Numerical Modelling

Jonathan GULA
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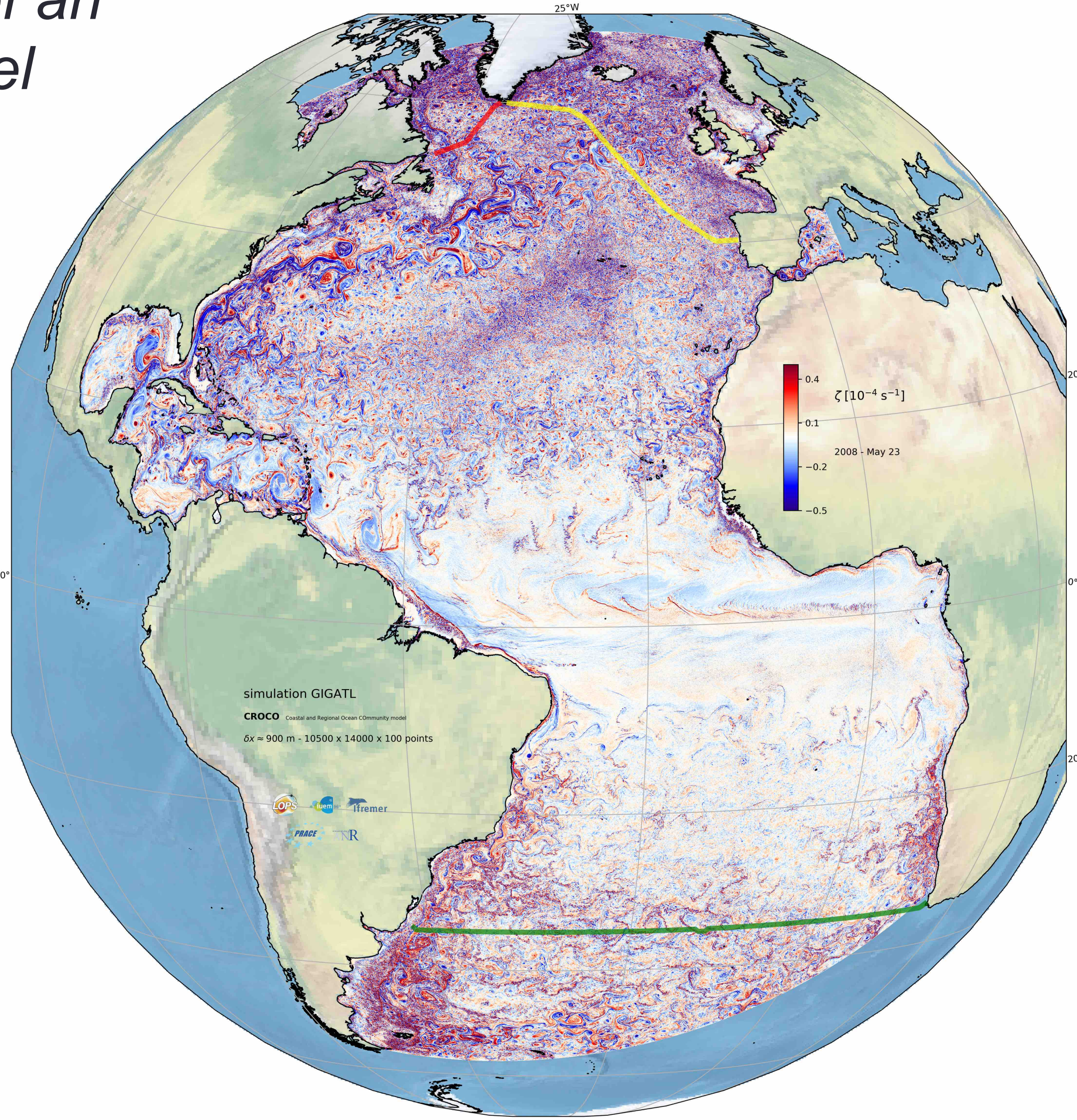
the anatomy of an ocean model



- **Lesson 1 :** [D109]
  - Introduction
  - Equations of motions
  - *Activity 1 [run an ocean model]*
- **Lesson 2 :** [D109]
  - Horizontal Discretization
  - *Activity 2 [Dynamics of an ocean gyre]*
- **Lesson 3 :** [D109]
  - *Activity 2 [Dynamics of an ocean gyre]*
  - Dynamics of the ocean gyre
- **Lesson 4 :** [D109]
  - Numerical schemes
  - *Activity 3 [Impacts of numerics]*
- **Lesson 5 :** [D109]
  - Vertical coordinates
  - *Activity 4 [Impact of topography]*
- **Lesson 6 :** [D109]
  - Boundary Forcings
  - Presentation of the model CROCO
  - *Activity 5 [Design a realistic simulation]*
- **Lesson 7 :** [D109]
  - Diagnostics and validation
  - *Activity 6 [Analyze a realistic simulation]*
- **Lesson 8 :** [D109]
  - *Project*

Presentations and material will be available at :

jgula.fr/ModNum/

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Evaluation

- The evaluation is based on a project, which consists in setting up a realistic configuration of the region of your choice, run the experiment and perform some analysis.

- Written Report **due Feb. 13**

Useful references

Extensive courses:

- MIT: https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-950-atmospheric-and-oceanic-modeling-spring-2004/lecture-notes/
- Princeton: https://stephengriffies.github.io/assets/pdfs/GFM_lectures.pdf

Overview on ocean modelling and current challenges:

- Griffies et al., 2000, Developments in ocean climate modelling, Ocean Modelling.http://jgula.fr/ModNum/Griffiesetal00.pdf
- Griffies, 2006, "Some Ocean Model Fundamentals", In "Ocean Weather Forecasting: An Integrated View of Oceanography", 2006, Springer Netherlands. http://jgula.fr/ModNum/Griffies_Chapter.pdf
- Fox-Kemper et al, 19, “Challenges and Prospects in Ocean Circulation Models" http://jgula.fr/ModNum/FoxKemperetal19.pdf

ROMS/CROCO:

- https://www.croco-ocean.org/documentation/
- Shchepetkin, A., and J. McWilliams, 2005: The Regional Oceanic Modeling System (ROMS): A split-explicit, free-surface, topography-following- coordinate ocean model. Ocean Modell. http://jgula.fr/ModNum/ShchepetkinMcWilliams05.pdf

Introduction

Master's degree 2nd year Marine Physics

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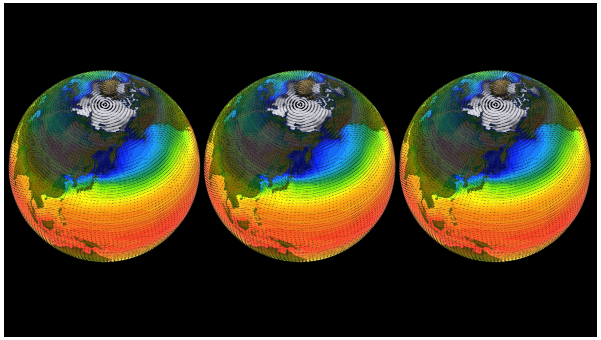
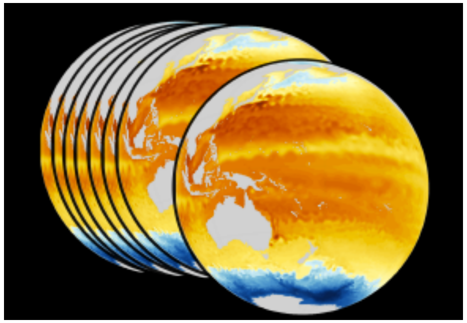
For what ocean models are used?

For what ocean models are used?

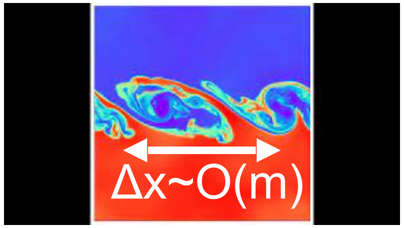
- Weather forecasts and Climate projections



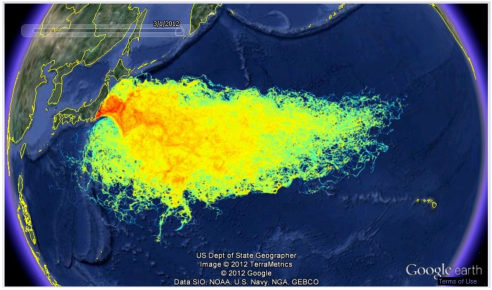
- Explore different states of the ocean and earth system



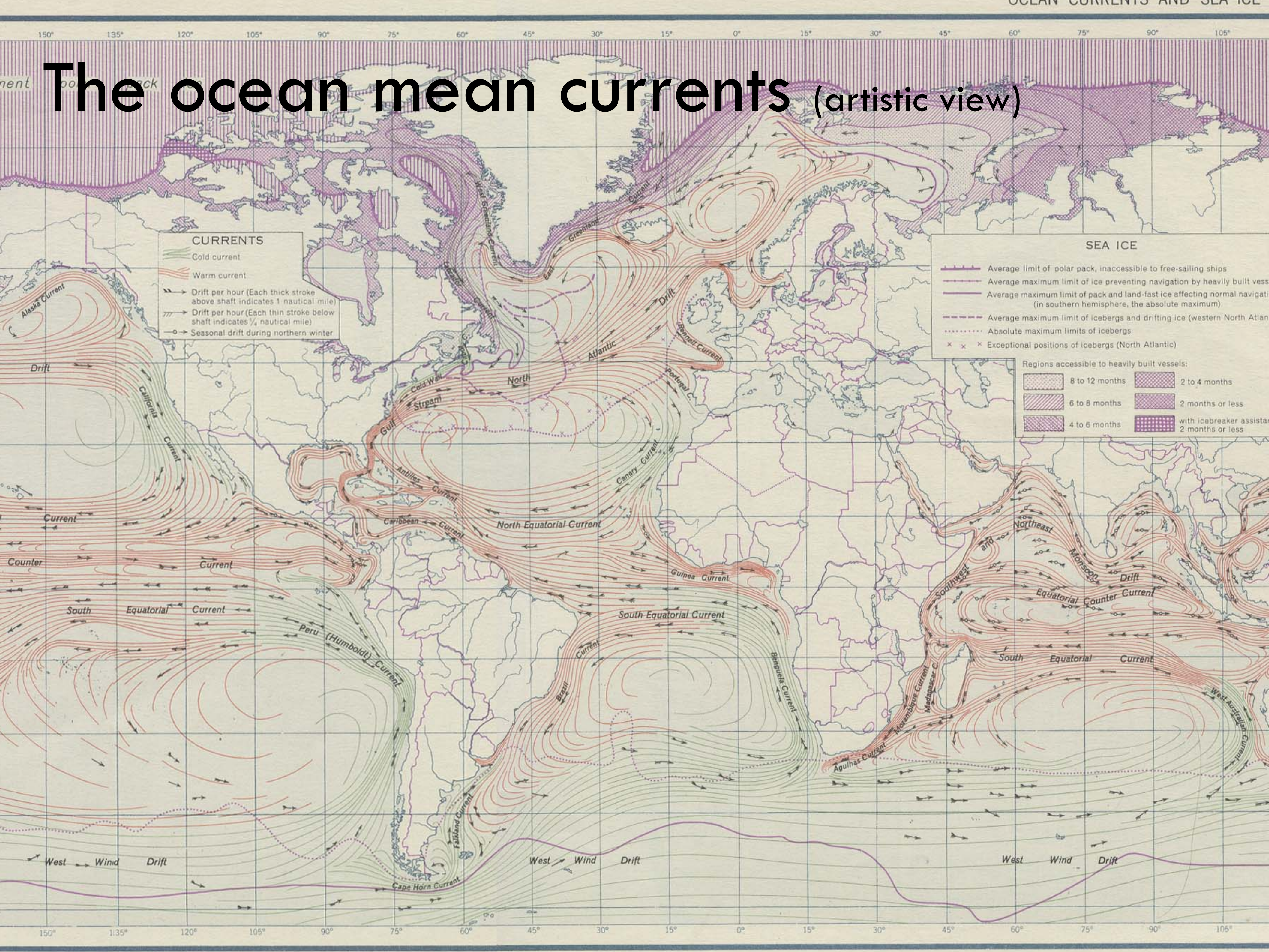
- Better understand the complex dynamics of geophysical flow, from small to large scales
- Testing (idealized) mathematical models



- Impact studies (e.g pollution, microplastics, larvae, etc.)



The ocean mean currents (artistic view)



Can be understood using theory and simplified models

THE WESTWARD INTENSIFICATION OF WIND-DRIVEN OCEAN CURRENTS

Henry Stommel

(Contribution No. 408, Woods Hole Oceanographic Instiution)

Abstract--A study is made of the wind-driven circulation in a homogeneous rectangular ocean under the influence of surface wind stress, linearised bottom friction, horizontal pressure gradients caused by a variable surface height, and Coriolis force.

An intense crowding of streamlines toward the western border of the ocean is discovered to be caused by variation of the Coriolis parameter with latitude. It is suggested that this process is the main reason for the formation of the intense currents (Gulf stream and others) observed in the actual oceans.

JOURNAL OF METEOROLOGY

ON THE WIND-DRIVEN OCEAN CIRCULATION

By *Walter H. Munk*

Institute of Geophysics and Scripps Institution of Oceanography, University of California[1]

(Manuscript received 24 September 1949)

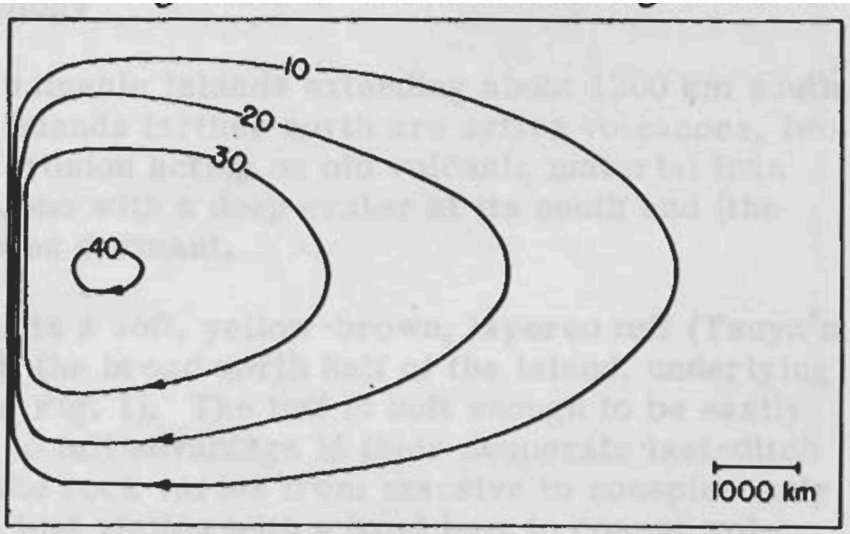


Fig. 5--Streamlines for the case where the Coriolis force is a linear function of latitude

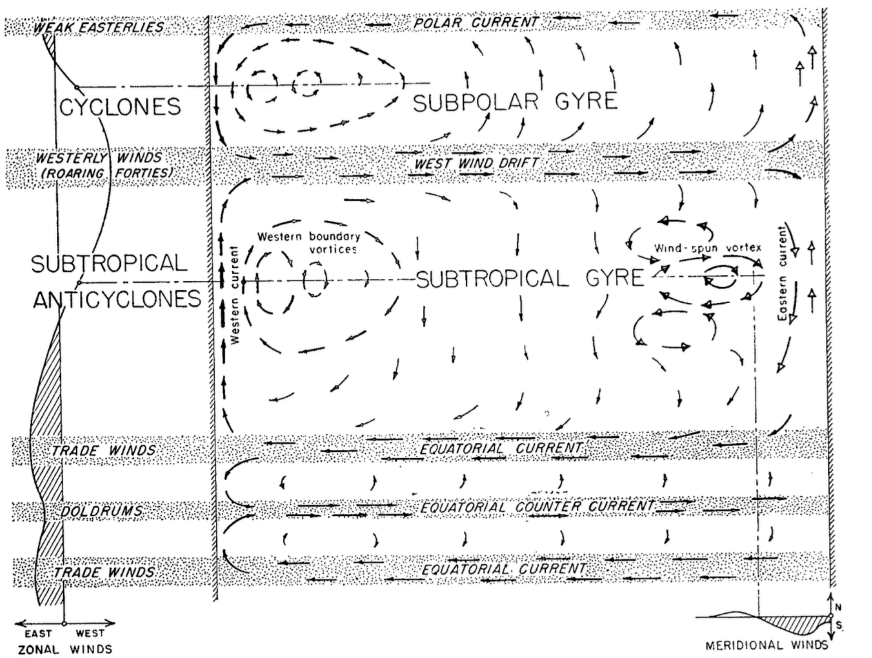
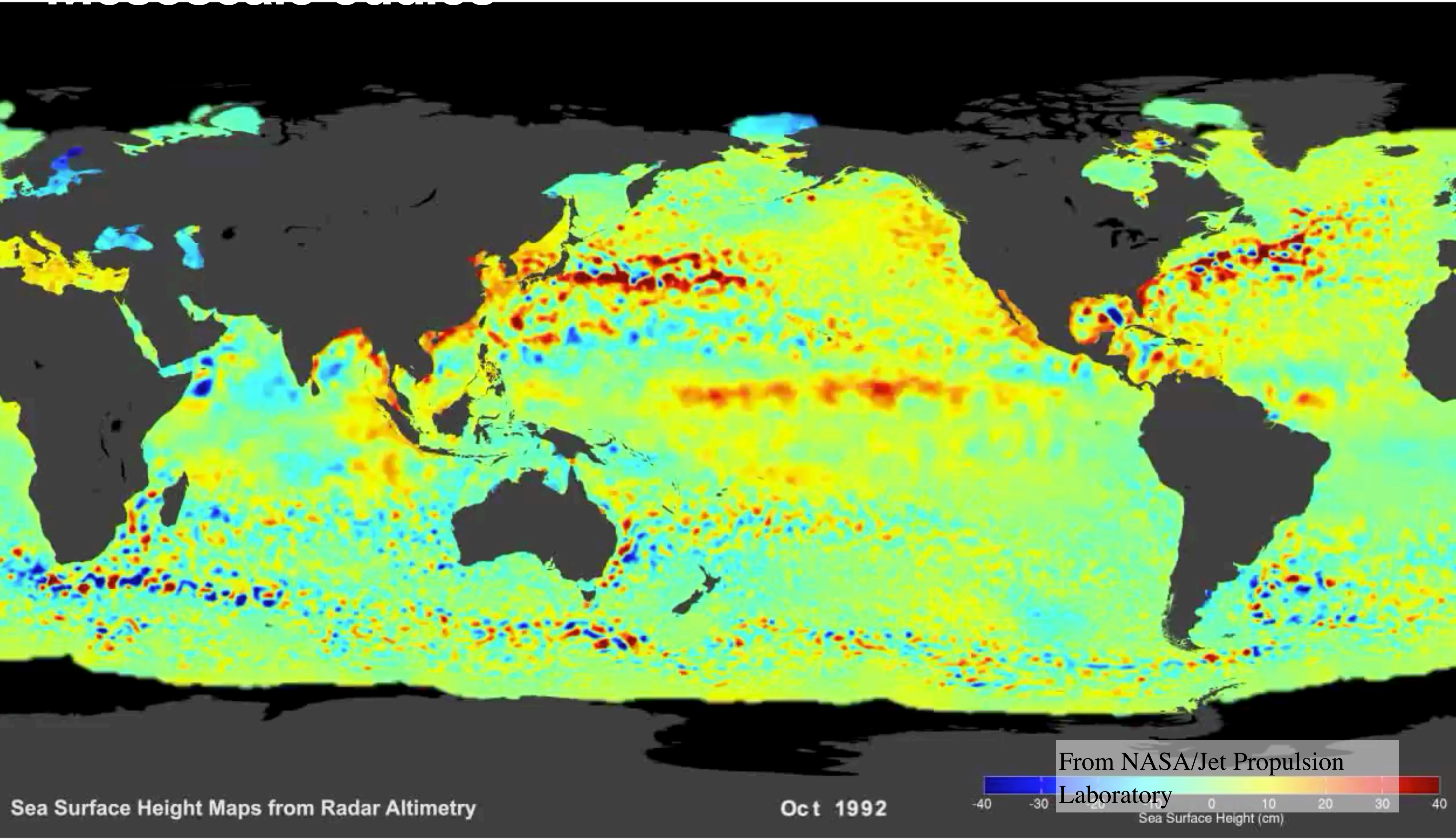


FIG. 8. Schematic presentation of circulation in a rectangular ocean resulting from zonal winds (filled arrowheads), meridional winds (open arrowheads), or both (half-filled arrowheads). The width of the arrows is an indication of the strength of the currents. The nomenclature applies to either hemisphere, but in the Southern Hemisphere the subpolar gyre is replaced largely by the Antarctic Circumpolar Current (west wind drift) flowing around the world. Geographic names of the currents in various oceans are summarized in table 3.

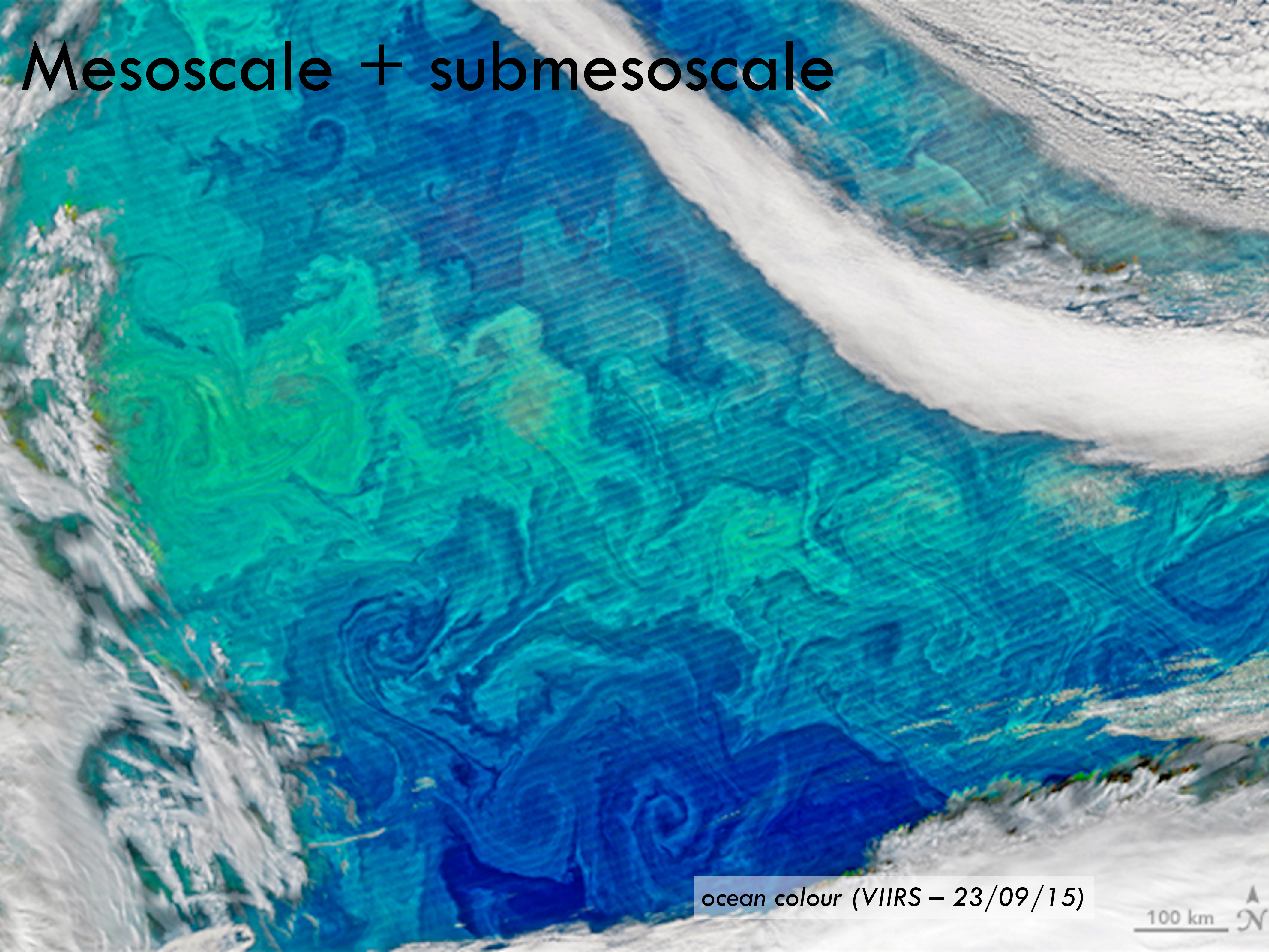
[Stommel, 1948](#) [Munk, 1950](#)

But reality is much more turbulent:



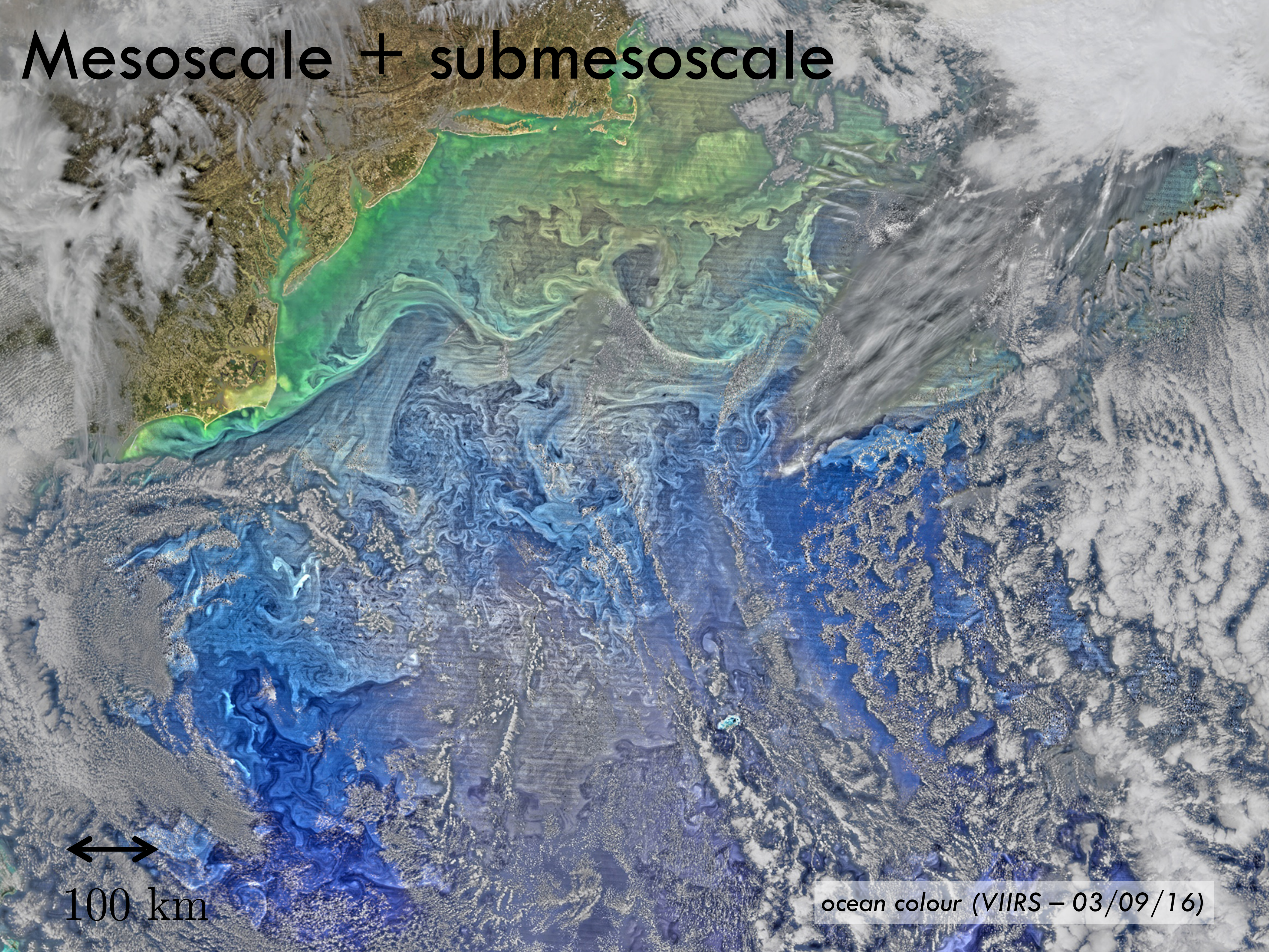
From NASA/Jet Propulsion Laboratory

Mesoscale + submesoscale

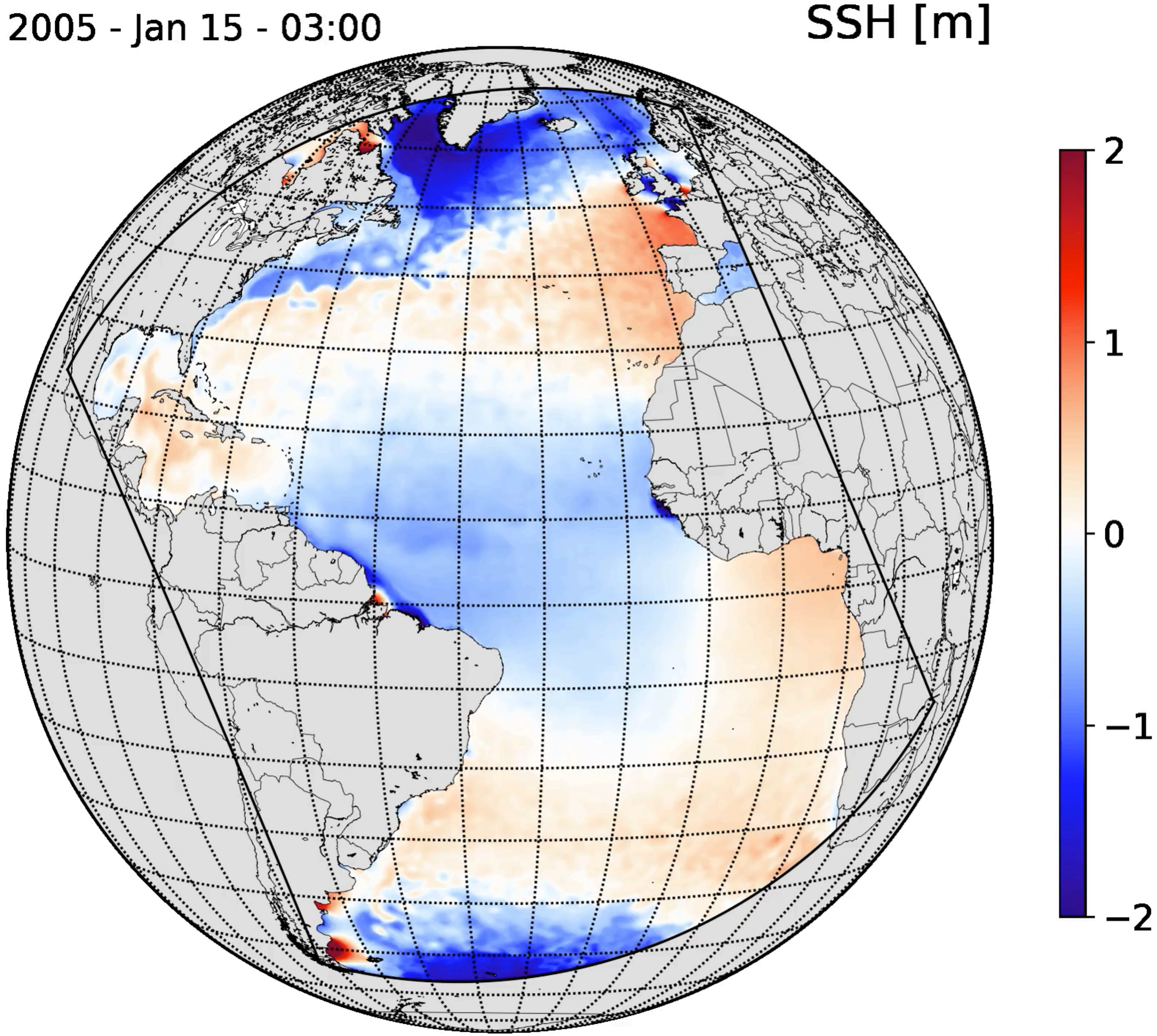


ocean colour (VIIRS – 23/09/15)

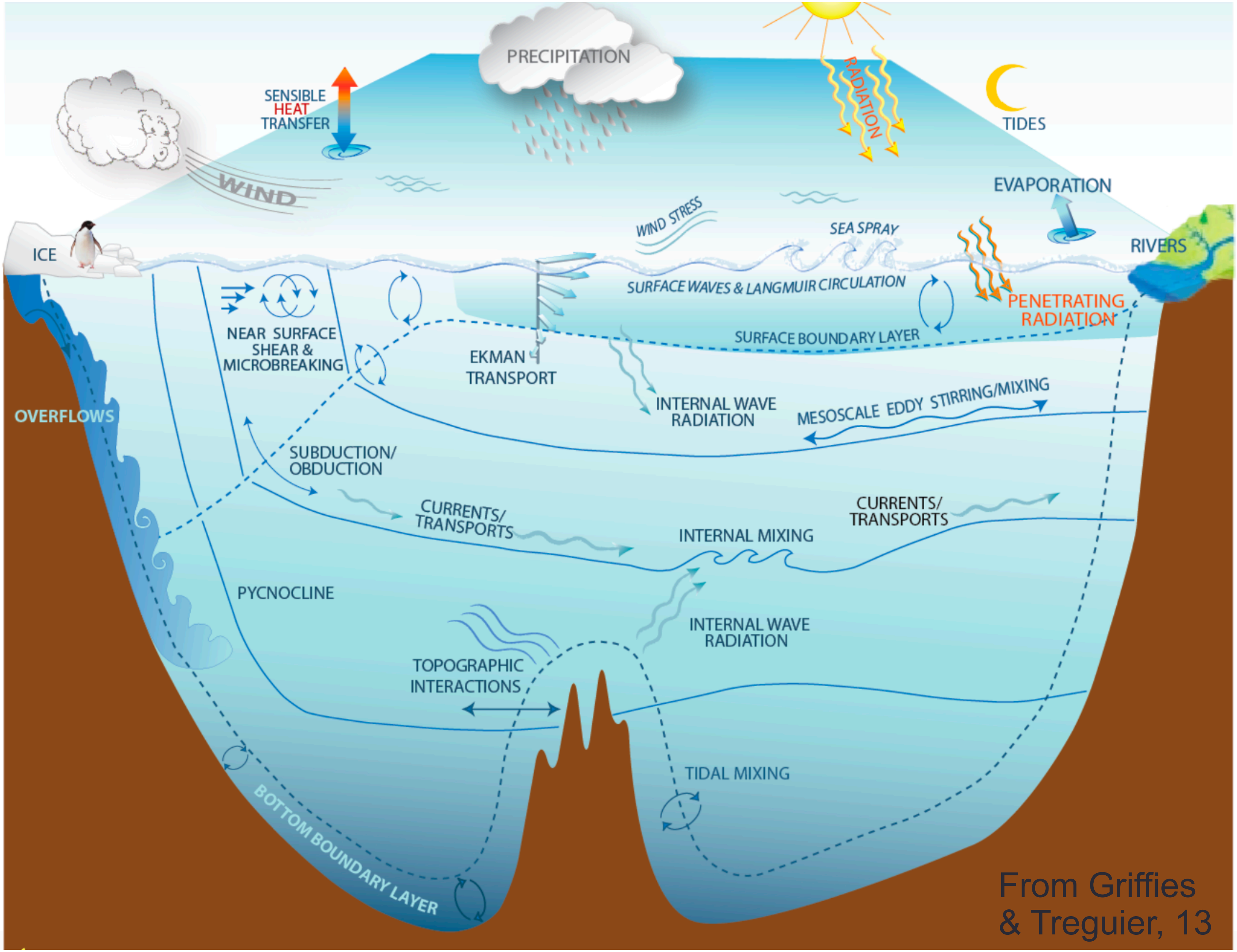
Mesoscale + submesoscale



And HF motions like tides:

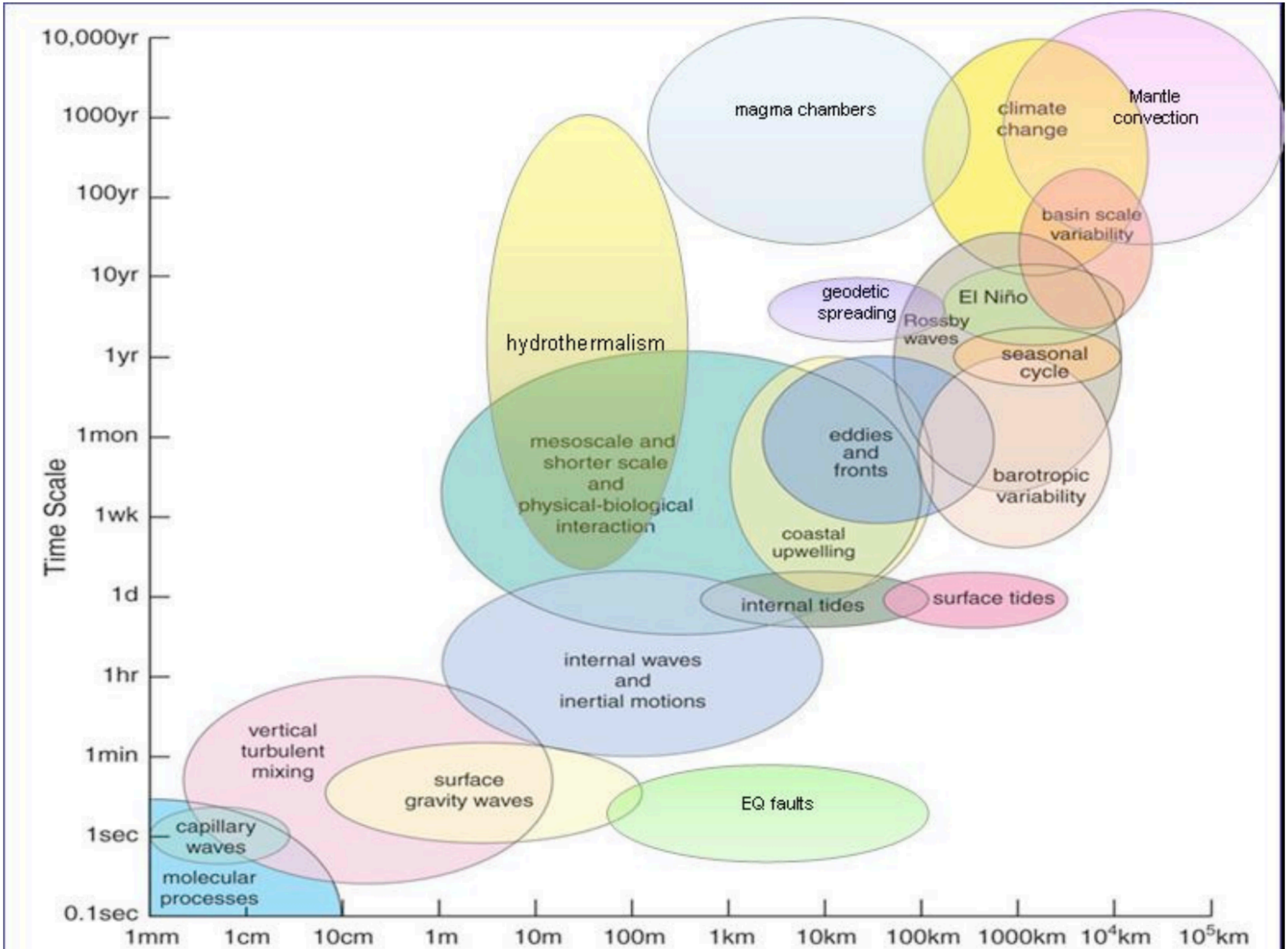


A zoo of processes and scales



From Griffies & Treguier, 13

Temporal and spatial scales in the ocean



From Chelton et al, 11

Ocean modeling principle

The problem of weather prediction, considered from the viewpoints of mechanics and physics

VILHELM BJERKNES

If it is true, as any scientist believes[E1]*, that subsequent states of the atmosphere develop from preceding ones according to physical laws, one will agree that the necessary and sufficient conditions for a rational solution of the problem of meteorological prediction are the following:*

1. *One has to know with sufficient accuracy the state of the atmosphere at a given time.*
2. *One has to know with sufficient accuracy the laws according to which one state of the atmosphere develops from another.*

Bjerknes, 1904

Original citation:
BJERKNES, V., 1904: Das Problem der Wettervorhersage, betrachtet vom Standpunkte der Mechanik und der Physik. – Meteorol. Z. **21**, 1–7.

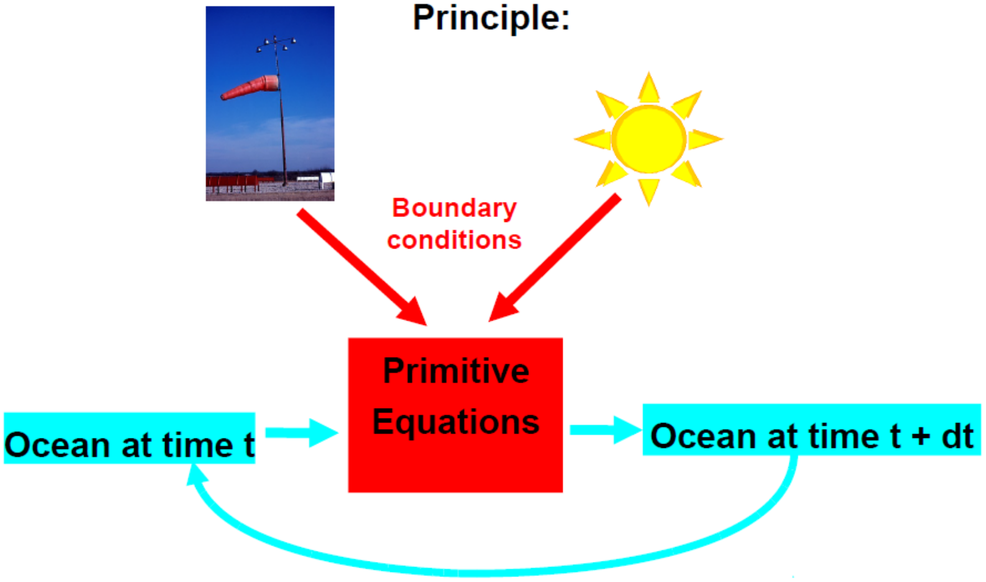
Ocean modeling principle

If we know:

- The **ocean state at time *t* :**

  u, v, w, T, S, ...

- **Boundary conditions :**
  surface, bottom, lateral sides



→ We can **compute the ocean state at time *t+dt*** by solving the **fluid dynamics equations**

Ocean modeling principle

The problem of weather prediction, considered from the viewpoints of mechanics and physics

VILHELM BJERKNES

For calculating these parameters, we can propose the following equations:

1. The three hydrodynamic equations of motion. These are differential relations among the three velocity components, density and air pressure.
2. The continuity equation, which expresses the principle of the conservation of mass during motion. This equation is also a differential relation, namely between the velocity components and the density.
3. The equation of state for the atmosphere, which is a f nite relation among density, air pressure, temperature, and humidity of a given air mass.
4. The two fundamental laws of thermodynamics, which allow us to write two differential relations that specify how energy and entropy of any air mass change in a change of state. These equations do not introduce

We can therefore set up seven equations independent from each other with the seven normally occurring variables. As far as it is possible to have an overview of the problem now, we must conclude that our knowledge of the laws of atmospheric processes is suff cient to serve as a basis for a rational weather prediction. However, it must be admitted that we may have overlooked important factors due to our incomplete knowledge. The interference of unknown cosmic effects is possible. Furthermore, the major atmospheric phenomena are accompanied by a long list of side effects, such as those of an electrical and optical nature. The question is to what extent such side effects could have considerable effects on the development of atmospheric processes. Such effects evidently do exist. The rainbow, for instance, will result in a modif ed distribution of incoming radiation and it is well known that electrical charges inf uence condensation processes. However, evidence is still lacking on whether processes of this kind have an impact on major atmospheric processes. At any rate, the scientif c method is to begin with the simplest formulation of the problem, which is the problem posed above with seven variables and seven equations.

Original citation:

BJERKNES, V., 1904: Das Problem der Wettervorhersage, betrachtet vom Standpunkte der Mechanik und der Physik. – Meteorol. Z. **21**, 1–7.

We know the equations:

Momentum equations [Navier-Stokes]

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}}$$

Conservation of mass (continuity)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Conservation of heat *(internal energy)*

$$\frac{DT}{Dt} = \mathcal{S}_T$$

Conservation of salinity *(or humidity for the atmosphere)*

$$\frac{DS}{Dt} = \mathcal{S}_S$$

Equation of state

$$\rho = \rho(T, S, p)$$

We know the equations:

Momentum equations [Navier-Stokes]

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}}$$

Conservation of mass (continu

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Conservatio energy)

Conservatio humidity for the atmosphere)

$$Dt$$

But we don't know the solutions…

Equation of state

$$\rho = \rho(T, S, p)$$

We know the equations:

Millennium Prize problems

Navier–Stokes existence and smoothness

The Clay Mathematics Institute in May 2000 made this problem one of its seven Millennium Prize problems in mathematics. It offered a US $1,000,000 prize to the first person providing a solution for a specific statement of the problem:[1]

Prove or give a counter-example of the following statement:

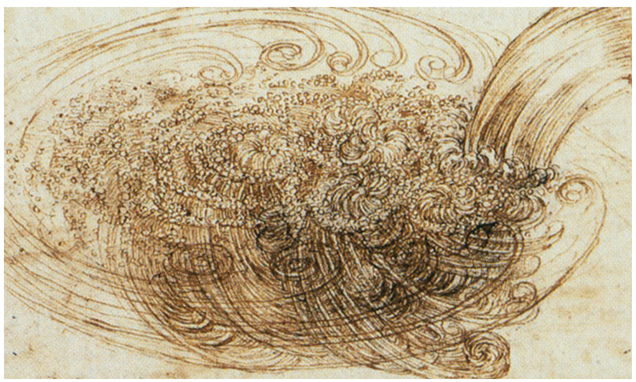
In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

We know the equations:

Navier-Stokes
[Momentum equations]

$$\frac{\partial \vec{u}}{\partial t} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}}$$

Non-linear terms = Turbulence



turbolenza by da Vinci [1507]

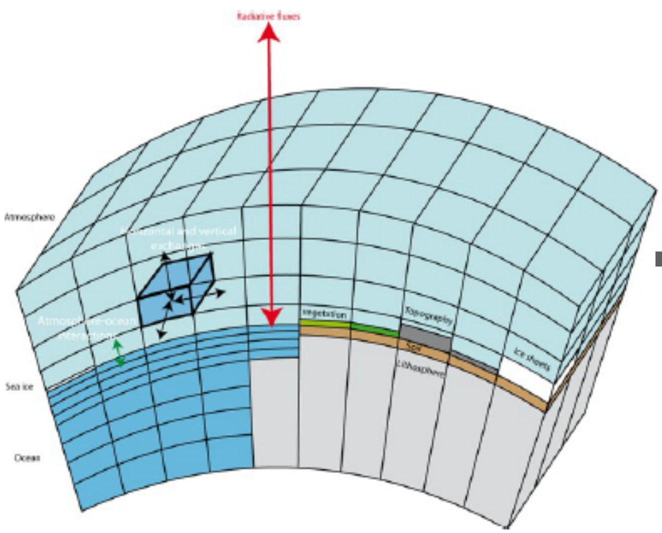
Big whorls have little whorls,
which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity.
L.F. Richardson [1922]

Turbulent water jet (Re = 2300) [Van Dyke, 82]

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}}$$

Navier Stokes equations

+



Grid Discretization

+

Supercomputer (Curie – CEA)

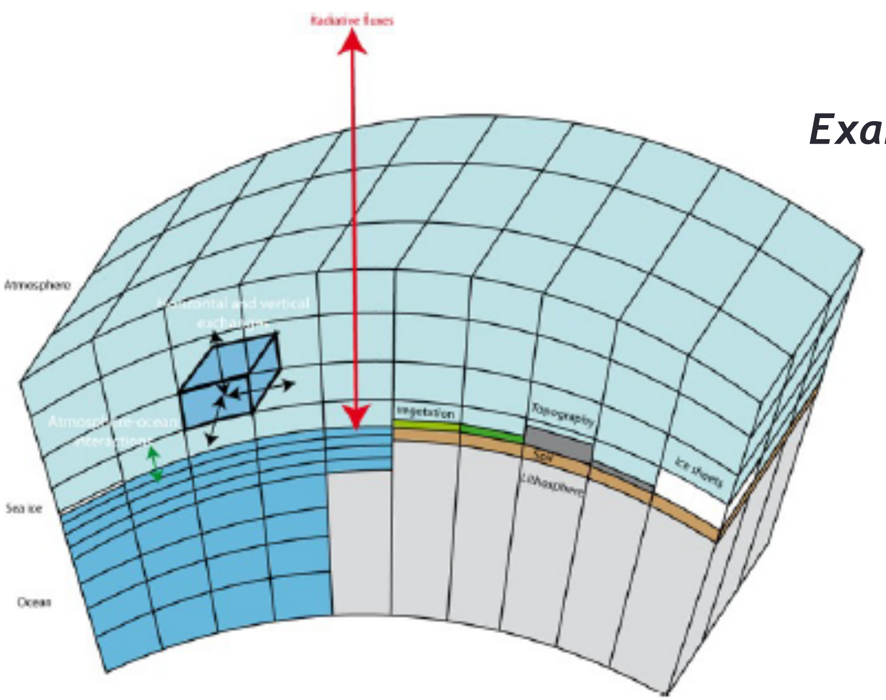
One way to solve them (approximately): **Numerical modelling**

Ocean modeling principle

An ocean model is simplified representation of physical processes that take place in the ocean.

