Master OFFWIND

Coastal Dynamics

Problem 1: Baroclinic Coastal Kelvin Wave (two-layer ocean)

Consider a straight vertical coastline at $x = 0$ with the ocean occupying $x > 0$. The fluid is on an f-plane (constant Coriolis parameter f). The ocean has two homogeneous layers: an upper layer of thickness H_1 and density ρ_1, over a deeper layer of thickness H_2 and density $\rho_2 > \rho_1$. Use linearized two-layer shallow-water theory (no background flow), with alongshore coordinate y (positive north) and offshore coordinate x (positive seaward). Neglect friction.

Let $\eta_1(x, y, t)$ and $\eta_2(x, y, t)$ be the layer-thickness perturbations (or equivalently interface displacement for internal motions). You may use the reduced gravity

$$g' = g\frac{\rho_2 - \rho_1}{\rho_2}. \tag{1}$$

(a) Starting from the linearized two-layer shallow-water momentum and continuity equations, derive the governing equations for small-amplitude internal (baroclinic) motions in the long-wave limit. Show that there exists a trapped coastal Kelvin wave solution in which alongshore velocity and interface displacement propagate alongshore with no cross-shore velocity, and all fields decay exponentially offshore.

(b) Find the dispersion relation for the baroclinic Kelvin wave and show that the alongshore phase speed equals

$$c = \sqrt{g'\,\frac{H_1 H_2}{H_1 + H_2}}.$$

Show that the offshore decay scale is $L_d = c/f$. Write the solution as

$$\eta(x, y, t) = \hat{\eta}\, e^{-x/L_d}\, e^{i(ky-\omega t)}.$$

(c) Explain why baroclinic Kelvin waves are trapped more strongly than barotropic waves. Discuss the role of g' and stratification.

(d) Using $H_1 = 50$ m, $H_2 = 150$ m, $\rho_2 - \rho_1 = 2$ kg/m^3, $\rho_2 = 1025$ kg/m^3, and latitude 45° ($f = 2\Omega \sin\phi$ with $\Omega = 7.2921 \times 10^{-5}$ s^{-1}), compute g', c, and L_d.

Solution: Baroclinic Coastal Kelvin Wave (two-layer ocean)

Setup and notation

We consider linearized dynamics for a two homogeneous layer ocean on an f–plane. Let the upper and lower layer mean thicknesses be H_1 and H_2, densities ρ_1 and ρ_2 ($\rho_2 > \rho_1$). Introduce the reduced gravity

$$g' \equiv g\frac{\rho_2 - \rho_1}{\rho_2}.$$

For the baroclinic (internal) long wave the appropriate equivalent depth is the harmonic combination

$$H_e \equiv \frac{H_1 H_2}{H_1 + H_2}.$$

We denote (u, v) the perturbation velocities in the offshore (x) and alongshore (y) directions and $\eta(x, y, t)$ the interfacial displacement (positive upward). The linearized long-wave (non-rotating) internal wave speed is

$$c = \sqrt{g' H_e}.$$

(a) Governing equations and existence of a trapped Kelvin wave

The linearized two-layer shallow water equations for the internal mode (long-wave approximation) reduce to the reduced-gravity shallow water system for the interface displacement η and baroclinic velocities (u, v):

$$\frac{\partial u}{\partial t} - fv = -g'\frac{\partial \eta}{\partial x}, \tag{2}$$

$$\frac{\partial v}{\partial t} + fu = -g'\frac{\partial \eta}{\partial y}, \tag{3}$$

$$\frac{\partial \eta}{\partial t} + H_e\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{4}$$

Seek a coastal trapped solution for $x \geq 0$ with boundary condition of no normal flow at the wall ($u(x=0)=0$) and decay offshore ($x \to \infty$). For a Kelvin wave we assume

$$u \equiv 0, \qquad \eta(x,y,t) = \hat{\eta}\, e^{-x/L_d} e^{i(ky-\omega t)}, \qquad v(x,y,t) = \hat{v}\, e^{-x/L_d} e^{i(ky-\omega t)},$$

with real alongshore wavenumber k, frequency ω, and offshore e-folding scale $L_d > 0$. Substituting $u = 0$ into radial momentum (2) gives

$$-fv = -g'\frac{\partial \eta}{\partial x} = -g'\left(-\frac{1}{L_d}\right)\eta \quad \Longrightarrow \quad fv = \frac{g'}{L_d}\,\eta. \tag{5}$$

