TURBULENCE

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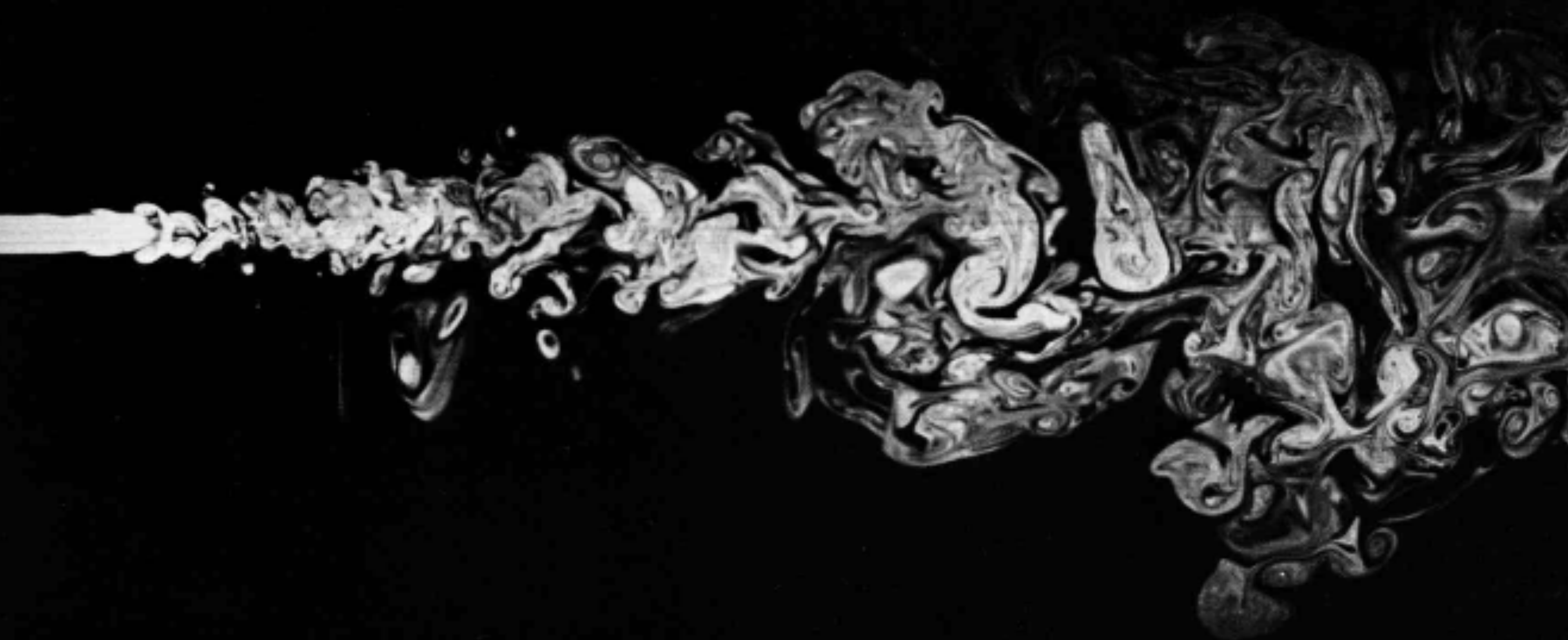

- **Lesson 1 :** [D109]
  - Introduction
  - Properties of turbulence
- **Lesson 2 :** [D109]
  - 3D turbulence: The Kolmogorov theory
  - 2D turbulence
- **Lesson 3 :**[D109]
  - 2D turbulence (activity)
  - Geostrophic turbulence
  - Surface QG turbulence
- **Lesson 4 :**[D109]
  - Ocean turbulence (activity)
  - Turbulent diffusion

Presentations and material will be available at :

jgula.fr/Turb/

Numerical Modelling

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Evaluation

- The evaluation will be based on a report, based on 2 numerical activities and a discussion of research articles

- Written Report will be **due Mar. 22**

References:

- Vallis G.K., Atmospheric and Oceanic Fluid Dynamics.
- MIT online course: *https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-820-turbulence-in-the-ocean-and-atmosphere-spring-2007/lecture-notes/*
- LaCasce J.H., Turbulence in the Atmosphere and Ocean.

TURBULENCE

INTRODUCTION

- **Lesson 1 - Introduction:**
  - *What is turbulence?*
  - *Why do we care about turbulence?*
  - *Where does it come from?*
  - Properties of turbulence:
    - Unpredictability
    - Scale interactions
  - The closure problem

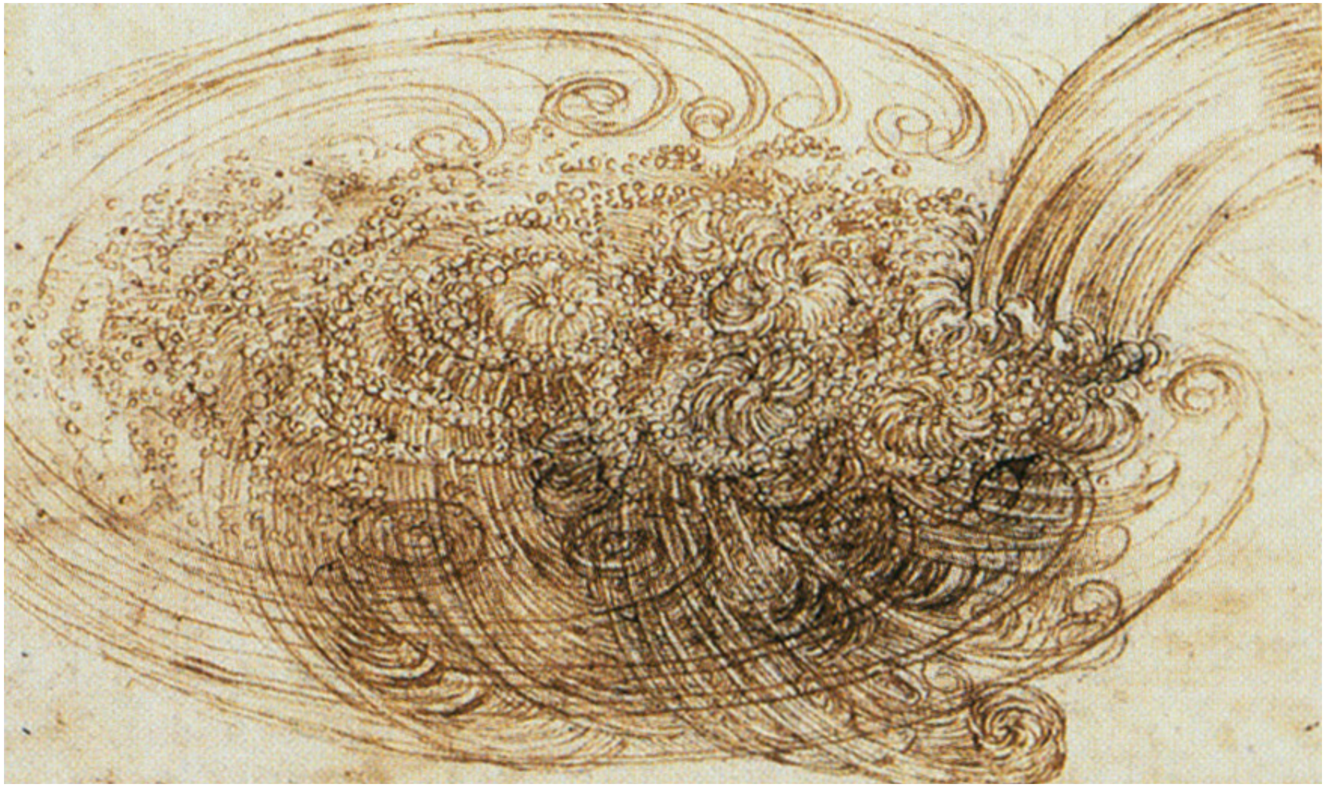
What is turbulence?

Werner Heisenberg
(1901 - 1976)

"When I meet God, I'm going to ask him two questions: why relativity? And why turbulence?

I really believe he'll have an answer for the first."

What is turbulence?



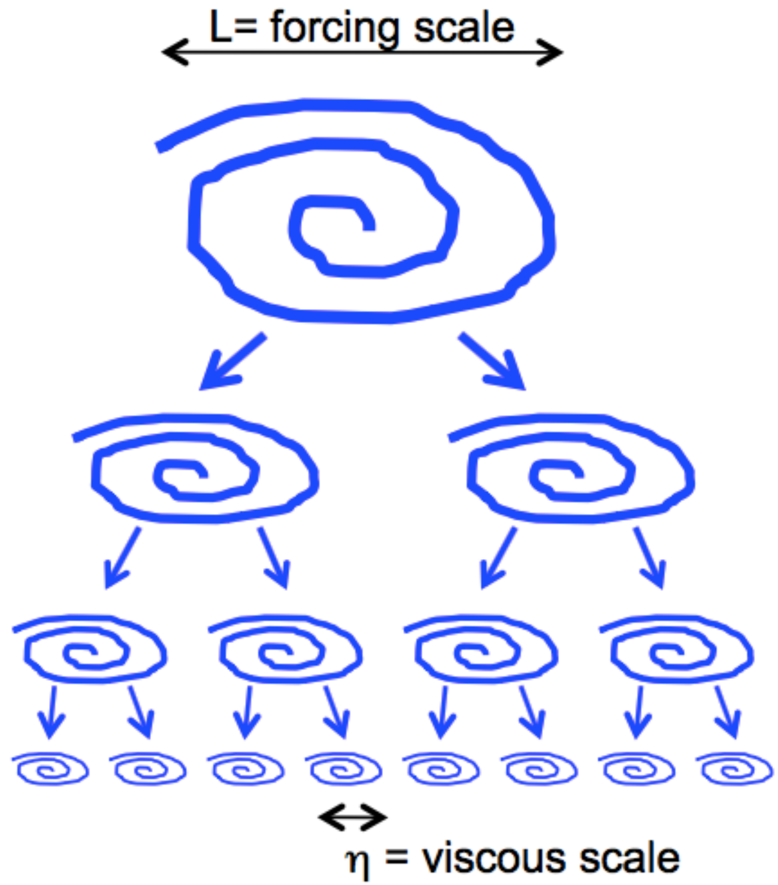
turbolenza by da Vinci [1507]

"…the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large."

What is turbulence?

No formal definition of turbulence. One description as a poem from by L.F. Richardson, in 1922:

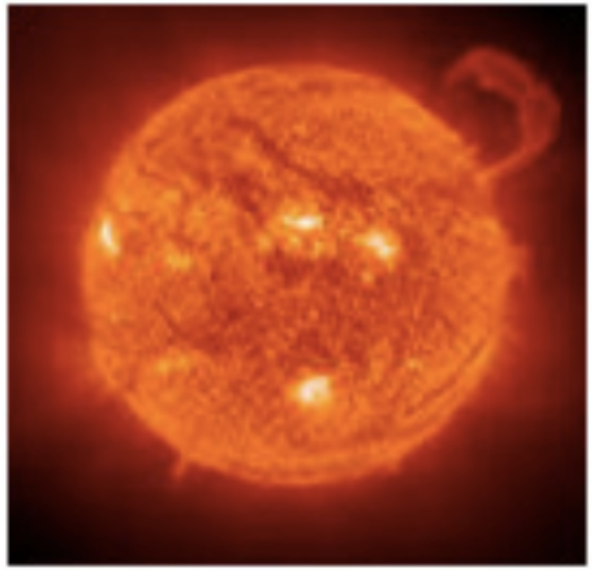
Big whirls have little whirls,
which feed on their velocity;
And little whirls have lesser whirls,
And so on to viscosity.

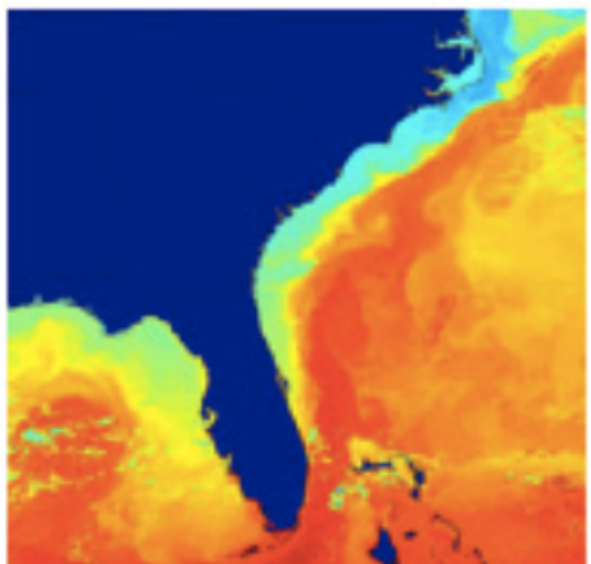


Turbulence in nature

(a) water flow from a faucet, (b) water from a garden hose, (c) flow past a curved wall, and (d) and (e) whitewater rapids whose turbulent fluctuations are so intense that air is entrained by the flow and produces small bubbles that diffusely reflect light and cause the water to appear white.

Turbulence in nature





Examples of turbulent flows at the surface of the Sun, in the Earth's atmosphere, in the Gulf Stream at the ocean surface, and in a volcanic eruption.

Turbulence in nature

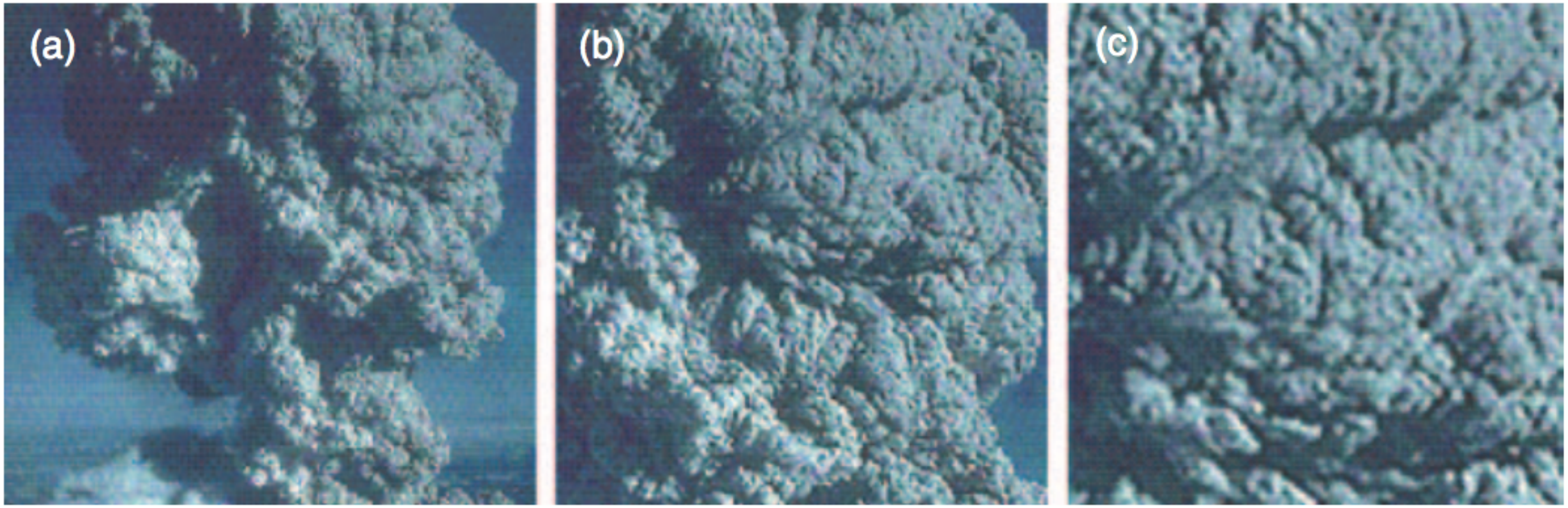
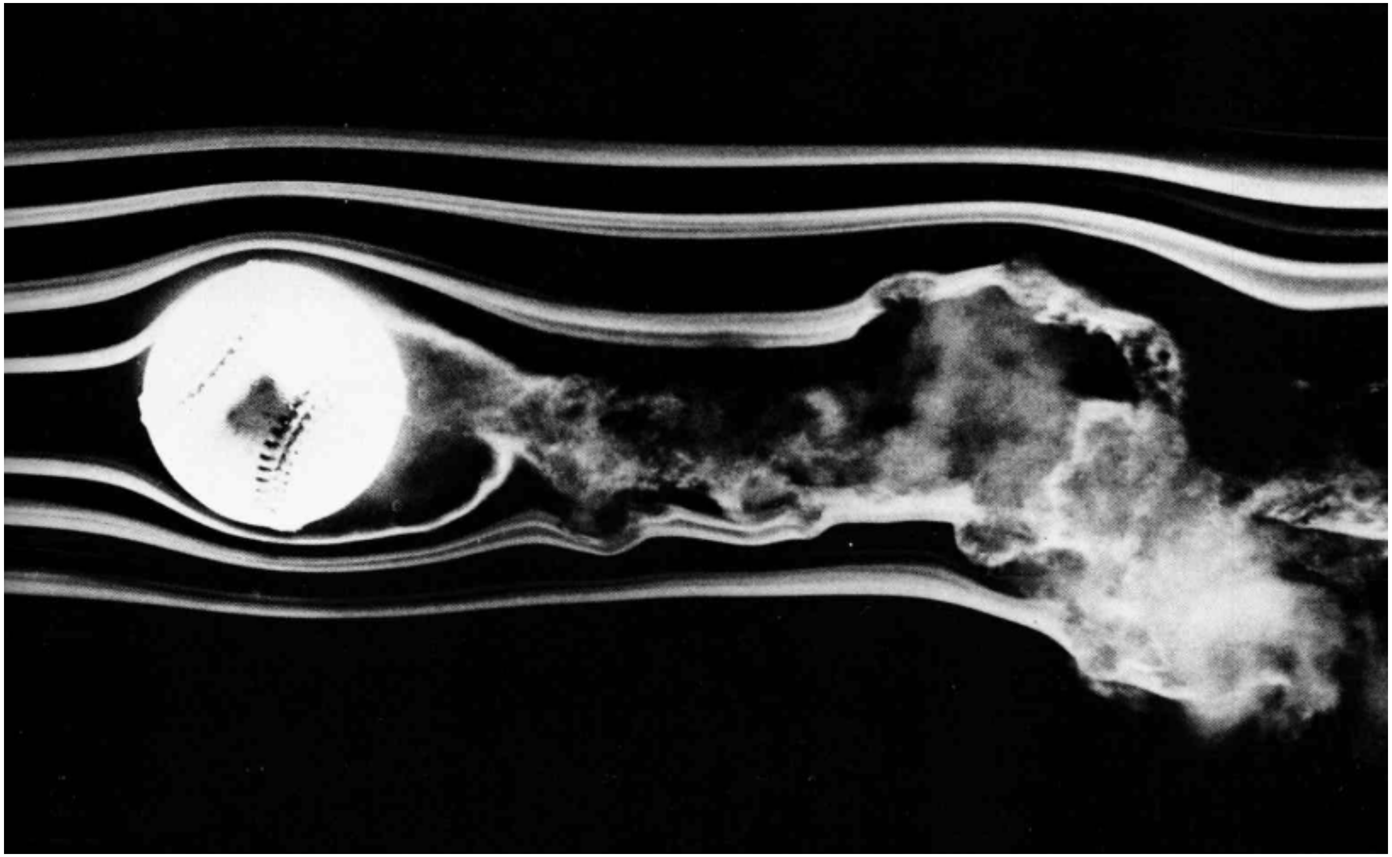


Figure 2. Scale-Independence in Turbulent Flows
The turbulent structure of the pyroclastic volcanic eruption of Mt. St. Helens shown in (a) is expanded by a factor of 2 in (b) and by another factor of 2 in (c). The characteristic scale of the plume is approximately 5 km. Note that the expanded images reveal the increasingly finer scale structure of the turbulent flow. The feature of scale independence, namely, that spatial images or temporal signals look the same (statistically) under increasing magnification is called self-similarity.

