

Space-Time Scales of Internal Waves: A Progress Report

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We present a revised model for the distribution of internal wave energy in wave number frequency space. The model is empirical, guided by the following measurements: moored spectra and moored coherences for horizontal and vertical separations (MS, MHC, MVC as functions of frequency), towed spectra and towed vertical and time-lagged coherences (TS, TVC, TLC as functions of horizontal wave number), and dropped spectra and dropped horizontal and lagged coherences (DS, DHC, DLC as functions of vertical wave number). Measurements are available for all but TLC and DHC. There is some indication of universality, suggesting perhaps a saturation limit.

Classical oceanography is concerned with fluctuations below the inertial frequency. The measurements are often obscured by high-frequency internal waves, with their large vertical and horizontal displacements (typically 10 m and 1 km). But one man's noise is another man's signal, and the forthcoming symposium is devoted to the state-of-the-art of measuring and describing internal waves.

The internal wave spectrum occupies a continuum in scales: from the Brunt-Väisälä period ($\frac{1}{4}$ hour in the thermocline) to the inertial period (~ 1 day), as many octaves as on a piano, all horizontal wavelengths, vertical wavelengths possibly from a few centimeters to the depth of the ocean. An experiment to sort out the scales requires a tapered three-dimensional array, and this has been out of reach until the advent of the internal waves experiment (IWEX) (appropriately located on the longitude of Woods Hole and the latitude of Scripps). But the principal source of data is still a variety of experimental configurations, involving moored sensors, sensors that are towed horizontally, dropped vertically, or made to yo-yo at middepth.

Two years ago we asked the question whether the diverse evidence could be reconciled with a single model and contrived an empiricism that seemed to do the job (after overlooking some inconvenient evidence). We call this model GM72 [Garrett and Munk, 1972]. The initials suggest some planned obsolescence and permit us to bring out new models from time to time. GM72 cuts off sharply beyond some specified wave numbers; this top-hat model was chosen as the most explicit admission of ignorance concerning small scales.

GM72 relied heavily on moored observations and horizontal tows. In addition to the interpretation of time series from individual sensors we used coherences between sensor pairs. The evidence from coherences between sensor pairs is often the more critical, since decoherence time (or space) is the reciprocal of frequency (wave number) bandwidth. At separations of the order of the reciprocal bandwidth the different wave numbers begin to interfere destructively and coherence is lost.

Our revision of GM72 includes a reduction of the wave number bandwidth associated with moored horizontal and

vertical coherence (MHC and MVC) but uses the same frequency dependence for these as before. The frequency spectra (MS) for moored observations are unchanged.

The major change stems from accepting the hypothesis that much observed fine structure in the oceans is due to internal waves distorting a smooth profile rather than due to persistent layers arising from some other mechanism. This assumption enables us to replace the top hat of GM72 with a tapered cutoff, chosen to be consistent in slope with the dropped spectra (DS) of Millard [1972]. The revised model then agrees well with the towed spectra (TS) of Katz [1973] and is reasonably consistent with preliminary data on towed vertical coherence (TVC) and dropped lagged coherence (DLC).

Measurements of dropped horizontal coherences (DHC) have been made, but the results are not yet available. There has been no experiment to measure the towed lagged coherence (TLC); this requires two ships following each other at various set time intervals, each recording the displacement of a particular isopycnal. Predictions for TLC and DHC are given in Figures 4 and 6, respectively.

MODEL

The following description closely follows the discussion in GM72.

Scaling. All quantities are nondimensionalized with reference to the buoyancy scale depth b (1.3 km) and the Väisälä frequency n_0 (3 cph) at the top of the thermocline (extrapolated from abyssal depths).

Dispersion. Horizontal wave number $\alpha(\alpha_1, \alpha_2)$ and vertical wave number β are related to the mode number j (1, 2, ...) and frequency ω according to the approximate formulas (GM72)

$$\alpha = j\pi(\omega^2 - \omega_i^2)^{1/2} \quad \beta = j\pi n(z)$$

where ω_i is the inertial frequency.