Equation (3) becomes (with $u = 0$)

$$\frac{\partial v}{\partial t} = -g'\frac{\partial \eta}{\partial y} \quad \Longrightarrow \quad (-i\omega)\, v = -g'\,(ik)\,\eta \quad \Longrightarrow \quad \omega v = g'k\eta. \tag{6}$$

From continuity (4) (with $u = 0$ and the exponential ansatz),

$$(-i\omega)\eta + H_e\,(ikv) = 0 \quad \Longrightarrow \quad \omega\eta = H_e k v. \tag{7}$$

Equations (6) and (7) are consistent if the dispersion relation and the coupling between v and η hold (see below). Eliminating v between (5) and (7) will give the decay scale L_d and the phase speed.

Thus a solution with $u = 0$ and fields decaying offshore exists: it is the baroclinic coastal Kelvin wave.

(b) Dispersion relation and offshore decay scale

From continuity (7) we have

$$v = \frac{\omega}{H_e k}\,\eta.$$

Using this expression in (6) yields

$$\omega\left(\frac{\omega}{H_e k}\eta\right) = g'k\eta \quad \Longrightarrow \quad \frac{\omega^2}{H_e k} = g'k \quad \Longrightarrow \quad \omega^2 = g'H_e k^2.$$

Therefore the dispersion relation is

$$\omega = \pm c\,k, \qquad c \equiv \sqrt{g'H_e}.$$

The physical Kelvin wave propagates alongshore in one direction (choose the sign according to coast orientation and sign of f).

Next, use (5) and the expression for v from continuity:

$$fv = \frac{g'}{L_d}\eta \quad \Longrightarrow \quad f\left(\frac{\omega}{H_e k}\eta\right) = \frac{g'}{L_d}\eta.$$

Cancel η and substitute $\omega = ck$:

$$f\frac{ck}{H_e k} = \frac{g'}{L_d} \quad \Longrightarrow \quad f\frac{c}{H_e} = \frac{g'}{L_d}.$$

Use $c^2 = g'H_e$ (hence $g' = c^2/H_e$) to simplify the right-hand side:

$$f\frac{c}{H_e} = \frac{c^2/H_e}{L_d} \quad \Longrightarrow \quad f\frac{c}{H_e} = \frac{c^2}{H_e L_d} \quad \Longrightarrow \quad 1 = \frac{c}{fL_d}.$$

Thus the offshore e-folding decay scale is

$$\boxed{L_d = \frac{c}{f}}.$$

So the full modal solution is

$$\boxed{\eta(x,y,t) = \hat{\eta}\, e^{-x/L_d} e^{i(ky-\omega t)}, \qquad v(x,y,t) = \hat{v}\, e^{-x/L_d} e^{i(ky-\omega t)}, \qquad u \equiv 0,}$$

with dispersion $\omega = ck$ and amplitude relation

$$\hat{v} = \frac{c}{H_e}\,\hat{\eta}.$$

(One may also write $\hat{v} = \dfrac{g'}{fL_d}\,\hat{\eta}$, which is algebraically identical using $L_d = c/f$ and $c^2 = g'H_e$.)

(c) Physical interpretation

Baroclinic Kelvin waves represent internal (interface) motions where the restoring buoyancy force is reduced compared with barotropic surface gravity waves. The internal reduced gravity g' and equivalent depth H_e produce a smaller phase speed $c = \sqrt{g'H_e}$ than the barotropic speed $\sqrt{gH}$ for a full-depth mode; consequently the Rossby radius of the internal mode $L_d = c/f$ is much smaller than the barotropic Rossby radius. That smaller L_d means stronger trapping (exponential decay over a few km rather than tens to hundreds of km). Stratification (via g' and the partitioning of H_1, H_2 in H_e) controls c and thus the trapping scale: weaker stratification (smaller $\rho_2 - \rho_1$) reduces g', reduces c, and produces tighter coastal trapping.

(d) Numerical evaluation

Given

$$H_1 = 50 \text{ m}, \quad H_2 = 150 \text{ m}, \quad \rho_2 - \rho_1 = 2.0 \text{ kg m}^{-3}, \quad \rho_2 \approx 1025 \text{ kg m}^{-3},$$

latitude $\phi = 45°$, and $\Omega = 7.2921 \times 10^{-5} \text{ s}^{-1}$.