Turbulence in nature

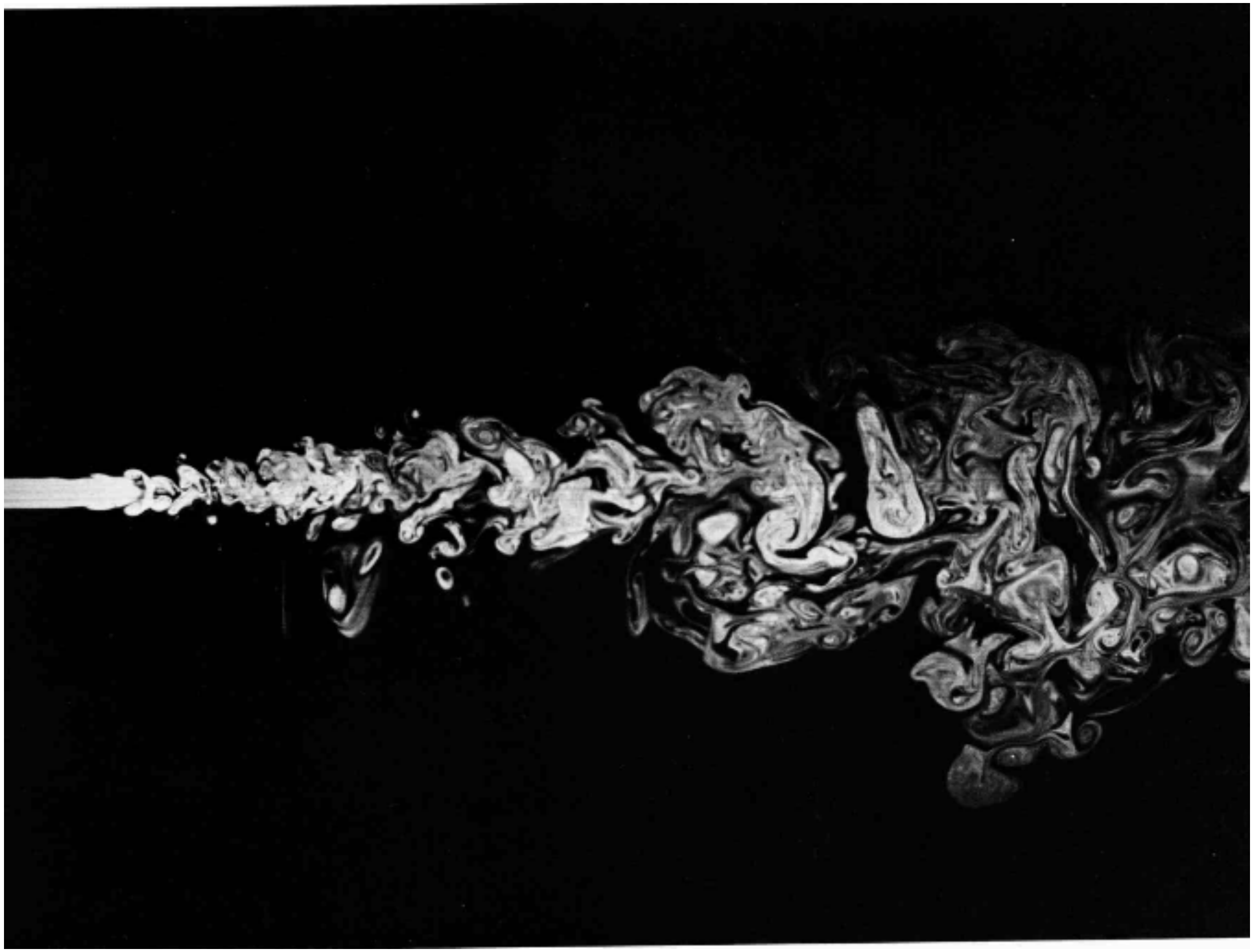
A panoramic Hubble image of the Carina nebula showing the turbulent effects of the stellar winds and ionizing radiation from massive stars on the molecular cloud out of which the stars were born. The width of the image is 16 pc. [https://www.nature.com/articles/s41550-020-01277-w]

Turbulence in the lab



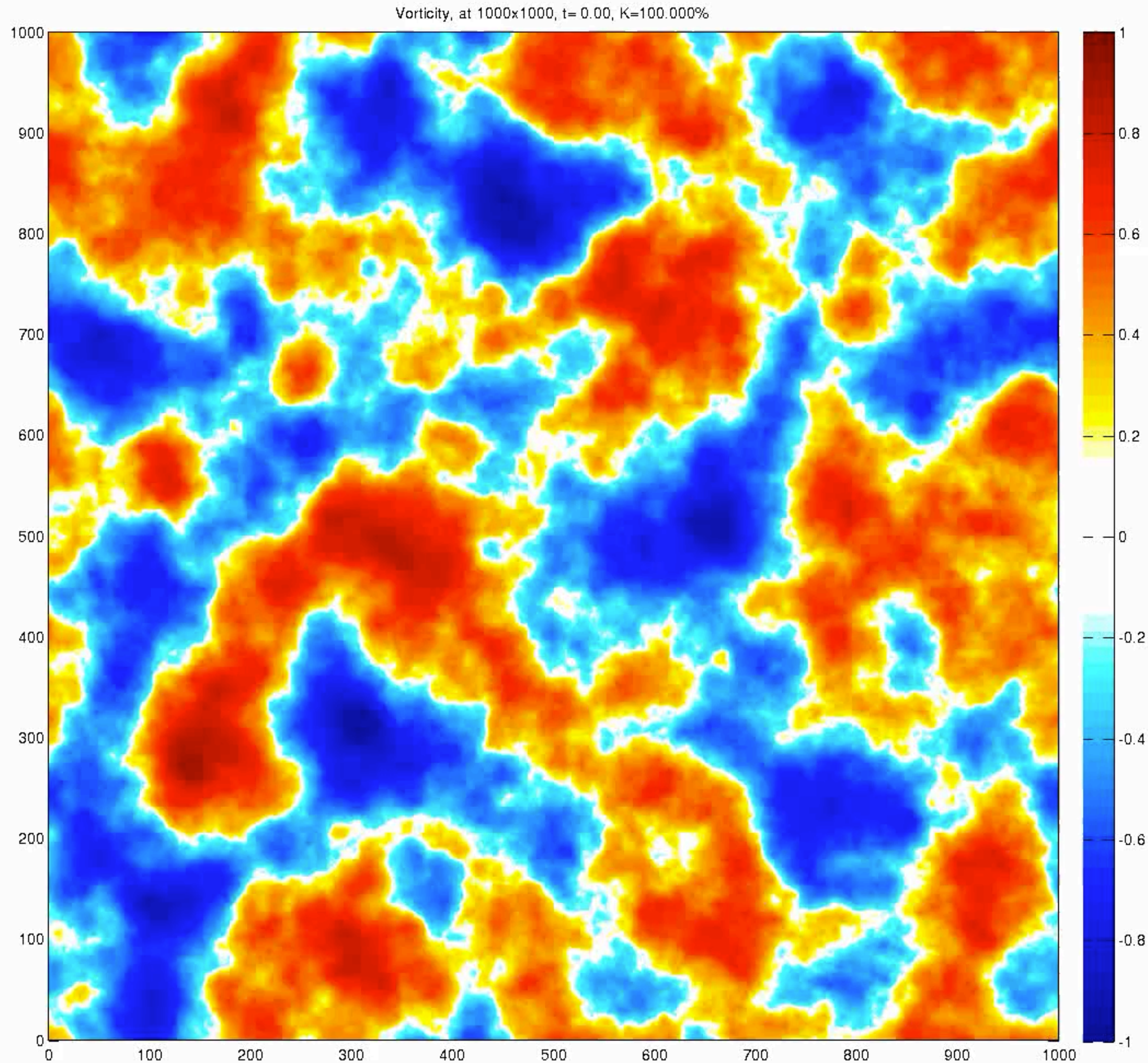
Spinning Baseball [Van Dyke, 82]

Turbulence in the lab

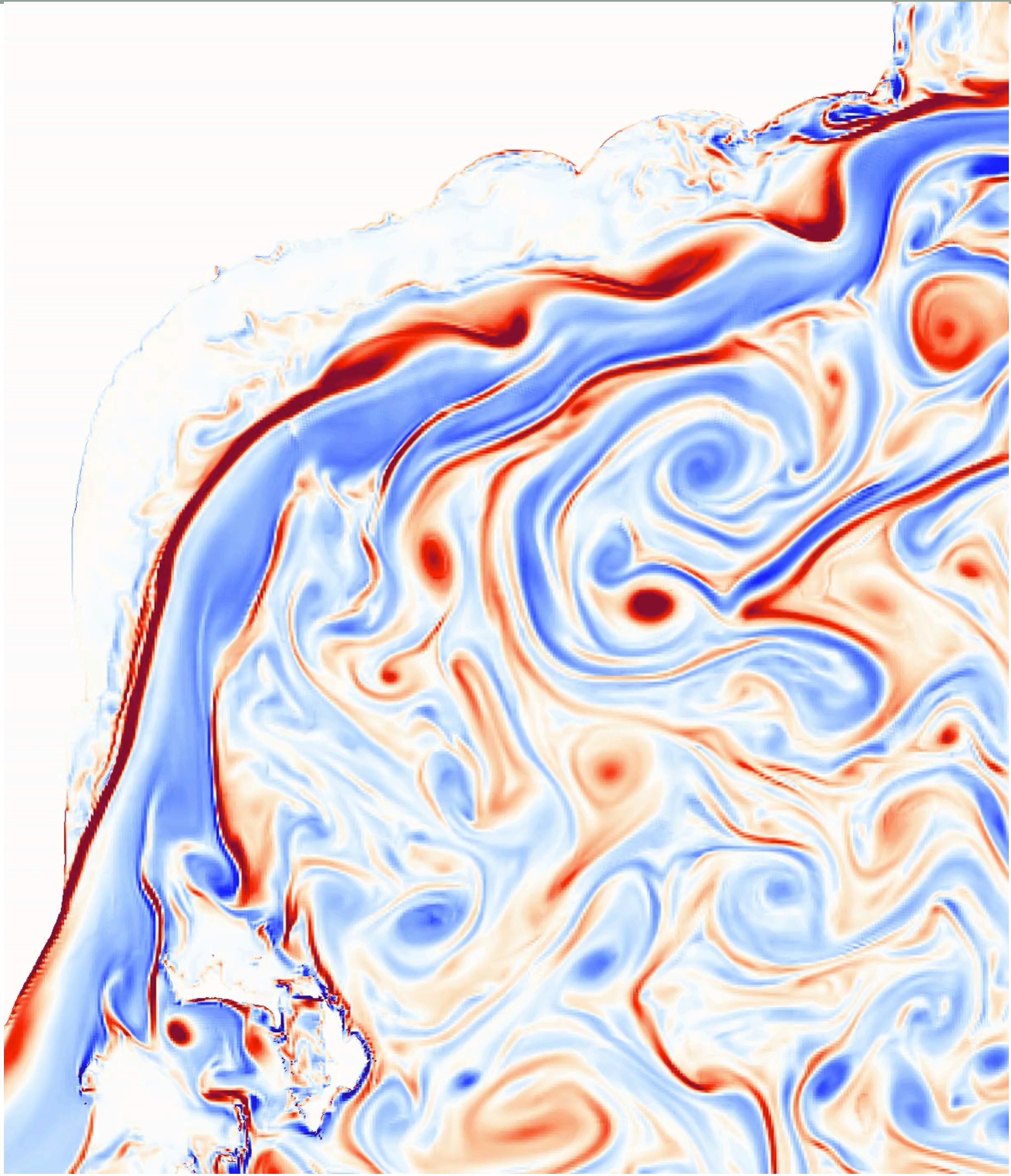


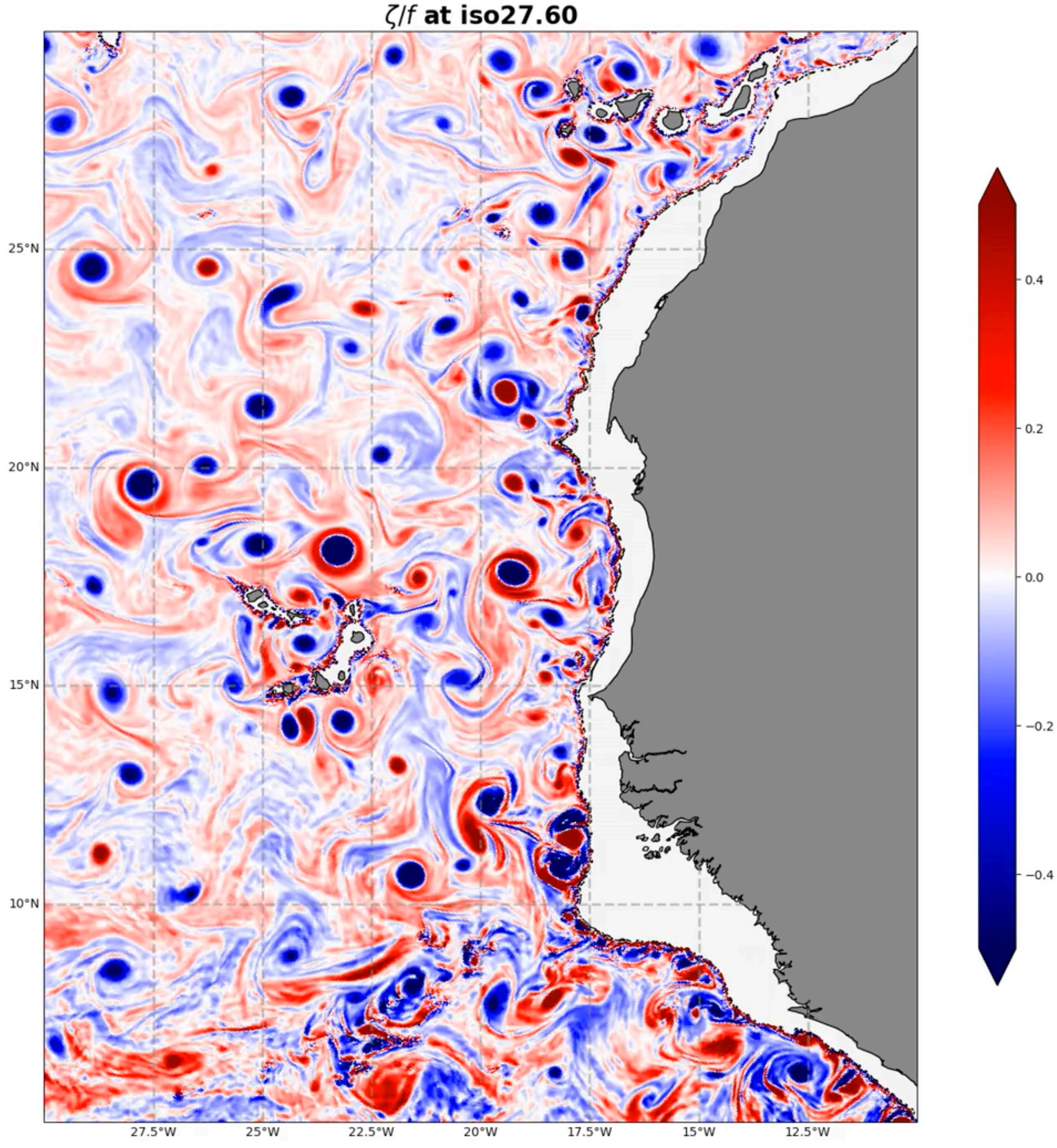
Turbulent water jet (Re = 2300) [Van Dyke, 82]

2d free-decay turbulence [G. Roullet]

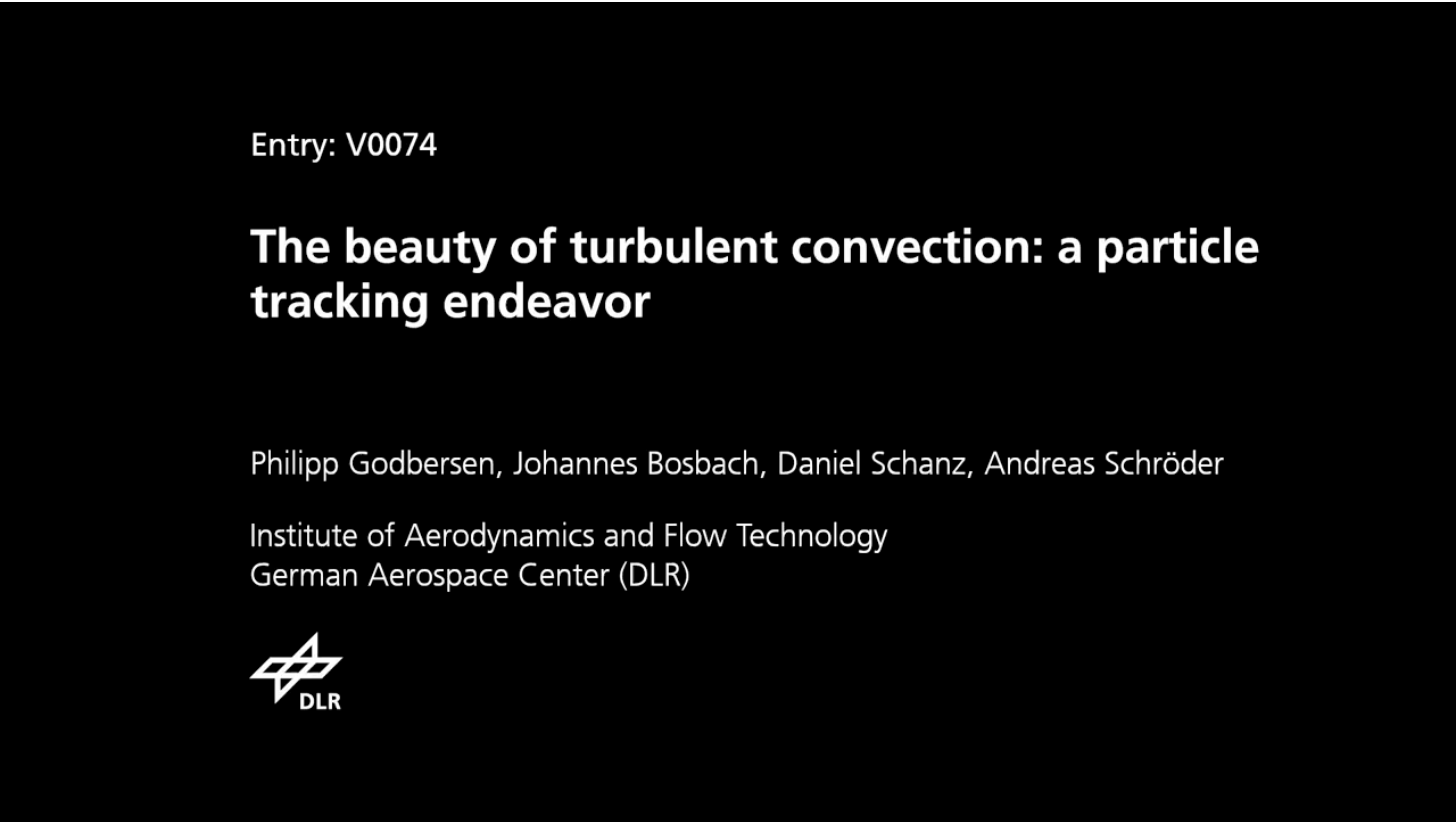


Turbulence in a realistic ocean model

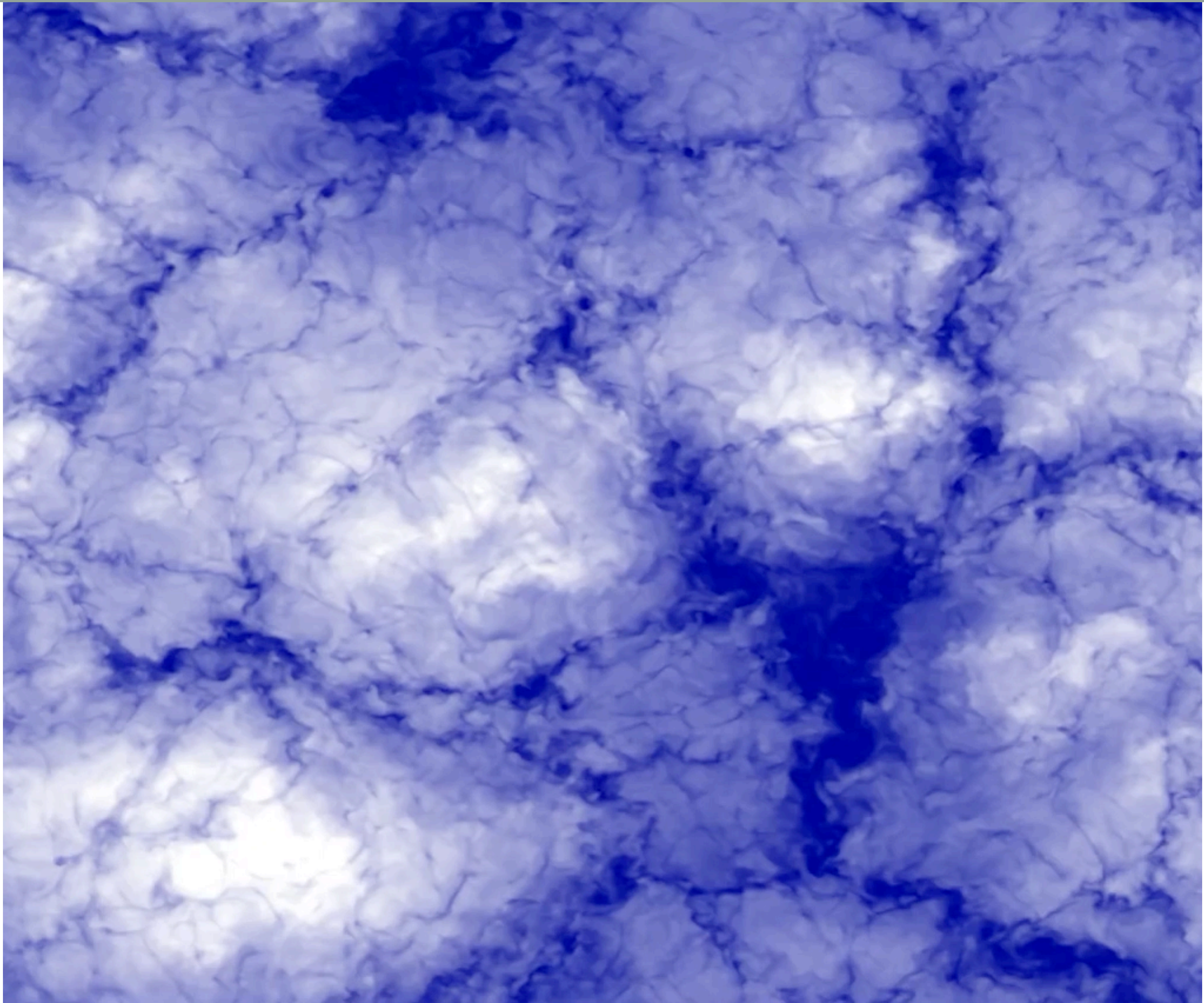




ζ/f at iso27.60
25°N
20°N
15°N
10°N
27.5°W
25°W
22.5°W
20°W
17.5°W
15°W
12.5°W
0.4
0.2
0.0
−0.2
−0.4

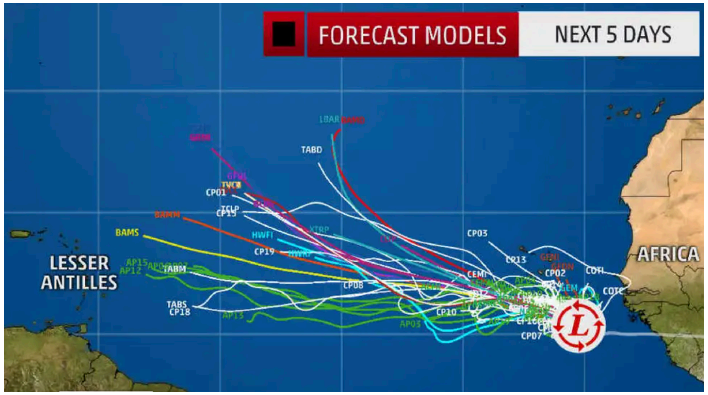


https://gfm.aps.org/meetings/dfd-2020/5f5fe77f199e4c091e67bfe8

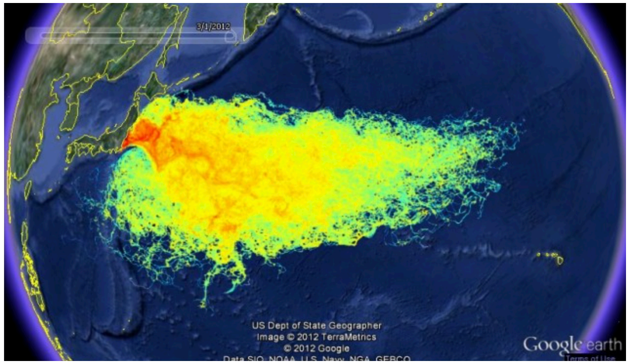


Why do we care about turbulence?