Wave functions. Mean square horizontal and vertical displacements vary with depth proportional to

$$X_1^2 + X_2^2 = X^2 = n\omega^{-4}(\omega^2 + \omega_i^2) \quad Z^2 = n^{-1}\omega^{-2}(\omega^2 - \omega_i^2)$$

The corresponding expressions for components of mean square velocity are $\omega^2 X^2$ and $\omega^2 Z^2$, respectively.

Normalization. When horizontal isotropy is assumed, various forms of the energy spectrum are related according to

$$\begin{aligned} \iiint E(\alpha_1, \alpha_2, \beta) d\alpha_1 d\alpha_2 d\beta &= \iint E(\alpha, \beta) d\alpha d\beta \\ &= \iint E(\alpha, \omega) d\alpha d\omega = \iint E(\beta, \omega) d\beta d\omega = E \end{aligned}$$

where E is a dimensionless constant related to the energy per unit area. These definitions, together with the dispersion relations, permit the transformation into various frequency wave number spaces. For example,

$$E(\beta, \omega) = E(\alpha, \omega) \cdot d\alpha/d\beta = E(\alpha, \omega) \cdot (\omega^2 - \omega_i^2)^{1/2}/n(z)$$

Mean square horizontal and vertical displacements are

$$[\langle \xi^2 \rangle, \langle \zeta^2 \rangle] = \iint [X^2(\omega), Z^2(\omega)] E(\alpha, \omega) d\alpha d\omega$$

and so

$$F_u(\alpha, \omega) = \omega^2 X^2(\omega) E(\alpha, \omega) \quad F_z(\alpha, \omega) = Z^2(\omega) E(\alpha, \omega)$$

are the spectra of horizontal velocity and vertical displacement.

Spectra. The moored, towed, and dropped spectra of vertical displacement, say, are

$$MS_T(\omega) = \int Z^2(\omega) E(\alpha, \omega) d\alpha$$

$$TS_T(\alpha_1) = \iint Z^2(\omega) E(\alpha_1, \alpha_2, \omega) d\alpha_2 d\omega$$

$$DS_T(\beta) = \int Z^2(\omega) E(\beta, \omega) d\omega$$

Clearly, a frequency analysis of a moored time series cannot distinguish between the contributions from various wave numbers to a given frequency band, and this leads to the foregoing expression for the MS. Similarly, for a dropped spectrum a given β band contains contributions from all frequencies. For the towed spectrum the integration is performed by setting $\alpha_2 = \alpha_1 \tan \theta$ and integrating over θ holding α_1 constant. Because of the assumed isotropy we have $TS_T(\alpha_1) = TS_T(\alpha_2)$.

Cross spectra. As an example, the moored coherence of vertical displacement for a (small) vertical separation Y is given by

$$MVC_T(\omega, Y) = \frac{\int Z^2(\omega) E(\beta, \omega) \cos \beta Y d\beta}{\int Z^2(\omega) E(\beta, \omega) d\beta}$$

and this approaches unity as $Y \rightarrow 0$. We refer to GM72 for further discussions.

Proposed spectrum. We designate a mode number scale j_* and associated wave numbers

$$\alpha_* = j_* \pi (\omega^2 - \omega_i^2)^{1/2} \quad \beta_* = j_* \pi n(z) \quad j_* = 6$$

Most of the energy is contained in wave numbers less than α_* and β_* according to

$$A(\lambda) = (t - 1)(1 + \lambda)^{-t} \quad \lambda = \alpha/\alpha_* = \beta/\beta_* \quad t = 2.5$$

(The numerical values to the right are the ones here selected.) Further, set

$$B(\omega) = 2\pi^{-1} \omega_i \omega^{-2} \gamma^{-1} \quad \gamma = (1 - \omega_i^2/\omega^2)^{1/2}$$

for $\omega_i \leq \omega \leq n(z)$, and zero otherwise. The functions A and B are so normalized that

$$\int_0^\infty A(\lambda) d\lambda = 1 \quad \int_0^\infty B(\omega) d\omega = 1$$

The energy spectrum is taken as

$$E(\alpha, \omega) = EA(\alpha/\alpha_*)B(\omega)/\alpha_* \quad E = 6.3 \times 10^{-6}$$

