Equations for linear internal waves.

$$\begin{cases} u_t - \beta v = -\dfrac{1}{\rho^*} p'_x & (1) \\ v_t + \beta u = -\dfrac{1}{\rho^*} p'_y & (2) \\ w_t + \dfrac{\rho'}{\rho^*} g = -\dfrac{1}{\rho^*} p'_z & (3) \\ u_x + v_y + w_z = 0 & (4) \\ -\dfrac{g}{\rho_0} \rho'_t + N^2 w = 0 & (5) \end{cases}$$

- remove p':

$\partial_z(1) - \partial_x(3)$: $$u_{zt} - \beta v_z - w_{xt} - \frac{\rho'_x}{\rho^*} g = 0 \quad (6)$$

$\partial_z(2) - \partial_y(3)$: $$v_{zt} + \beta u_z - w_{yt} - \frac{\rho'_y}{\rho^*} g = 0 \quad (7)$$

$\partial_y(1) - \partial_x(2)$: $$\begin{cases} u_{yt} - \beta v_y - v_{xt} - \beta u_x = 0 \\ (u_y - v_x)_t - \beta \underbrace{(u_x + v_y)}_{(4)\; -w_z} = 0 \quad (8) \end{cases}$$

∂_{yt} 7 $+ \partial_{xt}$ 6 :

$$v_{yztt} + \beta u_{zyt} - w_{yytt} - \frac{\rho'_{yyt}}{\rho^*} g$$
$$+ u_{xztt} - \beta v_{zxt} - w_{xxtt} - \frac{\rho'_{xxt}}{\rho^*} g$$
$$= 0$$

$$\underbrace{(u_x + v_y)}_{4}{}_{ztt} + \beta \underbrace{(u_y - v_x)}_{8}{}_{zt} - w_{xxtt} - w_{yytt} - \underbrace{\frac{(\rho'_{xxt} + \rho'_{yyt})g}{\rho^*}}_{5} = 0$$

$$0 = -w_{zztt} - \beta^2 w_{zz} - w_{xxtt} - w_{yytt} - N^2(w_{xx} + w_{yy})$$

Finally :

$$\underbrace{(w_{xx} + w_{yy} + w_{zz})}_{\nabla^2 w}{}_{tt} + \beta^2 w_{zz} + N^2 \underbrace{(w_{xx} + w_{yy})}_{\nabla_h^2 w} = 0$$

Method of characteristics:

$$(N^2-\omega^2)\,\hat{\omega}_{xx} - (\omega^2-\beta^2)\,\hat{\omega}_{zz} = 0$$

$$\Longleftrightarrow \quad \hat{\omega}_{xx} - \left(\frac{\omega^2-\beta^2}{N^2-\omega^2}\right)\hat{\omega}_{zz} = 0$$

change of variables $\quad \mu_\pm = \pm\left(\dfrac{\omega^2-\beta^2}{N^2-\omega^2}\right)^{1/2}$

$$\xi_\pm = \mu_\pm x - z$$

we can write $\quad \hat{\omega}(x,z) = \bar{\omega}(\xi_+,\xi_-)$

such that:

$$\frac{\partial\hat{\omega}}{\partial x} = \frac{\partial\bar{\omega}}{\partial\xi_+}\frac{\partial\xi_+}{\partial x} + \frac{\partial\bar{\omega}}{\partial\xi_-}\frac{\partial\xi_-}{\partial x}$$

$$= \mu_+\frac{\partial\bar{\omega}}{\partial\xi_+} + \mu_-\frac{\partial\bar{\omega}}{\partial\xi_-}$$

$$\frac{\partial\hat{\omega}}{\partial z} = -\frac{\partial\bar{\omega}}{\partial\xi_+} - \frac{\partial\bar{\omega}}{\partial\xi_-}$$

