TURBULENCE

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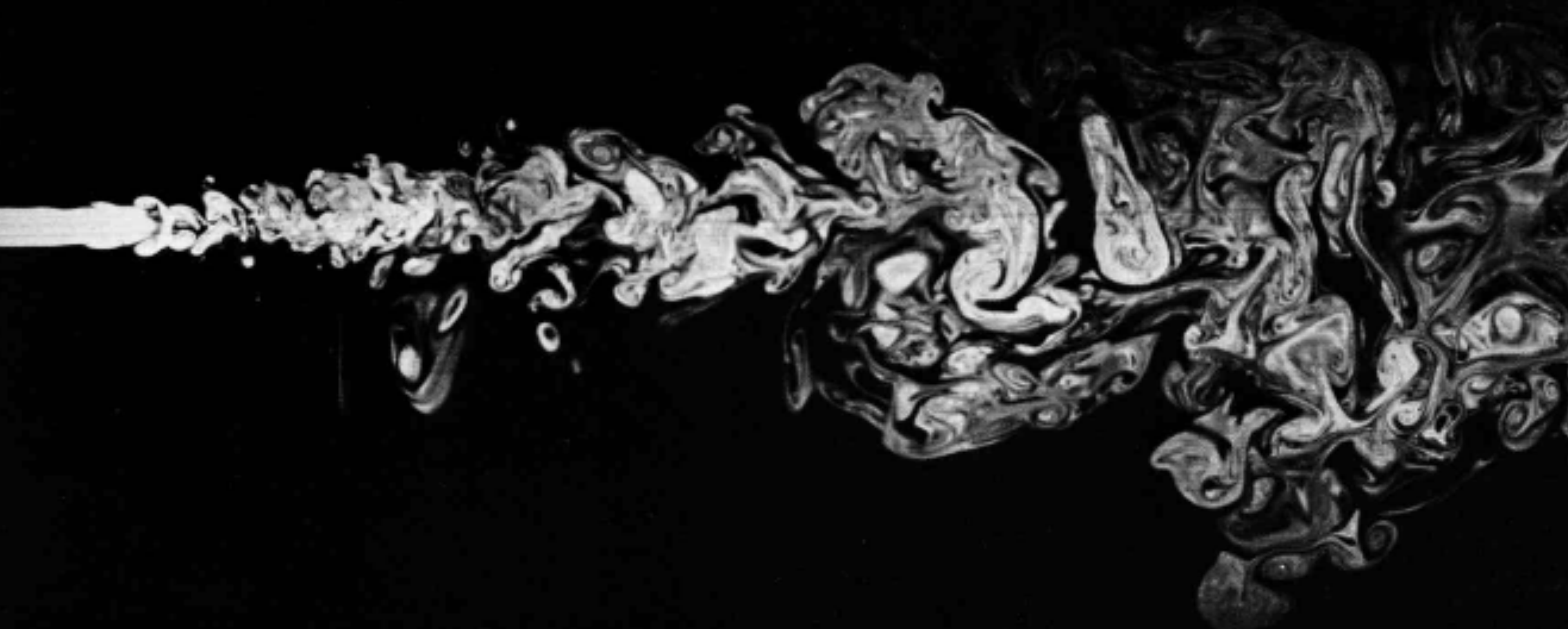


TURBULENCE

2. 3D TURBULENCE

- **Lesson 1 :** [D109]
  - Introduction
  - Properties of turbulence
- **Lesson 2 :** [D109]
  - 3D turbulence: The Kolmogorov theory
  - 2D turbulence
- **Lesson 3 :**[D109]
  - 2D turbulence (activity)
- **Lesson 4 :**[D109]
  - Geostrophic turbulence
  - Surface QG turbulence
  - Ocean turbulence (activity)
  - Turbulent diffusion

Presentations and material will be available at :

jgula.fr/Turb/

References:

- Vallis G.K., Atmospheric and Oceanic Fluid Dynamics.
- MIT online course: https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-820-turbulence-in-the-ocean-and-atmosphere-spring-2007/lecture-notes/
- LaCasce J.H., Turbulence in the Atmosphere and Ocean.

- The equation of motions are deterministic, but their solutions have many attributes of random processes.
- Individual trajectories are erratic and difficult to predict. However ensemble of trajectories reflects the mean structure of the attractor and can be predicted.
- Hence the need for a **statistical description of turbulence**.

The Search for Universal Properties and the Kolmogorov Scaling Laws

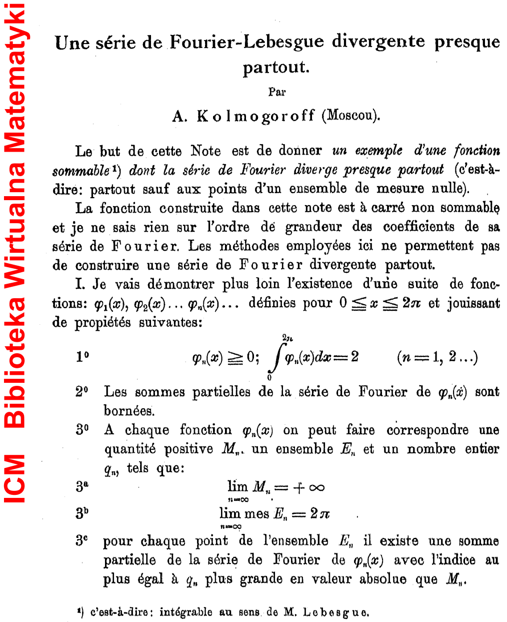
The foundation of many theories of turbulence is the spectral theory of Kolmogorov.

Andrey Nikolaevich Kolmogorov

Soviet Mathematician (1903 - 1987)

Kolmogorov

In 1922, Kolmogorov gained international recognition for constructing a Fourier series that diverges almost everywhere.



Une série de Fourier-Lebesgue divergente presque partout.

Par

A. Kolmogoroff (Moscou).

Le but de cette Note est de donner *un exemple d'une fonction sommable*[1]) *dont la série de Fourier diverge presque partout* (c'est-à-dire: partout sauf aux points d'un ensemble de mesure nulle).

La fonction construite dans cette note est à carré non sommable et je ne sais rien sur l'ordre de grandeur des coefficients de sa série de Fourier. Les méthodes employées ici ne permettent pas de construire une série de Fourier divergente partout.

I. Je vais démontrer plus loin l'existence d'une suite de fonctions: $\varphi_1(x), \varphi_2(x) \dots \varphi_n(x) \dots$ définies pour $0 \leqq x \leqq 2\pi$ et jouissant de propiétés suivantes:

1° $\varphi_n(x) \geqq 0; \quad \int_0^{2\pi} \varphi_n(x)dx = 2 \qquad (n = 1, 2 \dots)$

2° Les sommes partielles de la série de Fourier de $\varphi_n(x)$ sont bornées.

3° A chaque fonction $\varphi_n(x)$ on peut faire correspondre une quantité positive M_n, un ensemble E_n et un nombre entier q_n, tels que:

3ª $\lim_{n=\infty} M_n = +\infty$

3ᵇ $\lim_{n=\infty} \text{mes}\, E_n = 2\pi$

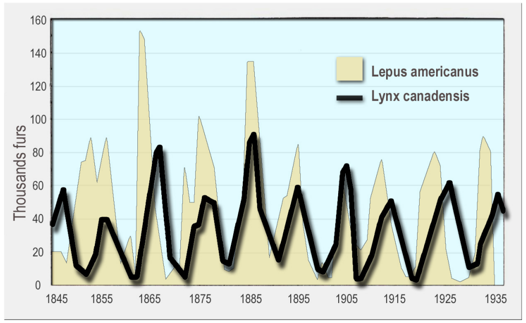
3ᶜ pour chaque point de l'ensemble E_n il existe une somme partielle de la série de Fourier de $\varphi_n(x)$ avec l'indice au plus égal à q_n plus grande en valeur absolue que M_n.

[1]) c'est-à-dire: intégrable au sens de M. Lebesgue.

In 1925, he came interested in probabilistic theory. In 1933, he published: **Foundations of the Theory of Probability**

He also contributed to the field of ecology and generalized the Lotka–Volterra model of **predator–prey systems**.

$$\frac{dx}{dt} = \alpha x - \beta x y,$$
$$\frac{dy}{dt} = \delta x y - \gamma y,$$



In a 1938, he established the basic theorems for smoothing and predicting stationary **stochastic processes**. *[contributed to the Russian war effort by developing a scheme of stochastic distribution of barrage balloons intended to help protect Moscow from German bombers]*

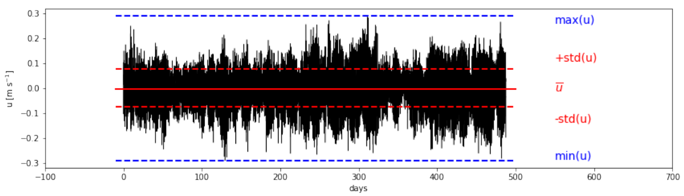
Kolmogorov

After 1941, he focused his research on **turbulence**.

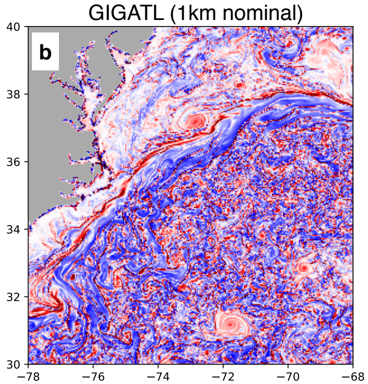
“I took an interest in the study of turbulent flows of liquids and gases in the late thirties. It was clear to me from the very beginning that the main mathematical instrument in this study must be the theory of random functions of several variables (random fields) which had only then originated. Moreover, it soon became clear to me that there was no chance of developing a purely closed mathematical theory. Due to the lack of such a theory, it was necessary to use some hypotheses based on the results of the treatment of the experimental data.”

And constructed a probabilistic description of turbulence.

Random variable = velocity of the fluid at each point



in time and space



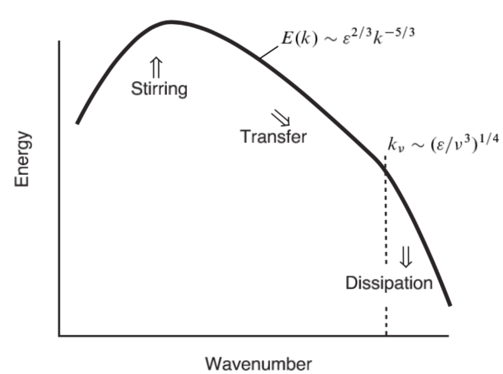
Kolmogorov

- [1] The local structure of turbulence in an incompressible fluid at very high Reynolds numbers. Dokl. Acad. Nauk USSR, 30 (1941), p. 299-303.
- [2] The logarithmically normal distribution of the size of particles under the fragmentation. Dokl. Acad. Nauk USSR, 31 (1941), p. 99-101.
- [3] The decay of isotropic turbulence in an incompressible viscous fluid. Dokl. Acad. Nauk USSR 31 (1941), p. 538-541.
- [4] Energy dissipation in locally isotropic turbulence. Dokl. Acad. Nauk USSR, 32 (1941), p. 19-21.

The Search for Universal Properties and the Kolmogorov Scaling Laws

The **"Law of two-thirds"** is the pearl of the first investigations by A.N. Kolmogorov. This is a universal law of the turbulence nature, supported by the experiments made for the fluids with high Reynolds numbers.

This theory does not close the equations, but provides a prediction for the **energy spectrum of a turbulent flow** *(how much energy is present at a particular spatial scale)* by suggesting a relationship between the energy spectrum (a second order quantity in velocity) and the spectral energy flux (a third order quantity).



Equations for the flow:

- Central in what follows are **two conservation laws**.
- These are for **energy** and **enstrophy**.