The ocean is divided into boxes : **discretization**

The NS equations can be solved on the grid using **numerical methods**



Example of a finite difference grid

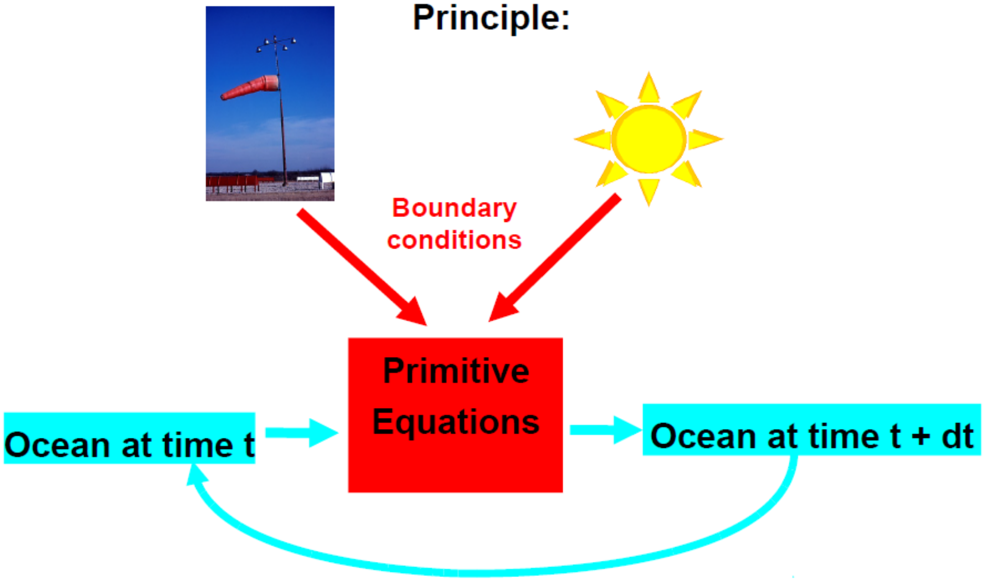
Ocean modeling principle

If we know:

- The **ocean state at time *t* :**

  u, v, w, T, S, …

- **Boundary conditions :**
  surface, bottom, lateral sides



→ We can **compute the ocean state at time *t+dt*** using **numerical approximations of fluid dynamics equations**

Ocean modeling principle

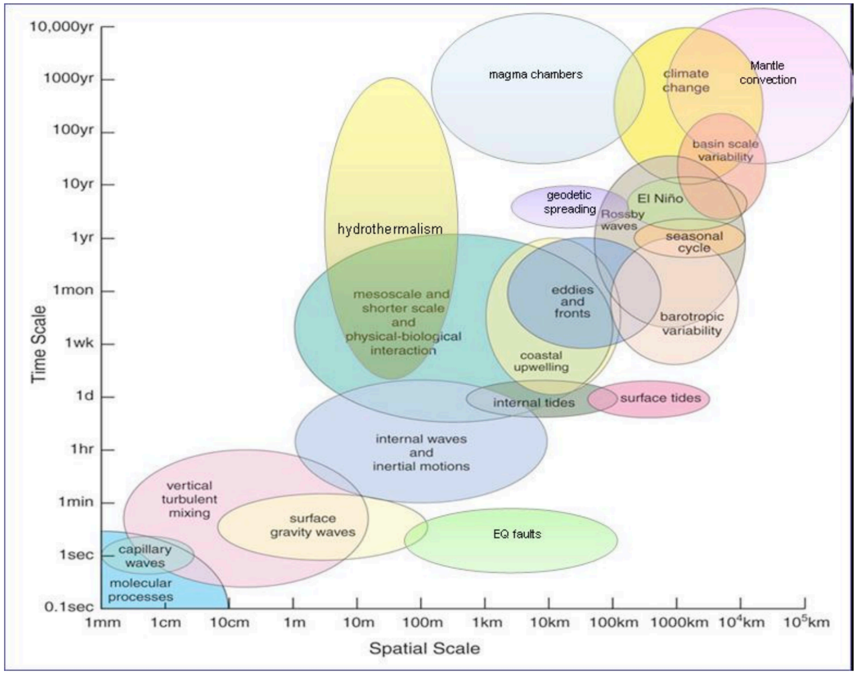
One major Problem:

Setting the model's grid scale to the Kolmogorov length (to resolve diffusive processes) $\Delta = 10^{-3}$m over a global ocean domain of volume 1.3×10^{18} m^3 requires 1.3×10^{27} discrete grid cells.

And Direct Numerical Simulation (DNS) of the global ocean climate requires 3×10^{10} time steps of one second (1000 years).

So we will be dust long before DNS of global ocean climate is possible:

In the meantime, we must use **subgrid scale parameterizations** to simulate the ocean.



Ocean Circulation Models

- ROMS ➔ CROCO https://www.croco-ocean.org/
- NEMO http://www.nemo-ocean.eu/
- MITgcm http://mitgcm.org/
- HYCOM http://hycom.org/
- POP http://www.cesm.ucar.edu/models/cesm1.0/pop2/
- OFES http://www.jamstec.go.jp/esc/ofes/eng/
- MOM http://www.gfdl.noaa.gov/ocean-model
- POM http://www.ccpo.odu.edu/POMWEB/
- Oceananigans https://github.com/CliMA/Oceananigans.jl
- ICON https://www.icon-model.org

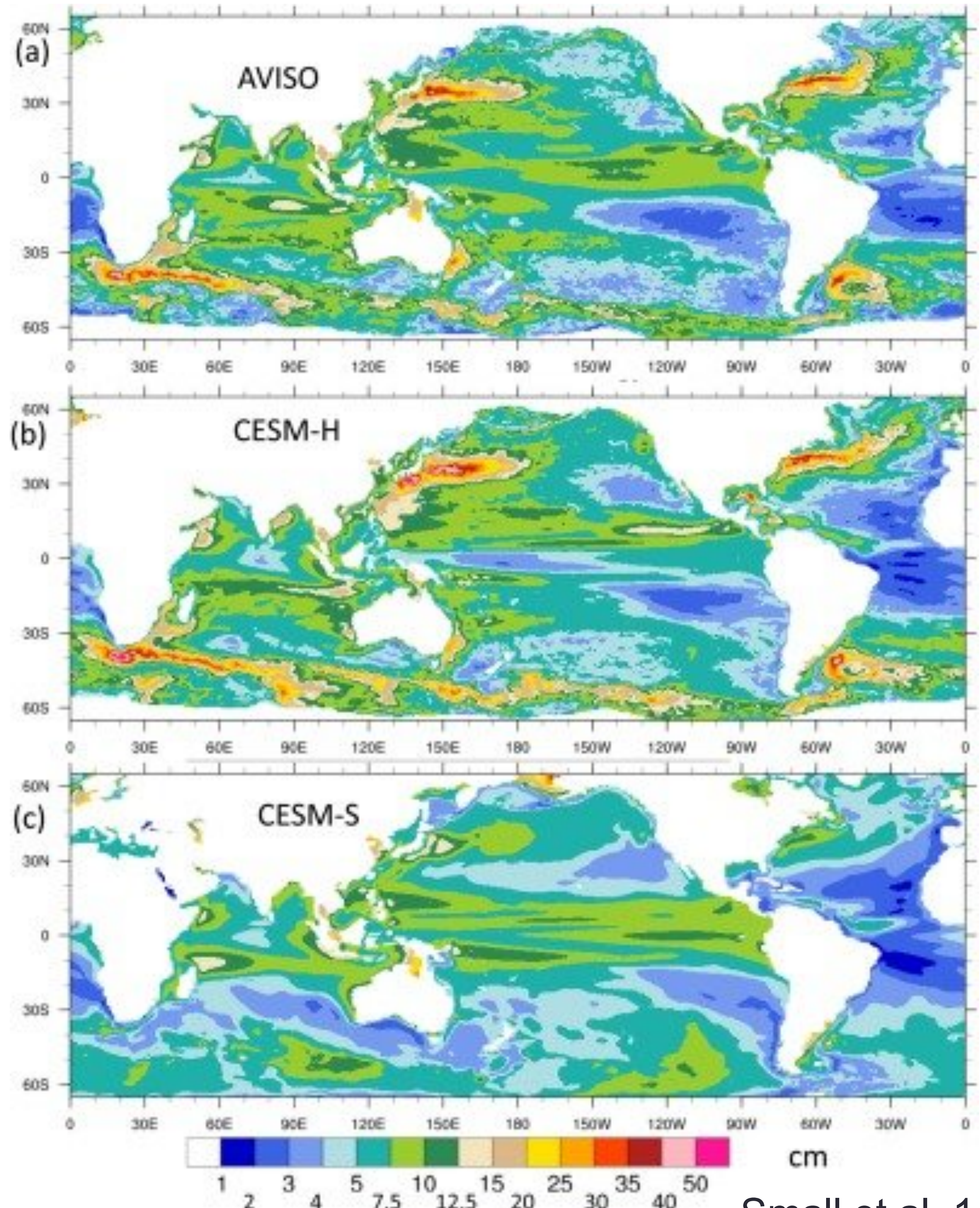
Ocean Circulation Models

- Mechanistic studies of ocean and climate processes:
  - Process studies using fine resolution (≤ 1 km) simulations (**MITgcm, SUNTANS, CROCO-NH, Oceananigans**)
  - Mechanisms for coastal and shelf processes (≤ 10 km) (**ROMS/CROCO, MARS3D, HYCOM**)
  - Mechanisms for climate variability (basin to global) (**MOM6, NEMO**)
- Operational predictions and state estimation
  - Coastal forecasting India **INCOIS**
  - Coastal forecasting USA **NCEP**
  - Ocean state estimation **ECCO**
- Projections for future climate change
  - IPCC-class simulations with anthropogenic forcing (**CMIP**)

Ocean Modelling example:

IPCC global run

Typical length = 100 - 1000 years



0.1 deg

1 deg

Small et al, 14

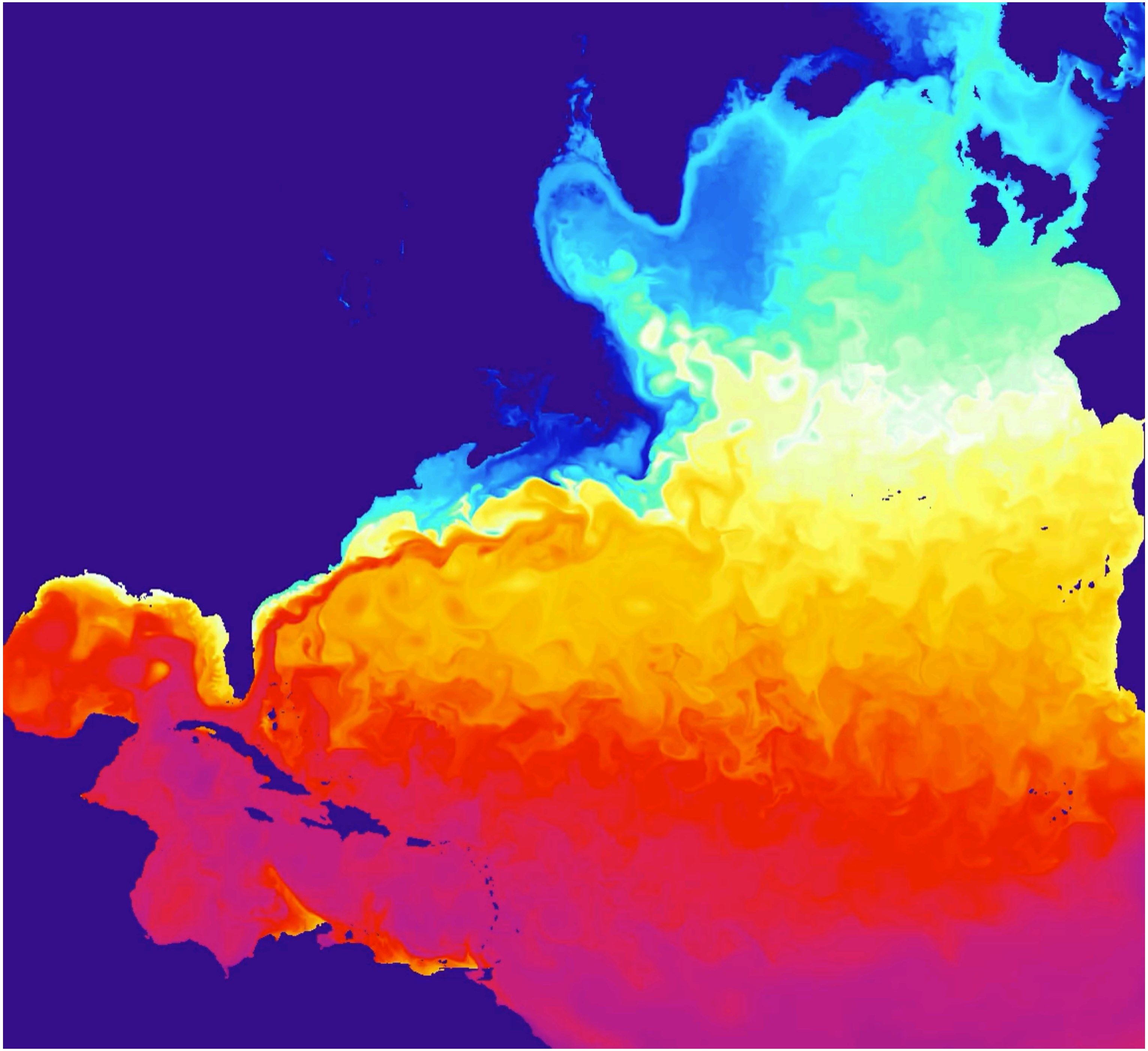
Ocean Modelling example:

Mesoscale-resolving basin-scale simulation

Typical length = 10 - 100 years

North-Atlantic coupled simulations:

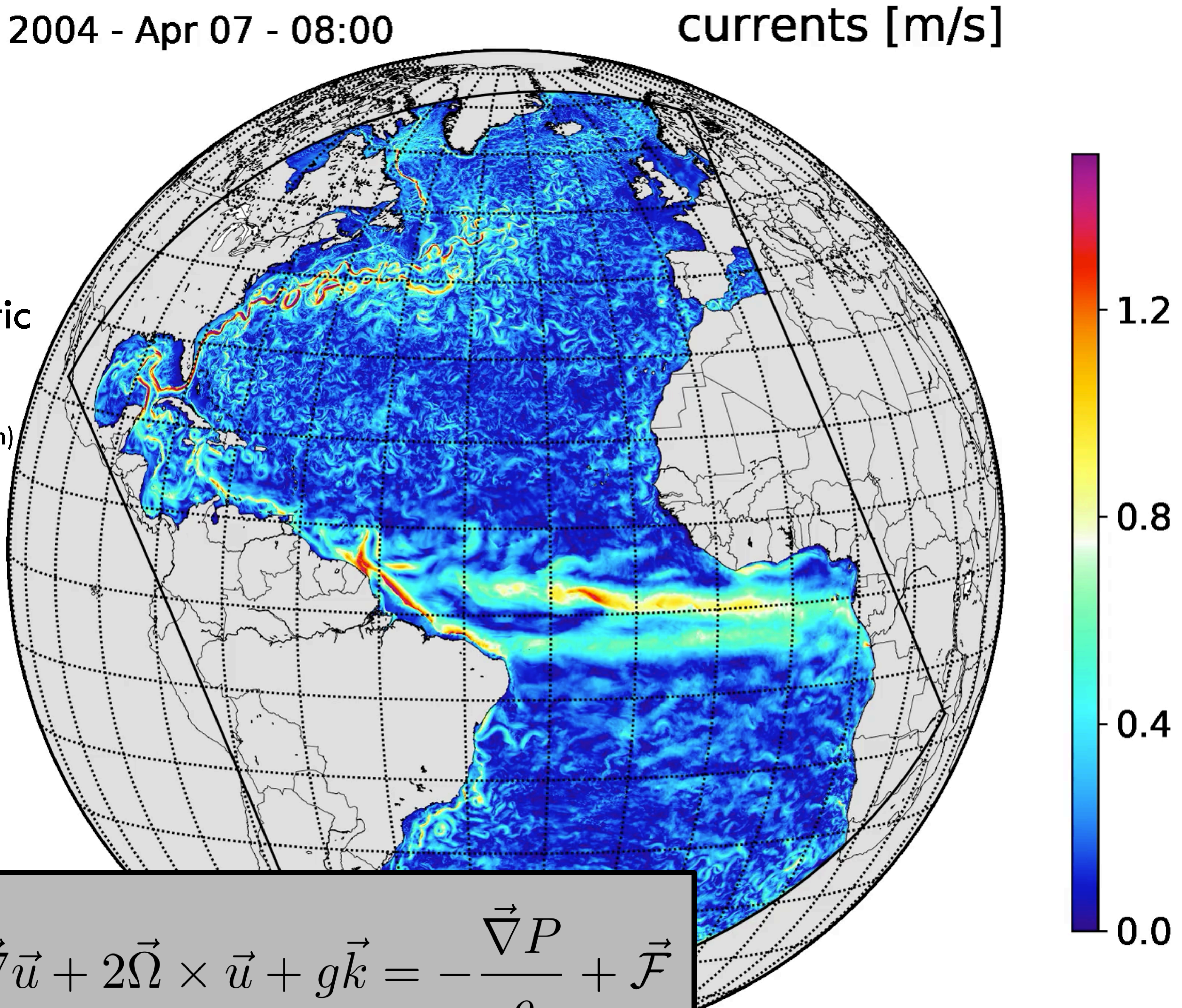
- oceanic model (6 km)
- atmospheric model (18 km)



Realistic Modelling:

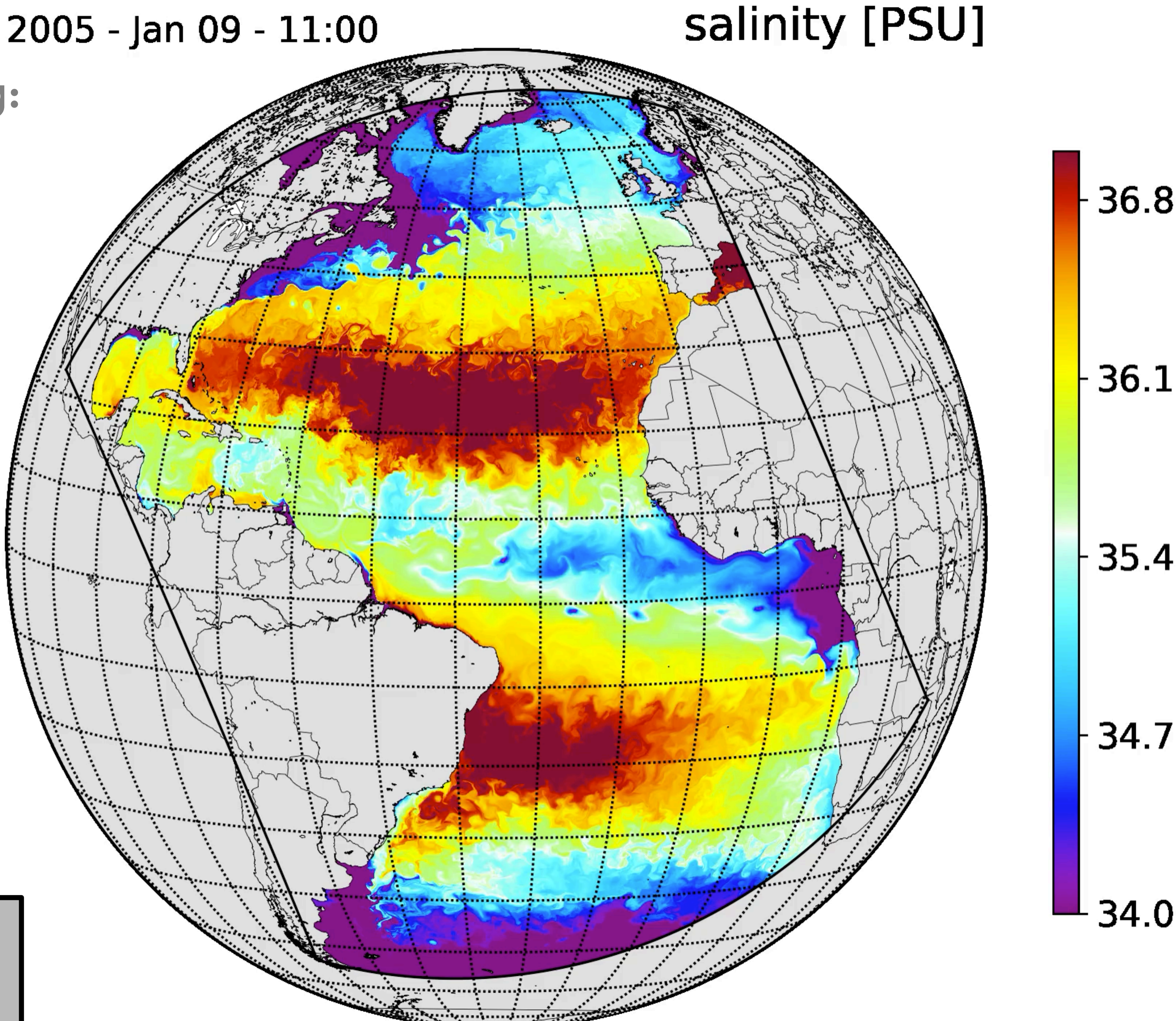
Forced Atlantic simulations:

- oceanic model (3 km)



$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}}$$

Realistic Modelling:

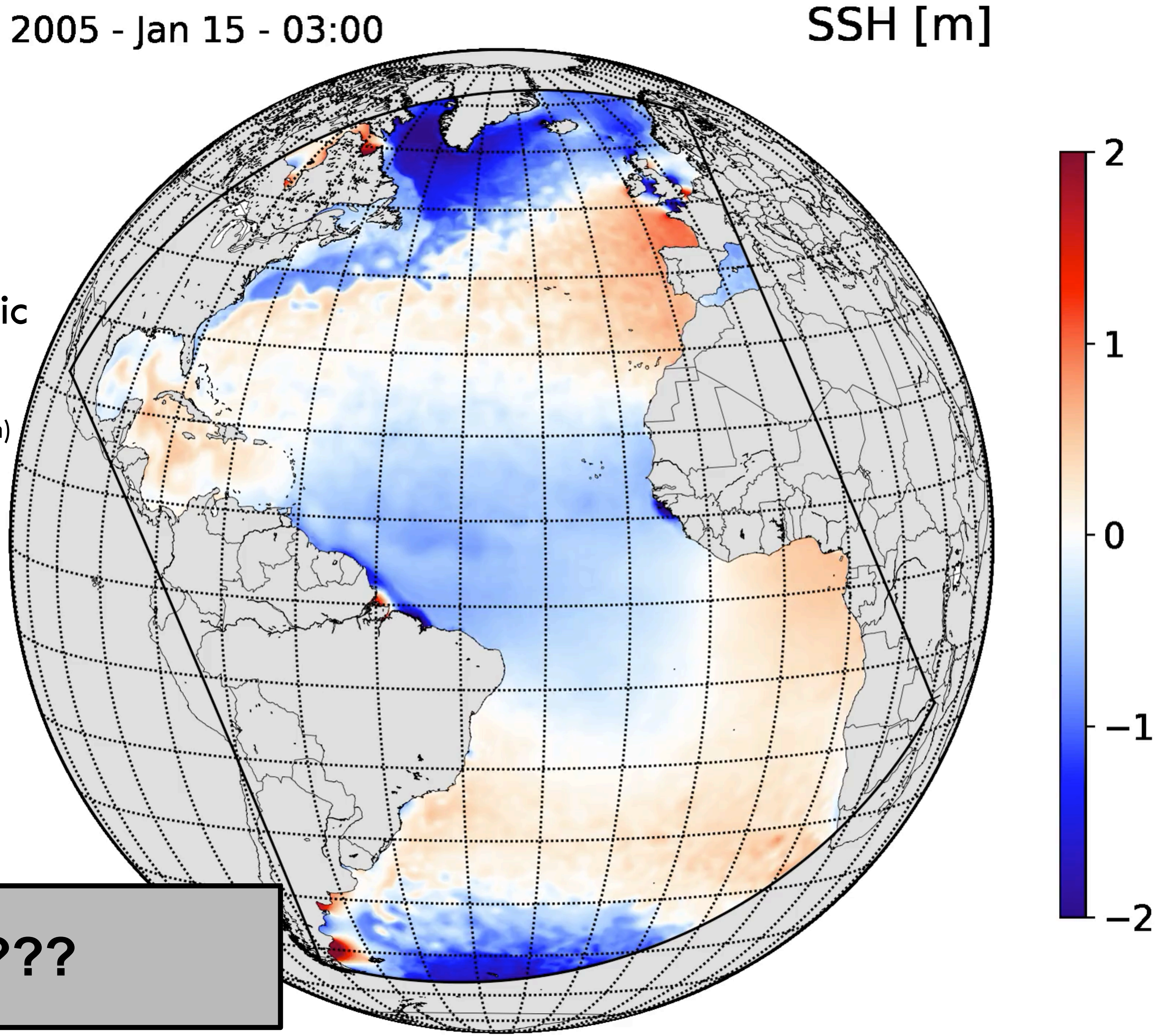


$$\frac{DS}{Dt} = \mathcal{S}_S$$

Realistic Modelling:

Forced Atlantic simulations:

- oceanic model (3 km)



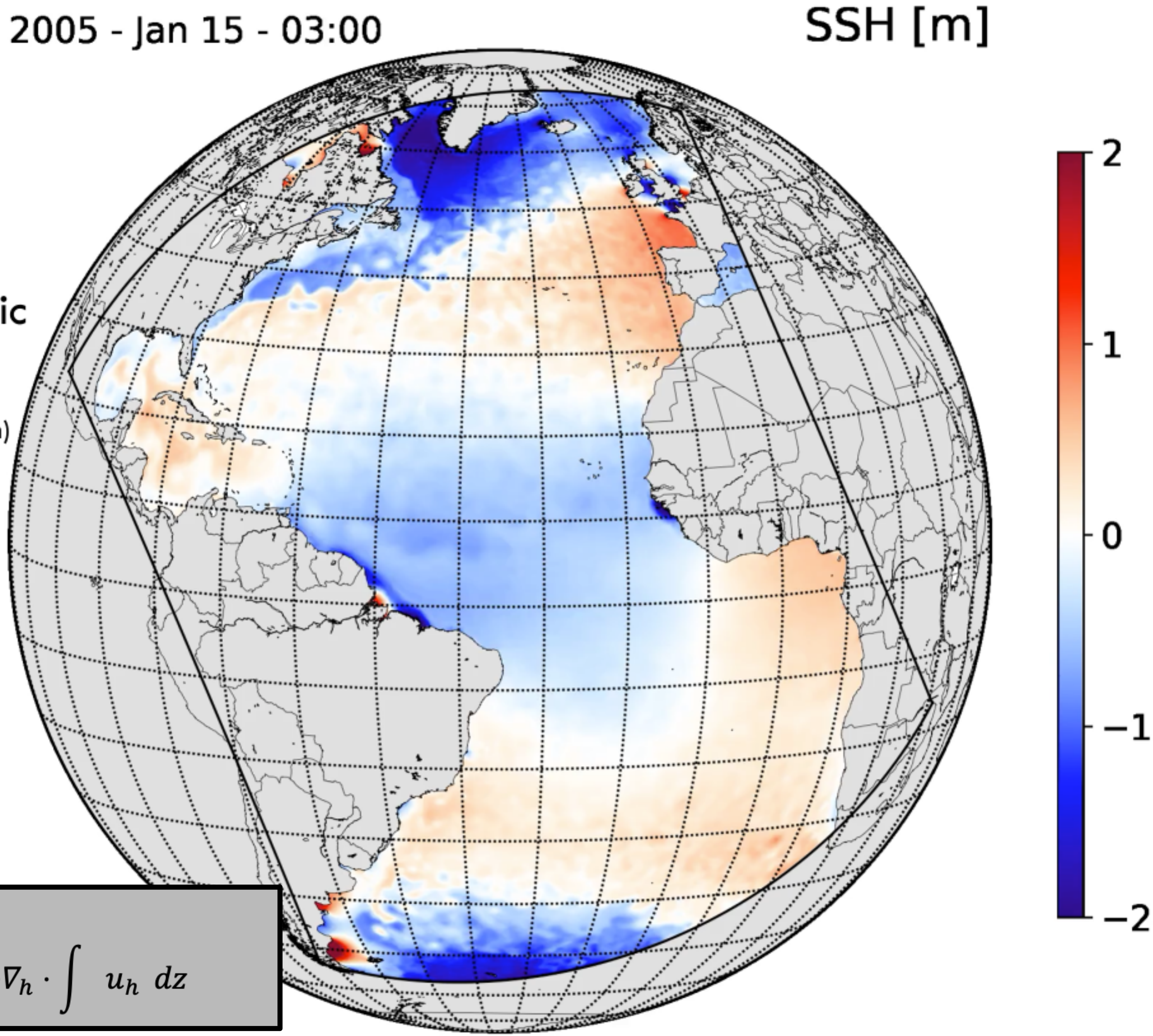
???

