

# TURBULENCE

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## 2. 2D TURBULENCE

- **Lesson 1 :**

- Introduction
  - *What is turbulence?*
- Properties of turbulence
  - *Where does it come from?*
  - *What does it do?*
- The closure problem
- The Kolmogorov theory

- **Lesson 2 :**

- 2D turbulence

- **Lesson 3 :**

- Geostrophic turbulence

# References:

- Vallis G.K., Atmospheric and Oceanic Fluid Dynamics.
- MIT online course: <https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-820-turbulence-in-the-ocean-and-atmosphere-spring-2007/lecture-notes/>
- LaCasce J.H., Turbulence in the Atmosphere and Ocean.

# Conservation laws

- Equation for kinetic energy

$$\frac{\partial}{\partial t} \frac{1}{2} |\vec{u}^2| + \nabla \cdot (\vec{u} \frac{1}{2} |\vec{u}^2|) = -\nabla \cdot [\vec{u} \left( \frac{p}{\rho_0} + gz \right)] + \vec{u} \cdot \mathcal{F} + \nu \vec{u} \cdot \nabla^2 \vec{u}$$

- Energy integrated over a closed (or periodic) domain

$$E = \iiint \frac{1}{2} |\vec{u}^2| dV \quad \frac{d}{dt} E = \iiint \vec{u} \cdot \mathcal{F} dV + \nu \iiint \vec{u} \cdot \nabla^2 \vec{u} dV$$

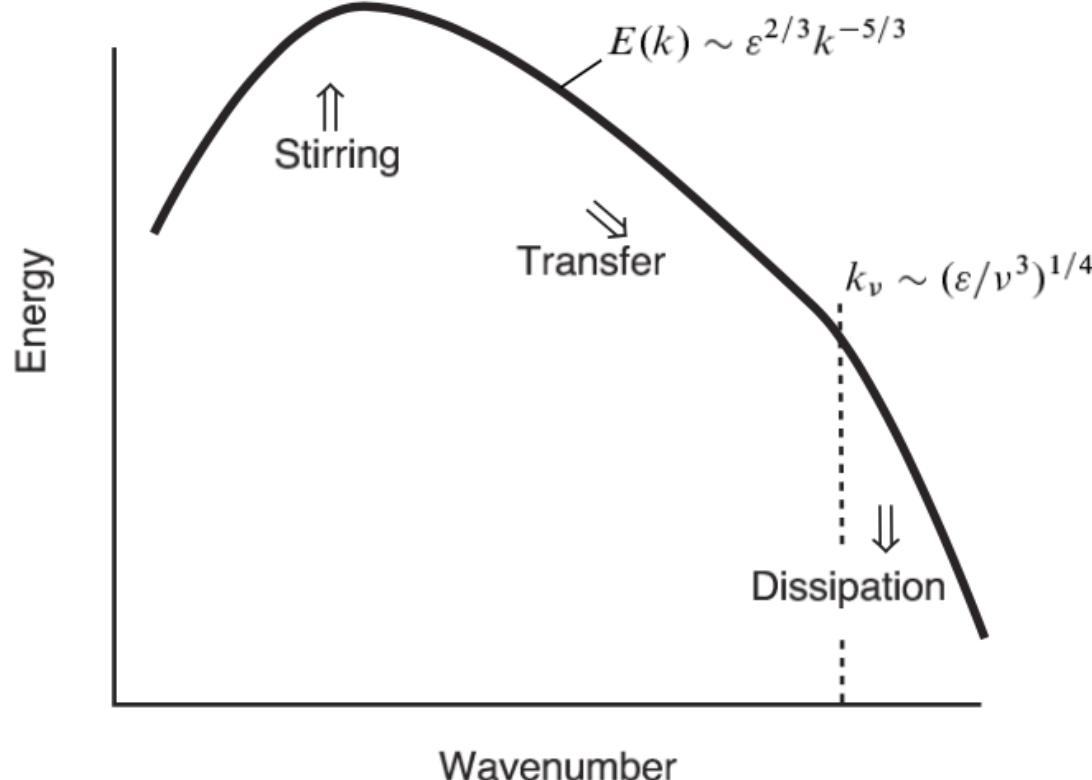
- the inertial terms in the momentum equation conserve energy (redistribute in the domain)

# Conservation laws

$$\frac{d}{dt} E = \underset{\text{forcing}}{\iiint \vec{u} \cdot \mathcal{F} dV} - \nu \underset{\text{dissipation}}{\iiint |\vec{\omega}|^2 dV}$$

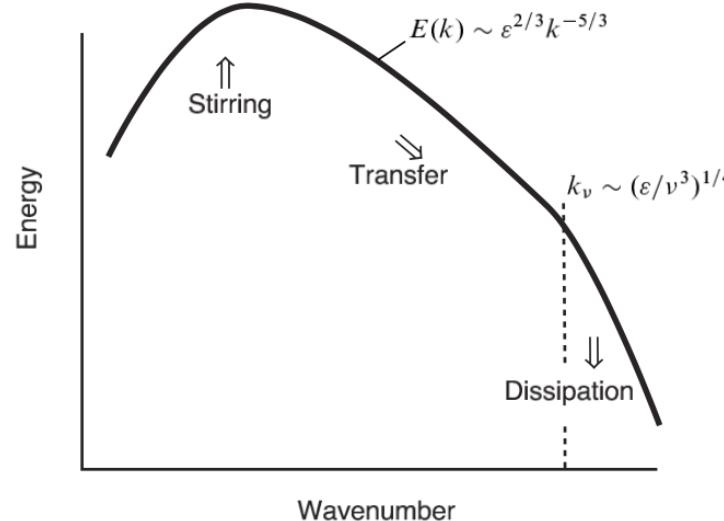
- Forcing puts energy in the system and dissipation removes it.
- Dissipation is proportional to the integral of the squared vorticity, also known as the **enstrophy**
- $E$  is an **inviscid invariant** (invariant when no forcing and viscosity)

# 3D turbulence



# 3D turbulence

- The energy dissipation rate is equal to the energy cascade rate.
- It is *independent of the viscosity*. As viscosity tends to zero, the scale at which viscous effects become important becomes smaller in just such a way as to preserve the constancy of the energy dissipation.



- there must be *production* of enstrophy in the absence of forcing

# Activity 1

- Write the vorticity equation (*starting from NS equations below, with constant rotation and incompressibility*).

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

- A few useful identities:

$$\begin{aligned}
 \vec{\omega} &= \vec{\nabla} \times \vec{u} \\
 (\vec{u} \cdot \vec{\nabla}) \vec{u} &= \frac{1}{2} \vec{\nabla}(|\vec{u}|^2) + \vec{\omega} \times \vec{u} \\
 \vec{\nabla} \times (\vec{u} \times \vec{\omega}) &= -\vec{\omega}(\vec{\nabla} \cdot \vec{u}) + (\vec{\omega} \cdot \vec{\nabla}) \vec{u} - (\vec{u} \cdot \vec{\nabla}) \vec{\omega} \\
 \vec{\nabla} \cdot \vec{\omega} &= 0 \\
 \vec{\nabla} \times \vec{\nabla} \Phi &= 0
 \end{aligned}$$

# Vorticity equation

- Taking the curl of

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

- We get:  $\frac{\partial}{\partial t} \vec{\omega} + \vec{u} \cdot \nabla \vec{\omega}_a + \vec{\omega}_a (\nabla \cdot \vec{u}) = \vec{\omega}_a \cdot \nabla \vec{u} + \nabla \times \vec{\mathcal{F}} + \nu \nabla^2 \vec{\omega}$

With the absolute vorticity  $\vec{\omega}_a = \nabla \times \vec{u} + f \vec{k}$

- Which gives (with constant rotation and incompressibility):

$$\frac{\partial}{\partial t} \vec{\omega} + \nabla \cdot (\vec{u} \circ \vec{\omega}) = \vec{\omega}_a \cdot \nabla \vec{u} + \nabla \times \vec{\mathcal{F}} + \nu \nabla^2 \vec{\omega}$$

# Enstrophy equation

- Dot product with vorticity gives

$$\frac{1}{2} \frac{\partial}{\partial t} |\vec{\omega}|^2 + \frac{1}{2} \nabla \cdot (\vec{u} |\vec{\omega}|^2) = \vec{\omega} \cdot (\vec{\omega}_a \cdot \nabla \vec{u}) + \nu \vec{\omega} \cdot \nabla^2 \vec{\omega}$$

