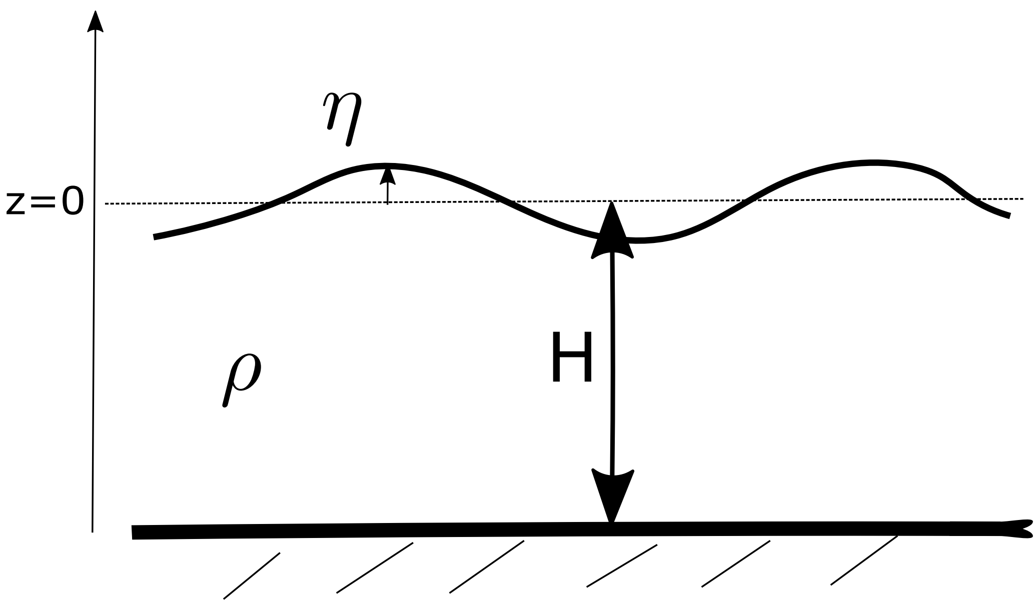
Internal waves - Notes

November 12, 2020

1 The one-layer shallow-water model



We start from the Navier-Stokes equations for momentum (without viscosity nor forcings):

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}.\vec{\nabla}\vec{u} + f\vec{k} \times \vec{u} = -\frac{\vec{\nabla} P}{\rho} \tag{1}$$

with $\vec{u} = (u, v, w)$, ρ the density and $P(x, y)$ the pressure.

We consider a flow which is **quasi - horizontal** and neglect vertical variations of horizontal velocities ($u_z = v_z = 0$), such that equations for the horizontal components of momentum become:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -\frac{1}{\rho}\frac{\partial P}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -\frac{1}{\rho}\frac{\partial P}{\partial y}
\end{aligned}
$$

In the following we use the following notations $\frac{\partial u}{\partial t} = u_t$, $\frac{\partial u}{\partial x} = u_x$, $\frac{\partial u}{\partial y} = u_y$, etc., such that:

$$
\begin{aligned}
u_t + uu_x + vu_y - fv &= -\frac{1}{\rho}P_x \\
v_t + uv_x + vv_y + fu &= -\frac{1}{\rho}P_y
\end{aligned} \tag{2}
$$

We consider that the flow is in **hydrostatic balance**:

$$
p(z) = P_a + \rho g(\eta - z) \tag{3}
$$

where P_a is the atmospheric pressure, η the free surface height, and z the depth (increasing from the surface to the bottom).

We also consider that density is constant over the layer of fluid ($\rho = cste$), such that the horizontal pressure gradients reduce to:

$$
\begin{aligned}
p_x &= \rho g \eta_x \\
p_y &= \rho g \eta_y
\end{aligned} \tag{4}
$$

So the momentum equations become:

$$
\begin{aligned}
u_t + uu_x + vu_y - fv &= -g\eta_x \\
v_t + uv_x + vv_y + fu &= -g\eta_y
\end{aligned} \tag{5}
$$