This involves the similarity assumption that the shape of the spectrum as a function of wave number is invariable, but for a scale factor, $\alpha_*(\omega)$. Following the rules for transformation,

$$E(\beta, \omega) = EA(\beta/\beta_*)B(\omega)/\beta_*$$

$$E(\alpha, \beta) = \frac{2\pi^{-1} \omega_i E n(z) (\beta/\beta_*) A(\beta/\beta_*)}{n^2(z) \alpha^2 + \omega_i^2 \beta^2}$$

$$0 \leq \alpha \leq \beta [1 - \omega_i^2/n^2(z)]^{1/2}$$

Figure 1 is a presentation of the model with log-log-log coordinates. This choice helps to clarify regions where the spectral energy asymptotically obeys simple power laws, but it is difficult to estimate the relative contributions from different space-time scales. To obtain the spectrum of vertical displacement, say, one needs to multiply by $Z^2 = n^{-1} \omega^{-2} (\omega^2 - \omega_i^2)$, and this would remove the discontinuous cliff at ω_i .

Diagrams like those of Figure 1 are helpful in a preliminary investigation of other models. We have explored a variety of possibilities with different values of the exponents p, q, r of GM72 and the exponent t introduced here, but in this report we discuss only the model that gives the best overall representation of the different types of data.

We will now consider various cuts; for ease of comparison with observations we shall revert to dimensional coordinates. (As is done in GM72, we take the inertial frequency as 0.04 cph, corresponding to 29° latitude. Data from other latitudes may be incorporated into our coherence graphs by converting back to variables $\beta(\omega_i/n)X$ for DHC and $\omega_i T$ for DLC (Figure 6). For both TVC and TLC (Figure 4) the abscissa should be expressed as $\alpha_1(j_* \omega_i)^{-1}$; the ordinates should be $j_* nY$ and $\omega_i \tau$, respectively. Variation of spectral energy levels with latitude has not yet been considered but may offer an important clue to the overall dynamics.)

MOORED SPECTRA

The most popular experiments have been those using moored instrumentation. The pioneering work at site D (39°N, 70°W) is in a class by itself, and much of the GM72 model was based on these measurements [Fofonoff, 1969; Webster, 1972]. But there are some difficulties with bottom-moored experiments. One is Doppler smearing: internal waves move slowly, so that in the presence of a variable current the wave packets are convected past the transducers with frequencies of encounter that may significantly differ from the intrinsic frequencies (relative to the water). Another difficulty is microstructure contamination: in the presence of a jagged vertical temperature profile a smooth vertical motion due to internal waves appears as a jagged temperature record with high-frequency components that are related to microstructure rather than internal waves [Garrett and Munk, 1971]. The principal effect of Doppler smearing and microstructure contamination is to spread some of the spectral energy into the 'forbidden frequencies' beyond the Väisälä frequency. Another effect is the reduction of coherence.

