Internal Waves in the Ocean

Master 2 — Physique de l'Océan et du Climat

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Outline

1. **A general introduction to ocean waves**
    - **1.1. Overview of ocean waves**
    - **1.2. Equations**
    - **1.3. Surface waves**
2. What are internal waves ? Why do we study internal waves ?
3. Internal waves in the two-layer shallow-water model
4. Internal waves in the continuously-stratified model
5. Generation of internal waves
6. Propagation of internal waves
7. Dissipation of internal waves and impacts

Outline

- The course is largely based on:
  - Gerkema- Zimmerman (2008). *An introduction to internal waves* https://www.jgula.fr/Ondes/gerkema.pdf

- Other useful references are:
  - Leblond-Mysak (1977) *: Waves in the ocean*
  - Whitham (1974) *: Linear and nonlinear waves*
  - Gill (1982) *: Atmosphère-Ocean Dynamics*
  - Kundu-Cohen (1987). *Fluid Mechanics. Third edition*
  - Cushman-Roisin*. Introduction to geophysical fluid Dynamics*

- **Some important research articles and reviews are available here: https://www.jgula.fr/Ondes/**

Outline

1. **A general introduction to ocean waves**
    1. **Waves in the oceans**
    2. **Equations**
    3. **Surface waves**

2. What are internal waves ? Why do we study internal waves ?

3. Internal waves in the two-layer shallow-water model

4. Internal waves in the continuously-stratified model

5. Generation of internal waves

6. Propagation of internal waves

7. Dissipation of internal waves and impacts

1. A general introduction to ocean waves

Surface waves

1. A general introduction to ocean waves

Surface waves

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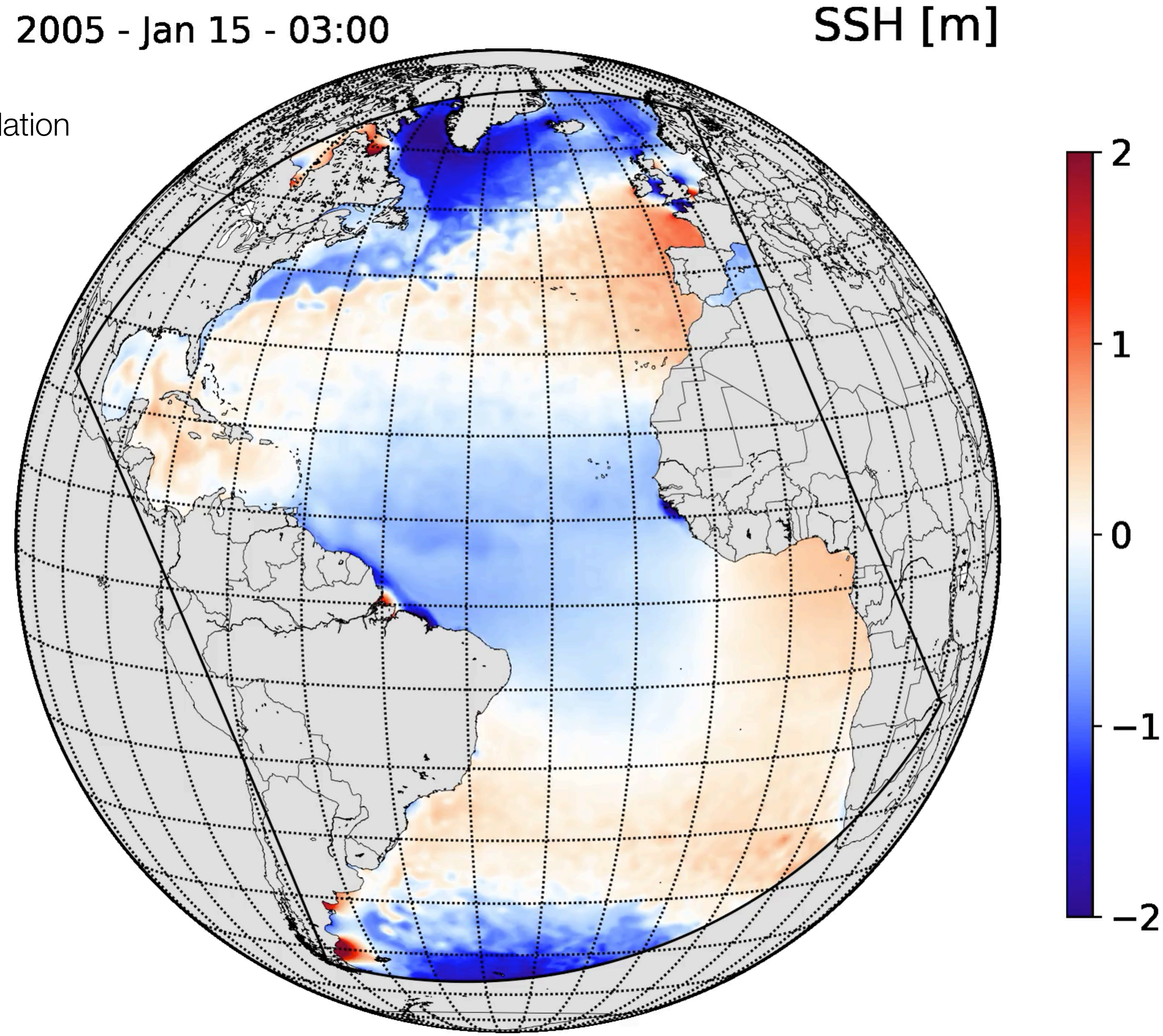
1. A general introduction to ocean waves

Surface capillary waves

1. A general introduction to ocean waves

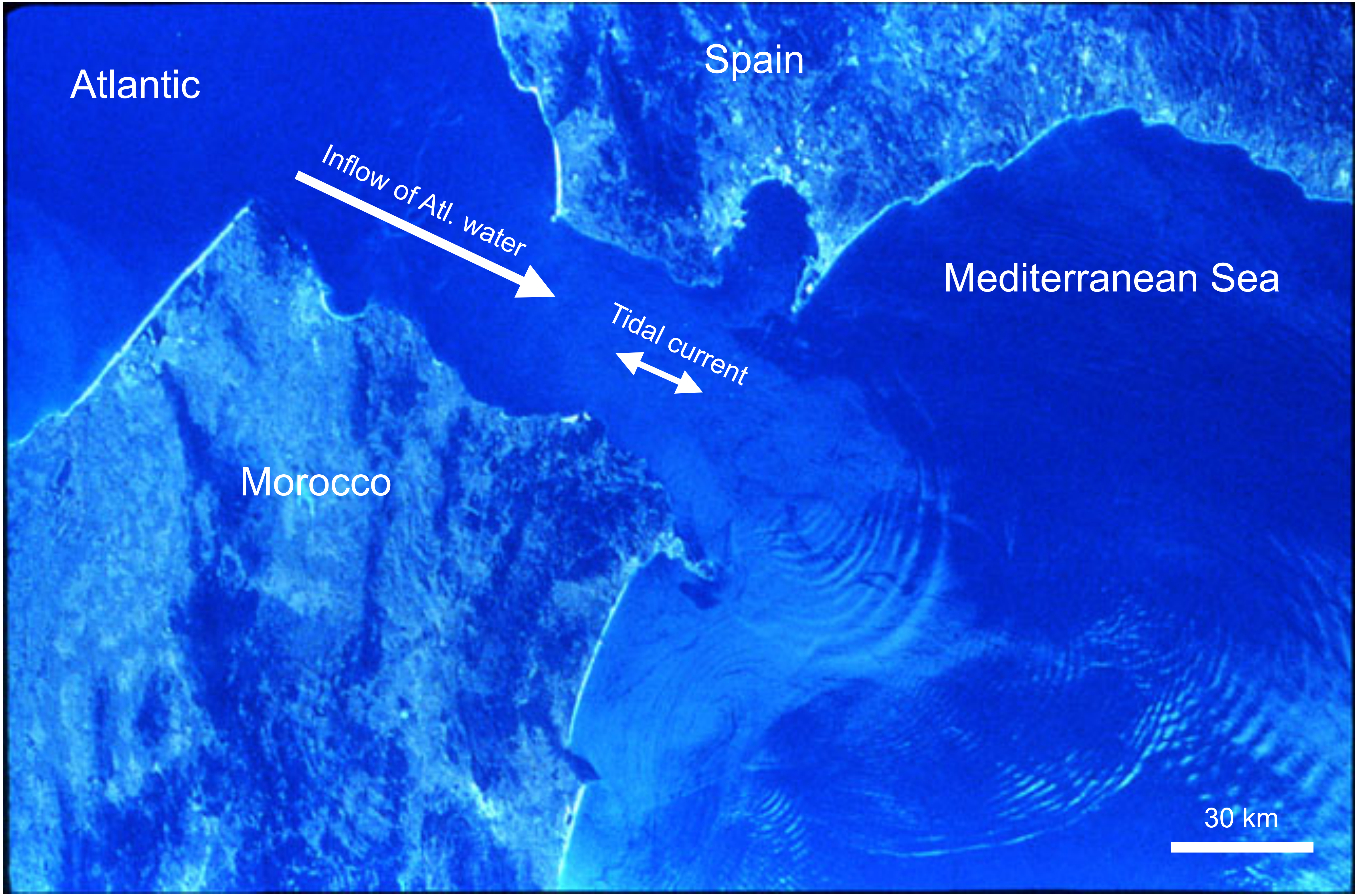
GIGATL simulation



Barotropic tidal wave


ENEOS
龍 泉 洞
Ryusendo Cave
51Km
三 王 岩
Sannoiwa Rocks
20Km
浄 土 ヶ 浜
Jodogahama Beach
4Km
シートピアなあど
Seatopia Naado
1Km

1. A general introduction to ocean waves



Internal wave (soliton) with a surface signature

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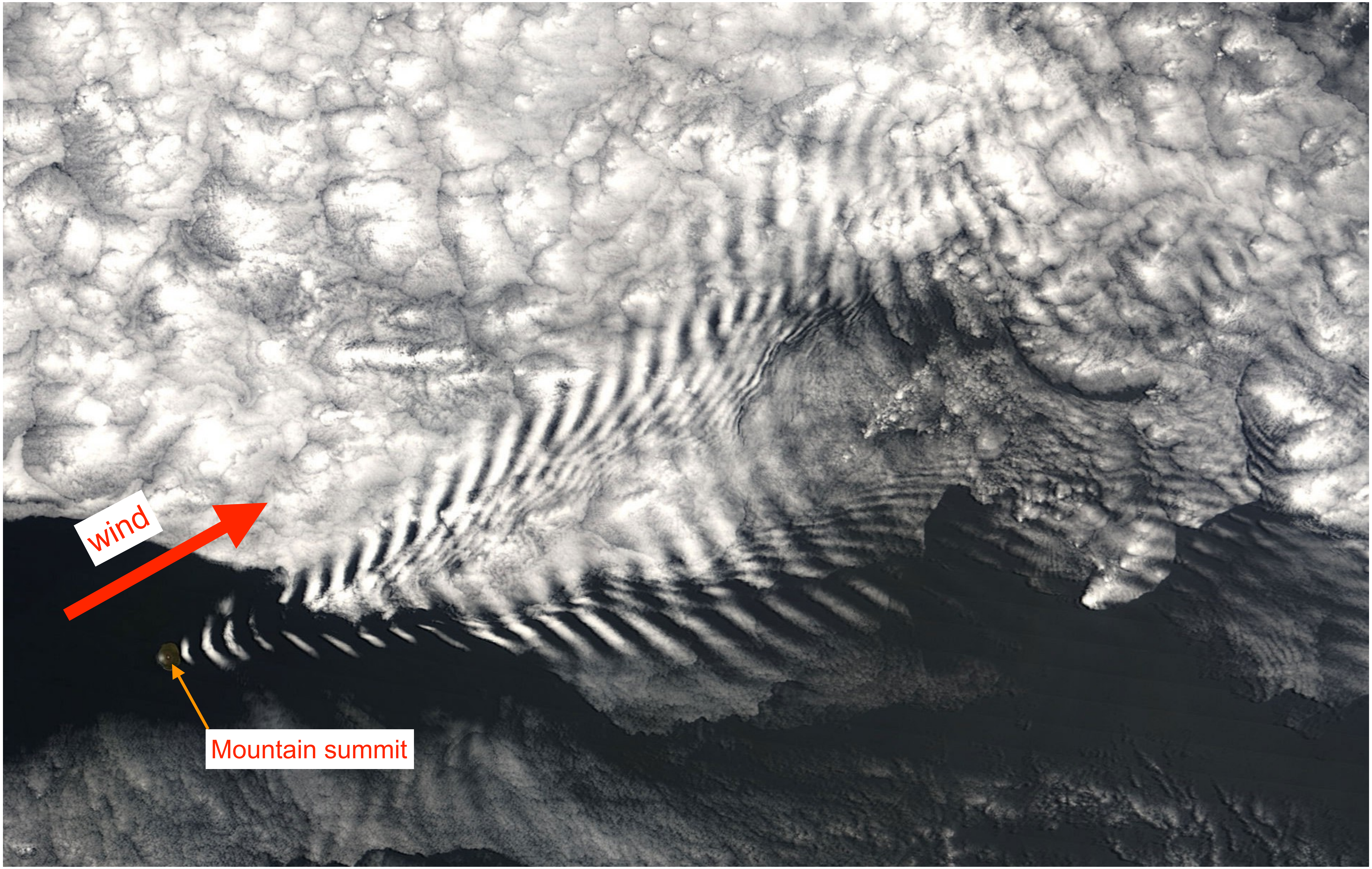
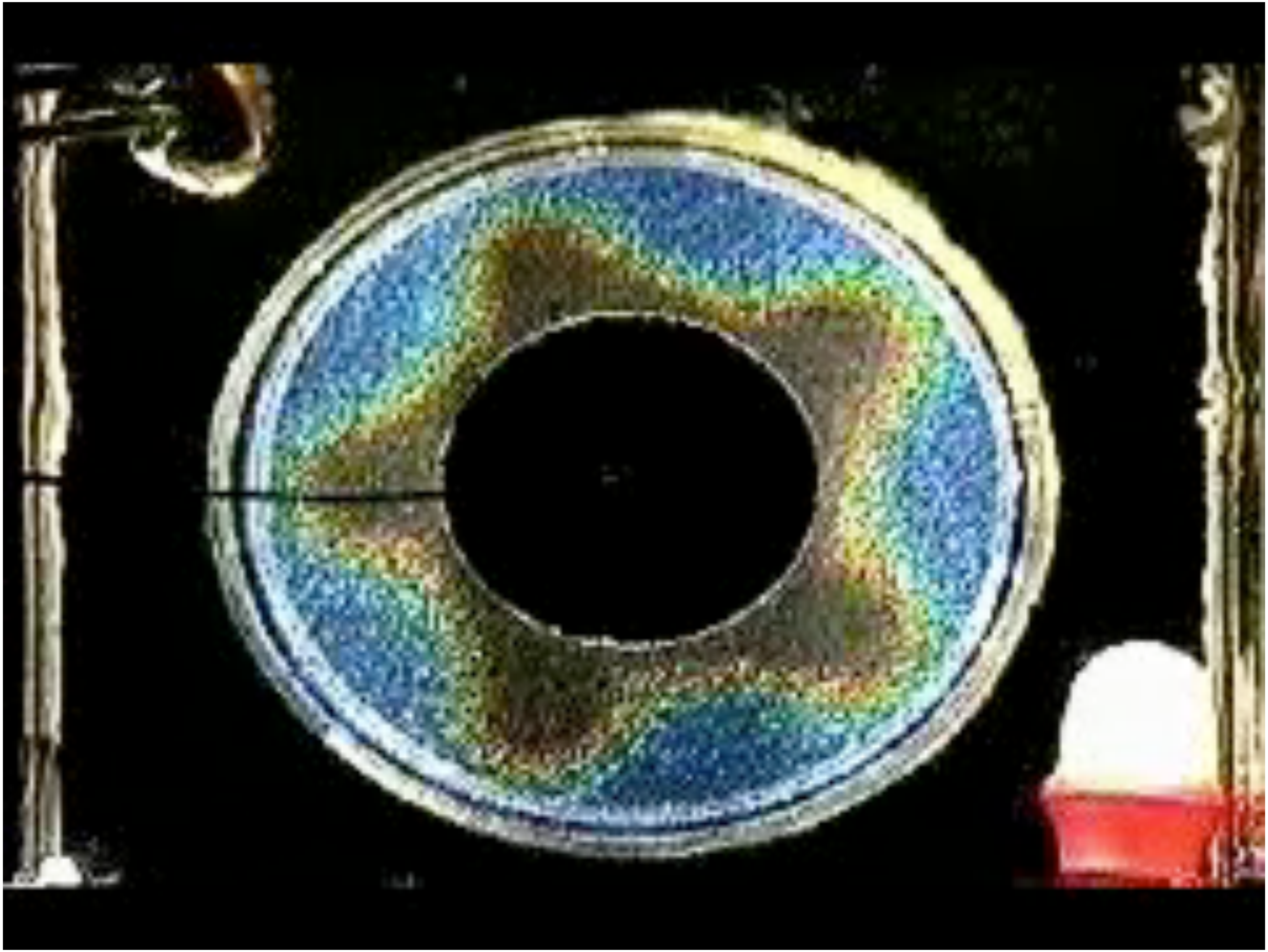
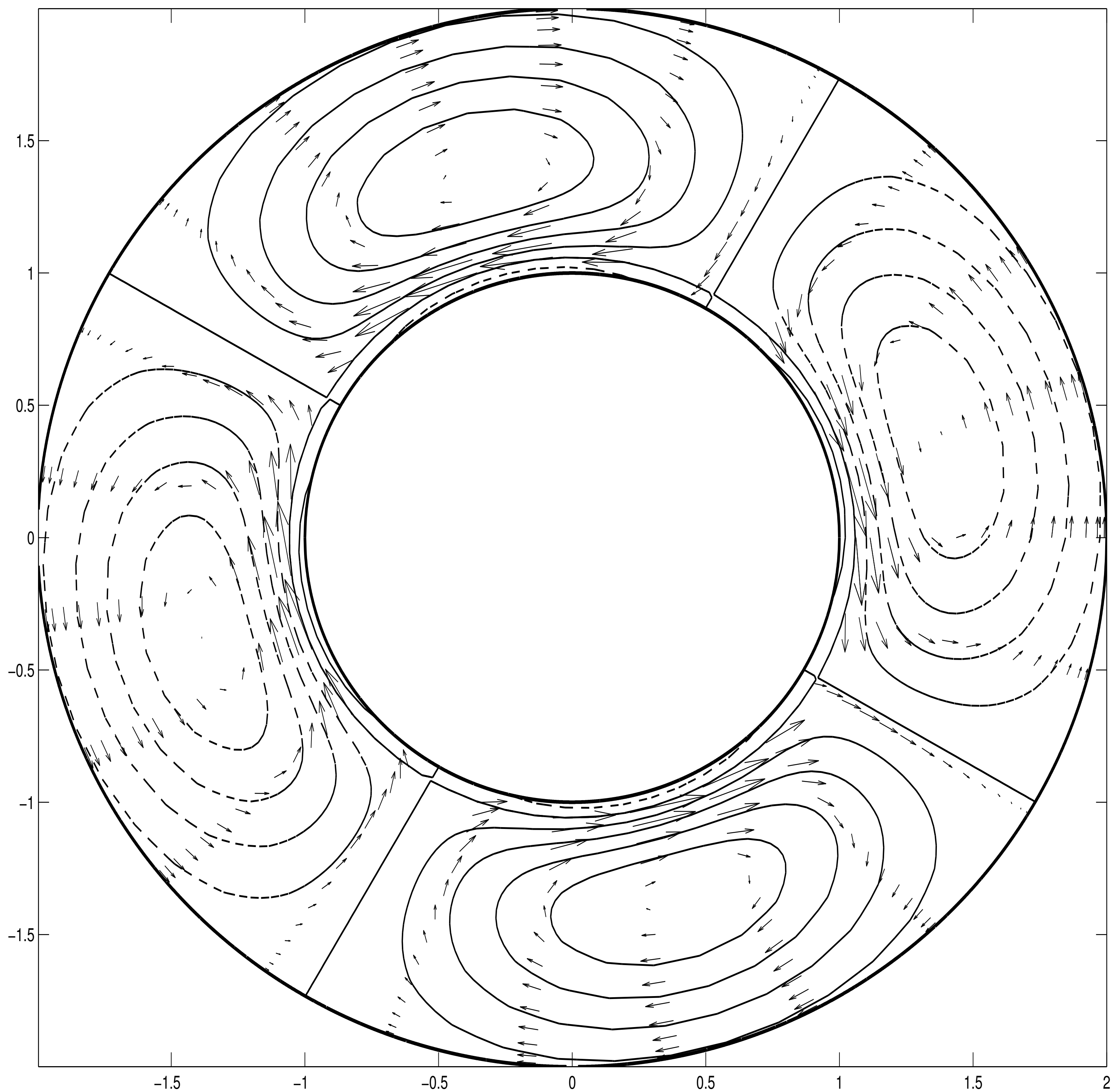
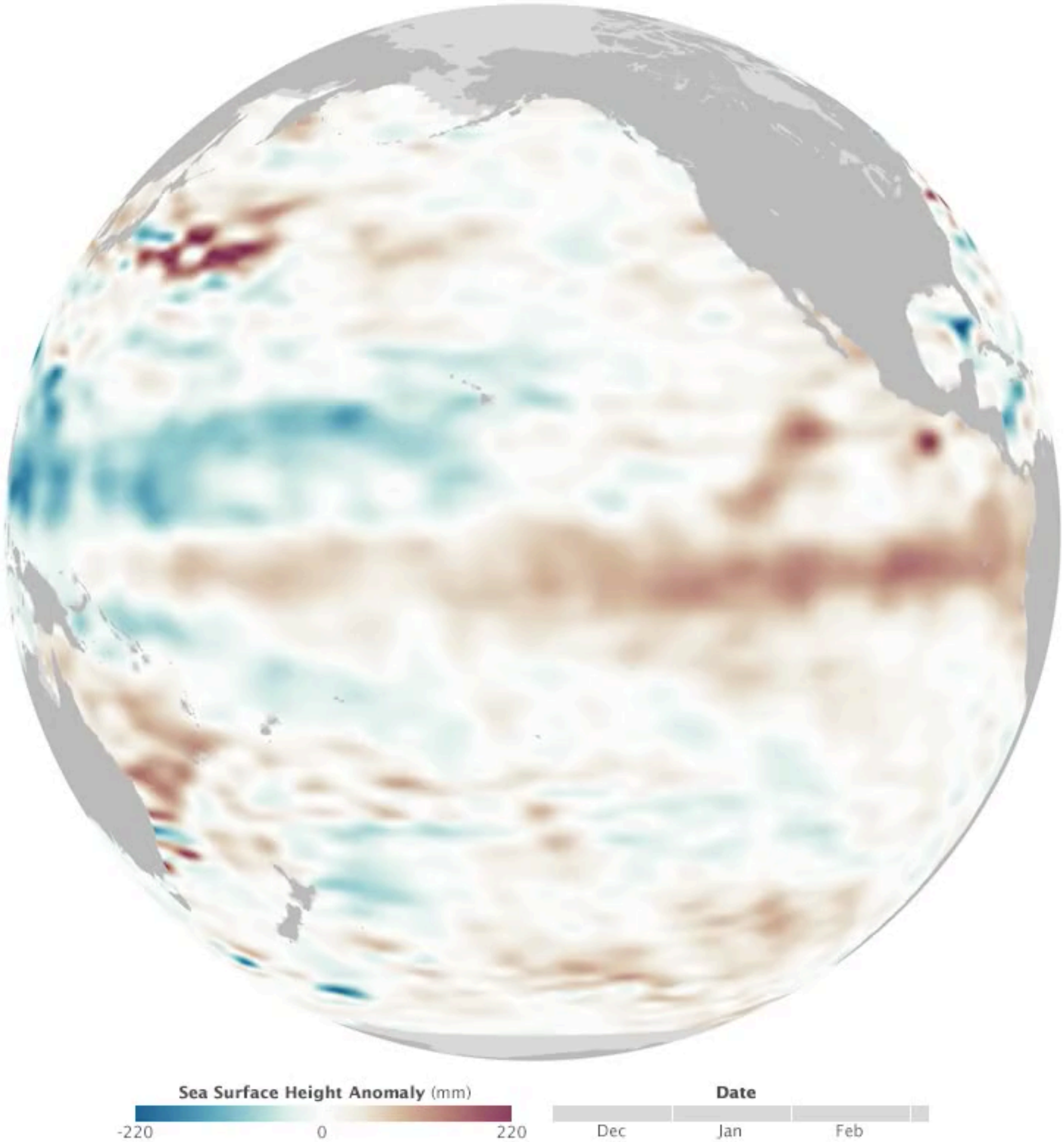


