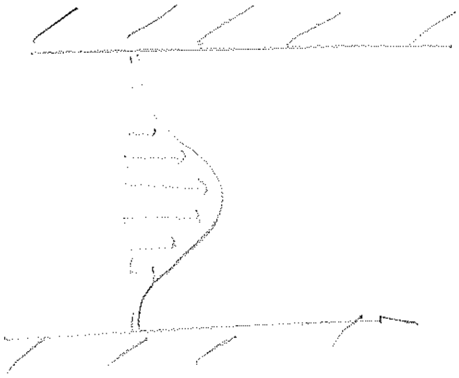
II.3 Parallel Flows

Instability of a plane-parallel barotropic incompressible flow.



background velocity shear $U(y)$

- Equations: [vorticity equation for a 2D incompressible flow.]

$$\frac{D\vec{u}}{Dt} = -\frac{\vec{\nabla} p}{\rho} + \vec{F} + \nu \nabla^2 \vec{u} \qquad \text{(no rotation)}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\frac{\vec{\nabla} p}{\rho} + \vec{F} + \nu \nabla^2 \vec{u}$$

for vorticity $\vec{\omega} = \vec{\nabla}\wedge\vec{u}$

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla}\wedge\left((\vec{u}\cdot\vec{\nabla})\vec{u}\right) = \vec{\nabla}\wedge\vec{F} + \underbrace{\vec{\nabla}\wedge\left(\nu\vec{\nabla}^2\vec{u}\right)}_{=\nu\vec{\nabla}^2\vec{\omega}} - \underbrace{\vec{\nabla}\wedge\left(\frac{\vec{\nabla} p}{\rho}\right)}_{=0}$$

$$\begin{cases} (\vec{u}\cdot\vec{\nabla})\vec{u} = \frac{1}{2}\vec{\nabla}(|\vec{u}|^2) + \vec{\omega}\wedge\vec{u} \\ \vec{\nabla}\wedge(\vec{\omega}\wedge\vec{u}) = \underbrace{\vec{\omega}\,(\vec{\nabla}\cdot\vec{u})}_{=\vec{0}} - (\vec{\omega}\cdot\vec{\nabla})\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{\omega} \end{cases}$$

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{\omega} = (\vec{\omega}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}\wedge\vec{F} + \nu\vec{\nabla}^2\vec{\omega}$$

with no viscosity / forcings:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\vec{\nabla})\vec{u} \quad \text{in 3d}$$

$$\boxed{\frac{D\zeta}{Dt} = 0} \quad \text{in 2d} \qquad \vec{\omega} = \begin{vmatrix} 0 \\ 0 \\ \zeta = v_x - u_y \end{vmatrix}$$

$$\vec{u} = \begin{vmatrix} u \\ v \\ 0 \end{vmatrix}$$

we introduce a streamfunction ψ such that:

$$u, v = -\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x} \qquad \zeta = v_x - u_y = \nabla^2\psi$$

so $\quad \frac{D}{Dt}(\nabla^2\psi) = 0 \qquad u(\nabla^2\psi)_x + v(\nabla^2\psi)_y$

$$\frac{\partial}{\partial t}\nabla^2\psi + (\vec{u}\cdot\vec{\nabla})\nabla^2\psi = 0$$

$$\nabla^2\psi_t + J(\nabla^2\psi, \psi) = 0$$

$$\to (\nabla^2\psi)_x\psi_y - (\nabla^2\psi)_y\psi_x$$

- linearization: $\psi = \psi_0 + \phi$ with $U(y) = -\frac{\partial\psi_0}{\partial y}$

$$\nabla^2\phi_t + U\nabla^2\phi_x - \phi_x U''(y) = 0$$

- Fourier transformation

$$\phi(x,y,t) = \hat{\phi}(y)\, e^{ik(x-ct)} \qquad \text{with } c = c_r + i c_i$$

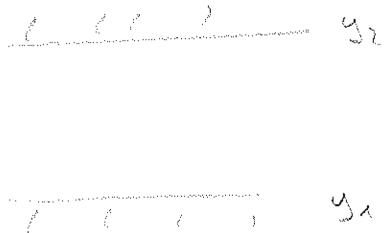
which gives:

$$\left(-k^2\hat{\phi} + \hat{\phi}''\right)(-ikc) + U_{(y)}(ik)\left(-k^2\hat{\phi} + \hat{\phi}''\right) - ik\hat{\phi}\,U''_{(y)} = 0$$

$$\left(U_{(y)} - c\right)\left(\hat{\phi}'' - k^2\hat{\phi}\right) - U''_{(y)}\hat{\phi} = 0$$

$$\boxed{\hat{\phi}'' - \left[k^2 + \frac{U''_{(y)}}{U_{(y)} - c}\right]\hat{\phi} = 0}$$

with boundary conditions



$$v\big|_{y=y_{1,2}} = \hat{\phi}_x\big|_{y=y_{1,2}} = 0$$

$$\boxed{\hat{\phi}\big|_{y=y_1,y_2} = 0}$$

Difficult to solve analytically - + singularities when $U - c = 0$ => critical layers

