



The energy flux spectrum of internal waves generated by turbulent convection

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We present three-dimensional direct numerical simulations of internal waves excited by turbulent convection in a self-consistent, Boussinesq and Cartesian model of mixed convective and stably stratified fluids. We demonstrate that in the limit of large Rayleigh number ($Ra \in [4 \times 10^7, 10^9]$) and large stratification (Brunt–Väisälä frequencies $f_N \gg f_c$, where f_c is the convective frequency), simulations are in good agreement with a theory that assumes waves are generated by Reynolds stresses due to eddies in the turbulent region as described in Lecoanet & Quataert (*Mon. Not. R. Astron. Soc.*, vol. 430 (3), 2013, pp. 2363–2376). Specifically, we demonstrate that the wave energy flux spectrum scales like $k_\perp^4 f^{-13/2}$ for weakly damped waves (with k_\perp and f the waves' horizontal wavenumbers and frequencies, respectively), and that the total wave energy flux decays with z , the distance from the convective region, like $z^{-13/8}$.

Key words: geophysical and geological flows, internal waves, turbulent convection

1. Introduction

Internal gravity waves can be found in planetary atmospheres and oceans, radiative envelopes of stars, and planetary fluid cores. Numerous studies have shown that internal gravity waves can drive strong large-scale flows (Bordes *et al.* 2012; Grisouard & Bühler 2012; van den Bremer & Sutherland 2018; Couston *et al.* 2018) and enhance turbulence and mixing (Staquet, & Sommeria 2002; Kunze 2017; Thorpe 2018). Through their effects on mean flows and properties of the fluid in which they propagate, internal waves can have a significant impact on the long-term evolution of global systems, such as Earth's climate (Alexander *et al.* 2010) or the differential rotation of stars (Rogers, Lin & Lau 2012).

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The energy flux of internal waves is required to accurately predict the influence of waves on global dynamics, but is difficult to estimate, especially when the generating source is turbulent. Nevertheless, turbulent sources account for a significant fraction of the internal wave energy in the atmosphere, stars and planetary interiors, so they can hardly be neglected. Turbulent motions in the bottom Ekman layer (Taylor & Sarkar 2007) or the surface mixed layer of the ocean (Munroe & Sutherland 2014), turbulent buoyant plumes in the atmosphere (Ansong & Sutherland 2010), fingering convection (Garaud 2018) and penetrative convection in stars (Pinçon, Belkacem & Goupil 2016) are examples of turbulent sources that can generate internal gravity waves.

Past studies of turbulence-driven internal waves have focused on extracting scaling laws for the total wave flux with respect to problem parameters. Information on the energy flux spectrum, i.e. the distribution of energy for different wavenumbers k and frequencies f of internal waves, is often beyond reach or difficult to interpret because of experimental or numerical constraints. For instance, the simulations of Rogers *et al.* (2013) were restricted to two dimensions because of this difficulty. However, knowledge of the energy flux spectrum is essential in predicting wave effects, since both wave propagation, damping and interaction with the background state depend strongly on k and f .

Several theoretical approaches have been put forward to predict the wave spectrum due to turbulent motions. Under the linearized rapid distortion theory (e.g. Carruthers & Hunt 1986), the wave amplitude is proportional to the convective velocity. We instead assume the wave amplitude is linearly proportional to the Reynolds stress (i.e., the square of the convective velocity), which can accurately predict waves in direct numerical simulations (DNS) (Lecoanet *et al.* 2015). Two theoretical descriptions of the Reynolds stresses in turbulent convection are the ‘eddy’ and ‘plume’ theories. In the ‘eddy’ theory, the convection is decomposed into eddies which follow Kolmogorov statistics and eddy stresses are used as generating sources of internal waves in the stable region (Goldreich & Kumar 1990; Lecoanet & Quataert 2013). In the ‘plume’ theory, the convection is seen as an ensemble of uncorrelated plumes impinging at the base of the stable layer (Pinçon *et al.* 2016). The ‘plume’ theory predicts the wave energy flux spectrum to be Gaussian in k and f ; the ‘eddy’ theory predicts power laws.

Here we validate for the first time the prediction of the wave flux spectrum based on the ‘eddy’ theory in the limit of strong stable stratification (Lecoanet & Quataert 2013) using three-dimensional state-of-the-art DNS of a mixed convective and stably stratified fluid model. We note that comparing the theory with DNS results is challenging because the DNS must resolve a fully turbulent flow interacting with a strongly stratified layer and because we need data at high spatial/temporal resolution to construct the spectra.

2. DNS model

The fluid model we use is similar to the one used in Couston *et al.* (2017), though here it is three-dimensional (figure 1). We solve the Navier–Stokes equations under the Boussinesq approximation, i.e. the fluid density is $\rho = \rho_0 + \delta\rho$, where ρ_0 is the (constant) reference density and $\delta\rho \ll \rho_0$ are small variations due to temperature fluctuations. We consider constant kinematic viscosity ν , acceleration due to gravity g and thermal diffusivity κ . The governing equations for the velocity $\mathbf{u} = (u, v, w)$ and temperature T in a Cartesian (x, y, z) frame of reference ($\hat{\mathbf{z}}$ is the upward-pointing

