Master OFFWIND

Coastal Dynamics #1

Exercise 1: Bottom pressure

(a) Find the deep-water speed and wavelength of a wave of period 12 s.

(b) Find the speed and wavelength of a wave of period 12 s in water of depth 3 m using the general full dispersion relation (you will need to solve it numerically).

(c) Compare with the shallow-water approximation.

Solution:

(a) Deep-Water Wave

For deep water, the dispersion relation is

$$\omega^2 = gk$$

where $\omega = \dfrac{2\pi}{T}$ and $k = \dfrac{2\pi}{\lambda}$.

$$\omega = \frac{2\pi}{12} = 0.5236 \text{ rad/s}$$

$$k = \frac{\omega^2}{g} = \frac{(0.5236)^2}{9.81} = 0.0279 \text{ m}^{-1}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.0279} = 225 \text{ m}$$

$$c = \frac{\lambda}{T} = \frac{225}{12} = 18.8 \text{ m/s}$$

$$\boxed{\lambda_{\text{deep}} = 225 \text{ m}}, \quad \boxed{c_{\text{deep}} = 18.8 \text{ m/s}}$$

(b) Finite Depth $h = 3$ m

The dispersion relation for finite depth is:

$$\omega^2 = gk \tanh(kh)$$

Substituting the known values:

$$(0.5236)^2 = 9.81\,k\ \tanh(3k)$$

$$0.274 = 9.81\,k\ \tanh(3k)$$

Solving this numerically gives:

$$k \approx 0.098\ \mathrm{m}^{-1}$$

Then,

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.098} = 64.1\ \mathrm{m}$$

$$c = \frac{\lambda}{T} = \frac{64.1}{12} = 5.34\ \mathrm{m/s}$$

$$\boxed{\lambda_{h=3} = 64\ \mathrm{m}}, \quad \boxed{c_{h=3} = 5.3\ \mathrm{m/s}}$$

(c) Shallow-Water Approximation

In shallow water ($kh \ll 1$), the phase speed is:

$$c_{\text{shallow}} = \sqrt{gh} = \sqrt{9.81 \times 3} = 5.42\ \mathrm{m/s}$$

This value is very close to the exact finite-depth result $c = 5.3$ m/s, confirming that the shallow-water approximation is valid.

Exercise 2: Tides in the Bay of Fundy

The Bay of Fundy in New Brunswick (Canada) has a very large tidal range (17 m), a depth $H = 75$ m and a length $L \approx 320$ km. Gravity is $g = 9.8$ m s^{-2}.

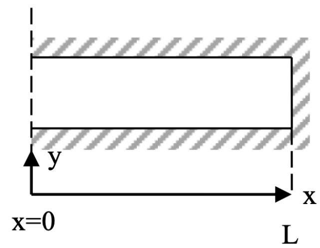

Figure 1: An illustration of the Bay of Fundy.

The main assumptions here are that the wave solution is linear and independent of y. Recall that the linearized long gravity wave equations are:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}, \tag{1}$$
$$\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0, \tag{2}$$

where η is the elevation of the free surface.

The flow in the bay is forced at the entrance of the bay by a (small) tide of O(1 m) coming from the open ocean, so:

$$\eta(x = 0, t) = \eta_0 \cos(\omega \mathrm{t})$$

where the forcing period above is $T = 12h25$' for the principal lunar constituent called M2.

(i) Derive the wave equation in terms of the elevation η.

(ii) Look for a forced solution of the form $\eta(x, t) = F(x)\cos(\omega \mathrm{t})$ which has the same shape (in time) as the forcing: derive a differential equation for F and then find $F(x)$ given boundary conditions at $x = 0$ and $x = L$.

(iii) Show that the solution can become infinitely large for certain frequencies to be found, illustrating a resonance phenomenon. Discuss.

(iv) Calculate the period of the gravest resonant mode (longest wavelength, longest period) for the Bay of Fundy when the amplitude in (iii) becomes infinite. Compare with the period of the M2 tide and discuss. Estimate the amplification factor (ratio of maximum amplitude in the Bay to amplitude at the mouth) for the M2 tide. Estimate the wavelength and draw a sketch of the surface elevation along the Bay (as a function of x).

Solution:

(i) Wave equation for η

Differentiate (2) w.r.t. t and substitute u_t from (1):

$$\eta_{tt} + H u_{xt} = 0 \quad \Longrightarrow \quad \eta_{tt} + H(-g\,\eta_x)_x = 0.$$

Thus

$$\boxed{\eta_{tt} - gH\,\eta_{xx} = 0}$$

or equivalently

$$\eta_{tt} = gH\,\eta_{xx},$$

the one-dimensional wave equation with wave speed $c = \sqrt{gH}$.

