Numerical Modelling

the anatomy of an ocean model

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Outline

- **Lesson 1 :** [D109]
  - Introduction
  - Equations of motions
  - *Activity 1 [run an ocean model]*
- **Lesson 2 :** [D109]
  - Subgrid-scale parameterization
  - Dynamics of the ocean gyre
  - *Activity 2 [Dynamics of an ocean gyre]*
- **Lesson 3 :** [D109]
  - Horizontal Discretization
  - Vertical coordinates
  - *Activity 2 [Dynamics of an ocean gyre]*
  - *Activity 3 [Impacts of numerics / topography]*
- **Lesson 4 :** [D109]
  - Numerical schemes
  - Presentation of the model CROCO
  - *Activity 3 [Impacts of numerics / topography]*
- **Lesson 5 :** [D109]
  - Boundary Forcings
  - *Activity 5 [Design a realistic simulation]*
- **Lesson 6 :** [D109]
  - Diagnostics and validation
  - *Activity 6 [Analyze a realistic simulation]*
- **Lesson 7 :** [D109]
  - *Project*

Presentations and material will be available at :

jgula.fr/ModNum/

#3 Discretization

Master's degree 2nd year Marine Physics

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Useful references

Extensive courses:

•MIT: https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-950-atmospheric-and-oceanic-modeling-spring-2004/lecture-notes/

•Princeton: https://stephengriffies.github.io/assets/pdfs/GFM_lectures.pdf

Overview on ocean modelling and current challenges:

•Griffies et al., 2000, Developments in ocean climate modelling, Ocean Modelling.http://jgula.fr/ModNum/Griffiesetal00.pdf

•Griffies, 2006, "Some Ocean Model Fundamentals", In "Ocean Weather Forecasting: An Integrated View of Oceanography", 2006, Springer Netherlands. http://jgula.fr/ModNum/Griffies_Chapter.pdf

•Fox-Kemper et al, 19, "Challenges and Prospects in Ocean Circulation Models" http://jgula.fr/ModNum/FoxKemperetal19.pdf

CROCO/CROCO:

•https://www.myCROCO.org/wiki/

•Shchepetkin, A., and J. McWilliams, 2005: The Regional Oceanic Modeling System (CROCO): A split-explicit, free-surface, topography-following- coordinate ocean model. Ocean Modell. http://jgula.fr/ModNum/ShchepetkinMcWilliams05.pdf

Discretization

We cannot solve the continuous equations.

How to represent them with a finite set of numbers?

Discretization

We cannot solve the continuous equations.

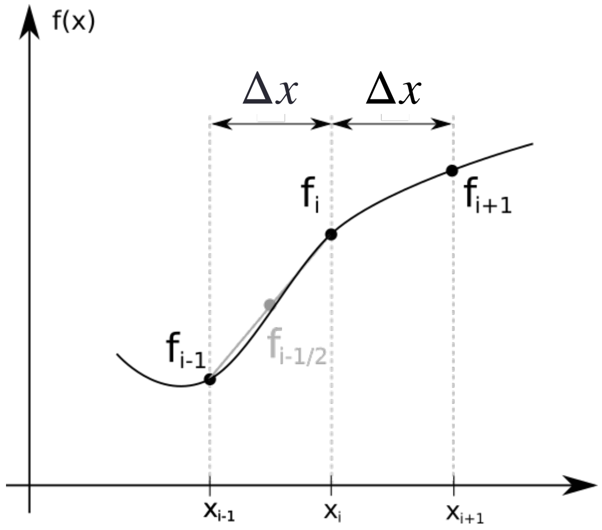
How to represent them with a finite set of numbers?

Two basic strategies:

- **Grid-point methods** (finite difference, finite volume)
- Series expansion methods (spectral, finite element)

Discretization

<u>Grid-point methods</u>:



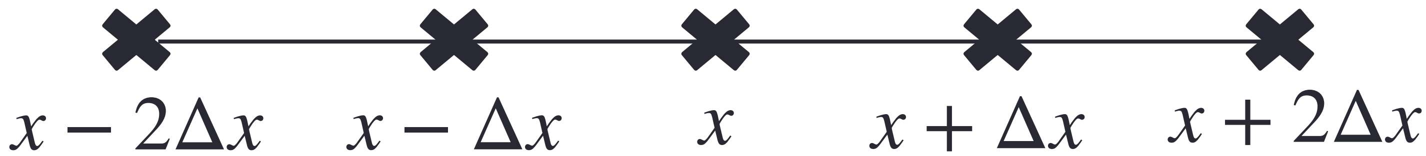
- Use of Taylor series to estimate truncation errors

$$f(x_{i+1}) = f(x_i) + \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \ldots + \frac{f^{(n)}(x)}{n!}\Delta x^n + R_n$$

- Order of accuracy = **lower order** of the error of a scheme

Discretization

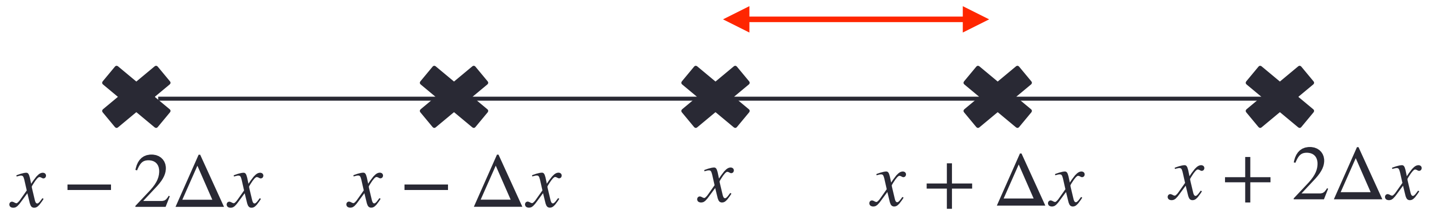
- Constructing a difference operator using Taylor series:



$$f(x + \Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \ldots + \frac{f^{(n)}(x)}{n!}\Delta x^n + R_n$$

Discretization

- Constructing a difference operator using Taylor series:



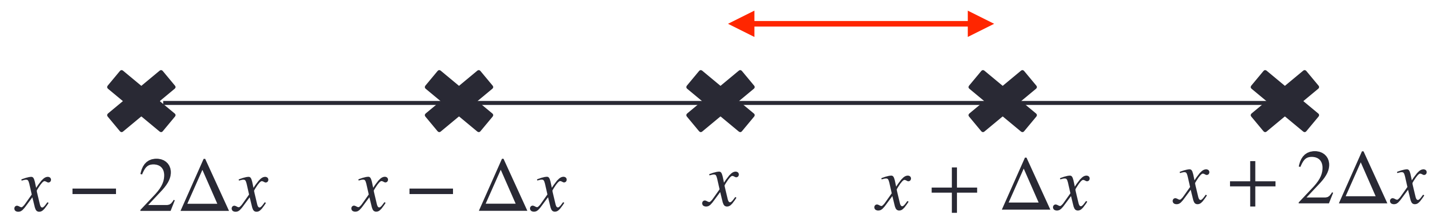
$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \ldots + \frac{f^{(n)}(x)}{n!}\Delta x^n + R_n$$

- 1st order approx. for derivative:

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f^{(2)}(x)}{2!}\Delta x + \ldots$$

Discretization

- Constructing a difference operator using Taylor series:



$$f(x + \Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \ldots + \frac{f^{(n)}(x)}{n!}\Delta x^n + R_n$$

- 1st order approx. for derivative (Downstream):

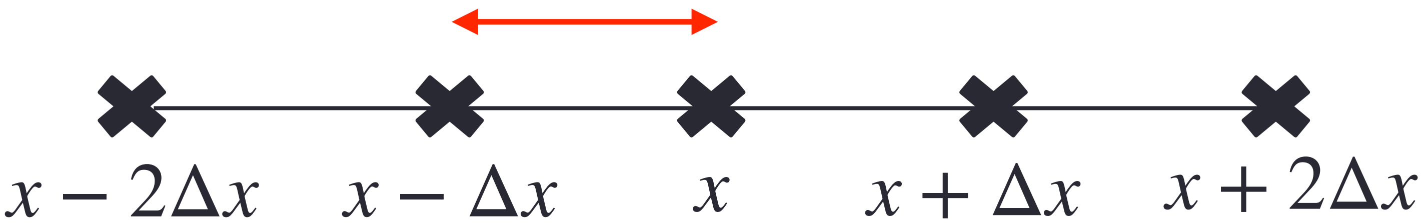
Truncation Error

$$\boxed{f'(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x}} - \frac{f^{(2)}(x)}{2!}\Delta x + \ldots$$

$$O(\Delta x)$$

Discretization

- Constructing a difference operator using Taylor series:



$$f(x - \Delta x) = f(x) - \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \ldots + (-1)^n \frac{f^{(n)}(x)}{n!}\Delta x^n + R_n$$

- 1st order approx. for derivative (upstream):

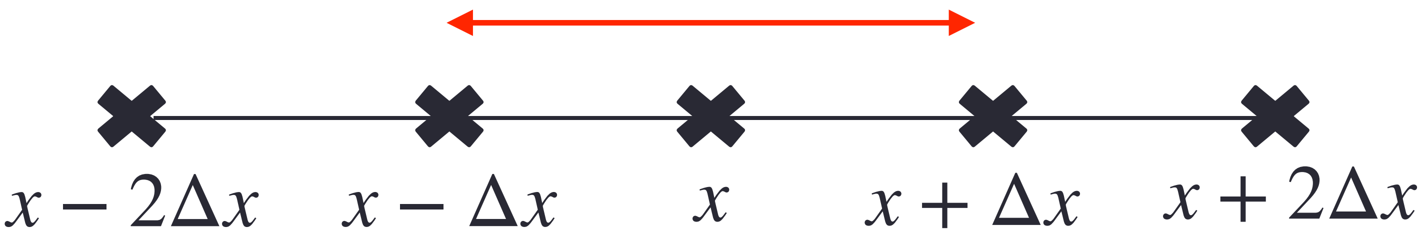
Truncation Error

$$\boxed{f'(x) = \frac{f(x) - f(x - \Delta x)}{\Delta x}} + \frac{f^{(2)}(x)}{2!}\Delta x + \ldots$$

$$O(\Delta x)$$

Discretization

- Constructing a difference operator using Taylor series:



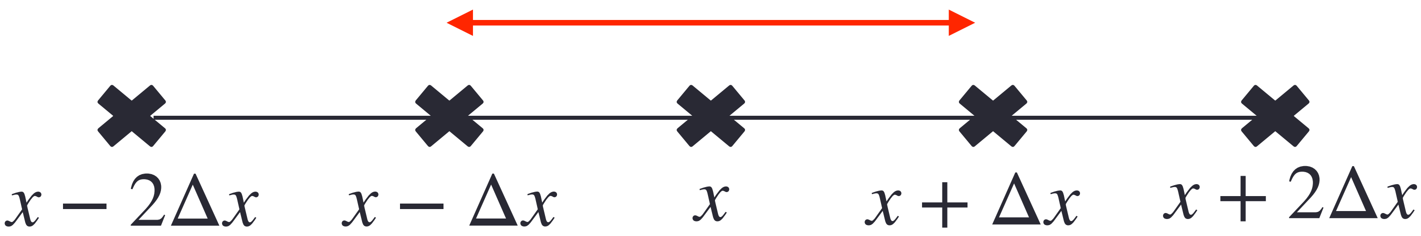
$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \ldots + \frac{f^{(n)}(x)}{n!}\Delta x^n + R_n$$

$$f(x-\Delta x) = f(x) - \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \ldots + (-1)^n\frac{f^{(n)}(x)}{n!}\Delta x^n + R_n$$

- 2nd order approx. for derivative?

Discretization

- Constructing a difference operator using Taylor series:



$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1\,!}\Delta x + \frac{f^{(2)}(x)}{2\,!}\Delta x^2 + \ldots + \frac{f^{(n)}(x)}{n\,!}\Delta x^n + R_n$$

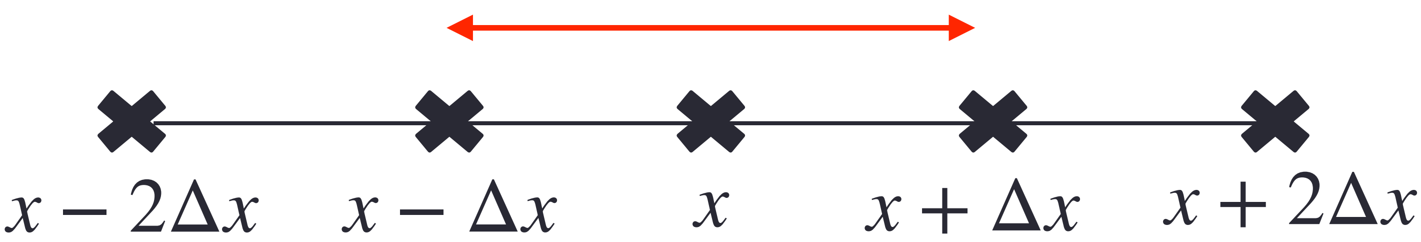
$$f(x-\Delta x) = f(x) - \frac{f'(x)}{1\,!}\Delta x + \frac{f^{(2)}(x)}{2\,!}\Delta x^2 + \ldots + (-1)^n\frac{f^{(n)}(x)}{n\,!}\Delta x^n + R_n$$

- 2nd order approx. for derivative?

