INTERNAL WAVES

2. CONTINUOUS STRATIFICATION

I.3.2 Internal waves

I.3.2.1 Generalities about internal waves

I.3.3.2 Internal waves in the two-layer model

I.3.3.3 Internal waves with continuous stratification

Bibliography

- Gerkema- Zimmerman (2008). *An introduction to internal waves*
  - http://stockage.univ-brest.fr/~gula/Ondes/gerkema.pdf
- Gill (1982) *: Atmosphère-Ocean Dynamics*
- Kundu-Cohen (1987). *Fluid Mechanics. Third edition*
- Cushman-Roisin*. Introduction to geophysical fluid Dynamics*

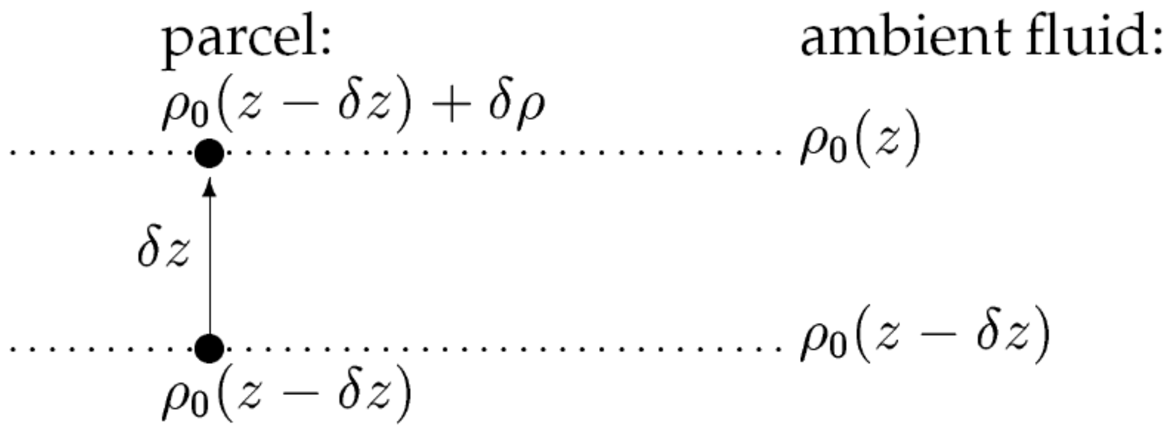
- **1.2 : Internal waves with continuous stratification**
  - Equations
  - Method of vertical modes
  - Method of characteristics

Local Static Stability

- Stratification: $$\rho = \rho_0(z)$$
- A fluid is
  - **stably stratified** if a displaced parcel tends to return to its original position,
  - **unstably stratified** if it tends to move further away from its original position
  - **neutrally stratified** if it tends to stay where it is.

Local Static Stability

- Let's move a parcel:



- Buoyancy force:

$$\rho_0(z)\ddot{\delta z} = g\left(\rho_0(z) - \rho_0(z-\delta z) - \delta\rho\right)$$

Local Static Stability

- With background density variation:

$$\rho_0(z) - \rho_0(z - \delta z) = \frac{d\rho_0}{dz}\delta z$$

- And parcel density variation?
  - *From thermodynamics, if entropy and salinity are conserved during displacement:*

$$\delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_{\eta,S} \delta p = c_s^{-2}\delta p$$

Where c_s is the speed of sound

Local Static Stability

- So we get:

$$\rho_0(z)\ddot{\delta z} = g\left(\frac{d\rho_0}{dz}\delta z + \frac{\rho_0 g \delta z}{c_s^2}\right)$$

Local Static Stability

- Simple Harmonic oscillator:

$$\ddot{\delta z} - \underbrace{\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)}_{N^2}\delta z = 0$$

- Brunt-Vaisala frequency: N^2

-

$$\ddot{\delta z} + N^2 \delta z = 0$$

Local Static Stability

- Simple Harmonic oscillator:

$$\ddot{\delta z} + N^2 \delta z = 0$$

- Solutions: $e^{\pm \mathrm{i} N t}$

- Stable if $N^2 > 0$

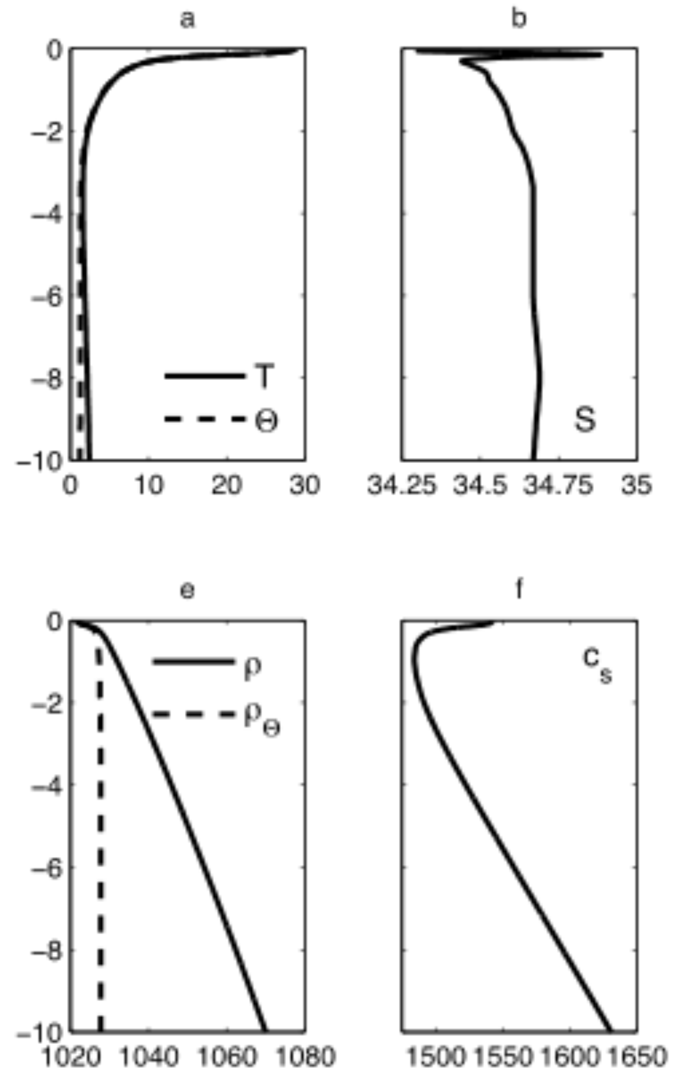
The parcel oscillates vertically at frequency *N* about its equilibrium position.

Local Static Stability

- Brunt-Vaisala frequency:

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)$$

The effect of compressibility is often neglected in the upper ocean but it is not true in general.



Vertical profiles of T, S, in-situ and potential density, and speed of sound in the Mindanao Trench.

