TURBULENCE

- **Lesson 1 :** [D109]
  - Introduction
  - The Kolmogorov theory

- **Lesson 2 :** [D109]
  - 2D turbulence

- **Lesson 3 :**[B014]
  - Geostrophic turbulence
  - Real ocean turbulence

Turbulence

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Evaluation

- Oral presentation of a scientific article
  - By groups of 1 or 2 students
  - Pick an article from http://mespages.univ-brest.fr/~gula/Turb/Articles/
  - 15 minutes oral presentations [March 27^{th}, 2 pm, Amphi D]

TURBULENCE

1. 3D TURBULENCE

- **Lesson 1 :**

  - Introduction
    - *What is turbulence?*

  - Properties of turbulence
    - *Where does it come from?*
    - *What does it do?*

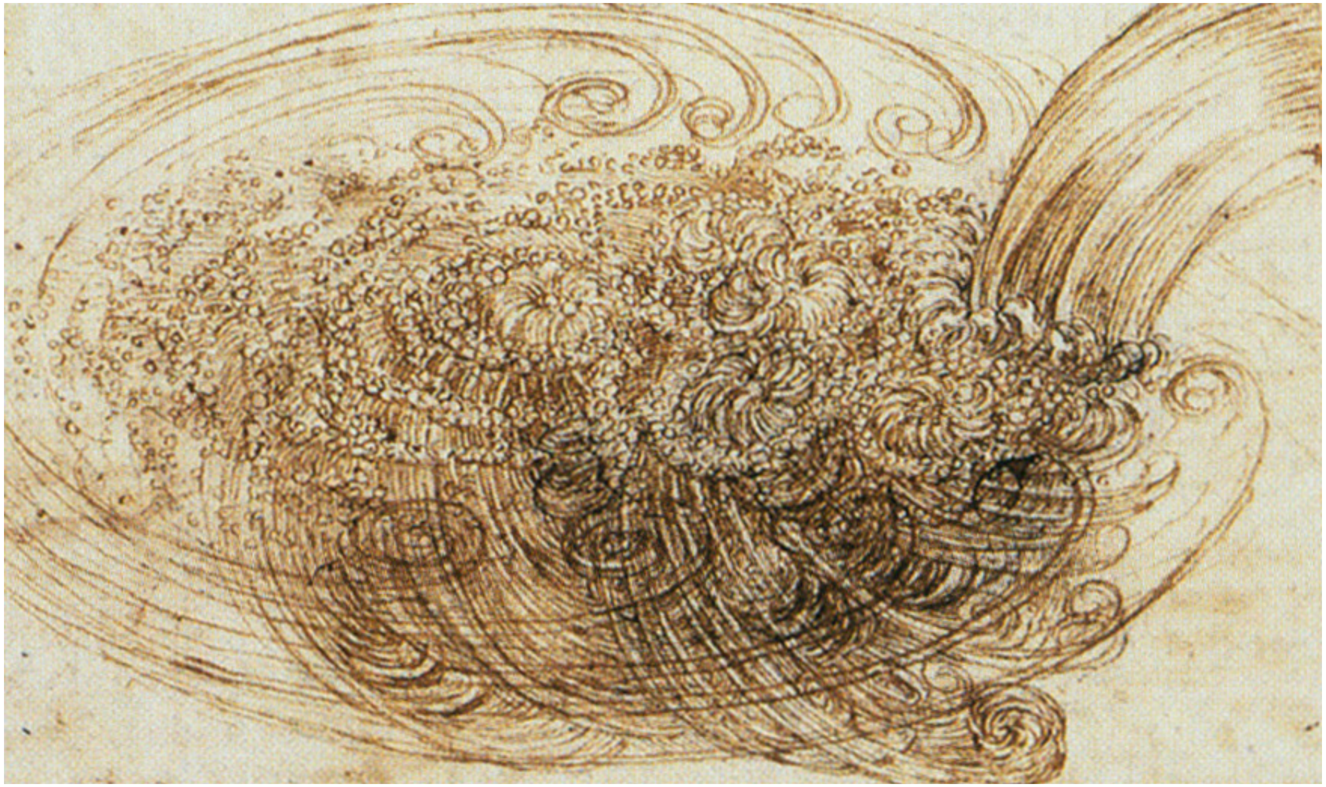
  - The closure problem

  - The Kolmogorov theory

References:

- Vallis G.K., Atmospheric and Oceanic Fluid Dynamics.
- MIT online course: https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-820-turbulence-in-the-ocean-and-atmosphere-spring-2007/lecture-notes/
- LaCasce J.H., Turbulence in the Atmosphere and Ocean.

What is turbulence?



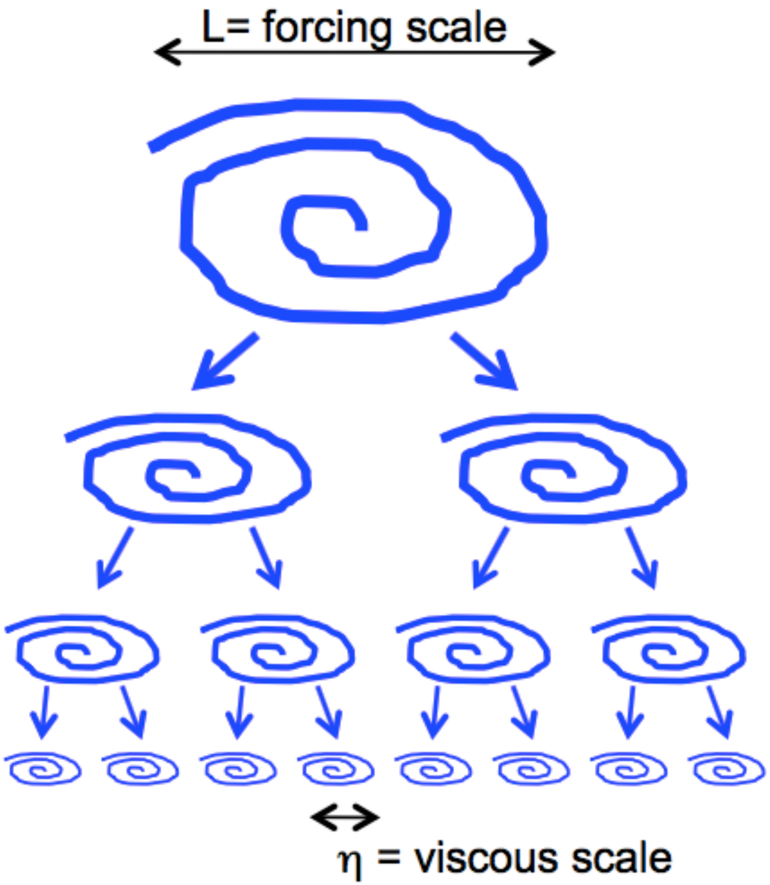
turbolenza by da Vinci [1507]

"…the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large."

What is turbulence?

No formal definition of turbulence. One of the best is by L.F. Richardson, in 1922:

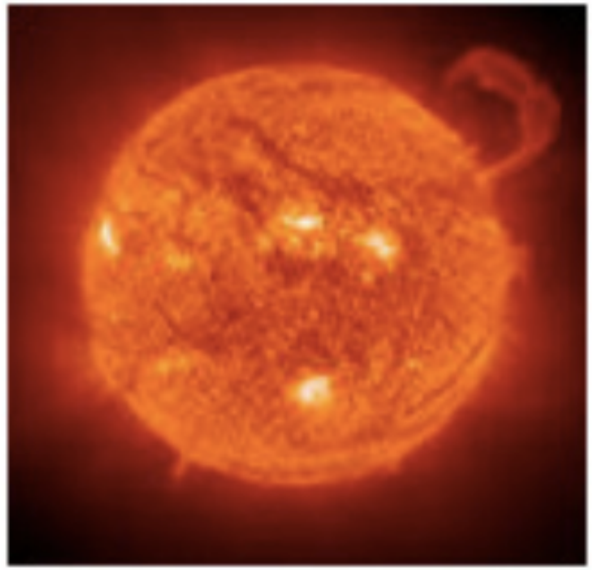
Big whorls have little whorls,
which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity.

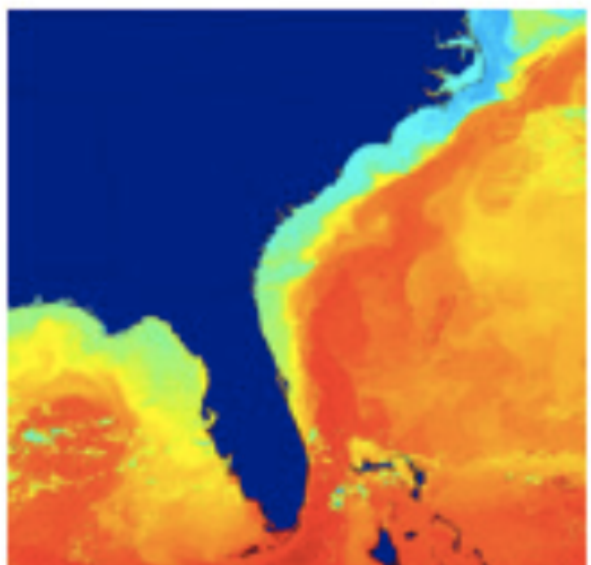


Turbulence in nature

(a) water flow from a faucet, (b) water from a garden hose, (c) flow past a curved wall, and (d) and (e) whitewater rapids whose turbulent fluctuations are so intense that air is entrained by the flow and produces small bubbles that diffusely reflect light and cause the water to appear white.

Turbulence in nature





Examples of turbulent flows at the surface of the Sun, in the Earth's atmosphere, in the Gulf Stream at the ocean surface, and in a volcanic eruption.

Turbulence in nature

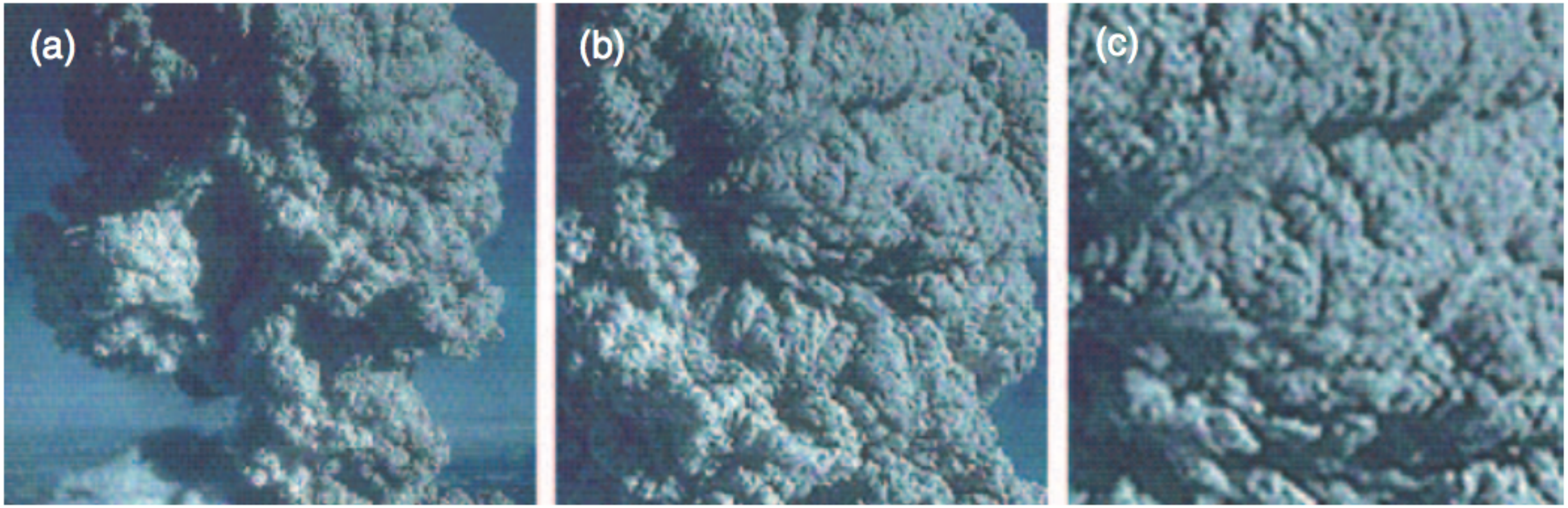
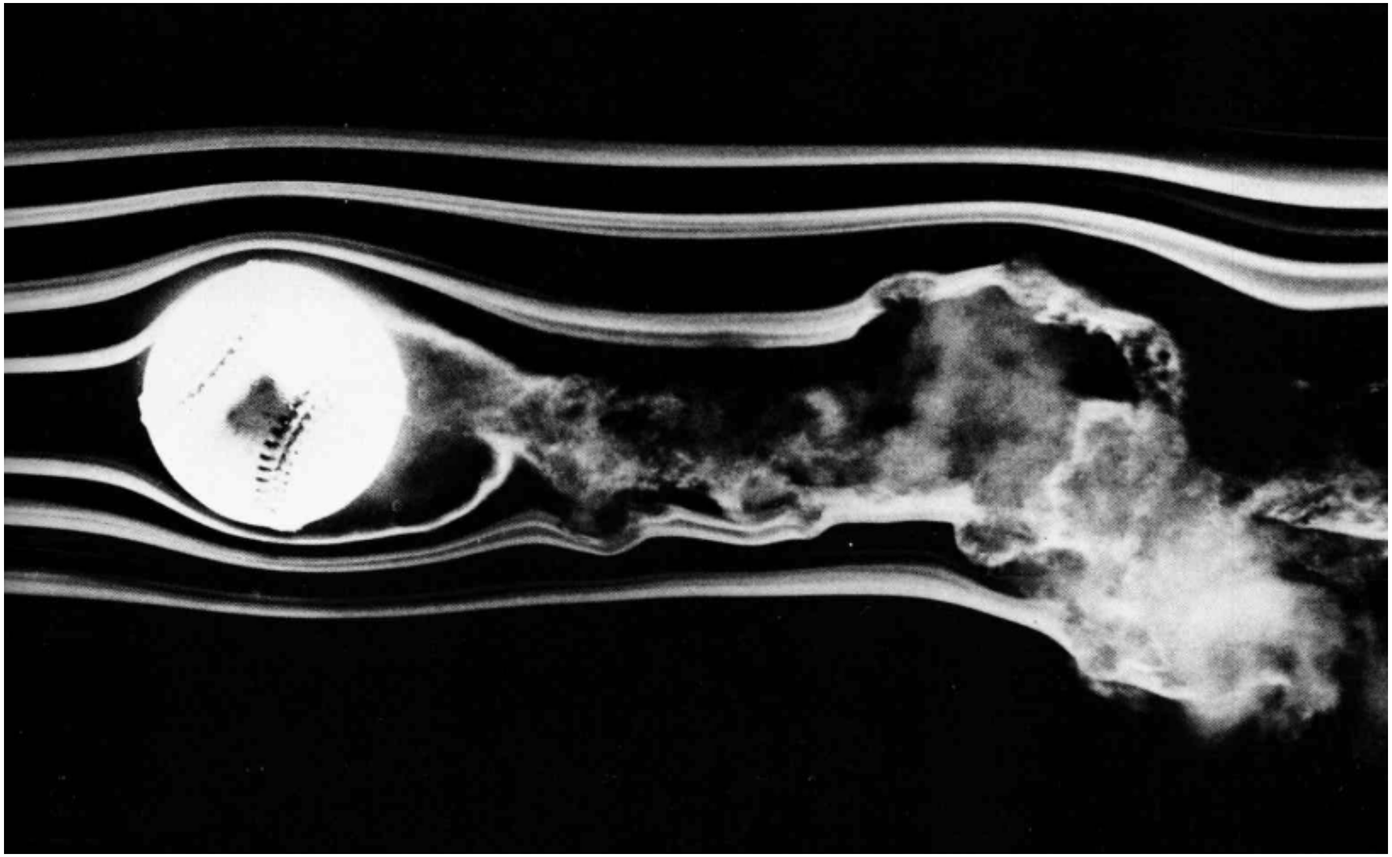


