Numerical Modelling

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the anatomy of an ocean model

#3 Discretization

Master's degree 2nd year Marine Physics

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Useful references

Courses:

https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-950-atmospheric-and-oceanic-modeling-spring-2004/lecture-notes/

GENERAL

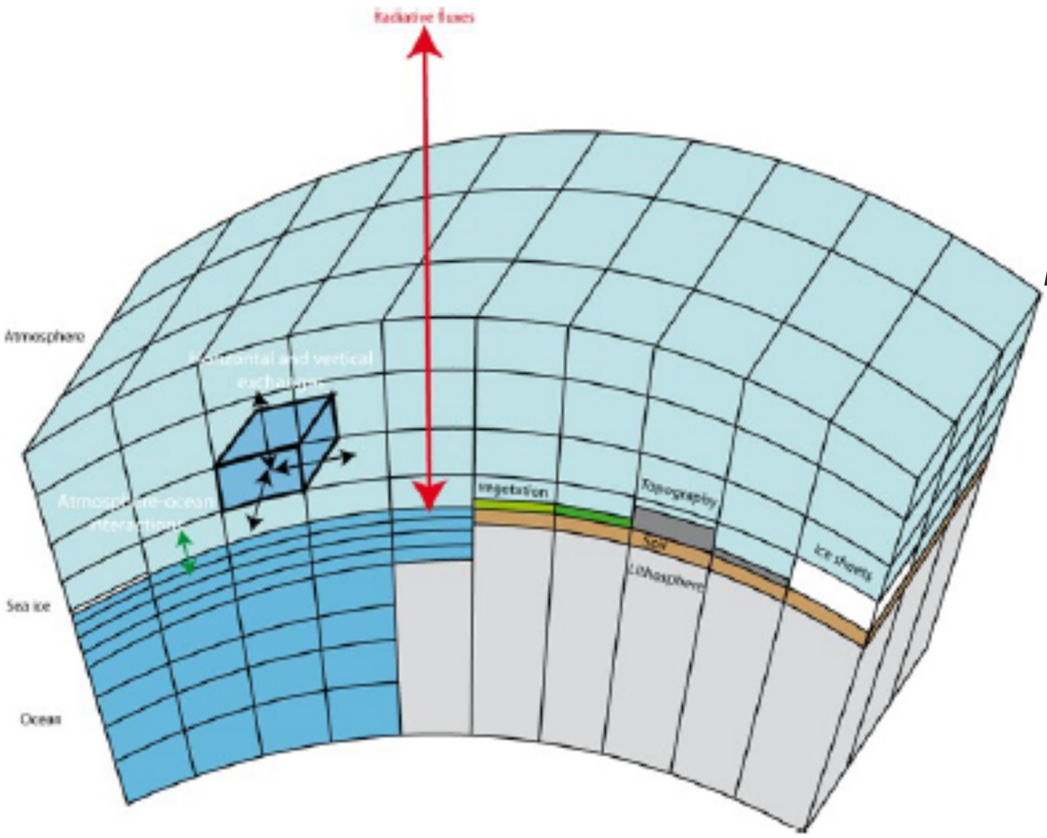
- Griffies et al., 2000, Developments in ocean climate modelling, Ocean Modelling.http://jgula.fr/ModNum/Griffiesetal00.pdf

ROMS:

- Shchepetkin, A., and J. McWilliams, 2005: The Regional Oceanic Modeling System (ROMS): A split-explicit, free-surface, topography-following- coordinate ocean model. Ocean Modell. http://jgula.fr/ModNum/ShchepetkinMcWilliams05.pdf
- https://www.myroms.org/wiki/

Discretization

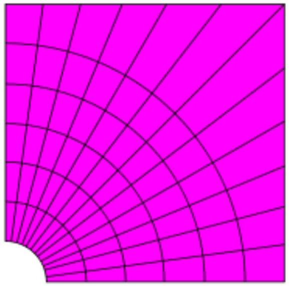
The ocean is divided into boxes : discretization



Example of a finite difference grid

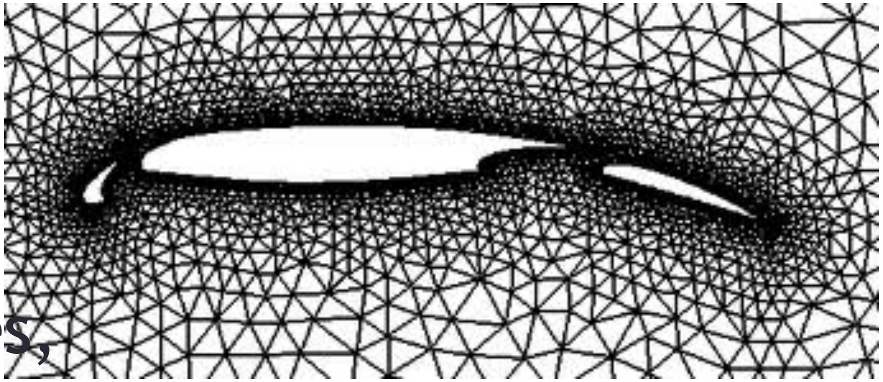
Discretization

Structured grids
Identified by regular connectivity
= can be adressed by (i,j,k)



Unstructured grids
The domain is tiled using more general geometrical shapes (triangles, …) pieced together to optimally fit details of the geometry.

- ✓ Good for tidal modeling, engineering applications.
- ✓ Problems:
  geostrophic balance accuracy,
  conservation and positivity properties,
  …



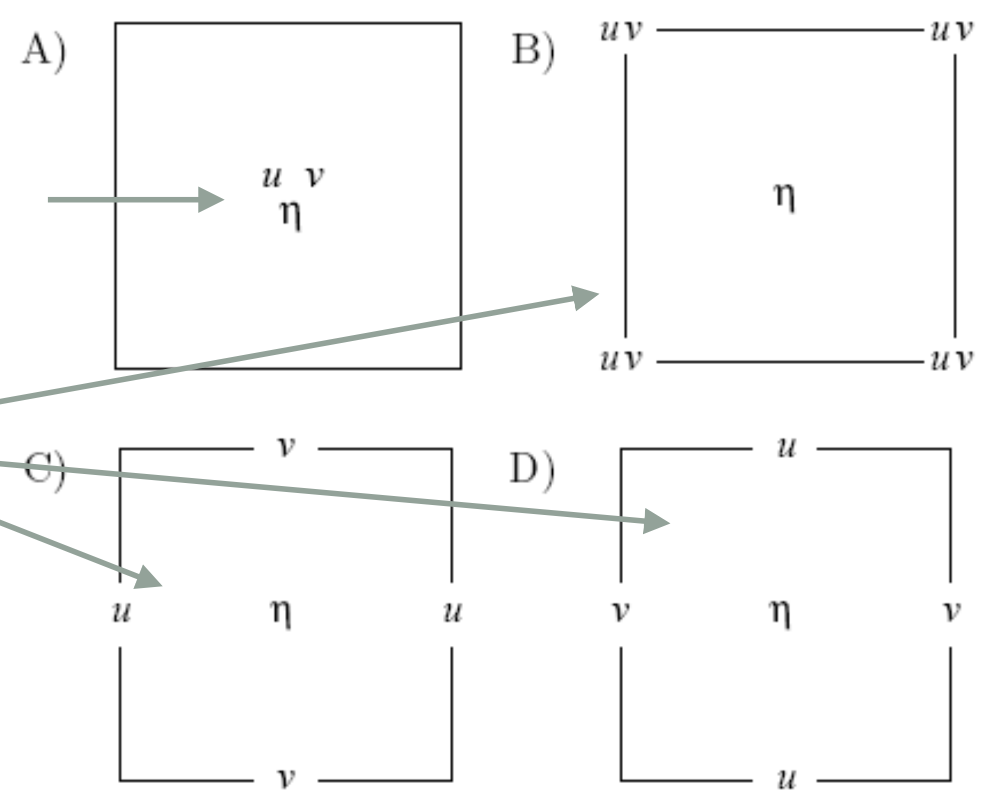
Horizontal discretization

Different types of Horizontal Grids (Arakawa Grids):

Non-staggered
(= collocated variables)

Or

Staggered



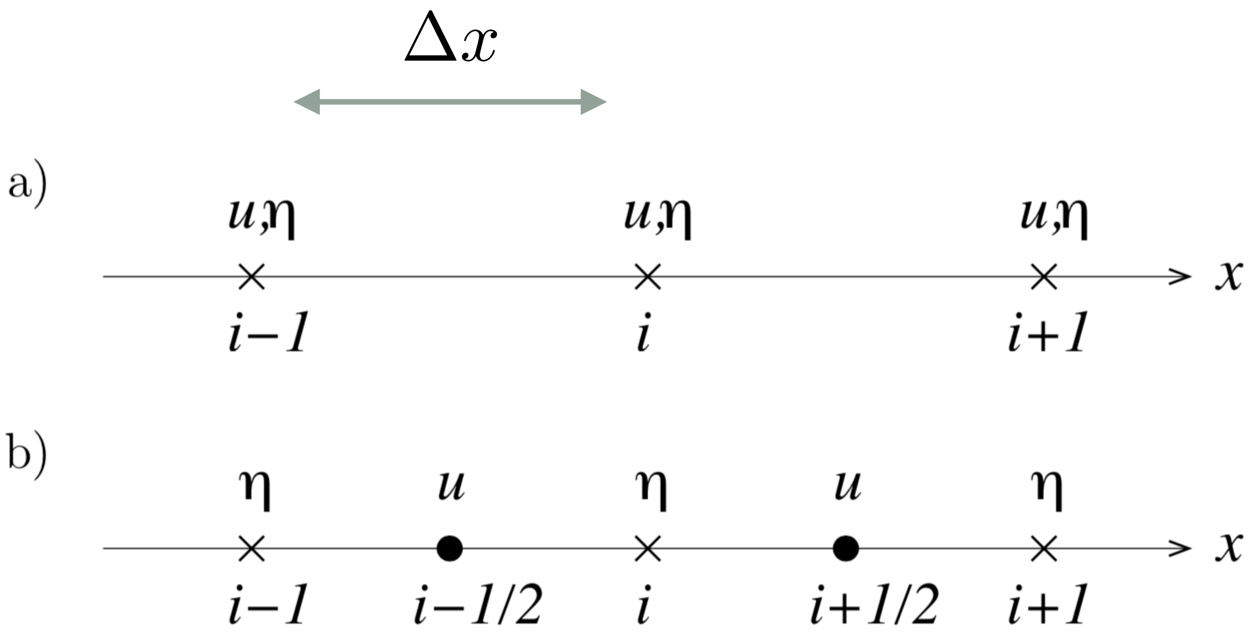
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

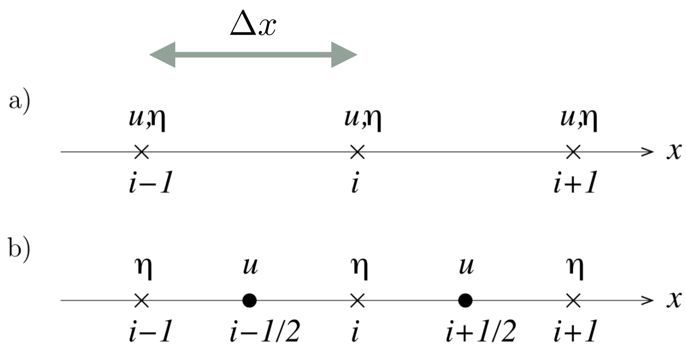
Non-staggered

Vs

Staggered



Horizontal discretization



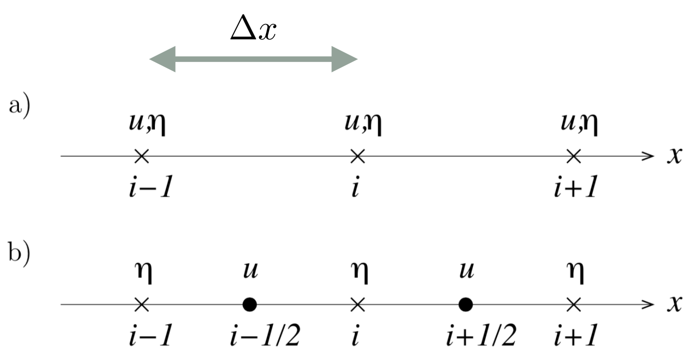
Staggered Vs unstaggered : the 1D problem

1. The advection equation $\partial_t \theta + c \partial_x \theta = 0$

Solutions of the continuous equations are non-dispersive waves

$\theta(x,t) = \theta_o e^{i(kx - \omega t)}$ *with dispersion relation* $\omega = ck$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

1. The advection equation $\partial_t \theta + c \partial_x \theta = 0$

Solutions of the ***continuous equations*** *are non-dispersive waves*

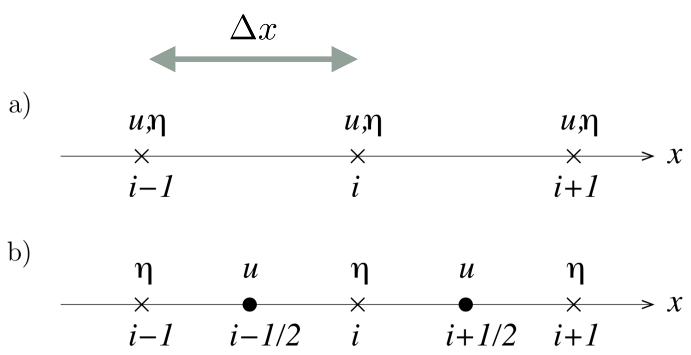
$\theta(x,t) = \theta_o e^{i(kx - \omega t)}$ *with dispersion relation* $\omega = ck$

Discretized equations with the centered second order derivative are:

$$d_t \theta + \frac{c}{\Delta x} \delta_i \overline{\theta}^i = 0$$

$$d_t \theta_i + \frac{c}{2\Delta x} \left(\theta_{i+1} - \theta_{i-1}\right) = 0$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

1. The advection equation $\partial_t \theta + c\partial_x \theta = 0$

Substituting in our solution:

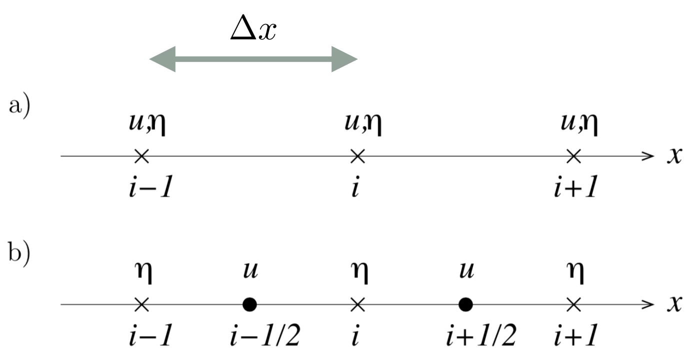
$$\theta_i(x,t) = \theta_0 e^{i(kx - \omega t)}$$

$$\theta_{i-1}(x,t) = \theta_0 e^{i(k(x-\Delta x) - \omega t)}$$

$$\theta_{i+1}(x,t) = \theta_0 e^{i(k(x+\Delta x) - \omega t)}$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem



1. The advection equation $\partial_t\theta + c\partial_x\theta = 0$

Substituting in our solution gives:

$$\begin{aligned} -i\omega &= -\frac{c}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) \\ &= -\frac{ci}{\Delta x}\sin k\Delta x \end{aligned}$$

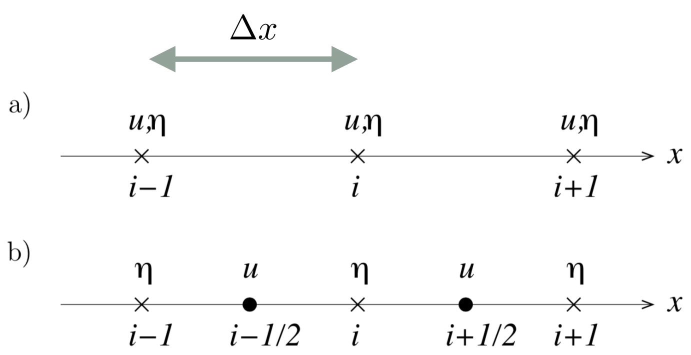
Now the solution is **dispersive!!!**

Even it will converge to the non-dispersive solution in the limit of small Δx

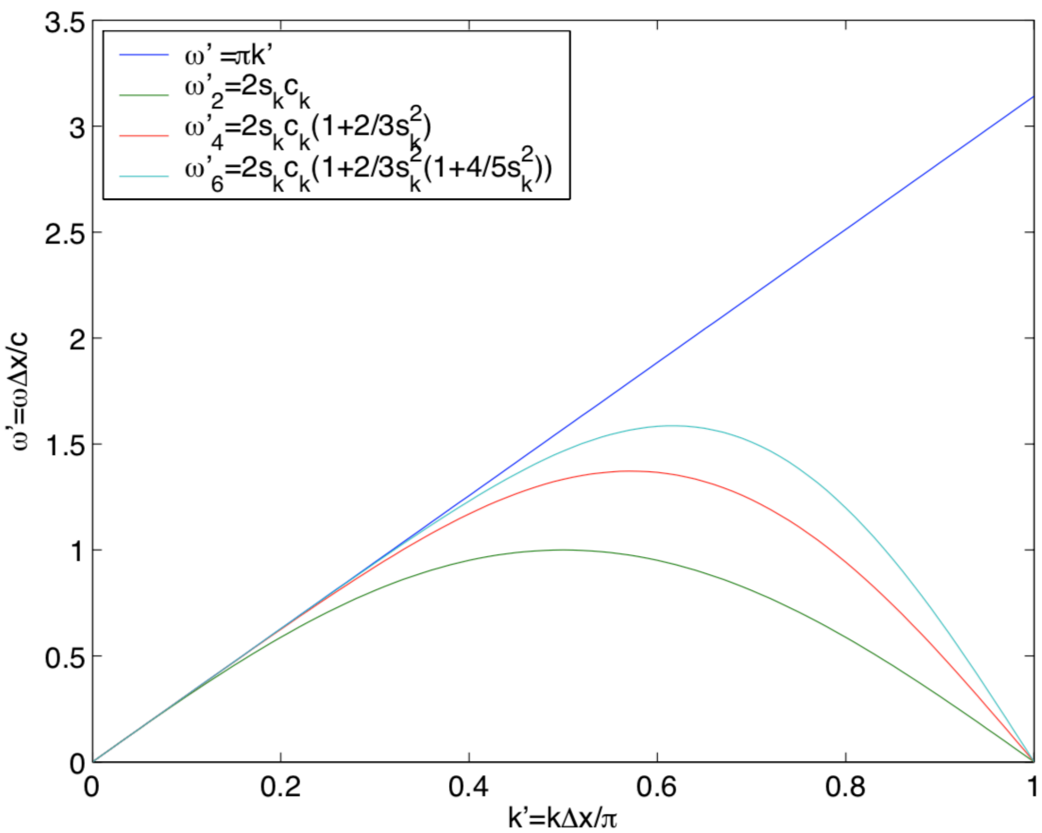
$$\omega = \frac{c}{\Delta x}\sin k\Delta x \overset{\Delta x \to 0}{=} ck$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

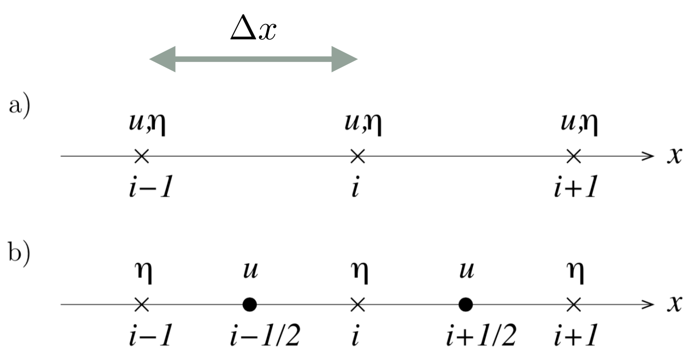


1. The advection equation $$\partial_t \theta + c\partial_x \theta = 0$$



Dispersion relations for constant flow advection using second, fourth, and sixth order spatial differences.

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

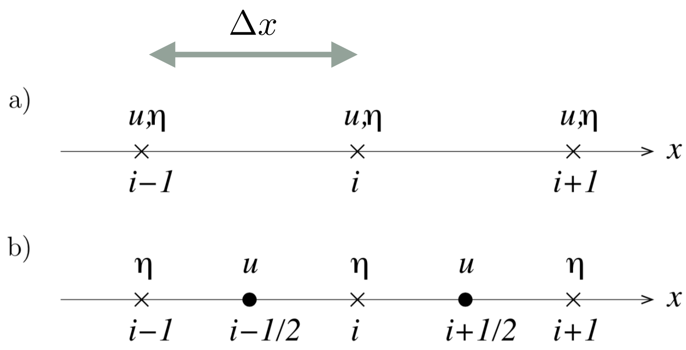
2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Solutions of the continuous equations are non-dispersive waves

$\eta = \eta_o e^{i(kx-\omega t)}$ *with dispersion relation* $\omega = \pm\sqrt{gH}k$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Solutions of the continuous equations are non-dispersive waves

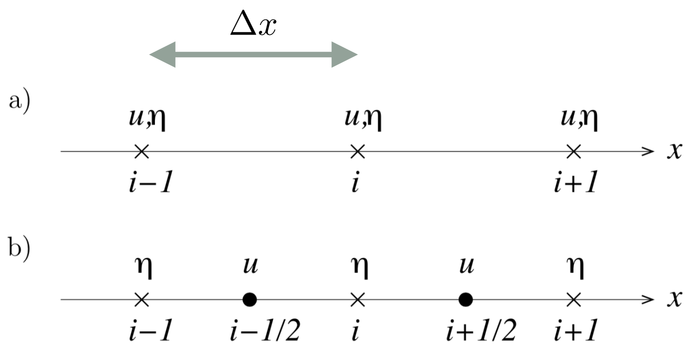
$\eta = \eta_o e^{i(kx-\omega t)}$ *with dispersion relation* $\omega = \pm\sqrt{gH}k$

Discretized equations with the centered second order derivative on the **unstaggered grid** are:

$$\begin{aligned} \partial_t u &= -\frac{g}{\Delta x}\delta_i \overline{\eta}^i \\ \partial_t \eta &= -\frac{H}{\Delta x}\delta_i \overline{u}^i \end{aligned}$$

$$\longrightarrow \partial_{tt}\eta = \frac{gH}{\Delta x^2}\delta_{ii}\overline{\eta}^{ii}$$ with $$\delta_{ii}\overline{\eta}^{ii} = \frac{1}{4}\left(\eta_{i-2} - 2\eta_i + \eta_{i+2}\right)$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

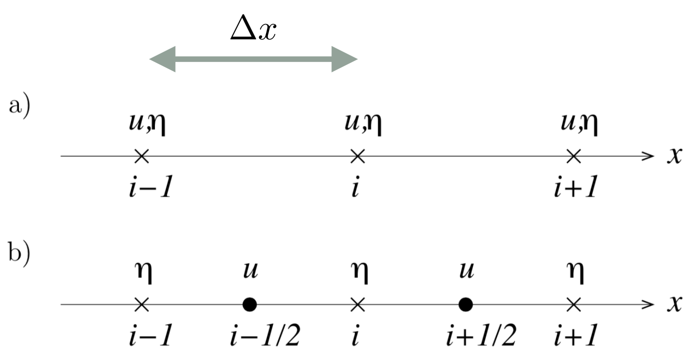
2. Gravity waves

$$\begin{aligned}\partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u\end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Substituting in our solution on the unstaggered grid gives :

$$\begin{aligned}-\omega^2 &= \frac{gH}{4\Delta x^2}\left(e^{-i2k\Delta x} - 2 + e^{i2k\Delta x}\right) \\ &= \frac{gH}{4\Delta x^2}\left(2\cos 2k\Delta x - 2\right) \\ &= -\frac{4gH}{\Delta x^2}\sin^2\frac{k\Delta x}{2}\cos^2\frac{k\Delta x}{2}\end{aligned}$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

2. Gravity waves

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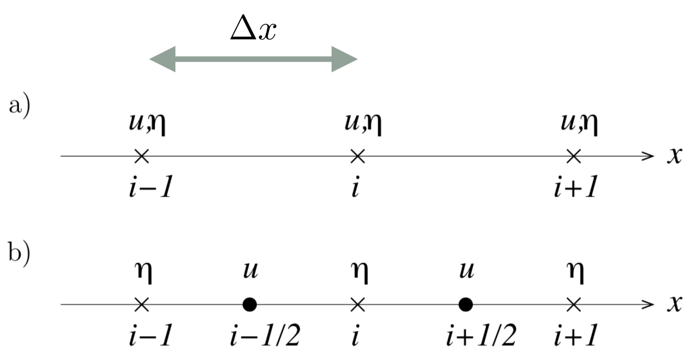
- Question:
  - What is the dispersion relation on the staggered grid?

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Discretized equations with the centered second order derivative on the **staggered grid** are:

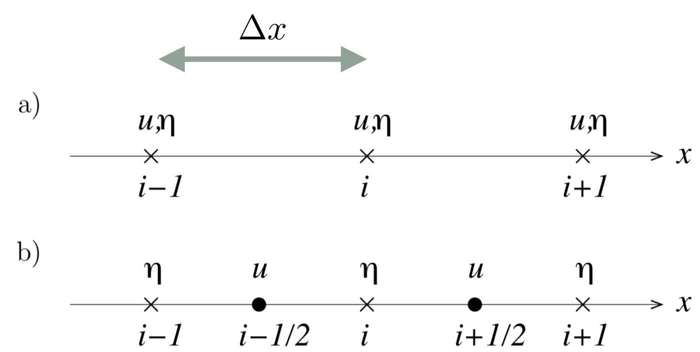
$$\begin{aligned} \partial_t u &= -\frac{g}{\Delta x}\delta_i \eta \\ \partial_t \eta &= -\frac{H}{\Delta x}\delta_i u \end{aligned}$$

This can be written as a system:

$$\begin{pmatrix} \partial_t & \frac{g}{\Delta x}\delta_i \\ \frac{H}{\Delta x}\delta_i & \partial_t \end{pmatrix} \begin{pmatrix} u \\ \eta \end{pmatrix} = 0 \qquad \begin{pmatrix} -i\omega & \frac{2ig}{\Delta x}\sin\frac{k\Delta x}{2} \\ \frac{2iH}{\Delta x}\sin\frac{k\Delta x}{2} & -i\omega \end{pmatrix} \begin{pmatrix} u \\ \eta \end{pmatrix} = 0$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem



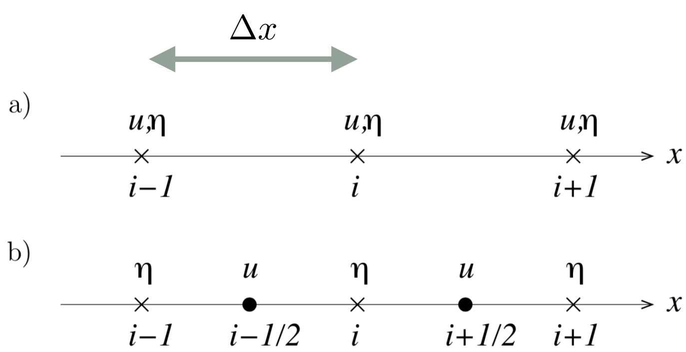
2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Substituting in our solution on the staggered grid gives :

$$\omega^2 - \frac{4gH}{\Delta x^2}\sin^2\frac{k\Delta x}{2} = 0$$

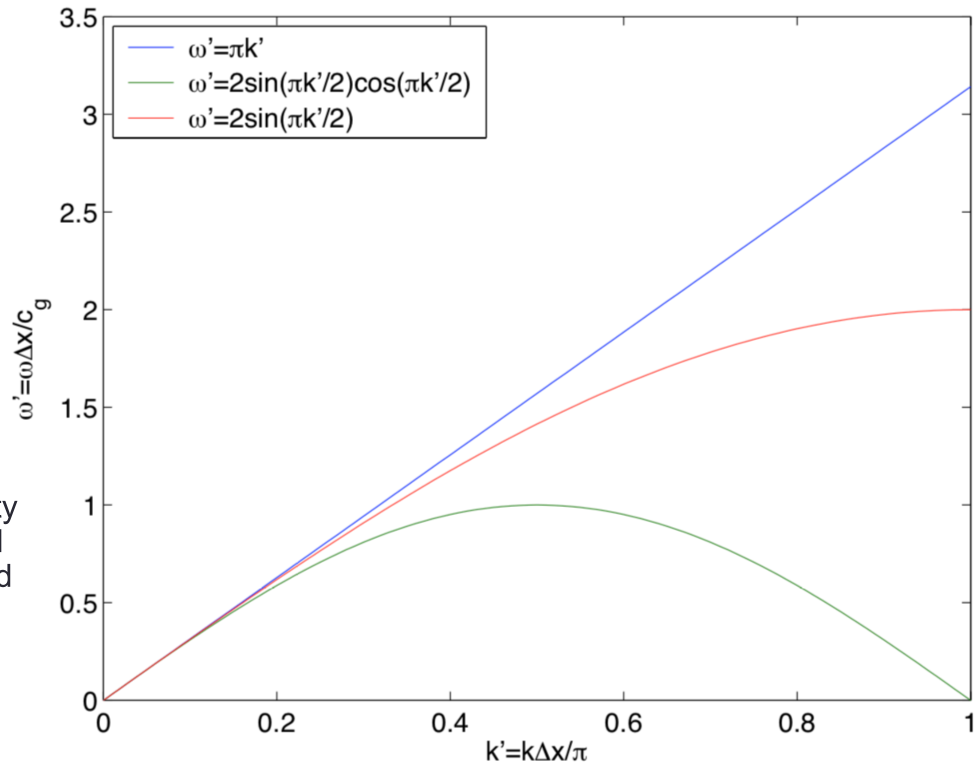
Horizontal discretization



Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$



Dispersion of numerical gravity wave for the unstaggered grid (green) and the staggered grid (red). The continuum (= k) is plotted for comparison (blue).