(ii) Forced harmonic solution $\eta(x,t) = F(x)\cos(\omega t)$

Assume

$$\eta(x,t) = F(x)\cos(\omega t), \qquad u(x,t) = U(x)\sin(\omega t),$$

(the phase choice ensures the time derivatives match signs conveniently). Substitute into the linear equations.

From the continuity equation $\eta_t + Hu_x = 0$:

$$-\omega F(x)\sin(\omega t) + HU'(x)\sin(\omega t) = 0 \quad \Rightarrow \quad HU'(x) = \omega F(x).$$

From the momentum equation $u_t = -g\eta_x$:

$$\omega U(x)\cos(\omega t) = -gF'(x)\cos(\omega t) \quad \Rightarrow \quad \omega U(x) = -gF'(x).$$

Differentiate the last identity with respect to x or eliminate U using $U' = \omega F/H$:

$$U' = -\frac{g}{\omega}F'' \quad \text{and} \quad U' = \frac{\omega}{H}F$$

so

$$-\frac{g}{\omega}F'' = \frac{\omega}{H}F \quad \Longrightarrow \quad F'' + \frac{\omega^2}{gH}F = 0.$$

Define

$$k^2 = \frac{\omega^2}{gH}, \qquad k = \frac{\omega}{\sqrt{gH}}.$$

Thus F satisfies

$$\boxed{F'' + k^2 F = 0.}$$

Boundary conditions. At the mouth $x = 0$: $F(0) = \eta_0$. At the head $x = L$: the bay is closed (no normal flow) so $u(L,t) = 0$ for all t. From $u(x,t) = U(x)\sin(\omega t)$ and $\omega U = -gF'$ we get

$$U(L) = 0 \quad \Longrightarrow \quad F'(L) = 0.$$

General solution and constants. General solution:

$$F(x) = A\cos(kx) + B\sin(kx).$$

From $F(0) = \eta_0$ we get $A = \eta_0$. From $F'(L) = 0$:

$$F'(x) = -Ak\sin(kx) + Bk\cos(kx) \quad \Rightarrow \quad -Ak\sin(kL) + Bk\cos(kL) = 0,$$

hence

$$B = A\tan(kL) = \eta_0\tan(kL).$$

Therefore

$$\boxed{F(x) = \eta_0\big[\cos(kx) + \tan(kL)\sin(kx)\big]} \quad \text{, with } k = \frac{\omega}{\sqrt{gH}}.$$

(iii) Resonance (singular solution) and discussion

The solution above becomes singular (formally infinite) when $\tan(kL)$ goes to infinity. That occurs when

$$kL = \left(n + \tfrac{1}{2}\right)\pi, \qquad n = 0, 1, 2, \ldots$$

At these values the linear forced solution (with no damping/friction) predicts arbitrarily large amplitude: this is *linear resonance* (standing-wave resonance) of the bay. Physically, real bays have friction, radiative losses, nonlinearities, and geometry variations that limit growth; in the ideal linear, inviscid, perfectly reflecting model the amplitude diverges. The resonant frequencies (or periods) are therefore those that satisfy the condition above.

Resonant frequencies and periods

From $k = \omega/\sqrt{gH}$ and $kL = (2n+1)\pi/2$,

$$\omega_n = \frac{(2n+1)\pi}{2L}\sqrt{gH}.$$

Hence the resonant periods are

$$T_n = \frac{2\pi}{\omega_n} = \frac{4L}{(2n+1)\sqrt{gH}}.$$

The *gravest* (longest) resonant mode is $n = 0$, giving

$$\boxed{T_0 = \frac{4L}{\sqrt{gH}}.}$$

(iv) Numerical estimates for the Bay of Fundy

Compute the characteristic speed:

$$\sqrt{gH} = \sqrt{9.8 \times 75} = \sqrt{735} \approx 27.1109\,\mathrm{m\,s^{-1}}.$$

Gravest resonant period

$$T_0 = \frac{4L}{\sqrt{gH}} = \frac{4 \times 320000}{27.1109} \approx 47\,216\,\mathrm{s} \approx 13.12\,\mathrm{h}.$$

So the gravest resonant period of the idealized rectangular bay is about **13.12 hours**.