(i) Reduced gravity

$$g' = g\frac{\rho_2 - \rho_1}{\rho_2} = 9.81 \, \frac{2.0}{1025} \approx 0.01914146 \text{ m s}^{-2}.$$

(ii) Equivalent depth and internal phase speed

$$H_e = \frac{H_1 H_2}{H_1 + H_2} = \frac{50 \times 150}{50 + 150} = \frac{7500}{200} = 37.5 \text{ m}.$$

$$c = \sqrt{g' H_e} = \sqrt{0.01914146 \times 37.5} \approx 0.84723 \text{ m s}^{-1}.$$

(iii) Coriolis parameter and offshore e-folding scale

$$f = 2\Omega \sin\phi = 2(7.2921 \times 10^{-5}) \sin(45°) \approx 1.03125867 \times 10^{-4} \text{ s}^{-1}.$$

$$L_d = \frac{c}{f} = \frac{0.84723366}{1.03125867 \times 10^{-4}} \approx 8215.53 \text{ m} \approx 8.22 \text{ km}.$$

Problem 2: Tides on an Ocean-Covered Planet

Imagine a spherical planet with radius $r = 4727$ km, mass $M = 2.6792 \times 10^{24}$ kg, entirely covered by ocean. The planet rotates once every 20 hours. A satellite of mass $m = 4.63 \times 10^{21}$ kg is in a circular orbit in the same direction as rotation. The orbital distance is $d = 500{,}000$ km, and its period is 200 hours.

The universal gravitational constant is $G = 6.672 \times 10^{-11}$ N m^{2}kg^{-2}.

1. Show that g at the surface is 8 N/kg.
2. Find the period (h) and wavelength (km) of the tide at the equator.
3. With ocean depth $H = 3200$ m, compute the maximum speed of surface gravity waves.
4. Find the speed at the equator needed to maintain equilibrium tide.
5. What is the lowest latitude for equilibrium tide?
6. If the orbit were elliptical, which tidal properties would change? Why?
7. The planet orbits a star of mass $M_\star = 5 \times 10^{30}$ kg at distance 3.0×10^8 km. Is the stellar tide stronger than the satellite tide?

Solution: Tides on an Ocean-Covered Planet

1) Gravitational acceleration at the surface

The gravitational acceleration at the surface is

$$g = \frac{GM}{r^2}.$$

Substitute the numbers (showing the arithmetic step-by-step):

$$r^2 = (4.727 \times 10^6)^2 = (4.727)^2 \times 10^{12} = 22.349 \ \times 10^{12} = 2.2349 \times 10^{13} \text{ m}^2.$$

Now

$$GM = (6.672 \times 10^{-11})(2.6792 \times 10^{24}) = 6.672 \times 2.6792 \times 10^{13}.$$

Compute the coefficient:

$$6.672 \times 2.6792 = 17.87199984 \approx 17.872.$$

So

$$GM \approx 17.872 \times 10^{13} = 1.7872 \times 10^{14}\ \mathrm{m^3 s^{-2}}.$$

Therefore

$$g = \frac{1.7872 \times 10^{14}}{2.2349 \times 10^{13}} \approx 7.99999964\ \mathrm{m\,s^{-2}} \approx 8.0\ \mathrm{N/kg}.$$

$\boxed{g \approx 8\ \mathrm{N/kg}}$.

2) Period and wavelength of the tide at the equator

The tide forcing seen on the rotating planet is determined by the relative angular speed between the satellite's orbital motion and the planet's rotation. Using periods in hours:

Planet spin frequency (per hour): $1/T_{\mathrm{rot}} = 1/20 = 0.05\ \mathrm{h^{-1}}$.

Satellite orbital frequency: $1/T_{\mathrm{orb}} = 1/200 = 0.005\ \mathrm{h^{-1}}$.

The apparent tidal forcing frequency (in the planet frame) is the absolute difference:

$$\frac{1}{T_{\mathrm{tide}}} = \left|\frac{1}{T_{\mathrm{orb}}} - \frac{1}{T_{\mathrm{rot}}}\right| = |0.005 - 0.05| = 0.045\ \mathrm{h^{-1}}.$$

Thus the tidal period is

$$T_{\mathrm{tide}} = \frac{1}{0.045}\ \mathrm{h} \approx 22.2222222\ \mathrm{h}.$$

Convert to seconds:

$$T_{\mathrm{tide}} = 22.2222222 \times 3600\ \mathrm{s} = 80\,000\ \mathrm{s}.$$

(So $T_{\mathrm{tide}} = 22.2222\ \mathrm{h} = 80{,}000\ \mathrm{s}$.)