When compared to the continuum we see that the numerical modes are still dispersive on the staggered grid, but:

there is no false extrema, unlike the non-staggered grid,

the group speed is of the correct sign everywhere, even if reduced.

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

$$
\begin{aligned}
\partial_t u - f v + g \partial_x \eta &= 0 \\
\partial_t v + f u &= 0 \\
\partial_t \eta + H \partial_x u &= 0
\end{aligned}
$$

Solutions of the continuous equations are waves following the dispersion relation:

$$
\left| \begin{pmatrix} -i\omega & -f & gik \\ f & -i\omega & 0 \\ Hik & 0 & -i\omega \end{pmatrix} \right| = 0 \Rightarrow \begin{cases} \omega = 0 \\ \omega^2 = f^2 + gHk^2 \end{cases}
$$

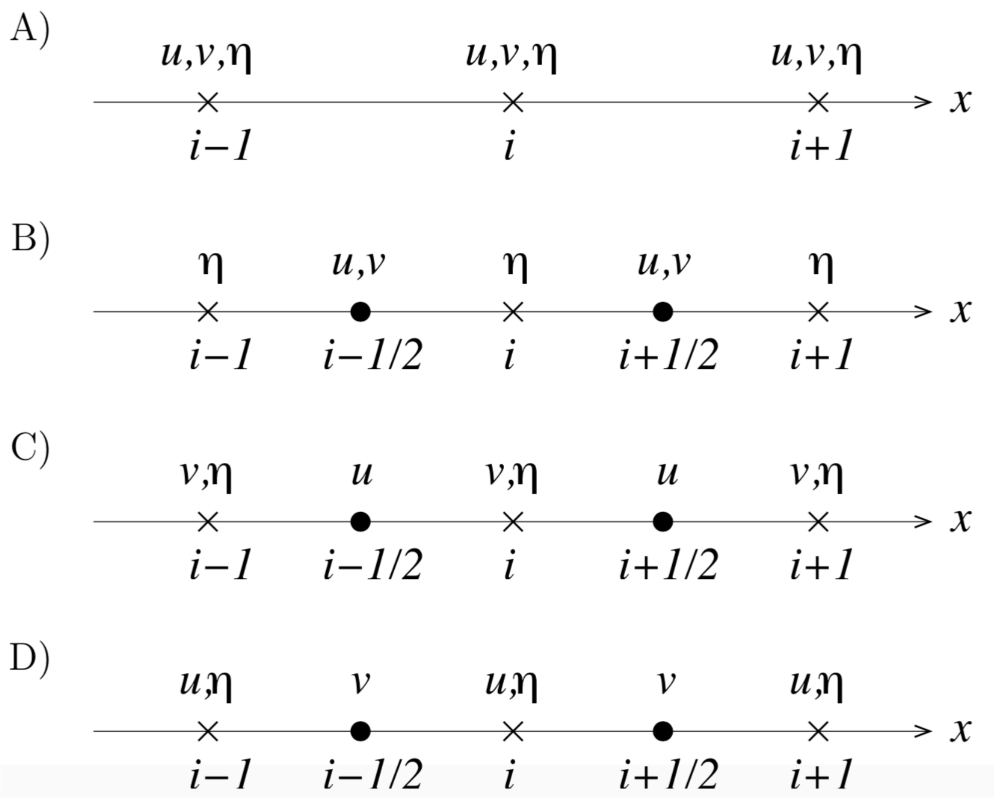
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

$$
\begin{aligned}
\partial_t u - fv + g\partial_x \eta &= 0 \\
\partial_t v + fu &= 0 \\
\partial_t \eta + H\partial_x u &= 0
\end{aligned}
$$

Now, 4 different grids are possible:



Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

- A-grid model

$$
\begin{aligned}
\partial_t u - fv + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\
\partial_t v + fu &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i &= 0
\end{aligned}
$$

- B-grid model

$$
\begin{aligned}
\partial_t u - fv + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + fu &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u &= 0
\end{aligned}
$$

- C-grid model

$$
\begin{aligned}
\partial_t u - f\overline{v}^i + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f\overline{u}^i &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u &= 0
\end{aligned}
$$

- D-grid model

$$
\begin{aligned}
\partial_t u - f\overline{v}^i + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\
\partial_t v + f\overline{u}^i &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i &= 0
\end{aligned}
$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

The corresponding dispersion relations are :

$$\text{A:} \quad \frac{\omega^2}{f^2} = 1 + \frac{4L_d^2}{\Delta x^2} s_k^2 c_k^2$$

$$\text{B:} \quad \frac{\omega^2}{f^2} = 1 + \frac{4L_d^2}{\Delta x^2} s_k^2$$

$$\text{C:} \quad \frac{\omega^2}{f^2} = c_k^2 + \frac{4L_d^2}{\Delta x^2} s_k^2$$

$$\text{D:} \quad \frac{\omega^2}{f^2} = c_k^2 + \frac{4L_d^2}{\Delta x^2} s_k^2 c_k^2$$

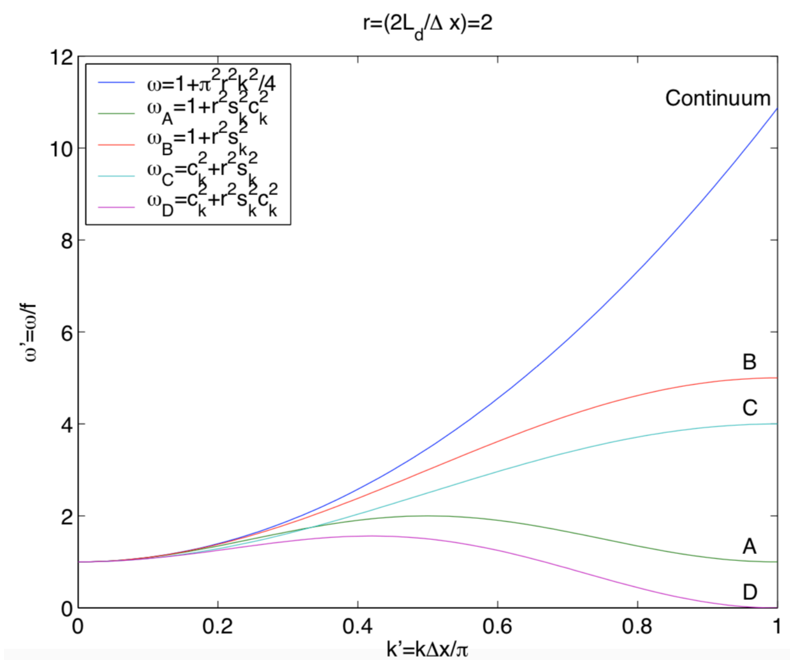
$$s_k = \sin \frac{k\Delta x}{2} \qquad c_k = \cos \frac{k\Delta x}{2}$$

$$L_d = \sqrt{gH}/f$$

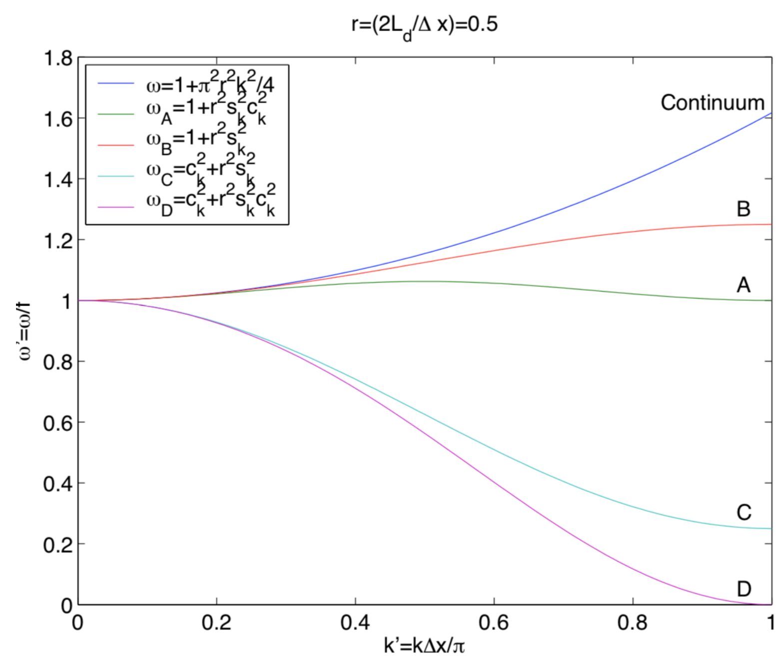
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves



deformation radius is resolved

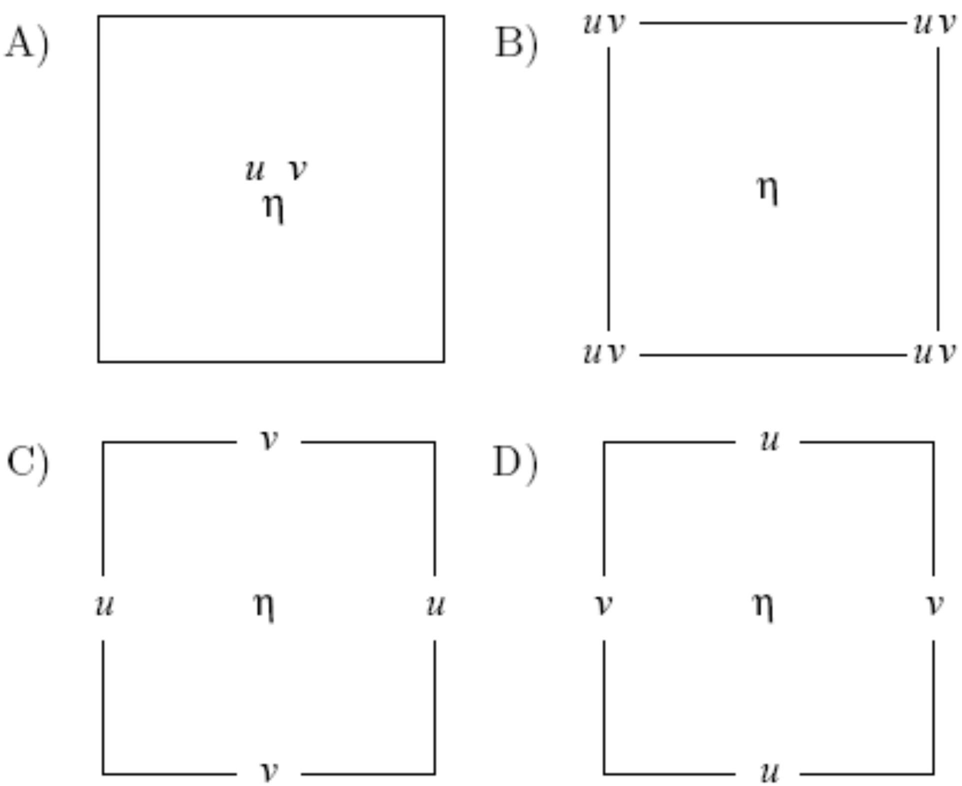


deformation radius is not resolved

Staggering variables in the form of the B grid is most likely to avoid computational modes when solving one-dimensional shallow water equations.

Horizontal discretization

Horizontal Arakawa Grids:

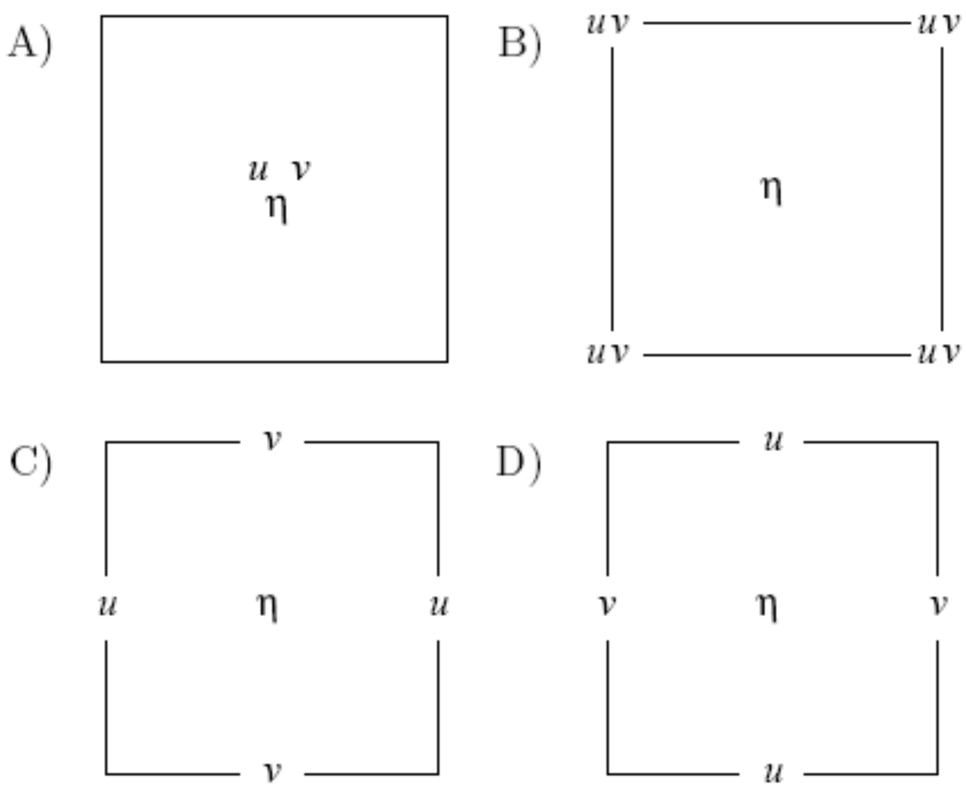


Linear shallow water equation:

$$
\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f u + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_y \eta &= 0
\end{aligned}
$$

Horizontal discretization

Horizontal Arakawa Grids:

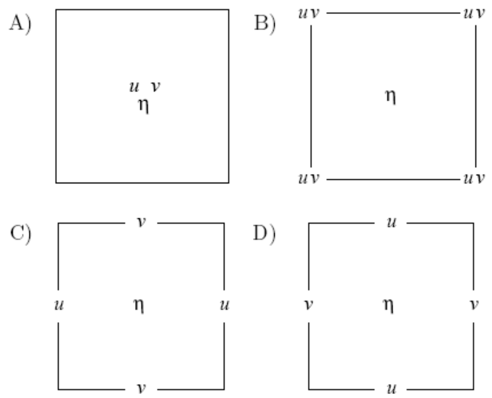


Linear shallow water equation:

$$
\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f u + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_y \eta &= 0
\end{aligned}
$$

- Question:
  - Which grid minimises the number of averaging between points when solving linear SW equations in 2d?

Horizontal discretization



- A grid:

$$
\begin{aligned}
\partial_t u - fv + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\
\partial_t v + fu + \frac{g}{\Delta y}\delta_j \overline{\eta}^j &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i + \frac{H}{\Delta y}\delta_j \overline{v}^j &= 0
\end{aligned}
$$

- B grid:

$$
\begin{aligned}
\partial_t u - fv + \frac{g}{\Delta x}\delta_i \overline{\eta}^j &= 0 \\
\partial_t v + fu + \frac{g}{\Delta y}\delta_j \overline{\eta}^i &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^j + \frac{H}{\Delta y}\delta_j \overline{v}^i &= 0
\end{aligned}
$$

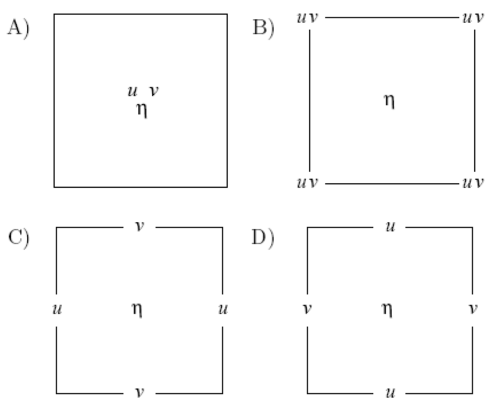
- C grid:

$$
\begin{aligned}
\partial_t u - f\overline{v}^{ij} + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f\overline{u}^{ij} + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_j v &= 0
\end{aligned}
$$

- D grid:

$$
\begin{aligned}
\partial_t u - f\overline{v}^{ij} + \frac{g}{\Delta x}\delta_i \overline{\eta}^{ij} &= 0 \\
\partial_t v + f\overline{u}^{ij} + \frac{g}{\Delta y}\delta_j \overline{\eta}^{ij} &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^{ij} + \frac{H}{\Delta y}\delta_j \overline{v}^{ij} &= 0
\end{aligned}
$$

Horizontal discretization



Response of each operator:

$$
\begin{aligned}
R(\delta_i \phi) &= 2i \sin \frac{k\Delta x}{2} = 2i s_k \\
R(\delta_j \phi) &= 2i \sin \frac{l\Delta y}{2} = 2i s_l \\
R(\overline{\phi}^i) &= \cos \frac{k\Delta x}{2} = c_k \\
R(\overline{\phi}^j) &= \cos \frac{l\Delta y}{2} = c_l
\end{aligned}
$$

Dispersion relations:

- A grid:

$$
\begin{aligned}
\omega^2 &= f^2 + \frac{4gH}{\Delta x^2} s_k^2 c_k^2 + \frac{4gH}{\Delta y^2} s_l^2 c_l^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= 1 + r_x^2 s_k^2 c_k^2 + r_y^2 s_l^2 c_l^2
\end{aligned}
$$

- B grid:

$$
\begin{aligned}
\omega^2 &= f^2 + \frac{4gH}{\Delta x^2} s_k^2 c_l^2 + \frac{4gH}{\Delta y^2} s_l^2 c_k^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= 1 + r_x^2 s_k^2 c_l^2 + r_y^2 s_l^2 c_k^2
\end{aligned}
$$

- C grid:

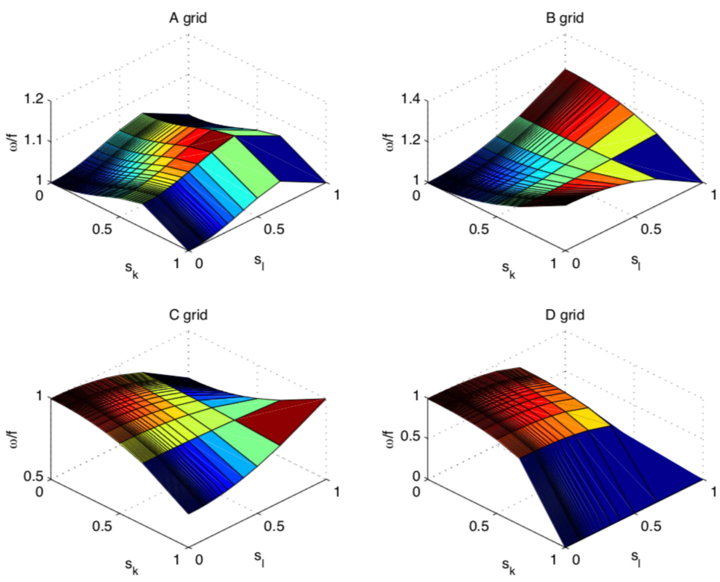
$$
\begin{aligned}
\omega^2 &= f^2 c_k^2 c_l^2 + \frac{4gH}{\Delta x^2} s_k^2 + \frac{4gH}{\Delta y^2} s_l^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= c_k^2 c_l^2 + r_x^2 s_k^2 + r_y^2 s_l^2
\end{aligned}
$$

- D grid:

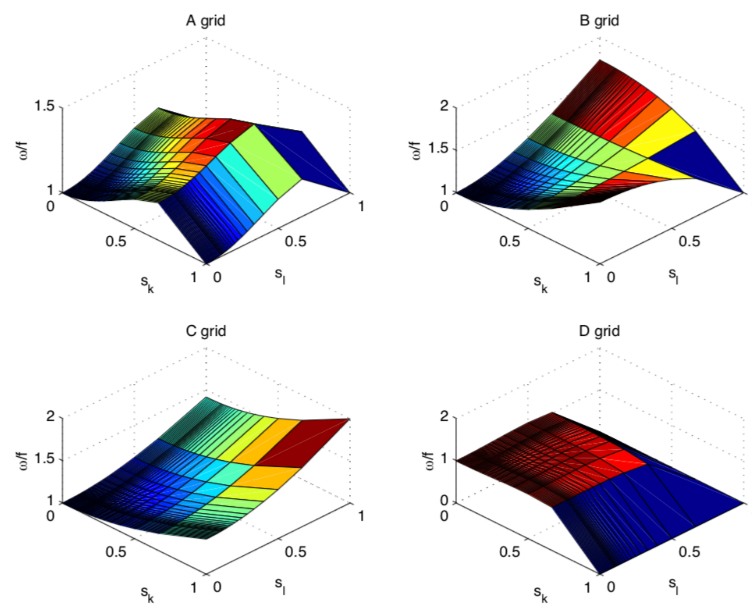
$$
\begin{aligned}
\omega^2 &= f^2 c_k^2 c_l^2 + \frac{4gH}{\Delta x^2} s_k^2 c_k^2 c_l^2 + \frac{4gH}{\Delta y^2} s_l^2 c_k^2 c_l^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= (1 + r_x^2 s_k^2 + r_y^2 s_l^2) c_k^2 c_l^2
\end{aligned}
$$

Horizontal discretization

Coarse resolution:



High resolution:

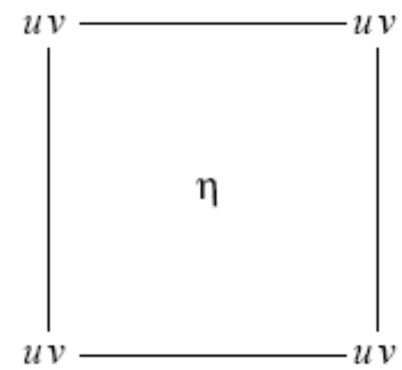


D is always bad.
B underestimates frequency for short two-dimensional waves
C is the only grid with monotonically increasing frequency (i.e. right sign of group velocity) at high res.

Horizontal discretization

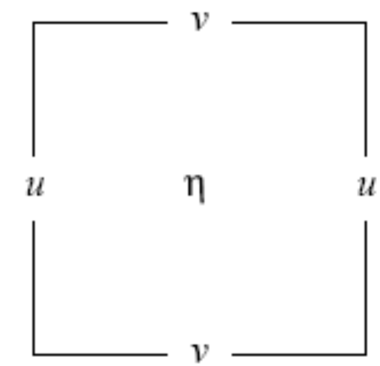
- B grid is prefered at coarse resolution, when Coriolis is important:
  - Superior for poorly resolved inertia-gravity waves.
  - Good for Rossby waves: collocation of velocity points.
  - Bad for gravity waves: computational checkerboard mode

B)



- C grid is prefered at fine resolution, when Coriolis is less important
  - Superior for gravity waves.
  - Good for well resolved inertia-gravity waves.
  - Bad for poorly resolved waves: Rossby waves (computational checkerboard mode) and inertia-gravity waves due to averaging the Coriolis force.