To get a third equation for the free surface height η, we use the continuity equation (conservation of mass) :

$$
\vec{\nabla} \cdot \vec{u} = u_x + v_y + w_z = 0. \tag{6}
$$

that we integrate vertically:

$$
\int_{-H}^{\eta} (u_x + v_y + w_z)\, dz = (H + \eta)(u_x + v_y) + [w]_{-H}^{\eta} \tag{7}
$$

with

$$
\begin{aligned}
(H+\eta)\,(u_x+v_y) &= [(H+\eta)u]_x + [(H+\eta)v]_y - u(H+\eta)_x - v(H+\eta)_y \\
&= [(H+\eta)u]_x + [(H+\eta)v]_y - u\eta_x - v\eta_y = 0
\end{aligned}
$$

and

$$
[w]_{-H}^{\eta} = w(\eta) - w(-H)
$$

and the kinematic conditions at the surface and bottom (assuming a flat bottom):

$$
\begin{aligned}
w(\eta) &= \frac{D\eta}{Dt} = \eta_t + u\eta_x + v\eta_y \\
w(-H) &= \frac{D(-H)}{Dt} = -H_t - uH_x - vH_y = 0
\end{aligned}
$$

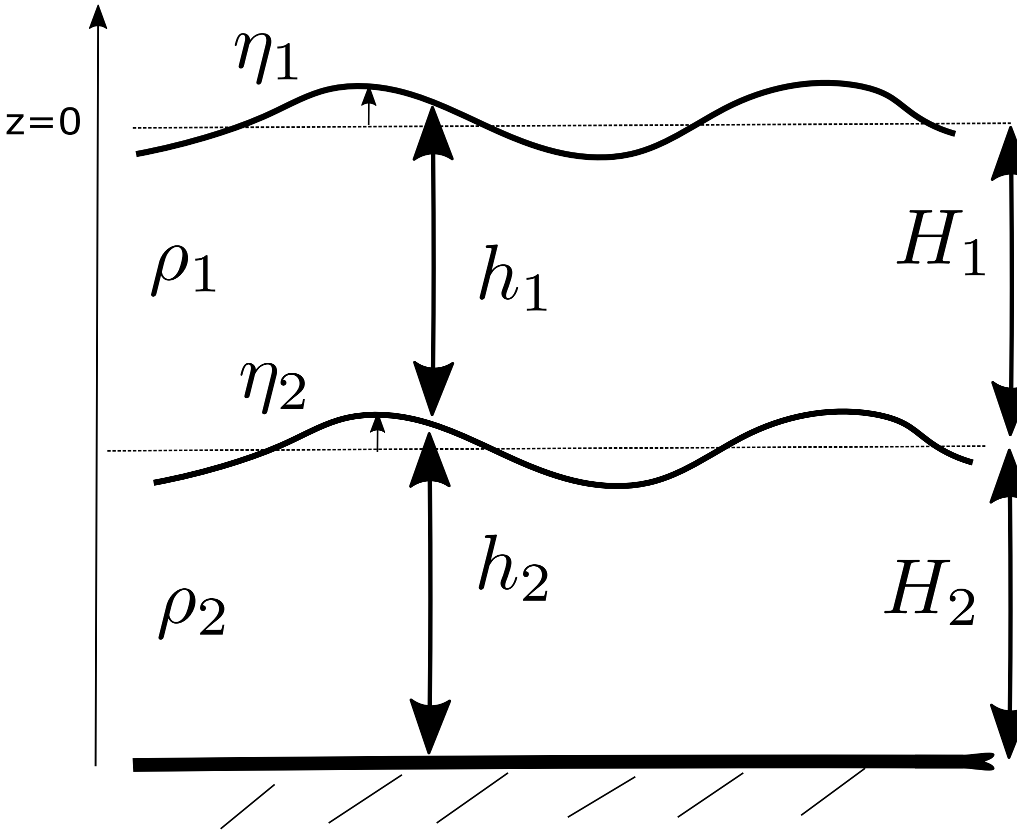
Such that Equ. 8 becomes:

$$
\eta_t + [(H+\eta)u]_x + [(H+\eta)v]_y = 0
$$

And finally the one-layer shallow-water equations, for u, v, η, are:

$$
\begin{aligned}
u_t + uu_x + vu_y - fv &= -g\eta_x \\
v_t + uv_x + vv_y + fu &= -g\eta_y \\
\eta_t + [(H+\eta)u]_x + [(H+\eta)v]_y &= 0
\end{aligned}
\tag{8}
$$