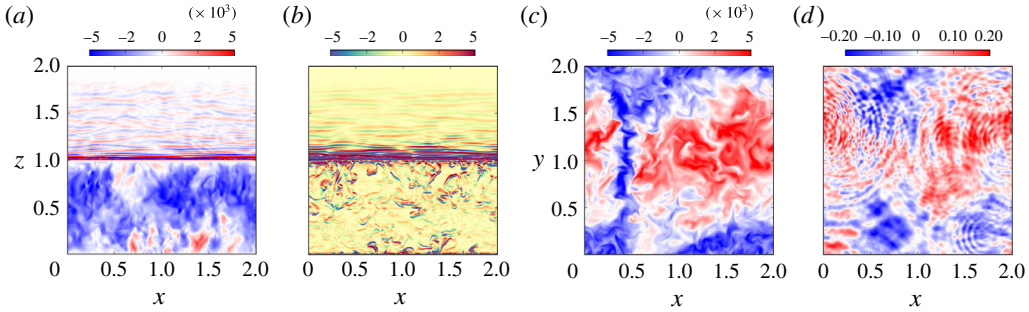


FIGURE 1. Snapshots of (a) $w(y=0)$, (b) $T_z(y=0) - \overline{T_z}$ (overbar denotes the horizontal average), (c) $w(z=0.7)$, (d) $w(z=1.3)$ for case C_8^{400} at steady-state. Variables in the wave region ($z > 1$) in (a), (b) have been multiplied by 10^4 , 10^3 , respectively. A movie is available as Supplementary Material at <https://doi.org/10.1017/jfm.2018.669>.

unit vector of the z -axis) read

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} - g \delta \rho / \rho_0 \hat{\mathbf{z}} - \mathbf{u} / \tau, \quad (2.1a)$$

$$\partial_t T + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T, \quad (2.1b)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.1c)$$

where p is the pressure and $-\mathbf{u} / \tau$ is a damping term only significant close to the upper boundary and used to prevent wave reflection from the top. The density anomaly $\delta \rho$ is related to T through the equation of state

$$\delta \rho / \rho_0 = -\alpha(T)(T - T_i) = -\alpha_s H(T - T_i)(T - T_i) + \alpha_s S H(T_i - T)(T - T_i), \quad (2.2)$$

with H the Heaviside function; $\alpha_s > 0$ is the thermal expansion coefficient for $T > T_i$, with T_i the inversion temperature at which the density anomaly $\delta \rho / \rho_0 = 0$ is maximum, and $S > 0$ is the stiffness parameter. We assume periodicity in the horizontal directions and impose free-slip and fixed temperatures conditions on the top ($T = T_t$) and bottom boundaries ($T = T_b$), with $T_b > T_i > T_t$ (note that the use of no-slip boundary condition at the bottom boundary is not expected to significantly change the results at the interface). Then, because (2.2) is non-monotonic, the lower part of the fluid is convectively unstable, the upper part is stably stratified and S indicates how sharp or ‘stiff’ the transition is at the interface (Couston *et al.* 2017).

We use the characteristic length scale H , the thermal diffusion time H^2 / κ , and the temperature difference $\Delta T = T_b - T_i$ in order to define the non-dimensional variables:

$$(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{(x, y, z)}{H}, \quad \tilde{t} = \frac{t \kappa}{H^2}, \quad \tilde{\mathbf{u}} = \frac{\mathbf{u} H}{\kappa}, \quad \tilde{T} = \frac{T - T_i}{\Delta T}, \quad \tilde{p} = \frac{p H^2}{\kappa^2}, \quad \tilde{\rho} = \frac{\delta \rho}{\rho_0 \alpha_s \Delta T}. \quad (2.3a-f)$$

Substituting (2.3) in (2.1)–(2.2) and dropping tildes we obtain the dimensionless equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + Pr \nabla^2 \mathbf{u} - Pr Ra \rho \hat{\mathbf{z}} - \mathbf{u} / \tau, \quad (2.4a)$$

$$\partial_t T + (\mathbf{u} \cdot \nabla) T = \nabla^2 T, \quad (2.4b)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.4c)$$

$$\rho = -TH(T) + STH(-T), \quad (2.4d)$$

with $Pr = \nu/\kappa$ the Prandtl number and $Ra = \alpha_s g \Delta T H^3 / (\nu \kappa)$ the Rayleigh number. We denote L_x, L_y, L_z the dimensionless lengths of the domain in the x, y, z directions and $T_{top} < 0$ the dimensionless temperature at $z = L_z$ ($T = 1$ at $z = 0$ and the inversion temperature is $T = 0$ in dimensionless space). The dimensionless z -dependent damping coefficient used to prevent wave reflection from the top boundary is $\tau^{-1}(z) = 0.5 f_{N^*} \{\tanh[(z - L_z + 0.15)/0.05] + 1\}$ with f_{N^*} the target buoyancy frequency (see next section).

We solve (2.4) using the open-source pseudo-spectral code DEDALUS (Burns *et al.* 2018). We are interested in DNS results at statistical steady state and in the limit of fully turbulent convection (large Ra) and strong stable stratification (large \mathcal{S}). For ease of presentation, we want the convection to extend from $z = 0$ to $z = z_i \approx 1$ at thermal equilibrium, i.e. such that H is approximately the size of the convection zone in dimensional variables and Ra is the effective Rayleigh number of the convection. z_i is an output of our simulations, but it can be controlled by iteratively adjusting T_{top} (all other parameters fixed). At leading order, $z_i \approx 1$ when T_{top} is chosen such that the measured heat flux $Q = \overline{wT - T_z}$ in the convection (overbar denotes the horizontal average) equals the expected diffusive heat flux of the stable region $\approx -T_{top}/(L_z - z_i)$ with $z_i = 1$. Because Q in the convection zone changes with T_{top} , finding T_{top} is carried out iteratively. We fix all problem parameters except T_{top} , make an initial guess for T_{top} , and run a simulation for a few convective turnover times, initialized with a temperature of $T = (1 - z)H(1 - z) + T_{top}(z - 1)/(L_z - 1)H(z - 1)$ plus low-amplitude noise. We then update our guess for T_{top} using $T_{top} = -Q(L_z - z_i)$ with $z_i = 1$, where Q is the average heat flux at $z = 0$, and we reiterate until T_{top} does not change by more than about 10 %. The final T_{top} gives an estimate for the buoyancy frequency $f_{N^*} = \sqrt{-PrRaST_{top}/(L_z - 1)}/(2\pi)$ used in the expression for the damping term τ in (2.4a). The actual buoyancy frequency $f_N = N/(2\pi)$ is computed using $N = \sqrt{PrRa\mathcal{S}\overline{Q}}$ with Q averaged from $z = 0$ to $z = 1$ and over the simulation time Δt . In all the simulations, we set $L_x = L_y = L_z = L$ with $L = 2$ or $L = 3$. Therefore, the ratio between the horizontal and vertical extent of the convective zone is larger than 2, which ensures that confinement effects associated with the horizontal periodic boundary conditions are not dominating the dynamics. We set $Pr = 1$ for simplicity. Input and output parameters of the eight simulations discussed in this paper are reported in table 1.