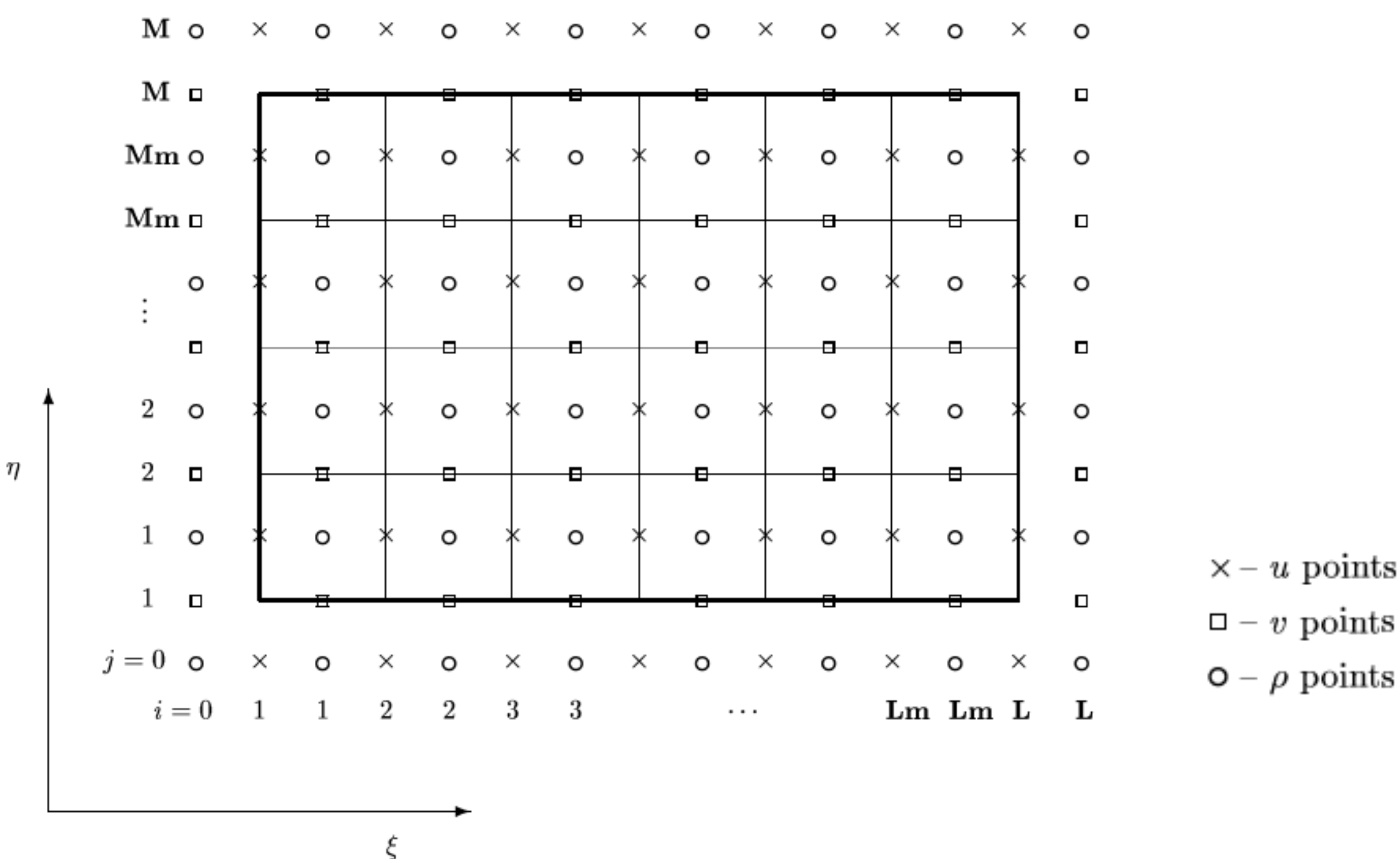
C)



ROMS

Horizontal discretization

ROMS: Arakawa C-grid



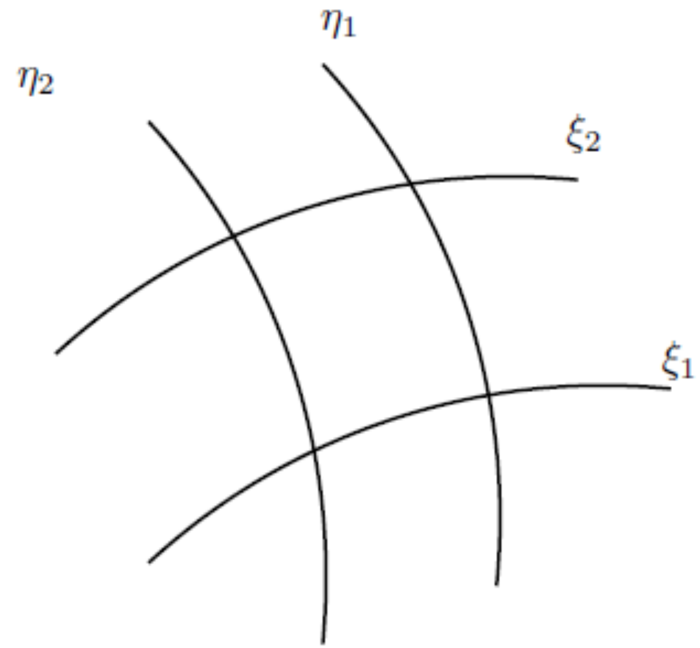
Horizontal curvilinear grid

- **ROMS:** is formulated in general horizontal curvilinear coordinates:

$$(ds)_\xi = \left(\frac{1}{m}\right) d\xi$$

$$(ds)_\eta = \left(\frac{1}{n}\right) d\eta$$

m, *n*: scale factors relating the differential distances to the physical arc lengths



$$\vec{v} \cdot \hat{\xi} = u$$

$$\vec{v} \cdot \hat{\eta} = v$$

Horizontal curvilinear grid

- **ROMS:** is formulated in general horizontal curvilinear coordinates:

$$(ds)_\xi = \left(\frac{1}{m}\right) d\xi$$

$$(ds)_\eta = \left(\frac{1}{n}\right) d\eta$$

With classical formulas for div, grad, curl and lap in curvilinear coordinates:

$$\nabla \phi = \hat{\xi} m \frac{\partial \phi}{\partial \xi} + \hat{\eta} n \frac{\partial \phi}{\partial \eta}$$

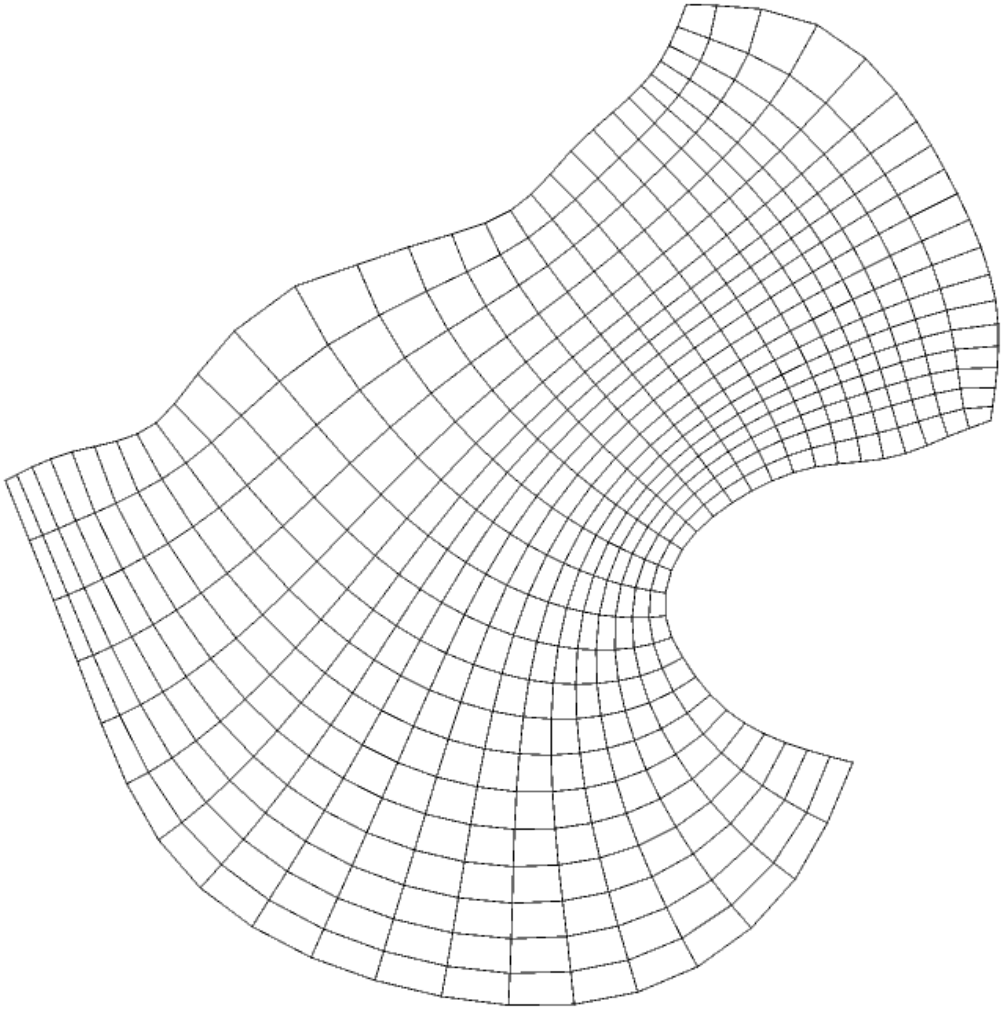
$$\nabla \cdot \vec{a} = mn \left[\frac{\partial}{\partial \xi}\left(\frac{a}{n}\right) + \frac{\partial}{\partial \eta}\left(\frac{b}{m}\right) \right]$$

$$\nabla \times \vec{a} = mn \begin{vmatrix} \frac{\hat{\xi}_1}{m} & \frac{\hat{\xi}_2}{n} & \hat{k} \\ \frac{\partial}{\partial \xi} & \frac{\partial}{\partial \eta} & \frac{\partial}{\partial z} \\ \frac{a}{m} & \frac{b}{n} & c \end{vmatrix}$$

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = mn \left[\frac{\partial}{\partial \xi}\left(\frac{m}{n} \frac{\partial \phi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{n}{m} \frac{\partial \phi}{\partial \eta}\right) \right]$$

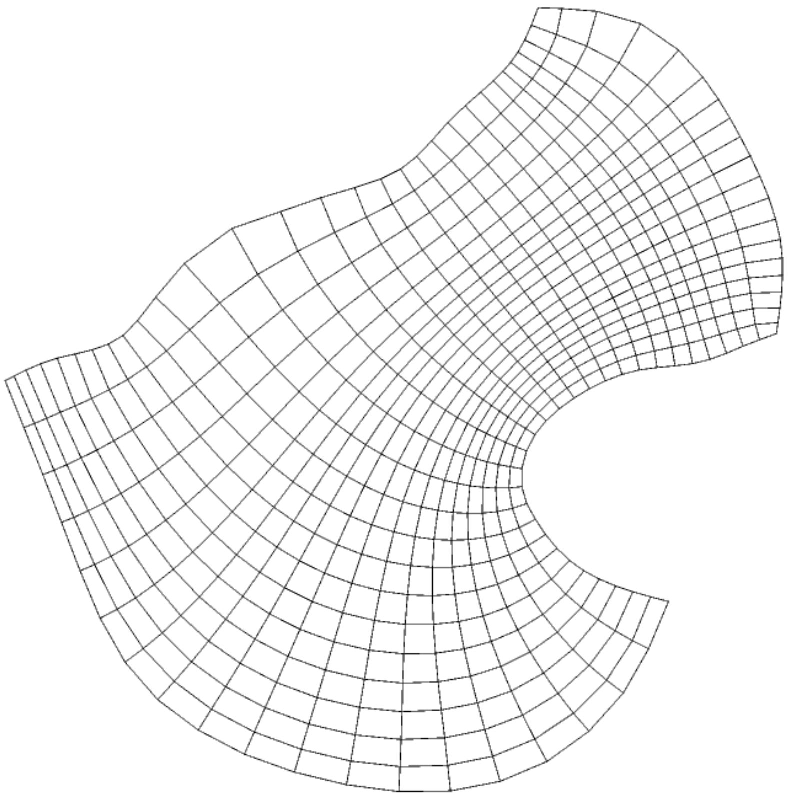
Horizontal curvilinear grid

- This is a possible grid:



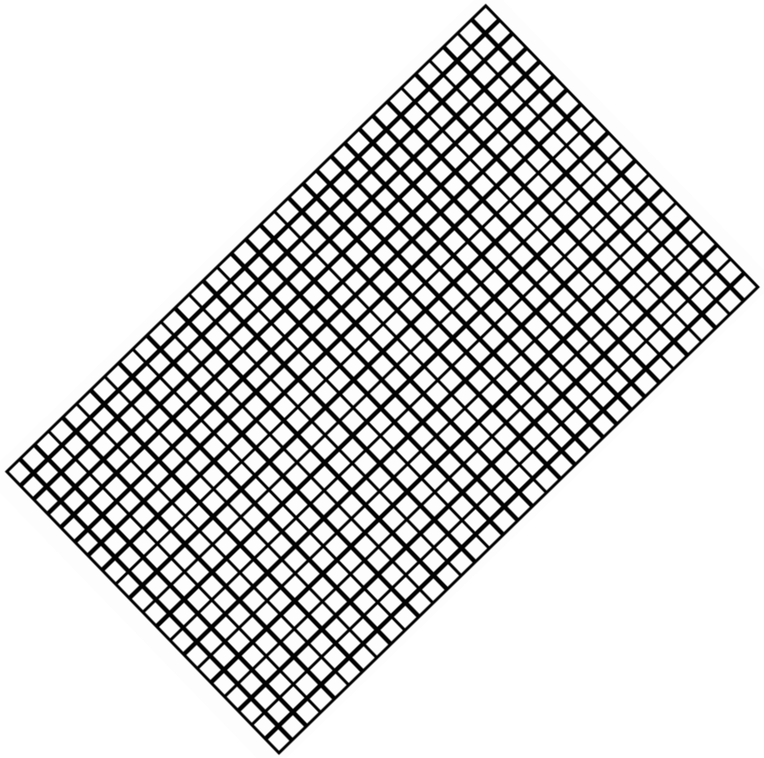
Horizontal curvilinear grid

- This is a possible grid:



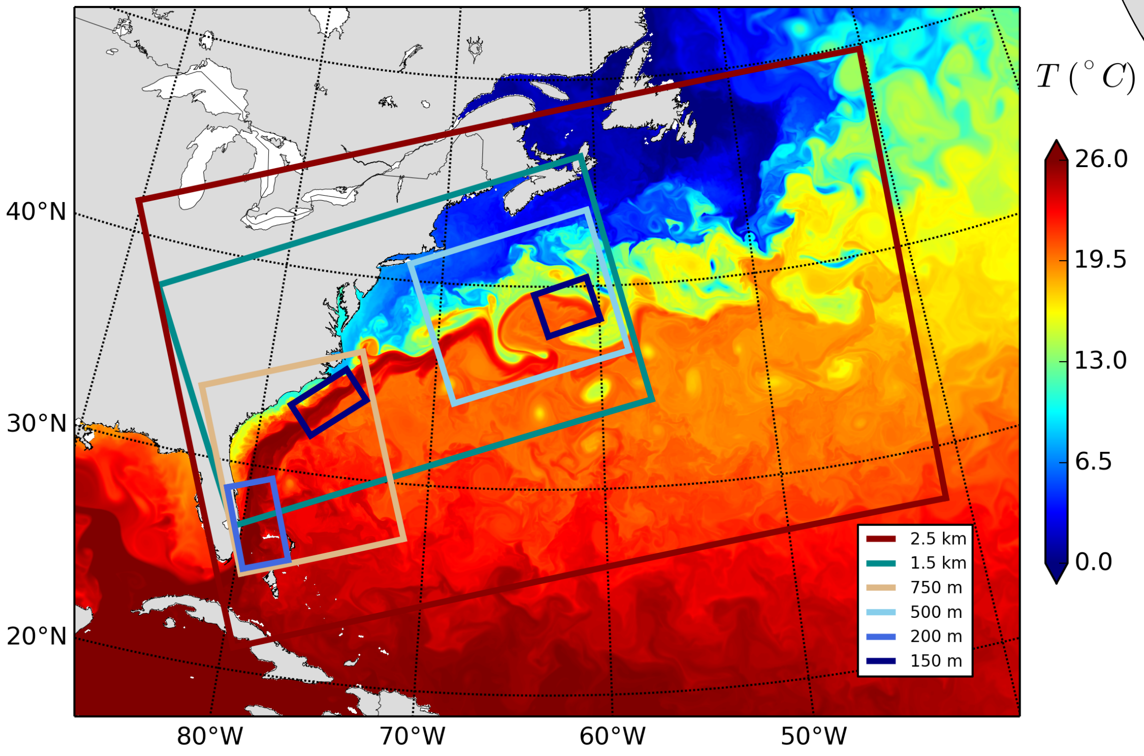
In practice variations in *dx* and *dy* should be minimized to minimize errors and optimize computation time.

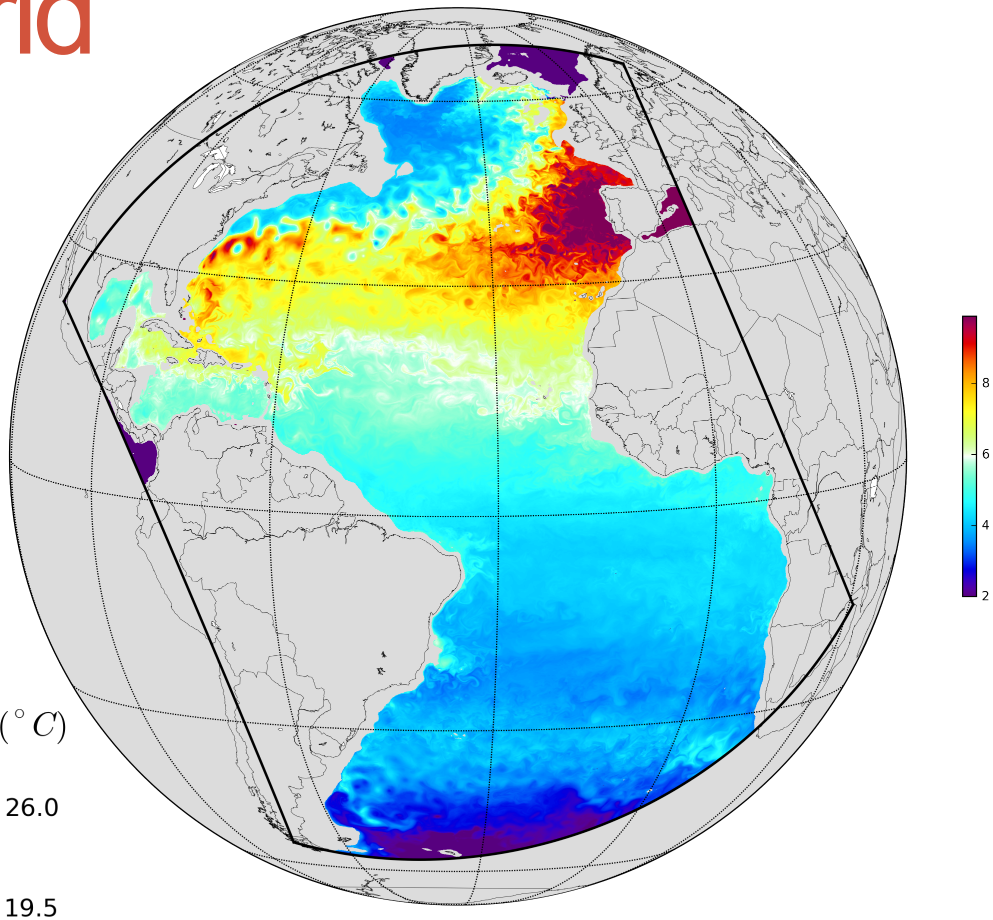
So avoir extreme distorsions and be as close as rectangular grids as possible (+ use land masks)



Horizontal curvilinear grid

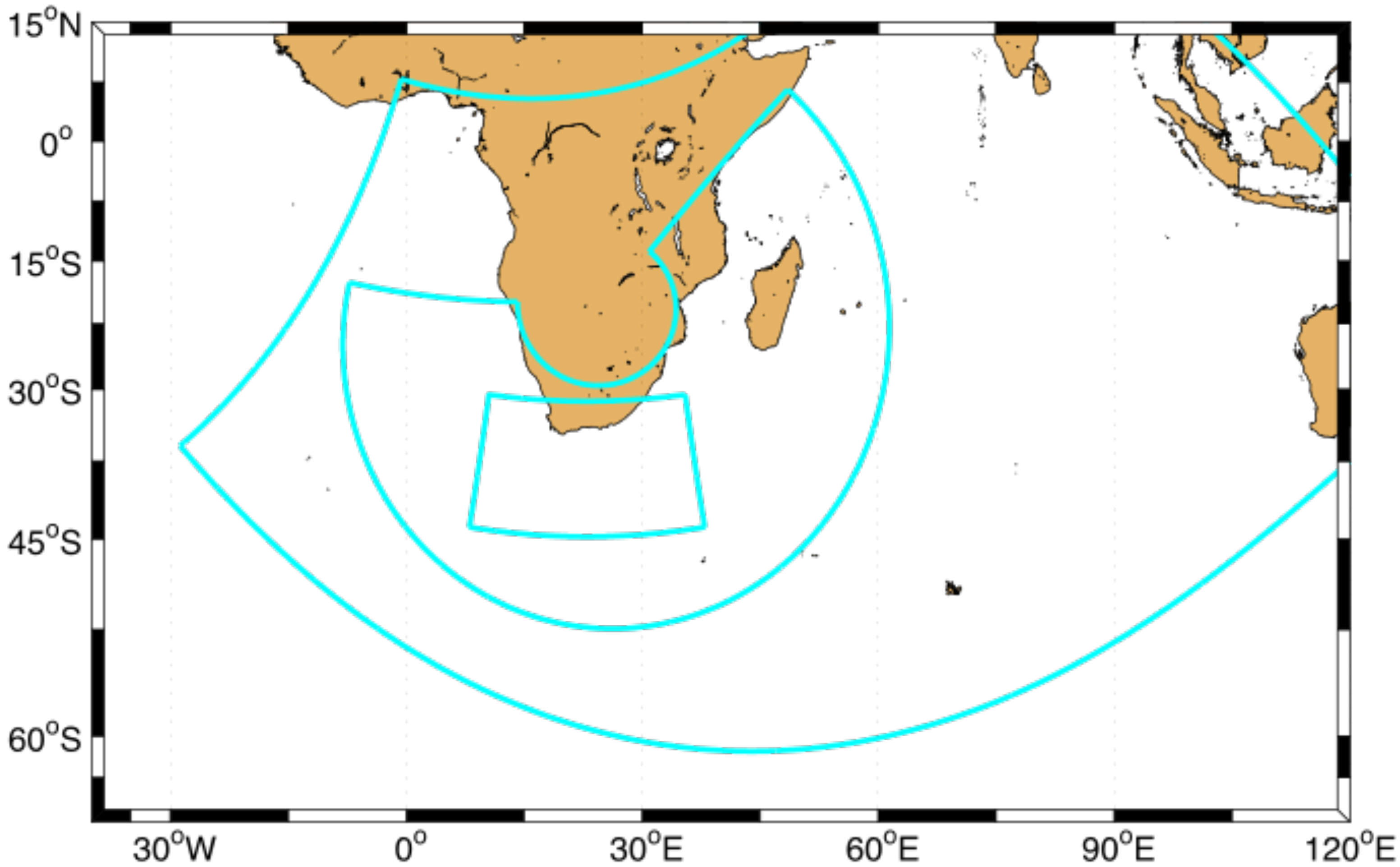
- Example of realistic domains:





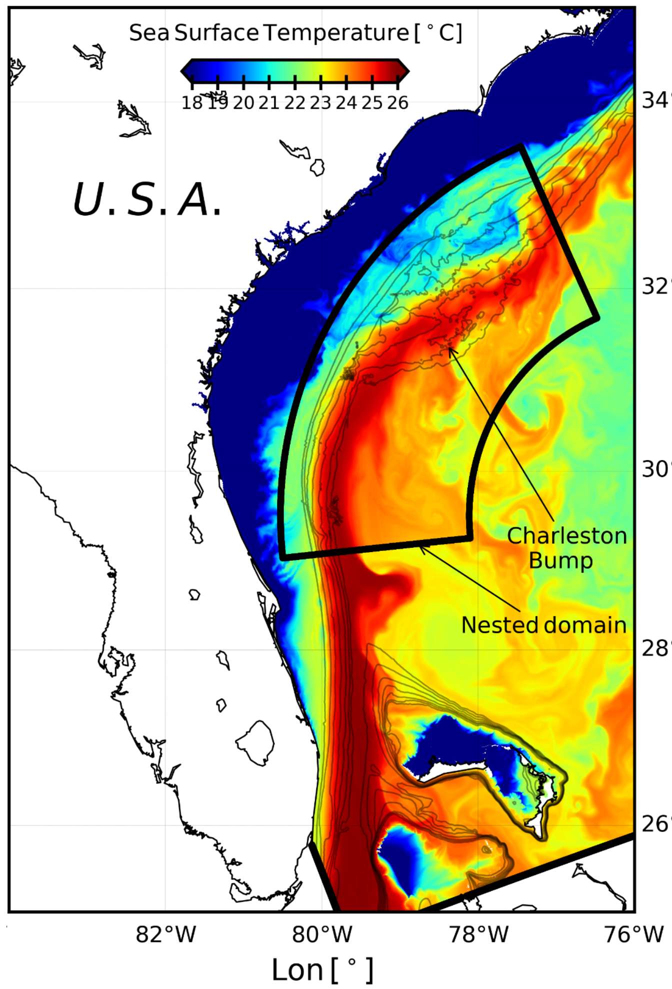
Horizontal curvilinear grid

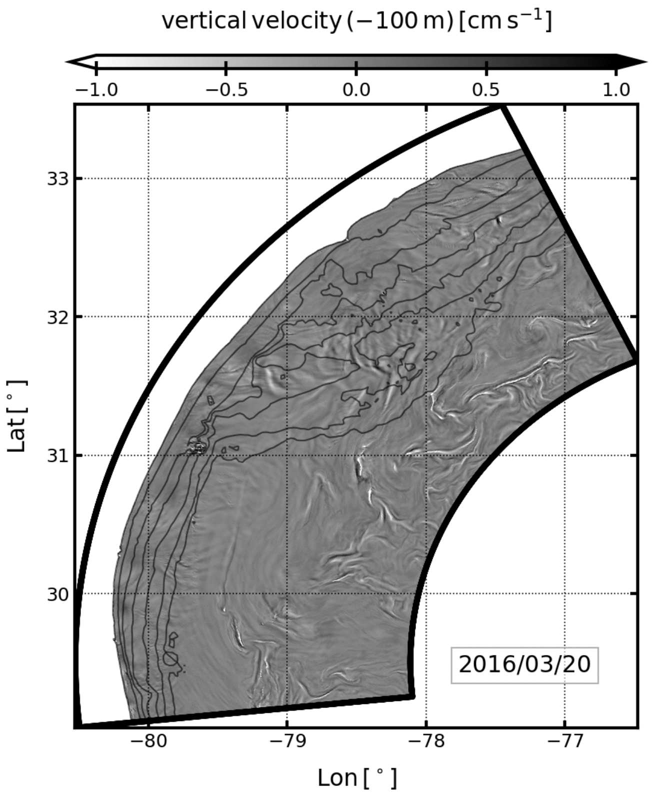
- Example of realistic domains (with gentle bendings):



Horizontal curvilinear grid

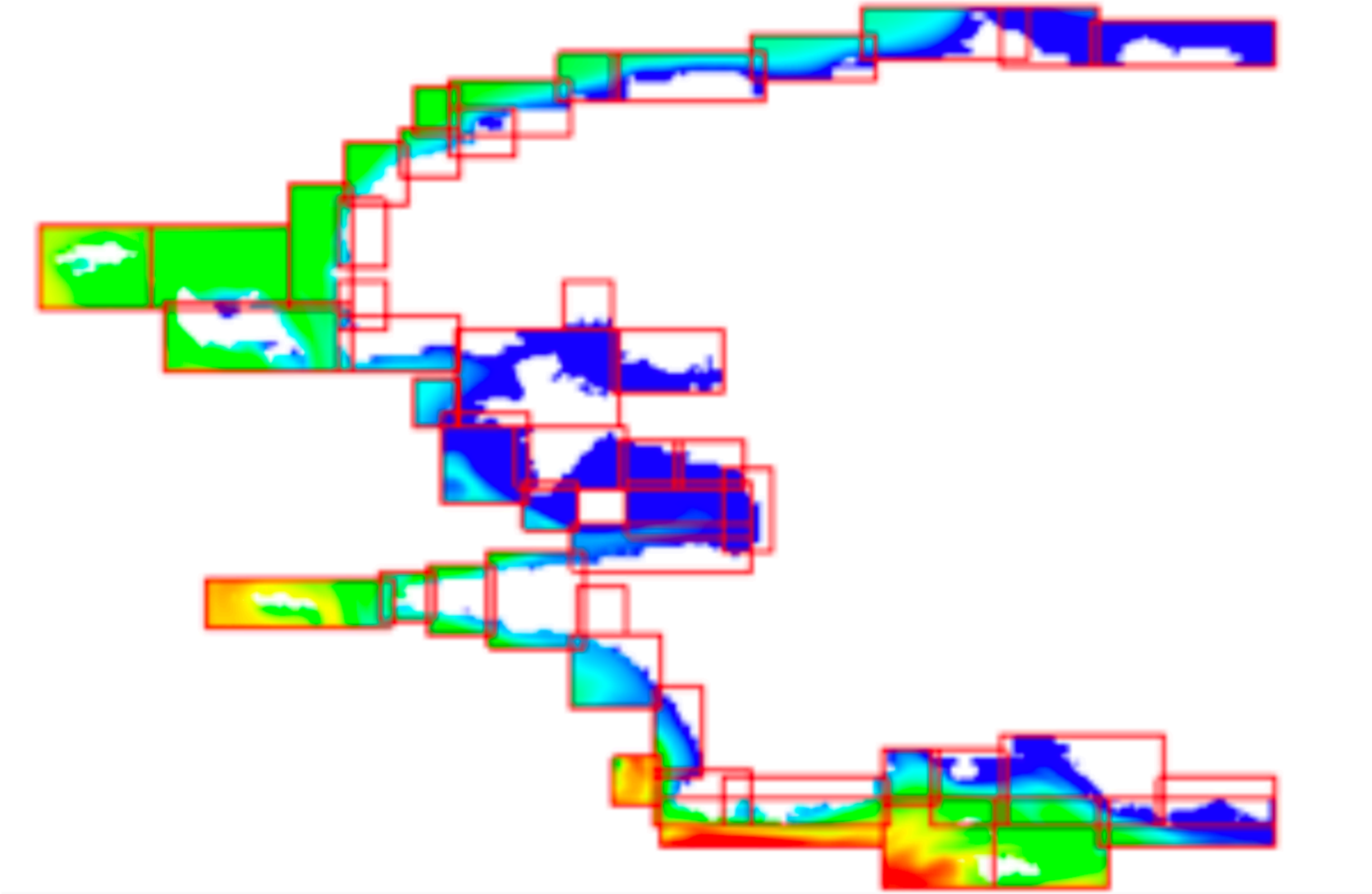
- Example of realistic domains (with gentle bendings):