Equations for the flow:

- We start from the incompressible Navier-Stokes equations:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

- The energy integrated over a closed (or periodic) domain is

$$E = \iiint \frac{1}{2} |\vec{u}^2| \, dV$$

Activity 3

- Starting from Incompressible NS equations: $\vec{\nabla} \cdot \vec{u} = 0$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

- Write the energy equation integrated over a closed (or periodic) domain.

$$\frac{d}{dt}E = \qquad \text{with} \qquad E = \iiint \frac{1}{2} |\vec{u}^2| \, dV$$

- *You can use:*

$$\vec{u} \cdot \vec{\nabla} \vec{u} = (\vec{\nabla} \times \vec{u}) \times \vec{u} + \frac{1}{2} \vec{\nabla}(\|\vec{u}\|^2)$$

Conservation laws

- Equation for kinetic energy

$$\frac{\partial}{\partial t}\frac{1}{2}|\vec{u}^2| + \nabla \cdot (\vec{u}\frac{1}{2}|\vec{u}^2|) = -\nabla \cdot [\vec{u}(\frac{p}{\rho_0} + gz)] + \vec{u} \cdot \mathcal{F} + \nu \vec{u} \cdot \nabla^2 \vec{u}$$

- Energy integrated over a closed (or periodic) domain:

$$E = \iiint \frac{1}{2}|\vec{u}^2|\, dV \qquad \frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F}\, dV + \nu \iiint \vec{u} \cdot \nabla^2 \vec{u}\, dV$$

- the inertial terms in the momentum equation conserve energy (redistribute in the domain)

Conservation laws

$$\frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F}\, dV + \nu \iiint \vec{u} \cdot \nabla^2 \vec{u}\, dV$$

- Can be rewritten using identities:

$$\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u}) = -\nabla \times \vec{\omega}$$

$$\vec{u} \cdot \nabla^2 \vec{u} = -\vec{u} \cdot (\nabla \times \omega) = -\vec{\omega} \cdot (\nabla \times \vec{u}) + \nabla \cdot (\vec{\omega} \times \vec{u})$$

$$\nu \iiint \vec{u} \cdot \nabla^2 \vec{u}\, dV = -\nu \iiint \vec{\omega} \cdot (\nabla \times \vec{u})\, dV = -\nu \iiint |\vec{\omega}|^2\, dV$$

forcing dissipation

$$\boxed{\frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F}\, dV - \nu \iiint |\vec{\omega}|^2\, dV}$$

Conservation laws

forcing dissipation

$$\frac{d}{dt}E = \iiint \vec{u}\cdot\mathcal{F}\,dV - \nu \iiint |\vec{\omega}|^2\,dV$$

- Forcing puts energy in the system and dissipation removes it.
- Dissipation is proportional to the integral of the squared vorticity, also known as the ***enstrophy***

Conservation laws

forcing dissipation

$$\boxed{\frac{d}{dt}E = \iiint \vec{u}\cdot\mathcal{F}\,dV - \nu \iiint |\vec{\omega}|^2\,dV}$$

- How is the energy transferred from the forcing scales to the dissipative scales?

Conservation laws

$$\boxed{\frac{d}{dt}E = \underbrace{\iiint \vec{u}\cdot\mathcal{F}\,dV}_{\text{forcing}} - \underbrace{\nu \iiint |\vec{\omega}|^2\,dV}_{\text{dissipation}}}$$

- **One important experimental law of turbulence:**
  - Dissipation stays finite, even for $Ro \to \infty \;\; (\nu \to 0)$

Cascade of energy

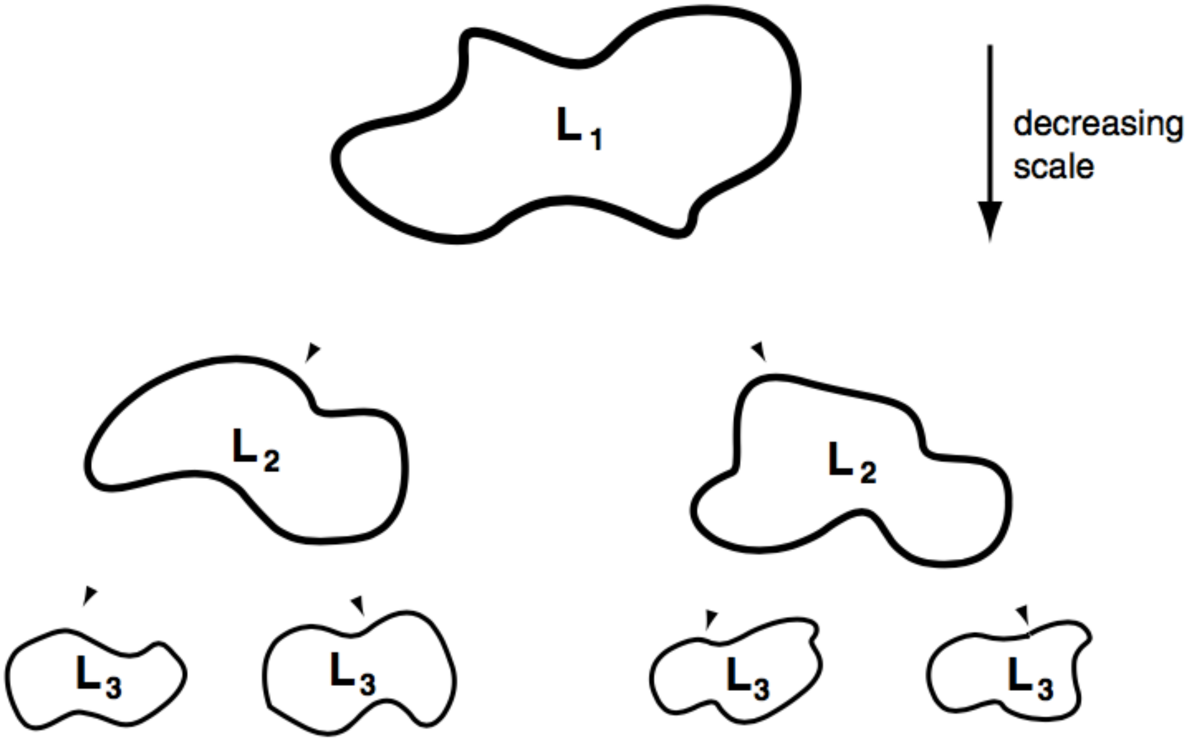


Fig. 8.2 Schema of a 'cascade' of energy to smaller scales: eddies at a large scale break up into smaller scale eddies, thereby transferring energy to smaller scales. If the transfer occurs between eddies of similar sizes (i.e., if it is spectrally local) the transfer is said to be a cascade. The eddies in reality are embedded within each other.

Kolmogorov's inertial range

- *Thus forcing puts energy into the system and dissipation removes it. We assume the forcing happens at much larger scales than the dissipation, which happens on molecular scales, and that there is a range of scales in between where neither forcing or dissipation are important.*

- Kolmogorov proposed a theory in 1941 for this transfer, which has become known as the *inertial range*, with a few assumptions:

  - Turbulence is *isotropic*—the same in all directions.
  - Turbulence is *homogeneous*—the same at all locations in space.
  - Triad interactions are *local* (in spectral space)

Kolmogorov's inertial range

- Details of forcing and dissipation don't matter in the inertial range.

- The *only* important parameter in the inertial range is the rate at which energy is transferred downscale. We call this the energy flux: ϵ

- If we decompose velocities in Fourier space:

$$u(x,y,z,t) = \sum_{k^x,k^y,k^z} \tilde{u}(k^x,k^y,k^z,t)\mathrm{e}^{i(k^x x+k^y y+k^z z)}$$

- The energy can be written (Parseval theorem)

with energy spectral density

$$\hat{E} = \int E\,\mathrm{d}V = \frac{1}{2}\int (u^2+v^2+w^2)\,dV$$
$$= \frac{1}{2}\sum(|\tilde{u}|^2+|\tilde{v}|^2+|\tilde{w}|^2)\,\mathrm{d}\boldsymbol{k}$$

$$\hat{E} \equiv \int \mathcal{E}(k)\,\mathrm{d}k$$

Kolmogorov's inertial range

- If the rate of energy input per unit volume by stirring is equal to ϵ then if we are in a steady state there must be a flux of energy from large scales to small also equal to ϵ, and an energy dissipation rate, also ϵ
- So the general form of energy spectral density is

$$\mathcal{E}(k) = g(\varepsilon, k, k_0, k_\nu)$$

Forcing wavenumber

Dissipation wavenumber

- In the inertial range (Assuming locality), we don't feel forcing and dissipation so:

$$\mathcal{E}(k) = g(\varepsilon, k)$$

Activity 4

- Find the function g such that: $\mathcal{E}(k) = g(\varepsilon, k)$

Kolmogorov's inertial range

Dimensions and the Kolmogorov Spectrum

Quantity	**Dimension**
Wavenumber, k	1/L
Energy per unit mass, E	$U^2 = L^2/T^2$
Energy spectrum, $\mathcal{E}(k)$	$EL = L^3/T^2$
Energy Flux, ε	$E/T = L^2/T^3$

If $\mathcal{E} = f(\varepsilon, k)$ then the only dimensionally consistent relation for the energy spectrum is

$$\mathcal{E} = \mathcal{K}\varepsilon^{2/3}k^{-5/3}$$

where $\mathcal{K}$ is a dimensionless constant.

- The parameter K is a dimensionless constant, undetermined by the theory. It is known as Kolmogorov's constant and experimentally it is found to be approximately 1.5.

Kolmogorov's inertial range

- At some small length-scale we should expect viscosity to become important and the scaling theory we have just set up will fail.

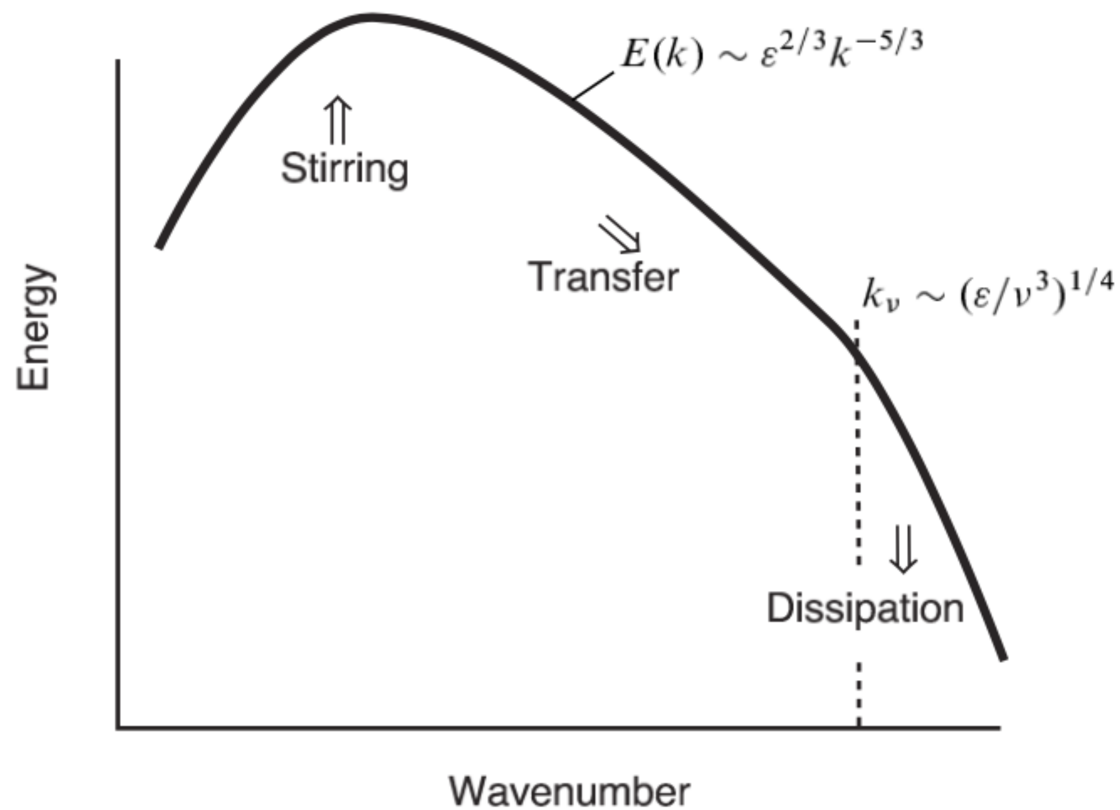
- This is given by the Kolmogorov Length scale:

$$k_\nu \sim \left(\frac{\varepsilon}{\nu^3}\right)^{1/4}, \qquad L_\nu \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

Kolmogorov's inertial range

- This is the famous '**Kolmogorov -5/3 spectrum**'. Cornerstone of turbulence theory

Figure 8.3 Schema of energy spectrum in three-dimensional turbulence, in the theory of Kolmogorov. Energy is supplied at some rate ε; it is transferred ('cascaded') to small scales, where it is ultimately dissipated by viscosity. There is no systematic energy transfer to scales larger than the forcing scale, so here the energy falls off.



