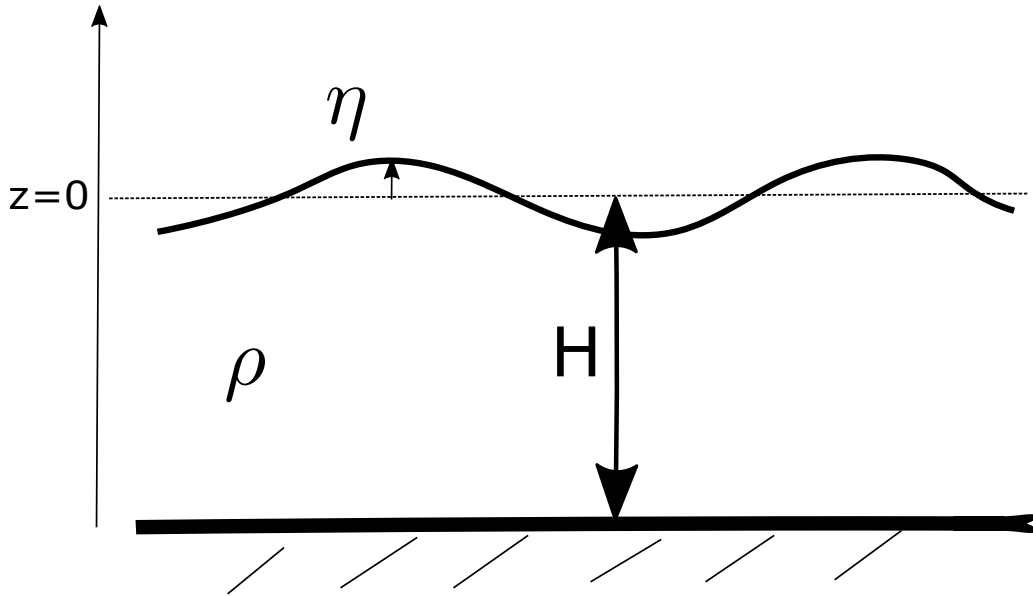


# Internal waves in the 2-layer shallow-water model

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## 1 The one-layer shallow-water model



We start from the Navier-Stokes equations for momentum (without viscosity nor forcings):

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} = -\frac{\vec{\nabla} P}{\rho} \quad (1)$$

with  $\vec{u} = (u, v, w)$ ,  $\rho$  the density and  $P(x, y)$  the pressure.

We consider a flow which is **quasi - horizontal** and neglect vertical variations of horizontal velocities ( $u_z = v_z = 0$ ), such that equations for the horizontal components of momentum become:

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -\frac{1}{\rho}\frac{\partial P}{\partial x} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -\frac{1}{\rho}\frac{\partial P}{\partial y}\end{aligned}$$

In the following we use the following notations  $\frac{\partial u}{\partial t} = u_t$ ,  $\frac{\partial u}{\partial x} = u_x$ ,  $\frac{\partial u}{\partial y} = u_y$ , etc., such that:

$$\begin{aligned}u_t + uu_x + vu_y - fv &= -\frac{1}{\rho}P_x \\ v_t + uv_x + vv_y + fu &= -\frac{1}{\rho}P_y\end{aligned}\tag{2}$$

We consider that the flow is in **hydrostatic balance**:

$$p(z) = P_a + \rho g(\eta - z)\tag{3}$$

where  $P_a$  is the atmospheric pressure,  $\eta$  the free surface height, and  $z$  the depth (increasing from the surface to the bottom).

We also consider that density is constant over the layer of fluid ( $\rho = cste$ ), such that the horizontal pressure gradients reduce to:

$$\begin{aligned}p_x &= \rho g\eta_x \\ p_y &= \rho g\eta_y\end{aligned}\tag{4}$$

So the momentum equations become:

$$\begin{aligned}u_t + uu_x + vu_y - fv &= -g\eta_x \\ v_t + uv_x + vv_y + fu &= -g\eta_y\end{aligned}\tag{5}$$