Standing wave in the atmosphere (also exists in the ocean)




1.5
1
0.5
0
−0.5
−1
−1.5
−1.5
−1
−0.5
0
0.5
1
1.5
2

1. A general introduction to ocean waves



Equatorial Kelvin wave (planetary wave)

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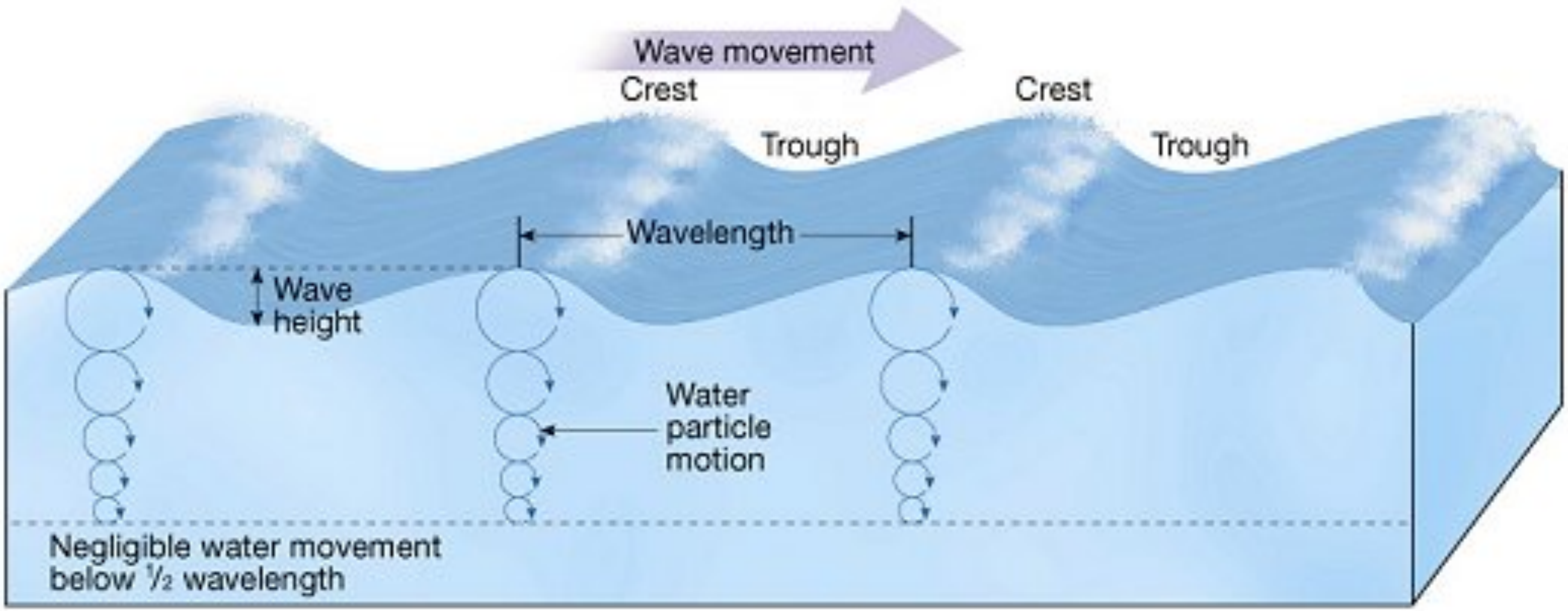
Waves in physics are difficult to define without referring to equations… For example :

“A wave is a **recognizable** signal that is transferred from one part of the medium to another with a **recognizable** velocity of propagation. The signal may be any feature of the disturbance, such as a maximum or an abrupt change in some quantity, provided that it can be clearly **recognized** and its location at any time can be determined.” (*Linear and nonlinear waves*, Whitham 1974)

1. A general introduction to ocean waves

A wave results from the displacement of a fluid parcel from its position of equilibrium. The restoration of the parcel to its position of equilibrium produces a movement of the fluid back and forth relative to this position. The parcel's trajectory forms the "wave orbit."



A linear wave carries energy but do not transport material

1. A general introduction to ocean waves

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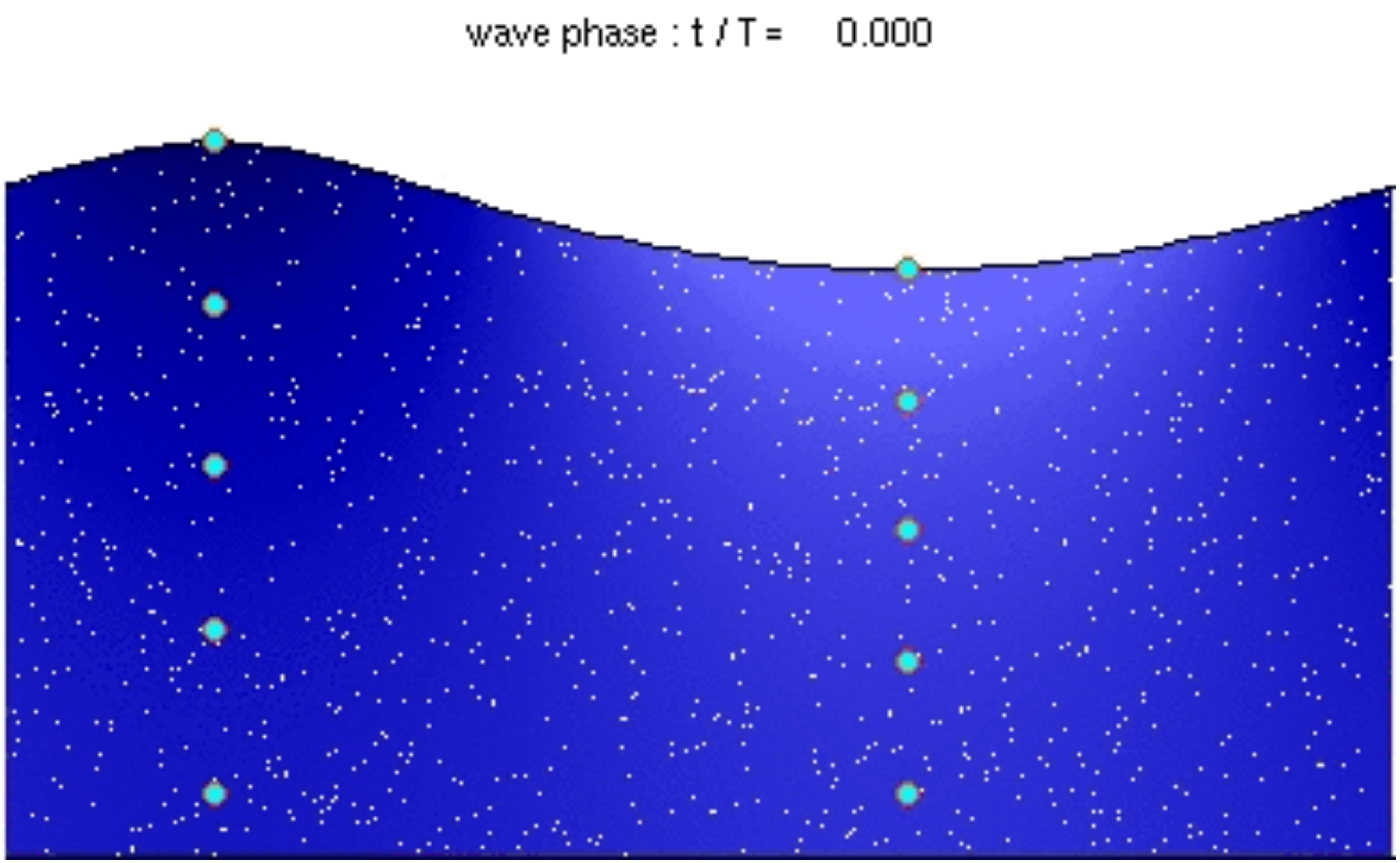

A **nonlinear wave** also carries energy and can lead to a net transport of material (for example, Stokes drift at the ocean surface)

Animation from https://en.wikipedia.org/wiki/Wind_wave

1. A general introduction to ocean waves

Different types of waves can be discriminated on the basis of:

- Disturbing force
- Restoring force
- Properties: wavelength, frequency, amplitude, …
- Freely propagating vs forced wave

1. A general introduction to ocean waves

Type of waves	Disturbing force	Restoring force	Horizontal wavelength	Period
Acoustic wave				
Capillary wave				
Surface gravity wave				
Rossby wave				
Kelvin wave				
Internal wave				

1. A general introduction to ocean waves

Type of waves	Disturbing force	Restoring force	Horizontal wavelength	Period
Acoustic wave	Any motion	Pressure (compressibility)	1 mm - 10 km	< 1 s
Capillary wave				
Surface gravity wave				
Rossby wave				
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Internal wave				

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Surface gravity wave				
Swells / Wind waves				
Tsunamis				
Tides				

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Surface gravity wave		Gravity	1 cm - 100 km	1 s - 1 day
Swells / Wind waves	Wind	Gravity	1 m - 100 m	1 s - 10 s
Tsunamis	Earthquakes	Gravity	10 km - 1000 km	1 min - 1 hr
Tides	lunar/sun gravitation	Gravity	10 km - 1000 km	1 hr - 24 h

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Surface gravity wave	Wind, boat, earthquake, moon/ sun gravitation…	Gravity	1 cm - 100 km	1 s - 1 day
Rossby wave				
Kelvin wave				
Internal wave				

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Rossby wave	Atmospheric forcing (wind, heat flux)	Potential vorticity (variation of Coriolis with latitude)	100 km - 1000 km	Days - months
Kelvin wave				
Internal wave				

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Internal wave				

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Kelvin wave	Atmospheric forcing	Pressure gradient + Coriolis	10 km - 100 km	Days - months
Internal wave	Tides, winds, current instability, …	Gravity (stratification) + Coriolis	1 m - 100 km	10 s - 1 day

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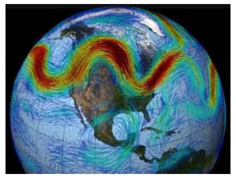
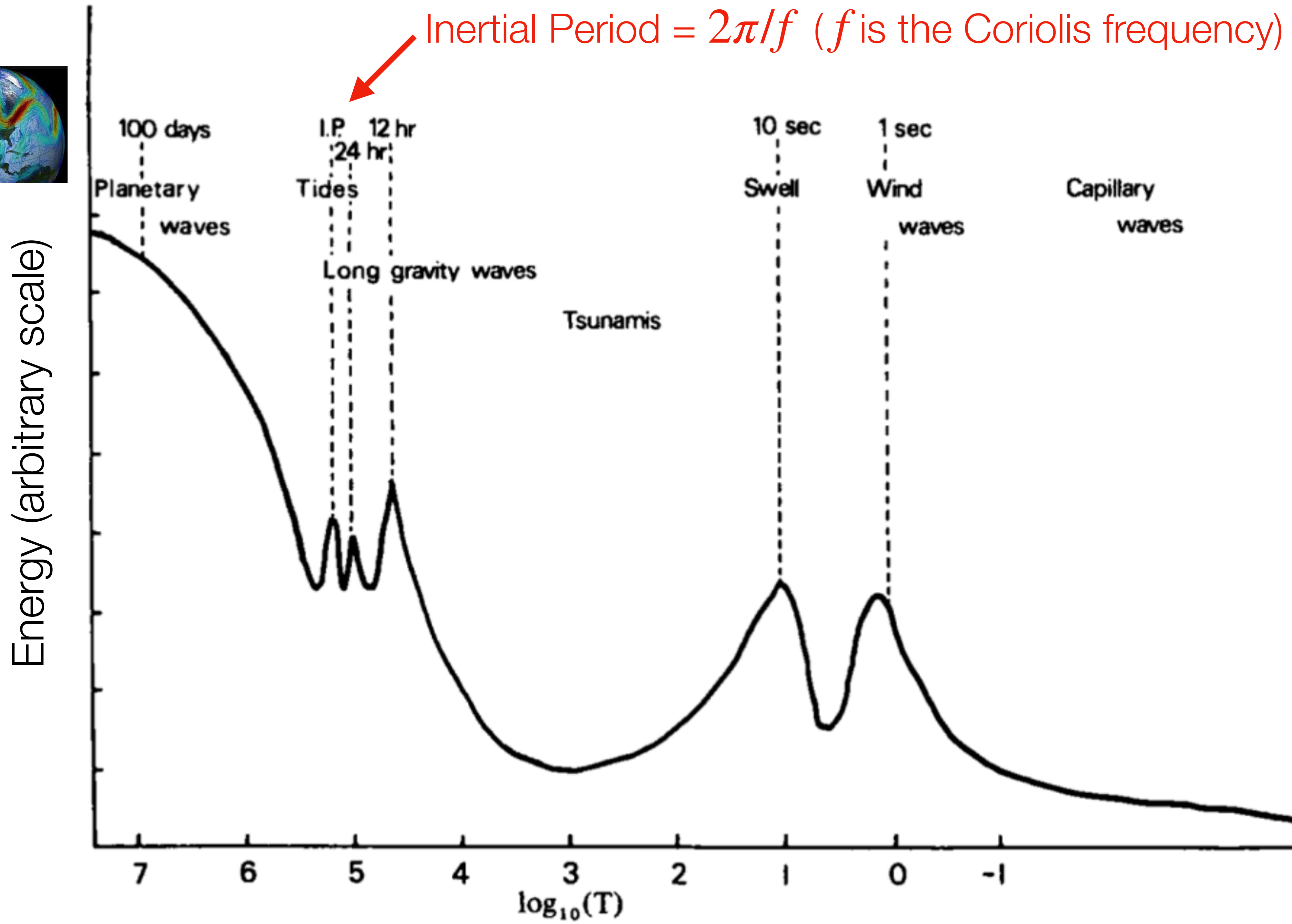
Inertial Period = $2\pi/f$ (f is the Coriolis frequency)





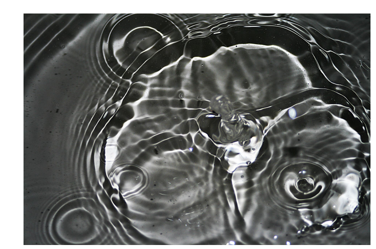


Fig. 2.1. Schematic energy spectrum of oceanic variability, showing the different types of waves occurring in the ocean. I.P. denotes the inertial period and is defined as $\pi/\Omega|\sin\phi|$, where Ω = magnitude of the Earth's rotation vector and ϕ is the geographic latitude (Section 3). In this picture I.P. = 35 hours, corresponding to a latitude of ± 20°. The relative amplitudes of the various parts of the spectrum do not necessarily reflect actual conditions.