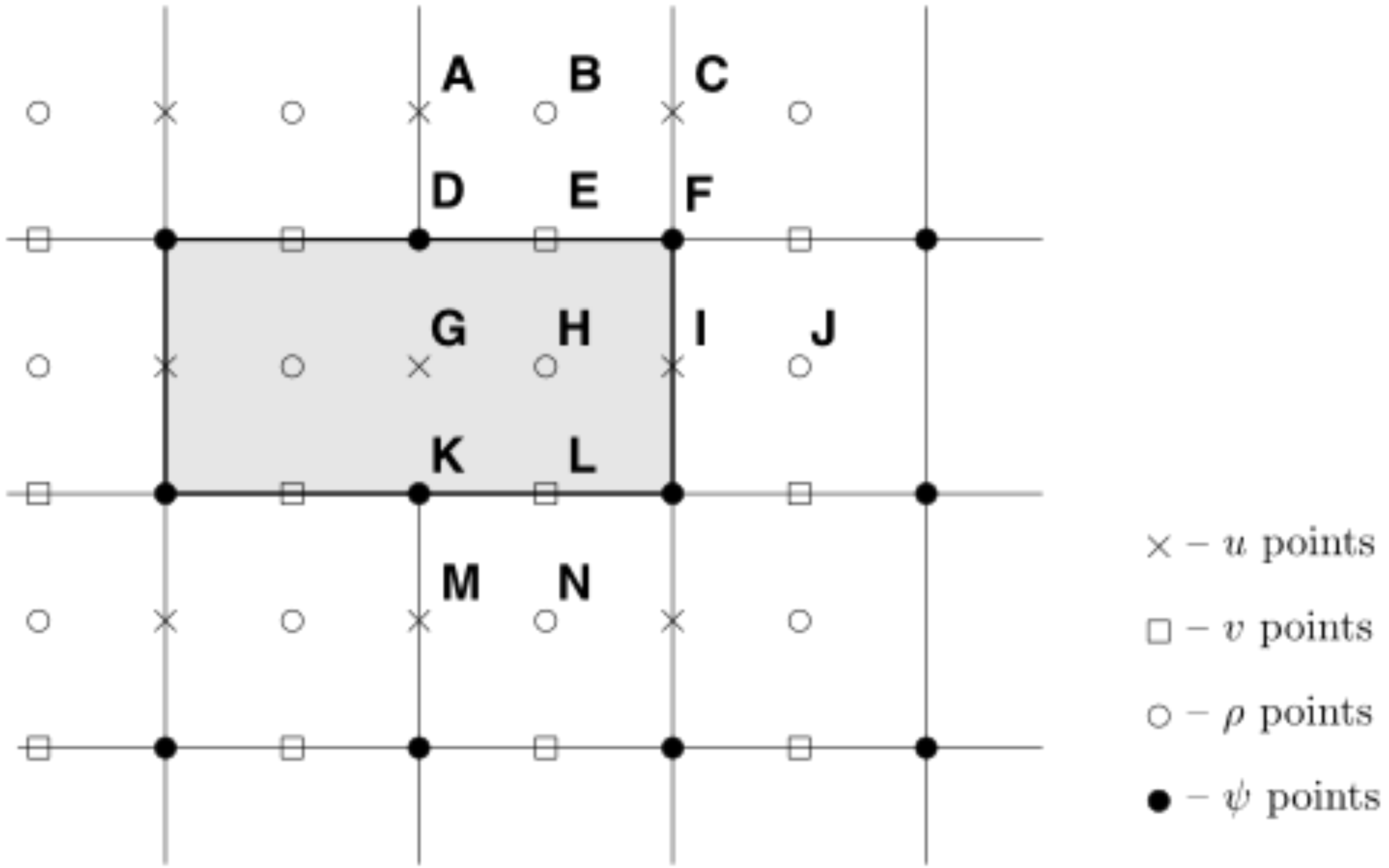
Horizontal curvilinear grid

- Another method = massive multigrain



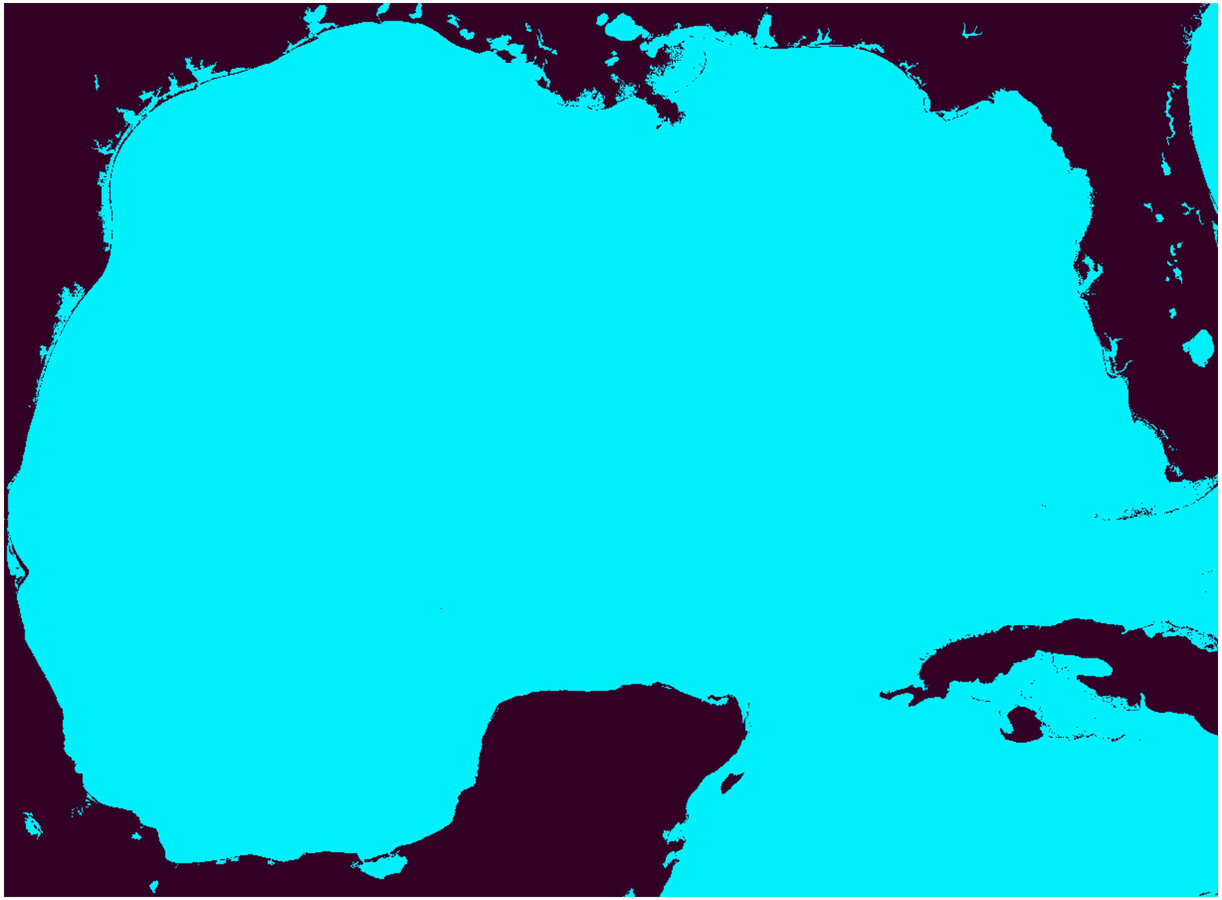
Land/sea Mask

Variables within the masked region are set to zero by multiplying by the mask for either the u, v or rho points :



Land/sea Mask

Variables within the masked region are set to zero by multiplying by the mask for either the u, v or rho points :



Land/sea Mask

See the code routines:

```
#ifdef MASKING
# define SWITCH *
#else
# define SWITCH !
#endif
```

```
!###########################
        do k=1,N
          do i=IstrU,Iend
            u(i,j,k,nnew)=(DC(i,k)-DC(i,0)) SWITCH umask(i,j)
```

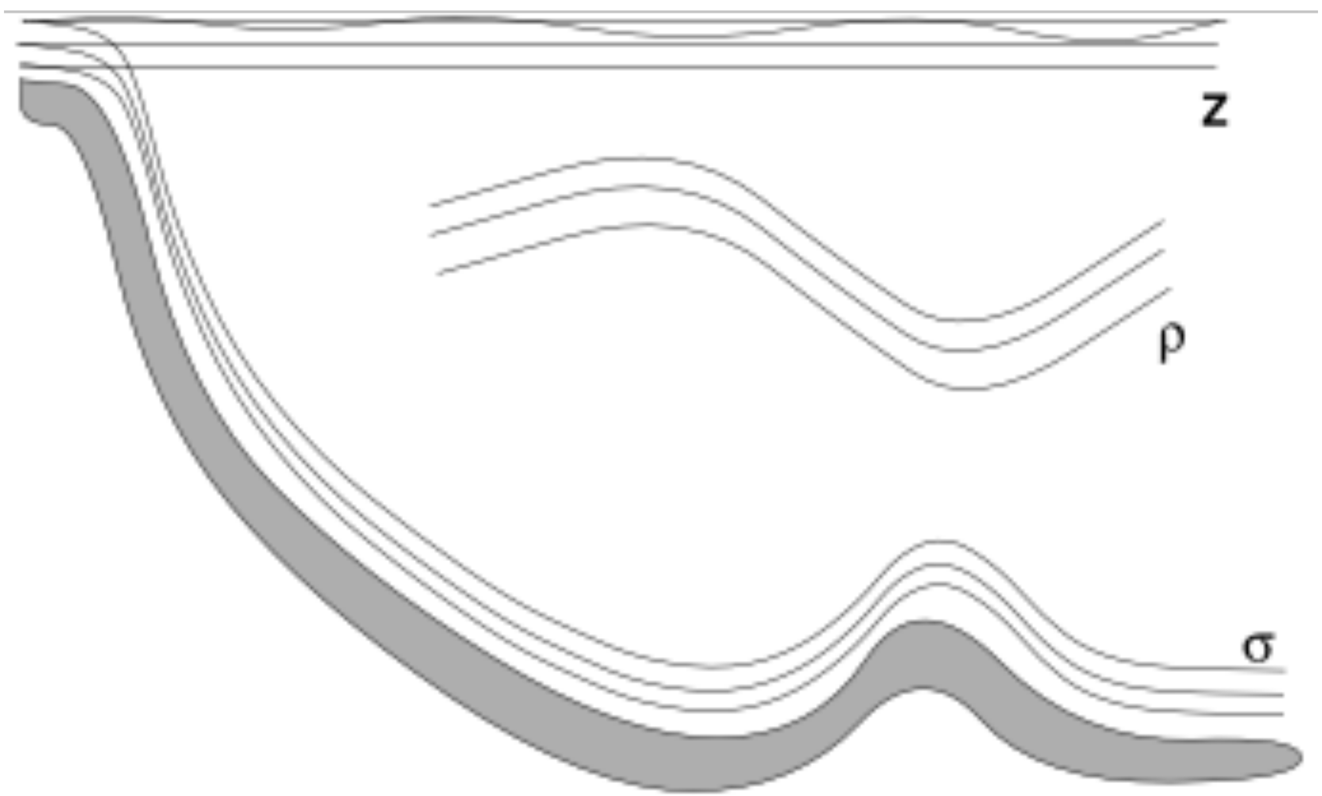
#4 Vertical Discretization

Master's degree 2nd year Marine Physics

Jonathan GULA
gula@univ-brest.fr

Vertical discretization

Depending on the depth, motions in the ocean will be mostly aligned along **geopotentials, isopycnal or topography.**



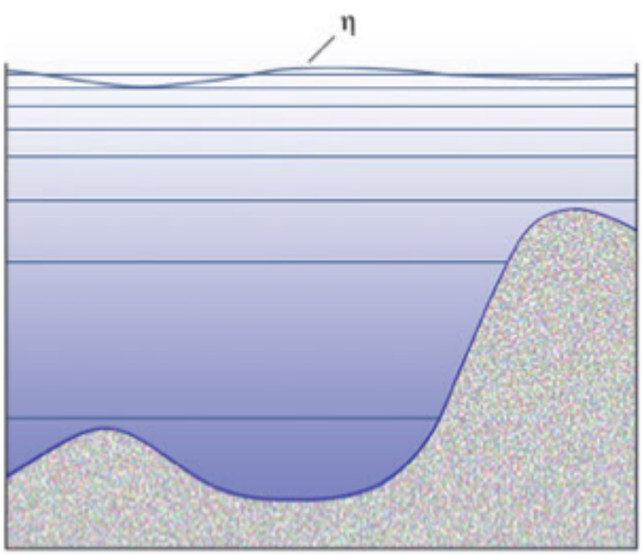
Surface mixed layer

Interior

Bottom

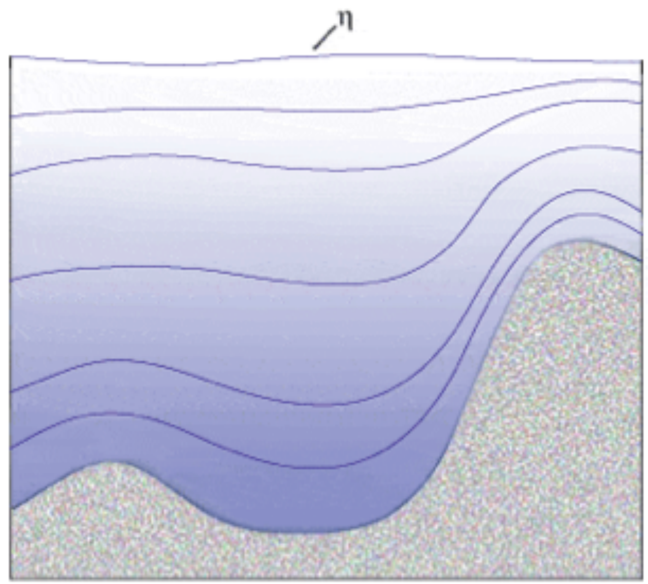
Vertical discretization

Several choices are possible to define the vertical coordinates system:



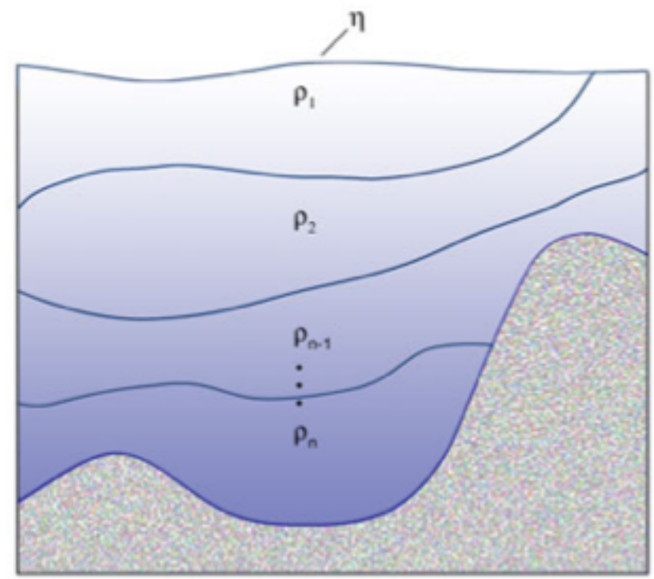
z-coordinates

Vertical coordinate is height (or depth)



sigma-coordinates

The vertical coordinate follows the bathymetry



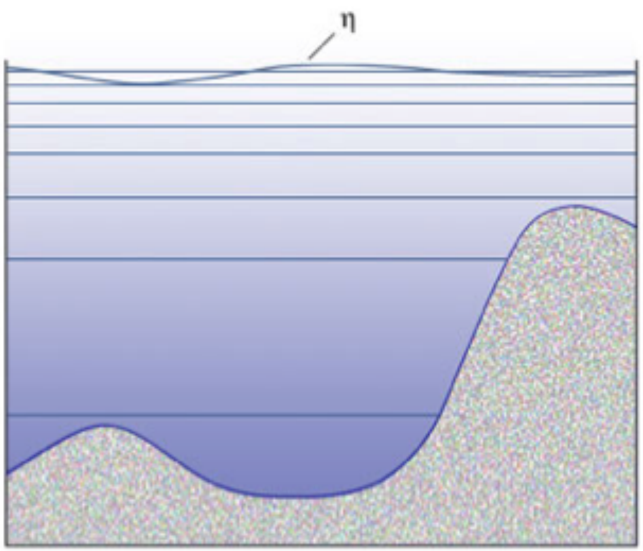
isopycnal-coordinates

The vertical coordinate is the potential density

The vertical coordinate is the major difference between models.

Vertical discretization

z-coordinates



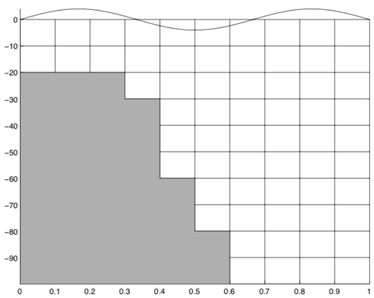
PROS

- Natural in the upper ocean and for mixed-layer processes
- Ideal to compute horizontal (pressure) gradients
- Easier to implement and use

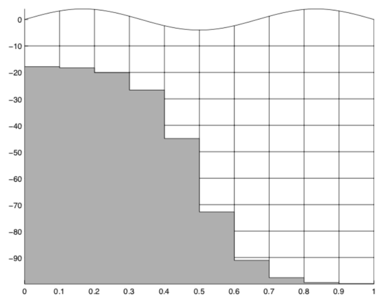
CONS

- The representation of tracer advection and diffusion along inclined density surfaces in the ocean interior is cumbersome.
- Representation of bottom topography is difficult.
- Need for bottom and lateral conditions
- Representation and parameterization of the BBL is unnatural.

a)

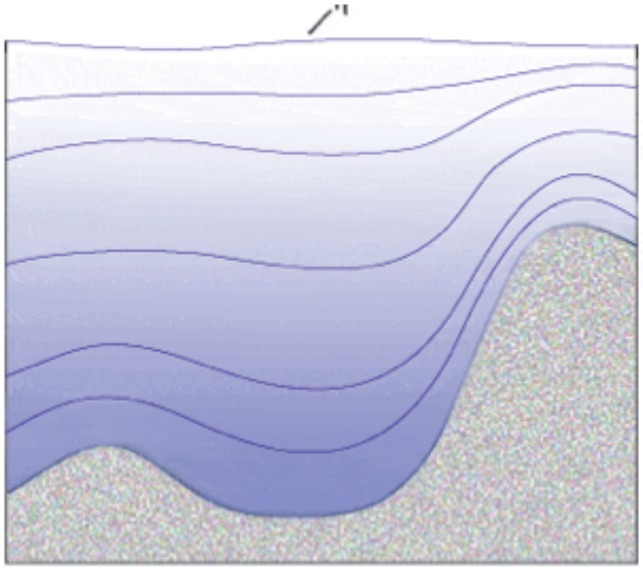


a)



Vertical discretization

sigma-coordinates



PROS

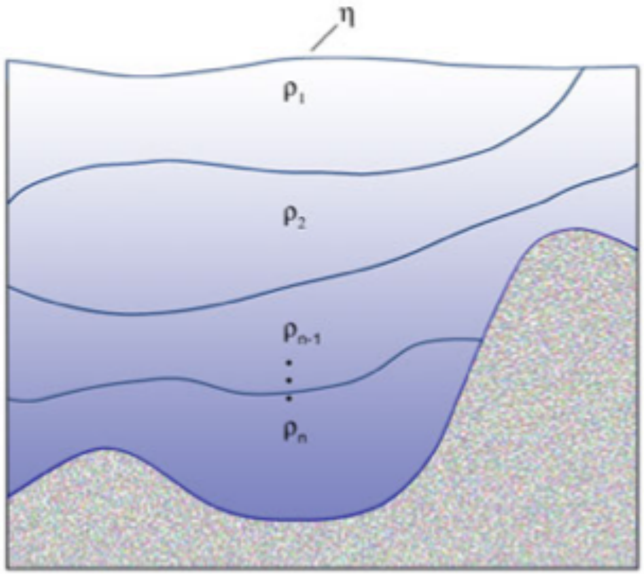
- Representation of bottom topography is natural (only bottom boundary condition)
- Representation and parameterization of the BBL is natural (more vertical res. in BBL)

CONS

- The representation of tracer advection and diffusion along inclined density surfaces in the ocean interior is cumbersome.
- Pressure gradient errors can be a problem

Vertical discretization

isopycnal-coordinates



PROS

- The representation of tracer advection and diffusion along inclined density surfaces in the ocean interior is natural.
- Water mass characteristics are preserved over long time scales

CONS

- Representing the effects of a realistic (non-linear) equation of state is cumbersome.
- Inappropriate for representing the surface mixed layer or BBL which are mostly unstratified.
- Non-hydrostatic effects/dynamics are not possible.
- Vertical and horizontal resolution are tightly connected in regions where isopycnals outcrop. This can lead to inadequate horizontal resolution in regions such as the ACC.

Vertical discretization

- Equations for a generalized coordinate system:

Consider a general vertical coordinate, r, which is assumed to be a monotonic function of height, z.

Any variable can be written in the new coordinate system: $$A = A(x, y, z(x, y, r, t), t)$$

Vertical derivatives can be written $$\frac{\partial A}{\partial r} = \frac{\partial z}{\partial r}\frac{\partial A}{\partial z} \qquad \frac{\partial A}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial A}{\partial r}$$

And other derivatives (for a horizontal coordinate s = x,y,t) can be written using the chain rule:

$$\left.\frac{\partial A}{\partial s}\right|_z = \left.\frac{\partial A}{\partial s}\right|_r - \frac{\partial A}{\partial z}\left.\frac{\partial z}{\partial s}\right|_r$$

So $$\left.\frac{\partial A}{\partial s}\right|_z = \left.\frac{\partial A}{\partial s}\right|_r - \frac{\partial r}{\partial z}\frac{\partial A}{\partial r}\left.\frac{\partial z}{\partial s}\right|_r$$

Vertical discretization

- Equations for a generalized coordinate system:

Using the chain rule, you can write the horizontal gradient: $\nabla_z A = \nabla_r A - \dfrac{\partial A}{\partial r}\dfrac{\partial r}{\partial z}\nabla_r z$

And the vertical velocity can be written:

$$\begin{aligned} w = D_t z &= \left.\frac{\partial z}{\partial t}\right|_r + \left.\frac{\partial z}{\partial x}\right|_r D_t x + \left.\frac{\partial z}{\partial y}\right|_r D_t y + \frac{\partial z}{\partial r} D_t r \\ &= \left.\frac{\partial z}{\partial t}\right|_r + \vec{v} \cdot \nabla_r z + \dot{r}\frac{\partial z}{\partial r} \end{aligned}$$

Vertical discretization

- Equations for a generalized coordinate system:

And the total derivative becomes:

$$\begin{aligned}
D_t A &= \left.\frac{\partial A}{\partial t}\right|_z + \vec{v}\cdot\nabla_z A + w\frac{\partial A}{\partial z} \\
&= \left.\frac{\partial A}{\partial t}\right|_r + \vec{v}\cdot\nabla_r A + \left(w - \left.\frac{\partial z}{\partial t}\right|_r - \vec{v}\cdot\nabla_r z\right)\frac{\partial r}{\partial z}\frac{\partial A}{\partial r} \\
&= \left.\frac{\partial A}{\partial t}\right|_r + \vec{v}\cdot\nabla_r A + \dot{r}\frac{\partial A}{\partial r}
\end{aligned}$$

And the horizontal pressure gradient:

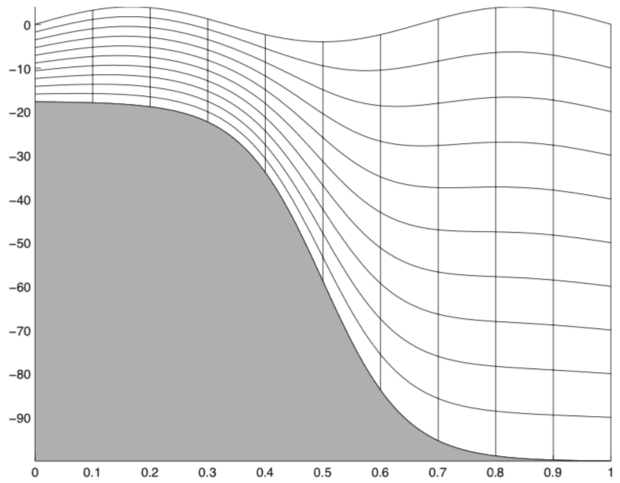
$$\begin{aligned}
\nabla_z p &= \nabla_r p - \frac{\partial p}{\partial z}\nabla_r z \\
&= \nabla_r p + \rho\nabla_r g z
\end{aligned}$$

Vertical grid : σ generalized coordinate

- Sigma-coordinate system:

Sigma-coordinates are terrain-following coordinates defined as:

$$\sigma = \frac{z}{H_z(x,y)}$$



$$\frac{\partial \sigma}{\partial z} = \frac{1}{H_z}$$

Vertical grid : σ generalized coordinate

- Horizontal and vertical derivatives can be written:

$$\left(\frac{\partial}{\partial x}\right)_z = \left(\frac{\partial}{\partial x}\right)_\sigma - \left(\frac{1}{H_z}\right)\left(\frac{\partial z}{\partial x}\right)_\sigma \frac{\partial}{\partial \sigma}$$

$$\left(\frac{\partial}{\partial y}\right)_z = \left(\frac{\partial}{\partial y}\right)_\sigma - \left(\frac{1}{H_z}\right)\left(\frac{\partial z}{\partial y}\right)_\sigma \frac{\partial}{\partial \sigma}$$

$$\frac{\partial}{\partial z} = \left(\frac{\partial s}{\partial z}\right)\frac{\partial}{\partial \sigma} = \frac{1}{H_z}\frac{\partial}{\partial \sigma}$$

Vertical grid : σ generalized coordinate

- Example: (Hydrostatic) Potential vorticity (PV) is defined as

$$q = \left(f\vec{k} + \vec{\nabla} \times \vec{u} \right) .\vec{\nabla} b$$

$$q = \left[f + \left.\frac{\partial v}{\partial x}\right|_z - \left.\frac{\partial u}{\partial y}\right|_z \right] \frac{\partial \rho}{\partial z} - \frac{\partial v}{\partial z} \left.\frac{\partial \rho}{\partial x}\right|_z + \frac{\partial u}{\partial z} \left.\frac{\partial \rho}{\partial y}\right|_z$$

