TURBULENCE

3. GEOSTROPHIC TURBULENCE

- **Lesson 1 :**
  - Introduction
    - *What is turbulence?*
  - Properties of turbulence
    - *Where does it come from?*
    - *What does it do?*
  - The closure problem
  - The Kolmogorov theory

- **Lesson 2 :**
  - 2D turbulence

- **Lesson 3 :**
  - Geostrophic turbulence

References:

- Vallis G.K., Atmospheric and Oceanic Fluid Dynamics.
- MIT online course: *https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-820-turbulence-in-the-ocean-and-atmosphere-spring-2007/lecture-notes/*
- LaCasce J.H., Turbulence in the Atmosphere and Ocean.

Geostrophic turbulence

= turbulence in **stably-stratified flow** that is in **near-geostrophic balance** (name from Charney, 1971)

- The flow in the atmosphere and ocean are affected by planetary rotation, stratification and bottom topography.

- How come energy spectra resemble those in pure 2-D turbulence?

Impact of Rotation

- The vorticity equation in 2D is:

$$\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla(\zeta + f) = \nu \nabla^2 \zeta$$

- What happens if f is constant?

Impact of Rotation

- The vorticity equation in 2D is:

$$\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla(\zeta + f) = \nu \nabla^2 \zeta$$

- *What happens if f is constant?*

Nothing! A constant Coriolis parameter has no effect on 2-D turbulence.

Impact of Rotation

- The vorticity equation in 2D is:

$$\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla(\zeta + f) = \nu\nabla^2\zeta$$

- Let's see what happens when f varies with latitude.

- We use the Beta-approximation: $f = f_0 + \beta y$

Impact of Rotation

- The equation for vorticity is (we neglect viscosity for now):

$$\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla\zeta + \beta v = 0$$

- *The fundamental difference here is that meridional motions can induce changes in the relative vorticity.*

- Activity: Analyze the linear equation

  1. Write the linear equation in terms of a 2D streamfunction ψ
  2. Solve for a wave solution

Impact of Rotation

- The equation can be written: $$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = 0$$

- Using a wave solution: $$\psi = \hat{\psi}e^{ikx+ily-i\omega t}$$

- We get the dispersion relation for Rossby waves:

$$\omega = -\frac{\beta k}{k^2+l^2} \qquad c_x = \frac{\omega}{k} = -\frac{\beta}{k^2+l^2}$$

Impact of Rotation

- If we put back advection in the equation, the question is:

At which scales is the flow is turbulent (dominated by non-linearities) and at which scales is it dominated by linear waves?

- Let's look at the scaling of the different terms:

$$\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla\zeta + \beta v = 0$$

$$\frac{U}{LT} \qquad \frac{U^2}{L^2} \qquad \beta U$$

Impact of Rotation

$$\frac{\partial}{\partial t}\zeta + \vec{u}\cdot\nabla\zeta + \beta v = 0$$

$$\frac{1}{\beta L T} \qquad \frac{U}{\beta L^2} \qquad 1$$

- The advective terms scale as a Rossby number $\dfrac{U}{\beta L^2}$

Impact of Rotation

- So the threshold between non-linear and linear is given by the length scale:

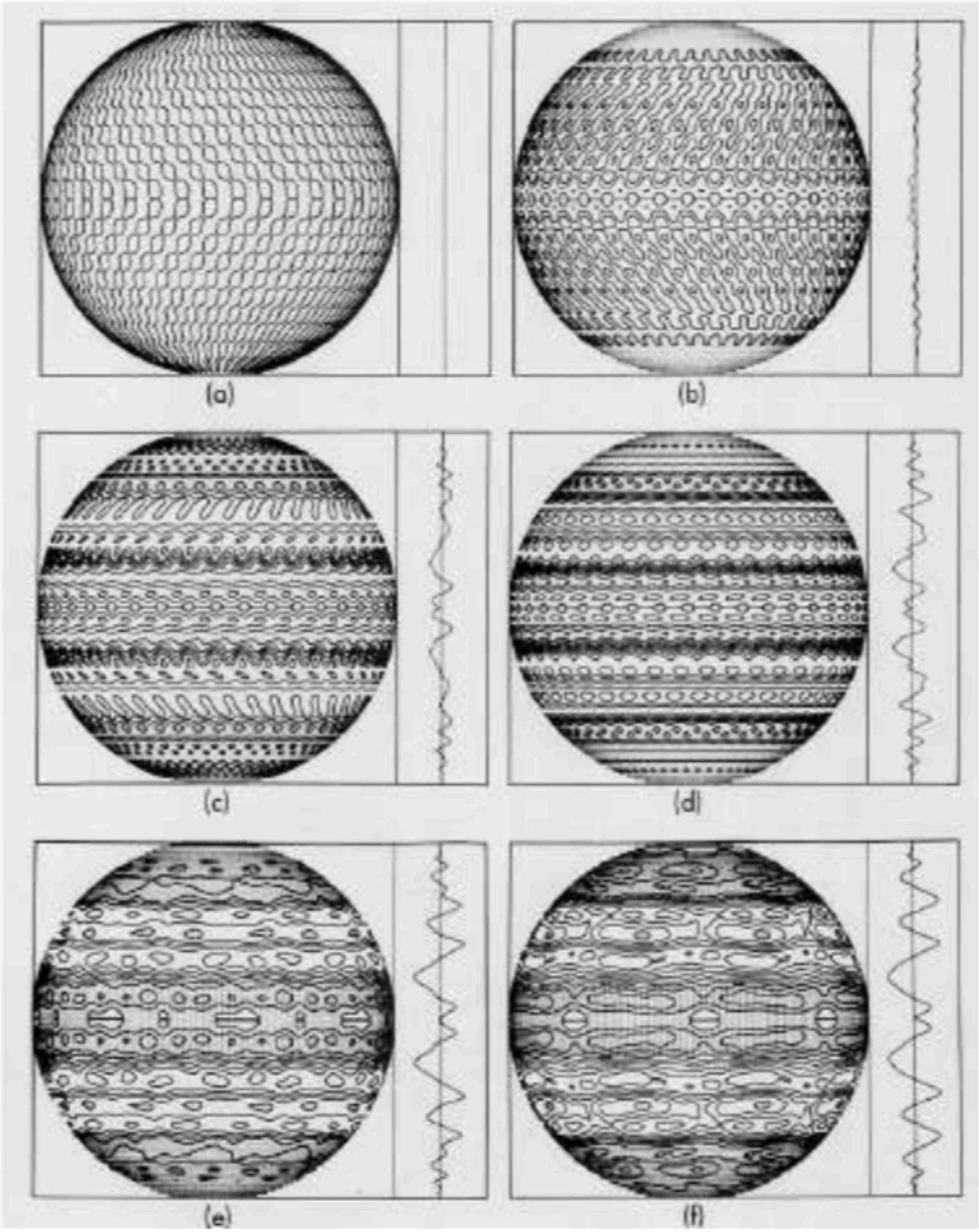
$$L_\beta = \sqrt{\frac{U}{\beta}}$$

which is called **Rhines scale** [Rhines, 1975]

- The inverse cascade of energy will be halted near the Rhines scale due to beta effect. At larger scales linear Rossby waves will dominate the flow.

Simulation of a barotropic fluid on a sphere [Williams, 1978]:

- Arrest of the energy cascade in the y-direction, but not in the x-direction = **anisotropic due to beta**
- Result is a banded structure



Impact of Rotation

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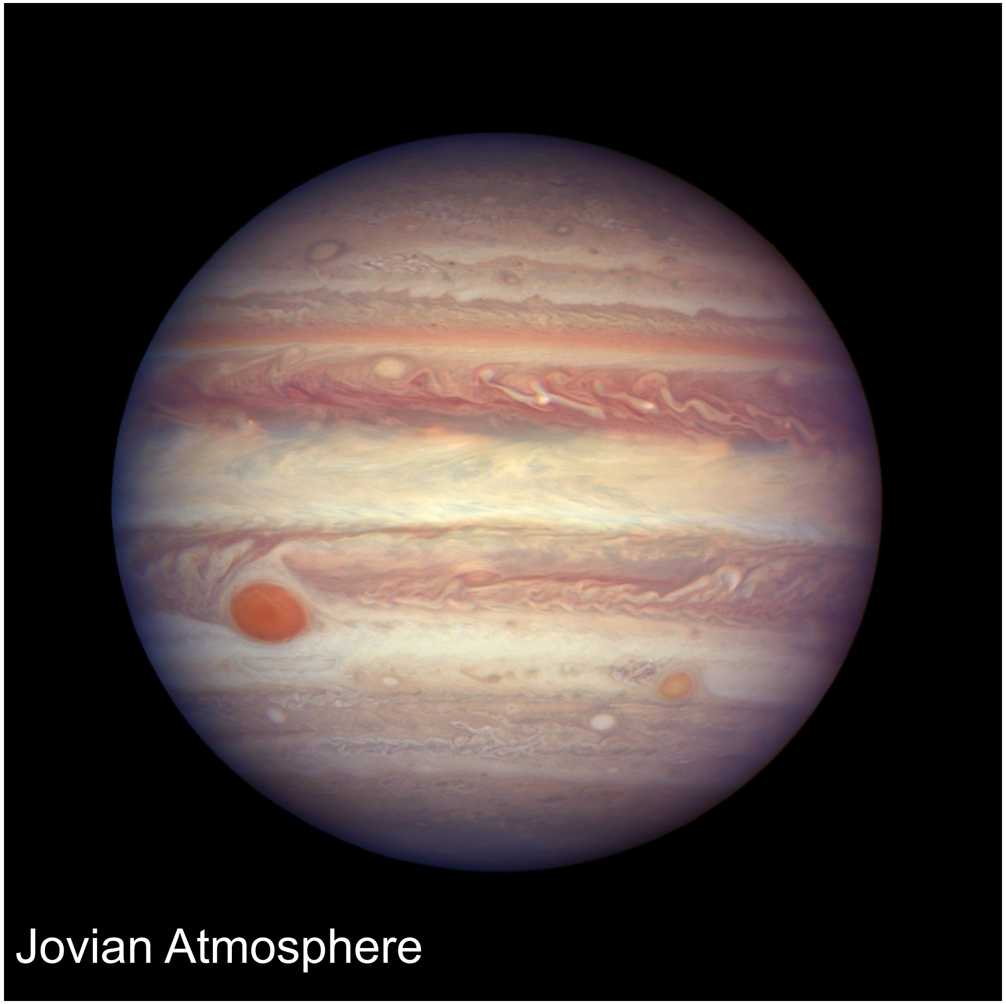


Jovian Atmosphere

Impact of Rotation

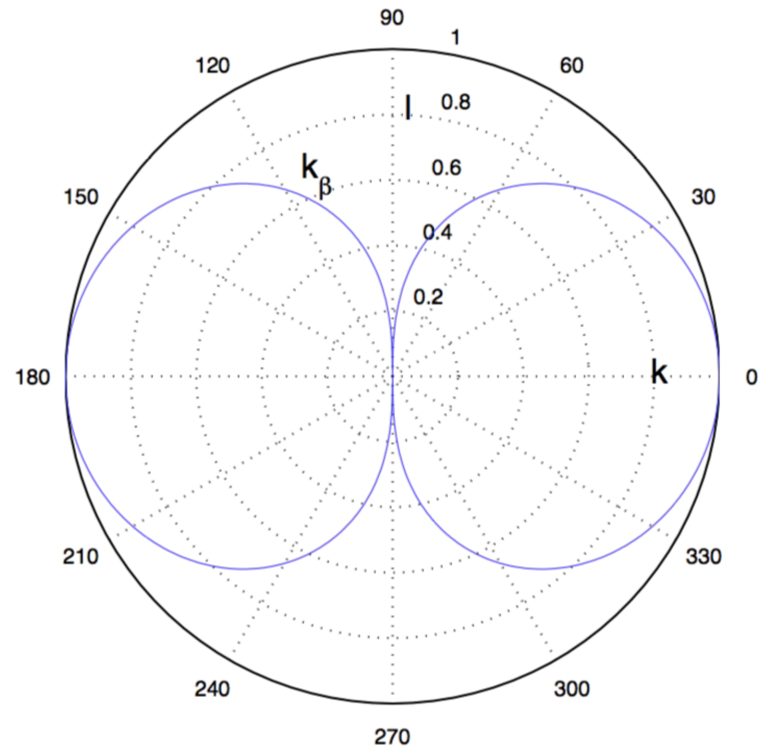
Anisotropy of turbulence:

- We can write the "wave" time scale as: $$\tau_R \propto |\omega^{-1}| = \frac{k^2 + l^2}{\beta k}$$
- We introduce the angle: $$(k, l) = [\kappa cos(\theta), \kappa sin(\theta)]$$
- And the time scale becomes: $$\tau_R = \frac{\kappa^2}{\beta \kappa cos(\theta)} = \frac{\kappa}{\beta cos(\theta)}$$
- The "turbulent" time scale is $$\tau = \epsilon^{-1/3} \kappa^{-2/3}$$
- And the two are equal when: $$\epsilon^{-1/3} \kappa^{-2/3} = \frac{\kappa}{\beta cos(\theta)}$$

Impact of Rotation

Anisotropy of turbulence:

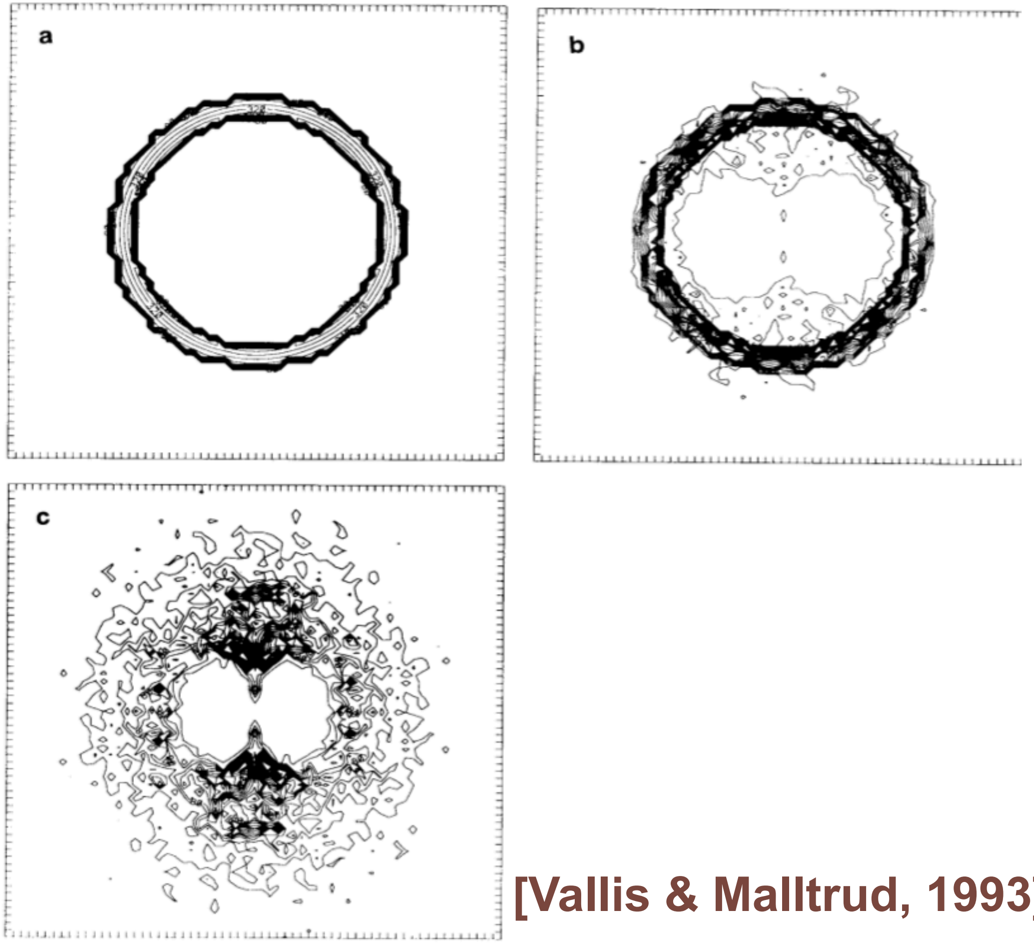
- Which means: $$(k_\beta, l_\beta) = [(\frac{\beta^3}{\epsilon})^{1/5} cos^{8/5}(\theta), (\frac{\beta^3}{\epsilon})^{1/5} cos^{3/5}(\theta) sin(\theta)]$$
- The result is an arrest boundary in *(k,l)* space:



Impact of Rotation

Anisotropy of turbulence:

- Evolution of the energy spectrum in a freely-evolving 2D simulation on the *beta*-plane.