- Weather prediction



- Dispersion of polluants



- Aeronautics

And many more …

Why do we care about turbulence?

Turbulent cascades in foreign exchange markets

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THE availability of high-frequency data for financial markets has made it possible to study market dynamics on timescales of less than a day[1]. For foreign exchange (FX) rates Müller *et al.*[2] have shown that there is a net flow of information from long to short timescales: the behaviour of long-term traders (who watch the markets only from time to time) influences the behaviour of short-term traders (who watch the markets continuously). Motivated by this hierarchical feature, we have studied FX market

dynamics in more detail, and report here an analogy between these dynamics and hydrodynamic turbulence[3–8]. Specifically, the relationship between the probability density of FX price changes (Δx) and the time delay (Δt) (Fig. 1*a*) is much the same as the relationship between the probability density of the velocity differences (Δv) of two points in a turbulent flow and their spatial separation Δr (Fig. 1*b*). Guided by this similarity we claim that there is an information cascade in FX market dynamics that corresponds to the energy cascade in hydrodynamic turbulence. On the basis of this analogy we can now rationalize the statistics of FX price differences at different time delays, which is important for, for example, option pricing. The analogy also provides a conceptual framework for understanding the short-term dynamics of speculative markets.

https://www.nature.com/articles/381767a0

A flow of energy from large to small scales is one of the main characteristics of fully developed homogeneous isotropic turbulence in three spatial dimensions[24]. It provides a mechanism for dissipating large amounts of energy in a viscous fluid. Energy is pumped into the system at large scales of the order of, say, metres (by a moving car or a flying aeroplane) or kilometres (by meteorological events), transferred to smaller scales through a hierarchy of eddies of decreasing sizes, and dissipated at the smallest scale—of the order of millimetres in the above examples. This cascade of kinetic energy extending over several orders of magnitude generates a scaling behaviour of the eddies and manifests itself in a scaling of the moments $\langle(\Delta v)^n\rangle$ of Δv as $(\Delta r)^{\zeta_n}$ (refs 5,6). Here the angle brackets $\langle\rangle$ denote the mean value of the enclosed quantity and Δv is the difference of the velocity component in the direction of the spatial separation of length Δr. Under the assumption that the eddies of each size are space-filling and that the downward energy flow is homogeneous, $\zeta_n = n/3$ (ref. 8). The probability densities $P_{\Delta r}(\Delta v)$ are then scale invariant. This means that if the velocity differences are normalized by their respective standard deviation, the resulting standardized probability densities do not depend on Δr. But for $n > 3$, experimentally determined values of ζ_n follow a concave curve definitely below the ($\zeta_n = n/3$)-line (Fig. 2*b*). The dependence of the standardized probability density on Δr also provides evidence that eddies of a given size are not space-filling but rather fluctuating in space and time in a typical intermittent way (Fig. 1*b*).

Our analyses of FX markets are based on a data set provided by Olsen and Associates containing all worldwide 1,472,241 bid–ask quotes for US dollar–German mark exchange rates which have emanated from the interbank Reuters network from 1 October 1992 until 30 September 1993. From these data we have determined the probability densities of price changes $P_{\Delta t}(\Delta x)$ with time delays Δt varying from five minutes up to approximately two days, which are displayed in Fig. 1*a*. In comparison, Fig. 1*b* shows the analogous turbulent probability densities $P_{\Delta r}(\Delta v)$, which exhibit the same characteristic features. Using the probability density $P_{\Delta t}(\Delta x)$, the information acquired by observing the market after a time Δt can be quantified as $I(\Delta t) = -\int P_{\Delta t}(\Delta x)\log P_{\Delta t}(\Delta x)\,d(\Delta x)$. It turns out that the dependence of this information on Δt is directly related to the scaling of the variance of Δx with Δt. In turbulence, on the other hand, the variance of the velocity differences at a distance Δr is proportional to the mean energy which is contained in an eddy of size Δr. This further supports the proposed analogy between energy and information.

Given the analogy between turbulence and FX market dynamics, we expect the moments of FX price changes to scale with the time delay as $\langle(\Delta x)^n\rangle \propto (\Delta t)^{\zeta_n}$. Scaling has already been reported for the mean absolute values of FX returns in ref. 9 and by Evertsz in ref. 1, and for the second moments of the variations of the Standard & Poor's 500 economic index in ref. 10. For FX

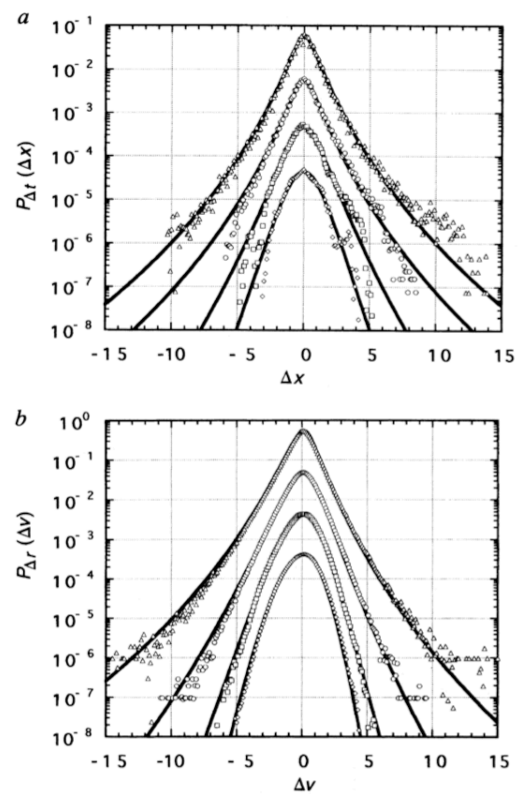

FIG. 1 *a*, Data points: standardized probability density $P_{\Delta t}(\Delta x)$ of price changes $\Delta x = x(t) - x(t+\Delta t)$ for time delays $\Delta t = 640$ s, 5,120 s, 40,960 s, 163,840 s (from top to bottom). The middle prices $x(t) = (x_{bid}(t) + x_{ask}(t))/2$ have been used (data provided by Olsen & Associates (see text)). The probability density has been calculated in a similar way as ref. 10. Full lines: results of (least-squares) fits carried out according to ref. 7; $\lambda^2 = 0.25$, 0.23, 0.13, 0.06 (from top to bottom). For better visibility, the curves have been vertically shifted with respect to each other. Note the systematic change in shape of the densities. *b*, Data points: standardized probability density $P_{\Delta v}$ of velocity differences $P_{\Delta r}(\Delta v)$ for a turbulent flow with $\Delta r = 3.3\eta$, 18.5η, 138η, 325η (data taken from ref. 13, $R_\lambda = 598$. Here η is the Kolmogorov scale, where viscous dissipation occurs. Full lines: results of (least-squares) fits carried out according to ref. 7; $\lambda^2 = 0.19$, 0.10, 0.04, 0.01.

- **Lesson 1 - Introduction:**

  - *What is turbulence?*
  - *Why do we care about turbulence?*
  - ***Where does it come from?***

  - Properties of turbulence:
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  - The closure problem

Navier-Stokes Equations

The Navier-Stokes equations probably contain all of turbulence.

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Momentum equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation
(no source/sink)

Kinematic viscosity:

$$\nu_{water} \approx 10^{-6} \mathrm{m}^2 \mathrm{s}^{-1}$$

$$\nu_{air} \approx 1.5 \times 10^{-5} \mathrm{m}^2 \mathrm{s}^{-1}$$

Navier-Stokes Equations

Millennium Prize problems:

Since understanding the Navier–Stokes equations is considered to be the first step to understanding the elusive phenomenon of turbulence, the Clay Mathematics Institute in May 2000 made this problem one of its seven Millennium Prize problems in mathematics. It offered a US $1,000,000 prize to the first person providing a solution for a specific statement of the problem:[1]

Prove or give a counter-example of the following statement:

In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

See https://www.jgula.fr/Turb/Articles/Robert.pdf

Navier-Stokes Equations

+ Boussinesq approximation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

Navier-Stokes Equations

+ Boussinesq approximation

$$\boxed{\frac{\partial \vec{u}}{\partial t}} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + \boxed{f \vec{k} \times \vec{u}} + \boxed{g \vec{k}} = \boxed{-\frac{\vec{\nabla} P}{\rho_0}} + \boxed{\nu \vec{\nabla}^2 \vec{u}} + \boxed{\vec{\mathcal{F}}}$$

Time variation | Advection (inertia) | Rotation | Gravity | Pressure gradient | Dissipation (viscosity) | Forcings

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

Navier-Stokes Equations

+ Boussinesq approximation

- Turbulence arises from the non-linear terms

$$\frac{\partial \vec{u}}{\partial t} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Advection
(inertia)

- Only one here = advection *(quadratic nonlinearity)*
- Turbulence results from the nonlinear nature of advection, which enables interaction between motions on different spatial scales.

Navier-Stokes Equations

+ Boussinesq approximation

- x-momentum equation:

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u}\cdot\vec{\nabla} u - f v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad fU \qquad \frac{P}{\rho_0 L} \qquad \frac{\nu U}{L^2} \qquad F$$

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad fU \qquad \frac{P}{\rho_0 L} \qquad \frac{\nu U}{L^2} \qquad F$$

$$\frac{L^2}{\nu T} \qquad \boxed{\frac{UL}{\nu}} \qquad \frac{fL^2}{\nu} \qquad \frac{PL}{\rho_0 \nu U} \qquad 1 \qquad \frac{FL^2}{\nu U}$$

Equations scalings

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

$$\boxed{\frac{UL}{\nu}}$$

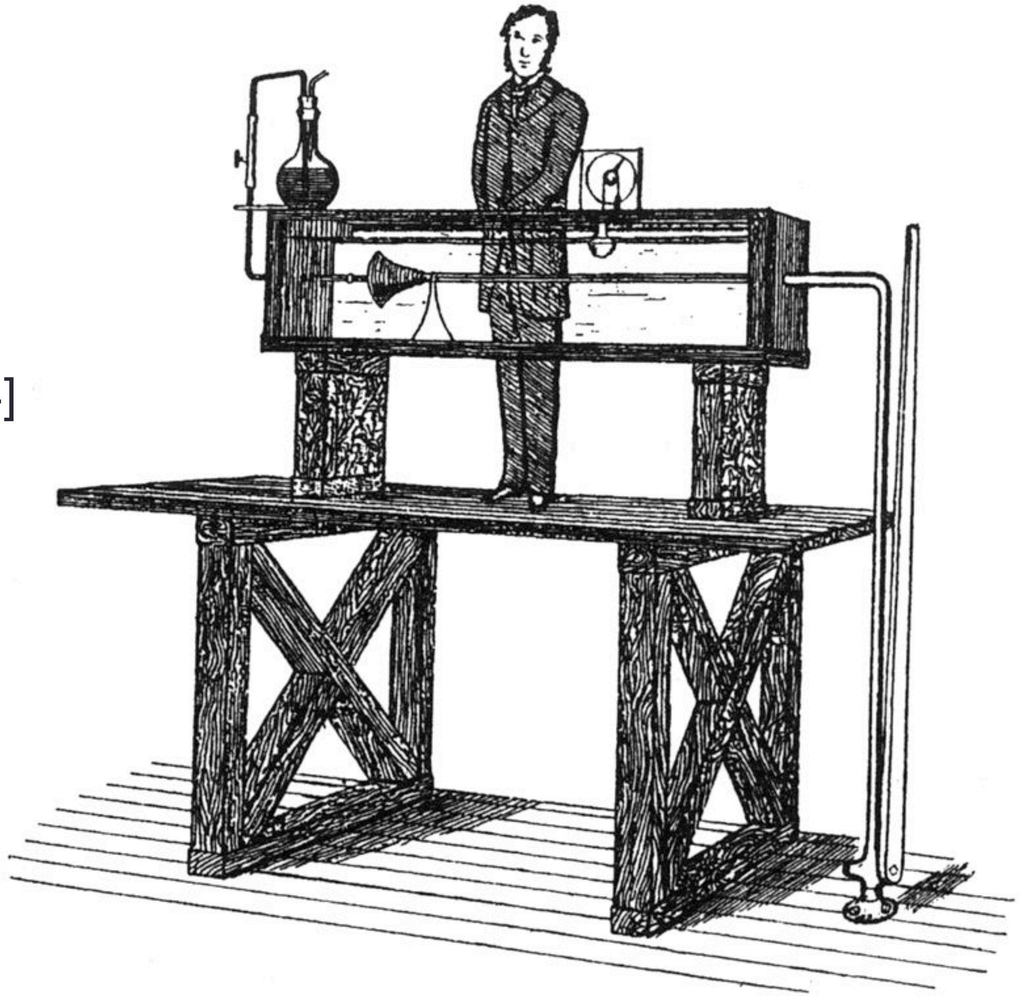
= Reynolds Number

the ratio of the non-linear terms to the viscous terms

Reynolds Number

$$Re = \frac{UL}{\nu}$$

Osborne Reynolds experiment [1884]



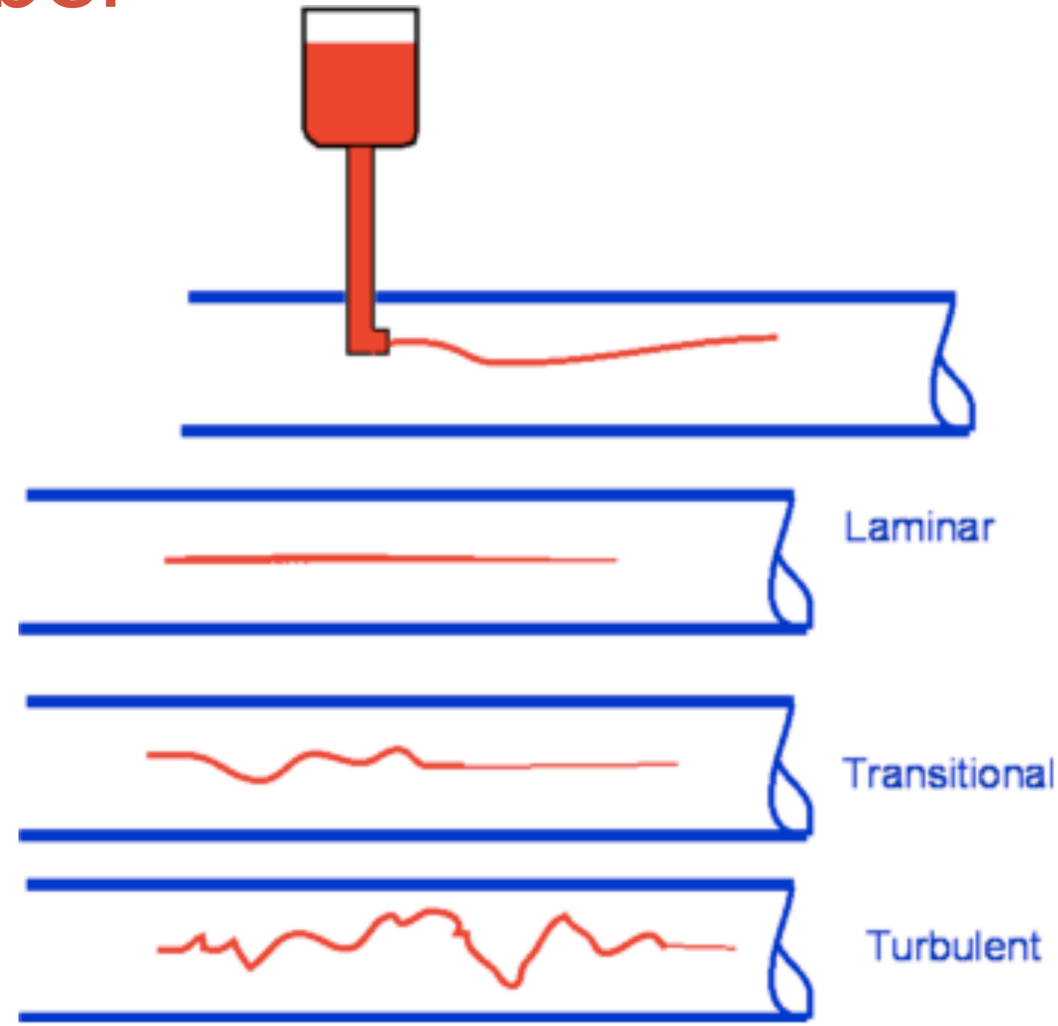
An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous

Reynolds Number

$$Re = \frac{UL}{\nu}$$

Re < 2000 = laminar

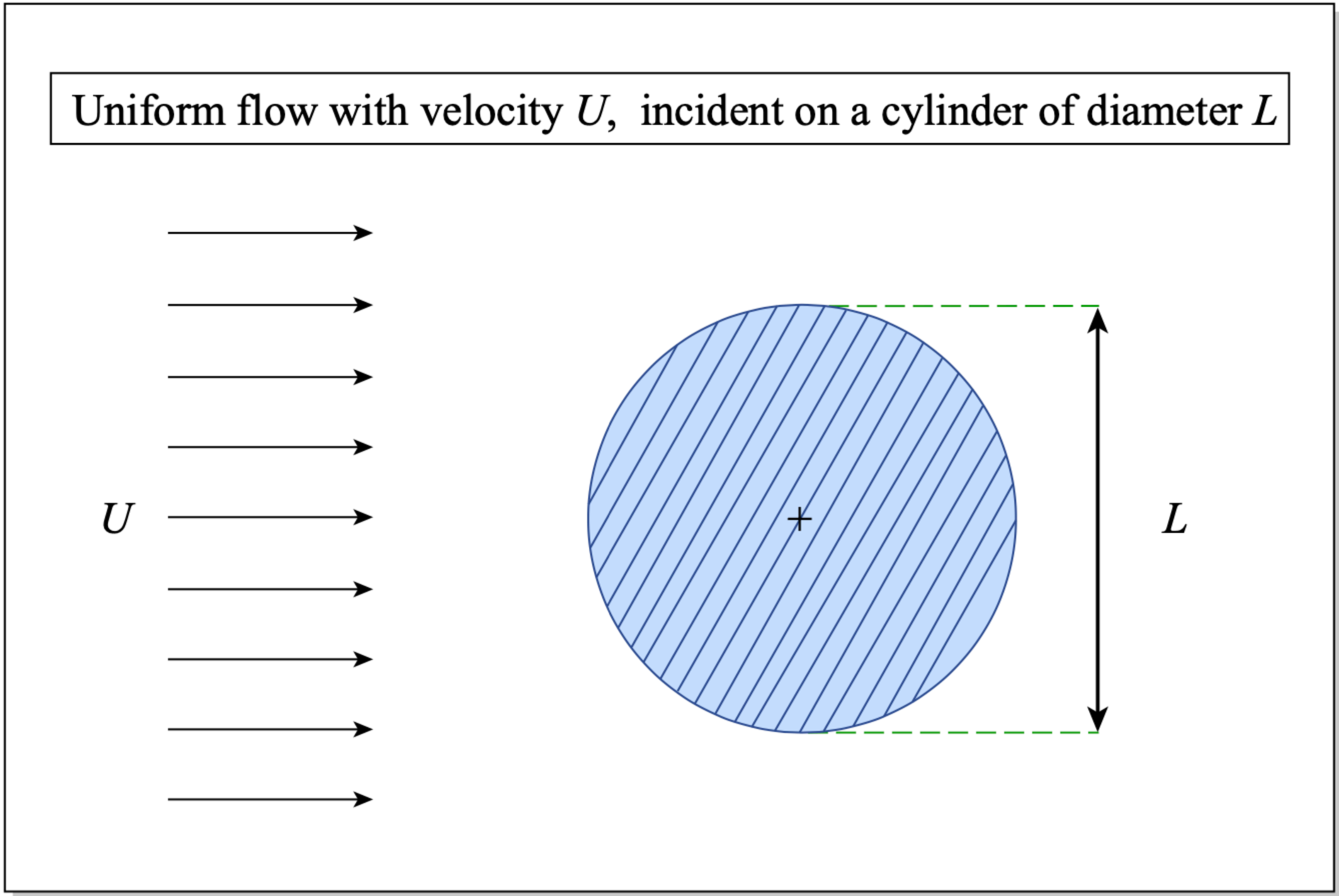
Re > 4000 = turbulent



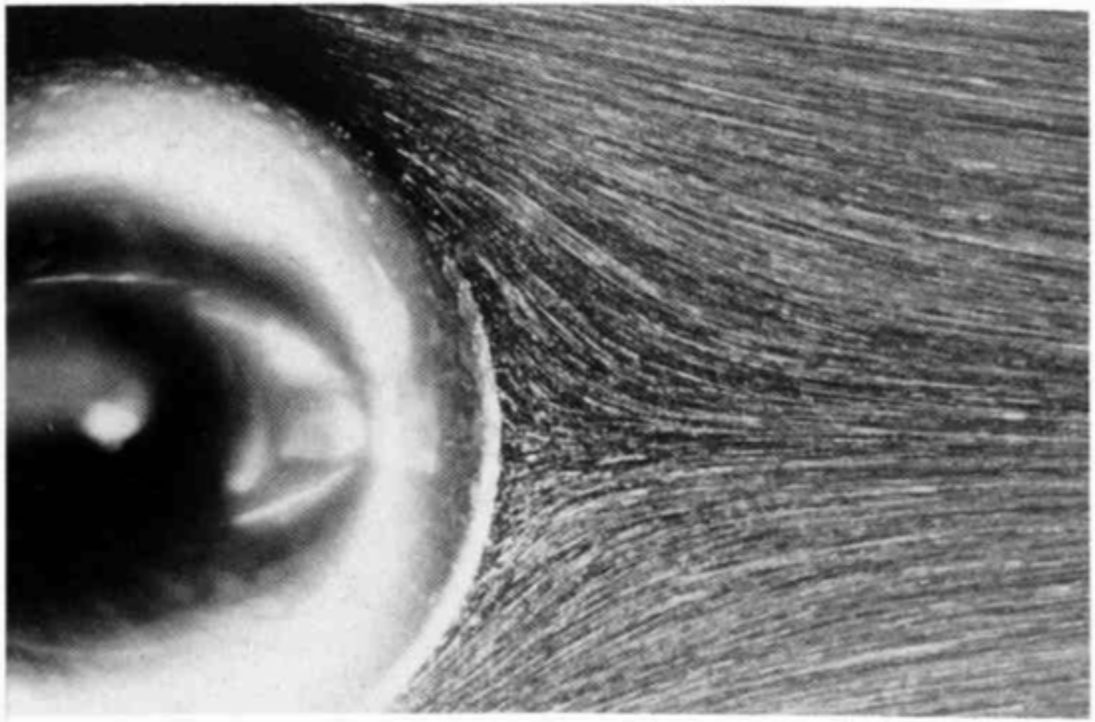
Reynolds experiment

http://www-mdp.eng.cam.ac.uk/web/library/enginfo/aerothermal_dvd_only/aero/fprops/pipeflow/node8.html