Realistic Modelling:

2005 - Jan 15 - 03:00

Forced Atlantic simulations:

- oceanic model (3 km)

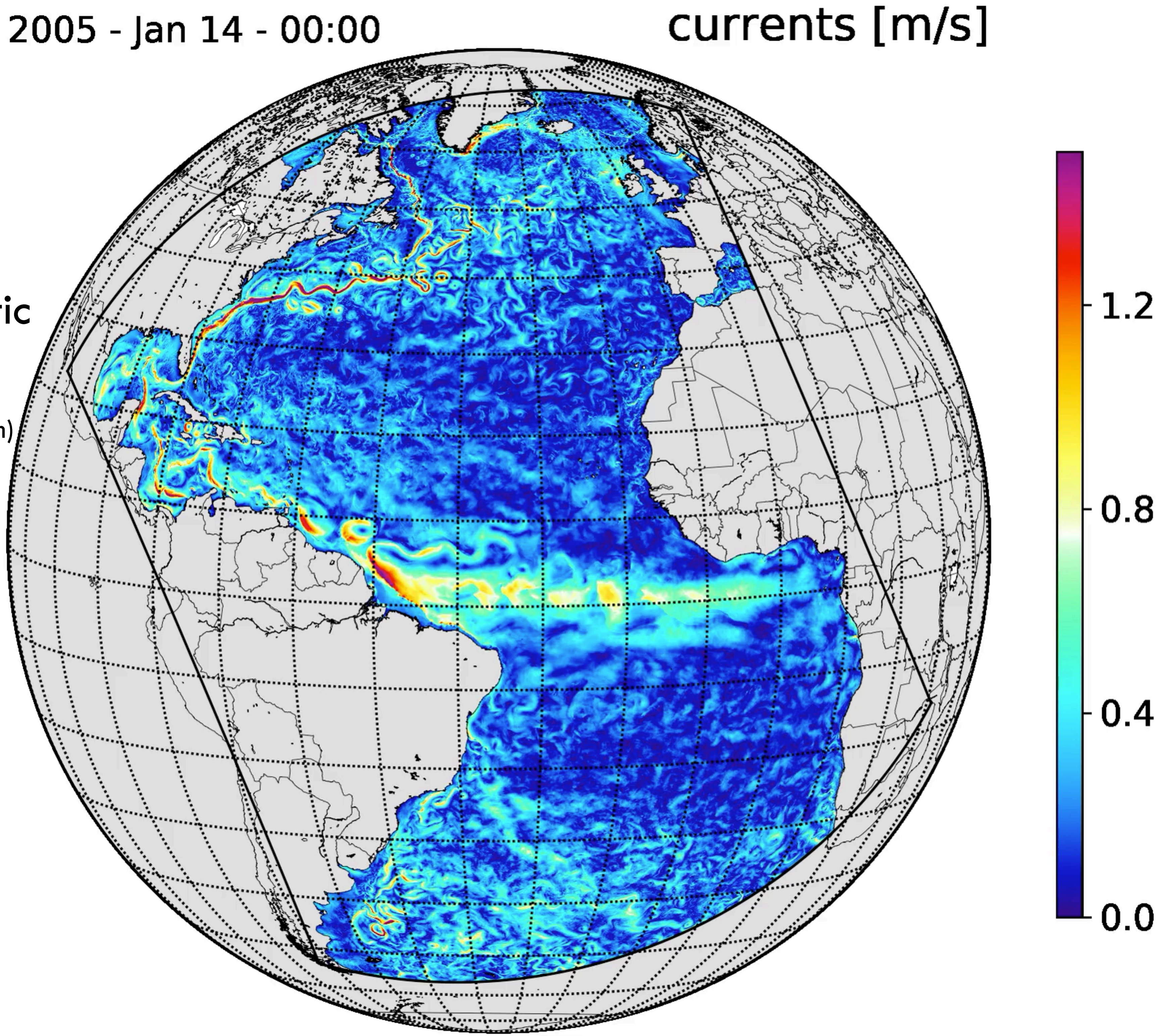


$$\partial_t \eta = -\nabla_h \cdot \int u_h \, dz$$

Realistic Modelling:

Forced Atlantic simulations:

- oceanic model (3 km)



Basin-scale configuration **GIGATL**

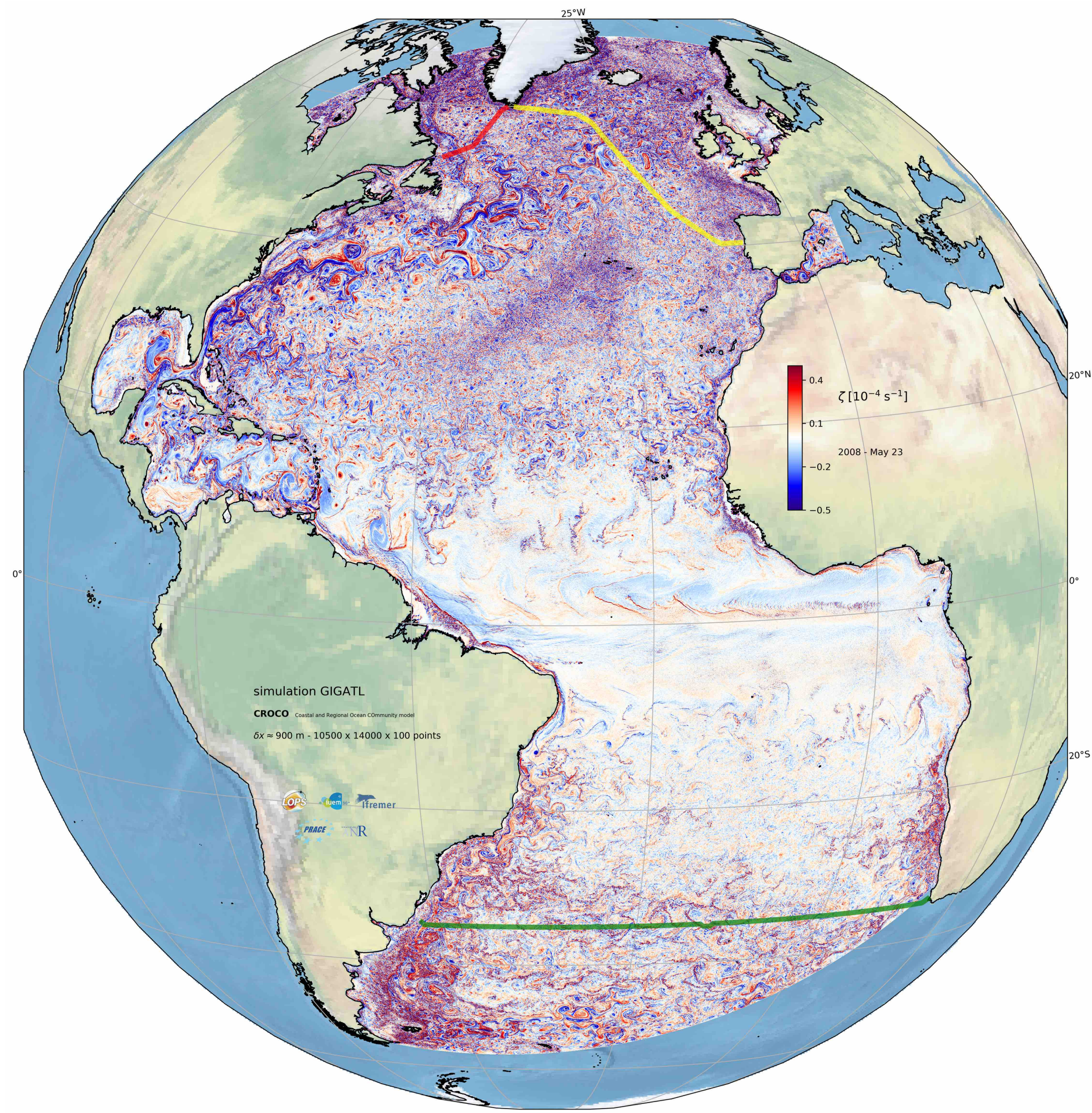
Submesoscale-permitting basin-scale simulation

Typical length = 1 year

Horizontal resolutions up to **0.75 - 1 km** with 100 topography following levels (refined at the bottom)
= 10500 x 14000 x 100 points

including **hourly surface forcings and tides**.

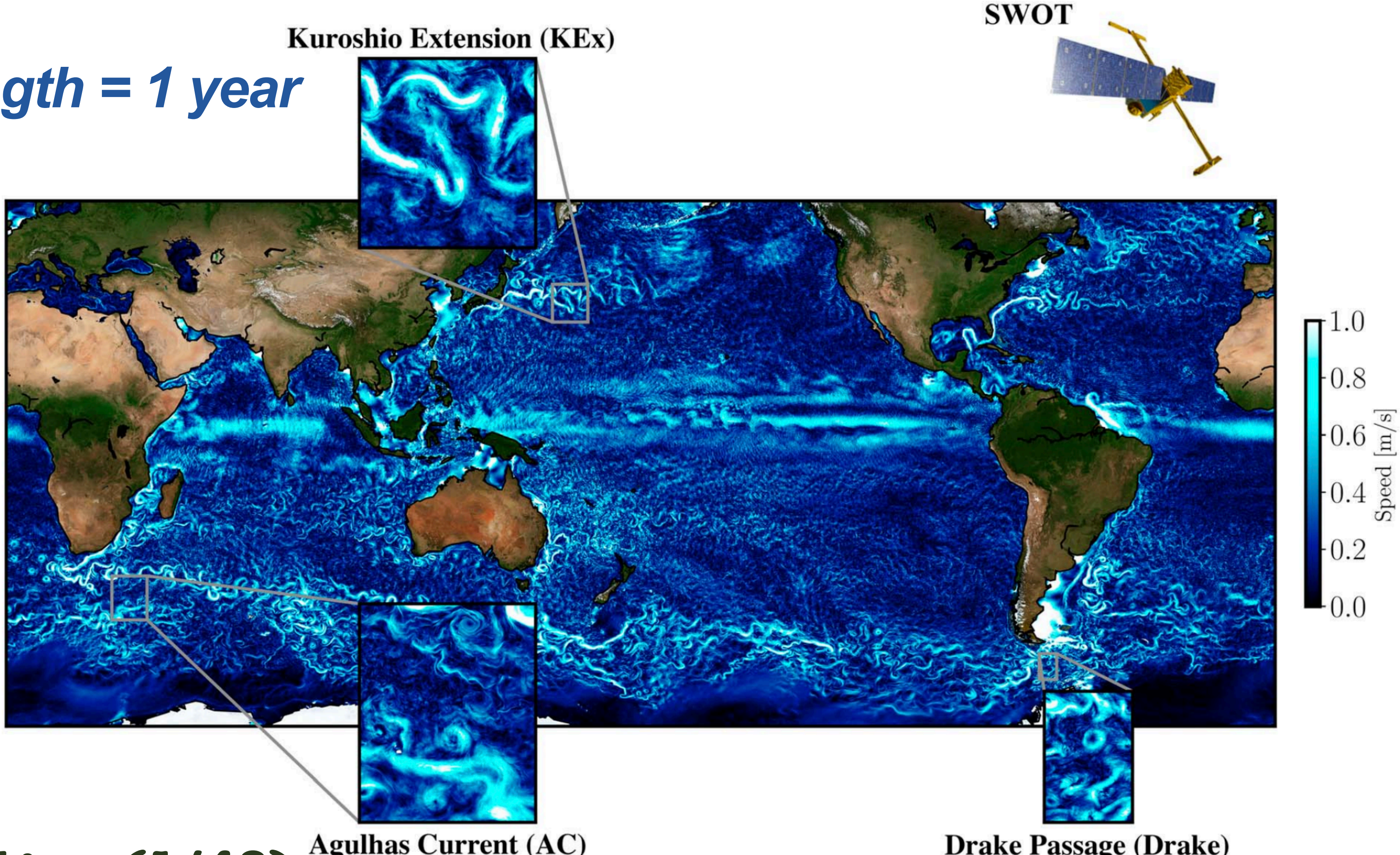
Run on 10000 processors - 40M cpu-hours - About 4PB of data



Global configuration LLC4320

Submesoscale-permitting global simulation

Typical length = 1 year

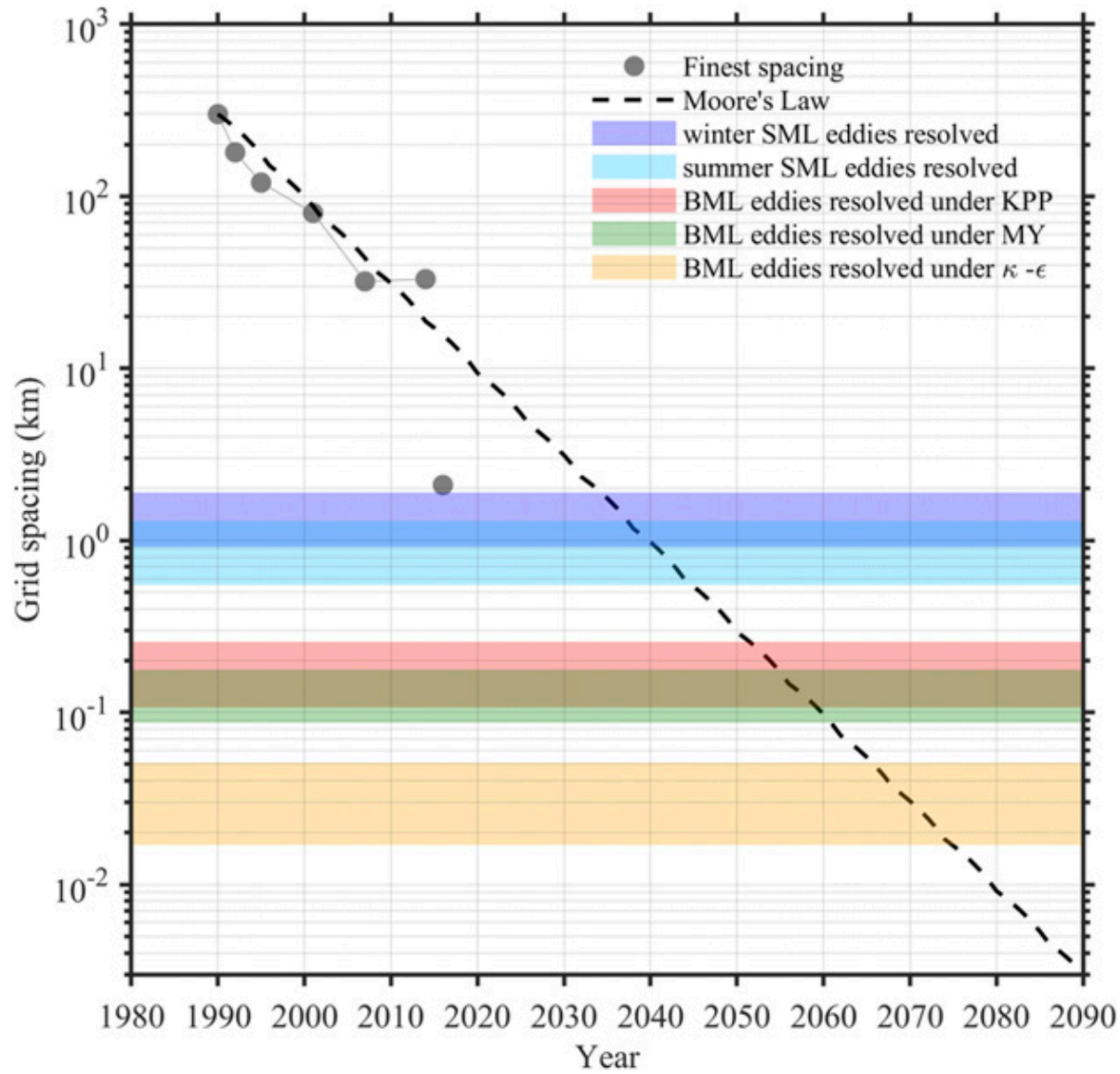


Model MITgcm (1/48)

[Torres et al, 19]

Estimate of horizontal grid spacings of the IPCC ocean models

- The gray dots denote the finest grid spacings reported by the IPCC reports by year of publication, except the latest one from the ECCO MITgcm LLC4320 simulation.
- The black line denotes the estimate predicted by Moore's Law, while the shaded regions denote the grid spacing intervals resolving 50% and 90% of surface mixed-layer eddies globally based on the observations and bottom mixed-layer eddies based on simulations.

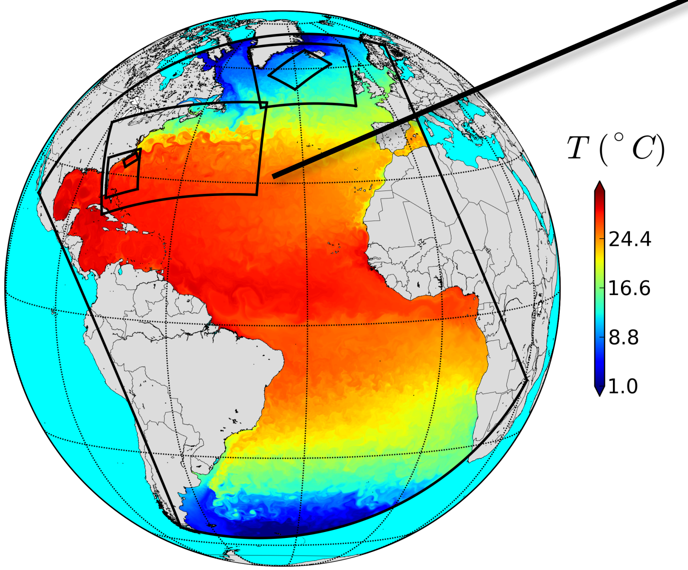


Dong et al. (2020)

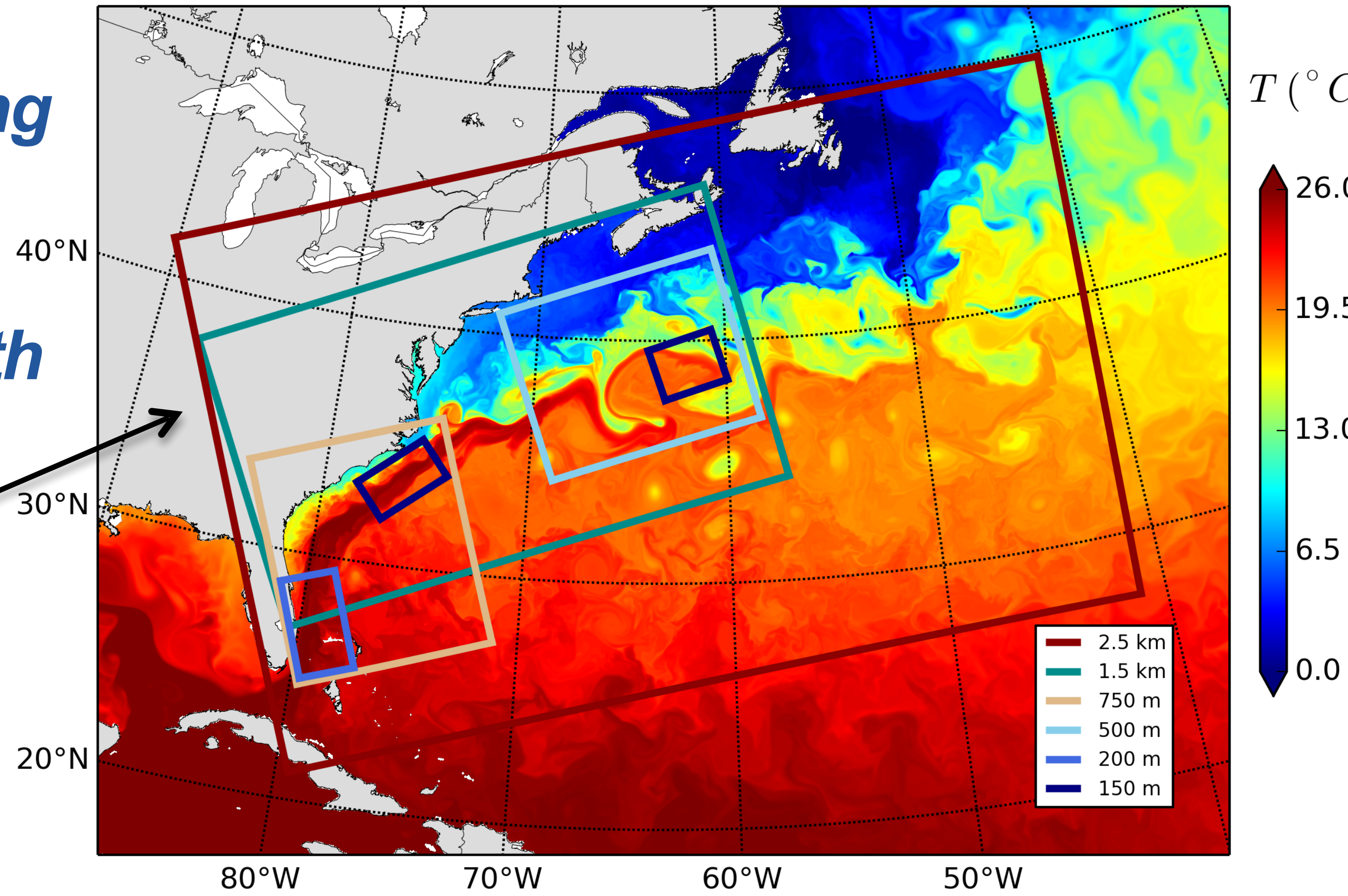
Ocean Modelling examples:

Submesoscale-resolving regional simulation

Typical length = 1 month
10 years

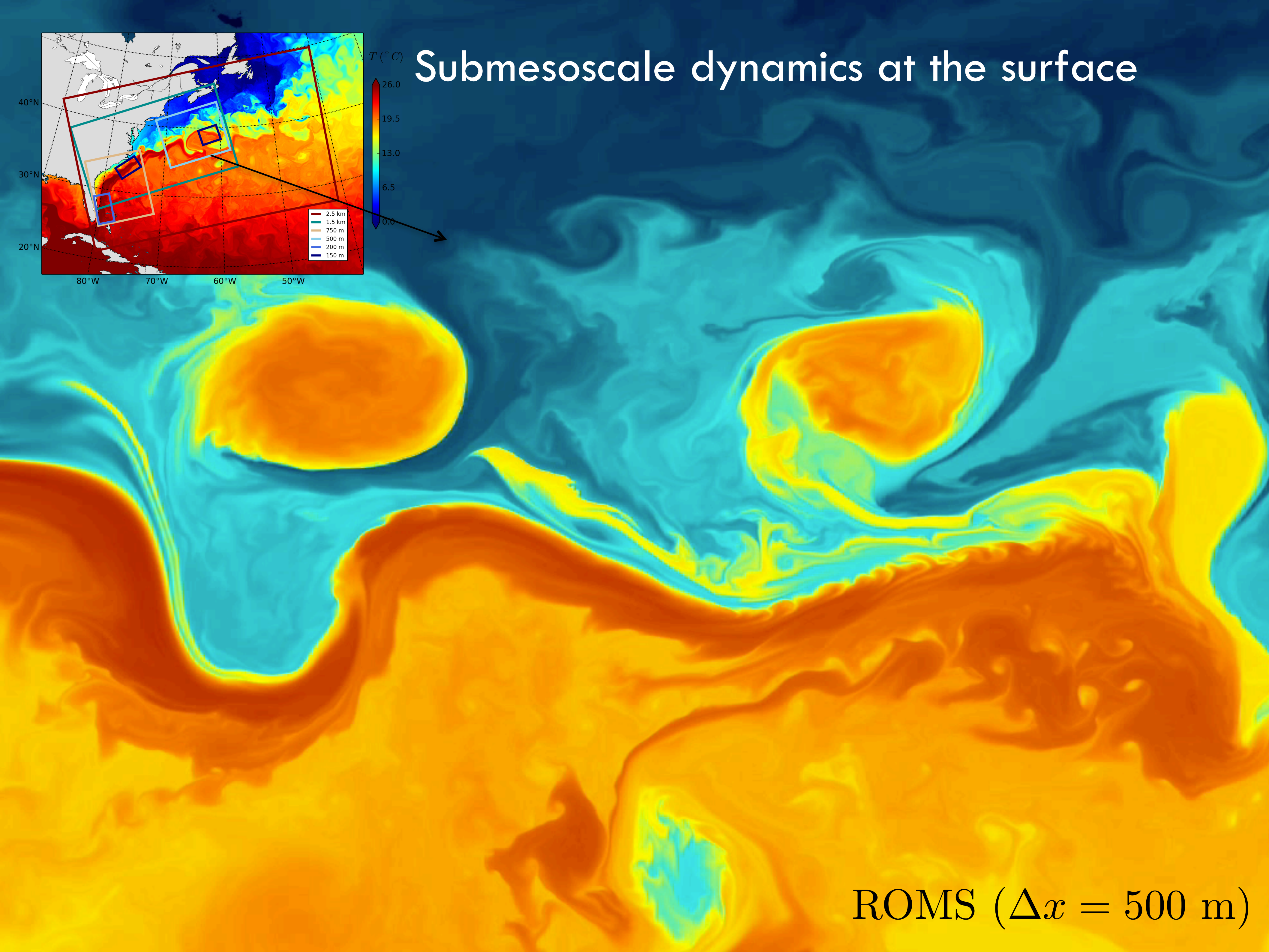


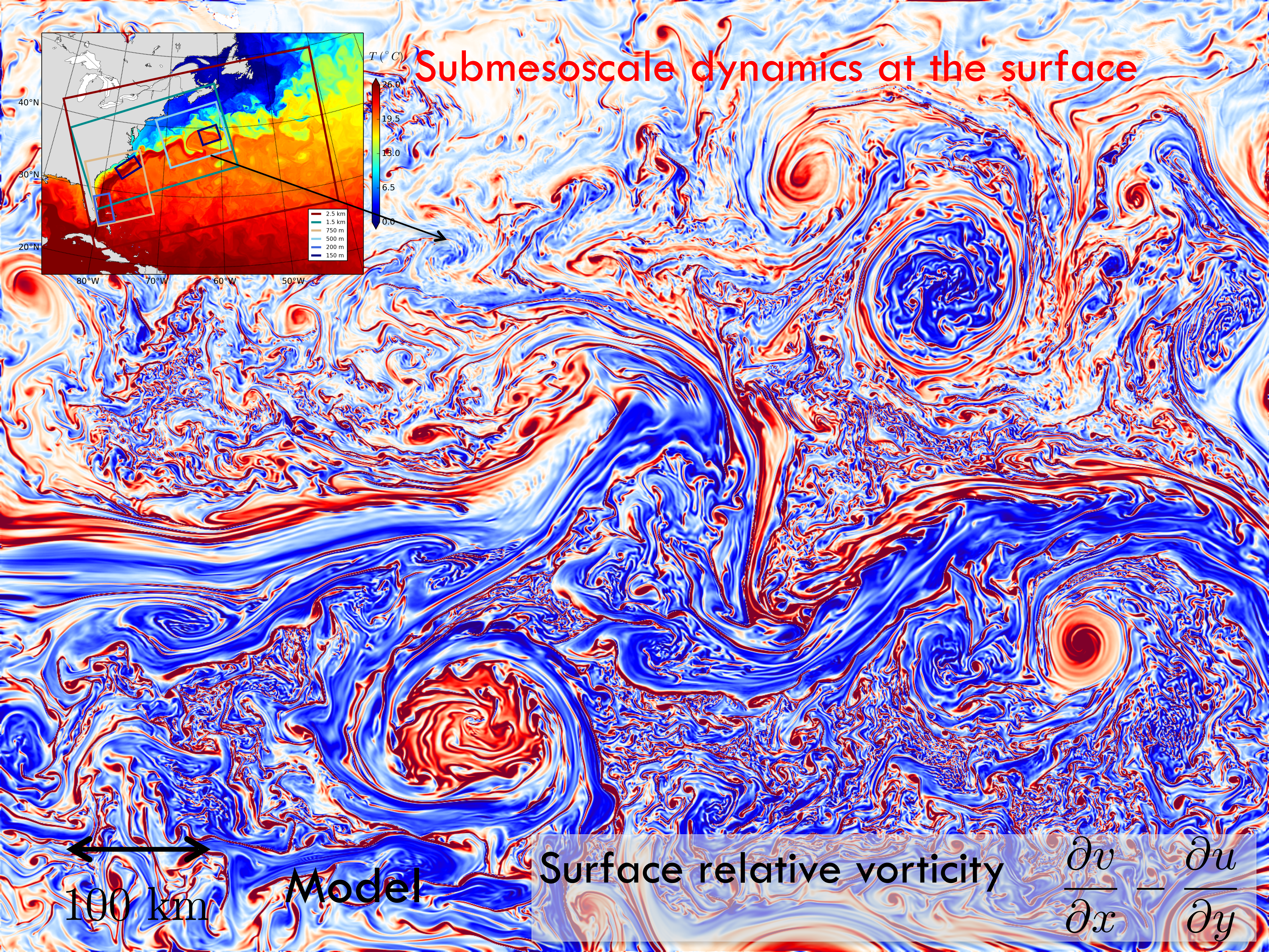
$$\Delta x = 6 \rightarrow 0.15\ \mathrm{km}$$



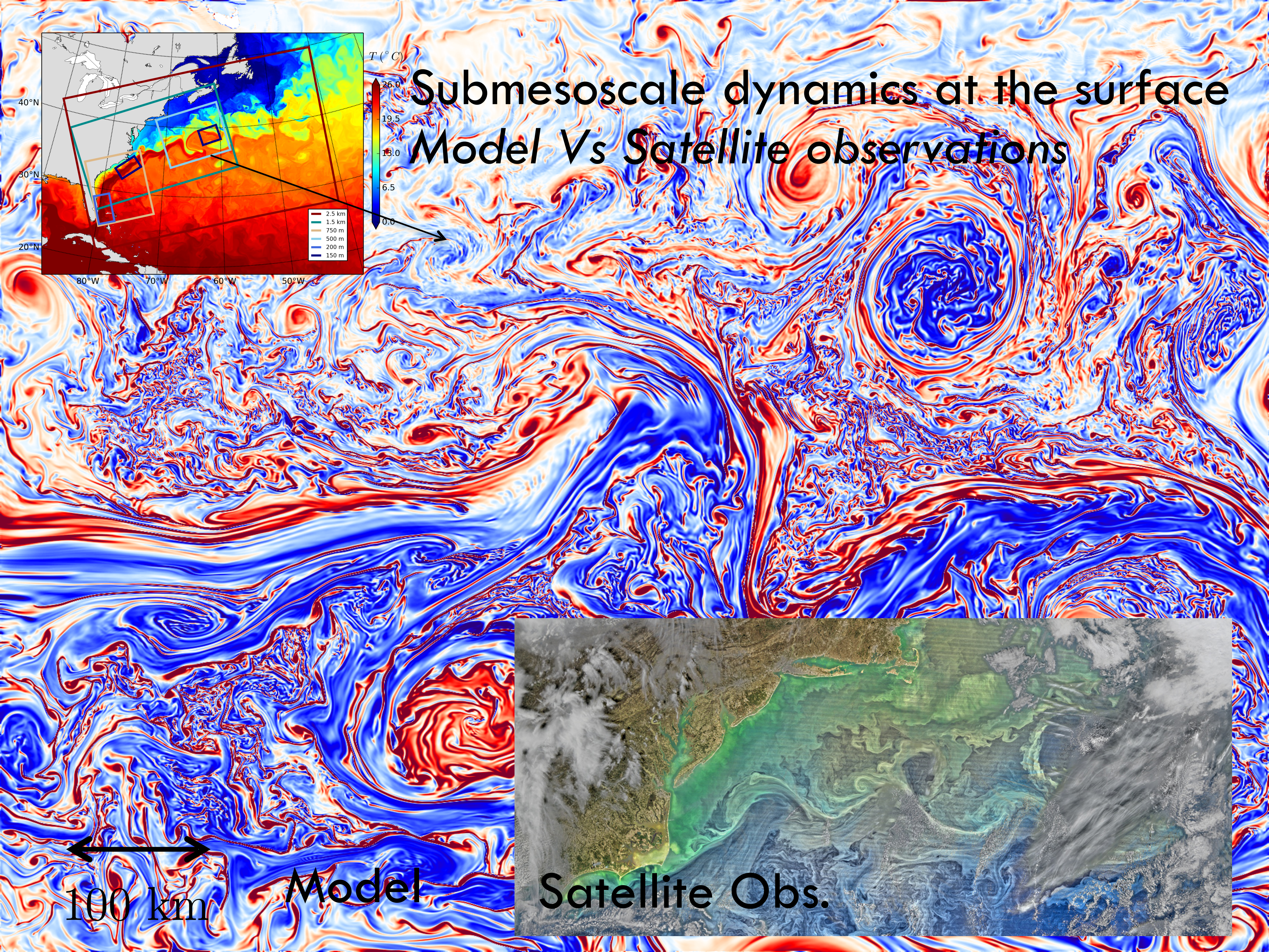
A portion of the Atlantic domain showing mean SST and several (1-way) nested grids:

Submesoscale dynamics at the surface



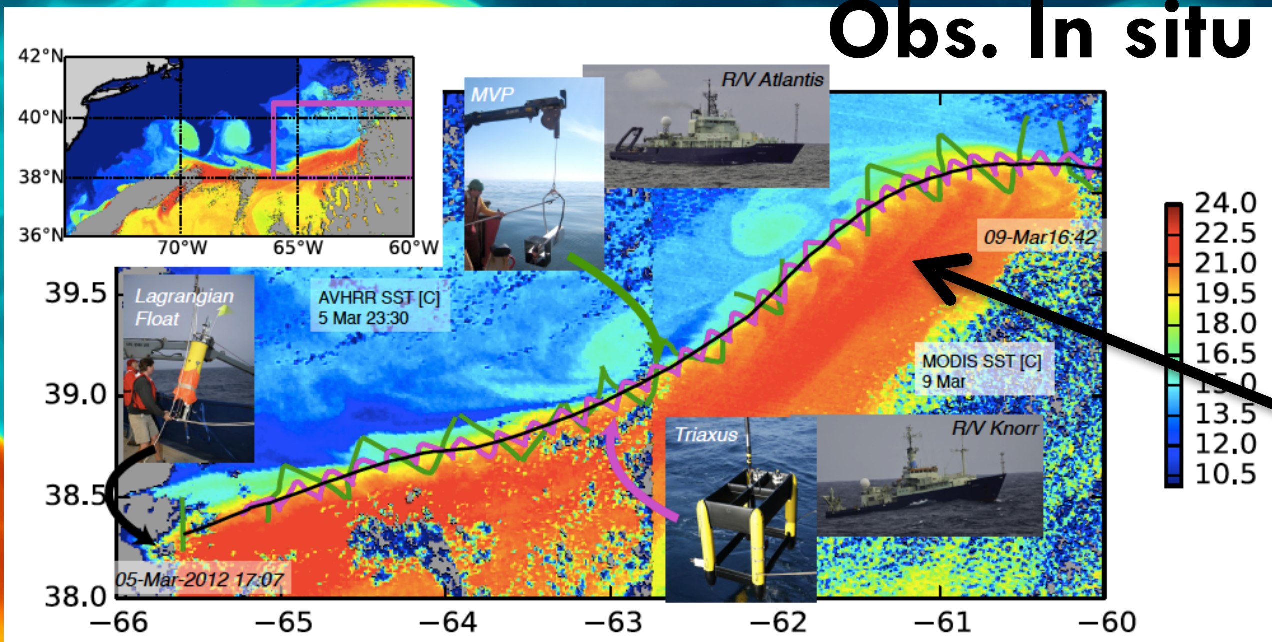


Submesoscale dynamics at the surface
T(°C)
26.0
19.5
13.0
6.5
0.0
2.5 km
1.5 km
750 m
500 m
200 m
150 m
40°N
30°N
20°N
80°W
70°W
60°W
50°W
100 km
Model
Surface relative vorticity $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$


Submesoscale dynamics at the surface
Model Vs Satellite observations
$T(^\circ C)$
26.0
19.5
13.0
6.5
0.0
40°N
30°N
20°N
80°W
70°W
60°W
50°W
2.5 km
1.5 km
750 m
500 m
200 m
150 m
100 km
Model
Satellite Obs.

Submesoscale dynamics at the surface
Model Vs Satellite observations

Obs. In situ



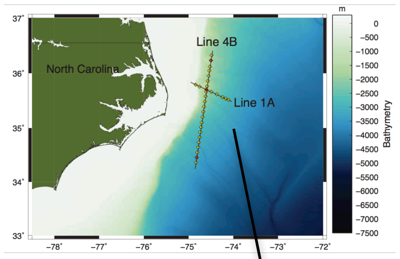
LATMIX 2012 Campaign

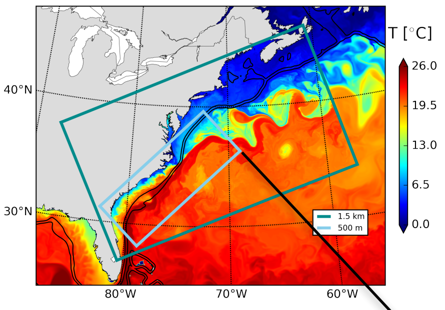
(Scalable Lateral Mixing and Coherent Turbulence)

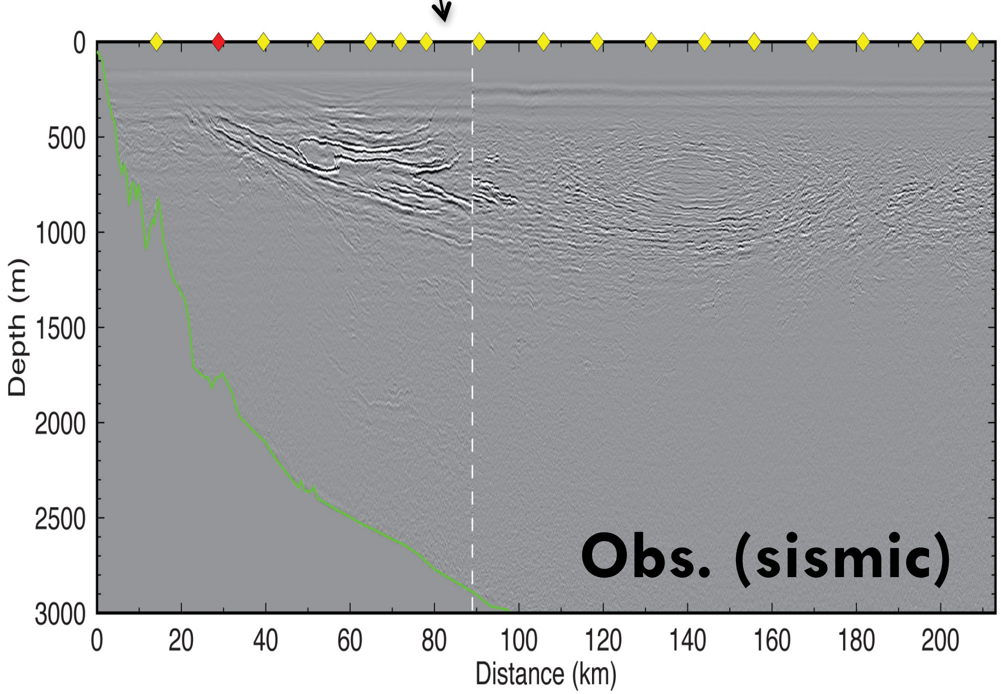
Model

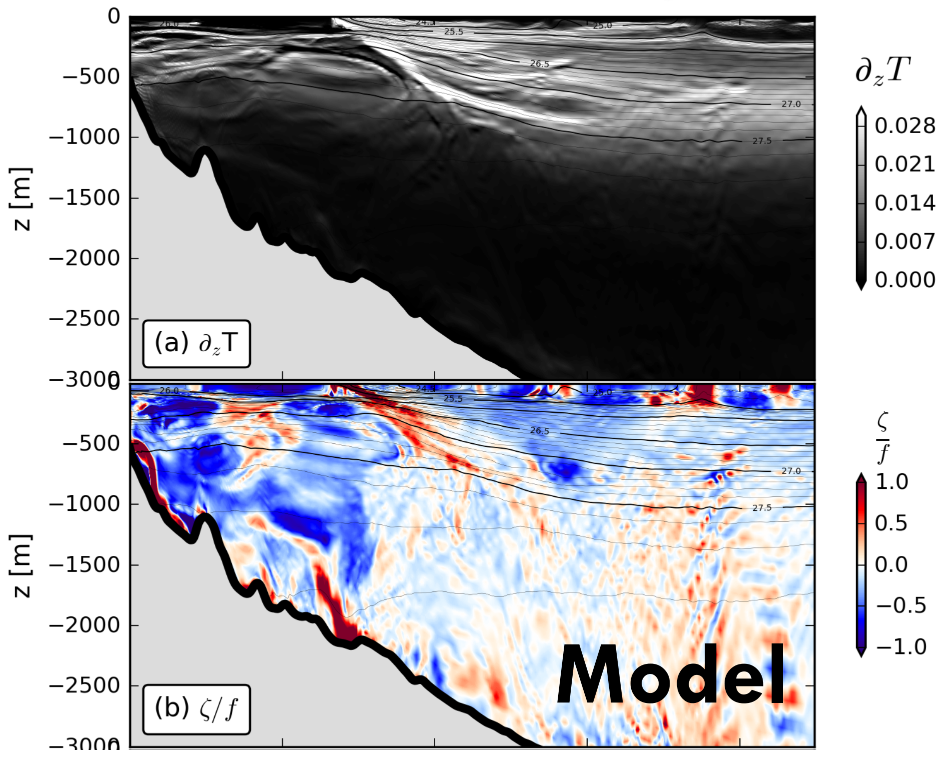
Submesoscale dynamics in the interior

- Generation of submesoscale coherent vortices:

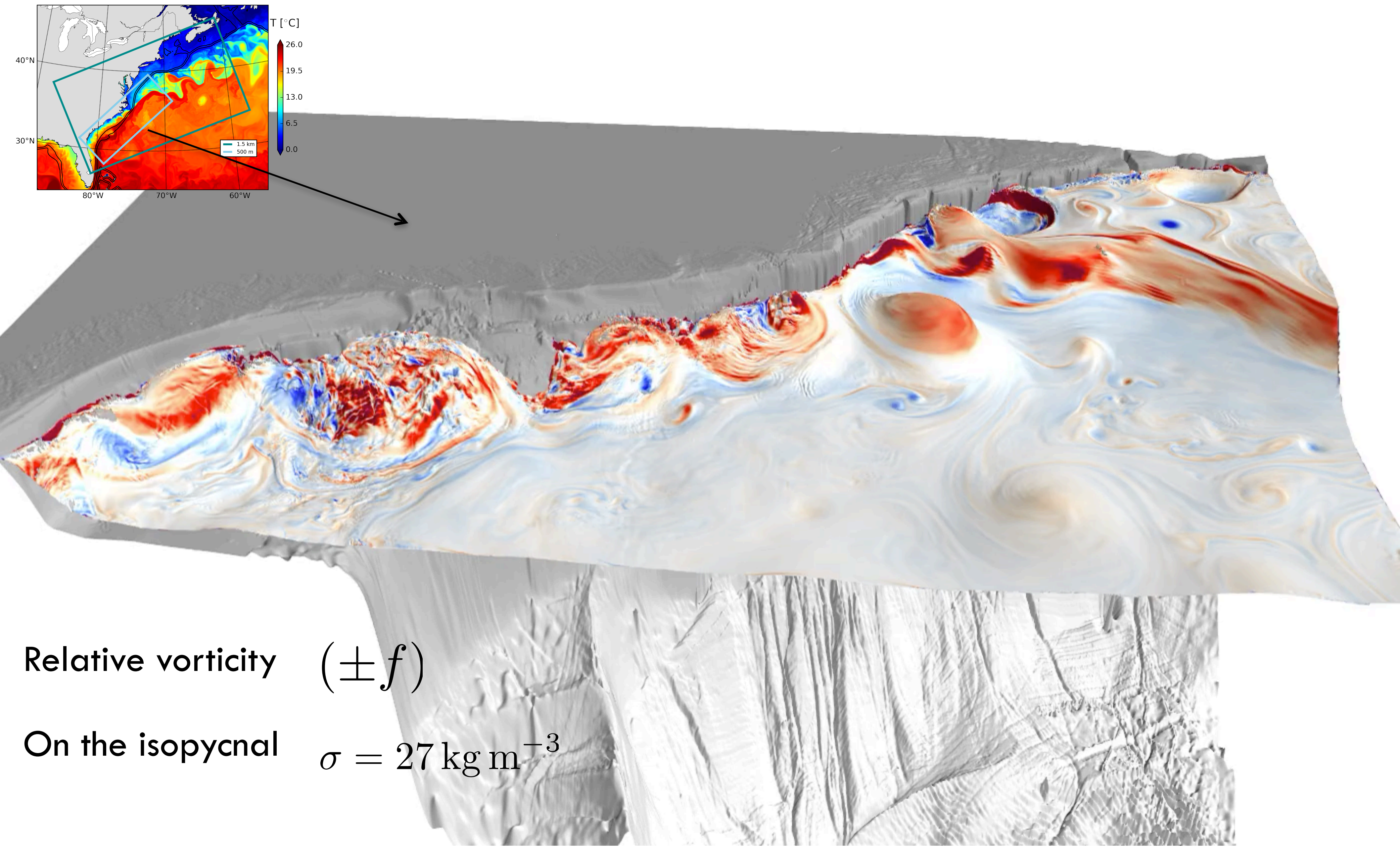








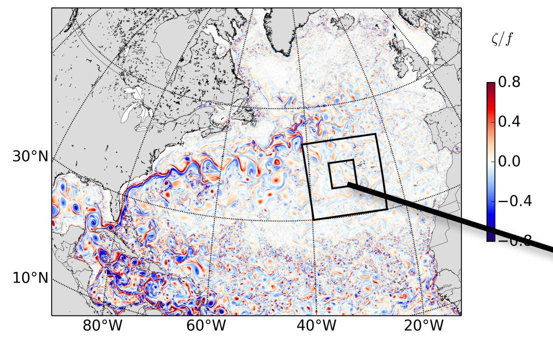
Submesoscale dynamics in the interior



Relative vorticity $(\pm f)$

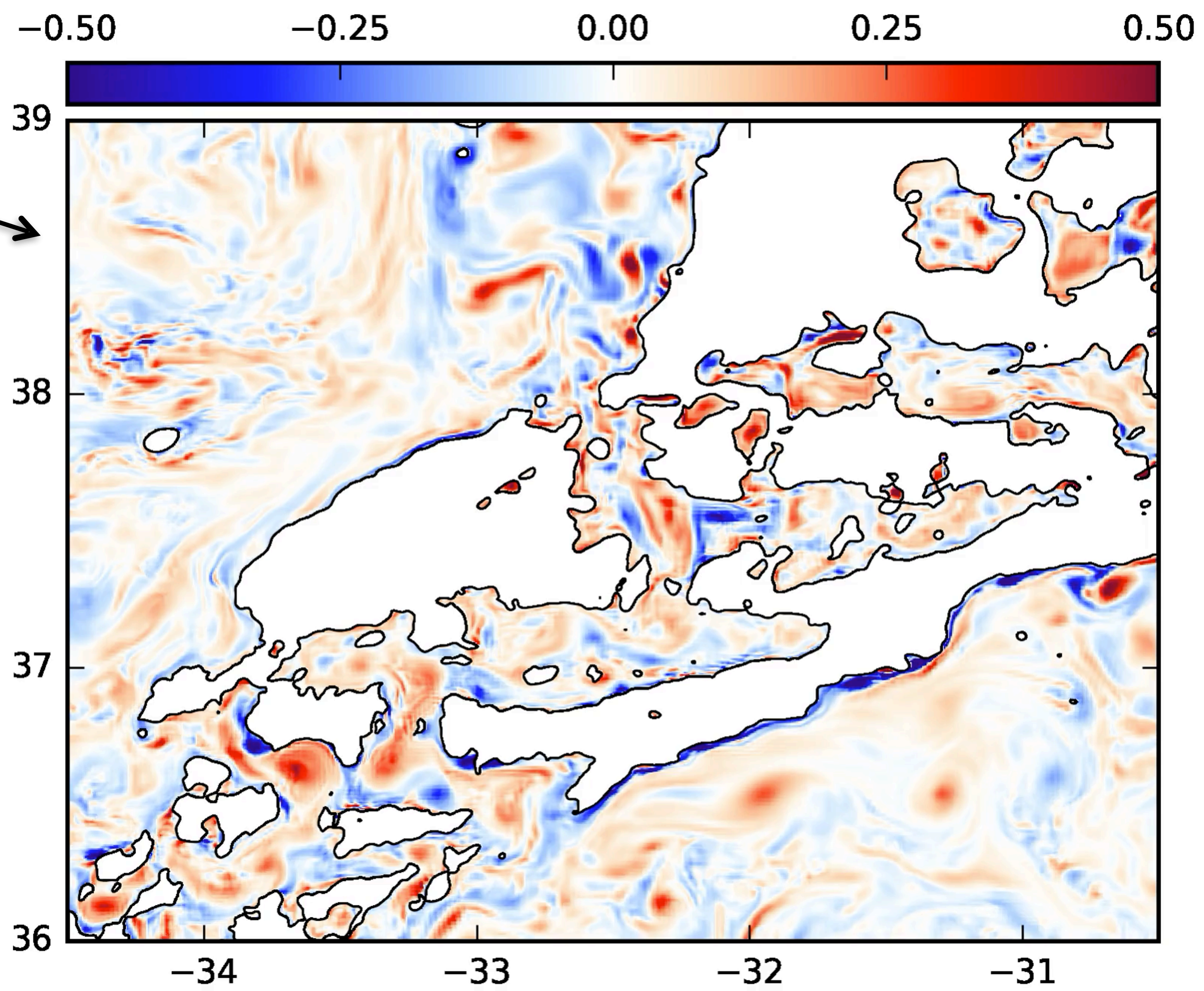
On the isopycnal $\sigma = 27\,\mathrm{kg\,m^{-3}}$

Submesoscale dynamics in the Abyss!