Equations for a stratified flow

- Navier-Stokes Equations:

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}} + \vec{\mathcal{D}}$$

Momentum equations

$$\frac{D\rho}{Dt} + \rho\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{DP}{Dt}$$

Thermodynamic
'energy' equation
(no diabatic effects)

Equations for a stratified flow

- Approximations for the momentum equation:

No forcings/dissipation

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \cancel{\vec{\mathcal{F}} + \vec{D}}$$

Equations for a stratified flow

- Boussinesq Approximation:

Density perturbations small compared to mean background value:

$$\rho - \rho^* << \rho^*$$

Linearize all terms involving a product with density, except the gravity term which is already linear:

$$\rho\vec{u} \rightarrow \rho^*\vec{u}$$

$$\rho g \rightarrow \rho g$$

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- Traditional Approximation:

= neglect horizontal Coriolis term

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

$$\downarrow$$

$$\frac{D\vec{u}}{Dt} + f\vec{k} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- We think as internal waves as a perturbation of a (known) static background state that has only vertical dependences

$$P = p_0(z) + p'(t, \vec{x})$$
$$\rho = \rho_0(z) + \rho'(t, \vec{x})$$

- And linearize momentum equations:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} + \frac{\rho'}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Equations for a stratified flow

- For the thermodynamic equation: $$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{DP}{Dt}$$
- We can write: $$\frac{D\rho}{Dt} = \frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z}$$
- And: $$\frac{1}{c_s^2}\frac{DP}{Dt} = \frac{1}{c_s^2}\frac{Dp'}{Dt} + \frac{1}{c_s^2}w\frac{\partial p_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$
- So : $$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

Equations for a stratified flow

- We linearize:

$$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

$$\Downarrow$$

$$\frac{\partial\rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

- We can do a scale analysis and remove small terms using the following scalings and relations:

$$\boxed{U \ll C \ll c_{sf} \ll c_s\,.}$$

$$U \sim O(10^{-1})\,;\quad C \sim O(1)\,;\quad c_{sf} \sim O(10^1\,, 10^2)\,;\quad c_s \sim O(10^3)\,\mathrm{m\,s^{-1}}\,.$$

- Which denote the particle velocity and phase speed of internal waves, the phase speed of surface waves, and the speed of sound in seawater, respectively.

Equations for a stratified flow

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

$$\Downarrow$$

- We get:

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = -w\frac{\rho_0 g}{c_s^2}$$

- So we can write:

$$-\frac{g}{\rho_0}\frac{\partial \rho'}{\partial t} + N^2 w = 0$$

Equations for a stratified flow

- For the mass equation:

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0$$

- We also linearize

$$\rho\nabla\cdot\vec{u} \quad = \quad -\frac{1}{c_s^2}\frac{Dp'}{Dt} + w\frac{\rho_0 g}{c_s^2}$$

- remove small terms and get:

$$\vec{\nabla}\cdot\vec{u} = 0$$

Equations for a stratified flow

- Linearized Equations are:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} + \frac{\rho'}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

$$-\frac{g}{\rho_0}\frac{\partial \rho'}{\partial t} + N^2 w = 0$$

Thermodynamic
'energy' equation
(no diabatic effects)

Activity:

- Starting from linearized Equations :

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} + \frac{\rho'}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

$$\vec{\nabla} \cdot \vec{u} = 0 \qquad -\frac{g}{\rho_0}\frac{\partial \rho'}{\partial t} + N^2 w = 0$$

- Activity:
- Write an equation for w alone.

Equations for a stratified flow

- Finally we get an equation for *w* alone:

$$\boxed{\frac{\partial^2}{\partial t^2}(\nabla^2 w) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0}$$

Solutions of the equation

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0$$

- We suppose waves to be sinusoidal in time:

$$w = \hat{w} e^{-\mathrm{i}\omega t}$$

- And get:

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

Solutions of the equation

Two methods to solve the equation:

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

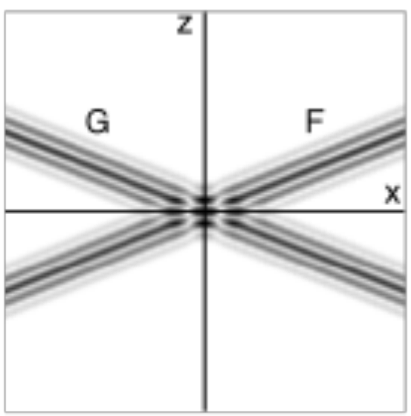
1. The method of characteristics
2. The method of modes

Solutions of the equation

1. Method of characteristics

- N = cste
- Arbitrary boundary conditions
- Solution of the form:

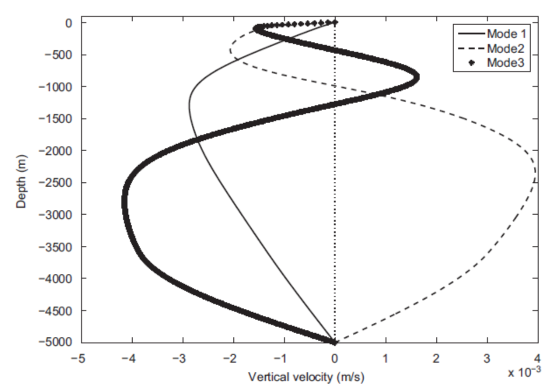
$$\hat{w} = F(\mu_+ x - z) + G(\mu_- x - z)$$



2. Method of Modes

- N = N(z)
- Horizontal surface/bottom
- Solution of the form:

$$w = W(z)e^{-i\omega t + ikx + ily}$$



Solutions: Method of characteristics

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

- For simplicity we take $\dfrac{\partial .}{\partial y} = 0$
- So that: $(N^2 - \omega^2)\hat{w}_{xx} - (\omega^2 - f^2)\hat{w}_{zz} = 0$
-

Solutions: Method of characteristics

- Equation coefficients are all constant (if *N=cste*)

- The solution is :

$$\hat{w} = F(\mu_+ x - z) + G(\mu_- x - z)$$

where *F* and *G* are arbitrary functions and

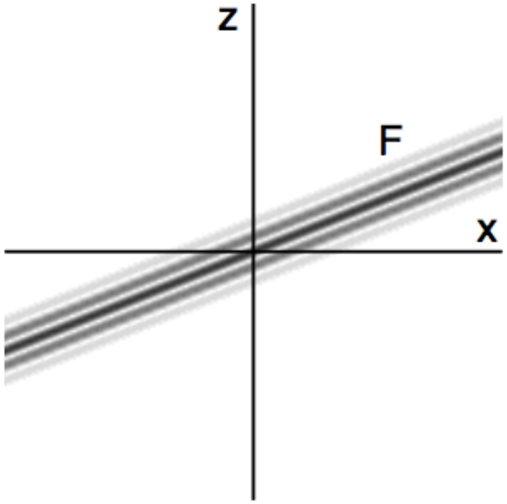
$$\mu_\pm = \pm \left(\frac{\omega^2 - f^2}{N^2 - \omega^2} \right)^{1/2}$$

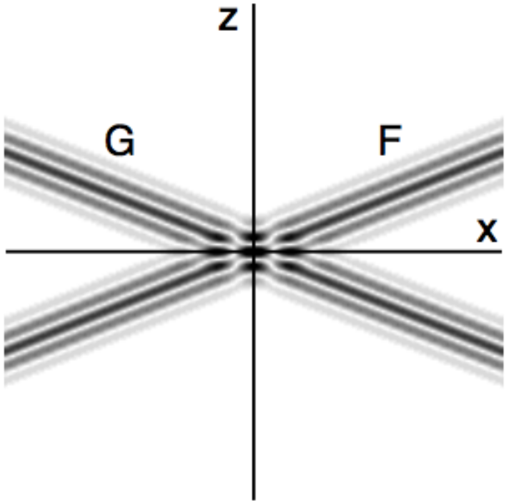
Solutions: Method of characteristics

- For example we use: $$F(\xi) = \exp(-\xi^2)\exp(ik\xi)$$

- The solution is then, with $\xi_\pm = \mu_\pm x - z$

$$w = \exp(-\xi_+^2)\cos(k\xi_+ - \omega t) + \exp(-\xi_-^2)\cos(k\xi_- - \omega t)$$





