Internal Waves in the Ocean

Master 2 — Physique de l'Océan et du Climat

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Outline

1. A general introduction to ocean waves
2. What are internal waves ? Why do we study internal waves ?
3. Internal waves in the two-layer shallow-water model
4. **Internal waves in the continuously-stratified model**
5. Generation of internal waves
6. Propagation of internal waves
7. Dissipation of internal waves and impacts

Local Static Stability

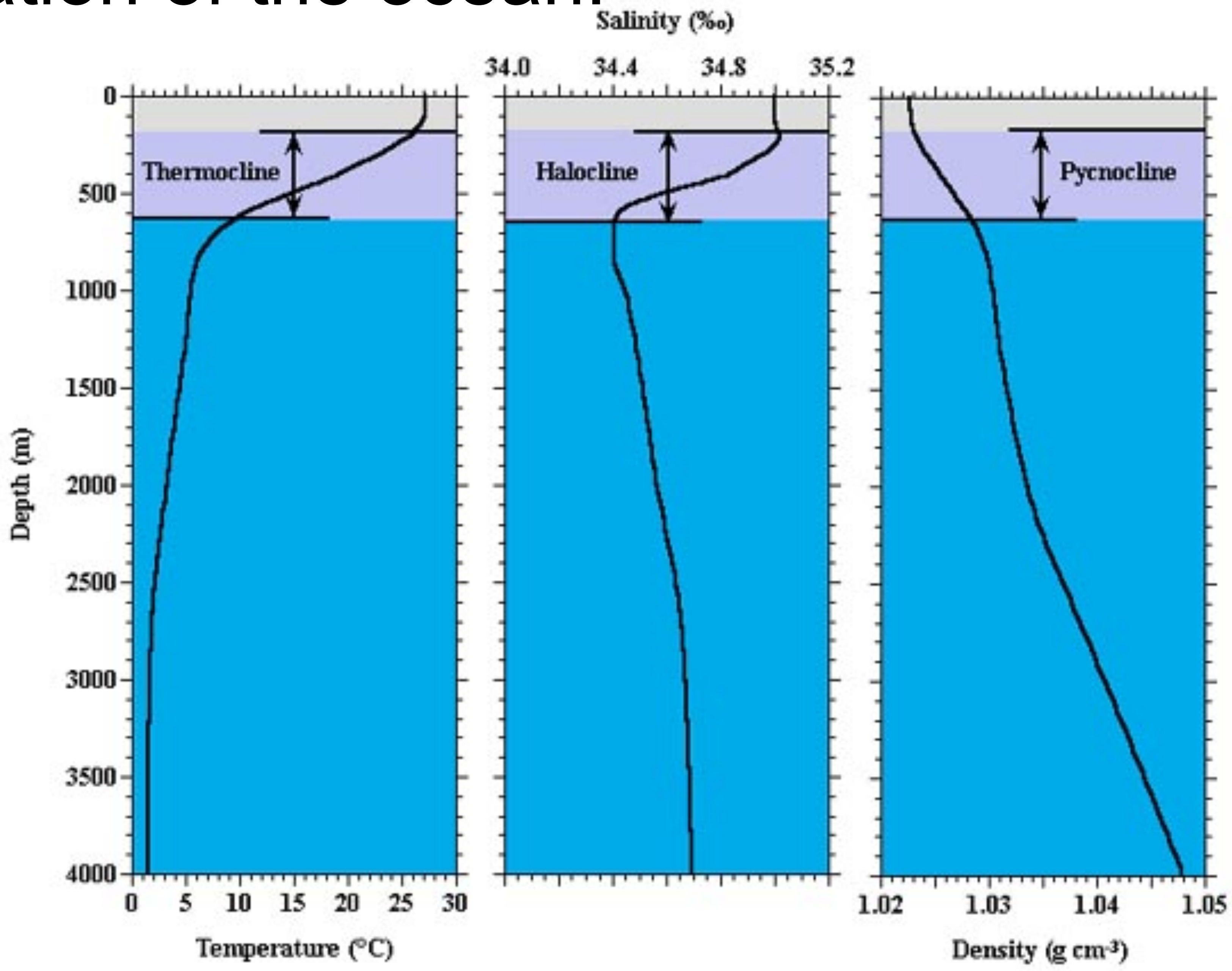
- Stratification:

$$\rho = \rho_0(z)$$

- A fluid is
  - stably stratified if a displaced parcel tends to return to its original position,
  - unstably stratified if it tends to move further away from its original position
  - neutrally stratified if it tends to stay where it is.

Local Static Stability

Stratification of the ocean:

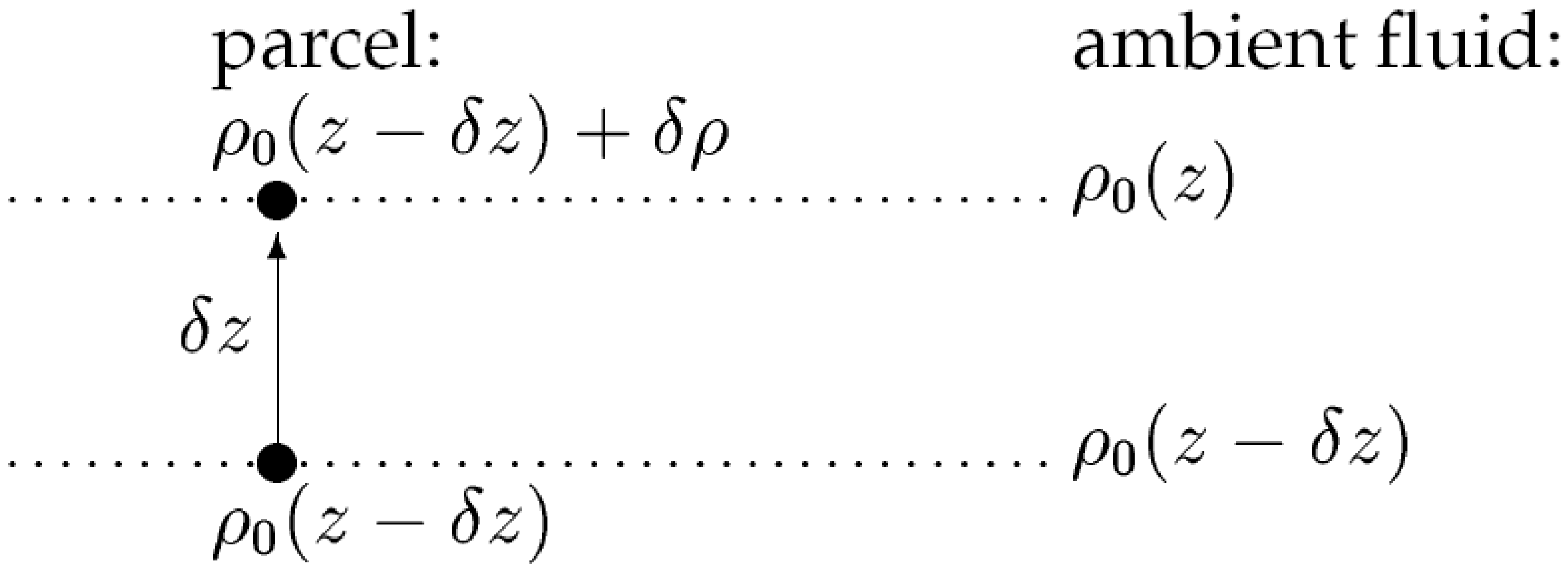


with Equation of state :

$$\rho = \rho(T, S, p)$$

Local Static Stability

- Let’s move a parcel:



- Buoyancy force:

$$\rho_0(z)\ddot{\delta z} = g\left(\rho_0(z) - \rho_0(z-\delta z) - \delta\rho\right)$$

Local Static Stability

- With background density variation:

$$\rho_0(z) - \rho_0(z - \delta z) = \frac{d\rho_0}{dz}\delta z$$

- And parcel density variation?
  - *From thermodynamics, if entropy and salinity are conserved during displacement:*

$$\delta \rho = \left(\frac{\partial \rho}{\partial p}\right)_{\eta, S} \delta p = c_s^{-2}\delta p$$

Where c_s is the speed of sound

Local Static Stability

- So we get:

$$\rho_0(z)\ddot{\delta z} = g\left(\frac{d\rho_0}{dz}\delta z + \frac{\rho_0 g \delta z}{c_s^2}\right)$$

Local Static Stability

- Simple Harmonic oscillator:

$$\underbrace{\ddot{\delta z} - \frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)}_{N^2}\delta z = 0$$

- Brunt-Vaisala frequency:

- Solutions: $e^{\pm \mathrm{i} N t}$

$$\ddot{\delta z} + N^2 \delta z = 0$$

- Stable if $N^2 > 0$

Local Static Stability

- Simple Harmonic oscillator:

$$\ddot{\delta z} + N^2 \delta z = 0$$

- Solutions: $e^{\pm \mathrm{i} N t}$

- Stable if $N^2 > 0$

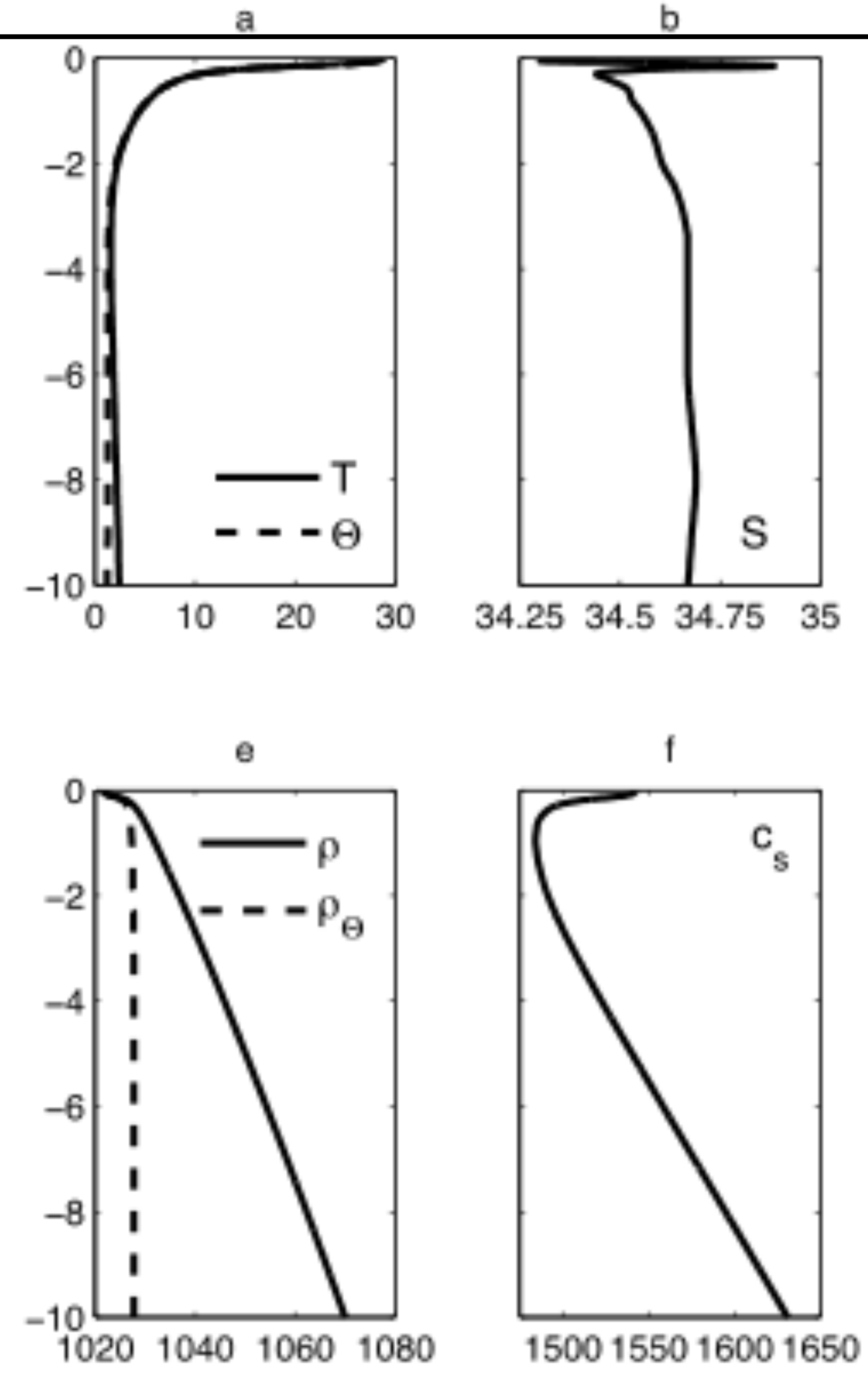
The parcel oscillates vertically at frequency *N* about its equilibrium position.

Local Static Stability

- Brunt-Vaisala frequency:

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)$$

The effect of compressibility is often neglected in the upper ocean but it is not true in general.



Vertical profiles of T, S, in-situ and potential density, and speed of sound in the Mindanao Trench.

Local Static Stability

- How do you connect density and stability?

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)$$

- It is convenient to define **the generalized potential density,** corresponding to the density that the parcel would attain if moved from z to a reference level z_r, under conservation of its entropy and salinity.

- We compute it by vertically integrating:
-

$$\delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_{\eta,S}\delta p = c_s^{-2}\delta p$$

Local Static Stability

- Which gives the generalized potential density:

$$\rho_r(z_r, z) = \rho_0(z) + g \int_{z_r}^{z} \frac{\rho_0(z')}{c_s^2(z, z')} dz'$$

- Such that:

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_r}{dz}(z_r, z) + \frac{g^2}{\rho_0}\int_{z_r}^{z} \rho_0(z')\frac{\partial c_s^{-2}}{\partial z}(z, z')dz'$$

- and for $z_r = z$:

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_r}{dz}(z, z)$$

Local Static Stability

- In practice we use potential density, where z_r is fixed. For example at the surface $(z_r = 0)$:

$$\rho_\Theta(z) = \rho_0(z) + g \int_0^z \frac{\rho_0(z')}{c_s^2(z,z')} dz'$$

Which is the density that the parcel would acquire if adiabatically brought to the surface.

- Such that:

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_\Theta}{dz} + \frac{g^2}{\rho_0}\int_0^z \rho_0(z')\frac{\partial c_s^{-2}}{\partial z}(z,z')dz'$$

Equations for a stratified flow

- Navier-Stokes Equations:

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}} + \vec{\mathcal{D}}$$

Momentum equations

$$\frac{D\rho}{Dt} + \rho\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation (no source/sink)

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{DP}{Dt}$$

Thermodynamic ‘energy’ equation
 (no diabatic effects)

Equations for a stratified flow

- Approximations for the momentum equation:

No forcings/dissipation

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} \cancel{+ \vec{\mathcal{F}} + \vec{D}}$$

Equations for a stratified flow

- Boussinesq Approximation:

Density perturbations small compared to mean background value:

$$\rho - \rho^* << \rho^*$$

Linearize all terms involving a product with density, except the gravity term which is already linear:

$$\rho \vec{u} \rightarrow \rho^* \vec{u}$$

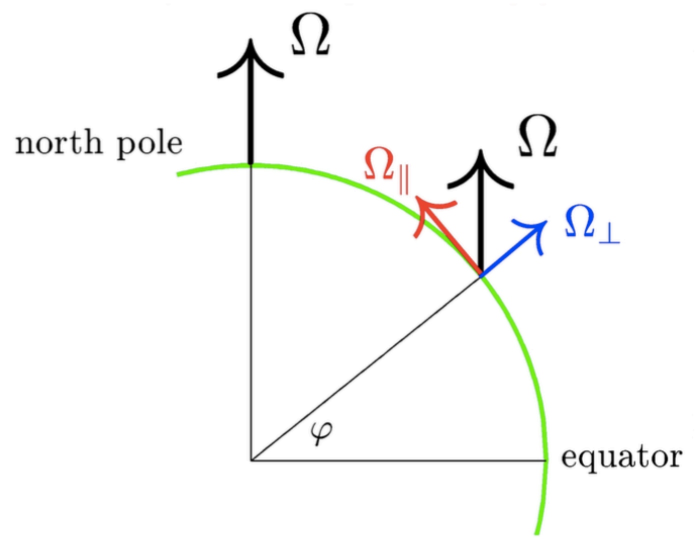
$$\rho g \rightarrow \rho g$$

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- Traditional Approximation:

= neglect horizontal Coriolis term



$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

$$\frac{D\vec{u}}{Dt} + f\vec{k} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- We think as internal waves as a perturbation of a (known) static background state that has only vertical dependences

$$P = p_0(z) + p'(t, \vec{x})$$
$$\rho = \rho_0(z) + \rho'(t, \vec{x})$$

with $$\frac{dp_0}{dz} = -\rho_0 g$$

- And linearize momentum equations:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} + \frac{\rho'}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Equations for a stratified flow

- For the thermodynamic equation:

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{DP}{Dt}$$

- We can write:

$$\frac{D\rho}{Dt} = \frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z}$$

- And:

$$\frac{1}{c_s^2}\frac{DP}{Dt} = \frac{1}{c_s^2}\frac{Dp'}{Dt} + \frac{1}{c_s^2} w\frac{\partial p_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

- So :

$$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

Equations for a stratified flow

- We linearize:

$$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

$$\Big\downarrow$$

$$\frac{\partial\rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

-

Equations for a stratified flow

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

- We can show that the pressure time tendencies are small using the scalings:

$$= \underbrace{\frac{1}{c_s^2}\frac{\partial p'}{\partial t}}_{P/(Tc_s^2)} \quad \underbrace{-w\frac{\rho_0 g}{c_s^2}}_{W\rho^* g/c_s^2}$$

- with scalings for variables p, t, w, u, ρ_0 written as P, T, W, U, ρ^*
- and we have $P \sim \rho^* CU$ assuming inertia-gravity waves, with $C = L/T$ the phase speed of the wave
- and $W \sim UH/L$ based on the divergence equation

Equations for a stratified flow

- The ratio between the terms is:

$$\frac{\dfrac{1}{c_s^2}\dfrac{\partial p'}{\partial t}}{w\dfrac{\rho_0 g}{c_s^2}} \sim \frac{P/(Tc_s^2)}{W\rho^* g/c_s^2} = \frac{C^2}{gH} = \frac{C^2}{c_{sf}^2}$$

- with the fluid velocity U, the internal wave phase speed C, the surface gravity wave phase speed $c_{sf} = \sqrt{gH}$, and the sound speed c_s
- They scale as:

$$U \sim O(10^{-1}); \quad C \sim O(1); \quad c_{sf} \sim O(10^1, 10^2); \quad c_s \sim O(10^3)\,\mathrm{m\,s^{-1}}.$$

- such that $\boxed{U \ll C \ll c_{sf} \ll c_s .}$ and thus: $\dfrac{C^2}{c_{sf}^2} \ll 1$