- Question: How do you compute PV in sigma-coordinates?
  - *Write the expression of q using only derivatives along sigma coordinates:*

$$\left.\frac{\partial}{\partial x}\right|_\sigma \qquad \left.\frac{\partial}{\partial y}\right|_\sigma \qquad \frac{\partial}{\partial \sigma}$$

Vertical grid : σ generalized coordinate

- Example: (Hydrostatic) Potential vorticity (PV) is defined as

$$q = \left[f + \left.\frac{\partial v}{\partial x}\right|_\sigma - \left.\frac{\partial u}{\partial y}\right|_\sigma \right] \frac{\partial \rho}{\partial z} - \frac{\partial v}{\partial z} \left.\frac{\partial \rho}{\partial x}\right|_\sigma + \frac{\partial u}{\partial z} \left.\frac{\partial \rho}{\partial y}\right|_\sigma$$

- Even if:

$$\left.\frac{\partial v}{\partial x}\right|_\sigma - \left.\frac{\partial u}{\partial y}\right|_\sigma \neq \left.\frac{\partial v}{\partial x}\right|_z - \left.\frac{\partial u}{\partial y}\right|_z$$

Vertical grid : isopycnal coordinates

- For isopycnal coordinates, the vertical coordinate is: $r = \rho$

- Potential vorticity (PV) is easily defined as :

$$q = \frac{1}{h}\left[f + \left.\frac{\partial v}{\partial x}\right|_{\rho} - \left.\frac{\partial u}{\partial y}\right|_{\rho}\right] \qquad \text{with} \qquad h = \frac{\partial z}{\partial \rho}$$

Vertical grid : σ generalized coordinate

ROMS: Generalized σ-Coordinate

Stretching & condensing of vertical resolution:

$$z(x,y,\sigma,t) = \zeta(x,y,t) + [\zeta(x,y,t) + h(x,y)]\, S(x,y,\sigma),$$

$$S(x,y,\sigma) = \frac{h_c\, \sigma + h(x,y)\, C(\sigma)}{h_c + h(x,y)}$$

$$S(x,y,\sigma) = \begin{cases} 0, & \text{if } \sigma = 0, \quad C(\sigma) = 0, \quad \text{at the free-surface;} \\ -1, & \text{if } \sigma = -1, \quad C(\sigma) = -1, \quad \text{at the ocean bottom.} \end{cases}$$

Surface refinement function:

$$C(\sigma) = \frac{1 - \cosh(\theta_S\, \sigma)}{\cosh(\theta_S) - 1}, \qquad \text{for } \theta_S > 0, \qquad C(\sigma) = -\sigma^2, \qquad \text{for } \theta_S \leq 0$$

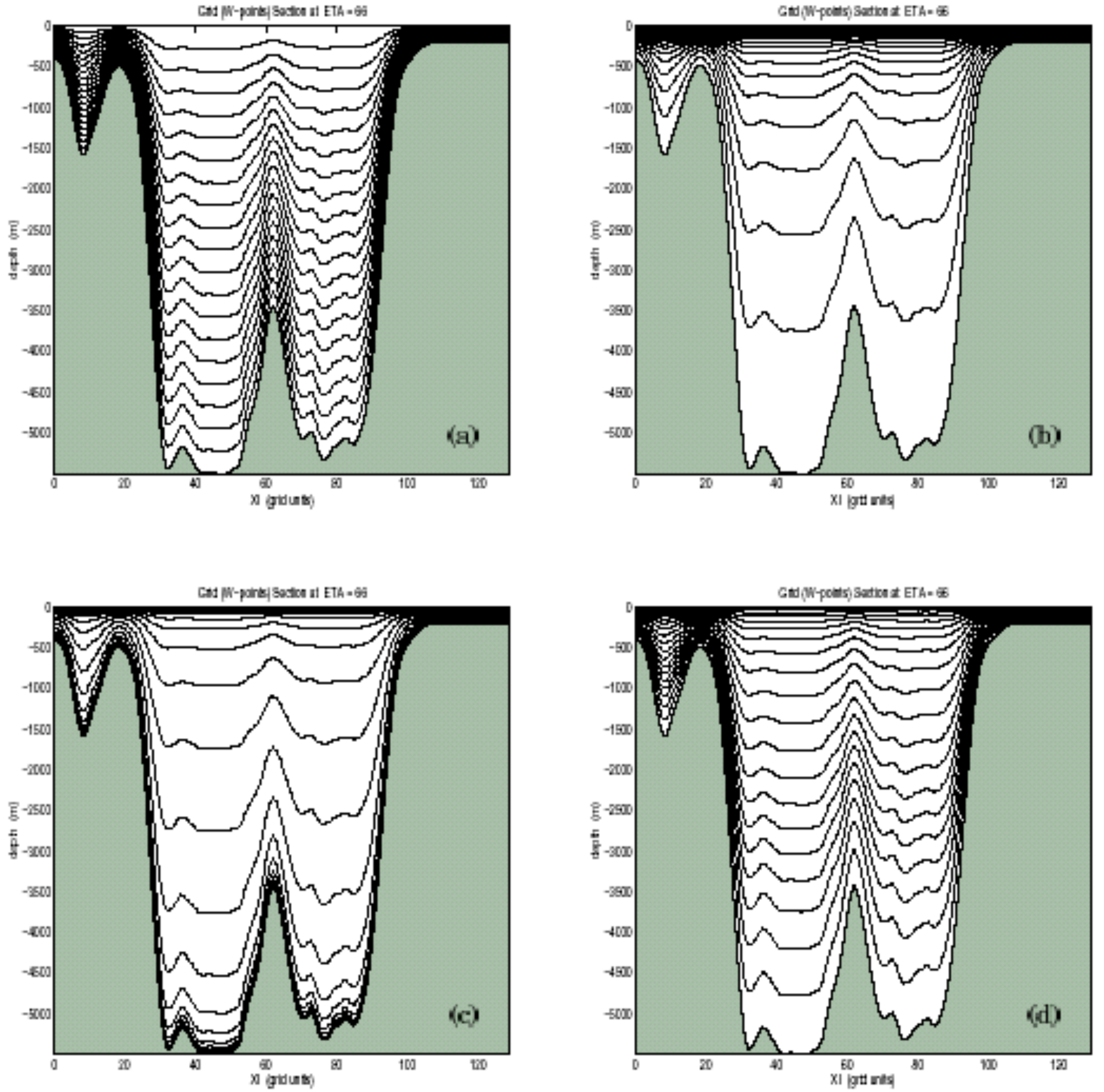
Bottom refinement function:

$$C(\sigma) = \frac{\exp(\theta_B\, C(\sigma)) - 1}{1 - \exp(-\theta_B)}, \qquad \text{for } \theta_B > 0$$

Vertical grid : σ generalized coordinate

ROMS: Generalized σ-Coordinate

Stretching & condensing of vertical resolution



θ and b: suface and bottom parameters:

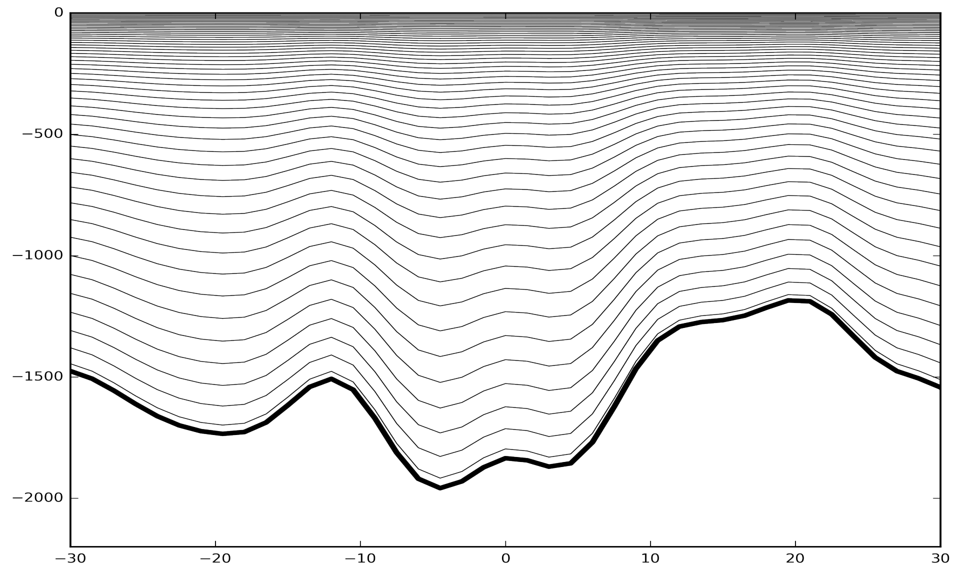
(a) θ=0, b=0
(b) θ=8, b=0
(c) θ=8, b=1
(d) θ=5, b=0.4

- !! Take care of the Pressure Gradient Error on steep slopes !!

Vertical grid : σ generalized coordinate

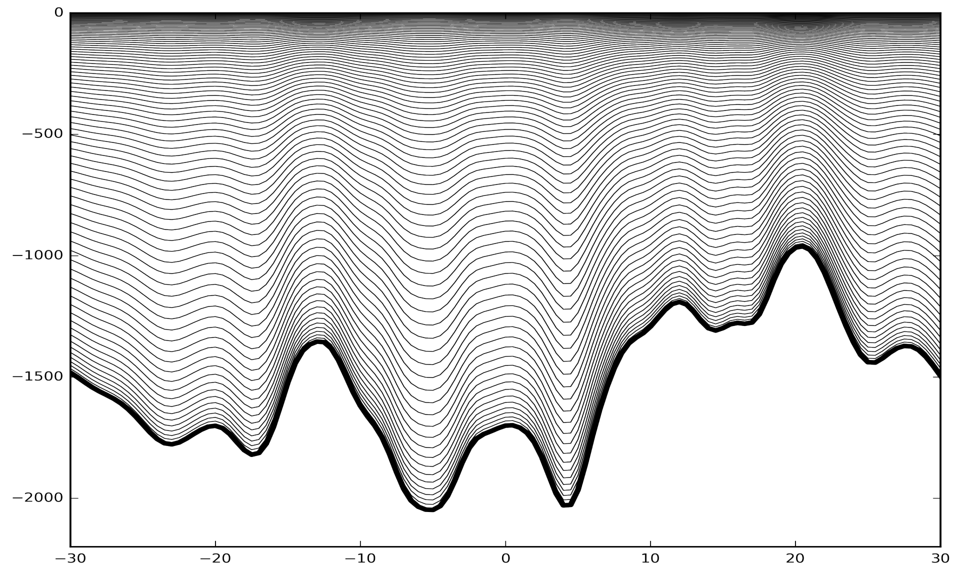
- 50 vertical levels

  $\theta=7$, $b=2$, $h_c=300$ m



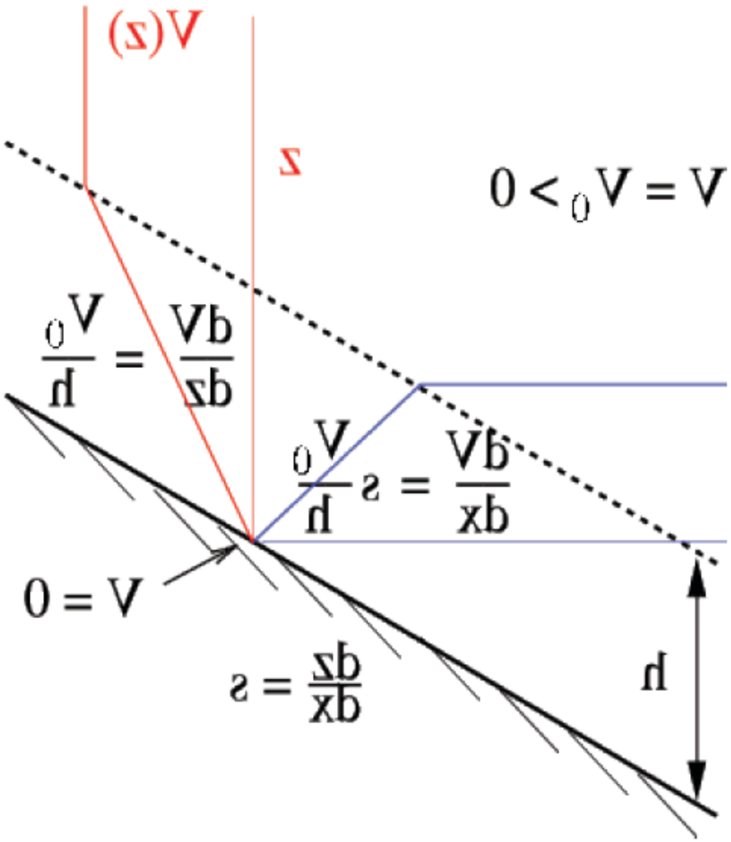
- 80 vertical levels

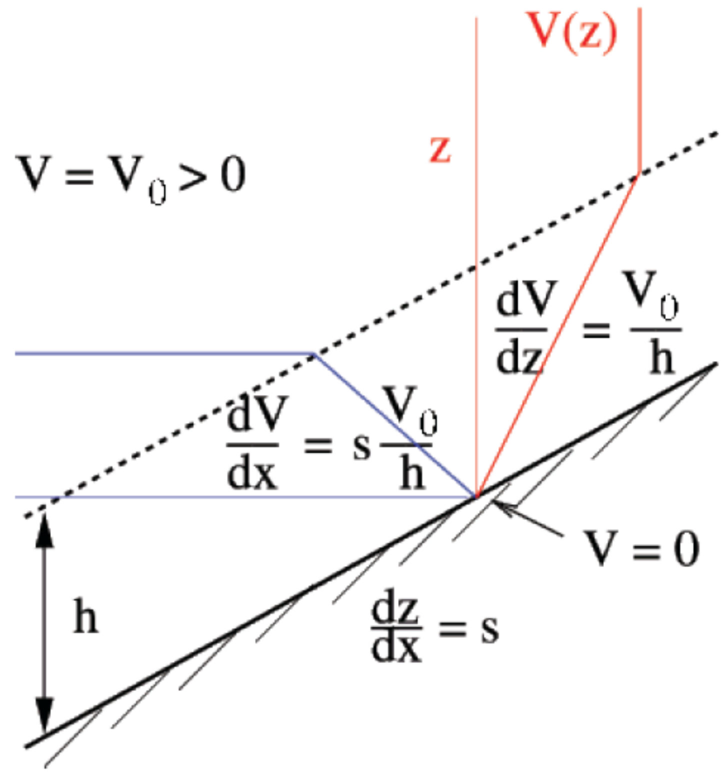
  $\theta=6$, $b=4$, $h_c=300$ m



Ex: Topographic vorticity generation

Generation of vertical vorticity within the bottom boundary layer:



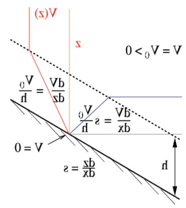


Current flowing with the coast on its left in the Northern hemisphere (opposite to Kelvin wave propagation)
= Positive vorticity generation

Current flowing with the coast on its right in the Northern hemisphere (same than Kelvin wave propagation)
= Negative vorticity generation

Ex: Topographic vorticity generation

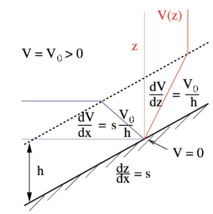
Positive vorticity generation :



- Horizontal shear instability
- Formation of submesoscale cyclones

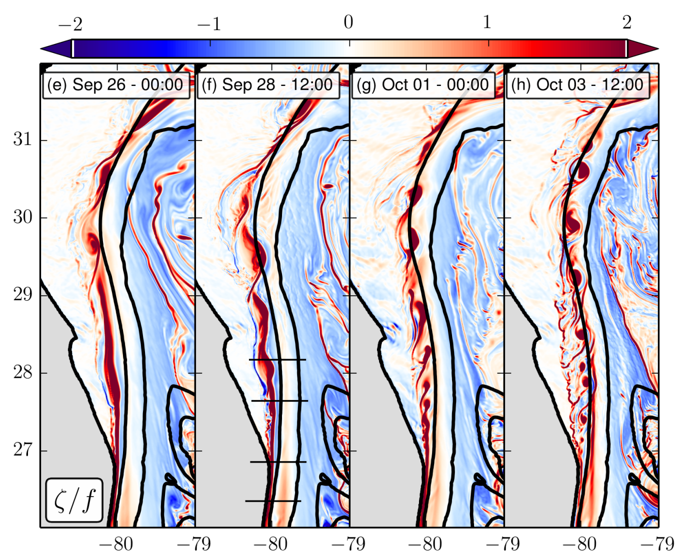
e.g.: Gulf Stream along the slope

Negative vorticity generation:

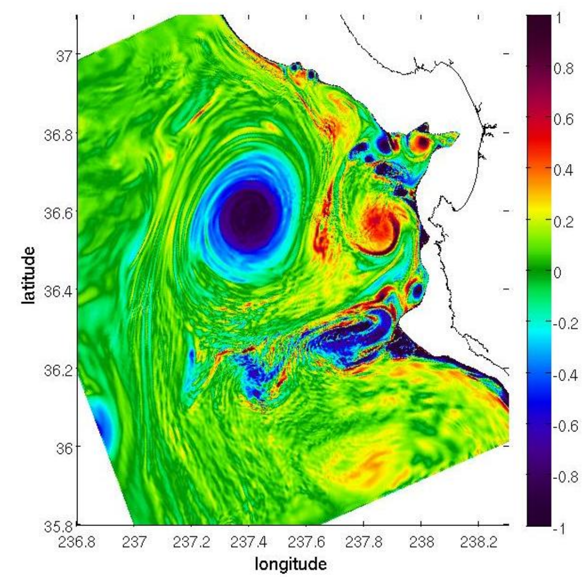


- Centrifugal instability
- Small-scale turbulence, Mixing and dissipation
- Formation of submesoscale anticyclones

e.g. California Undercurrent (formation of Cuddies)

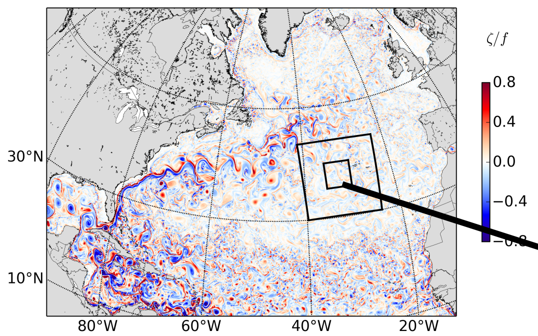


[Gula et al., GRL, 2015]



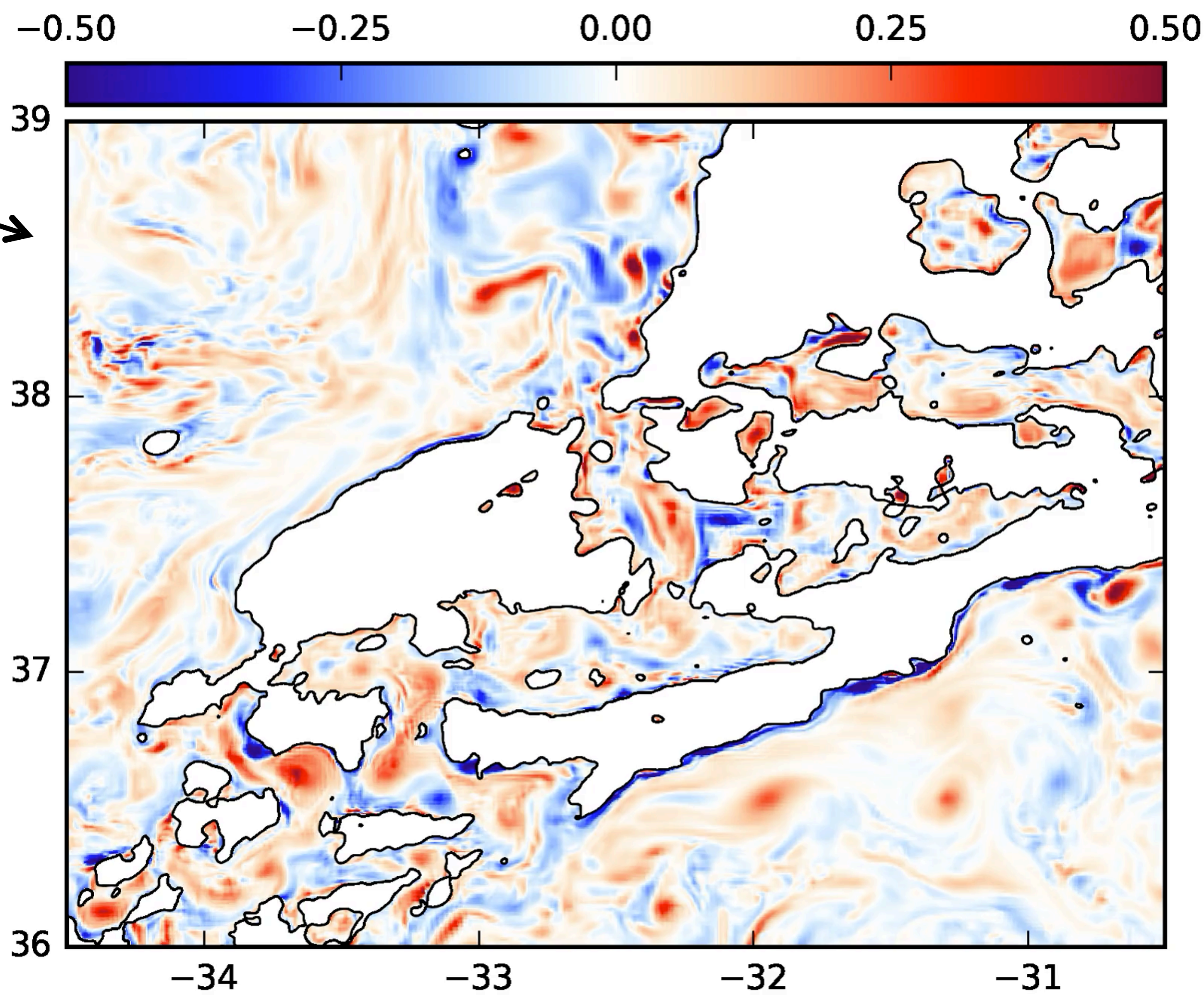
[Molemaker et al., JPO, 2015]

Ex: Topographic vorticity generation



$$\Delta x = 500\,\mathrm{m}$$

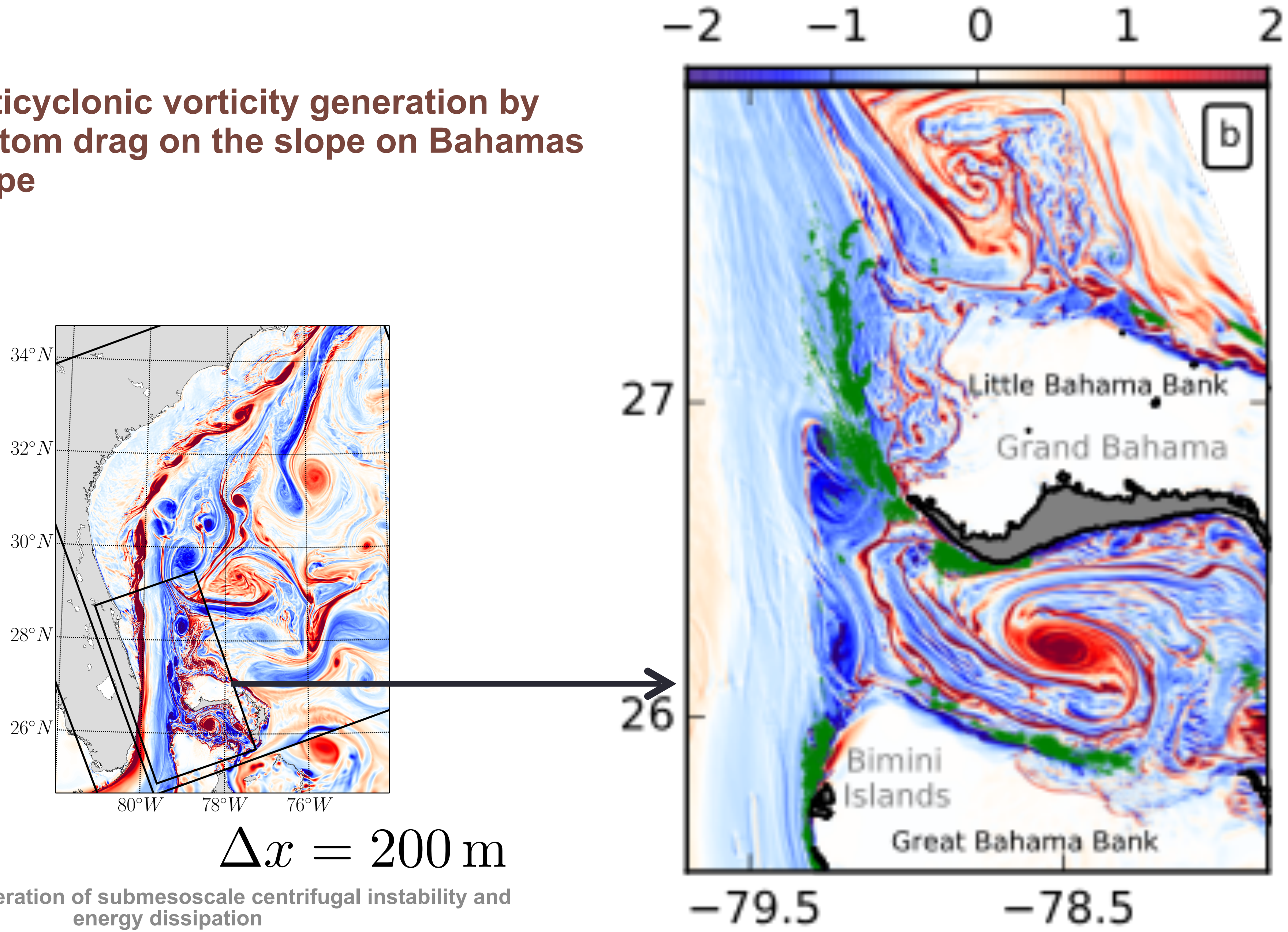
relative vorticity at -1800 m



- **Dispersion of hydrothermal effluents** and larvae from the Lucky Strike vent field
- High levels of **dissipation and mixing** [St Laurent & Thunrherr, Nat., 2006]