The variables of interest include the kinetic energy spectrum and the energy flux spectrum. The kinetic energy per unit surface area and unit (thermal) time reads

$$K(z) = \int \frac{|\mathbf{u}|^2}{2} \frac{dx dy dt}{L^2 \Delta t} \approx \sum_{f, k_\perp} \frac{|\hat{\mathbf{u}}|^2}{2\delta f \delta k_\perp} \delta f \delta k_\perp \equiv \sum_{f, k_\perp} dK \delta f \delta k_\perp \approx \int \frac{\partial^2 K}{\partial f \partial k_\perp} df dk_\perp, \quad (2.5)$$

with hat denoting the Fourier transform in (x, y, t) , Δt the period over which results are integrated in time, k_\perp and f are the horizontal wavenumber and frequency, δk_\perp and δf are the unit increments in spectral space, and dK is the discrete version of the kinetic energy spectrum $\partial^2 K / \partial f \partial k_\perp$. Similarly, we define the z -dependent vertical energy flux as

$$F(z) = \int w p \frac{dx dy dt}{L^2 \Delta t} \approx \sum_{f, k_\perp} \frac{\text{Re}(\hat{w} \hat{p}^*)}{\delta f \delta k_\perp} \delta f \delta k_\perp \equiv \sum_{f, k_\perp} dF \delta f \delta k_\perp \approx \int \frac{\partial^2 F}{\partial f \partial k_\perp} df dk_\perp, \quad (2.6)$$

where $*$ denotes the complex conjugate, dF is the discrete energy flux spectrum and Re denotes the real part. Note that throughout this paper, we assume isotropic motions

Name	$\frac{Ra}{10^8}$	\mathcal{S}	T_{top}	L	$n_x n_y \times (n_{zl} + n_{zu})$	$\frac{\delta t}{10^{-7}}$	$\frac{f_c}{10^3}$	$f_c \Delta t$	$\frac{f_c}{u_{rms}}$	$\frac{f_N}{f_c}$
C_7^{10}	0.4	10	-86	3	$256^2 \times (384 + 256)$	0.5	1.1	9.8	1.6	19
C_7^{100}	0.4	100	-35	2	$256^2 \times (384 + 128)$	2	1.2	10	1.5	50
C_7^{400}	0.4	400	-32	2	$256^2 \times (384 + 192)$	2	1.3	10.7	1.9	86
C_8^{10}	2	10	-140	3	$256^2 \times (512 + 256)$	0.2	2.6	7.1	1.6	23
C_8^{50}	2	50	-62	2	$256^2 \times (512 + 256)$	0.6	2.5	9.1	1.5	49
C_8^{100}	2	100	-60	2	$256^2 \times (512 + 256)$	0.6	2.4	12.5	1.4	71
C_8^{400}	2	400	-52	2	$512^2 \times (768 + 384)$	0.6	3.0	3.8	1.9	109
C_9^{100}	10	100	-100	2	$512^2 \times (768 + 256)$	0.1	5.4	5.4	1.7	90

TABLE 1. Simulation parameters: number of Fourier modes n_x, n_y in the x, y directions, Chebyshev modes n_{zl}/n_{zu} in the lower/upper regions of the compound z basis (stitched at $z = 1.2$), typical time steps δt , best-fit convective frequency f_c for $\mathcal{A} = 30$ (cf. § 4.2), time $f_c \Delta t$ of data collection, convective velocity u_{rms} and Brunt–Väisälä frequency f_N . $Pr = 1$ in all simulations.

in the x, y directions and express variables in spectral space in terms of the average horizontal wavenumber $k_\perp = (k_x^2 + k_y^2)^{1/2}$ (with k_x, k_y the wavenumbers in the x, y directions). Thus, $|\hat{\mathbf{u}}|^2 \equiv |\hat{\mathbf{u}}(k_\perp)|^2$ is the azimuthal average of $|\hat{\mathbf{u}}(k_x, k_y)|^2$ and $\text{Re}(\hat{\mathbf{w}}\hat{p}^*)$ is the azimuthal average of $\text{Re}[\hat{\mathbf{w}}(k_x, k_y)\hat{p}^*(k_x, k_y)]$ based on the modulus of vector (k_x, k_y) , i.e. averaging over all orientations. We apply a Hann windowing function to all variables before taking the time Fourier transform and we discard the modes $f = 0$ and $k_\perp = 0$ from the spectra.

3. Theoretical prediction

We now describe a heuristic model which predicts the wave energy flux spectrum at the interface, $F^i(f, k_\perp)$, between the convective and stably stratified layers. This is based on the calculation in Lecoanet & Quataert (2013) (specifically, the ‘discontinuous’ N^2 case), which itself is based on the earlier calculation in Goldreich & Kumar (1990). We first assume that turbulence driven by the convection can be decomposed into ‘eddies’, each of which has a characteristic length h and velocity u_h . The largest such ‘eddy’ is the large-scale circulation of the convection, which has length $\ell_c = 2\pi/k_c$ and velocity u_c . We imagine this large-scale circulation represents the injection scale of a Kolmogorov turbulent cascade and to recover the $k^{-5/3}$ energy spectrum (k is the three-dimensional wavenumber), we require that the typical velocity of eddies with size h scales like $u_h/u_c = (h/\ell_c)^{1/3}$. We assume each eddy is coherent for an eddy turnover time $\tau_h = h/u_h$, which implies that the power spectrum in the convection zone is peaked along the curve $\tau_h f_c = (h/\ell_c)^{2/3}$. This ignores the fact that small eddies can be advected over their size h on time scales shorter than τ_h by large-scale motions. This set of assumptions is very simplistic, but allows a theoretical derivation of the wave energy flux spectrum.