Schematic from textbook *Waves in the ocean,* LeBlond & Mysak (1981)

1. A general introduction to ocean waves

A quick reminder on Fourier transform and Power Spectral Density (PSD), from textbook *Modern observational physical oceanography* (Wunsch, 2015)

Suppose $x(t)$ is periodic (t is time) with period T:

$$x(t+T) = x(t).$$

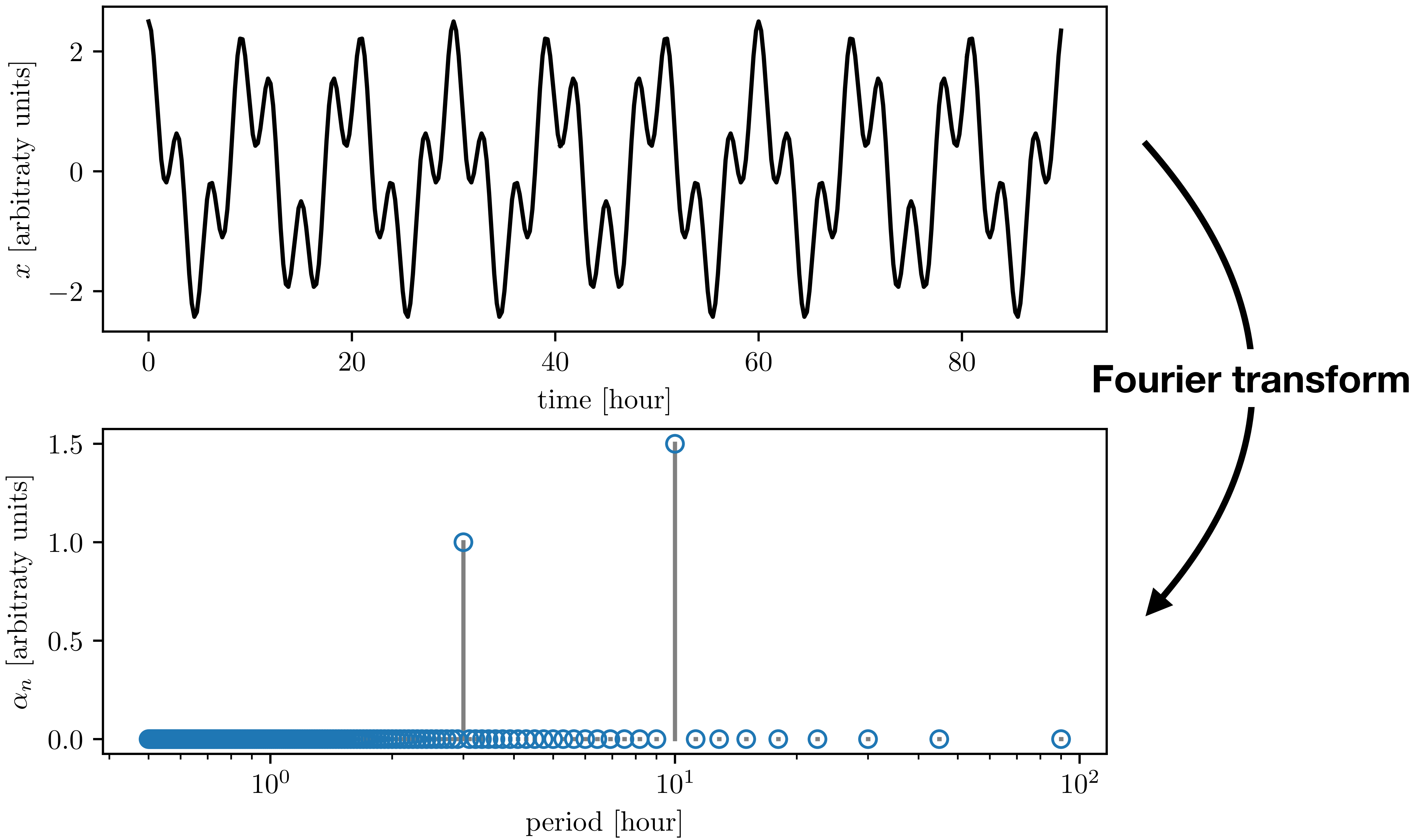
Under very general conditions (we do not go into mathematical details), x can be written as a Fourier series (set of functions with periods T/n):

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \exp\left(i\omega_n t\right) \qquad \text{with } \omega_n = \frac{2\pi n}{T}$$

With the Fourier coefficients α_n:

$$\alpha_n = \frac{1}{T}\int_0^T x(t)\exp\left(-i\frac{2\pi n}{T}t\right)$$

$A(t) = A_1 \cos(2\pi t/T_1) + A_2 \cos(2\pi t/T_2)$, with $A_1 = 1,\ T_1 = 3,\ A_2 = 1.5,\ T_2 = 10$



1. A general introduction to ocean waves

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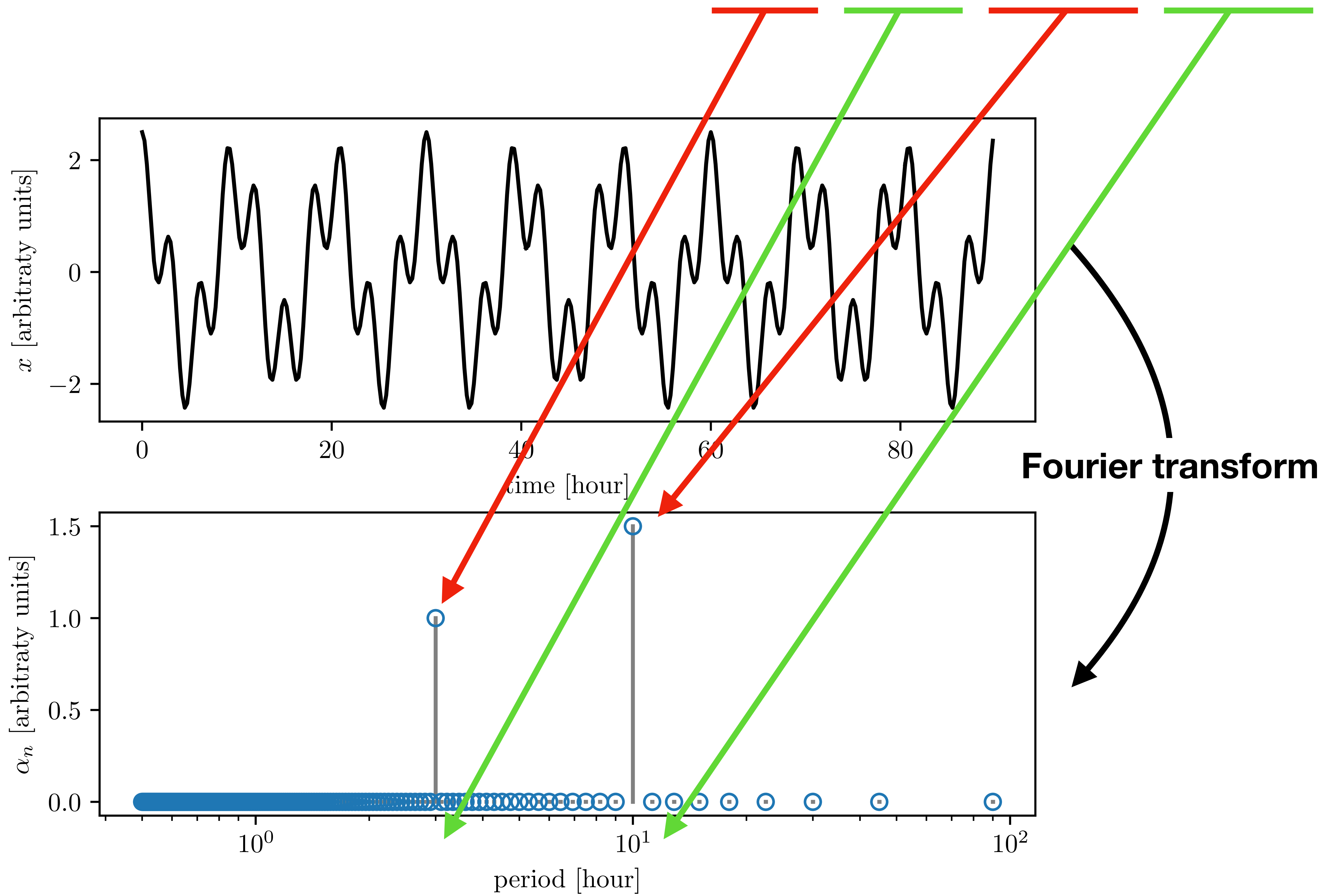

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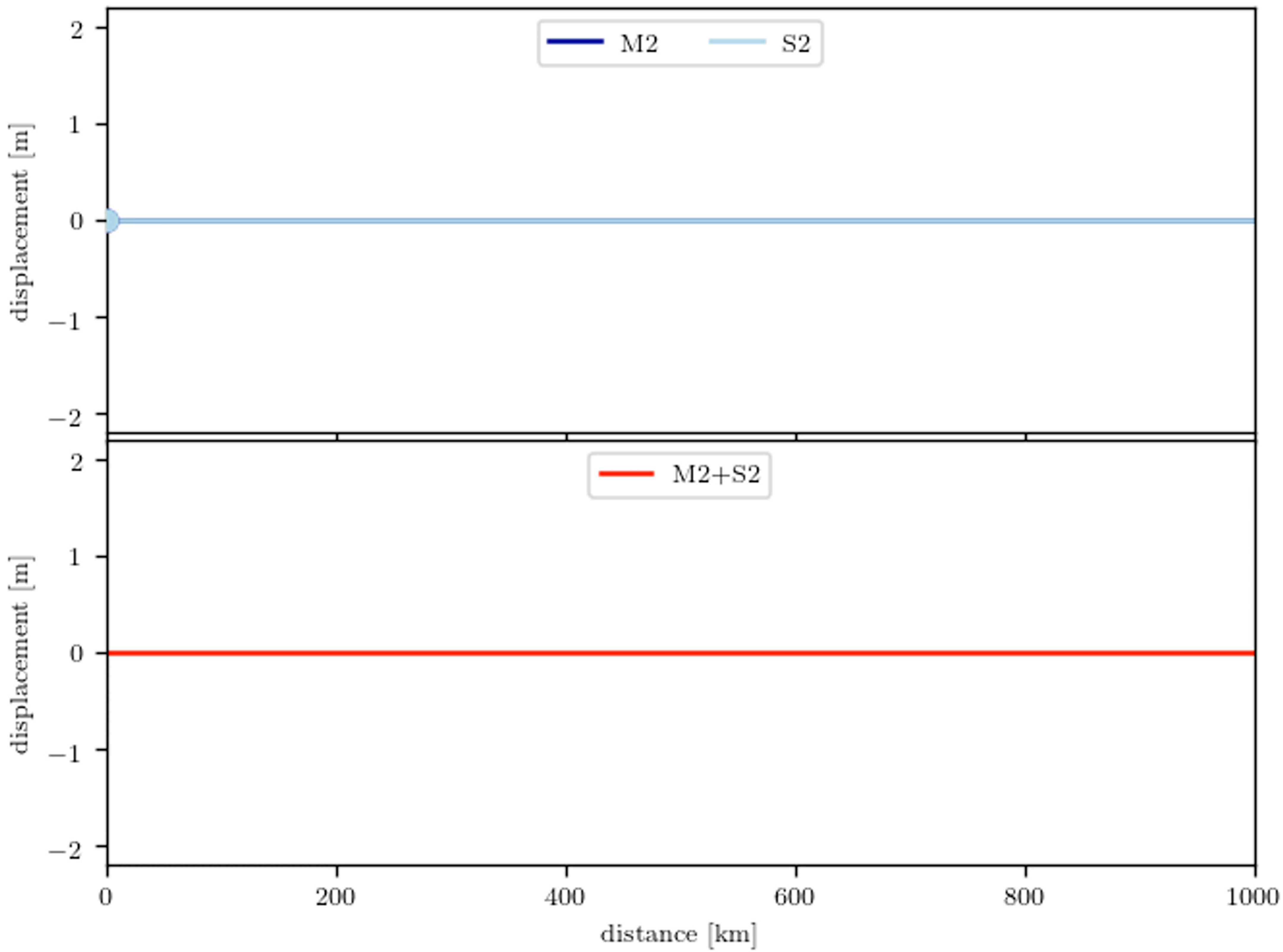
- **Exemple**: superposition of tidal waves,
  "M2 wave" (M=Moon): $\eta_{M2} = A_{M2} \exp\left(i(\omega_{M2} t - k_{M2} x)\right)$
  "S2 wave" (S=Sun): $\eta_{S2} = A_{S2} \exp\left(i(\omega_{S2} t - k_{S2} x)\right)$


M2
S2
M2+S2
displacement [m]
displacement [m]
2
1
0
−1
−2
2
1
0
−1
−2
0
200
400
600
800
1000
distance [km]

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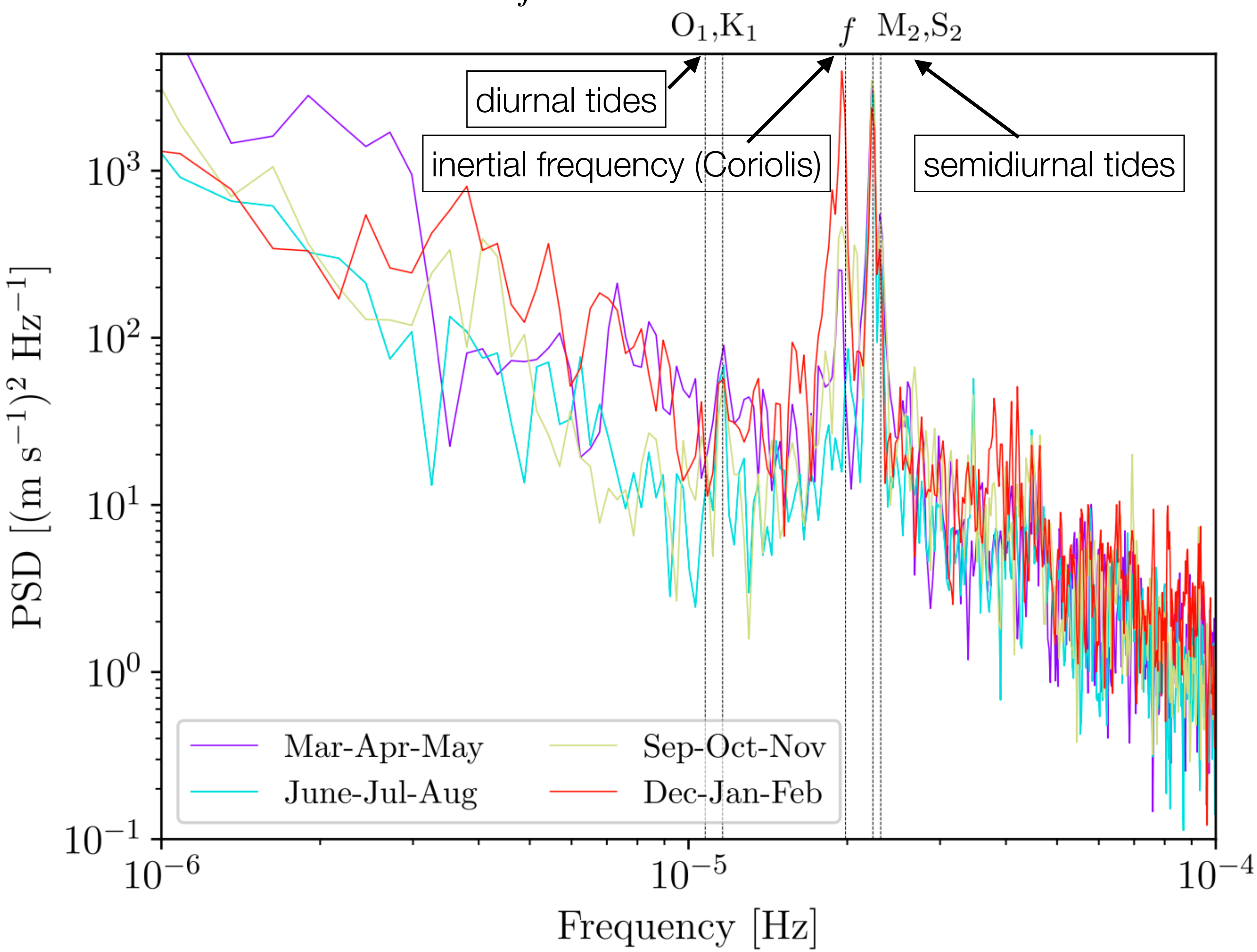
Example with a *real* time series $A(t) = \sum_j A_j \cos(2\pi t/T_j) + R(t)$ (residual)



Power Spectral Density (~Fourier transform) of horizontal currents above the Reykjanes Ridge (30.7˚W, 58.8˚N)

Vic et al. (JPO 2021)

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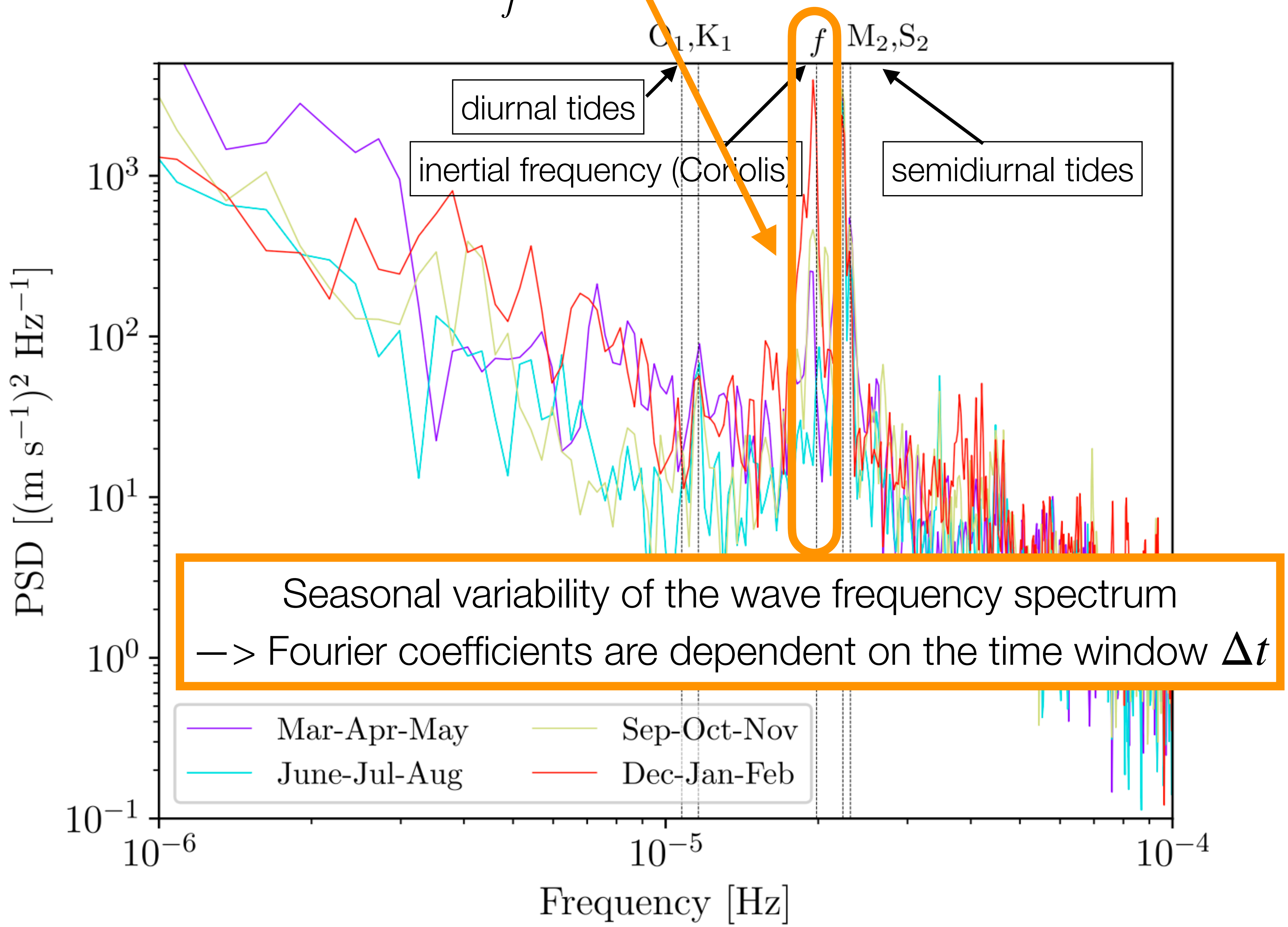
Example with a *real* time series $A(t) = \sum_j A_j(\Delta t) \cos(2\pi t/T_j) + R(t)$ (residual)