To get a third equation for the free surface height  $\eta$ , we use the continuity equation (conservation of mass) :

$$\vec{\nabla} \cdot \vec{u} = u_x + v_y + w_z = 0.\tag{6}$$

that we integrate vertically:

$$\int_{-H}^{\eta} (u_x + v_y + w_z) dz = (H + \eta) (u_x + v_y) + [w]_{-H}^{\eta} \quad (7)$$

with

$$\begin{aligned} (H + \eta) (u_x + v_y) &= [(H + \eta)u]_x + [(H + \eta)v]_y - u(H + \eta)_x - v(H + \eta)_y \\ &= [(H + \eta)u]_x + [(H + \eta)v]_y - u\eta_x - v\eta_y = 0 \end{aligned}$$

and

$$[w]_{-H}^{\eta} = w(\eta) - w(-H)$$

and the kinematic conditions at the surface and bottom (assuming a flat bottom):

$$\begin{aligned} w(\eta) &= \frac{D\eta}{Dt} = \eta_t + u\eta_x + v\eta_y \\ w(-H) &= \frac{D(-H)}{Dt} = -H_t - uH_x - vH_y = 0 \end{aligned}$$

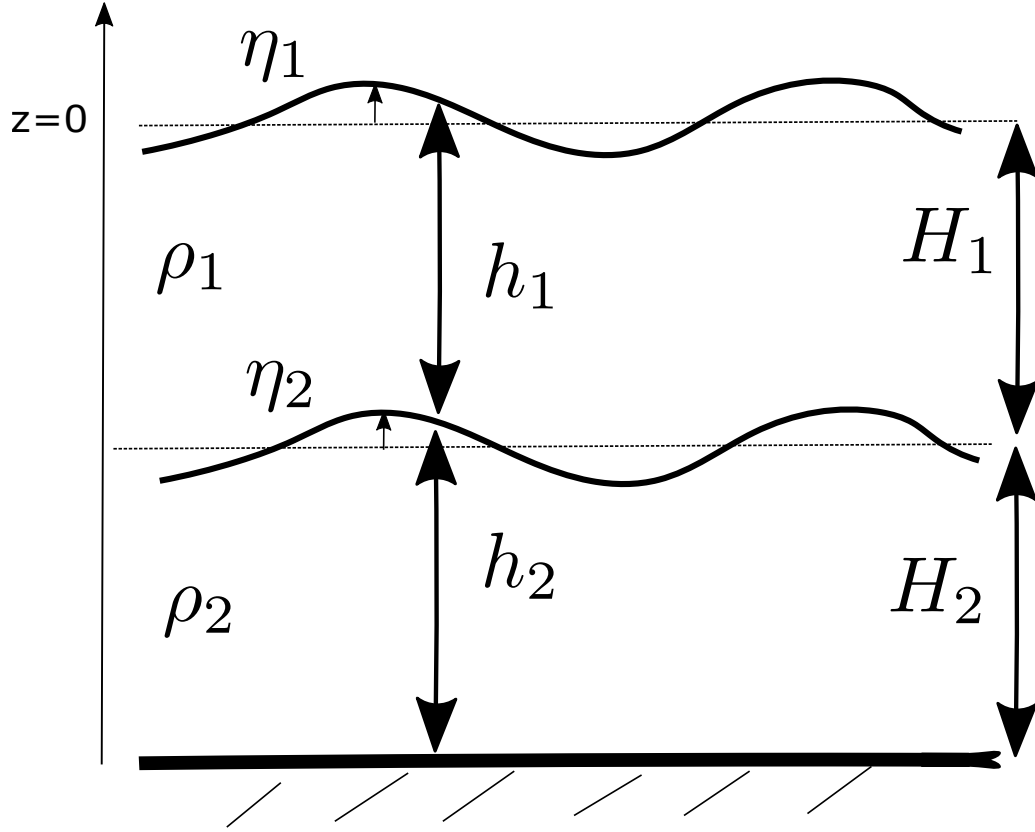
Such that Equ. 8 becomes:

$$\eta_t + [(H + \eta)u]_x + [(H + \eta)v]_y = 0$$

And finally the one-layer shallow-water equations, for  $u, v, \eta$ , are:

$$\begin{aligned} u_t + uu_x + vu_y - fv &= -g\eta_x \\ v_t + uv_x + vv_y + fu &= -g\eta_y \\ \eta_t + [(H + \eta)u]_x + [(H + \eta)v]_y &= 0 \end{aligned} \quad (8)$$