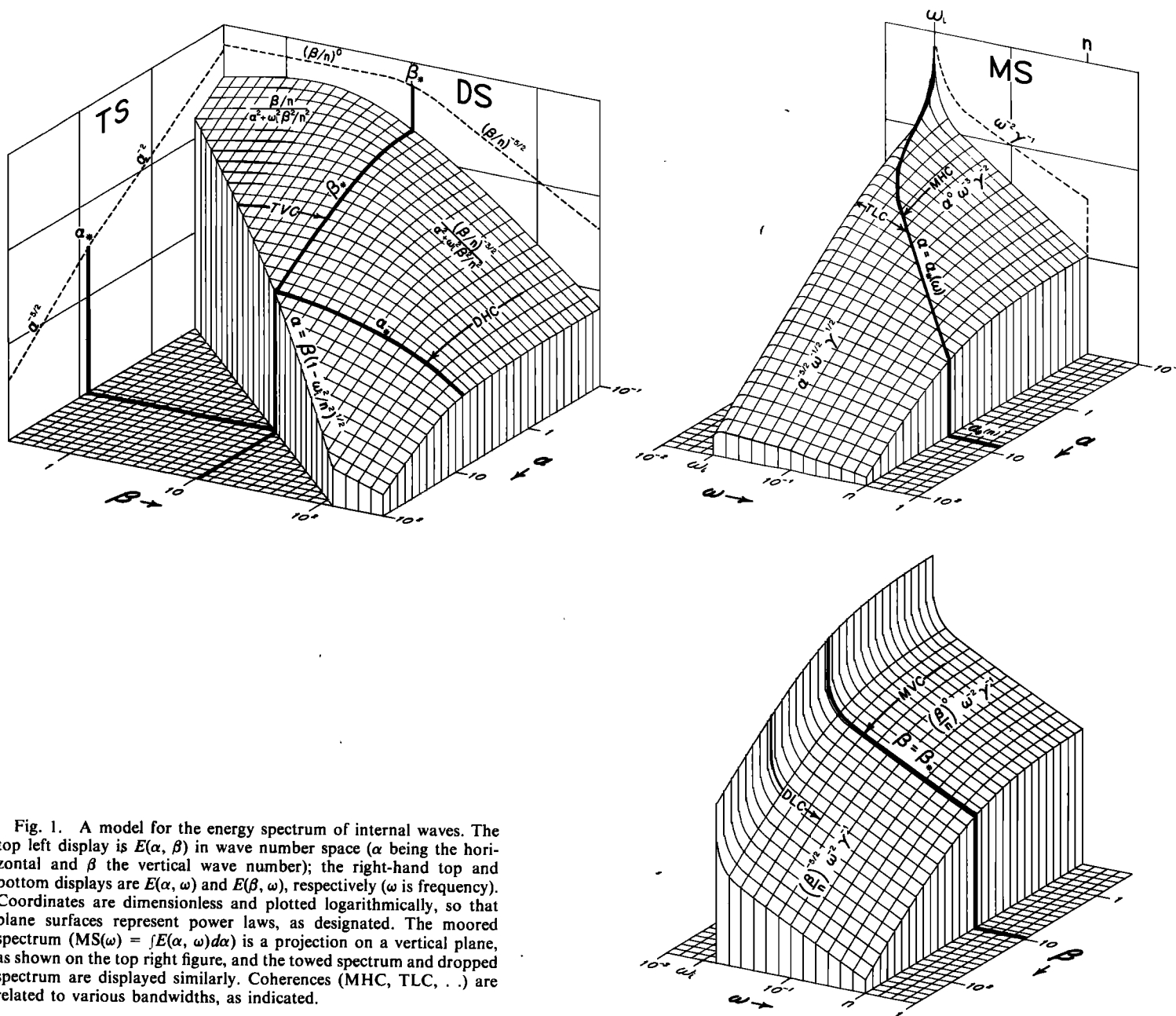


Fig. 1. A model for the energy spectrum of internal waves. The top left display is $E(\alpha, \beta)$ in wave number space (α being the horizontal and β the vertical wave number); the right-hand top and bottom displays are $E(\alpha, \omega)$ and $E(\beta, \omega)$, respectively (ω is frequency). Coordinates are dimensionless and plotted logarithmically, so that plane surfaces represent power laws, as designated. The moored spectrum ($MS(\omega) = \int E(\alpha, \omega) d\alpha$) is a projection on a vertical plane, as shown on the top right figure, and the towed spectrum and dropped spectrum are displayed similarly. Coherences (MHC, TLC, . . .) are related to various bandwidths, as indicated.

From our viewpoint, then, the traditional coordinate system attached to the sea floor is neither unique nor desirable; experiments where the instrumentation is moored to the water are attractive. Cairns [1974] has employed a Snodgrass capsule that yo-yos freely at midwater, continuously profiling the temperature and thus recording the elevation of fixed isotherms (rather than the temperature at fixed depths). In this way, both microstructure and Doppler contaminations are minimized. It is encouraging that with this configuration a sharp cutoff is observed at the local Väisälä frequency (Figure 2). The rise in the spectrum just below the local $n(z)$ can be associated with the precise solutions to the internal wave equations. (In our WKB approximation the oscillations at the cutoff are suppressed.) Pinkel [1974] also observes the n peak and subsequent sharp cutoff, as did Voorhis [1968] and Gould [1971]. (It is interesting that the peak and cutoff are displayed prominently in all recent reports though largely absent in earlier work. This occurrence demonstrates a great improvement in measurement schemes in the last few years.)