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{f^{(3)}(x)}{3\,!}\Delta x^2 + \ldots$$

Discretization

- Constructing a difference operator using Taylor series:



$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \ldots + \frac{f^{(n)}(x)}{n!}\Delta x^n + R_n$$

$$f(x-\Delta x) = f(x) - \frac{f'(x)}{1!}\Delta x + \frac{f^{(2)}(x)}{2!}\Delta x^2 + \ldots + (-1)^n\frac{f^{(n)}(x)}{n!}\Delta x^n + R_n$$

- 2nd order approx. for derivative?

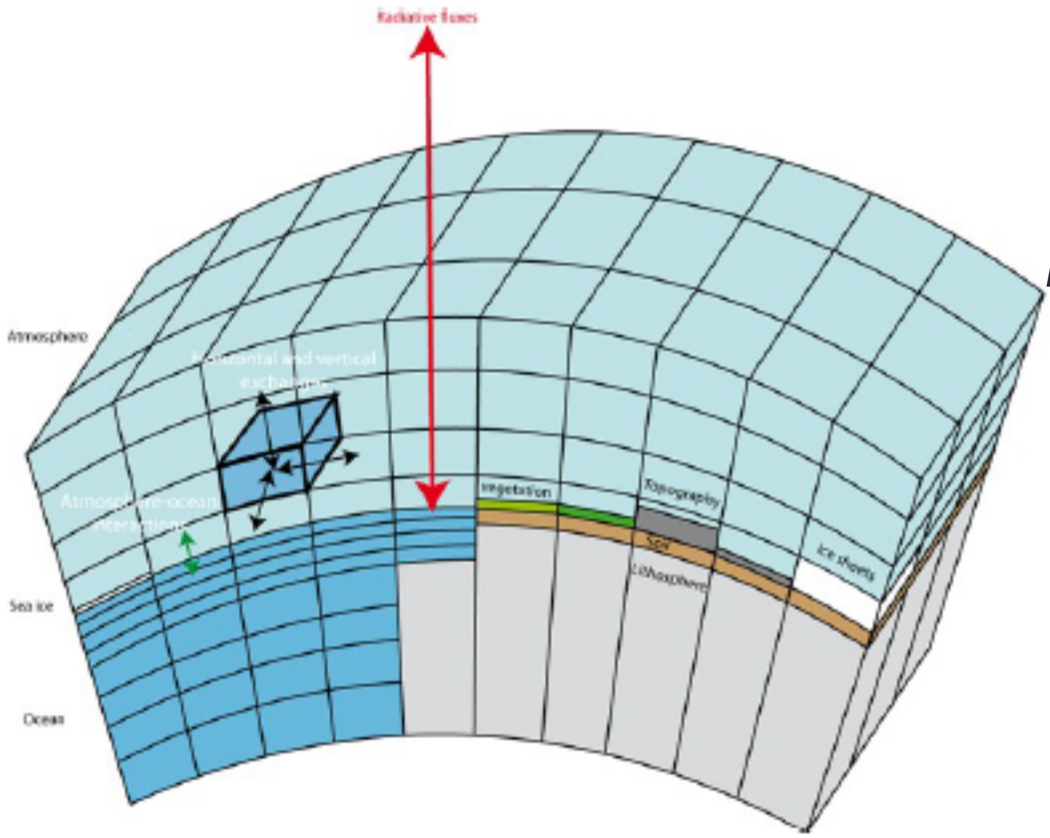
Truncation Error

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{f^{(3)}(x)}{3!}\Delta x^2 + \ldots$$

$$O(\Delta x^2)$$

Discretization

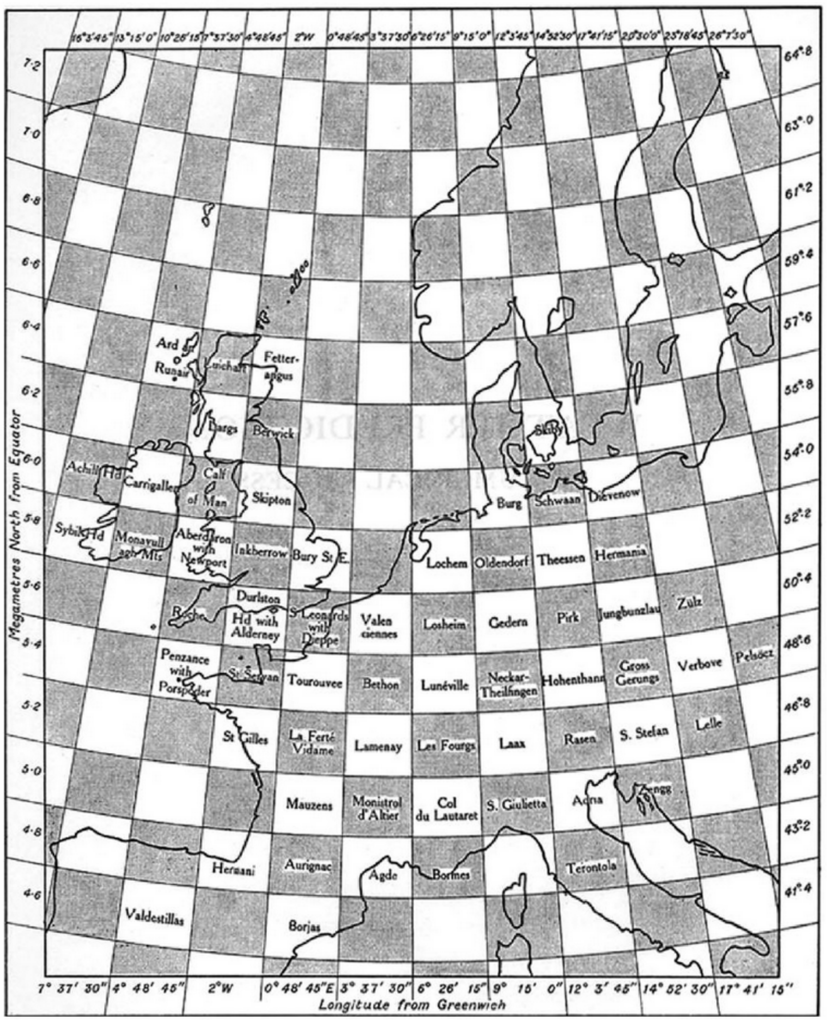
The ocean is divided into boxes : discretization



Example of a finite difference grid

Discretization

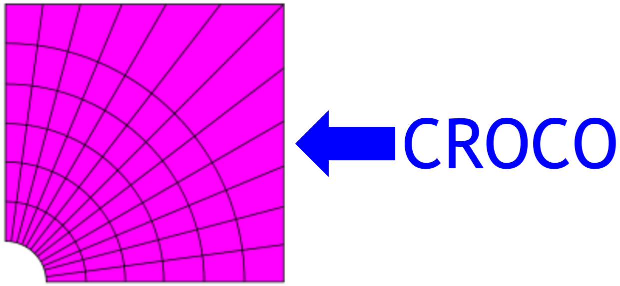
The ocean is divided into boxes : **discretization**



Richardson's 1922 first grid designed for weather prediction

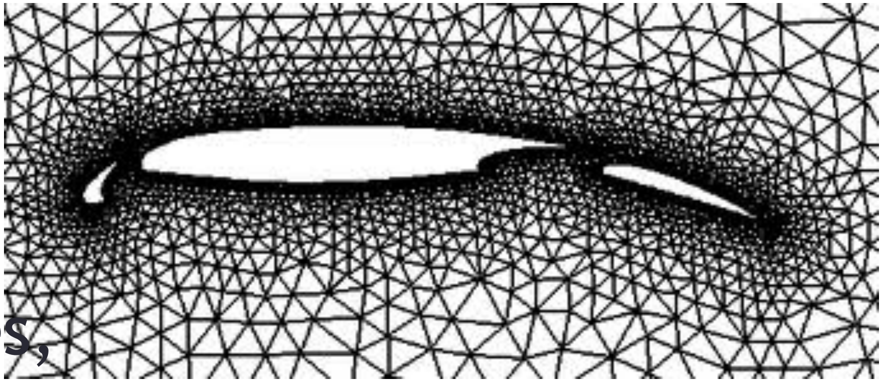
Discretization

Structured grids
Identified by regular connectivity
= can be adressed by (i,j,k)



Unstructured grids
The domain is tiled using more general geometrical shapes (triangles, ...) pieced together to optimally fit details of the geometry.

- ✓ Good for tidal modeling, engineering applications.
- ✓ Problems:
  geostrophic balance accuracy,
  conservation and positivity properties,
  ...



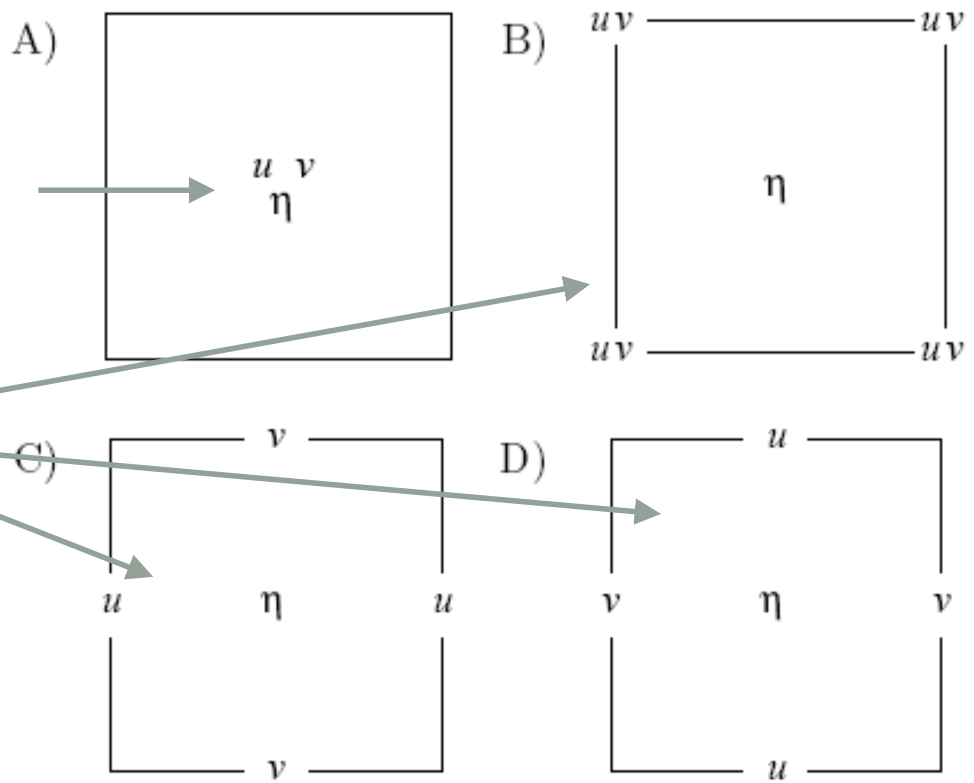
Horizontal discretization

Different types of Horizontal Grids (Arakawa Grids):

Non-staggered
(= collocated variables)

Or

Staggered



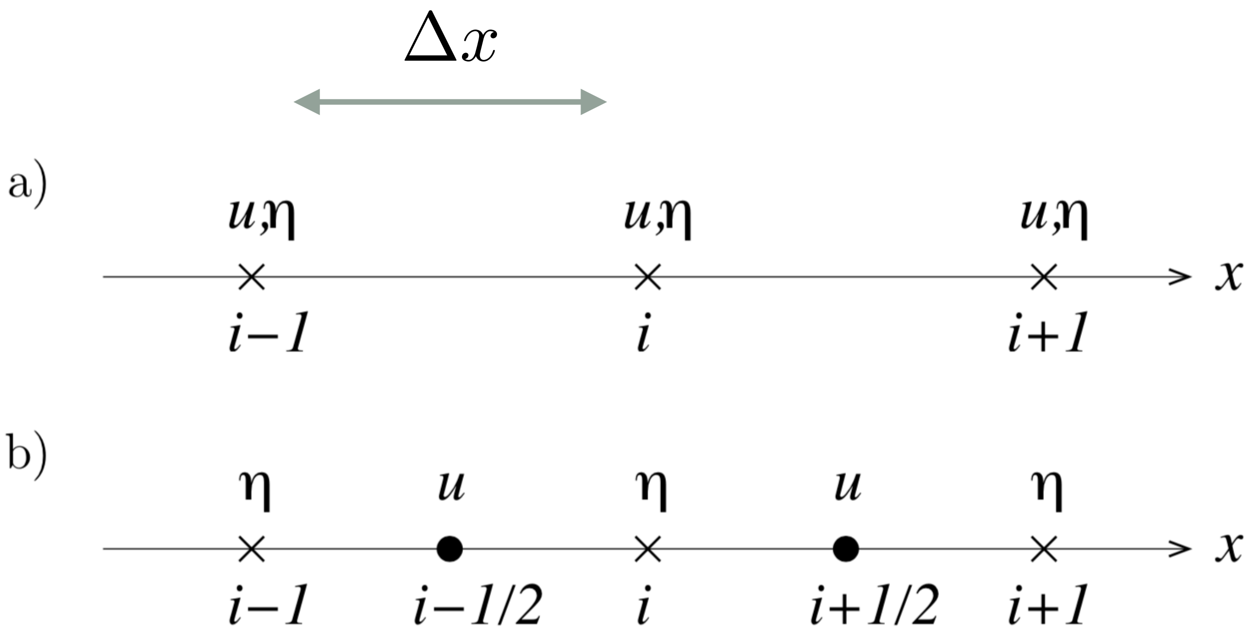
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

Non-staggered

Vs

Staggered



Use of staggered or unstaggered grids strongly affects stability, accuracy, and spurious modes.