Ex: Topographic vorticity generation

Anticyclonic vorticity generation by bottom drag on the slope on Bahamas slope



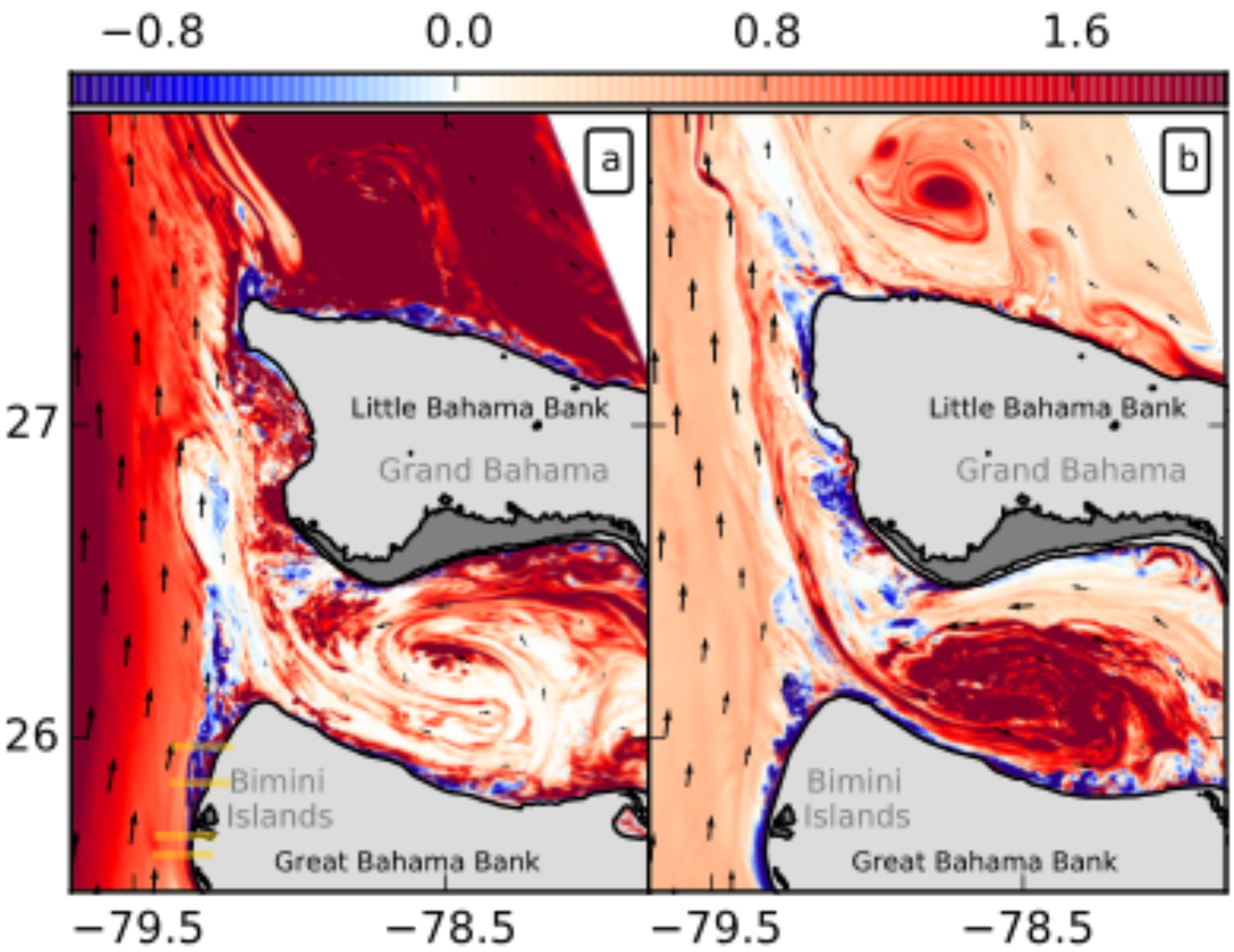
$$\Delta x = 200\,\mathrm{m}$$

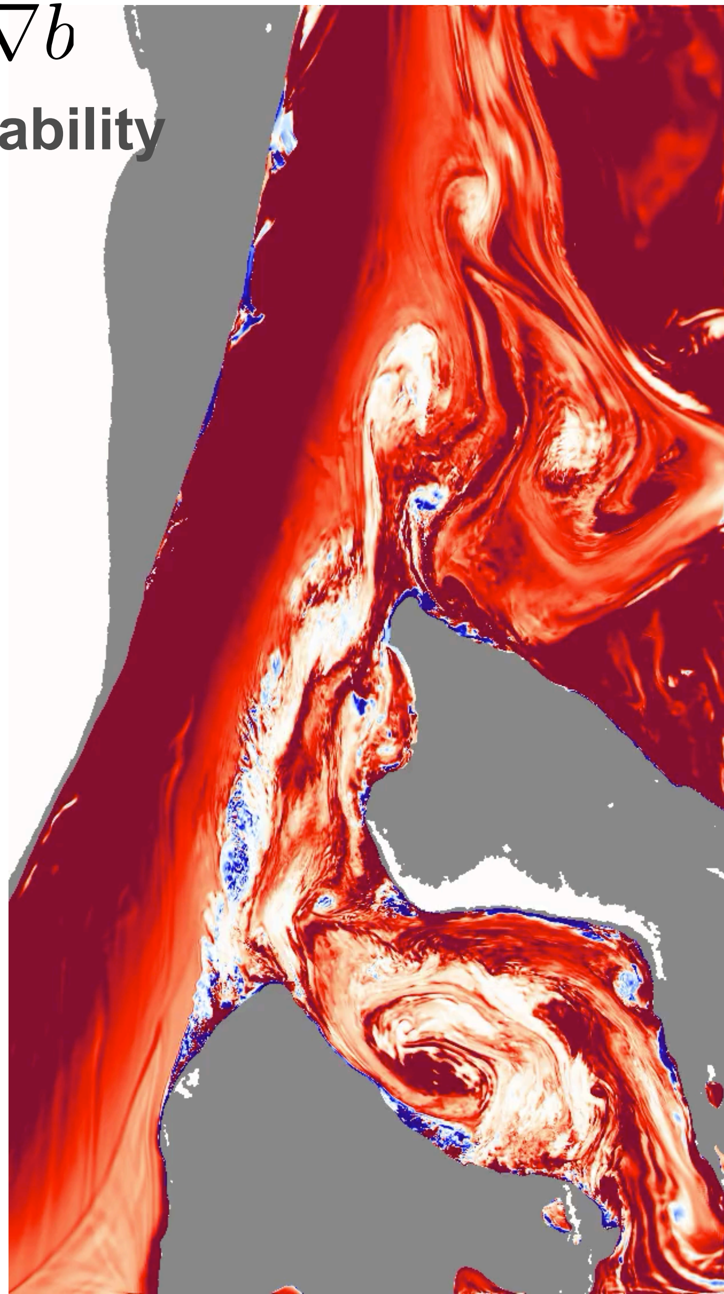
Topographic generation of submesoscale centrifugal instability and energy dissipation

Ex: Topographic vorticity generation

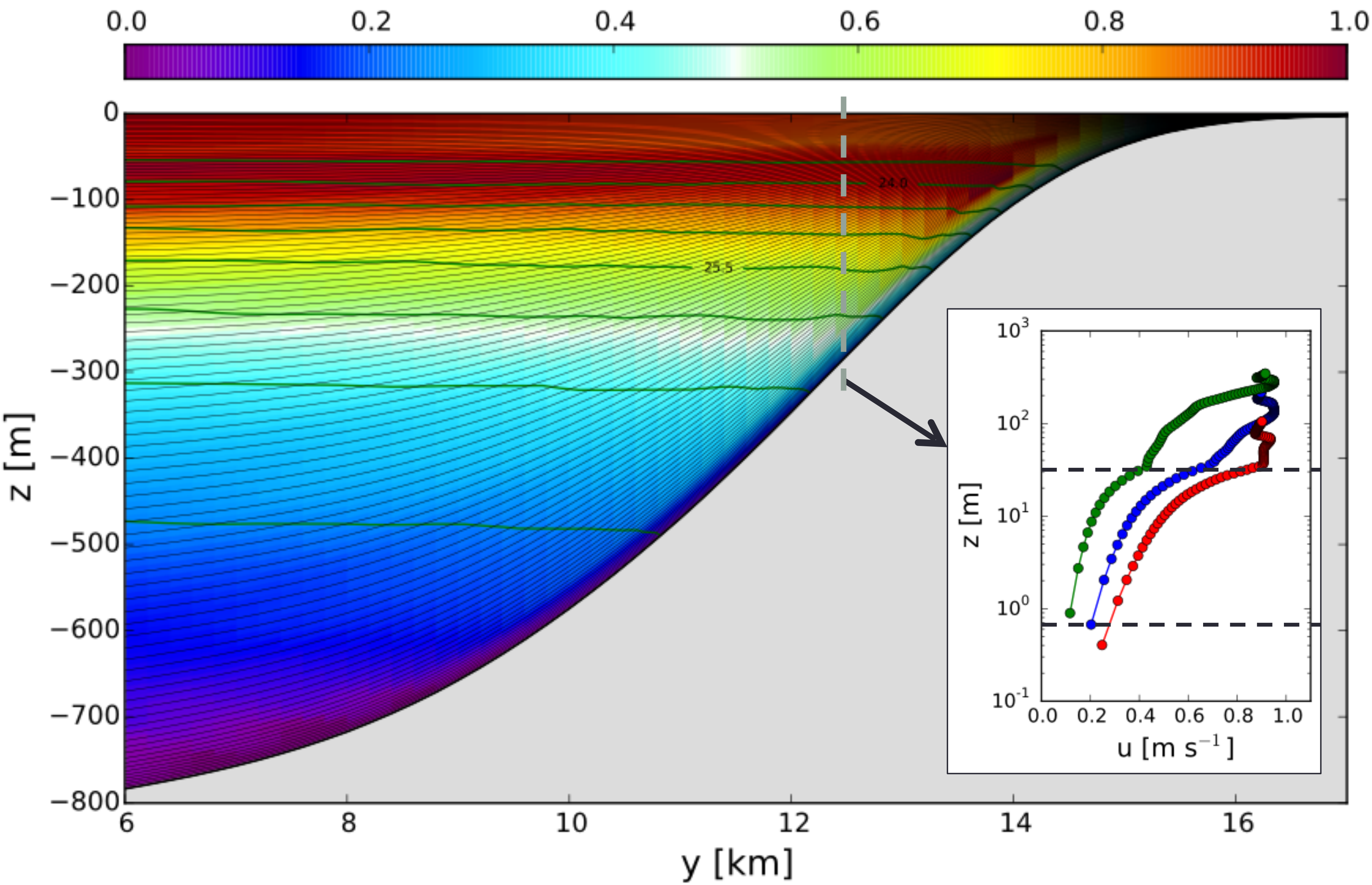
$$q = (f\vec{z} + \vec{\nabla} \times \vec{u}) \cdot \vec{\nabla} b$$

Negative PV (f q<0) = centrifugal (inertial) instability

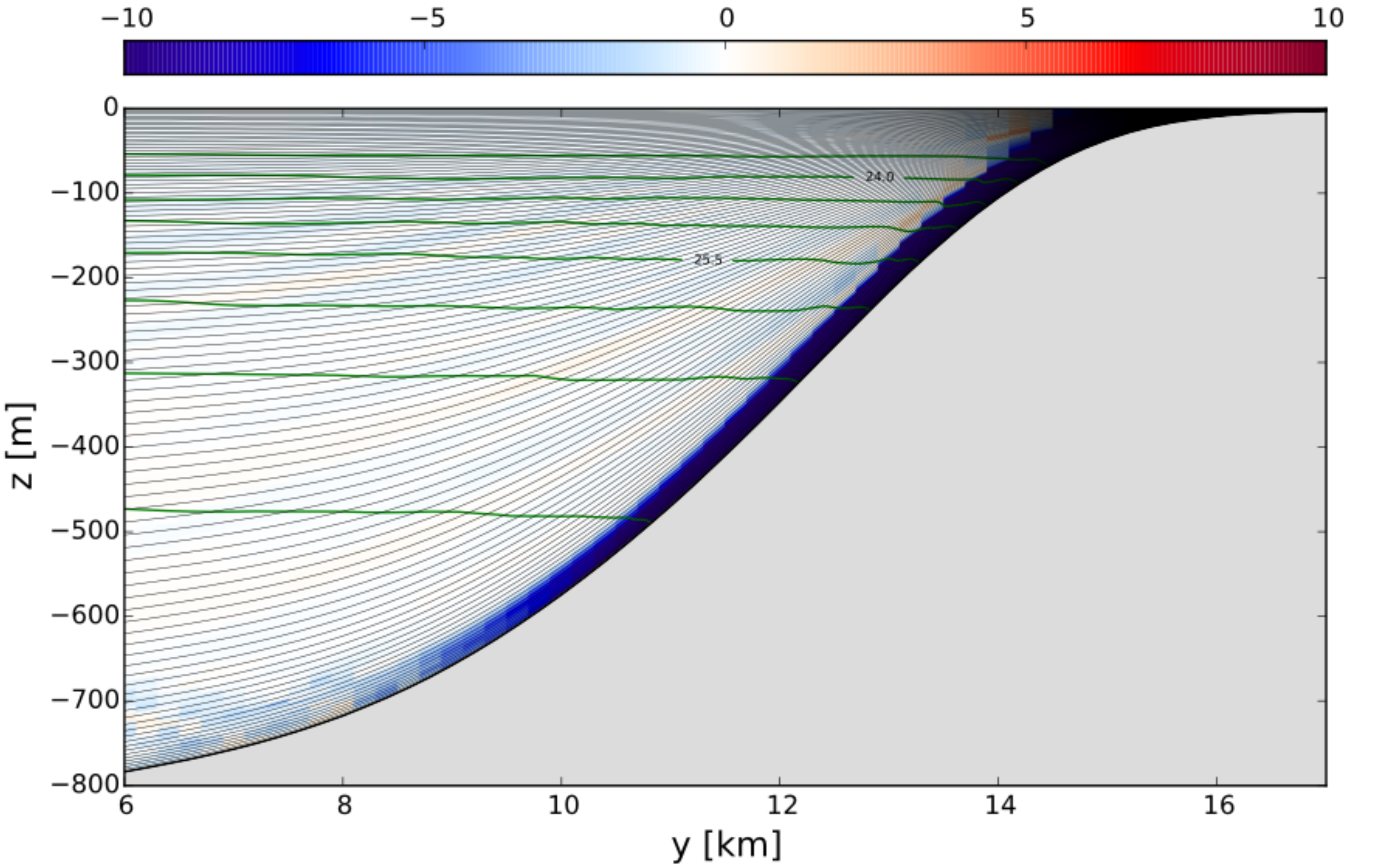




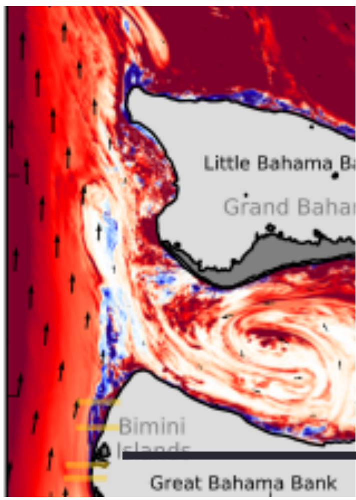
Ex: Topographic vorticity generation

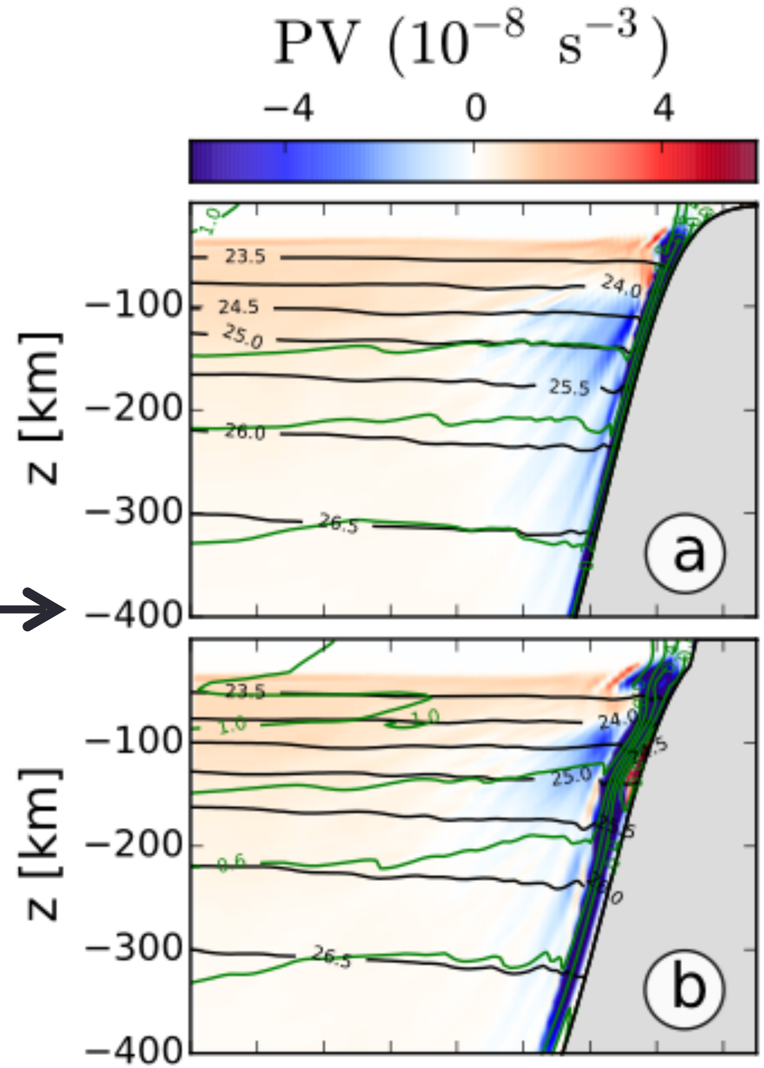


Ex: Topographic vorticity generation

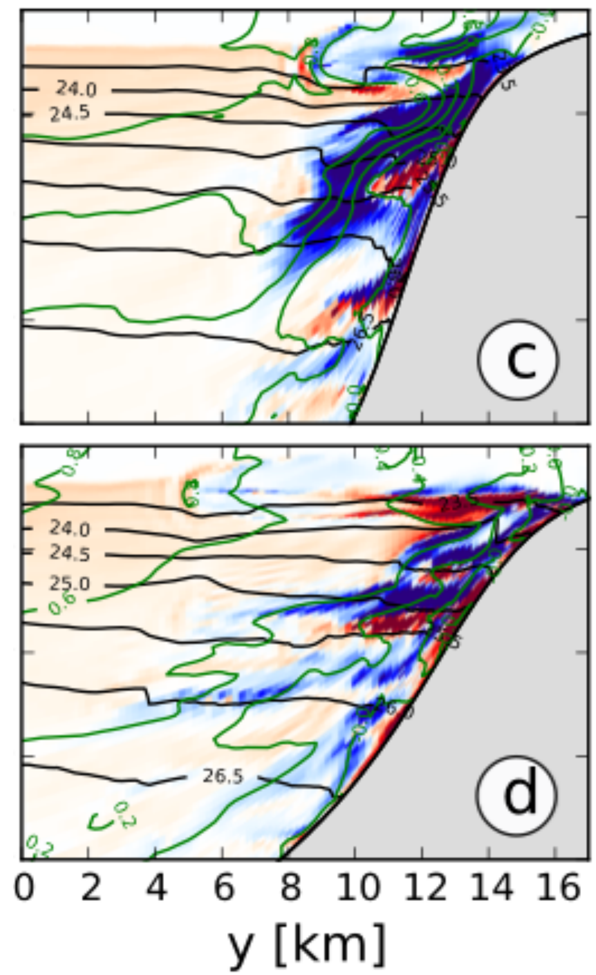


Ex: Topographic vorticity generation

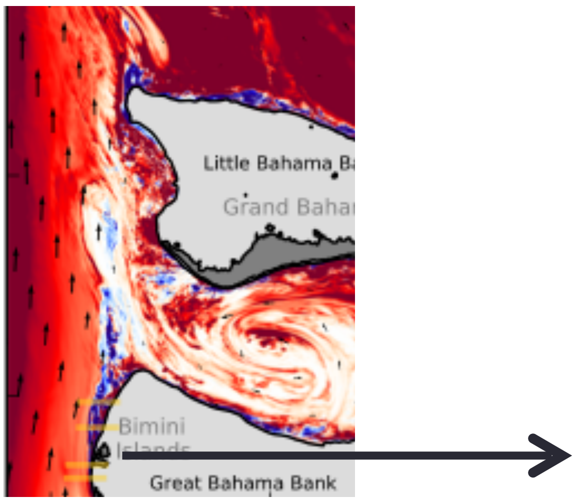


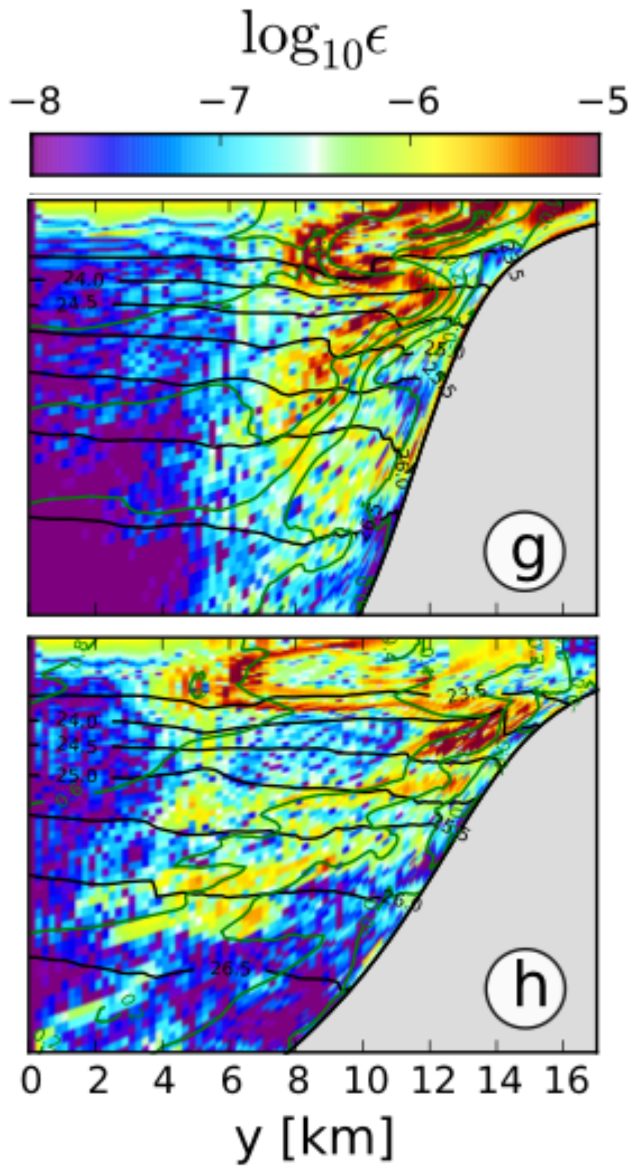


Centrifugal Instability =
Intense mixing and dissipation

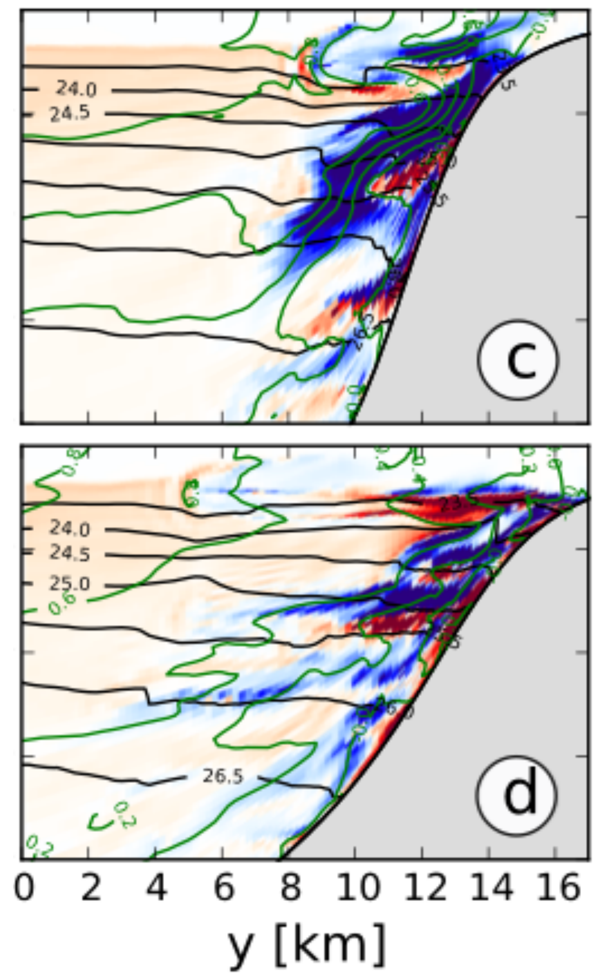


Ex: Topographic vorticity generation





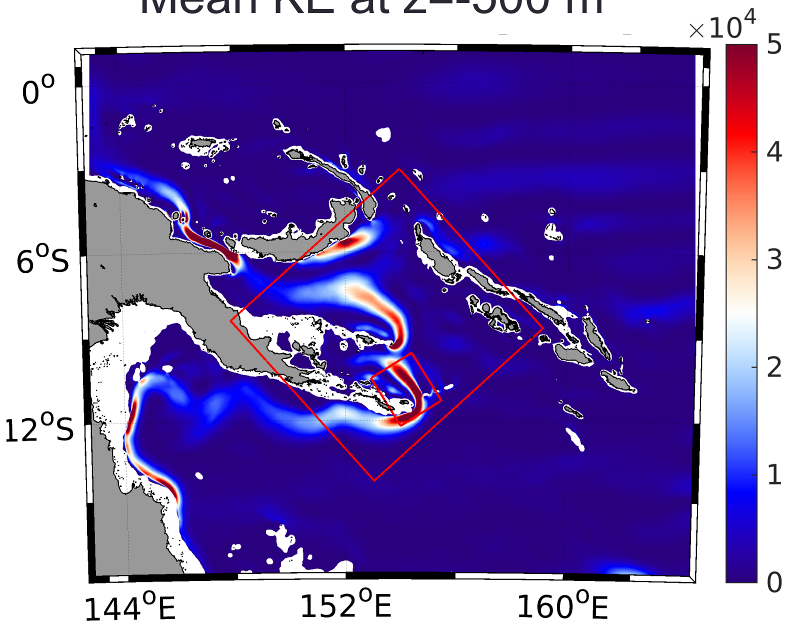
Centrifugal Instability =
Intense mixing and dissipation