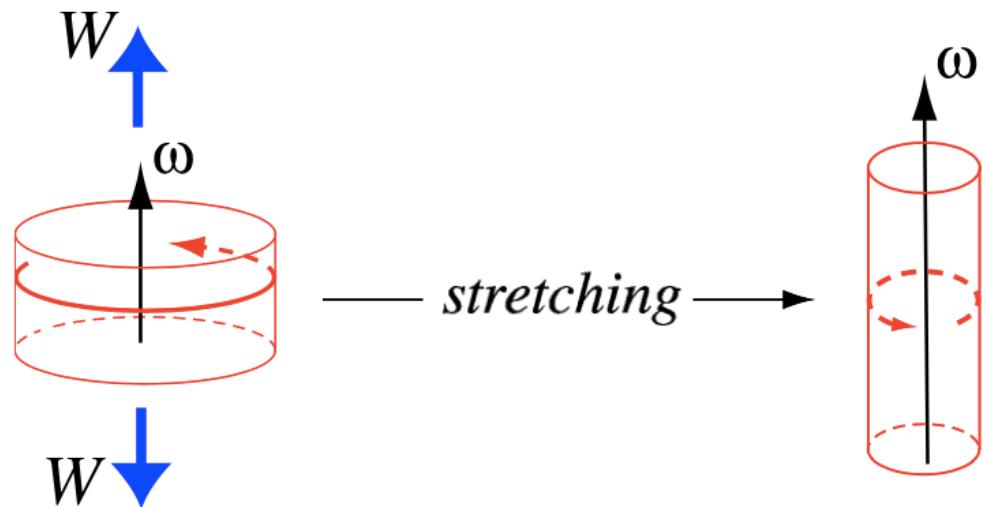
- Note that domain averaging is equivalent to an ensemble averaging as we hypothesize statistical homogeneity of the turbulence = all divergences integrate to zero
- Finally:

$$\frac{d}{dt} \iiint \frac{1}{2} |\vec{\omega}|^2 dV = \boxed{\iiint \vec{\omega} \cdot (\vec{\omega}_a \cdot \nabla \vec{u}) dV} - \boxed{\nu \iiint |\nabla \times \vec{\omega}|^2 dV}$$

Production of enstrophy due to  
vortex stretching (undetermined sign)

dissipation

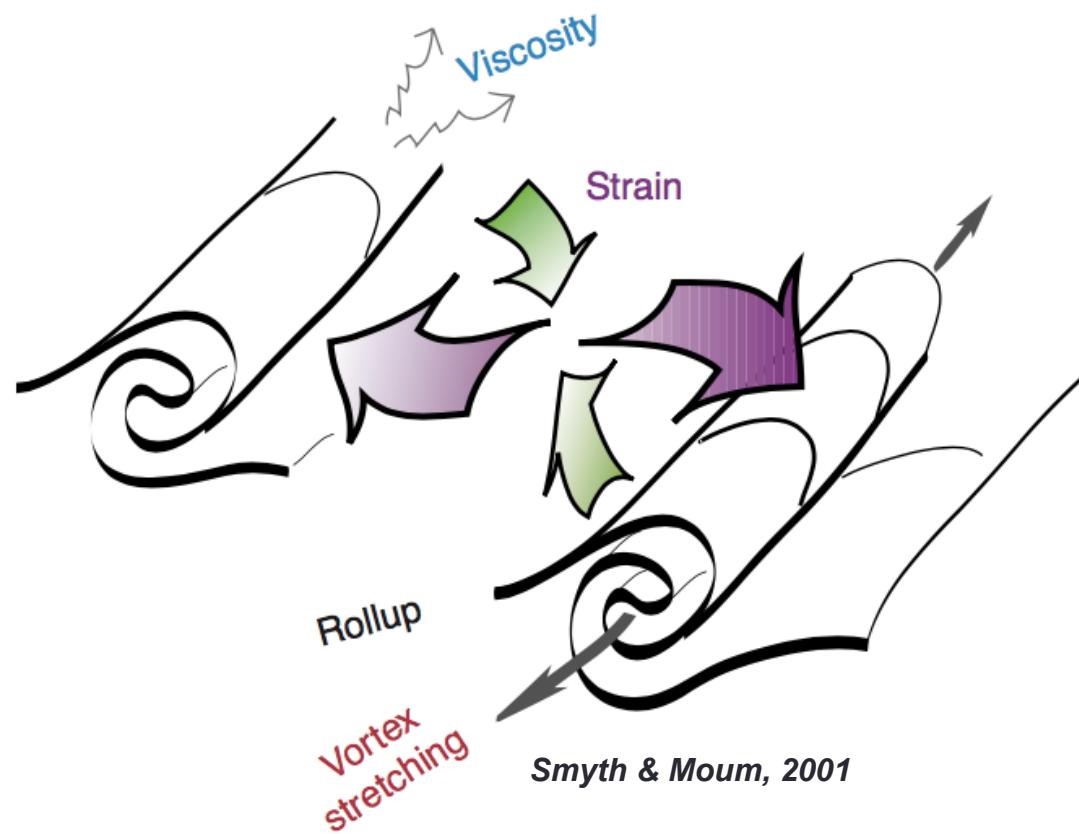
# 3d turbulence



**Fig. 4.5** **Stretching** of material lines distorts the cylinder of fluid as shown. Vorticity is tied to material lines, and so is amplified in the direction of the **stretching**. However, because the volume of fluid is conserved, the end surfaces shrink, the material lines through the cylinder ends converge, and the integral of vorticity over a material surface (the circulation) remains constant, as discussed in section 4.3.2.

# 3d turbulence

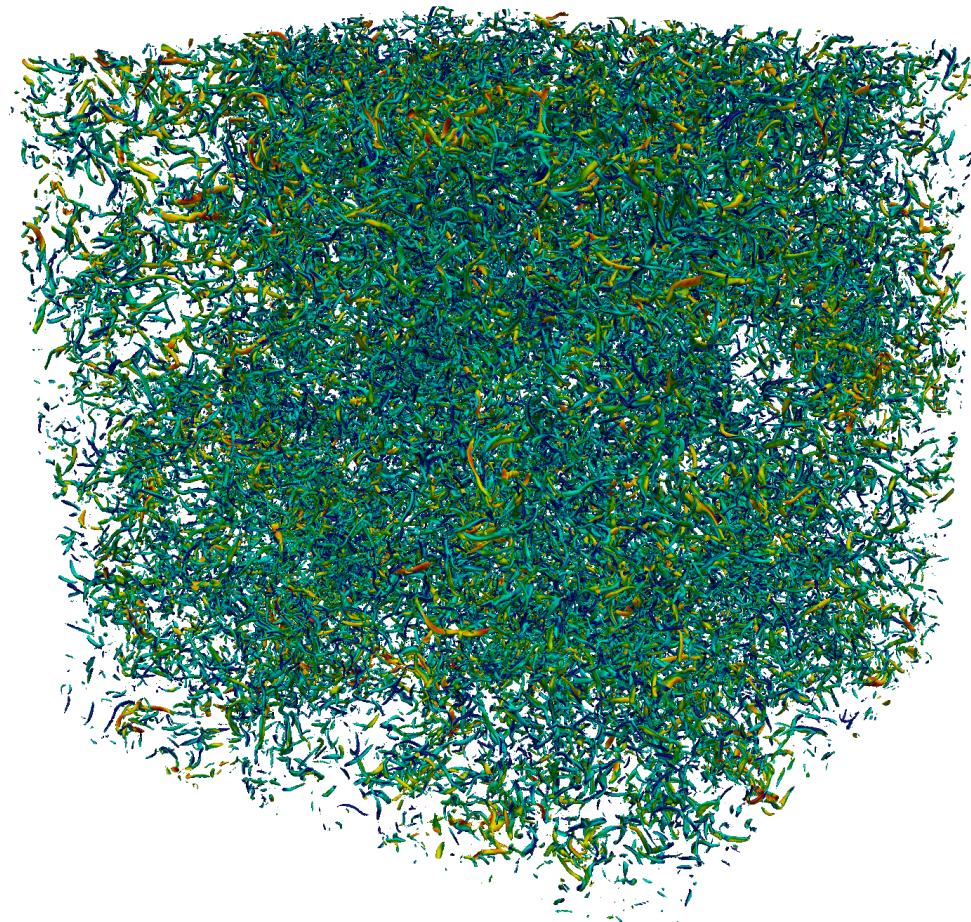
- energy is cascaded to small scales via vortex stretching



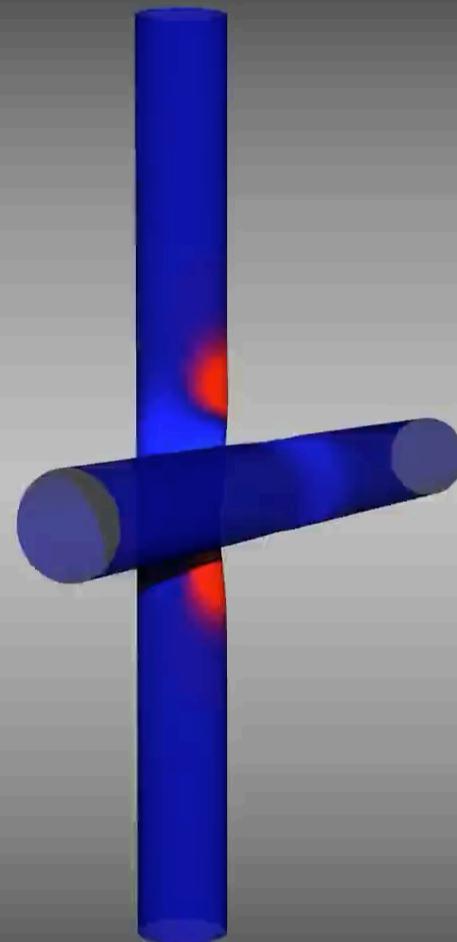
*Schematic illustration of line vortices and strained regions in turbulent flow. Fluid parcels in the vortex interiors rotate with only weak deformation. In contrast, fluid parcels moving between the vortices are rapidly elongated in the direction of the purple arrows and compressed in the direction of the green arrows.*