Solutions: Method of characteristics

- Energy propagates along the lines: $\mu_{\pm} x - z = cste$

which are the characteristic coordinates, and which are diagonals in the x, z-plane

Solutions: Method of characteristics

$$(N^2 - \omega^2)\hat{w}_{xx} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

Activity:

- Assuming a solution of the form:

$$\hat{w} = w_0 e^{\mathrm{i}(kx+mz)}$$

- Write the dispersion relation $\omega(k)$

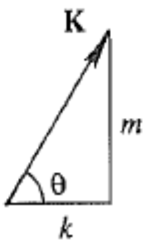
Solutions: Method of characteristics

- The dispersion relation is:

$$\boxed{\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2 + m^2}}$$

Solutions: Method of characteristics

- The dispersion relation: $$\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2 + m^2}$$



can be simplified by using polar coordinates:

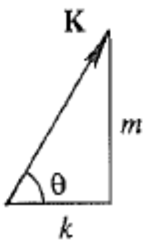
$$\vec{k} = (k, m) = \kappa(\cos\theta, \sin\theta)\,; \quad \kappa = (k^2 + m^2)^{1/2}\,,$$

Such that:

Solutions: Method of characteristics

- The dispersion relation:

$$\omega^2 = \frac{N^2 k^2 + f^2 m^2}{k^2 + m^2}$$



can be simplified by using polar coordinates:

$$\vec{k} = (k, m) = \kappa(\cos\theta, \sin\theta)\,; \qquad \kappa = (k^2 + m^2)^{1/2}\,,$$

Such that:

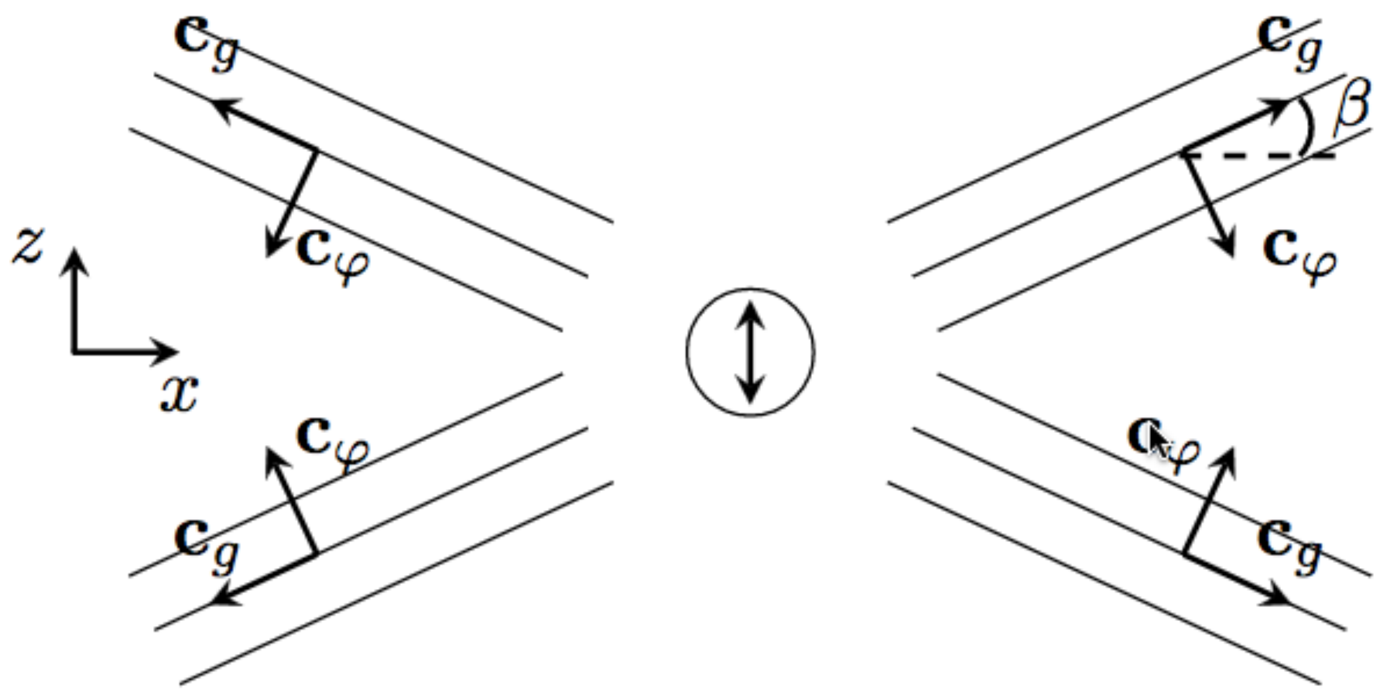
$$\boxed{\omega^2 = N^2 \cos^2\theta + f^2 \sin^2\theta}$$

The wave frequency depends only on the direction θ of propagation and of the local N and f.

if the frequency is imposed (e.g., by tidal forcing), all waves propagate at fixed angles from the horizontal.

Solutions: Method of characteristics

- The four possible configurations of internal wave beams created by an oscillating body at fixed frequency:



A famous experiment

J. Fluid Mech. (1967), *vol.* 28, *part* 1, *pp.* 1–16 1
Printed in Great Britain

A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density stratified liquid

By D. E. MOWBRAY and B. S. H. RARITY†
Department of the Mechanics of Fluids, University of Manchester

L'expérience de MOWBRAY et RARITY (1967) conçue pour vérifier la relation de dispersion est à ce stade particulièrement intéressante pour bien comprendre les conséquences de résultats inhabituels. Il s'agit d'observer dans une cuve rectangulaire les ondes générées par un barreau oscillant verticalement à une fréquence donnée ω. La cuve est remplie d'eau salée de manière à obtenir une stratification linéaire (l'eau plus salée et donc plus dense se situe par conséquent en bas) et les ondes sont observées par ombroscopie, en utilisant le fait que l'indice optique dépend de la concentration en sel. Si des mouvements sont forcés à la pulsation ω_f dans un fluide de gradient de masse volumique constant, toute onde de vecteur d'onde faisant un angle $\beta = \arcsin(\omega_f / N)$ avec la verticale est théoriquement excitée. Comme les rayons sont donnés par les directions de la vitesse de groupe, on voit donc apparaître *une croix* (*cf.* figure 3), avec quatre branches, chacune faisant un angle β avec l'horizontale. On est donc bien loin de l'onde circulaire que donnerait une émission acoustique localisée par exemple.