• Criteria for instability - following Rayleigh's work:

we multiply by conjugate ϕ^* and integrate between y_1 and y_2:

$$\int_{y_1}^{y_2}\left(\hat{\phi}^*\hat{\phi}'' - \left(k^2 + \frac{U''}{U-c}\right)\hat{\phi}^*\hat{\phi}\right)dy = 0$$

$$\int_{y_1}^{y_2} \hat{\phi}^* \hat{\phi}'' \, dy = \underbrace{\left[\hat{\phi}' \hat{\phi}^* \right]_{y_1}^{y_2}}_{=0} - \int_{y_1}^{y_2} \hat{\phi}^{*\prime} \hat{\phi}' \, dy$$

$$= - \int_{y_1}^{y_2} | \hat{\phi}' |^2 \, dy$$

So

$$\int_{y_1}^{y_2} \left(|\hat{\phi}'|^2 + \left(k^2 + \frac{U''(y)}{U-c} \right) |\hat{\phi}|^2 \right) dy = 0$$

The imaginary part is: $\quad (c = c_r + i c_i)$

$$c_i \int \frac{U''}{|U-c|^2} |\hat{\phi}|^2 \, dy = 0$$

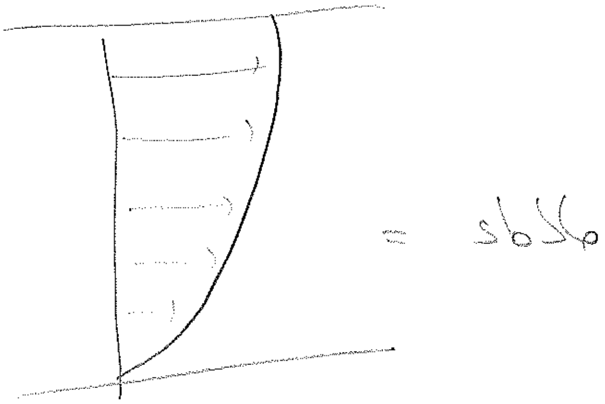
So in the absence of critical layers $(U-c \neq 0)$ if the flow is unstable $(c_i \neq 0)$, we must have

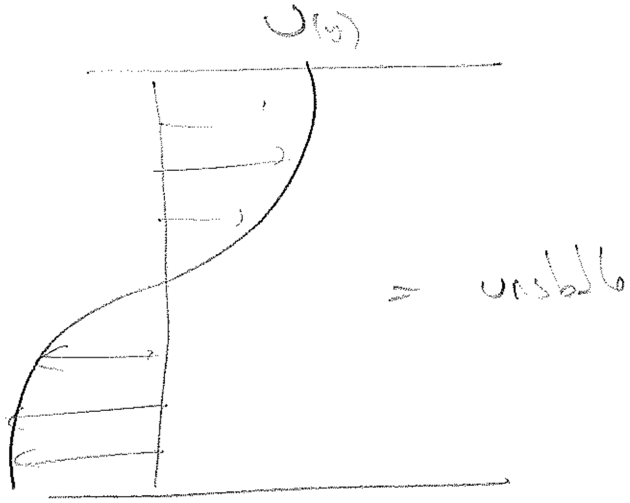
$$\boxed{U'' = 0}$$

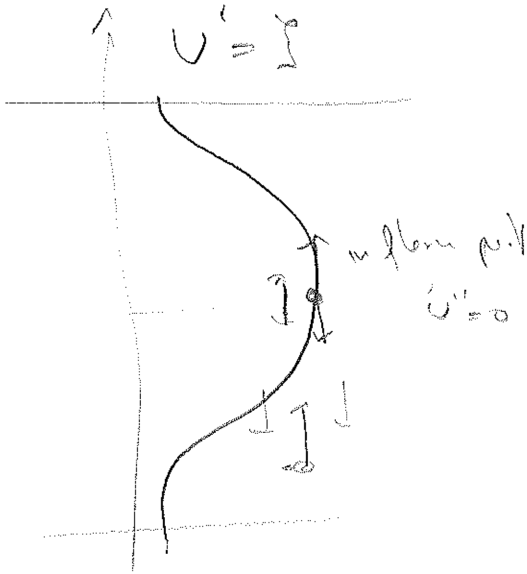
somewhere in the flow –

= we need an inflexion point in the velocity profile

= Rayleigh's necessary condition for instability




U(y)