2 The two-layer shallow-water model



2.1 Momentum equations

If we now consider 2 layers of fluid, each with constant densities ρ_1 and ρ_2 ($\rho_2 > \rho_1$). The momentum equations 2 can be used for velocities inside each layer. The differences lie in the pressure term, which will be different for both layers:

$$p_1(z) = P_a + \rho_1 g(\eta_1 - z) \tag{9}$$

$$\begin{aligned} p_2(z) = P_a &+ \rho_1 g(H_1 + \eta_1 - \eta_2) \\ &+ \rho_2 g(\eta_2 - (z + H_1)) \end{aligned} \tag{10}$$

and the corresponding pressure gradients are:

$$\frac{p_{1x}}{\rho_1} = g\eta_{1x} \tag{11}$$

$$\frac{p_{2x}}{\rho_2} = \frac{\rho_1}{\rho_2}g(\eta_{1x} - \eta_{2x}) + g\eta_{2x} \tag{12}$$

$$= \frac{\rho_1}{\rho_2}g\eta_{1x} + (1 - \frac{\rho_1}{\rho_2})g\eta_{2x}. \tag{13}$$

We introduce the *reduced gravity*: $g' = (1 - \frac{\rho_1}{\rho_2})g$, and write:

$$\frac{p_{2x}}{\rho_2} = (g - g')\eta_{1x} + g'\eta_{2x}. \tag{14}$$

In the following we'll use the fact that $g \gg g'$ to write:

$$\frac{p_{1x}}{\rho_1} = g\eta_{1x} \tag{15}$$

$$\frac{p_{2x}}{\rho_2} = g\eta_{1x} + g'\eta_{2x}. \tag{16}$$

2.2 Coastal ocean example

Let's compute the values of g and g' using typical values for the costal ocean, considering a two-layer flow separated by a thermocline.

We can define density variations as $\rho \approx \rho_0(1 - \alpha T)$, with $\alpha = 2\,.10^{-4}$ K^{-1}.

So $g' \approx \frac{\rho_2 - \rho_1}{\rho_0} = \alpha(T_1 - T_2)g$

For a temperature gradient $\Delta T = T_1 - T_2 = 10$ K, we get: $g' = 2.10^{-3}g \ll g$.

2.3 Continuity equations

For the 2 layers of fluid, the continuity equations can be written as previsously, but we need to take into account the depth of the upper layer, which is $h_1 = H_1 + \eta_1 - \eta_2$ and the depth of the bottom layer, which is $h_2 = H_2 + \eta_2$.

Our system of equations for the 2-layer shallow water model is thus:

$$
\begin{aligned}
u_{1t} + u_1 u_{1x} + v_1 u_{1y} - f v_1 &= -g\eta_{1x} \\
v_{1t} + u_1 v_{1x} + v_1 v_{1y} + f u_1 &= -g\eta_{1y} \\
(\eta_1 - \eta_2)_t + [(H_1 + \eta_1 - \eta_2)u_1]_x + [(H_1 + \eta_1 - \eta_2)v_1]_y &= 0 \\
u_{2t} + u_2 u_{2x} + v_2 u_{2y} - f v_2 &= -g\eta_{1x} - g'\eta_{2x} \\
v_{2t} + u_2 v_{2x} + v_2 v_{2y} + f u_2 &= -g\eta_{1y} - g'\eta_{2y} \\
\eta_{2t} + [(H_2 + \eta_2)u_2]_x + [(H_2 + \eta_2)v_2]_y &= 0 \qquad (17)
\end{aligned}
$$

2.4 Wave solution for the SW equations

We want to find solutions of the system in the form of linear waves. So we linearize the equations and consider only small perturbations to the background state:

$$
\begin{aligned}
u_{1t} - f v_1 &= -g\eta_{1x} && (18) \\
v_{1t} + f u_1 &= -g\eta_{1y} && (19) \\
(\eta_1 - \eta_2)_t + H_1 u_{1x} + H_1 v_{1y} &= 0 && (20) \\
u_{2t} - f v_2 &= -g\eta_{1x} - g'\eta_{2x} && (21) \\
v_{2t} + f u_2 &= -g\eta_{1y} - g'\eta_{2y} && (22) \\
\eta_{2t} + H_2 u_{2x} + H_2 v_{2y} &= 0 && (23)
\end{aligned}
$$

We look for solutions of the system in the form of monochromatic waves:

$$A = A_0 e^{i(\omega t - kx - ly)} \qquad (24)$$

where A can be u, v, η.

So we have the properties:

$$
\begin{aligned}
A_t &= i\omega A \\
A_x &= -ikA \\
A_y &= -ilA
\end{aligned}
$$

$$
\begin{aligned}
A_{tt} &= -\omega^2 A \\
A_{xx} &= -k^2 A \\
A_{yy} &= -l^2 A
\end{aligned}
$$

Now we want to find the dispersion relations of the waves in our system of equations. One way to proceed is first try to simplify the system, by reducing it to 2 equations for 2 variables (for example η_1 and η_2).