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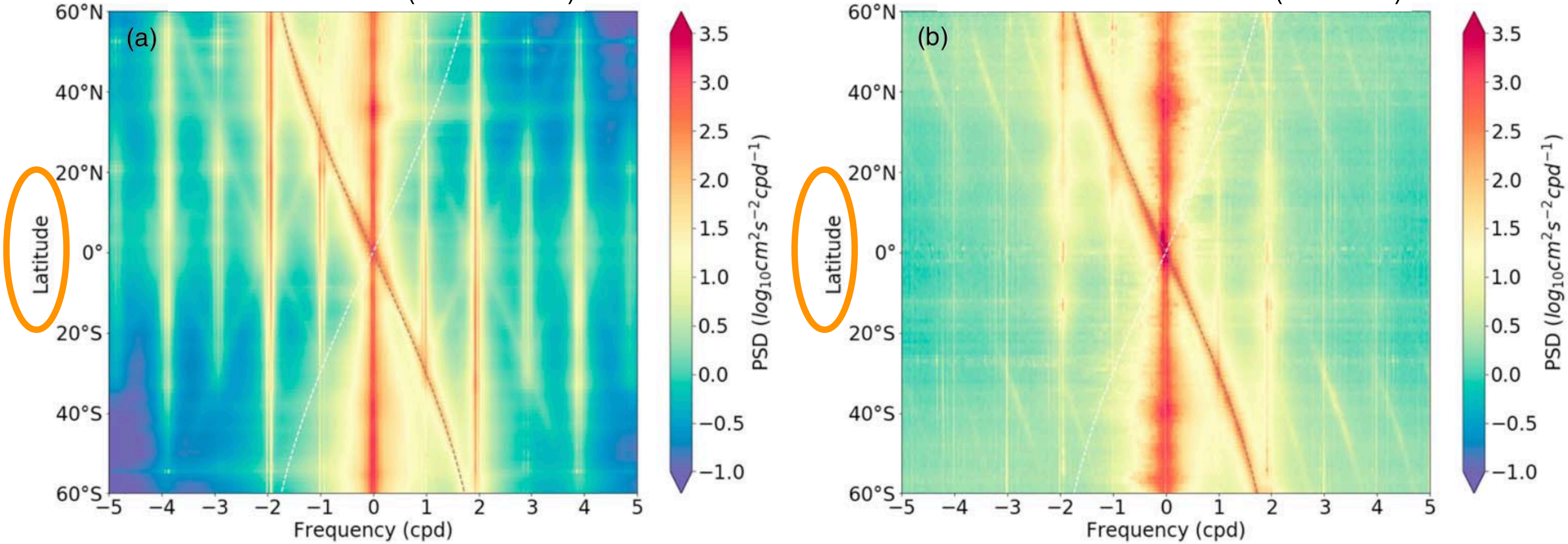
1. A general introduction to ocean waves

Example with a *real* time series $A(x, y, z, t) = \sum_j A_j(x, y, z, \Delta t) \cos(2\pi t / T_j) + R(t)$ (residual)

Global model (LLC4320) | in situ observations (drifters)



Spatial (latitude, longitude) variability of the wave frequency spectrum
—> Fourier coefficients are dependent on $x, y, z, \Delta t$

Yu et al. (GRL 2019)

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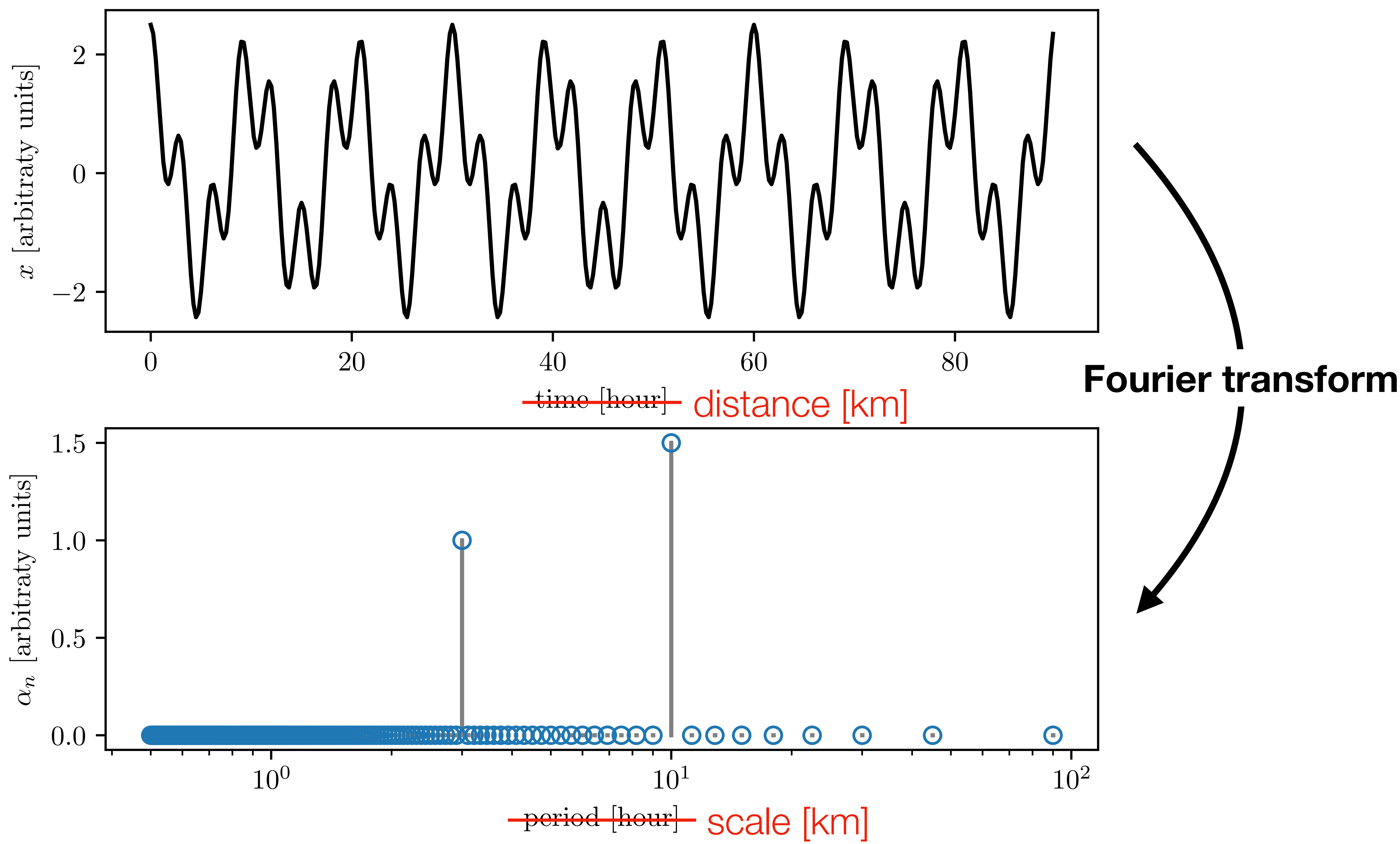
Spectra can also be computed in space: $A(x) = \sum_j A_j \cos(2\pi x/\lambda_j) + R(x)$ (residual)



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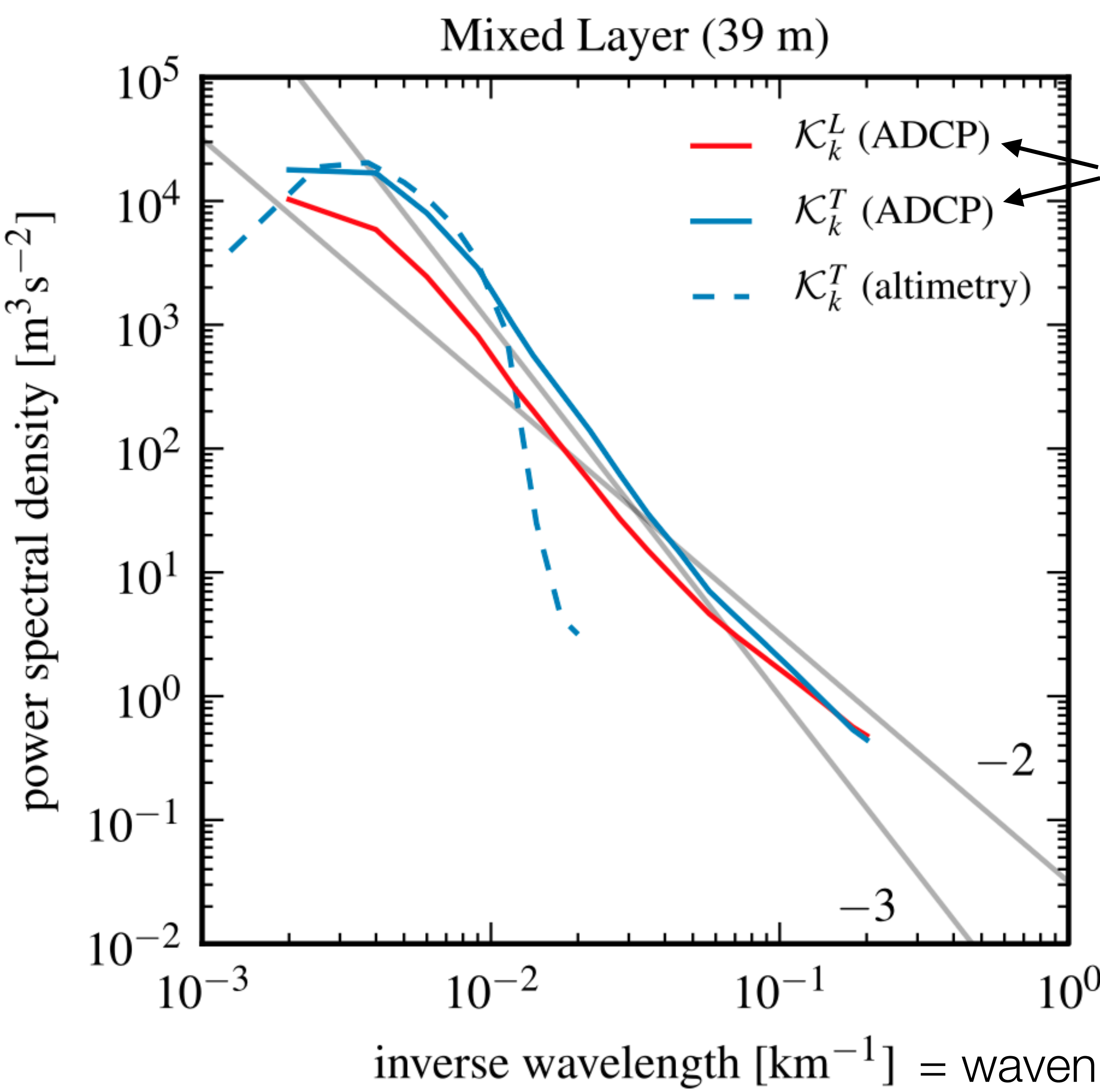

in two different directions

= wavenumber $k = \frac{2\pi}{\lambda}$, λ is the wavelength

NB: the **wavenumber PSD** (in space) looks qualitatively different from the **frequency PSD** (in time).
There are (usually) no marked peak.
The dynamics are more continuous in space than in time.

Callies and Ferrari (JPO 2013)

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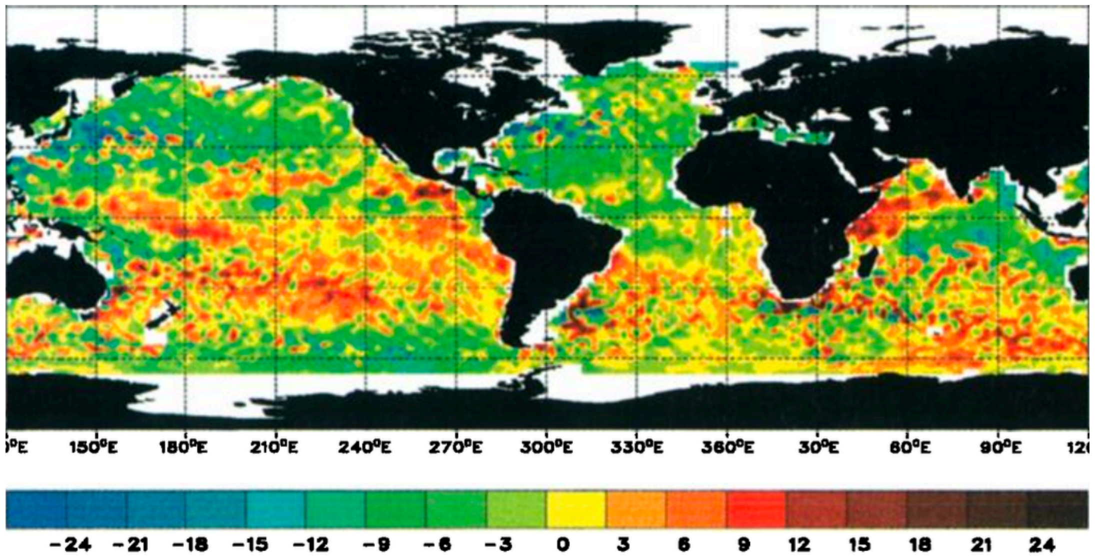
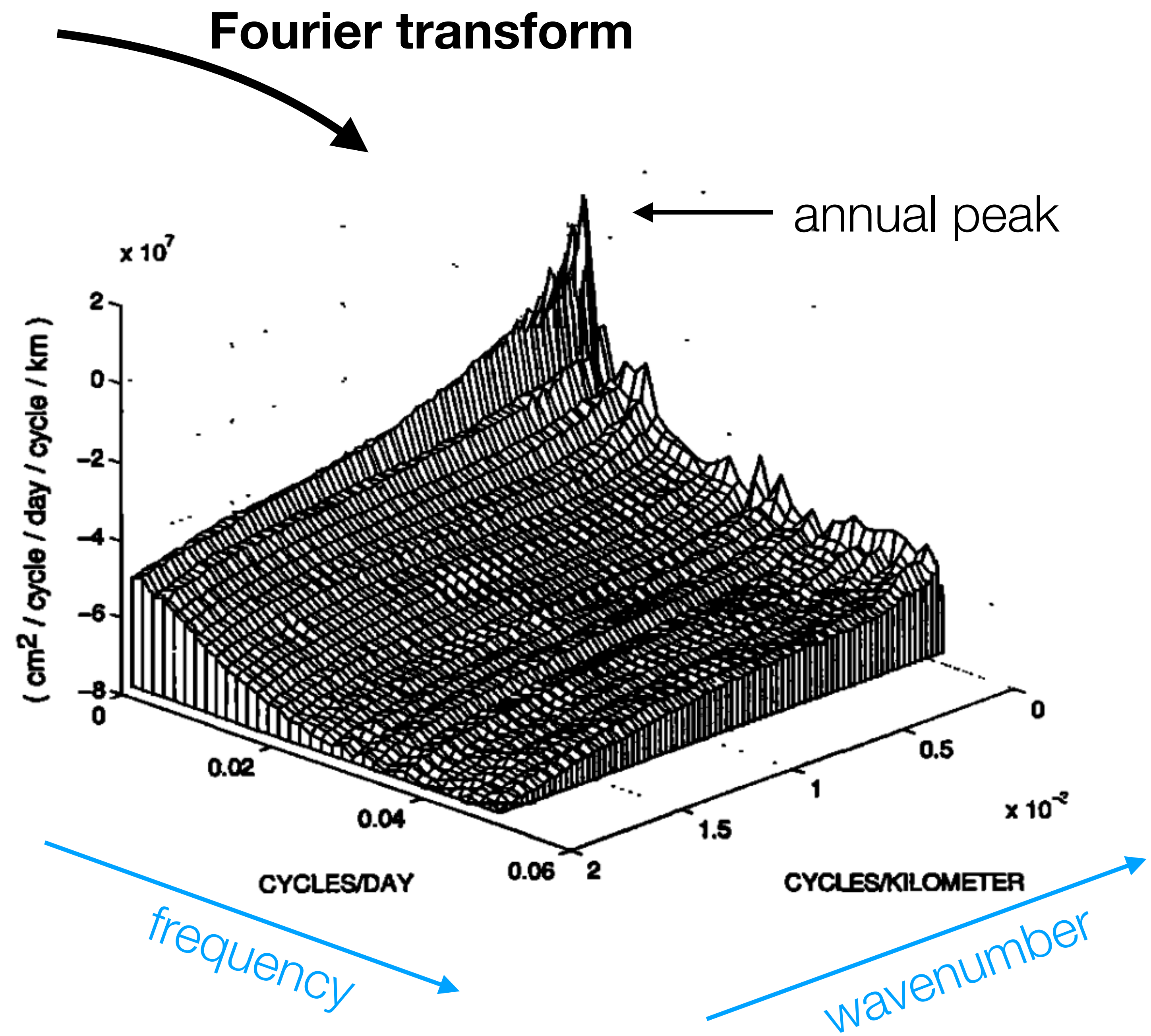
Spectra can ultimately be computed in space and time: **frequency-wavenumber spectra**



Sea Surface Height (SSH) observed from satellite altimetry for two years



Wunsch and Stammer (JGR 1995)

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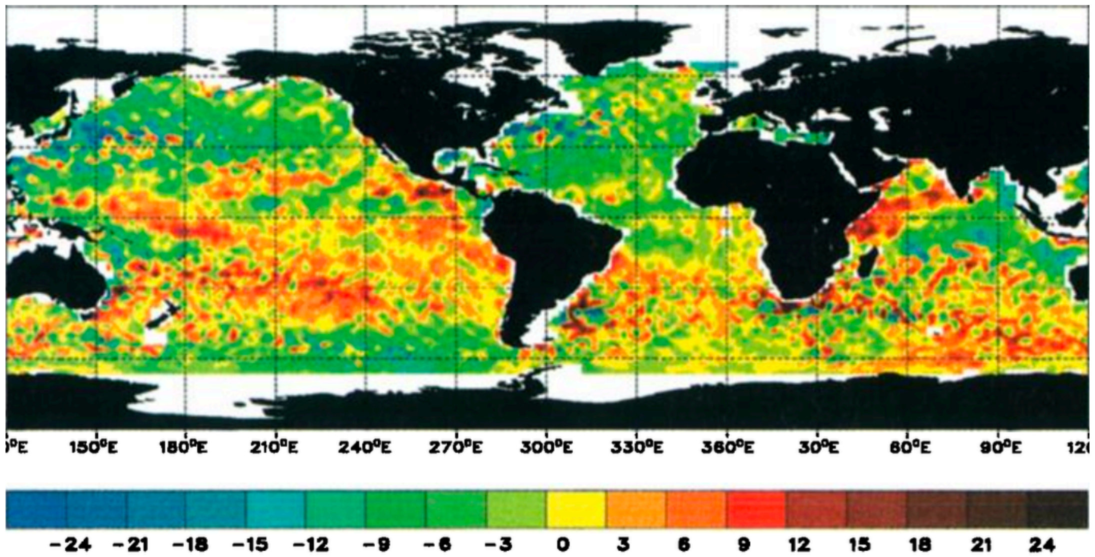

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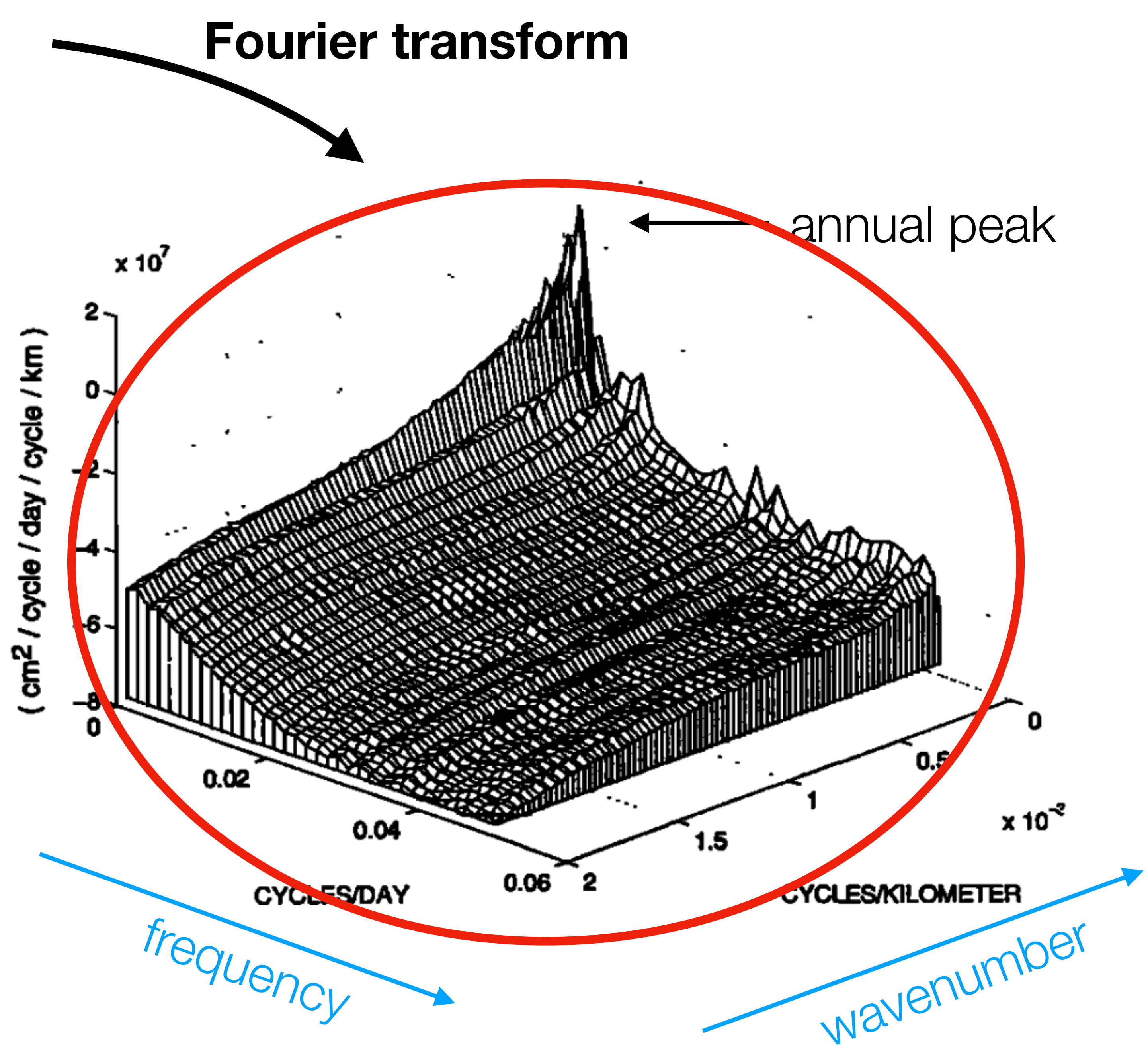