Compare with M2 period. M2 period $T_{\mathrm{M2}} = 12\,\mathrm{h}\,25' \approx 12.4167$ h. Thus the gravest resonant period 13.12 h is fairly close to the M2 period (difference ≈ 0.7 h, or $\sim 5.5\%$). This near-match explains why the Bay of Fundy exhibits a very large tidal amplification for the semidiurnal tide: the natural period of the bay is near the forcing period.

Amplification factor for a forcing at frequency ω. From the solution

$$F(x) = \eta_0 \big[\cos(kx) + \tan(kL)\sin(kx) \big]$$

the maximum magnitude of F (for fixed ω) is for $x = L$, as $F'(L) = 0$. Thus the *amplification factor* (ratio of typical interior amplitude to η_0) is

$$A(\omega) = \big[|\cos(kL) + \tan(kL)\sin(kL)| \big] = \frac{1}{|\cos(kL)|}$$

For the M2 tide compute numerically:

First compute T_{M2} in seconds:

$$T_{\mathrm{M2}} = 12.4167 \text{ h} \times 3600 \approx 44\,700 \,\mathrm{s},$$

so

$$\omega_{\mathrm{M2}} = \frac{2\pi}{T_{\mathrm{M2}}} \approx \frac{6.283185}{44700} \approx 1.405 \times 10^{-4} \text{ s}^{-1}.$$

Then

$$k = \frac{\omega}{\sqrt{gH}} \approx \frac{1.405 \times 10^{-4}}{27.1109} \approx 5.183 \times 10^{-6} \text{ m}^{-1}.$$

So

$$kL \approx 5.183 \times 10^{-6} \times 320000 \approx 1.6586 \text{ rad}.$$

Note $\frac{\pi}{2} \approx 1.5708$ rad, so kL is slightly larger than $\pi/2$. Compute

$$\cos(kL) \approx \cos(1.6586) \approx -0.0876$$

(here sign irrelevant for amplitude). Therefore

$$A_{\mathrm{M2}} = \frac{1}{|\cos(kL)|} \approx \frac{1}{0.0876} \approx 11.4.$$

Thus an $O(1 \text{ m}))$ open-ocean M2 tidal amplitude η_0 would be amplified to about $\approx 11\eta_0$ in the bay according to this idealized linear model. If the ocean forcing amplitude were $\eta_0 \approx 1\,\mathrm{m}$ this yields an interior amplitude $\sim 11\,\mathrm{m}$, somewhat smaller than the observed $\sim 17\,\mathrm{m}$ tidal range; the remaining difference may be due to more realistic geometry (converging basin), baroclinic effects, nonlinear shoaling, funneling, and frictional phase shifts which can increase the local range and modify the effective resonant frequency.

Resonant wavelength (gravest mode). For the resonant mode $n = 0$ we had $kL = \pi/2$, so $k = \pi/(2L)$ and the wavelength

$$\lambda_0 = \frac{2\pi}{k} = \frac{2\pi}{\pi/(2L)} = 4L \approx 4 \times 320\ \text{km} = 1280\,\text{km}.$$

This very large wavelength (compared with the bay length) is typical of long (shallow-water) gravity waves: the resonant standing mode in the bay corresponds to a quarter of a wavelength fitting in the bay (node at the mouth, antinode at the head).

Sketch of the surface elevation along the bay. For frequencies near the gravest resonance the spatial shape (up to an overall amplitude factor) is

$$F(x) \propto \sin(kx)$$

when $kL \approx \pi/2$ (use $\cos(k(x - L)) = \cos(kx - kL)$ and $kL \approx \pi/2$). Thus the elevation is approximately sinusoidal in x starting with a small value at $x = 0$ (the mouth) and growing toward a maximum at the head $x = L$. In the exact resonant limit the amplitude grows without bound in the linear model; in practice one obtains a large antinode near the head and a node or small amplitude near the mouth.

Exercise 3: Short Waves

You know that a storm has occurred on January 1, 2018 in the North Atlantic. You observe waves on a beach in Brest which have a 12" period. The next day you observe waves that have an 8" period. Estimate the distance of the storm (assuming of course that the waves that you observe have been forced by the same storm).