So let's first to express u_1, v_1, u_2, v_2 as functions of η_1 and η_2. This can be done relatively simply by taking time derivatives of momentum equations (18,19,21,22).
For the upper layer:

$$\begin{aligned} u_{1tt} - f v_{1t} &= -g\eta_{1xt} \\ v_{1tt} + f u_{1t} &= -g\eta_{1yt} \end{aligned}$$

Then we replace time derivatives in the Coriolis terms using equations 18 and 19:

$$\begin{aligned} u_{1tt} - f(-f u_1 - g\eta_{1y}) &= -g\eta_{1xt} \\ v_{1tt} + f(f v_1 - g\eta_{1x}) &= -g\eta_{1yt} \end{aligned}$$

And we can express:

$$\begin{aligned} u_{1tt} + f^2 u_1 &= -gf\eta_{1y} - g\eta_{1xt} \\ v_{1tt} + f^2 v_1 &= gf\eta_{1x} - g\eta_{1yt} \end{aligned}$$

Using a wave solution we get:

$$\begin{aligned} -\omega^2 u_1 + f^2 u_1 &= -gf\eta_{1y} - g\eta_{1xt} \\ -\omega^2 v_1 + f^2 v_1 &= gf\eta_{1x} - g\eta_{1yt} \end{aligned}$$

$$u_1 = \frac{-gf\eta_{1y} - g\eta_{1xt}}{f^2 - \omega^2} \tag{25}$$

$$v_1 = \frac{gf\eta_{1x} - g\eta_{1yt}}{f^2 - \omega^2} \tag{26}$$

Similarly for the bottom layer, we get:

$$u_2 = \frac{-gf\eta_{1y} - g'f\eta_{2y} - g\eta_{1xt} - g'\eta_{2xt}}{f^2 - \omega^2} \tag{27}$$

$$v_2 = \frac{gf\eta_{1x} + g'f\eta_{2x} - g\eta_{1yt} - g'\eta_{2yt}}{f^2 - \omega^2} \tag{28}$$

We can now use equations 25- 28 to replace velocities in the continuity equations 29 and 29:

$$\begin{aligned}
(\eta_1 - \eta_2)_t + H_1 \frac{-gf\eta_{1xy} - g\eta_{1xxt}}{f^2 - \omega^2} + H_1 \frac{gf\eta_{1xy} - g\eta_{1yyt}}{f^2 - \omega^2} &= 0 \\
\eta_{2t} + H_2 \frac{-gf\eta_{2xy} - g\eta_{1xxt} - g'\eta_{2xxt}}{f^2 - \omega^2} + H_2 \frac{gf\eta_{2xy} - g\eta_{1yyt} - g'\eta_{2yyt}}{f^2 - \omega^2} &= 0
\end{aligned}$$

which simplifies as:

$$\begin{aligned}
(f^2 - \omega^2)(\eta_1 - \eta_2)_t + H_1(-g\eta_{1xxt} - g\eta_{1yyt}) &= 0 \\
(f^2 - \omega^2)\eta_{2t} + H_2(-g\eta_{1xxt} - g'\eta_{2xxt} - g\eta_{1yyt} - g'\eta_{2yyt}) &= 0
\end{aligned}$$

then:

$$\begin{aligned}
(f^2 - \omega^2)(\eta_1 - \eta_2) + H_1 g\eta_1(k^2 + l^2) &= 0 \\
(f^2 - \omega^2)\eta_2 + H_2 g\eta_1(k^2 + l^2) + H_2 g'\eta_2(k^2 + l^2)) &= 0
\end{aligned}$$

We introduce the amplitude of the horizontal wavenumber $K = \sqrt{k^2 + l^2}$, and we finally have a system of 2 equations for variables η_1 and η_2:

$$\begin{aligned}
(f^2 - \omega^2 + gH_1K^2)\eta_1 - (f^2 - \omega^2)\eta_2 &= 0 \\
gH_2K^2\eta_1 + (f^2 - \omega^2 + g'H_2K^2)\eta_2 &= 0
\end{aligned}$$