The wavelength λ of the tidal wave at the equator (treating it as a long surface gravity wave) is

$$\lambda = c\,T_{\mathrm{tide}},$$

where c is the shallow-water gravity-wave speed $c = \sqrt{gH}$. We compute c in part (3) below. Using the result $c \approx 160$ m/s (see part 3),

$$\lambda = 160\ \mathrm{m\,s^{-1}} \times 80{,}000\ \mathrm{s} = 12{,}800{,}000\ \mathrm{m} = 12{,}800\ \mathrm{km}.$$

$\boxed{T_{\mathrm{tide}} \approx 22.22\ \mathrm{h}, \qquad \lambda \approx 12{,}800\ \mathrm{km}.}$

3) Maximum speed of the surface gravity wave

The maximum (linear) phase speed for a shallow-water surface gravity wave is

$$c = \sqrt{gH}.$$

We computed $g \approx 7.99999964 \ \mathrm{m\,s^{-2}}$ and $H = 3200$ m. Compute:

$$gH \approx 7.99999964 \times 3200 = 25{,}600.0 \ \mathrm{m^2 s^{-2}},$$

hence

$$c = \sqrt{25{,}600.0} \ \mathrm{m\,s^{-1}} = 160.00 \ \mathrm{m\,s^{-1}}.$$

$\boxed{c \approx 160 \ \mathrm{m/s}.}$

4) Speed needed at the equator to maintain the equilibrium tide

The tidal forcing pattern (the tidal bulge) moves at the linear speed

$$U_{\mathrm{eq}} = r\,\Delta\Omega,$$

where $\Delta\Omega = |\Omega_{\mathrm{rot}} - \Omega_{\mathrm{orb}}|$ is the difference between planetary spin angular velocity and satellite orbital angular velocity. Compute angular velocities:

$$\Omega_{\mathrm{rot}} = \frac{2\pi}{T_{\mathrm{rot}}} = \frac{2\pi}{20 \times 3600} \ \mathrm{s^{-1}} = \frac{2\pi}{72{,}000} \ \mathrm{s^{-1}} \approx 8.72664626 \times 10^{-5} \ \mathrm{s^{-1}},$$

$$\Omega_{\mathrm{orb}} = \frac{2\pi}{T_{\mathrm{orb}}} = \frac{2\pi}{200 \times 3600} \ \mathrm{s^{-1}} = \frac{2\pi}{720{,}000} \ \mathrm{s^{-1}} \approx 8.72664626 \times 10^{-6} \ \mathrm{s^{-1}}.$$

Thus

$$\Delta\Omega \approx 8.72664626 \times 10^{-5} - 8.72664626 \times 10^{-6} = 7.853981634 \times 10^{-5} \ \mathrm{s^{-1}}.$$

Now

$$U_{\mathrm{eq}} = r\,\Delta\Omega = (4.727 \times 10^{6} \ \mathrm{m})(7.853981634 \times 10^{-5} \ \mathrm{s^{-1}}) \approx 371.26 \ \mathrm{m\,s^{-1}}.$$

So the tidal pattern moves at about 371 m/s at the equator. To maintain an *equilibrium* (quasi-static) tide, the ocean wave speed c should be comparable to or exceed the forcing speed U_{eq}. Here

$$c \approx 160 \ \mathrm{m/s} \ll U_{\mathrm{eq}} \approx 371 \ \mathrm{m/s},$$

so the river/sea cannot respond in quasi-static equilibrium at the equator: the tidal bulge moves faster than surface gravity waves can adjust.

$$\boxed{U_{\mathrm{eq}} \approx 371 \text{ m/s.}}$$

5) Lowest latitude at which the tide can be in equilibrium

At latitude φ the linear speed of the forcing pattern around a latitude circle is

$$U(\varphi) = r\cos\varphi\,\Delta\Omega.$$

The condition for equilibrium-like behavior is $U(\varphi) \lesssim c$. Solving $U(\varphi) = c$ for φ gives the limiting (lowest) latitude $\varphi_{\min}$ such that for $|\varphi| \geq \varphi_{\min}$ the local forcing speed is $\leq c$:

$$r\cos\varphi_{\min}\,\Delta\Omega = c \quad\Longrightarrow\quad \cos\varphi_{\min} = \frac{c}{r\,\Delta\Omega}.$$