Transition to turbulence

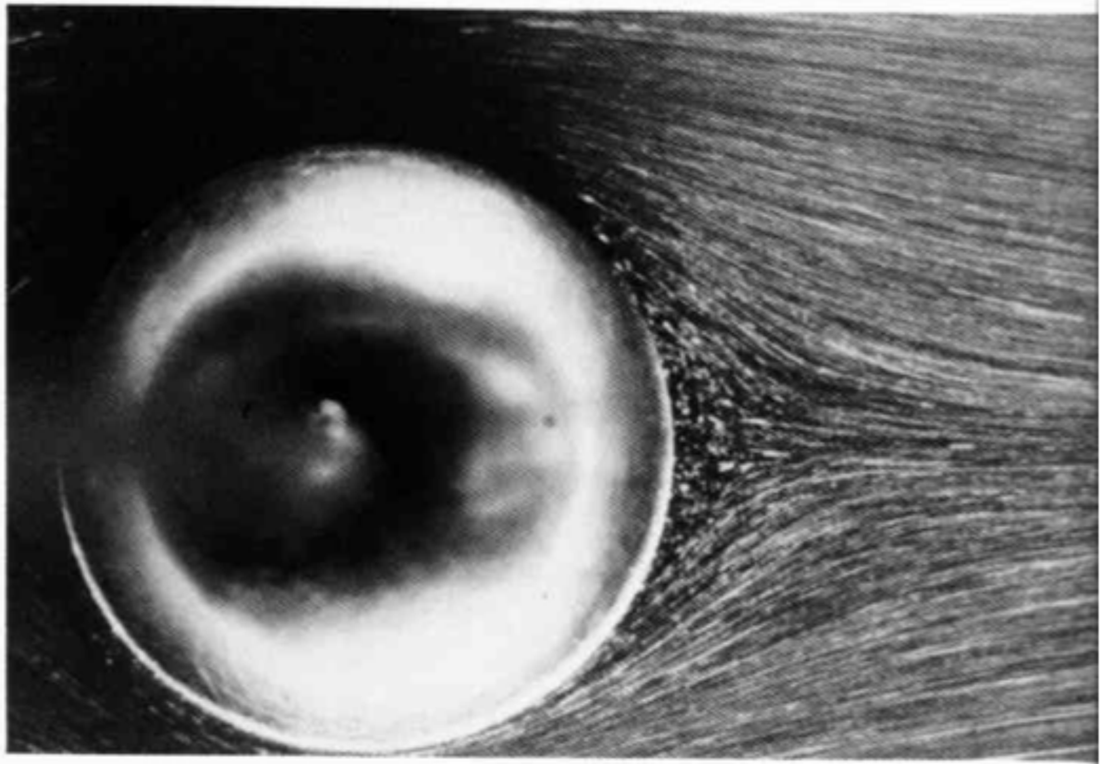


Transition to turbulence



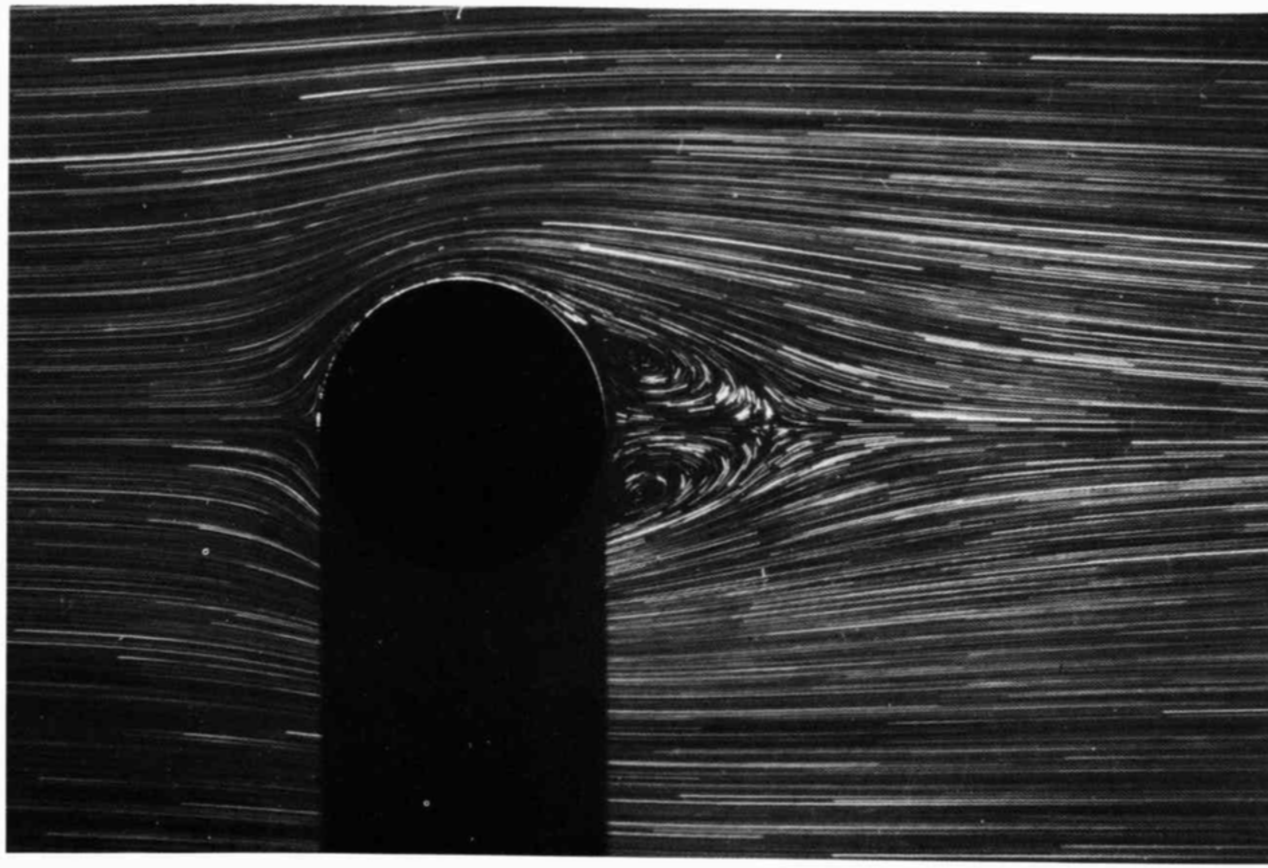
49. Sphere at R=25.5. Although it is not obvious, the flow is believed just to have separated at the rear at this Reynolds number, in contrast to the unseparated flow of figure 27. Aluminum dust is illuminated in water. *Taneda 1956b*

Transition to turbulence



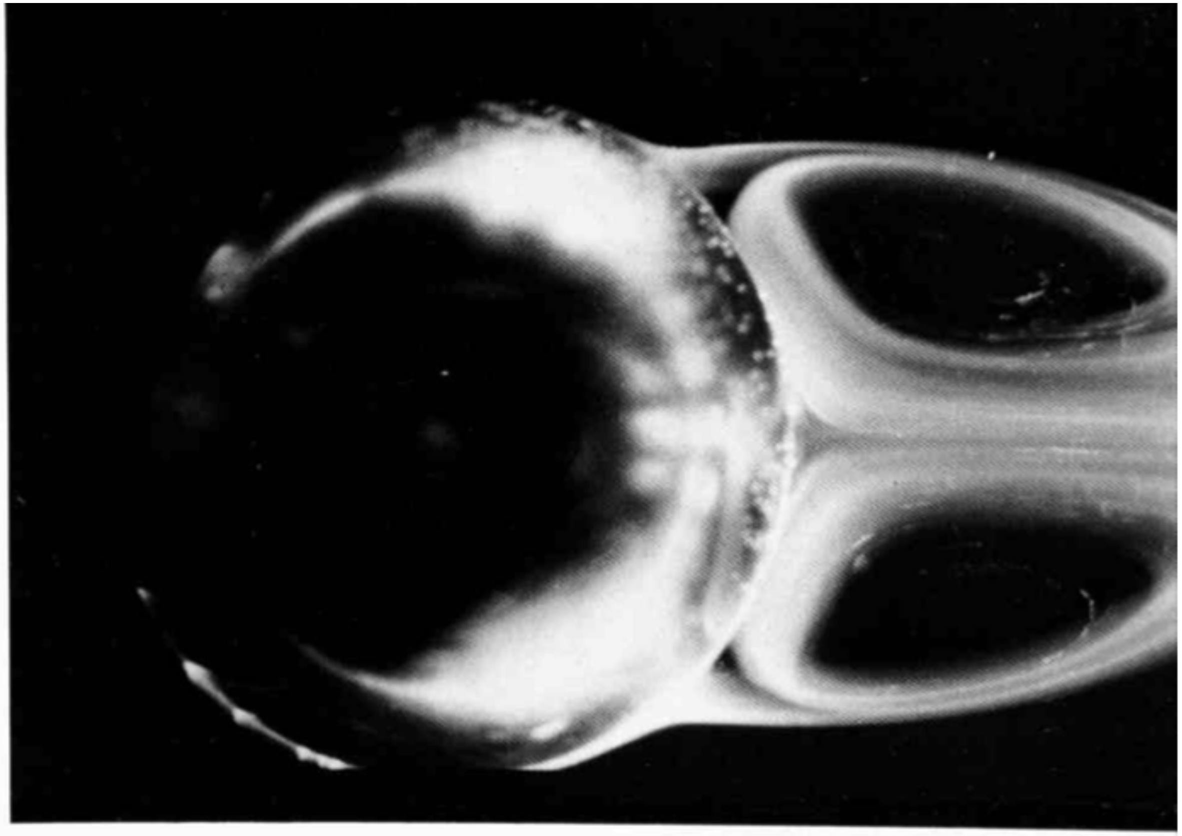
50. **Sphere at $R=26.8$.** At this slightly higher speed the flow has clearly separated over the rear of the sphere, to form a thin standing vortex ring. Aluminum dust is illuminated in water. *Taneda* 1956b

Transition to turbulence



51. Sphere at $R = 56.5$**.** As in figure 8, the sphere is falling steadily down the axis of a tube filled with oil, but here so large that the influence of the walls is negligible. Magnesium cuttings are illuminated by a sheet of light, which casts the shadow of the sphere. *Archives de l'Académie des Sciences de Paris. Payard & Coutanceau 1974*

Transition to turbulence

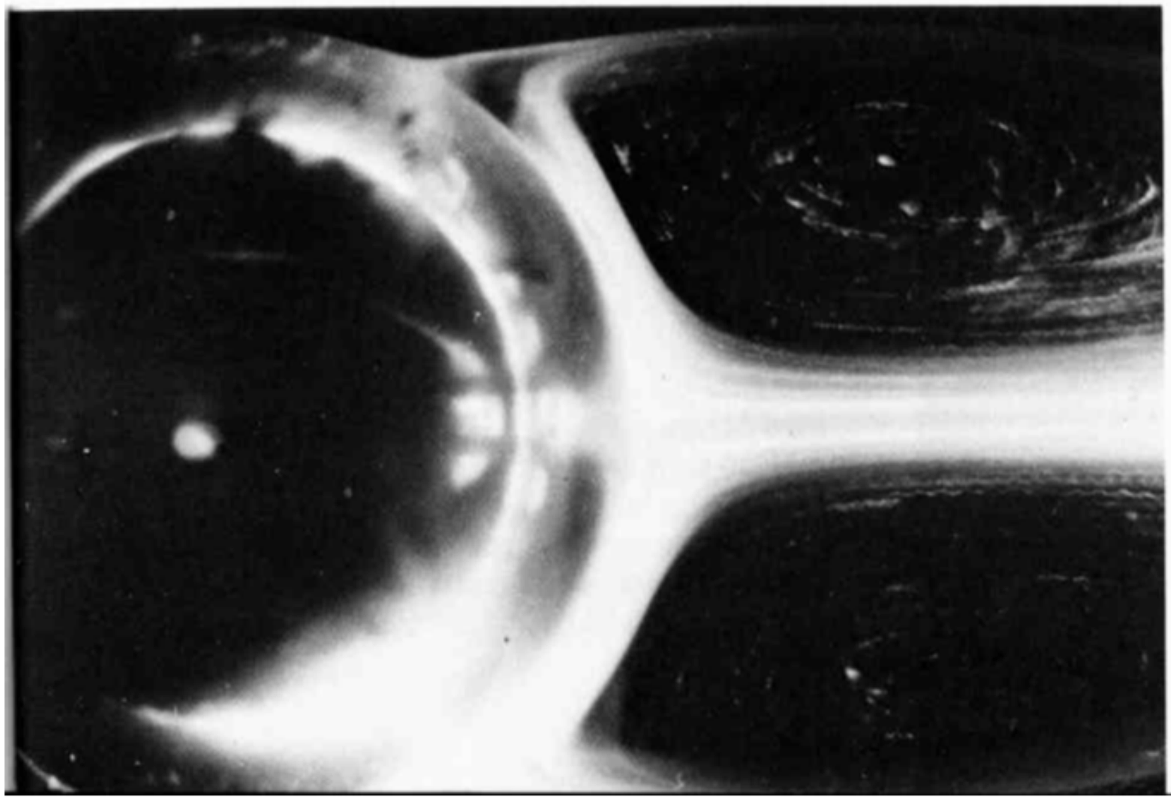


52. Sphere at R**=104.** At this Reynolds number the recirculating wake extends a full diameter downstream, but is perfectly steady, as for the circle in figure 44. Visualization is by a thin coating of condensed milk on the sphere, which gradually melts and is carried into the stream of water. *Taneda 1956b*

Transition to turbulence

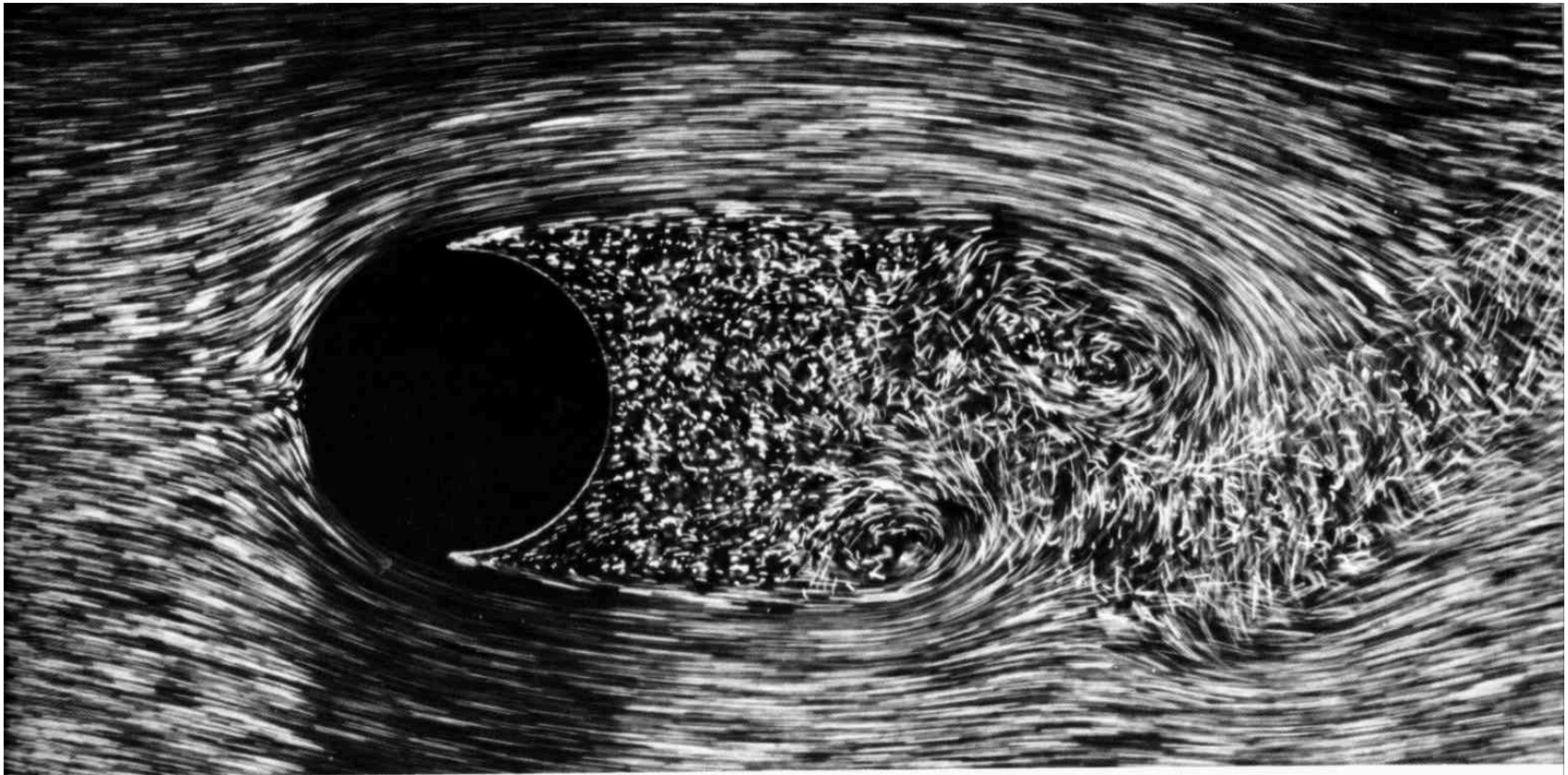
53. Sphere at R**=118.** The wake grows more slowly in axisymmetric than plane flow. These photographs have shown that the length of the recirculating region is proportional to the logarithm of the Reynolds number, whereas it grows linearly with Reynolds number for a cylinder. Aluminum dust shows the flow of water. *Taneda 1956b*

Transition to turbulence



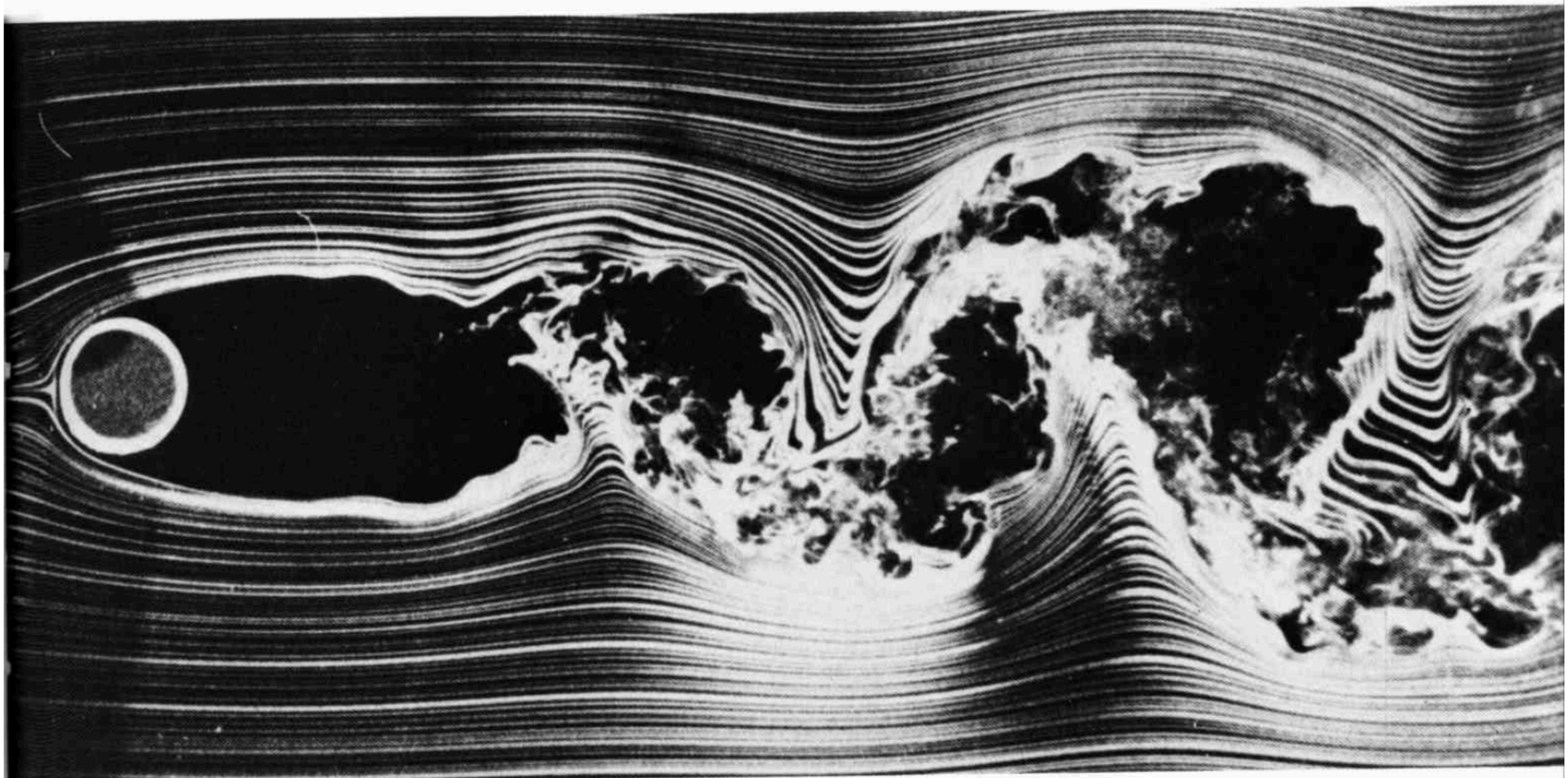
54. Sphere at R**=202.** The rear of the recirculating region behind a sphere begins to oscillate slowly at a Reynolds number of about 130, but the flow is still perfectly laminar at this higher speed. Visualization is by condensed milk in water. *Taneda* 1956b