Kolmogorov's inertial range

- Experimental validation

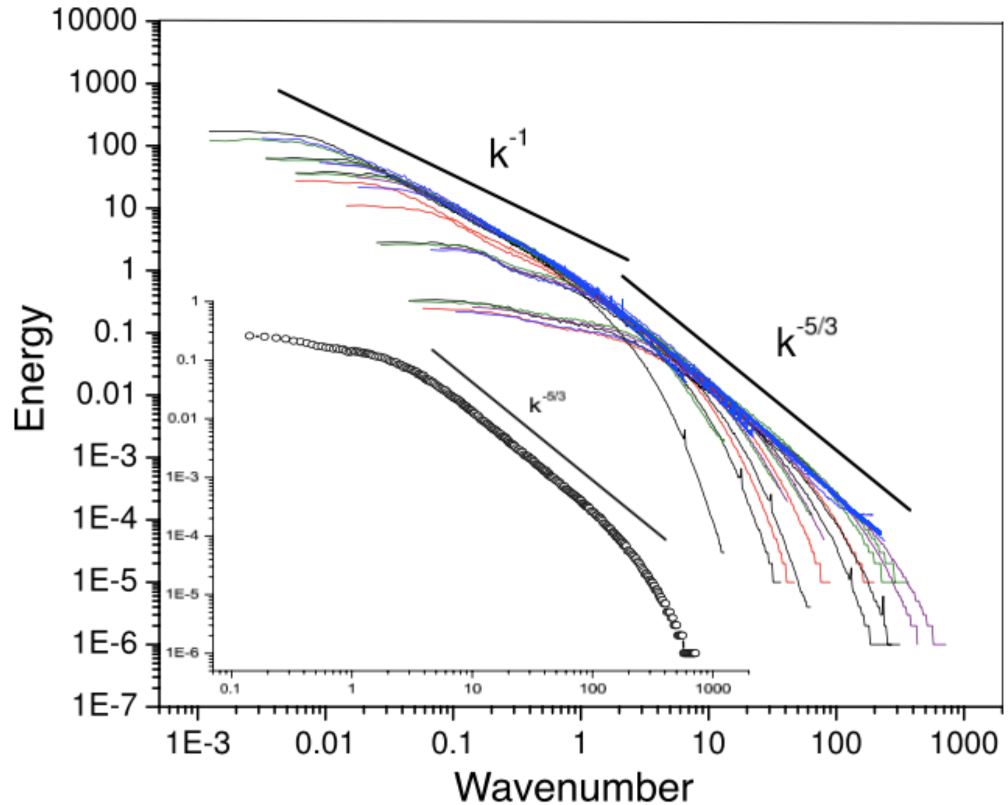


Fig. 8.4 The energy spectrum of 3D turbulence measured in some experiments at the Princeton Superpipe facility.[5] The outer plot shows the spectra from a large-number of experiments at different Reynolds numbers, with the magnitude of their spectra appropriately rescaled. Smaller scales show a good -5/3 spectrum, whereas at larger scales the eddies feel the effects of the pipe wall and the spectra are a little shallower. The inner plot shows the spectrum in the centre of the pipe in a single experiment at $Re \approx 10^6$.

Kolmogorov’s inertial range

- Experimental validation

measurements in a jet in the laboratory

-

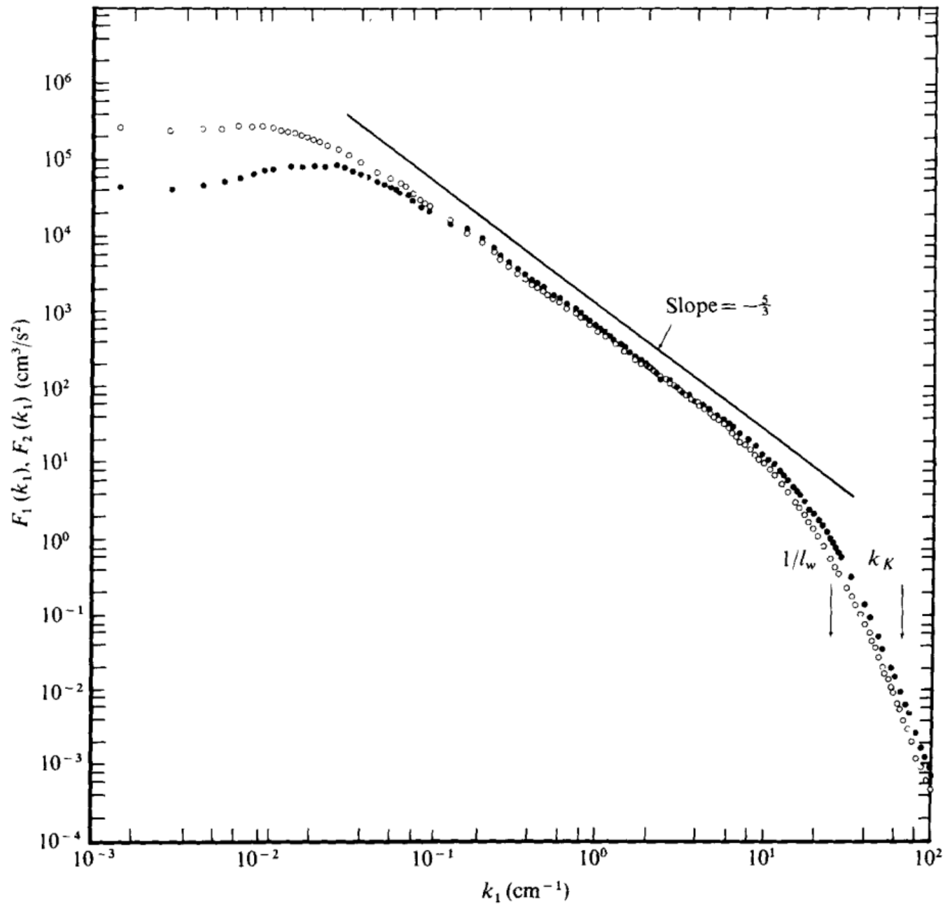


FIGURE 15. One-dimensional spectra of streamwise- and lateral-component velocity fluctuations for an axisymmetric jet; $Re = 3{\cdot}7 \times 10^5$, $x/d = 70$, $r/d = 0$. ○, $F_1(k_1)$; ●, $F_2(k_1)$.

Champagne, 78

Kolmogorov's inertial range

- What is the **"Law of two-thirds"** ?
  - Kolmogorov (1941) did not actually derive the form of the energy spectrum. Rather, he derived relations for the **velocity structure functions.**
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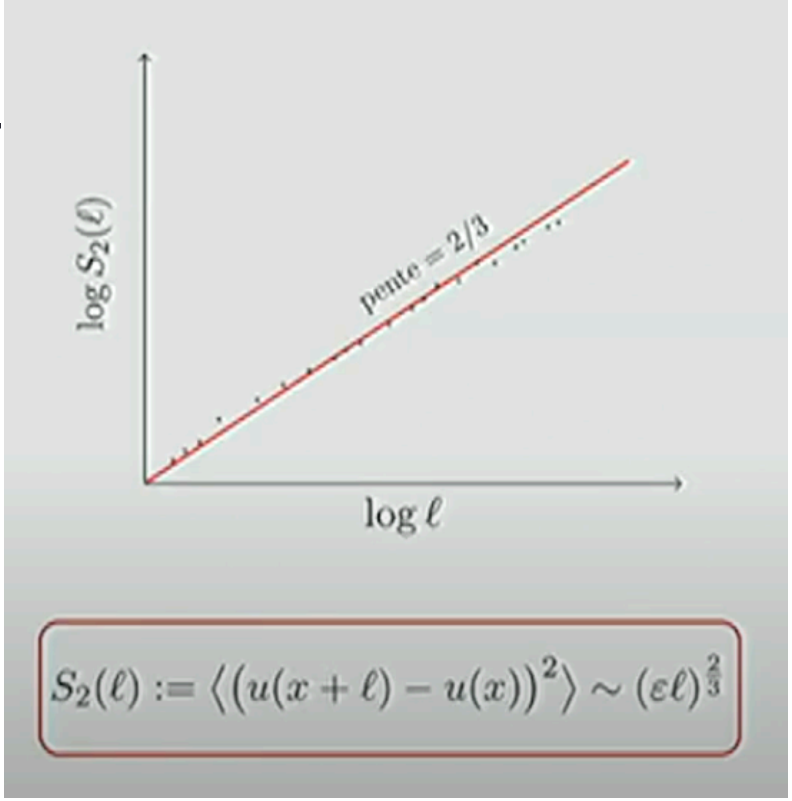


Conservation laws

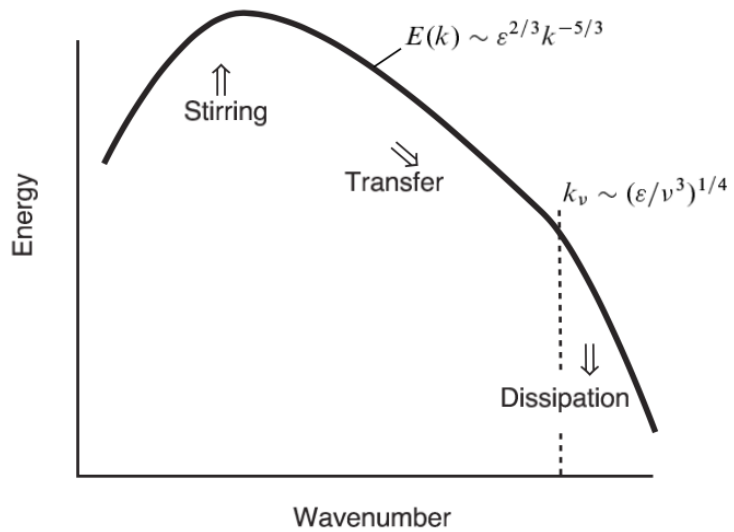
forcing dissipation

$$\boxed{\frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F}\, dV - \nu \iiint |\vec{\omega}|^2\, dV}$$

- Forcing puts energy in the system and dissipation removes it.
- Dissipation is proportional to the integral of the squared vorticity, also known as the ***enstrophy***
- E is an **inviscid invariant** (invariant when no forcing and viscosity)

3D turbulence

- The energy dissipation rate is equal to the energy cascade rate.
- **It is *independent of the viscosity***. *As viscosity tends to zero,* the scale at which viscous effects become important *becomes smaller in just such a way as to preserve the constancy of the energy dissipation.*



- there must be ***production* of enstrophy** in the absence of forcing

Activity 4

- Write the vorticity equation *(starting from NS equations below, with constant rotation and incompressibility)*.

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

- A few useful identities:

$$\begin{aligned}
\vec{\omega} &= \vec{\nabla} \times \vec{u} \\
(\vec{u} \cdot \vec{\nabla})\vec{u} &= \frac{1}{2}\vec{\nabla}(|\vec{u}|^2) + \vec{\omega} \times \vec{u} \\
\vec{\nabla} \times (\vec{u} \times \vec{\omega}) &= -\vec{\omega}(\vec{\nabla} \cdot \vec{u}) + (\vec{\omega} \cdot \vec{\nabla})\vec{u} - (\vec{u} \cdot \vec{\nabla})\vec{\omega} \\
\vec{\nabla} \cdot \vec{\omega} &= 0 \\
\vec{\nabla} \times \vec{\nabla}\Phi &= 0
\end{aligned}$$

Vorticity equation

- Taking the curl of

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

- We get: $\frac{\partial}{\partial t}\vec{\omega} + \vec{u} \cdot \nabla \vec{\omega}_a + \vec{\omega}_a (\nabla \cdot \vec{u}) = \vec{\omega}_a \cdot \nabla \vec{u} + \nabla \times \mathcal{F} + \nu \nabla^2 \vec{\omega}$

With the absolute vorticity $\vec{\omega_a} = \nabla \times \vec{u} + f\vec{k}$

- Which gives (with constant rotation and incompressibility):

$$\frac{\partial}{\partial t}\vec{\omega} + \nabla \cdot (\vec{u} \circ \vec{\omega}) = \vec{\omega}_a \cdot \nabla \vec{u} + \nabla \times \mathcal{F} + \nu \nabla^2 \vec{\omega}$$

Enstrophy equation

- Dot product with vorticity gives the enstrophy equation.

$$\frac{1}{2}\frac{\partial}{\partial t}|\vec{\varpi}|^2+\frac{1}{2}\nabla\cdot(\vec{u}|\vec{\varpi}|^2)=\vec{\varpi}\cdot(\vec{\varpi}_a\cdot\nabla\vec{u})+\nabla\times\mathscr{F}+\nu\vec{\varpi}\cdot\nabla^2\vec{\varpi}$$