Gostiaux, Dauxois, BUP868

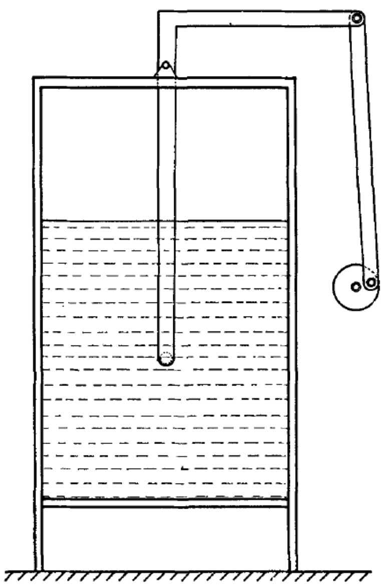


FIGURE 4. A sketch of the apparatus.

Journal of Fluid Mechanics, Vol. 28, part 1 *Plate 1*

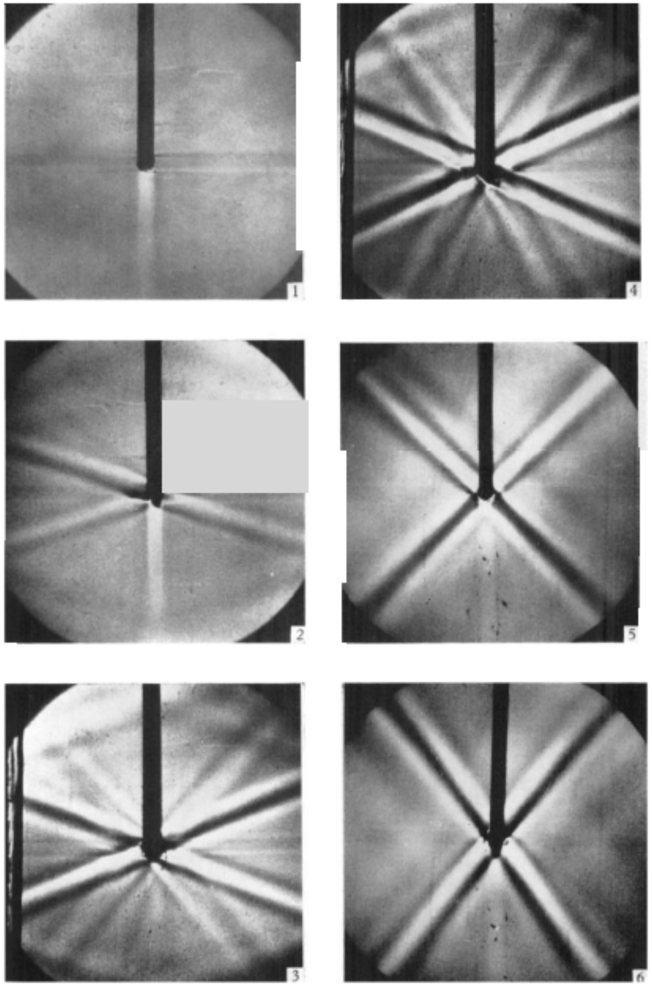


PLATE 1. (1) The image of the undisturbed fluid. (2) $\omega/\omega_0 = 0{\cdot}318$. (3) $\omega/\omega_0 = 0{\cdot}366$. (4) $\omega/\omega_0 = 0{\cdot}419$. (5) $\omega/\omega_0 = 0{\cdot}615$. (6) $\omega/\omega_0 = 0{\cdot}699$.

6. Conclusion

The predictions of the small amplitude theory of the phase configurations of internal waves in a stratified fluid have been tested and confirmed.

http://stockage.univ-brest.fr/~gula/Ondes/MowbrayRarity67.pdf

Solutions: Method of characteristics

- The group velocity: $$\vec{c}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial m}\right)$$

- Can be written using the new coordinates:

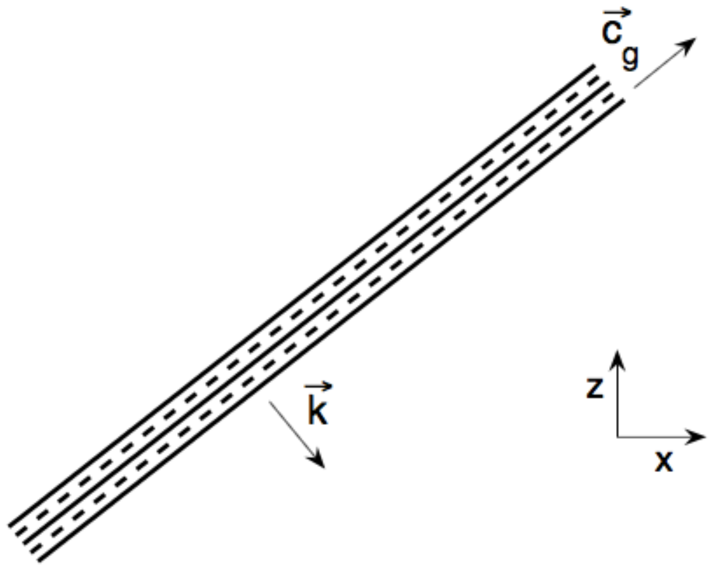
$$\omega(k(\kappa,\theta), m(\kappa,\theta)) = \bar{\omega}(\kappa,\theta)$$

- Such that $$\frac{\partial \bar{\omega}}{\partial \kappa} = \frac{\partial \omega}{\partial k}\frac{\partial k}{\partial \kappa} + \frac{\partial \omega}{\partial m}\frac{\partial m}{\partial \kappa} = \vec{c}_g \cdot \vec{k}/\kappa$$

- *ω* does not depend on *κ* so: $$\vec{c}_g \cdot \vec{k} = 0$$ $$\boxed{\vec{c}_g \perp \vec{k}}$$

The group velocity is always perpendicular to the phase speed and makes an angle θ with the vertical.

