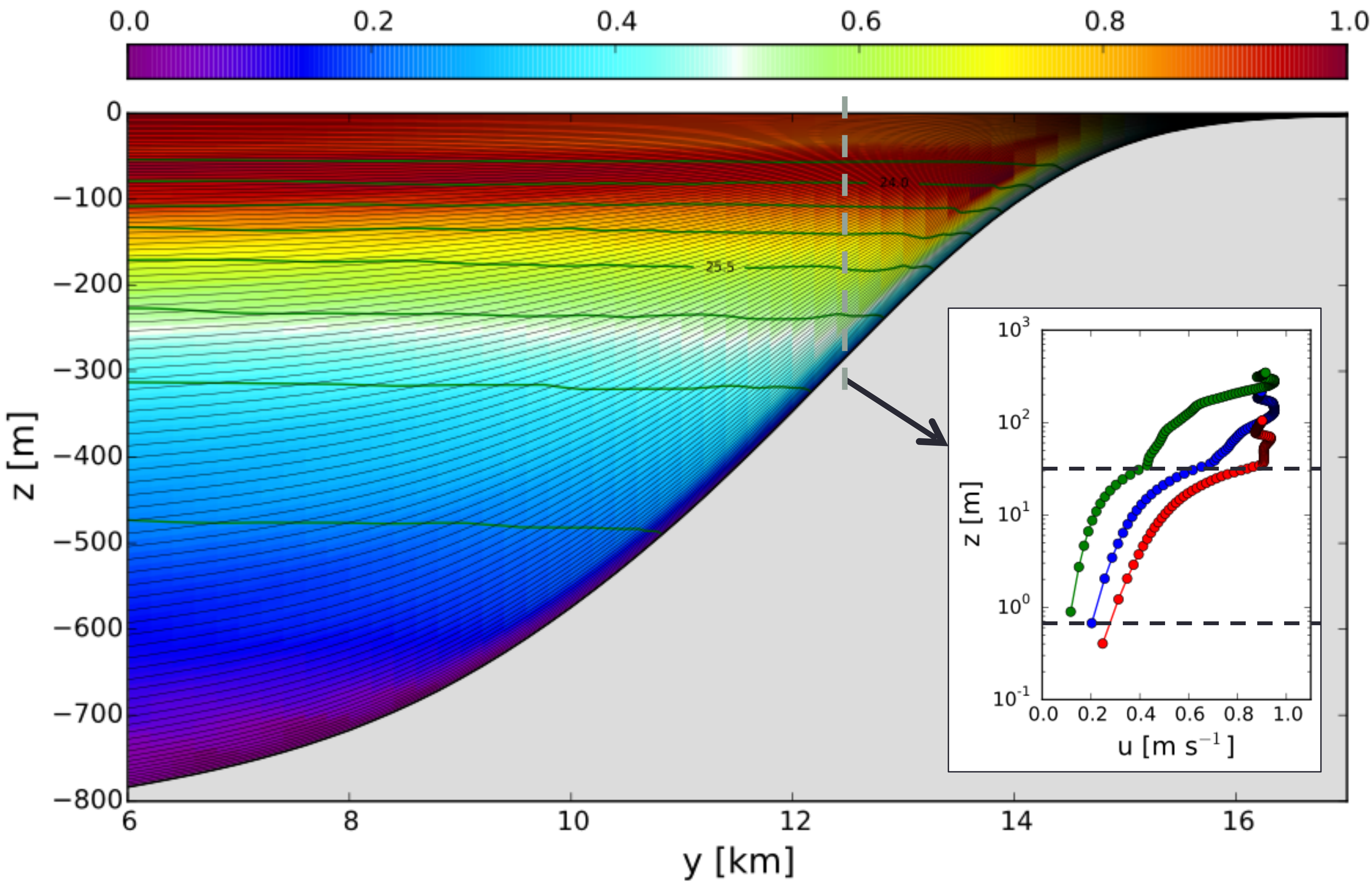


$$\Delta x = 200\,\mathrm{m}$$

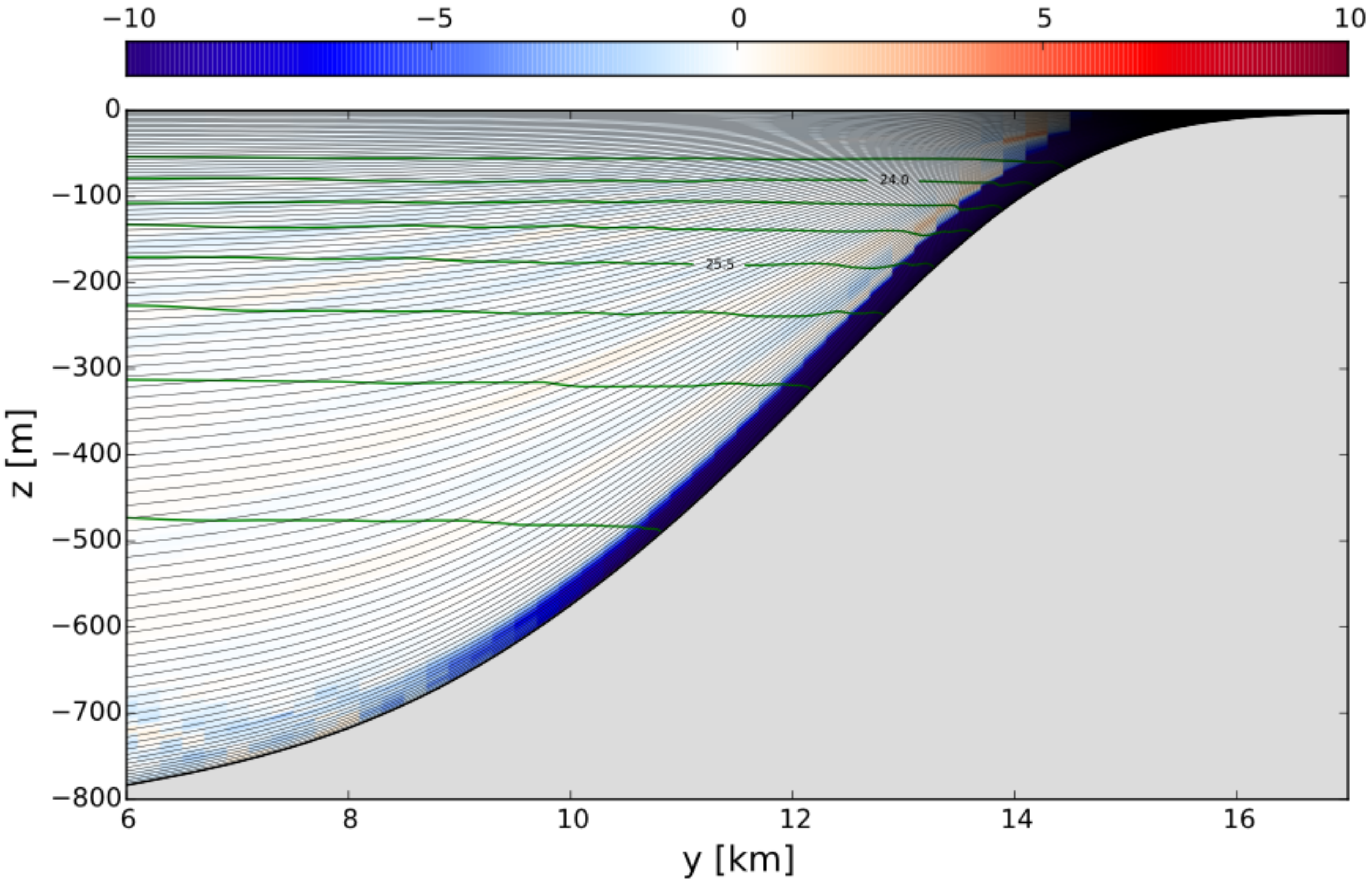


Topographic generation of submesoscale centrifugal instability and energy dissipation