By contouring the time history of two isotherms, separated on the average by $Y = 100$ m, Cairns obtained the moored vertical coherence (Figure 2). The observations are consistent with our assumption that the coherence does not depend on frequency (unlike the Webster's [1972] rule $\omega Y = \text{const}$ for fixed MVC). The numerical value for the coherence is a measure of the modal bandwidth according to

$$\text{MVC}(Y) = \int_0^\infty A(\lambda) \cos(j_* \pi n \lambda Y) d\lambda \quad \lambda = j/j_*$$

For the top-hat model ($A = 1$ for $0 \leq \lambda \leq 1$ and zero beyond) the Cairns results give $j_* = 8$, much reduced from 20 in GM72. For the tapered model, $j_* = 3$. (The interpretation of j_* for top-hat and tapered models is not the same.)

The moored coherence at a horizontal separation X is

$$\text{MHC}(\omega, X) = \int_0^\infty A(\lambda) J_0(\alpha_* X \lambda) d\lambda$$

Some work by Siedler [1974] at site D corresponds to $j_* = 9$ to 12 for the top-hat model or $j_* = 5$ to 7 for the tapered model. Abyssal internal wave measurements conducted by W. S. Brown (unpublished manuscript, 1974) during MODE (Mid-Ocean Dynamics Experiment) lead to similar values.

We were surprised by the low j_* value obtained by Cairns, but on balance one is likely to pay more attention to experiments that give high coherence (low j_*), since nearly anything that can go wrong (and usually does) will lower the coherence. As this is written, Cairns is obtaining additional and much longer time series, but we have temporarily settled on $j_* = 6$ for the tapered model.

TOWED SPECTRA

Figure 3 shows the results of recent work by Katz [1973]. The GM72 model leads to n -dependent cutoffs, in disagreement with the towed observations, if one accepts that Katz's spectrum at high wave numbers is indeed due to internal waves. Our new model is in good agreement with the TS. The TVC for the tapered model (Figure 4) decreases with increasing separation or wave number and is roughly consistent with measurements by Sambuco [1974] made in conjunction with the Katz tows. This represents an improvement over GM72, which predicted TVC approximately independent of wave number. The TLC has not yet been measured.

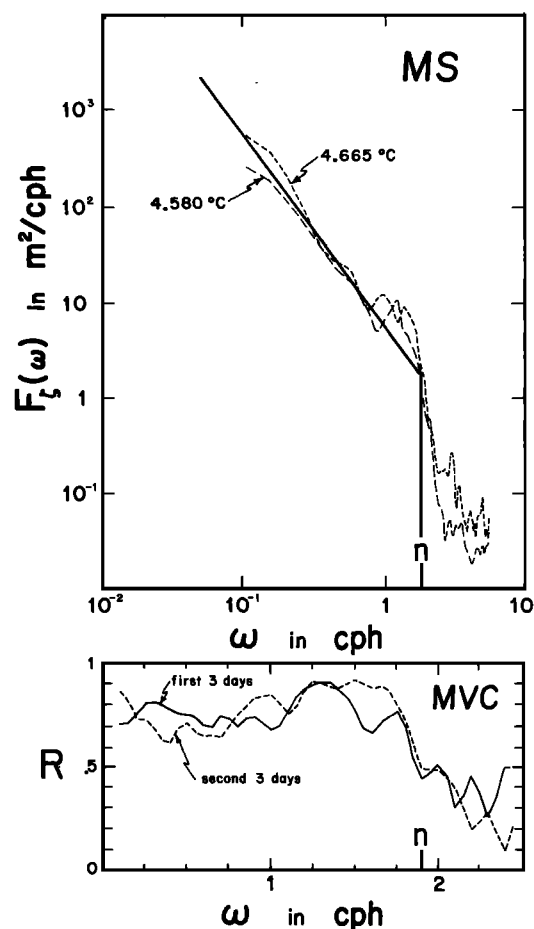


Fig. 2. The spectrum of vertical displacement and the coherence for a vertical separation of 100 m at 30°20'N, 121°20'W, the upper sensor being at about 750-m depth [from Cairns, 1974]. The solid curve with the abrupt n cutoff is from the internal wave model (consistent also with GM72). The model predicts an MVC independent of frequency, consistent with the observations.