Horizontal discretization

Example of numerical dispersion:

1. The advection equation $\partial_t \theta + c\partial_x \theta = 0$

Solutions of the continuous equations are non-dispersive waves
$\theta(x,t) = \theta_o e^{i(kx-\omega t)}$ *with dispersion relation* $\omega = ck$

Horizontal discretization

Example of numerical dispersion:

1. The advection equation $\partial_t \theta + c\partial_x \theta = 0$

*Solutions of the **continuous equations** are non-dispersive waves*
$\theta(x,t) = \theta_o e^{i(kx-\omega t)}$ *with dispersion relation* $\omega = ck$

Discretized equations with the centered second order derivative are:

$$d_t\theta + \frac{c}{\Delta x}\delta_i \overline{\theta}^i = 0$$

$$d_t\theta_i + \frac{c}{2\Delta x}\left(\theta_{i+1} - \theta_{i-1}\right) = 0$$

Horizontal discretization

Example of numerical dispersion:

1. The advection equation $\partial_t \theta + c\partial_x \theta = 0$

Substituting in our solution:

$$\theta_i(x,t) = \theta_0 e^{i(kx-\omega t)}$$

$$\theta_{i-1}(x,t) = \theta_0 e^{i(k(x-\Delta x)-\omega t)}$$

$$\theta_{i+1}(x,t) = \theta_0 e^{i(k(x+\Delta x)-\omega t)}$$

Horizontal discretization

Example of numerical dispersion:

1. The advection equation $\partial_t \theta + c\partial_x \theta = 0$

Substituting in the equation:

$$d_t \theta_i + \frac{c}{2\Delta x}\left(\theta_{i+1} - \theta_{i-1}\right) = 0$$

gives

$$\begin{aligned} -i\omega &= -\frac{c}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) \\ &= -\frac{ci}{\Delta x}\sin k\Delta x \end{aligned}$$

Horizontal discretization

Example of numerical dispersion:

1. The advection equation $\partial_t \theta + c\partial_x \theta = 0$

$$\begin{aligned} -i\omega &= -\frac{c}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) \\ &= -\frac{ci}{\Delta x}\sin k\Delta x \end{aligned}$$

Now the solution is **dispersive!!!**

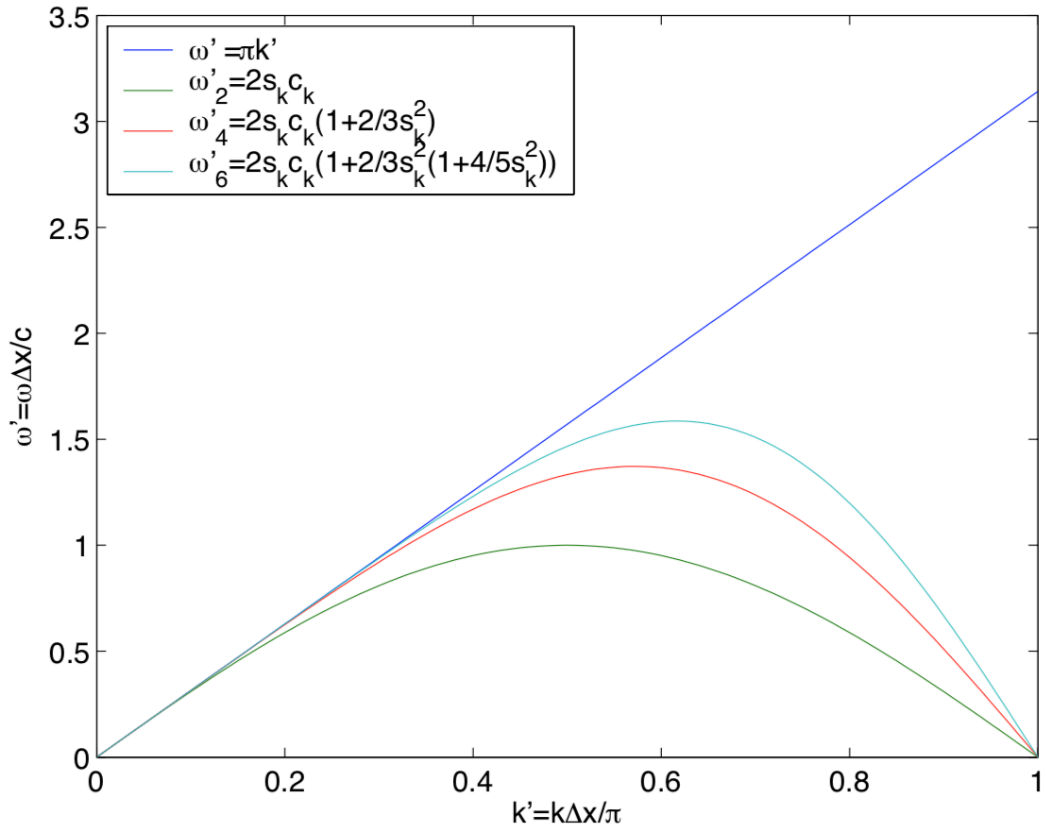
Even it will converge to the non-dispersive solution in the limit of small Δx

$$\omega = \frac{c}{\Delta x}\sin k\Delta x \overset{\Delta x \to 0}{=} ck$$

Horizontal discretization

Example of numerical dispersion:

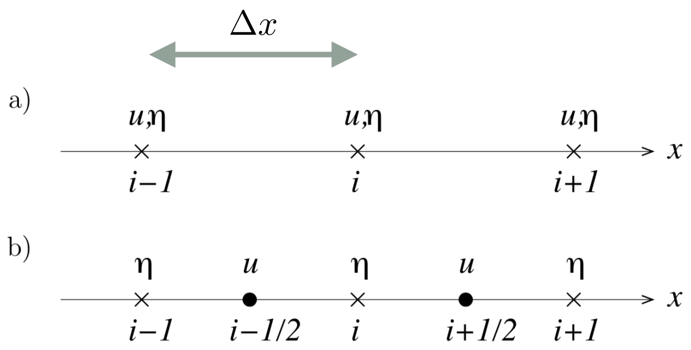
1. The advection equation $\partial_t\theta + c\partial_x\theta = 0$



Dispersion relations for constant flow advection using second, fourth, and sixth order spatial differences.

Horizontal discretization

Staggered Vs unstaggered : the 1D problem



2. Gravity waves

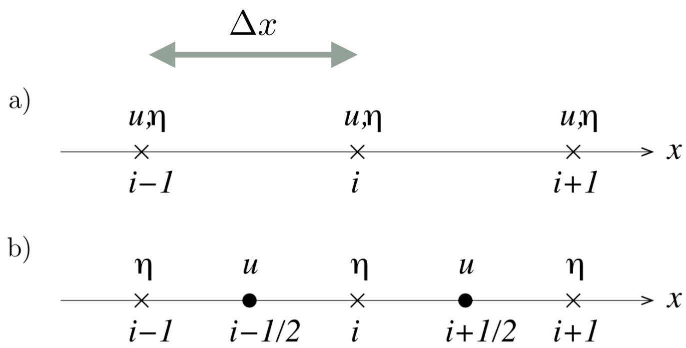
$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Solutions of the continuous equations are non-dispersive waves

$\eta = \eta_o e^{i(kx-\omega t)}$ *with dispersion relation* $\omega = \pm\sqrt{gH}k$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem



2. Gravity waves

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Solutions of the continuous equations are non-dispersive waves

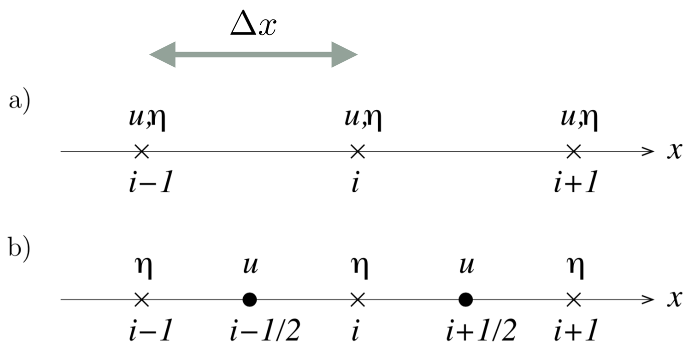
$\eta = \eta_o e^{i(kx-\omega t)}$ *with dispersion relation* $\omega = \pm\sqrt{gH}k$

Discretized equations with the centered second order derivative on the **unstaggered grid** are:

$$\begin{aligned}\partial_t u &= -\frac{g}{\Delta x}\delta_i\overline{\eta}^i \\ \partial_t \eta &= -\frac{H}{\Delta x}\delta_i\overline{u}^i\end{aligned}$$

$$\longrightarrow \partial_{tt}\eta = \frac{gH}{\Delta x^2}\delta_{ii}\overline{\eta}^{ii}$$ with $$\delta_{ii}\overline{\eta}^{ii} = \frac{1}{4}\left(\eta_{i-2} - 2\eta_i + \eta_{i+2}\right)$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$\begin{aligned}\partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u\end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

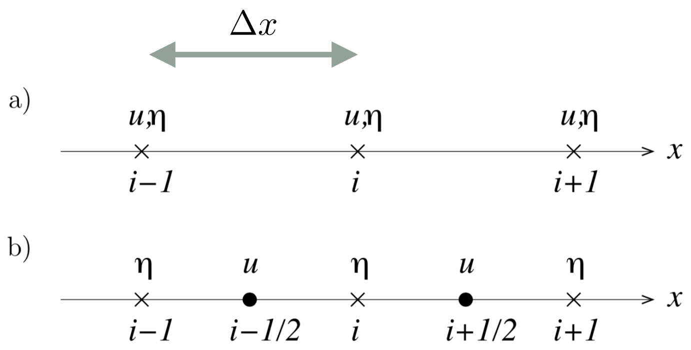
Substituting in our solution on the unstaggered grid gives :

$$\begin{aligned}-\omega^2 &= \frac{gH}{4\Delta x^2}\left(e^{-i2k\Delta x} - 2 + e^{i2k\Delta x}\right) \\ &= \frac{gH}{4\Delta x^2}(2\cos 2k\Delta x - 2) \\ &= -\frac{4gH}{\Delta x^2}\sin^2\frac{k\Delta x}{2}\cos^2\frac{k\Delta x}{2}\end{aligned}$$

- Question:
  - What is the dispersion relation on the staggered grid?

Horizontal discretization

Staggered Vs unstaggered : the 1D problem



2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

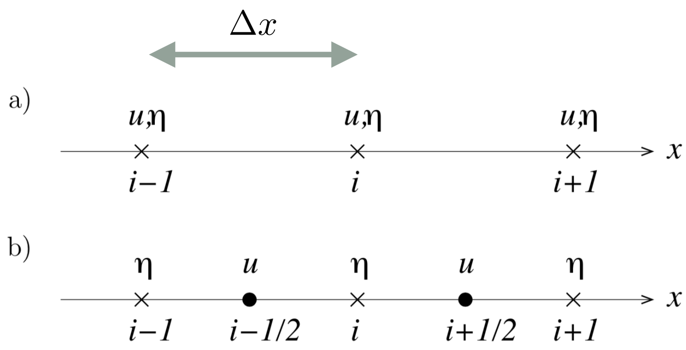
Discretized equations with the centered second order derivative on the **staggered grid** are:

$$\begin{aligned} \partial_t u &= -\frac{g}{\Delta x}\delta_i \eta \\ \partial_t \eta &= -\frac{H}{\Delta x}\delta_i u \end{aligned}$$

This can be written as a system:

$$\begin{pmatrix} \partial_t & \frac{g}{\Delta x}\delta_i \\ \frac{H}{\Delta x}\delta_i & \partial_t \end{pmatrix} \begin{pmatrix} u \\ \eta \end{pmatrix} = 0 \qquad \begin{pmatrix} -i\omega & \frac{2ig}{\Delta x}\sin\frac{k\Delta x}{2} \\ \frac{2iH}{\Delta x}\sin\frac{k\Delta x}{2} & -i\omega \end{pmatrix} \begin{pmatrix} u \\ \eta \end{pmatrix} = 0$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

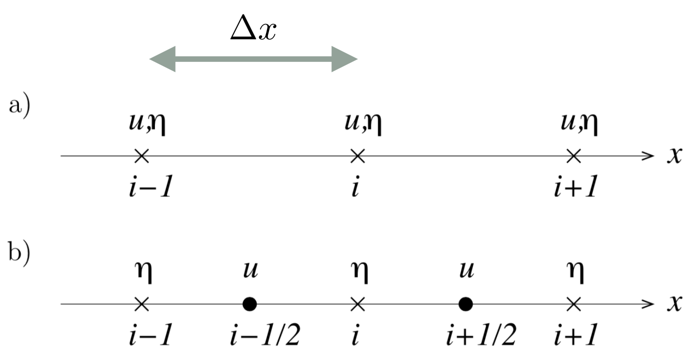
2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Substituting in our solution on the staggered grid gives :

$$\omega^2 - \frac{4gH}{\Delta x^2}\sin^2\frac{k\Delta x}{2} = 0$$

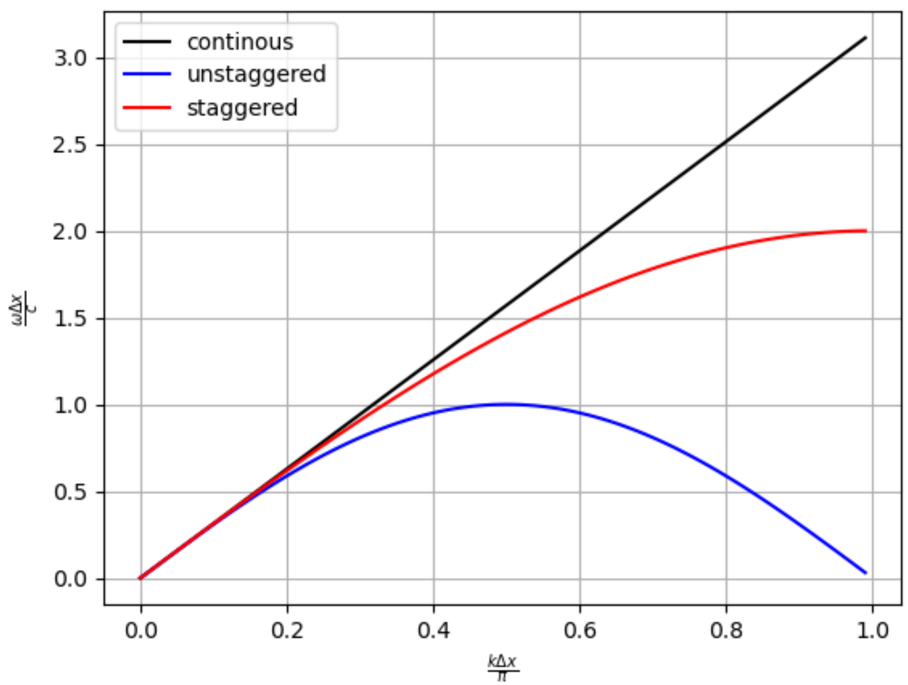
Horizontal discretization



Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$
\begin{aligned}
\partial_t u &= -g\partial_x \eta \\
\partial_t \eta &= -H\partial_x u
\end{aligned}
\longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta
$$



Dispersion of numerical gravity wave for the unstaggered grid (green) and the staggered grid (red). The continuum (= k) is plotted for comparison (blue).

