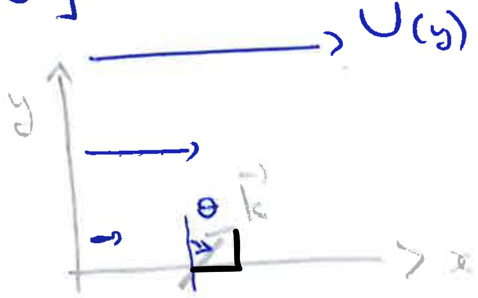
1.4C Wave refraction by a horizontally sheared flow (see LeBlond & Mysak, p334)

current : $\vec{U} = [\underline{U(y)}, 0, 0]$

wave : $\vec{k} = [k_x, k_y, 0]$

$$\begin{cases} k_x = |\vec{k}| \sin\theta \\ k_y = k\cos\theta \end{cases}$$



intrinsic frequency : $\omega_0 = kc_0$

frequency : $\omega = \boxed{kc_0 + k_x U}$

in a time independent flow, ω is constant.

$$\Rightarrow \quad \omega = kc_0 + k_x U$$
$$= \boxed{kc_0 + k\sin\theta\, U = cste}$$

no variation in the x direction : $\left(\frac{\partial U}{\partial x} = 0\right)$

$$\Rightarrow \boxed{\frac{dk_x}{dt} = 0} \Leftrightarrow \boxed{k\sin\theta = cste}$$

$$\Leftrightarrow \quad \frac{\omega}{k} = c_0 + U\sin\theta$$

$$\frac{\omega}{k \sin\theta} = \frac{c_0}{\sin\theta} + U = \text{cste}$$

initial values : c_e, U_e, θ_e

so
$$\boxed{\frac{c_0}{\sin\theta} + U = \frac{c_e}{\sin\theta_e} + U_e}$$

(= loi de Snell - Descartes)

ex : for long surface gravity waves :

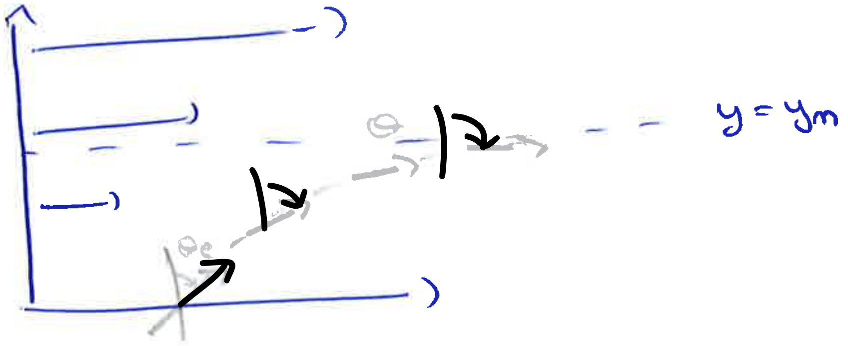
$$\begin{cases} c_0 = \sqrt{gH} \\ H = \text{cste} \\ U_e = U(y=0) = 0 \end{cases}$$

$$\Leftrightarrow \boxed{\frac{c_0}{\sin\theta} + U(y) = \frac{c_0}{\sin\theta_e}}$$

$$\sin\theta = \frac{1}{\frac{1}{\sin\theta_e} - \frac{U}{c_0}}$$

$$\boxed{\sin\theta = \frac{\sin\theta_e}{1 - \frac{U(y)\sin\theta_e}{\sqrt{gH}}}}$$

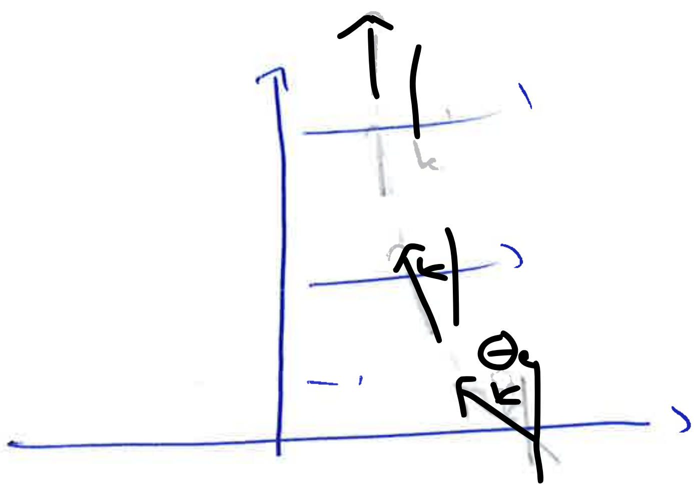
- if $0 < \Theta_e < \pi/2$: $\Theta(H) > \Theta_e$



internal reflection.