Equations for a stratified flow

- We get:

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = -w\frac{\rho_0 g}{c_s^2}$$

- So we can write:

$$\frac{\partial \rho'}{\partial t} + \left[\frac{\partial \rho_0}{\partial z} + \frac{\rho_0 g}{c_s^2}\right] w = 0$$

- Which gives

$$-\frac{g}{\rho^*}\frac{\partial \rho'}{\partial t} + N^2 w = 0$$

- With

$$N^2 = -\frac{g}{\rho^*}\left(\frac{\partial \rho_0}{\partial z} + \frac{\rho_0 g}{c_s^2}\right)$$

Equations for a stratified flow

- And finally introducing buoyancy: $$b = -g\frac{\rho'}{\rho^*}$$
-

$$\frac{\partial b}{\partial t} + N^2 w = 0$$

Equations for a stratified flow

- For the mass equation:

$$\frac{D\rho}{Dt} + \rho\vec{\nabla} \cdot \vec{u} = 0$$

- We rewrite:

- $$\rho\nabla \cdot \vec{u} = -\frac{D\rho}{Dt} = -\frac{1}{c_s^2}\frac{Dp'}{Dt} + w\frac{\rho_0 g}{c_s^2}$$

Equations for a stratified flow

- We look at the scales of the different terms:

$$\overbrace{\rho\frac{\partial u}{\partial x}}^{\rho_* U/L} + \overbrace{\rho\frac{\partial v}{\partial y}}^{\rho_* U/L} + \overbrace{\rho\frac{\partial w}{\partial z}}^{\rho_* W/H} = -\overbrace{\frac{1}{c_s^2}\frac{\partial p'}{\partial t}}^{P/(Tc_s^2)} - \overbrace{\frac{u}{c_s^2}\frac{\partial p'}{\partial x}}^{UP/(Lc_s^2)} - \overbrace{\frac{v}{c_s^2}\frac{\partial p'}{\partial y}}^{UP/(Lc_s^2)} - \overbrace{\frac{w}{c_s^2}\frac{\partial p'}{\partial z}}^{WP/(Hc_s^2)} + \overbrace{w\frac{\rho_0 g}{c_s^2}}^{\rho_* W g/c_s^2}$$

- We assume again a separation in time scales:

$$\boxed{U \ll C \ll c_{sf} \ll c_s\,.}$$

$$U \sim O(10^{-1})\,;\quad C \sim O(1)\,;\quad c_{sf} \sim O(10^1\,,10^2)\,;\quad c_s \sim O(10^3)\,\mathrm{m\,s^{-1}}\,.$$

- If we remove small terms we simply get: $\vec{\nabla}\cdot\vec{u} = 0$

Equations for a stratified flow

- Linearized Equations are:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} - b\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation (no source/sink)

$$\frac{\partial b}{\partial t} + N^2 w = 0$$

Thermodynamic ‘energy’ equation
(no diabatic effects)

4. The continuously-stratified model

We have a set of 5 equations for 5 variables:

$$u_t - fv = -\frac{p_x}{\rho^*} \quad (1)$$

$$v_t + fu = -\frac{p_y}{\rho^*} \quad (2)$$

$$w_t - b = -\frac{p_z}{\rho^*} \quad (3)$$

$$u_x + v_y + w_z = 0 \quad (4)$$

$$b_t + N^2 w = 0 \quad (5)$$

- Activity:
- Write an equation for w alone.

Equations for a stratified flow

- Finally we get an equation for *w* alone:

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0$$

Solutions of the equation

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0$$

- We suppose waves to be sinusoidal in time:

$$w = \hat{w} e^{-\mathrm{i}\omega t}$$

- And get:

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

Solutions of the equation

Two methods to solve the equation:

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

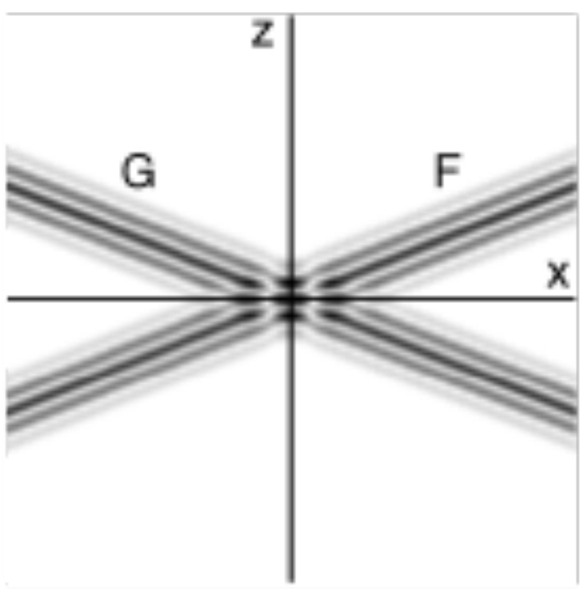
1. The method of characteristics
2. The method of modes

Solutions of the equation

1. Method of characteristics

- N = cste
- Arbitrary boundary conditions
- Solution of the form:

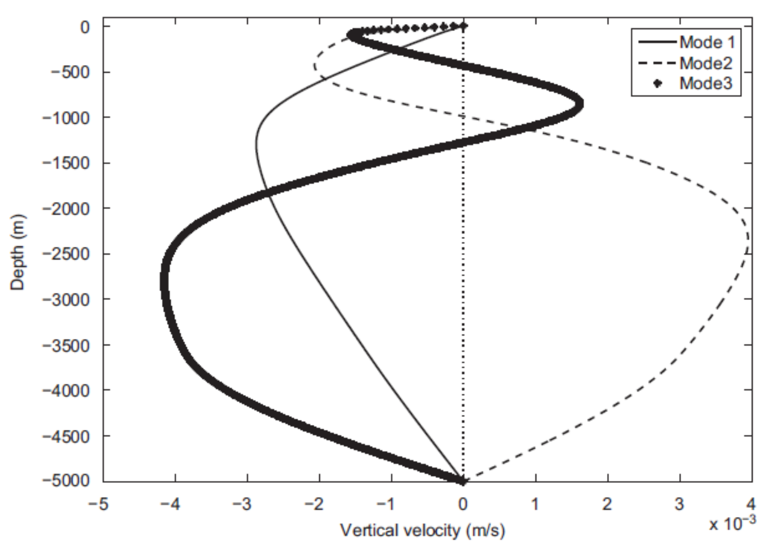
$$\hat{w} = F(\mu_+ x - z) + G(\mu_- x - z)$$



2. Method of Modes

- N = N(z)
- Horizontal surface/bottom
- Solution of the form:

$$w = W(z)e^{-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}ly}$$



Method of characteristics

Assumptions:

- stratification is constant, $N = cst$
- the domain is infinite, that is, there is no boundary effect
- waves are sinusoidal in time and propagate in the $x - z$ plane (no variation in the y direction):

  $$w(x, y, z, t) = \hat{w}(x, z)\ \exp(i\omega t)$$

The equation for w becomes: $\boxed{\hat{w}_{xx} - \dfrac{\omega^2 - f^2}{N^2 - \omega^2}\hat{w}_{zz} = 0}$ (10)

Method of characteristics

The equation for w becomes: $\hat{w}_{xx} - \dfrac{\omega^2 - f^2}{N^2 - \omega^2}\hat{w}_{zz} = 0$

The method of characteristics consists in changing the variables x, z into ξ_+, ξ_- :

$$\xi_\pm = \mu_\pm x - z \text{ with } \mu_\pm = \pm\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}.$$

NB: we assume that $f < \omega < N$ so that $\mu_\pm$ exists.

Method of characteristics

Now we use $\hat{w}(x,z) = \bar{w}(\xi_+, \xi_-)$ with the properties:

$$\frac{\partial \hat{w}}{\partial x} = \frac{\partial \bar{w}}{\partial \xi_+}\frac{\partial \xi_+}{\partial x} + \frac{\partial \bar{w}}{\partial \xi_-}\frac{\partial \xi_-}{\partial x}$$, with $\frac{\partial \xi_+}{\partial x} = \mu_+$ and $\frac{\partial \xi_-}{\partial x} = \mu_-$.

After some reorganising,

$$\frac{\partial^2 \hat{w}}{\partial x^2} = \mu_+^2 \left(\frac{\partial^2 \bar{w}}{\partial \xi_+^2} + \frac{\partial^2 \bar{w}}{\partial \xi_-^2} - 2\frac{\partial^2 \bar{w}}{\partial \xi_+ \partial \xi_-} \right)$$ and

$$\frac{\partial^2 \hat{w}}{\partial z^2} = \frac{\partial^2 \bar{w}}{\partial \xi_+^2} + \frac{\partial^2 \bar{w}}{\partial \xi_-^2} + 2\frac{\partial^2 \bar{w}}{\partial \xi_+ \partial \xi_-}.$$

Method of characteristics

The equation for $\hat{w}$ reduces to:

$\hat{w}_{xx} - \mu_+^2 \hat{w}_{zz} = \ldots = \boxed{-4\mu_+^2 \dfrac{\partial^2 \bar{w}}{\partial \xi_+ \partial \xi_-} = 0.}$ That is the new PDE to solve.

Any function of the form $\bar{w}(\xi_+, \xi_-) = F(\xi_+) + G(\xi_-)$ is a solution of the PDE,
with F and G arbitrary functions.

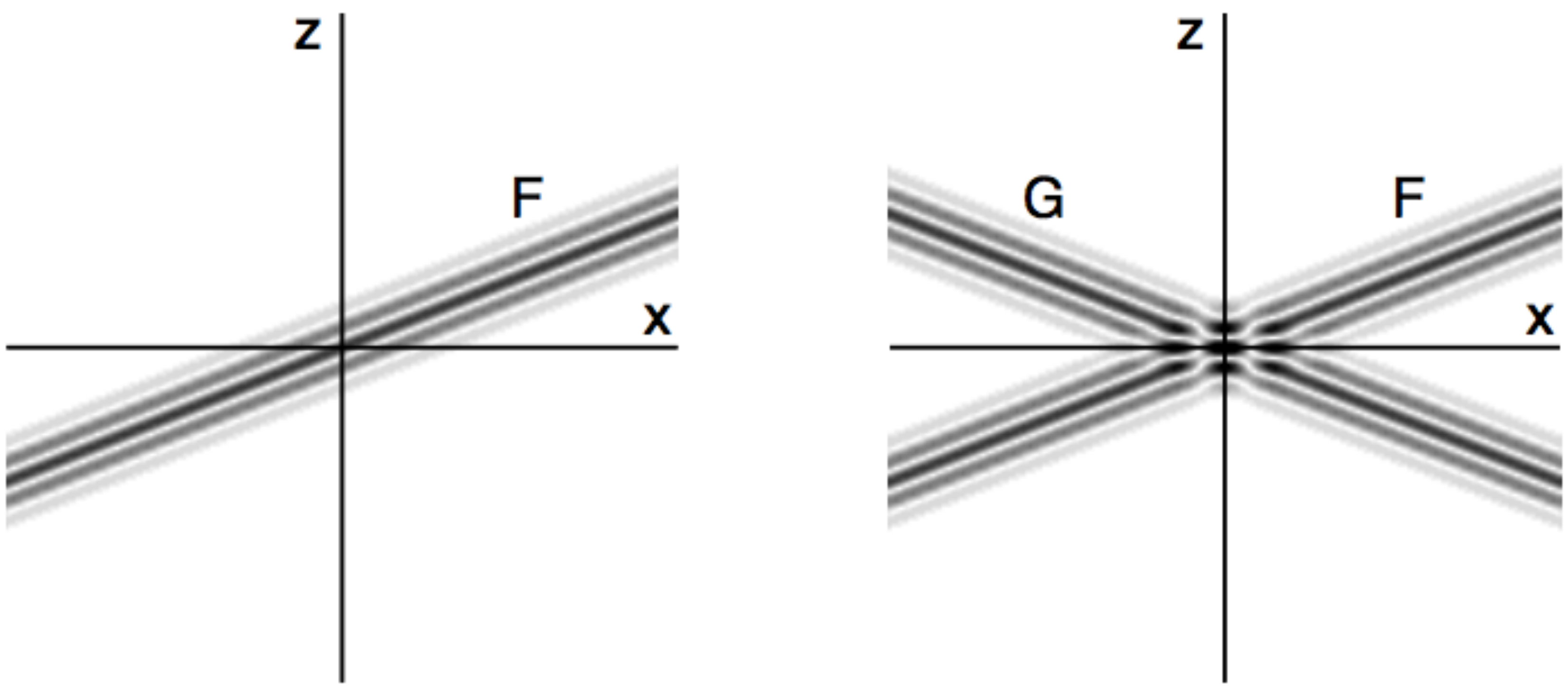
The general solution is thus $\boxed{\hat{w}(x, z) = F(\mu_+ x - z) + G(\mu_- x - z)\ .}$

Solutions: Method of characteristics

- For example we use: $$F(\xi) = \exp(-\xi^2)\exp(ik\xi)$$

- The solution is then, with $$\xi_\pm = \mu_\pm x - z$$

$$w = \exp(-\xi_+^2)\cos(k\xi_+ - \omega t) + \exp(-\xi_-^2)\cos(k\xi_- - \omega t)$$



Solutions: Method of characteristics

- Energy propagates along the lines: $$\mu_{\pm} x - z = cste$$

which are the characteristic coordinates, and which are diagonals in the x, z-plane

Solutions: Method of characteristics

$$(N^2 - \omega^2)\hat{w}_{xx} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

Activity:

- Assuming a solution of the form: $\hat{w} = w_0 e^{\mathrm{i}(kx+mz)}$
- Write the dispersion relation $\omega(k, m)$

Solutions: Method of characteristics

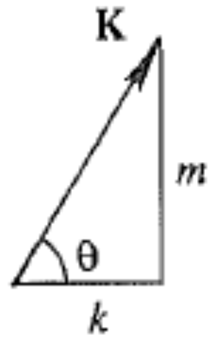
- The dispersion relation is:

$$\boxed{\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2 + m^2}}$$

Solutions: Method of characteristics

- The dispersion relation:

$$\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2 + m^2}$$



can be simplified by using polar coordinates:

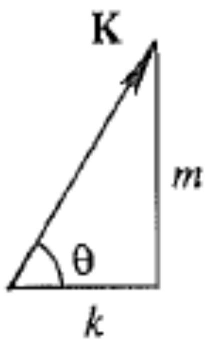
$$\vec{k} = (k, m) = \kappa(\cos\theta, \sin\theta)\,; \quad \kappa = (k^2 + m^2)^{1/2}\,,$$

Such that:

Solutions: Method of characteristics

- The dispersion relation:

$$\omega^2 = \frac{N^2 k^2 + f^2 m^2}{k^2 + m^2}$$



can be simplified by using polar coordinates:

$$\vec{k} = (k, m) = \kappa(\cos\theta, \sin\theta)\,; \quad \kappa = (k^2 + m^2)^{1/2}\,,$$

Such that:

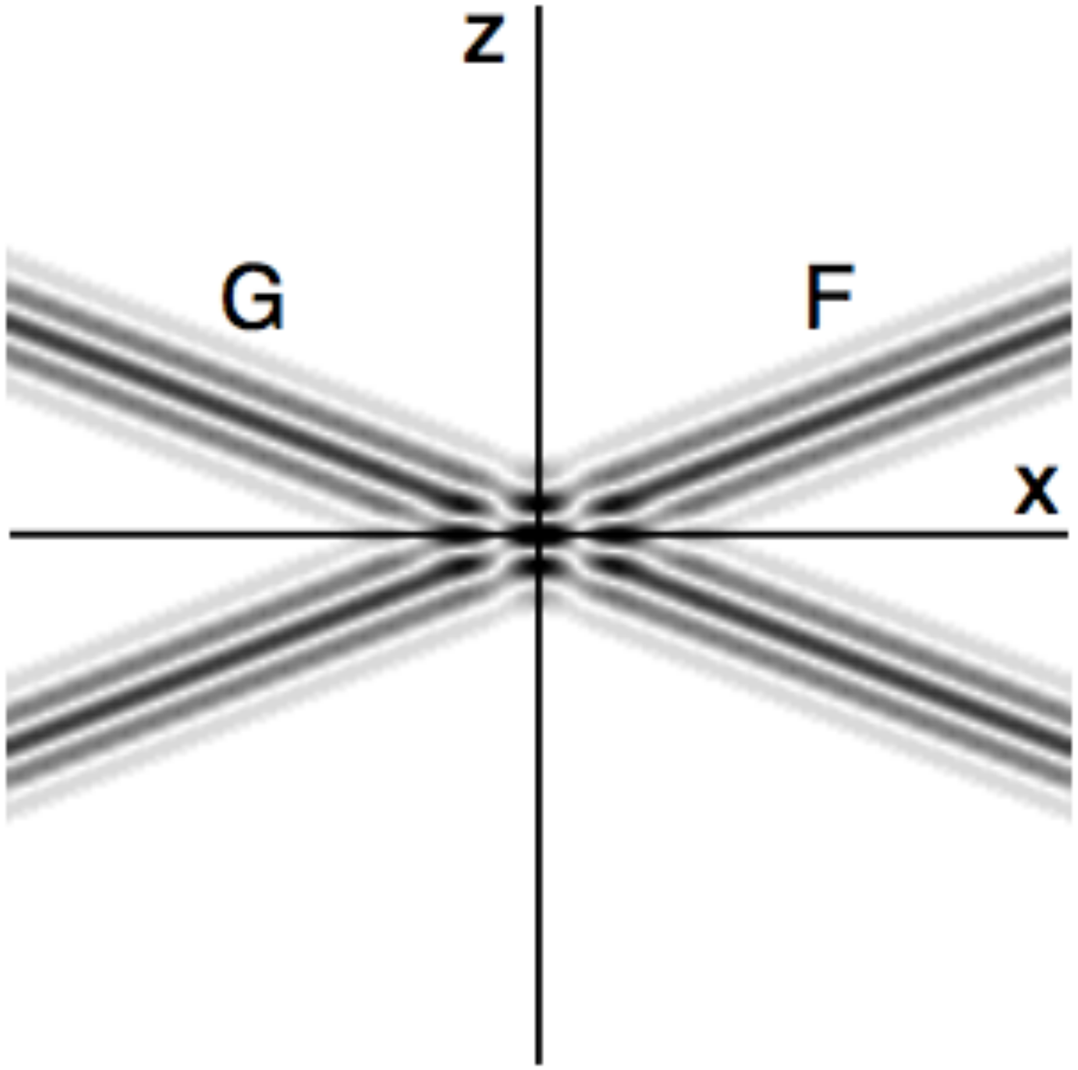
$$\boxed{\omega^2 = N^2 \cos^2\theta + f^2 \sin^2\theta}$$

The wave frequency depends only on the direction θ of propagation and of the local N and f.

if the frequency is imposed (e.g., by tidal forcing), all waves propagate at fixed angles from the horizontal.