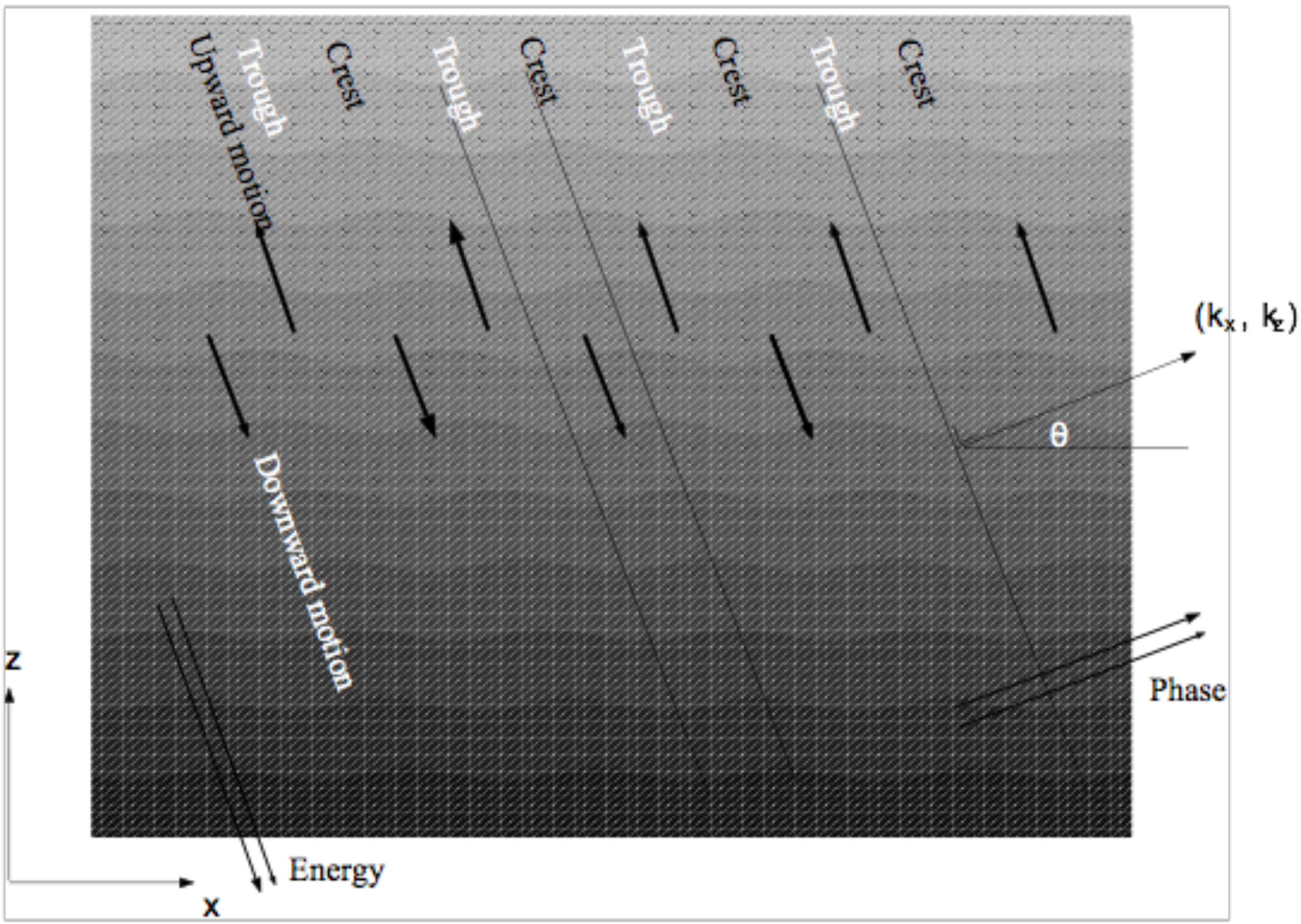
Solutions: Method of characteristics



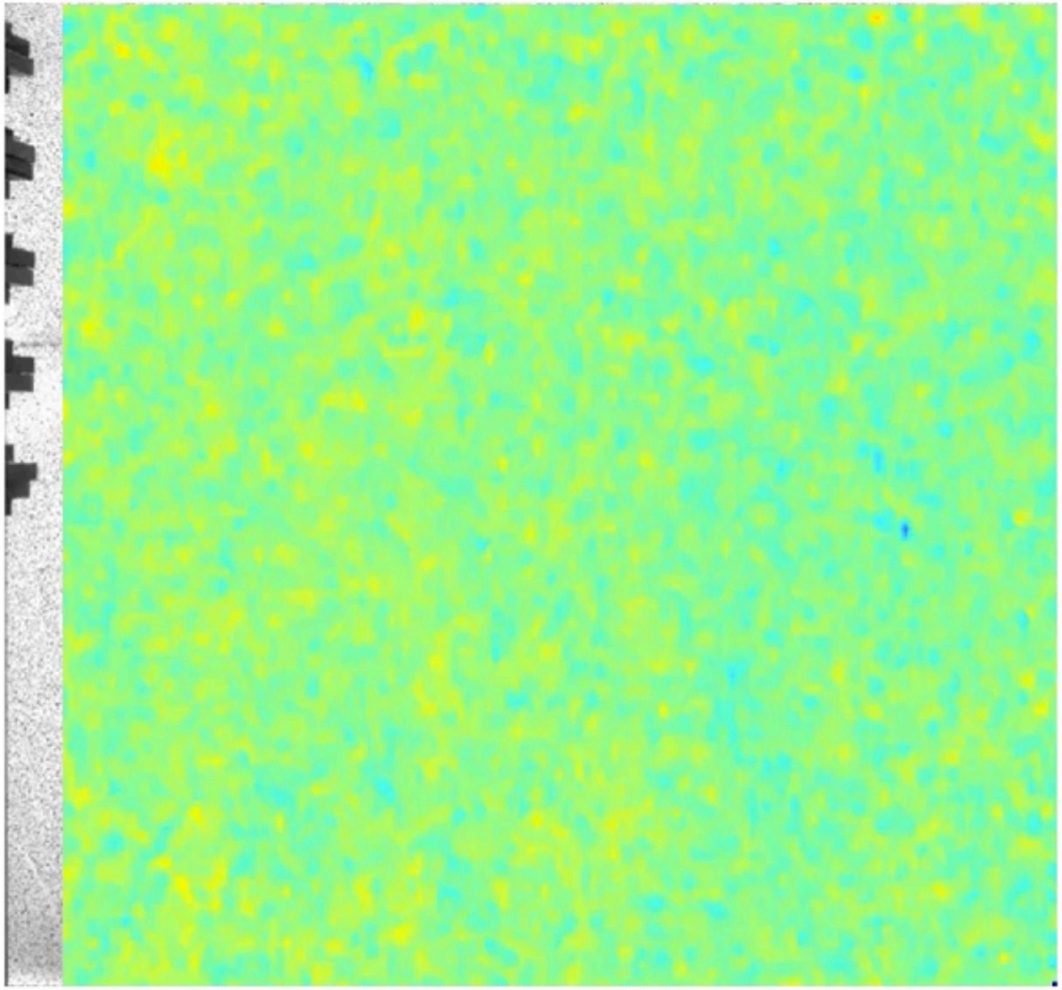
The group velocity is always perpendicular to the phase speed and makes an angle θ with the vertical.