## 2 The two-layer shallow-water model



### 2.1 Momentum equations

If we now consider 2 layers of fluid, each with constant densities  $\rho_1$  and  $\rho_2$  ( $\rho_2 > \rho_1$ ). The momentum equations 2 can be used for velocities inside each layer. The differences lie in the pressure term, which will be different for both layers:

$$p_1(z) = P_a + \rho_1 g(\eta_1 - z) \quad (9)$$

$$p_2(z) = P_a + \rho_1 g(H_1 + \eta_1 - \eta_2) + \rho_2 g(\eta_2 - (z + H_1)) \quad (10)$$

and the corresponding pressure gradients are:

$$\frac{p_{1x}}{\rho_1} = g\eta_{1x} \quad (11)$$

$$\frac{p_{2x}}{\rho_2} = \frac{\rho_1}{\rho_2}g(\eta_{1x} - \eta_{2x}) + g\eta_{2x} \quad (12)$$

$$= \frac{\rho_1}{\rho_2}g\eta_{1x} + \left(1 - \frac{\rho_1}{\rho_2}\right)g\eta_{2x}. \quad (13)$$

We introduce the *reduced gravity*:  $g' = \left(1 - \frac{\rho_1}{\rho_2}\right)g$ , and write:

$$\frac{p_{2x}}{\rho_2} = (g - g')\eta_{1x} + g'\eta_{2x}. \quad (14)$$

In the following we'll use the fact that  $g \gg g'$  to write:

$$\frac{p_{1x}}{\rho_1} = g\eta_{1x} \quad (15)$$

$$\frac{p_{2x}}{\rho_2} = g\eta_{1x} + g'\eta_{2x}. \quad (16)$$

## 2.2 Coastal ocean example

Let's compute the values of  $g$  and  $g'$  using typical values for the coastal ocean, considering a two-layer flow separated by a thermocline.

We can define density variations as  $\rho \approx \rho_0(1 - \alpha T)$ , with  $\alpha = 2 \cdot 10^{-4} \text{ K}^{-1}$ .

So  $g' \approx \frac{\rho_2 - \rho_1}{\rho_0} = \alpha(T_1 - T_2)g$

For a temperature gradient  $\Delta T = T_1 - T_2 = 10 \text{ K}$ , we get:  $g' = 2 \cdot 10^{-3}g \ll g$ .

## 2.3 Continuity equations

For the 2 layers of fluid, the continuity equations can be written as previously, but we need to take into account the depth of the upper layer, which is  $h_1 = H_1 + \eta_1 - \eta_2$  and the depth of the bottom layer, which is  $h_2 = H_2 + \eta_2$ .

Our system of equations for the 2-layer shallow water model is thus:

$$\begin{aligned}
u_{1t} + u_1 u_{1x} + v_1 u_{1y} - f v_1 &= -g \eta_{1x} \\
v_{1t} + u_1 v_{1x} + v_1 v_{1y} + f u_1 &= -g \eta_{1y} \\
(\eta_1 - \eta_2)_t + [(H_1 + \eta_1 - \eta_2) u_1]_x + [(H_1 + \eta_1 - \eta_2) v_1]_y &= 0 \\
u_{2t} + u_2 u_{2x} + v_2 u_{2y} - f v_2 &= -g \eta_{1x} - g' \eta_{2x} \\
v_{2t} + u_2 v_{2x} + v_2 v_{2y} + f u_2 &= -g \eta_{1y} - g' \eta_{2y} \\
\eta_{2t} + [(H_2 + \eta_2) u_2]_x + [(H_2 + \eta_2) v_2]_y &= 0 \tag{17}
\end{aligned}$$