$$\frac{\partial^2\hat{\omega}}{\partial x^2} = \mu_+^2\frac{\partial^2\bar{\omega}}{\partial\xi_+^2} - \mu_+^2\frac{\partial^2\bar{\omega}}{\partial\xi_+\partial\xi_-} + \mu_+^2\frac{\partial^2\bar{\omega}}{\partial\xi_-^2} - \mu_+^2\frac{\partial^2\bar{\omega}}{\partial\xi_+\partial\xi_-}$$

$$-\mu_+^2\frac{\partial^2\hat{\omega}}{\partial z^2} = -\mu_+^2\frac{\partial^2\bar{\omega}}{\partial\xi_+^2} - \mu_+^2\frac{\partial^2\bar{\omega}}{\partial\xi_+\partial\xi_-} - \mu_+^2\frac{\partial^2\bar{\omega}}{\partial\xi_-^2} - \mu_+^2\frac{\partial^2\bar{\omega}}{\partial\xi_+\partial\xi_-}$$

the equation becomes:

$$\frac{\partial^2 \overline{\omega}}{\partial \xi_+ \partial \xi_-} = 0$$

solution is any function of the form:

$$\overline{\omega}(\xi_+, \xi_-) = F(\xi_+) + G(\xi_-)$$

Methods of modes:

$$W'' + k^2 \underbrace{\frac{N^2-\omega^2}{\omega^2-f^2}}_{m^2} W = 0$$

solution $$W = A\cos(mz) + B\sin(mz)$$

$$W(0) = 0 \iff A = 0$$

$$W(H) = 0 \iff B\sin(mH) = 0$$

$$\iff m_n H = \pm n\pi$$

$$\iff m_n = \pm \frac{n\pi}{H}$$

relation de dispersion: $$\boxed{k_n = \pm \frac{n\pi}{H}\left(\frac{\omega^2-f^2}{N^2-\omega^2}\right)}$$

Solutions: $$W_n = a_n \sin\left(\frac{n\pi z}{H}\right), \quad n = 1, 2, 3, \ldots$$

$$W = \sum_n a_n \sin\left(\frac{n\pi z}{H}\right)\cos(k_n x - \omega t)$$

1.2 Lee Waves (see Gill, p142, C-R, p414)

- equ. $\frac{\partial^2}{\partial t^2}(\nabla^2 w) + N^2 \nabla_h^2 w = 0$ (no rotation)

- BC: $w(z=0) = U\frac{\partial h}{\partial x} = UkH\cos(kx - \omega t)$

sol: $w(z) = w_0 \exp(i(k_x x + k_z z - \omega t))$

$$+\omega^2(k_x^2 + k_z^2) + N^2 k_x^2 = 0$$

$$\Longleftrightarrow \quad \omega^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2} \quad \Longleftrightarrow \quad \boxed{k_z^2 = \frac{N^2}{U^2} - k_x^2}$$

$\boxed{\text{case 1}}$ radiating waves $\frac{N}{U} > k_x$

= strong stratification $(N > k_x U)$
and/or
long topo wavelength $(k_x < \frac{N}{U})$

$k_z = \pm\sqrt{\frac{N^2}{U^2} - k_x^2}$ (choose $k_z > 0$ so energy propagates upward)

$$\begin{cases} w = k_x U H \cos(k_x x + k_z z - \omega t) \\ u = -k_z U H \cos(k_x x + k_z z - \omega t) & \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\right) \\ p' = \rho_0 k_z U^2 H \cos(k_x x + k_z z - \omega t) & \left(\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}\right) \end{cases}$$

$\boxed{\text{Case 2}}$. $k_x > \frac{N}{U}$ = trapped waves

= weak stratification
and/or
short waves

$$\gamma = \sqrt{k_x^2 - \frac{N^2}{U^2}} \Rightarrow \left| \begin{aligned} w &= k_x U H e^{-\gamma z} \cos(k_x x - \omega t) \\ u &= -\gamma U H e^{-\gamma z} \sin(k_x x - \omega t) \\ p' &= -\rho_0 \gamma U^2 H e^{-\gamma z} \sin(k_x x - \omega t) \end{aligned} \right.$$