When compared to the continuum we see that the numerical modes are still dispersive on the staggered grid, but:

there is no false extrema, unlike the non-staggered grid,

the group speed is of the correct sign everywhere, even if reduced.

$$v_g = \partial_k \omega$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

$$\begin{aligned} \partial_t u - fv + g\partial_x \eta &= 0 \\ \partial_t v + fu &= 0 \\ \partial_t \eta + H\partial_x u &= 0 \end{aligned}$$

Solutions of the continuous equations are waves following the dispersion relation:

$$\left| \begin{pmatrix} -i\omega & -f & gik \\ f & -i\omega & 0 \\ Hik & 0 & -i\omega \end{pmatrix} \right| = 0 \Rightarrow \begin{cases} \omega = 0 \\ \omega^2 = f^2 + gHk^2 \end{cases}$$

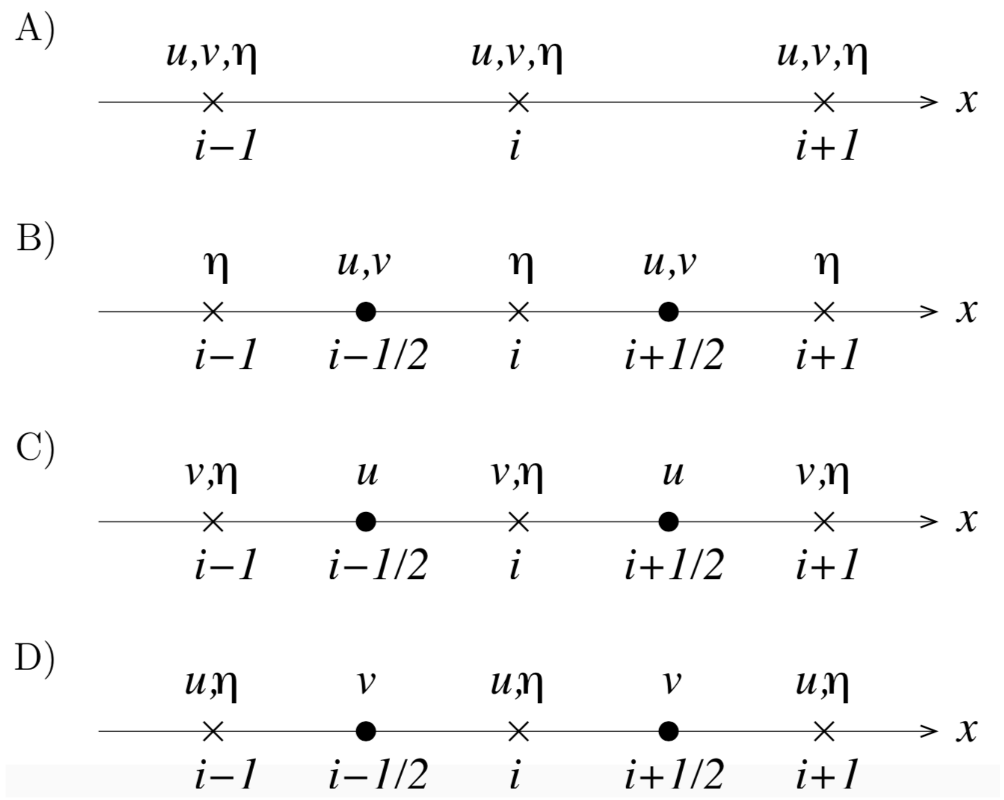
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

$$
\begin{aligned}
\partial_t u - fv + g\partial_x \eta &= 0 \\
\partial_t v + fu &= 0 \\
\partial_t \eta + H\partial_x u &= 0
\end{aligned}
$$

Now, 4 different grids are possible:



Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

- A-grid model

$$
\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\
\partial_t v + f u &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i &= 0
\end{aligned}
$$

- B-grid model

$$
\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f u &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u &= 0
\end{aligned}
$$

- C-grid model

$$
\begin{aligned}
\partial_t u - f \overline{v}^i + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f \overline{u}^i &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u &= 0
\end{aligned}
$$

- D-grid model

$$
\begin{aligned}
\partial_t u - f \overline{v}^i + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\
\partial_t v + f \overline{u}^i &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i &= 0
\end{aligned}
$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

The corresponding dispersion relations are :

A: $$\frac{\omega^2}{f^2} = 1 + \frac{4L_d^2}{\Delta x^2} s_k^2 c_k^2$$

B: $$\frac{\omega^2}{f^2} = 1 + \frac{4L_d^2}{\Delta x^2} s_k^2$$

C: $$\frac{\omega^2}{f^2} = c_k^2 + \frac{4L_d^2}{\Delta x^2} s_k^2$$

D: $$\frac{\omega^2}{f^2} = c_k^2 + \frac{4L_d^2}{\Delta x^2} s_k^2 c_k^2$$

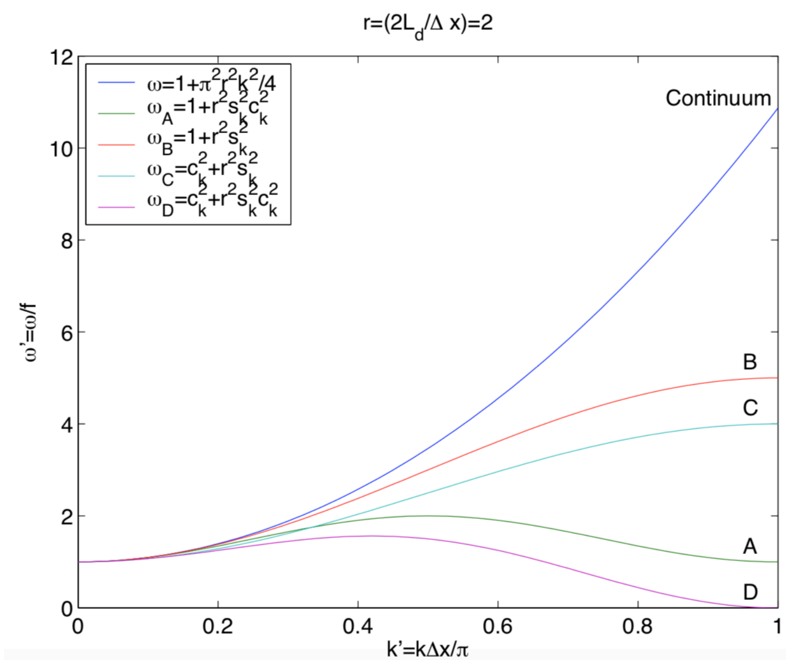
$$s_k = \sin \frac{k\Delta x}{2} \qquad c_k = \cos \frac{k\Delta x}{2}$$

$$L_d = \sqrt{gH}/f$$

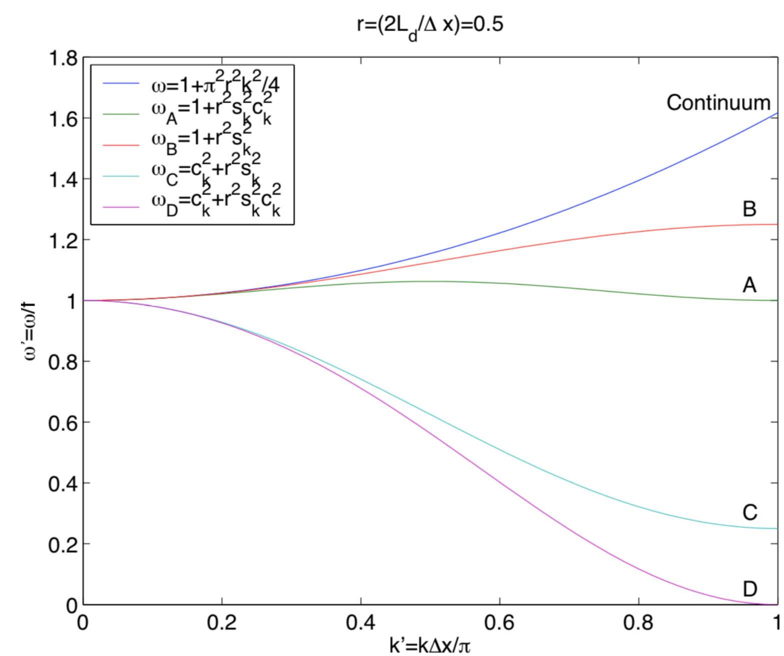
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves



deformation radius is resolved

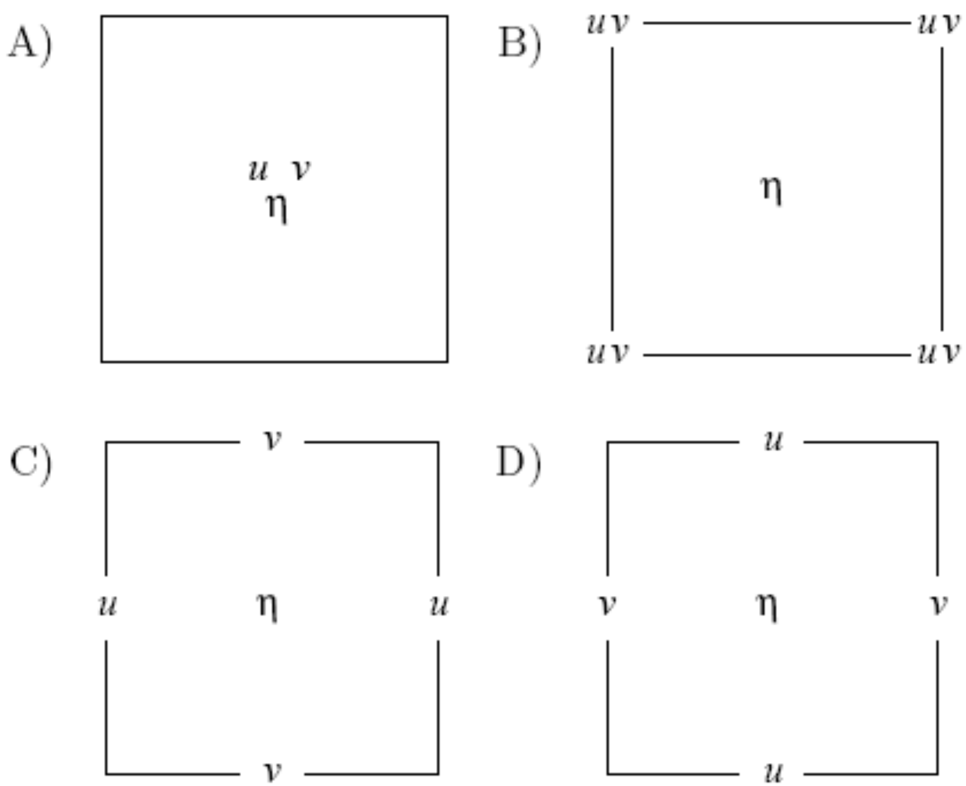


deformation radius is not resolved

Staggering variables in the form of the B grid is most likely to avoid computational modes when solving one-dimensional shallow water equations.