Transition to turbulence



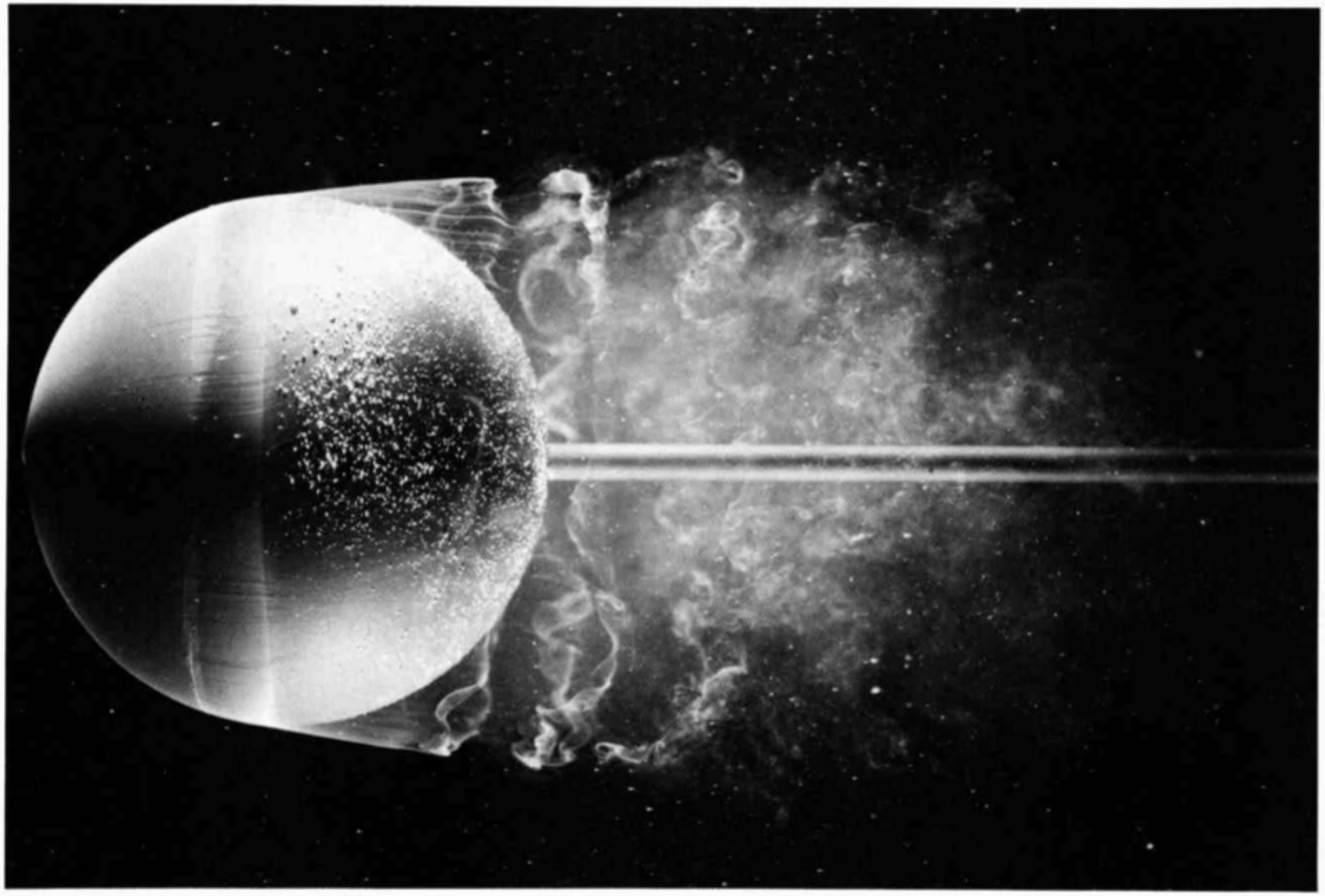
47. Circular cylinder at R=2000. At this Reynolds number one may properly speak of a boundary layer. It is laminar over the front, separates, and breaks up into a turbulent wake. The separation points, moving forward as the Reynolds number is increased, have now attained their upstream limit, ahead of maximum thickness. Visualization is by air bubbles in water. *ONERA photograph, Werlé & Gallon 1972*

Transition to turbulence



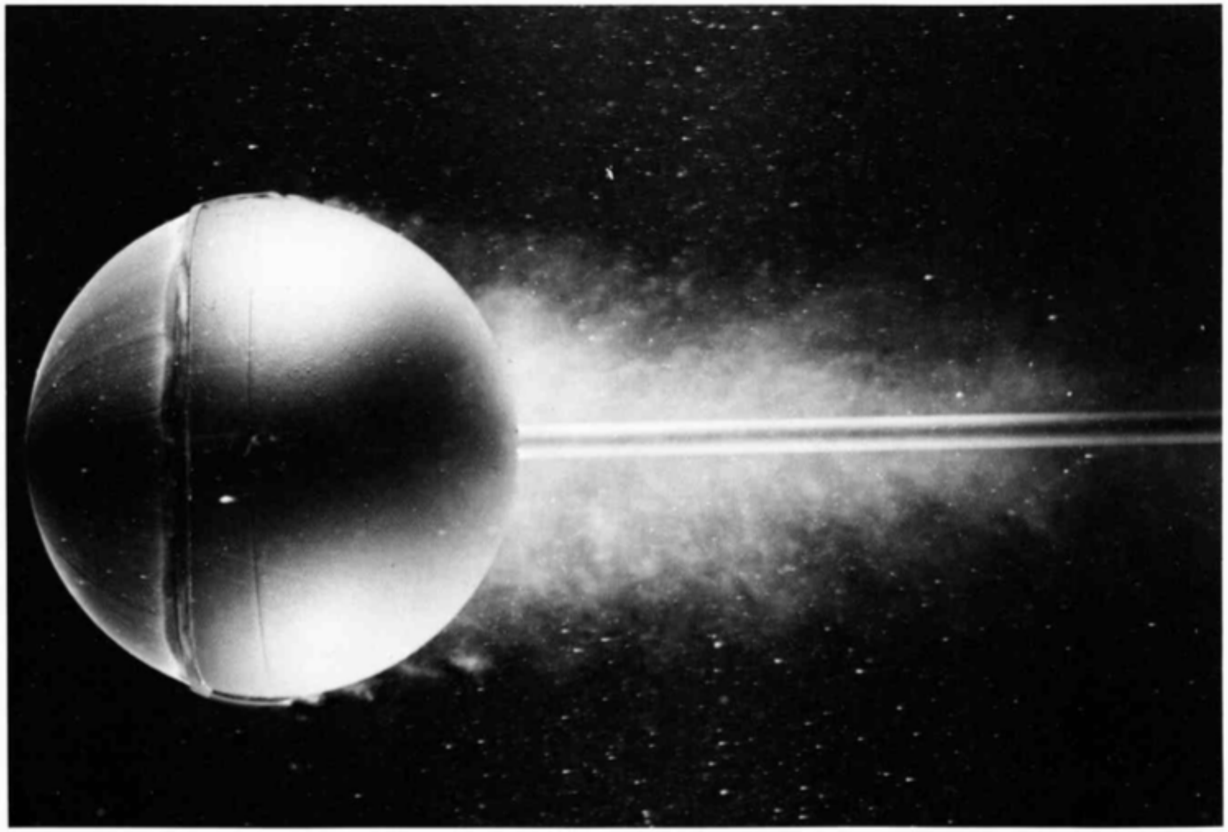
48. Circular cylinder at R**=10,000.** At five times the speed of the photograph at the top of the page, the flow pattern is scarcely changed. The drag coefficient consequently remains almost constant in the range of Reynolds number spanned by these two photographs. It drops later when, as in figure 57, the boundary layer becomes turbulent at separation. *Photograph by Thomas Corke and Hassan Nagib*

Transition to turbulence



55. Instantaneous flow past a sphere at R=15,000. Dye in water shows a laminar boundary layer separating ahead of the equator and remaining laminar for almost one radius. It then becomes unstable and quickly turns turbulent. *ONERA photograph, Werlé 1980*

Transition to turbulence



57. Instantaneous flow past a sphere at R=30,000 with a trip wire. A classical experiment of Prandtl and Wieselsberger is repeated here, using air bubbles in water. A wire hoop ahead of the equator trips the boundary layer. It becomes turbulent, so that it separates farther rearward than if it were laminar (opposite page). The drag is thereby dramatically reduced, in a way that occurs naturally on a smooth sphere only at a Reynolds number ten times as great. *ONERA photograph, Werlé 1980*

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- Dissipative time scale = *time scale that is required for molecular friction to bring the motion at scale L to rest*

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- Dissipative time scale *= time scale that is required for molecular friction to bring the motion at scale L to rest*

$$\frac{L^2}{\nu T} = \frac{T_\nu}{T} \qquad T_\nu = \frac{L^2}{\nu}$$

Activity 1:

Kinematic viscosity:

$$\nu_{water} \approx 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$$
$$\nu_{air} \approx 1.5 \times 10^{-5}\mathrm{m}^2\mathrm{s}^{-1}$$

- Reynolds number for a mesoscale vortex in the ocean?
- Reynolds number for a storm system in the atmosphere?

- How long would it take for molecular dissipation to halt a storm system?
- How long would it take for molecular dissipation to halt a stirred coffee?

Properties of turbulence

- No formal definition for turbulence.
- It can be defined as the attempt to bring together our understanding of the laws that govern fluid dynamics (the Navier-Stokes equations) with the irregular nature of real flows.
- A useful approach is to list what properties must be present to consider a flow turbulent.

Properties of turbulence

- **Broadband spectrum in space and time**
  - *Turbulent flows are characterized by structures on a broad range of spatial and temporal scales, even given smooth or periodic initial conditions and forcing.*
-

Properties of turbulence

- **Broadband spectrum in space and time**

- **Dominated by nonlinearity**
  - *A field of non-interacting linear internal waves with many different frequencies and wavenumbers can also have a large range of length scales, but it is not turbulent. Why not? In a turbulent flow the different scales interact, through the nonlinear terms in the equations of motion. And these nonlinear interactions are responsible for the presence of structure on many different scales.*

-

Properties of turbulence

- **Broadband spectrum in space and time**
- **Dominated by nonlinearity**
- **Unpredictable in space and time**
  - *Turbulent flows are predictable for only short times and short distances. Even though we know the equations that govern the evolution of the fluid, we cannot make predictions about the details of the flow due to its sensitive dependence on initial and boundary conditions. Predictability, however, can be recovered in a statistical sense. The sensitive dependence on initial and boundary conditions is a fundamental property of chaotic systems.*

Properties of turbulence

- **Broadband spectrum in space and time**
- **Dominated by nonlinearity**
- **Unpredictable in space and time**
- **Time irreversible**
  - *Turbulent motions are not time reversible. As time goes on, turbulent motions tend to forget their initial conditions and reach some equilibrated state. Turbulence mixes stuff up, it does not unmix it.*

Non-linearities and scale interactions

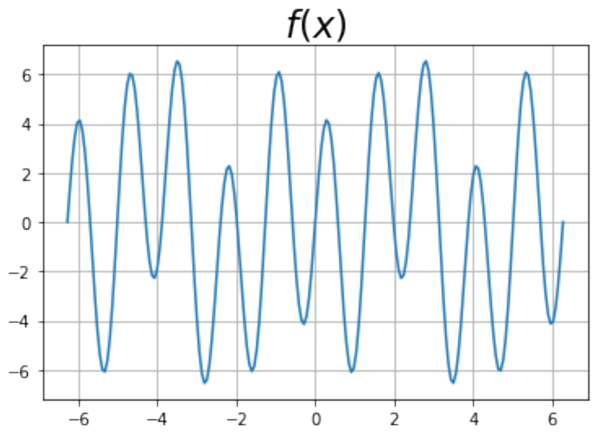
- Fourier decomposition: $$f(x) = \sum_{k_x} \tilde{f}(k_x) e^{ik_x x}$$

- We can extract the component at a single frequency/ wavenumber by Fourier transforming :

$$\tilde{f}(k_x) = \int f(x) e^{-ik_x x} dx$$

Non-linearities and scale interactions

- For example



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Non-linearities and scale interactions

- For example
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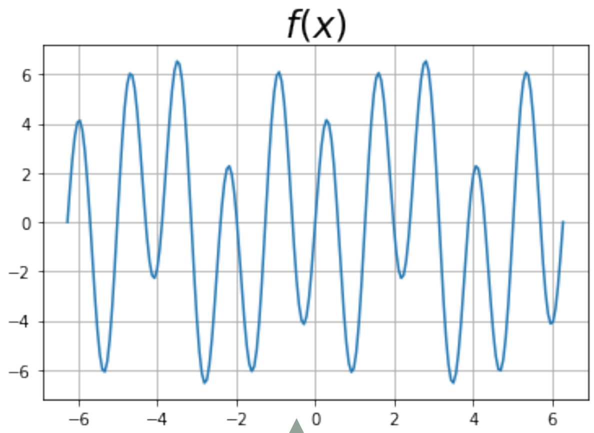
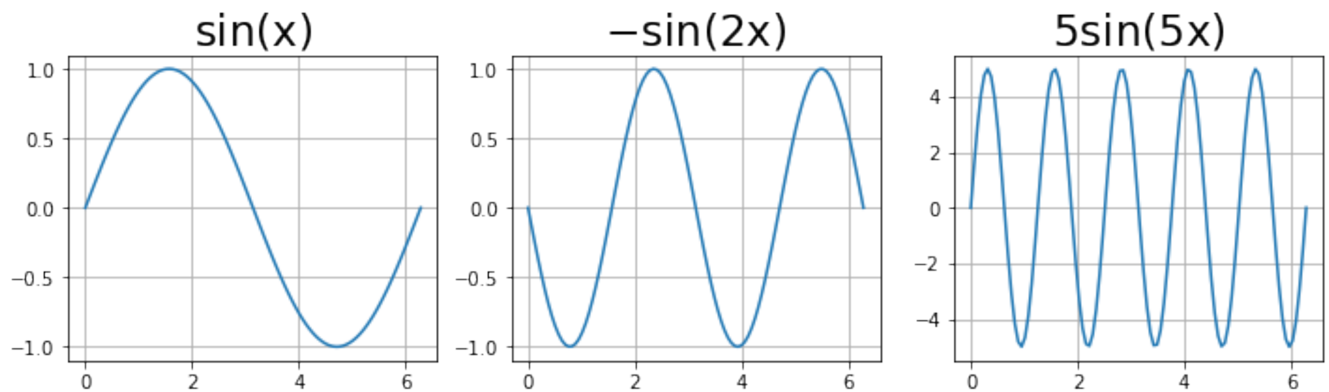
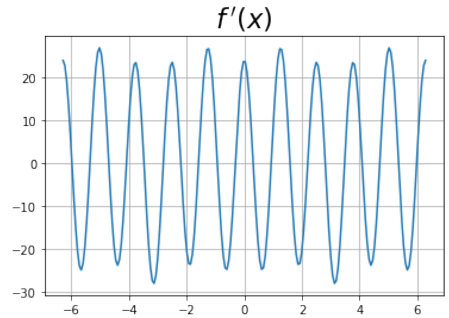
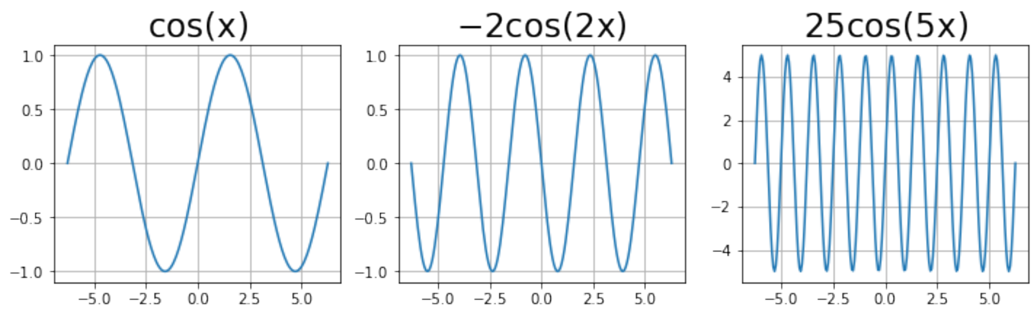




Non-linearities and scale interactions

- Linear combinations / derivate of the function still have the same components:



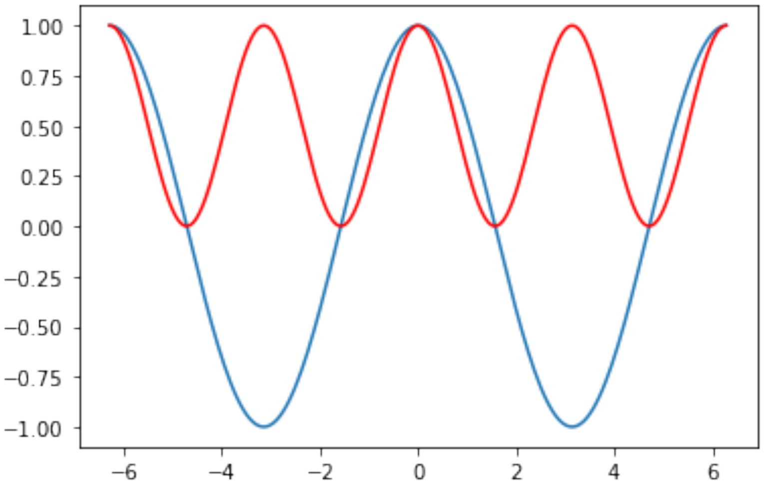


- So for a **linear system of equations**, if you have forcings/ initial condition at a given scale (e.g. k_x), **the solution will have the same frequency/wavenumbers.**

Non-linearities and scale interactions

- But the product of two components can give rise to a new scale in the flow.

- For example: $$f^2(x) = \cos^2(\mathrm{x}) = \frac{1}{2}(1 + \cos(2\mathrm{x}))$$



Triads interactions

- The nonlinear term in the equations of motion leads to interactions among different length scales
- *Let's write an equation for an incompressible 2 dimensional flow (for simplicity)*

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \zeta, \quad \zeta = \nabla^2 \psi.$$

With $u, v = -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}$ *and* $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Triads interactions

- *In the Fourier space:*

$$\psi(x,y,t) = \sum_{\boldsymbol{k}} \tilde{\psi}(\boldsymbol{k},t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \zeta(x,y,t) = \sum_{\boldsymbol{k}} \tilde{\zeta}(\boldsymbol{k},t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $\boldsymbol{k} = \mathbf{i}k^x + \mathbf{j}k^y$, $\tilde{\zeta} = -k^2\tilde{\psi}$ where $k^2 = {k^x}^2 + {k^y}^2$

Triads interactions

- The equation (without forcings and dissipation for simplicity):

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = 0$$

- Becomes:

$$\frac{\partial}{\partial t}\sum_{\boldsymbol{k}} \tilde{\zeta}(\boldsymbol{k}, t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} = -\sum_{\boldsymbol{p}} p^x \tilde{\psi}(\boldsymbol{p}, t)\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^y \tilde{\zeta}(\boldsymbol{q}, t)\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}}$$
$$+\sum_{\boldsymbol{p}} p^y \tilde{\psi}(\boldsymbol{p}, t)\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^x \tilde{\zeta}(\boldsymbol{q}, t)\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}}.$$

Triads interactions

$$\frac{\partial}{\partial t}\sum_{\boldsymbol{k}}\tilde{\zeta}(\boldsymbol{k},t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} = -\sum_{\boldsymbol{p}} p^x\tilde{\psi}(\boldsymbol{p},t)\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^y\tilde{\zeta}(\boldsymbol{q},t)\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} + \sum_{\boldsymbol{p}} p^y\tilde{\psi}(\boldsymbol{p},t)\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^x\tilde{\zeta}(\boldsymbol{q},t)\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}}.$$

- We multiply by $\exp(-i\boldsymbol{k}\cdot\boldsymbol{x})$
- And use the fact that the Fourier modes are orthogonal;

$$\int \mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}}\,\mathrm{d}A = \frac{1}{L^2}\delta(\boldsymbol{p}+\boldsymbol{q}).$$

where $\delta(\boldsymbol{p}+\boldsymbol{q})$ equals unity if $\boldsymbol{p} = -\boldsymbol{q}$ and is zero otherwise.