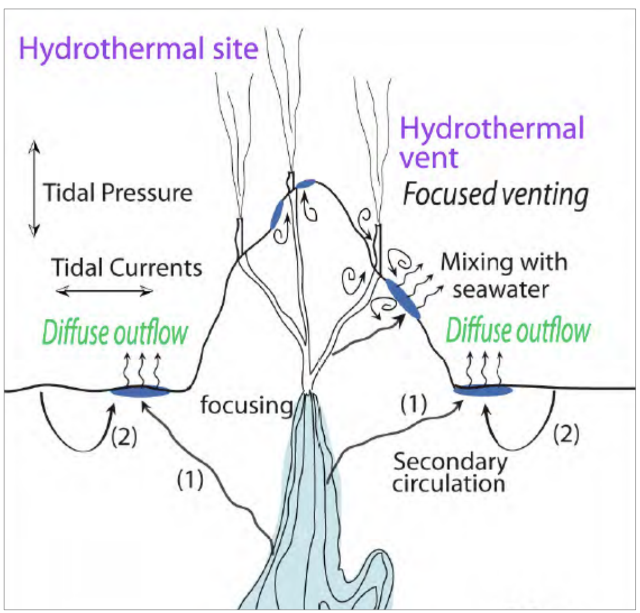
$$\Delta x = 500\,\mathrm{m}$$

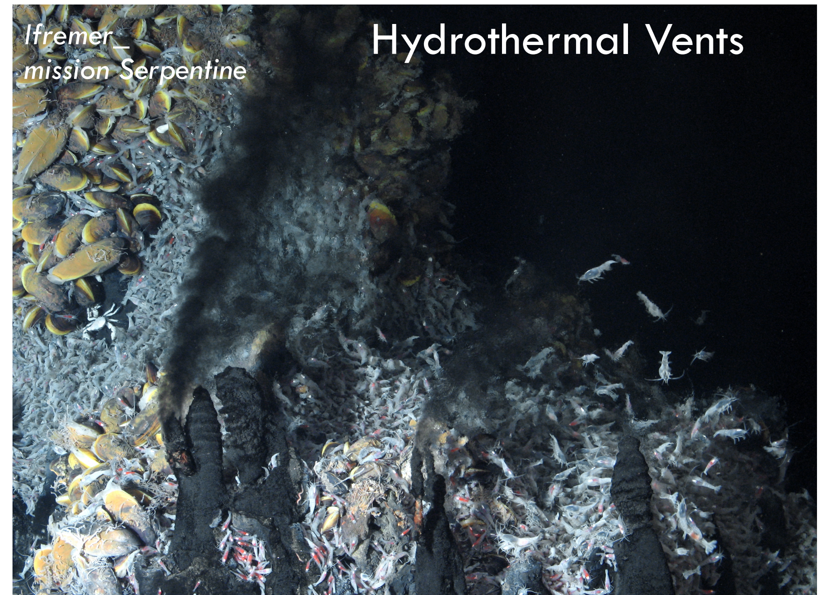
relative vorticity at -1800 m

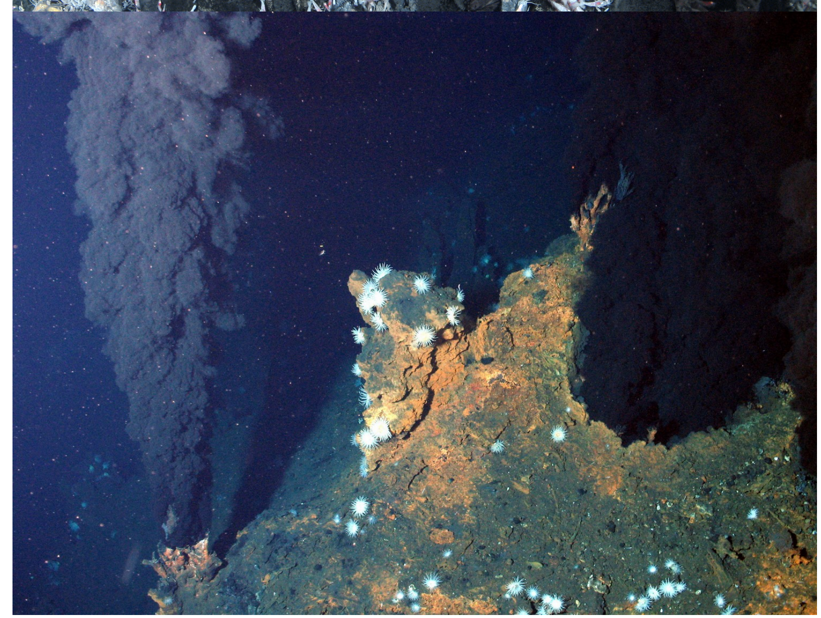


And the abyssal turbulence impacts a lot of things:

- *Dispersion of hydrothermal effluents*
- *Transport of biogeochemical tracers*
- *Connectivity between deep ecosystems*

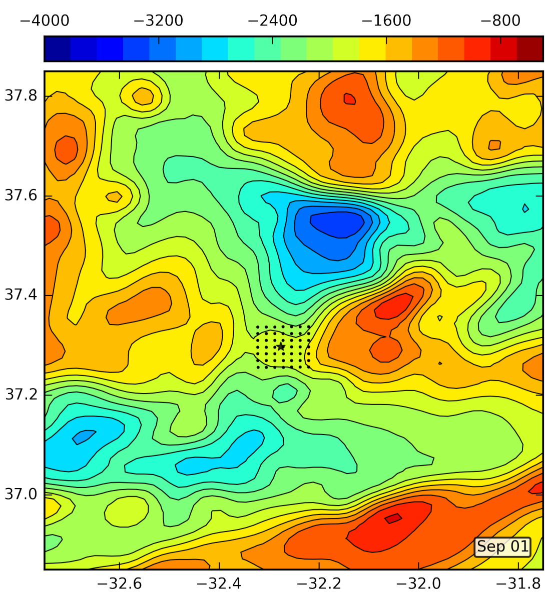


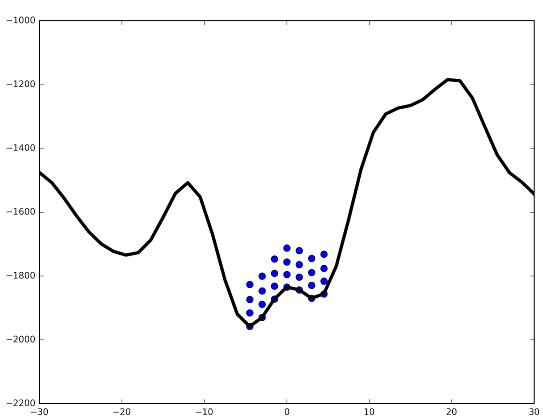


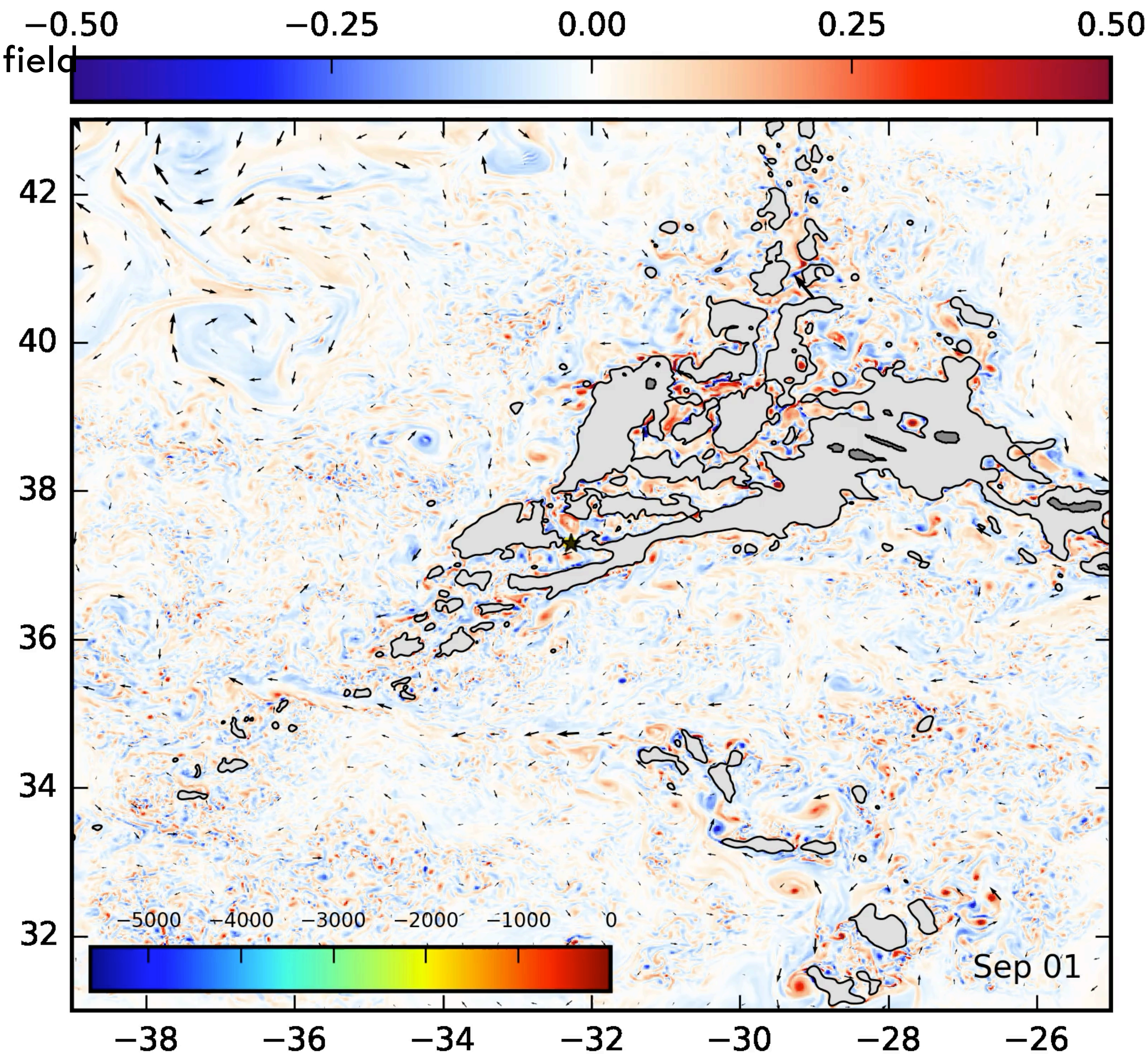


Dispersion of larvae by abyssal turbulence

Continuous release of Lagrangian particles near the Lucky Strike vent field (1.5 km res)







Dispersion of larvae by abyssal turbulence

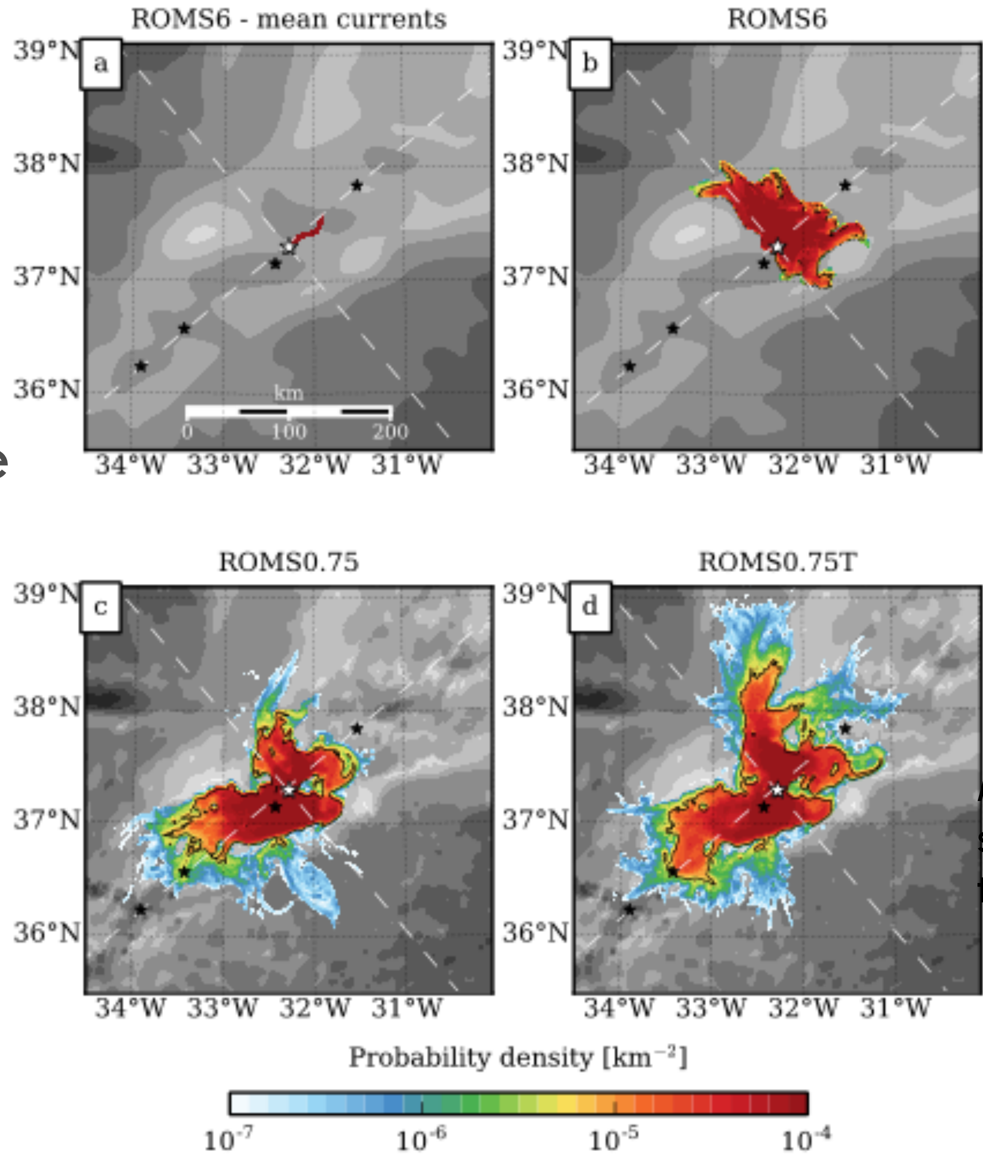
Larvae dispersion from the Lucky Strike vent after 30 days.

Mean currents only

Mesoscale

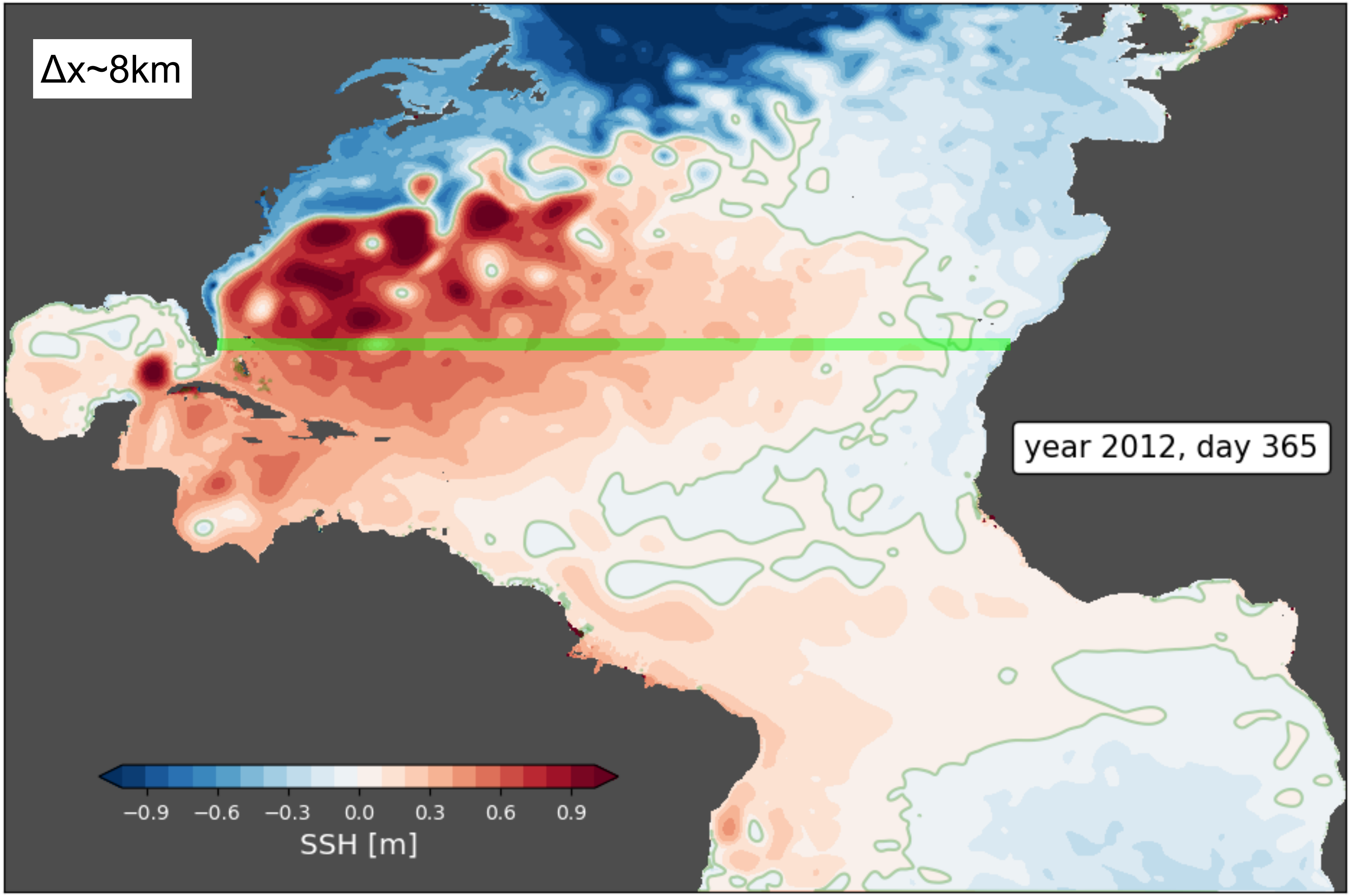
Mesoscale + submesoscale

Mesoscale + submesoscale + tides



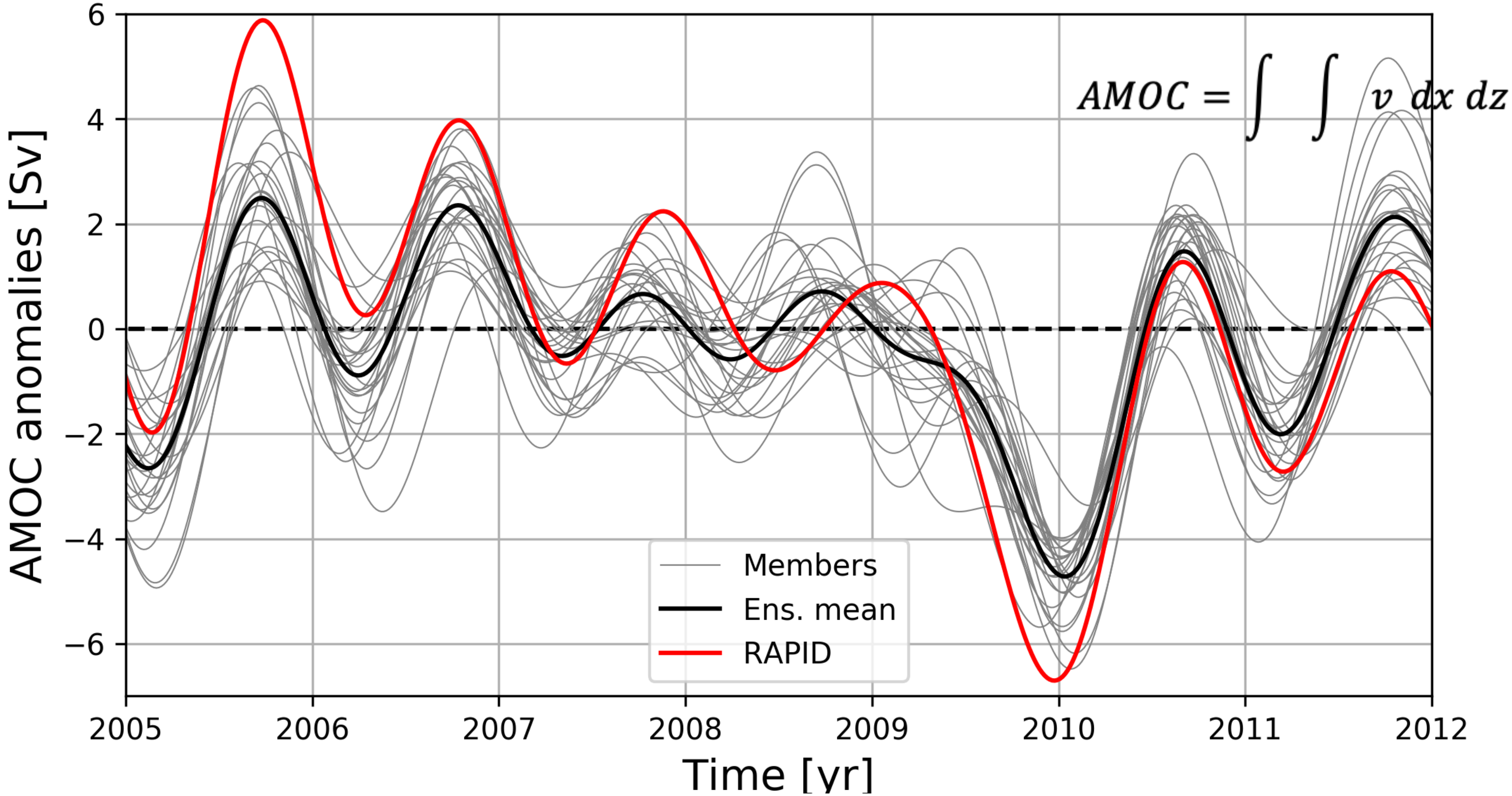
Model vs. Observations

→ The ocean is a chaotic system



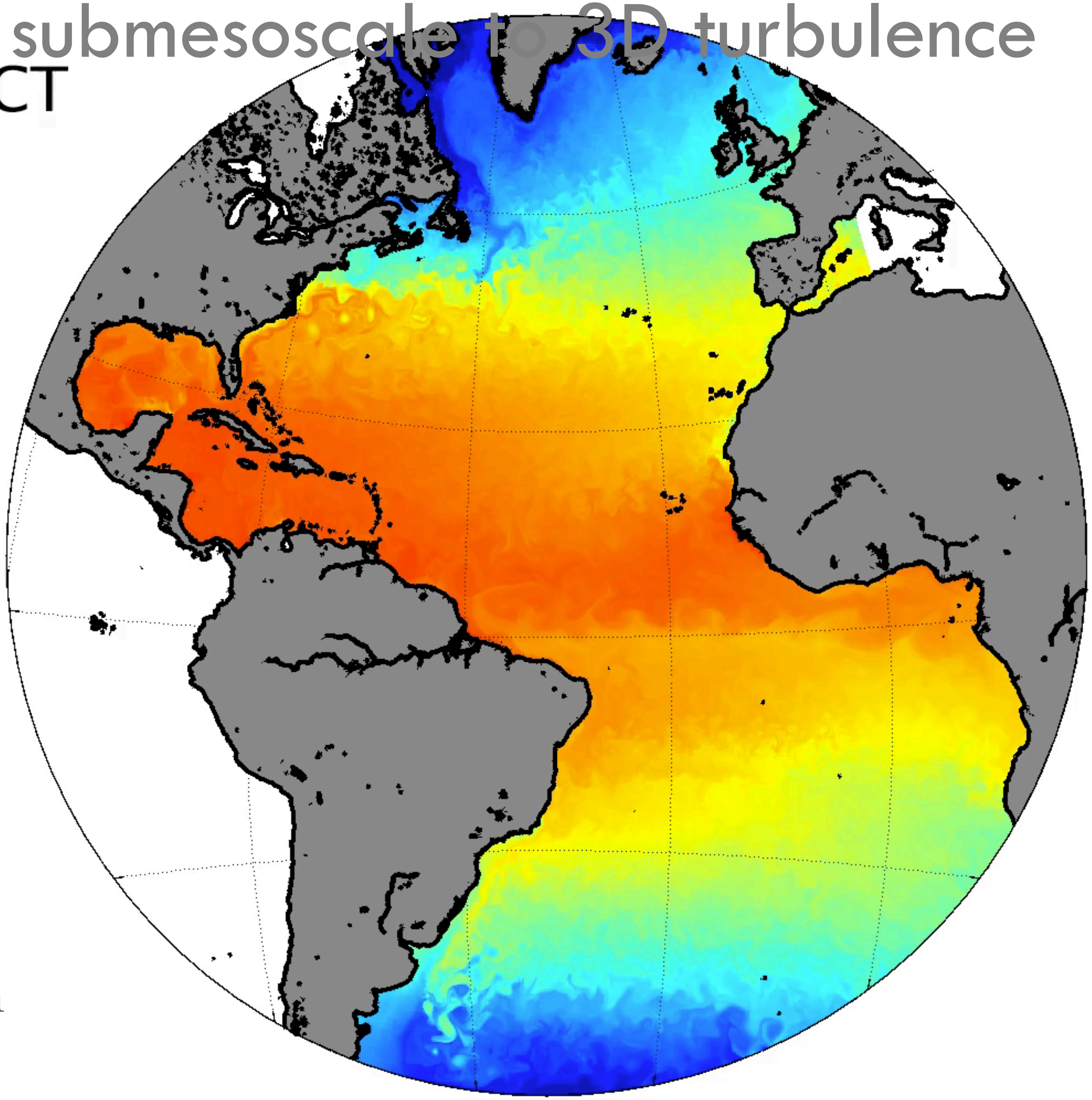
Model *vs.* Observations

→ The ocean is a chaotic system



24 equi-valid model reproductions of AMOC time series @ 26.5°N.