  - *Note that domain averaging is equivalent to an ensemble averaging as we hypothesize statistical homogeneity of the turbulence = all divergences integrate to zero*

- Finally

$$\frac{d}{dt}\iiint\frac{1}{2}|\vec{\varpi}|^2dV=\boxed{\iiint\vec{\varpi}\cdot(\vec{\varpi}_a\cdot\nabla\vec{u})dV}+\boxed{\iiint\nabla\times\mathscr{F}dV}-\boxed{\nu\iiint|\nabla\times\vec{\varpi}|^2dV}$$

Production of enstrophy due to vortex stretching (undetermined sign)

Forcing

dissipation

Enstrophy equation

- So if we consider what happens at small scales (far from forcings) :

$$\frac{d}{dt}\iiint \frac{1}{2}|\vec{\omega}|^2\, dV = \boxed{\iiint \vec{\omega}\cdot(\vec{\omega}_a\cdot\nabla\vec{u})\, dV} - \boxed{\nu \iiint |\nabla\times\vec{\omega}|^2\, dV}$$

Production of enstrophy due to vortex stretching (undetermined sign)

dissipation

3d turbulence

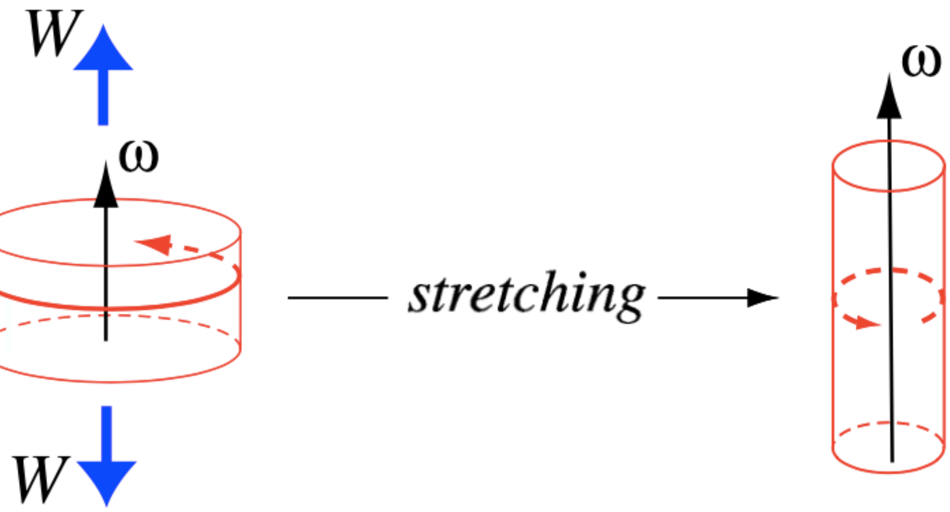
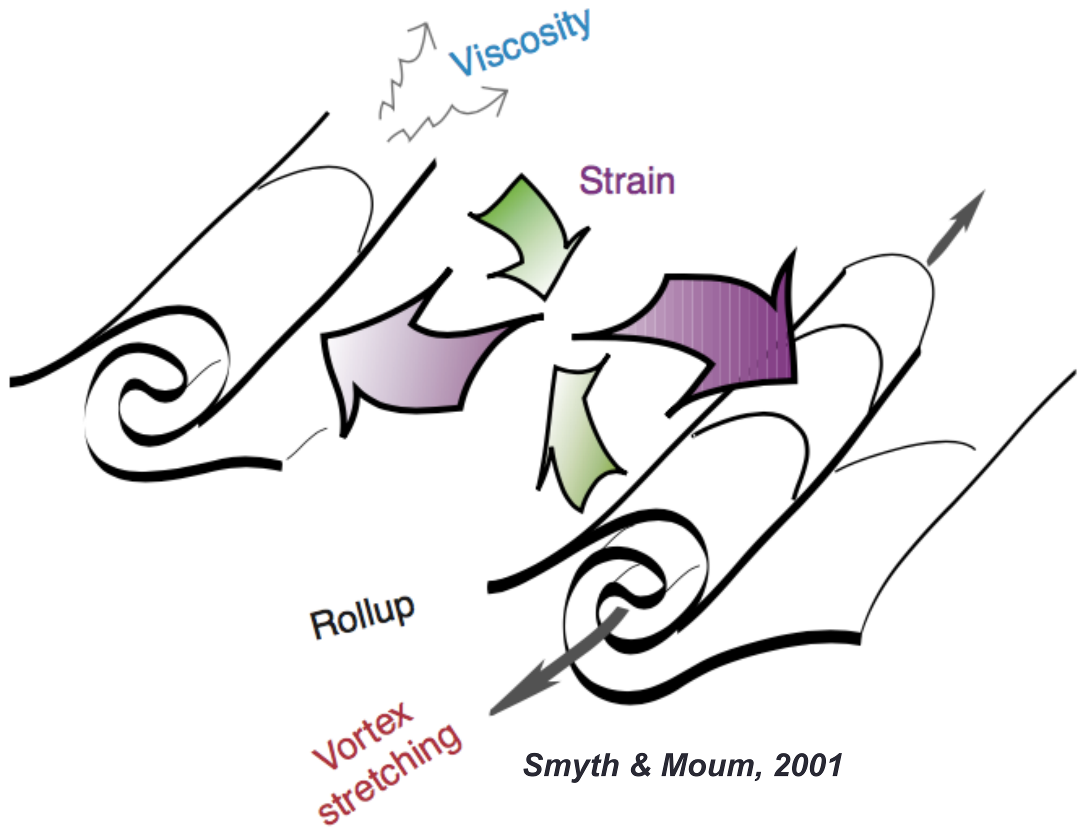


Fig. 4.5 Stretching of material lines distorts the cylinder of fluid as shown. Vorticity is tied to material lines, and so is amplified in the direction of the stretching. However, because the volume of fluid is conserved, the end surfaces shrink, the material lines through the cylinder ends converge, and the integral of vorticity over a material surface (the circulation) remains constant, as discussed in section 4.3.2.

3d turbulence

- energy is cascaded to small scales via vortex stretching

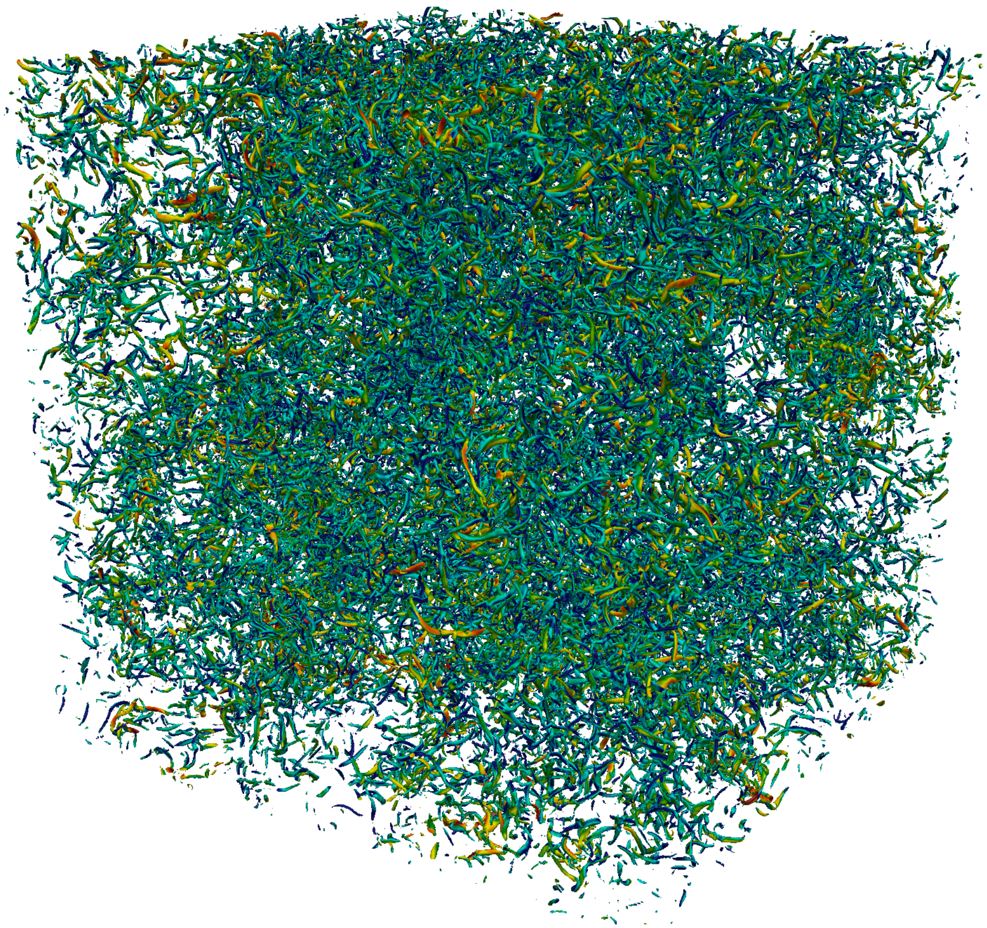


Smyth & Moum, 2001

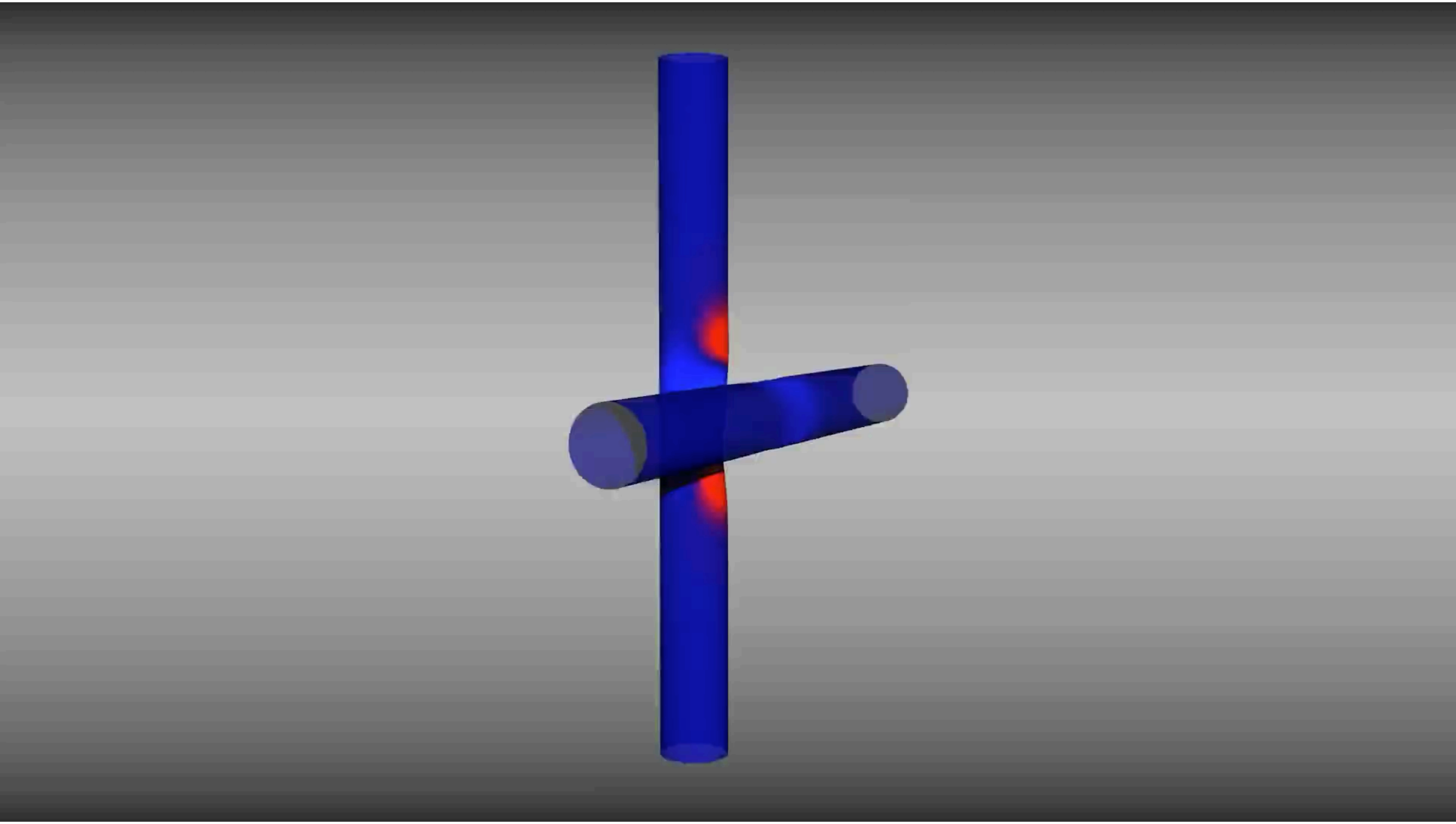
Schematic illustration of line vortices and strained regions in turbulent flow. Fluid parcels in the vortex interiors rotate with only weak deformation. In contrast, fluid parcels moving between the vortices are rapidly elongated in the direction of the purple arrows and compressed in the direction of the green arrows.