[Vallis & Malltrud, 1993]

Impact of Rotation

What happens in a closed basin?

- Rossby waves become a superposition of a propagating wave and a stationary envelope:

$$\psi = A cos(kx - \omega t) sin(\frac{m\pi x}{L_x}) sin(\frac{n\pi y}{L_y})$$

- The dispersion relation becomes

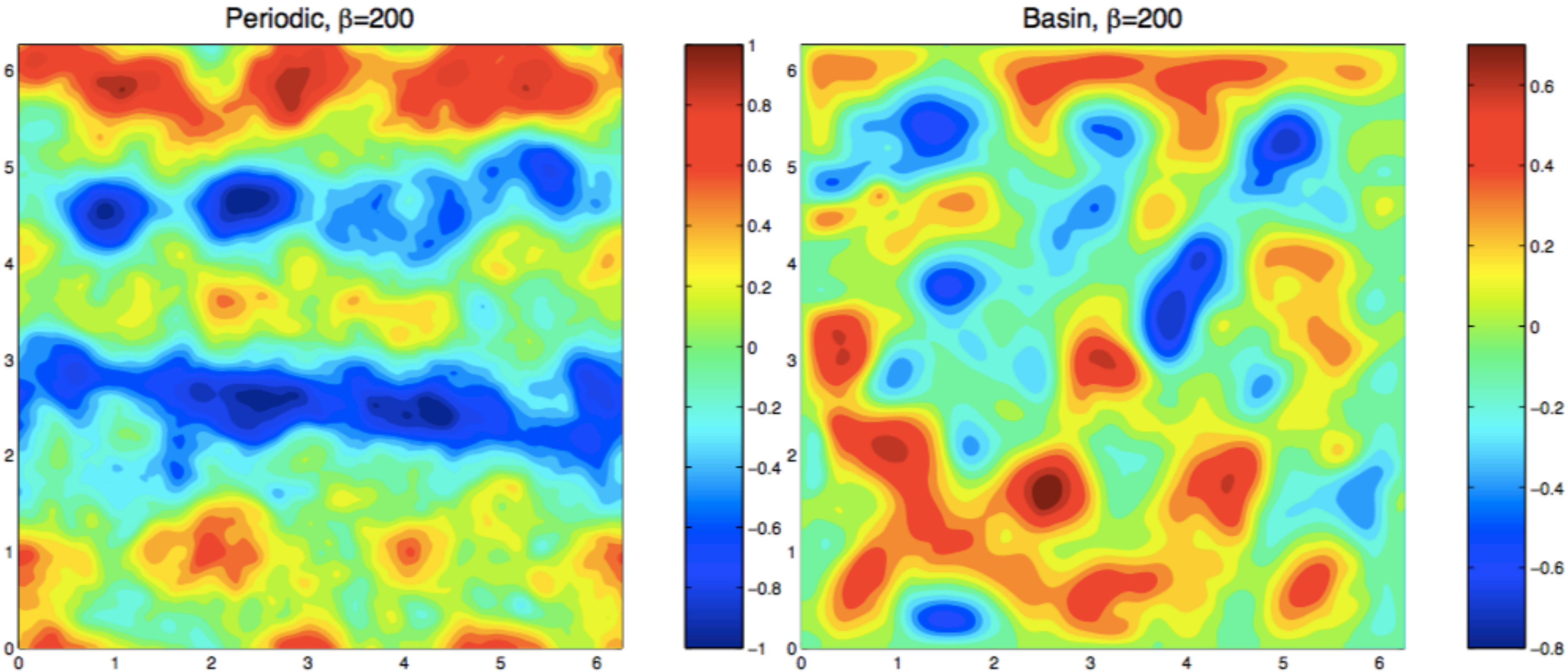
$$\omega = \omega_{mn} = -\frac{\beta}{2\pi(m^2/L_x^2 + n^2/L_y^2)^{1/2}}$$

- And we get an “arrest” scale with no angle dependance!

$$\kappa_\beta = \beta^{3/5}\epsilon^{-1/5}$$

Impact of Rotation

What happens in a closed basin?



Impact of Topography

[see Lacasce, p 83]

- Note that a bottom slope would act like similarly to a beta-effect. But instead of limiting N-S motions, it will *inhibits motion across the depth contours*

= *An inverse cascade would be expected to generate jets over a topographic slope!*

Impact of Topography

- Conservation of PV for a shallow-water model:

$$\frac{d}{dt}[\frac{\zeta + f}{H + \eta}] = 0$$

- Using a QG approximation (small variations of H)

$$q \equiv \frac{\zeta + f}{H} = \frac{\zeta + f_0 + \beta y}{D - h} = \frac{f_0}{D}(\frac{1 + \zeta/f_0 + \beta y/f_0}{1 - h/D})$$

$$q \approx (1 + \frac{\zeta}{f_0} + \frac{\beta y}{f_0} + \frac{h}{D}) \frac{f_0}{D}$$

Impact of Topography

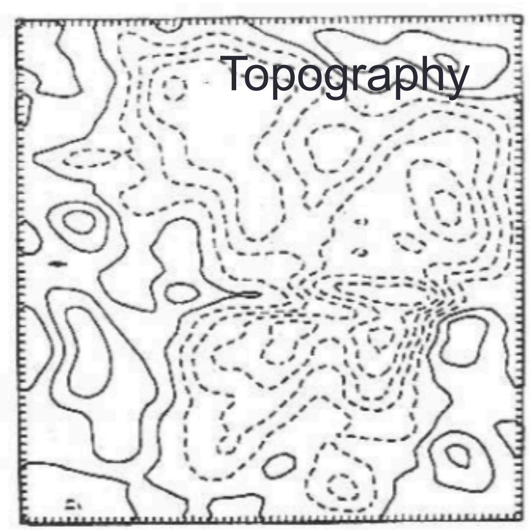
- Finally the equation of vorticity is:

$$\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla(\zeta + \beta y + h) = \nu \nabla^2 \zeta$$

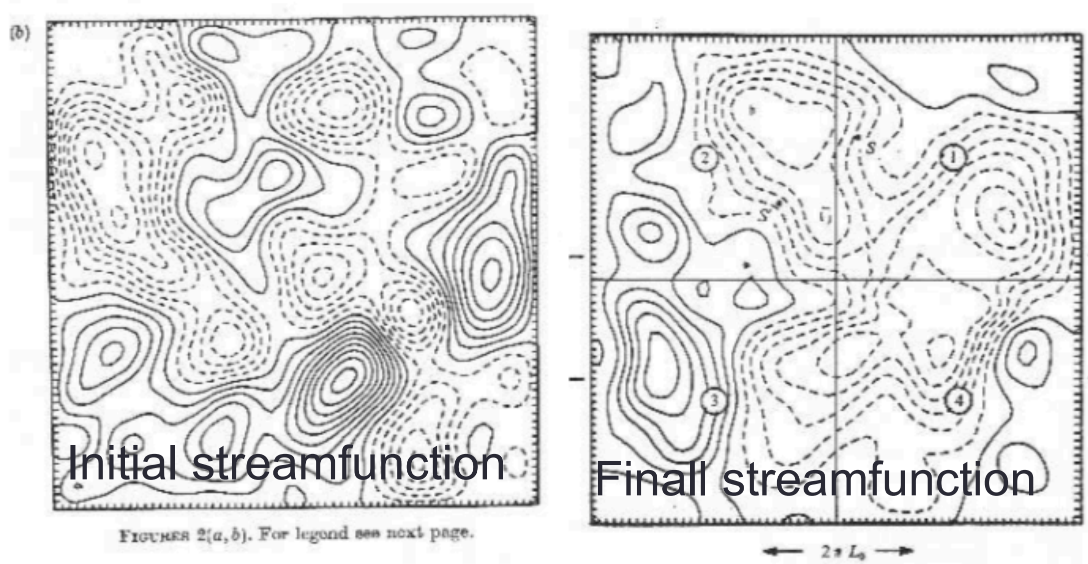
-

Impact of Topography

[see Lacasce, p 83]



[Bretherton & Haidvogel, 1976]



Simulation of Freely-evolving 2D turbulence with topography

= After some time the streamfunction ressembles the topography

Impact of Stratification

- Both atmosphere and ocean are stratified – which allows important aspects of their dynamics such as **baroclinic instability**

- In Barotropic turbulence, triads interactions are limited to interactions between horizontal wavenumbers. When adding a stratification, we will allow interactions between waves with **different vertical structure**.

Impact of Stratification

- Quasi-geostrophic equations with constant Coriolis parameter and constant stratification:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \qquad q = \nabla^2\psi + Pr^2\frac{\partial^2\psi}{\partial z^2},$$

Where Pr is the Prandtl ratio $Pr = f_0/N$

And $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\nabla$

And vertical boundary condition $\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial\psi}{\partial z}\right) = 0,$ at $z = 0, H.$

Impact of Stratification

- We can get equations for energy (integrated over the domain)

$$\frac{\mathrm{d}\hat{E}}{\mathrm{d}t} = 0, \qquad \hat{E} = \int_V \left[(\nabla\psi)^2 + Pr^2 \left(\frac{\partial \psi}{\partial z} \right)^2 \right] \mathrm{d}V,$$

$$\frac{\mathrm{d}\hat{Z}}{\mathrm{d}t} = 0, \qquad \hat{Z} = \int_V q^2 \, \mathrm{d}V = \int_V \left[\nabla^2\psi + Pr^2 \left(\frac{\partial^2 \psi}{\partial z^2} \right) \right]^2 \mathrm{d}V.$$

Impact of Stratification

- If we rescale the vertical coordinate as $z' = z/Pr$.

$$\hat{E} = \int (\nabla_3 \psi)^2 \, \mathrm{d}V, \qquad \hat{Z} = \int q^2 dV = \int (\nabla_3^2 \psi)^2 \, \mathrm{d}V$$

- With a 3D laplacian

$$\nabla_3 = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z'$$

- These 2 invariants imply the same properties than in 2D turbulence:

Energy moves to larger scale and enstrophy to smaller scales.

Impact of Stratification

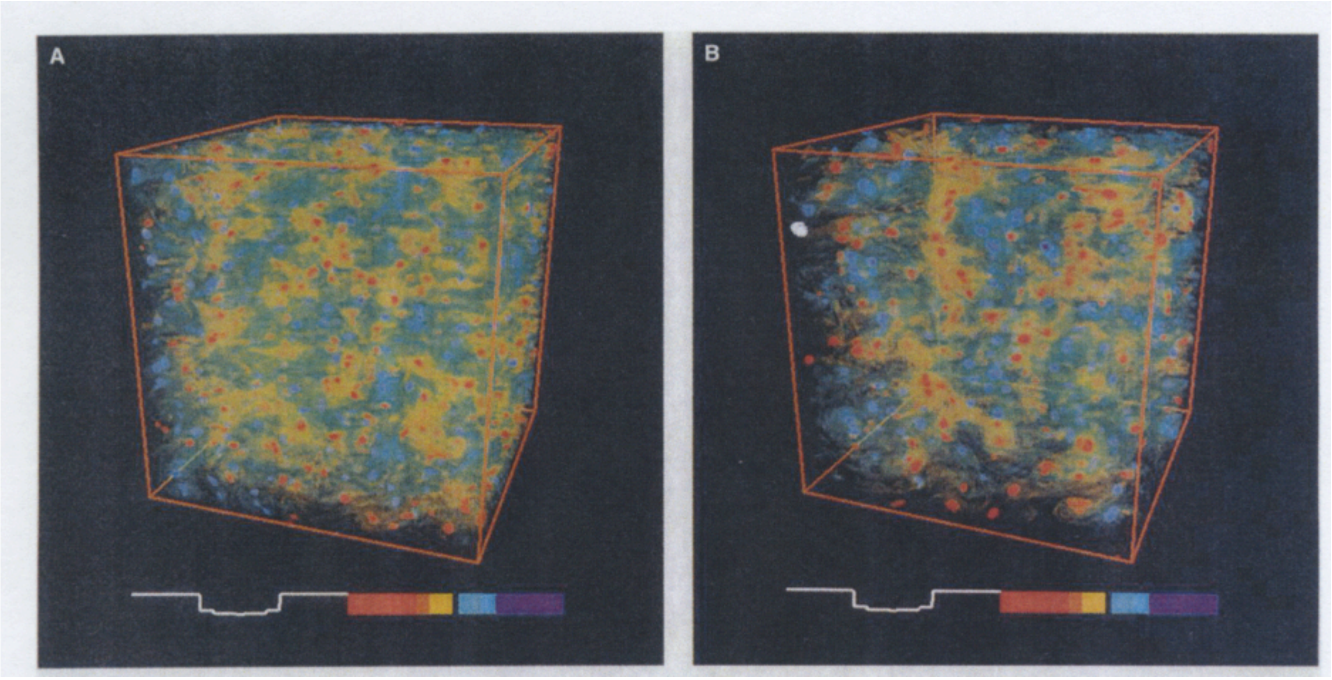
- These 2 invariants imply the same properties than in 2D turbulence: ***Energy moves to larger scale and enstrophy to smaller scales.***

- However, the wavenumber is now a *three-dimensional* wavenumber *(appropriately scaled by the Prandtl ratio in the vertical)*.

- The energy cascade to larger horizontal scales is accompanied by a cascade to larger vertical scales — **a *barotropization* of the flow.**

Impact of Stratification

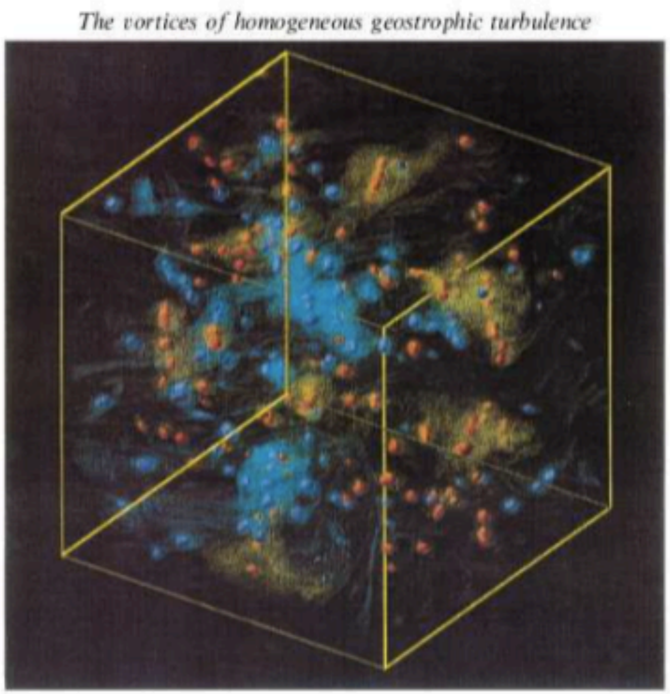
[McWilliams et al., 1999]

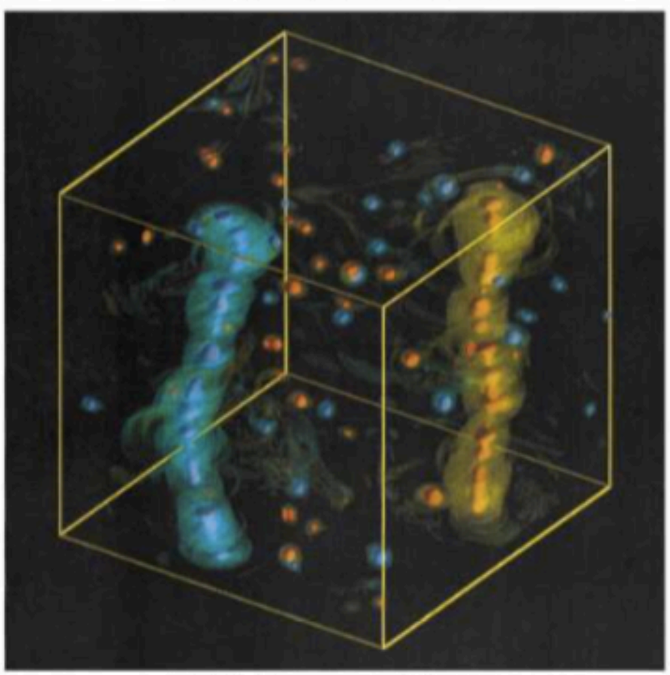


- Potential vorticity from a 3D QG simulation

Impact of Stratification

[McWilliams et al., 1999]





- Potential vorticity from a 3D QG simulation = **vertical alignment of vortex structures**

Impact of Stratification

- To understand the barotropization, we can write:

$$q = \nabla^2 \psi + \frac{f_0^2}{N^2} \frac{\partial^2}{\partial z^2} \psi$$

$$\frac{UL}{L^2} \qquad \frac{f_0^2 UL}{N^2 H^2}$$

$$1 \qquad \frac{f_0^2 L^2}{N^2 H^2}$$

- With $\frac{f_0^2 L^2}{N^2 H^2} = \frac{L^2}{L_d^2}$ and the deformation radius $L_d = \frac{NH}{f_0}$

Impact of Stratification

- If the vortex is larger than L_d, the stretching velocity dominates.
- If the vortex is smaller than L_d, the relative velocity dominates.