# 3d turbulence

- energy is cascaded to small scales via vortex stretching



# 3d turbulence



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- **Lesson 2 :**

- 2D turbulence

- **Lesson 3 :**

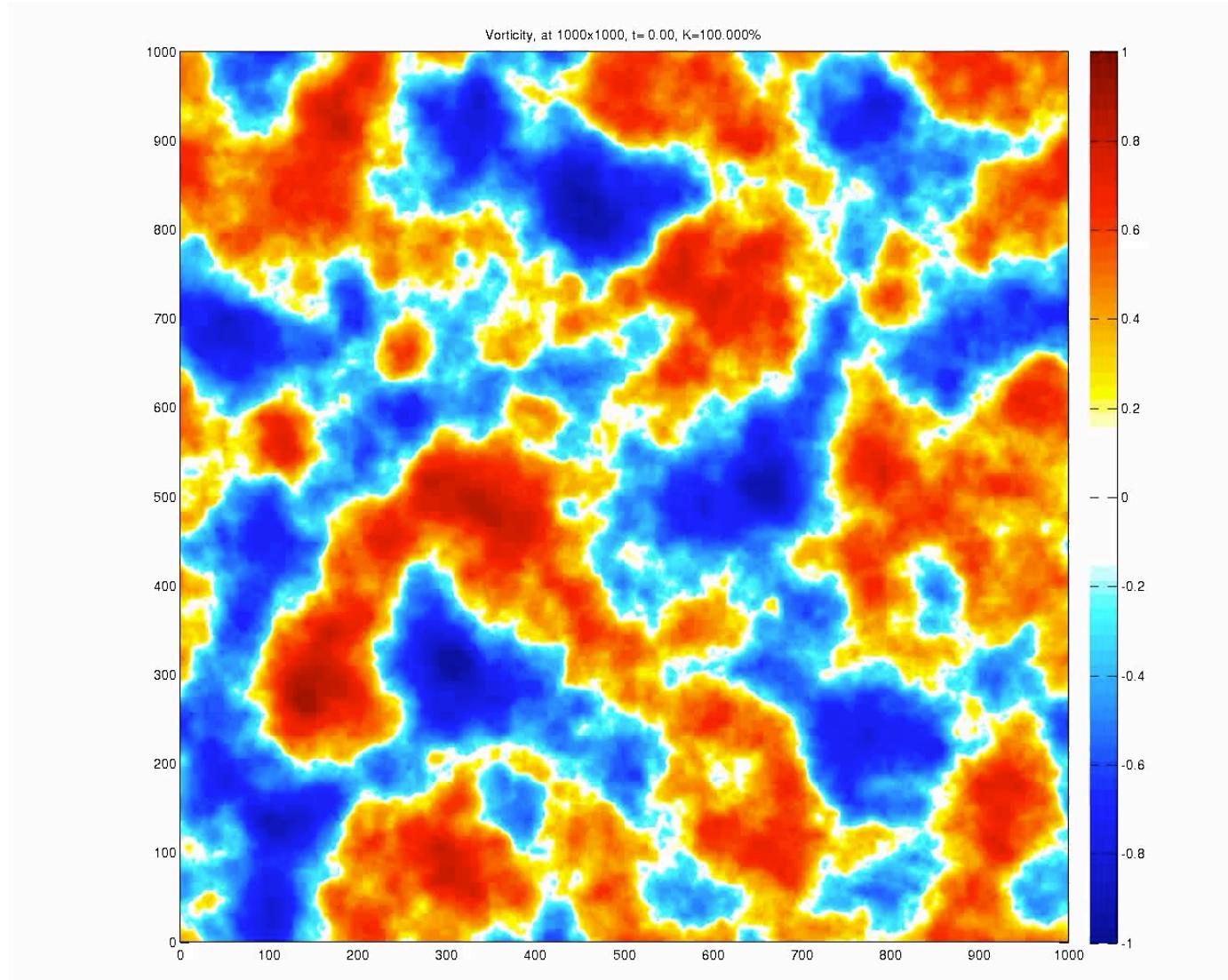
- Geostrophic turbulence

# 2d turbulence

- 2D homogeneous turbulence is relevant to geophysical turbulence on large horizontal scales because of the thinness of Earth's atmosphere and ocean (i.e.,  $H/L \ll 1$ ) and Earth's rotation (i.e.,  $Ro \ll 1$ ) and stable stratification (i.e.,  $Fr \ll 1$ ), both of which tend to suppress vertical flow and make the 2D horizontal velocity component dominant.



# 2d turbulence



# 2d turbulence

- What happens to the vorticity equation if the velocity is purely horizontal?

$$\vec{u} = (u, v, 0)$$

- The vorticity becomes purely vertical:  $\vec{\omega} = (0, 0, \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u) \equiv \zeta \hat{k}$

# 2d turbulence

- What happens to the vorticity equation if the velocity is purely horizontal?

$$\vec{u} = (u, v, 0)$$

- The vorticity becomes purely vertical:

$$\vec{\omega} = (0, 0, \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u) \equiv \zeta \hat{k}$$

- And the vorticity equation becomes

$$\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla(\zeta + f) = \nabla \times \mathcal{F} + \nu \nabla^2 \zeta$$

- the vortex stretching term disappears!

$$\omega_a \cdot \nabla \vec{u} = (\zeta + f)\hat{k} \cdot \nabla(u\hat{i} + v\hat{j}) = 0$$

- In 2D both energy and enstrophy are **inviscid invariants**

# 2d turbulence (some notations)

- *The vorticity equation can also be written:*

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \zeta, \quad \zeta = \nabla^2 \psi.$$

*By introducing the streamfunction such that:*

$$u, v = -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}$$

*And*       $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi$

*The equation can also be written:*

$$\frac{\partial \nabla^2 \psi}{\partial t} + J(\psi, \nabla^2 \psi) = F + \nu \nabla^4 \psi$$

# 2d turbulence (some notations)

- Energy and enstrophy are:

$$\hat{E} = \frac{1}{2} \int_A (u^2 + v^2) dA = \frac{1}{2} \int_A (\nabla \psi)^2 dA,$$

$$\hat{Z} = \frac{1}{2} \int_A \xi^2 dA = \frac{1}{2} \int_A (\nabla^2 \psi)^2 dA,$$

- And their spectra can be written:

$$\hat{E} = \int E(k) dk, \quad \hat{Z} = \int Z(k) dk = \int k^2 E(k) dk,$$

# Triad interaction in 2d turbulence

- Activity 2:  
Fjortoft example

Initially there are Energy  $E_0$  and enstrophy  $Z_0$  at wavenumber  $k_0$ .

There is no viscosity (or we do not feel it because we are in the inertial range)

All Energy  $E_0$  and enstrophy  $Z_0$  are transferred to wavenumbers  $k_1$  and  $k_2$ .

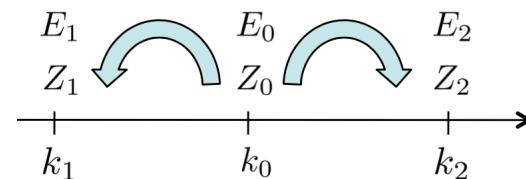
Based on Energy and enstrophy conservation, what will be the ratio between  $E_1/E_2$  and  $Z_1/Z_2$ ?



On the Changes in the Spectral Distribution of Kinetic Energy  
for Two-dimensional, Nondivergent Flow

By RAGNAR FJØRTOFT, University of Copenhagen

(Manuscript received April 25, 1953)



$$\begin{array}{ll} k_1 = 1/2 k_0 & \text{Large scale} \\ k_2 = 2 k_0 & \text{Small scale} \end{array}$$

# Triad interaction in 2d turbulence

- Fjortoft example:

It is the opposite of  
3d turbulence and  
corresponds to an  
**inverse cascade of  
energy!**

S V E N S K A   G E O F Y S I S K A   F Ö R E N I N G E N

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## On the Changes in the Spectral Distribution of Kinetic Energy for Twodimensional, Nondivergent Flow

By RAGNAR FJØRTOFT, University of Copenhagen

(Manuscript received April 25, 1953)

### *Abstract*

Total kinetic energy as well as total vorticity squared are integral quantities which cannot change in the course of time in a *twodimensional* flow of a homogeneous, nondivergent, and inviscid fluid when the fluid is isolated from the surroundings. The case is considered where the fluid is defined over the total region of the surface of a sphere. The nature of the changes in time of the spectral distribution of kinetic energy is discussed on the basis of the two conservation requirements mentioned above. It is found that only fractions of the initial energy can flow into smaller scales and that a greater fraction simultaneously has to flow to components with larger scales. The upper limits to the flow of kinetic energy into components with scales less than a given one are found. The conservation theorems are also used to discuss the stability of a certain stationary flow for a twodimensional motion which is not necessarily spherical. It is shown how important it is for the proof of stability that not only the kinetic energy of the disturbance is supposed to be small but also its vorticities.