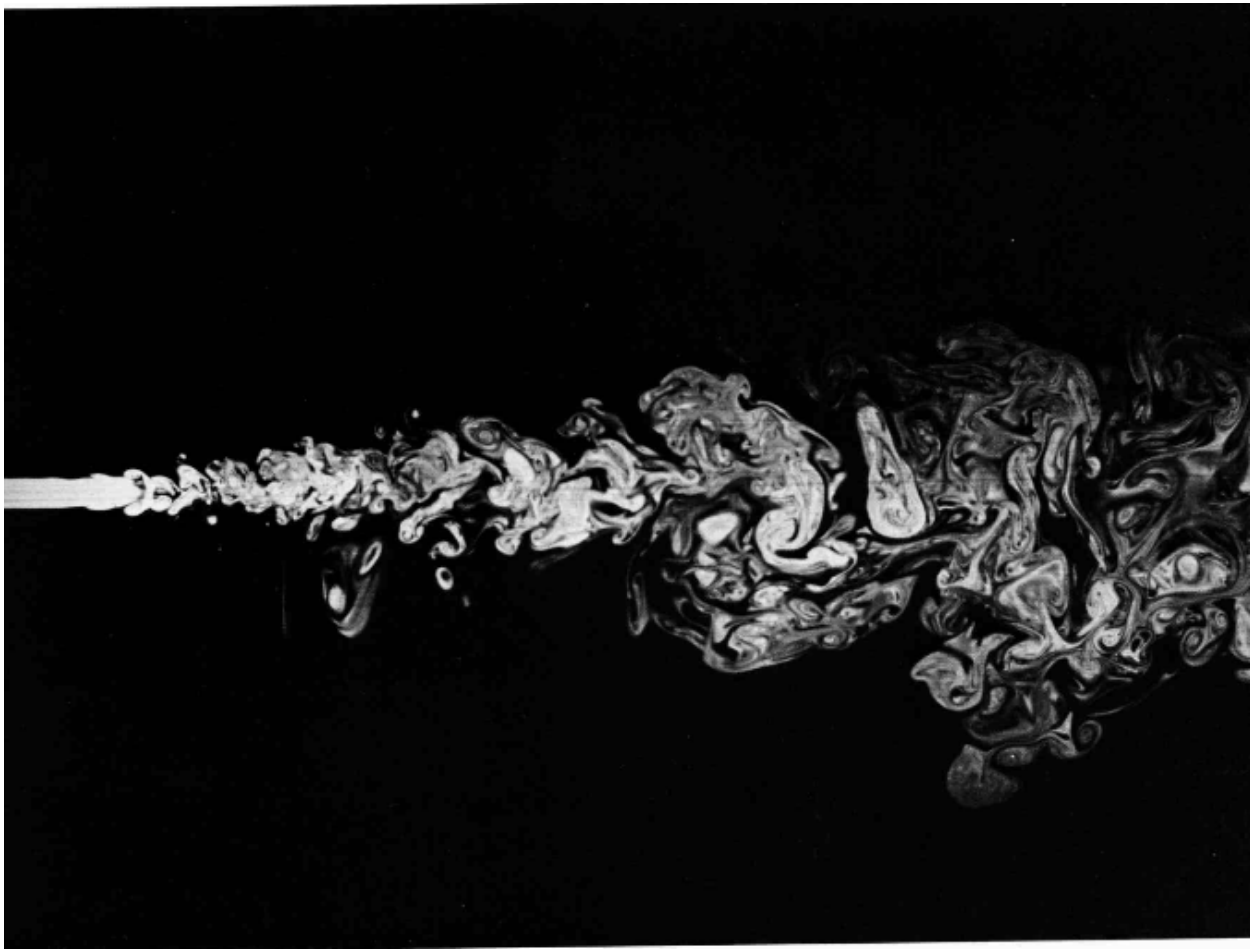
Figure 2. Scale-Independence in Turbulent Flows
The turbulent structure of the pyroclastic volcanic eruption of Mt. St. Helens shown in (a) is expanded by a factor of 2 in (b) and by another factor of 2 in (c). The characteristic scale of the plume is approximately 5 km. Note that the expanded images reveal the increasingly finer scale structure of the turbulent flow. The feature of scale independence, namely, that spatial images or temporal signals look the same (statistically) under increasing magnification is called self-similarity.

Turbulence in the lab



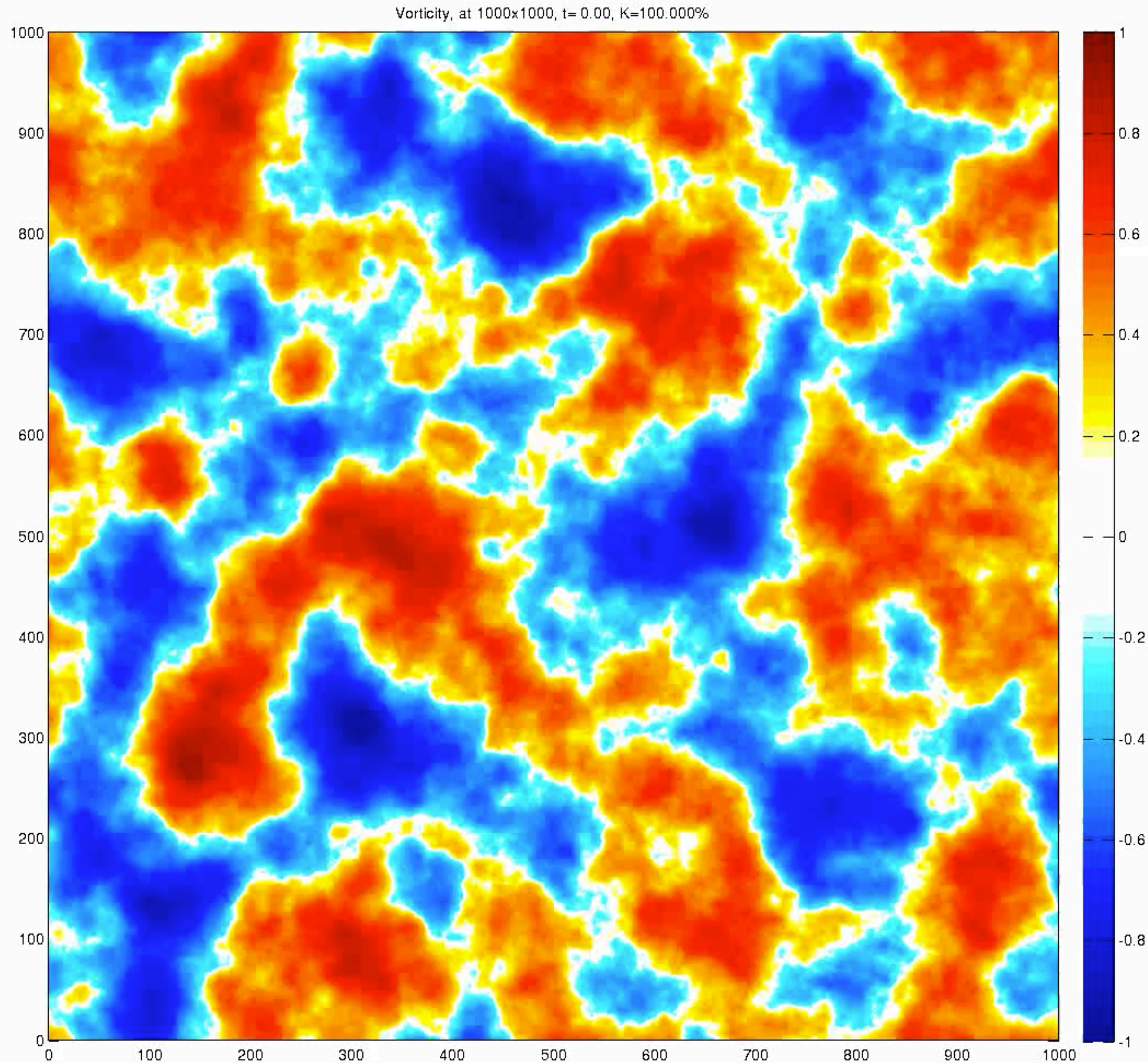
Spinning Baseball [Van Dyke, 82]