- Small vortices will behave just like vortices in 2-D turbulence on each vertical level = Like-sign vortices will merge, making larger vortices.

- As the vortices become larger, the stretching vorticity will be more important. The vortices will have greater vertical extent and will vertically *align* with one another = merging in the vertical.

Impact of Stratification

- *What is the role of baroclinic instability?*

- A simple model to study it is the two-layer geostrophic model [see Vallis, p403]

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0, \qquad i = 1, 2,$$

- Where:

$$q_1 = \nabla^2 \psi_1 + \frac{1}{2} k_d^2 (\psi_2 - \psi_1), \qquad q_2 = \nabla^2 \psi_2 + \frac{1}{2} k_d^2 (\psi_1 - \psi_2),$$

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial a}{\partial x}, \qquad \frac{1}{2} k_d^2 = \frac{2 f_0^2}{g' H} \equiv \frac{4 f_0^2}{N^2 H^2}.$$

Impact of Stratification

- *What is the role of baroclinic instability?*

- You can decompose into barotropic and baroclinic modes:

$$\psi \equiv \frac{1}{2}(\psi_1 + \psi_2), \qquad \tau \equiv \frac{1}{2}(\psi_1 - \psi_2).$$

- And look at possible triad interactions

$$(\psi, \psi) \to \psi, \quad (\tau, \tau) \to \psi, \quad (\psi, \tau) \to \tau.$$

Impact of Stratification

- Energy is:

$$\hat{T} = \int_A (\nabla\psi)^2\, dA, \qquad \frac{\mathrm{d}\hat{T}}{\mathrm{d}t} = \int_A \psi J(\tau, (\nabla^2 - k_\mathrm{d}^2)\tau)\,\mathrm{d}A$$

$$\hat{C} = \int_A [(\nabla\tau)^2 + k_\mathrm{d}^2\tau^2]\,\mathrm{d}A, \qquad \frac{\mathrm{d}\hat{C}}{\mathrm{d}t} = \int_A \tau J(\psi, (\nabla^2 - k_\mathrm{d}^2)\tau)\,\mathrm{d}A.$$

- With corresponding spectra:

$$\hat{T} = \int \mathcal{T}(k)\,\mathrm{d}k \quad \text{and} \quad \hat{C} = \int \mathcal{C}(k)\,\mathrm{d}k,$$

Impact of Stratification

Two types of triads are possible:

- 1. barotropic triads *(same than 2D turbulence)*

$$\text{Energy:} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{T}(k) + \mathcal{T}(p) + \mathcal{T}(q)\Big) = 0,$$

$$\text{Enstrophy:} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big(k^2\mathcal{T}(k) + p^2\mathcal{T}(p) + q^2\mathcal{T}(q)\Big) = 0.$$

- 2. Baroclinic triads (2 baroclinic modes: **p,q** + 1 barotropic mode: **k**)

$$\text{Energy:} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{T}(k) + C(p) + C(q)\right) = 0, \tag{9.38a}$$

$$\text{Enstrophy:} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big(k^2\mathcal{T}(k) + (p^2 + k_\mathrm{d}^2)C(p) + (q^2 + k_\mathrm{d}^2)C(q)\Big) = 0. \tag{9.38b}$$

  - *4 different subcases depending on scales of k,p,q,k_d*

Impact of Stratification

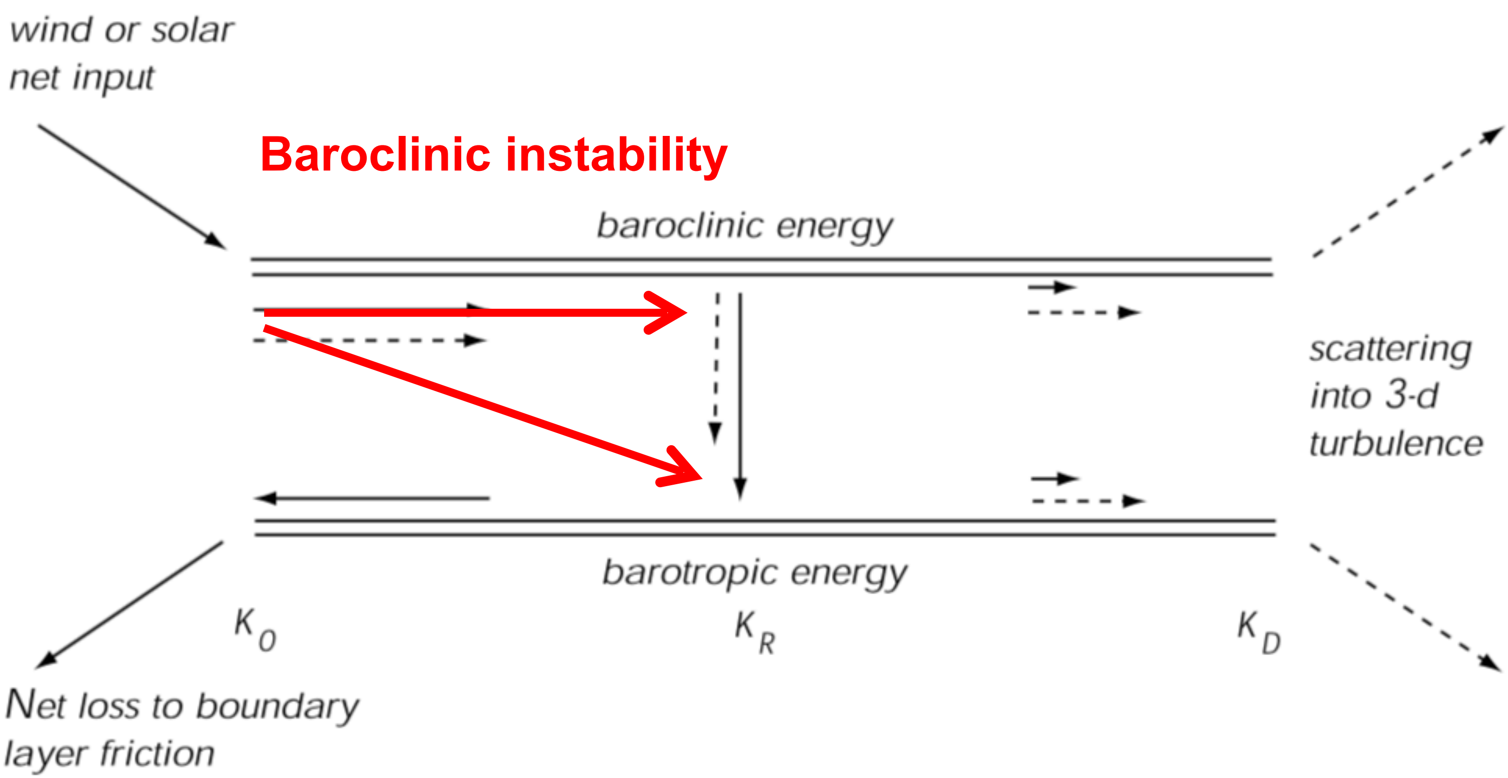
Baroclinic triads:

- A. $(p, q) >> k_d$ = analog to barotropic case
- B. $(k, p, q) << k_d$ = exchange between baroclinic modes only
- C. $(k, p, q) \sim k_d$ = barotropization
- D. $p << (k, q, k_d)$ = Baroclinic instability

The large scale baroclinic mode is the mean vertical shear in classic baroclinic instability model = **Non local interactions**

Impact of Stratification

Idealized two-layer baroclinic turbulence:



TURBULENCE

4. SQG TURBULENCE

Surface Quasi-geostrophic turbulence

= QG turbulence with a few twists!

- Regions close to boundaries, (ocean surface or atmospheric tropopause) behave differently than the interior.
- However QG dynamics is driven by large-scale interior PV contrasts, but not influenced by boundary anomalies:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \qquad q = \nabla^2\psi + Pr^2\frac{\partial^2\psi}{\partial z^2},$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial\psi}{\partial z}\right) = 0, \qquad \text{at } z = 0, H.$$

Surface Quasi-geostrophic turbulence

- SQG dynamics is entirely driven by the density anomaly evolution at the boundary: ***We take QG potential vorticity = zero, but we add a non-trivial surface temperature***

$$\left(\nabla_h^2 + \frac{f^2}{N^2}\partial_{zz}\right)\psi = 0\,, \qquad \partial_z\psi = \theta \text{ at } z = 0\,.$$

- We have 2 quadratic invariants (potential enstrophy is 0)
  - **Energy** (E)
  - **Surface temperature variance** *(equivalent to QG surface available potential energy density)* = *(*$E_{\theta s}$*)*

-

Surface Quasi-geostrophic turbulence

- Inertial ranges studied by Blumen (1978)
- Forward cascade of θ variance (and $E_{\theta s}$) toward small scales ($k^{-5/3}$) + conversion to kinetic energy
- Inverse cascade of kinetic energy ($k^{-5/3}$)
-

Surface Quasi-geostrophic turbulence

- SQG is driven by frontogenesis = source of KE at small scales

[Capet et al., 2008]

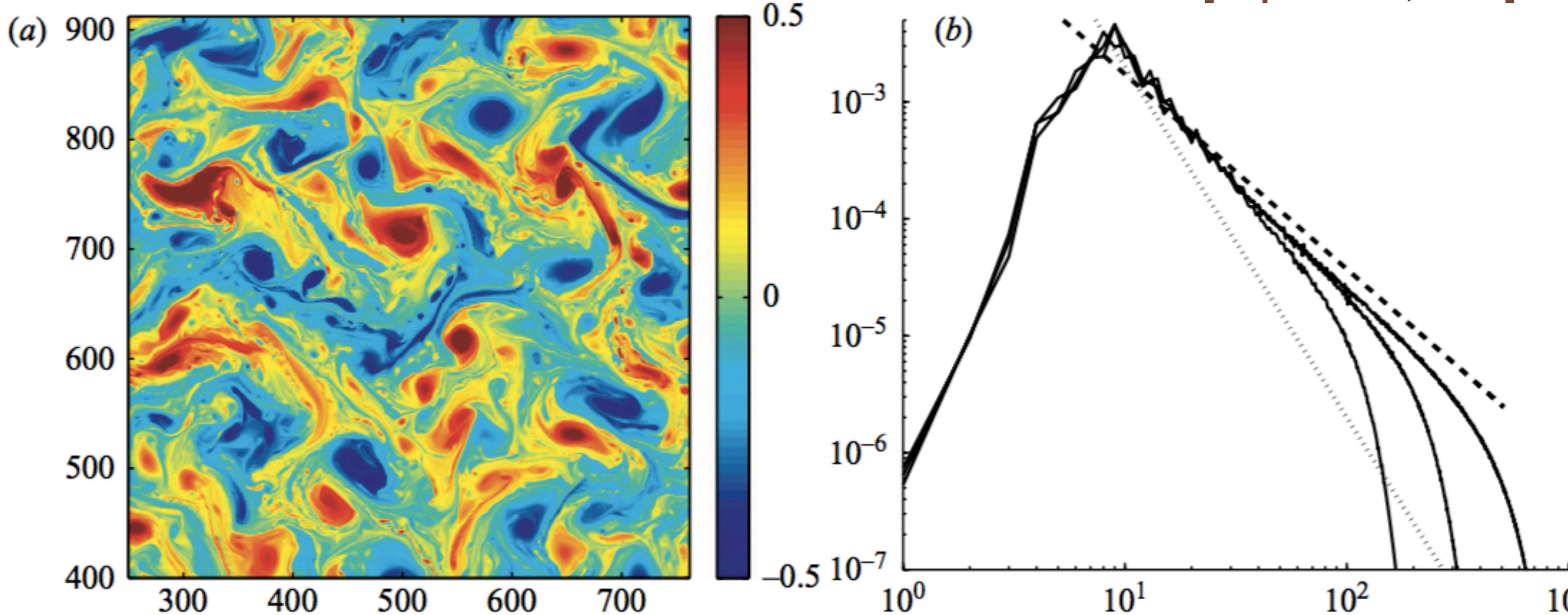


FIGURE 1. (*a*) Snapshot of surface density at $t = 8$ over a 512×512 points subregion for simulation HR. (*b*) LR, MR and HR spectra (continuous with high-wavenumber variance increasing with resolution) of surface density variance (i.e. also surface KE) averaged over the time interval [6.4 11.2]. The dashed line (resp. dotted) represents a $-5/3$ (resp. -3) slope.

Surface Quasi-geostrophic turbulence

- SQG is driven by frontogenesis = source of KE at small scales

[Capet et al., 2008]

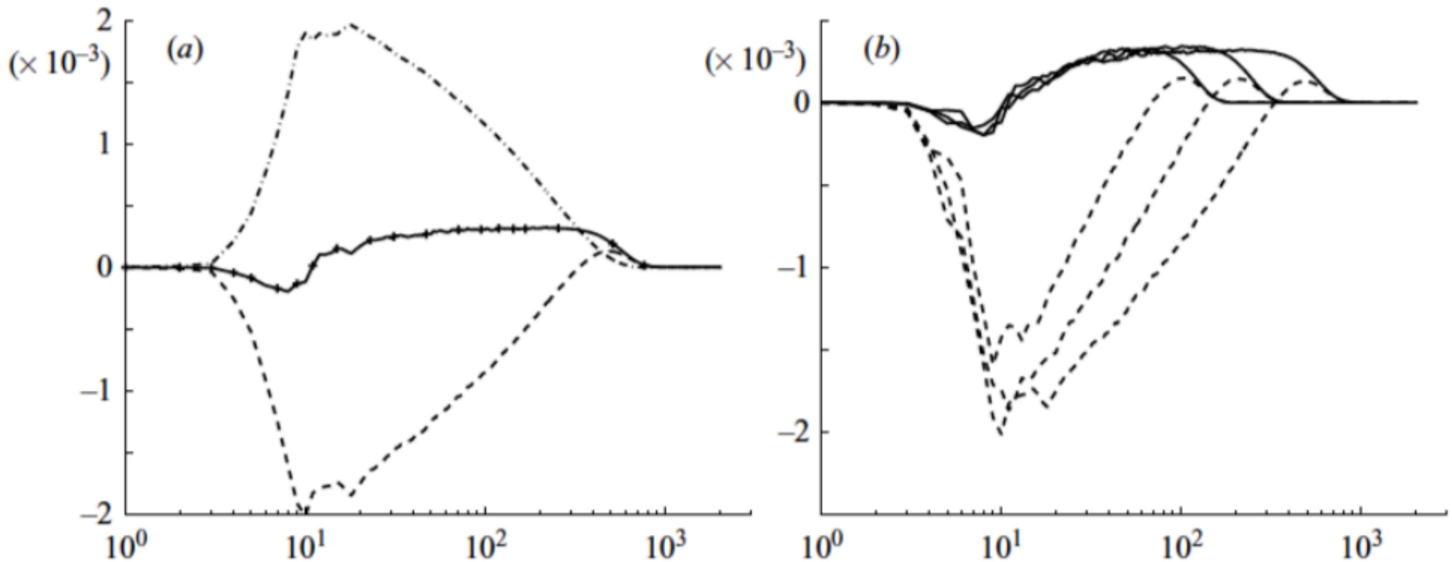


Figure 18: (a) Spectrum transfer functions Π in a freely decaying solution for SQG in a horizontally periodic domain with $N = 1024$ grid points in each direction: for surface temperature variance (Π_θ, solid line), surface kinetic energy due to horizontal geostrophic advection (Π_u, dashed line) and to horizontal ageostrophic advection (Π_a, dot-dashed line). (b) Π_u and Π_ρ for three values of $N = 256, 512, 1024$. (Capet *et al.*, 2008a)

Turbulence in the ocean

- 2D, QG, SQG turbulence all predict inverse kinetic energy cascades… ***So how is energy able to escape to reach smaller scales (and 3d turbulence regimes)?***
- Energy will forward cascade into **non-geostrophic types of turbulence** due to frontogenesis (which reach high Ro number at small scales), different type of submesoscale ageostrophic instabilities, spontaneous wave emissions, interactions with NIW, etc.