In chapter II molecular viscosity is taken into account for the spherical flow. Finally some conclusive remarks are offered regarding the fundamental difference between two- and three-dimensional flow.

# Triad interaction in 2d turbulence

- Batchelor argument:

Imagine we have a narrow initial energy spectrum centered on wavenumber  $k_e$

The wavenumber characterizing the spectral location of the energy is the centroid:

$$k_e = \frac{\int k \mathcal{E}(k) dk}{\int \mathcal{E}(k) dk}$$

And the spreading out of the energy distribution is formalized by setting:

$$I \equiv \int (k - k_e)^2 \mathcal{E}(k) dk$$

$I$  measures the width of the energy distribution.

# Triad interaction in 2d turbulence

- Batchelor argument:

Expanding the integral gives:

$$I \equiv \int (k - k_e)^2 \mathcal{E}(k) dk$$

$$\begin{aligned} I &= \int k^2 \mathcal{E}(k) dk - 2k_e \int k \mathcal{E}(k) dk + k_e^2 \int \mathcal{E}(k) dk \\ &= \int k^2 \mathcal{E}(k) dk - k_e^2 \int \mathcal{E}(k) dk, \end{aligned}$$

If we assume that the distribution spreads in time  
 (as expected from NL interactions):  $\frac{dI}{dt} > 0.$

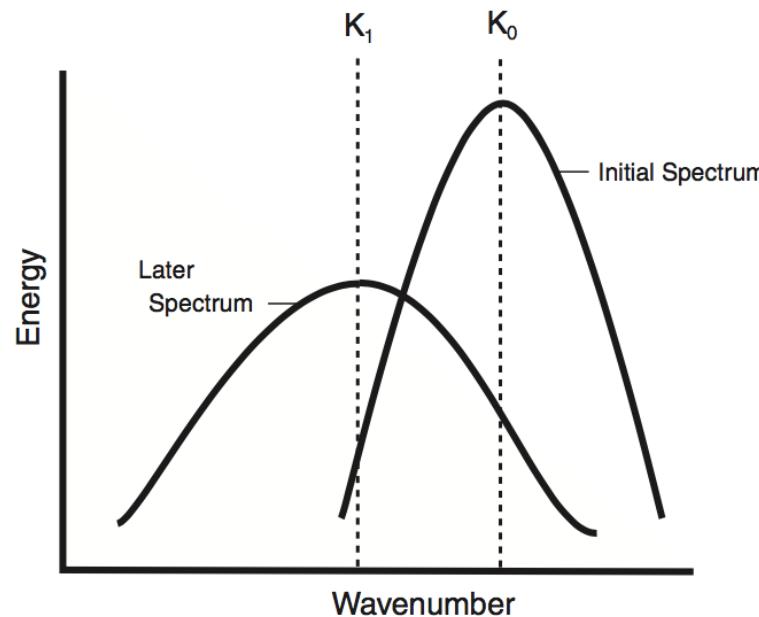
Conservation of E and Z gives:

$$\frac{dk_e^2}{dt} = -\frac{1}{\hat{E}} \frac{dI}{dt} < 0.$$

**the centroid of the distribution moves to smaller wavenumber and to larger scale !**

# Triad interaction in 2d turbulence

- Batchelor argument:



**Figure 8.6** In two-dimensional flow, the centroid of the energy spectrum will move to large scales (smaller wavenumber) provided that the width of the distribution increases, which can be expected in a nonlinear, eddying flow

**the centroid of the distribution moves to smaller wavenumber and to larger scale !**

# Triad interaction in 2d turbulence

- Batchelor argument:

For enstrophy we have a similar argument using the inverse wavenumber  $q = 1/k$

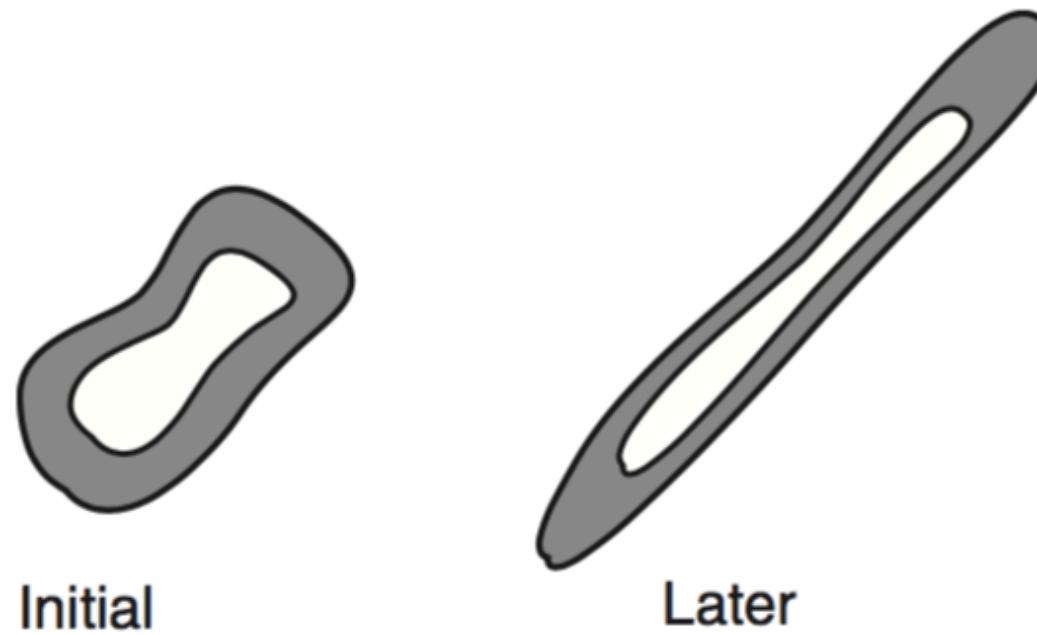
$$J = \int (q - q_e)^2 Z(q) dq, \quad \frac{dJ}{dt} > 0,$$

Which gives

$$\frac{dq_e^2}{dt} = -\frac{1}{\hat{Z}} \frac{dJ}{dt} < 0$$

Thus, the length scale characterizing the enstrophy distribution gets smaller, and the corresponding wavenumber gets larger; the enstrophy spectrum is shifting to the right, toward small scales.

# Triad interaction in 2d turbulence



**Fig. 8.5** In incompressible two-dimensional flow, a band of fluid will generally be elongated, but its area will be preserved. Since vorticity is tied to fluid parcels, the values of the vorticity in the hatched area (and in the hole in the middle) are maintained; thus, vorticity gradients will increase and the enstrophy is thereby, on average, moved to smaller scales.

# Inertial ranges in 2d turbulence

In a forced- dissipative two-dimensional fluid, energy is transferred to larger scales and enstrophy is transferred to small scales.

If we assume that the fluid is forced and that the spectrum is stationary as in the Kolmogorov case in 3-D, we get 2 inertial ranges: an *energy inertial range* carrying energy to larger scales, and an *enstrophy inertial range* carrying enstrophy to small scales

- The energy cascade range is as in the Kolmogorov case. The only difference is the direction of transfer, which is now *upscale*. It is known as the *inverse cascade*.
- Dimensionally, this is exactly the same as in the Kolmogorov case:

$$\mathcal{E}(k) = \mathcal{K}_\varepsilon \varepsilon^{2/3} k^{-5/3},$$

# Inertial ranges in 2d turbulence

- The eddy turnover time is the time taken for a parcel with velocity  $v_k$  to move a distance  $1/k$ ,  $v_k$  being the velocity associated with the (inverse) scale  $k$ . On dimensional consideration:

$$v_k = (\mathcal{E}(k)k)^{1/2}$$

$$\tau_k = (k^3 \mathcal{E}(k))^{-1/2}.$$

- The enstrophy cascades to smaller scales at a rate given by

$$\eta \sim \frac{k^3 \mathcal{E}(k)}{\tau_k}.$$

# Inertial ranges in 2d turbulence

- And the dimensional arguments give:

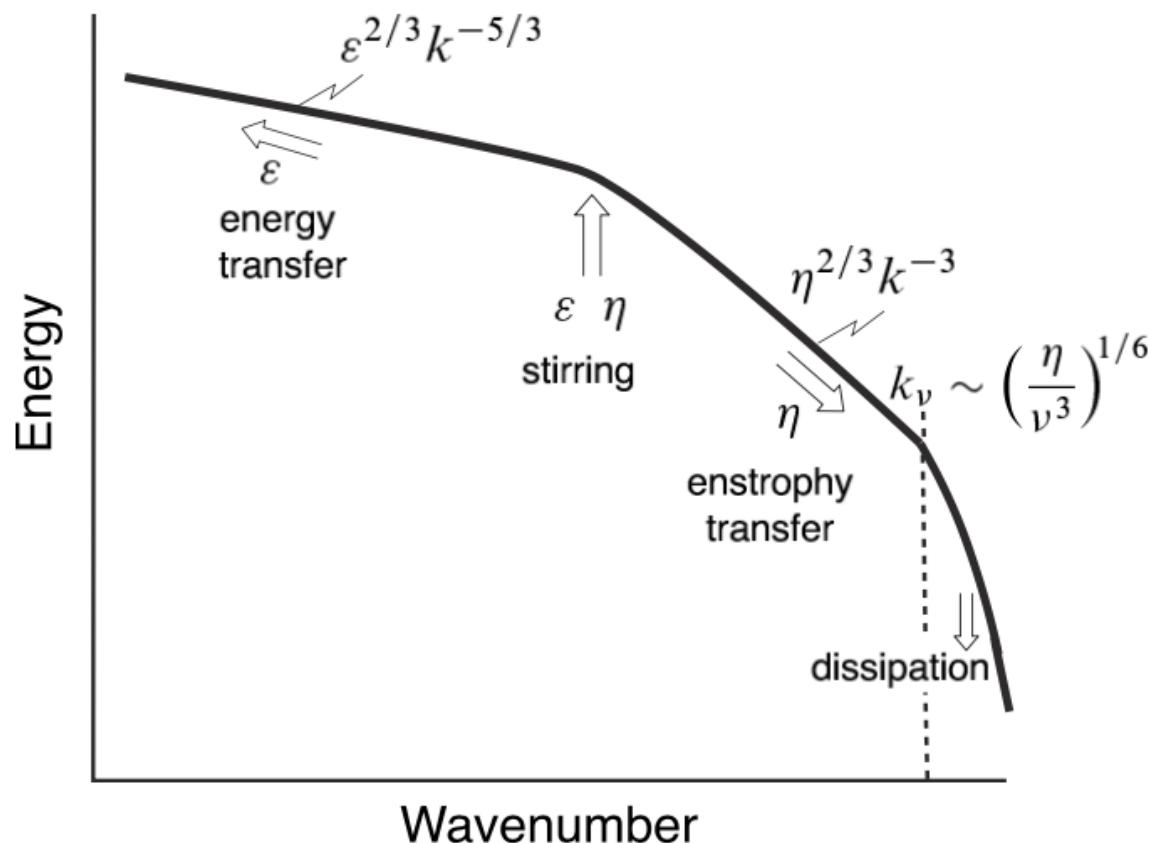
$$\mathcal{E}(k) = \mathcal{K}_\eta \eta^{2/3} k^{-3},$$

- Note that *the eddy turnover time* is:

$$t_k \sim l_k/v_k \sim \eta^{-1/3}$$

Thus, *the eddy turnover time in the enstrophy range of two-dimensional turbulence is length-scale invariant* (no dependance on  $k$  as in 3d turbulence). This time scale is determined by the largest eddies in the cascade range. As such, **the enstrophy cascade is non-local**—the smaller scales are stirred by the eddies at the top of the inertial range, which can be much larger.

# Inertial ranges in 2d turbulence

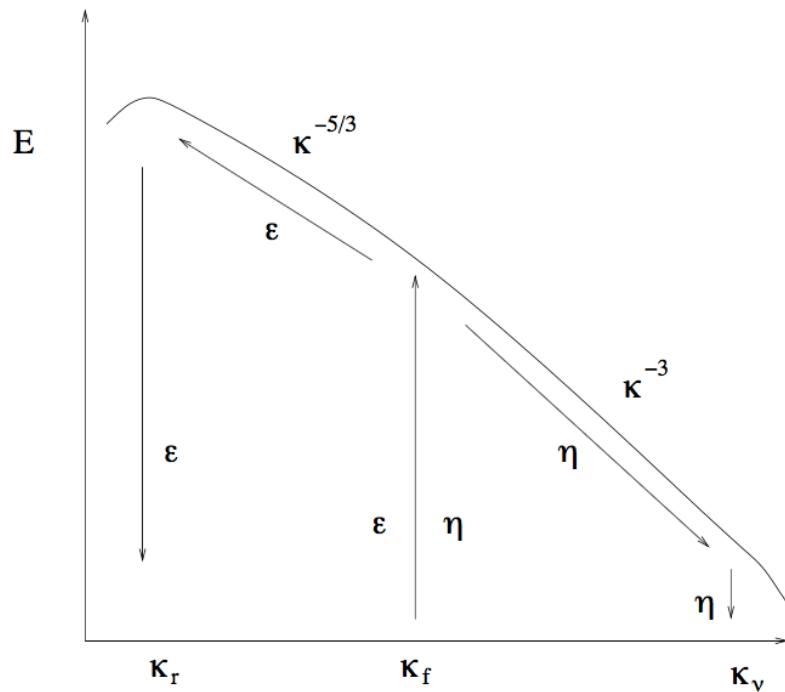


**Figure 8.7** The energy spectrum of two-dimensional turbulence. (Compare with Fig. 8.3.) Energy supplied at some rate  $\varepsilon$  is transferred to large scales, whereas enstrophy supplied at some rate  $\eta$  is transferred to small scales, where it may be dissipated by viscosity. If the forcing is localized at a scale  $k_f^{-1}$  then  $\eta \approx k_f^2 \varepsilon$ .

# Inertial ranges in 2d turbulence

- Energy piles up at large scale and we require dissipation which acts at large scales (e.g. *bottom friction*)

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F - r\zeta + \nu \nabla^2 \zeta.$$

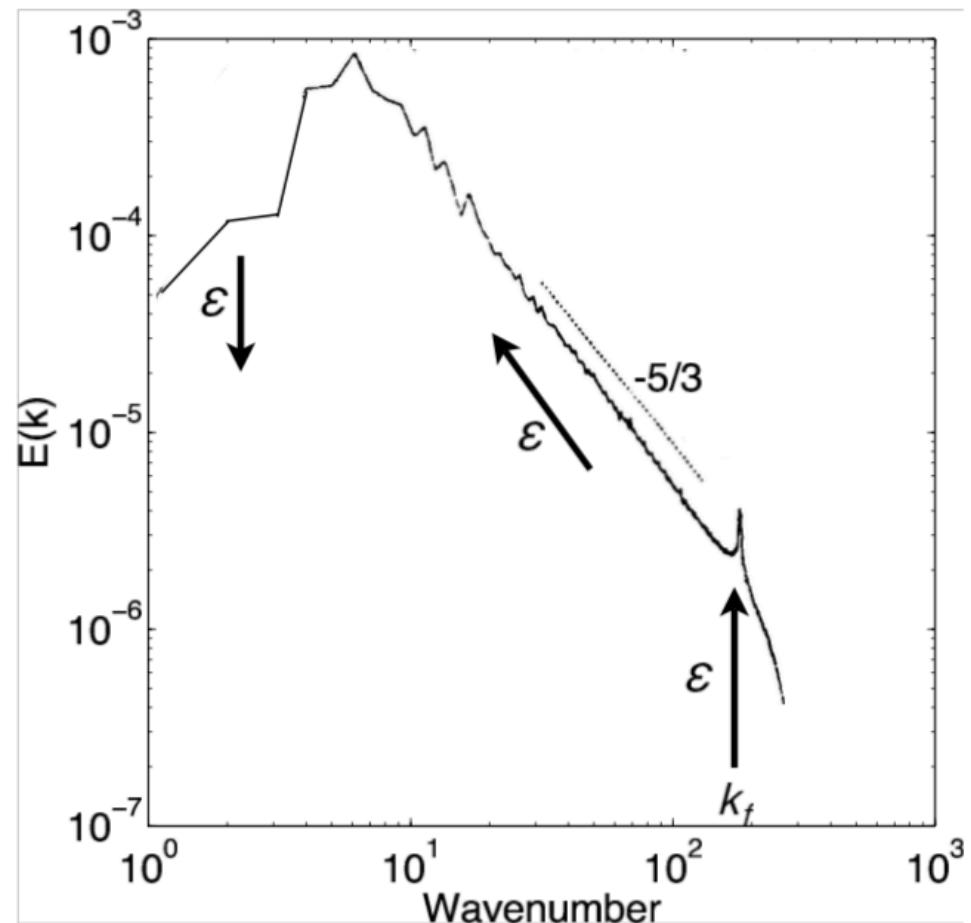


$$\begin{aligned}\frac{d\hat{E}}{dt} &= -2r\hat{E} - \int \psi F dx + \int \nu \zeta^2 dA \approx -2r\hat{E} - \int \psi F dA, \\ \frac{d\hat{Z}}{dt} &= -2r\hat{Z} + \int \xi F dA + D_Z \approx -2r\hat{Z} - k_f^2 \int \psi F dA + D_Z,\end{aligned}$$

$$\kappa_r = \left( \frac{r^3}{\epsilon} \right)^{1/2} \quad \kappa_\nu = \left( \frac{\eta^{1/3}}{\nu} \right)^{1/2}$$