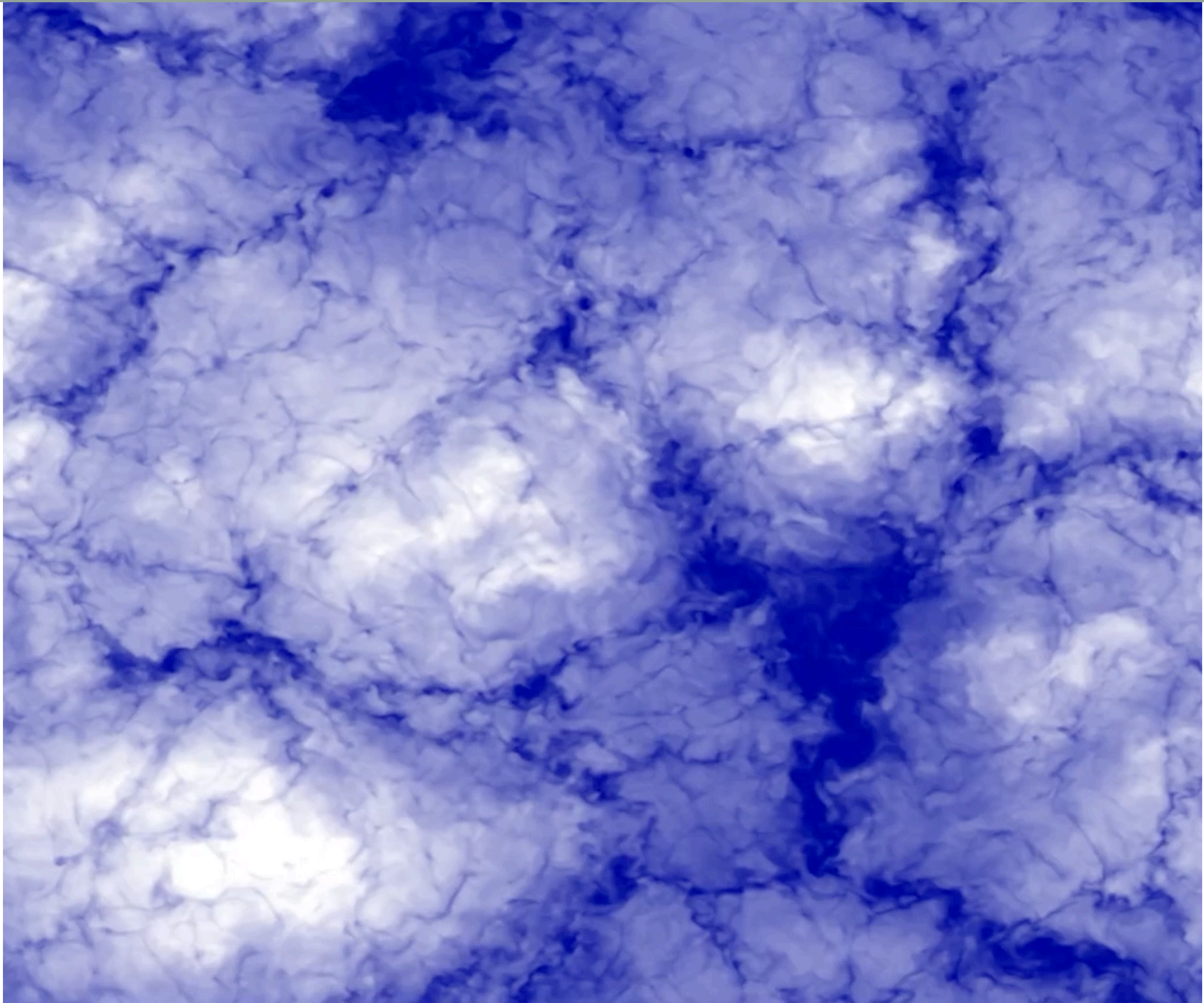
Turbulence in the lab



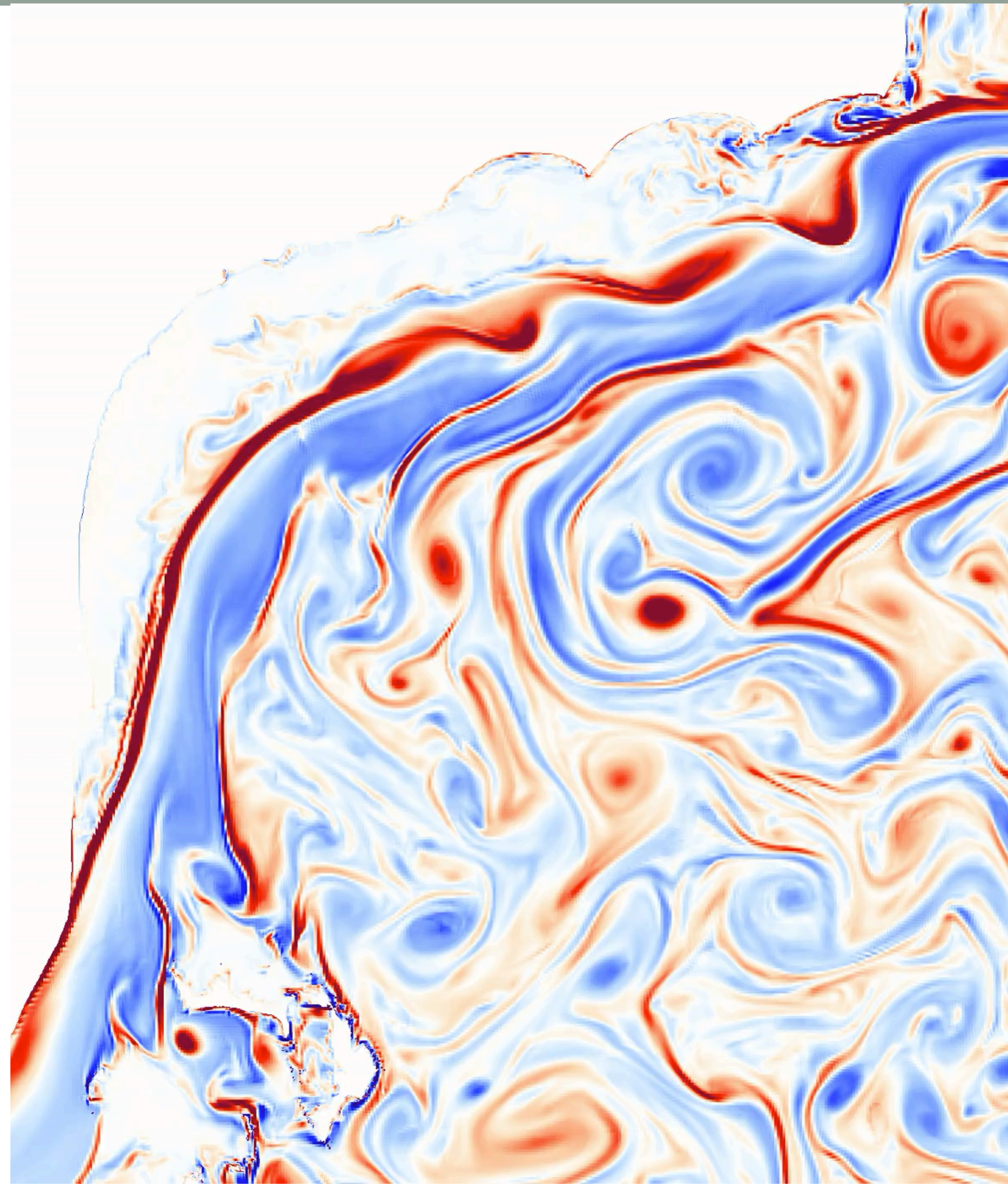
Turbulent water jet (Re = 2300) [Van Dyke, 82]

2d free-decay turbulence [G. Roullet]





Turbulence in a realistic ocean model

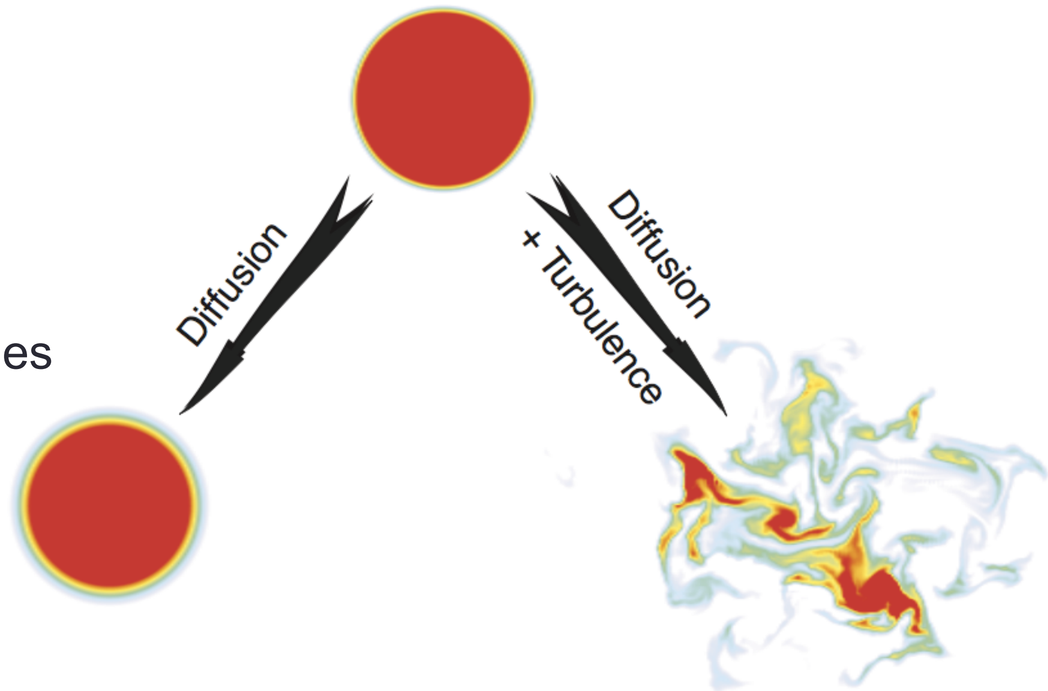


Role of turbulence

- Momentum transfer

- Scalar mixing

= homogenization of fluid properties by random molecular motions



*A comparison of **mixing enhanced by turbulence** with **mixing due to molecular processes alone**, as revealed by a numerical solution of the equations of motion.*

The initial state includes a circular region of dyed fluid in a white background. Two possible evolutions are shown: one in which the fluid is motionless (save for random molecular motions), and one in which the fluid is in a state of fully developed, two-dimensional turbulence. The mixed region (yellow/green) expands much more rapidly in the turbulent case.

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  - 2D turbulence

- **Lesson 3 :**
  - Geostrophic turbulence

Properties of turbulence

- **Broadband spectrum in space and time**
  - *Turbulent flows are characterized by structures on a broad range of spatial and temporal scales, even given smooth or periodic initial conditions and forcing.*
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- **Dominated by nonlinearity**
  - *A field of non-interacting linear internal waves with many different frequencies and wavenumbers can also have a large range of length scales, but it is not turbulent. Why not? In a turbulent flow the different scales interact, through the nonlinear terms in the equations of motion. And these nonlinear interactions are responsible for the presence of structure on many different scales.*
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- **Unpredictable in space and time**
  - *Turbulent flows are predictable for only short times and short distances. Even though we know the equations that govern the evolution of the fluid, we cannot make predictions about the details of the flow due to its sensitive dependence on initial and boundary conditions. Predictability, however, can be recovered in a statistical sense. The sensitive dependence on initial and boundary conditions is a fundamental property of chaotic systems.*

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- **Time irreversible**
  - *Turbulent motions are not time reversible. As time goes on, turbulent motions tend to forget their initial conditions and reach some equilibrated state. Turbulence mixes stuff up, it does not unmix it.*

Navier-Stokes Equations

The Navier-Stokes equations probably contain all of turbulence.

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Momentum equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation
(no source/sink)

Kinematic viscosity:

$$\nu_{water} \approx 10^{-6} \mathrm{m}^2 \mathrm{s}^{-1}$$

$$\nu_{air} \approx 1.5 \times 10^{-5} \mathrm{m}^2 \mathrm{s}^{-1}$$

Navier-Stokes Equations

Millennium Prize problems:

Since understanding the Navier–Stokes equations is considered to be the first step to understanding the elusive phenomenon of turbulence, the Clay Mathematics Institute in May 2000 made this problem one of its seven Millennium Prize problems in mathematics. It offered a US $1,000,000 prize to the first person providing a solution for a specific statement of the problem:[1]

Prove or give a counter-example of the following statement:

In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

Navier-Stokes Equations

+ Boussinesq approximation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

Navier-Stokes Equations

+ Boussinesq approximation

$$\boxed{\frac{\partial \vec{u}}{\partial t}} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + \boxed{f \vec{k} \times \vec{u}} + \boxed{g \vec{k}} = \boxed{-\frac{\vec{\nabla} P}{\rho_0}} + \boxed{\nu \vec{\nabla}^2 \vec{u}} + \boxed{\vec{\mathcal{F}}}$$

Time variation | Advection (inertia) | Rotation | Gravity | Pressure gradient | Dissipation (viscosity) | Forcings

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation (no source/sink)

Navier-Stokes Equations

+ Boussinesq approximation

- Turbulence arises from the non-linear terms

$$\frac{\partial \vec{u}}{\partial t} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Advection
(inertia)

- Only one here = advection *(quadratic nonlinearity)*
- Turbulence results from the nonlinear nature of advection, which enables interaction between motions on different spatial scales.

Navier-Stokes Equations

+ Boussinesq approximation

- x-momentum equation:

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad fU \qquad \frac{P}{\rho_0 L} \qquad \frac{\nu U}{L^2} \qquad F$$

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$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

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$$\frac{L^2}{\nu T} \qquad \boxed{\frac{UL}{\nu}} \qquad \frac{fL^2}{\nu} \qquad \frac{PL}{\rho_0 \nu U} \qquad 1 \qquad \frac{FL^2}{\nu U}$$