Turbulence in the ocean

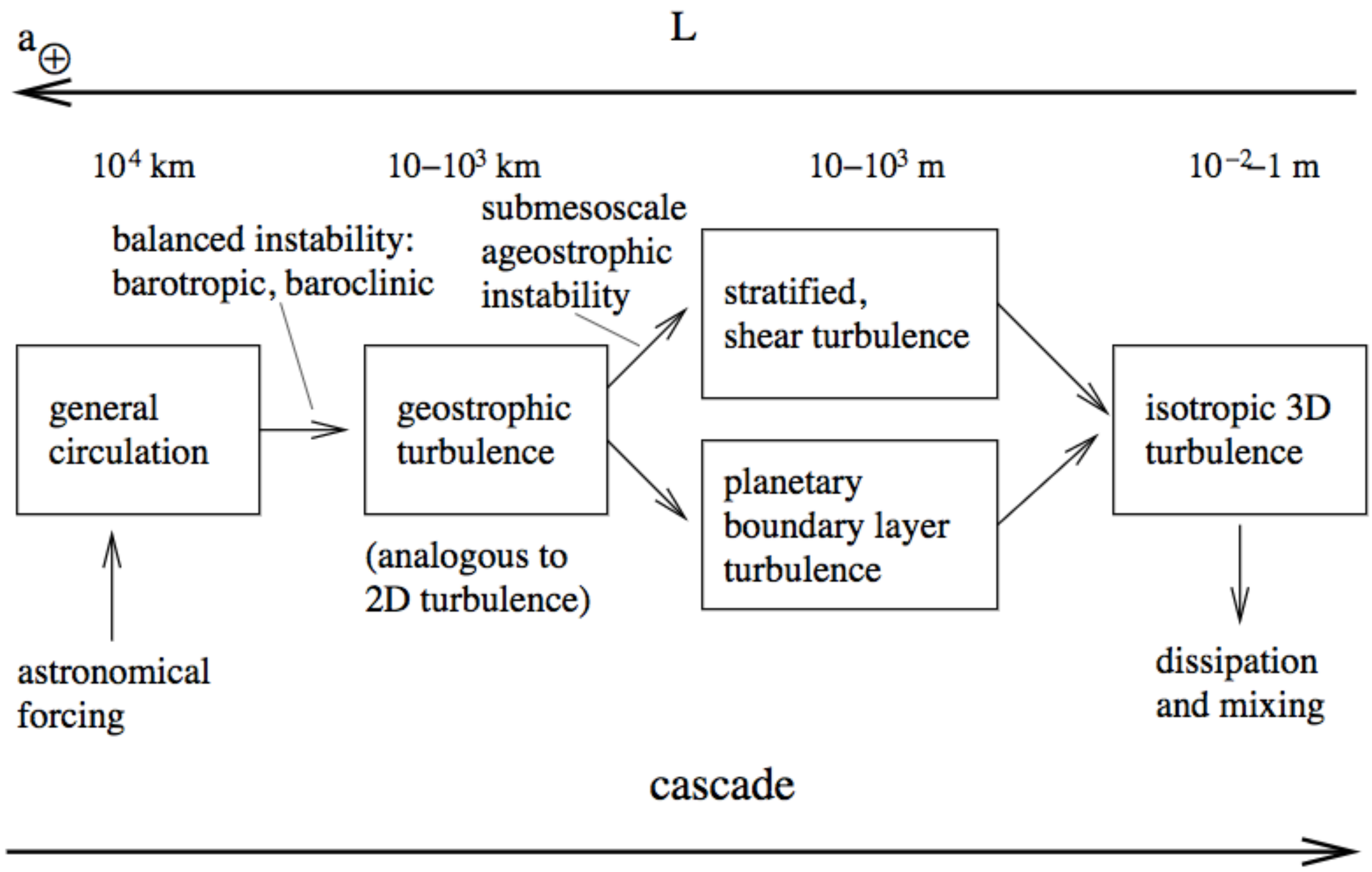


Figure 8: Schematic diagram of the regimes of turbulence in the atmosphere and ocean in a broad sweep of energy from the astronomically-forced planetary scale down to the microscale where mixing and dissipation occur.

TURBULENCE

5. REALISTIC TURBULENCE

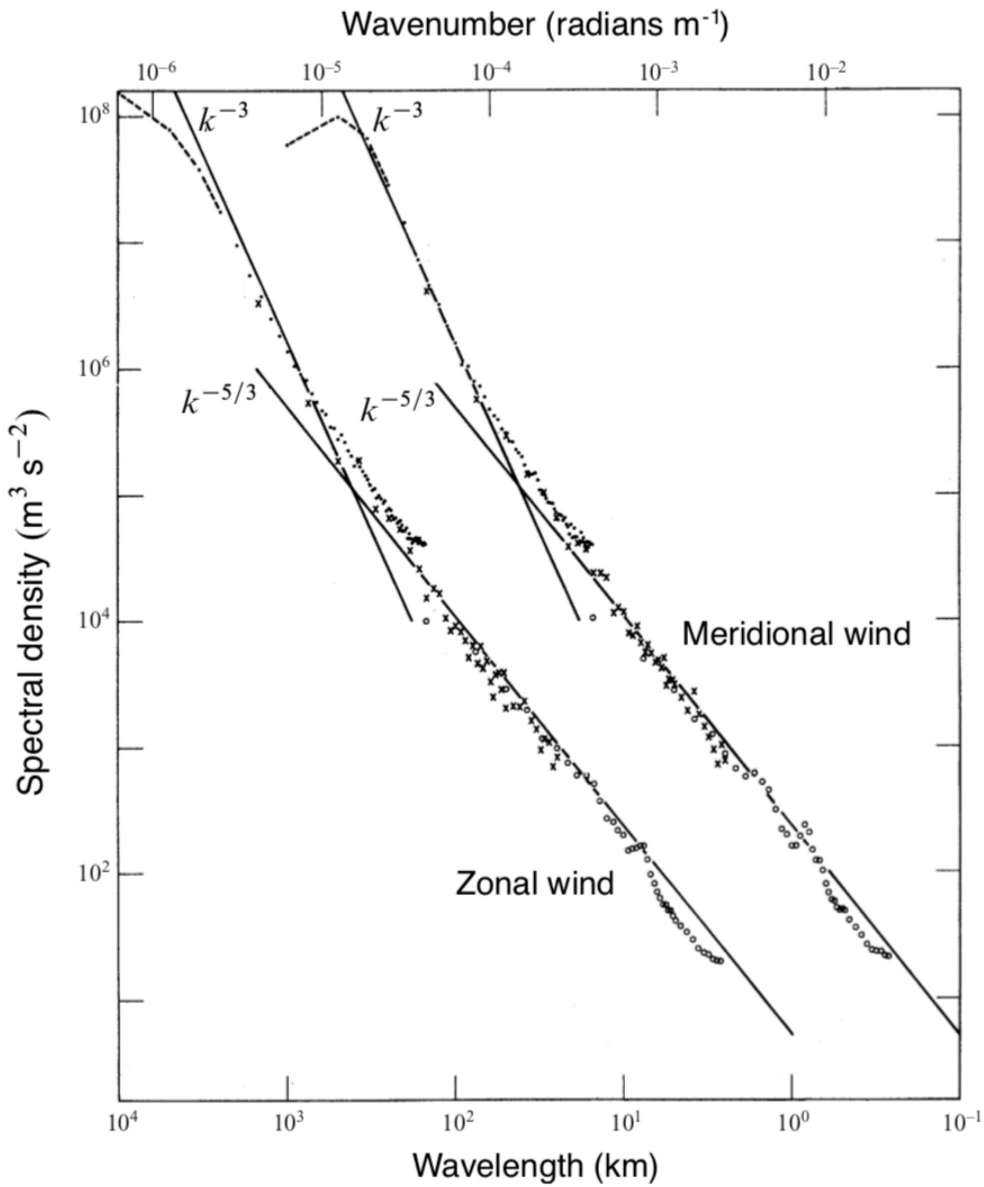
Atmosphere

Energy spectra of the zonal and meridional wind near the tropopause, from thousands of commercial aircraft measurements between 1975 and 1979. (from Gage and Nastrom 1986)

= direct enstrophy cascade (k^{-3}) between 2000 and 100 km

consistent with theory of QG turbulence.

The $k^{-5/3}$ nature is still debated (3d turbulence, inverse 2d cascade?)

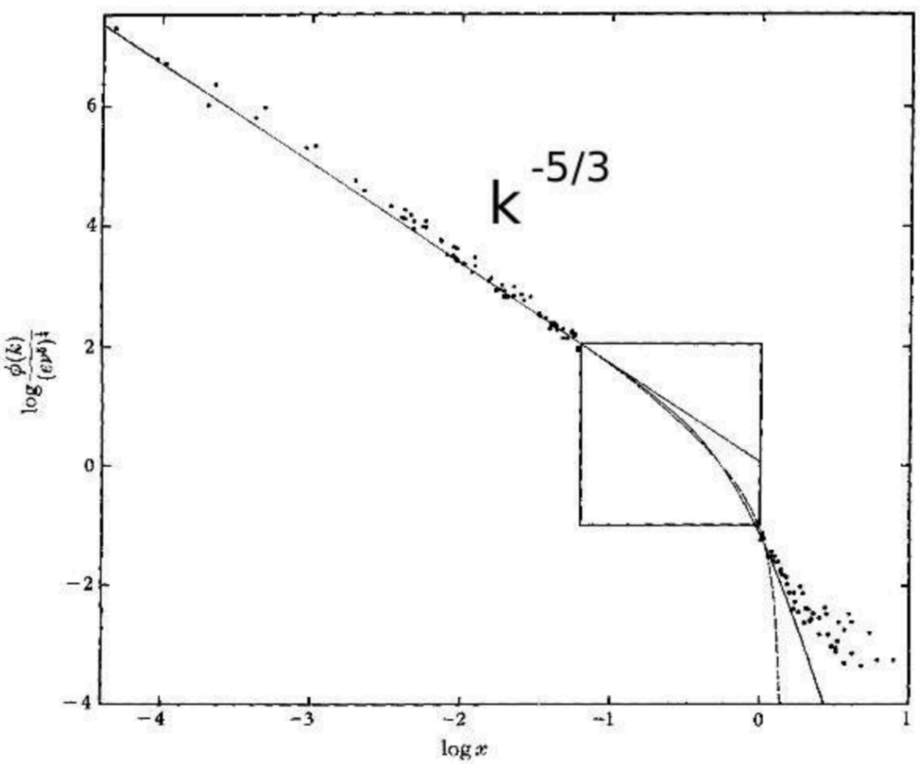


Ocean

Energy spectrum from towed measurements in a a tidally-mixed fjord on the west coast of the US by Grant et al. (1962).

= direct energy cascade ($k^{-5/3}$)

consistent with 3d turbulence



Ocean

Wavenumber Spectrum in the Gulf Stream from Shipboard ADCP Observations

= direct enstrophy cascade (k^{-3})

consistent with theory of QG turbulence

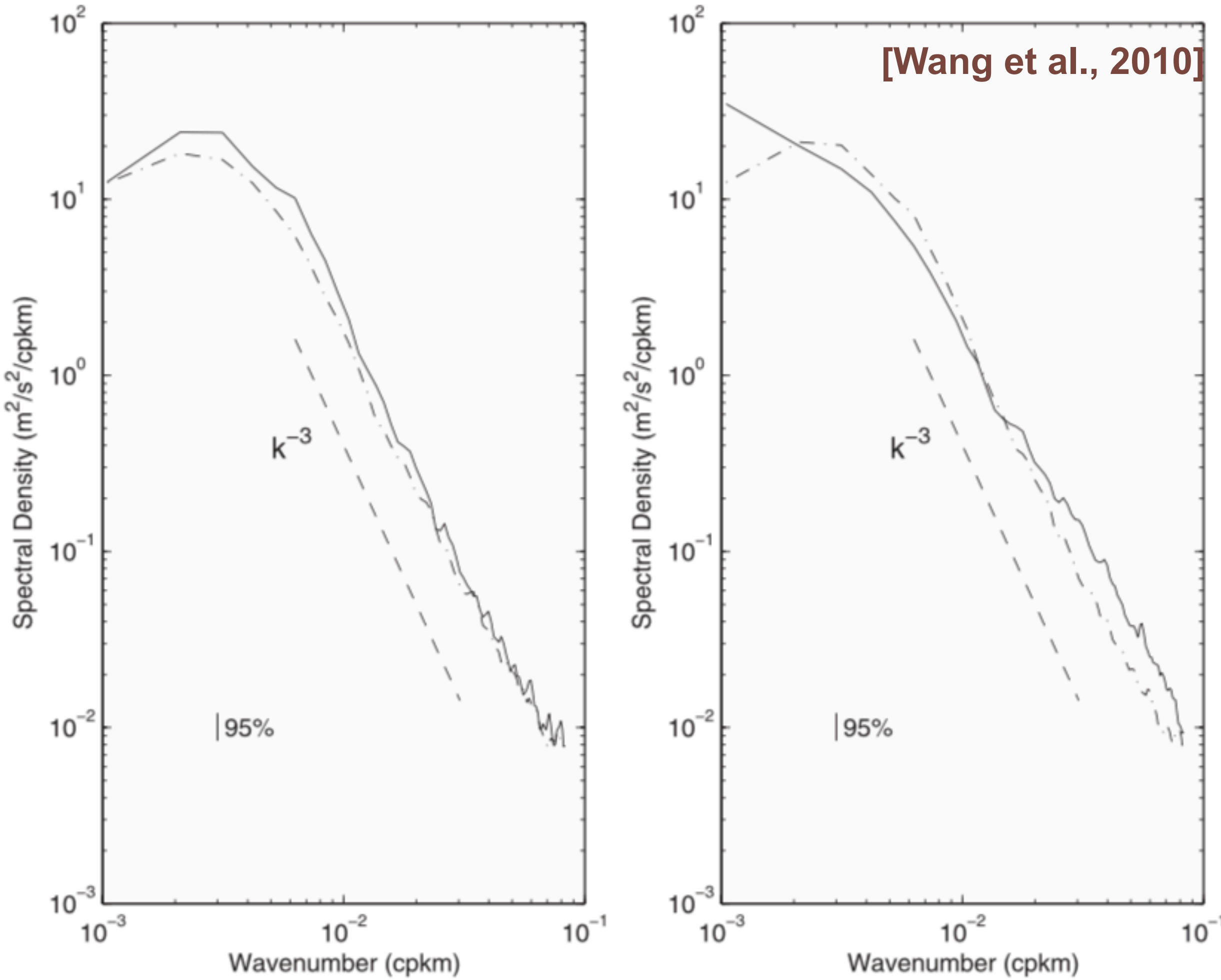
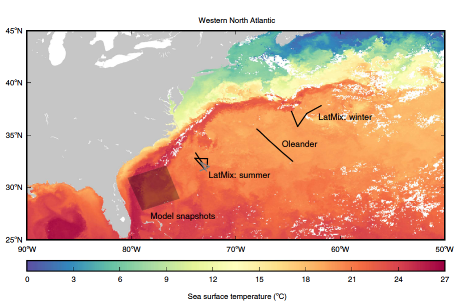


FIG. 2. (left) Zonal (solid) and meridional (dashed–dotted) velocity spectra and (right) potential energy (solid) and kinetic energy (dashed–dotted) spectra from the *Oleander* observations. Dashed lines indicate a −3 slope. The 95% confidence interval is marked.