In this hypothesis, the waves are generated by Reynolds stresses within the convection zone. We calculate the waves generated by a single eddy with size h and corresponding turnover time τ_h , and then sum up the wave energy flux over all eddies. An eddy with size h and turnover time τ_h can only efficiently excite a wave

with frequency f and wavenumber k_\perp if both

$$f \leq \tau_h^{-1}, \quad \ell_c(k_\perp/2\pi) \leq \ell_c/h = (\tau_h f_c)^{-3/2}. \quad (3.1a,b)$$

This is because we assume the spectral representation of an eddy's velocity field is peaked at wavenumbers around $2\pi/h$ and decreases exponentially at higher wavenumbers due to analyticity (and similarly for the temporal spectrum with time scale τ_h). Waves with $f > \tau_h^{-1}$ or $k_\perp > 2\pi/h$ then have exponentially small coupling to eddies of size h .

To connect the properties of the convection to the waves, one can write an equation for the wave velocity as a function of turbulent stresses as (Lecoanet & Quataert 2013),

$$\nabla^2 \partial_t^2 w + N^2(z)(\partial_x^2 + \partial_y^2)w = S, \quad (3.2)$$

where S is related to the Reynolds stresses associated with convective eddies. Note that we have neglected the effects of diffusion in the excitation process and that we have also neglected thermal stresses within the convection zone (as in Lecoanet *et al.* 2015). Lecoanet & Quataert (2013) derive the internal waves' Green's function from this equation assuming $N^2(z)$ is constant in the convective and stably stratified regions, and discontinuous at the interface. One can then estimate the convolution of this Green's function with the Reynolds stress of an eddy to find that the wave energy flux from eddies of size h is

$$\frac{dF^i}{d \log k_\perp d \log f d \log h} = \mathcal{A} u_h^3 \frac{f_c}{f_N} \frac{f}{f_c} \left(\frac{k_\perp h}{2\pi} \right)^4 = \mathcal{A} u_c^3 \frac{f_c}{f_N} \frac{f}{f_c} \frac{h}{\ell_c} \left(\frac{k_\perp h}{2\pi} \right)^4, \quad (3.3)$$

with \mathcal{A} a multiplicative constant, assumed universal, i.e. independent of all problem, wave or eddy parameters. To calculate the wave energy flux for a given frequency and wavenumber, we must sum up excitation from all eddies which satisfy (3.1). Because of the strong dependence on the eddy size in (3.3), the wave energy flux is dominated by the largest possible eddies which can excite a wave. The wave energy flux depends on whether the wave frequency is higher or lower than the frequency of the largest generating eddy $f^*(k_\perp) = f_c[\ell_c k_\perp/(2\pi)]^{2/3}$. Waves with (k_\perp, f) satisfying $f < f^*(k_\perp)$ are not observed in the bulk of the stable region in DNS because they are strongly dissipated (see discussion after (3.5)), so we neglect their contribution to the energy flux in the theory. The expression for the flux of the waves that matter away from the interface, i.e. with frequency $f > f_c[\ell_c k_\perp/(2\pi)]^{2/3}$, is

$$\frac{dF^i}{d \log k_\perp d \log f} = \mathcal{A} u_c^3 \frac{f_c}{f_N} \left(\frac{f}{f_c} \right)^{-13/2} \left(\frac{k_\perp \ell_c}{2\pi} \right)^4. \quad (3.4)$$

We recall that (3.4) is a prediction for the energy flux spectrum of inviscid propagating wave modes at the interface between convective and stably stratified regions. In the DNS, however, the energy flux at the interface has contributions from both inviscid and viscous wave modes and overshooting plumes, such that it is not expected to match the prediction (3.4). In order to avoid this discrepancy, we compare the theory with DNS away from the interface, where overshooting plumes and strongly viscous modes can be neglected.

In order to extend the expression for the flux (3.4) at the interface to an expression valid in all of the stable layer we derive a wave decay rate from the linear equation

$$(\partial_t - \nabla^2) \nabla^2 w + N^2 (\partial_x^2 + \partial_y^2) w = 0, \quad (3.5)$$

valid for $Pr = 1$. Note this differs from the inviscid equation (3.2) used to derive the wave flux at the interface. Substituting a plane wave solution $w \sim e^{i(k_x x + k_y y - \omega t)} e^{i(k_z - \gamma)z}$ with $\omega > 0$ in (3.5) yields a cubic complex equation for $(ik_z - \gamma)^2$ whose six roots can be written in closed form. We assume that motions in the stable layer are dominated by waves propagating upward, i.e. with group velocity $C_{gz} = \partial\omega/\partial k_z > 0$, and whose amplitudes decrease with z . One of the three roots (k_{zj}, γ_j) with $C_{gzj} > 0$ has a much smaller decay rate γ_j than the other two, such that it is expected to be the dominant contribution to upward-propagating and decaying waves in DNS. Unsurprisingly, this solution is the ‘full-dissipation’ equivalent of the simpler Wentzel–Kramers–Brillouin (WKB) solution

$$k_z = \mp k_\perp N/\omega, \quad \gamma = \pm k_\perp^3 N^3/\omega^4, \quad (3.6a, b)$$

with $k_z < 0$ for upward-propagating waves, derived in the weak dissipation limit and valid when $\omega \gg k^2 \geq O(1)$ and $\omega^2 \ll N^2$. We neglect the other two more-rapidly decaying upward-propagating wave solutions of (3.5). Note that whether they capture some of the turbulent energy is beyond the scope of this work, and neglecting them is consistent with prediction (3.4) since the theory only considers the WKB mode (3.6) in its inviscid form. Given that a broad range of frequencies and wavenumbers is excited in the DNS, it may be expected that (3.6) does not in general accurately predict the decay rate of the dominant waves. However, we find that the WKB solution and the ‘full-dissipation’ solution for γ yield similar results for the theoretical spectra presented here, so for simplicity we will use (3.6) in the remainder of this paper (note that this suggests that the discrepancy between the ‘full’ and the WKB decay rate is significant only when both decay rates are so large that it affects waves whose amplitudes are so small that they can be neglected). Note for future discussions that the decay rate is strongest for low-frequency, high-wavenumber waves, as clearly shown by (3.6).