3d turbulence

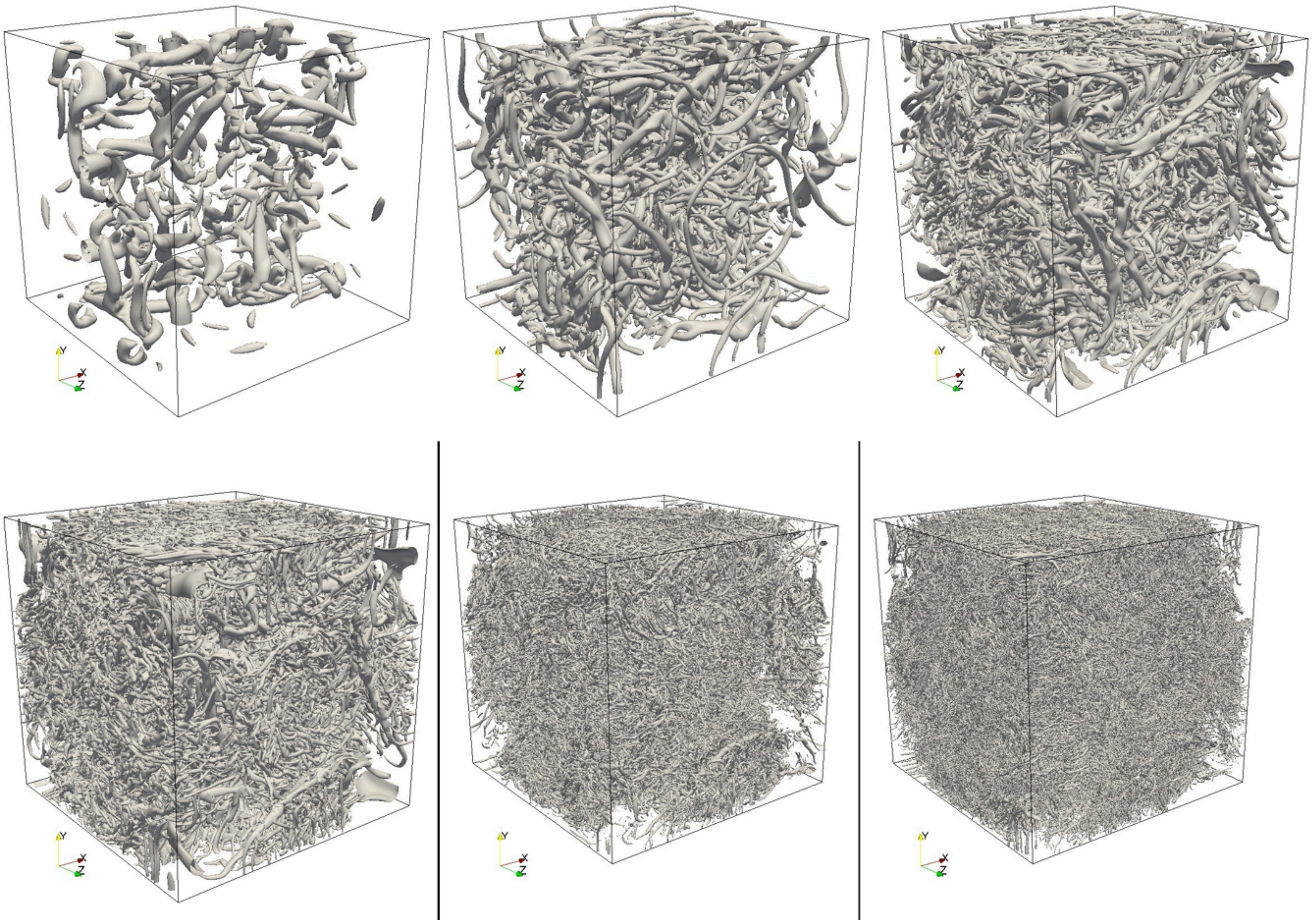
- energy is cascaded to small scales via vortex stretching



3d turbulence



3d turbulence



Taylor-Green flow, DNS, Re=1250→40000 [image E. Lamballais]

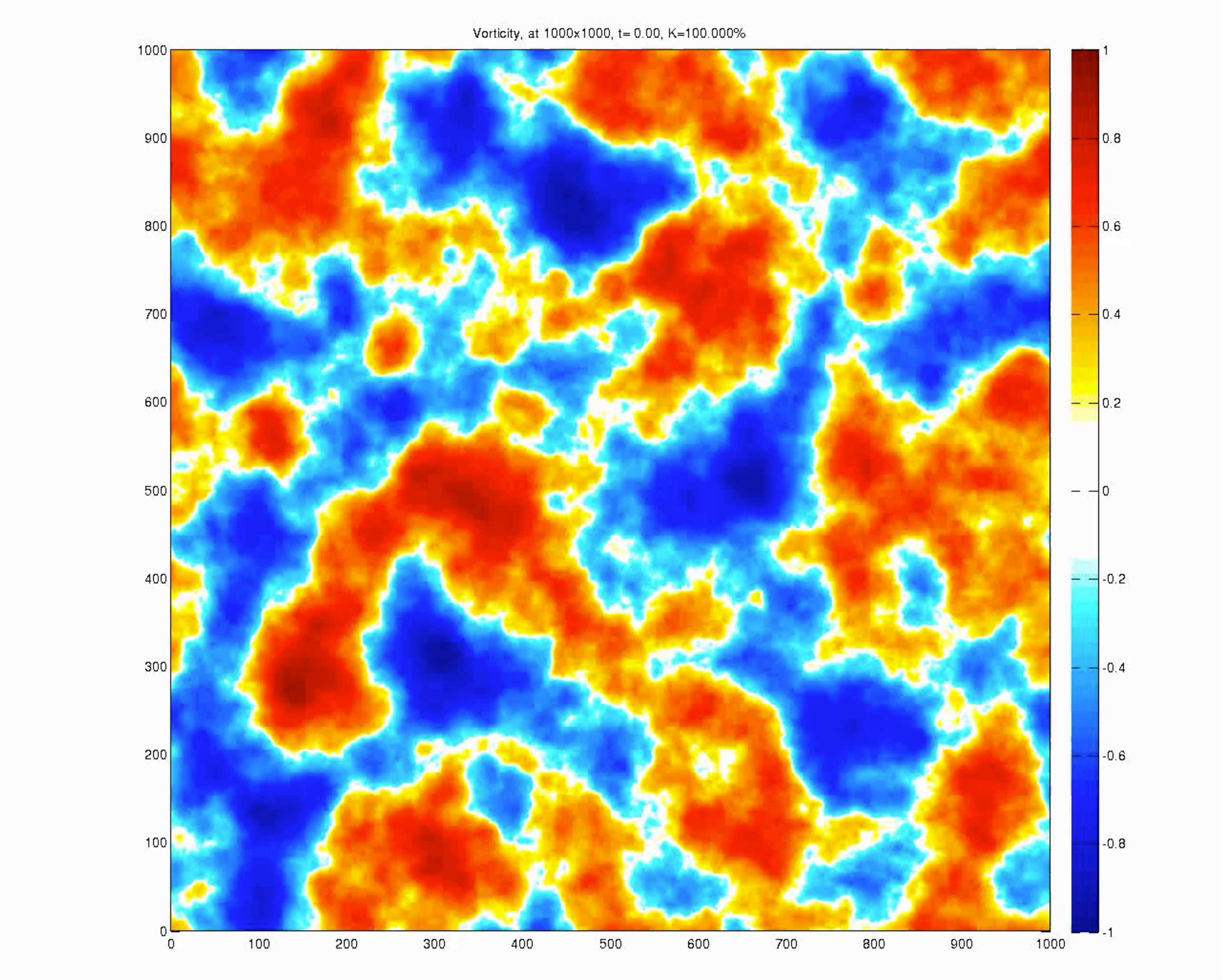
TURBULENCE

3. 2D TURBULENCE

2d turbulence

- 2D homogeneous turbulence is relevant to geophysical turbulence on large horizontal scales because of the thinness of Earth's atmosphere and ocean (i.e., $H/L \ll 1$) and Earth's rotation (i.e., $Ro \ll 1$) and stable stratification (i.e., $Fr \ll 1$), both of which tend to suppress vertical flow and make the 2D horizontal velocity component dominant.

2d turbulence



2d turbulence

- What happens to the vorticity equation if the velocity is purely horizontal?

$$\vec{u} = (u, v, 0)$$

- The vorticity becomes purely vertical: $\vec{\omega} = (0, 0, \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u) \equiv \zeta\hat{k}$

2d turbulence

- What happens to the vorticity equation if the velocity is purely horizontal?

$$\vec{u} = (u, v, 0)$$

- The vorticity becomes purely vertical: $\vec{\omega} = (0, 0, \frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u) \equiv \zeta \hat{k}$

- And the vorticity equation becomes

$$\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla(\zeta + f) = \nabla \times \mathcal{F} + \nu \nabla^2 \zeta$$

- Because the vortex stretching term disappears!

$$\omega_a \cdot \nabla \vec{u} = (\zeta + f)\hat{k} \cdot \nabla(u\hat{\imath} + v\hat{\jmath}) = 0$$

-

2d turbulence

- In 2D both energy and enstrophy are inviscid invariants

Forcing dissipation

$$\frac{d}{dt}E = \boxed{\iiint \vec{u}\cdot\mathcal{F}\,dV} \boxed{-\nu \iiint |\vec{\omega}|^2\,dV}$$

$$\frac{d}{dt}\iiint \frac{1}{2}|\vec{\omega}|^2 dV = \frac{d}{dt}Z = \boxed{\iiint \nabla\times\mathscr{F}\,dV} \boxed{-\nu\iiint |\nabla\times\vec{\omega}|^2 dV}$$

Forcing dissipation

2d turbulence (some notations)

- *The vorticity equation can also be written:*

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \zeta, \quad \zeta = \nabla^2 \psi.$$

By introducing the streamfunction such that: $u, v = -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}$

And $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi$

The equation can thus be written:

$$\frac{\partial \nabla^2 \psi}{\partial t} + J(\psi, \nabla^2 \psi) = F + \nu \nabla^4 \psi$$

2d turbulence (some notations)

- Energy and enstrophy are:

$$\hat{E} = \frac{1}{2}\int_A (u^2 + v^2)\,\mathrm{d}A = \frac{1}{2}\int_A (\nabla\psi)^2\,\mathrm{d}A,$$

$$\hat{Z} = \frac{1}{2}\int_A \zeta^2\,\mathrm{d}A = \frac{1}{2}\int_A (\nabla^2\psi)^2\,\mathrm{d}A,$$

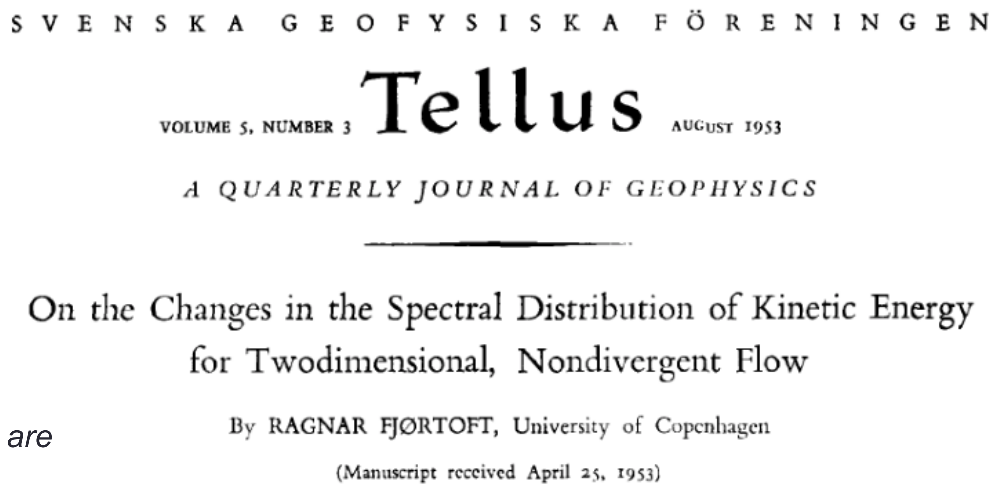
- And their spectra can be written:

$$\hat{E} = \int \mathcal{E}(k)\,\mathrm{d}k, \qquad \hat{Z} = \int \mathcal{Z}(k)\,\mathrm{d}k = \int k^2\mathcal{E}(k)\,\mathrm{d}k,$$

Triad interaction in 2d turbulence

- Activity 5:

Fjortoft example


SVENSKA GEOFYSISKA FÖRENINGEN

VOLUME 5, NUMBER 3 **Tellus** AUGUST 1953

A QUARTERLY JOURNAL OF GEOPHYSICS

On the Changes in the Spectral Distribution of Kinetic Energy for Twodimensional, Nondivergent Flow

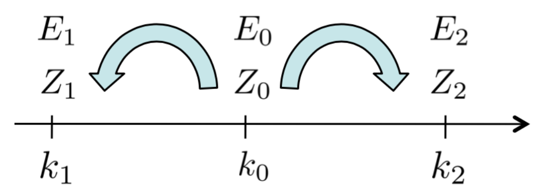
By RAGNAR FJØRTOFT, University of Copenhagen

(Manuscript received April 25, 1953)

Initially there are Energy E_0 and enstrophy Z_0 at wavenumber k_0.