From submesoscale to 3D turbulence

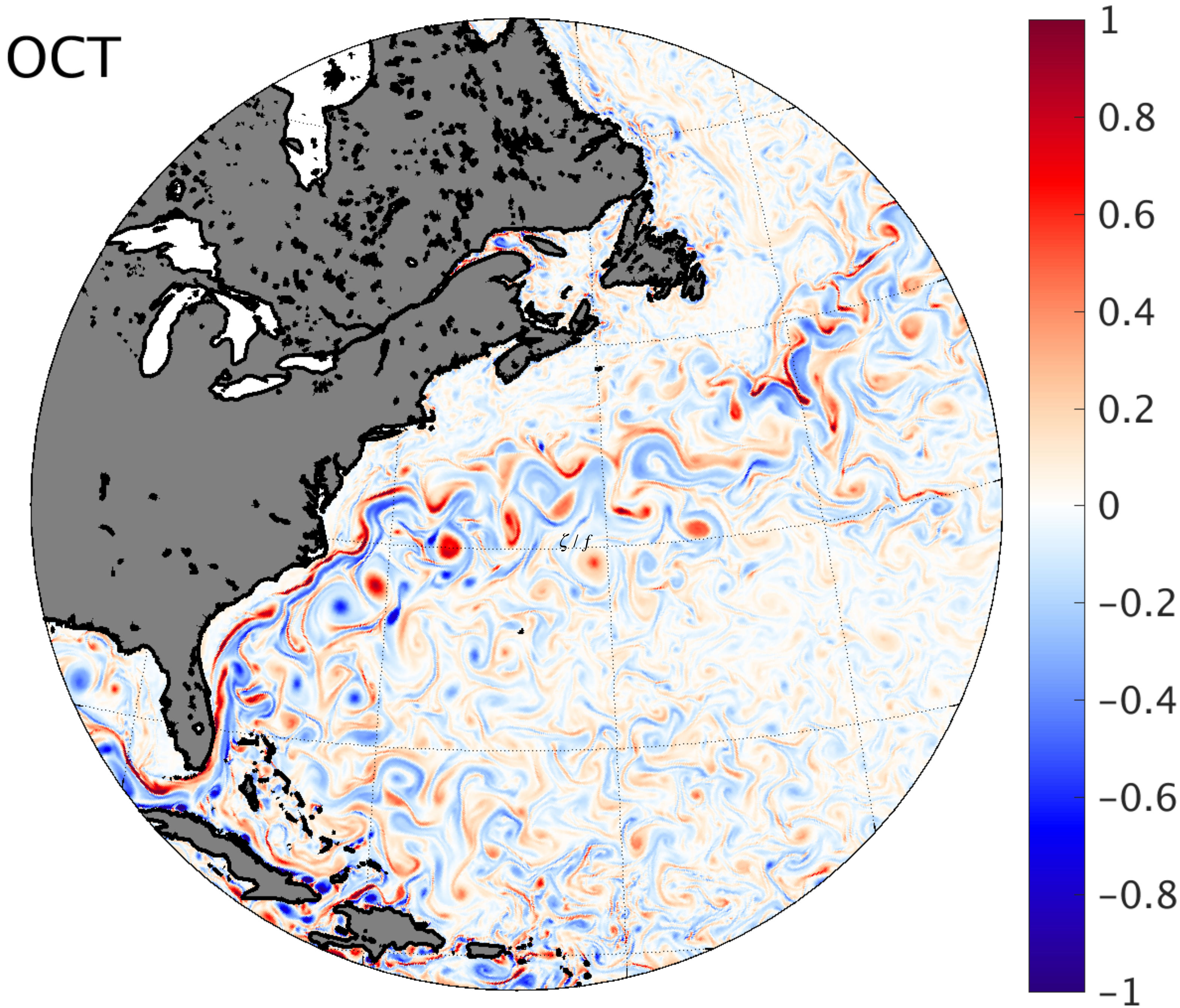


dx = 5-7 km

SST

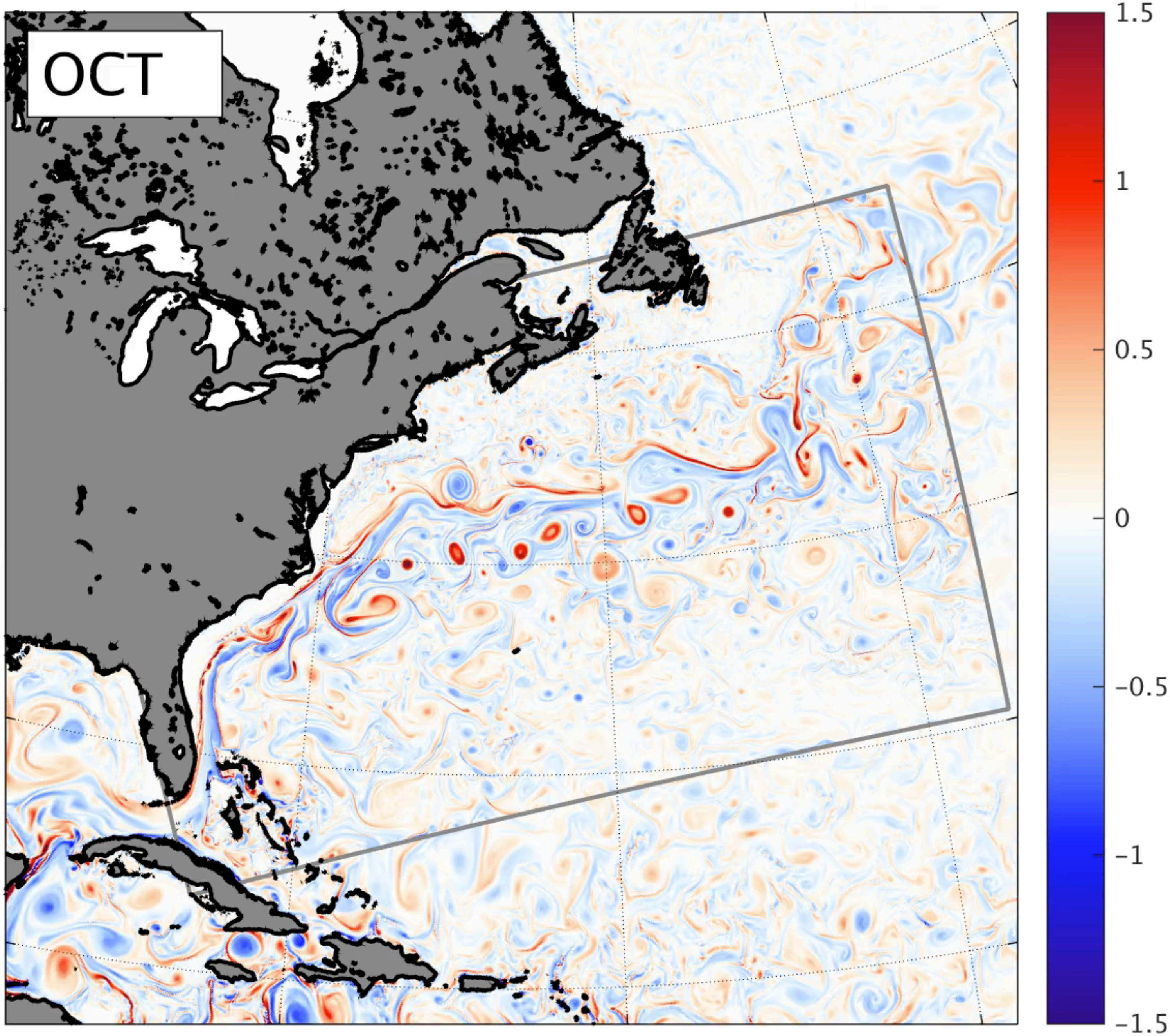
Simulations using ROMS: Regional is a relative concept

Animation from J. Molemaker



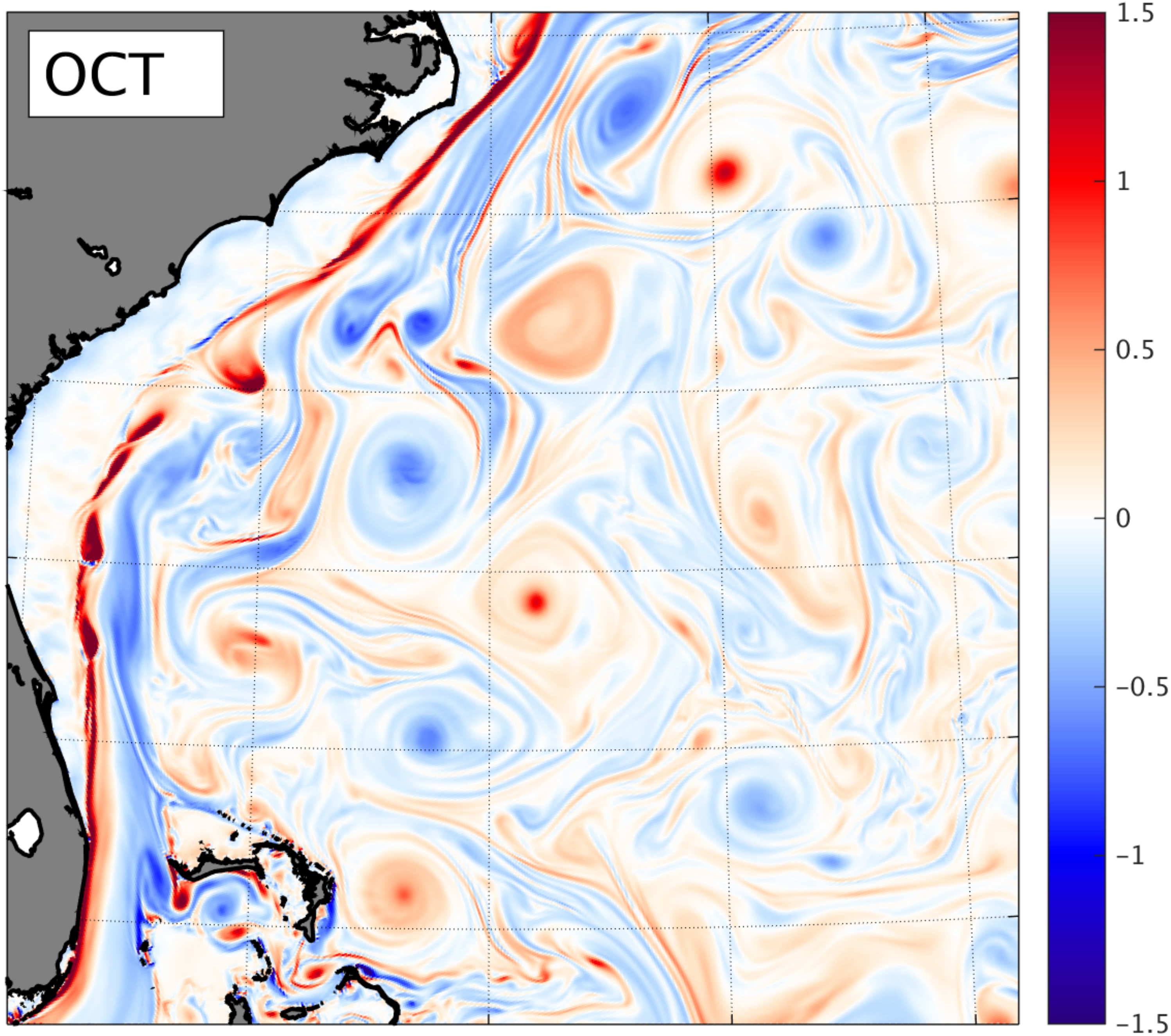
Normalized relative vorticity, or $Ro = \zeta / f$