Finally, the z -dependent theoretical wave flux spectrum becomes

$$\frac{dF^{th}}{d \log k_\perp d \log f} = \frac{dF^i}{d \log k_\perp d \log f} e^{-2\gamma(z-z_i)} = \mathcal{A} u_c^3 \frac{2\pi f_c}{N} \left(\frac{f}{f_c}\right)^{-13/2} \left(\frac{k_\perp \ell_c}{2\pi}\right)^4 e^{-2\gamma(z-z_i)}, \quad (3.7)$$

with γ given by (3.6) and z_i the interface height ($z_i \approx 1$ here). A useful approximate expression can be obtained for the total flux as a function of z by integrating (3.7) in the limit $f_N \gg f_c \gg (z - z_i)k_c^3/\pi$ and sufficiently far from the interface, i.e. $z - z_i \gg [\gamma(k_c, f_c)]^{-1}$. The derivation is tedious but straightforward, yielding (cf. appendix A)

$$F^{th} = \int_{f_c}^{f_N} \int_{k_c}^{k_c(f/f_c)^{3/2}} \frac{dF^{th}}{df dk_\perp} dk_\perp df \approx \mathcal{A} u_c^3 \frac{f_c^{15/2}}{f_N^{47/8}} \frac{5\Gamma(\frac{5}{8}) \ell_c^{39/8}}{28(8\pi^2)^{13/8}} (z - z_i)^{-13/8} \equiv \tilde{\mathcal{F}} (z - z_i)^{-13/8}. \quad (3.8)$$

The domain of validity of the asymptotic solution is expected to encompass all of the wave region in DNS. In the next section we show that the wave flux spectrum in DNS follows the trends predicted by the theory, i.e. (3.4) and (3.8).

4. Results and discussion

4.1. Statistics of turbulent convection

The theory described in §3 assumes that the spatial correlations of the turbulent convection follow the standard Kolmogorov prediction for homogeneous and isotropic turbulence. Figure 2(a) shows that the frequency-integrated kinetic energy spectrum

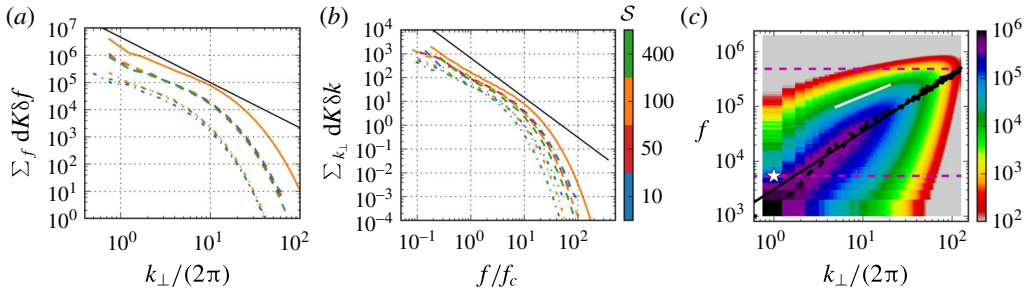


FIGURE 2. Integrated kinetic energy at $z=0.7$ as a function of (a) k_\perp and (b) f . Solid, dash–dash and dash–dot lines correspond to $Ra = 4 \times 10^7$, 2×10^8 and 10^9 and different colours correspond to different stiffnesses. Black solid lines are power laws with slope $-5/3$. (c) Kinetic energy spectrum $k_\perp f dK$ at $z=0.7$ for C_9^{100} . The white star shows the wavenumber–frequency pair (ℓ_c^{-1}, f_c) of the large-scale circulation that allows a good agreement of the DNS with the theory. The pair (ℓ_c^{-1}, f_c) is not unreasonably far from the line $f(k_\perp)$ of maximum energy, which is shown by the black dash–dash line (with the black solid line showing the linear fit). Dashed purple lines show f_c, f_N . The solid white line shows slope $+2/3$.

$\Sigma_f dK \delta f$ indeed has a slope relatively close to $k_\perp^{-5/3}$ for all simulations, thus validating the use of the Kolmogorov hypothesis. The theory also requires prescription of the temporal correlations of the turbulence fluctuations. Here, the theory uses the eddy turnover time as the relevant time scale, which is known as the straining hypothesis. However, one could argue that the temporal correlations of small-scale turbulent features seen at a fixed point in space are affected by the advection by large-scale motions, and that the sweeping hypothesis should be considered instead (Tennekes 1975; Chen & Kraichnan 1989). The wavenumber-integrated kinetic energy spectrum in figure 2(b) shows a frequency scaling somewhat weaker than $f^{-5/3}$, which is the typical scaling for turbulent flows in Eulerian frames with sweeping (Chevallard *et al.* 2005; Canet *et al.* 2017), in agreement with convection experiments (Sano, Wu & Libchaber 1989). The straining mechanism would instead lead to a steeper f^{-2} scaling, which is recovered when computing temporal correlations along Lagrangian trajectories in homogeneous turbulence (Chevallard *et al.* 2005) and in Rayleigh–Bénard convection (Liot *et al.* 2016). Figure 2(c) shows $k_\perp f dK$ for simulation C_9^{100} in (k_\perp, f) space, thus providing a more detailed picture of the turbulence statistics of the most turbulent simulation (similar results are obtained for other simulations). Again, it can be seen that the line of maximum $k_\perp f dK$ is reasonably well approximated by a power fit of the form $f \sim k_\perp^a$ (solid black line), with a close to 1 for all simulations, suggesting strong sweeping effects. Isocontours at higher frequencies, however, have a milder slope $f \sim k_\perp^b$ with $b < 1$, possibly suggesting that the eddy dominant frequency does scale like $\sim k_\perp^{2/3}$ (solid white line) and that sweeping does not affect all eddies. We will show that there is a good agreement between DNS results and the theory based on the straining hypothesis despite the presence of a strong large-scale flow. It should be noted that the theory in § 3 changes with the choice of temporal correlations, as is already the case for other wave generation mechanisms by turbulence (for sound waves, see e.g. Zhou & Rubinstein 1996; Favier, Godefert & Cambon 2010).