Equations scalings

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

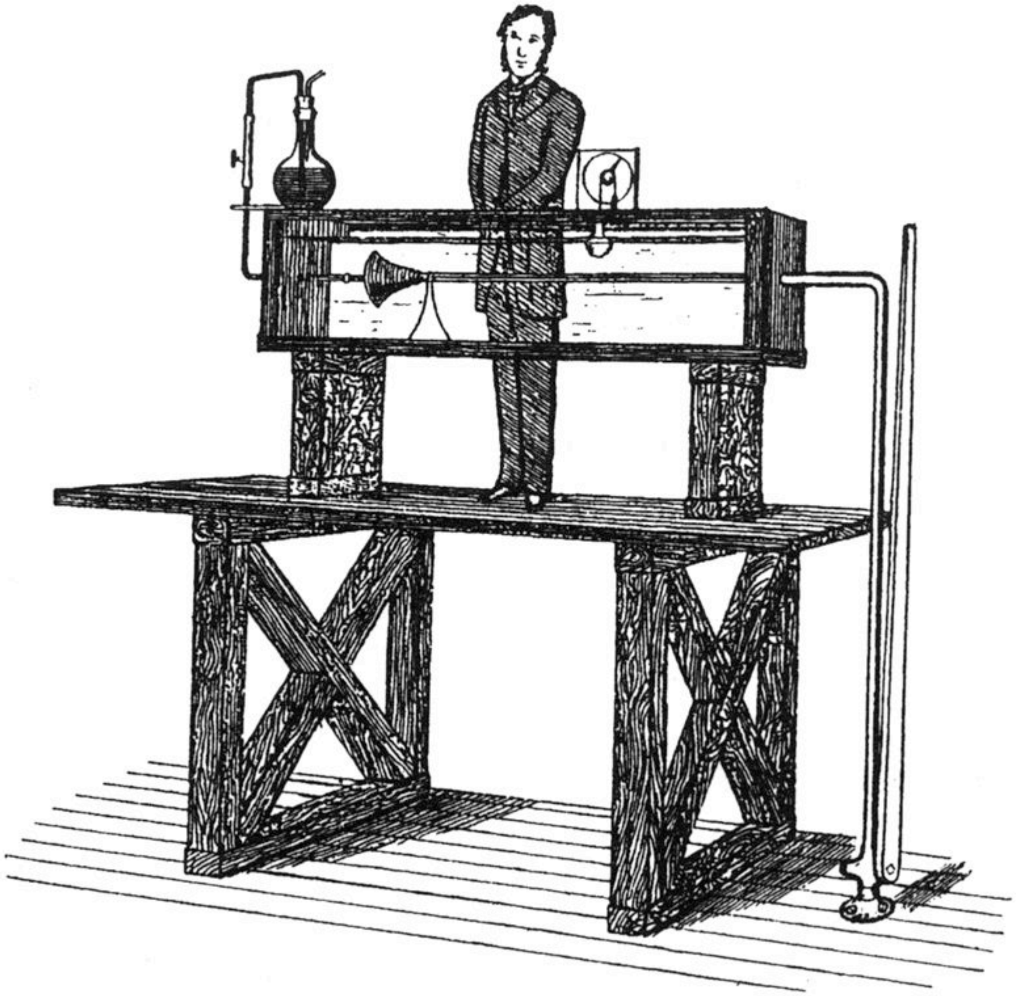
$$\boxed{\frac{UL}{\nu}}$$

= Reynolds Number

the ratio of the non-linear terms to the viscous terms

Reynolds Number

$$Re = \frac{UL}{\nu}$$



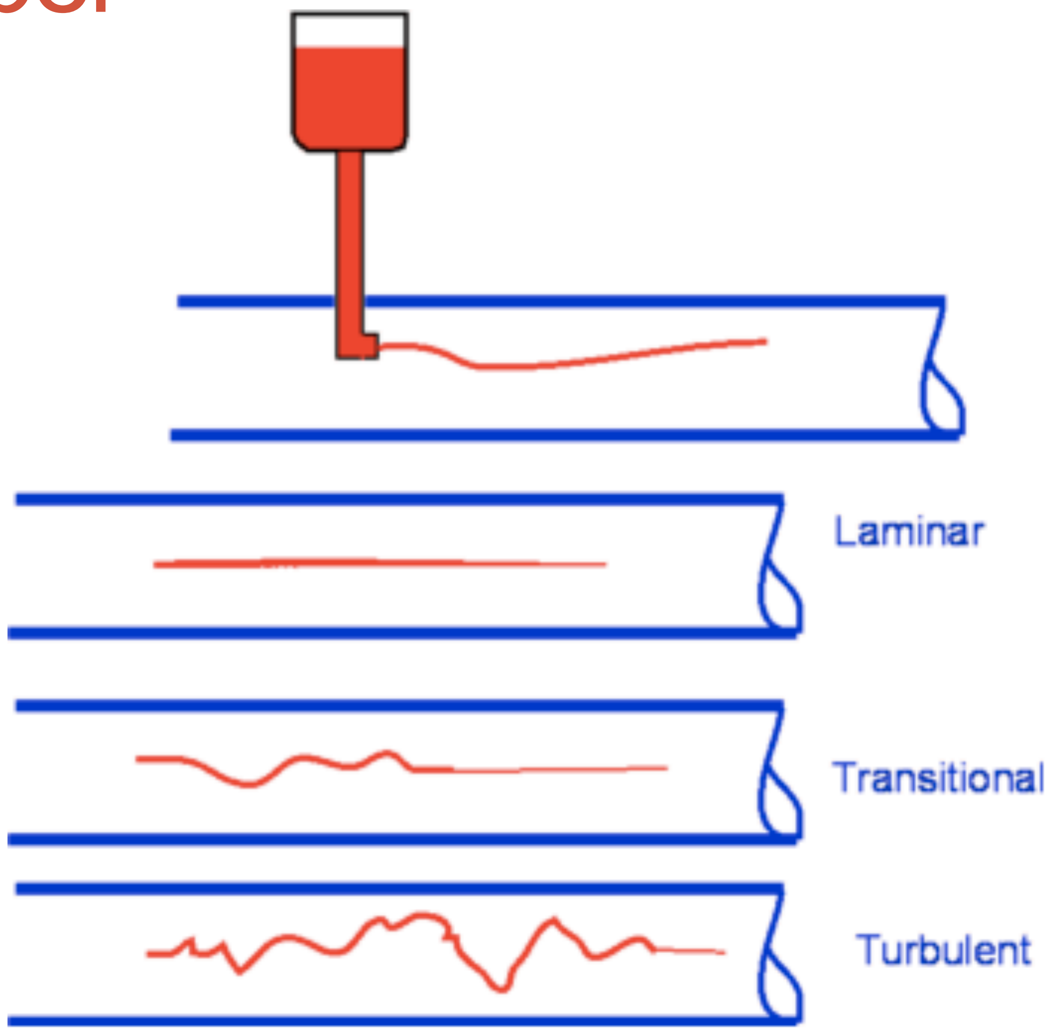
Osborne Reynolds experiment [1984]

Reynolds Number

$$Re = \frac{UL}{\nu}$$

Re < 2000 = laminar

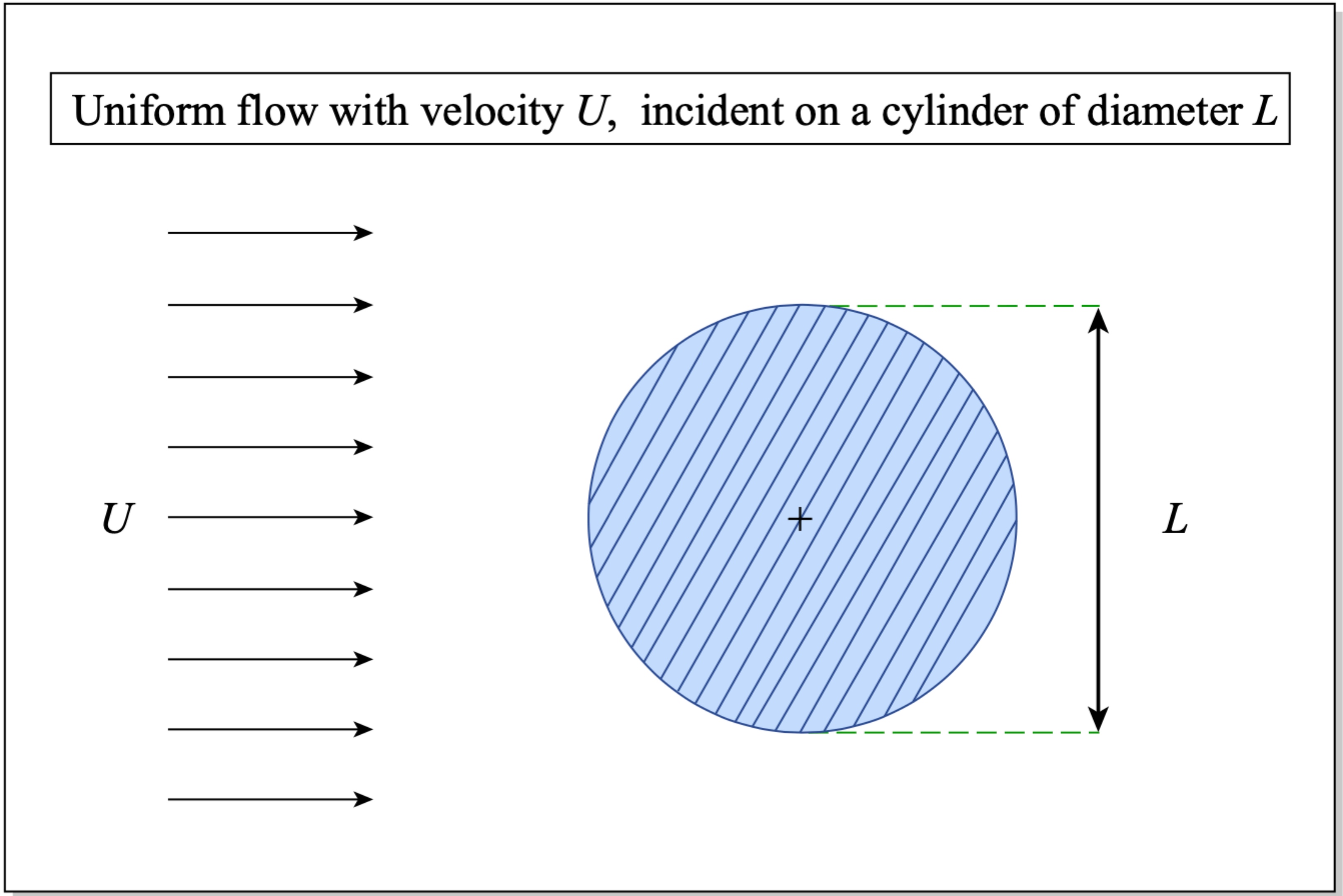
Re > 4000 = turbulent



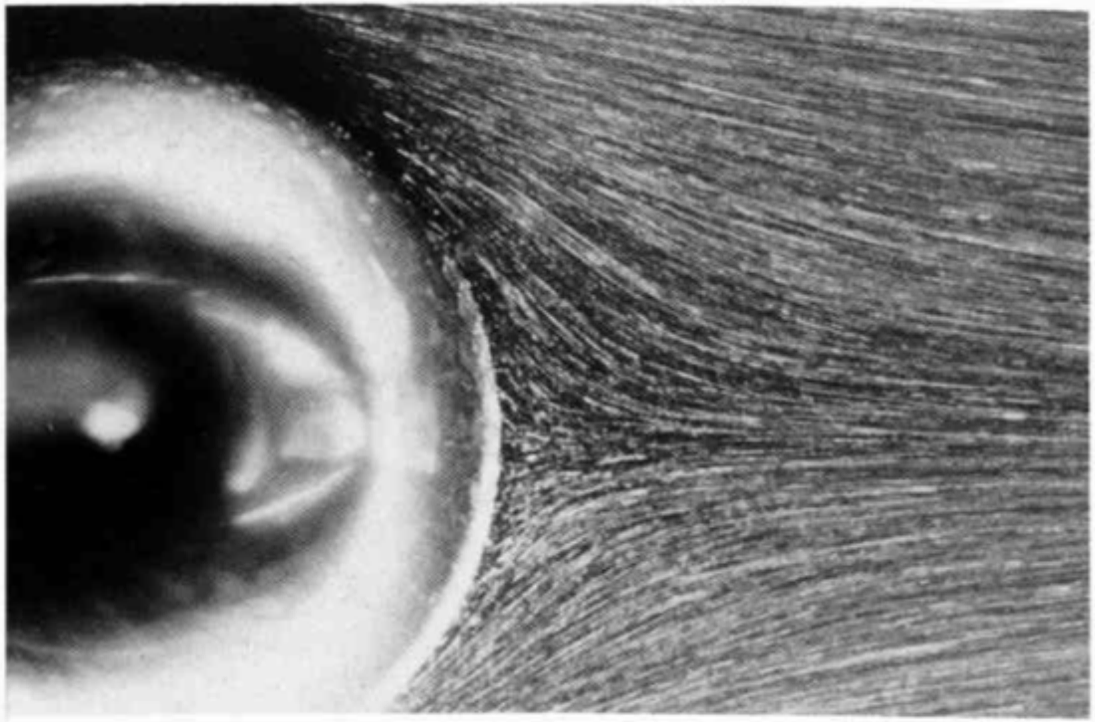
Reynolds experiment

http://www-mdp.eng.cam.ac.uk/web/library/enginfo/aerothermal_dvd_only/aero/fprops/pipeflow/node8.html

Transition to turbulence

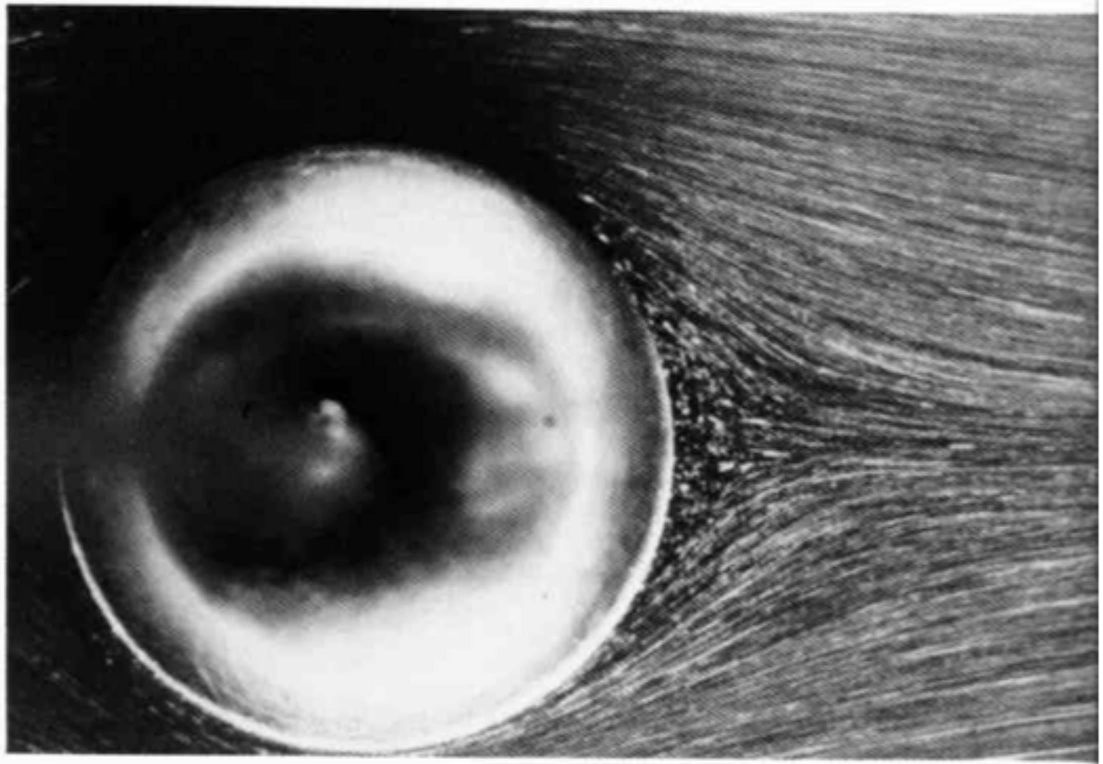


Transition to turbulence



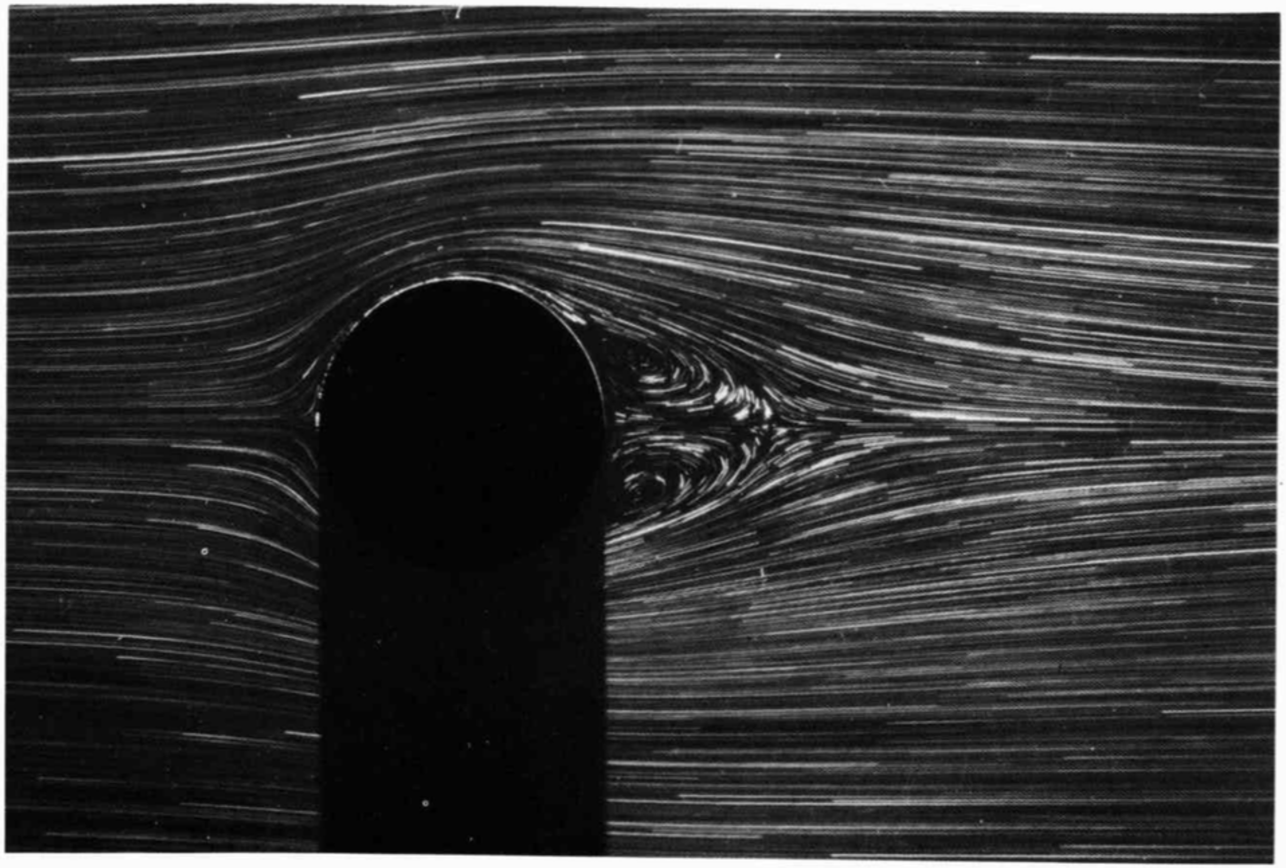
49. Sphere at $R=25.5$. Although it is not obvious, the flow is believed just to have separated at the rear at this Reynolds number, in contrast to the unseparated flow of figure 27. Aluminum dust is illuminated in water. *Taneda* 1956b