- A stronger condition is given by the real part:

$$(1) \quad \int_{y_1}^{y_2} U'' \frac{U - c_r}{|U-c|^2} |\hat{\phi}|^2 \, dy = - \int \left(|\hat{\phi}'|^2 + k^2 |\hat{\phi}|^2 \right) dy < 0$$

So if $c_i \neq 0$ (unstable flow) somewhere we have

$$(c_r - U_i) \int_{y_1}^{y_2} \frac{U''}{|U-c|^2} |\hat{\phi}|^2 \, dy = 0 \quad (2)$$

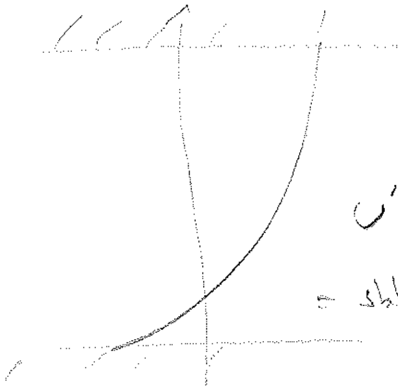
with $U_i = U(y = y_i)$ the inflexion point

So (1) + (2) gives the condition:

$$\int_{y_1}^{y_2} (U - U_i) \frac{U''}{|U-c|^2} |\hat{\phi}|^2 \, dy < 0$$

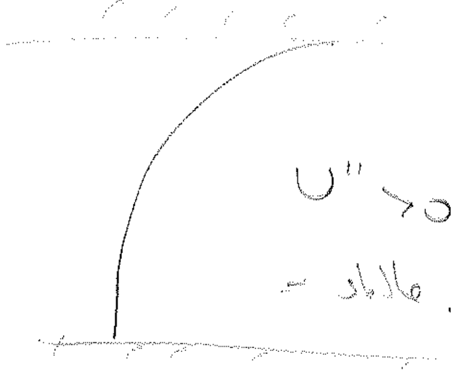
~~So if $U(y)$ is a monotonic function with one inflexion point~~, a necessary condition for instability is that $U''(U - U_i) < 0$ somewhere in the flow

= theorem of Rayleigh-Fjørtoft



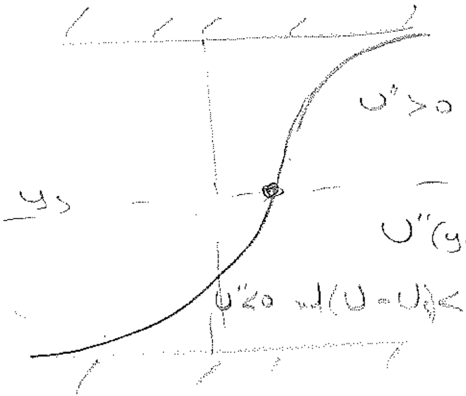
$U'' < 0$

= stable



$U'' > 0$

= stable.