Solutions: Method of characteristics

- The four possible configurations of internal wave beams created by an oscillating body at fixed frequency:



Solutions: Method of characteristics

- The group velocity: $$\vec{c}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial m}\right)$$

- Can be written using the new coordinates:

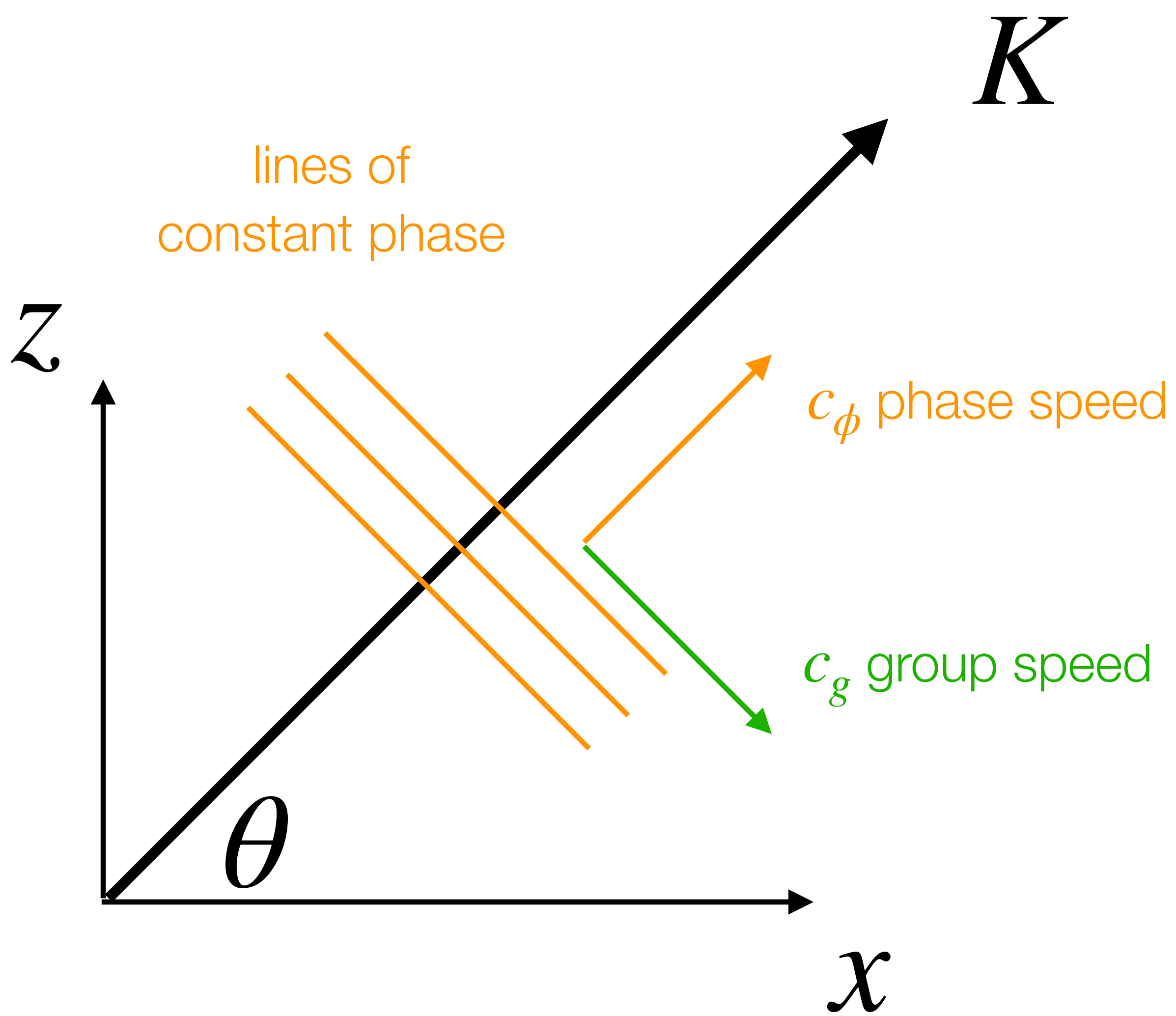
$$\omega(k(\kappa,\theta), m(\kappa,\theta)) = \bar{\omega}(\kappa,\theta)$$

- Such that $$\frac{\partial \bar{\omega}}{\partial \kappa} = \frac{\partial \omega}{\partial k}\frac{\partial k}{\partial \kappa} + \frac{\partial \omega}{\partial m}\frac{\partial m}{\partial \kappa} = \vec{c}_g \cdot \vec{k}/\kappa$$

- ω does not depend on κ so: $$\vec{c}_g \cdot \vec{k} = 0$$ $$\boxed{\vec{c}_g \perp \vec{k}}$$

The group velocity is always perpendicular to the phase speed and makes an angle θ with the vertical.

Solutions: Method of characteristics

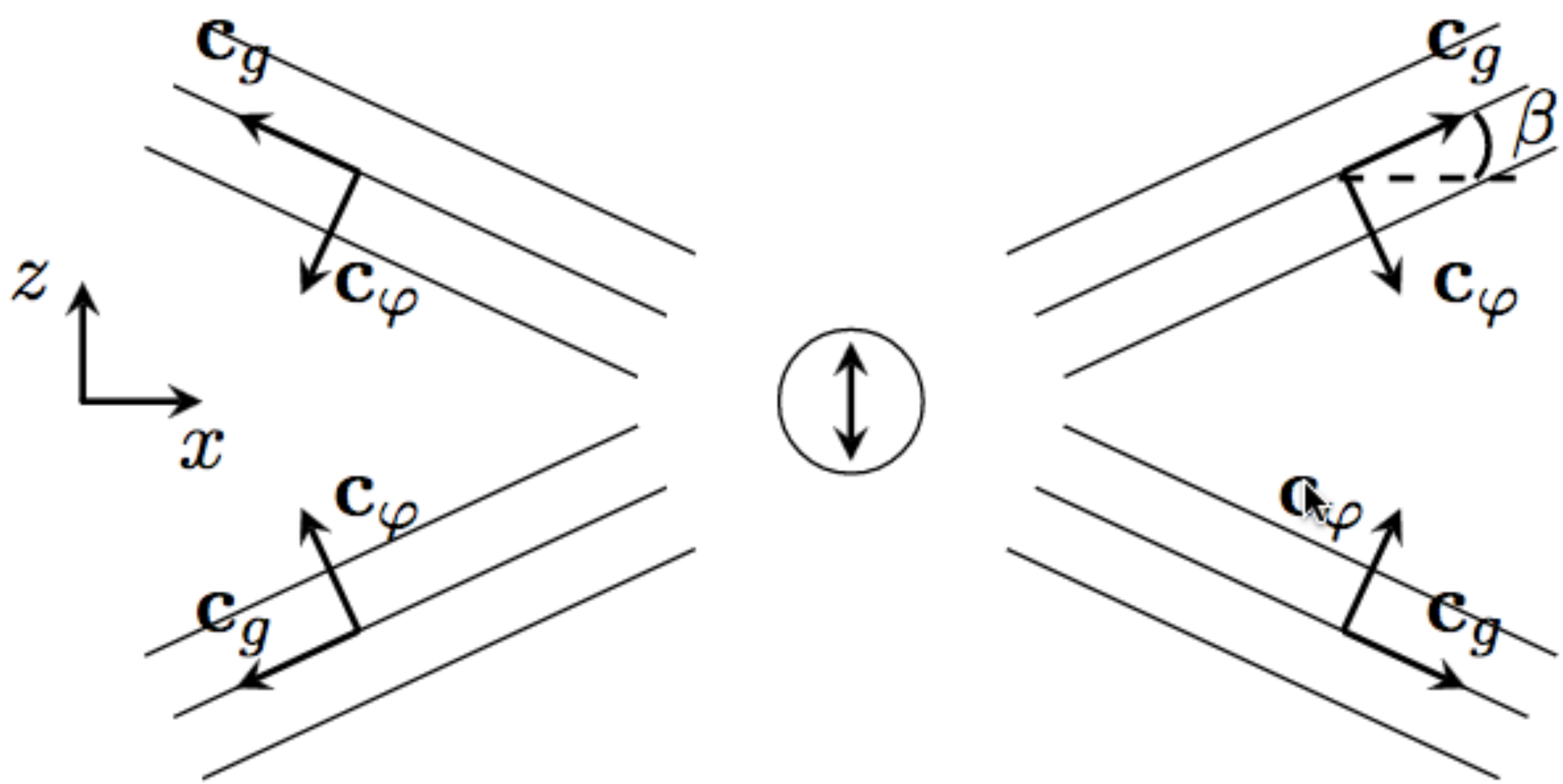


$$\tan\theta = \pm\left(\frac{N^2-\omega^2}{\omega^2-f^2}\right)^{1/2}$$

The group velocity is always perpendicular to the phase speed and makes an angle θ with the vertical.

Solutions: Method of characteristics

- The four possible configurations of internal wave beams created by an oscillating body at fixed frequency:



A famous experiment

J. Fluid Mech. (1967), *vol.* 28, *part* 1, *pp.* 1–16 1
Printed in Great Britain

A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density stratified liquid

By D. E. MOWBRAY AND B. S. H. RARITY†
Department of the Mechanics of Fluids, University of Manchester

L'expérience de MOWBRAY et RARITY (1967) conçue pour vérifier la relation de dispersion est à ce stade particulièrement intéressante pour bien comprendre les conséquences de résultats inhabituels. Il s'agit d'observer dans une cuve rectangulaire les ondes générées par un barreau oscillant verticalement à une fréquence donnée ω. La cuve est remplie d'eau salée de manière à obtenir une stratification linéaire (l'eau plus salée et donc plus dense se situe par conséquent en bas) et les ondes sont observées par ombroscopie, en utilisant le fait que l'indice optique dépend de la concentration en sel. Si des mouvements sont forcés à la pulsation ω_f dans un fluide de gradient de masse volumique constant, toute onde de vecteur d'onde faisant un angle $\beta = \arcsin(\omega_f / N)$ avec la verticale est théoriquement excitée. Comme les rayons sont donnés par les directions de la vitesse de groupe, on voit donc apparaître *une croix* (*cf.* figure 3), avec quatre branches, chacune faisant un angle β avec l'horizontale. On est donc bien loin de l'onde circulaire que donnerait une émission acoustique localisée par exemple.

Gostiaux, Dauxois, BUP868

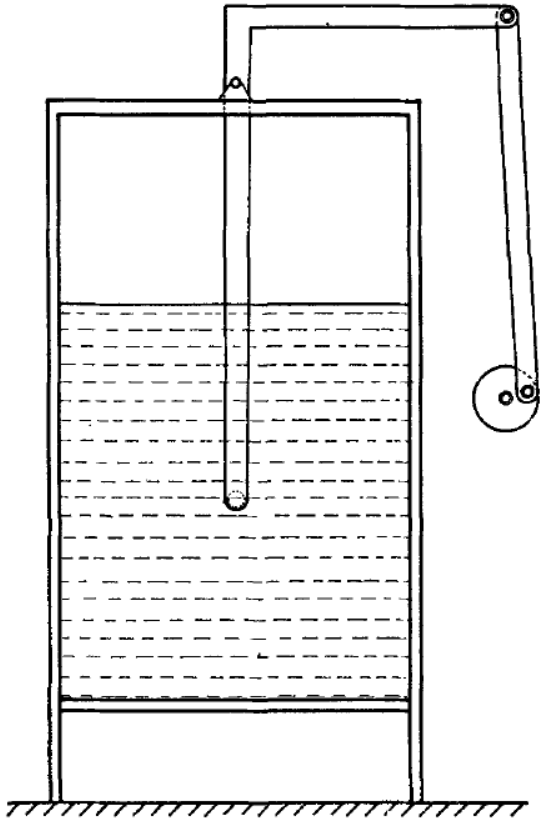


FIGURE 4. A sketch of the apparatus.

Journal of Fluid Mechanics, Vol. 28, part 1 *Plate 1*

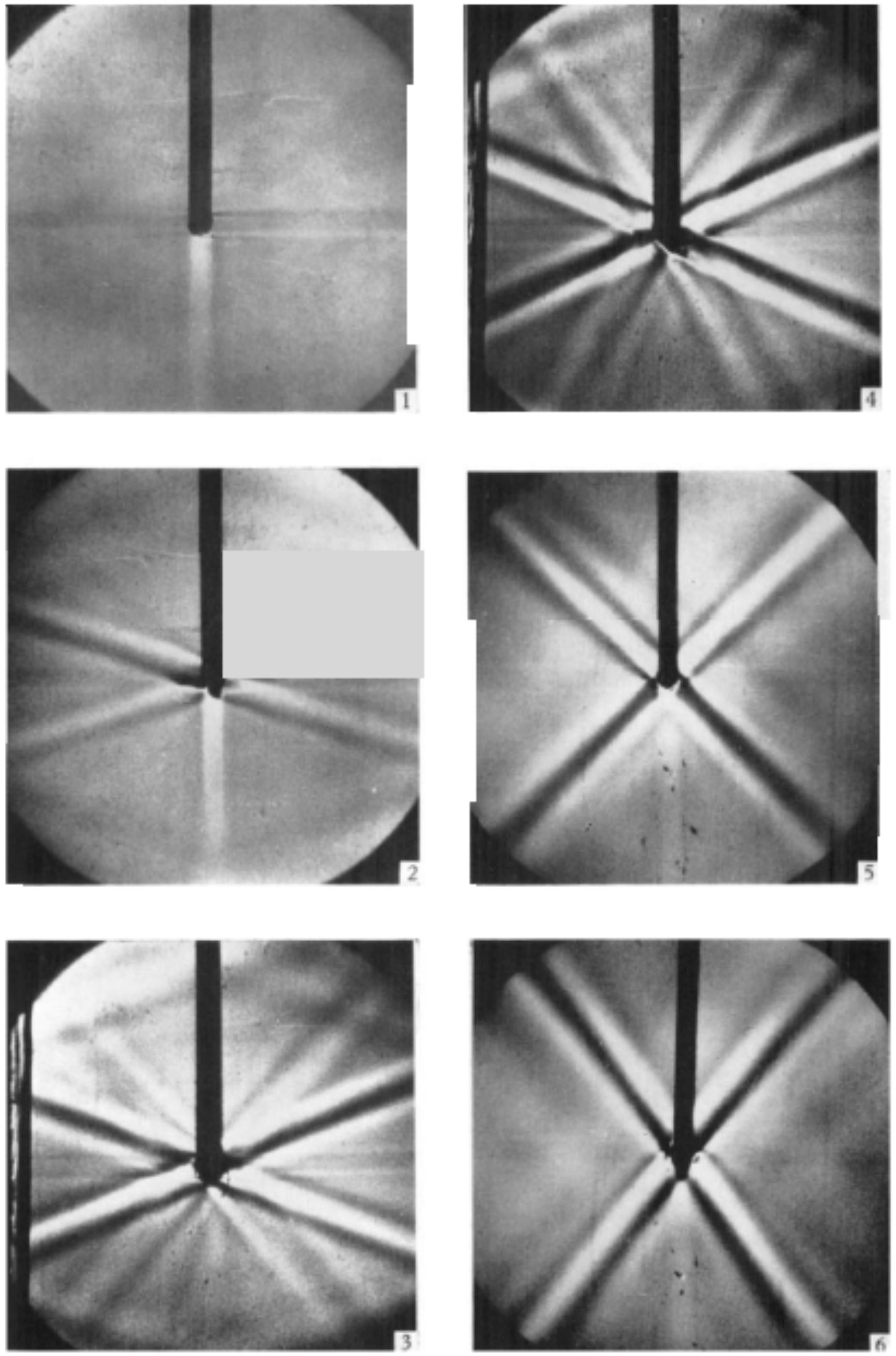


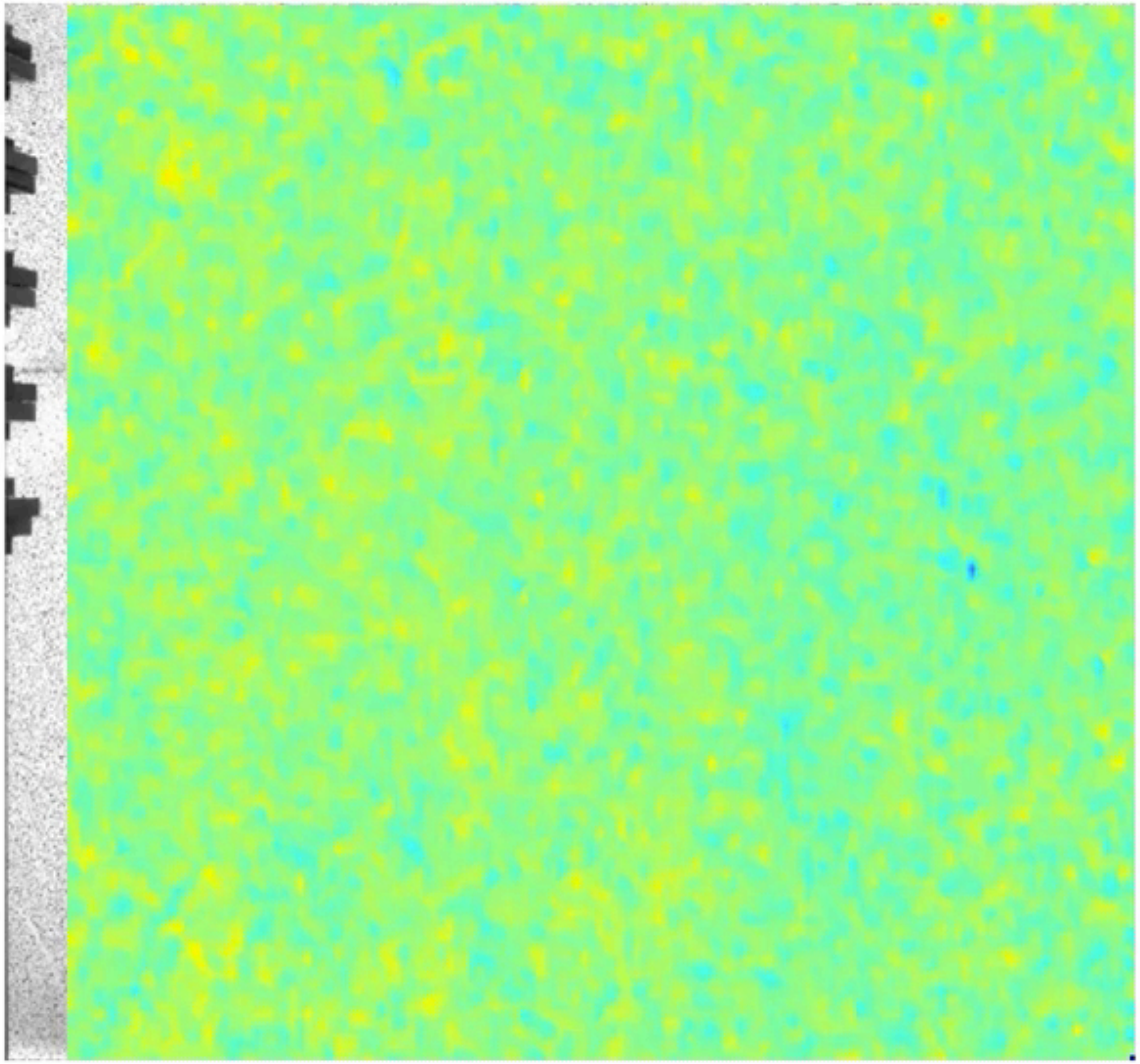
PLATE 1. (1) The image of the undisturbed fluid. (2) $\omega/\omega_0 = 0{\cdot}318$. (3) $\omega/\omega_0 = 0{\cdot}366$. (4) $\omega/\omega_0 = 0{\cdot}419$. (5) $\omega/\omega_0 = 0{\cdot}615$. (6) $\omega/\omega_0 = 0{\cdot}699$.