Horizontal discretization

Horizontal Arakawa Grids:

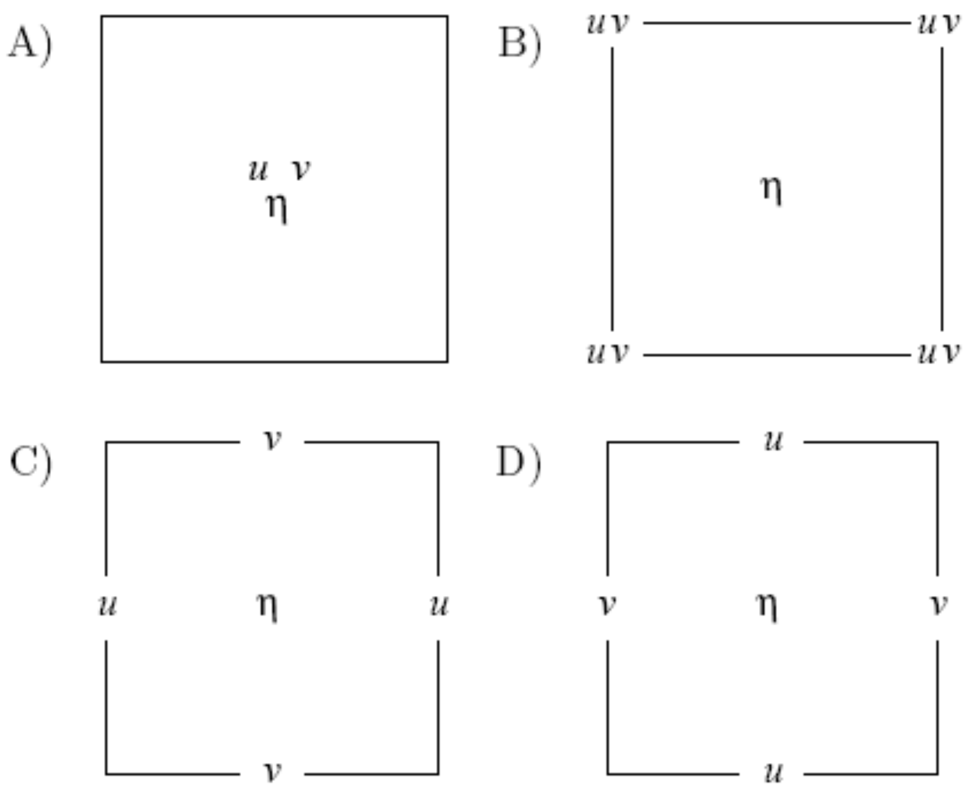


Linear shallow water equation:

$$
\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f u + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_y \eta &= 0
\end{aligned}
$$

Horizontal discretization

Horizontal Arakawa Grids:

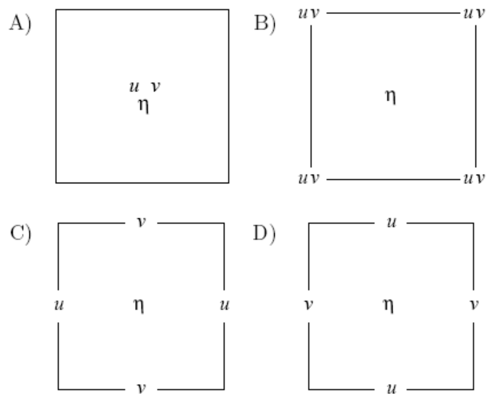


Linear shallow water equation:

$$
\begin{aligned}
\partial_t u - fv + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + fu + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_y \eta &= 0
\end{aligned}
$$

- Question:
  - Which grid minimises the number of averaging between points when solving linear SW equations in 2d?

Horizontal discretization



- A grid:

$$\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\
\partial_t v + f u + \frac{g}{\Delta y}\delta_j \overline{\eta}^j &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i + \frac{H}{\Delta y}\delta_j \overline{v}^j &= 0
\end{aligned}$$

- B grid:

$$\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \overline{\eta}^j &= 0 \\
\partial_t v + f u + \frac{g}{\Delta y}\delta_j \overline{\eta}^i &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^j + \frac{H}{\Delta y}\delta_j \overline{v}^i &= 0
\end{aligned}$$

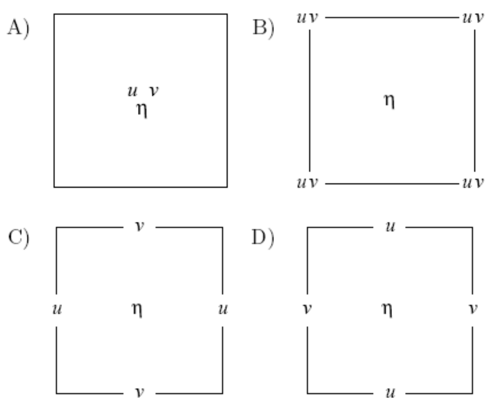
- C grid:

$$\begin{aligned}
\partial_t u - f \overline{v}^{ij} + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f \overline{u}^{ij} + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_j v &= 0
\end{aligned}$$

- D grid:

$$\begin{aligned}
\partial_t u - f \overline{v}^{ij} + \frac{g}{\Delta x}\delta_i \overline{\eta}^{ij} &= 0 \\
\partial_t v + f \overline{u}^{ij} + \frac{g}{\Delta y}\delta_j \overline{\eta}^{ij} &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^{ij} + \frac{H}{\Delta y}\delta_j \overline{v}^{ij} &= 0
\end{aligned}$$

Horizontal discretization



Response of each operator:

$$
\begin{aligned}
R(\delta_i\phi) &= 2i\sin\frac{k\Delta x}{2} = 2is_k \\
R(\delta_j\phi) &= 2i\sin\frac{l\Delta y}{2} = 2is_l \\
R(\overline{\phi}^i) &= \cos\frac{k\Delta x}{2} = c_k \\
R(\overline{\phi}^j) &= \cos\frac{l\Delta y}{2} = c_l
\end{aligned}
$$

Dispersion relations:

- A grid:

$$
\begin{aligned}
\omega^2 &= f^2 + \frac{4gH}{\Delta x^2}s_k^2c_k^2 + \frac{4gH}{\Delta y^2}s_l^2c_l^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= 1 + r_x^2s_k^2c_k^2 + r_y^2s_l^2c_l^2
\end{aligned}
$$

- B grid:

$$
\begin{aligned}
\omega^2 &= f^2 + \frac{4gH}{\Delta x^2}s_k^2c_l^2 + \frac{4gH}{\Delta y^2}s_l^2c_k^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= 1 + r_x^2s_k^2c_l^2 + r_y^2s_l^2c_k^2
\end{aligned}
$$

- C grid:

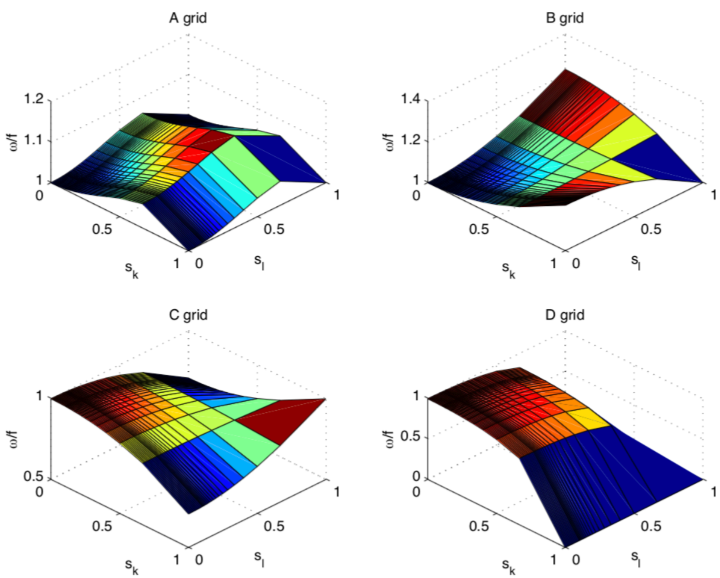
$$
\begin{aligned}
\omega^2 &= f^2c_k^2c_l^2 + \frac{4gH}{\Delta x^2}s_k^2 + \frac{4gH}{\Delta y^2}s_l^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= c_k^2c_l^2 + r_x^2s_k^2 + r_y^2s_l^2
\end{aligned}
$$

- D grid:

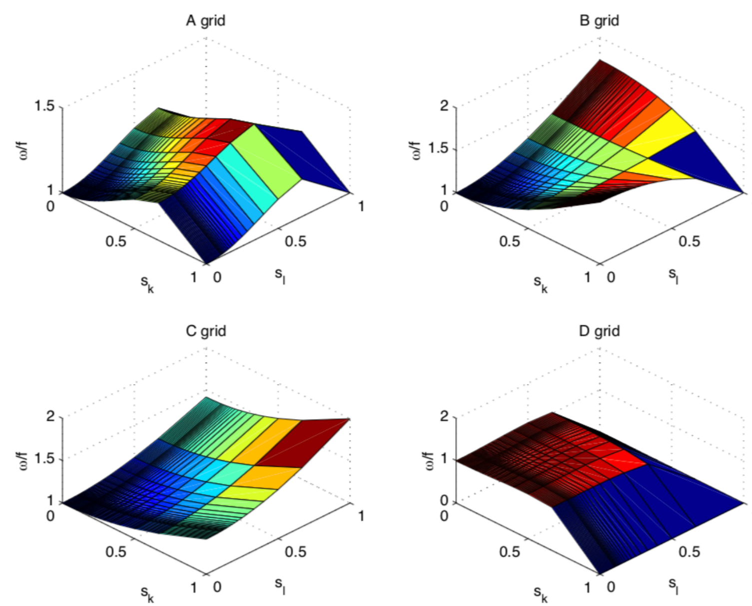
$$
\begin{aligned}
\omega^2 &= f^2c_k^2c_l^2 + \frac{4gH}{\Delta x^2}s_k^2c_k^2c_l^2 + \frac{4gH}{\Delta y^2}s_l^2c_k^2c_l^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= (1 + r_x^2s_k^2 + r_y^2s_l^2)c_k^2c_l^2
\end{aligned}
$$

Horizontal discretization

Coarse resolution:



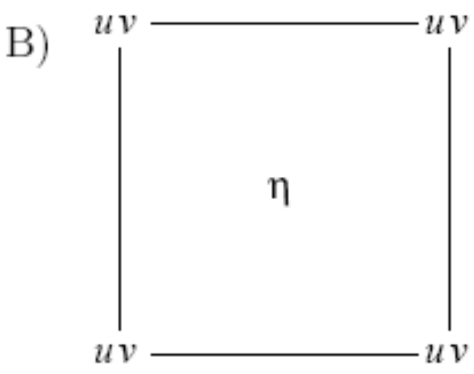
High resolution:



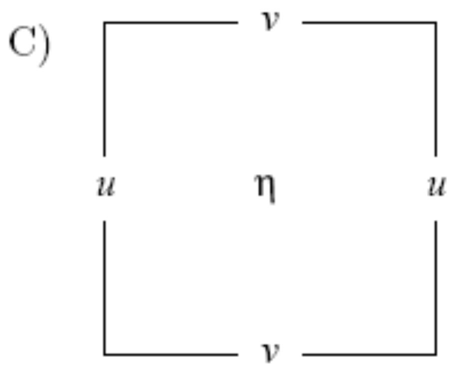
D is always bad.
B underestimates frequency for short two-dimensional waves
C is the only grid with monotonically increasing frequency (i.e. right sign of group velocity) at high res.

Horizontal discretization

- B grid is prefered at coarse resolution, when Coriolis is important:
  - Superior for poorly resolved inertia-gravity waves.
  - Good for Rossby waves: collocation of velocity points.
  - Bad for gravity waves: computational checkerboard mode



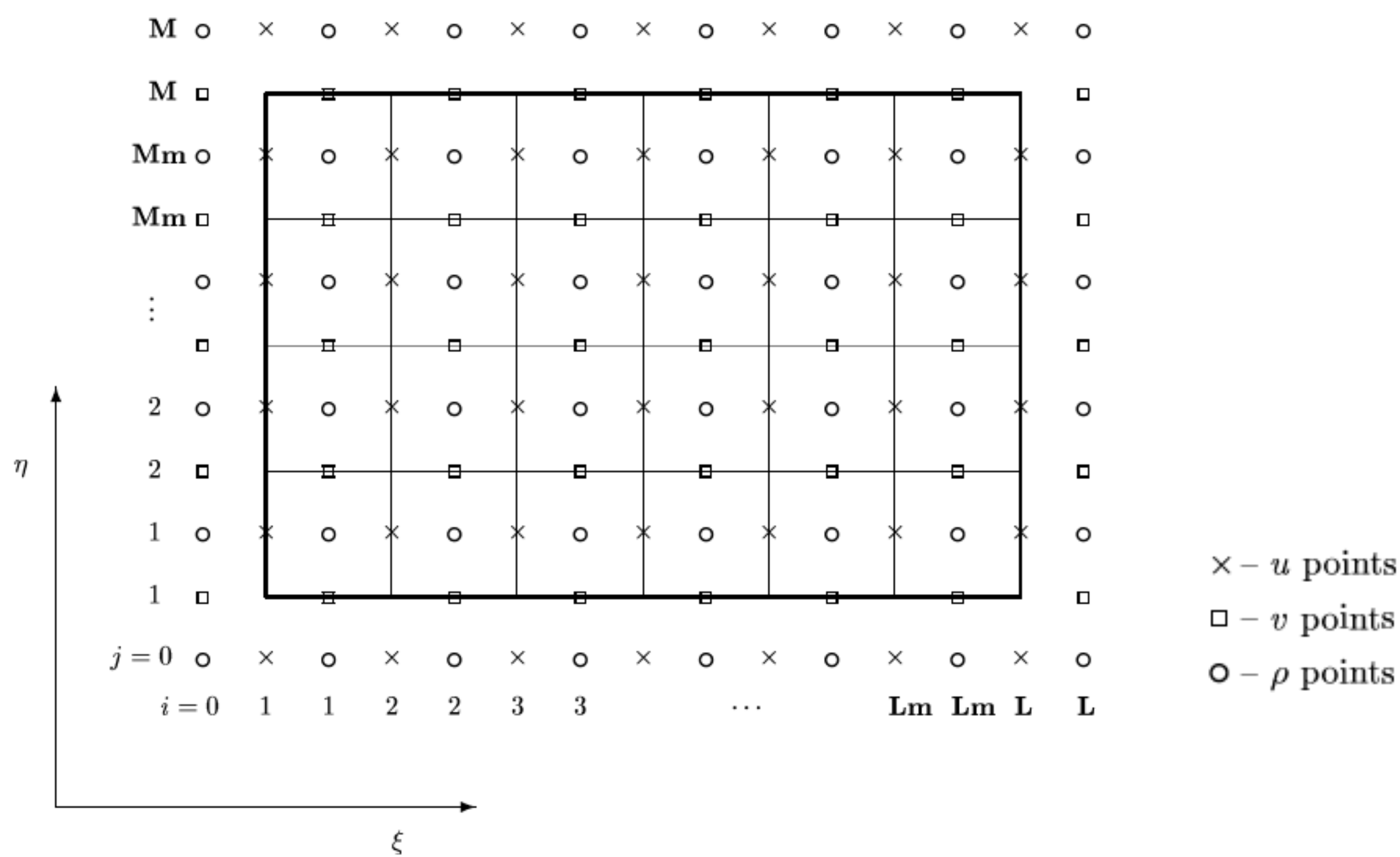
- C grid is prefered at fine resolution, when Coriolis is less important
  - Superior for gravity waves.
  - Good for well resolved inertia-gravity waves.
  - Bad for poorly resolved waves: Rossby waves (computational checkerboard mode) and inertia-gravity waves due to averaging the Coriolis force.