There is no viscosity (or we do not feel it because we are in the inertial range)

All Energy E_0 and enstrophy Z_0 are transferred to wavenumbers k_1 and k_2 .



$k_1 = 1/2\ k_0$ Large scale

$k_2 = 2\ k_0$ Small scale

Based on Energy and enstrophy conservation, what will be the ratio between E_1/E_2 and Z_1/Z_2?

Triad interaction in 2d turbulence

- Fjortoft example:

It is the opposite of 3d turbulence and corresponds to an inverse cascade of energy!

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Abstract

Total kinetic energy as well as total vorticity squared are integral quantities which cannot change in the course of time in a *twodimensional* flow of a homogeneous, nondivergent, and inviscid fluid when the fluid is isolated from the surroundings. The case is considered where the fluid is defined over the total region of the surface of a sphere. The nature of the changes in time of the spectral distribution of kinetic energy is discussed on the basis of the two conservation requirements mentioned above. It is found that only fractions of the initial energy can flow into smaller scales and that a greater fraction simultaneously has to flow to components with larger scales. The upper limits to the flow of kinetic energy into components with scales less than a given one are found. The conservation theorems are also used to discuss the stability of a certain stationary flow for a twodimensional motion which is not necessarily spherical. It is shown how important it is for the proof of stability that not only the kinetic energy of the disturbance is supposed to be small but also its vorticities.

In chapter II molecular viscosity is taken into account for the spherical flow. Finally some conclusive remarks are offered regarding the fundamental difference between two- and threedimensional flow.

Triad interaction in 2d turbulence

- Batchelor argument:

Imagine we have a narrow initial energy spectrum centered on wavenumber k_e

The wavenumber characterizing the spectral location of the energy is the centroid:

$$k_e = \frac{\int k \mathcal{E}(k)\, dk}{\int \mathcal{E}(k)\, \mathrm{d}k}$$

And the spreading out of the energy distribution is formalized by setting:

I measures the width of the energy distribution.

$$I \equiv \int (k - k_e)^2 \mathcal{E}(k)\, \mathrm{d}k$$

Triad interaction in 2d turbulence

- Batchelor argument:

Expanding the integral gives:

$$I \equiv \int (k - k_e)^2 \mathcal{E}(k)\, dk$$

$$I = \int k^2 \mathcal{E}(k)\, dk - 2k_e \int k \mathcal{E}(k)\, dk + k_e^2 \int \mathcal{E}(k)\, dk$$
$$= \int k^2 \mathcal{E}(k)\, dk - k_e^2 \int \mathcal{E}(k)\, dk,$$

If we assume that the distribution spreads in time (as expected from NL interactions):

$$\frac{dI}{dt} > 0.$$

Conservation of E and Z gives:

$$\frac{dk_e^2}{dt} = -\frac{1}{\hat{E}} \frac{dI}{dt} < 0.$$

the centroid of the distribution moves to smaller wavenumber and to larger scale !

Triad interaction in 2d turbulence

- Batchelor argument:

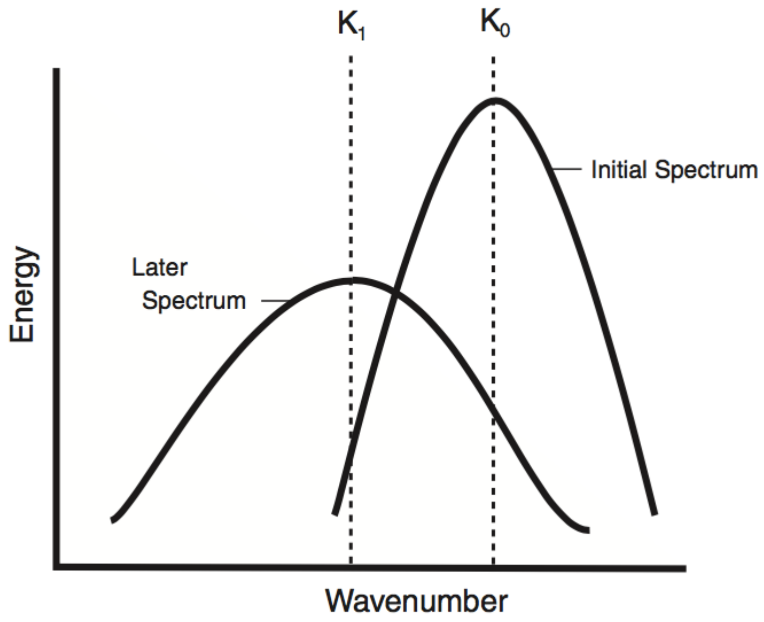


Figure 8.6 In two-dimensional flow, the centroid of the energy spectrum will move to large scales (smaller wavenumber) provided that the width of the distribution increases, which can be expected in a nonlinear, eddying flow

the centroid of the distribution moves to smaller wavenumber and to larger scale !

Triad interaction in 2d turbulence

- Batchelor argument:

For enstrophy we have a similar argument using the inverse wavenumber $q = 1/k$

$$J = \int (q - q_e)^2 Z(q)\,\mathrm{d}q, \qquad \frac{\mathrm{d}J}{\mathrm{d}t} > 0,$$

Which gives

$$\frac{\mathrm{d}q_e^2}{\mathrm{d}t} = -\frac{1}{\hat{Z}}\frac{\mathrm{d}J}{\mathrm{d}t} < 0$$

Thus, the length scale characterizing the enstrophy distribution gets smaller, and the corresponding wavenumber gets larger; the enstrophy spectrum is shifting to the right, toward small scales.

Triad interaction in 2d turbulence

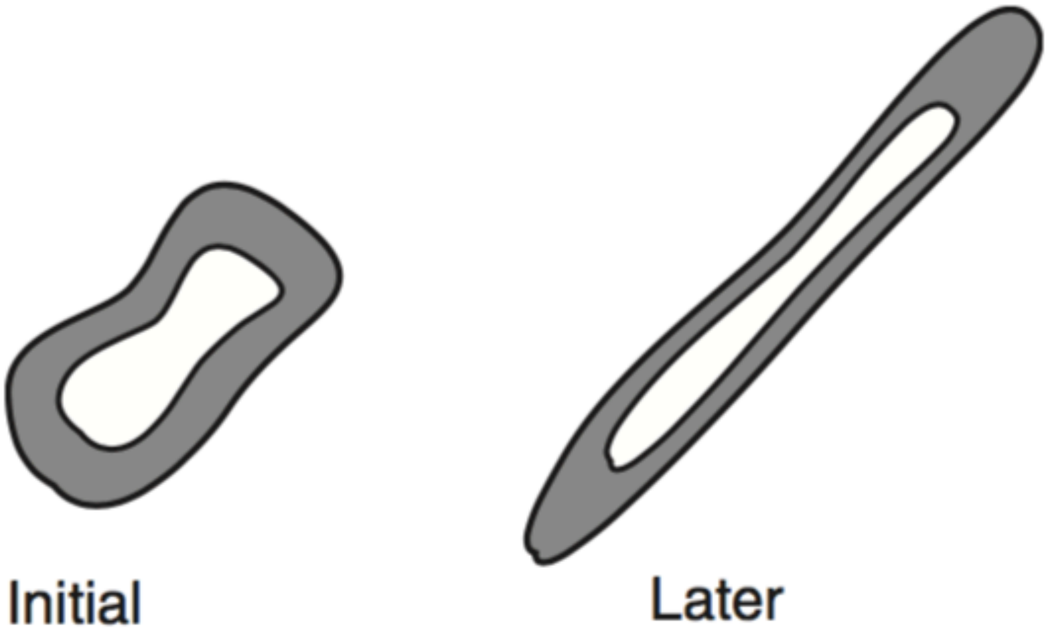


Fig. 8.5 In incompressible two-dimensional flow, a band of fluid will generally be elongated, but its area will be preserved. Since vorticity is tied to fluid parcels, the values of the vorticity in the hatched area (and in the hole in the middle) are maintained; thus, vorticity gradients will increase and the enstrophy is thereby, on average, moved to smaller scales.

Inertial ranges in 2d turbulence

Batchelor’s self similar spectrum:

In a forced- dissipative two-dimensional fluid, **energy is transferred to larger scales and enstrophy is transferred to small scales.**

If we assume that the fluid is forced and that the spectrum is stationary as in the Kolmogorov case in 3-D, we get 2 inertial ranges:

an *energy inertial range* carrying energy to larger scales,

and an *enstrophy inertial range* carrying enstrophy to small scales

Inertial ranges in 2d turbulence

The *energy inertial range* :

- The energy cascade range is as in the Kolmogorov case. The only difference is the direction of transfer, which is now *upscale*. It is known as the *inverse cascade.*

- Dimensionally, this is exactly the same as in the Kolmogorov case:

$$\boxed{\mathcal{E}(k) = \mathcal{K}_\varepsilon \varepsilon^{2/3} k^{-5/3}},$$

Inertial ranges in 2d turbulence

The *enstrophy inertial range* :

- Assuming a constant enstrophy cascade rate : η
- The dimensional arguments now give:
-

$$\boxed{\mathcal{E}(k) = \mathcal{K}_\eta \eta^{2/3} k^{-3}},$$

Inertial ranges in 2d turbulence

- The eddy turnover time is the time taken for a parcel with velocity v_k to move a distance 1/k, v_k being the velocity associated with the (inverse) scale k. On dimensional consideration:

$$\tau_k = (k^3 \mathcal{E}(k))^{-1/2}.$$

- So using: $\boxed{\mathcal{E}(k) = \mathcal{K}_\eta \eta^{2/3} k^{-3}}$,

- Gives *the eddy turnover time* : $t_k \sim l_k / v_k \sim \eta^{-1/3}$

Thus, *the eddy turnover time in the enstrophy range of two-dimensional turbulence* ***is length-scale invariant*** *(no dependance on k as in 3d turbulence).* This time scale is determined by the largest eddies in the cascade range. As such, **the enstrophy cascade is non-local**—the smaller scales are stirred by the eddies at the top of the inertial range, which can be much larger.