DROPPED SPECTRA

Millard [1972] has reported dropped temperature spectra for different depths using CTD data. We have divided these by the square of the local mean potential temperature gradient to obtain $DS_T(\theta)$ and then normalized these in the manner appropriate to internal waves (Figure 5). The exponent $t = 2.5$ of our tapered model is then chosen to fit the slope of Millard's spectra. The level of the spectrum depends on values of E and j_* that are already specified by the moored data, and so the rough agreement of the model with the data does not contradict the assumption that the observed fine structure is due to internal waves. (The variability of the normalized spectra in Figure 5 may arise from temperature not being a good signature of density at all depths.)

T. B. Sanford (personal communication, 1974) reports DS of horizontal velocities consistent with the tapered model.

S. P. Hayes (personal communication, 1974) has made CTD yo-yo profiles in the main thermocline (600–900 m) at the IWEX site. He was able to obtain time series for about 12 hours, yo-yo samples being taken once every 12 min. His preliminary results for DLC, together with those of Sanford [1975] for velocity, are in reasonable accord with calculations for our model (Figure 6). It is too early to say whether discrepancies between theory and observation indicate a failure of the model. The model has a DLC independent of vertical

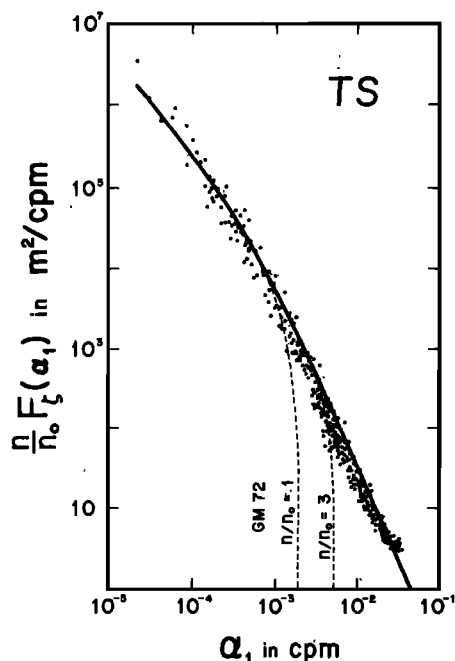


Fig. 3. The normalized towed spectrum of vertical displacement according to Katz [1974]. The solid line corresponds to the tapered internal wave model, the dashed curves to GM72.

wave number, displacement coherences falling off more rapidly than current coherences due to reduced energy at the inertial frequency, and hence increased bandwidth, for the displacement. For current measurements a more logical quantity to compare with observations may be the dropped lagged rotary coherence (DLRC), where the coherence is calculated between the current in one direction at time zero and the current in a direction rotated clockwise through $\omega_1 T$ at time T later. This calculation keeps the purely inertial oscillations completely coherent. Theoretical formulas and comparison with data will be presented in due course. Comparison of DHC with observations has not yet been possible.

DISCUSSION

The main respect in which the model proposed in this paper differs from GM72 is in the high wave number part of the spectra. The model is consistent with the DS of Millard [1972] and the TS of Katz [1973], on the assumption that both sets of data are due to internal waves. It is still possible, though, that the DS arise from layering associated with some other mechanism and that the TS are due to fine structure contamination, as discussed by Garrett and Munk [1971]. We emphasize that an unambiguous determination of the internal wave spectrum at high wave numbers ($\alpha > 10^{-2}$ cpm or $\beta > 10^{-2}$ cpm) remains a problem of great importance.