## 2.4 Wave solution for the SW equations

We want to find solutions of the system in the form of linear waves. So we linearize the equations and consider only small perturbations to the background state:

$$u_{1t} - f v_1 = -g \eta_{1x} \tag{18}$$

$$v_{1t} + f u_1 = -g \eta_{1y} \tag{19}$$

$$(\eta_1 - \eta_2)_t + H_1 u_{1x} + H_1 v_{1y} = 0 \tag{20}$$

$$u_{2t} - f v_2 = -g \eta_{1x} - g' \eta_{2x} \tag{21}$$

$$v_{2t} + f u_2 = -g \eta_{1y} - g' \eta_{2y} \tag{22}$$

$$\eta_{2t} + H_2 u_{2x} + H_2 v_{2y} = 0 \tag{23}$$

We look for solutions of the system in the form of monochromatic waves:

$$A = A_0 e^{i(\omega t - kx - ly)} \tag{24}$$

where  $A$  can be  $u, v, \eta$ .

So we have the properties:

$$A_t = i\omega A$$

$$A_x = -ikA$$

$$A_y = -ilA$$

$$A_{tt} = -\omega^2 A$$

$$A_{xx} = -k^2 A$$

$$A_{yy} = -l^2 A$$

Now we want to find the dispersion relations of the waves in our system of equations. One way to proceed is first try to simplify the system, by reducing it to 2 equations for 2 variables (for example  $\eta_1$  and  $\eta_2$ ).

So let's first to express  $u_1, v_1, u_2, v_2$  as functions of  $\eta_1$  and  $\eta_2$ . This can be done relatively simply by taking time derivatives of momentum equations (18,19,21,22).

For the upper layer:

$$\begin{aligned} u_{1tt} - f v_{1t} &= -g \eta_{1xt} \\ v_{1tt} + f u_{1t} &= -g \eta_{1yt} \end{aligned}$$

Then we replace time derivatives in the Coriolis terms using equations 18 and 19:

$$\begin{aligned} u_{1tt} - f(-f u_1 - g \eta_{1y}) &= -g \eta_{1xt} \\ v_{1tt} + f(f v_1 - g \eta_{1x}) &= -g \eta_{1yt} \end{aligned}$$

And we can express:

$$\begin{aligned} u_{1tt} + f^2 u_1 &= -g f \eta_{1y} - g \eta_{1xt} \\ v_{1tt} + f^2 v_1 &= g f \eta_{1x} - g \eta_{1yt} \end{aligned}$$

Using a wave solution we get:

$$\begin{aligned} -\omega^2 u_1 + f^2 u_1 &= -g f \eta_{1y} - g \eta_{1xt} \\ -\omega^2 v_1 + f^2 v_1 &= g f \eta_{1x} - g \eta_{1yt} \end{aligned}$$

$$u_1 = \frac{-g f \eta_{1y} - g \eta_{1xt}}{f^2 - \omega^2} \quad (25)$$

$$v_1 = \frac{g f \eta_{1x} - g \eta_{1yt}}{f^2 - \omega^2} \quad (26)$$

Similarly for the bottom layer, we get:

$$u_2 = \frac{-gf\eta_{2y} - g\eta_{1xt} - g'\eta_{2xt}}{f^2 - \omega^2} \quad (27)$$

$$v_2 = \frac{gf\eta_{2x} - g\eta_{1yt} - g'\eta_{2yt}}{f^2 - \omega^2} \quad (28)$$

We can now use equations 25- 28 to replace velocities in the continuity equations 29 and 29:

$$\begin{aligned} (\eta_1 - \eta_2)_t + H_1 \frac{-gf\eta_{1xy} - g\eta_{1xxt}}{f^2 - \omega^2} + H_1 \frac{gf\eta_{1xy} - g\eta_{1yyt}}{f^2 - \omega^2} &= 0 \\ \eta_{2t} + H_2 \frac{-gf\eta_{2xy} - g\eta_{1xxt} - g'\eta_{2xxt}}{f^2 - \omega^2} + H_2 \frac{gf\eta_{2xy} - g\eta_{1yyt} - g'\eta_{2yyt}}{f^2 - \omega^2} &= 0 \end{aligned}$$

which simplifies as:

$$\begin{aligned} (f^2 - \omega^2)(\eta_1 - \eta_2)_t + H_1(-g\eta_{1xxt} - g\eta_{1yyt}) &= 0 \\ (f^2 - \omega^2)\eta_{2t} + H_2(-g\eta_{1xxt} - g'\eta_{2xxt} - g\eta_{1yyt} - g'\eta_{2yyt}) &= 0 \end{aligned}$$

then:

$$\begin{aligned} (f^2 - \omega^2)(\eta_1 - \eta_2) + H_1g\eta_1(k^2 + l^2) &= 0 \\ (f^2 - \omega^2)\eta_2 + H_2g\eta_1(k^2 + l^2) + H_2g'\eta_2(k^2 + l^2) &= 0 \end{aligned}$$