6. Conclusion

The predictions of the small amplitude theory of the phase configurations of internal waves in a stratified fluid have been tested and confirmed.

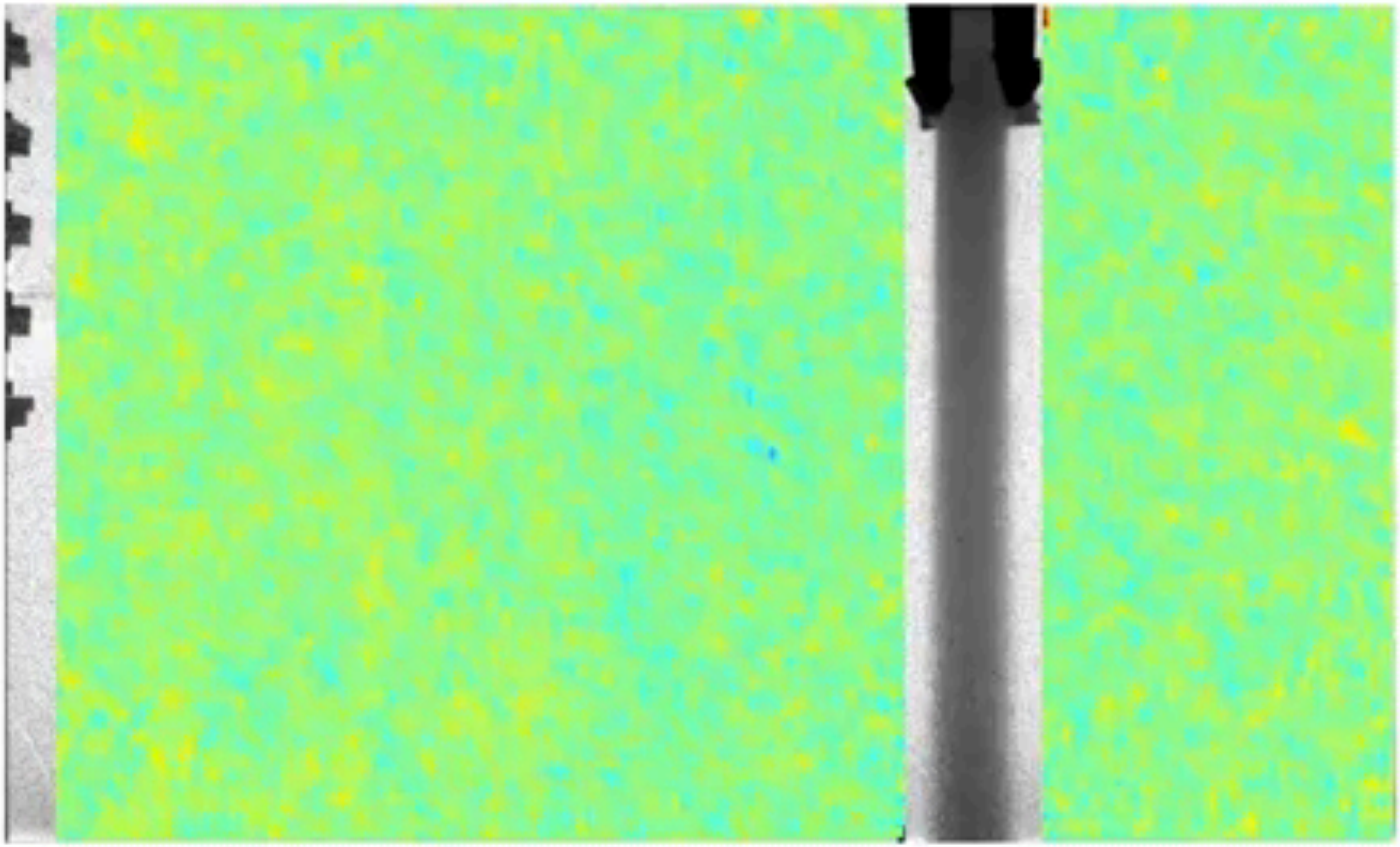
https://www.jgula.fr/Ondes/MowbrayRarity67.pdf

Solutions: Method of characteristics



[(c) E. Horne]

Solutions: Method of characteristics



[(c) E. Horne]

Solutions: Method of characteristics

- Solutions exist only in a range of allowable internal-wave frequencies:

$$\text{(I)} \quad N \leq \omega \leq |f| \quad \text{or} \quad \text{(II)} \quad |f| \leq \omega \leq N$$

- 2 extreme cases can occur:

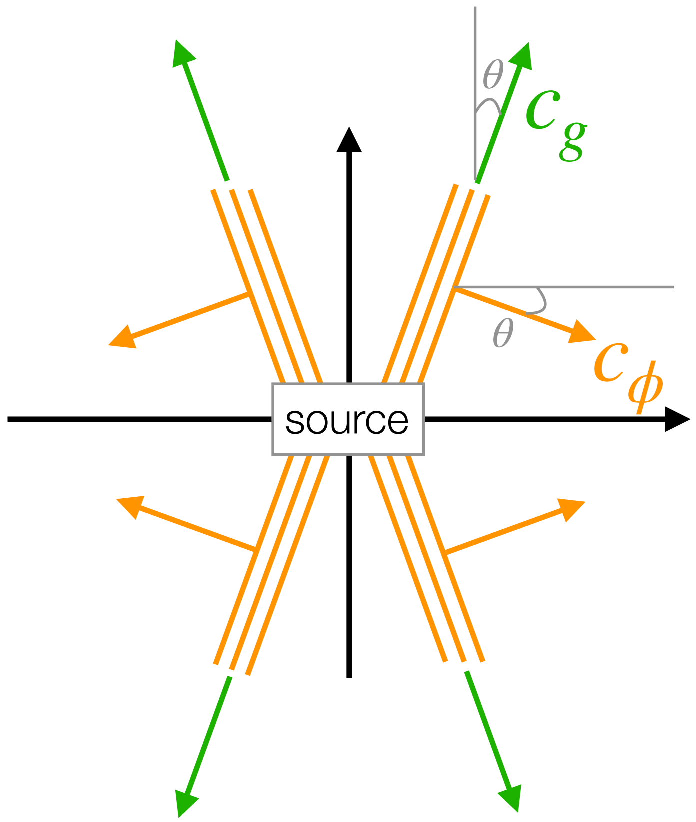
$\omega \to f$	$\omega \to N$
inertial waves	gravity waves

Solutions: Method of characteristics

The two extreme cases are:

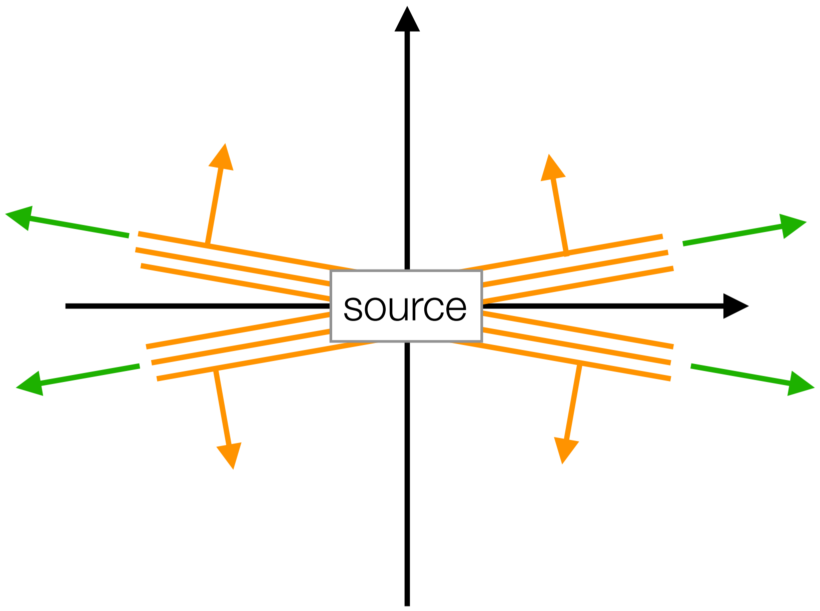
- $\omega \to N \Leftrightarrow \theta \to 0$

  rapidly oscillating waves propagate almost vertically



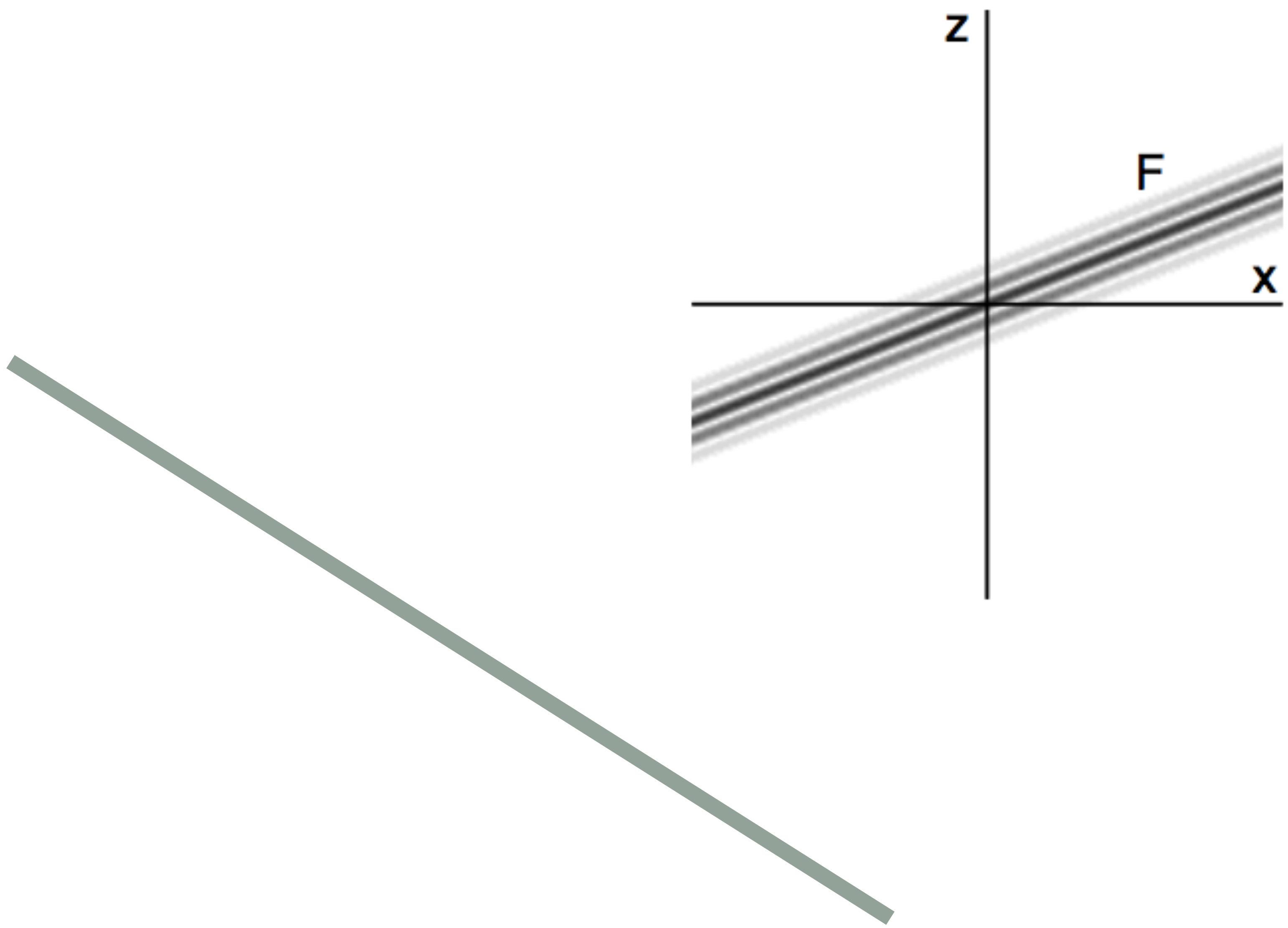
- $\omega \to f \Leftrightarrow \theta \to \dfrac{\pi}{2}$

  “near-inertial” waves propagate almost horizontally



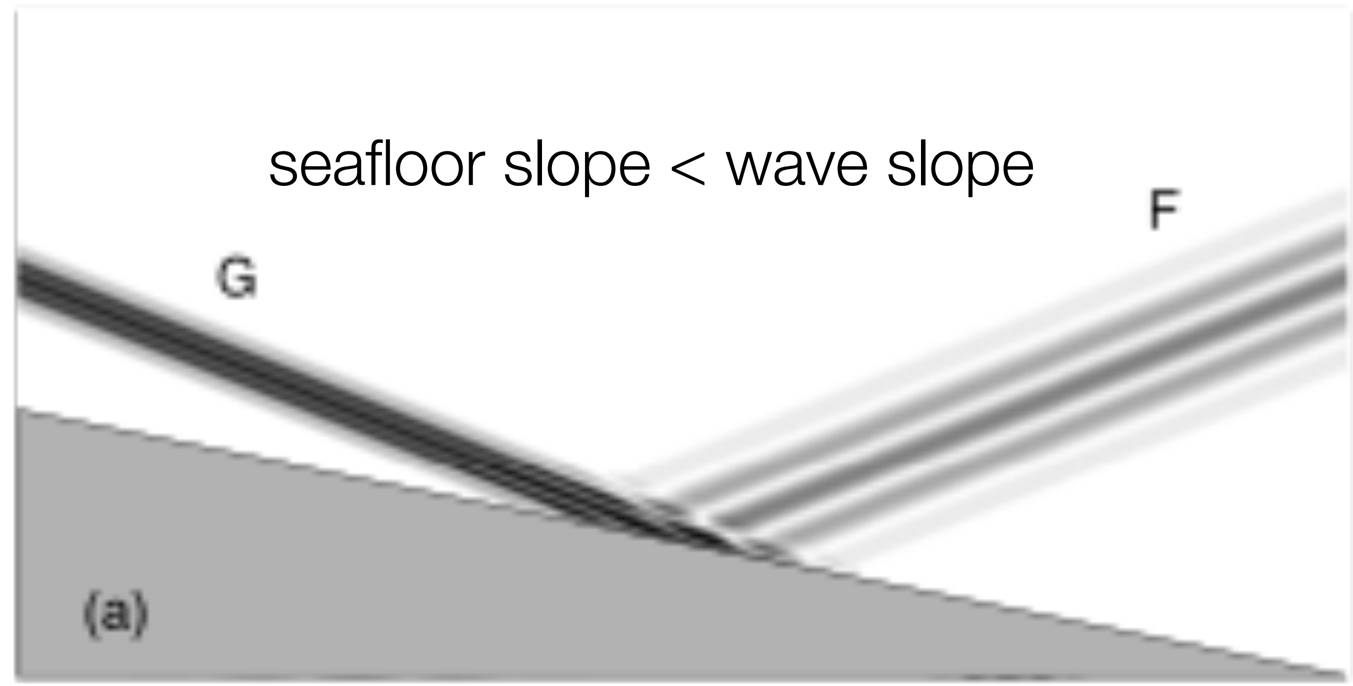
Solutions: Method of characteristics

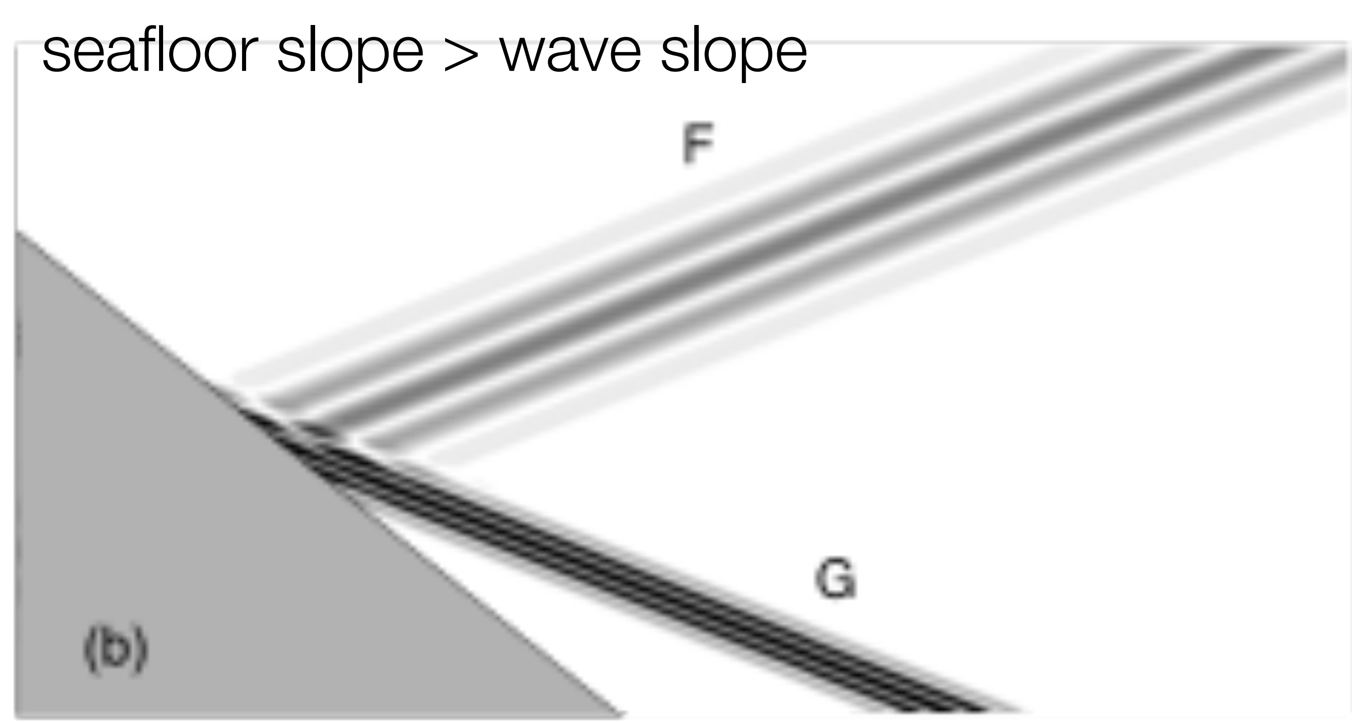
Wave Reflection?