Solutions: Method of characteristics

- Using the continuity equation we can see that $\vec{u} \perp \vec{k}$

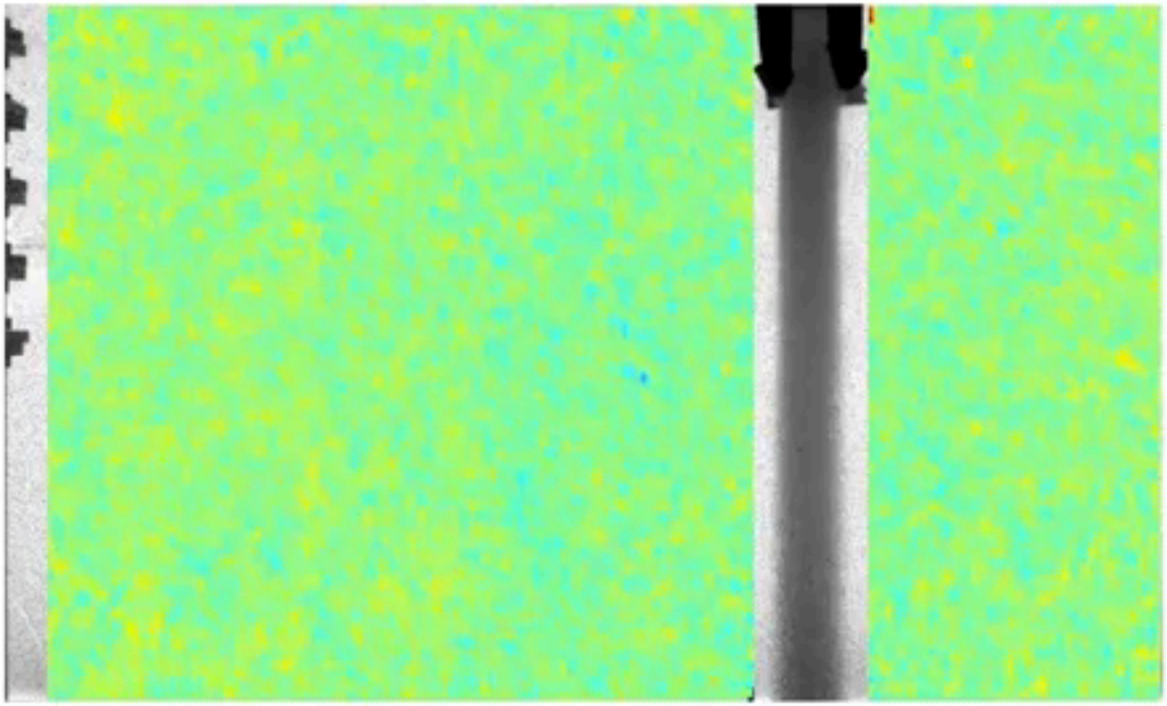


Solutions: Method of characteristics



[Animation E. Horne]

Solutions: Method of characteristics



[Animation E. Horne]

Solutions: Method of characteristics

- Solutions exist only in a range of allowable internal-wave frequencies:

$$(\mathrm{I}) \quad N \leq \omega \leq |f| \quad \text{or} \quad (\mathrm{II}) \quad |f| \leq \omega \leq N$$

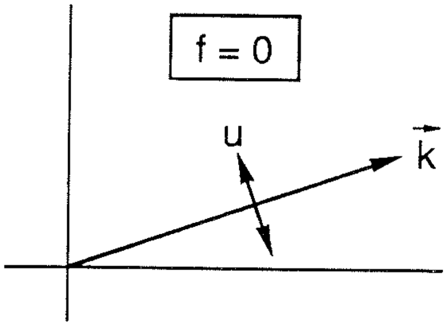
- 2 extreme cases can occur:

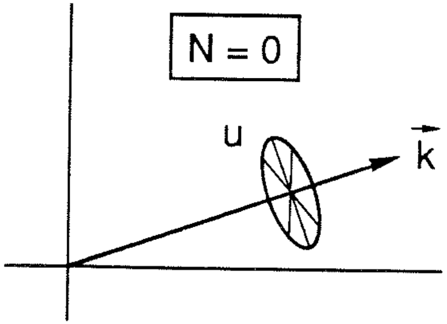
$\omega \rightarrow f$ k becomes vertical, and the group velocity horizontal

$\omega \rightarrow N$ k becomes horizontal and the group velocity vertical

Solutions: Method of characteristics

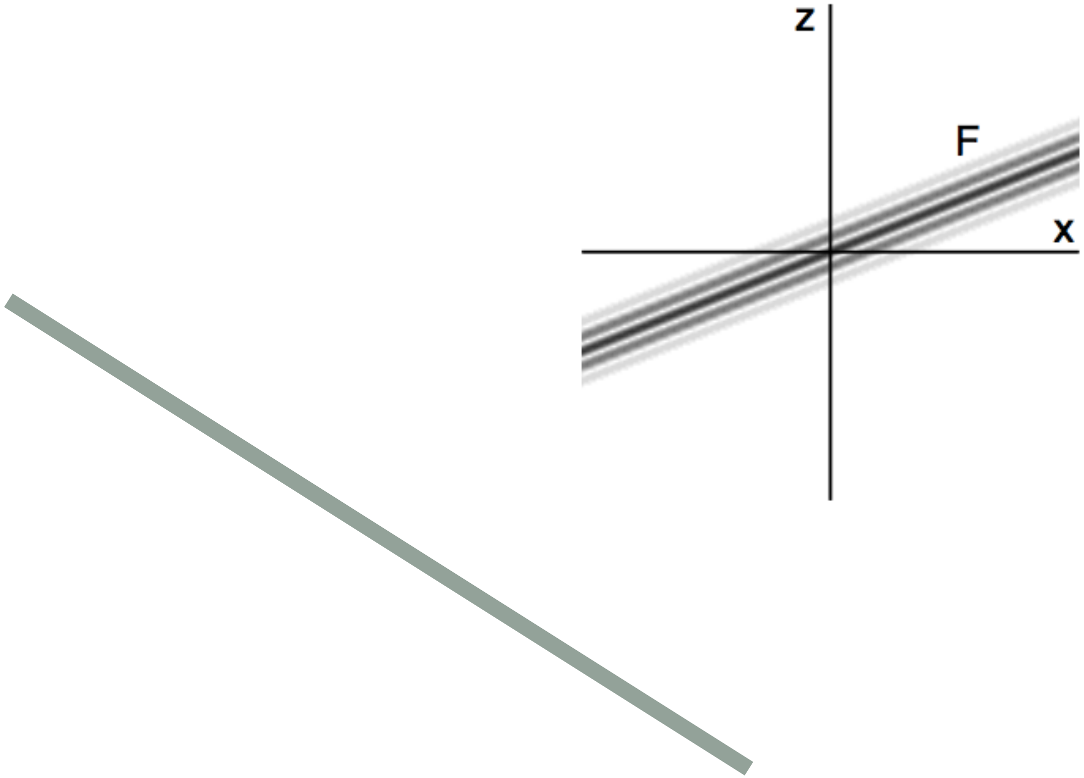
- The motion is along straight lines if f=0, and inertial circles if N=0 .





Solutions: Method of characteristics

Wave Reflection?