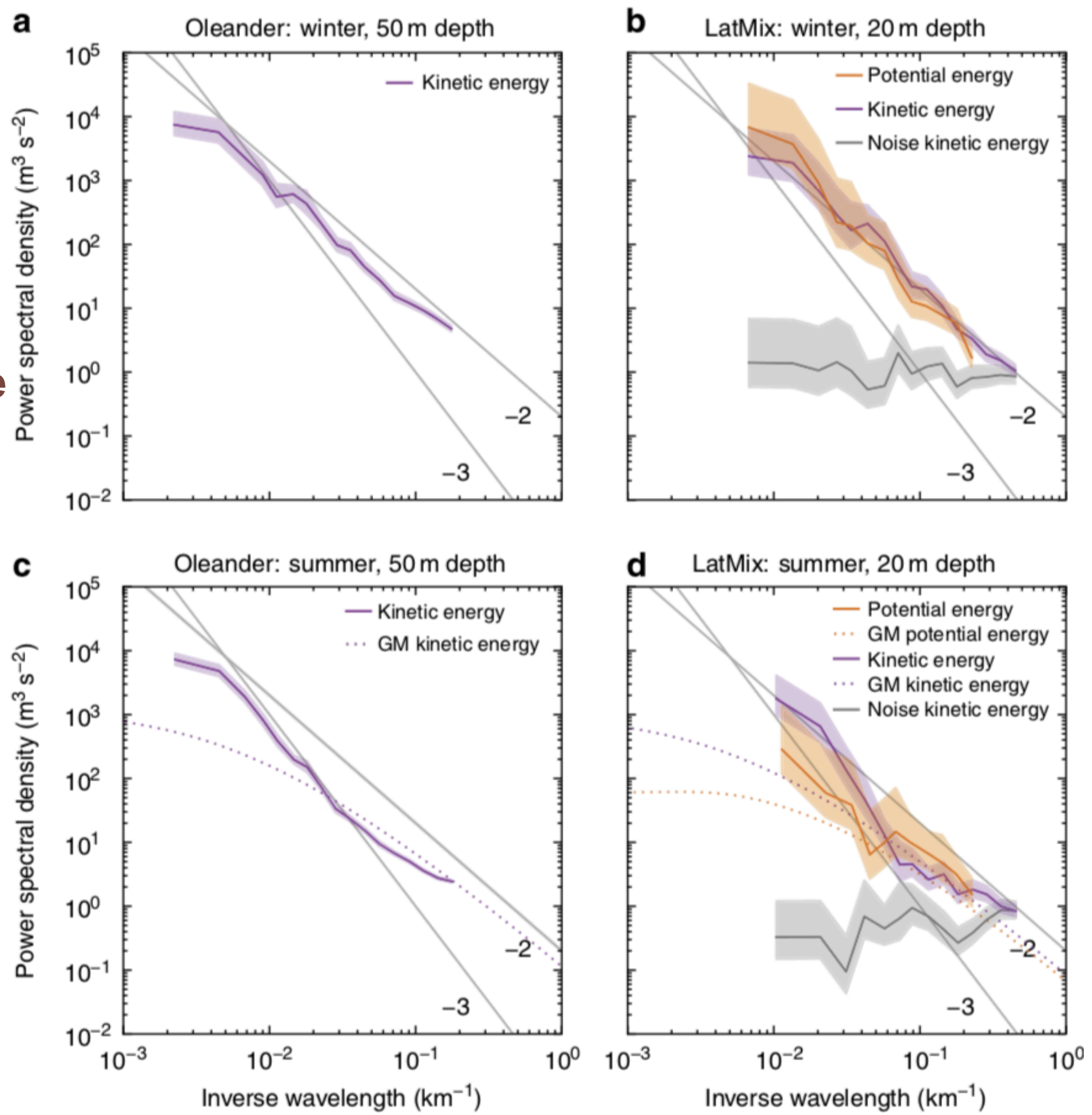


Ocean

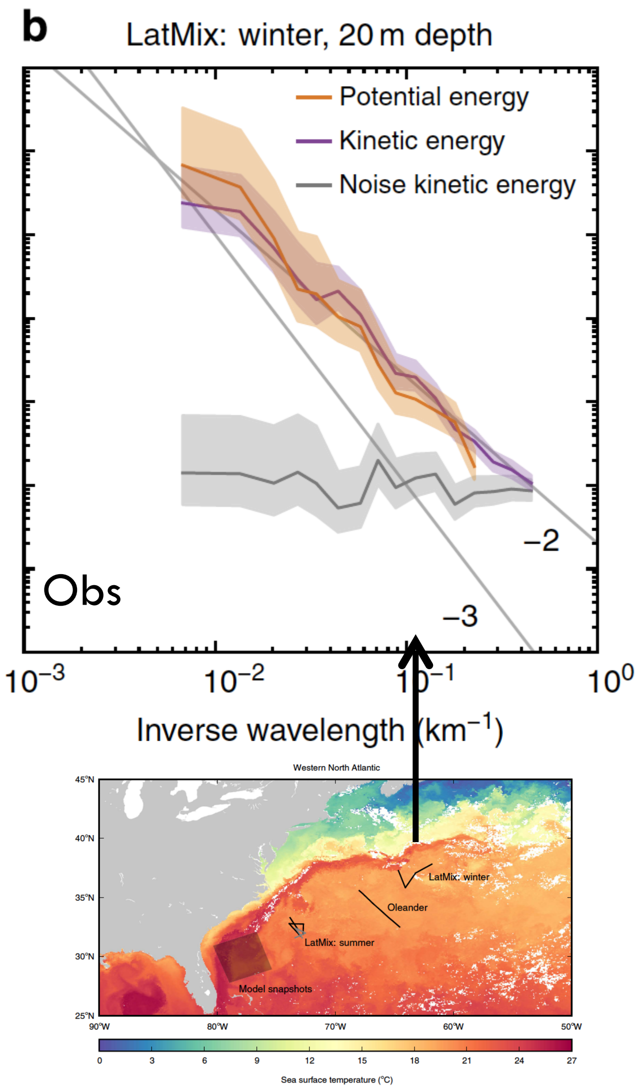
Seasonal cycle in the mixed-layer:

= direct enstrophy cascade (k^{-3}) in summer

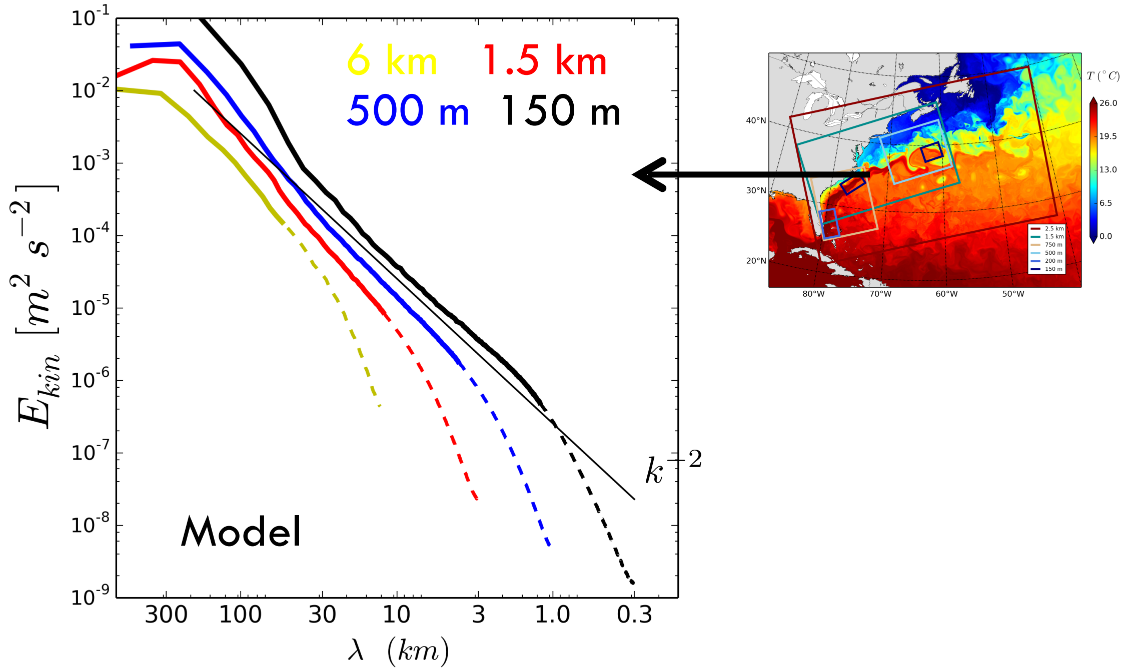
= energization by frontogenesis and mixed-mixed-layer submesoscale instabilities in winter (k^{-2})



Ocean



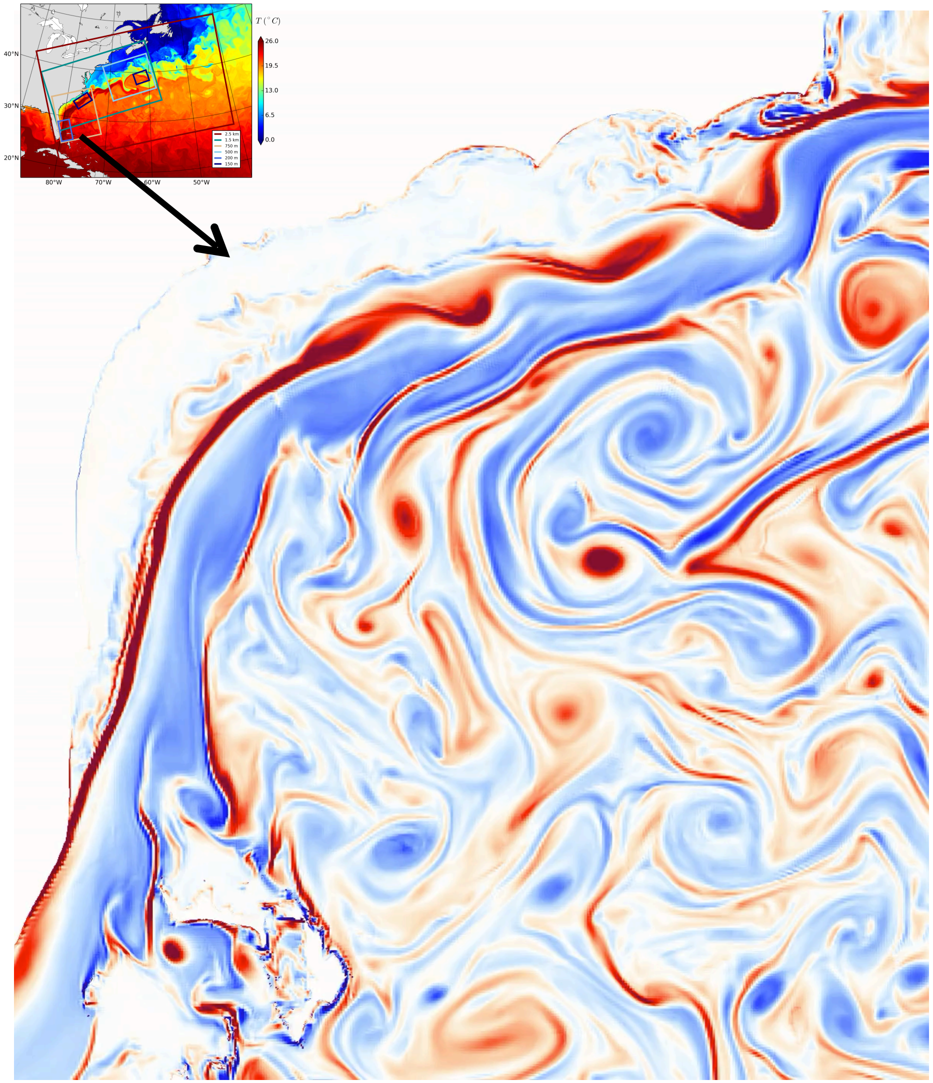
Seasonality in submesoscale turbulence
[Callies et al., 2015]



Azimuthally-averaged 2D wavenumber kinetic energy spectra in a sequence of simulations for the **Gulf Stream after separation in winter.**

Dashed lines indicate dissipation range.

Ocean



Surface relative vorticity $(\pm f)$

1 year of ROMS Simulation

$$\Delta x = 750\,\mathrm{m}$$

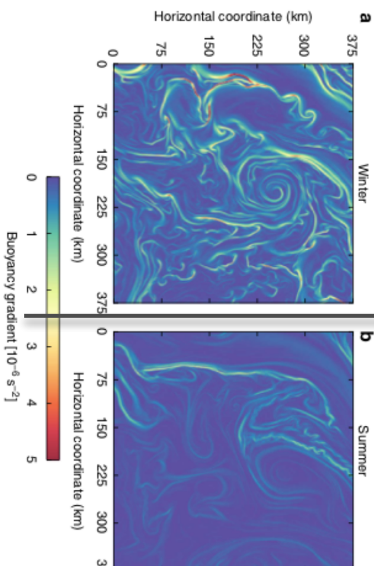
Monthly wind forcings / No tides

Ocean

Seasonal cycle

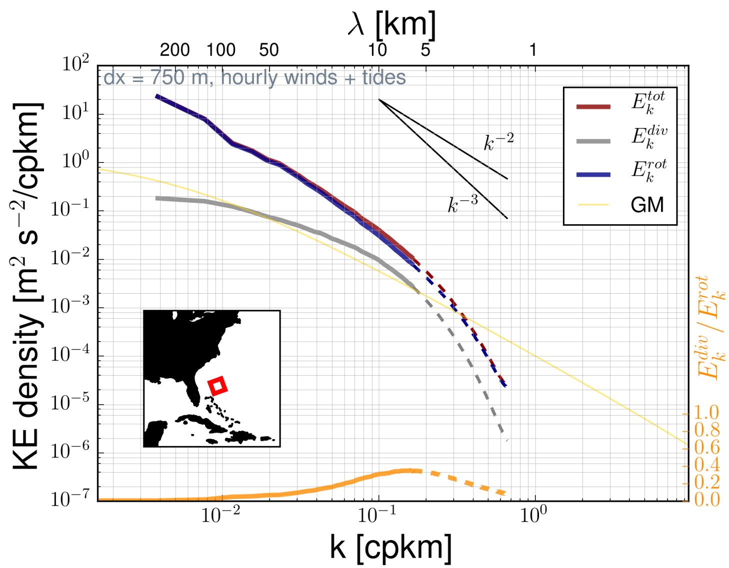
in the Sargasso Sea

WINTER



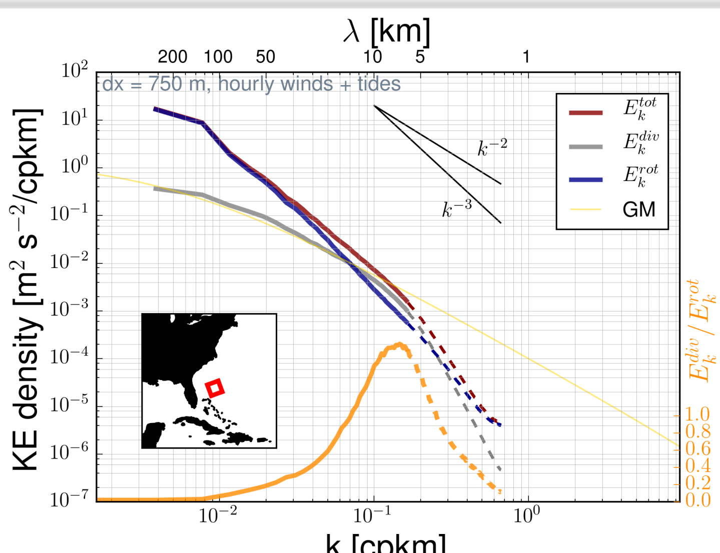
Callies et al, 2015

SUMMER



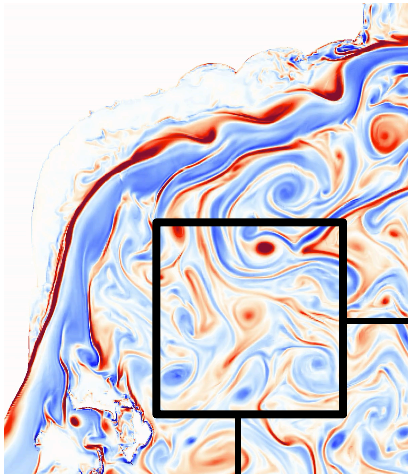
During winter the mixed-layer is deeper and feed energy to the partially balanced submesocale currents through mixed-layer instability

In the summer season the ratio becomes larger than 0.1 at scales smaller than about 100 km.