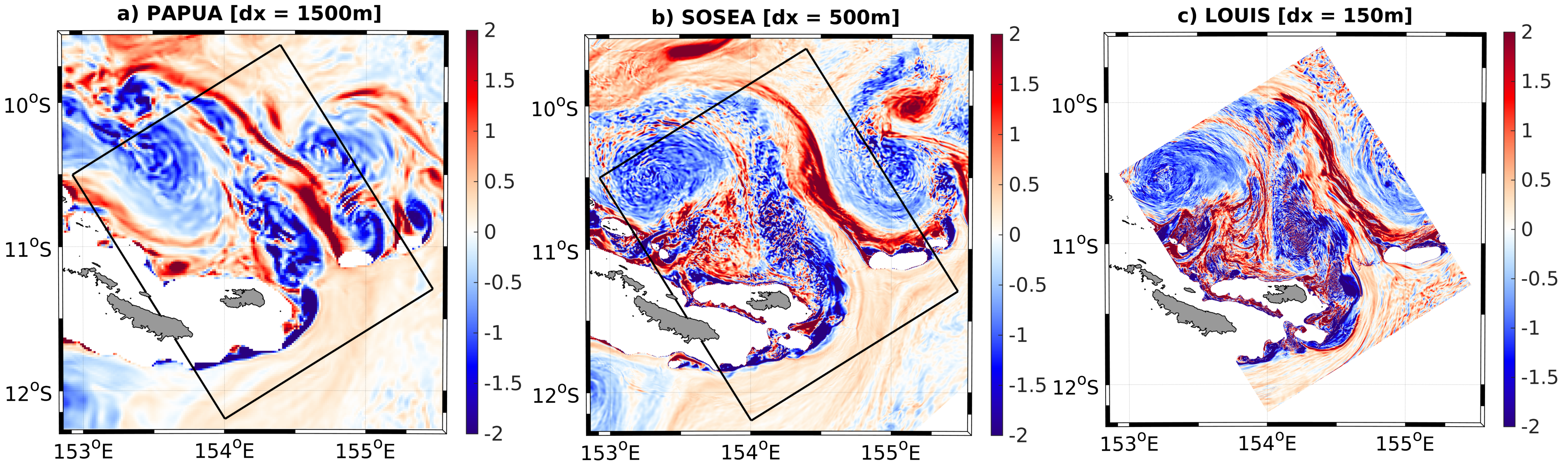
Ex: Topographic vorticity generation

Gulf of Papua Undercurrent

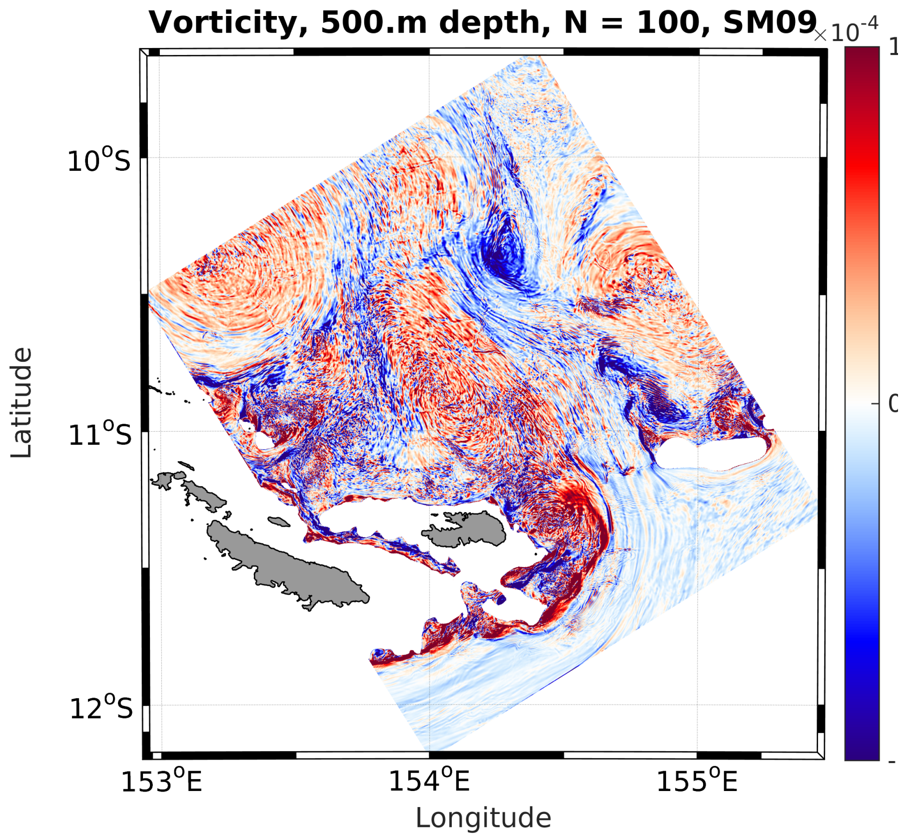
Mean KE at z=-500 m

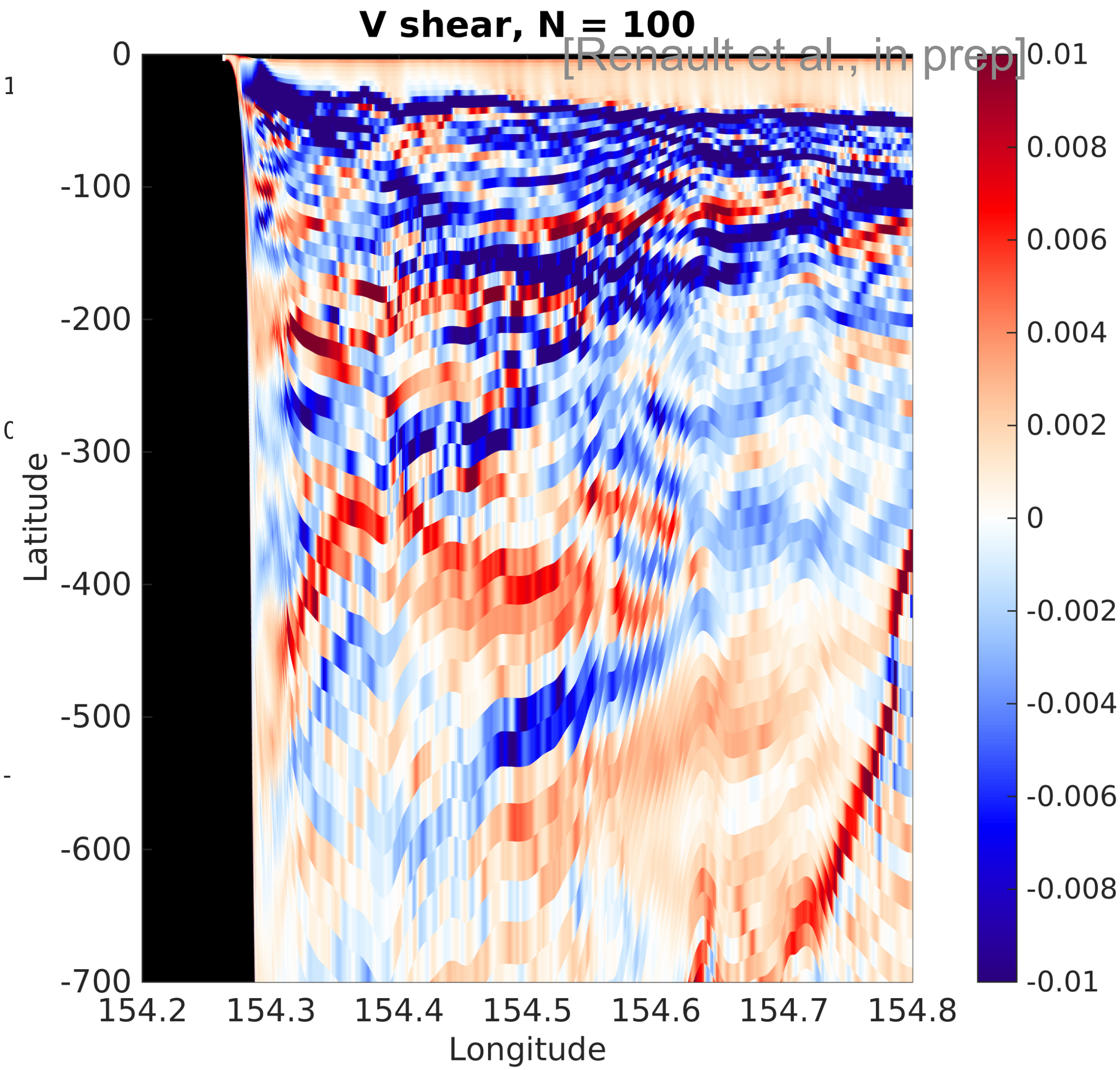


Relative vorticity at z=-500 m



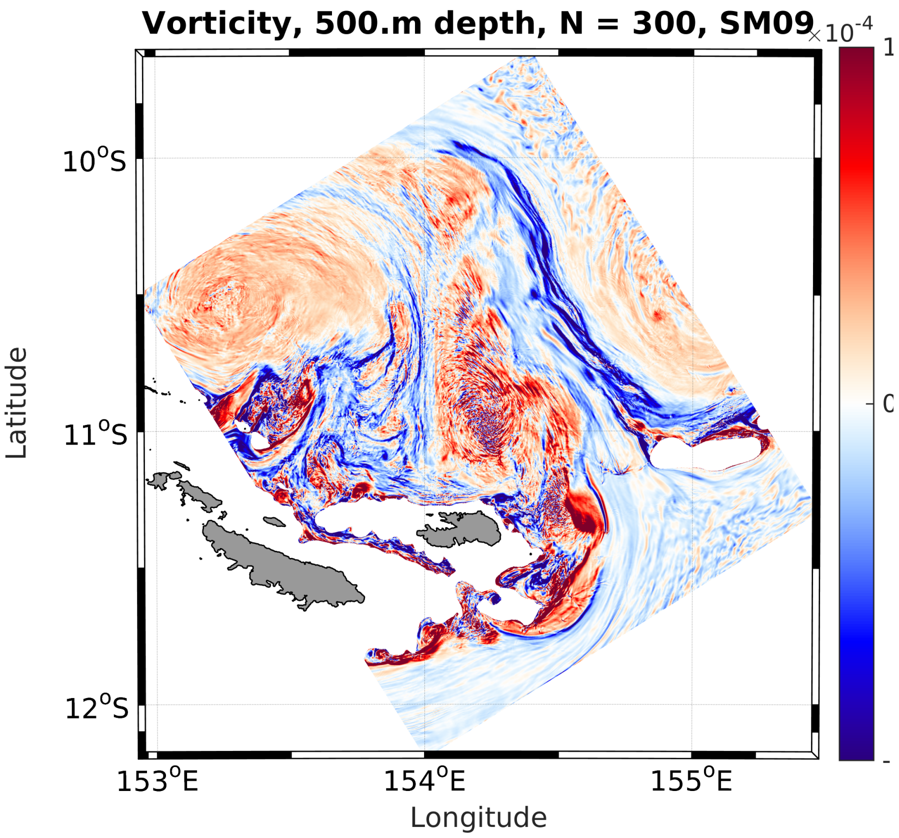
Ex: Need in Vertical resolution?

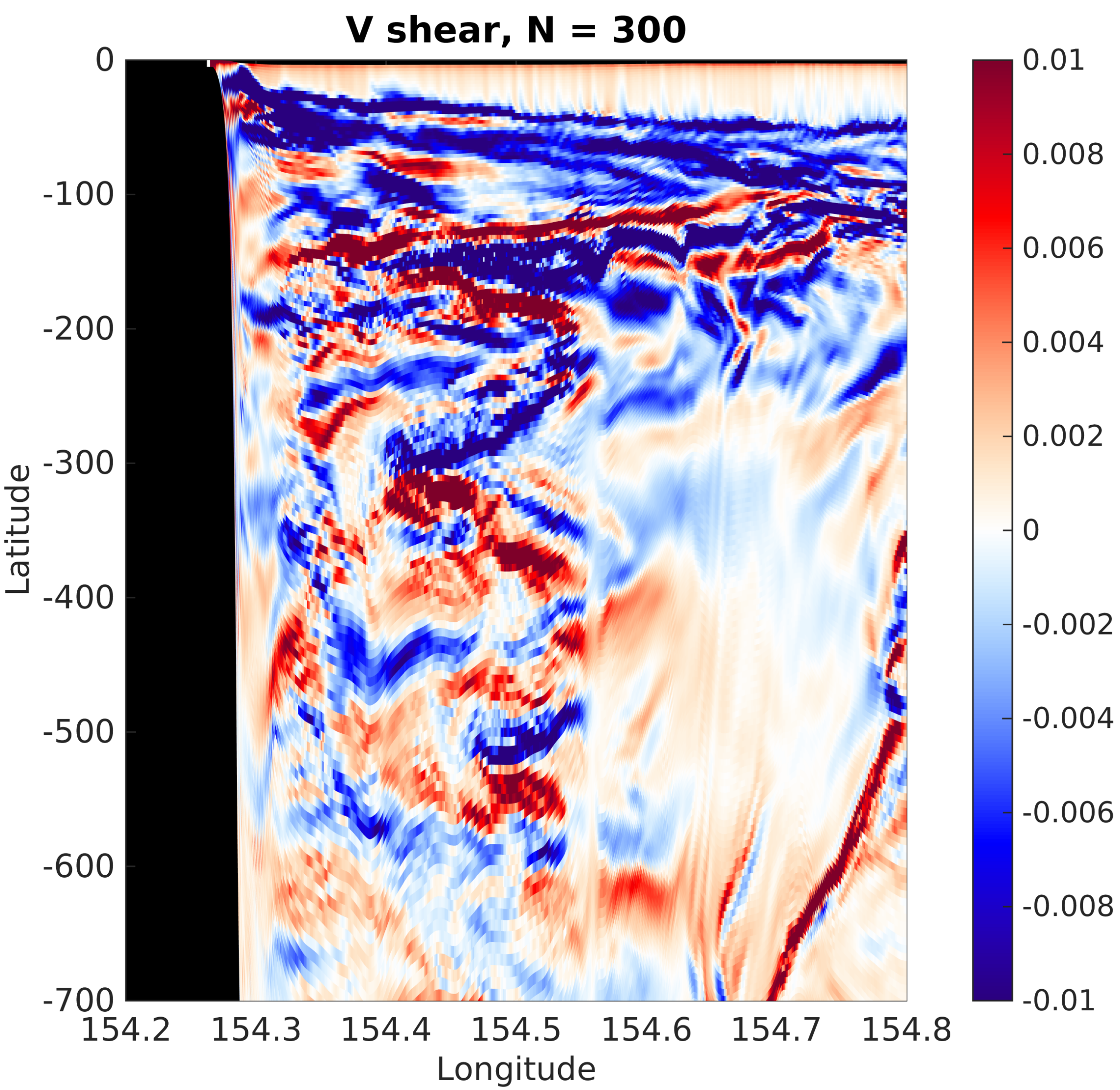




Ex: What happens when underresolving vertical scales with HR simulations (Solomon Sea)

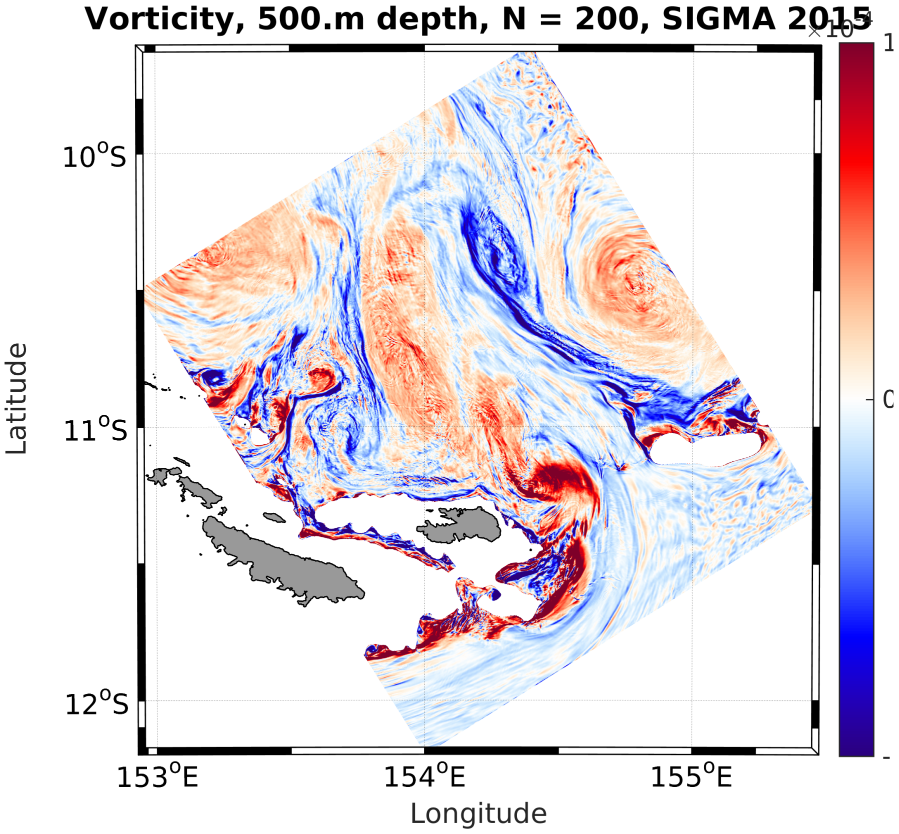
Ex: Need in Vertical resolution?

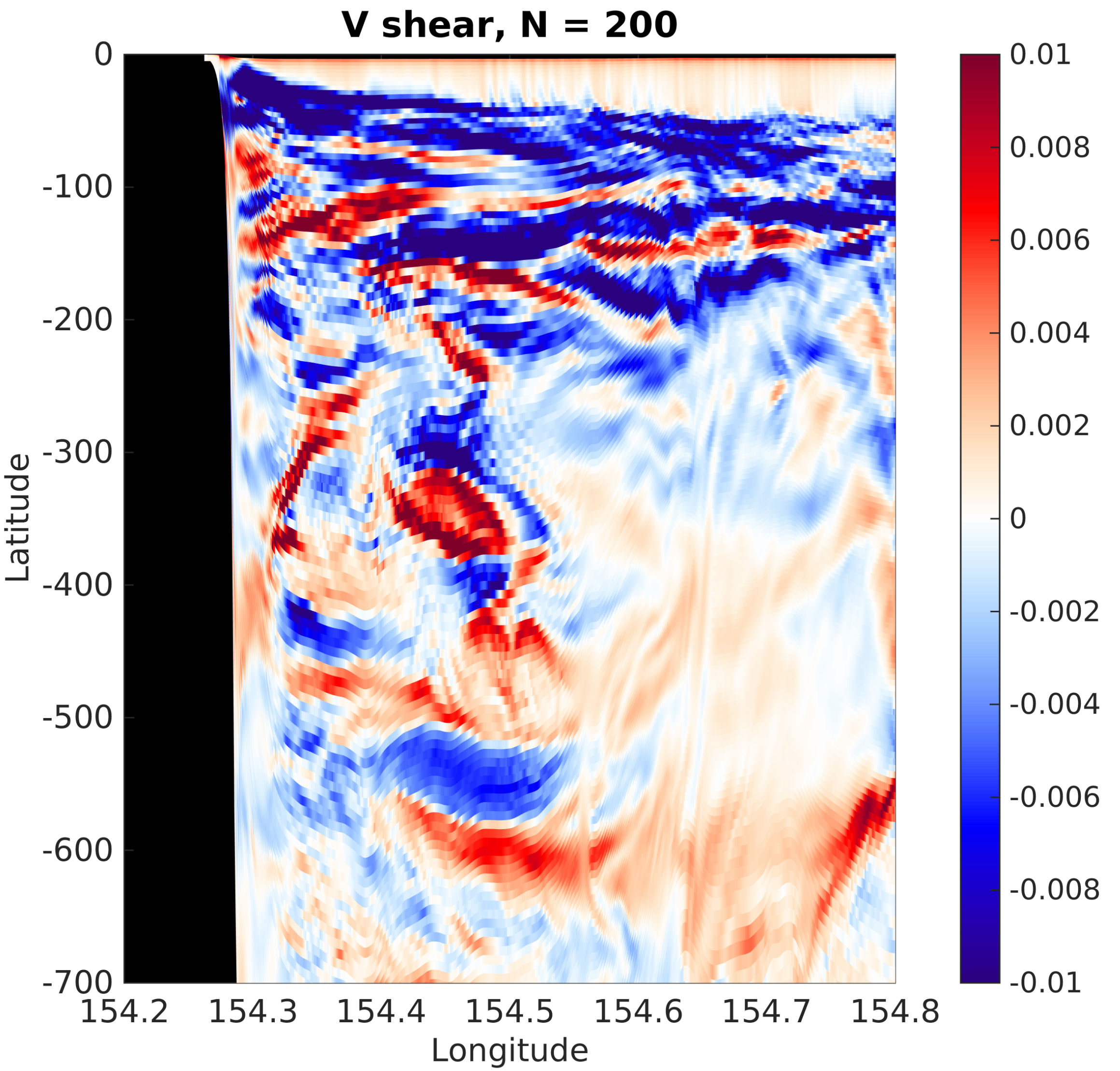




From N=100 levels to N=300 levels…

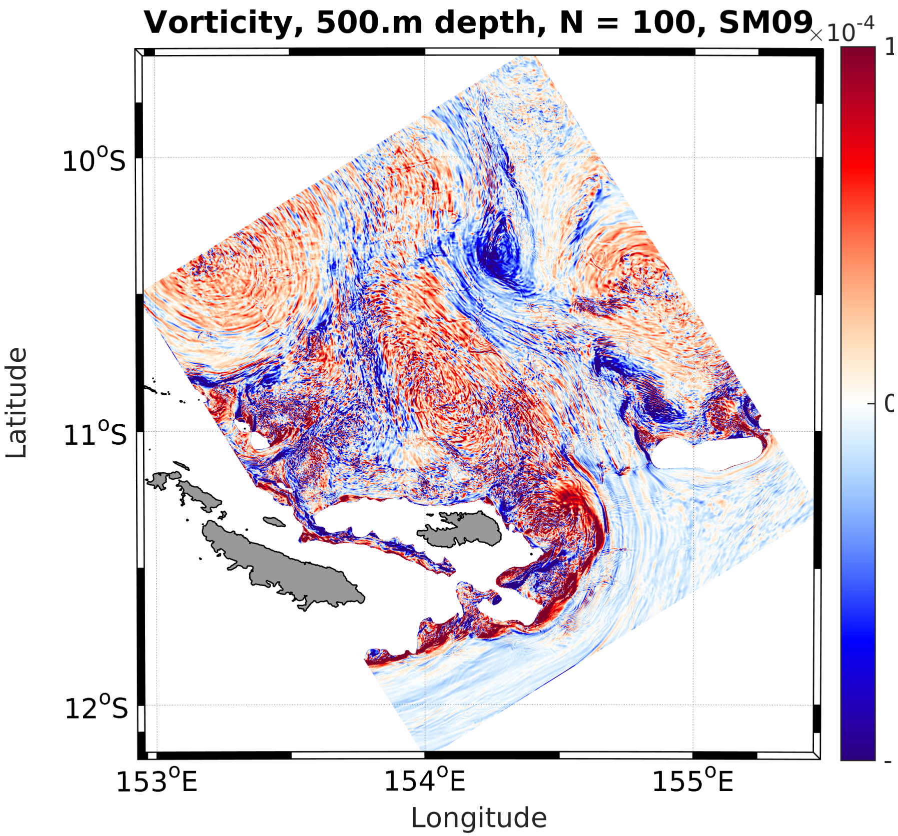
Ex: Need in Vertical resolution?

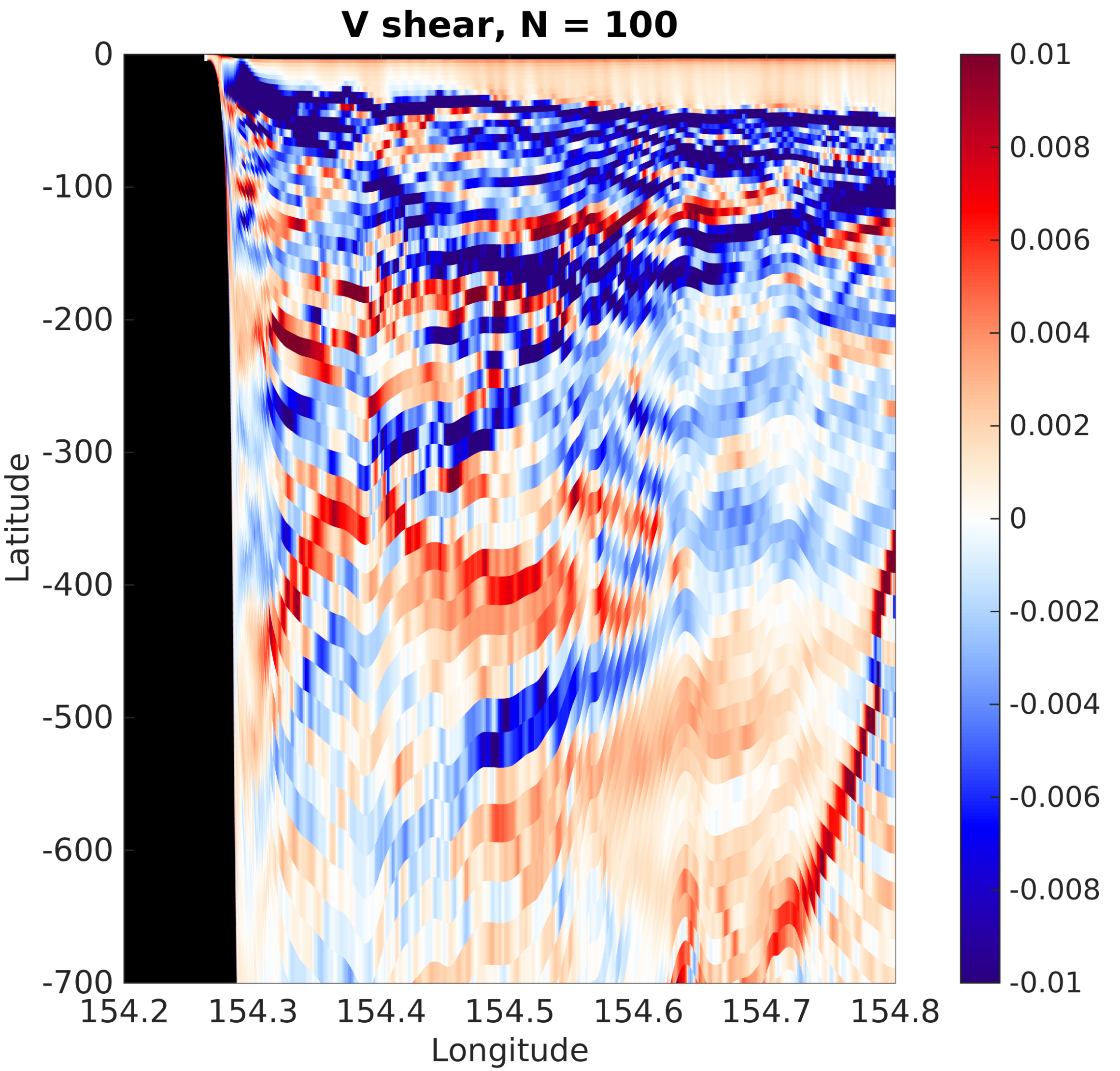




N=200 levels with a more balanced distribution

Ex: Need in Vertical resolution?