This system has a non-trivial solution (other than $u = v = \eta = 0$) only if the determinant is zero:

$$(f^2 - \omega^2 + gH_1K^2)(f^2 - \omega^2 + g'H_2K^2) + gH_2K^2(f^2 - \omega^2) \;\; = \;\; 0$$

which is a second order polynomial function in $(f^2 - \omega^2)$:

$$(f^2 - \omega^2)^2 + (g(H_1 + H_2)K^2 + g'H_2K^2)(f^2 - \omega^2) + gg'H_1H_2K^4 \;\; = \;\; 0$$

We can neglect the term $g'H_2K^2 \ll g(H_1 + H_2)K^2$, because we still assume that $g' \ll g$:

$$(f^2 - \omega^2)^2 + g(H_1 + H_2)K^2(f^2 - \omega^2) + gg'H_1H_2K^4 \;\; = \;\; 0$$

which has a discriminant:

$$\Delta = K^4(g^2H^2 - 4gg'H_1H_2) > 0$$

with $H = H_1 + H_2$.

And solutions:

$$f^2 - \omega^2 = \frac{-gHK^2 \pm K^2\sqrt{g^2H^2 - 4gg'H_1H_2}}{2}$$

With, at the first order, using the fact that $g'/g \ll 1$:

$$\sqrt{g^2H^2 - 4gg'H_1H_2} \approx g^2H^2\left[1 - 2\frac{g'H_1H_2}{gH^2}\right]$$

And finally 2 solutions are:

$$\begin{aligned}\omega_t^2 - f^2 &= gHK^2\left[1 - \frac{g'H_1H_2}{gH^2}\right]\\ \omega_c^2 - f^2 &= \frac{g'H_1H_2}{H}K^2\end{aligned}$$

$$\begin{aligned}\omega_t^2 &= f^2 + gHK^2\left[1 - \frac{g'H_1H_2}{gH^2}\right]\\ \omega_c^2 &= f^2 + \frac{g'H_1H_2}{H}K^2\end{aligned}$$

The first solution: $\omega_t^2 = f^2 + gHK^2\left[1 - \frac{g'H_1H_2}{gH^2}\right]$ corresponds to the *barotropic mode* (= external wave). It is almost the same solution than the one we get for the one-layer shallow water model (= shallow-water inertia-gravity waves / long waves: $\omega^2 = f^2 + gHK^2$), but with a small correction due to the baroclinic effects: $\left[1 - \frac{g'H_1H_2}{gH^2}\right]$, with $g'/g \ll 1$.

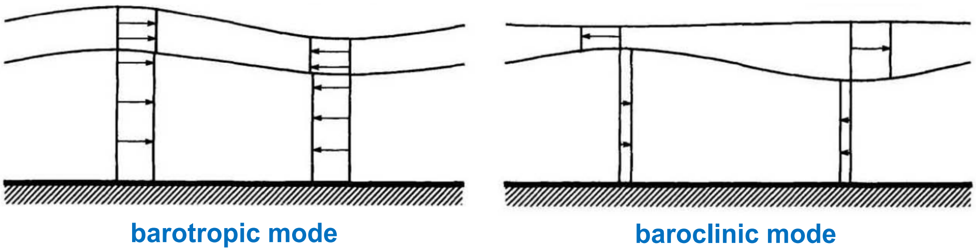
The corresponding phase speed is:

$$c_t^2 = \frac{\omega_t^2}{K^2} \quad = \quad gH\left[\frac{1 - \frac{g'H_1H_2}{gH^2}}{1 - \frac{f^2}{\omega^2}}\right]$$

The second solution: $\omega_c^2 = f^2 + \frac{g'H_1H_2}{H}K^2$ corresponds to the *baroclinic mode* (= internal wave). It does not exist in the one-layer shallow-water model because it requires a vertical density gradient between the 2-layers ($g' \neq 0$). The corresponding phase speed is:

$$c_c^2 = \frac{\omega_c^2}{K^2} \quad = \quad \frac{g'H_1H_2}{H(1 - \frac{f^2}{\omega^2})}$$

which is much slower than the phase speed of the barotropic wave (remember that $g' \ll g$).



2.5 Surface signature of an internal wave in the coastal ocean

We use typical values for the costal ocean, considered as two-layer flow separated by a thermocline. The upper layer thickness is $H_1 = 30$ m, the lower layer thickness is $H_1 = 70$ m, and $g' = 2.10^{-3}g$

What is the amplitude of the surface elevation (η_1) for an internal wave with a thermocline elevation ($\eta_2 = 10$ m)?