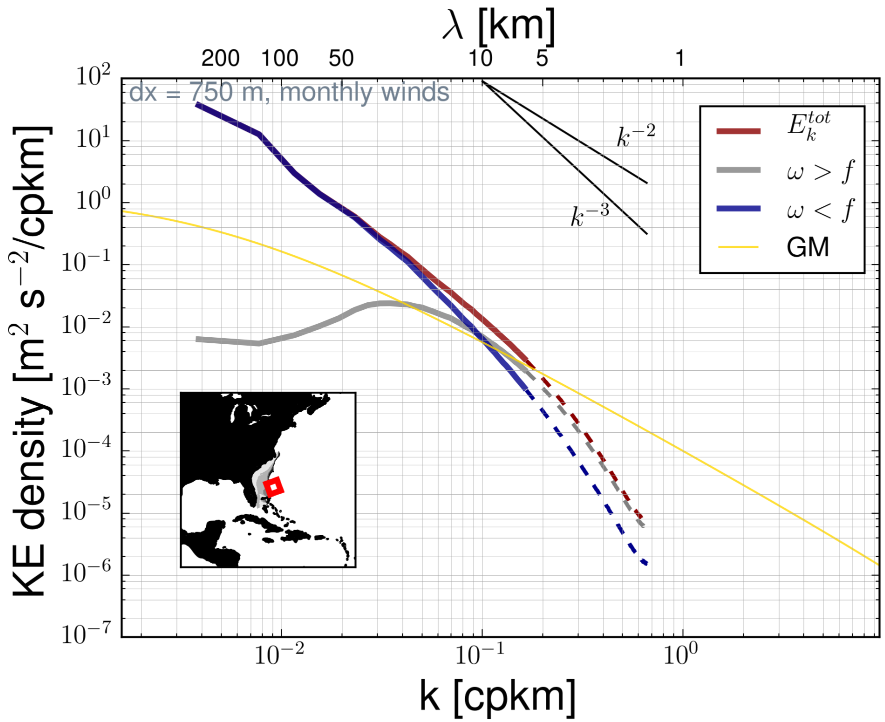


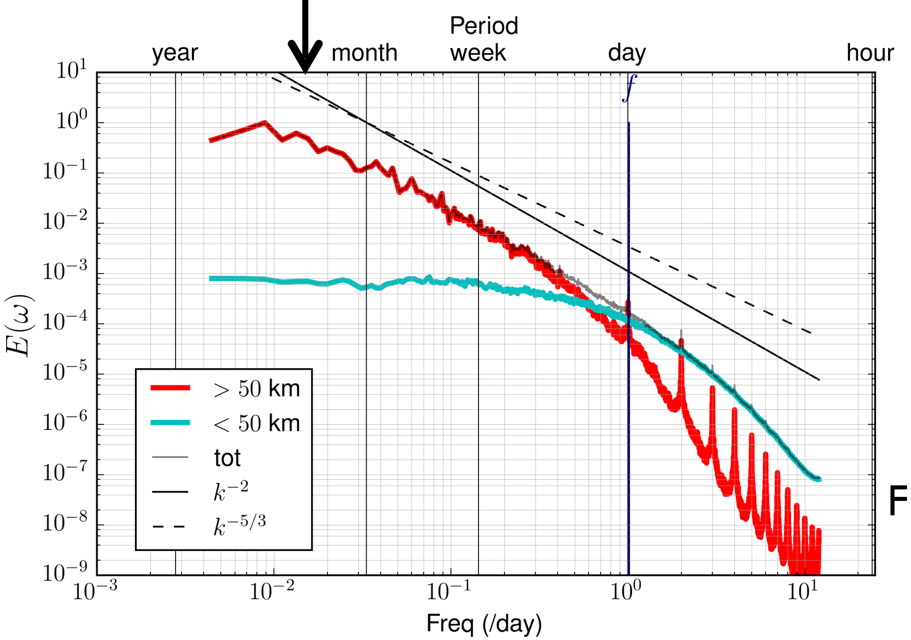
Qualitatively similar to the seasonal cycle observed in other regions of the world such as the Northwestern Pacific [Rocha et al., 2016a; Qiu et al., 2017] or the Drake Passage [Rocha et al., 2016]. See global census in [Qiu et al., 2017]