Solutions: Method of characteristics

When a wave impinges on a seafloor slope, its frequency ω is conserved, hence its propagating slope is conserved.





Solutions: Method of characteristics

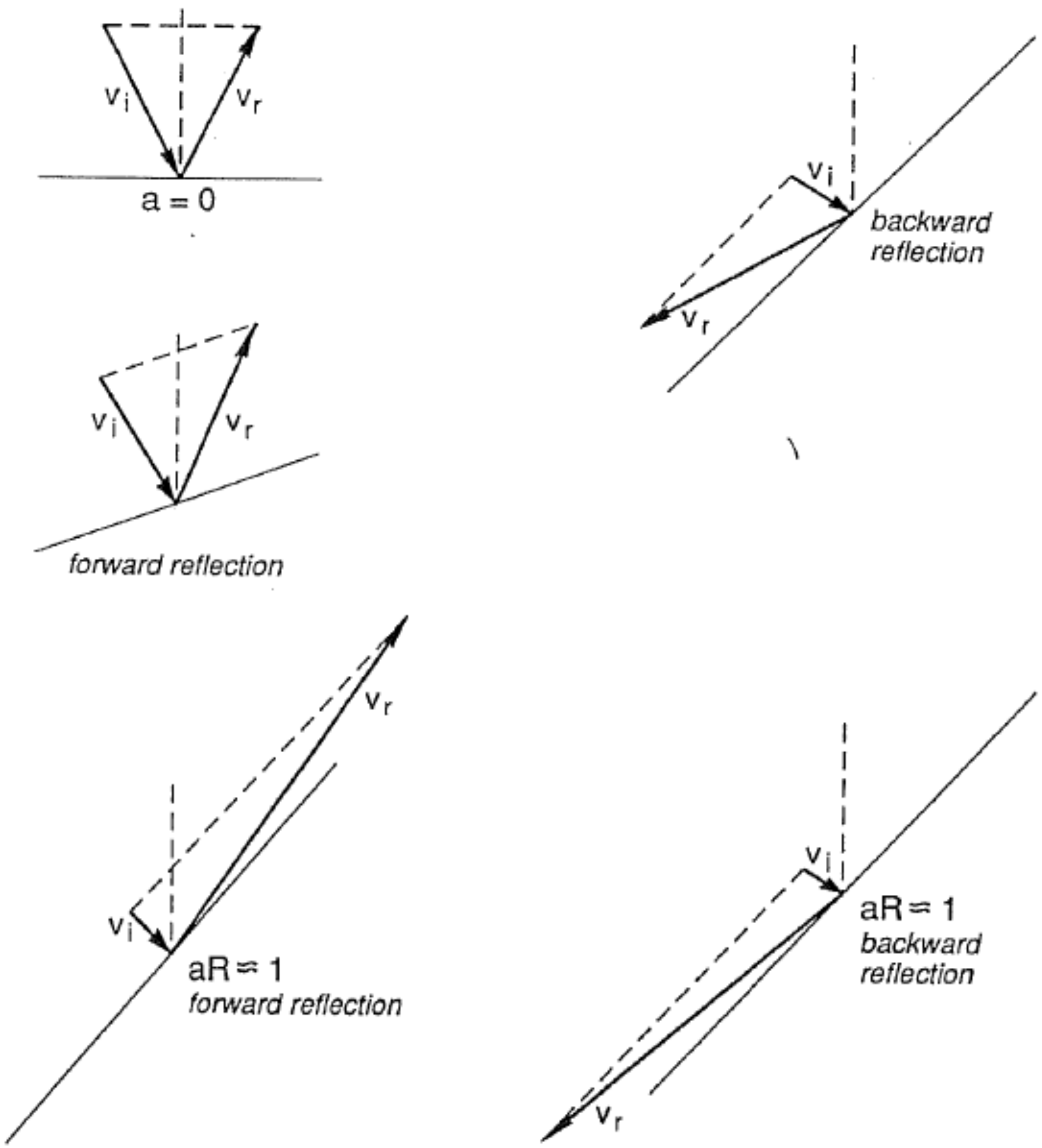
Wave Reflection

Several possible cases:

- For a flat bottom, energy is reflected with the same angle with the vertical
- Depending on the bottom slope, the reflection can be forward or backward

- The critical angle corresponds to a slope with the same angle than the ray:

$$\tan \theta = \pm \left(\frac{N^2 - \omega^2}{\omega^2 - f^2} \right)^{1/2}$$

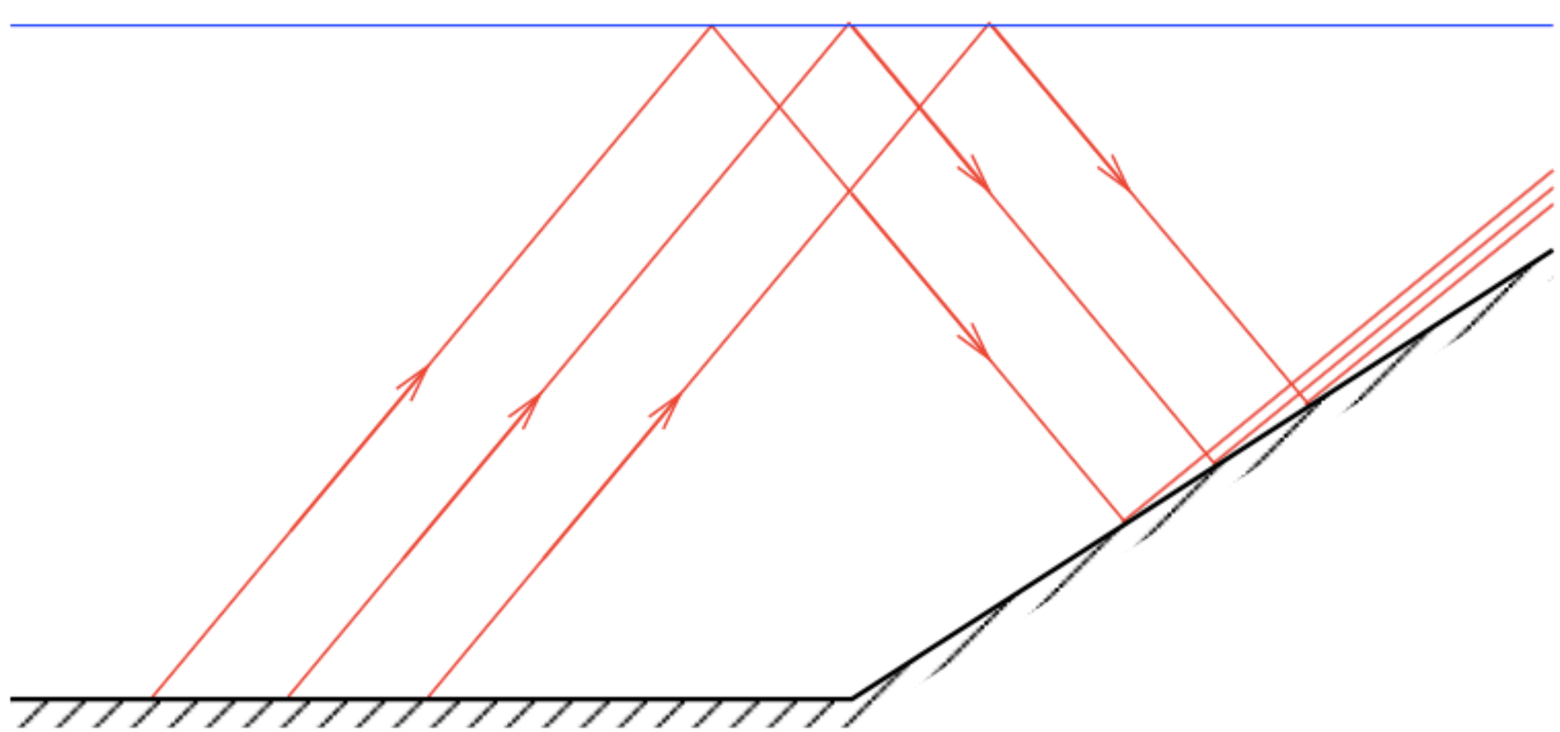


Solutions: Method of characteristics

Method of characteristics

When a wave impinges on a seafloor slope, its frequency ω is conserved, hence its propagating slope is conserved.

Energy can focus in narrow bands.



Solutions: Method of characteristics

Method of characteristics

Specific geometries can work as wave attractors.

Observation of an internal wave attractor in a confined, stably stratified fluid

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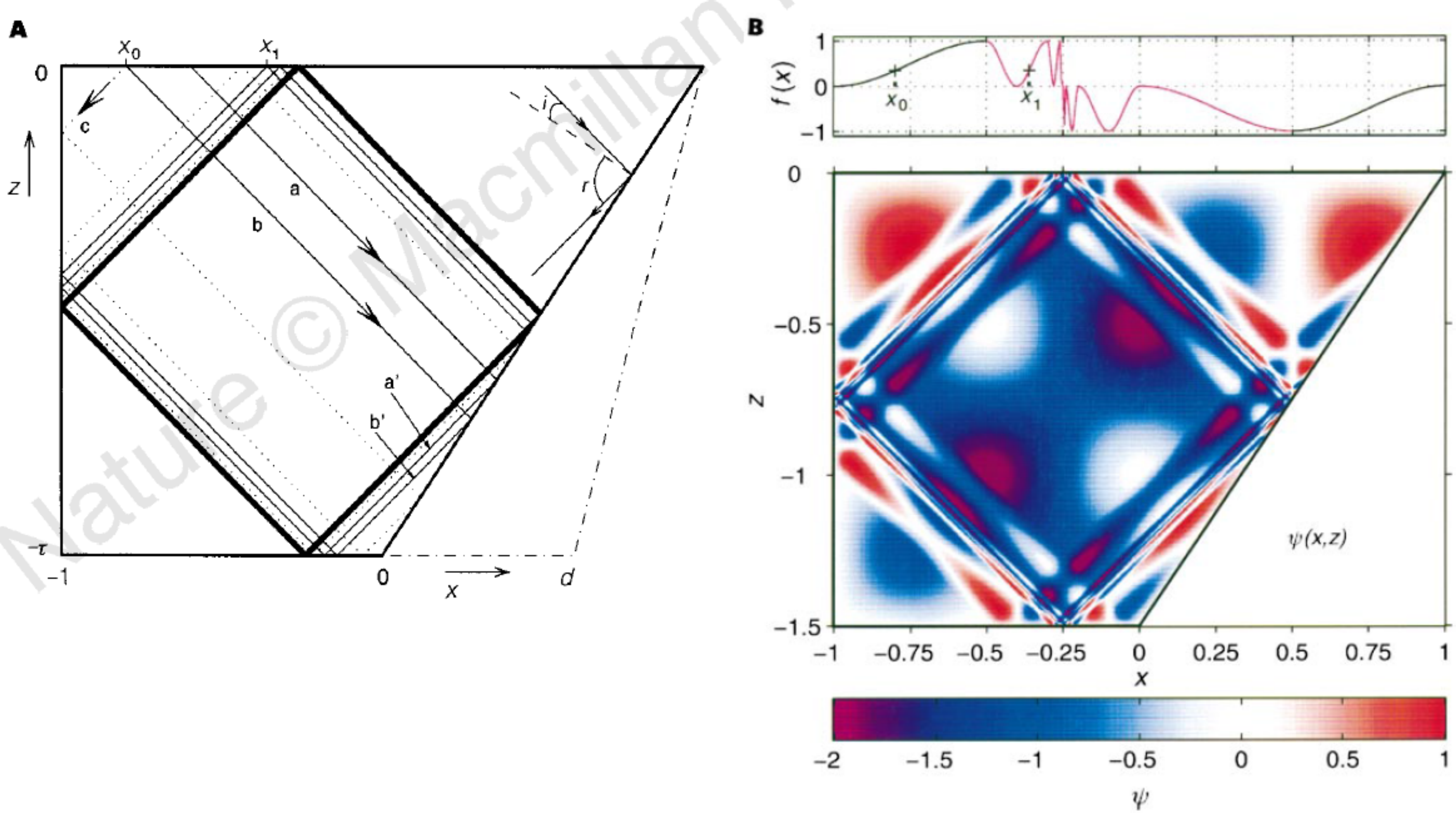


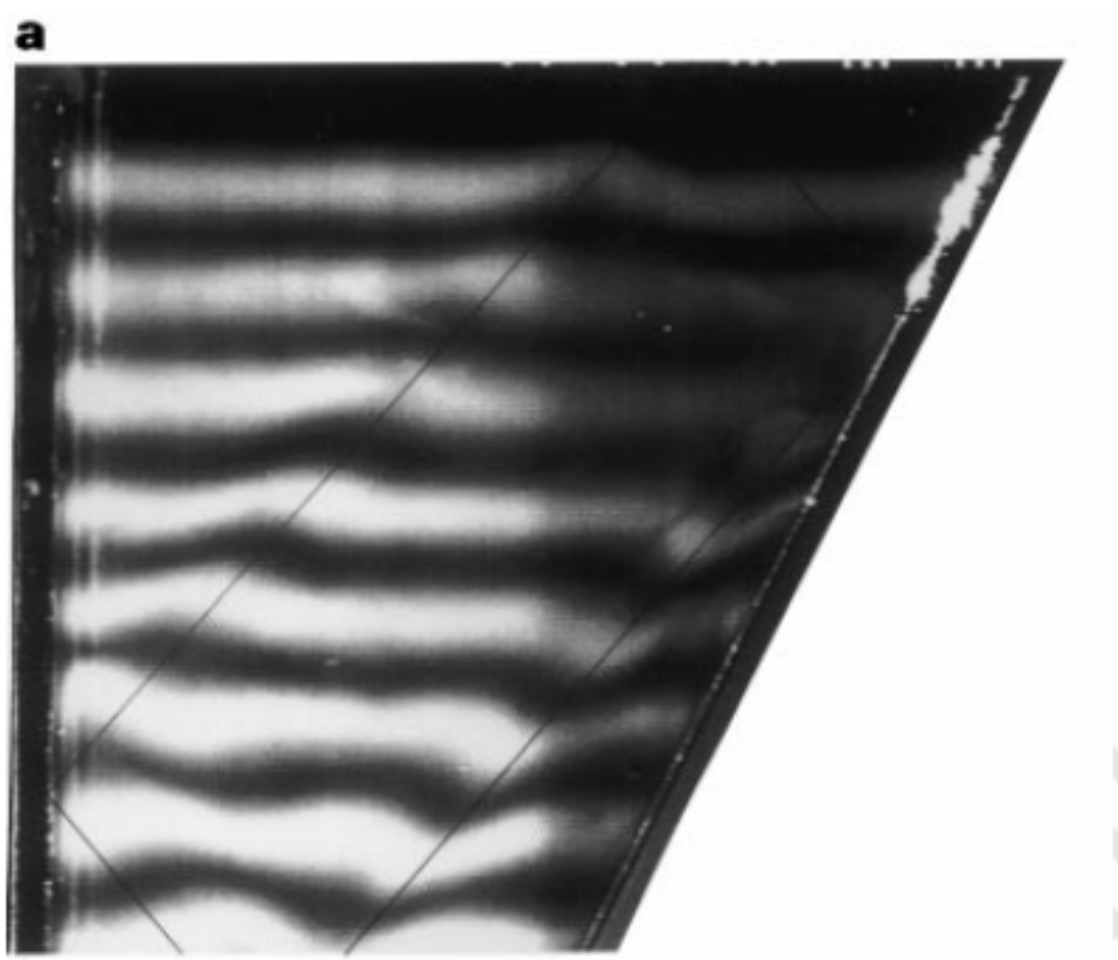
https://www.jgula.fr/Ondes/maasetal97.pdf

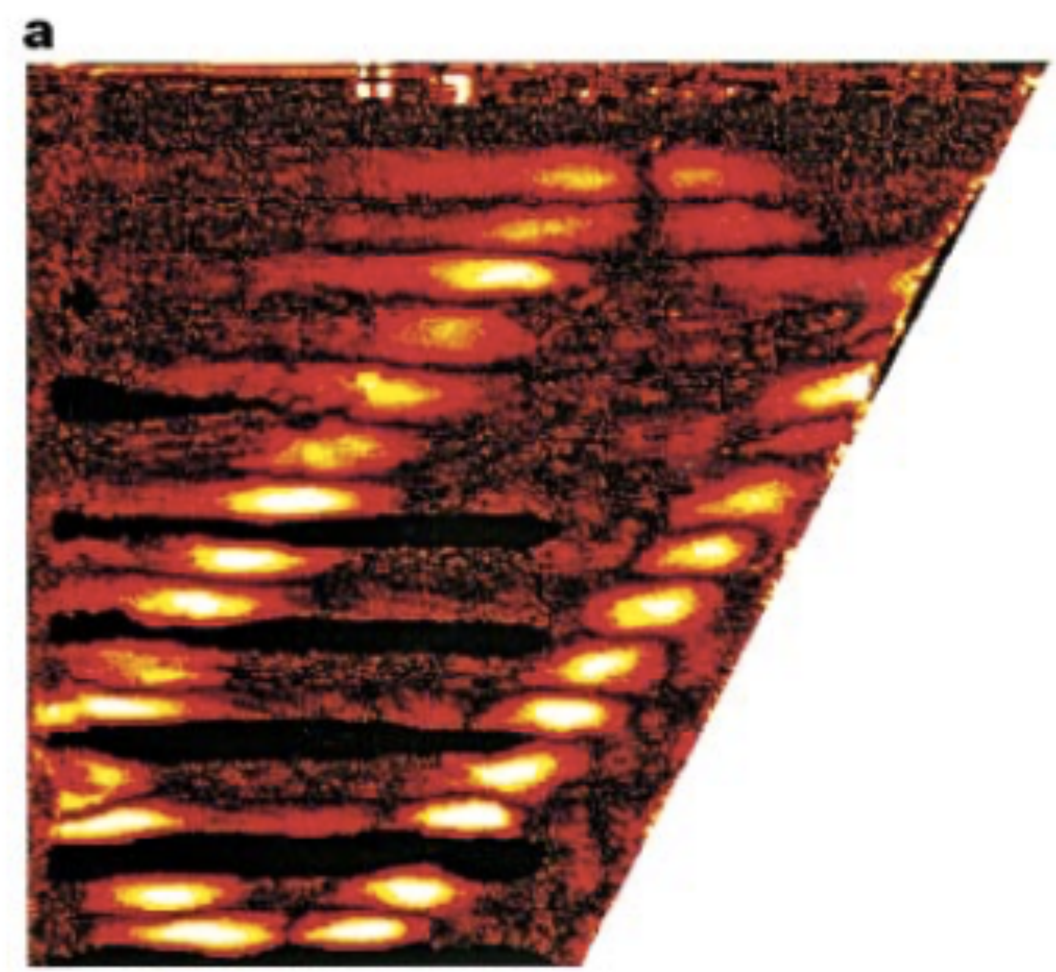
Maas et al. 1997

Solutions: Method of characteristics

Method of characteristics

Specific geometries can work as wave attractors.