Solution:

Estimate of storm distance

Observations:

$$T_1 = 12 \text{ s} \quad \text{(arrives first)}, \qquad T_2 = 8 \text{ s} \quad \text{(arrives 24 h later)}$$

Time difference:

$$\Delta t = 24 \text{ h} = 24 \times 3600 = 86400 \text{ s}.$$

For deep water the group velocity is

$$c_g(T) = \frac{gT}{4\pi},$$

so a wave group of period T emitted by the storm at distance D arrives after time

$$t(T) = \frac{D}{c_g(T)} = \frac{4\pi D}{gT}.$$

Hence the observed time delay between the T_2 and T_1 groups satisfies

$$\Delta t = t(T_2) - t(T_1) = \frac{4\pi D}{g}\left(\frac{1}{T_2} - \frac{1}{T_1}\right).$$

Solving for D,

$$D = \frac{g\,\Delta t}{4\pi\left(\frac{1}{T_2} - \frac{1}{T_1}\right)}.$$

Insert numbers ($g = 9.81 \text{ m}\,\text{s}^{-2}$, $T_1 = 12$ s, $T_2 = 8$ s):

$$D = \frac{9.81 \times 86400}{4\pi\left(\frac{1}{8} - \frac{1}{12}\right)} = \frac{9.81 \times 86400}{4\pi\left(\frac{3-2}{24}\right)} = \frac{9.81 \times 86400 \times 24}{4\pi} \approx 1.62 \times 10^6 \text{ m}.$$

Thus the storm is approximately

$$\boxed{D \approx 1.62 \times 10^6 \text{ m} \approx 1620 \text{ km}}$$

from the observation point.

Exercise 4: Long wave

The free surface of a long gravity wave is given by : $\eta(x,t) = a\,cos(kx - \omega t)$ in a fluid of depth H.

1 Compute the horizontal velocity u. Is there a phase lag with the pressure field ? Compare the amplitude of u with the phase velocity c. Derive a criterion for the validity of the linear approximation in term of the amplitude a.

2 Compute the tendency (time rate of change) $\partial\eta/\partial t$ and $\partial u/\partial t$ and try to explain the propagation of the wave through a sketch (drawing).

3 Compute the vertical velocity and discuss its variation as a function of z.

Solution:

1. Horizontal velocity, phase, and linearity criterion

From the linearized momentum equation

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x},$$

we obtain

$$u(x,t) = -\frac{gak}{\omega}\cos(kx - \omega t) = -c\frac{a}{H}\cos(kx - \omega t),$$

where $c = \sqrt{gH}$ is the long-wave (shallow-water) phase speed.

Thus, the horizontal velocity is π-out of phase with the pressure field ($p' = \rho g \eta$). The velocity amplitude is $u_{\max} = c\,a/H$.

$$\boxed{\text{Linearity criterion: } a \ll H.}$$

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2. Time tendencies and interpretation

$$\frac{\partial \eta}{\partial t} = -a\omega\sin(kx - \omega t), \qquad \frac{\partial u}{\partial t} = gak\sin(kx - \omega t).$$

The flow converges ($u_x < 0$) where η increases ($\eta_t > 0$), which raises the free surface — this produces wave propagation at speed c.

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3. Vertical velocity

From continuity $u_x + w_z = 0$ and bottom impermeability $w(-H) = 0$,

$$w(z, x, t) = -(z + H)u_x(x, t) = -(z + H)\frac{gak^2}{\omega}\sin(kx - \omega t).$$

At the surface,

$$w(0, x, t) = -H\frac{gak^2}{\omega}\sin(kx - \omega t) = -a\omega\sin(kx - \omega t),$$

consistent with the kinematic condition $\eta_t + w = 0$ at $z = 0$.

The vertical velocity grows linearly with height, from zero at the bottom to maximum at the surface.

Exercise 5: Seiche

Calculate the period of free oscillations (the basin modes are called seiches in this context) of a narrow lake of length L and depth h. ' Narrow ' implies that the modes are taken to vary along x only, (with x the coordinate axis along the length of the lake). Compare your solutions with the longest period of a few lakes which have been observed :

- Geneva lake : L=70 km, h=160m and T=73.5 mn
- Loch Earn (Scotland) : L=10 km, h=60m, T=14.5 mn
- Lake Baikal : L=665 km, h=680m, T=4.64 h

Solution:

Derivation of the modal periods

For long (shallow-water) gravity waves in a one-dimensional basin the linearized equations reduce to the wave equation for the free-surface elevation $\eta(x,t)$:

$$\eta_{tt} = gh\,\eta_{xx},$$

where subscripts denote partial derivatives. Seek separable harmonic modes

$$\eta(x,t) = F(x)\cos(\omega t).$$

This leads to the ordinary differential equation

$$F'' + \frac{\omega^2}{gh}\,F = 0.$$

Boundary conditions for a *closed* narrow lake (no normal flow at both ends) are

$$u(0,t) = u(L,t) = 0.$$

From the linear momentum relation $u_t = -g\eta_x$ (harmonic form) one finds $u \propto F'$, so $u = 0$ at the ends implies

$$F'(0) = F'(L) = 0,$$

i.e. Neumann boundary conditions for F. The eigenfunctions satisfying $F'(0) = F'(L) = 0$ are

$$F_n(x) = \cos\bigl(k_n x\bigr), \qquad k_n = \frac{n\pi}{L}, \quad n = 0, 1, 2, \ldots$$

($n = 0$ is the trivial uniform mode; the first nontrivial mode is $n = 1$).