Transition to turbulence



50. **Sphere at R=26.8.** At this slightly higher speed the flow has clearly separated over the rear of the sphere, to form a thin standing vortex ring. Aluminum dust is illuminated in water. *Taneda* 1956b

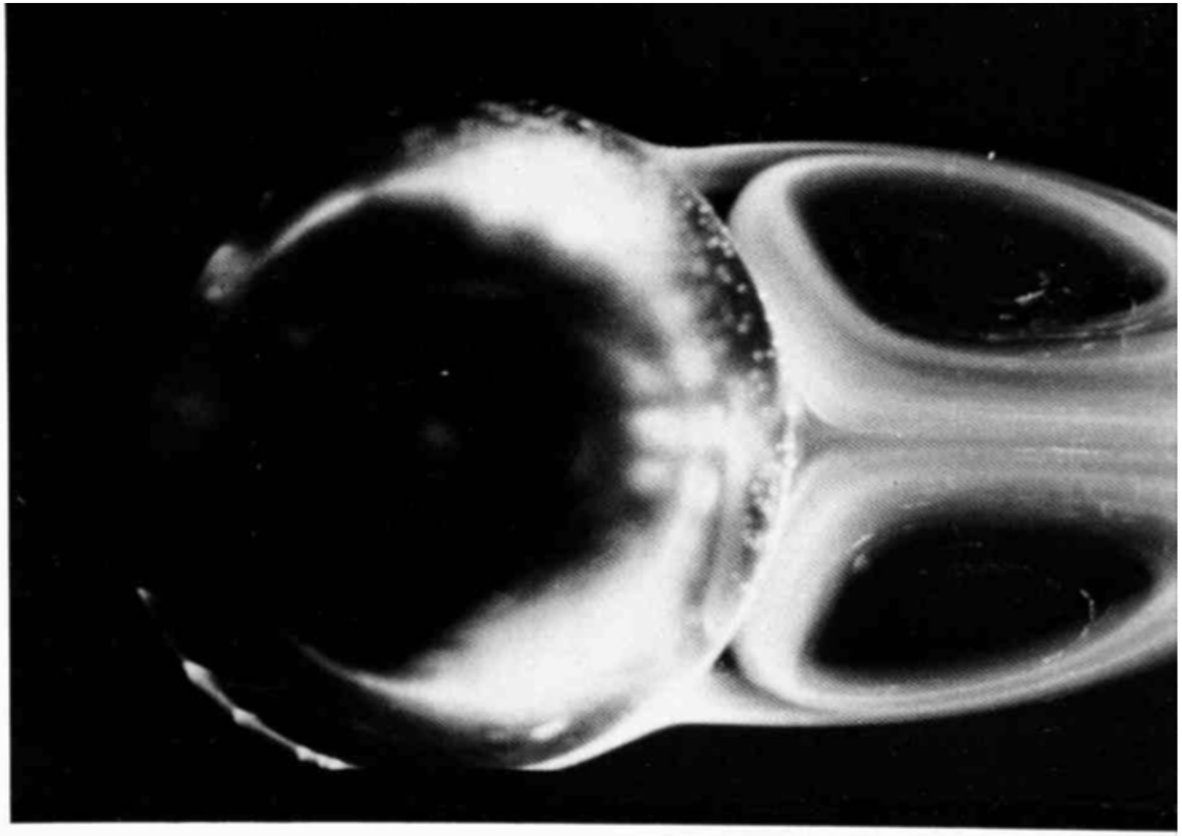
Transition to turbulence



51. Sphere at R**=56.5.** As in figure 8, the sphere is falling steadily down the axis of a tube filled with oil, but here so large that the influence of the walls is negligible. Magnesium cuttings are illuminated by a sheet of light, which casts the shadow of the sphere. *Archives de l'Académie des Sciences de Paris. Payard & Coutanceau 1974*

Transition to turbulence

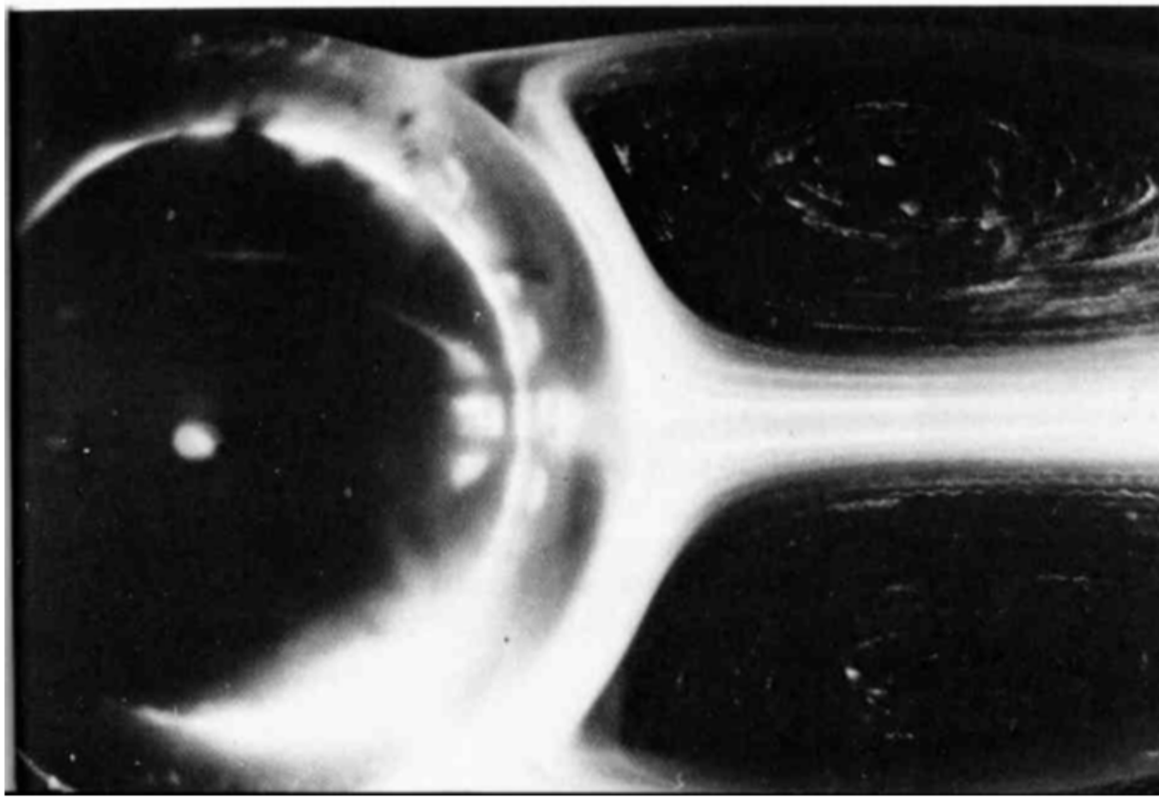
52. Sphere at $R=104$. At this Reynolds number the recirculating wake extends a full diameter downstream, but is perfectly steady, as for the circle in figure 44. Visualization is by a thin coating of condensed milk on the sphere, which gradually melts and is carried into the stream of water. *Taneda 1956b*



Transition to turbulence

53. Sphere at R=118. The wake grows more slowly in axisymmetric than plane flow. These photographs have shown that the length of the recirculating region is proportional to the logarithm of the Reynolds number, whereas it grows linearly with Reynolds number for a cylinder. Aluminum dust shows the flow of water. *Taneda 1956b*

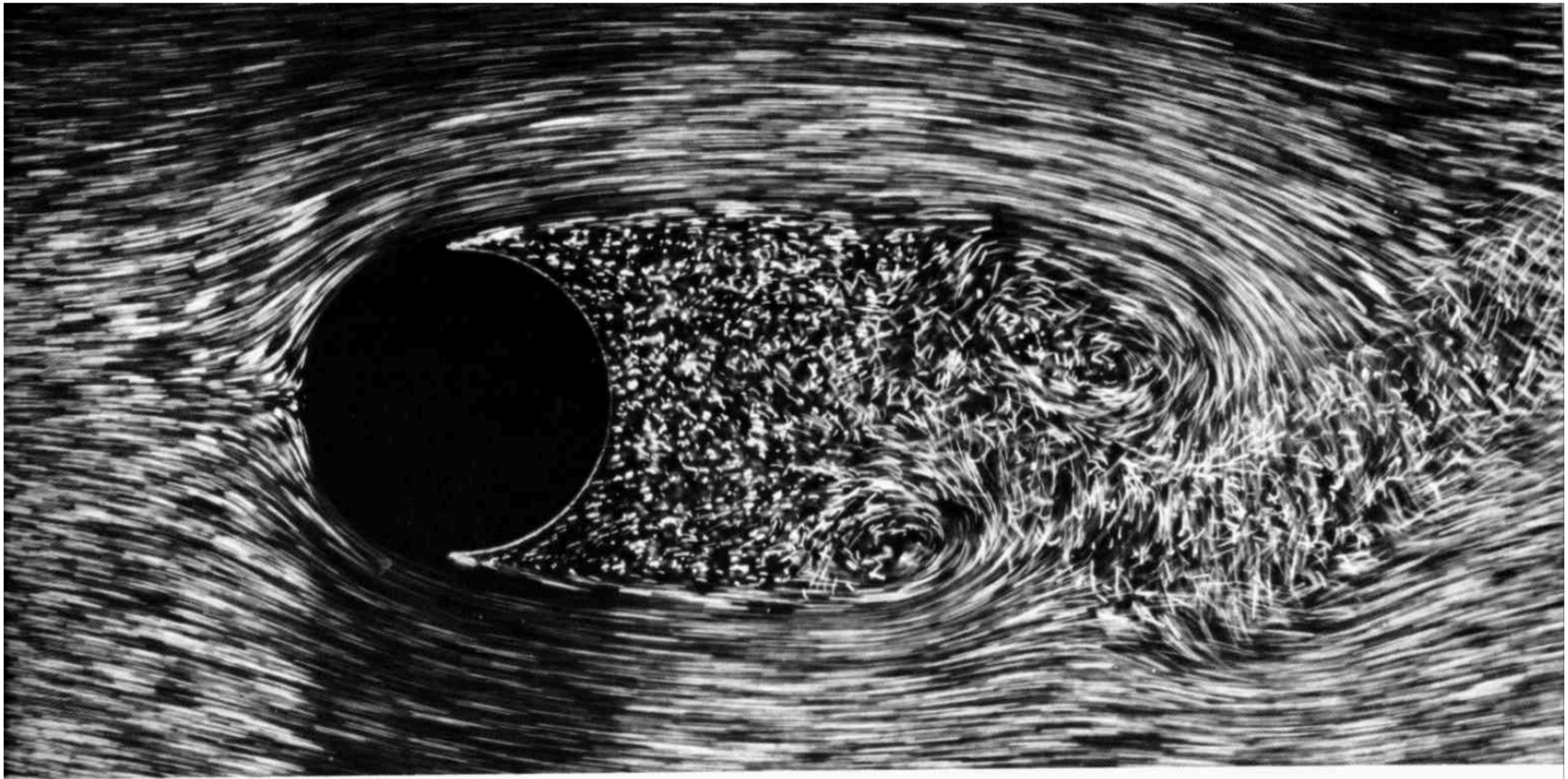
Transition to turbulence



54. Sphere at R**=202.** The rear of the recirculating region behind a sphere begins to oscillate slowly at a Reynolds number of about 130, but the flow is still perfectly laminar at this higher speed. Visualization is by condensed milk in water. *Taneda 1956b*