Ex: What happens when underresolving vertical scales with HR simulations (Solomon Sea)

Vertical grid : σ generalized coordinate

- Equations for ROMS/CROCO become:

$$\frac{\partial u}{\partial t} - fv + \vec{v}\cdot\nabla u = -\frac{\partial \phi}{\partial x} - \left(\frac{g\rho}{\rho_o}\right)\frac{\partial z}{\partial x} - g\frac{\partial \zeta}{\partial x} + \frac{1}{H_z}\frac{\partial}{\partial \sigma}\left[\frac{K_m}{H_z}\frac{\partial u}{\partial \sigma}\right] + \mathcal{F}_u + \mathcal{D}_u$$

$$\frac{\partial v}{\partial t} + fu + \vec{v}\cdot\nabla v = -\frac{\partial \phi}{\partial y} - \left(\frac{g\rho}{\rho_o}\right)\frac{\partial z}{\partial y} - g\frac{\partial \zeta}{\partial y} + \frac{1}{H_z}\frac{\partial}{\partial \sigma}\left[\frac{K_m}{H_z}\frac{\partial v}{\partial \sigma}\right] + \mathcal{F}_v + \mathcal{D}_v$$

$$\frac{\partial C}{\partial t} + \vec{v}\cdot\nabla C = \frac{1}{H_z}\frac{\partial}{\partial \sigma}\left[\frac{K_C}{H_z}\frac{\partial C}{\partial \sigma}\right] + \mathcal{F}_T + \mathcal{D}_T$$

$$\rho = \rho(T, S, P)$$

$$\frac{\partial \phi}{\partial \sigma} = \left(\frac{-gH_z\rho}{\rho_o}\right)$$

$$\frac{\partial H_z}{\partial t} + \frac{\partial (H_z u)}{\partial x} + \frac{\partial (H_z v)}{\partial y} + \frac{\partial (H_z \Omega)}{\partial \sigma} = 0$$

Vertical grid + curvilinear grid

$$\frac{\partial}{\partial t}\left(\frac{H_z u}{mn}\right)+\frac{\partial}{\partial \xi}\left(\frac{H_z u^2}{n}\right)+\frac{\partial}{\partial \eta}\left(\frac{H_z uv}{m}\right)+\frac{\partial}{\partial \sigma}\left(\frac{H_z u\Omega}{mn}\right)$$
$$-\left\{\left(\frac{f}{mn}\right)+v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right)-u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right\}H_z v=$$
$$-\left(\frac{H_z}{n}\right)\left(\frac{\partial \phi}{\partial \xi}+\frac{g\rho}{\rho_o}\frac{\partial z}{\partial \xi}+g\frac{\partial \zeta}{\partial \xi}\right)+\frac{1}{mn}\frac{\partial}{\partial \sigma}\left[\frac{K_m}{H_z}\frac{\partial u}{\partial \sigma}\right]+\frac{H_z}{mn}\left(\mathcal{F}_u+\mathcal{D}_u\right)$$

$$\frac{\partial}{\partial t}\left(\frac{H_z v}{mn}\right)+\frac{\partial}{\partial \xi}\left(\frac{H_z uv}{n}\right)+\frac{\partial}{\partial \eta}\left(\frac{H_z v^2}{m}\right)+\frac{\partial}{\partial \sigma}\left(\frac{H_z v\Omega}{mn}\right)$$
$$+\left\{\left(\frac{f}{mn}\right)+v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right)-u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right\}H_z u=$$
$$-\left(\frac{H_z}{m}\right)\left(\frac{\partial \phi}{\partial \eta}+\frac{g\rho}{\rho_o}\frac{\partial z}{\partial \eta}+g\frac{\partial \zeta}{\partial \eta}\right)+\frac{1}{mn}\frac{\partial}{\partial \sigma}\left[\frac{K_m}{H_z}\partial v \partial \sigma\right]+\frac{H_z}{mn}\left(\mathcal{F}_v+\mathcal{D}_v\right)$$

$$\frac{\partial}{\partial t}\left(\frac{H_z C}{mn}\right)+\frac{\partial}{\partial \xi}\left(\frac{H_z uC}{n}\right)+\frac{\partial}{\partial \eta}\left(\frac{H_z vC}{m}\right)+\frac{\partial}{\partial \sigma}\left(\frac{H_z \Omega C}{mn}\right)=$$
$$\frac{1}{mn}\frac{\partial}{\partial s}\left[\frac{K_C}{H_z}\frac{\partial C}{\partial \sigma}\right]+\frac{H_z}{mn}\left(\mathcal{F}_C+\mathcal{D}_C\right)$$

$$\rho=\rho(T,S,P)$$
$$\frac{\partial \phi}{\partial \sigma}=-\left(\frac{gH_z\rho}{\rho_o}\right)$$
$$\frac{\partial}{\partial t}\left(\frac{H_z}{mn}\right)+\frac{\partial}{\partial \xi}\left(\frac{H_z u}{n}\right)+\frac{\partial}{\partial \eta}\left(\frac{H_z v}{m}\right)+\frac{\partial}{\partial \sigma}\left(\frac{H_z \Omega}{mn}\right)=0.$$

Vertical grid : σ generalized coordinate

- The vertical velocity in sigma-coordinates is:

$$\Omega(x, y, \sigma, t) = \frac{1}{H_z}\left[w - \left(\frac{z+h}{\zeta+h}\right)\frac{\partial \zeta}{\partial t} - u\frac{\partial z}{\partial x} - v\frac{\partial z}{\partial y}\right]$$

- And the “true” vertical velocity:

$$w = \frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} + \Omega H_z.$$

Vertical grid : σ generalized coordinate

- Boundary conditions become:

top ($\sigma = 0$)

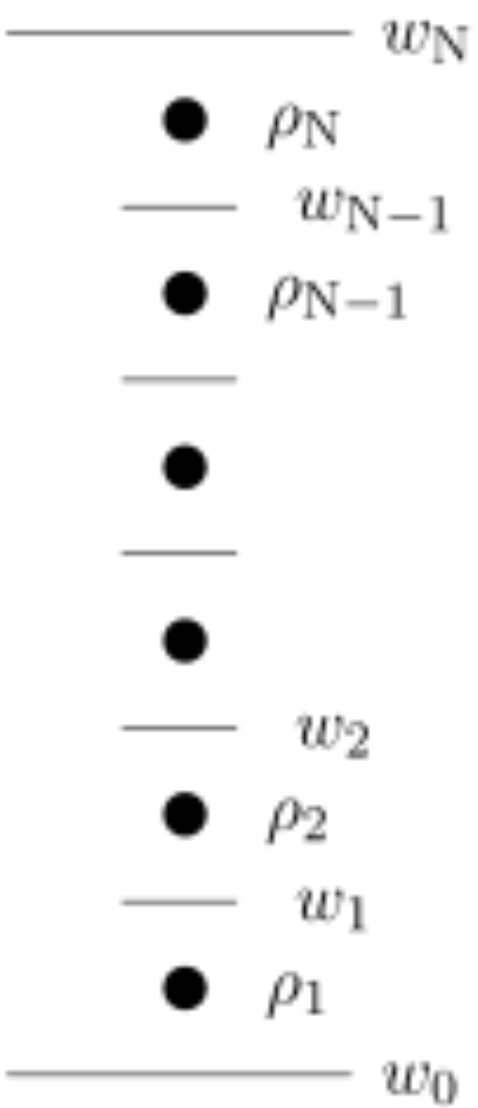
$$\left(\frac{K_m}{H_z}\right)\frac{\partial u}{\partial \sigma} = \tau_s^x(x,y,t)$$
$$\left(\frac{K_m}{H_z}\right)\frac{\partial v}{\partial \sigma} = \tau_s^y(x,y,t)$$
$$\left(\frac{K_C}{H_z}\right)\frac{\partial C}{\partial \sigma} = \frac{Q_C}{\rho_o c_P}$$
$$\Omega = 0$$

and bottom ($\sigma = -1$)

$$\left(\frac{K_m}{H_z}\right)\frac{\partial u}{\partial \sigma} = \tau_b^x(x,y,t)$$
$$\left(\frac{K_m}{H_z}\right)\frac{\partial v}{\partial \sigma} = \tau_b^y(x,y,t)$$
$$\left(\frac{K_C}{H_z}\right)\frac{\partial C}{\partial s} = 0$$
$$\Omega = 0.$$

Vertical discretization

ROMS: Staggered vertical grid

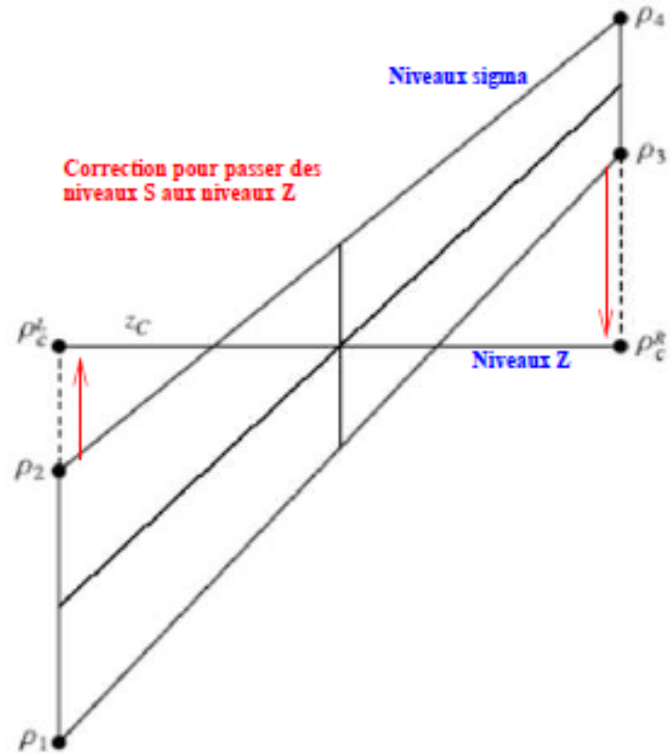


Pressure Gradient force

- Truncation errors are made from calculating the baroclinic pressure gradients across sharp topographic changes such as the continental slope

- Difference between 2 large terms

- Errors can appear in the unforced flat stratification experiment



$$\boxed{-\frac{1}{\rho_0}\left.\frac{\partial P}{\partial x}\right|_z = -\frac{1}{\rho_0}\left.\frac{\partial P}{\partial x}\right|_s + \frac{1}{\rho_0}\cdot\frac{\partial P}{\partial z}\cdot\left.\frac{\partial z}{\partial x}\right|_s,}$$

$$\epsilon \equiv \frac{\left|\left.\frac{\partial P}{\partial x}\right|_s - \frac{\partial P}{\partial z}\cdot\left.\frac{\partial z}{\partial x}\right|_s\right|}{\left|\left.\frac{\partial P}{\partial x}\right|_s\right| + \left|\frac{\partial P}{\partial z}\cdot\left.\frac{\partial z}{\partial x}\right|_s\right|} \ll 1,$$

Reducing PGF Truncation Errors

- Smoothing the topography using a nonlinear filter and a criterium: $r = \Delta h / h < 0.2$

- Using a "density formulation"

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial x}\bigg|_z = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}\bigg|_{z=\zeta} - \frac{g}{\rho_0}\int_z^{\zeta}\frac{\partial \rho}{\partial x}\bigg|_z dz'$$

$$= -\frac{g\rho(\zeta)}{\rho_0}\frac{\partial \zeta}{\partial x} - \frac{g}{\rho_0}\int_z^{\zeta}\left[\frac{\partial \rho}{\partial x}\bigg|_s - \frac{\partial \rho}{\partial z'}\frac{\partial z'}{\partial x}\bigg|_s\right]dz',$$

- Using high order schemes to reduce the truncation error (4th order, McCalpin, 1994)

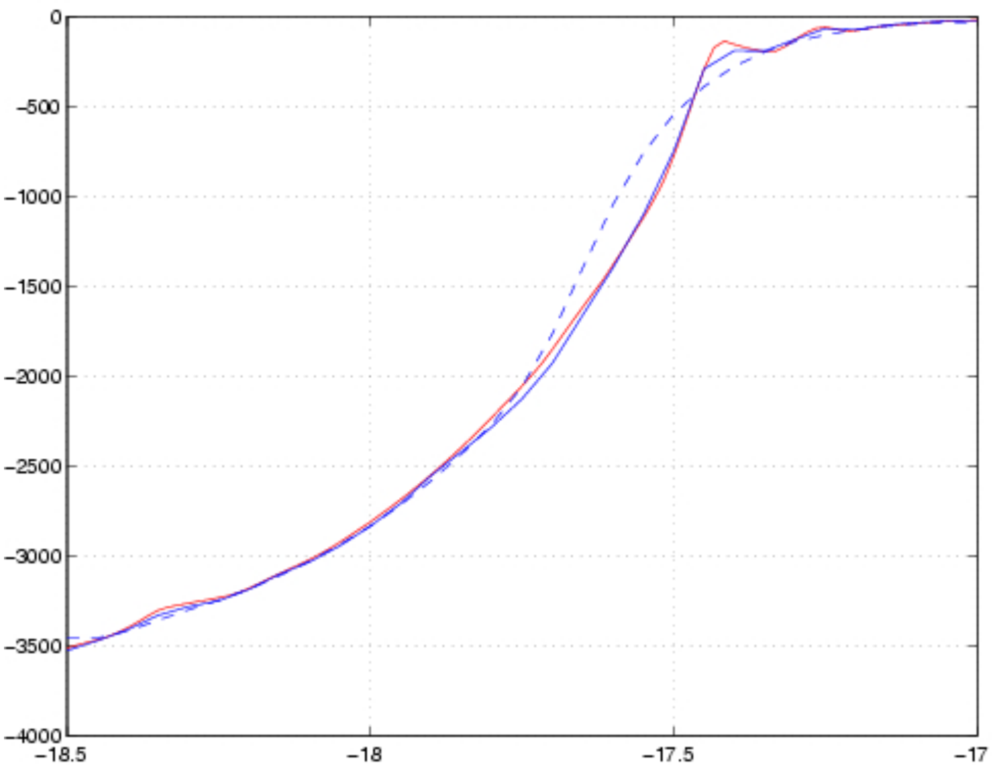
- Gary, 1973: substracting a reference horizontal averaged value from density ($\rho' = \rho - \rho_a$) before computing pressure gradient

- Rewritting Equation of State: reduce passive compressibility effects on pressure gradient

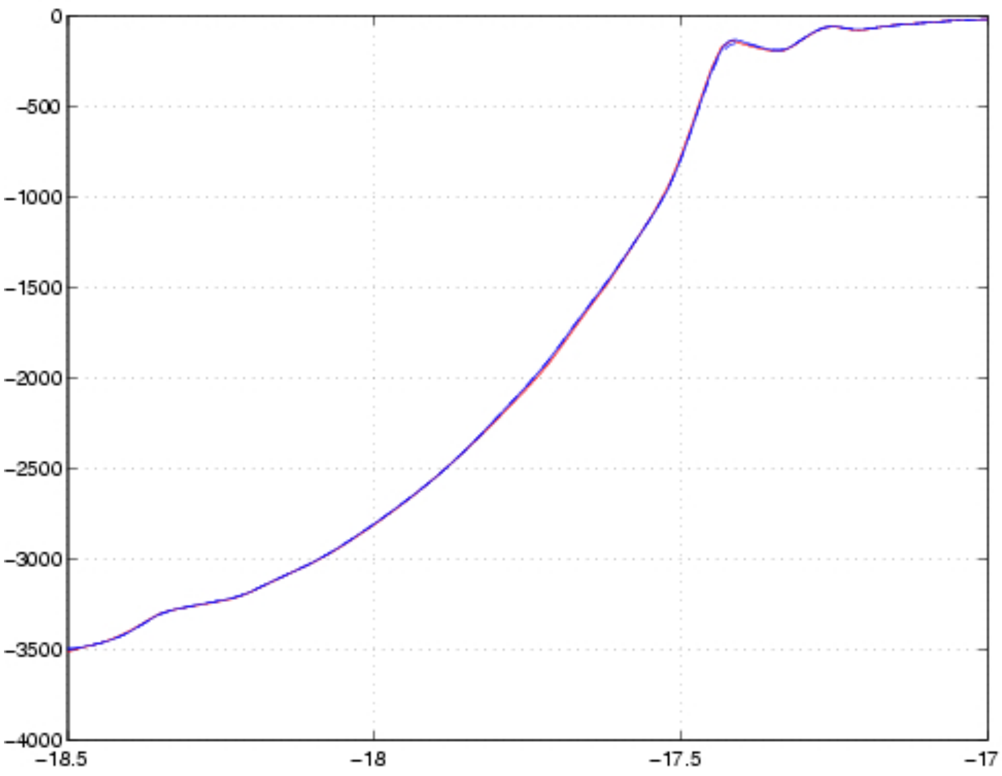
Smoothing methods

- ***r* = *Δh* / *h*** is the slope of the logarithm of *h*
- One method (ROMS) consists of smoothing *ln(h)* until $r < r_{max}$

Res: 5 km ; r_{max} = 0.25

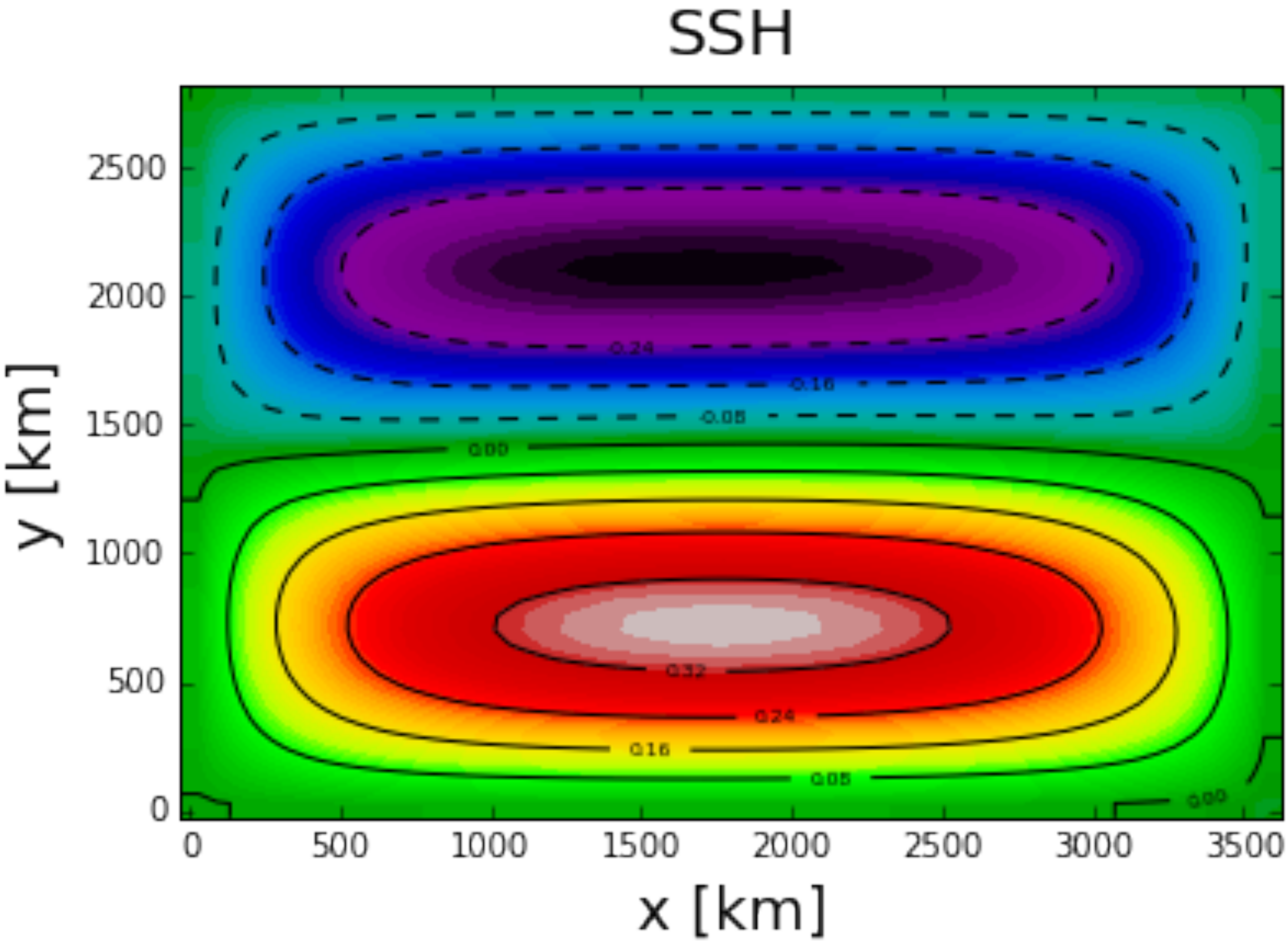


Res: 1 km ; r_{max} = 0.25



How to work with ROMS/CROCO outputs?

Activity 2 – Run an idealized ocean basin



Bottom friction parametrization

1. Linear friction, with r ***friction velocities*** [m/s] $\rightarrow$ $$(\tau_b^x, \tau_b^y) = -r\,(u_b, v_b)$$

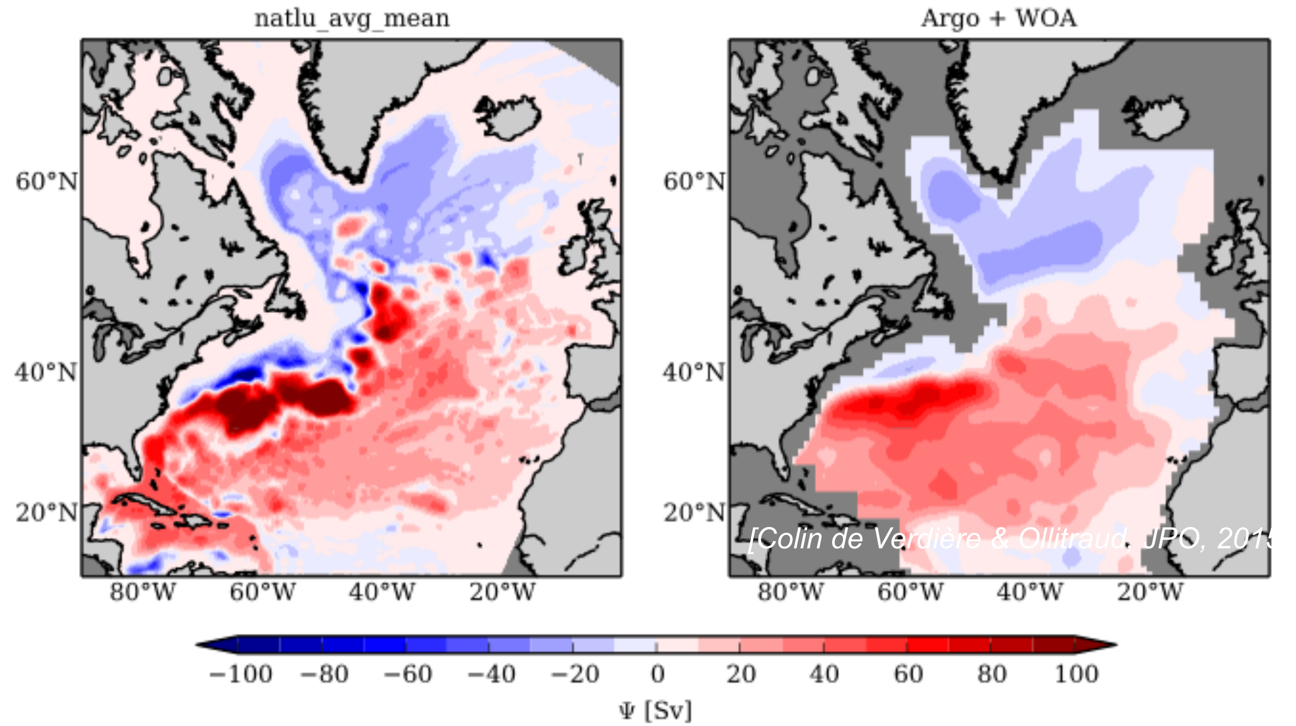
2. Quadratic friction, controled by a constant drag coefficient **Cd** $\rightarrow$ $$(\tau_b^x, \tau_b^y) = C_d\sqrt{u_b^2 + v_b^2}\,(u_b, v_b)$$

3. Quadratic friction coefficient, using variable **Cd** (Von Karman log. layer) $\rightarrow$
$$\begin{cases} (\tau_b^x, \tau_b^y) = C_d\sqrt{u_b^2 + v_b^2}\,(u_b, v_b) \\ C_d = \left(\frac{\kappa}{\log[\Delta z_b / z_r]}\right)^2 \text{ si } C_d^{min} < C_d < C_d^{max} \\ C_d = C_d^{min} \text{ ou } C_d^{max} \\ \kappa = 0.41 \\ z_r \text{ = Roughness Length} \\ \Delta z_b \text{ = thickness of the first bottom level} \end{cases}$$

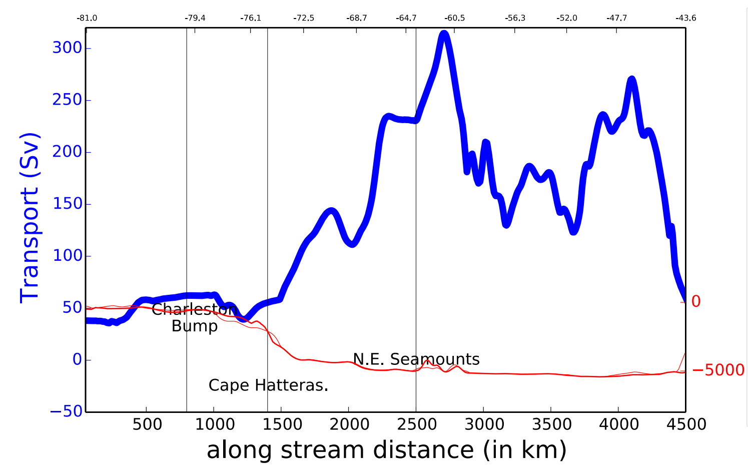
Bottom friction parametrization

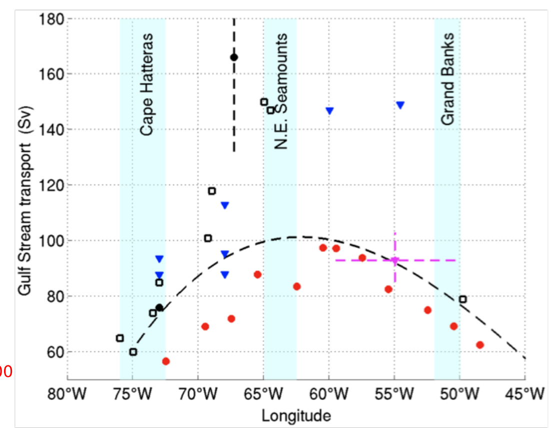
Example: What happens if you take a small roughness length

North-Atlantic ROMS-WRF Simulations

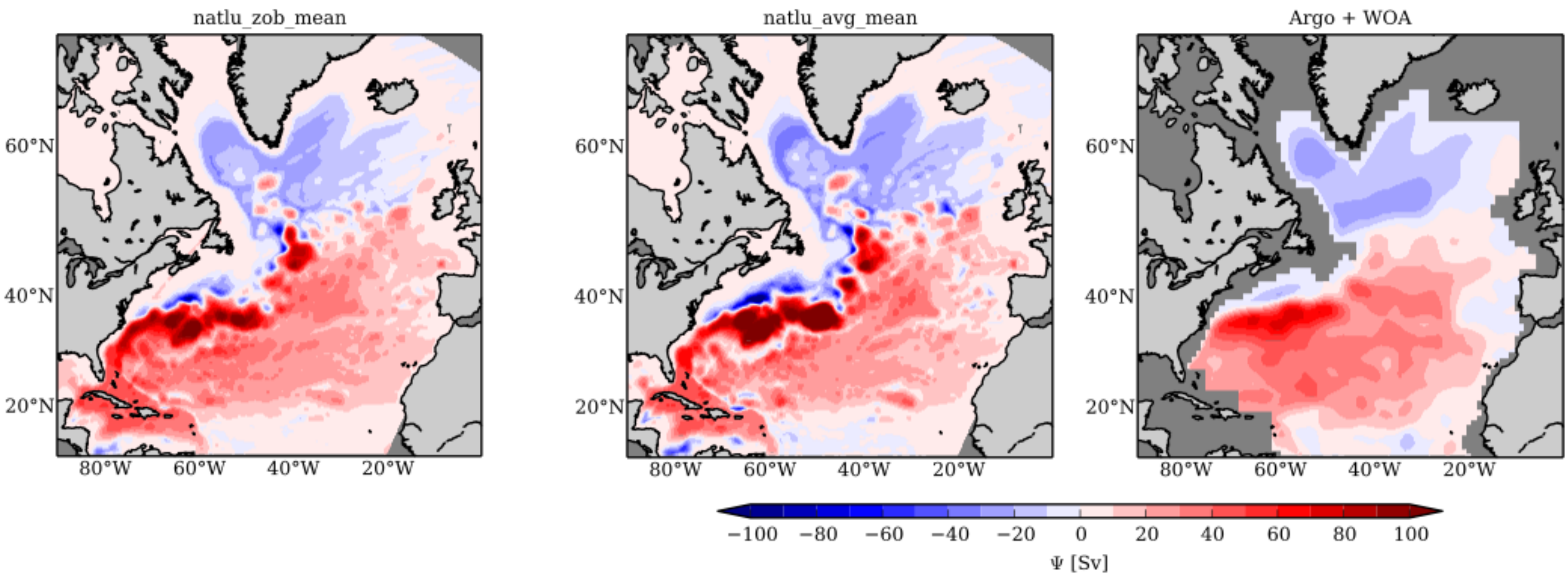


$$z_{0b} = 0.0001\mathrm{m}$$





Bottom friction parametrization



$z_{0b} = 0.01\mathrm{m}$

$z_{0b} = 0.0001\mathrm{m}$