https://www.jgula.fr/Ondes/maasetal97.pdf

Maas et al. 1997

Solutions of the equation

Method of characteristics

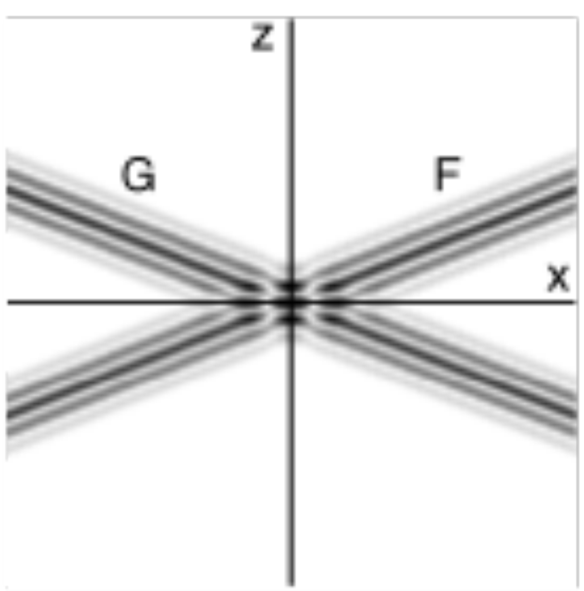
Main limit of the method of characteristics: no top or bottom boundary condition is specified. The ocean is supposed to be “infinite” ! What happens with a finite-depth ocean ? → **method of modes**

Solutions of the equation

1. Method of characteristics

- N = cste
- Arbitrary boundary conditions
- Solution of the form:

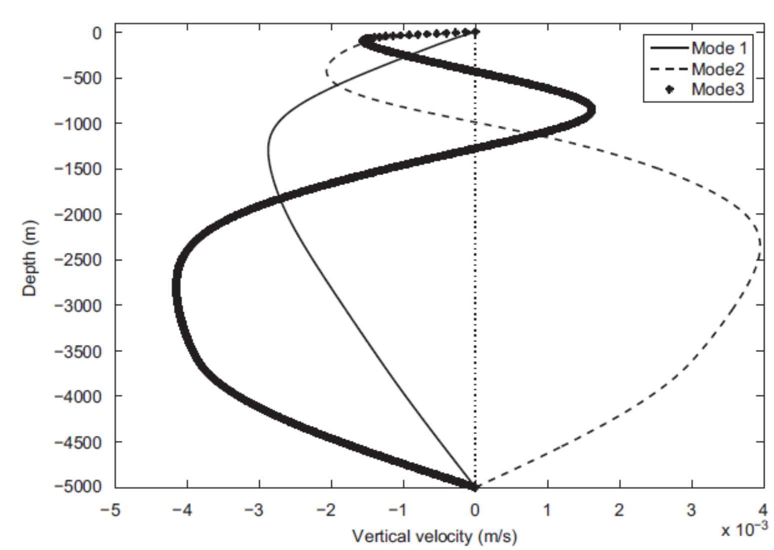
$$\hat{w} = F(\mu_+ x - z) + G(\mu_- x - z)$$



2. Method of Modes

- N = N(z)
- Horizontal surface/bottom
- Solution of the form:

$$w = W(z)e^{-\mathrm{i}\omega t+\mathrm{i}kx+\mathrm{i}ly}$$



Solutions: Method of modes

Assumptions:

- $N = N(z)$
- flat sea surface height (*rigid lid*) and flat seafloor

We start back from equation (9): $(\nabla^2 w)_{tt} + f^2 w_{zz} + N^2 \nabla_h^2 w = 0$,

and assume a solution in the form: $w = W(z)\ \exp(i(\omega t - kx))$, that is, the vertical structure is decoupled from the horizontal structure (+ no variation in y).

The equation for w becomes $W_{zz} + k^2 \dfrac{N(z)^2 - \omega^2}{\omega^2 - f^2} W = 0$, which can be written as:

$$\boxed{W_{zz} + m^2 W = 0} \quad \text{with} \quad \boxed{m^2(z) = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}}$$

and boundary conditions: $W(-H) = W(0) = 0$

Solutions: Method of modes

- These equations form a Sturm-Liouville problem, which for fixed ω has an infinite number of solutions W_n (eigenfunctions, vertical modes) with corresponding eigenvalues k_n.

- The general solution will be the superposition:

$$w = \sum_n W_n(z) \left[a_n^{\pm} \exp i(k_n^{\pm} x - \omega t) \right]$$

Solutions: Method of modes

- Two types of solution depending on the sign of:

$$m^2(z) = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}$$

- Oscillatory if $m^2(z) \geq 0.$

$$\boxed{\text{(I)} \quad N(z) \leq \omega \leq |f| \quad \text{or} \quad \text{(II)} \quad |f| \leq \omega \leq N(z)\,.}$$

- Exponential decay otherwise.

Solutions: Method of modes

- Case 1 - Constant N:

$$W'' + m^2 W = 0 \qquad m^2(z) = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}$$

- Activity: Solve the equation for the case of constant N

1. Find the dispersion relation

2. Find the general solution

Solutions: Method of modes

If $N(z) = cst$, the solution can be written: $W(z) = A\cos(mz) + B\sin(mz)$.

The top and bottom boundary conditions are: $w(0) = w(-H) = 0$.

They give: $m = \pm n\dfrac{\pi}{H}$, which translates into the dispersion relation:

$$\boxed{k_n = \pm\frac{n\pi}{H}\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}} \quad \text{or} \quad \boxed{\omega^2 = \frac{N^2 k_n^2 + \frac{n^2\pi^2}{H^2} f^2}{k_n^2 + \frac{n^2\pi^2}{H^2}}}$$

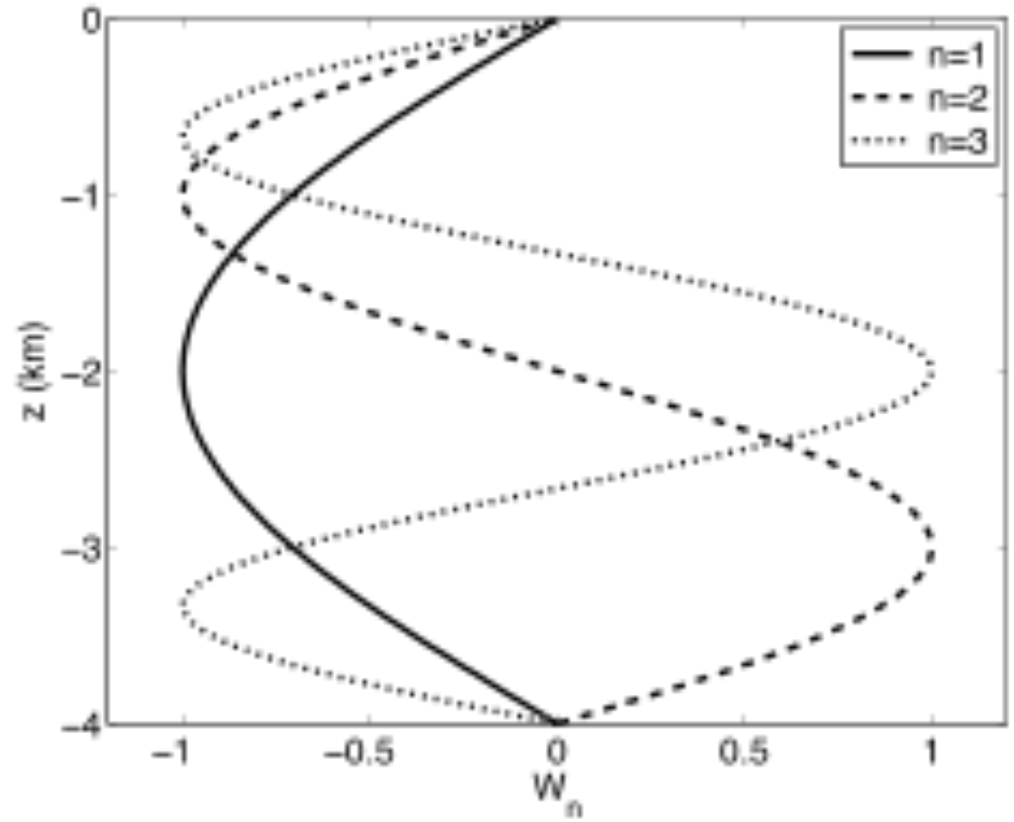
We recover the dispersion relation from the method of characteristics.

Solutions: Method of modes

The general solution for w is the superposition of modes:

$$w(x, z, t) = \sum_n W_n(z) a_n \cos(k_n x - \omega t).$$

If $N = cst$, $w(x, z, t) = \sum_n a_n \sin\left(\frac{n\pi z}{H}\right)\cos(k_n x - \omega t).$



Gerkema & Zimmerman (2008)

Solutions: Method of modes

- We can also write:

$$\omega^2 = \frac{N^2k^2 + f^2(\frac{n\pi}{H})^2}{k^2 + (\frac{n\pi}{H})^2}\,.$$

  - Horizontal phase speed:

$$\begin{aligned} c &= \frac{[N^2k^2 + f^2(\frac{n\pi}{H})^2]^{1/2}}{k[k^2 + (\frac{n\pi}{H})^2]^{1/2}} \\ &= \pm\left(\frac{H\omega}{n\pi}\right)\left(\frac{N^2-\omega^2}{\omega^2-f^2}\right)^{1/2}. \end{aligned}$$

  - Horizontal group speed:

$$\begin{aligned} c_g &= \frac{k(\frac{n\pi}{H})^2(N^2-f^2)}{[N^2k^2 + f^2(\frac{n\pi}{H})^2]^{1/2}\,[k^2 + (\frac{n\pi}{H})^2]^{3/2}} \\ &= \pm\left(\frac{H}{n\pi}\right)\frac{(\omega^2-f^2)^{1/2}(N^2-\omega^2)^{3/2}}{\omega(N^2-f^2)}\,. \end{aligned}$$

- **And see that higher vertical modes propagate more slowly**

Solutions: Method of modes

From the initial set of equations, we get the vertical structure of u, v, p, b:

$$U(z) = \frac{i}{k}\frac{\partial W}{\partial z}\ ;\ V(z) = \frac{f}{\omega k}\frac{\partial W}{\partial z}\ ;\ P(z) = i\rho_0 \frac{\omega^2 - f^2}{\omega k^2}\frac{\partial W}{\partial z}\ ;$$

$$B(z) = -\frac{iN^2}{\omega}W(z)$$

Gerkema & Zimmerman (2008)

Solutions: Method of modes

- which gives

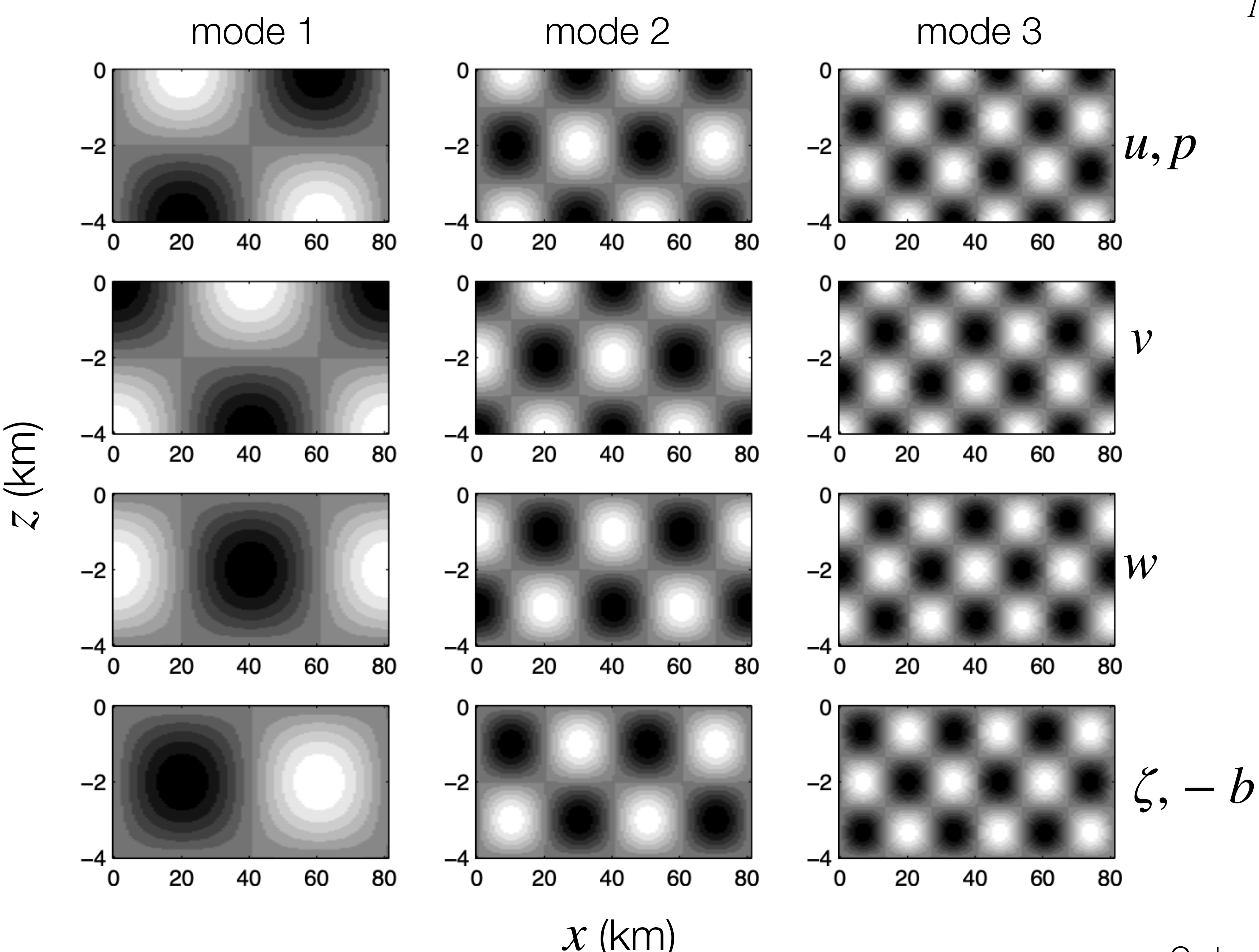
$$k_n = \pm \frac{n\pi}{H}\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}, \qquad n = 1, 2, 3, \cdots .$$

$$\begin{aligned}
u &= -\sum_n a_n \frac{n\pi}{k_n H} \cos\left(\frac{n\pi z}{H}\right) \sin(k_n x - \omega t) \\
v &= \frac{f}{\omega} \sum_n a_n \frac{n\pi}{k_n H} \cos\left(\frac{n\pi z}{H}\right) \cos(k_n x - \omega t) \\
p &= -\rho_* \frac{\omega^2 - f^2}{\omega} \sum_n a_n \frac{n\pi}{k_n^2 H} \cos\left(\frac{n\pi z}{H}\right) \sin(k_n x - \omega t) \\
b &= \frac{N^2}{\omega} \sum_n a_n \sin\left(\frac{n\pi z}{H}\right) \sin(k_n x - \omega t) \, .
\end{aligned}$$

Solutions: Method of modes

Method of modes

$N(z) = cst$



ζ is the vertical displacement of isopycnals.