Ex: Topographic vorticity generation

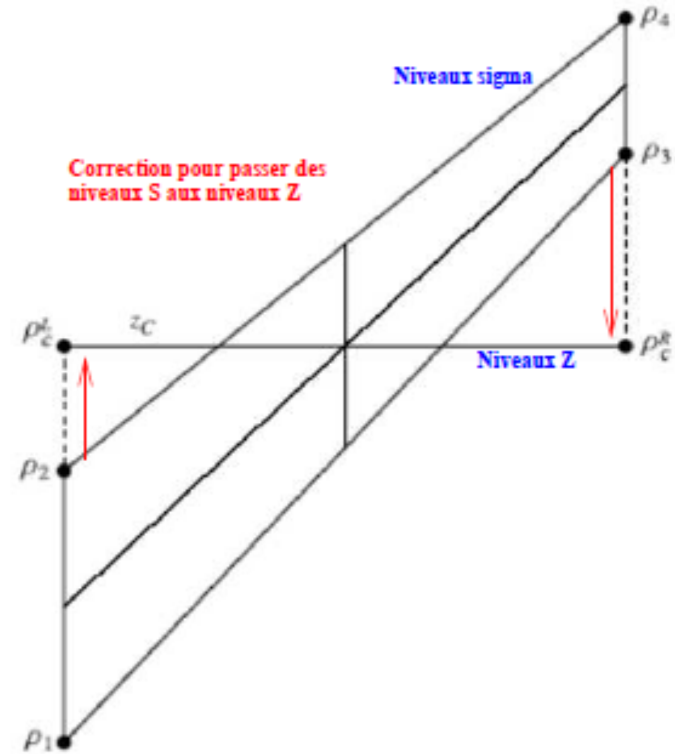


Ex: Topographic vorticity generation



Pressure Gradient force

- Truncation errors are made from calculating the baroclinic pressure gradients across sharp topographic changes such as the continental slope



- Difference between 2 large terms

$$\boxed{-\frac{1}{\rho_0}\left.\frac{\partial P}{\partial x}\right|_z = -\frac{1}{\rho_0}\left.\frac{\partial P}{\partial x}\right|_s + \frac{1}{\rho_0}\cdot\frac{\partial P}{\partial z}\cdot\left.\frac{\partial z}{\partial x}\right|_s,}$$

- Errors can appear in the unforced flat stratification experiment

$$\epsilon \equiv \frac{\left|\left.\frac{\partial P}{\partial x}\right|_s - \frac{\partial P}{\partial z}\cdot\left.\frac{\partial z}{\partial x}\right|_s\right|}{\left|\left.\frac{\partial P}{\partial x}\right|_s\right| + \left|\frac{\partial P}{\partial z}\cdot\left.\frac{\partial z}{\partial x}\right|_s\right|} \ll 1,$$

Reducing PGF Truncation Errors

- Smoothing the topography using a nonlinear filter and a criterium:

$$\boxed{r = \Delta h / h < 0.2}$$

- Using a "density formulation"

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial x}\bigg|_z = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}\bigg|_{z=\zeta} - \frac{g}{\rho_0}\int_z^{\zeta}\frac{\partial \rho}{\partial x}\bigg|_z \mathrm{d}z'$$

$$= -\frac{g\rho(\zeta)}{\rho_0}\frac{\partial \zeta}{\partial x} - \frac{g}{\rho_0}\int_z^{\zeta}\left[\frac{\partial \rho}{\partial x}\bigg|_s - \frac{\partial \rho}{\partial z'}\frac{\partial z'}{\partial x}\bigg|_s\right]\mathrm{d}z',$$

- Using high order schemes to reduce the truncation error (4th order, McCalpin, 1994)