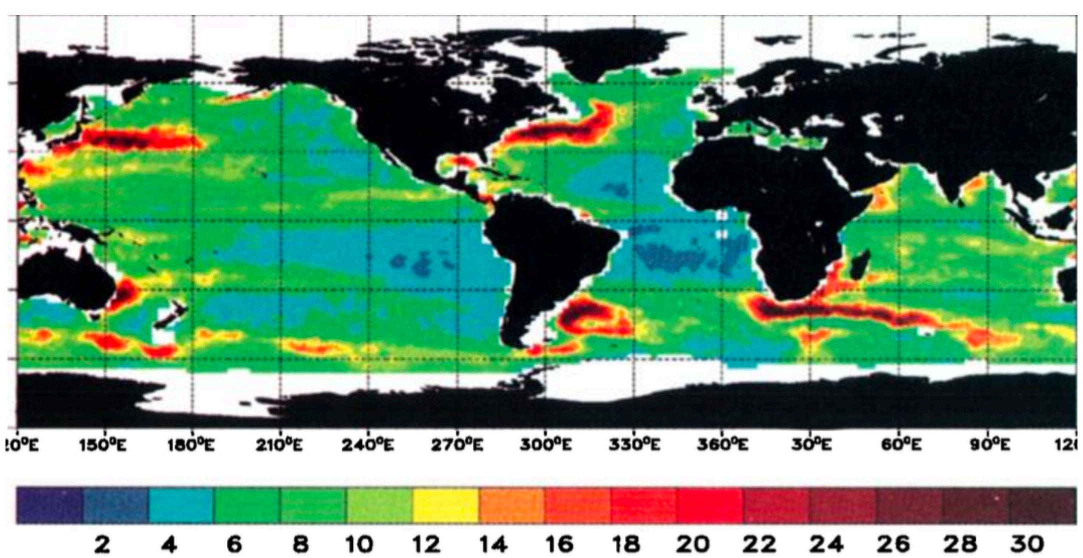

SSH variance of the whole signal, that is, through all frequencies and wavenumbers

Wunsch and Stammer (JGR 1995)

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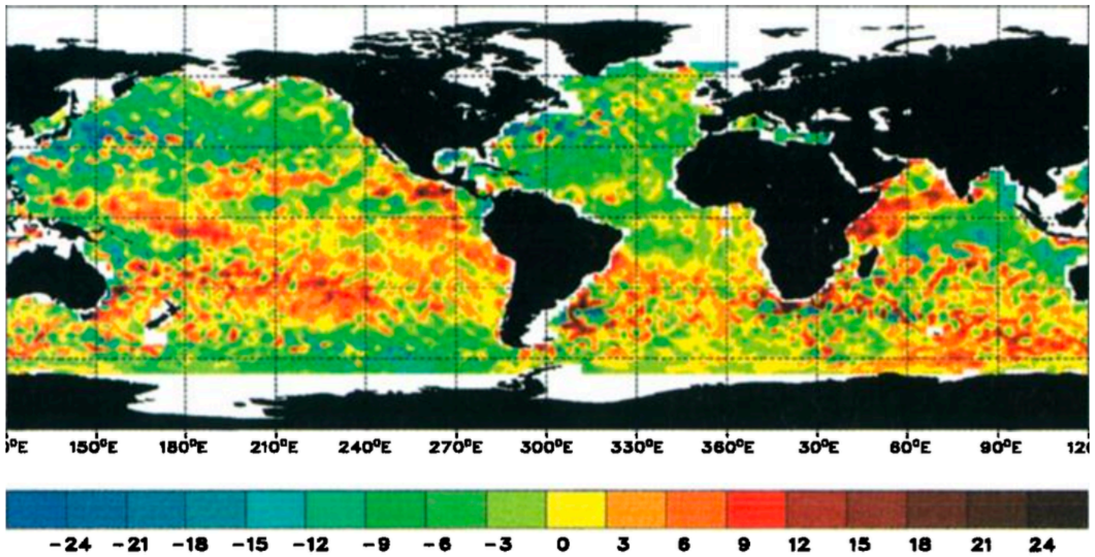
Spectra can ultimately be computed in space and time: **frequency-wavenumber spectra**



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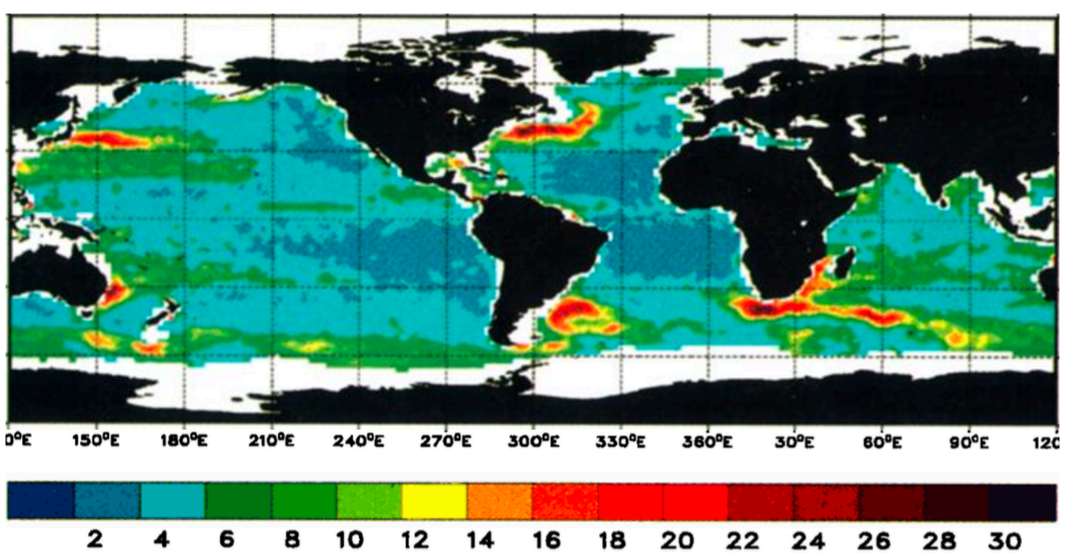
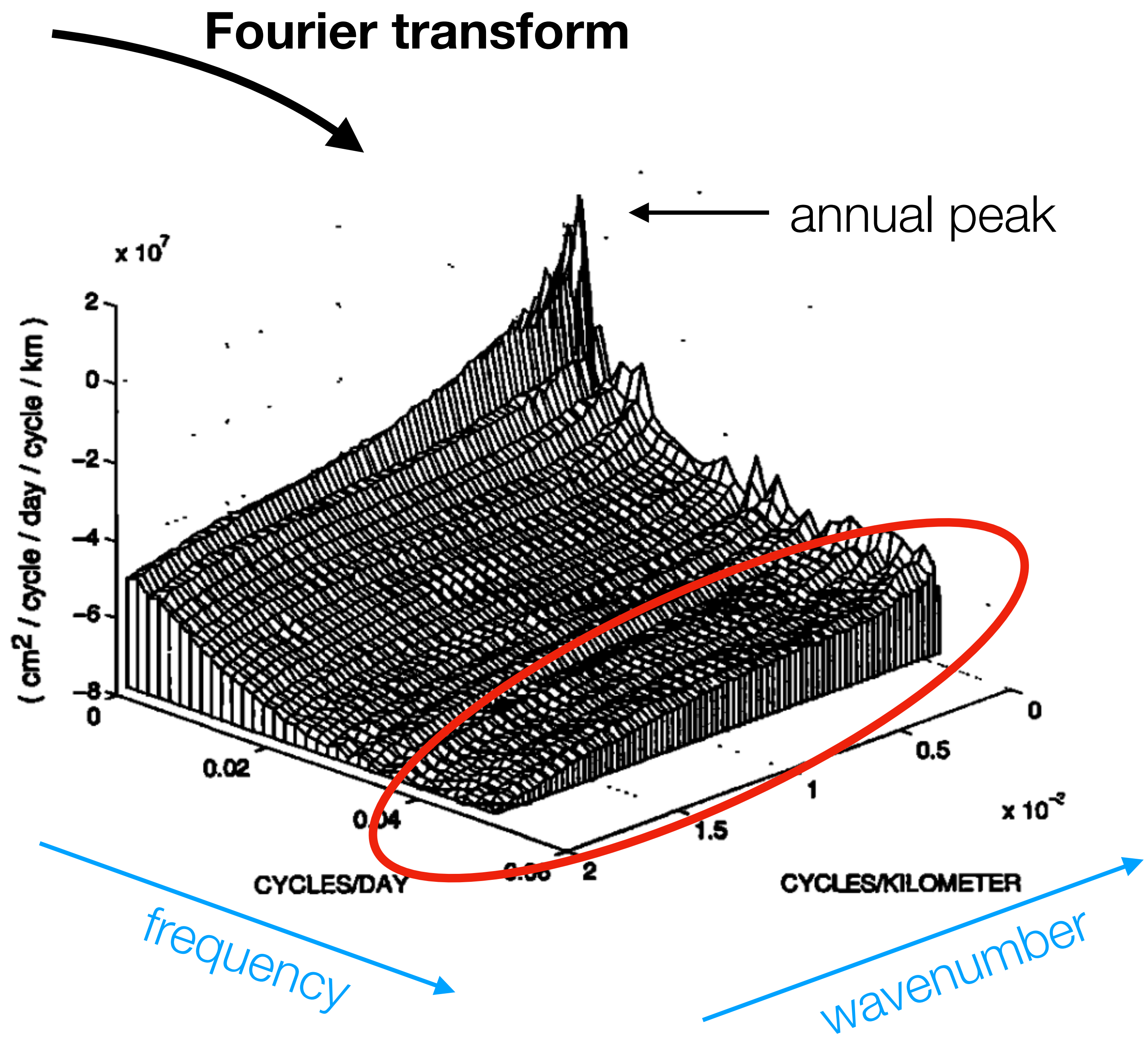


SSH variance of the signal for all wavenumbers and timescales < 150 days (~0.04 cycles/day)



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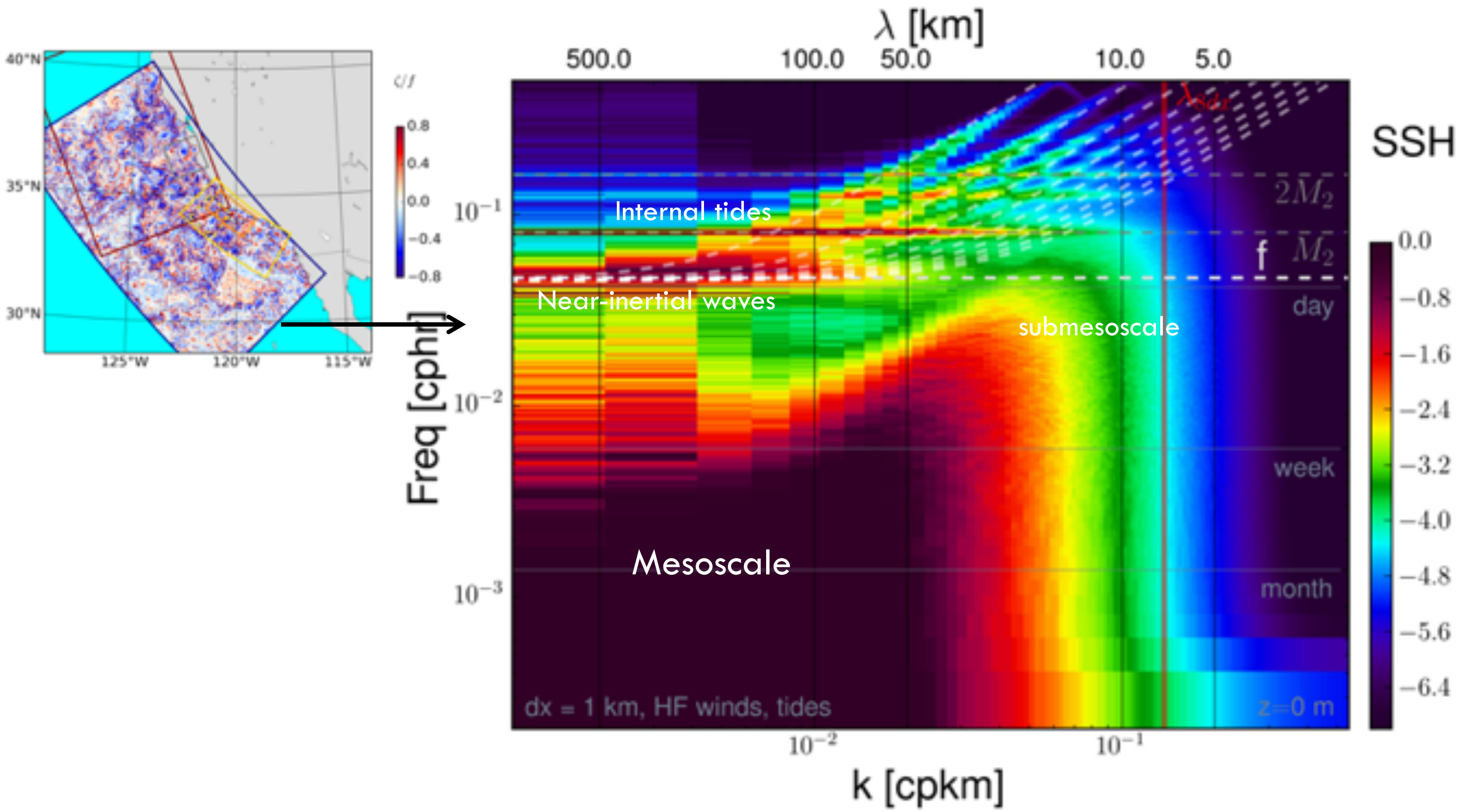
Spectra can ultimately be computed in space and time: **frequency-wavenumber spectra**



Azimuthally-averaged 2D frequency-wavenumber spectra for SSH in California Current

Horizontal wavenumber frequency spectrum of SSH variance (numerical model).

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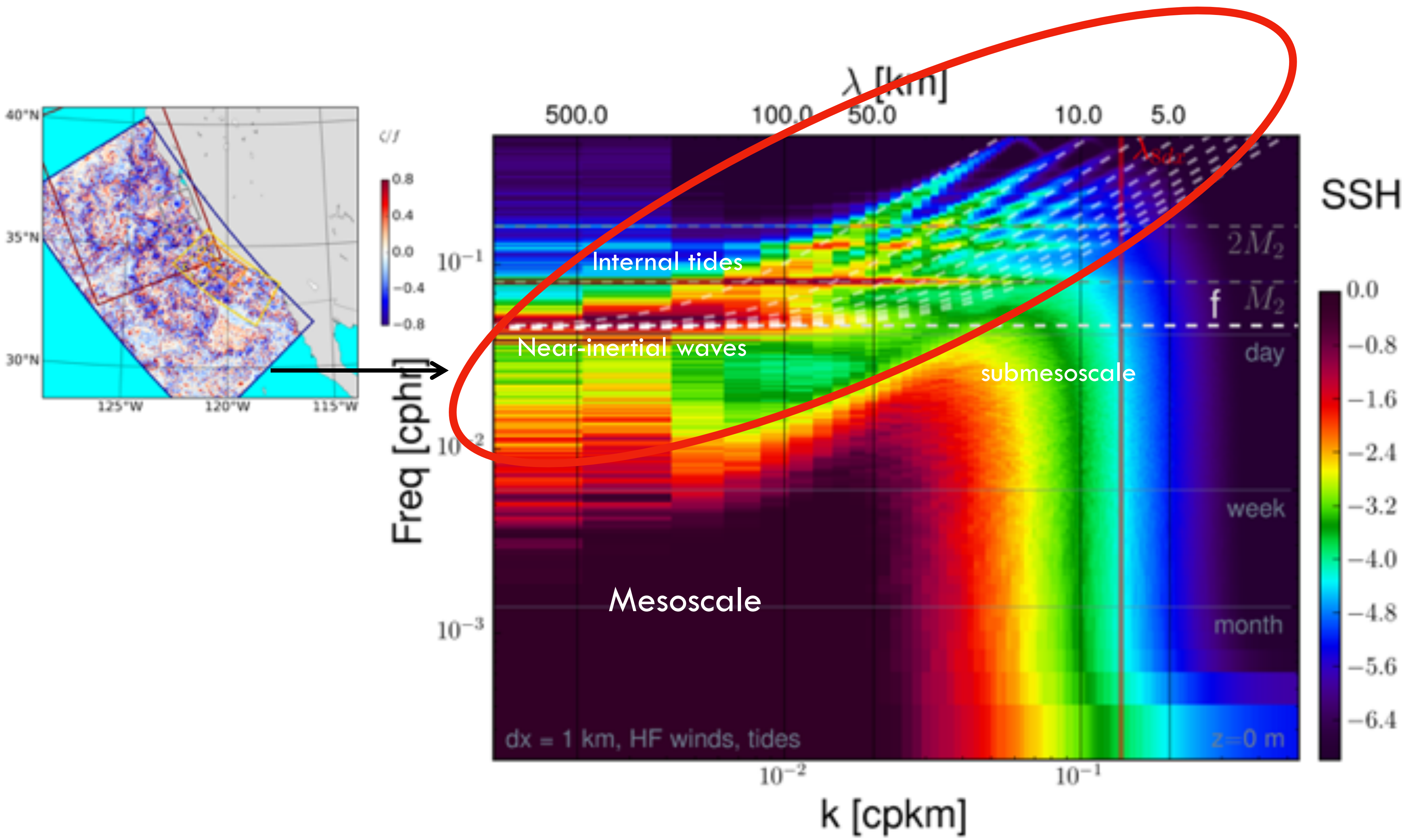
Spectra can ultimately be computed in space and time: **frequency-wavenumber spectra**



Azimuthally-averaged 2D frequency-wavenumber spectra for SSH in California Current

There is a coupling between wavenumbers and frequencies

—> typical of wave motions, which are described through a **dispersion relation**.

Horizontal wavenumber frequency spectrum of SSH variance (numerical model).