Inertial ranges in 2d turbulence

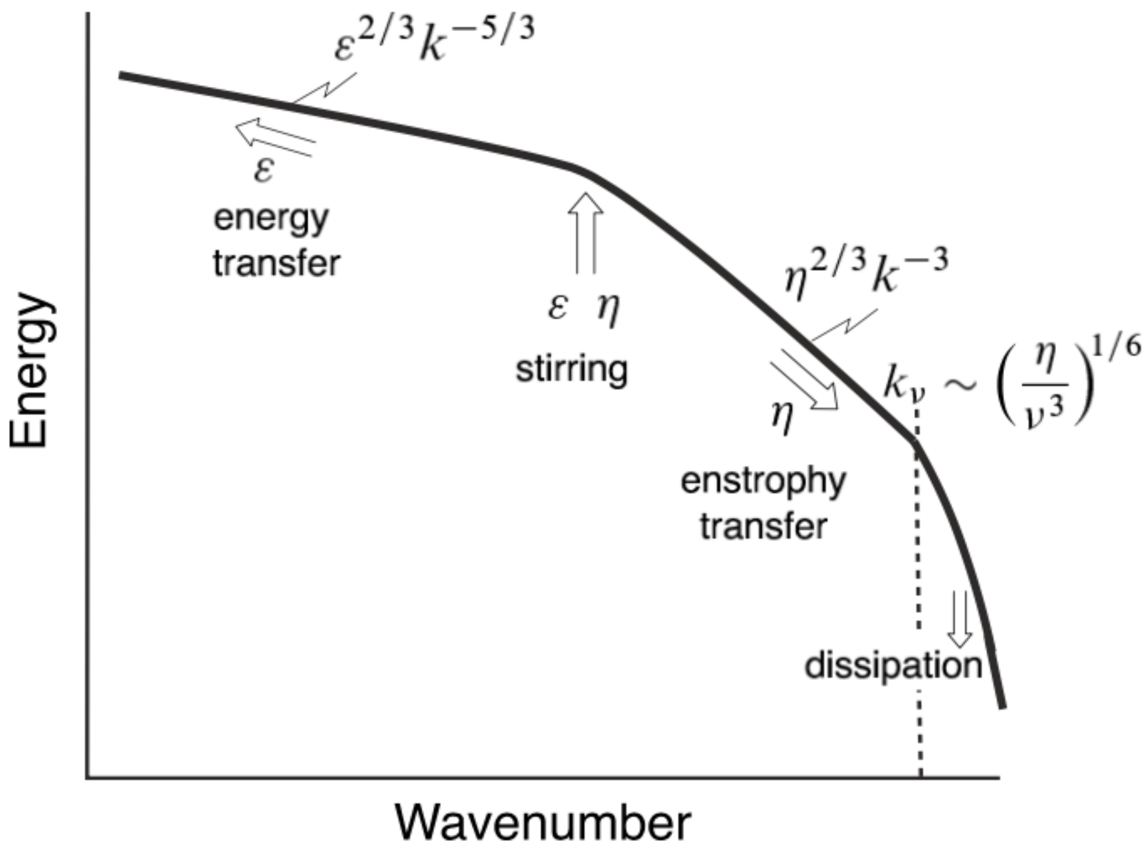


Figure 8.7 The energy spectrum of two-dimensional turbulence. (Compare with Fig. 8.3.) Energy supplied at some rate ε is transferred to large scales, whereas enstrophy supplied at some rate η is transferred to small scales, where it may be dissipated by viscosity. If the forcing is localized at a scale k_f^{-1} then $\eta \approx k_f^2 \varepsilon$.

2d turbulence

- In 2D both energy and enstrophy are inviscid invariants

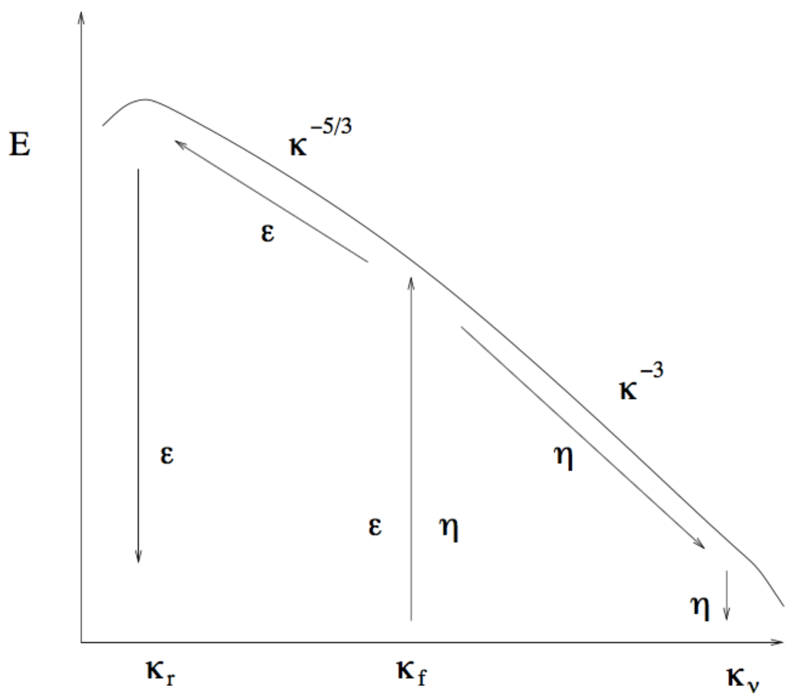
$$\frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F}\, dV - \nu \boxed{\iiint |\vec{\omega}|^2\, dV}$$

- So what happens if ν goes to 0? (Re goes to ∞)
  - **Dissipation goes to 0 !** Because enstrophy cannot become infinite as in 3d *(no production of enstrophy - only reduction due to viscosity)*

Inertial ranges in 2d turbulence

- Energy piles up at large scale and we require dissipation which acts at large scales (e.g. *bottom friction*)

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F \boxed{- r\zeta} + \nu \nabla^2 \zeta.$$



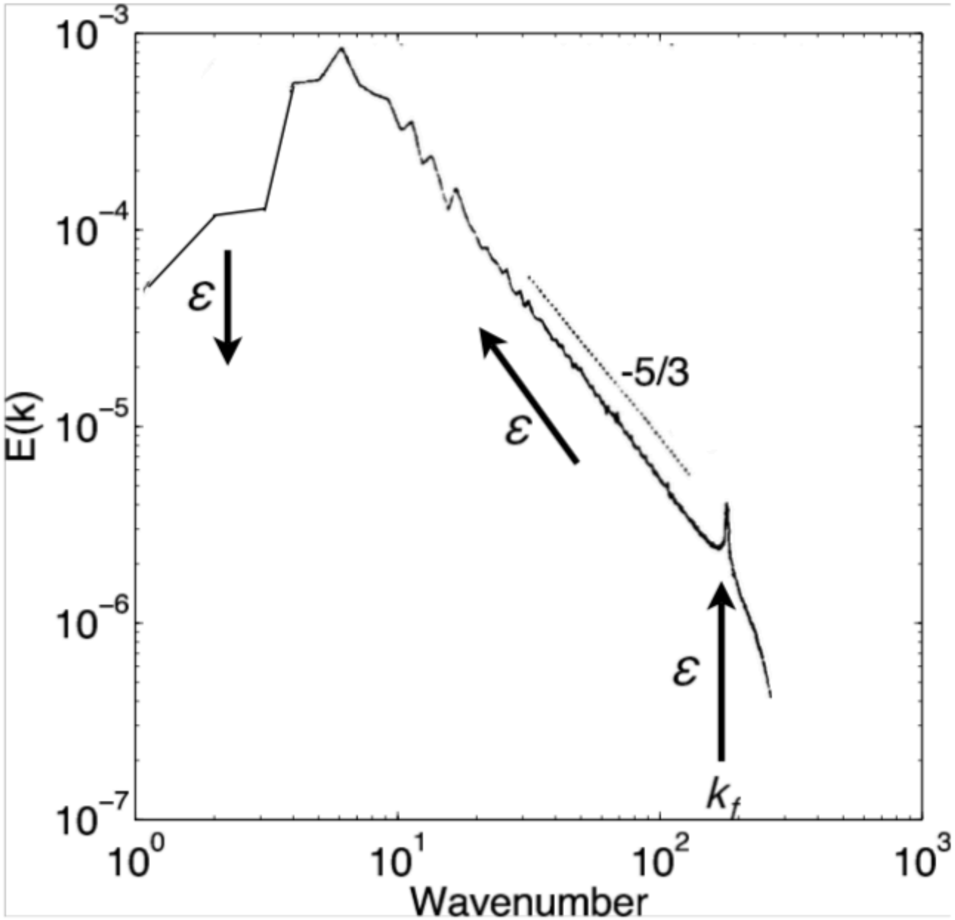
$$\frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} = -2r\widehat{E} - \int \psi F \,\mathrm{d}\boldsymbol{x} + \int \nu\zeta^2 \,\mathrm{d}A \approx -2r\widehat{E} - \int \psi F \,\mathrm{d}A,$$

$$\frac{\mathrm{d}\widehat{Z}}{\mathrm{d}t} = -2r\widehat{Z} + \int \zeta F \,\mathrm{d}A + D_Z \approx -2r\widehat{Z} - k_f^2 \int \psi F \,\mathrm{d}A + D_Z,$$

$$\kappa_r = \left(\frac{r^3}{\epsilon}\right)^{1/2} \qquad \kappa_\nu = \left(\frac{\eta^{1/3}}{\nu}\right)^{1/2}$$

Inertial ranges in 2d turbulence

Figure 8.10 The energy spectrum in a numerical simulation of forced-dissipative two-dimensional turbulence. The fluid is stirred at wavenumber k_f and dissipated at large scales with a linear drag, and there is an $k^{-5/3}$ spectrum at intermediate scales. The arrows schematically indicate the direction of the energy flow.[12]



Problems with the theory

[see Davidson (p585)]

- Hypothesis leading to Batchelor's self similar spectrum seen previously is that **eddy interactions are localized** (in Fourier space)

- Instead a k^{-3} law means that interactions are mostly **non-local**

- Simulations show the existence of **long lived coherent vortices** (long lived = longer than eddy turnover time l/u) = a form of intermittency.