The mean square shear

$$\langle s^2 \rangle = \int_{\omega_1}^{\omega_2} d\omega \int_0^\infty d\beta U^2 \beta^2 E(\beta, \omega)$$

has a β integrand that behaves like $\beta^{-1/2}$ for large β . To avoid infinite shear, we require a high wave number cutoff, at $\beta = \Lambda \beta_*$, say, and we assume Λ to be independent of frequency; $\langle s^2 \rangle$ is then proportional to $\Lambda^{1/2}$. Gregg *et al.* [1973] observed a change in the slope of the DS at about 2 cpm. Taking this to be the cutoff β in the upper part of the thermocline gives $\Lambda = 870$, and the Richardson number based on $\langle s^2 \rangle$ and the local mean

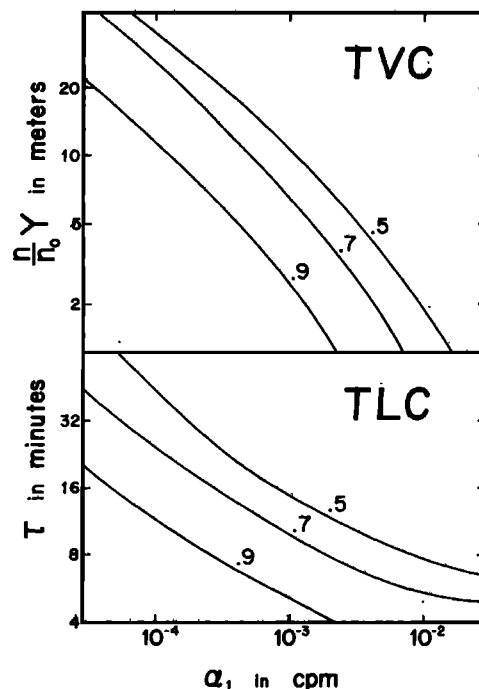


Fig. 4. The coherence contours $R = 0.5, 0.7, 0.9$, for towed vertical and lagged coherences, according to the internal wave model. Here, Y is the vertical separation, τ is time lag.

Brunt-Väisälä frequency is

$$(Ri) = 0.4(n_0/n)$$

This is close to the critical Richardson number of 0.25 for instability of stratified shear flows and suggests a cascade of energy to high wave numbers, limited by shear instability.

The present model has four other underlying assumptions, none of which are correct:

1. We assume that the observed motion is wavelike, as distinguished from turbulent fluctuations. Turbulence not dominated by buoyancy eventually takes over at the higher wave numbers [Monin, 1973].
2. We deal with the internal wave field as a random phase superposition of elementary wave trains. This is an

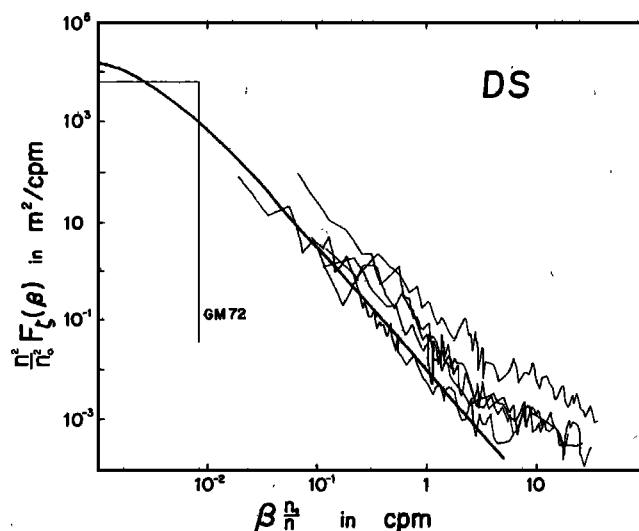


Fig. 5. The normalized dropped spectrum of vertical displacement according to Millard [1972]. The heavy solid curve is from the tapered internal wave model; the top hat corresponds to GM72.