Triads interactions

- And get:

$$\frac{\partial}{\partial t}\tilde{\psi}(\boldsymbol{k},t) = \sum_{\boldsymbol{p},\boldsymbol{q}} A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\tilde{\psi}(\boldsymbol{p},t)\tilde{\psi}(\boldsymbol{q},t) + \tilde{F}(\boldsymbol{k}) - \nu k^4 \tilde{\psi}(\mathbf{k},t),$$

where $A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = (q^2/k^2)(p^x q^y - p^y q^x)\delta(\boldsymbol{p}+\boldsymbol{q}-\mathbf{k})$

- Only vectors with $\mathbf{p}+\mathbf{q}-\mathbf{k}=0$ have a non-zero contribution = **Triads interactions**

Triads interactions

- Triads interactions can be local or non-local:

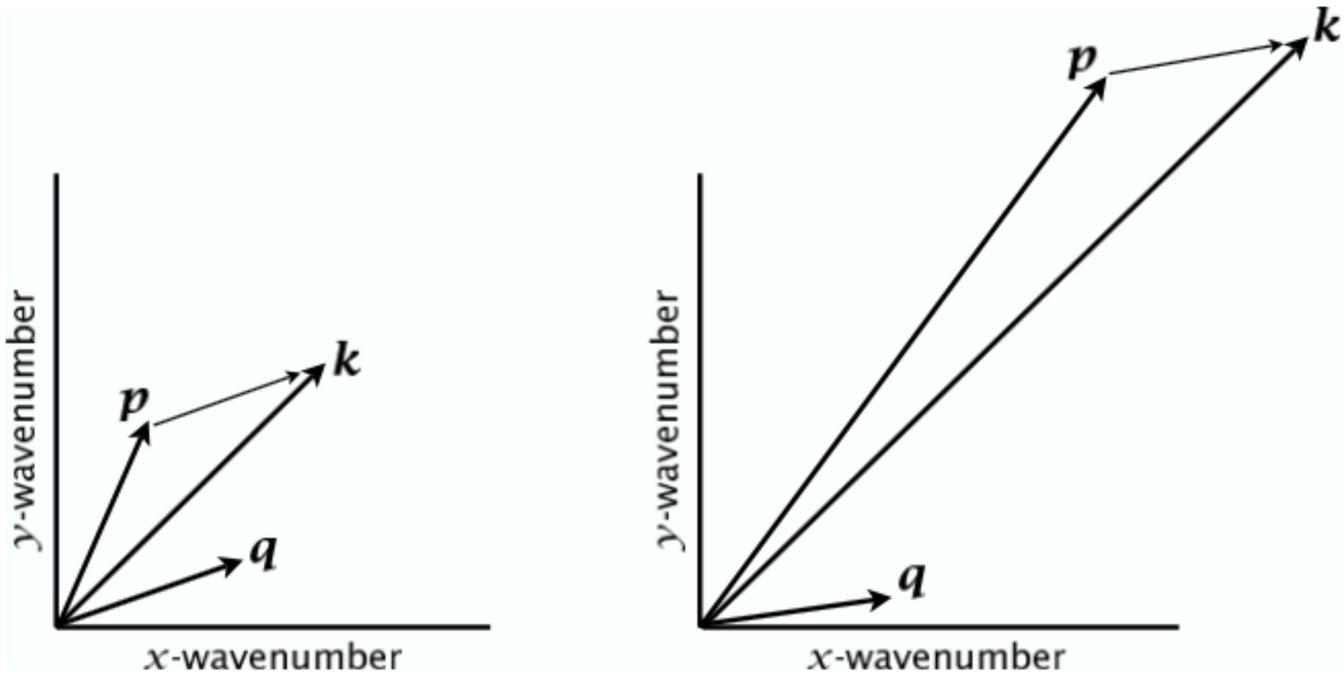
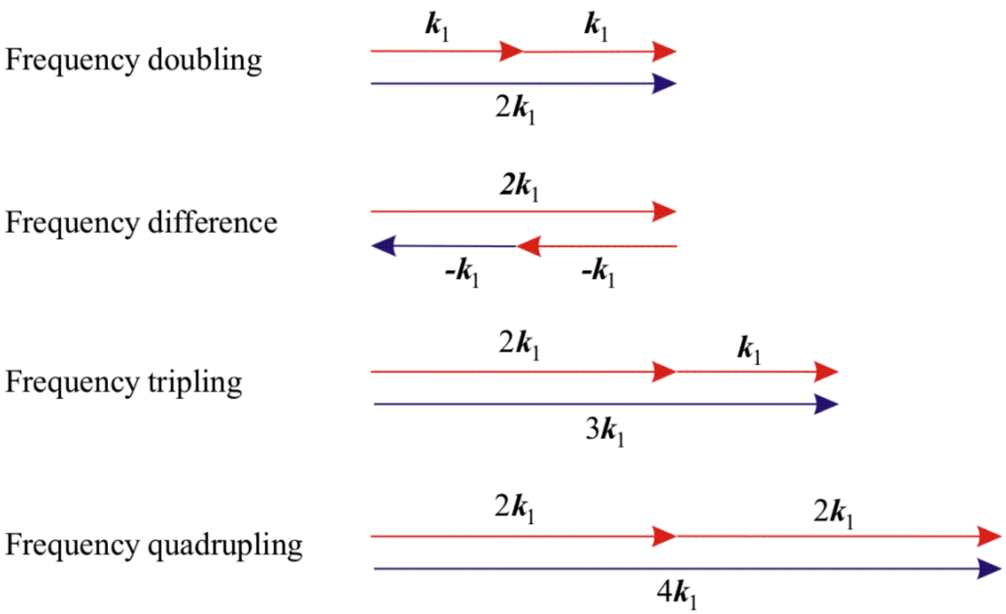


Fig. 8.1 Two interacting triads, each with $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$. On the left, a local triad with $k \sim p \sim q$. On the right, a nonlocal triad with $k \sim p \gg q$.

Triads interactions

- Starting with only 2 Fourier modes you can fill the entire spectrum due to scale interactions
- 

Triads interactions

- Forcing is fixed at a certain scale(s) and dissipation acts on each Fourier mode with a coefficient that increases with wavenumber and therefore that preferentially affects small scales.

$$\frac{\partial}{\partial t}\tilde{\psi}(\boldsymbol{k},t) = \sum_{\boldsymbol{p},\boldsymbol{q}} A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\tilde{\psi}(\boldsymbol{p},t)\tilde{\psi}(\boldsymbol{q},t) + \tilde{F}(\boldsymbol{k}) - \nu k^4 \tilde{\psi}(\mathbf{k},t),$$

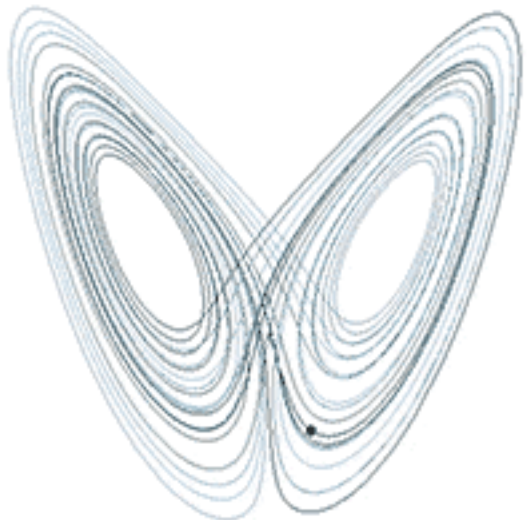
Chaotic behavior:

- **Chaos:** When the present determines the future, but the approximate present does not approximately determine the future *[E. Lorenz]*

A double rod pendulum showing chaotic behavior [https://en.wikipedia.org/wiki/Chaos_theory]

Chaotic behavior:

- **Chaos:** When the present determines the future, but the approximate present does not approximately determine the future *[E. Lorenz]*



Solutions in the Lorenz attractor
(originally a model for atmospheric convection)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y,$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$

A chaotic example:

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- Let's play around with a simpler version containing a non-linear term (with a Reynolds number r), a source and sink:

$$\frac{du}{dt} + r u^2 = -u + 1$$

- We will see how irregular and apparently stochastic solutions appear in from deterministic equations.

Properties of turbulence: unpredictability

$$\frac{du}{dt} + ru^2 = -u + 1$$

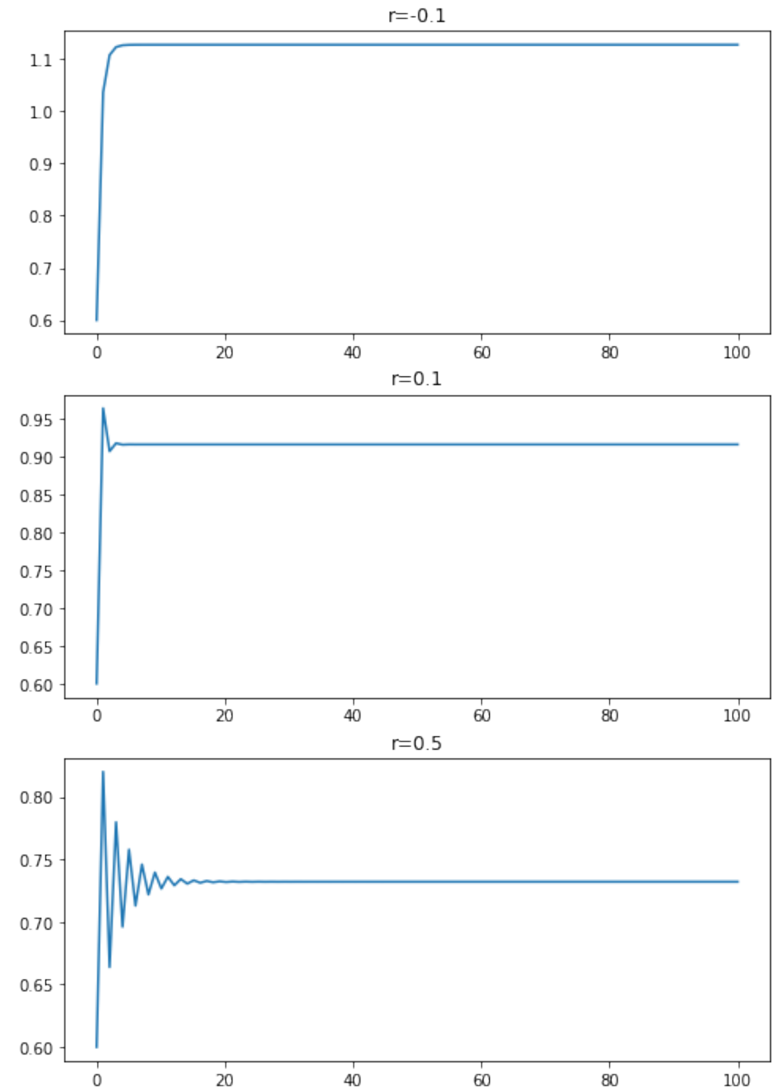
With $u_0 = 0.6$

- Fixed points:

$$\frac{d}{dt}u = -ru^2 + 1 - u = 0$$

$$u = -\frac{1}{2r} \pm \frac{\sqrt{1+4r}}{2r}$$

-



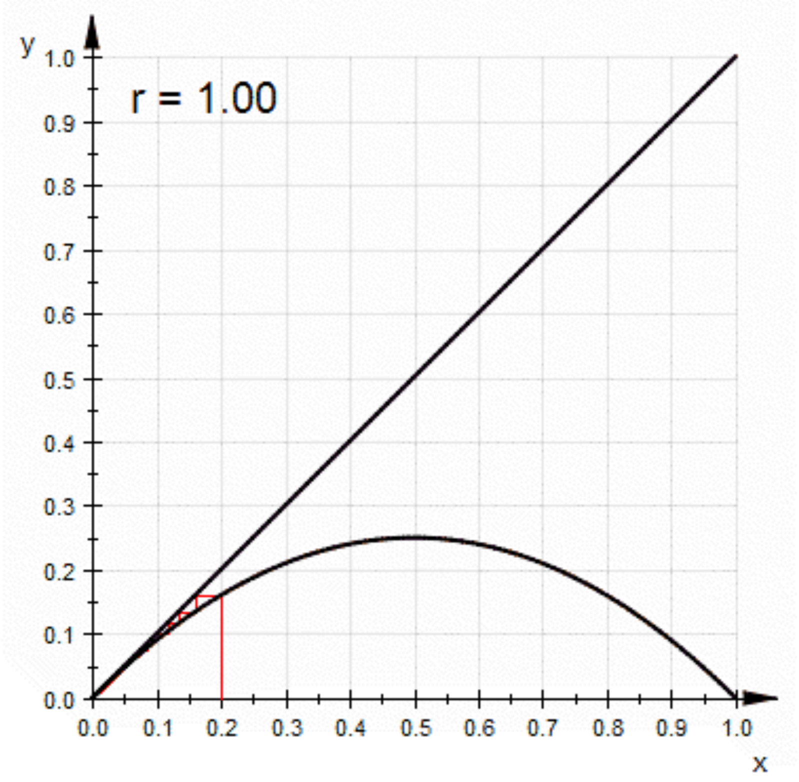
Activity 2:

$$\frac{du}{dt} + ru^2 = -u + 1$$

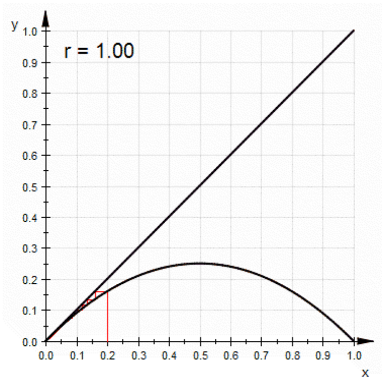
- Discretize using a simple forward Euler scheme (with dt=1) to get $u(t+1) = F[u(t)]$
- Using the discretized equation and u(0) = 0.6, plot u(t) for t=[0:100] and r = [0.1, 0.5, 0.8, 1.3, 2].
- Check the sensitivity to initial conditions (u(0) or u(0) + epsilon)
- Plot the pdf for r=2
- Plot the power spectra of u for r = [0.1, 0.5, 0.8, 1.3, 2],

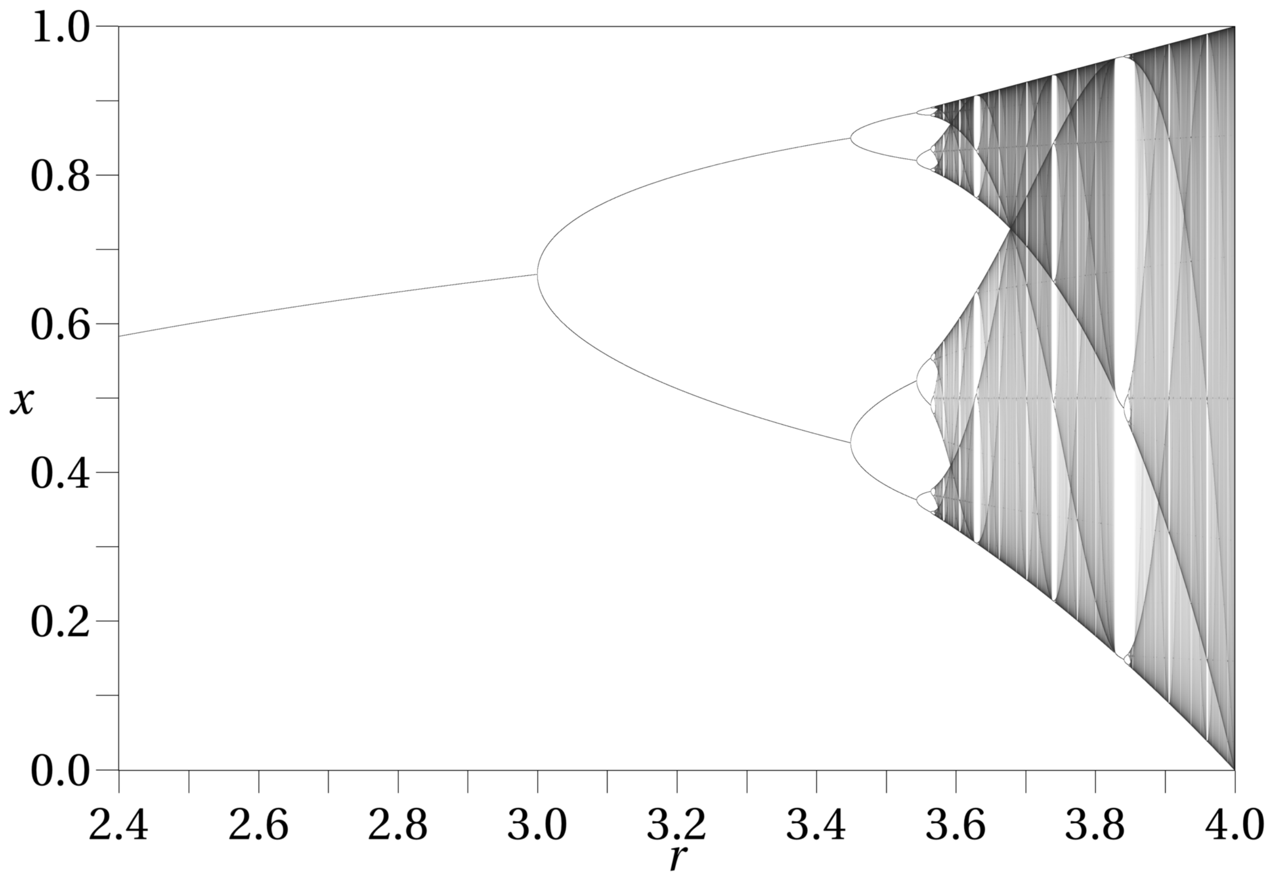
- The equation we've seen is similar to the **logistic map** by May (1976) [jgula.fr/Turb/Articles/May76.pdf] for the growth of population:

$$x_{n+1} = r x_n (1 - x_n)$$

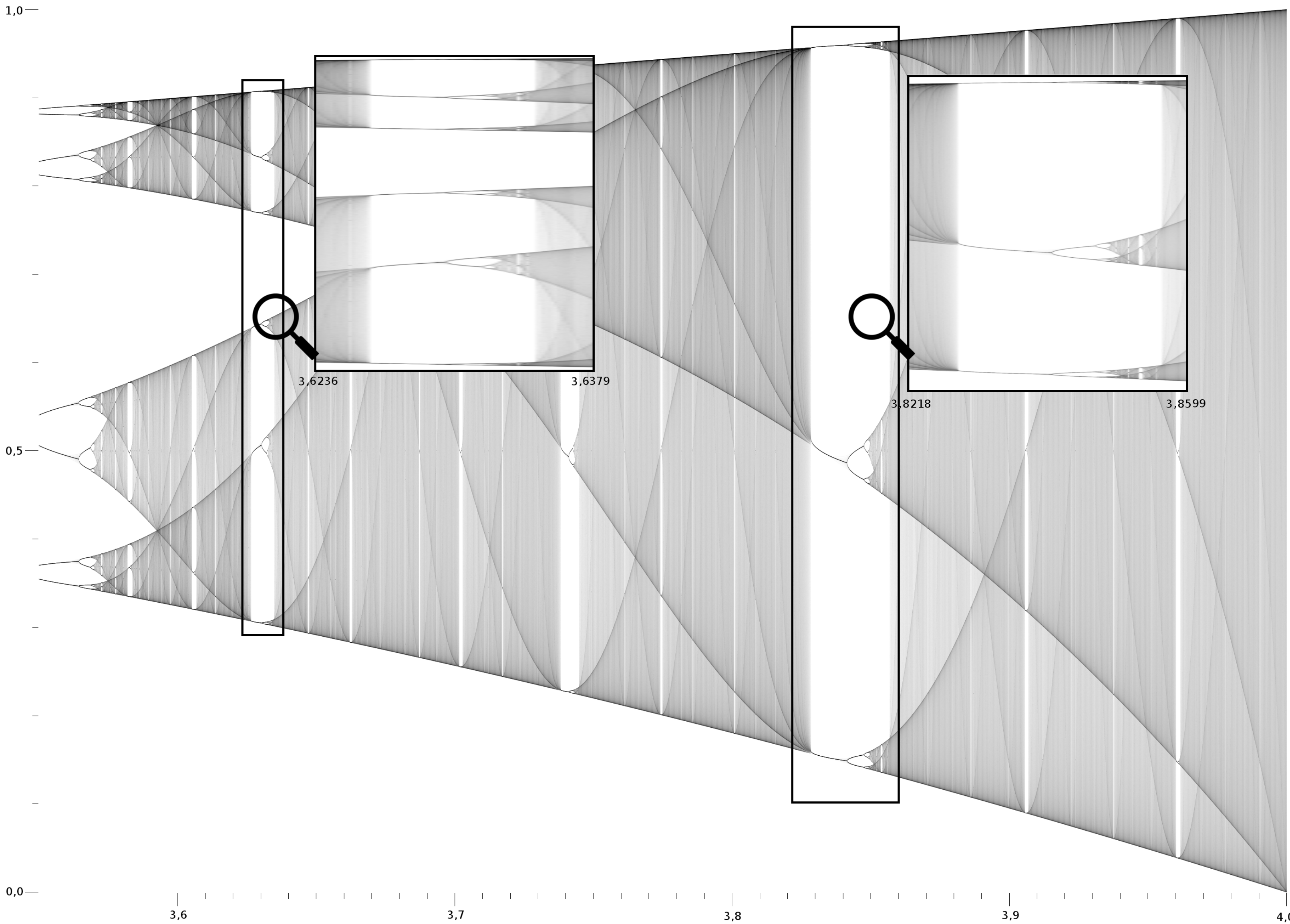


- From “laminar” to chaotic





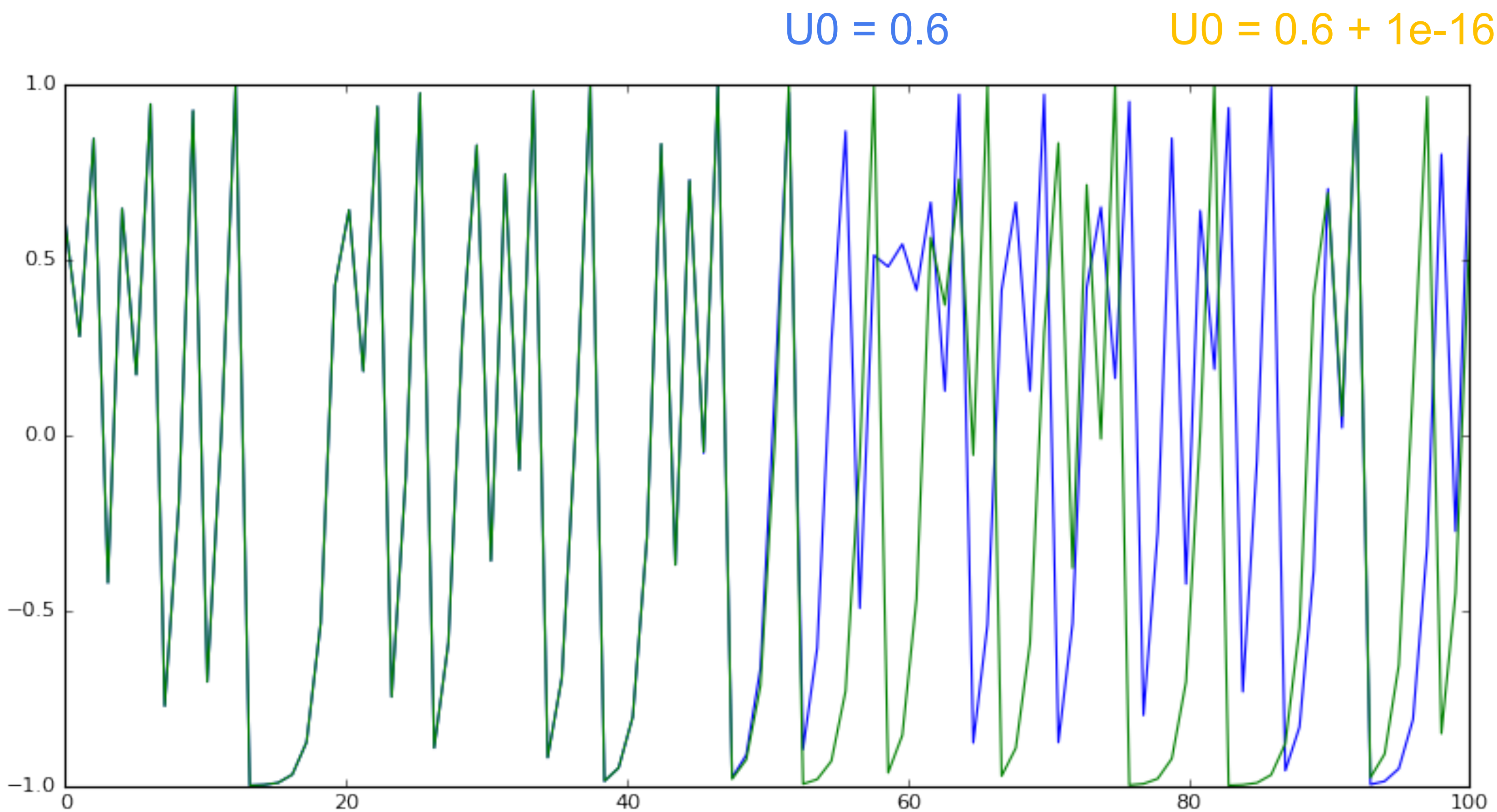
- With islands of stability:



Properties of turbulence: unpredictability

- Sensitivity to initial conditions

$$\frac{du}{dt} + ru^2 = -u + 1$$



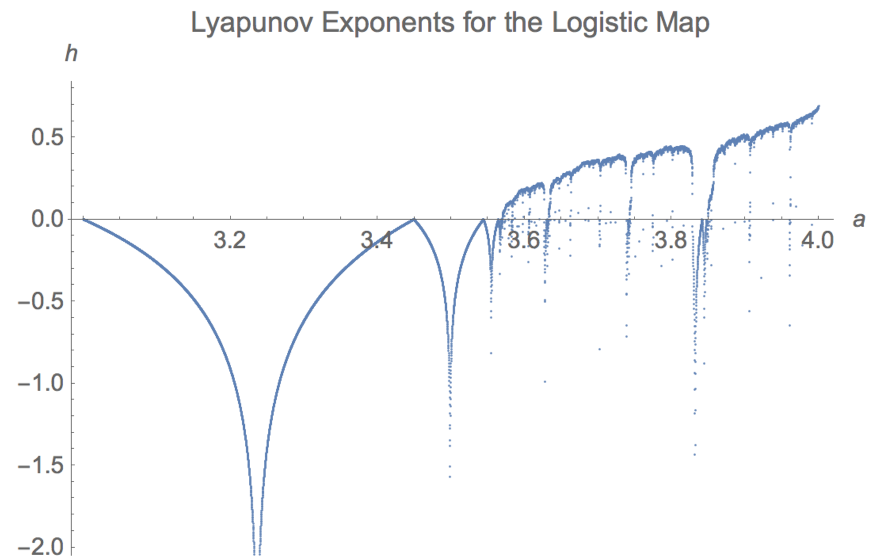
- **Sensitivity to initial conditions:**

In general, for an initial difference x_0 and $x_0 + \Delta x$

We expect the difference after n iteration to grow as $|\delta x_n| = \Delta x e^{n\lambda(x_0)}$

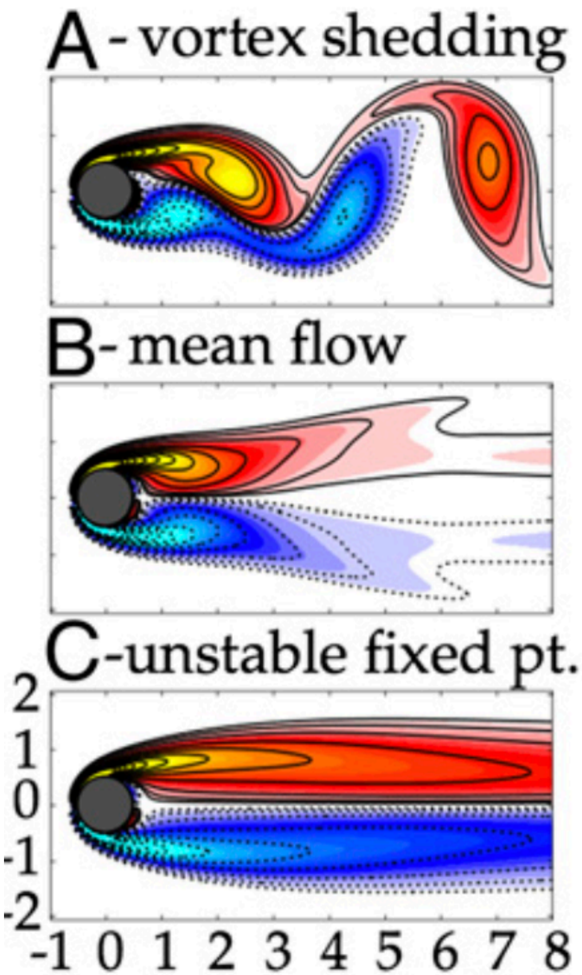
Where $\lambda(x_0)$ is the so-called **Lyapunov exponent** for the initial condition x_0

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\Delta x\to 0} \frac{1}{n} \log \left| \frac{F^n(x_0+\Delta x) - F^n(x_0)}{\Delta x} \right| = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{dF^n(x_0)}{dx_0} \right|$$

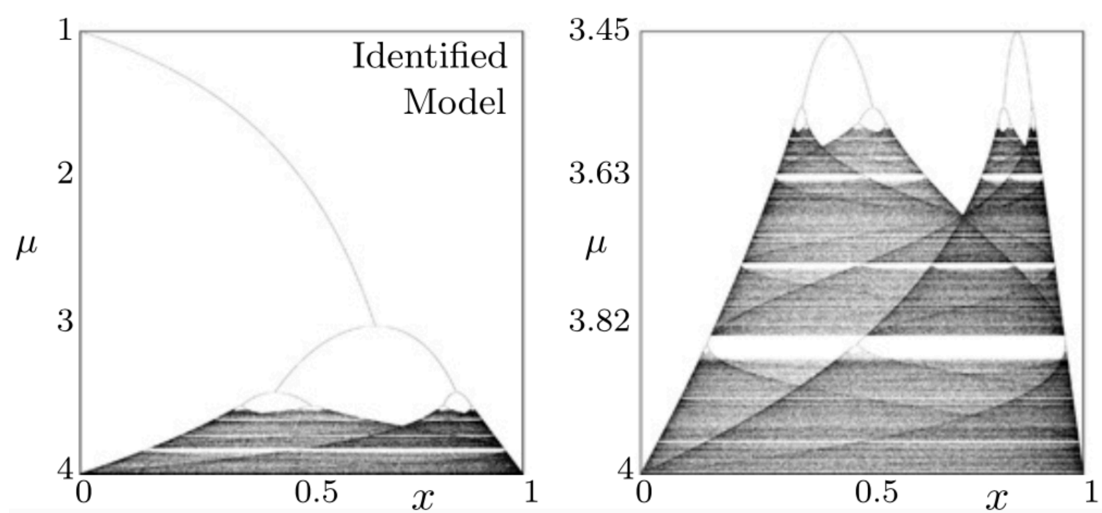


A positive Lyapunov exponent is a signature of chaos …

- fluid vortex shedding behind an obstacle :



[Brunton et al, 2016]



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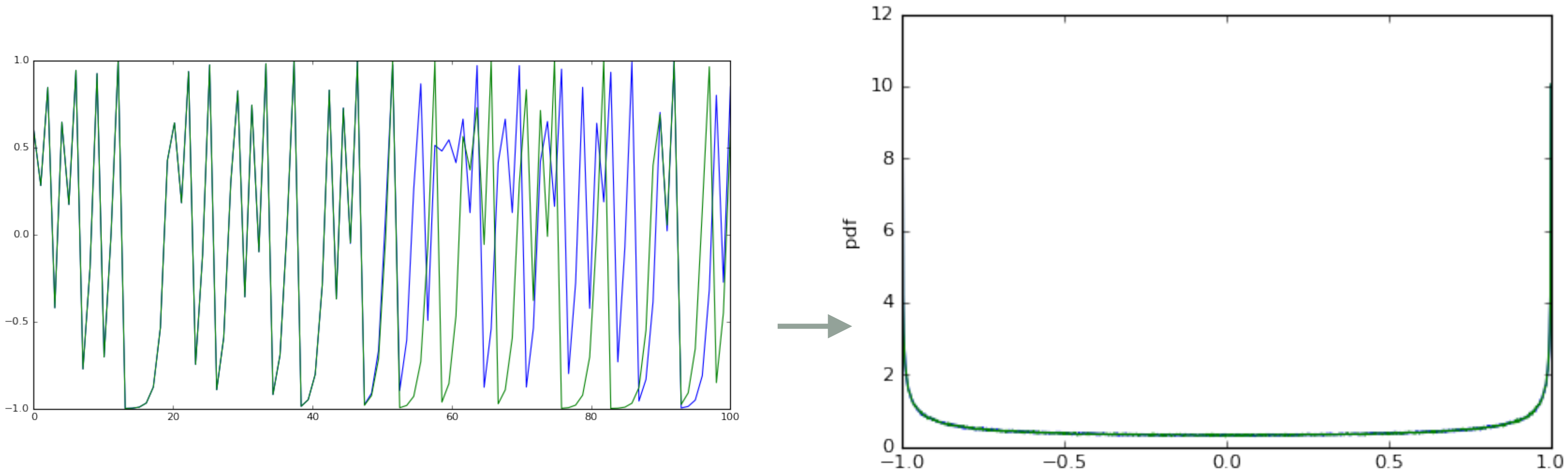
- Hence the need for a statistical description of turbulence:

- Individual trajectories are erratic and difficult to predict. However ensemble of trajectories reflects the mean structure of the attractor and can be predicted.



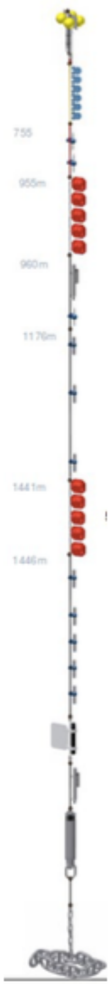
- statistical properties of the flow can be quite reproducible (pdf, mean, moments, etc.)

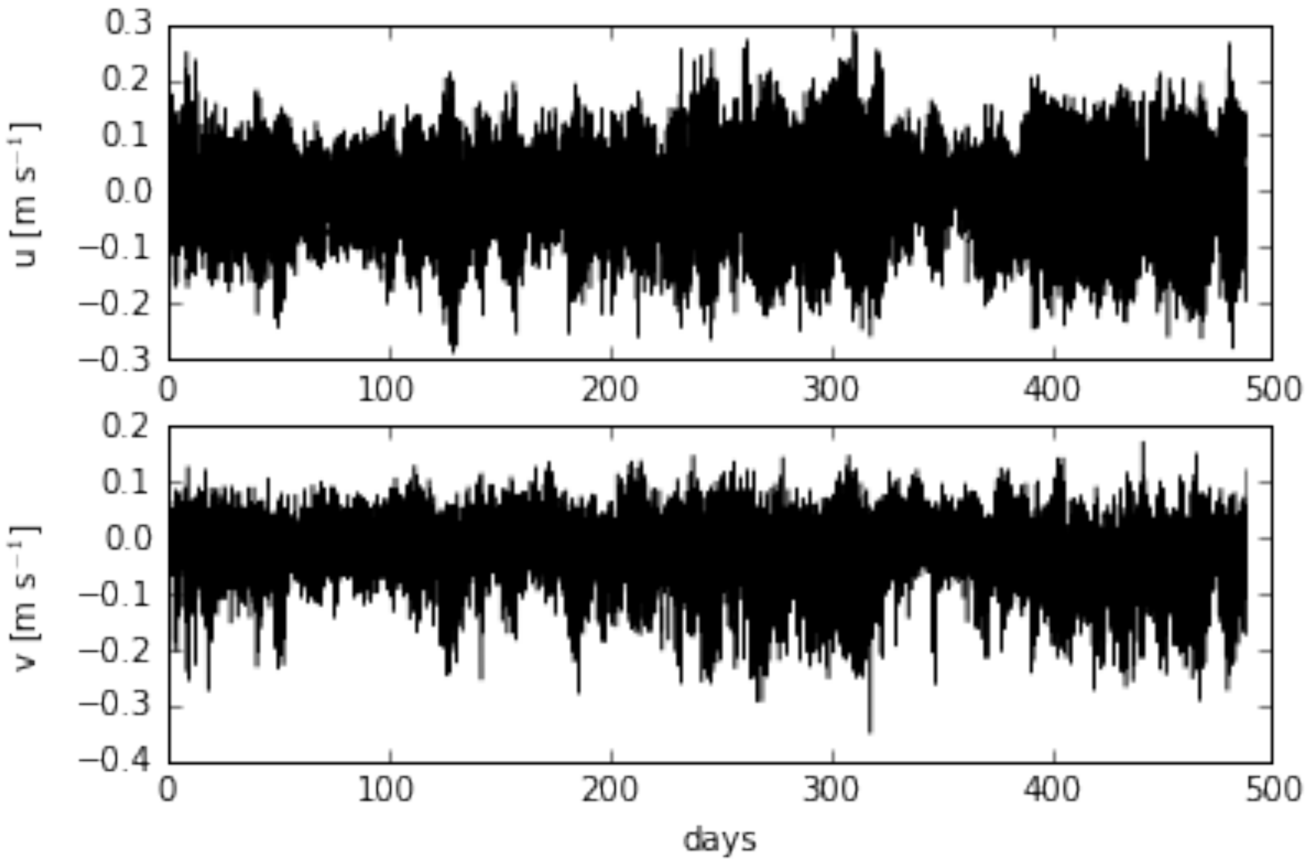
-

Properties of turbulence: unpredictability

- The equation of motions are deterministic, but their solutions have many attributes of random processes.
- Individual trajectories are erratic and difficult to predict. However ensemble of trajectories reflects the mean structure of the attractor and can be predicted.
- Hence the need for a **statistical description of turbulence**.

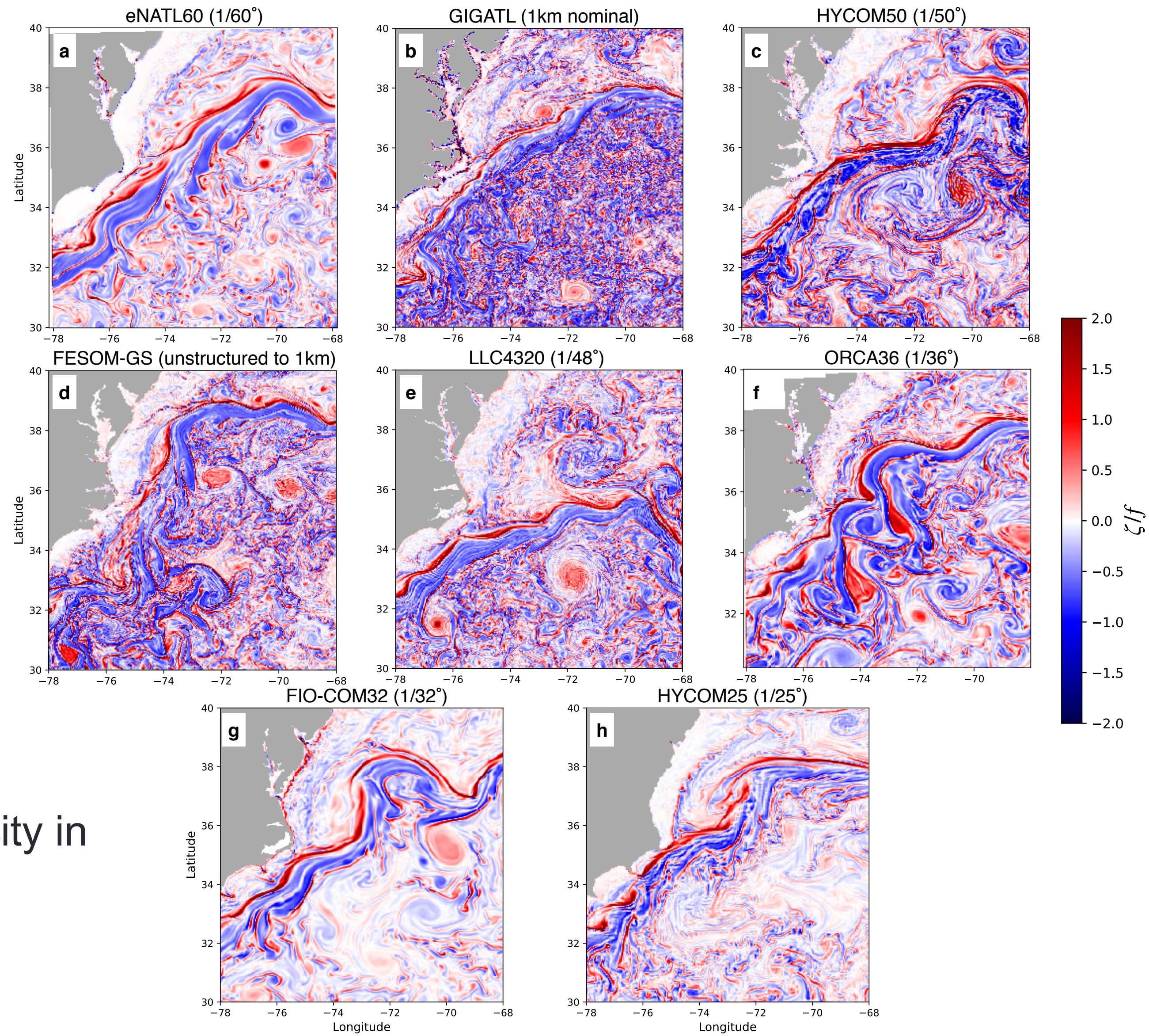
How to characterize properties of turbulence?





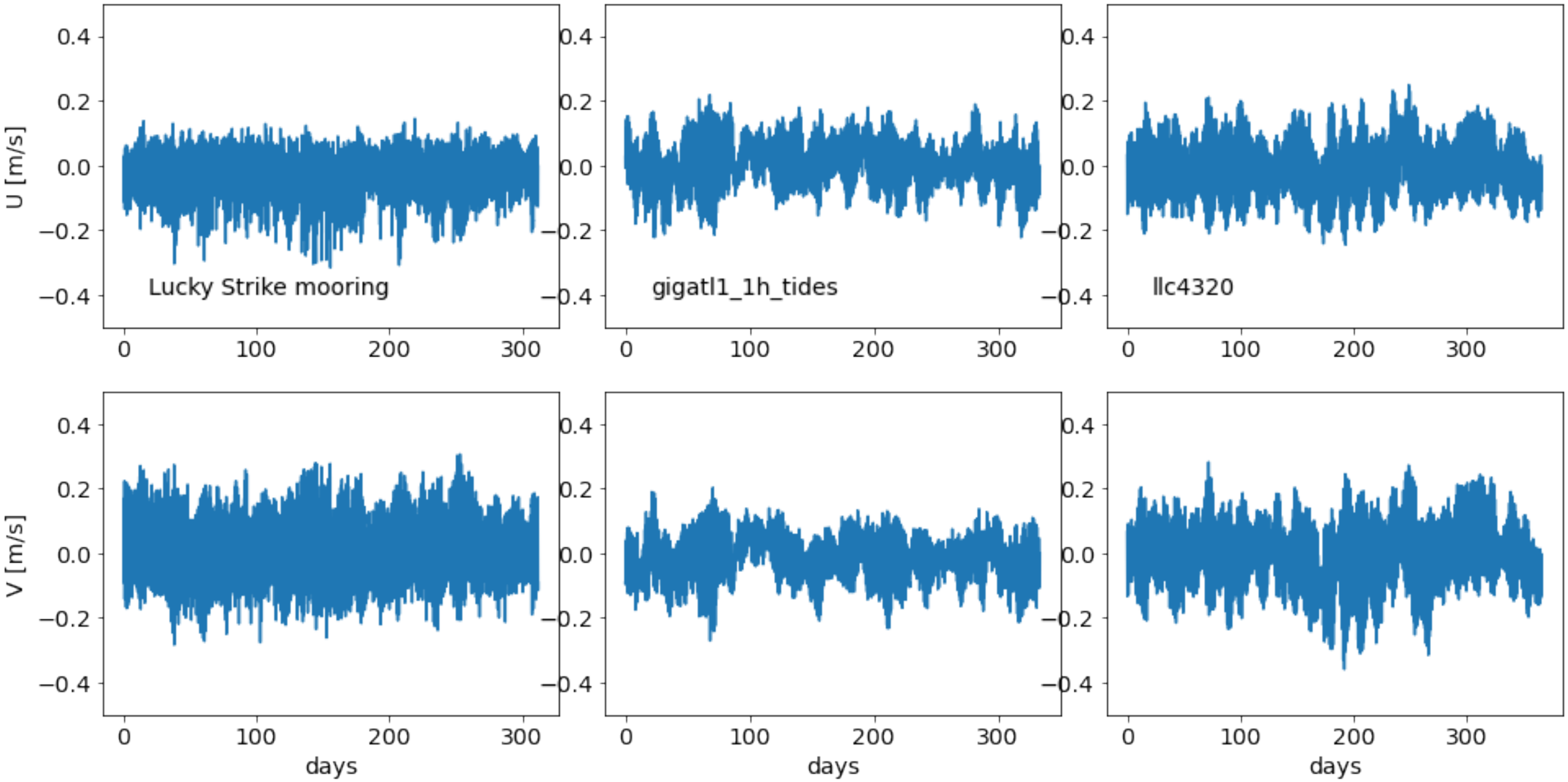
Velocities at 1700 m depths from currentometers at the Lucky Strike site

How to characterize properties of turbulence?