2.5.1 answer:

The relation between the surface (η_1) and the thermocline (η_2) elevations for the baroclinic mode can be computed using Equations 29 :

$$\eta_1 = \frac{\eta_2}{1 + \frac{gH_1K^2}{f^2 - \omega^2}}$$

and the dispersion relation: $\omega^2 = f^2 + \frac{g' H_1 H_2}{H} K^2$, to get:

$$\eta_1 = \frac{\eta_2}{1 - \frac{gH}{g' H_2}}$$

The numerical application gives:

$$\eta_1 = -\frac{\eta_2}{713}$$

So $\eta_2 = 10$ m corresponds to $\eta_1 = 1.4$ cm. The surface signature of an internal wave is very small.

3 Internal Waves in a continuously stratified flow

3.1 Equations

We use the Boussinesq approximation ($\rho - \rho^* \ll \rho^*$) and traditional approximation ($2\vec{\Omega} = f\vec{k}$, and linearize the equations ($\rho' \ll \rho$):
the linearized equations,

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} + \frac{\rho'}{\rho^*} g\vec{k} &= -\frac{\vec{\nabla} p'}{\rho^*} \\ \vec{\nabla} \cdot \vec{u} &= 0 \\ -\frac{g}{\rho^*}\frac{\partial \rho'}{\partial t} + N^2 w &= 0 \end{aligned}$$

We can use buoyancy: $b = -g\frac{\rho'}{\rho^*}$

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} - b\vec{k} &= -\frac{\vec{\nabla} p'}{\rho^*} \\ \vec{\nabla} \cdot \vec{u} &= 0 \\ \frac{\partial b}{\partial t} + N^2 w &= 0 \end{aligned}$$

We have now 5 equations for 5 variables (u, v, w, b, p):

$$u_t - fv = -\frac{p'_x}{\rho^*} \tag{29}$$

$$v_t + fu = -\frac{p'_y}{\rho^*} \tag{30}$$

$$w_t - b = -\frac{p'_z}{\rho^*} \tag{31}$$

$$u_x + v_y + w_z = 0 \tag{32}$$

$$b_t + N^2 w = 0 \tag{33}$$

We want to reduce the system to 1 equation for 1 variable (w) that we can then use to study the dynamics of gravity waves in this system.

The first step is to eliminate p' by combining the 3 momentum equations:

$\partial_z(29) - \partial_x(31)$ gives: $(u_t - fv)_z - (w_t - b)_x = -\frac{p'_{xz}}{\rho^*} + \frac{p'_{zx}}{\rho^*} = 0$
$\partial_z(30) - \partial_y(31)$ gives: $(v_t + fu)_z - (w_t - b)_y = -\frac{p'_{yz}}{\rho^*} + \frac{p'_{zy}}{\rho^*} = 0$
$\partial_y(29) - \partial_x(30)$ gives: $(u_t - fv)_y - (v_t + fu)_x = -\frac{p'_{xy}}{\rho^*} + \frac{p'_{yx}}{\rho^*} = 0$

$$u_{tz} - fv_z - w_{tx} + b_x = 0 \tag{34}$$
$$v_{tz} + fu_z - w_{ty} + b_y = 0 \tag{35}$$
$$u_{ty} - fv_y - v_{tx} - fu_x = 0 \tag{36}$$

Equ. 36 can be simplified using Equ. 32:

$$\begin{aligned} u_{ty} - fv_y - v_{tx} - fu_x &= 0 \\ (u_y - v_x)_t - f(u_x + v_y) &= 0 \\ (u_y - v_x)_t + fw_z &= 0 \end{aligned} \tag{37}$$

Then we get rid of u, v by combining Equ. 34 and 35, and using the fact that vertical vorticity and divergence can be expressed using w (though Equ. 32 and Equ. 37). More precisely $\partial_{yt}(35) + \partial_{xt}(34)$ gives:

$$(v_{tz} + fu_z - w_{ty} + b_y)_{yt} - (u_{tz} - fv_z - w_{tx} + b_x)_{xt} = 0$$
$$(u_x + v_y)_{ttz} + f(u_y - v_x)_{zt} - w_{ttyy} - w_{ttxx} + b_{yyt} + b_{xxt} = 0$$

Using Equ. 32, 33, 37, we eliminate vorticity, divergence and buoyancy:

$$(-w_z)_{ttz} + f(-fw_z)_z - w_{ttyy} - w_{ttxx} - N^2(w_{yy} + w_{xx}) = 0$$

Which gives:

$$\boxed{(\underbrace{w_{xx} + w_{yy} + w_{zz}}_{\nabla^2 w})_{tt} + f^2 w_{zz} + N^2(\underbrace{w_{xx} + w_{yy}}_{\nabla_h^2 w}) = 0} \qquad (38)$$

3.2 Method of characteristics

To solve Equ. 38 with a constant N, we can use the method of characteristics.