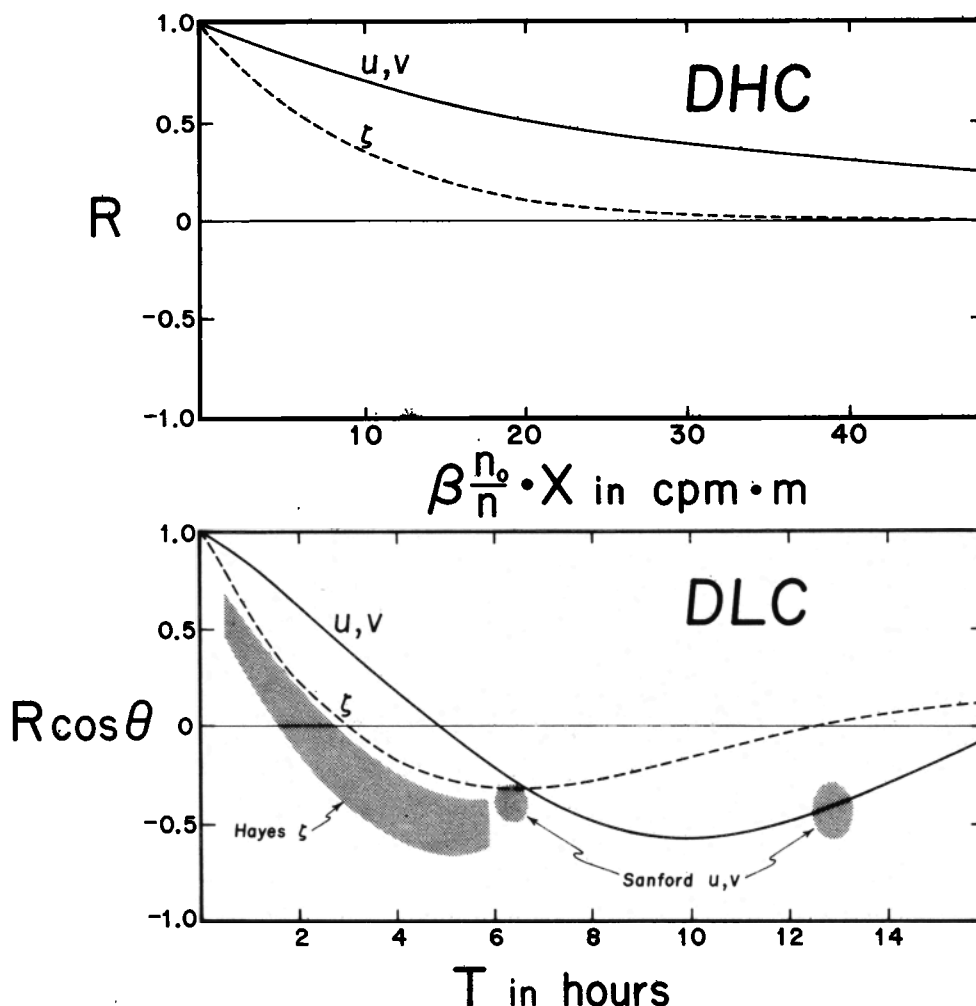


Fig. 6. Dropped horizontal and lagged coherences for current components u, v (solid curve) and vertical displacement ζ (dashed curve). For the lagged coherences we have plotted the in-phase component (normalized cospectrum); according to the internal wave model this is independent of vertical wave number. Some preliminary results by S. P. Hayes (personal communication, 1974) and Sanford [1975] are indicated by the shaded areas.

overlinearized view of a situation where nonlinear coupling processes are clearly important. We then use linearized wave functions to relate measurements of different dependent variables (u, v, ζ, \dots) and linearized wave dispersion to select any two of the three independent coordinates (α, β, ω).

3. We assume horizontal isotropy. This assumption is based on the statistics of current components by Fofonoff and Webster at site D and the insensitivity of Katz's towed spectra to the ship's course. Clearly, there are instances where a directional pattern is evident, as emphasized in recent Russian literature [Kitaigorodsky *et al.*, 1973], but in general the isotropic idealization would seem preferable to that of a pencil beam.

4. Finally, there is inherent in the model the assumption that observations at different times in different oceans can reasonably be related. The partial success of the model points toward a degree of universality, possibly as a result of some saturation phenomenon.

We wish to emphasize once again the descriptive nature of this undertaking. But there is some merit in describing a process in terms of empirical spectra even if the cause and effect is not worked out. The most rewarding aspect of this work has been the suggestion of new experiments and of a sampling strategy adequate to the task. A generation of

oceanographers has suffered the disappointments of ending up with uncorrelated measurements in what was intended as a coherent experiment because they failed to recognize the decoherency associated with broad band processes.

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