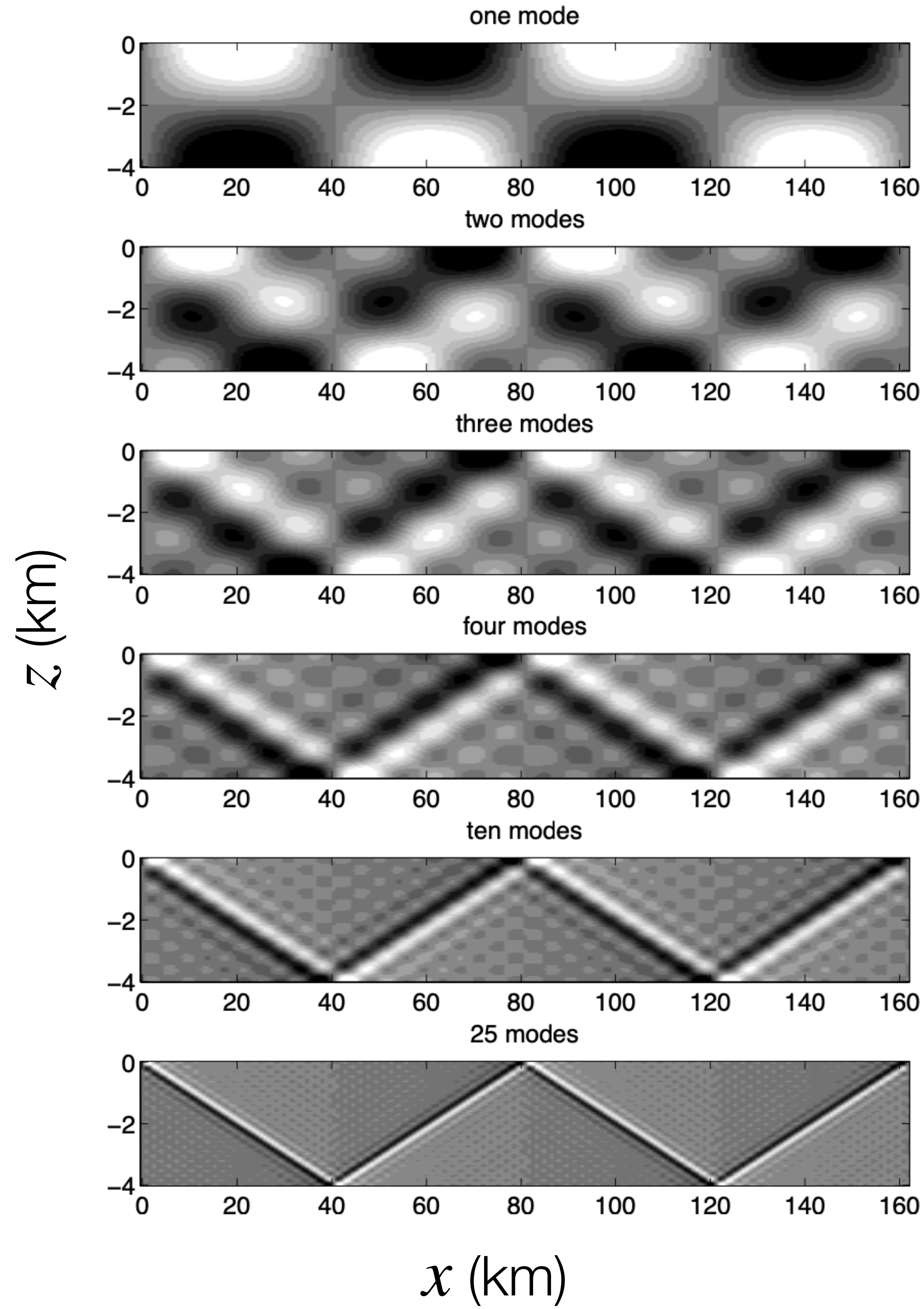
Gerkema & Zimmerman (2008)

Solutions: Method of modes

Method of modes

The superposition of modes shows up as beams

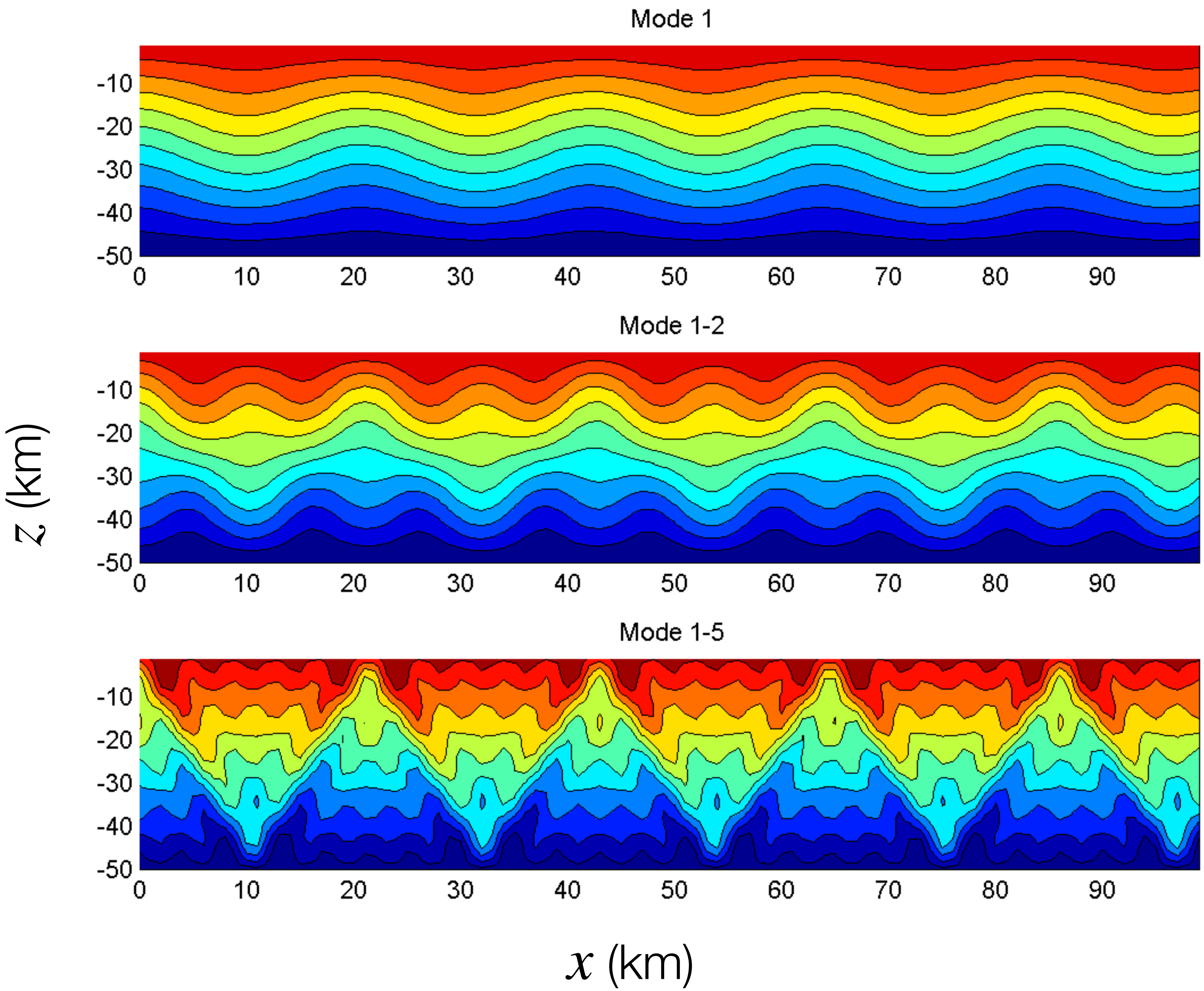
NB: $N(z) = cst$ in the figure



Gerkema & Zimmerman (2008)

Method of modes

$N(z) = cst$



Solutions: Method of modes

- **Case 2 - N "slowly" varying:**

$$m^2(z) = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}$$

- We can use the **WKB approximation**.

valid if $\left| \dfrac{dm}{dz} \right| \ll m^2(z),$

which gives the approximate solution:

$$w(z) \simeq \frac{C}{\sqrt{m(z)}} \sin\left(\int_{z_0}^{z} m(\zeta)\, d\zeta + \phi \right),$$

see e.g. https://www.jgula.fr/Ondes/LahayeGulaRoullet19.pdf

Solutions: Method of modes

Observations of an internal tidal beam:

Why is it curved?

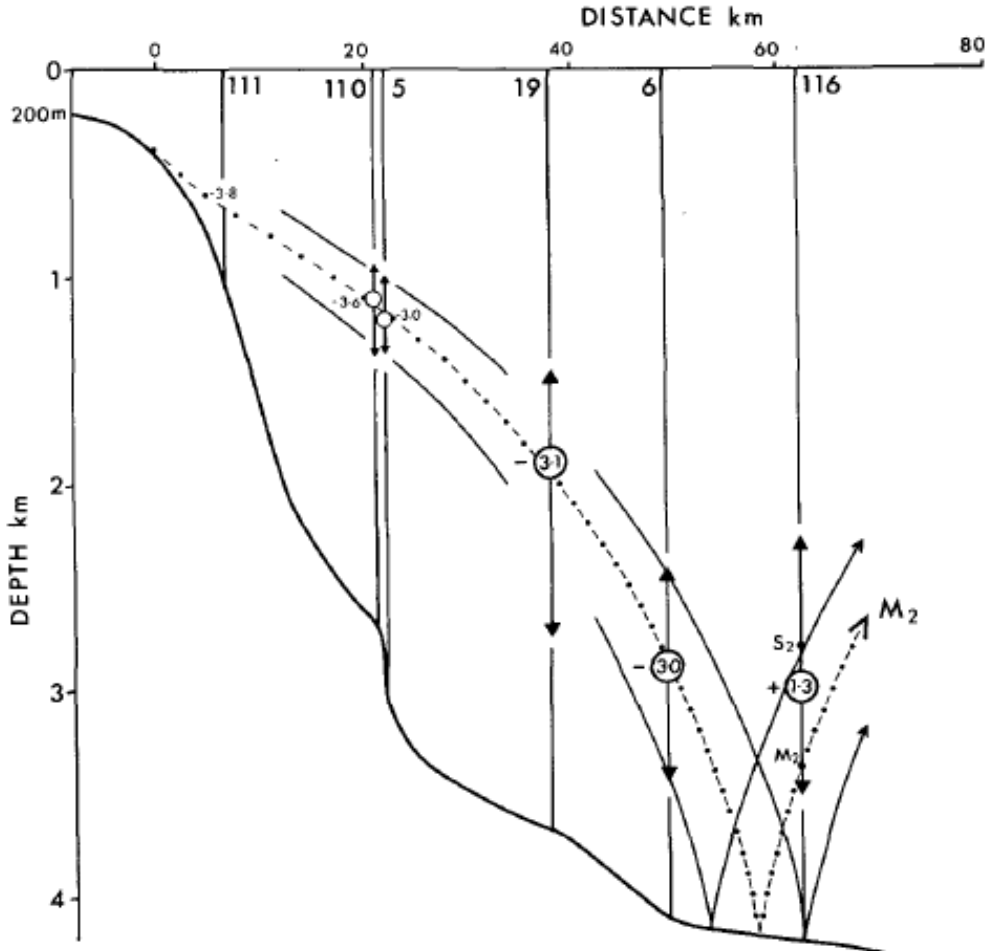


FIG. 9. Diagram showing the theoretical ray path (chained line) for a beam of internal tidal energy at the M_2 tidal frequency emanating from the critical depth (385 m) on the upper slopes and reflecting off the Biscay abyssal plain at a depth of about 4200 m, 58 km from the critical point. Also shown is a summary of the internal tidal oscillations obtained during the RRS *Challenger* cruises in 1988 (CH 31/88) and 1987 (CH 18/87). Vertical lines represent mooring and CTD station positions and are identified with numbers. CTD stations 5 and 6 and mooring 116 are from the 1988 cruise, whereas moorings 110 and 111 and CTD 19 were obtained in 1987. The depth of the maximum amplitude of the internal tidal oscillation found at each station is plotted as an open circle and the range where the amplitude is more than 70% of the maximum value is indicated by the arrows. Two further rays are shown (solid lines) passing through the 70% limits near mooring 110. The phase of the maximum upward displacement is given (within the circles) in hours with respect to HWP. A ray at the M_2 tidal frequency would intersect mooring 116 at the depth marked M_2; S_2 is the corresponding point for a ray at the S_2 tidal frequency. The topography is depicted by the bold line and is critical at 385 m; the horizontal distance scale is measured from the critical point.

Solutions: Method of modes

- For a 2-layer N:

The beam is slightly steeper in the lower layer due to the weakest stratification.

Internal reflections occur at the transition between the two layers,

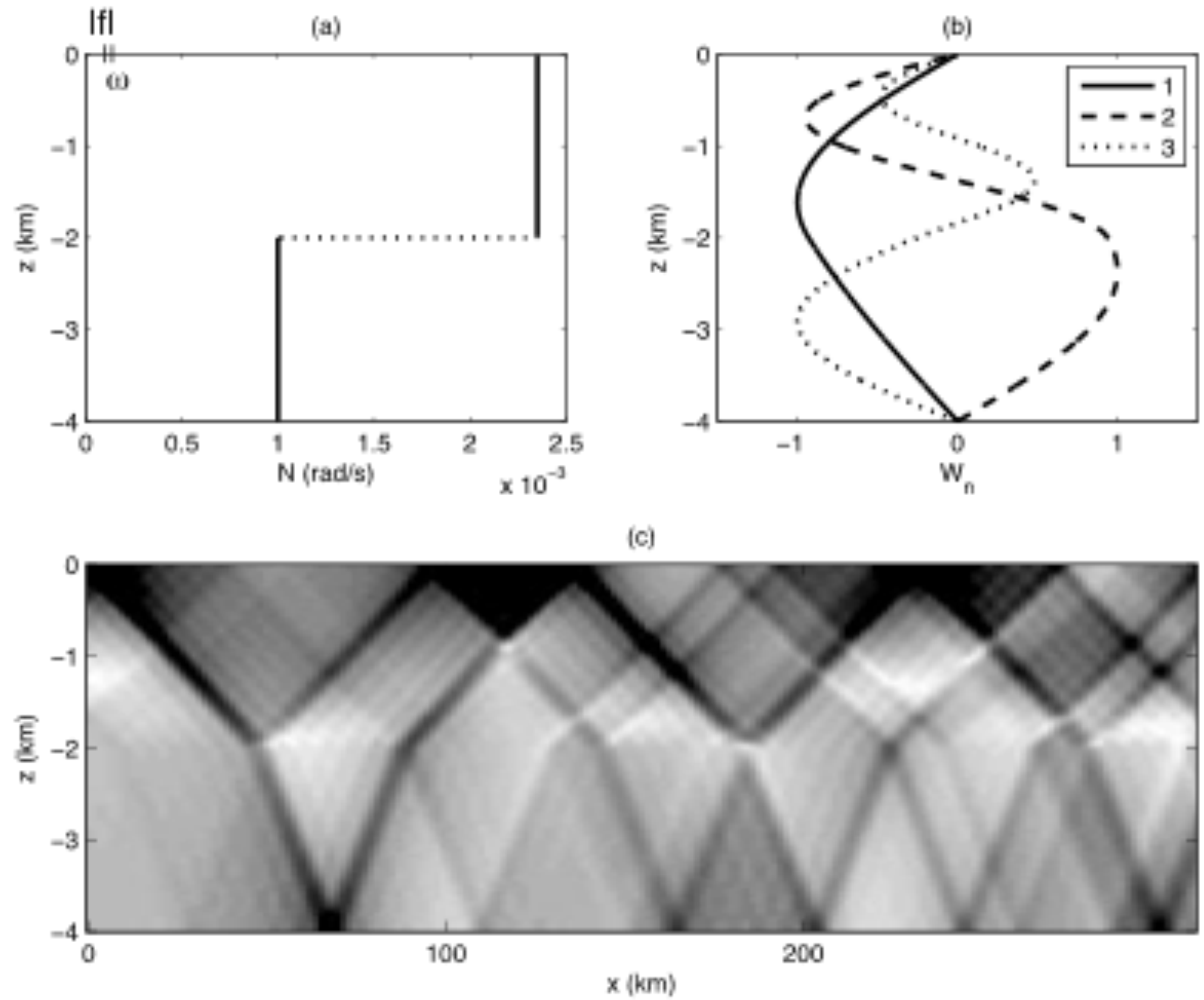


Fig. 5.7: Stratification with two layers of constant N, with $|f| < \omega < N_2 < N_1$ (**a**). Panel **b** shows the first three eigenmodes (5.30), with $C_{1,n}$ chosen such that their amplitudes are one. Modal coefficients are $a_n = 1/n$. The resulting superposition of 20 modes, representing the amplitude of u, is shown in **c**. White denotes zero; black, maximum values.

Solutions: Method of modes

- For a 2-layer N:

if: $|f| < N_2 < \omega < N_1$

High-frequency waves are trapped in the upper layer.

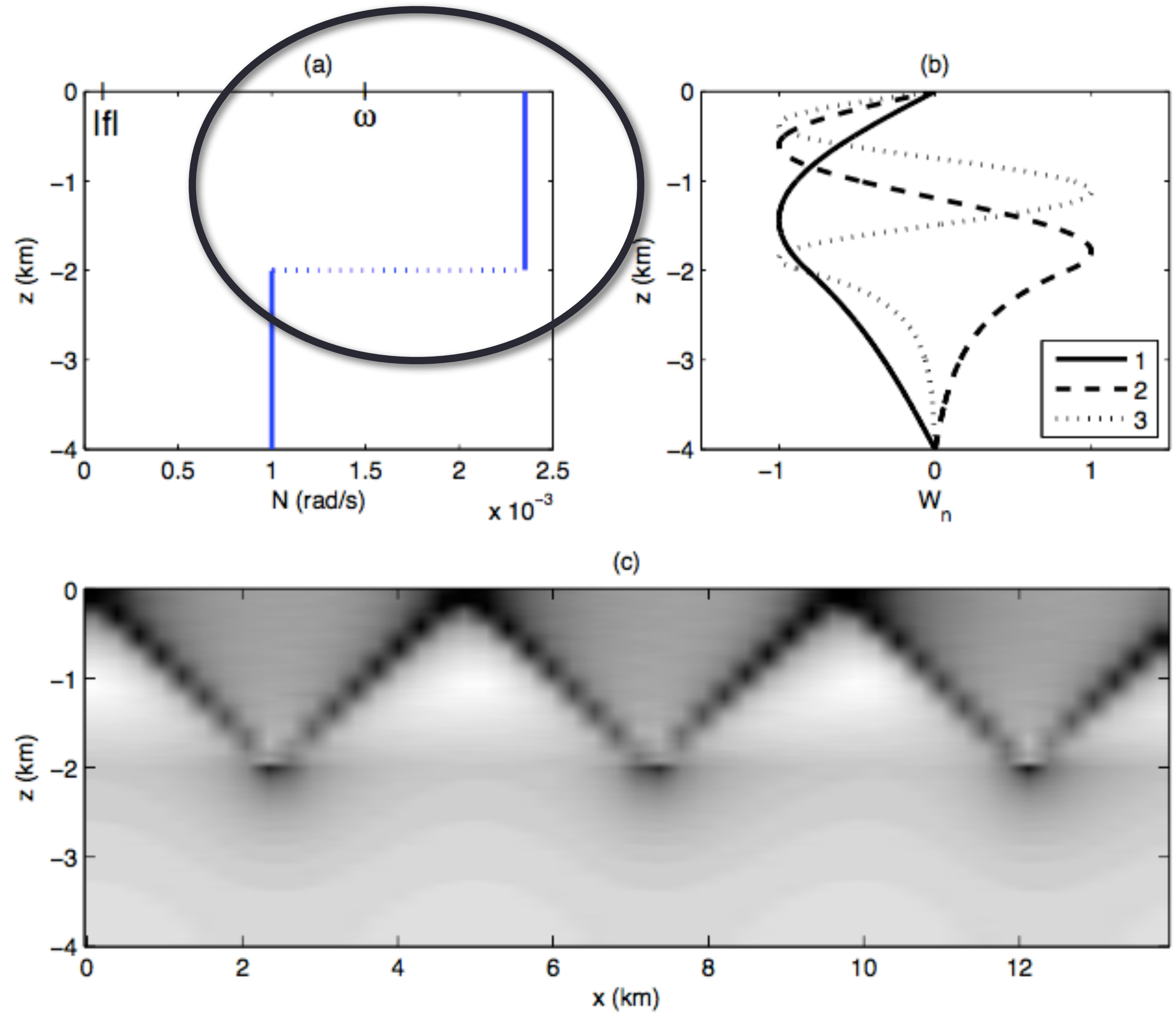


Fig. 5.9: Stratification with two layers of constant N; parameters are as in Figure 5.7, except for the wave frequency, which is now such that $|f| < N_2 < \omega < N_1$ (**a**). Panel **b** shows the first three eigenmodes (5.30), normalized to one. The resulting superposition of 15 modes, representing the amplitude of u, is shown in **c**. White denotes zero; black, maximum values.

Solutions: Method of modes

- For a 2-layer N:

if: $N_2 < \omega < |f| < N_1$

Low-frequency waves are trapped in the lower layer.