ζ / f

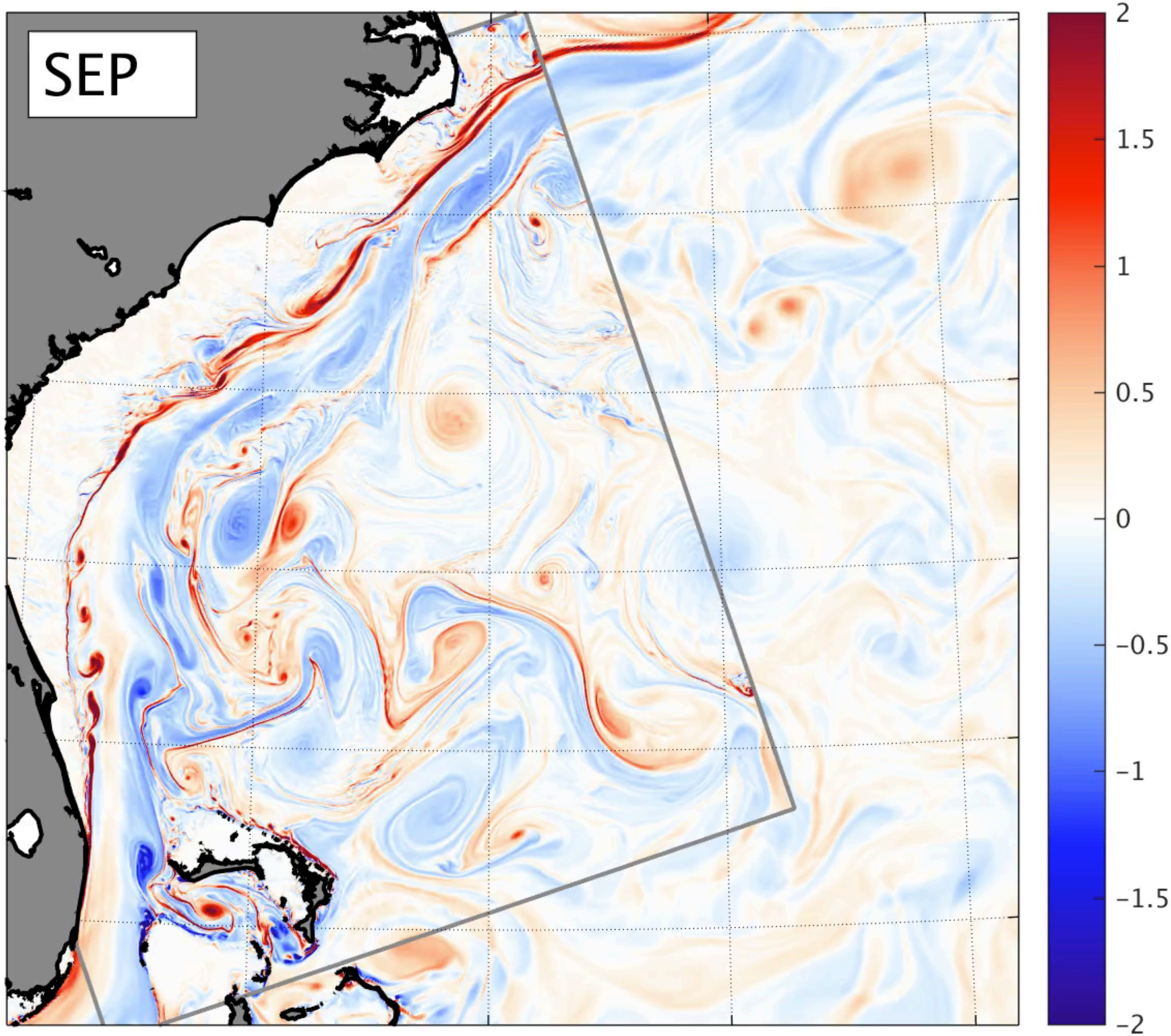


Nested domain with open boundaries with dx = 2.5 km

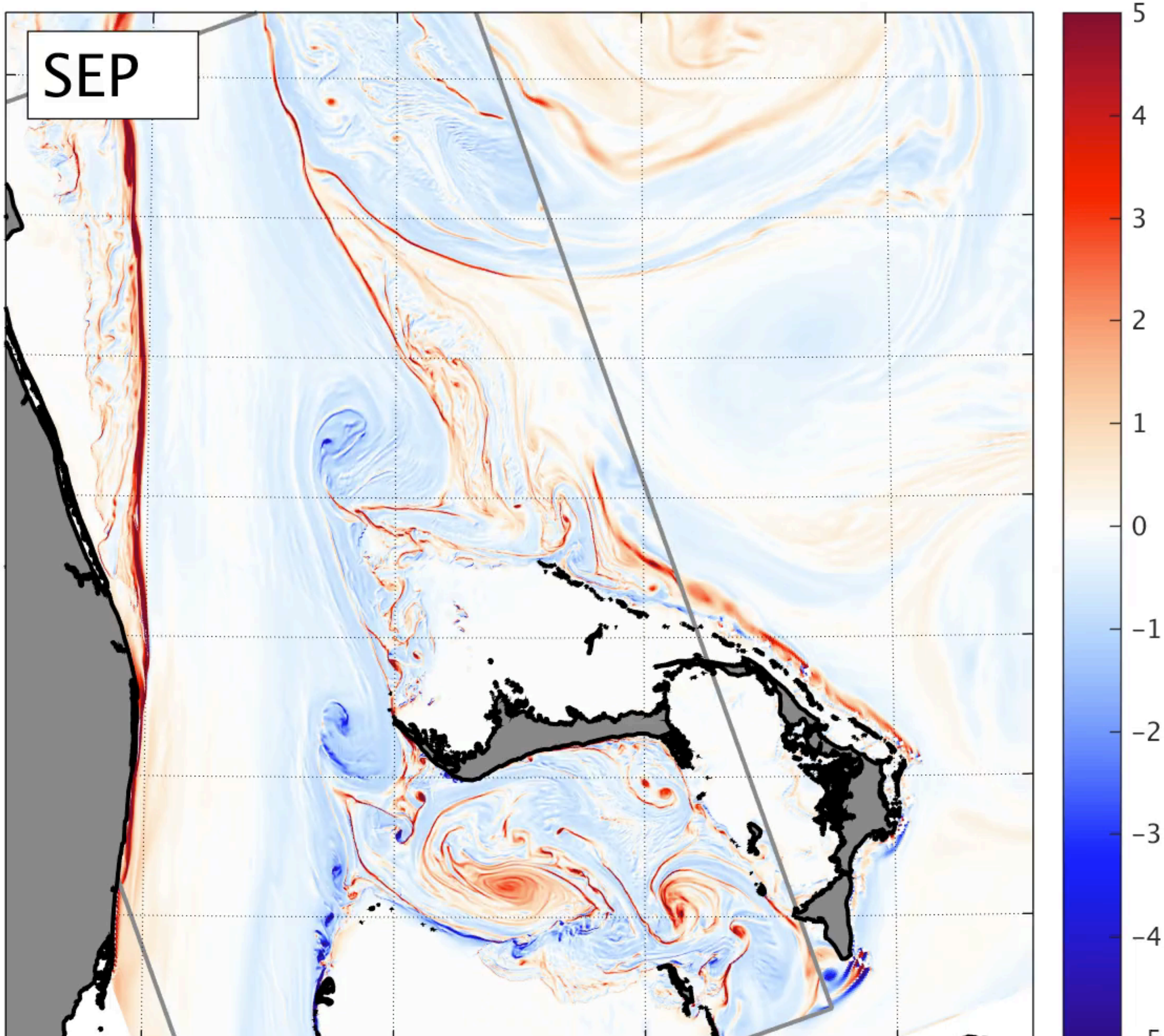
ζ / f



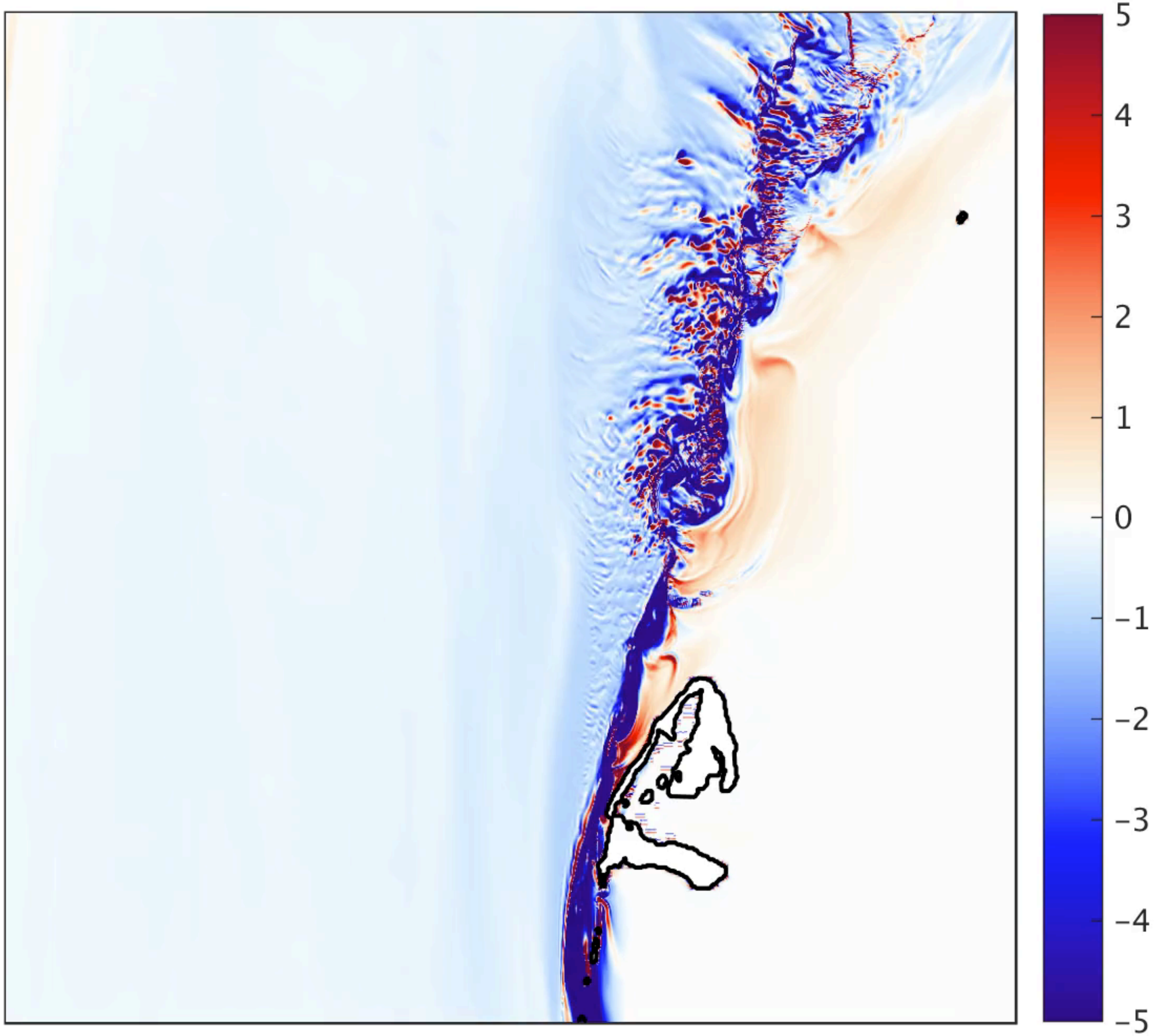
dx = 2.5 km



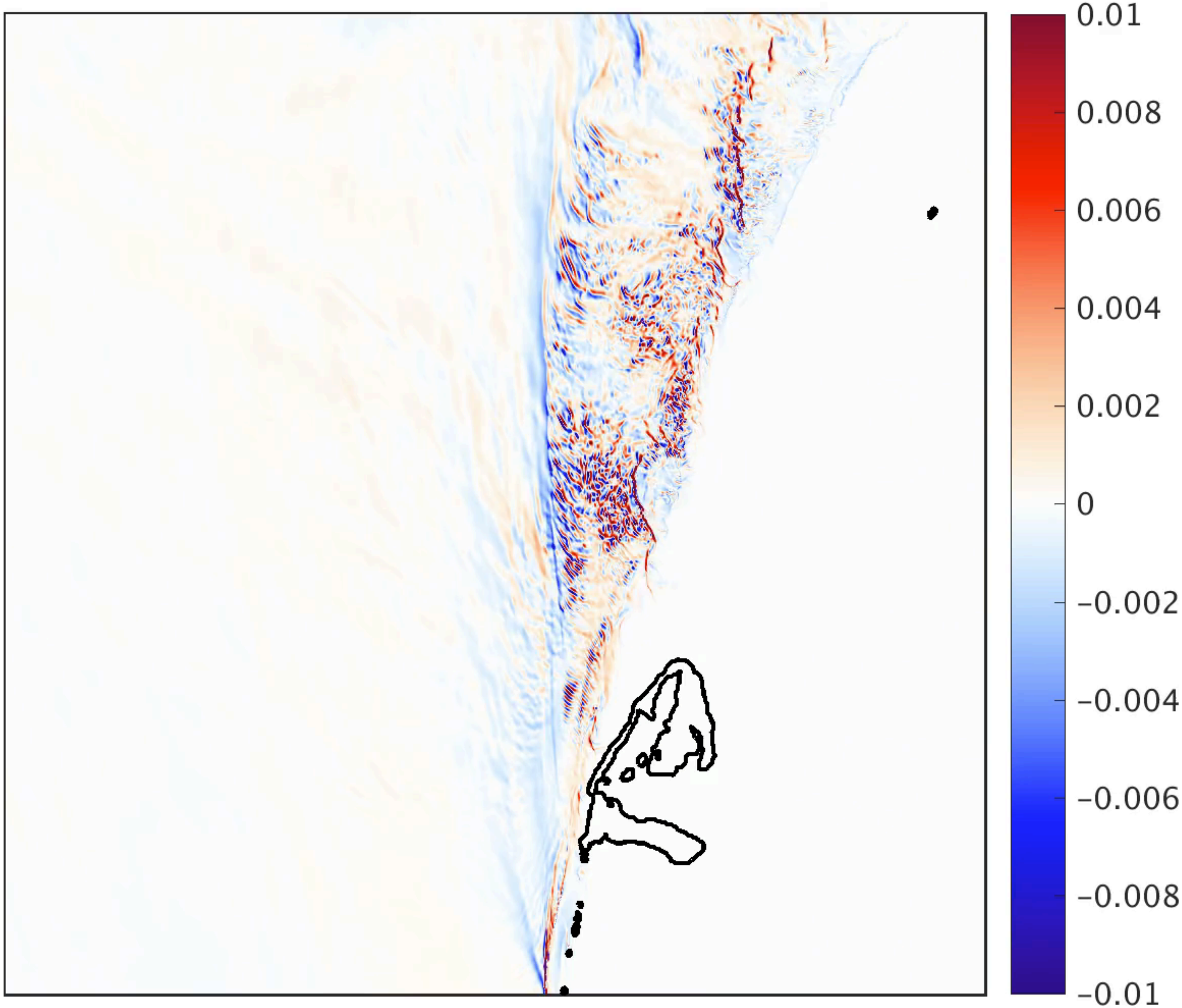
dx = 700 m



dx = 200 m

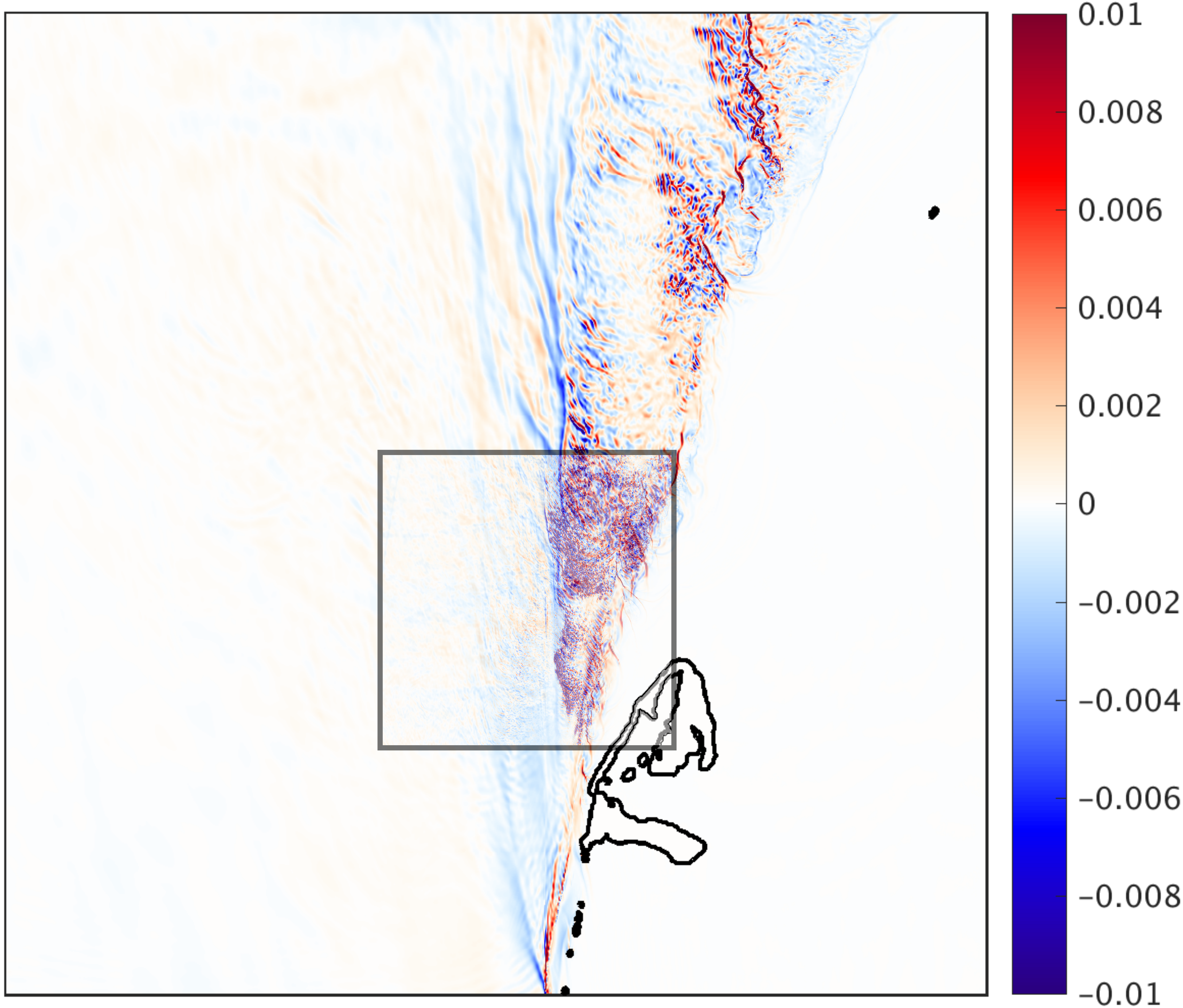


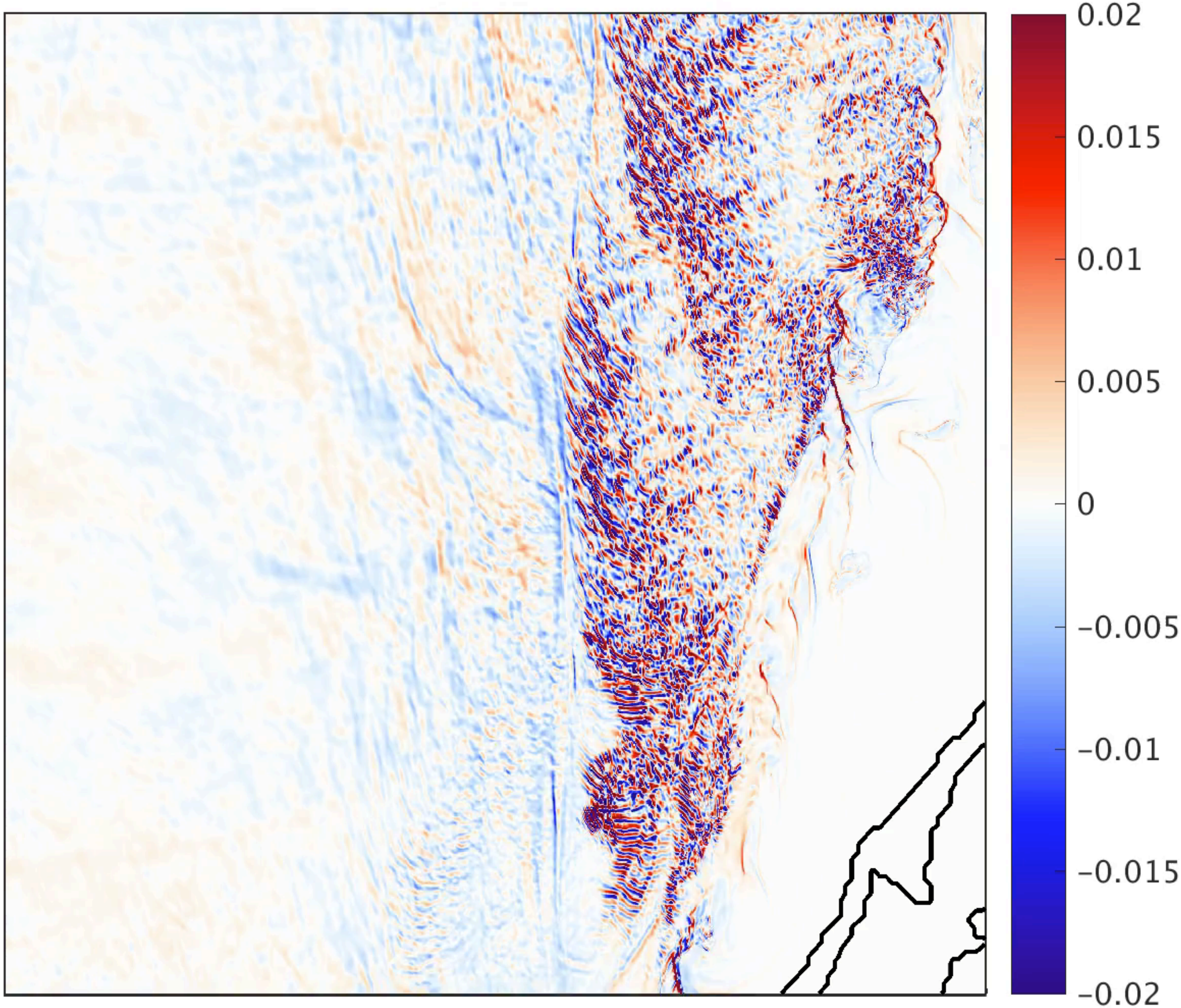
dx = 50 m, non-hydrostatic



dx = 50 m, non-hydrostatic

Surface layer vertical velocity


0.01
0.008
0.006
0.004
0.002
0
−0.002
−0.004
−0.006
−0.008
−0.01



dx = 15 m (NH)

Surface layer vertical velocity


Visible light, (credit NASA)

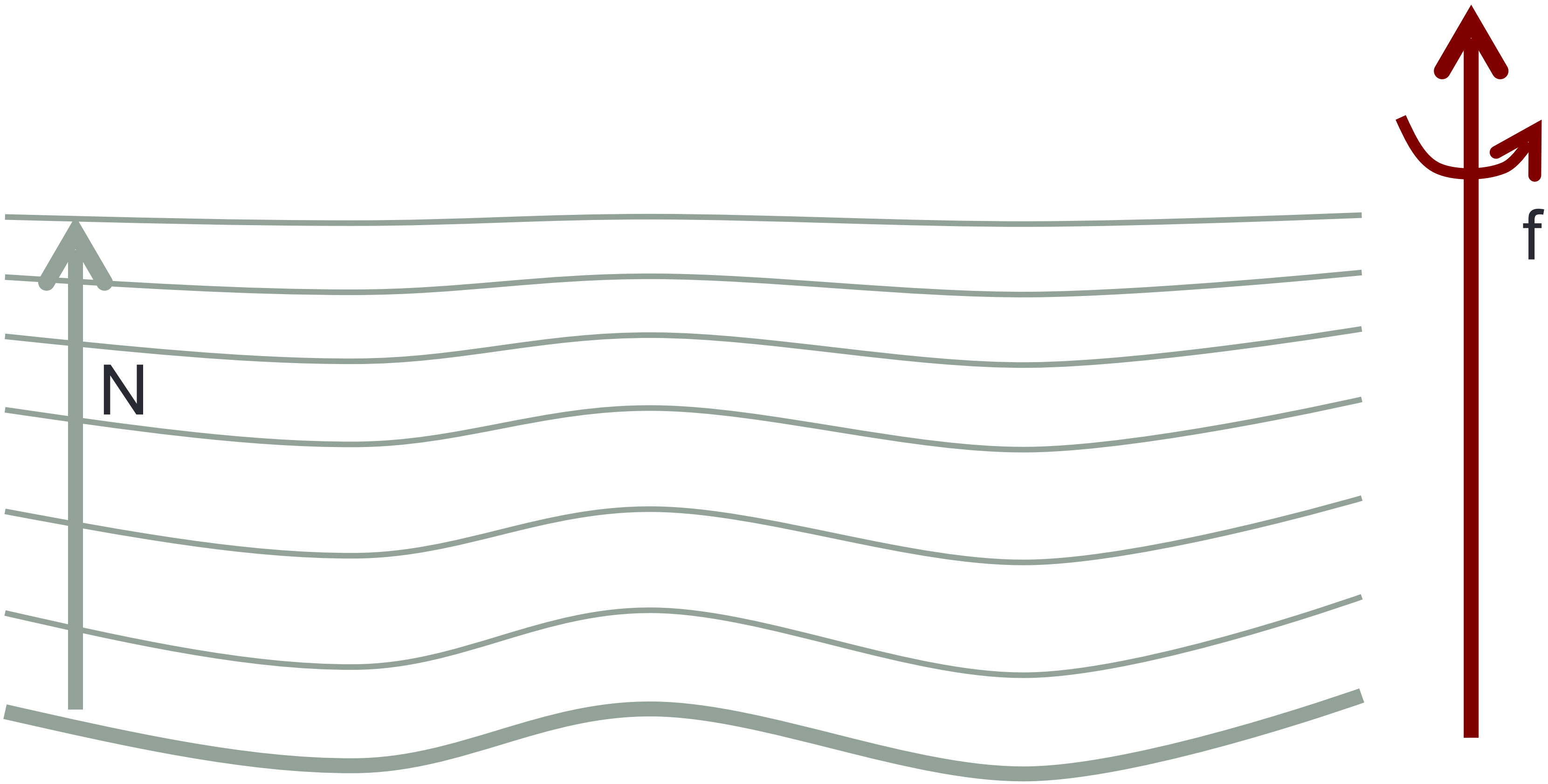
#1
Which Equations?

Master's degree 2nd year Marine Physics

Jonathan GULA
gula@univ-brest.fr

Which Equations?

Ingredients : rotation + stratification



(See chapter 3 of Cushman-Roisin and Beckers)

Which Equations?

- Momentum equations (3d) $$\frac{D\vec{u}}{Dt} = \ldots$$
- Conservation of mass $$\frac{D\rho}{Dt} = \mathcal{S}_\rho$$
- Conservation of heat $$\frac{DT}{Dt} = \mathcal{S}_T$$
- Conservation of salinity $$\frac{DS}{Dt} = \mathcal{S}_S$$
- Equation of state : $$\rho = \rho(T, S, p)$$

[7 equations for the 7 variables: u,v,w,p,T,S,ρ]

Which Equations?

- Momentum equations (3d) $$\frac{D\vec{u}}{Dt} = \ldots$$
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- Equation of state : $$\rho = \rho(T, S, p)$$

[7 equations for the 7 variables: u,v,w,p,T,S,ρ]

Equations for momentum/mass?

- Navier-Stokes Equations (NS)
- Non-hydrostatic Primitive Equations (NH)
- Hydrostatic Primitive Equations (PE)
- Shallow-water (SW)
- Quasi-geostrophic (QG)
- 2D Euler equations
- Etc.

Type of models

Navier Stokes

- DNS = Direct Numerical Simulation
- LES = Large Eddy Simulation

CFD

Process studies

PE

- PE = Primitive Equations models

Ocean Circulation Models

SW

- SW = Shallow-Water models

SQG

- SQG = Surface Quasi-Geostrophic models

QG

- QG = Quasi-Geostrophic models
- Etc.

Idealized models

Which Equations?

- The different approximations to obtain HPE:
  - Boussinesq approximation (Incompressible flow) $\rho(x, y, z, t) = \rho_0 + \rho'(x, y, z, t)$
  - Hydrostatic approximation $\delta = \frac{H}{L} = \frac{W}{U} \ll 1$
  - Traditional approximation (Horizontal Coriolis) $(\cancel{f}w, \cancel{f}u) \ll (fu, fv)$

Equations for momentum/mass?

Navier-Stokes Equations:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}}$$

Momentum equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation
(no source/sink)

Equations for momentum/mass?

Navier-Stokes Equations:

$$\boxed{\frac{\partial \vec{u}}{\partial t}} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + \boxed{2\vec{\Omega} \times \vec{u}} + \boxed{g\vec{k}} = \boxed{-\frac{\vec{\nabla} P}{\rho}} + \boxed{\vec{\mathcal{F}}}$$

Time variation

Advection (inertia)

Rotation

Gravity

Pressure gradient

Forcings + Dissipation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation (no source/sink)

Equations for momentum/mass?

- What are the fastest motions?
-

Equations for momentum/mass?

- What are the fastest motions?

Linearized momentum equations

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} P$$

Equations for momentum/mass?

- What are the fastest motions?

Linearized momentum equations

+ continuity equation

+ adiabatic motion :

= Acoustic modes (sound waves)

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} P$$

$$\frac{\partial P}{\partial t} = -\rho_0 c_s^2 \vec{\nabla} P \cdot \vec{u}$$

$$\partial_{tt} P = c_s^2 \nabla^2 P$$

Equations for momentum/mass?

- What are the fastest motions?

Linearized momentum equations

+ continuity equation

+ adiabatic motion :

= Acoustic modes (sound waves)

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} P$$

$$\frac{\partial P}{\partial t} = -\rho_0 c_s^2 \vec{\nabla} P \cdot \vec{u}$$

$$\partial_{tt} P = c_s^2 \nabla^2 P$$

With $c_s \approx 1500\,\mathrm{m\,s^{-1}}$ in water, a model requires a very small time-step to solve these equations.

Equations for momentum/mass?

Boussinesq Approximation:

Density perturbations small compared to mean background value:

$$\rho = \rho_0 + \rho' \qquad \rho' << \rho_0$$

Linearize all terms involving a product with density,

except the gravity term which is already linear:

$$\begin{aligned} \rho\vec{u} &\rightarrow \rho_0\vec{u} \\ \rho g &\rightarrow \rho g \end{aligned}$$

Equations for momentum/mass?

Boussinesq Approximation :

[+ incompressibility or adiabatic]

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

$$\Big\downarrow \quad \text{Continuity equation}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

Equations for momentum/mass?

Non hydrostatic boussinesq (NH):

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho_0} g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \frac{\vec{\mathcal{F}}}{\rho_0} + \frac{\vec{\mathcal{D}}}{\rho_0}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

Easier to solve than Navier-Stokes, but still requires to invert a 3d elliptic equation for P (computationally expensive)

Equations for momentum/mass?

Hydrostatic balance:

The vertical component of the Boussinesq momentum equations is

$$\frac{\partial w}{\partial t} + \vec{u} \cdot \vec{\nabla} w + 2\Omega \cos\phi v + \boxed{\frac{\rho}{\rho_0} g = -\frac{\partial_z P}{\rho_0}} + \frac{\mathcal{F}_w}{\rho_0} + \frac{\mathcal{D}_w}{\rho_0}$$

For long horizontal motions (L >> H) the dominant balance is

H ~3000 m
L ~3000 km

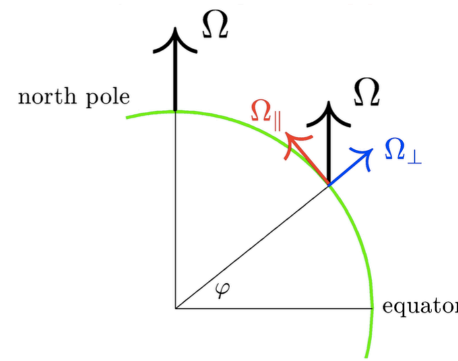
$$\frac{\partial P}{\partial z} = -\rho g$$

Such that pressure is just a vertical integral: $$P = \int_z^\eta g\rho dz$$

Equations for momentum/mass?

Traditional approximation

= neglect horizontal Coriolis term



$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho_0} g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \vec{\mathcal{F}} + \vec{\mathcal{D}}$$

$$\downarrow$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f\vec{k} \times \vec{u} + \frac{\rho}{\rho_0} g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \vec{\mathcal{F}} + \vec{\mathcal{D}}$$

Equations for momentum/mass?

Hydrostatic Primitive Equations (PE)

- 2d momentum with Boussinesq approximation:

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla}_H u + w \frac{\partial u}{\partial z} - f v = -\frac{\partial_x P}{\rho_0} + \mathcal{F}_u + \mathcal{D}_u$$

$$\frac{\partial v}{\partial t} + \vec{u} \cdot \vec{\nabla}_H v + w \frac{\partial v}{\partial z} + f u = -\frac{\partial_y P}{\rho_0} + \mathcal{F}_v + \mathcal{D}_v$$

- Hydrostatic:

$$\frac{\partial P}{\partial z} = -\rho g$$

- Continuity equation for an incompressible fluid: $\vec{\nabla} \cdot \vec{u} = 0$

Equations for PE Ocean Models

Hydrostatic Primitive Equations (PE)

- 2d momentum with Boussinesq approximation:

$$\begin{aligned}
\frac{\partial u}{\partial t} + \vec{u}\cdot\vec{\nabla}_H u + w\frac{\partial u}{\partial z} - fv &= -\frac{\partial_x P}{\rho_0} + \mathcal{F}_u + \mathcal{D}_u \\
\frac{\partial v}{\partial t} + \vec{u}\cdot\vec{\nabla}_H v + w\frac{\partial v}{\partial z} + fu &= -\frac{\partial_y P}{\rho_0} + \mathcal{F}_v + \mathcal{D}_v
\end{aligned}$$

- Hydrostatic: $$\frac{\partial P}{\partial z} = -\rho g$$
- Continuity equation for an incompressible fluid: $\vec{\nabla}\cdot\vec{u} = 0$
- Conservation of heat and salinity $\frac{DT}{Dt} = \mathcal{S}_T \qquad \frac{DS}{Dt} = \mathcal{S}_S$
- Equation of state : $\rho = \rho(T, S, z)$

Equations for PE Ocean Models

Hydrostatic Primitive Equations (PE)

- 4 prognostics equations for **u, v, T, S**
- 3 diagnostics equations for **w, ρ, P**

Equations for PE Ocean Models

Hydrostatic Primitive Equations (PE)

- 2d momentum with Boussinesq approximation:

?

$$\begin{aligned} \frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla}_H u + w \frac{\partial u}{\partial z} - fv &= -\frac{\partial_x P}{\rho_0} \boxed{+ \mathcal{F}_u + \mathcal{D}_u} \\ \frac{\partial v}{\partial t} + \vec{u} \cdot \vec{\nabla}_H v + w \frac{\partial v}{\partial z} + fu &= -\frac{\partial_y P}{\rho_0} \boxed{+ \mathcal{F}_v + \mathcal{D}_v} \end{aligned}$$

- Hydrostatic: $$\frac{\partial P}{\partial z} = -\rho g$$
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Equations for PE Ocean Models

Hydrostatic Primitive Equations (PE)

- 4 prognostics equations for **u, v, T, S**
- 3 diagnostics equations for **w, ρ, P**

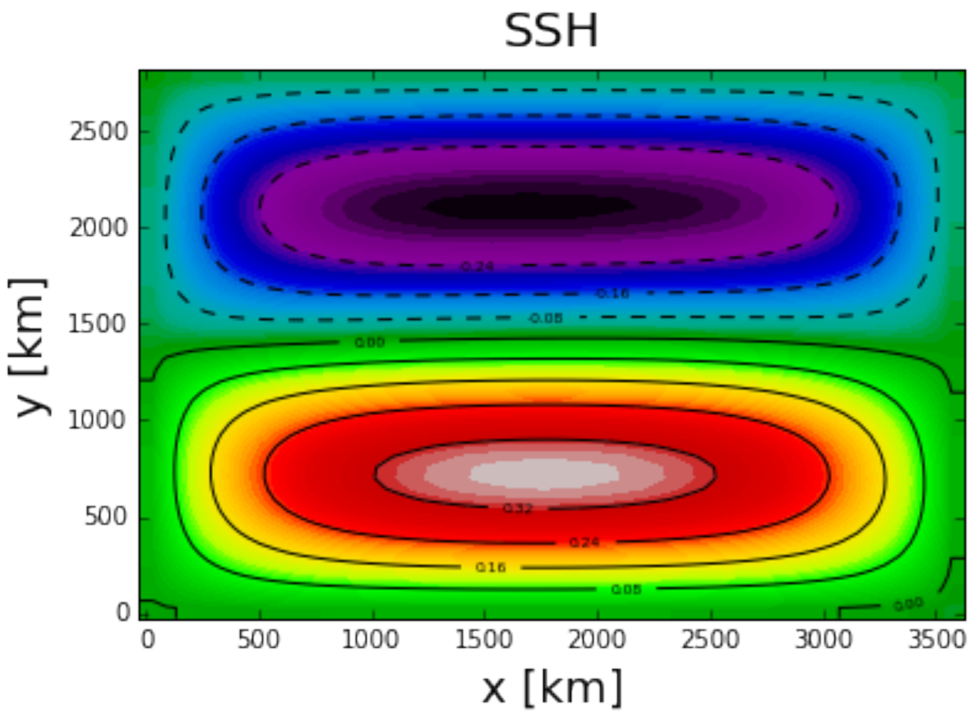
+ Forcings (wind, heat flux)

+ sub-grid scale parameterizations (bottom drag, mixing, etc.)

Activity 1 – Run an idealized ocean basin

- https://www.jgula.fr/ModNum/Activity1.html

- Utilisation du code océanique CROCO (https://www.croco-ocean.org) pour un cas idéalisé (gyre rectangulaire):



- Doc: https://croco-ocean.gitlabpages.inria.fr/croco_doc/

Activity 1 – Run an idealized ocean basin

- **Jobcomp** (compilation)
- **cppdefs.h** (Numerical/physical options)
- **param.h** (grid size/ parallelisation)
- **croco.in** (choice of variables, parameter values, etc.)

Homework

- For next time:
  - Read https://www.jgula.fr/ModNum/Stommel48.pdf
  - Read https://www.jgula.fr/ModNum/Munk50.pdf
  -