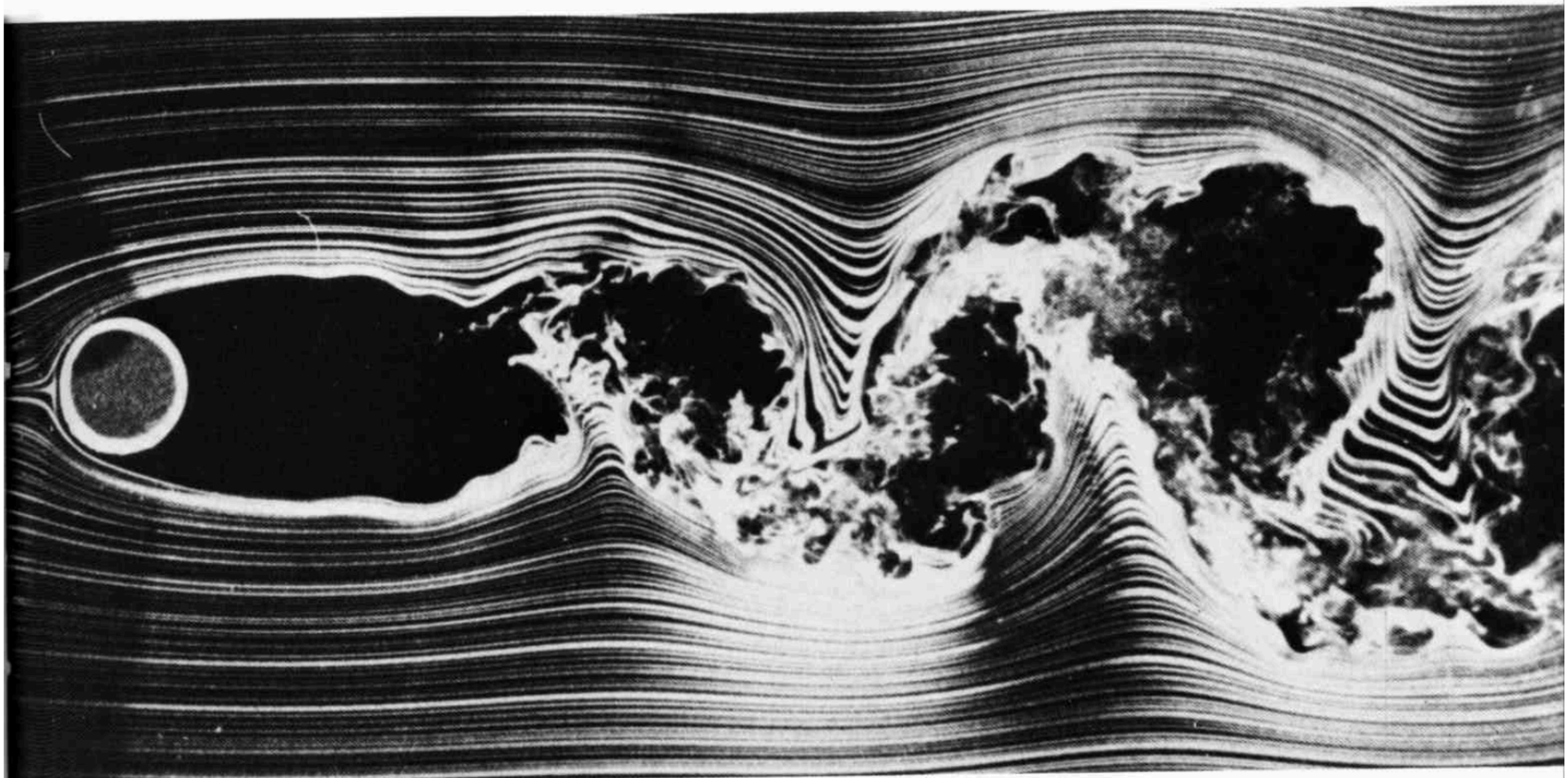
33

Transition to turbulence



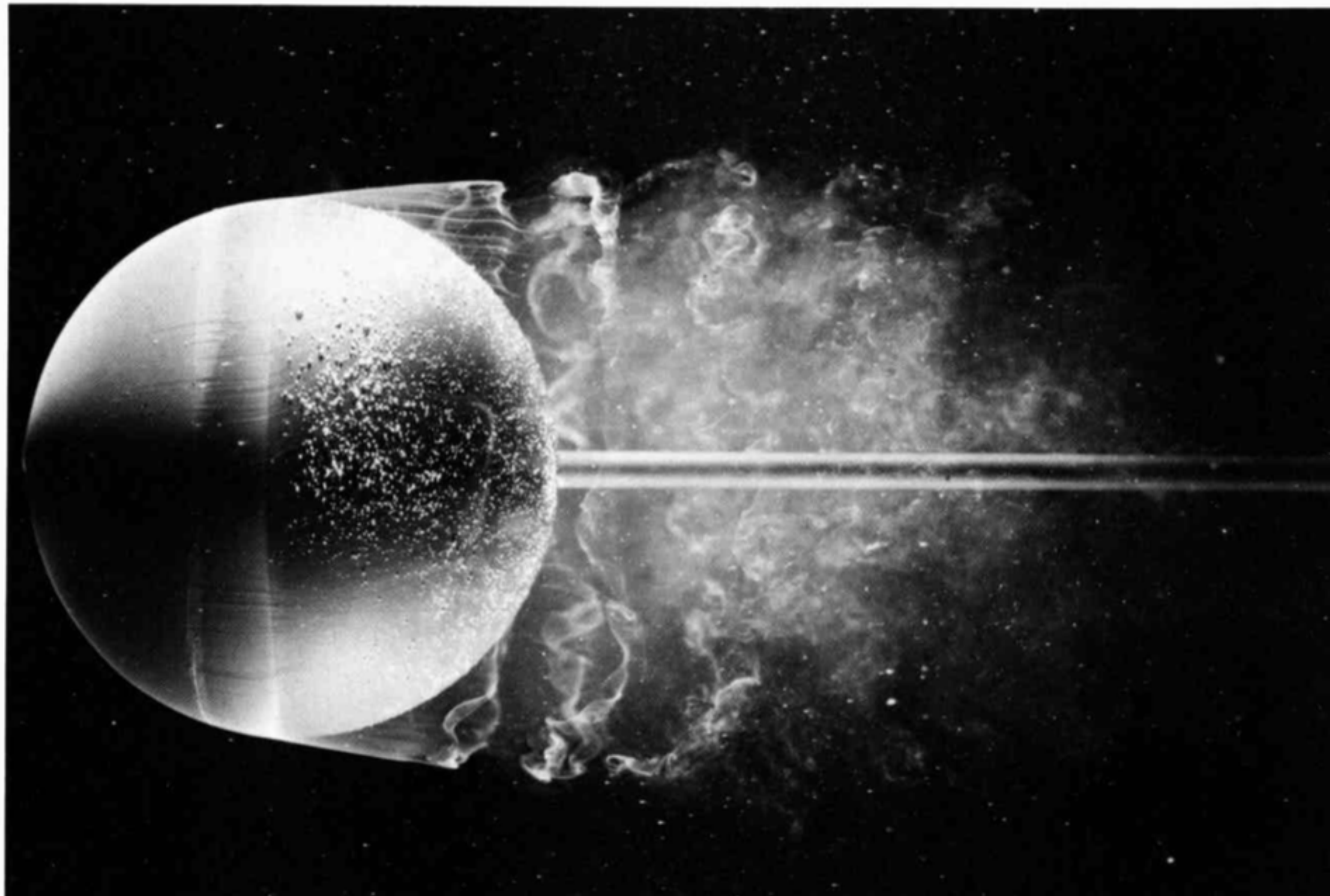
47. Circular cylinder at R**=2000.** At this Reynolds number one may properly speak of a boundary layer. It is laminar over the front, separates, and breaks up into a turbulent wake. The separation points, moving forward as the Reynolds number is increased, have now attained their upstream limit, ahead of maximum thickness. Visualization is by air bubbles in water. *ONERA photograph, Werlé & Gallon 1972*

Transition to turbulence



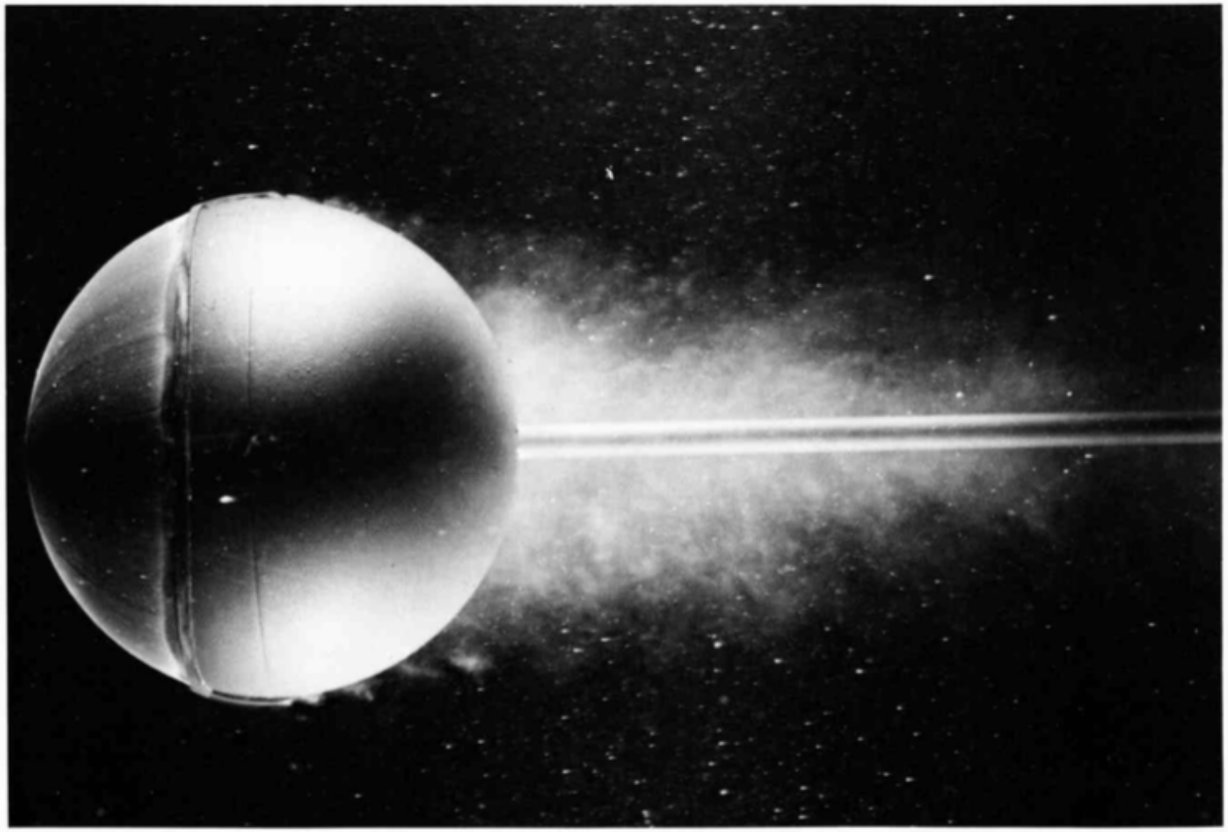
48. Circular cylinder at R=10,000. At five times the speed of the photograph at the top of the page, the flow pattern is scarcely changed. The drag coefficient consequently remains almost constant in the range of Reynolds number spanned by these two photographs. It drops later when, as in figure 57, the boundary layer becomes turbulent at separation. *Photograph by Thomas Corke and Hassan Nagib*

Transition to turbulence



55. Instantaneous flow past a sphere at R=15,000. Dye in water shows a laminar boundary layer separating ahead of the equator and remaining laminar for almost one radius. It then becomes unstable and quickly turns turbulent. *ONERA photograph, Werlé 1980*

Transition to turbulence



57. Instantaneous flow past a sphere at R=30,000 with a trip wire. A classical experiment of Prandtl and Wieselsberger is repeated here, using air bubbles in water. A wire hoop ahead of the equator trips the boundary layer. It becomes turbulent, so that it separates farther rearward than if it were laminar (opposite page). The drag is thereby dramatically reduced, in a way that occurs naturally on a smooth sphere only at a Reynolds number ten times as great. *ONERA photograph, Werlé 1980*

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- Dissipative time scale = *time scale that is required for molecular friction to bring the motion at scale L to rest*

$$\frac{L^2}{\nu T} = \frac{T_\nu}{T} \qquad T_\nu = \frac{L^2}{\nu}$$

Activity 1:

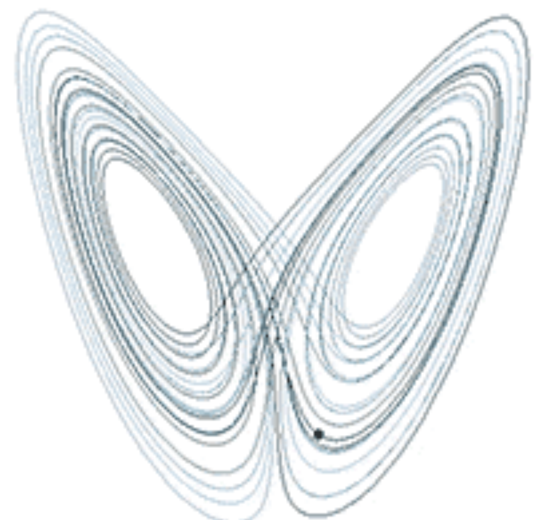
- Reynolds number for a mesoscale vortex in the ocean?
- Reynolds number for a storm system in the atmosphere?

- How long would it take for molecular dissipation to halt a storm system?
- How long would it take for molecular dissipation to halt a stirred coffee?