Assuming waves sinusoidal in time:

$$w = \hat{w}e^{\mathrm{i}\omega t}$$

and no variations in the y direction for simplicity, Equ. 38 reduces to:

$$\begin{aligned}(N^2 - \omega^2)\hat{w}_{xx} - (\omega^2 - f^2)\hat{w}_{zz} &= 0\\ \hat{w}_{xx} - \frac{\omega^2 - f^2}{N^2 - \omega^2}\hat{w}_{zz} &= 0\end{aligned} \tag{39}$$

A classical method to solve such equation is to make a change of variables by using ξ_+ and ξ_- instead of x and z, which are defined as:

$$\xi_\pm = \mu_\pm x - z$$

with

$$\mu_\pm = \pm\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}$$

Such that w can be expressed as:

$$\hat{w}(x, y) = \overline{w}(\xi_+, \xi_-)$$

We can verify that equation 39 greatly simplifies if we write it using $\overline{w}(\xi_+, \xi_-)$, because:

$$\begin{aligned}\frac{\partial \hat{w}}{\partial x} &= \frac{\partial \overline{w}}{\partial \xi_+}\frac{\partial \xi_+}{\partial x} + \frac{\partial \overline{w}}{\partial \xi_-}\frac{\partial \xi_-}{\partial x}\\ &= \mu_+\frac{\partial \overline{w}}{\partial \xi_+} + \mu_-\frac{\partial \overline{w}}{\partial \xi_-}\end{aligned}$$

and again:

$$\begin{aligned}
\frac{\partial^2 \hat{w}}{\partial x^2} &= \mu_+\left(\frac{\partial \frac{\partial \overline{w}}{\partial \xi_+}}{\partial \xi_+}\frac{\partial \xi_+}{\partial x} + \frac{\partial \frac{\partial \overline{w}}{\partial \xi_+}}{\partial \xi_-}\frac{\partial \xi_-}{\partial x}\right) + \mu_-\left(\frac{\partial \frac{\partial \overline{w}}{\partial \xi_-}}{\partial \xi_+}\frac{\partial \xi_+}{\partial x} + \frac{\partial \frac{\partial \overline{w}}{\partial \xi_-}}{\partial \xi_-}\frac{\partial \xi_-}{\partial x}\right) \\
&= \mu_+^2 \frac{\partial^2 \overline{w}}{\partial \xi_+^2} + \mu_+\mu_- \frac{\partial^2 \overline{w}}{\partial \xi_+ \partial \xi_-} + \mu_-\mu_+ \frac{\partial^2 \overline{w}}{\partial \xi_- \partial \xi_+} + \mu_-^2 \frac{\partial^2 \overline{w}}{\partial \xi_-^2} \\
&= \mu_+^2\left(\frac{\partial^2 \overline{w}}{\partial \xi_+^2} + \frac{\partial^2 \overline{w}}{\partial \xi_-^2} - 2\frac{\partial^2 \overline{w}}{\partial \xi_+ \partial \xi_-}\right)
\end{aligned}$$

Similarly we have:

$$\begin{aligned}
\frac{\partial \hat{w}}{\partial z} &= \frac{\partial \overline{w}}{\partial \xi_+}\frac{\partial \xi_+}{\partial z} + \frac{\partial \overline{w}}{\partial \xi_-}\frac{\partial \xi_-}{\partial z} \\
&= -\frac{\partial \overline{w}}{\partial \xi_+} - \frac{\partial \overline{w}}{\partial \xi_-}
\end{aligned}$$

and :

$$\begin{aligned}
\frac{\partial^2 \hat{w}}{\partial z^2} &= -\left(\frac{\partial \frac{\partial \overline{w}}{\partial \xi_+}}{\partial \xi_+}\frac{\partial \xi_+}{\partial z} + \frac{\partial \frac{\partial \overline{w}}{\partial \xi_+}}{\partial \xi_-}\frac{\partial \xi_-}{\partial z}\right) - \left(\frac{\partial \frac{\partial \overline{w}}{\partial \xi_-}}{\partial \xi_+}\frac{\partial \xi_+}{\partial z} + \frac{\partial \frac{\partial \overline{w}}{\partial \xi_-}}{\partial \xi_-}\frac{\partial \xi_-}{\partial z}\right) \\
&= \frac{\partial^2 \overline{w}}{\partial \xi_+^2} + \frac{\partial^2 \overline{w}}{\partial \xi_-^2} + 2\frac{\partial^2 \overline{w}}{\partial \xi_+ \partial \xi_-}
\end{aligned}$$