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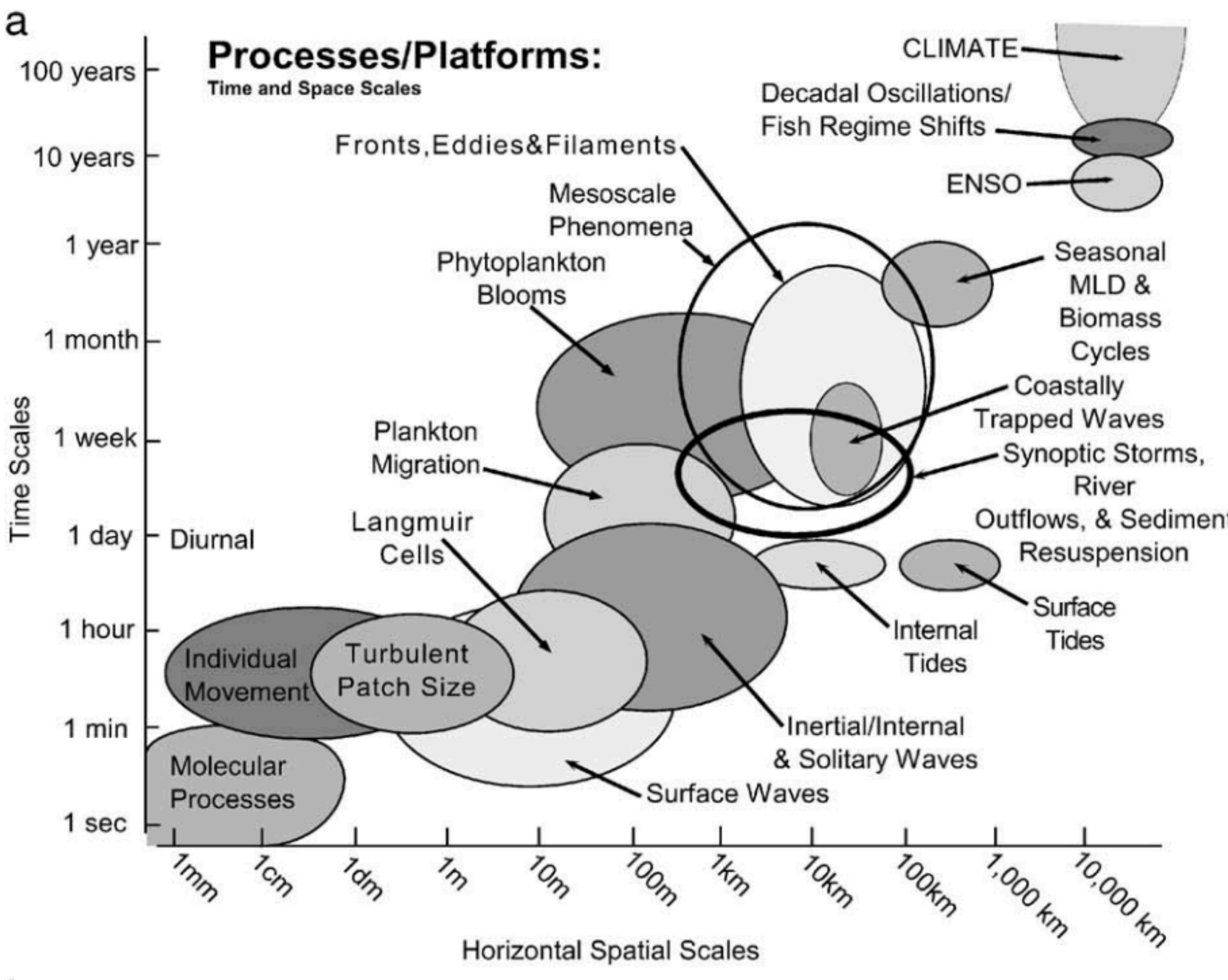
“Stommel diagram” of oceanic motions that illustrates the coupling between spatial and temporal scales.



1. A general introduction to ocean waves

Mathematically, there are two types of waves:

- **Hyperbolic waves** are described by hyperbolic partial differential equations (PDEs): in one dimension, $\frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial x^2} = 0$. The equation has the property that, if η and its first time derivative are arbitrarily specified initial data on the line $t = 0$ (with sufficient smoothness properties), then there exists a solution for all time t.
  General solutions have the form $\eta = f(x - ct) + g(x + ct)$.

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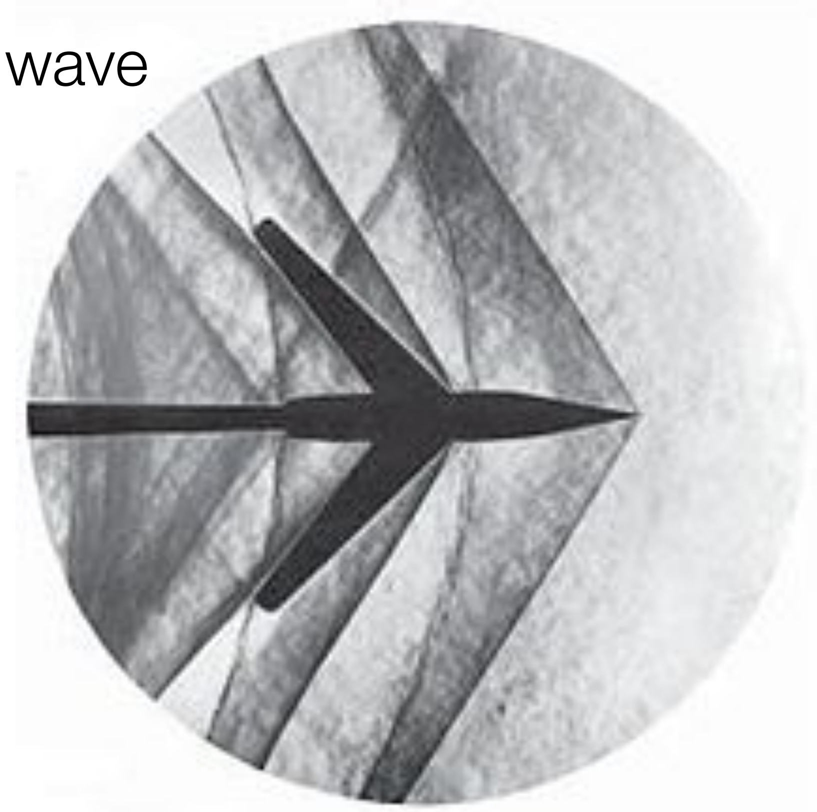
These waves are a way to propagate an initial disturbance.

Tidal bore

shock wave

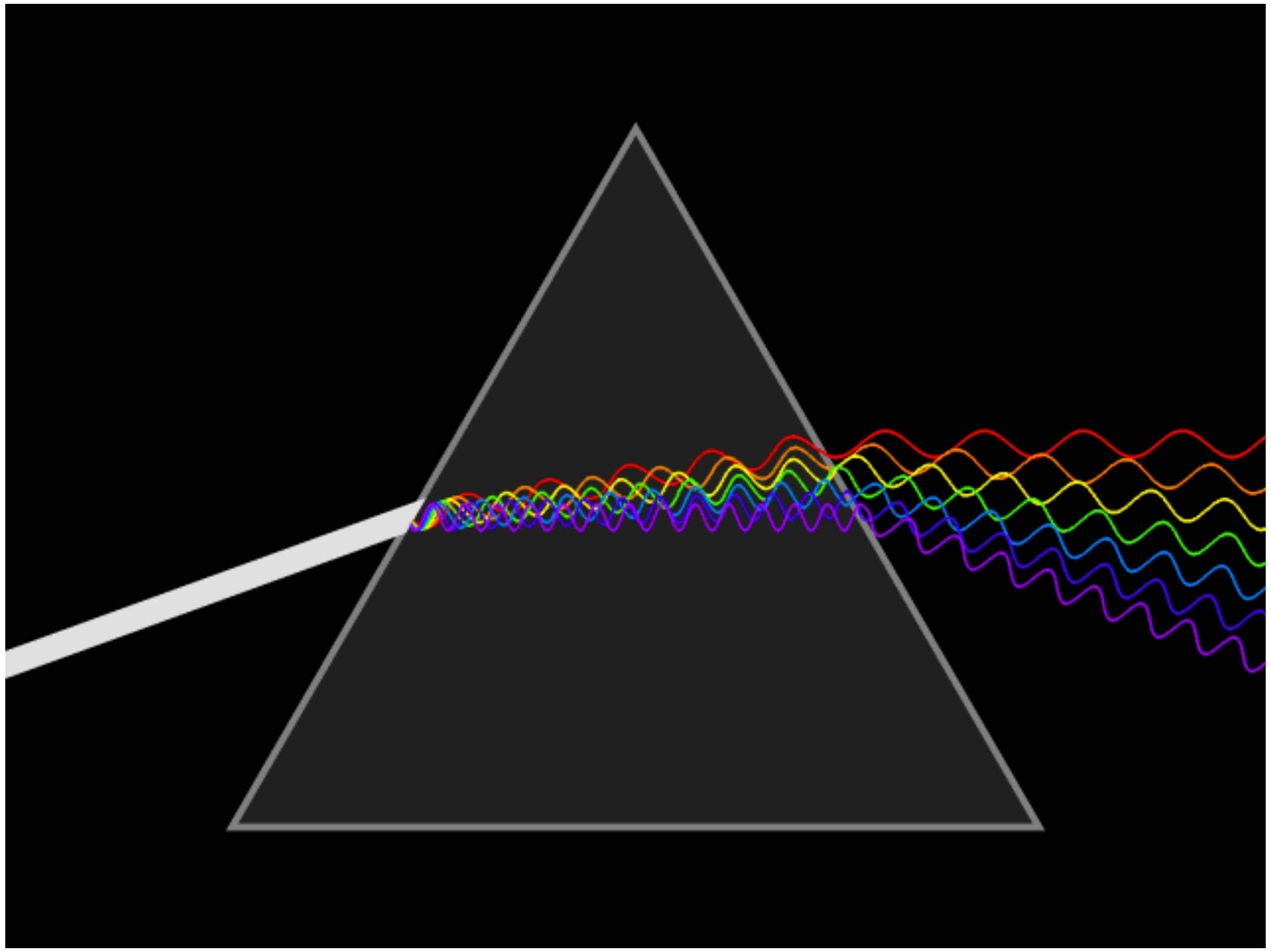


1. A general introduction to ocean waves

Mathematically, there are two types of waves:

- **Dispersive waves** come from a variety of PDEs and are characterised by their dispersion relation $\omega = f(k, \text{environment})$, with ω their frequency and k their wavenumber. They are observed as the superposition of waves where the Fourier components (corresponding to specific ω) propagate at different speeds.

  The wave (phase) speed is $c_\phi = \dfrac{\omega}{k}$ and the group speed is $c_g = \dfrac{\partial \omega}{\partial k}$.



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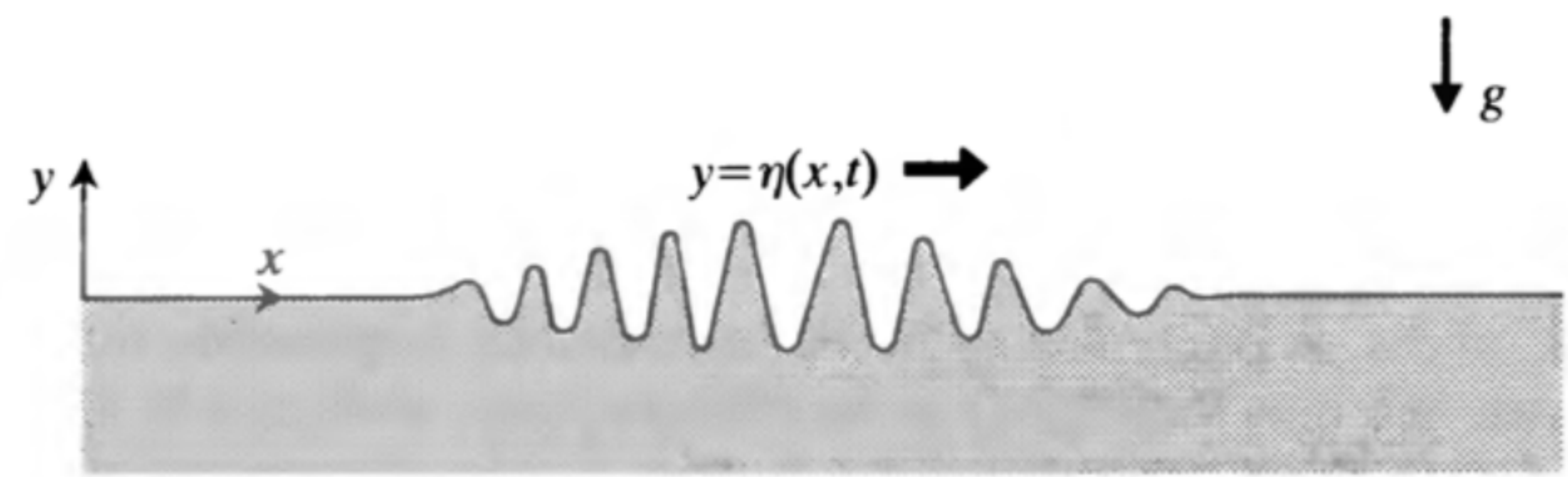
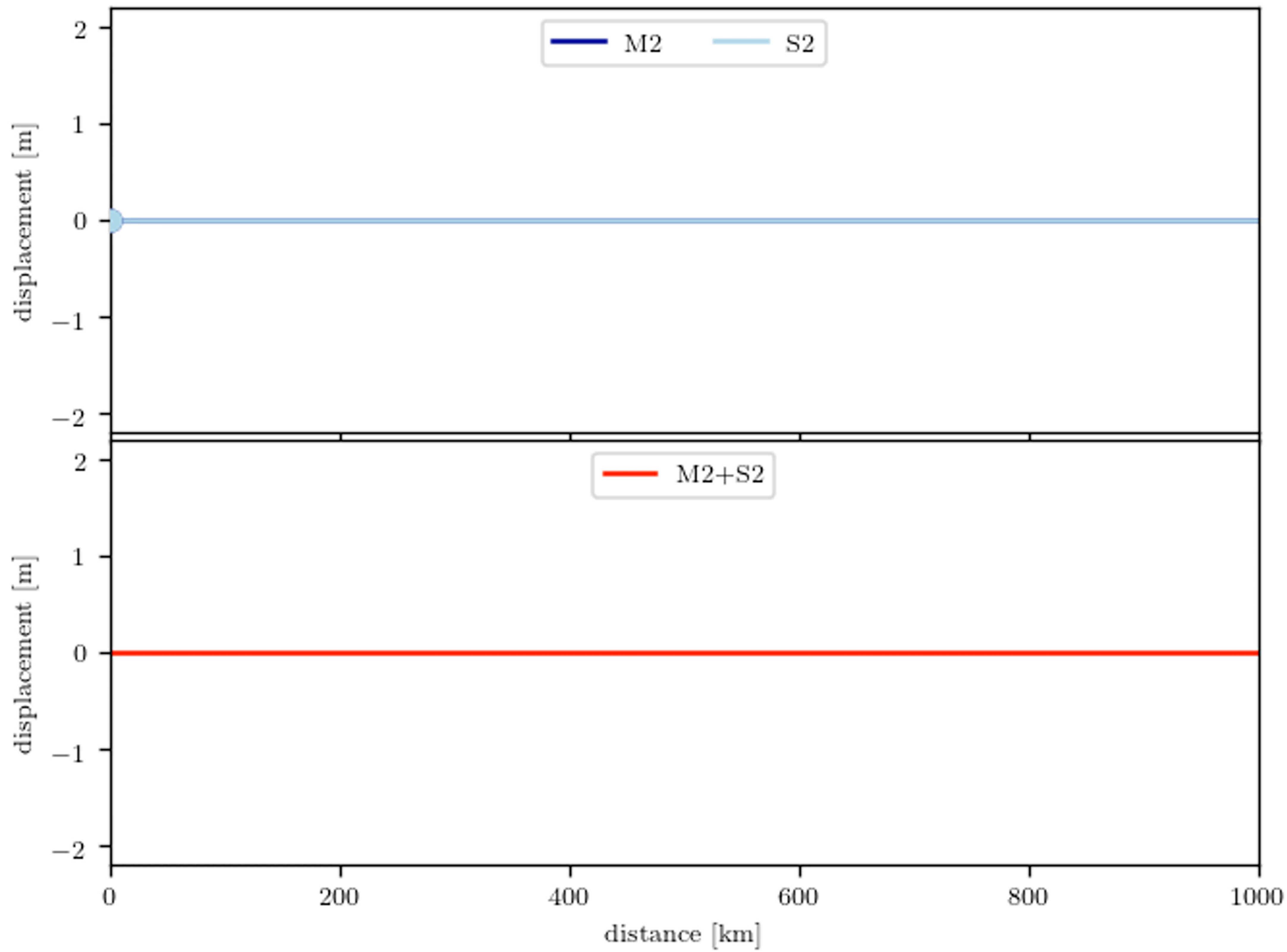
- They are visualized as a group of waves where the different Fourier components propagate at different speeds



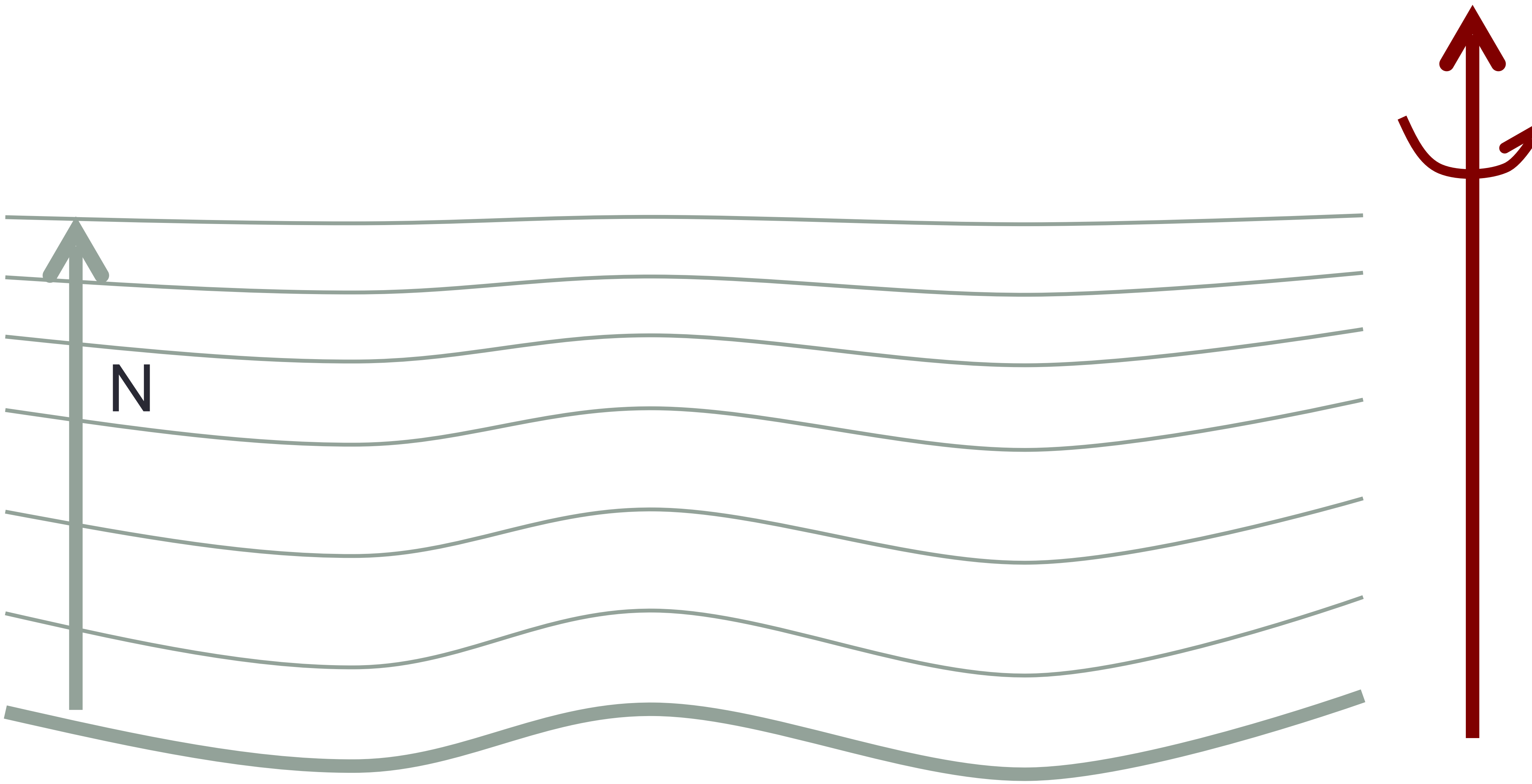
1. A general introduction to ocean waves