CROCO

Horizontal discretization

CROCO: Arakawa C-grid



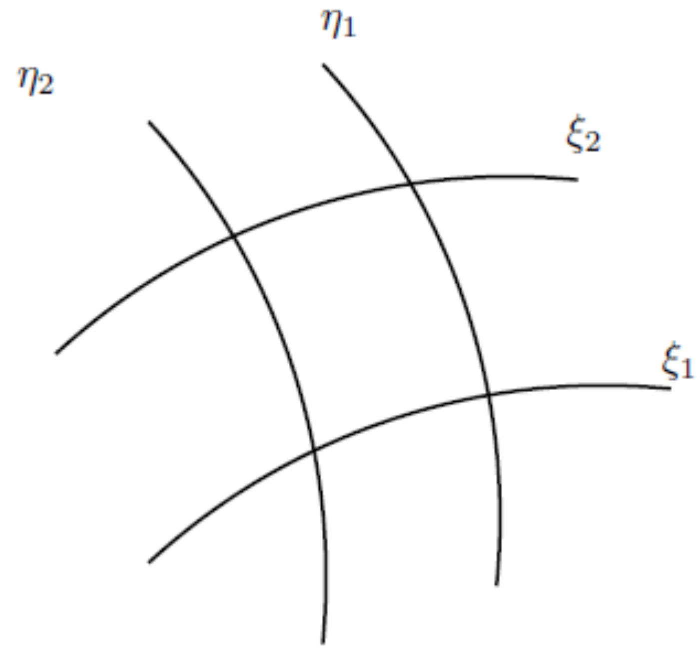
Horizontal curvilinear grid

- **CROCO:** is formulated in general horizontal curvilinear coordinates:

$$(ds)_\xi = \left(\frac{1}{m}\right) d\xi$$

$$(ds)_\eta = \left(\frac{1}{n}\right) d\eta$$

m, n: scale factors relating the differential distances to the physical arc lengths



$$\vec{v} \cdot \hat{\xi} = u$$

$$\vec{v} \cdot \hat{\eta} = v$$

Horizontal curvilinear grid

- **CROCO:** is formulated in general horizontal curvilinear coordinates:

$$(ds)_\xi = \left(\frac{1}{m}\right) d\xi$$

$$(ds)_\eta = \left(\frac{1}{n}\right) d\eta$$

With classical formulas for div, grad, curl and lap in curvilinear coordinates:

$$\nabla\phi = \hat{\xi} m \frac{\partial\phi}{\partial\xi} + \hat{\eta} n \frac{\partial\phi}{\partial\eta}$$

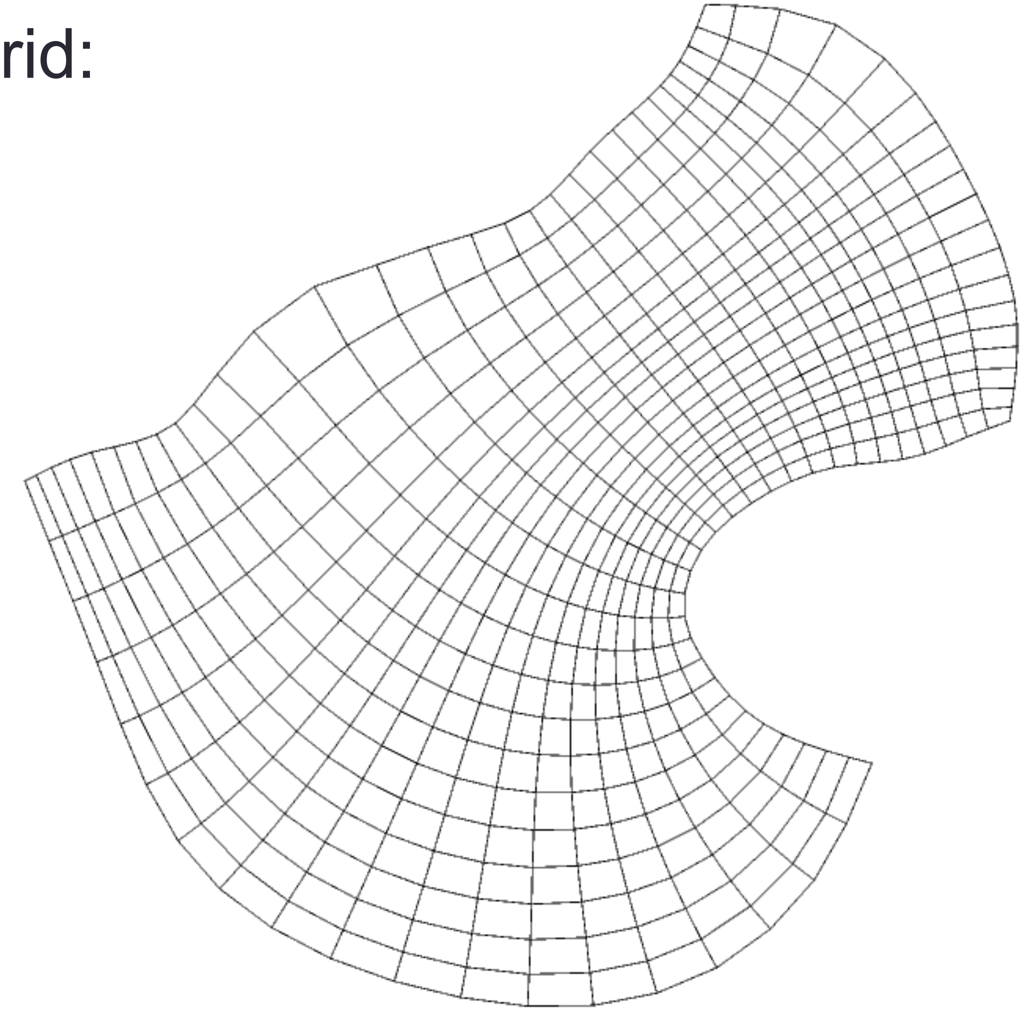
$$\nabla \cdot \vec{a} = mn \left[\frac{\partial}{\partial\xi}\left(\frac{a}{n}\right) + \frac{\partial}{\partial\eta}\left(\frac{b}{m}\right) \right]$$

$$\nabla \times \vec{a} = mn \begin{vmatrix} \frac{\hat{\xi}_1}{m} & \frac{\hat{\xi}_2}{n} & \hat{k} \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} & \frac{\partial}{\partial z} \\ \frac{a}{m} & \frac{b}{n} & c \end{vmatrix}$$

$$\nabla^2\phi = \nabla \cdot \nabla\phi = mn \left[\frac{\partial}{\partial\xi}\left(\frac{m}{n}\frac{\partial\phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{n}{m}\frac{\partial\phi}{\partial\eta}\right) \right]$$

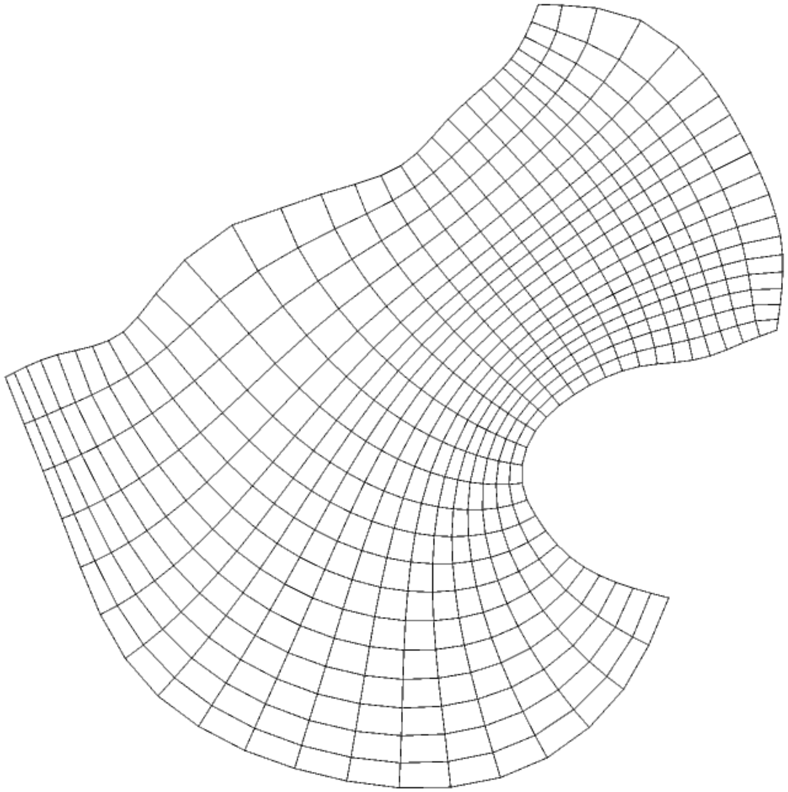
Horizontal curvilinear grid

- This is a possible grid:



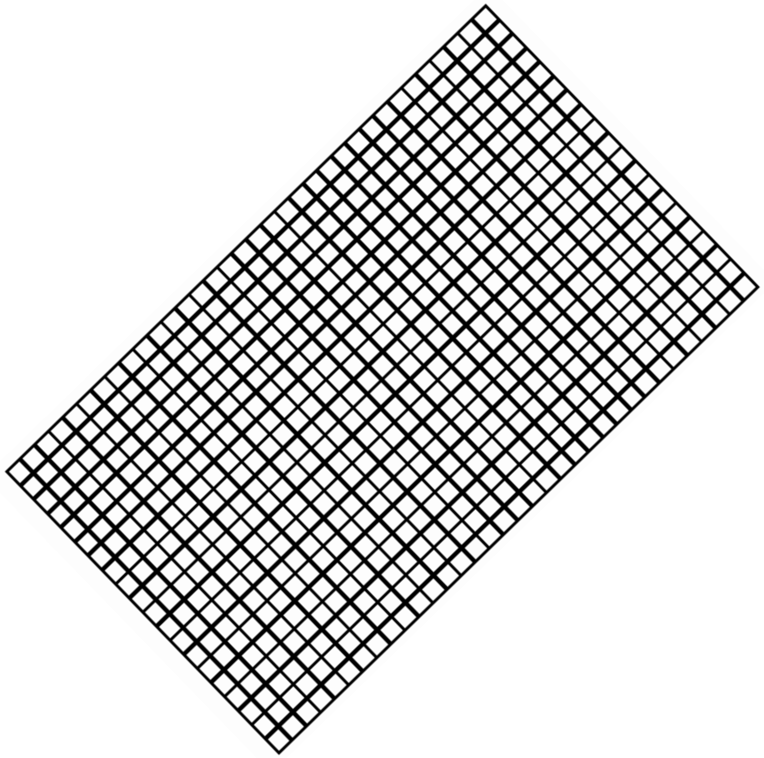
Horizontal curvilinear grid

- This is a possible grid:



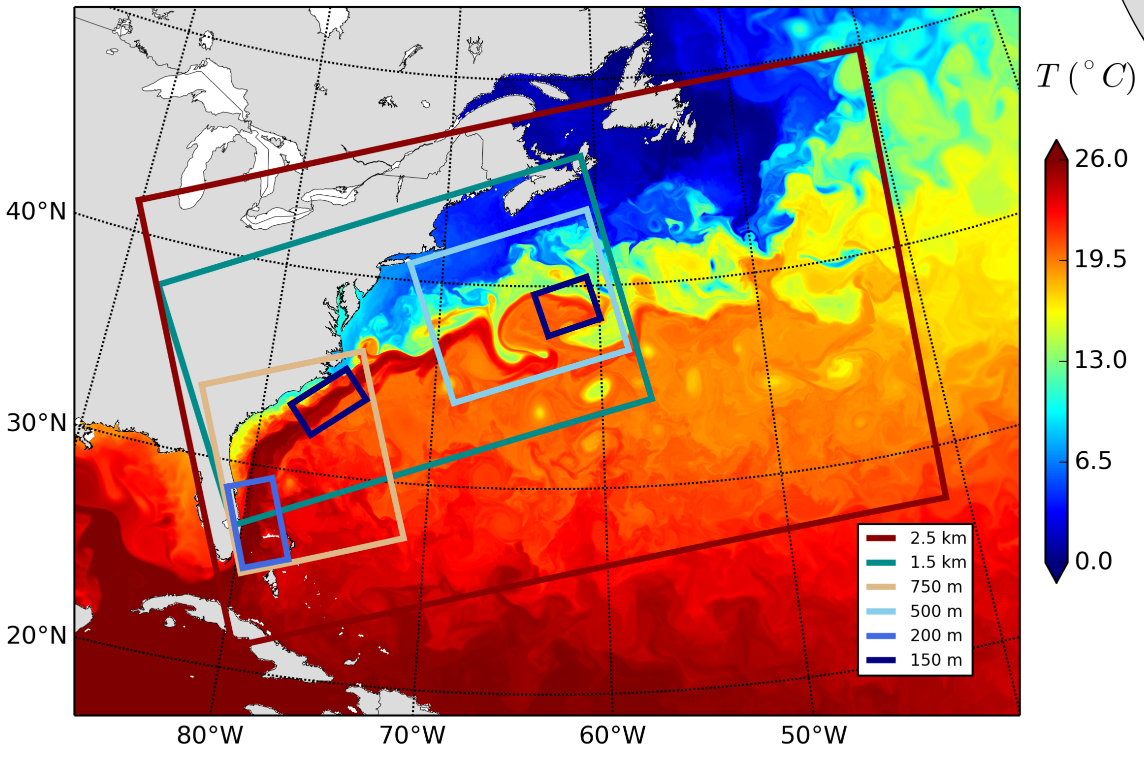
In practice variations in *dx* and *dy* should be minimized to minimize errors and optimize computation time.