Chaotic behavior:

- **Chaos:** When the present determines the future, but the approximate present does not approximately determine the future *[E. Lorenz]*

A double rod pendulum showing chaotic behavior [https://en.wikipedia.org/wiki/Chaos_theory]



Solutions in the Lorenz attractor (originally a model for atmospheric convection)

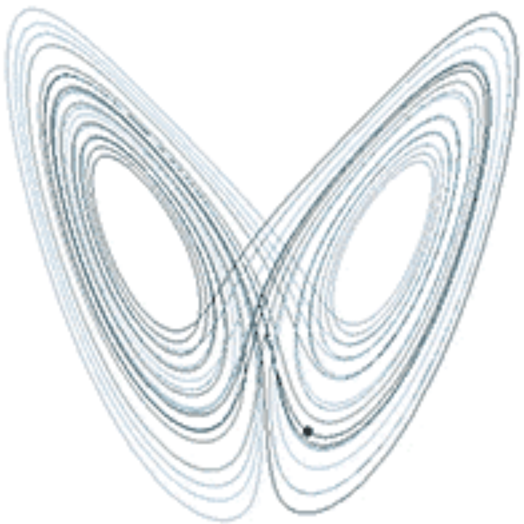
Chaotic behavior:

- **Chaos:** When the present determines the future, but the approximate present does not approximately determine the future *[E. Lorenz]*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y,$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$



Solutions in the Lorenz attractor
(originally a model for atmospheric convection)

A chaotic example:

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- Let’s play around with a simpler version containing a non-linear term (with a Reynolds number r), a source and sink:

$$\frac{du}{dt} + r u^2 = -u + 1$$

-

Properties of turbulence: unpredictability

$$\frac{du}{dt} + ru^2 = -u + 1$$

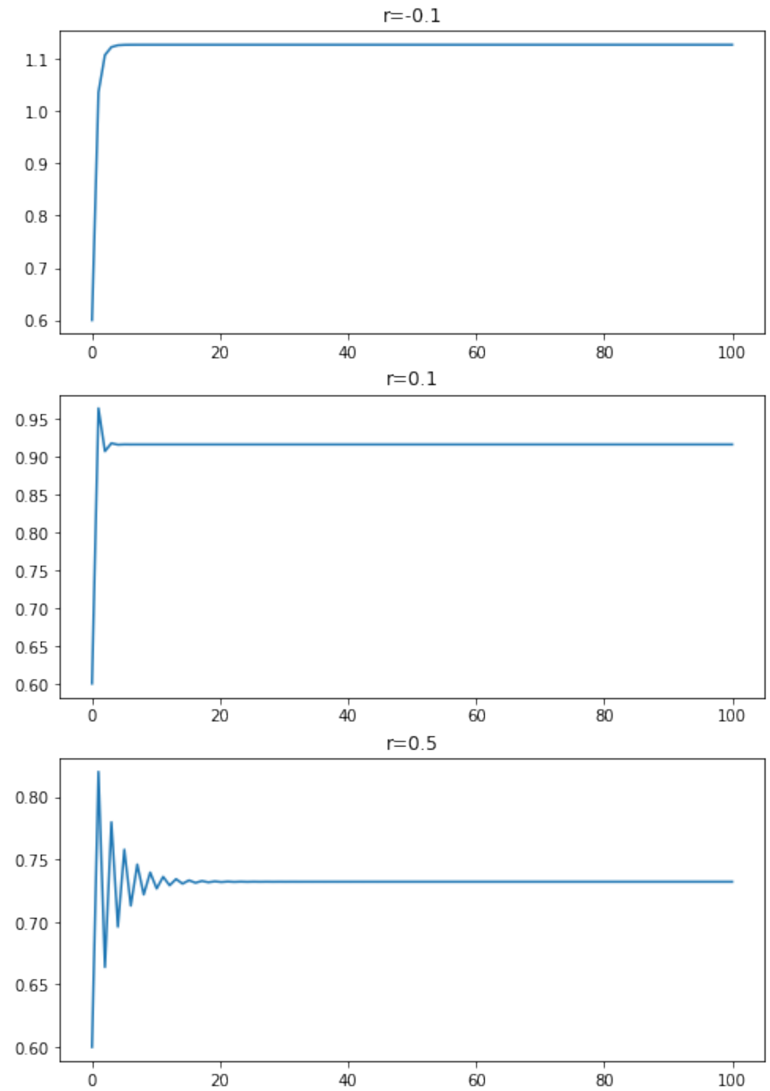
With $u_0 = 0.6$

- Fixed points:

$$\frac{d}{dt}u = -ru^2 + 1 - u = 0$$

$$u = -\frac{1}{2r} \pm \frac{\sqrt{1+4r}}{2r}$$

- Stable fixed points?



Activity 2:

$$\frac{du}{dt} + ru^2 = -u + 1$$

- Discretize using a simple forward Euler scheme (with dt=1) to get $u(t+1) = F[u(t)]$
- Using it and u(0) = 0.6, plot u(t) for t=[0:100] and r = [0.1, 0.5, 0.8, 1.3, 2].
- Check the sensitivity to initial conditions
- Plot the pdf for r=2
- Plot the power spectra of u for r = [0.1, 0.5, 0.8, 1.3, 2],

Computing Power Spectra

- We have a signal *(e.g. velocity u along dimension x)*
- *The total energy is* $$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx$$
- *Using the Fourier transform of the variable:*

$$\hat{u}(k) = \int_{-\infty}^{+\infty} e^{-2\pi i k x} u(x) dx$$

- *the power spectral density is* $$|\hat{u}(k)|^2$$
  - *It is the density of energy per unit wavenumber (or frequency)*

Computing Spectra

- Parseval's theorem states that:

$$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx = \int_{-\infty}^{+\infty} |\hat{u}(k)|^2 dk$$

  - *Integral of energy in the physical domain is equal to integral of spectral energy density over all wavenumbers.*

Computing Spectra

- In a finite and discrete domain:

$$\hat{u}(k) = \sum_{x=0}^{N-1} u(x) e^{-ikx\frac{2\pi}{N}}$$

$$\text{for } k = 0, \ldots, N-1$$

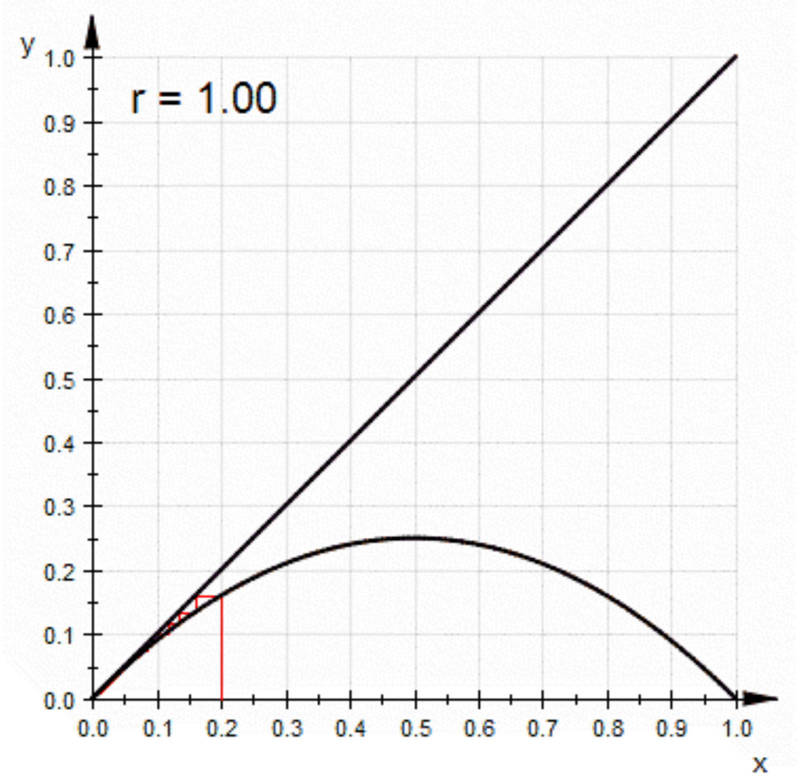
- And the power spectra is defined as: $$\frac{\hat{u}(k)\hat{u}^*(k)}{N}$$

Computing Spectra

```
def myfft(u):
    nx = u.shape[0]
    k = np.fft.rfftfreq(nx,d=1)
    psd = (np.abs(np.fft.rfft((u)))**2)/nx
    return k, psd
```

- The equation we've seen is similar to the **logistic map** by May (1976) for the growth of population:

$$x_{n+1} = r x_n (1 - x_n)$$



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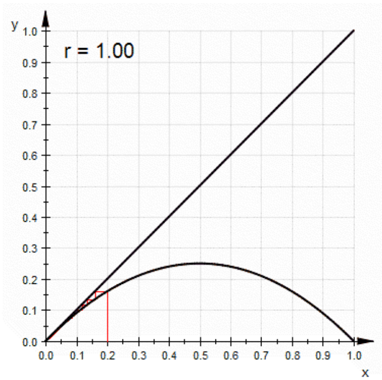