# Inertial ranges in 2d turbulence

**Figure 8.10** The energy spectrum in a numerical simulation of forced-dissipative two-dimensional turbulence. The fluid is stirred at wavenumber  $k_f$  and dissipated at large scales with a linear drag, and there is an  $k^{-5/3}$  spectrum at intermediate scales. The arrows schematically indicate the direction of the energy flow.<sup>12</sup>



# Problems with the theory

[see Davidson (p585)]

- Hypothesis leading to Batchelor's self similar spectrum seen previously is that **eddy interactions are localized** (in Fourier space)
- Instead a  $k^{-3}$  law means that interactions are mostly **non-local**
- Simulations show the existence of **long lived coherent vortices** (long lived = longer than eddy turnover time  $l/u$ )

# Inertial ranges in 2d turbulence

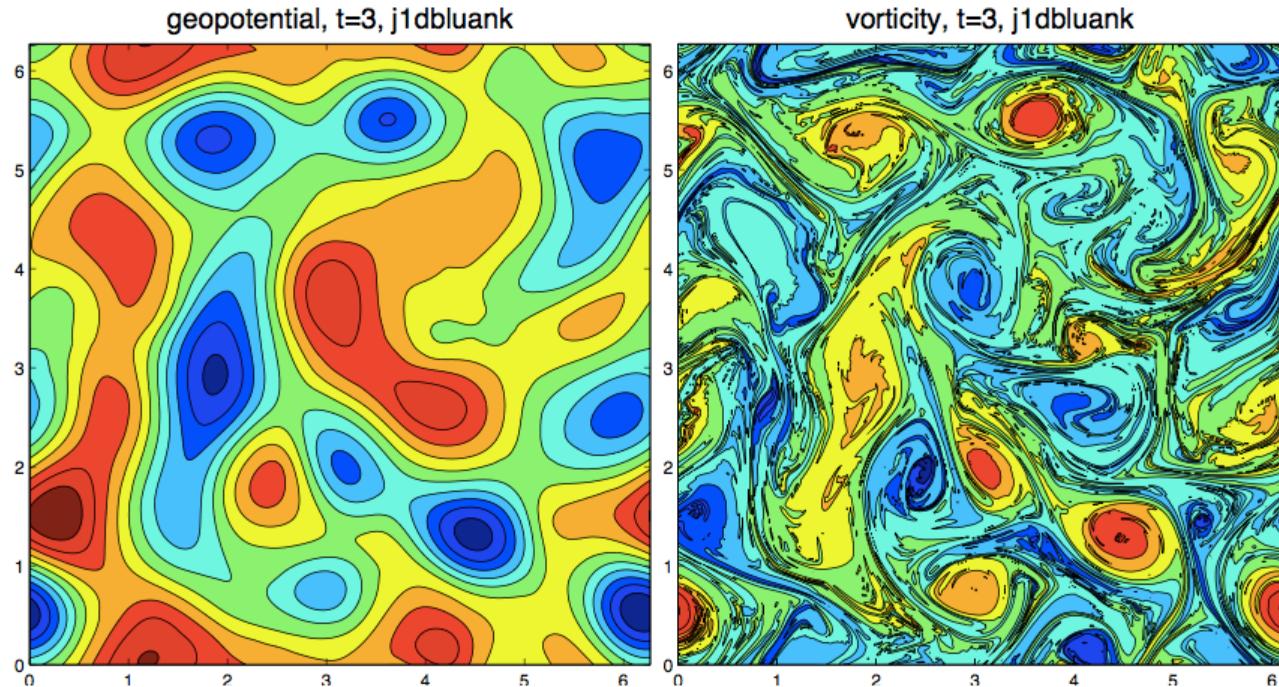
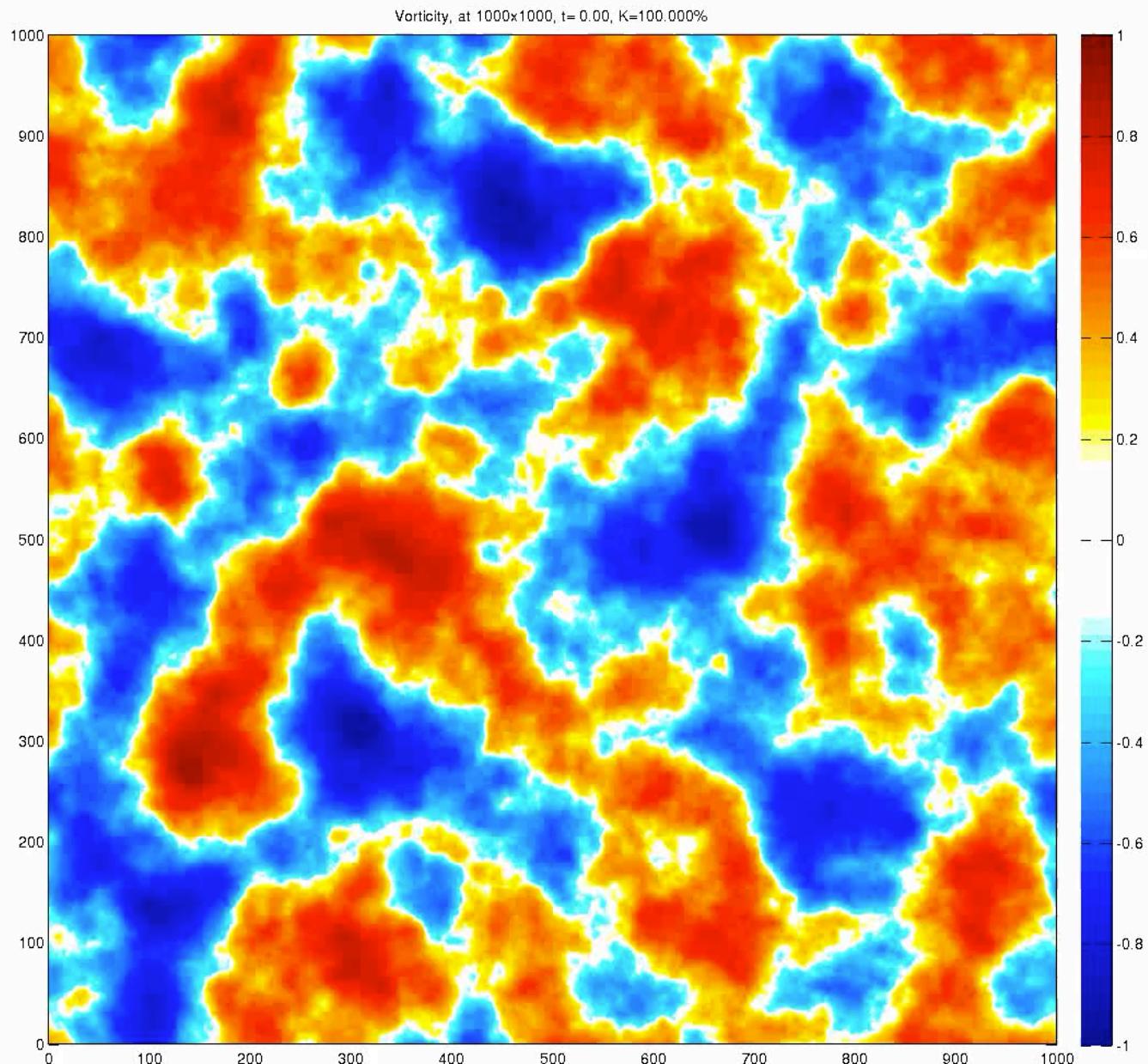


Figure 17: A snapshot of the streamfunction (left) and vorticity (right) from a 2-D turbulence simulation. Note the vorticity has much more small scale structure.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi$$