The dispersion relation for each mode is

$$\omega_n = k_n\sqrt{gh} = \frac{n\pi}{L}\sqrt{gh}.$$

Hence the period of the n-th mode is

$$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{(n\pi/L)\sqrt{gh}} = \frac{2L}{n\sqrt{gh}}.$$

The **fundamental (gravest) seiche** corresponds to $n = 1$ and has period

$$\boxed{T_1 = \frac{2L}{\sqrt{gh}}.}$$

(For reference: if one end were open and the other closed, the fundamental would be $T = 4L/\sqrt{gh}$; the factor depends on boundary conditions.)

Numerical estimates

We compute T_1 for each lake using

$$T_1 = \frac{2L}{\sqrt{gh}}.$$

Take $g = 9.81\,\mathrm{m\,s^{-2}}$.

Geneva lake:

$$L = 70 \times 10^3\,\mathrm{m}, \quad h = 160\,\mathrm{m},$$
$$\sqrt{gh} = \sqrt{9.81 \times 160} = \sqrt{1569.6} \approx 39.62\,\mathrm{m\,s^{-1}},$$
$$T_1 = \frac{2 \times 70000}{39.62} \approx 3533.7\,\mathrm{s} \approx \boxed{58.9\,\mathrm{min}}.$$

Observed: 73.5 min.

Loch Earn:

$$L = 10 \times 10^3\,\mathrm{m}, \quad h = 60\,\mathrm{m},$$
$$\sqrt{gh} = \sqrt{9.81 \times 60} = \sqrt{588.6} \approx 24.27\,\mathrm{m\,s^{-1}},$$
$$T_1 = \frac{2 \times 10000}{24.27} \approx 824.4\,\mathrm{s} \approx \boxed{13.74\,\mathrm{min}}.$$

Observed: 14.5 min.

Lake Baikal:

$$L = 665 \times 10^3\,\mathrm{m}, \quad h = 680\,\mathrm{m},$$
$$\sqrt{gh} = \sqrt{9.81 \times 680} = \sqrt{6670.8} \approx 81.67\,\mathrm{m\,s^{-1}},$$
$$T_1 = \frac{2 \times 665000}{81.67} \approx 16\,284.1\,\mathrm{s} \approx \boxed{4.52\,\mathrm{h}}.$$

Observed: 4.64 h.

Lake	L (km)	h (m)	T_1 (predicted) (minutes or hours)	T_{obs} (given)
Geneva	70	160	58.9 min	73.5 min
Loch Earn	10	60	13.74 min	14.5 min
Lake Baikal	665	680	4.52 h	4.64 h

Discussion of differences

The simple rectangular, depth-uniform model with perfectly reflecting ends gives an excellent estimate for Loch Earn and Lake Baikal (predicted periods within a few percent of observations). For Geneva lake the predicted period ($\sim$ 59 min) is substantially shorter than the observed $\sim$ 73.5 min. Causes for discrepancies (particularly for Geneva) include:

- **Non-uniform depth and basin geometry:** Real lakes are not rectangular and have varying cross-section and bathymetry. An effective depth or effective length different from the nominal h and L will change T.
- **Multiple basins / internal divisions:** Lakes with sills or multiple connected basins can have modes with longer effective wavelengths and periods.

- **Boundary conditions:** If one end behaves more like an open boundary (partial radiation) the modal structure and period change; partially reflecting ends shift frequencies.
- **Stratification and internal seiches:** Density stratification can support internal seiches with very different periods; if the observed mode is internal rather than surface, the surface period may not match the simple surface-wave estimate.
- **Friction and damping:** Friction and viscous effects slightly shift and broaden modal frequencies.
- **Measurement uncertainty / definition:** The reported observed period may correspond to a different modal index or an overtone rather than the fundamental.

In practice one can often improve agreement by (i) using a depth-averaged effective depth (or computing eigenmodes for the true bathymetry), and (ii) accounting for partially reflecting boundaries or basin narrowing.