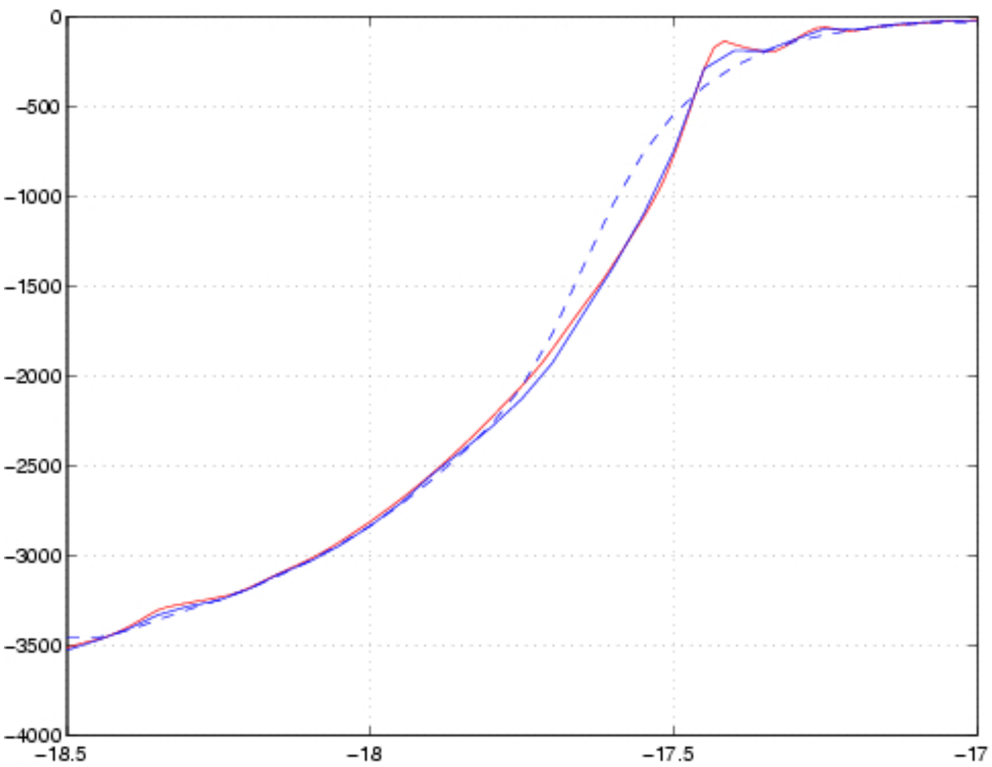
- Gary, 1973: substracting a reference horizontal averaged value from density ($\rho' = \rho - \rho_a$) before computing pressure gradient

- Rewritting Equation of State: reduce passive compressibility effects on pressure gradient