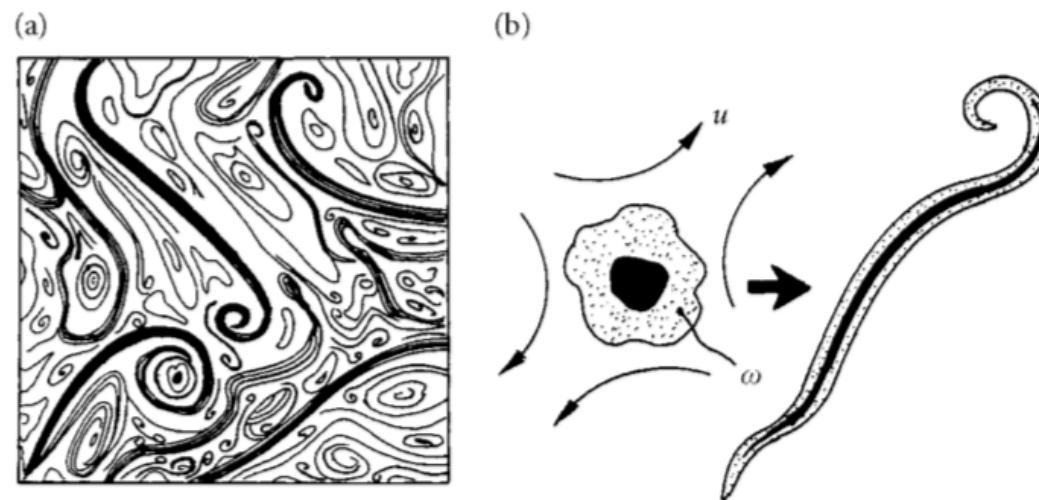
# 2d turbulence



2d turbulence

# Inertial ranges in 2d turbulence

- Enstrophy essentially behaves like a passive tracer (no source of enstrophy):

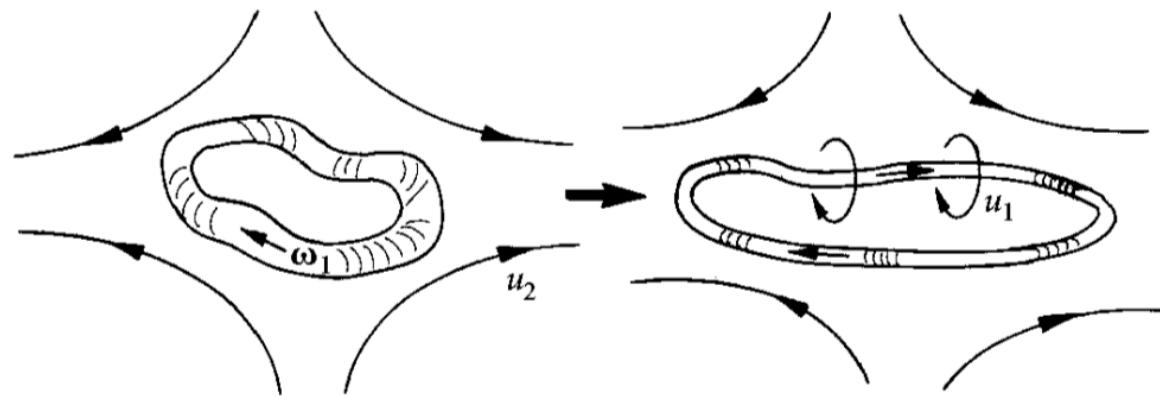


**Filamentation of the vorticity** - controlled by the inviscid, large-scale eddies and halted only when the vortex sheets are thin enough for viscosity to act, destroying enstrophy and diffusing vorticity.

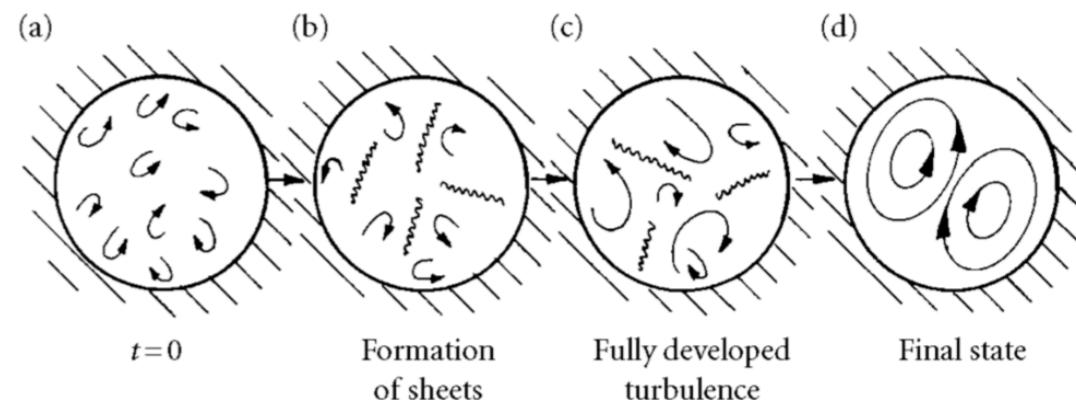
# Inertial ranges in 2d turbulence

- Freely- evolving (unforced) turbulence quickly evolves to a state where the vortices dominate the flow, as the vorticity between vortices is strained out and dissipated.
- Thereafter, the evolution is primarily a process of **mergers between vortices**.
- Positive vortices (cyclones) merge with other cyclones and negative vortices (anticyclones) merge with other anticyclones.
- The merged vortices are larger than the vortices which joined to make them. In this way, energy is shifted toward larger scales—the flow is dominated by fewer, larger vortices.

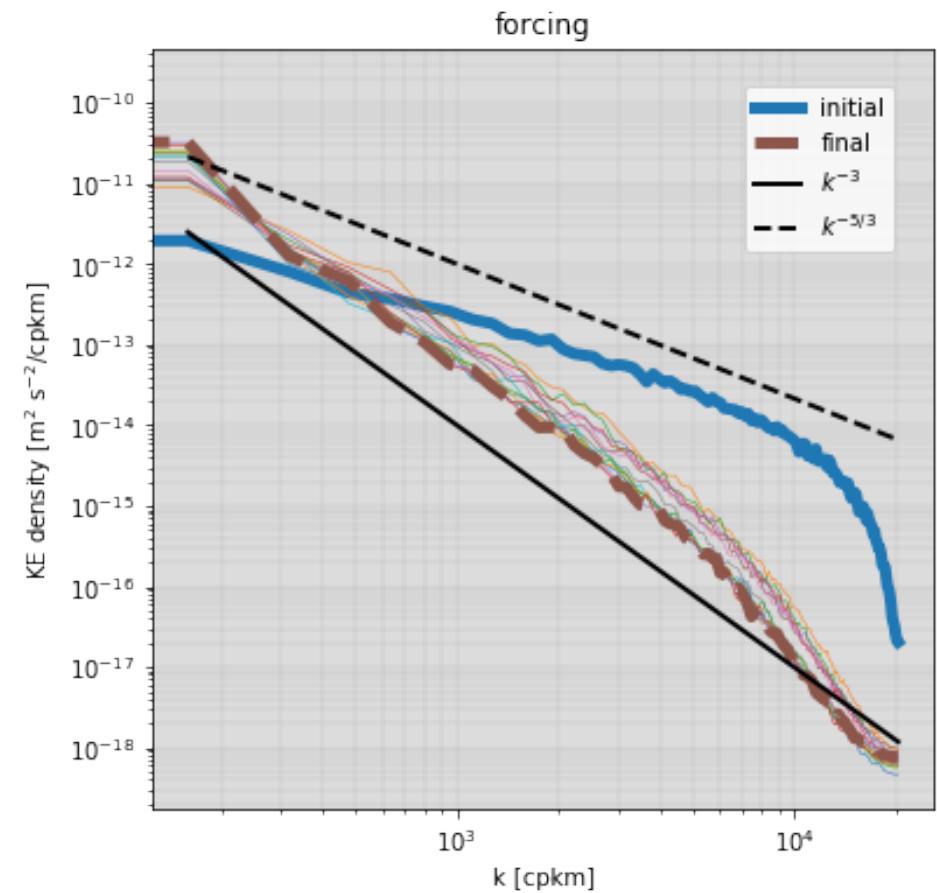
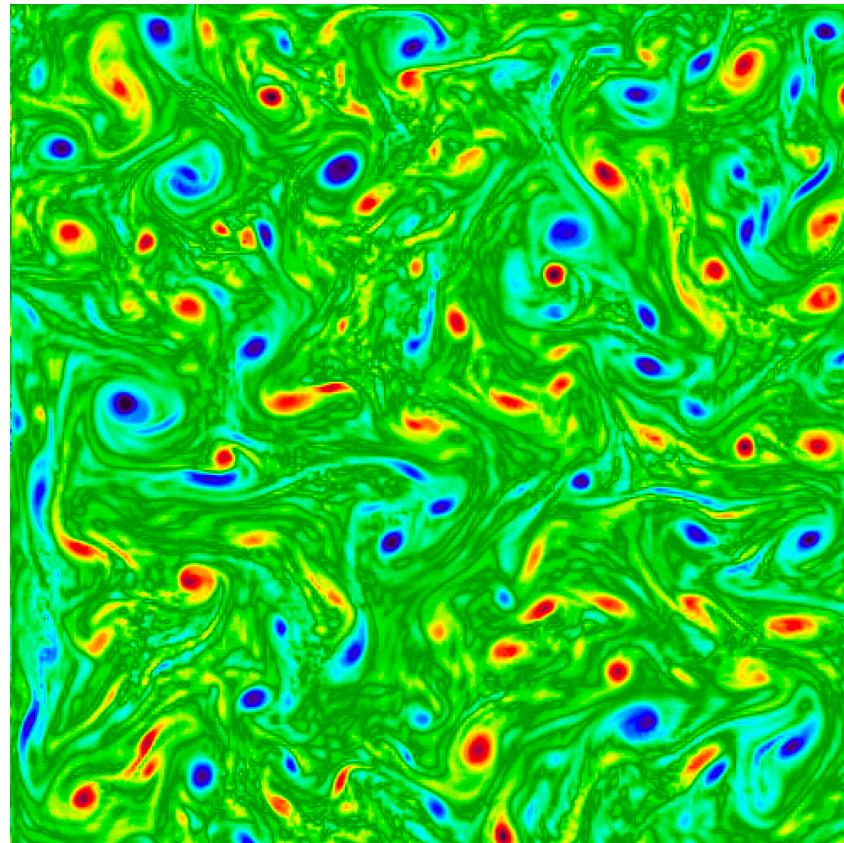
- 3D turbulence = **Vortex Stretching**