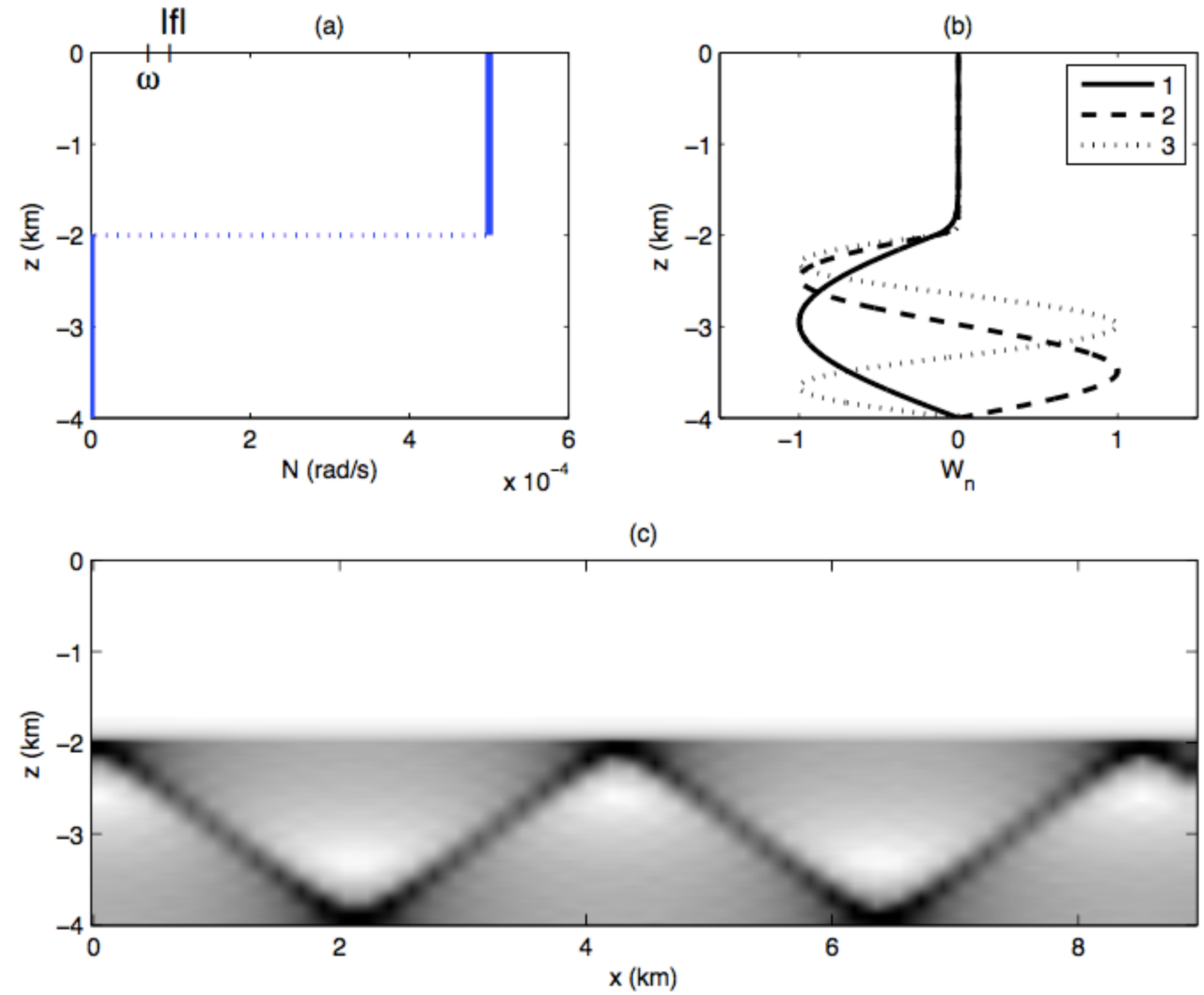


Fig. 5.10: Stratification with two layers of constant N, the lower layer being neutrally stable, $N_2 = 0$, with $N_2 < \omega < |f| < N_1$ (**a**). Panel **b** shows the first three eigenmodes (5.30), normalized to one. A superposition of 15 modes, representing the amplitude of u, is shown in **c**. White denotes zero; black, maximum values.

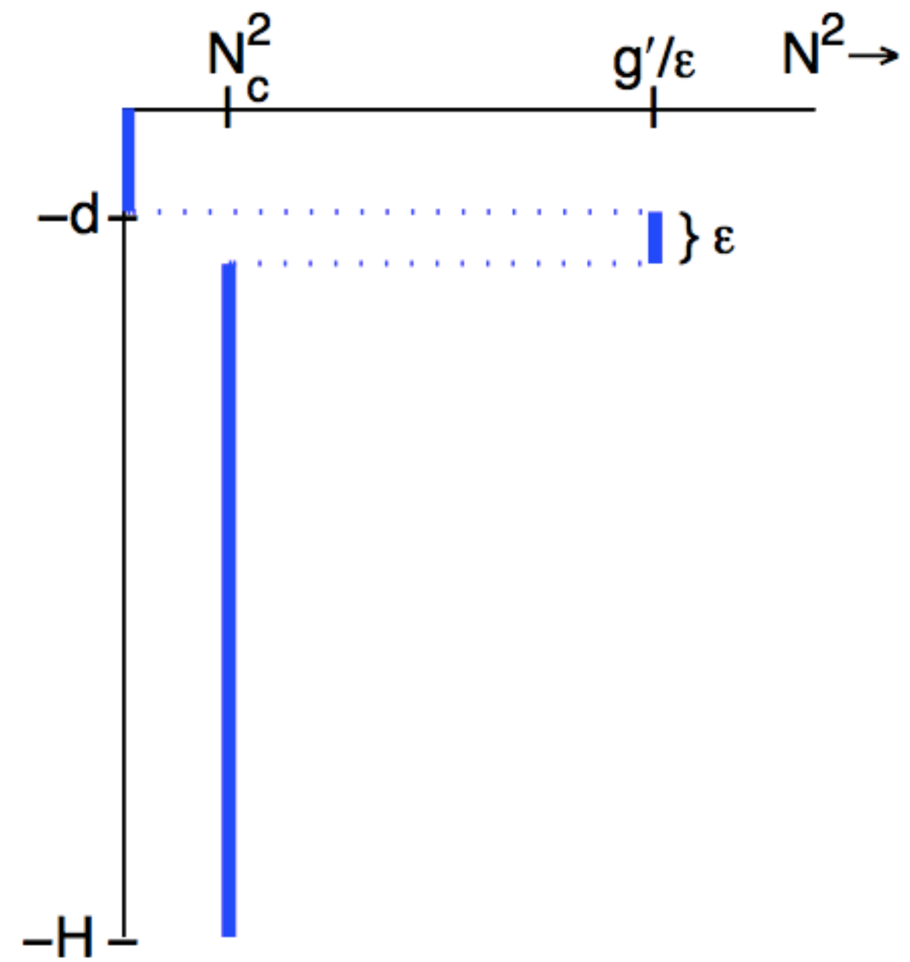
Solutions: Method of modes

- 3-layers N:

$$N^2(z) = \begin{cases} 0 & -d < z < 0 & \text{(mixed layer)} \\ g'/\epsilon & -d-\epsilon < z < -d & \text{(thermocline)} \\ N_c^2 & -H < z < -d-\epsilon & \text{(abyss)}\,. \end{cases}$$

We can get a more realistic stratification using 3 layers:

- a very weakly stratified upper mixed layer
- a seasonal thermocline
- a fairly weakly stratified abyssal ocean



Solutions: Method of modes

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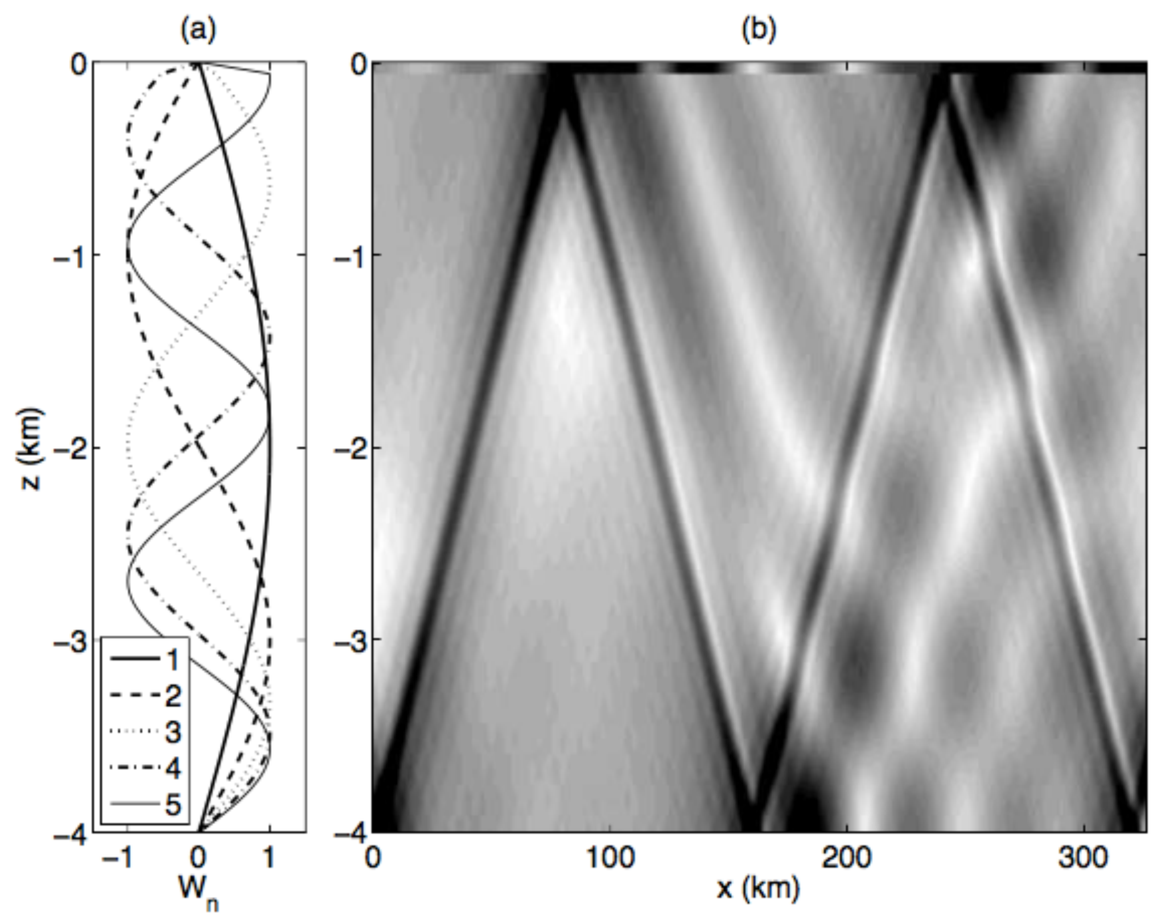


Fig. 5.12: The first five modes for the three-layer system (**a**), here with $g' = 0.005\,\mathrm{m\,s^{-2}}$. Other parameters are: $d = 60\,\mathrm{m}$ (mixed-layer depth), $N_c = 2 \times 10^{-3}$, $f = 1 \times 10^{-4}$ and $\omega = 1.4 \times 10^{-4}$, all in $\mathrm{rad\,s^{-1}}$; modal coefficients are $a_n = 1/n$. In **b**, a superposition of 25 modes, representing the amplitude of u. White denotes zero; black, maximum values.

Solutions: Method of modes

- For a linear N(z):

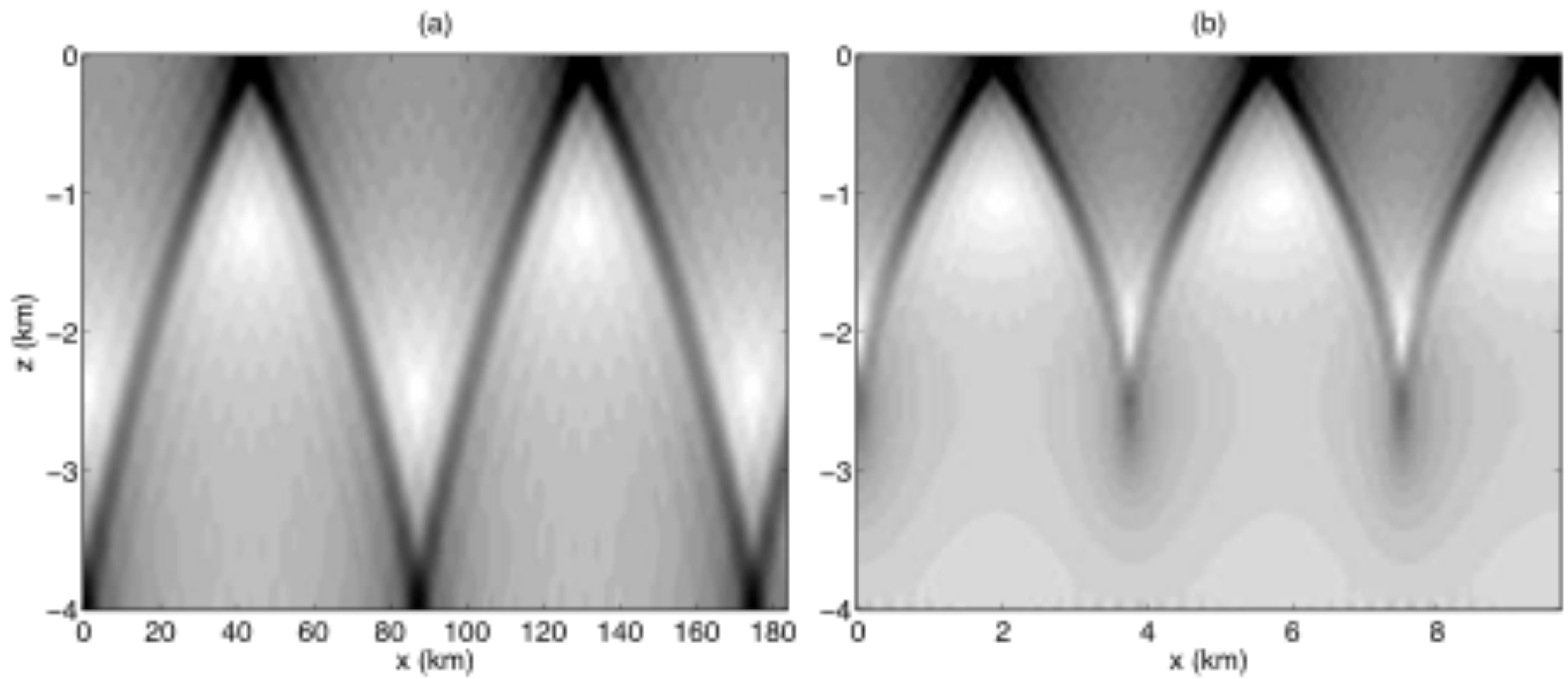
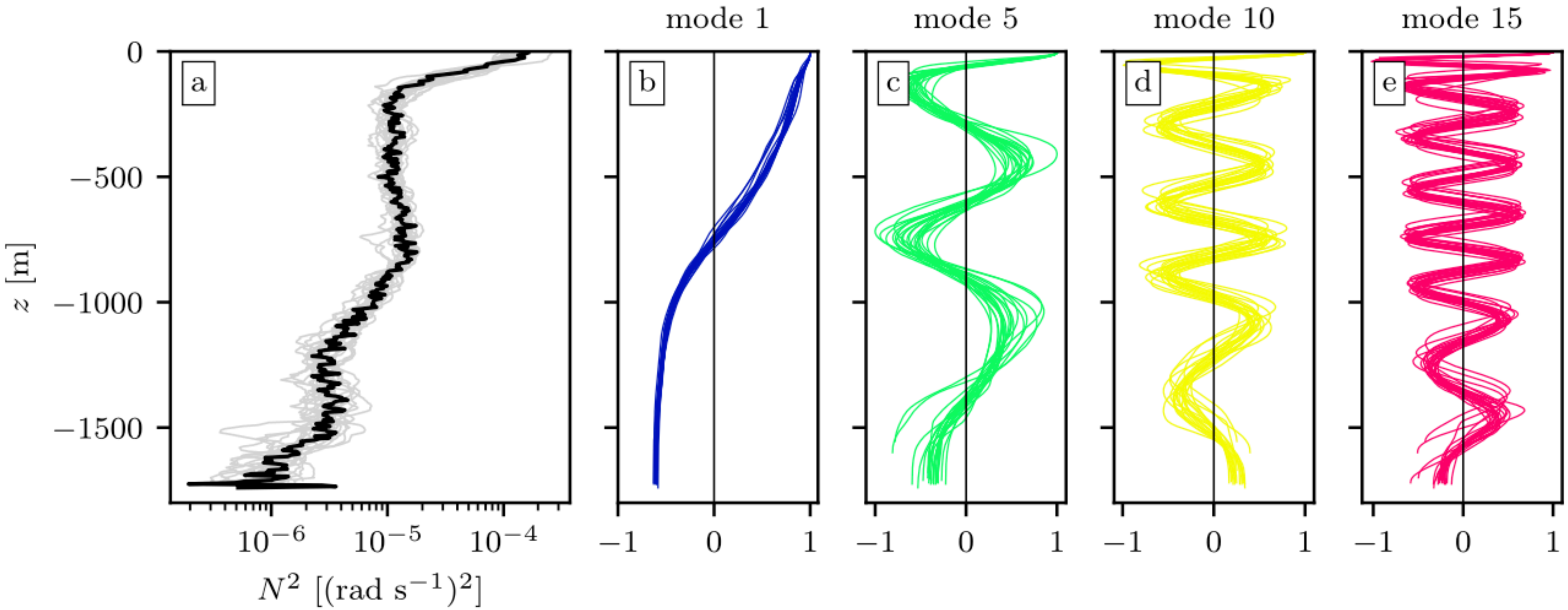


Fig. 5.16: Solution for a linearly varying $N(z)$; the amplitude of u is shown for a superposition 15 modes, with modal coefficients $a(n) = 1/n$. White denotes zero; black, maximum values. In **a**, internal-wave beams can propagate at any depth, but are refracted due to the decrease of N with depth. In **b**, a higher wave frequency is chosen, such that $|f| < N < \omega$ in the deeper part of the water column; hence, internal waves are trapped in the upper layer.

Solutions: Method of modes

Case of a real stratification $N(z)$ over the Mid-Atlantic Ridge

modes for u, v, p



Vic & Ferron (JGR 2023)

4. The continuously-stratified model

References used in this document:

- Gerkema & Zimmerman, 2008 textbook, *An introduction to internal waves*
- Maas et al., Nature 1997, *Observation of an internal wave attractor in a confined, stably stratified fluid*
- Vic & Ferron, JGR 2023, *Observed Structure of an Internal Tide Beam Over the Mid-Atlantic Ridge*