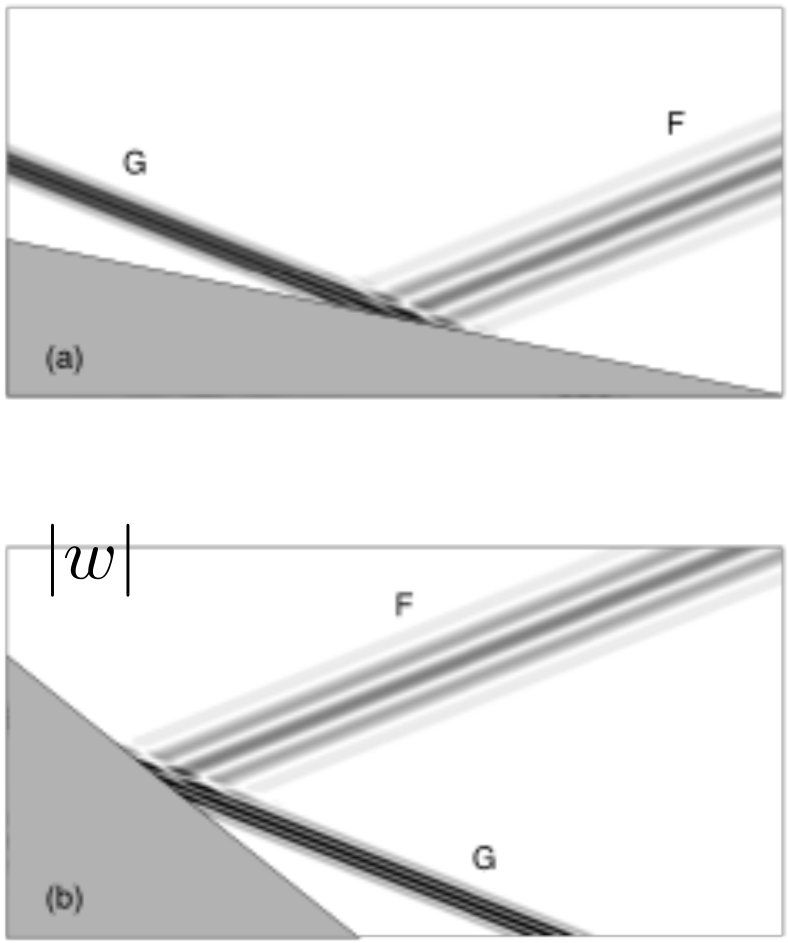


Solutions: Method of characteristics

Wave Reflection

The wave frequency is determined by the angle of propagation (θ). Vice versa, for a given frequency ω, the angle θ is fixed.

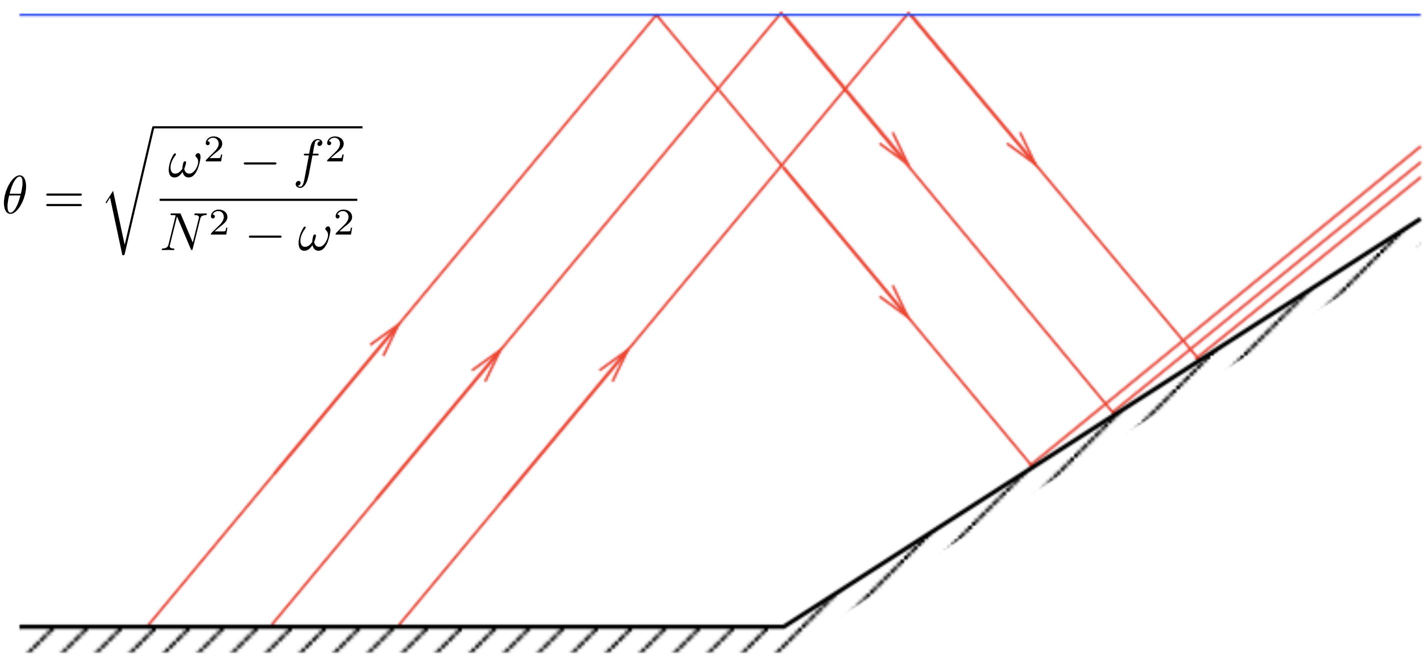
After reflection, energy must again propagate at an angle θ with the vertical, since the wave frequency has not changed.



Reflection of internal waves

Internal wave characteristics approaching a sloping boundary. After reflection, the energy in the waves is concentrated into a narrower band (energy density increases).

$$\tan\theta = \sqrt{\frac{\omega^2 - f^2}{N^2 - \omega^2}}$$



Another famous experiement:

Observation of an internal wave attractor in a confined, stably stratified fluid

http://stockage.univ-brest.fr/~gula/Ondes/maasetal97.pdf

Leo R. M. Maas*, Dominique Benielli†, Joël Sommeria† & Frans-Peter A. Lam*

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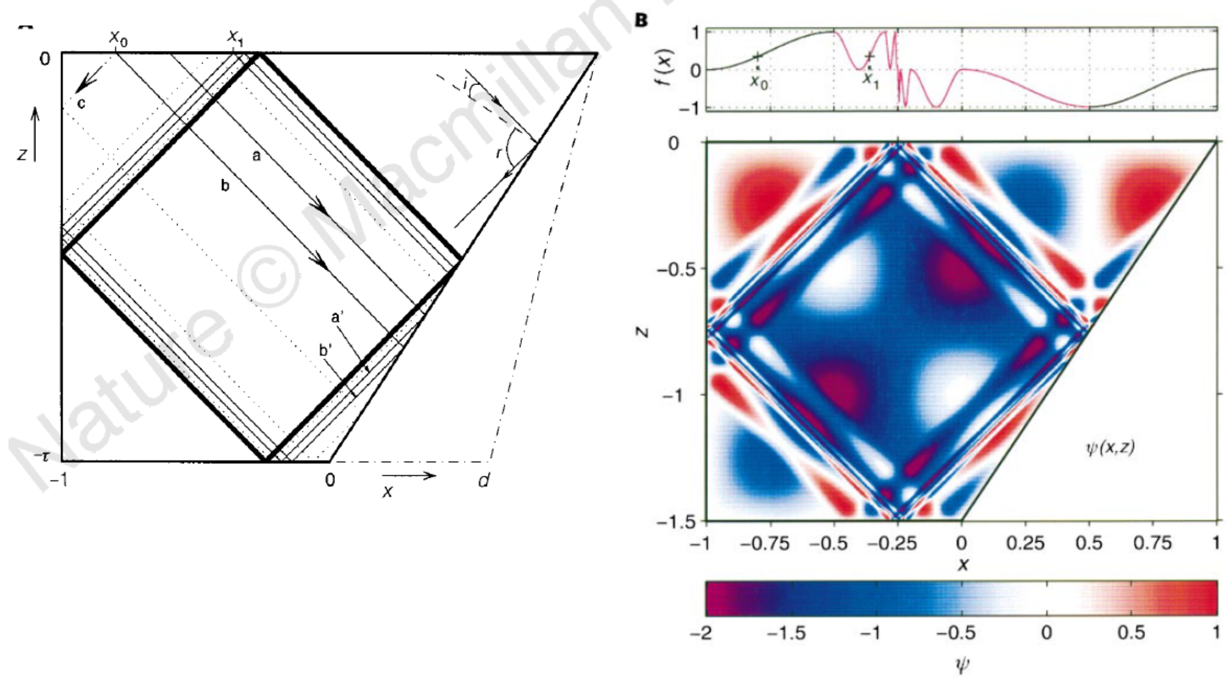