- 2D turbulence = **Vortex Merging**



# Activity 3: Numerical simulation of 2d turbulence



# Passive tracer spectra

- What happens for a passive tracer  $C$ :

$$\frac{\partial}{\partial t} C + \vec{u} \cdot \nabla C = \kappa \nabla^2 C$$

- If energy spectrum is  $E(k) = Ak^{-n}$ ,
- The tracer (variance) spectra is:

$$\mathcal{P}(k) = \mathcal{K}_\chi A^{-1/2} \chi k^{(n-5)/2}.$$

# Passive tracer spectra

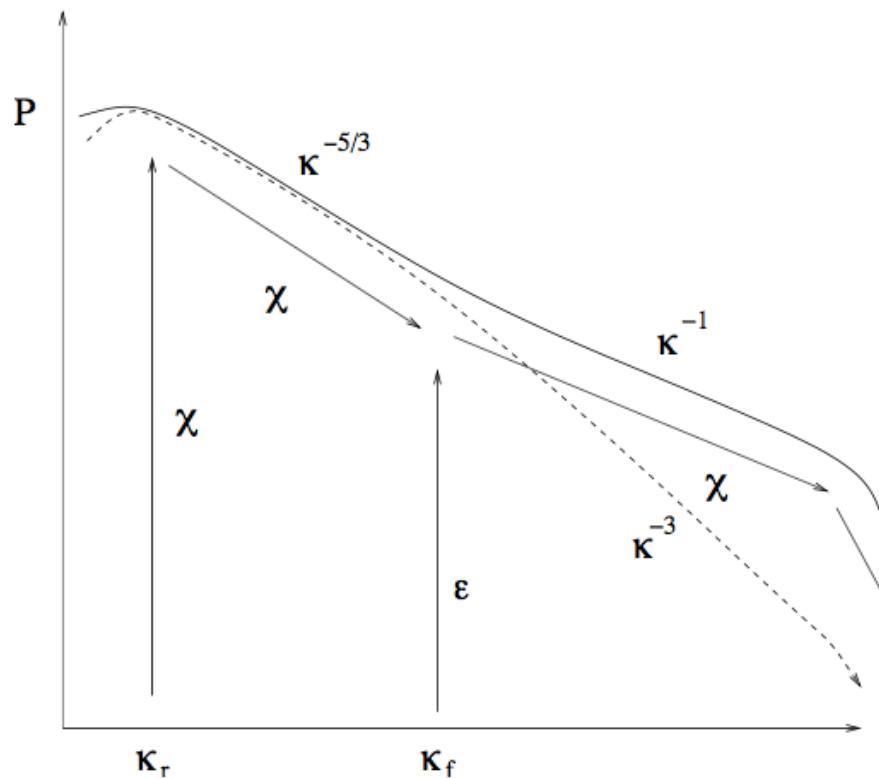


Figure 21: The passive energy spectrum in forced 2-D turbulence. The forcing is applied at  $\kappa_f$ , and the tracer is introduced at large scales, at  $\kappa_\chi$ . Note the tracer variance cascades downscale at all scales.

# Turbulence in the ocean

# Turbulence in the ocean

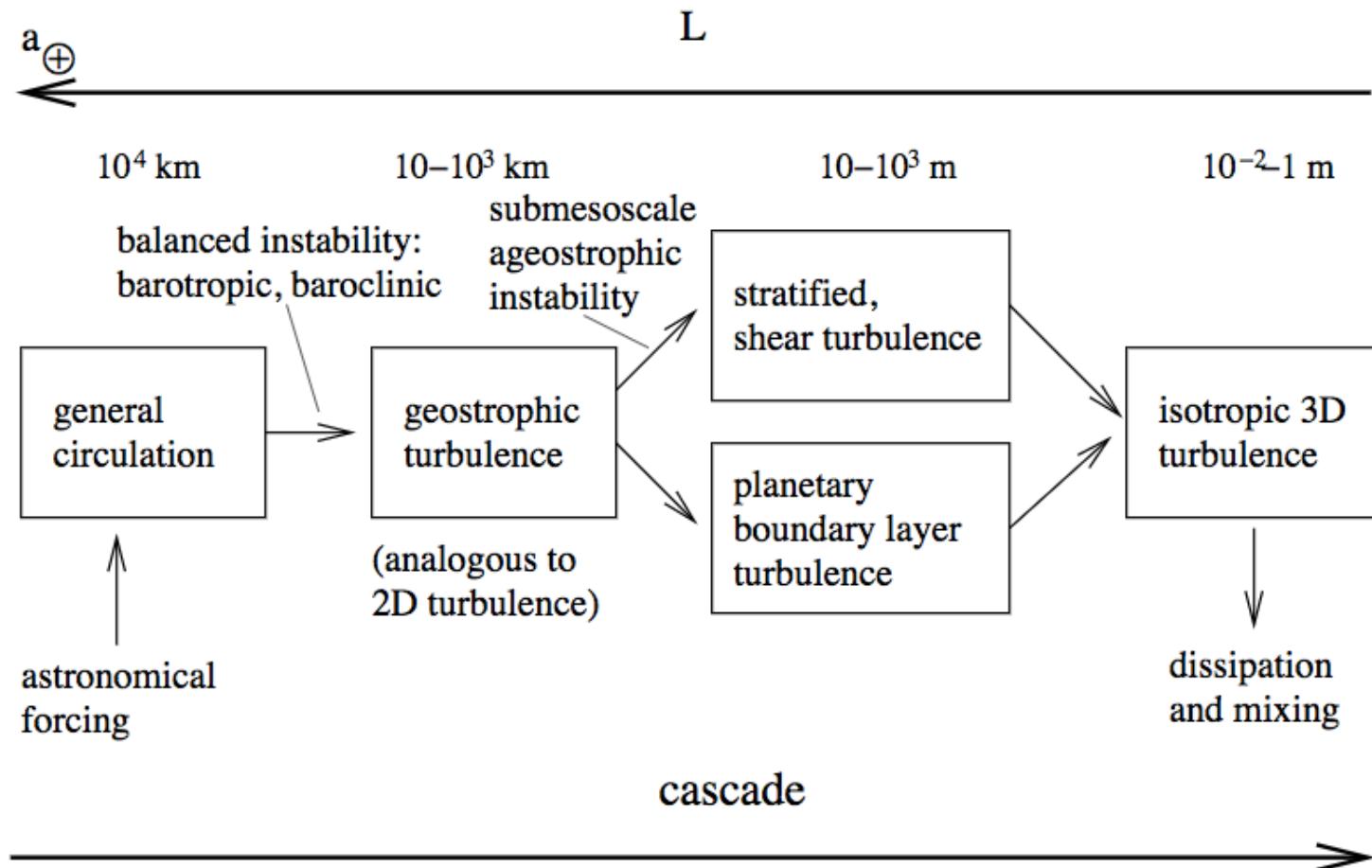


Figure 8: Schematic diagram of the regimes of turbulence in the atmosphere and ocean in a broad sweep of energy from the astronomically-forced planetary scale down to the microscale where mixing and dissipation occur.

# 2d turbulence

$$\mathbf{f} = 0 ; \nu = 0$$

2 inviscid invariants

**Energy**  $E = \frac{1}{2} \langle |\mathbf{u}|^2 \rangle$

**Enstrophy**  $Z = \frac{1}{2} \langle \omega^2 \rangle$

$$\mathbf{f} = 0 ; \nu \neq 0$$

No dissipative anomaly  
for kinetic energy

$$\frac{dE}{dt} = -2\nu Z = -\varepsilon_\nu$$

$$\frac{dZ}{dt} = -2\nu P = -\eta_\nu$$

**Palinstrophy**

$$P = \frac{1}{2} \langle |\nabla \omega|^2 \rangle$$

$$\mathbf{f} \neq 0 ; \nu \neq 0$$

Energy grows  
for  $t < L^2/\nu$

$$\frac{dE}{dt} = \varepsilon_f - \varepsilon_\nu$$

$$\frac{dZ}{dt} = \eta_f - \eta_\nu$$