y_s

$U'' > 0$ and $(U - U_s) > 0$

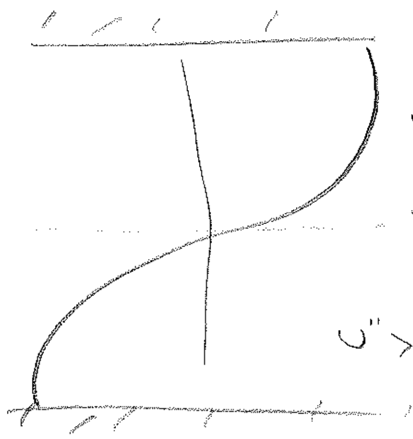
$U''(y_s) = 0$

$U'' < 0$ and $(U - U_s) < 0$

= stable

$U''(U - U_s) \geq 0$

extrema of shear are at the boundaries



$U'' < 0$ and $(U - U_s) > 0$

y_s

$U'' > 0$ and $(U - U_s) < 0$

$\Rightarrow$ possibly <u>unstable</u>

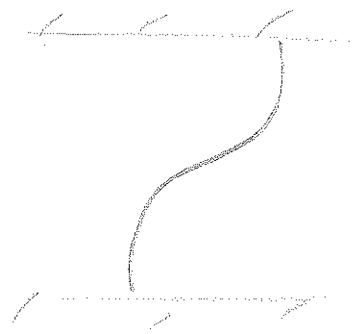
<u>Note:</u> if we add viscous effects

$$\frac{D\vec{u}}{Dt} = -\frac{\vec{\nabla} p}{\rho_0} + \nu \nabla^2 \vec{u}$$

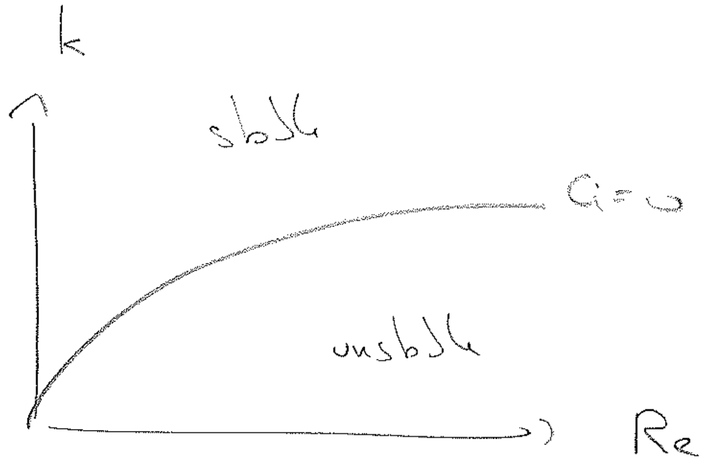
This is the Orr - Sommerfeld equation.

that can be solved numerically.

For example



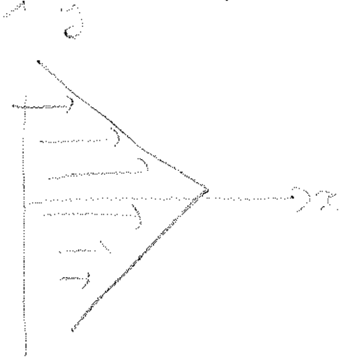
is unstable in the inviscid case.



$$Re = \frac{UL}{\nu}$$

II.3.2. Piecewise linear flows

It is simpler to consider piecewise linear flows:



with $\phi(x,y,t) = \hat{\phi}(y) e^{ik(x-ct)}$

$$(U-c)(\hat{\phi}'' - k^2 \hat{\phi}) - U'' \hat{\phi} = 0$$

→ if velocity is linear $U'' = 0$

$$\hat{\phi} = A e^{\pm ky} \quad (\text{if } U \neq c)$$

we need matching conditions:

ⓐ continuity of pressure:

we have $\dfrac{\partial u'}{\partial t} + U \dfrac{\partial u'}{\partial x} + v' \dfrac{\partial U}{\partial y} = -\dfrac{1}{\rho_0} \dfrac{\partial p'}{\partial x}$

$\Leftrightarrow$ at $y = a$

$$\Delta p = p_{y \to a^+} - p_{y \to a^-} = 0$$

assuming $u' = -\hat{\phi}_y e^{ik(x-ct)}$

$v' = \hat{\phi}_x e^{ik(x-ct)} = ik \hat{\phi} e^{ik(x-ct)}$

$\Leftrightarrow \quad ik(U-c)\hat{\phi}_y - ik \hat{\phi} U_y = -ik \hat{p}$

So the matching condition is

$$\boxed{\Delta\left[(U-c)\hat{\phi}_y - \hat{\phi}U_y\right] = 0}$$

(b) material interface condition:

$$v = \frac{D\eta}{Dt}$$

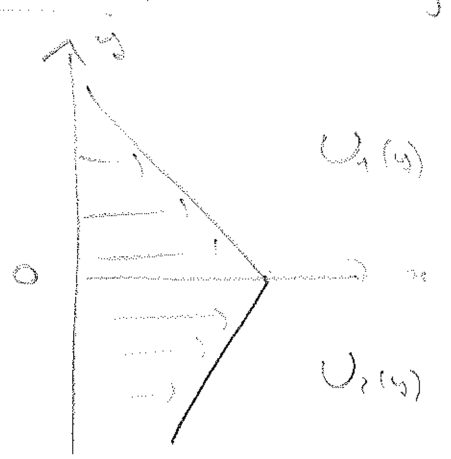
$$\Leftrightarrow \quad \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial x} \qquad \text{(linearised)}$$

$$\Rightarrow \quad (U-c)\hat{\eta} = \hat{\phi}$$

Second matching condition

$$\boxed{\Delta\left[\frac{\hat{\phi}}{U-c}\right] = 0}$$

1. <u>Stable case</u>: the edge waves.



$U_1(0) = U_2(0) = U_0$

$$\Rightarrow \quad \hat{\phi} = \begin{cases} \phi_1 e^{-ky} & \text{for } y>0 \\ \phi_2 e^{+ky} & \text{for } y<0 \end{cases}$$

we find c by applying the matching cond. 7

$$\phi_1 = \phi_2$$

$$-k(U_0 - c)\phi_1 - \phi_1 U_{1y} = k(U_0 - c)\phi_2 - \phi_2 U_{2y}$$

$$\Rightarrow \quad \boxed{c = U_0 + \frac{U_{1y} - U_{2y}}{2k}}$$

purely real = stable case with edge waves propagating on the interface.

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2. Unstable case :