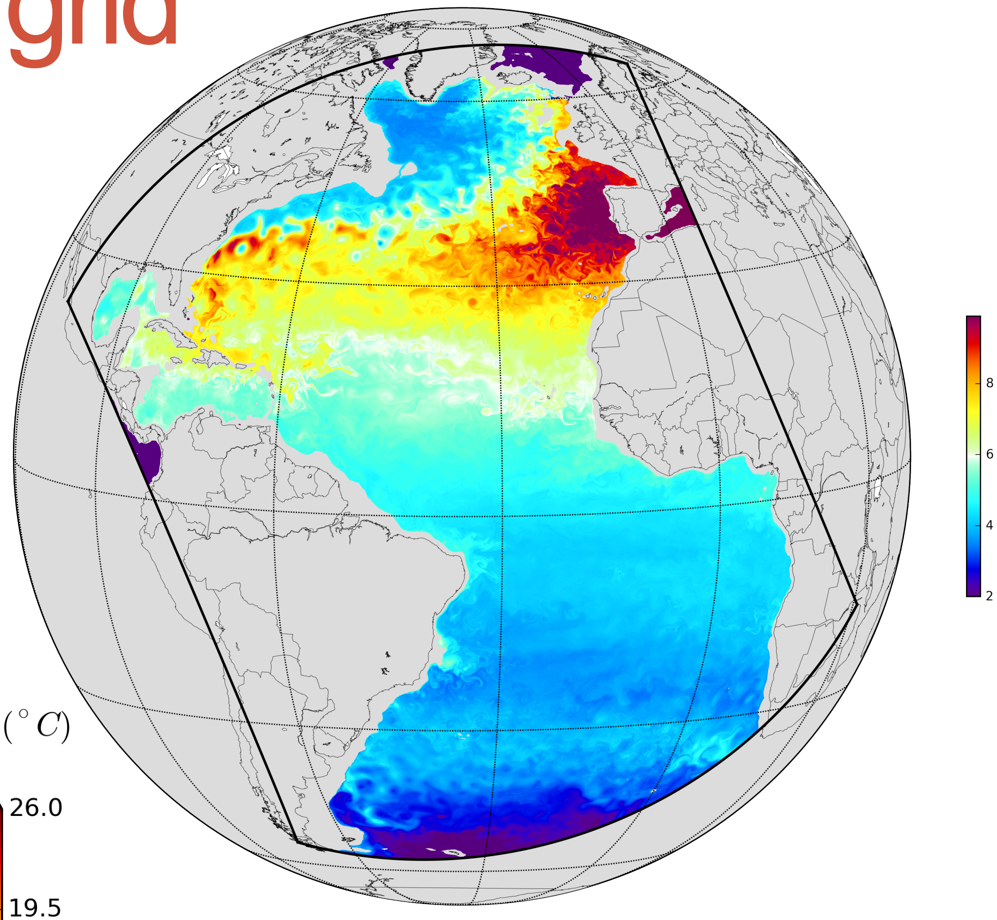
So avoir extreme distorsions and be as close as rectangular grids as possible (+ use land masks)



Horizontal curvilinear grid

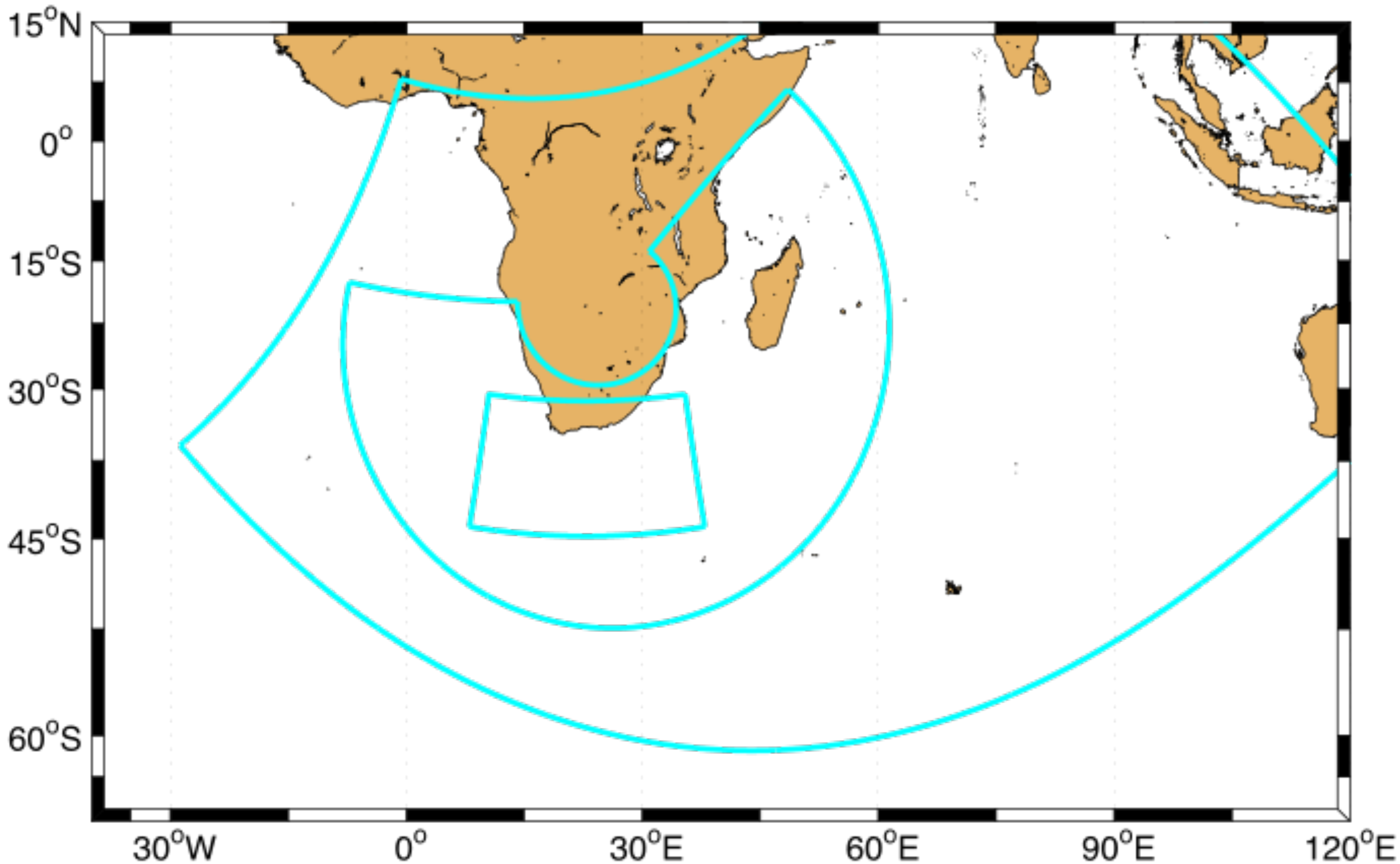
- Example of realistic domains:





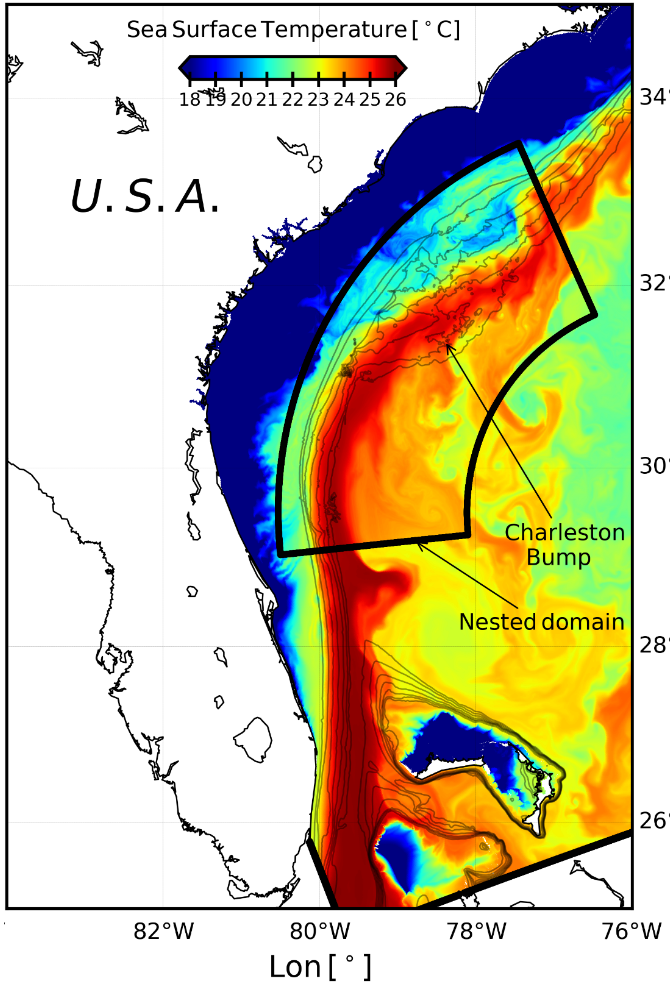
Horizontal curvilinear grid

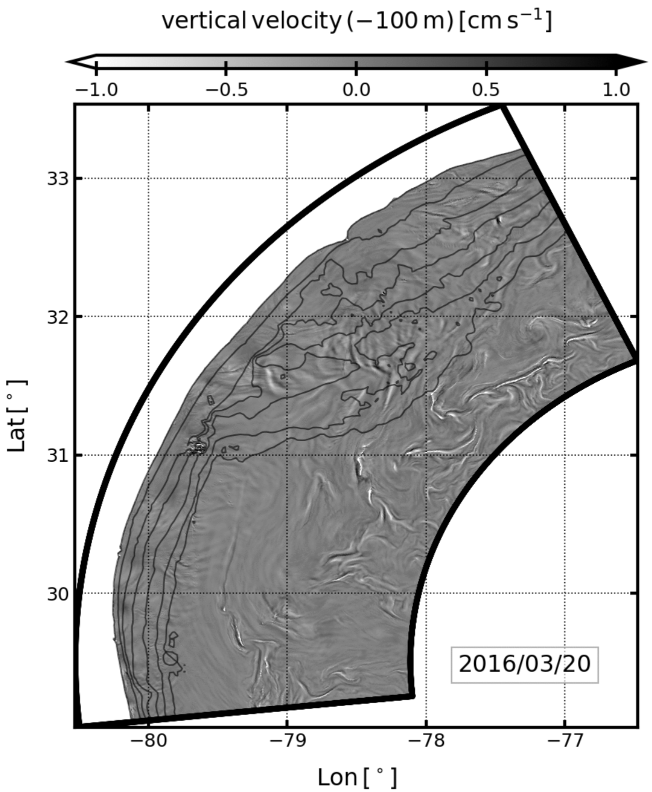
- Example of realistic domains (with gentle bendings):