We already have $c \approx 160.00$ m/s and $r\Delta\Omega \approx 371.2577$ m/s. Hence

$$\cos\varphi_{\min} = \frac{160.00}{371.2577} \approx 0.43096747.$$

Thus

$$\varphi_{\min} = \arccos(0.43096747) \approx 1.1255 \text{ rad} \approx 64.47^\circ.$$

So the tide can be (approximately) equilibrium only at sufficiently high latitudes, i.e.

$$\boxed{\varphi_{\min} \approx 64.5^\circ \text{ (from the equator).}}$$

Equivalently, inside the latitude band $|\varphi| < 64.5^\circ$ the tidal forcing moves faster than the gravity-wave speed c, so the ocean cannot maintain an equilibrium tide there.

6) If the satellite's orbit were elliptical, which tide characteristics would change and why?

An elliptical orbit makes the satellite–planet distance $d(t)$ time-dependent. Since tidal forcing scales (to leading order) as $\propto m/d^3$, an eccentric orbit produces:

- **Amplitude modulation:** the tidal amplitude would vary periodically with the orbital phase (stronger near perigee, weaker near apogee).
- **Non-sinusoidal signal and harmonic content:** the time-variation of $d(t)$ introduces harmonics of the orbital frequency into the tidal forcing (so the tide becomes less purely sinusoidal and contains multiple frequency components).
- **Time-varying frequency/phase:** if perigee precesses or if orbital speed varies, the instantaneous forcing frequency as seen on the rotating planet can vary, producing phase modulation of the tide.

- **Possible resonance effects:** if the modulation has components near natural modal frequencies of ocean basins, that could amplify some components intermittently.

In short: *amplitude*, *spectral content* (harmonics), and the *temporal structure* of the tide would be affected.

7) Is the stellar tide stronger than the satellite tide?

The (leading-order) tidal acceleration at the planet surface produced by a distant body of mass M' at distance D scales (up to constant factors and the planet radius r) like

$$a_{\text{tidal}} \propto \frac{M'}{D^3}.$$

We therefore compare

$$\frac{a_{\text{star}}}{a_{\text{sat}}} \approx \frac{M_\star/D_\star^3}{m/d_{\text{sat}}^3} = \frac{M_\star\, d_{\text{sat}}^3}{m\, D_\star^3}.$$

Given $M_\star = 5\times10^{30}$ kg and $D_\star = 3.0\times10^8$ km $= 3.0\times10^{11}$ m, and $m = 4.63\times10^{21}$ kg, $d_{\text{sat}} = 5.00 \times 10^8$ m, compute:

First compute the two cubic distances:

$$d_{\text{sat}}^3 = (5.00 \times 10^8)^3 = 1.25 \times 10^{26} \text{ m}^3,$$

$$D_\star^3 = (3.00 \times 10^{11})^3 = 27.0 \times 10^{33} = 2.70 \times 10^{34} \text{ m}^3.$$

Now compute the ratio:

$$\frac{a_{\text{star}}}{a_{\text{sat}}} \approx \frac{(5.0 \times 10^{30})\,(1.25 \times 10^{26})}{(4.63 \times 10^{21})\,(2.70 \times 10^{34})}.$$

Compute numerator and denominator coefficients:

$$\text{numerator } = 5.0 \times 1.25 = 6.25 \quad \Rightarrow \quad 6.25 \times 10^{56},$$

$$\text{denominator } = 4.63 \times 2.70 \approx 12.501 \quad \Rightarrow \quad 1.2501 \times 10^1 \times 10^{55} = 1.2501 \times 10^{56}.$$

Hence

$$\frac{a_{\text{star}}}{a_{\text{sat}}} \approx \frac{6.25 \times 10^{56}}{1.2501 \times 10^{56}} \approx 5.00.$$

Thus the stellar tide is about a factor of $\sim \mathbf{5}$ stronger than the satellite tide.

So the answer is: $\boxed{\text{Yes: the star's tidal effect is stronger } (\approx 5\times) \text{ than the satellite's.}}$