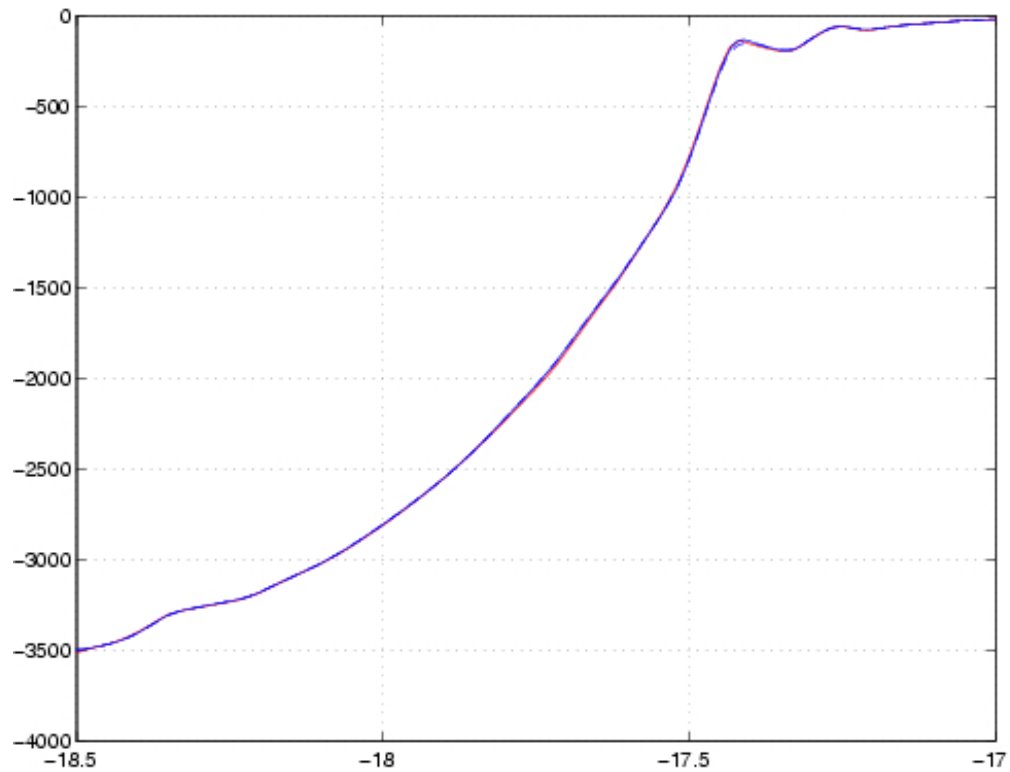
Smoothing methods

- ***r* = *Δh* / *h*** is the slope of the logarithm of *h*
- One method (CROCO) consists of smoothing *ln(h)* until $r < r_{max}$

Res: 5 km ; r_{max} = 0.25



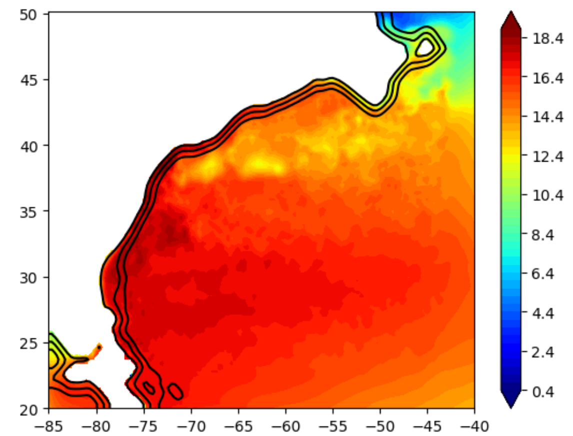
Res: 1 km ; r_{max} = 0.25



For Activity : Computing z and vertical interpolations

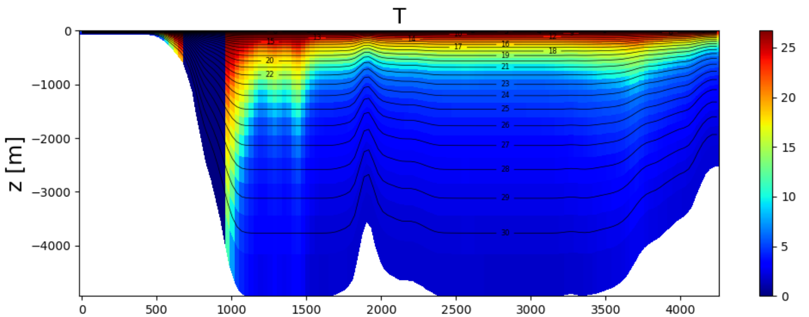
- See example in :

  https://github.com/Mesharou/mesharou.github.io/blob/master/ModNum/example_croco.ipynb




```
# some tools to interpolate croco outputs vertically

import tools as to
```

```
##############################################################################
#Load variables and parameters
##############################################################################

nc = Dataset(ncfile, 'r')

temp3d=np.array(nc.variables['temp'][-1,:,:,:])


#Load some parameters

zeta=nc.variables['zeta'][-1,:,:]
topo=nc.variables['h'][:]
pm=nc.variables['pm'][:]
pn=nc.variables['pn'][:]

hc = nc.hc
Cs_r = nc.Cs_r
Cs_w = nc.Cs_w

#close netcdf file
nc.close()

##############################################################################
#Compute vertical coordinates
##############################################################################

(z_r,z_w) =  to.zlevs(topo,zeta, hc, Cs_r, Cs_w)

##############################################################################
#Interpolate a variable on a given depth
##############################################################################

t400 = to.vinterp(temp3d,z_r,-500,topo=topo,cubic=1)
```



For Activity : Computing z and vertical interpolations

- An another using xarray in

  https://github.com/Mesharou/mesharou.github.io/blob/master/ModNum/example_croco_xarray.ipynb