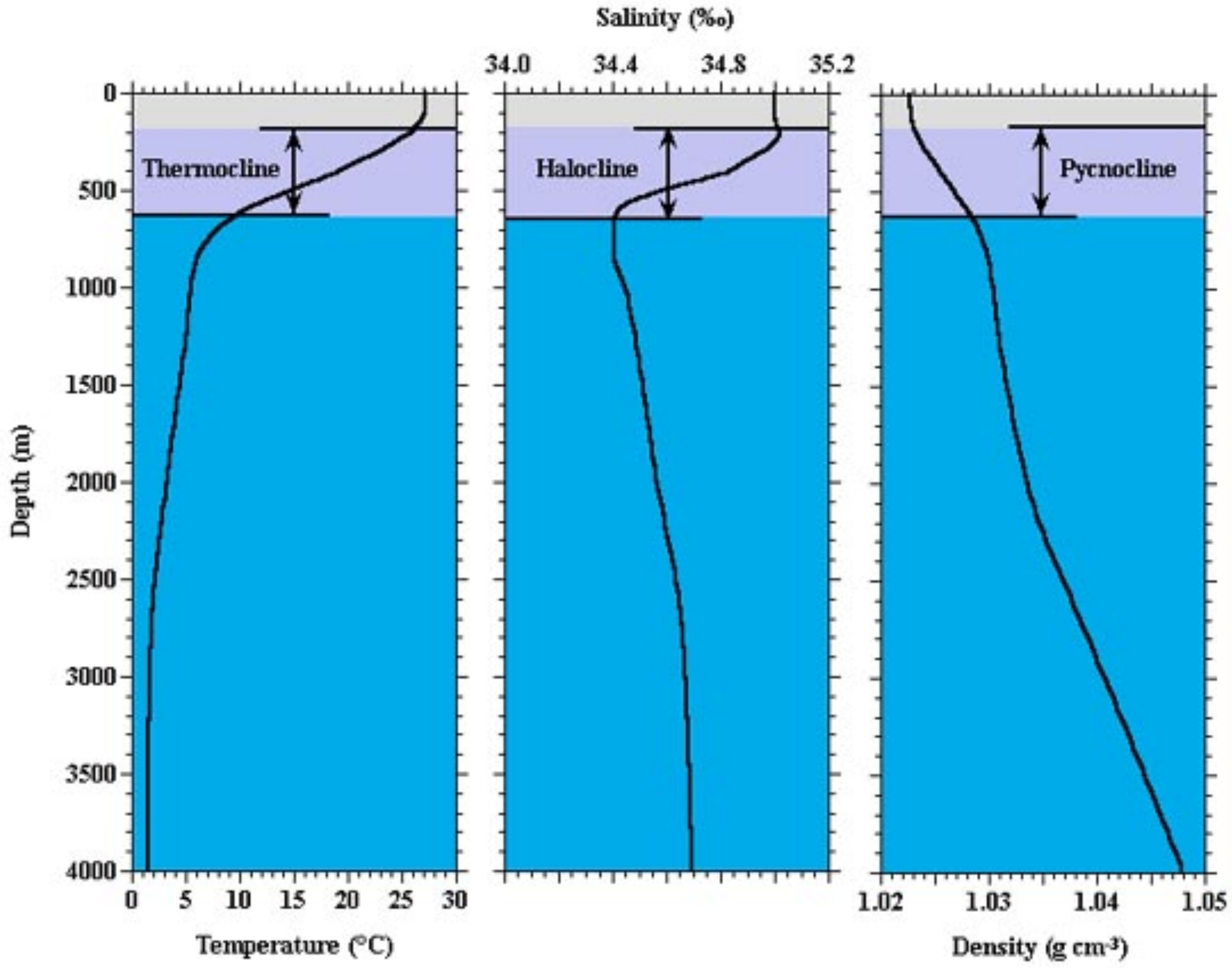
1.2 Equations

- Navier-Stokes Equations (NS)
- Non-hydrostatic Primitive Equations (NH)
- Hydrostatic Primitive Equations (PE)
- Shallow-water (SW)
- Quasi-geostrophic (QG)
- 2D Euler equations
- Etc.

2 important ingredients in Geophysical Fluid Dynamics (GFD): **rotation + stratification**

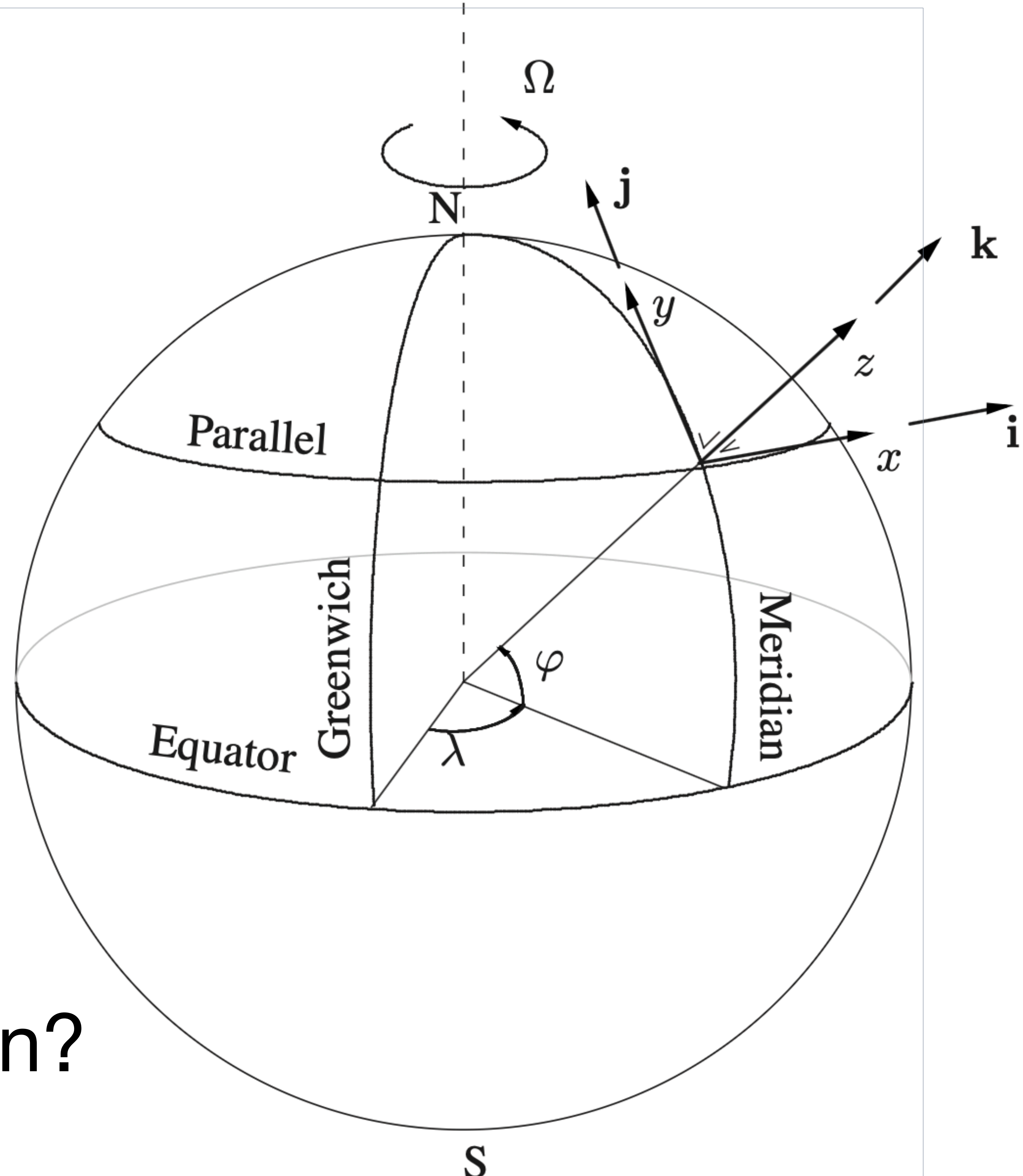


Stratification of the ocean:



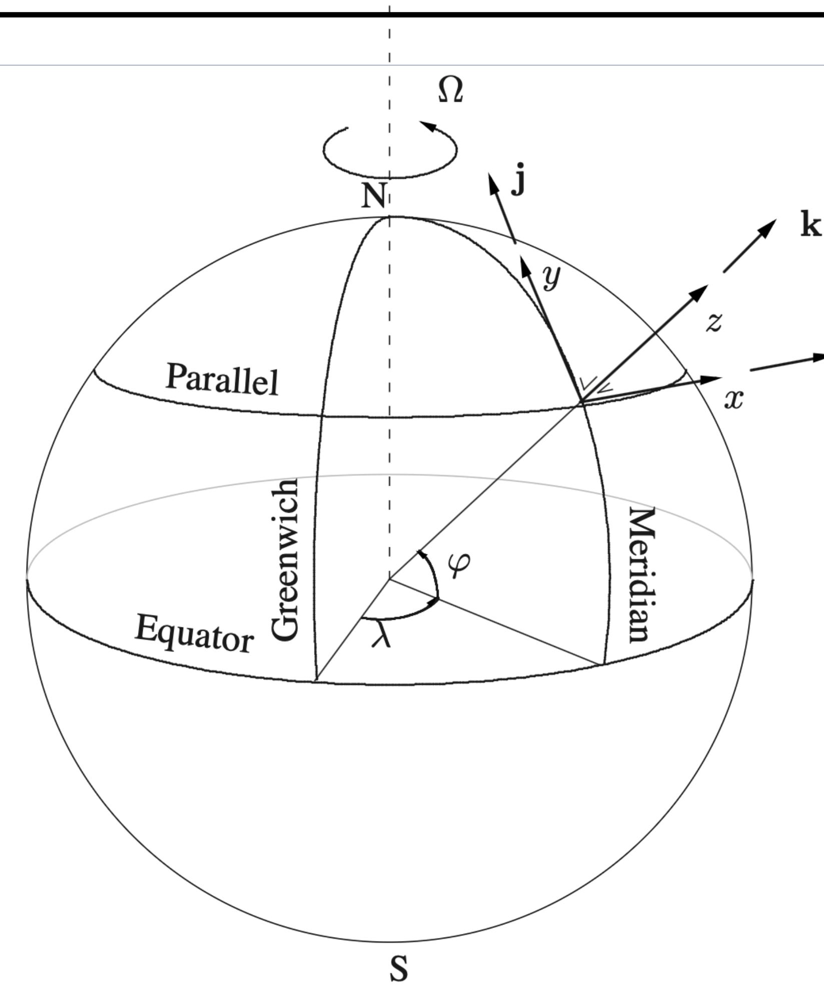
with Equation of state : $\rho = \rho(T, S, p)$

Rotation:



- Coriolis acceleration?

Rotation:

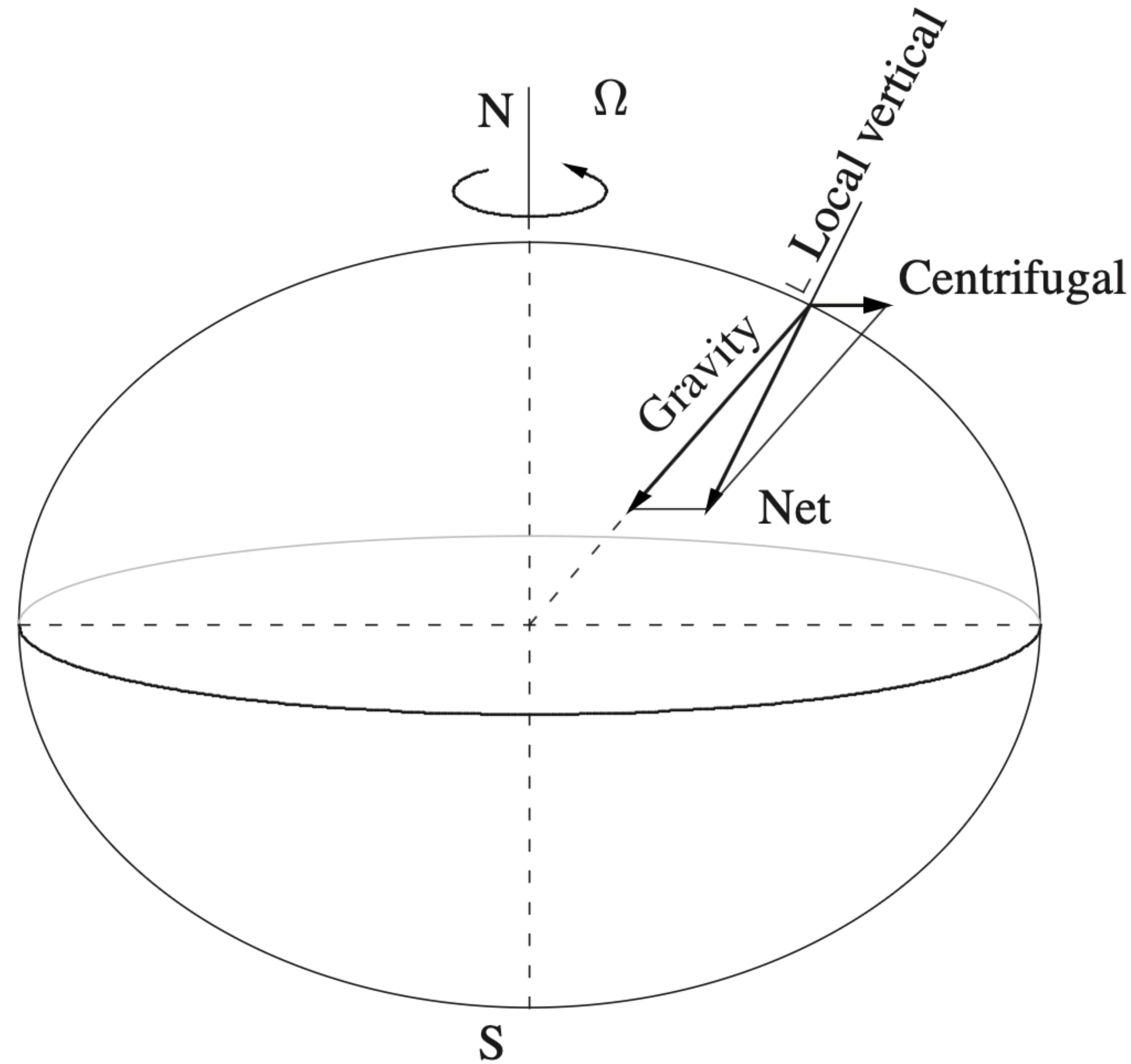


- Coriolis acceleration (or force): $2\overrightarrow{\Omega} \times \overrightarrow{v_r}$

- with the Coriolis parameter
- and the reciprocal Coriolis parameter

$$\begin{aligned} f &= 2\Omega \sin\varphi \\ f_* &= 2\Omega \cos\varphi. \end{aligned}$$

Rotation:



- Centrifugal force included in local gravity

Navier-Stokes Equations (with rotation + forcings):

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \overrightarrow{\nabla} \vec{u} + 2\overrightarrow{\Omega} \times \vec{u} + g\vec{k} = -\frac{\overrightarrow{\nabla} P}{\rho} + \nu \nabla^2 \vec{u} + \overrightarrow{\mathscr{F}}$$

Momentum equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation
(no source/sink)

Navier-Stokes Equations (with rotation + forcings):

$$\boxed{\frac{\partial \vec{u}}{\partial t}} + \boxed{\vec{u} \cdot \overrightarrow{\nabla} \vec{u}} + \boxed{2\overrightarrow{\Omega} \times \vec{u}} + \boxed{g\vec{k}} = \boxed{-\frac{\overrightarrow{\nabla} P}{\rho}} + \boxed{\nu \nabla^2 \vec{u}} + \boxed{\overrightarrow{\mathscr{F}}}$$

Time variation | Advection (inertia) | Rotation | Gravity | Pressure gradient | Viscosity | Forcings

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation (no source/sink)

Boussinesq Approximation:

Density perturbations small compared to mean background value:

$$\rho = \rho_0 + \rho' \qquad\qquad \rho' << \rho_0$$

Linearize all terms involving a product with density,

except the gravity term which is already linear:

$$\begin{aligned} \rho\vec{u} \quad &\rightarrow \quad \rho_0\vec{u} \\ \rho g \quad &\rightarrow \quad \rho g \end{aligned}$$

Boussinesq Approximation :

[+ incompressibility or adiabatic]

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

$$\Big\downarrow \quad \text{Continuity equation}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

Non hydrostatic boussinesq (NH):

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \overrightarrow{\nabla} \vec{u} + 2\overrightarrow{\Omega} \times \vec{u} + \frac{\rho}{\rho_0} g \vec{k} = -\frac{\overrightarrow{\nabla} P}{\rho_0} + \nu \nabla^2 \vec{u} + \overrightarrow{\mathscr{F}}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

Easier to solve than Navier-Stokes, but still requires to invert a 3d elliptic equation for P (computationally expansive)

Hydrostatic balance:

The vertical component of the Boussinesq momentum equations is

$$\frac{\partial w}{\partial t} + \vec{u} \cdot \overrightarrow{\nabla} w - 2\Omega cos\phi u + \boxed{\frac{\rho}{\rho_0} g = -\frac{\partial_z P}{\rho_0}} + \nu \nabla^2 w + \mathcal{F}_w$$

For long horizontal motions (L >> H) the dominant balance is

H ~10 m
L ~ 1 km

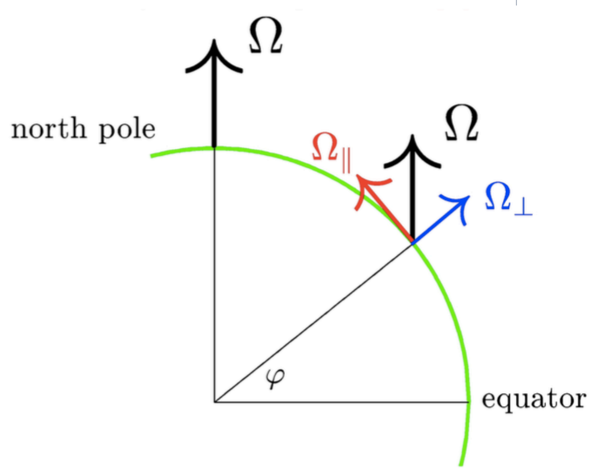
$$\frac{\partial P}{\partial z} = -\rho g$$

Such that pressure is just a vertical integral:

$$P = \int_z^\eta g \rho dz$$

Traditional approximation

= neglect horizontal Coriolis term



$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} + 2\vec{\Omega}\times\vec{u} + g\vec{k} = -\frac{\vec{\nabla}P}{\rho} + \nu\nabla^2\vec{u} + \vec{\mathscr{F}}$$

$$\downarrow$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} + f\vec{k}\times\vec{u} + g\vec{k} = -\frac{\vec{\nabla}P}{\rho} + \nu\nabla^2\vec{u} + \vec{\mathscr{F}}$$

1.2 Equations

Hydrostatic Primitive Equations (PE)

- 2d momentum with Boussinesq approximation:

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{\partial_x P}{\rho_0} + \nu \nabla^2 u + \mathcal{F}_u$$

$$\frac{\partial v}{\partial t} + \vec{u} \cdot \vec{\nabla} v + fu = -\frac{\partial_y P}{\rho} + \nu \nabla^2 v + \mathcal{F}_v$$

- Hydrostatic: $$\frac{\partial P}{\partial z} = -\rho g$$

- Continuity equation for an incompressible fluid: $$\vec{\nabla} \cdot \vec{u} = 0$$

1.2 Equations

- Ex: Geostrophic balance

$$\cancel{\frac{\partial u}{\partial t} + \vec{u} \cdot \overrightarrow{\nabla}\, u} - fv = -\frac{\partial_x P}{\rho_0} + \cancel{\nu \nabla^2 u + \mathscr{F}_u}$$

$$\cancel{\frac{\partial v}{\partial t} + \vec{u} \cdot \overrightarrow{\nabla}\, v} + fu = -\frac{\partial_y P}{\rho} + \cancel{\nu \nabla^2 v + \mathscr{F}_v}$$

1.2 Equations

- Ex: Linear Gravity waves

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v &= - \frac{\partial_x P}{\rho_0} + \nu \nabla^2 u + \mathscr{F}_u \\
\frac{\partial v}{\partial t} + \vec{u} \cdot \vec{\nabla} v + f u &= - \frac{\partial_y P}{\rho} + \nu \nabla^2 v + \mathscr{F}_v
\end{aligned}
$$

1.2 Equations

- Ex: Linear Inertia-Gravity waves

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \cancel{\vec{u} \cdot \vec{\nabla} u} - fv &= -\frac{\partial_x P}{\rho_0} + \cancel{\nu \nabla^2 u + \mathcal{F}_u} \\
\frac{\partial v}{\partial t} + \cancel{\vec{u} \cdot \vec{\nabla} v} + fu &= -\frac{\partial_y P}{\rho} + \cancel{\nu \nabla^2 v + \mathcal{F}_v}
\end{aligned}
$$

1.2 Equations

- Ex: Inertial motions and NIW

$$
\boxed{\begin{aligned}
\frac{\partial u}{\partial t} + \vec{u} \cdot \overrightarrow{\nabla} u - f v &= \\
\frac{\partial v}{\partial t} + \vec{u} \cdot \overrightarrow{\nabla} v + f u &=
\end{aligned}}
\begin{aligned}
&\cancel{- \frac{\partial_x P}{\rho_0} + \nu \nabla^2 u + \mathscr{F}_u} \\
&\cancel{- \frac{\partial_y P}{\rho} + \nu \nabla^2 v + \mathscr{F}_v}
\end{aligned}
$$

1.2 Equations

- Navier-Stokes Equations (NS)
- Non-hydrostatic Primitive Equations (NH)
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- Shallow-water (SW)
- Quasi-geostrophic (QG)
- 2D Euler equations
- Etc.