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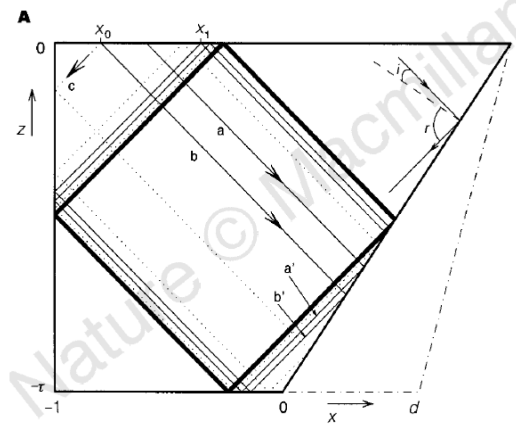

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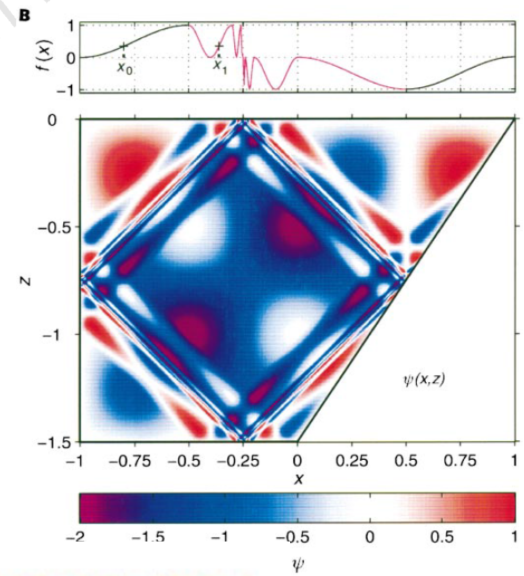

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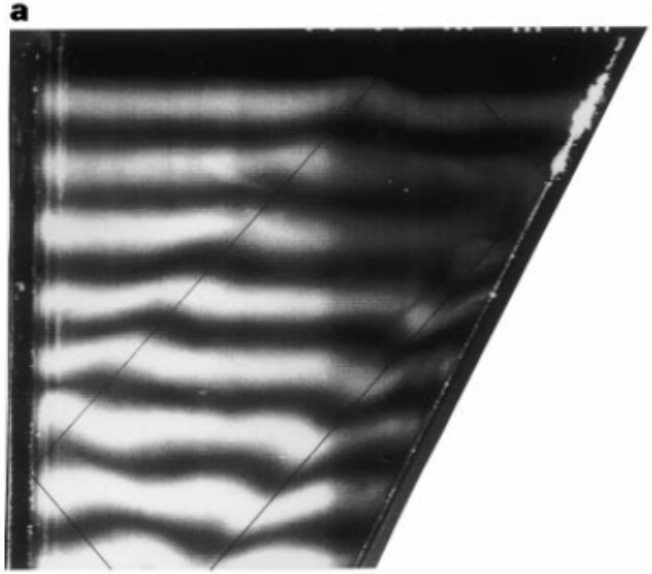

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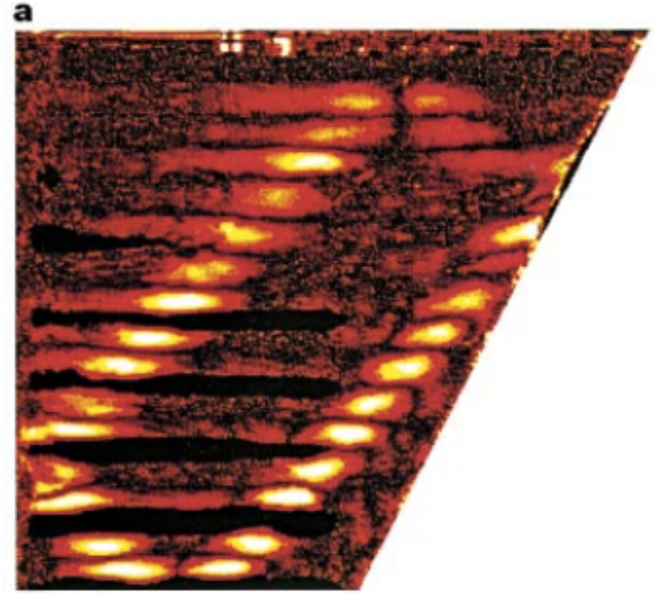


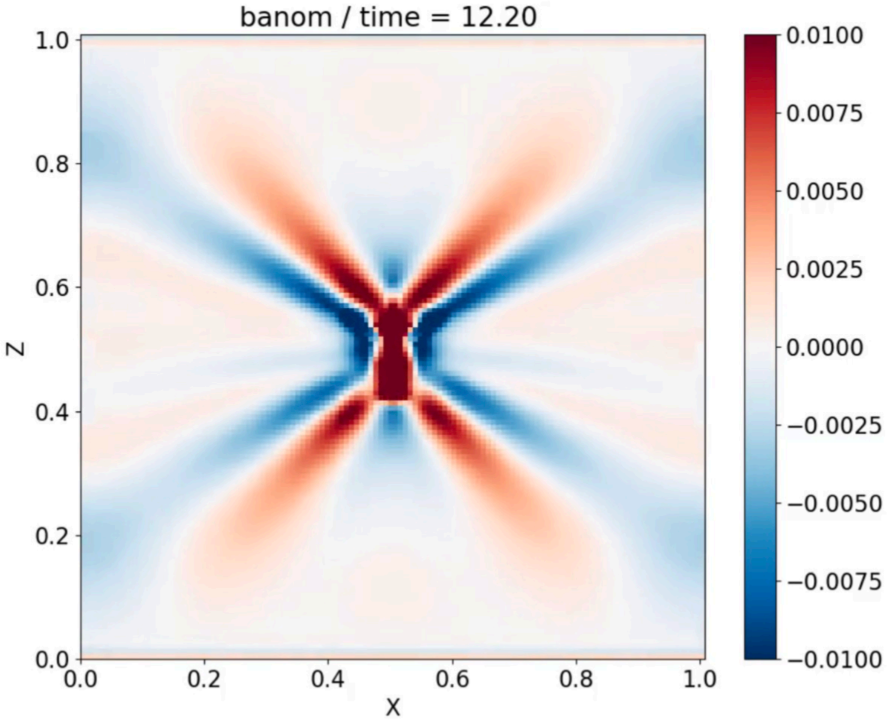
Activity: Numerical simulation of Internal waves

- Download and unzip Fluid2d:
  http://stockage.univ-brest.fr/~gula/Ondes/fluid2d.tar.gz

- Install
  - module load anaconda3
  - tar -xf fluid2d.tar.gz
  - cd fluid2d
  - make

  - bash
  - source activate.sh

Run the Internal wave case:
  - cd nhom/Internal_forced
  - Python forcedinternal.py



- *This is a localized periodic forcing*
- *Play with the forcing frequency (in forcedinternal.py) to generate evanescent waves*