Horizontal curvilinear grid

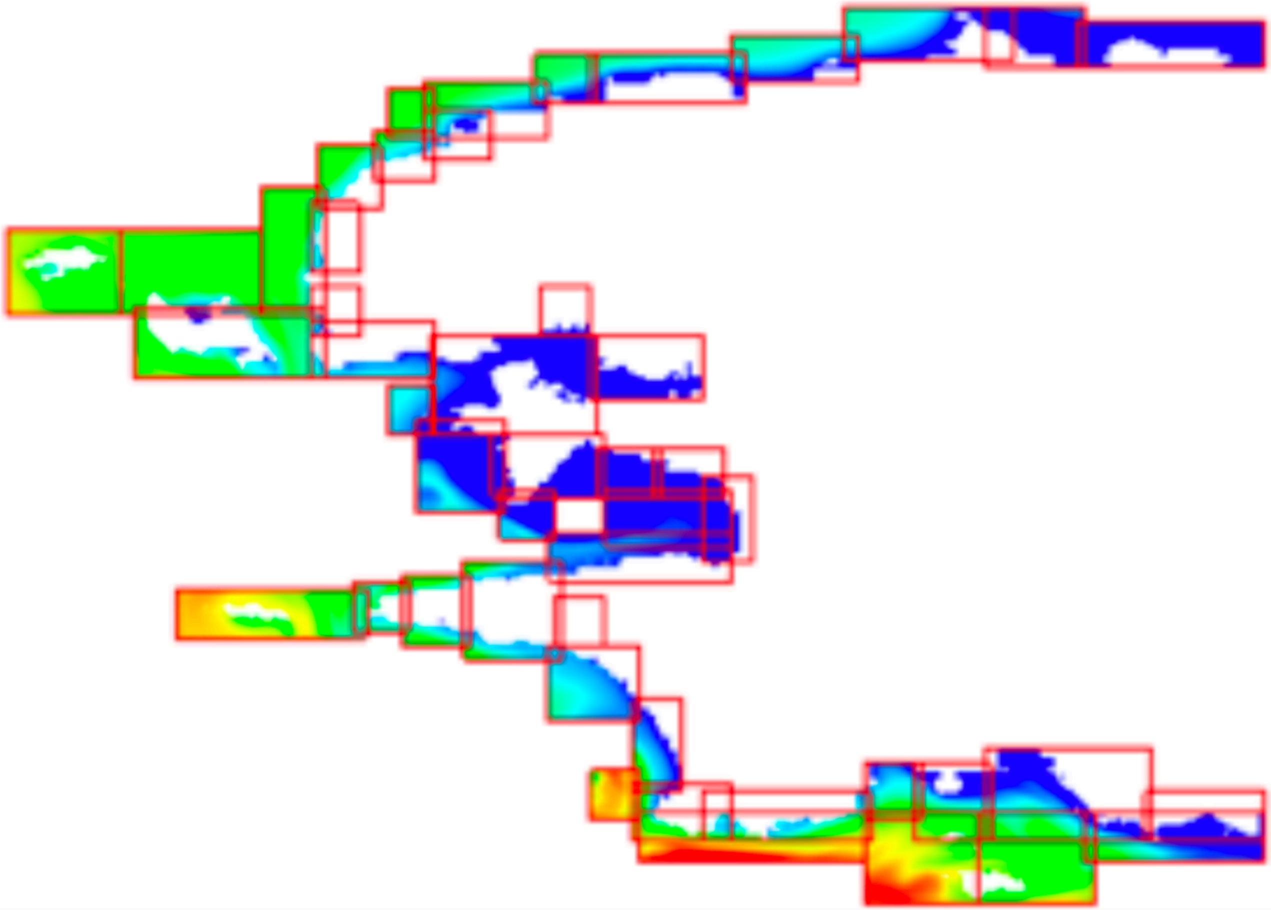
- Example of realistic domains (with gentle bendings):





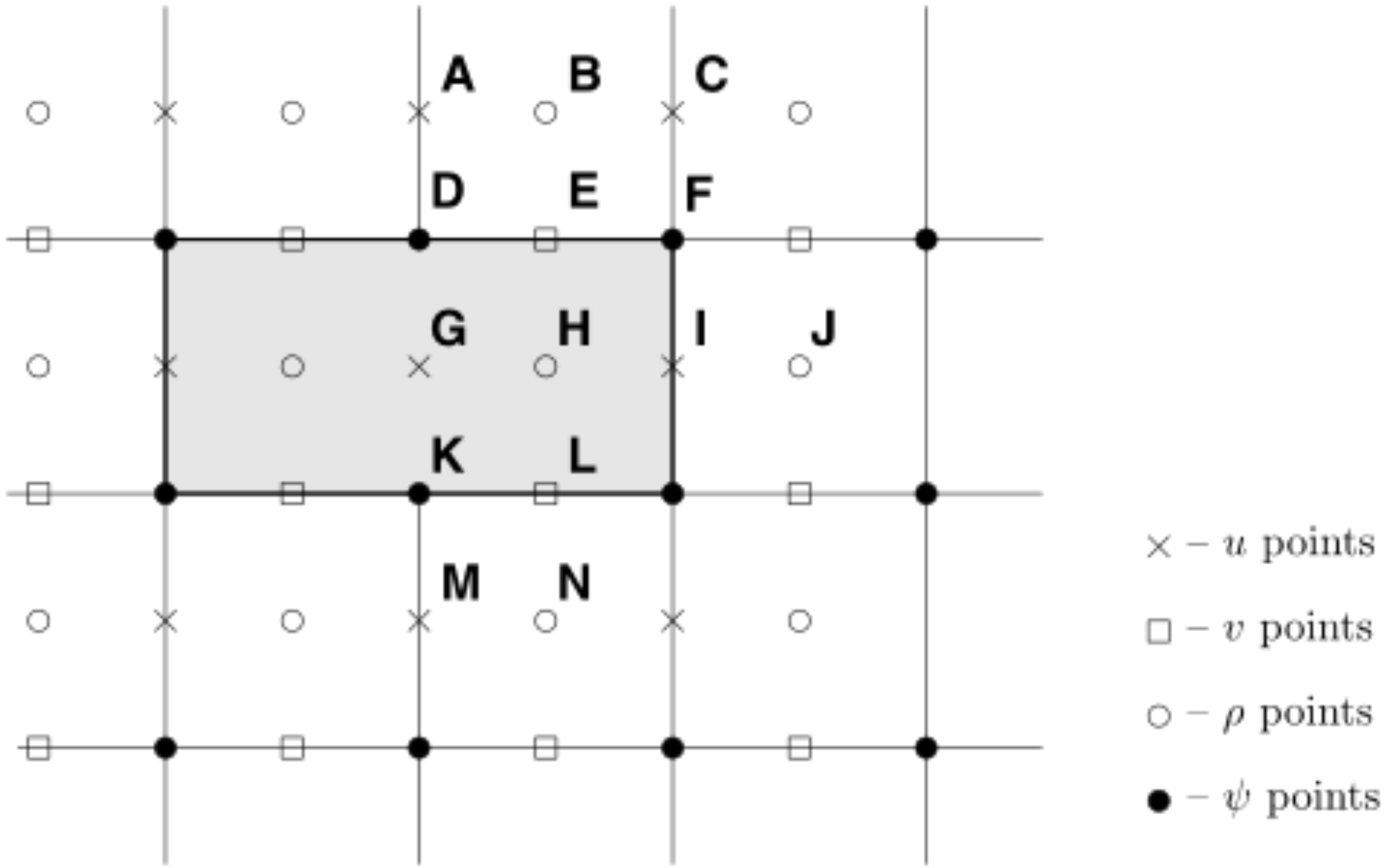
Horizontal curvilinear grid

- Another method = massive multigrain



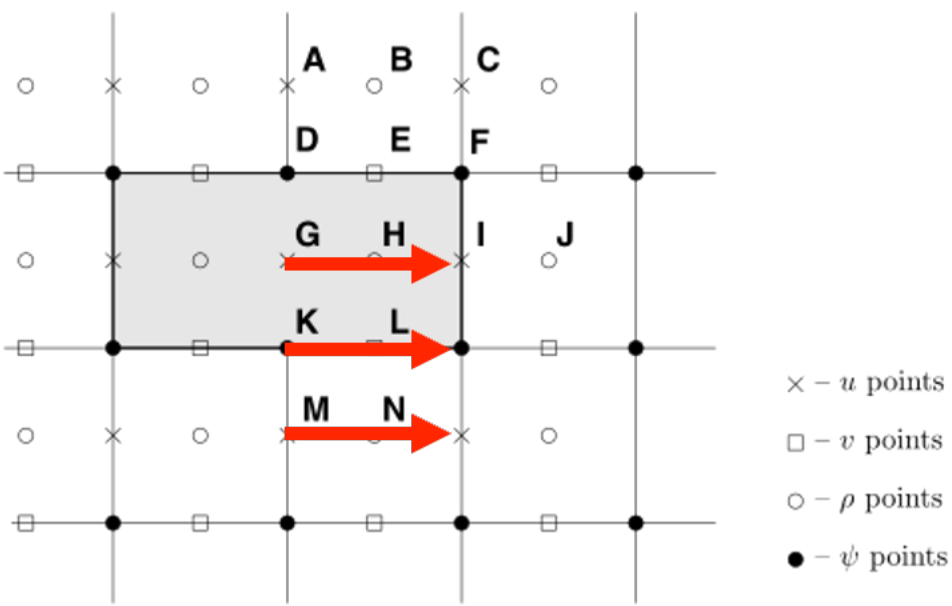
Land/sea Mask

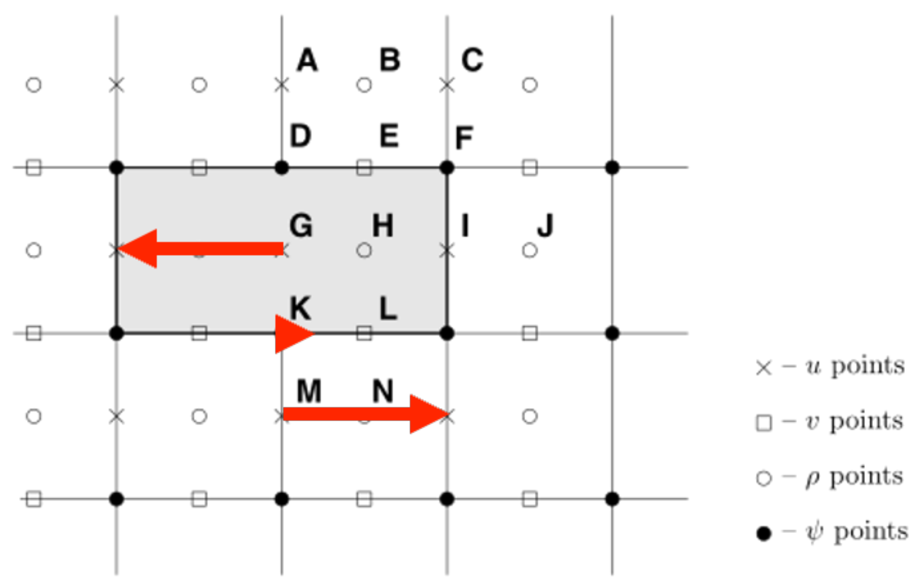
Variables within the masked region are set to zero by multiplying by the mask for either the u, v or rho points :



Land/sea Mask

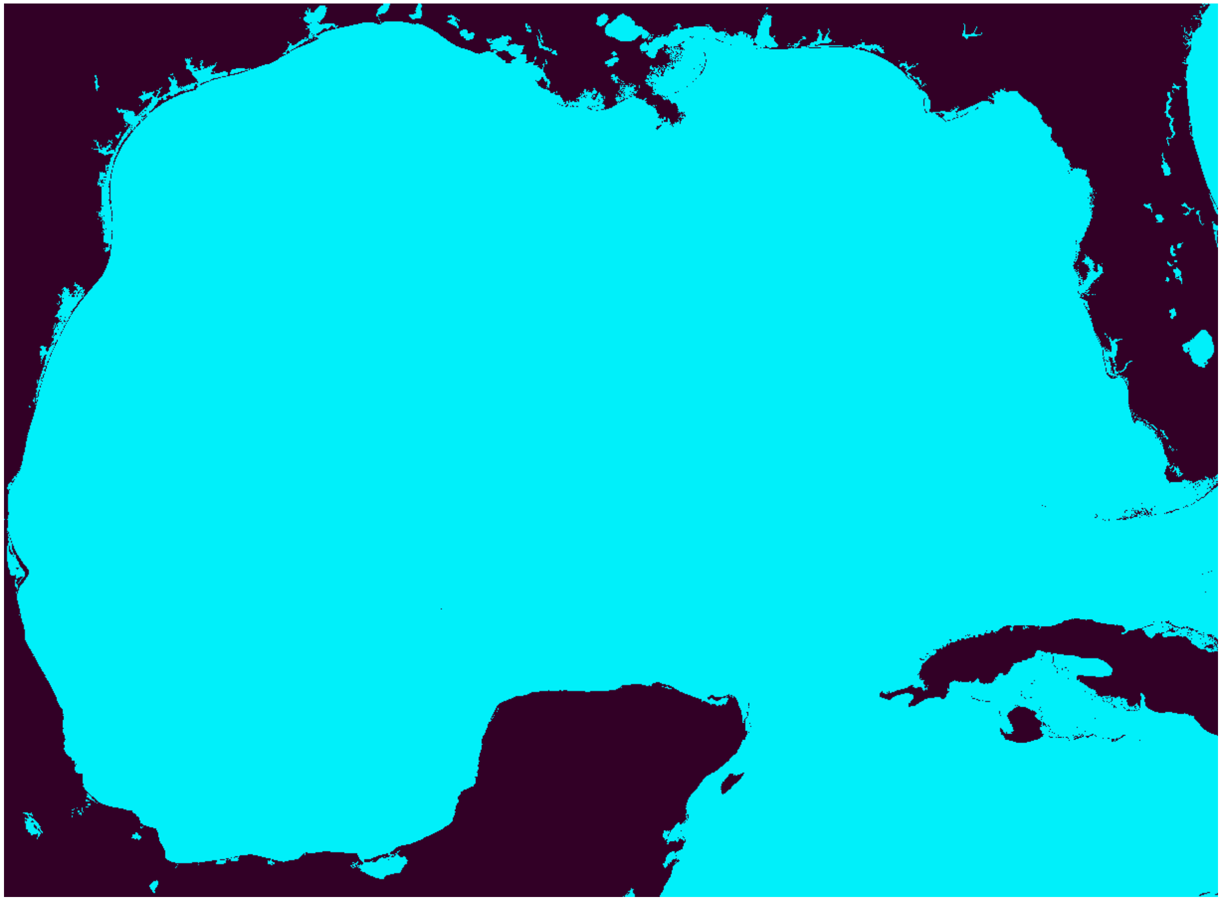
Free-slip versus No-Slip





Land/sea Mask

Variables within the masked region are set to zero by multiplying by the mask for either the u, v or rho points :



Land/sea Mask

See the code routines:

```
#ifdef MASKING
# define SWITCH *
#else
# define SWITCH !
#endif

!###########################
      do k=1,N
        do i=IstrU,Iend
          u(i,j,k,nnew)=(DC(i,k)-DC(i,0)) SWITCH umask(i,j)
```