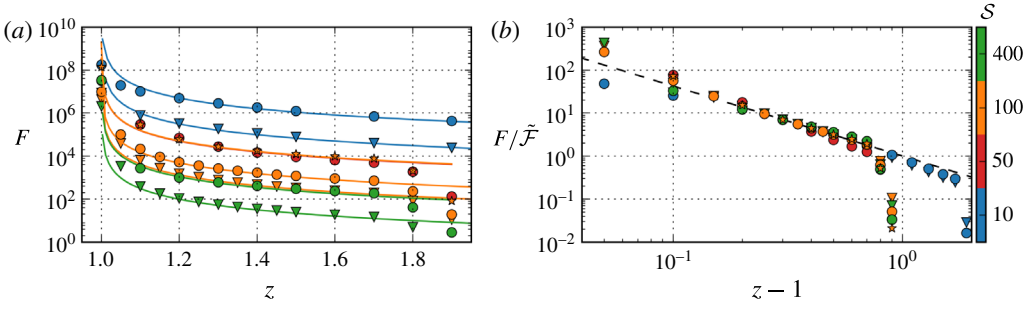


FIGURE 3. (a) Total flux in the wave region in DNS (triangles, circles, stars show results for $Ra = 4 \times 10^7, 2 \times 10^8, 10^9$) along with the prediction F^{th} ((3.8); solid lines) using the best-fit values for \mathcal{A} and f_c reported in table 1. (b) The normalized flux F/\tilde{F} is close to the prediction $F^{th}/\tilde{F} \approx (z-1)^{-13/8}$ (dashed line) for $(z-1) \in [0.1, 0.8]$.

4.2. Integrated wave energy flux

In this section we compare the total energy flux $F(z)$ obtained in DNS with the theoretical prediction $F^{th}(z)$ (cf. (3.8)), and we infer the free parameters of the theory \mathcal{A} and f_c from a best fit. The theoretical total flux F^{th} has a steep scaling with the buoyancy frequency f_N , the interface height z_i and the turbulence parameters ℓ_c , f_c and u_c . While f_N and z_i are relatively well-constrained, the turbulence parameters are challenging to estimate from DNS of turbulent convection, making F^{th} potentially highly sensitive to the definitions used. A slight variation of f_c by a factor of 2, due to a change of definition, can lead to a change of F^{th} by a factor of approximately 200 at all heights. Here, instead of estimating the values of the turbulence parameters using *ad hoc* definitions, we use a best-fit approach. For simplicity, we assume that $\ell_c = 1$ in all cases and thus take $u_c = f_c$. This leaves f_c as the only free parameter in the theory, besides \mathcal{A} , which is assumed to be the same for all simulations. *A posteriori*, we check that the best-fit f_c is not too different from the r.m.s. velocity $u_{rms} = \sqrt{2K}$ at $z = 0.7$ in the convection zone (cf. (2.5) and note that u_{rms} is relatively depth-invariant in the convective bulk).

We obtain f_c for each simulation case by requiring that the asymptotic prediction (3.8) equals the total flux in DNS at some height z^* in the wave region, i.e.

$$f_c^{21/2} = \frac{F(z^*)}{\mathcal{A}} f_N^{47/8} \frac{28(8\pi^2)^{13/8}}{5\Gamma(5/8)} (z^* - z_i)^{13/8}. \quad (4.1)$$

This definition of f_c may appear degenerate since f_c can take different values depending on \mathcal{A} or z^* . However, as discussed below (cf. also figure 3a), once \mathcal{A} is chosen, f_c does not depend on z^* provided that z^* is sufficiently far from the interface $z_i = 1$ (i.e. $z^* > 1.1$). With $\mathcal{A} \sim O(10)$, we find for all simulations that f_c is almost equal to u_{rms} (to within a factor of two), which can be considered as a reference value for the turnover time of the largest-scale eddies (Lecoanet *et al.* 2015). From now on we set the multiplicative constant arbitrarily to $\mathcal{A} = 30$. Other values could be chosen, but do not lead to significant changes as long as f_c is relatively close to u_{rms} .

We compare DNS results for $F(z)$ (symbols) with the theoretical prediction $F^{th}(z)$ (solid lines) in figure 3(a) for all simulation cases, using the parameter values listed in table 1 (with $\mathcal{A} = 30$). The total wave flux decreases with z in all cases and,

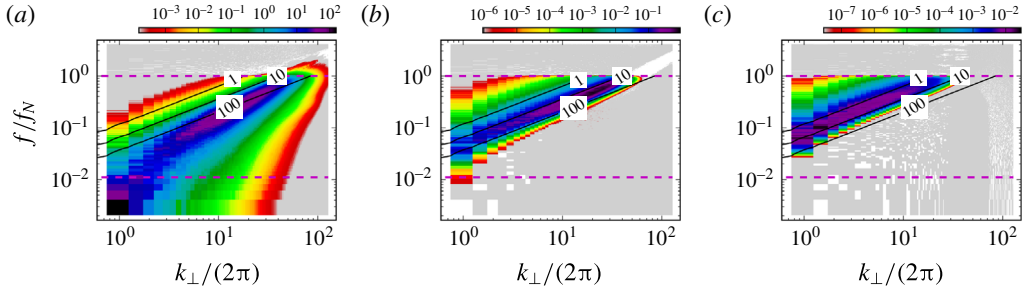


FIGURE 4. Wave flux spectrum $k_{\perp} f dF$ in DNS for case C_9^{100} at (a) $z=1.0$, (b) $z=1.1$, (c) $z=1.5$. The colour scale is shifted to lower values from (a) to (c) because the wave flux decreases with height. Solid black lines show isocontours of the decay rate 2γ . Dashed purple lines show f_c, f_N . White bins indicate $k_{\perp} f dF$ values that are negative.