Inertial ranges in 2d turbulence

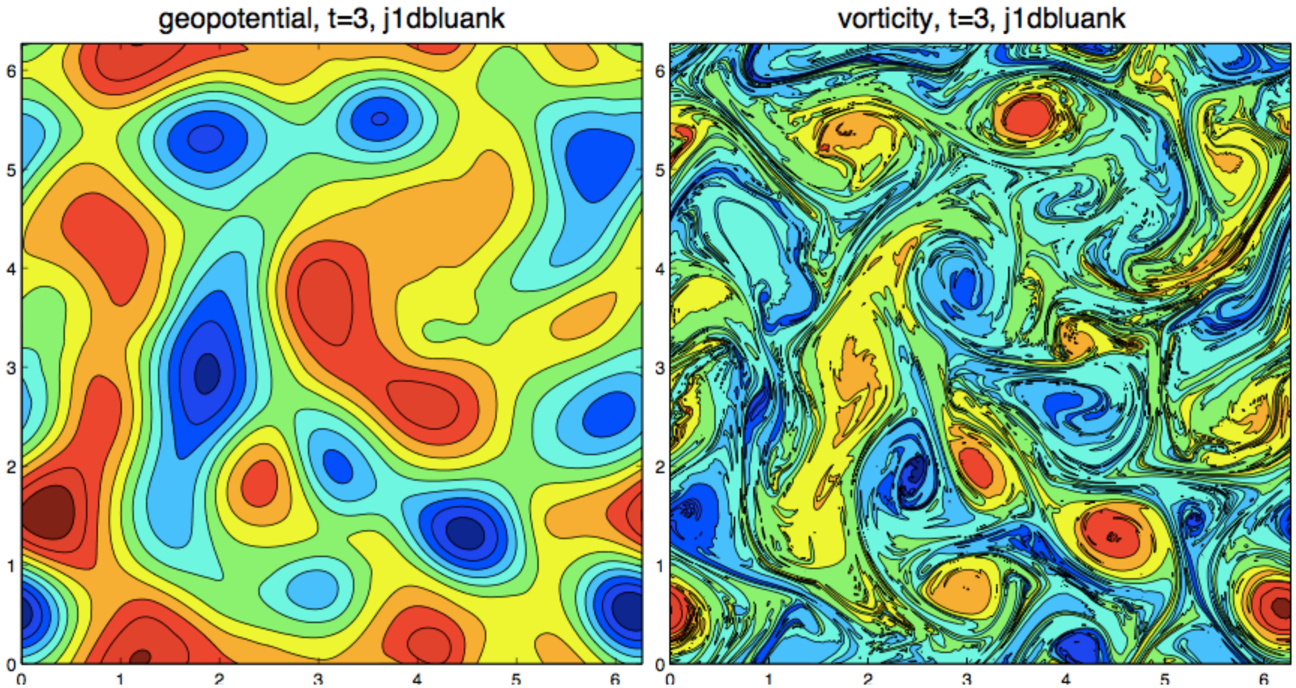
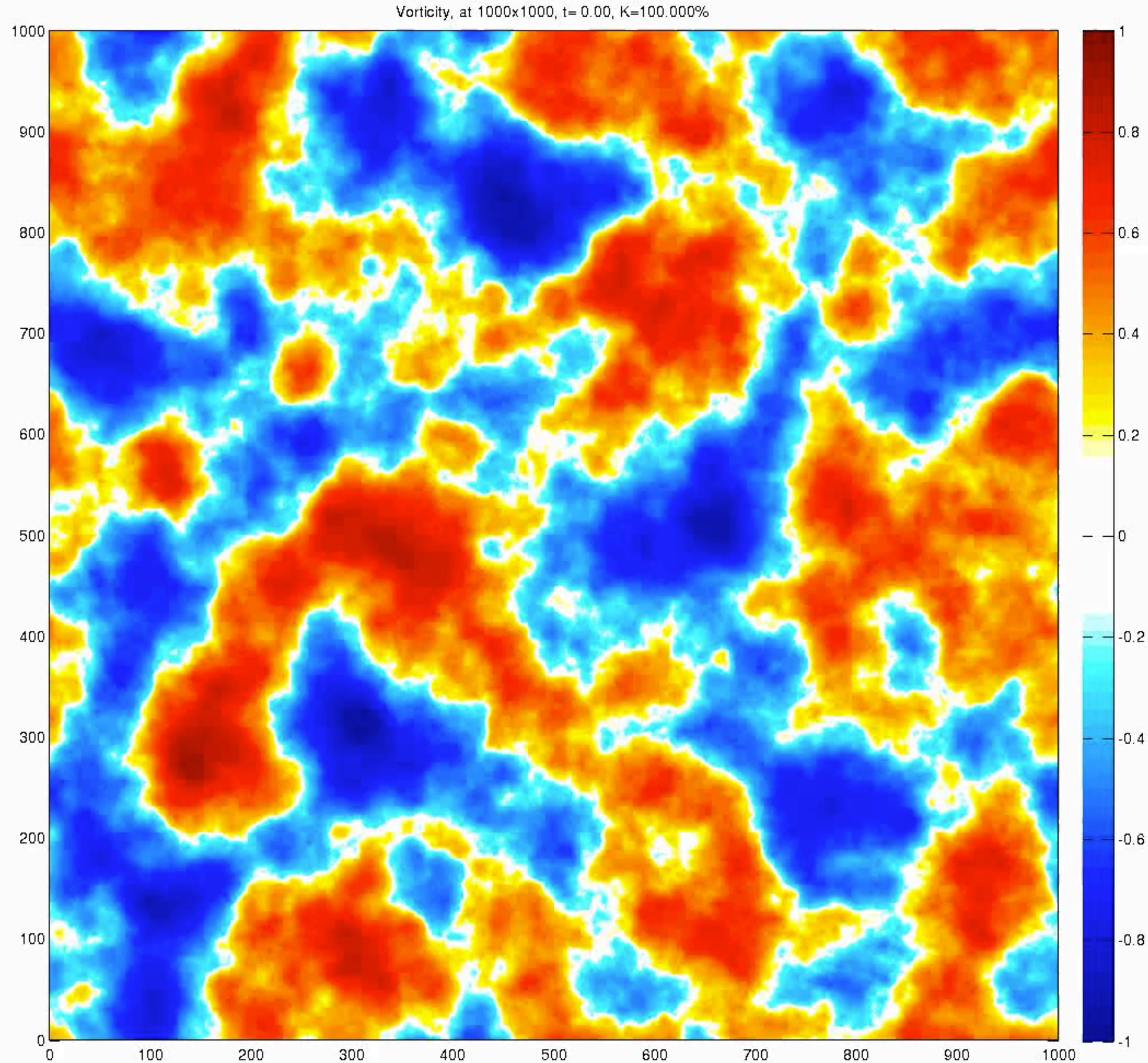


Figure 17: A snapshot of the streamfunction (left) and vorticity (right) from a 2-D turbulence simulation. Note the vorticity has much more small scale structure.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi$$

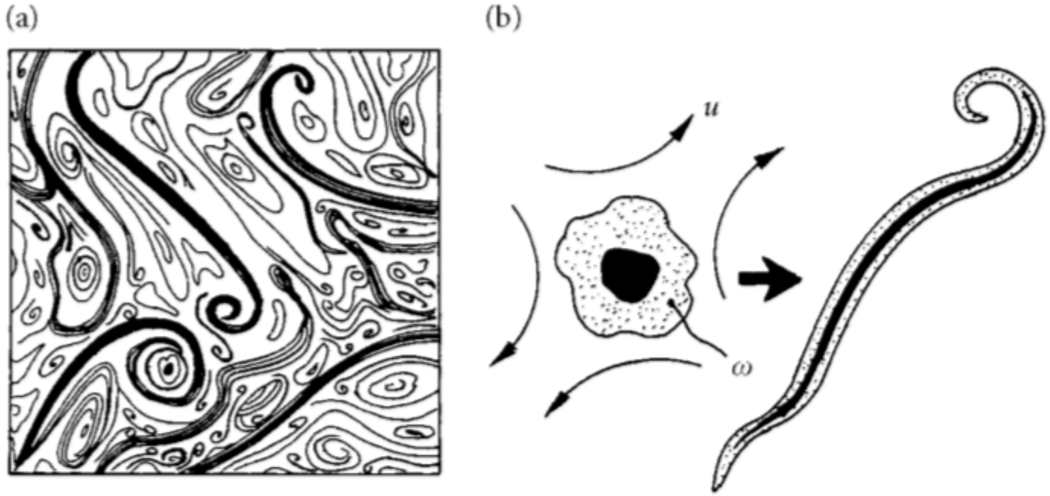
2d turbulence



2d turbulence

Inertial ranges in 2d turbulence

- Enstrophy essentially behaves like a passive tracer (no source of enstrophy):

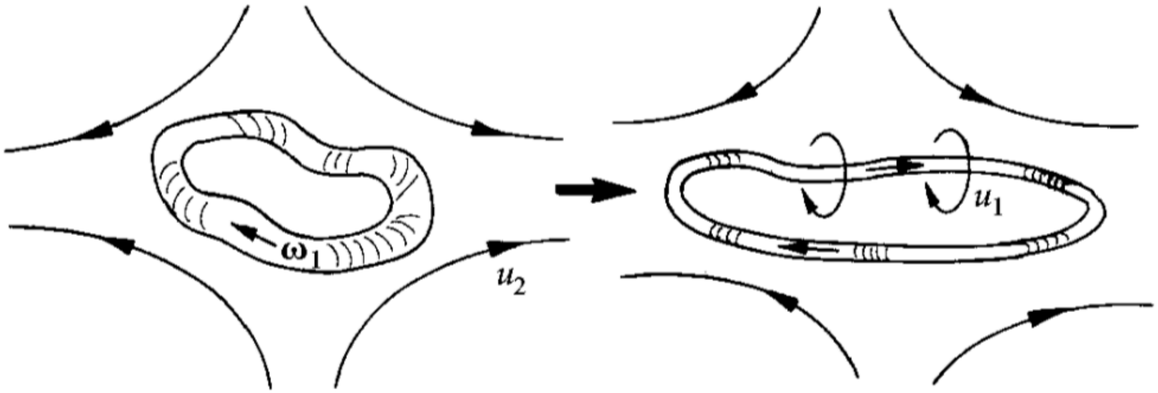


Filamentation of the vorticity - controlled by the inviscid, large-scale eddies and halted only when the vortex sheets are thin enough for viscosity to act, destroying enstrophy and diffusing vorticity.

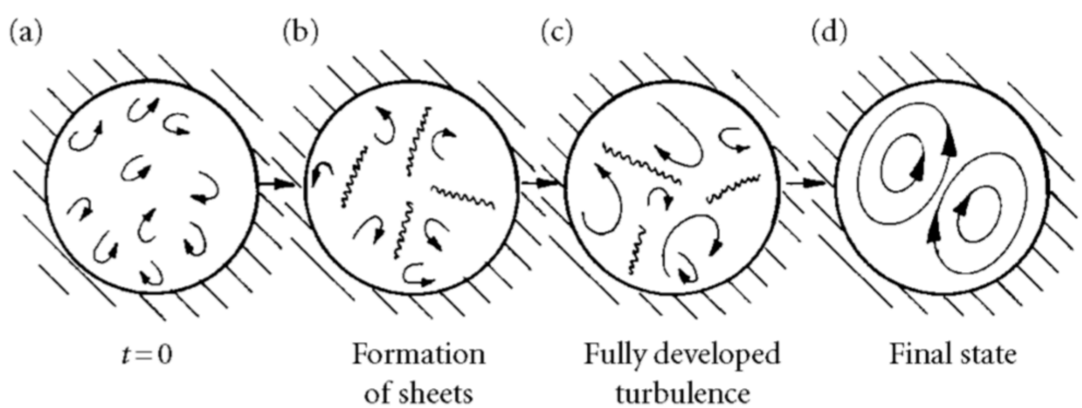
Inertial ranges in 2d turbulence

- Freely- evolving (unforced) turbulence quickly evolves to a state where the vortices dominate the flow, as the vorticity between vortices is strained out and dissipated.
- Thereafter, the evolution is primarily a process of ***mergers* between vortices**.
- Positive vortices (cyclones) merge with other cyclones and negative vortices (anticyclones) merge with other anticyclones.
- The merged vortices are larger than the vortices which joined to make them. In this way, energy is shifted toward larger scales—the flow is dominated by fewer, larger vortices.

- 3D turbulence = **Vortex Stretching**



- 2D turbulence = **Vortex Merging**



Passive tracer spectra

- What happens for a passive tracer C:

$$\frac{\partial}{\partial t}C + \vec{u} \cdot \nabla C = \kappa \nabla^2 C$$

- If energy spectrum is $\mathcal{E}(k) = A k^{-n}$,

- The tracer (variance) spectra is:

$$\boxed{\mathcal{P}(k) = \mathcal{K}_\chi A^{-1/2} \chi k^{(n-5)/2}}.$$

Passive tracer spectra

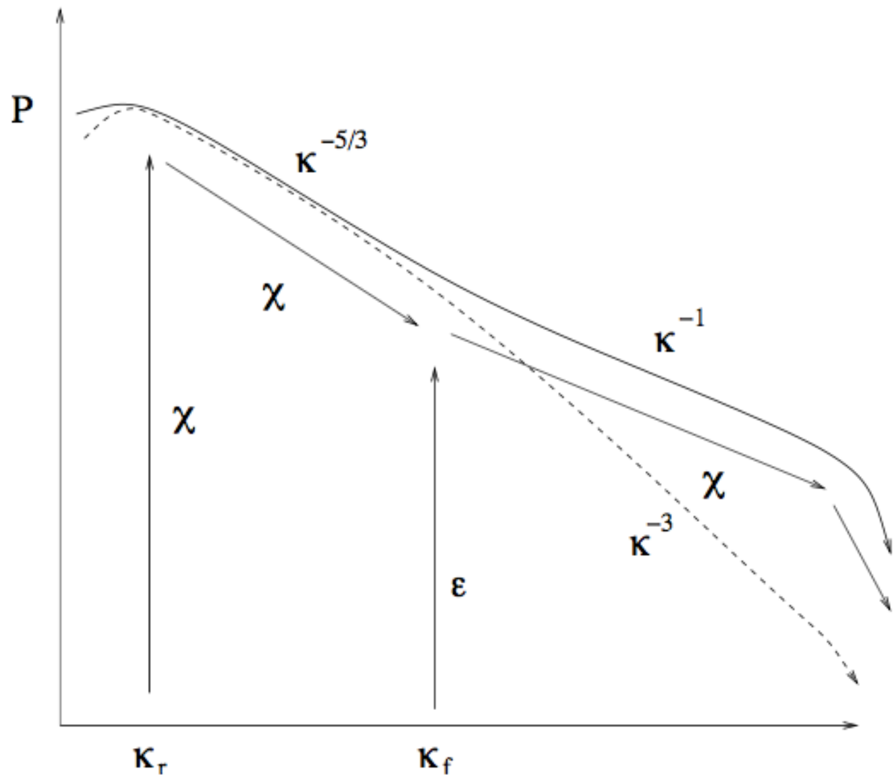


Figure 21: The passive energy spectrum in forced 2-D turbulence. The forcing is applied at κ_f, and the tracer is introduced at large scales, at κ_χ. Note the tracer variance cascades downscale at all scales.

Activity 6: Numerical simulation of 2d turbulence