Surface relative vorticity in the Gulf Stream

How do we compare results from a model to observations?



Obs. Numerical Models

Velocities at 1700 m depth at the Lucky Strike site

How to characterize properties of turbulence?

- Turbulent flow are chaotic and not predictable in their detailed properties, but their **statistical properties are reproducible.**

- A statistical measure is an average of some kind: over the symmetry coordinates, if any are available (e.g, a time average for stationary flows); over multiple realizations (e.g, an ensemble); or over the phase space of solutions if the dynamics are homogeneous.

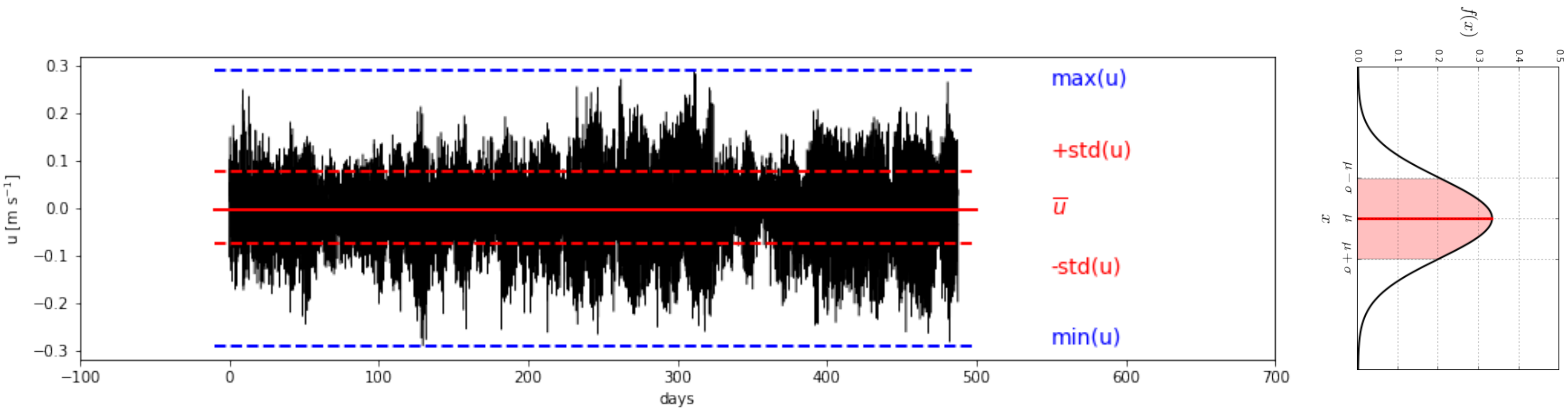
Statistics of turbulent fields:

We look at the values distribution (= **pdf**) and its moments:

The first one is the **mean.** $\mu = \int x f(x) dx$

The second one is the **variance** (= squared standard deviation / rms). It describes the spread of the pdf around the mean.

$$\sigma^2 = \int (x-\mu)^2 f(x)\, dx$$



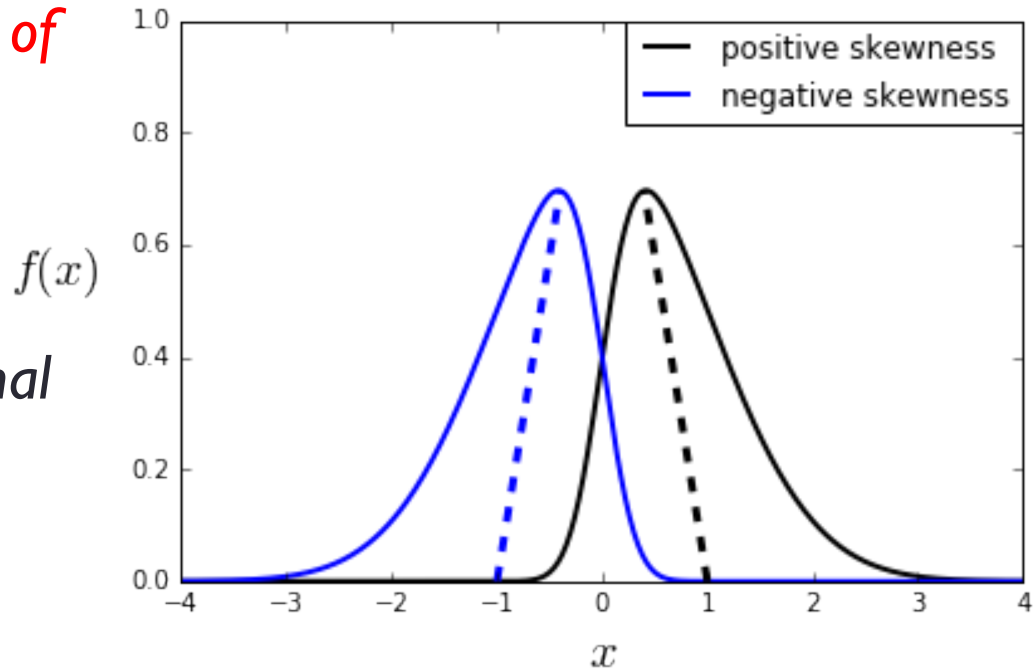
Statistics of turbulent fields:

The third one is the **skewness:**

$$\mu_3 = \frac{1}{\sigma^3}\int (x-\mu)^3 f(x)\, dx$$

It is a measure of the asymmetry of the pdf about its mean.

If the pdf is symmetric (e.g. normal distrib.), the skewness is 0



Statistics of turbulent fields:

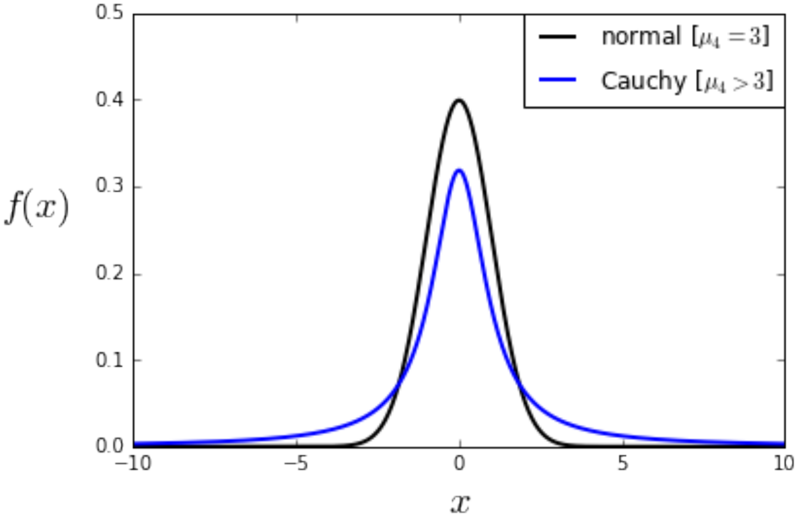
The fourth one is the **kurtosis:**

$$\mu_4 = \frac{1}{\sigma^4} \int (x-\mu)^4 f(x)\, dx$$

The kurtosis measures how fat are the tails of the pdf.

For the normal law the skewness is zero and the kurtosis is 3.

Values larger than 3 indicate more likely extreme values than the normal law.



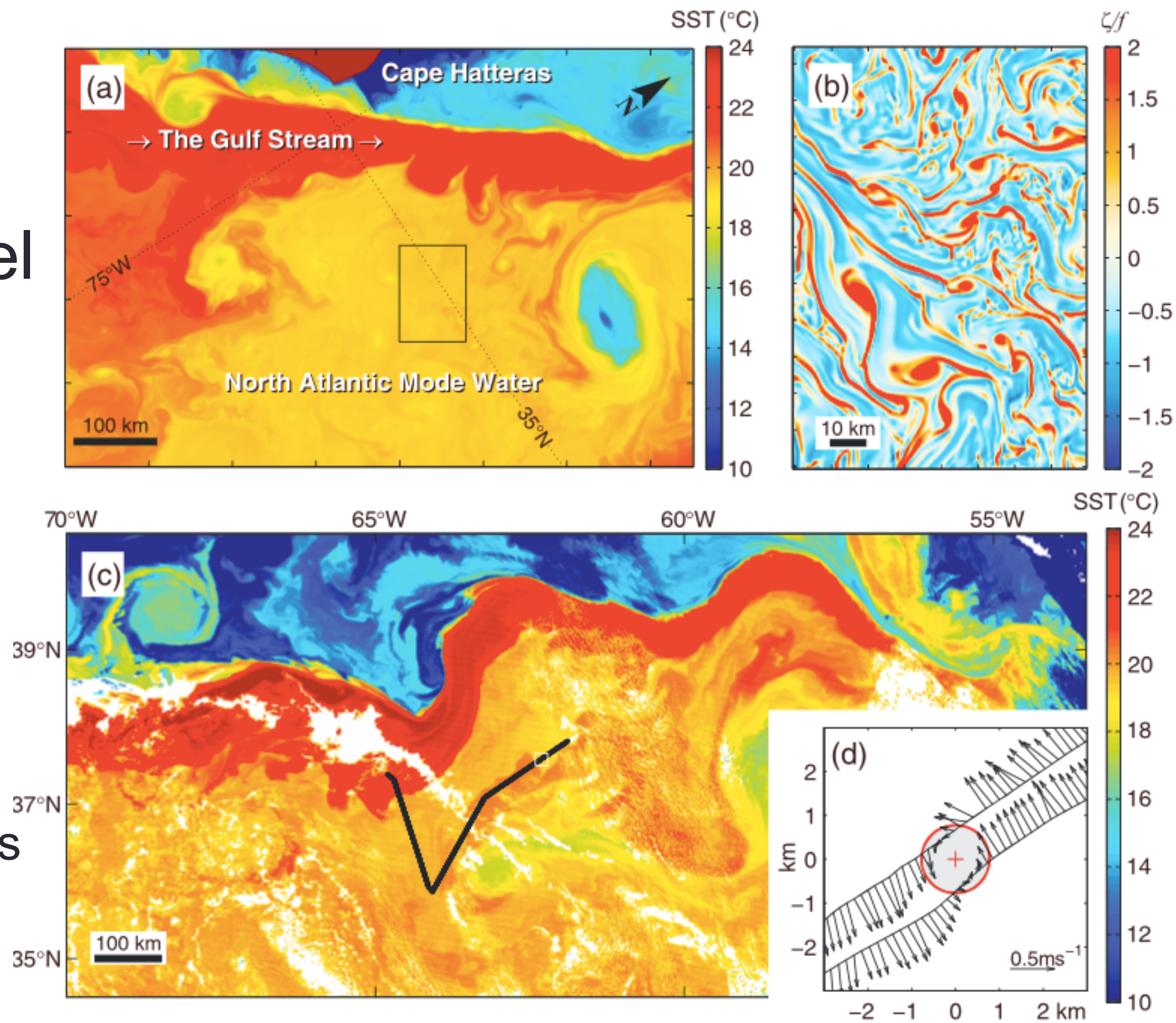
Properties of turbulence

Ex: How do we compare results from a model to observations:

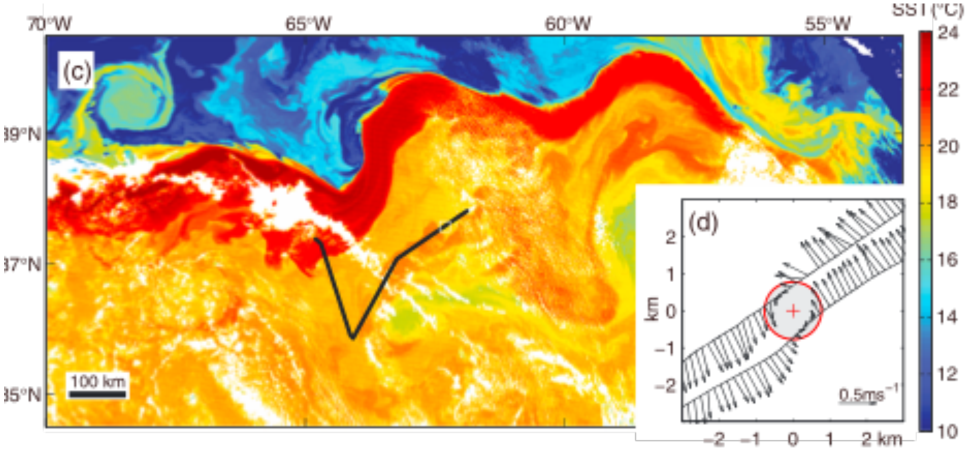
Numerical Model

Observations

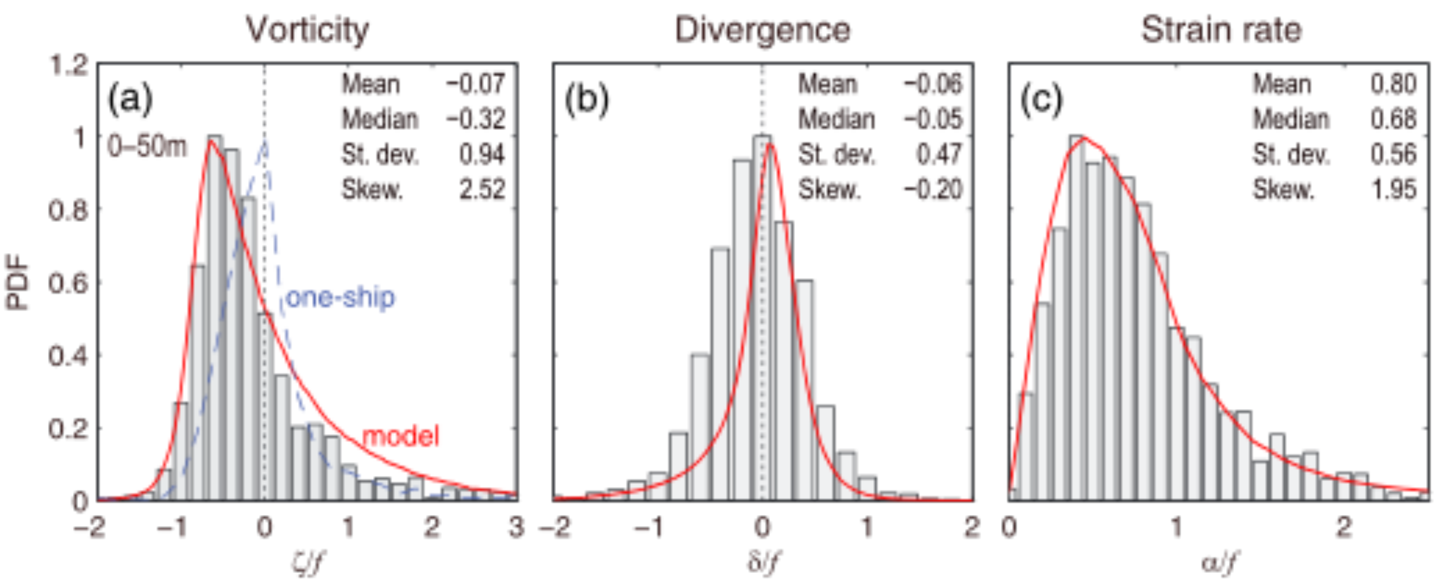
[satellite SST + velocity measurements from 2 parallel ships]



Ex: How do we compare results from a model to observations:

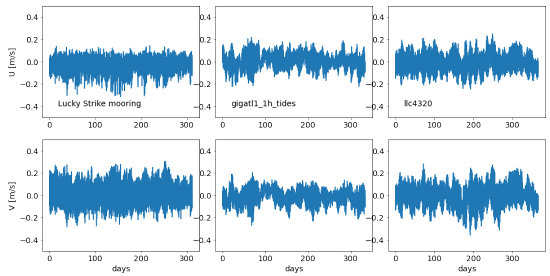


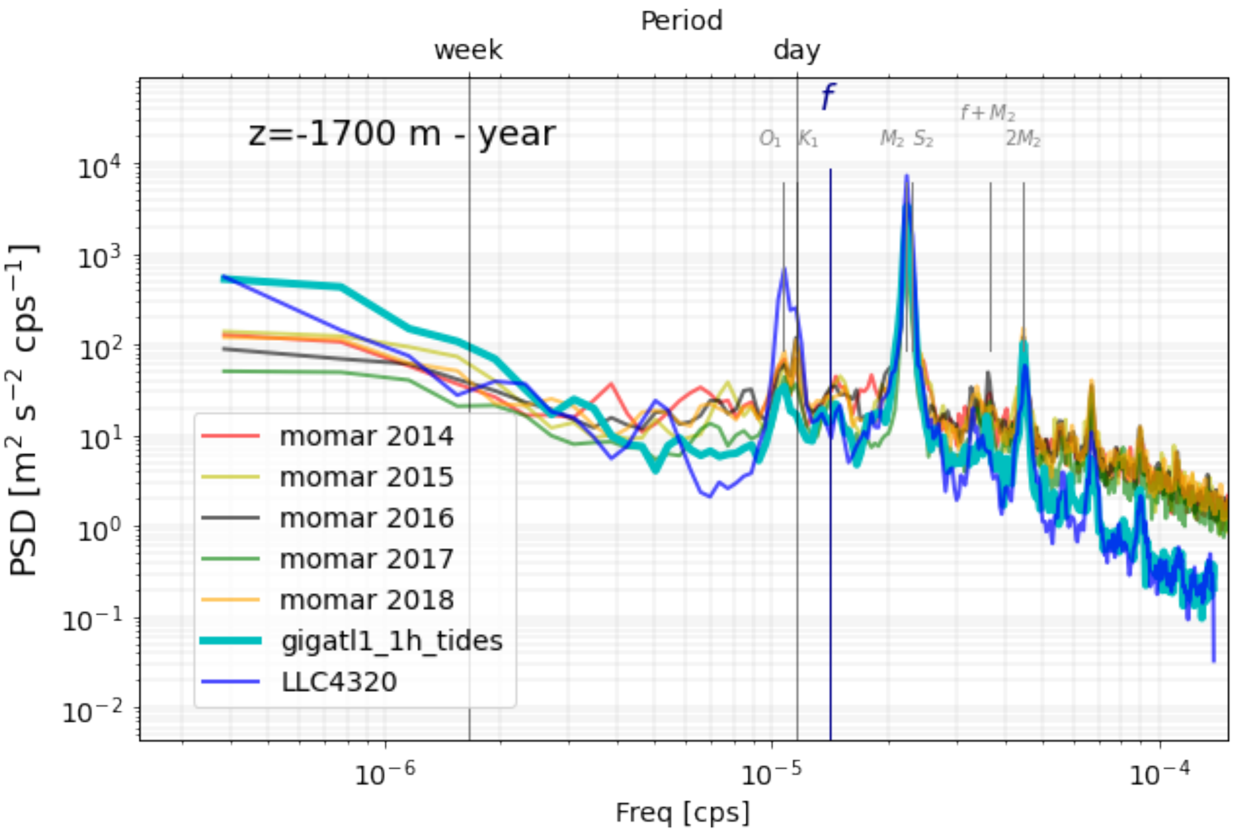
Ex: Shcherbina et al, 2015.



Statistics of turbulent fields:

Another way is to compute "**power spectral density**" (= spectra of energy or tracer variance)





Computing Power Spectra

- We have a signal *(e.g. velocity u along dimension x)*
- *The total energy is* $$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx$$
-

Computing Power Spectra

- We have a signal *(e.g. velocity u along dimension x)*
- *The total energy is* $$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx$$
- *Using the Fourier transform of the variable:*

$$\hat{u}(k) = \int_{-\infty}^{+\infty} e^{-2\pi ikx} u(x) dx$$

- *the power spectral density is* $$|\hat{u}(k)|^2$$
  - *It is the density of energy per unit wavenumber (or frequency)*

Computing Spectra

- Parseval's theorem states that:

$$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx = \int_{-\infty}^{+\infty} |\hat{u}(k)|^2 dk$$

  - *Integral of energy in the physical domain is equal to integral of spectral energy density over all wavenumbers.*

Computing Spectra

- In a finite and discrete domain:

$$\hat{u}(k) = \sum_{x=0}^{N-1} u(x) e^{-ikx\frac{2\pi}{N}}$$

$$\text{for } k = 0, \ldots, N-1$$

- And the power spectra is defined as: $$\frac{\hat{u}(k)\hat{u}^*(k)}{N}$$

Computing Spectra

A simple example in Python:

```
def myfft(u):
    nx = u.shape[0]
    k = np.fft.rfftfreq(nx,d=1)
    psd = (np.abs(np.fft.rfft((u)))**2)/nx
    return k, psd
```