as expected, is larger for higher Ra (for fixed S) and lower S (for fixed Ra) (Couston *et al.* 2017). With f_c obtained from a best fit at just one particular height, we find an excellent agreement between DNS and the theoretical prediction in all of the bulk of the wave region ($z \in [1.1, 1.8]$) for all simulations, which demonstrates that f_c would be similar for any $z^* \in [1.1, 1.8]$. The $(z-1)^{-13/8}$ scaling predicted by the asymptotic solution is well verified in figure 3(b), which shows the normalized wave flux F/\tilde{F} (cf. (3.8)). All DNS results collapse on the asymptotic solution $(z-1)^{-13/8}$ shown by the dashed line for $(z-1) \in [0.1, 0.8]$, demonstrating that the theoretical flux spectrum integrated over the full range of waves excited compares well with the total flux in DNS.

4.3. Wave energy flux spectrum

We now investigate in details the scalings of the wave flux spectrum $k_{\perp} f dF$ in DNS with k_{\perp} and f , and discuss how they compare with the theoretical prediction $\sim k_{\perp}^4 f^{-13/2}$. We first show in figure 4(a-c) the overall wave flux spectrum $k_{\perp} f dF$ in (k_{\perp}, f) space at heights $z=1, 1.1, 1.5$ for simulation C_9^{100} , which is the most turbulent case. The wave flux at $z=1$ can be difficult to interpret because it contains traces from both all types of waves and overshooting plumes, with one or the other type dominating depending on whether the convection zone extends slightly above or slightly below $z=1$. At $z=1.1, 1.5$, however, we are clearly in the propagating wave region and the spectrum has all of the features predicted by the theory: in the high-frequency, low-wavenumber range (top left corner), the wave flux spectrum decreases rapidly with frequency and increases with wavenumber up to a cutoff wavenumber. The line of maximum wave flux spectrum overlaps with an isocontour of the decay rate 2γ (cf. solid black lines and (3.6)), and that isocontour is closer to the top left corner as z increases because this corner corresponds to waves which are the least damped. The wave flux spectrum falls off quickly at frequencies $f > f_N$ (top dashed line), which is the maximum frequency for propagating waves. Note that while upward-going waves require $dF > 0$, few (f, k_{\perp}) modes can be seen to have azimuthally averaged $dF \leq 0$ in figures 4 (shown by white bins). However, they have negligible $|dF|$ compared to positive dF 's, which is why we restrict the colour map to $dF > 0$ only.

The wave flux spectrum is shown as a function of f for fixed wavenumber in figure 5(a,b), and as a function of k_{\perp} for fixed frequency in figure 5(c,d), providing a

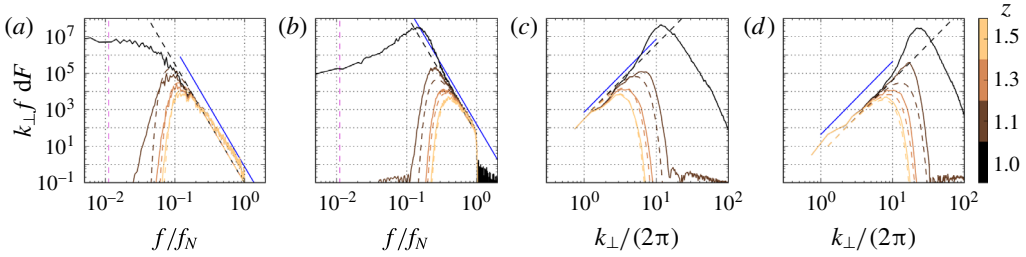


FIGURE 5. Wave flux spectrum $k_{\perp} f dF$ for fixed $k_{\perp}/(2\pi) = 1.7, 8.2$ (a,b) and fixed $f/f_N = 0.2, 0.4$ (c,d) for case C_9^{100} . Lighter colours show results at higher heights. Solid lines show DNS results and dashed lines show theoretical results using the convection parameters listed in table 1. The blue lines show the theoretical scalings. The vertical dashed purple lines in (a,b) show f_c/f_N .

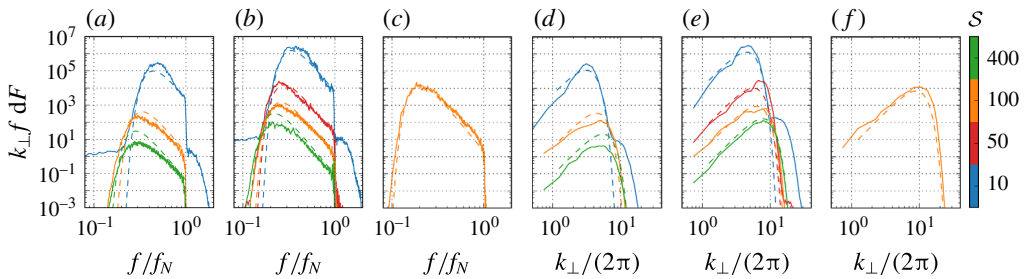


FIGURE 6. Wave flux spectrum $k_{\perp} f dF$ at $z = 1.3$ for fixed $k_{\perp}/(2\pi) = 3.7$ (a–c) and fixed $f/f_N = 0.4$ (d–f) with $Ra = 4 \times 10^7, 2 \times 10^8, 10^9$ increasing from (a) to (c) and from (d) to (f). Different colours correspond to different stiffnesses S . Solid lines show DNS results and dashed lines show theoretical results using the convection parameters in table 1.