Ocean



Wavenumber spectra



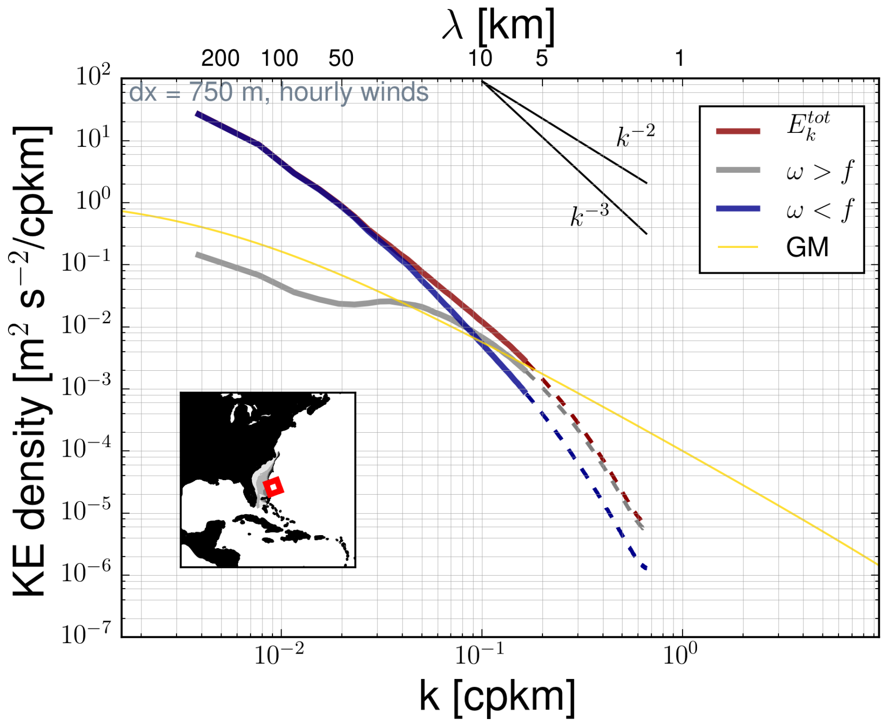


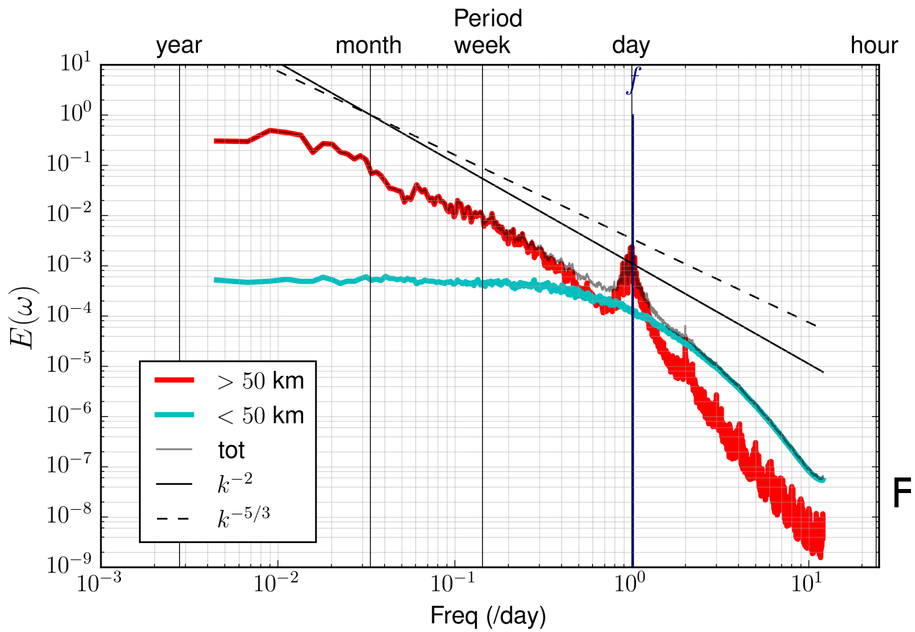
Monthly wind forcings / No tides

Frequency spectra

Ocean

Wavenumber spectra



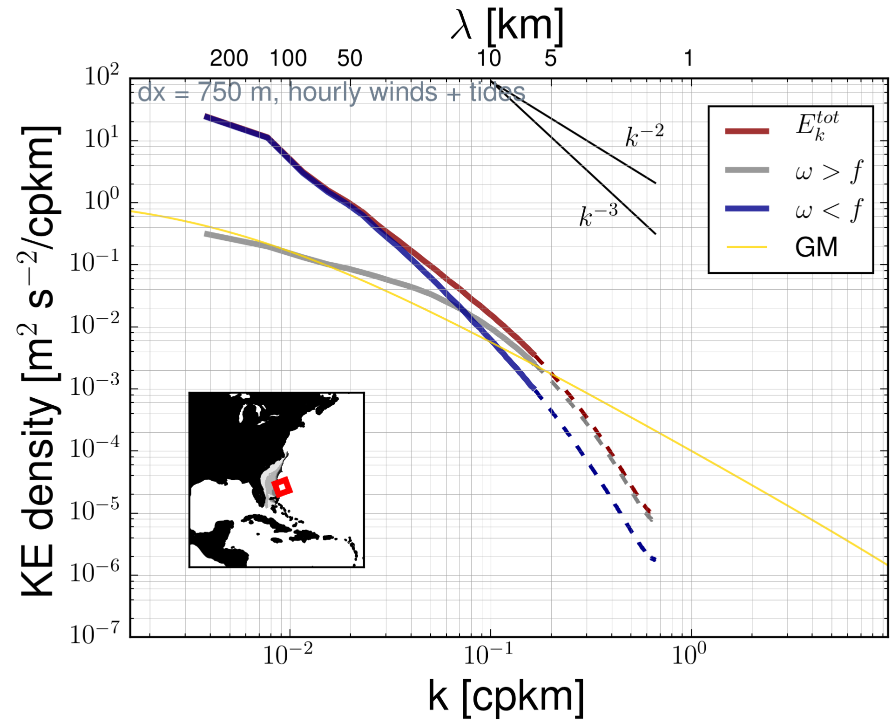


Hourly wind forcings / No tides

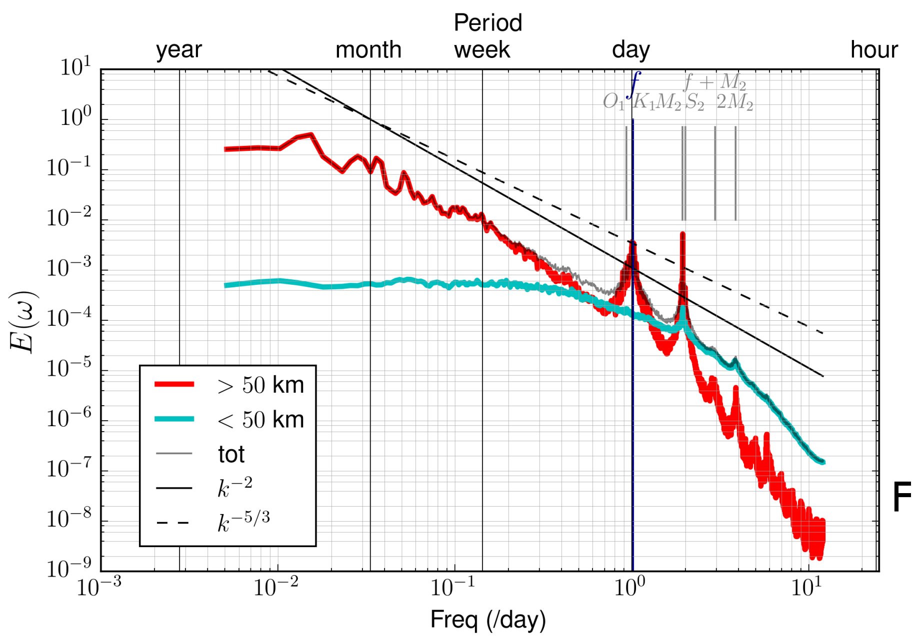
Frequency spectra

Ocean

Wavenumber spectra

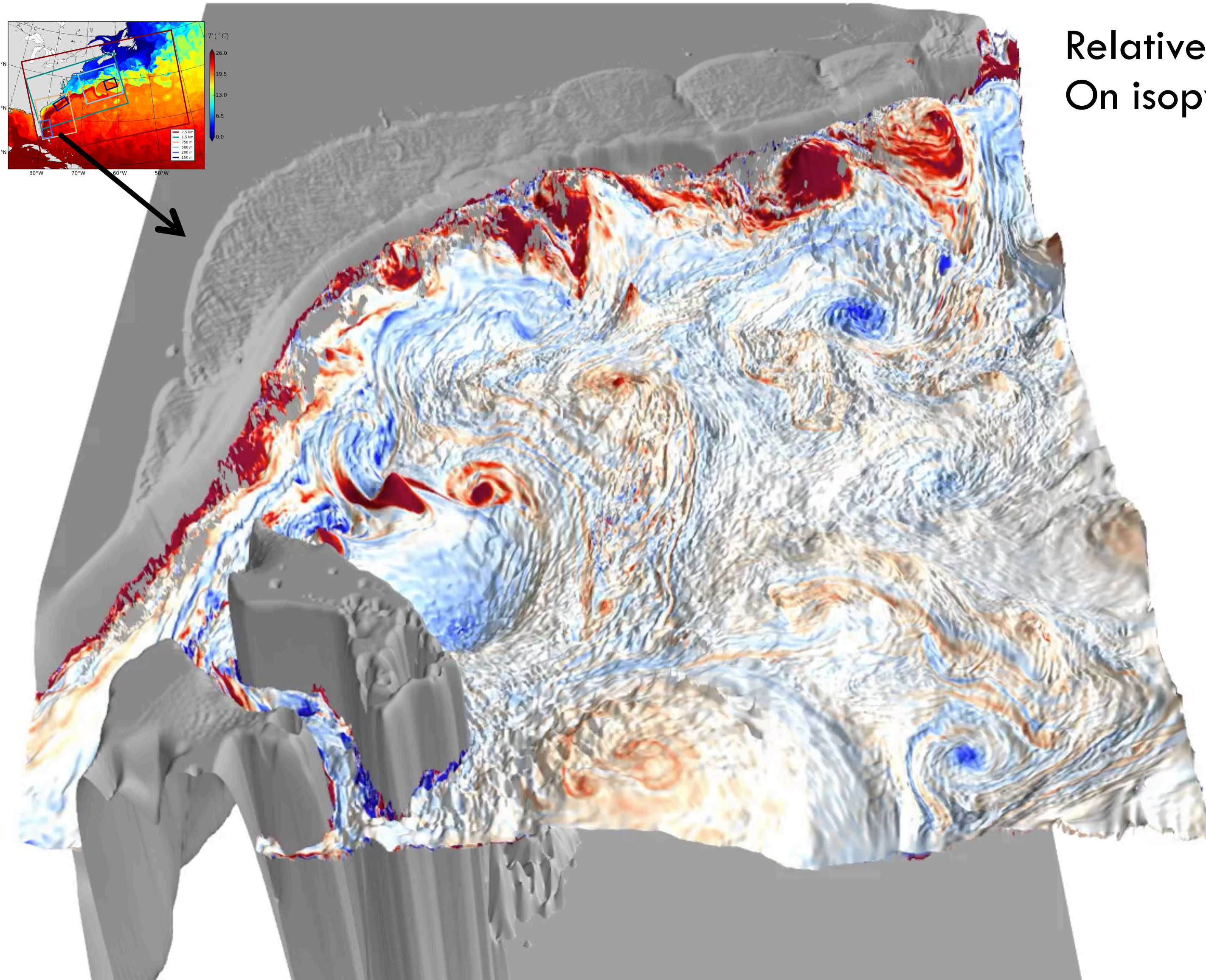


Hourly wind forcings / Tides



Frequency spectra

Ocean



Relative vorticity $(\pm f)$

On isopycnal $\sigma = 27\,\mathrm{kg\,m^{-3}}$

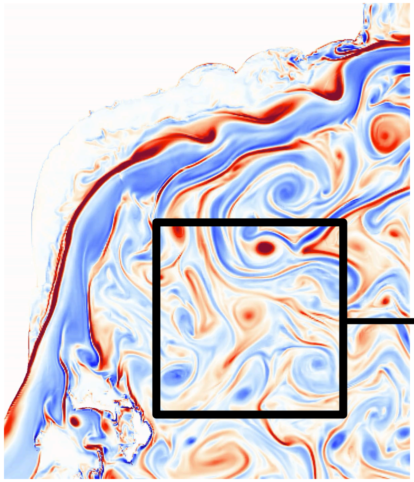
$$\Delta x = 750\,\mathrm{m}$$

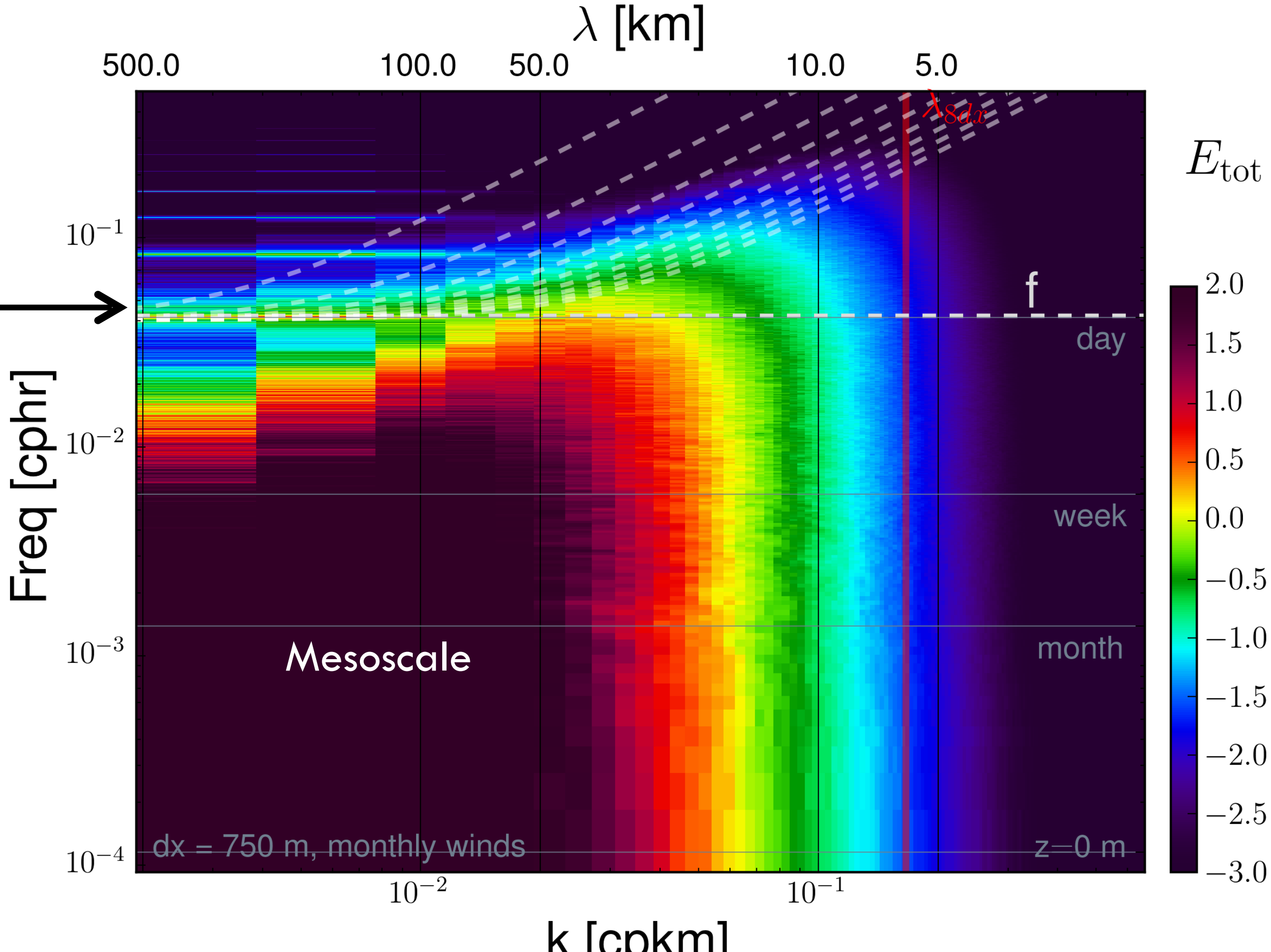
HF wind

+

Tides

Ocean



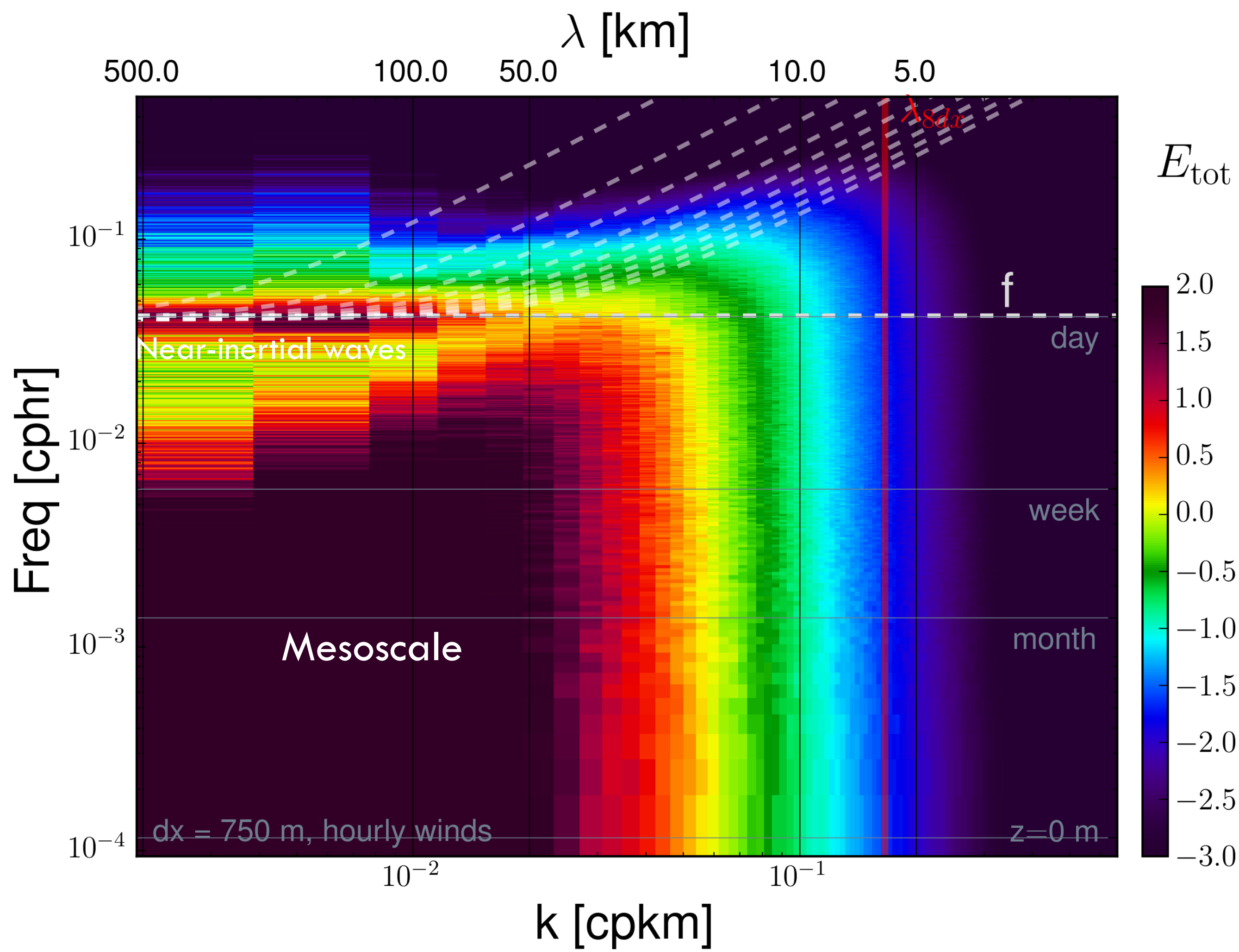


Azimuthally-averaged 2D frequency-wavenumber kinetic energy spectra in the Sargasso Sea.

Monthly winds
No tides

= deficient in internal waves

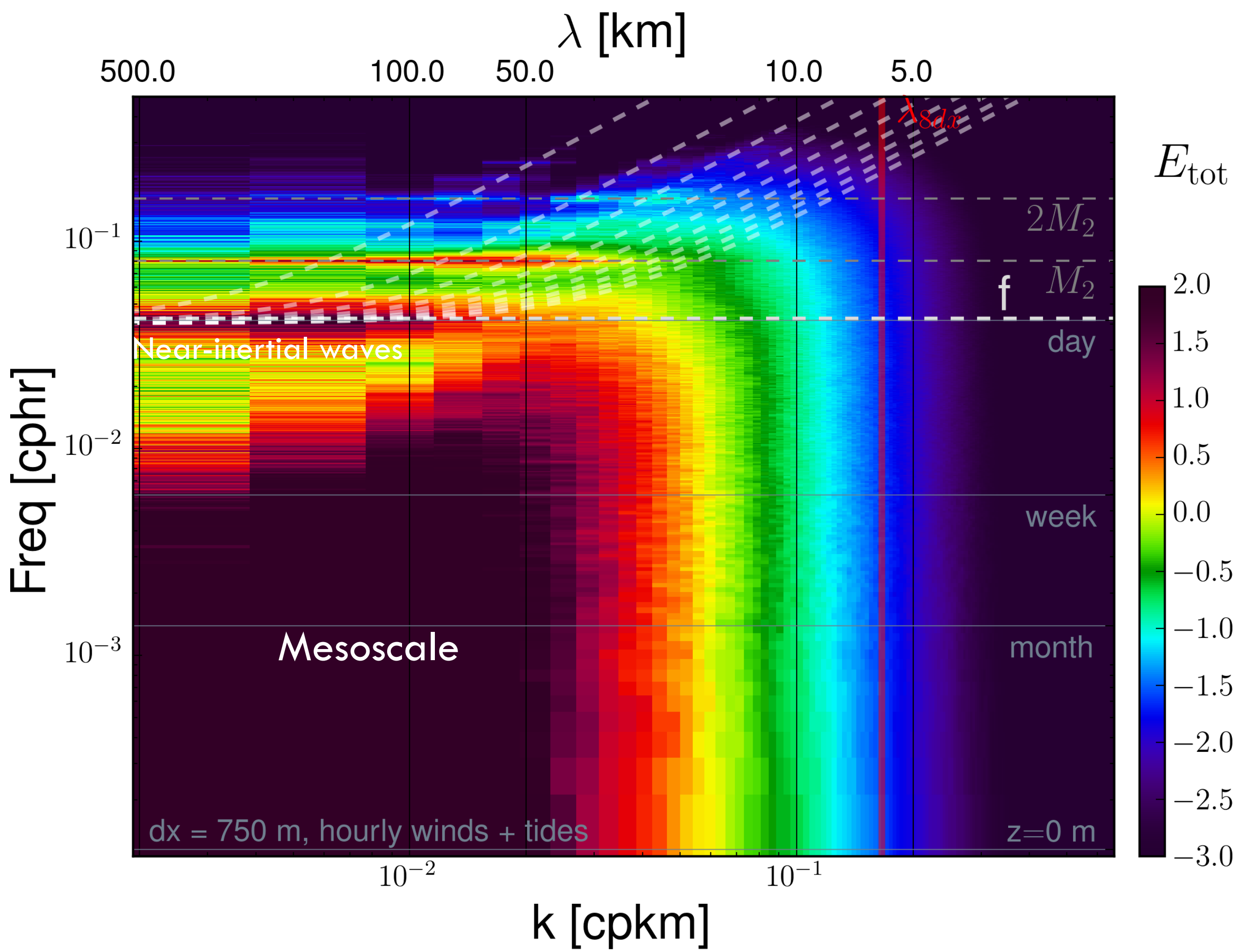
Ocean



Azimuthally-averaged 2D frequency-wavenumber kinetic energy spectra in the Sargasso Sea.

Add a source of internal waves =
High frequency wind

Ocean



Azimuthally-averaged 2D frequency-wavenumber kinetic energy spectra in the Sargasso Sea.

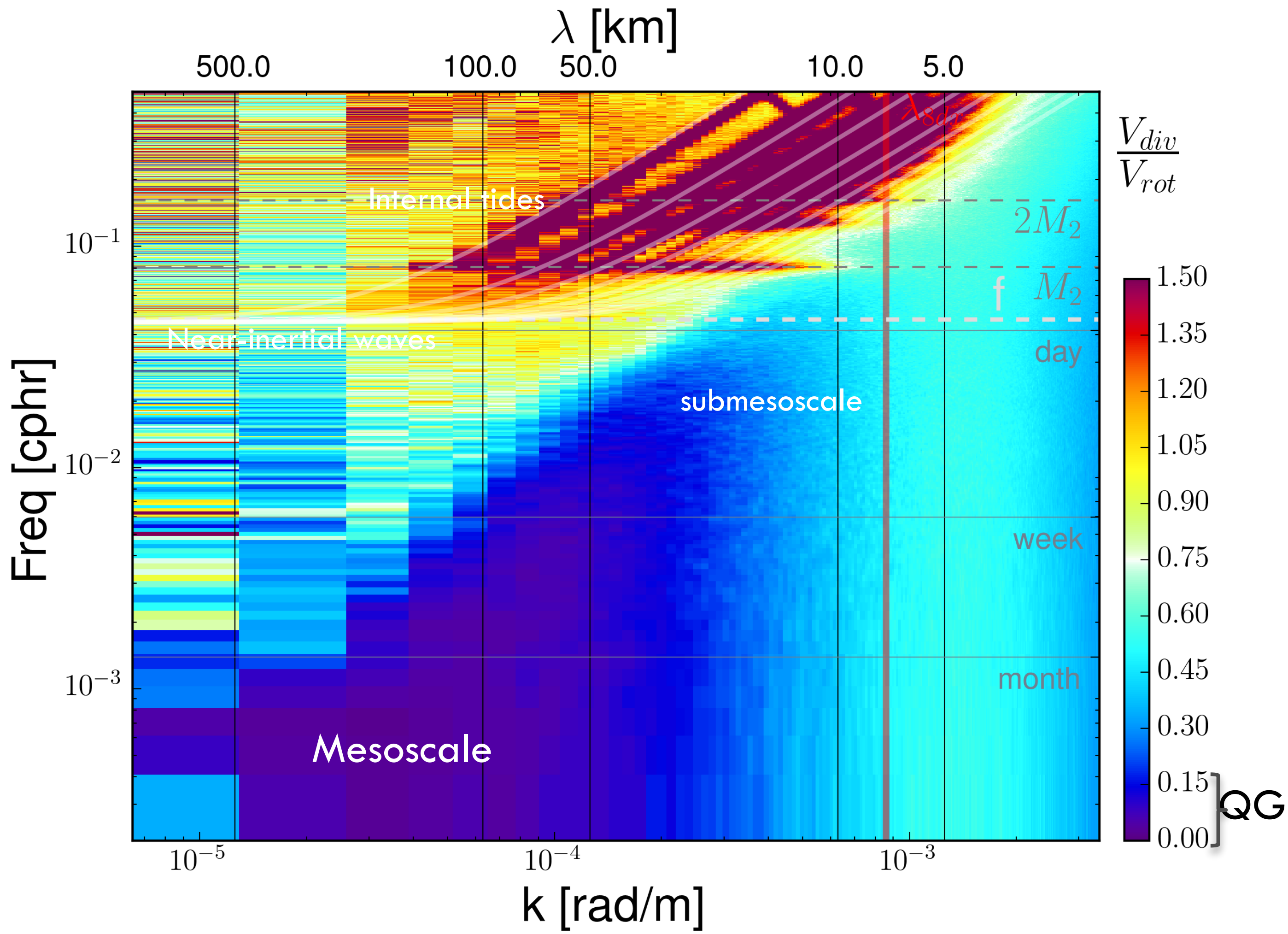
Add two sources of internal waves =
High frequency wind
+
Tidal forcings