So finally the equation reduces to:

$$\begin{aligned}
\hat{w}_{xx} - \frac{\omega^2 - f^2}{N^2 - \omega^2}\hat{w}_{zz} &= \hat{w}_{xx} - \mu_+^2 \hat{w}_{zz} \\
&= \mu_+^2\left(\frac{\partial^2 \overline{w}}{\partial \xi_+^2} + \frac{\partial^2 \overline{w}}{\partial \xi_-^2} - 2\frac{\partial^2 \overline{w}}{\partial \xi_+ \partial \xi_-}\right) - \mu_+^2\left(\frac{\partial^2 \overline{w}}{\partial \xi_+^2} + \frac{\partial^2 \overline{w}}{\partial \xi_-^2} + 2\frac{\partial^2 \overline{w}}{\partial \xi_+ \partial \xi_-}\right) \\
&= -4\mu_+^2 \frac{\partial^2 \overline{w}}{\partial \xi_+ \partial \xi_-}
\end{aligned}$$

Which means that the PDE to solve is just:

$$\frac{\partial^2 \overline{w}}{\partial \xi_+ \partial \xi_-} = 0.$$

Thus, any function in the form

$$\overline{w}(\xi_+, \xi_-) = F(\xi_+) + G(\xi_-)$$

will be a solution of the equation, with F and G arbitrary functions.

The general solution is thus:

$$\hat{w} = F(\mu_+ x - z) + G(\mu_- x - z)$$

where F and G are arbitrary functions and:

$$\mu_\pm = \pm \left(\frac{\omega^2 - f^2}{N^2 - \omega^2} \right)^{1/2}$$

3.3 Dispersion relation

Assuming a form of the solution:

$$\hat{w} = w_0 e^{\mathrm{i}(kx+mz)}$$

We get:

$$\begin{aligned}
(N^2 - \omega^2)\hat{w}_{xx} - (\omega^2 - f^2)\hat{w}_{zz} = 0 \\
(N^2 - \omega^2)(-k^2) - (\omega^2 - f^2)(-m^2) = 0 \\
\omega^2(k^2 + m^2) - (N^2k^2 + f^2m^2) = 0
\end{aligned}$$

And the dispersion relation is:

$$\boxed{\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2 + m^2}}$$

3.4 Method of modes

The method of modes can be used to solve the equation for a variable $N(z)$, but requires simple boundary conditions (flat bottom and surface).

We go back to equation:

$$(w_{xx} + w_{yy} + w_{zz})_{tt} + f^2 w_{zz} + N^2(w_{xx} + w_{yy}) = 0$$

And assume a solution in the form:

$$w = W(z)e^{-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}ly}$$

with the boundary conditions:

$$\begin{aligned} w(0) &= 0 \\ w(-H) &= 0 \end{aligned}$$

this give the equation for $W(z)$:

$$W'' + k^2 \frac{N(z)^2 - \omega^2}{\omega^2 - f^2} W = 0$$

which we can write:

$$W'' + m^2 W = 0$$

with

$$m^2(z) = k^2 \frac{N(z)^2 - \omega^2}{\omega^2 - f^2}$$

If N is constant, the general solution can be written in the form:

$$W(z) = A\mathrm{cos}(mz) + B\mathrm{sin}(mz)$$

and the boundary conditions give:

$$\begin{aligned} W(0) &= A = 0 \\ W(-H) &= B\mathrm{sin}(-mH) = 0 \end{aligned}$$

Such that we get the condition:

$$m = \pm n \frac{\pi}{H}$$

with n an integer, corresponding to an infinite number of solutions (modes n).

The dispersion relation is then:

$$k_n = \pm n \frac{\pi}{H} \left(\frac{\omega^2 - f^2}{N^2 - \omega^2} \right)^{1/2}$$

And the general solution is the superposition of all modes:

$$w(x, z, t) = \sum_n W_n(z) \cos(k_n x - \omega t)$$
$$w(x, z, t) = \sum_n a_n \sin(\frac{n\pi z}{H}) \cos(k_n x - \omega t)$$