Ex: Euler equations

- No forcings/dissipation/rotation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \overrightarrow{\nabla} \vec{u} + \frac{\rho}{\rho_0} g \vec{k} = -\frac{\overrightarrow{\nabla} P}{\rho_0}$$

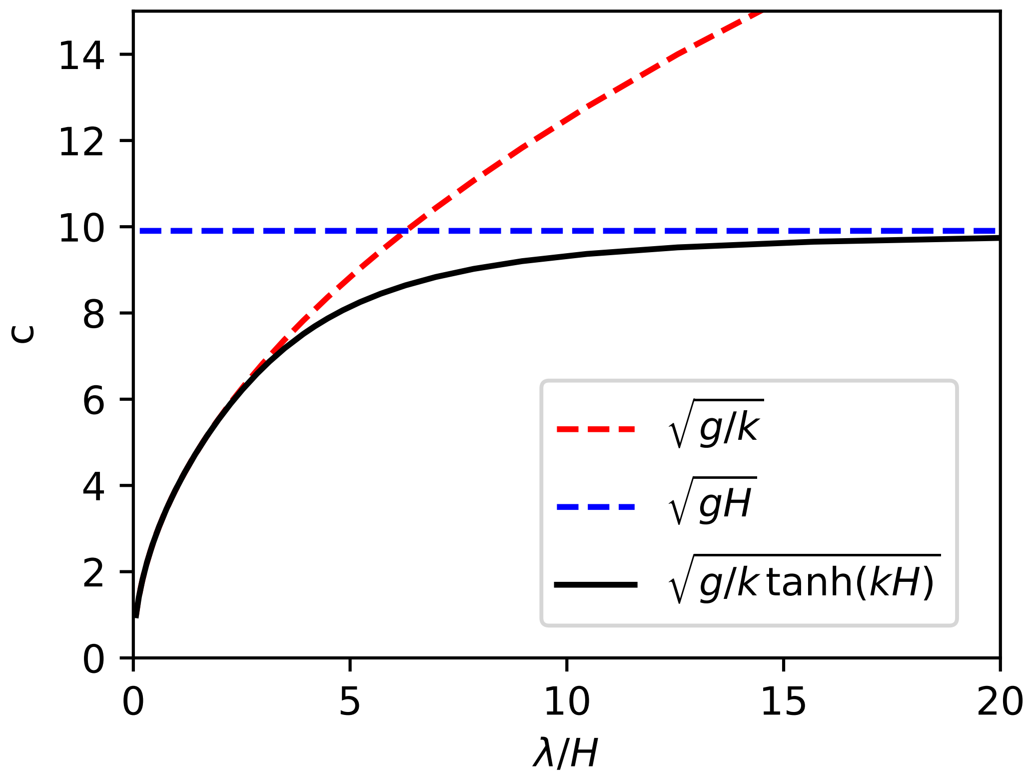
Ex: linearized incompressible Euler equations

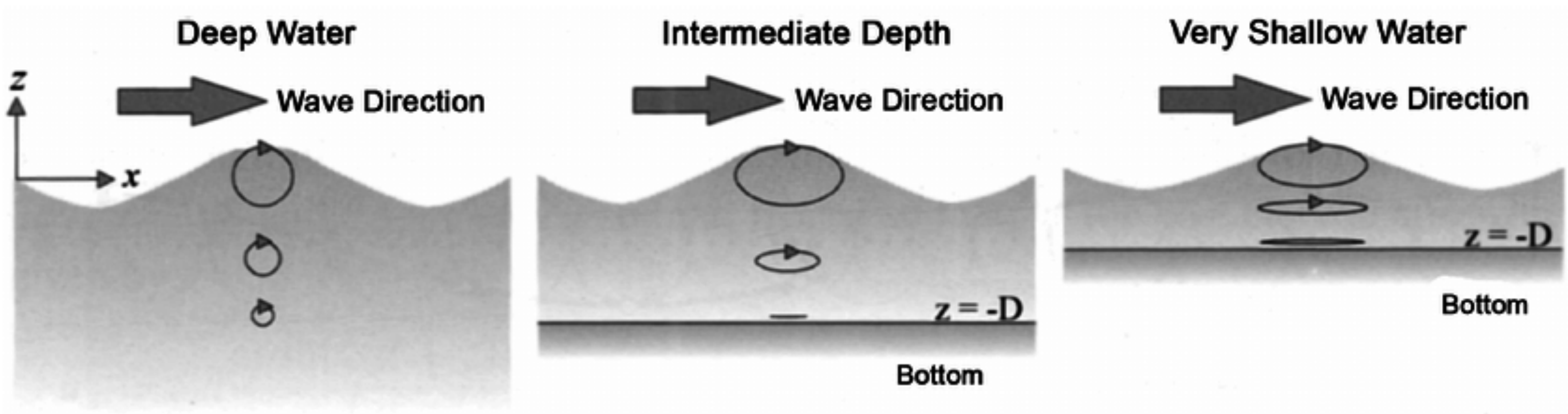
- No forcings/dissipation/rotation
- Incompressibility
- remove non-linear terms

$$
\begin{aligned}
u_t &= -p_x \\
v_t &= -p_y \\
w_t &= -p_z - g \\
0 &= u_x + v_y + w_z ,
\end{aligned}
$$

Ex: linearized incompressible Euler equations

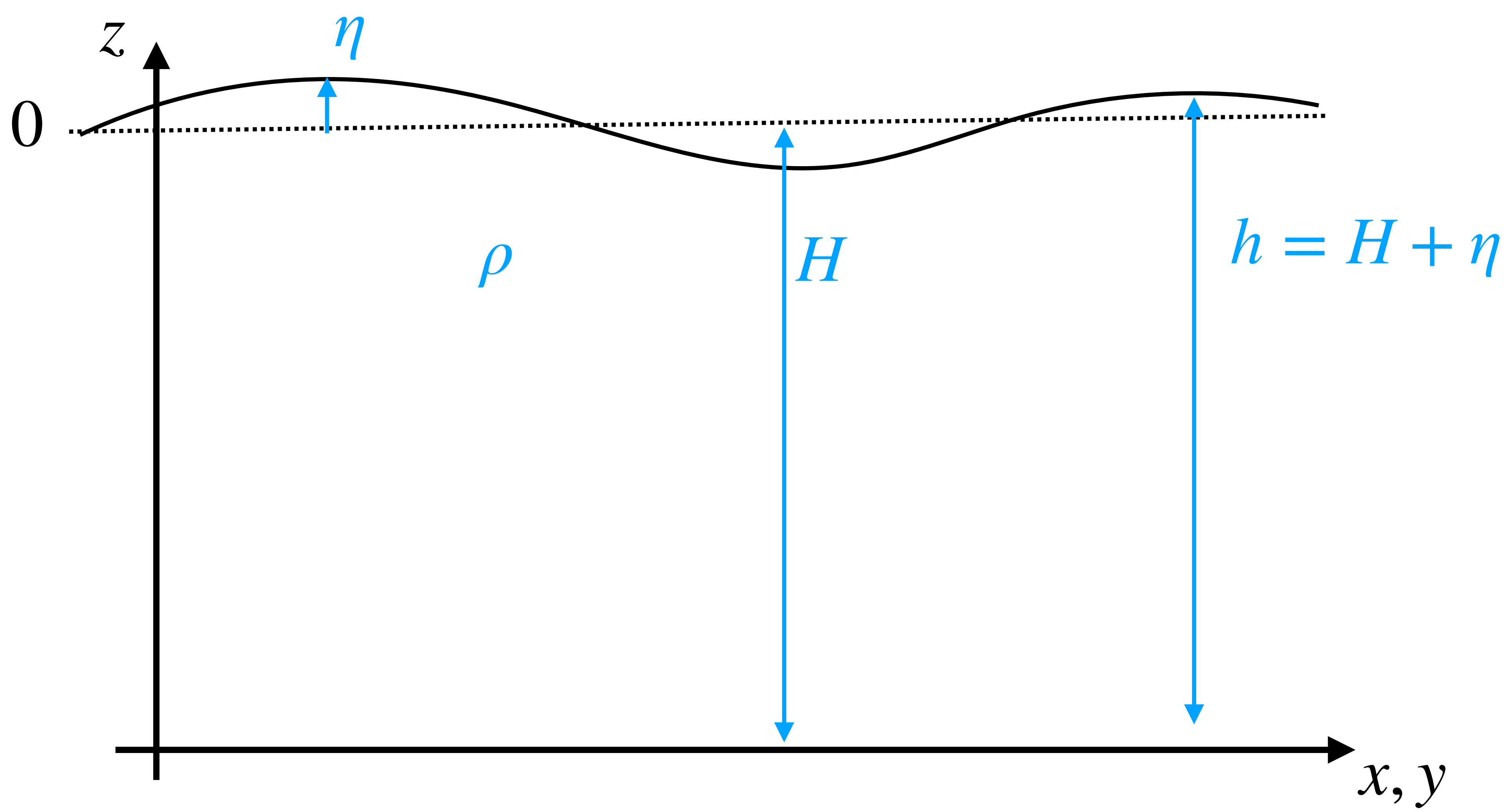
- Solutions give surface gravity waves:





Ex: Shallow-water equations

- No forcings/dissipation
- constant density / no vertical variations / hydrostatic



Ex: Shallow-water equations

- No forcings/dissipation
- constant density / no vertical variations / hydrostatic

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y},$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0,$$

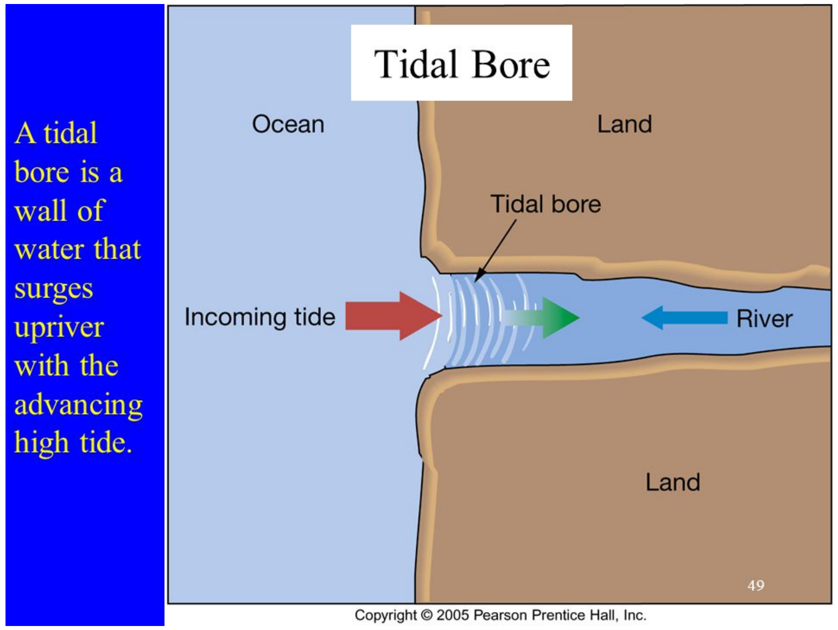
Ex: linearized Shallow-water equations

- No forcings/dissipation
- constant density / no vertical variations / hydrostatic
- remove non-linear terms

$$
\begin{aligned}
\frac{\partial u}{\partial t} - f v &= - g\frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} + f u &= - g\frac{\partial \eta}{\partial y}, \\
\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} + H\frac{\partial v}{\partial y} &= 0,
\end{aligned}
$$

Ex: linearized Shallow-water equations

- Solutions give long surface gravity waves:



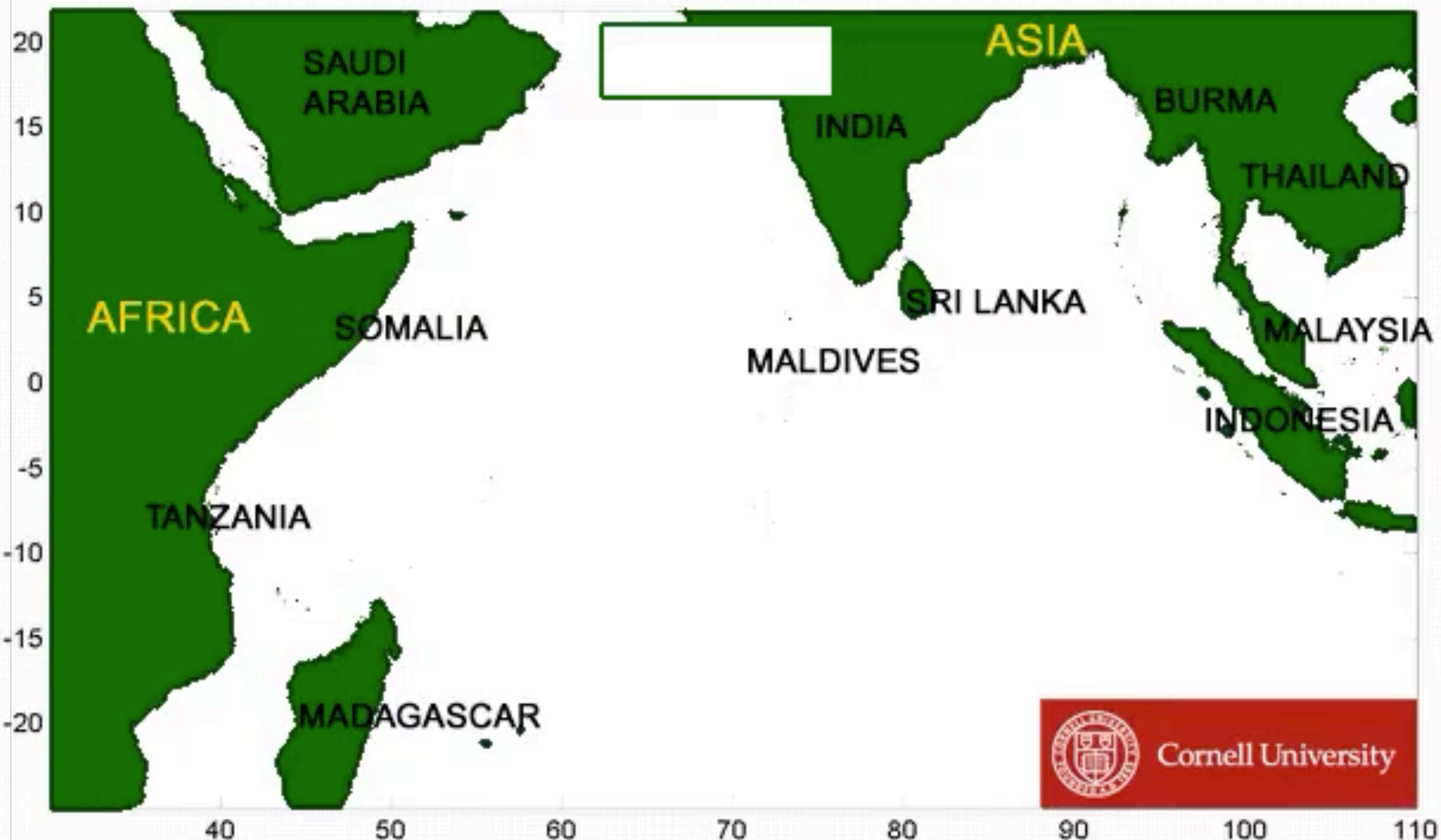
$$\omega = \sqrt{gH}k \qquad c = \sqrt{gH}$$

Ex: linearized Shallow-water equations

- Solutions give long surface gravity waves: $c = \sqrt{gH}$

Depth (H) (m)	c (m/s)	c (km/h)
10	9.9	36 km/h
100	31.3	113 km/h
1000	99.0	356 km/h
4000 (deep ocean)	198	713 km/h

Ex: Tsunamis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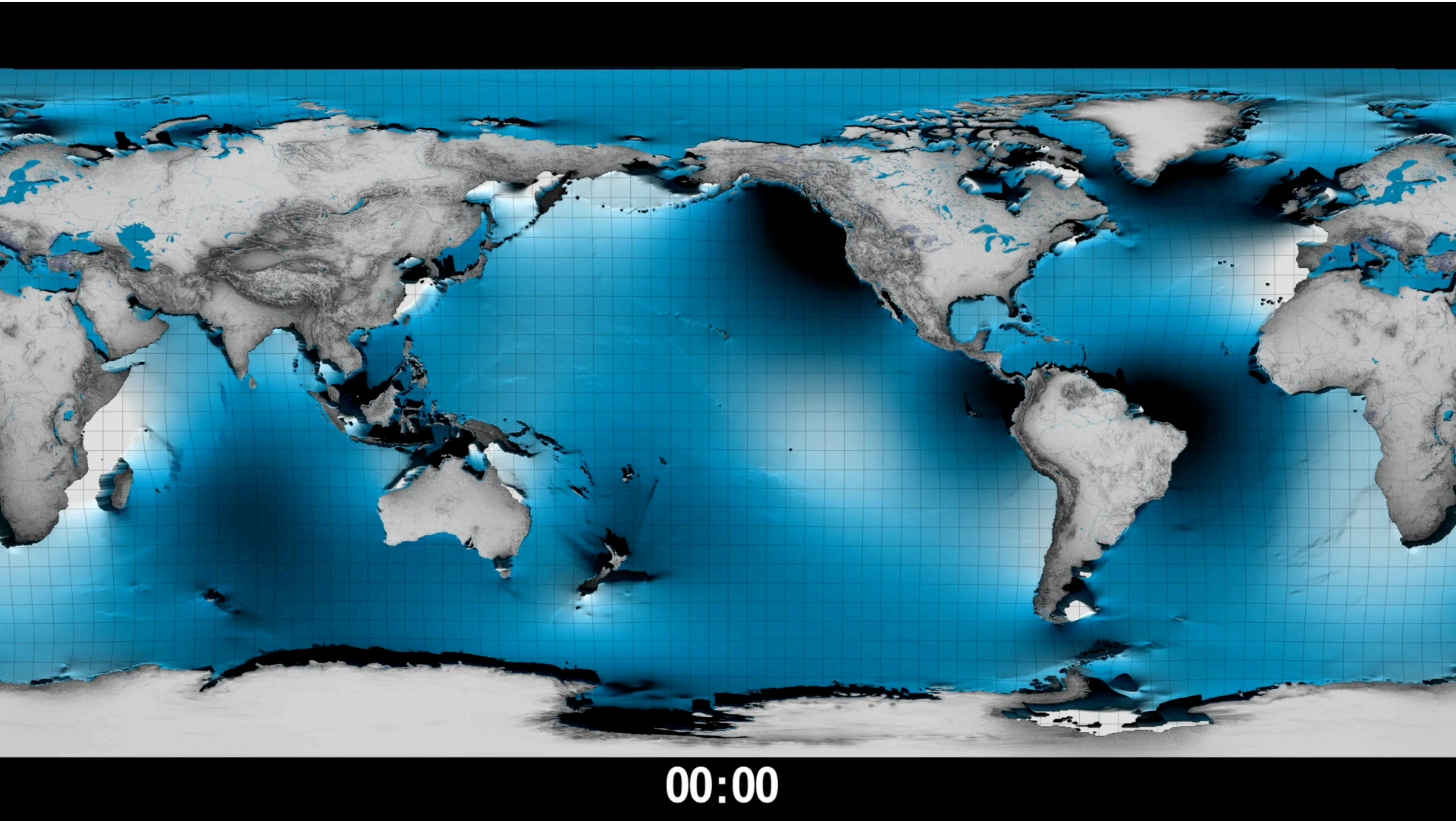

Ex: Barotropic tides

1. A general introduction to ocean waves

References used in this document:

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- https://en.wikipedia.org/wiki/Wave
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