Ocean

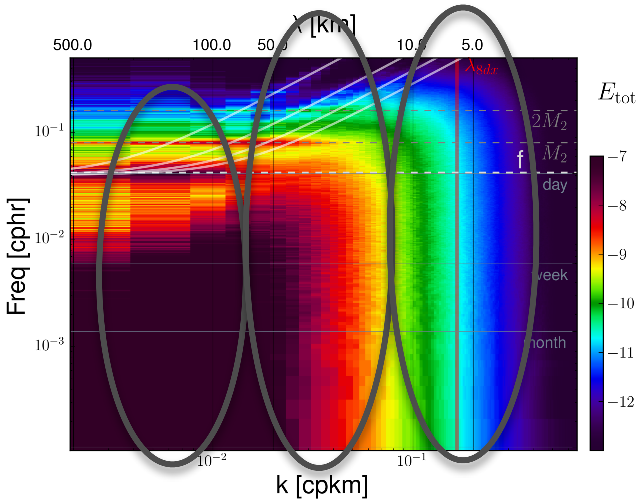
Ratio of divergent / rotational part of the kinetic energy

using Helmoltz decomposition of a 3d incompressible flow

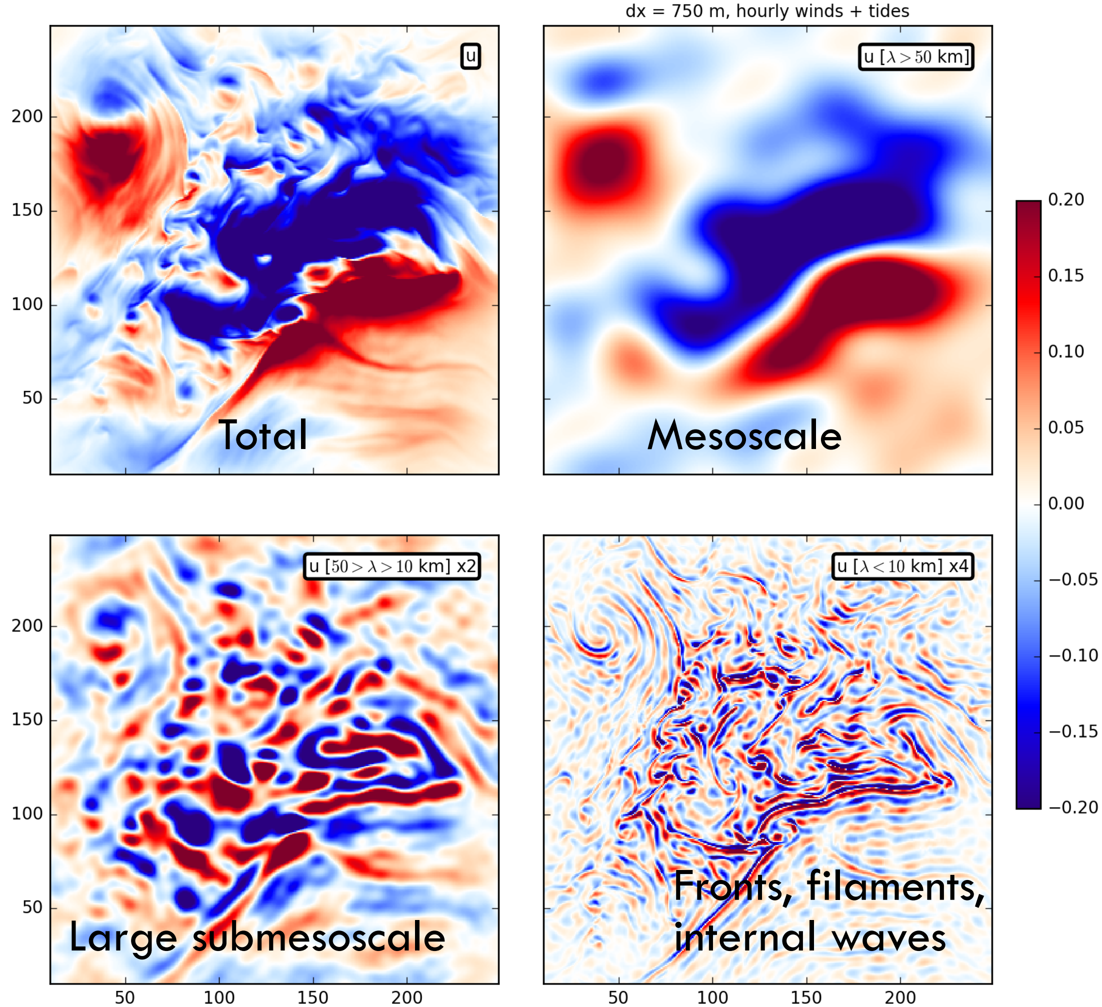
$$\mathbf{u}_h = \mathbf{u}_r + \mathbf{u}_d \,,$$
$$\nabla_h \cdot \mathbf{u}_r = 0$$
$$\hat{\mathbf{z}} \cdot \nabla_h \times \mathbf{u}_d = 0$$



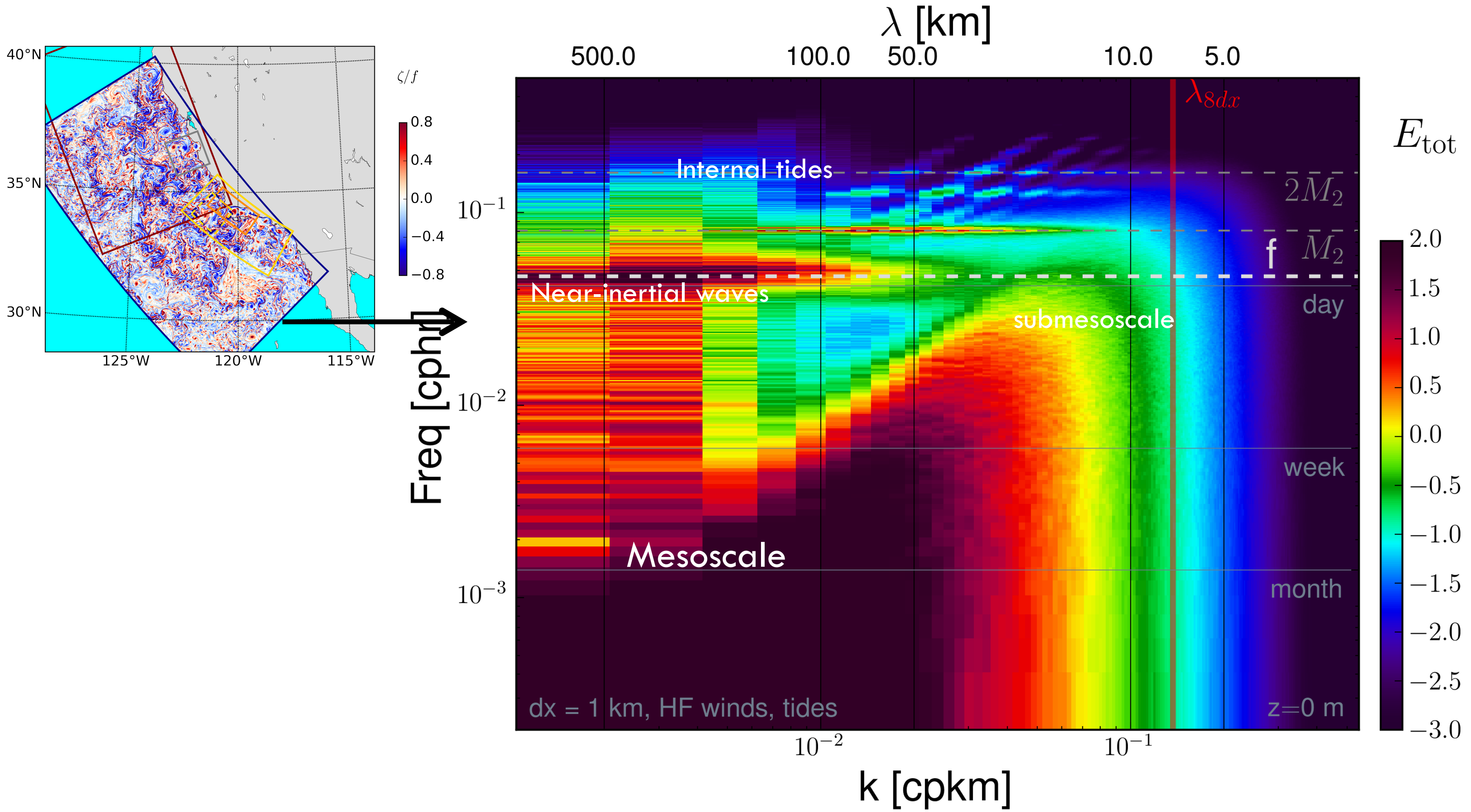
Ocean



Example of filtered zonal velocity

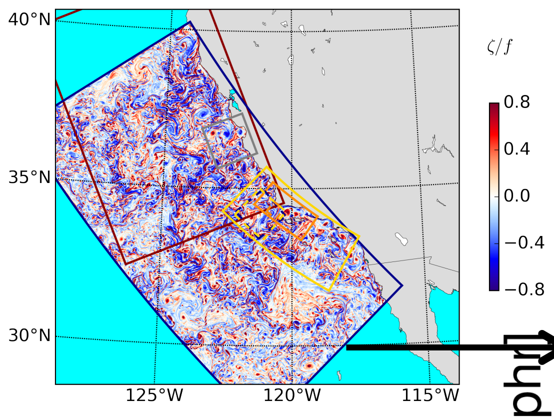


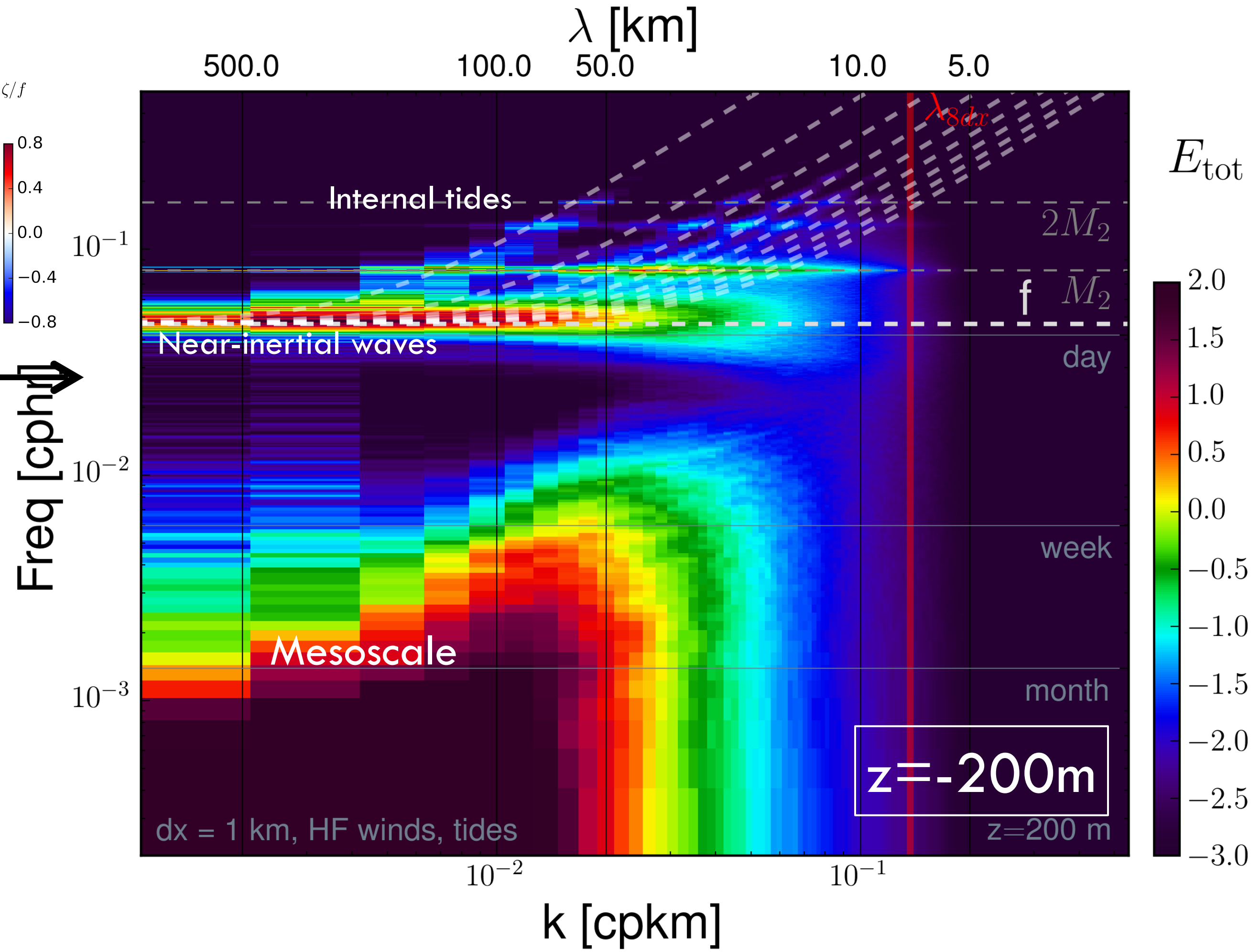
Ocean



Azimuthally-averaged 2D frequency-wavenumber kinetic energy spectra in the California Current.

Ocean





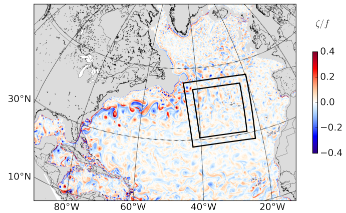
Below the mixed-layer, separation between internal waves and balanced dynamics is easy.

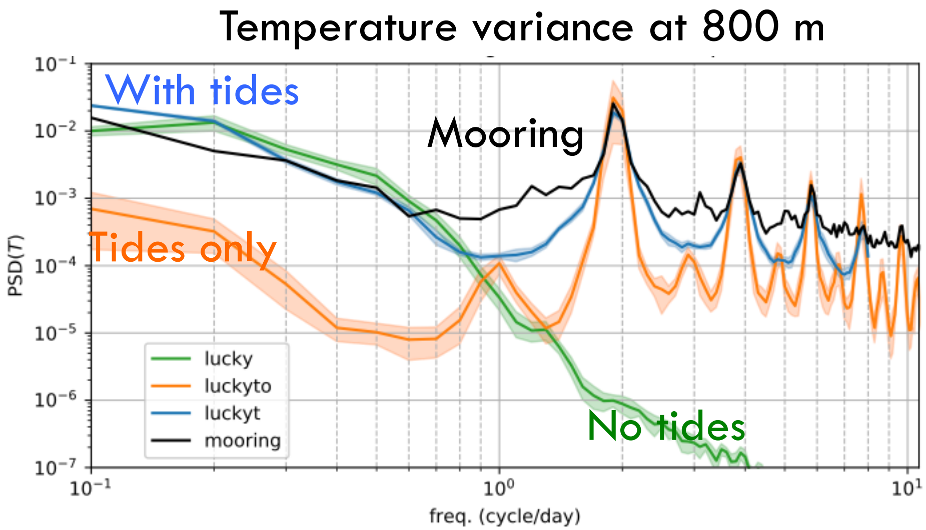
Azimuthally-averaged 2D frequency-wavenumber kinetic energy spectra in the California Current.

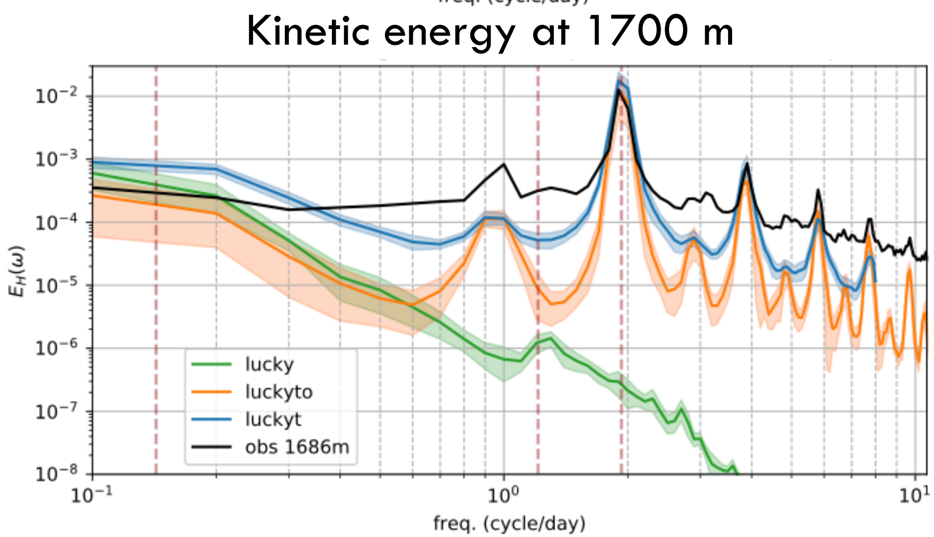
Ocean

Model vs. mooring

on the Lucky Strike hydrothermal vent:







Ocean

Ocean