quantitative comparison of DNS results (solid lines) with the theory (dashed lines) for C_9^{100} . The decrease of the wave flux spectrum in DNS with f is steep and close to the predicted $f^{-13/2}$ scaling (shown by the blue solid line). The increase of the wave flux spectrum with k_{\perp} closely follows the scaling k_{\perp}^4 , also predicted by the theory (shown by the blue solid line). Again, in figure 5(a,b) (respectively (c,d)) we note that the maximum flux moves towards higher frequencies (respectively lower wavenumbers) at higher z as a result of damping. We have checked that similar trends in frequency space and wavenumber space are obtained for other fixed low wavenumbers (within the inertial subrange) and fixed high frequencies ($f \gg u_{rms}$), suggesting an overall good agreement between DNS and theory for C_9^{100} .

We now demonstrate that a good agreement is obtained between DNS and the theory not only for C_9^{100} but for all simulations listed in table 1. In figure 6(a–c), we show the wave flux spectrum at $z = 1.3$ as a function of f for $k_{\perp} = 3.7$ with $Ra = 4 \times 10^7, 2 \times 10^8, 10^9$, respectively. Clearly, the DNS results (solid lines) overlap well with the theoretical predictions (dashed lines) obtained after estimating the convective frequency f_c from a best-fit approach (cf. § 4.2 and listed values in table 1). A good agreement is similarly obtained for the wave flux spectrum at $z = 1.3$ as a function of k_{\perp} and for $f/f_N = 0.4$ between DNS and the theory (cf. figure 6d–f).

5. Concluding remarks

We have shown that the scaling of the energy flux spectrum of internal waves generated by turbulent convection in DNS follows the $k_{\perp}^4 f^{-13/2}$ prediction in the high- \mathcal{S} regime (Lecoanet & Quataert 2013), and consistently, the total flux decreases as $(z-1)^{-13/8}$ for all simulations presented. As discussed in the previous section, we can match the flux predicted by the theory to the DNS if we multiply the analytical prediction by a universal constant \mathcal{A} and use a best fit for f_c . It is unsurprising that the theory needs be rescaled since numerical factors have been dropped off along the derivation, and it is equally unsurprising that the best-fit f_c equals a few times the reference frequency based on the r.m.s. velocity u_{rms} . It is worth pointing out, however, that there are some variations of f_c/u_{rms} across cases, which may mask some effects not included in the theory. Future work should explore the sweeping hypothesis, and include longer integration times with additional simulations or laboratory experiments in the high-turbulence, high- \mathcal{S} regime.

It is remarkable that there is such a good agreement between the DNS and an approximate-at-best theory based on second-order statistics of turbulent motions. The theory, as simple as it is, may prove useful in assessing the detectability of internal waves in geophysical and astrophysical fluids such as stars and in estimating wave-driven changes of large-scale fluids from the energy in turbulent sources.

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Appendix A

Here we present the key steps required to derive (3.8). Let us use the variables $f/f_c = \xi$, $k/k_c = \eta$, $2(2\pi)^2[k_c/(2\pi)]^3 f_N^3 f_c^{-4}(z - z_i) = z^*$. The integration of (3.8) in η results in

$$\begin{aligned} \frac{F(z)}{u_c^3 \frac{f_c}{f_N}} &= \int_1^{f_N/f_c} \int_1^{\xi^{3/2}} \frac{\xi^{-13/2} \eta^4}{\xi \eta} e^{-\eta^3 \xi^{-4} z^*} d\eta d\xi = \int_1^{f_N/f_c} \xi^{-15/2} \int_1^{\xi^{3/2}} \eta^3 e^{-\eta^3 \xi^{-4} z^*} d\eta d\xi \\ &= \int_1^{f_N/f_c} \left\{ \underbrace{\frac{1}{3z^*} [\xi^{-7/2} e^{-\xi^{-4} z^*} - \xi^{-2} e^{-\xi^{1/2} z^*}]}_{\mathcal{I}_1} \right. \\ &\quad \left. + \underbrace{\frac{\xi^{-13/6}}{9(z^*)^{4/3}} \left[\Gamma\left(\frac{1}{3}, \xi^{-4} z^*\right) - \Gamma\left(\frac{1}{3}, \xi^{1/2} z^*\right) \right]}_{\mathcal{I}_2} \right\} d\xi, \end{aligned} \quad (\text{A } 1)$$

with Γ the upper incomplete gamma function. Integrating \mathcal{I}_1 with respect to ξ yields

$$\mathcal{I}_1 = \left[\frac{\Gamma\left(\frac{5}{8}, z^* x^{-4}\right)}{12(z^*)^{13/8}} \right]_1^{f_N/f_c} - \left[\frac{z^*}{3} \text{Ei}\left(-z^* x^{1/2}\right) + \frac{e^{-z^* x^{1/2}} (z^* x^{1/2} - 1)}{3z^* x} \right]_1^{f_N/f_c} \approx \frac{\Gamma\left(\frac{5}{8}\right)}{12(z^*)^{13/8}}, \quad (\text{A } 2)$$

with Ei the exponential integral and the final expression is the leading-order approximation in the limits $f_N/f_c \gg 1$, $z^* \gg 1$ and $z^*(f_c/f_N)^4 \ll 1$. In the same limits, $\mathcal{I}_2 \approx (2/21)\Gamma(5/8)(z^*)^{-13/8}$, such that adding the leading-order approximations for \mathcal{I}_1 and \mathcal{I}_2 yields (3.8). Equation (3.8) is valid for roughly all simulation parameters and $(z - z_i) \in [0.01, 10]$.

Supplementary movie

Supplementary movie is available at <https://doi.org/10.1017/jfm.2018.669>.

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