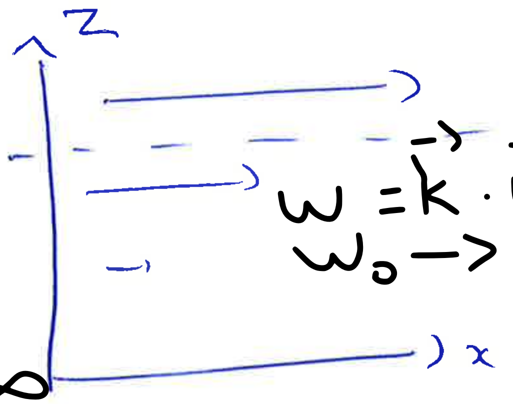
when $\Theta = \frac{\pi}{2} \iff \frac{U(y_m)}{\sqrt{gH}} = \frac{1}{\sin\Theta_e} - 1$

- if $\Theta_e < 0$: $\Theta(H) > \Theta_e$



1.4 D Critical layer (p 336)

- with $\cancel{f=0}$



$\omega = k \cdot U$

$\omega_0 \to 0$

$$\omega_0^2 = \frac{N^2 (k_x^2 + k_y^2)}{(k_x^2 + k_y^2 + k_z^2)} \quad k_z^2 \to +\infty$$

- $\boxed{\omega = \omega_0 + k_x U}$ (is cste along a ray)

- $$\boxed{c_{gz} = \frac{\partial \omega}{\partial k_z}} = - \frac{N^2 (k_x^2 + k_y^2)}{k_x^2 + k_y^2 + k_z^2} \times 2 k_z \times \frac{1}{2\omega_0}$$

$$= \frac{- k_z \omega_0}{(k_x^2 + k_y^2 + k_z^2)}$$

if $\omega_0 \to 0$: $\begin{cases} \boxed{c_{gz} \to 0} \\ \boxed{k_z \to +\infty} \end{cases}$

at the critical level k_z becomes unbounded

energy is absorbed by the background

[≠ reflection when when $c_{gz} = 0$]

$\begin{cases} k_z = 0 \\ \omega_0 = N \end{cases}$

$$c_{gz} = -\left(1-\frac{\omega_0^2}{N^2}\right)^{1/2} \frac{\omega_0^2}{N k_x} \qquad (\text{with } k_y = 0)$$

Taylor series near $z = z_c$:

$$\underline{\omega_i(z-z_c)} = \underline{\omega_i(z_c)} + \underline{(z-z_c)\frac{\partial \omega_i}{\partial z}} + \dots$$

$$\simeq \underline{(z-z_c)\, k_x \frac{\partial U}{\partial z}}$$

$$\Longleftrightarrow \quad c_{gz} \simeq -\frac{k_x}{N}\left(\frac{\partial U}{\partial z}\right)^2 (z-z_c)^2$$

$$\left(\text{with Ray equation } \frac{dz}{dt} = c_{gz}\right)$$

so integrating between z_1 and z_2, we get

$$t_2 - t_1 \simeq \frac{N}{k_x\left(\frac{\partial U}{\partial z}\right)^2}\left[\frac{1}{(z_2-z_c)} - \frac{1}{(z_1-z_c)}\right]$$

becomes unbounded as $z_2 \to z_c$

a wave group never reaches a critical level !!!

1.4 F Viscous dissipation

equations: $\vec{\nabla} \cdot \vec{u}' = 0$

$$\left\{ \begin{array}{l} \boxed{\dfrac{\partial \rho'}{\partial t} - \dfrac{\rho_0 N^2}{g} w} = \boxed{K_b \nabla^2 \rho'} \\[2ex] \dfrac{\partial \vec{u}'}{\partial t} + \dfrac{\vec{\nabla} p}{\rho^*} + \dfrac{\rho}{\rho^*} g \vec{k} = \boxed{K_M \nabla^2 \vec{u}'} \end{array} \right.$$

$$\left(\frac{\partial \cdot}{\partial t} - k_M \nabla^2 \cdot \right)$$

viscosity

$$\boxed{\left(\frac{\partial}{\partial t} - \kappa_B \nabla^2 \right) \left(\frac{\partial}{\partial t} - \kappa_M \nabla^2 \right) \nabla^2 w + N^2 \nabla_h^2 w = 0}$$

with $\boxed{w = w_0 \, e^{(\omega t + i \vec{k} \cdot \vec{x})}}$

$$(\omega + \kappa_B k^2)(\omega + \kappa_M k^2) k^2 + N^2 k_h^2 = 0$$

$$\omega^2 + \omega k^2 (\kappa_B + \kappa_M) + \kappa_B \kappa_M k^4 + N^2 \frac{k_h^2}{k^2} = 0$$

(with $N = \text{cst}$)

$$\Delta = \left(k^2 (k_B + k_M)\right)^2 - 4\left(k_B k_M k^4 + N^2 \frac{k_h^2}{k^2}\right)$$

$$= \left((k_B + k_M)^2 - 4 k_B k_M\right) k^4 - 4 N^2 \frac{k_h^2}{k^2}$$

$$\boxed{\omega = \frac{-k^2 (k_B + k_M) \pm i \sqrt{-\Delta}}{2}}$$

$$\omega = -\underbrace{\frac{k^2}{2}(k_B + k_M)}_{<0} \pm i \boxed{\frac{k_h N}{k}} \sqrt{1 - \frac{1}{4} \frac{k^6 (k_B - k_M)^2}{k_h^2 N^2}}$$

$= \gamma$

$e^{-\gamma t}$

waves are always <u>damped</u> !!

coef increase linearly with K_B, K_M

but quadratic <u>with k</u>

<u>short</u> waves are <u>more damped</u> than long waves

— Frequency smaller than in the non-dissipative case.

— case $\Delta > 0$: no waves

<u>viscous effects larger than N^2.</u>