We introduce the amplitude of the horizontal wavenumber  $K = \sqrt{k^2 + l^2}$ , and we finally have a system of 2 equations for variables  $\eta_1$  and  $\eta_2$ :

$$\begin{aligned} (f^2 - \omega^2 + gH_1K^2)\eta_1 - (f^2 - \omega^2)\eta_2 &= 0 \\ gH_2K^2\eta_1 + (f^2 - \omega^2 + g'H_2K^2)\eta_2 &= 0 \end{aligned}$$



This system has a non-trivial solution (other than  $u = v = \eta = 0$ ) only if the determinant is zero:

$$(f^2 - \omega^2 + gH_1K^2)(f^2 - \omega^2 + g'H_2K^2) + gH_2K^2(f^2 - \omega^2) = 0$$

which is a second order polynomial function in  $(f^2 - \omega^2)$ :

$$(f^2 - \omega^2)^2 + (g(H_1 + H_2)K^2 + g'H_2K^2)(f^2 - \omega^2) + gg'H_1H_2K^4 = 0$$

which has a discriminant:

$$\Delta = K^4(g^2H^2 - 4gg'H_1H_2) > 0$$

with  $H = H_1 + H_2$ .

And solutions:

$$f^2 - \omega^2 = \frac{-gHK^2 \pm K^2\sqrt{g^2H^2 - 4gg'H_1H_2}}{2}$$

With, at the first order, using the fact that  $g'/g \ll 1$ :

$$\sqrt{g^2H^2 - 4gg'H_1H_2} \approx g^2H^2 \left[ 1 - 2\frac{g'H_1H_2}{gH^2} \right]$$

And finally 2 solutions are:

$$\begin{aligned} \omega_t^2 - f^2 &= gHK^2 \left[ 1 - \frac{g'H_1H_2}{gH^2} \right] \\ \omega_c^2 - f^2 &= \frac{g'H_1H_2}{H^2} K^2 \end{aligned}$$

$$\begin{aligned}\omega_t^2 &= f^2 + gHK^2 \left[ 1 - \frac{g'H_1H_2}{gH^2} \right] \\ \omega_c^2 &= f^2 + \frac{g'H_1H_2}{H^2} K^2\end{aligned}$$

The first solution:  $\omega_t^2 = f^2 + gHK^2 \left[ 1 - \frac{g'H_1H_2}{gH^2} \right]$  corresponds to the *barotropic mode* (= external wave). It is almost the same solution than the one we get for the one-layer shallow water model (= shallow-water inertia-gravity waves / long waves:  $\omega^2 = f^2 + gHK^2$ ), but with a small correction due to the baroclinic effects:  $\left[ 1 - \frac{g'H_1H_2}{gH^2} \right]$ , with  $g'/g \ll 1$ .

The corresponding phase speed is:

$$c_t^2 = \frac{\omega_t^2}{K^2} = gH \left[ \frac{1 - \frac{g'H_1H_2}{gH^2}}{1 - \frac{f^2}{\omega^2}} \right]$$

The second solution:  $\omega_c^2 = f^2 + \frac{g'H_1H_2}{H^2} K^2$  corresponds to the *baroclinic mode* (= internal wave). It does not exist in the one-layer shallow-water model because it requires a vertical density gradient between the 2-layers ( $g' \neq 0$ ). The corresponding phase speed is:

$$c_c^2 = \frac{\omega_c^2}{K^2} = \frac{g'H_1H_2}{H(1 - \frac{f^2}{\omega^2})}$$

which is much slower than the phase speed of the barotropic wave (remember that  $g' \ll g$ ).