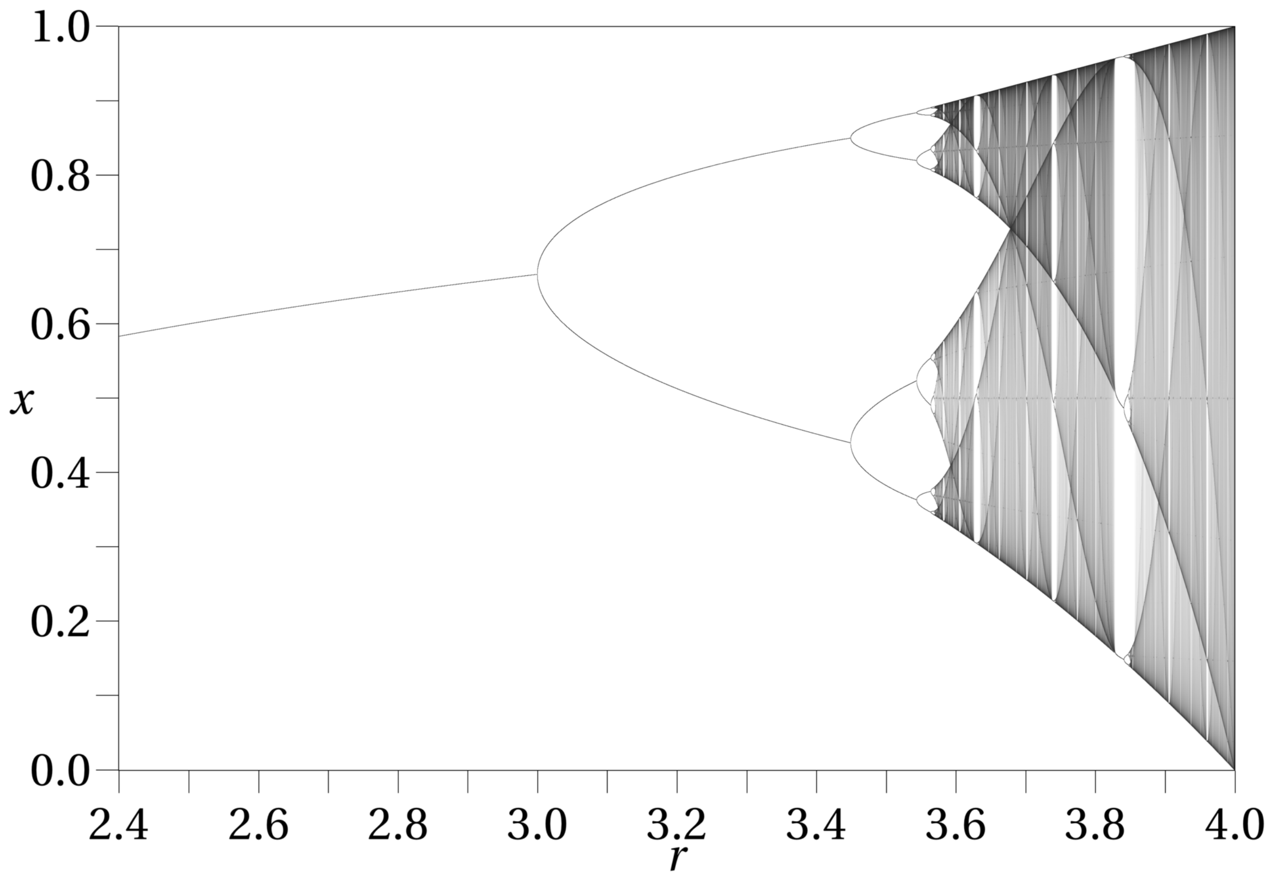

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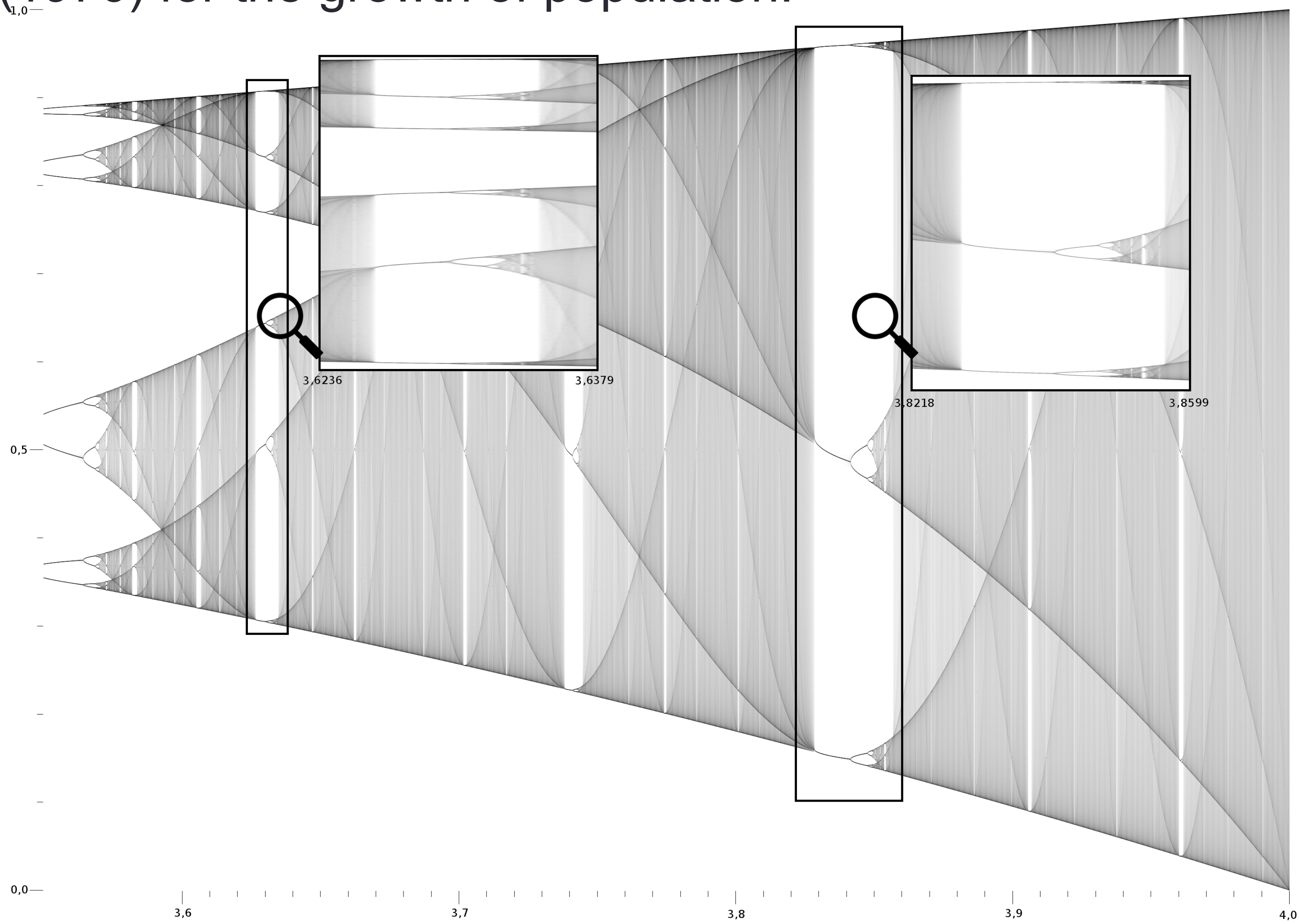
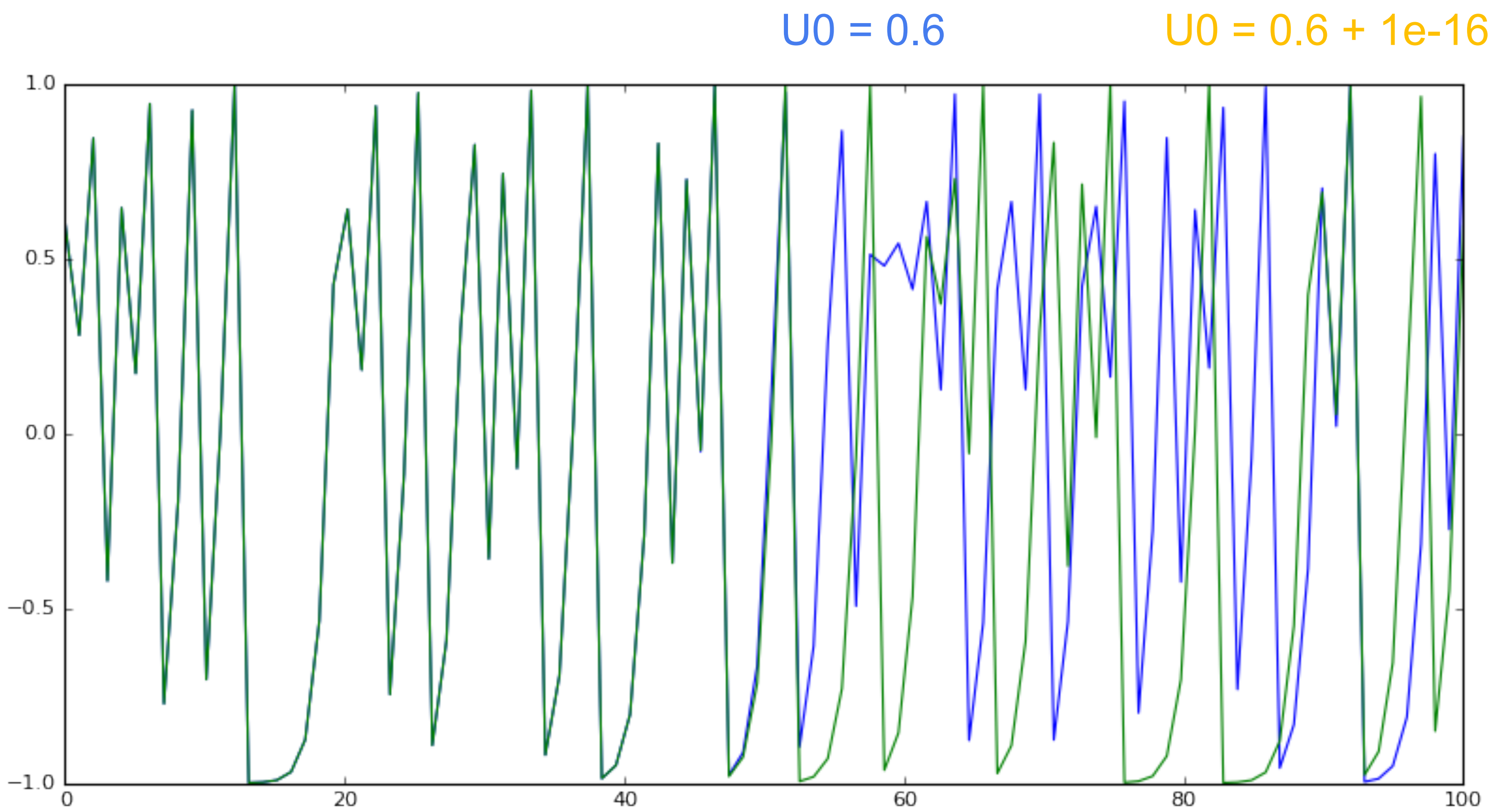


- Sensitivity to initial conditions

$$\frac{du}{dt} + ru^2 = -u + 1$$



Properties of turbulence: unpredictability

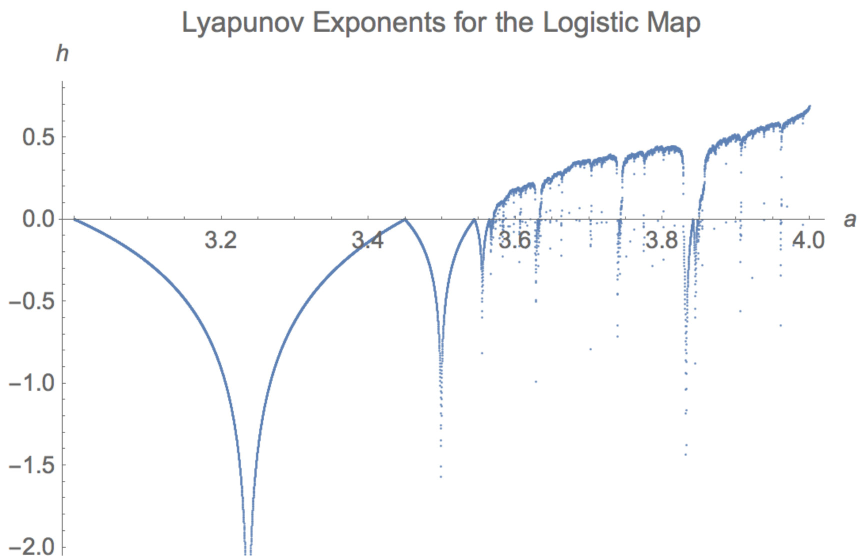
- **Sensitivity to initial conditions:**

In general, for an initial difference x_0 and $x_0 + \Delta x$

We expect the difference after n iteration to grow as $|\delta x_n| = \Delta x e^{n\lambda(x_0)}$

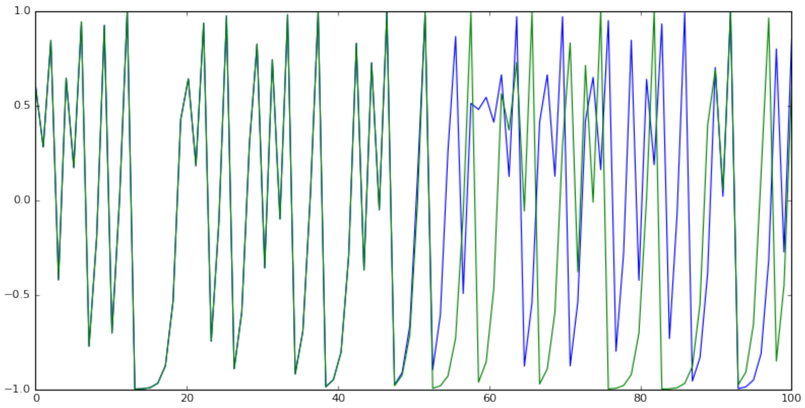
Where $\lambda(x_0)$ is the so-called **Lyapunov exponent** for the initial condition x_0

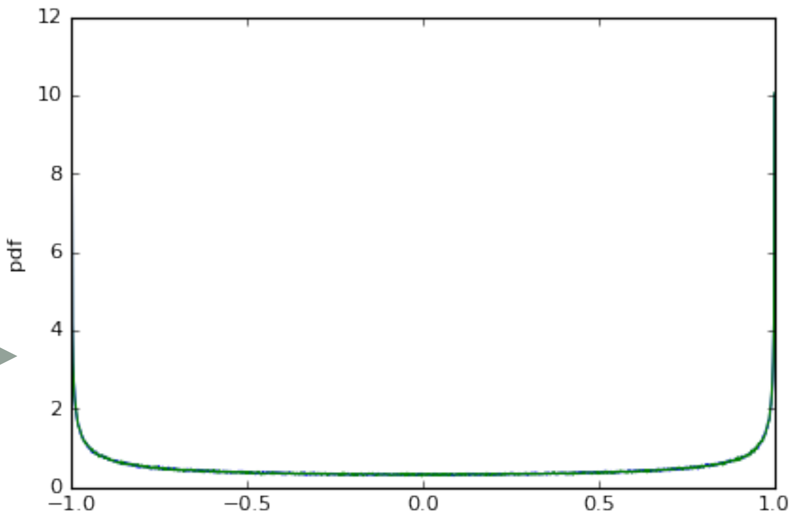
$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\Delta x\to 0} \frac{1}{n} \log \left| \frac{F^n(x_0+\Delta x) - F^n(x_0)}{\Delta x} \right| = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{dF^n(x_0)}{dx_0} \right|$$



A positive Lyapunov exponent is a signature of chaos …

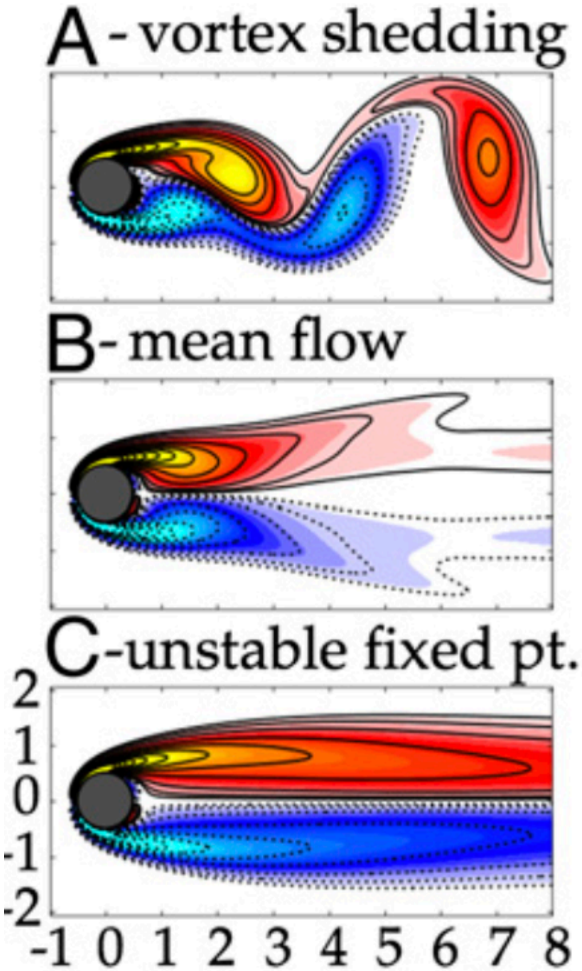
- Hence the need for a statistical description of turbulence:



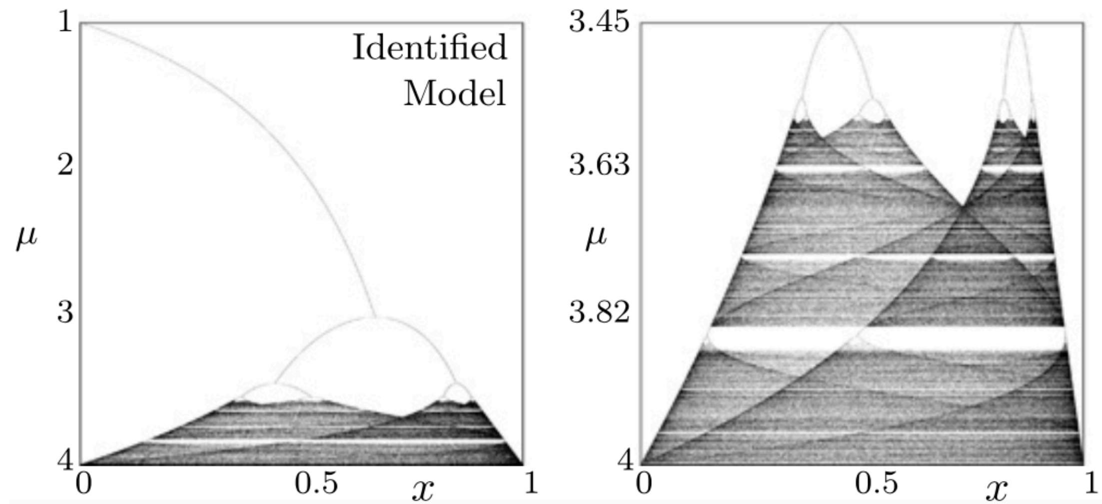


- statistical properties of the flow can be quite reproducible (pdf, mean, moments, etc.)
-

- fluid vortex shedding behind an obstacle :



[Brunton et al, 2016]



-

Triads interactions

- The nonlinear term in the equations of motion leads to interactions among different length scales
- *Let's write an equation for an incompressible 2 dimensional flow (for simplicity)*

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \zeta, \quad \zeta = \nabla^2 \psi.$$

With $u, v = -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}$ *and* $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Triads interactions

- *And work in Fourier space:*

$$\psi(x,y,t) = \sum_{\boldsymbol{k}} \widetilde{\psi}(\boldsymbol{k},t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \zeta(x,y,t) = \sum_{\boldsymbol{k}} \widetilde{\zeta}(\boldsymbol{k},t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $\boldsymbol{k} = \mathbf{i}k^x + \mathbf{j}k^y$, $\widetilde{\zeta} = -k^2\widetilde{\psi}$ where $k^2 = k^{x2} + k^{y2}$

Triads interactions

- The equation (without forcings and dissipation for now) is:

$$\frac{\partial}{\partial t}\sum_{\boldsymbol{k}}\tilde{\zeta}(\boldsymbol{k},t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} = -\sum_{\boldsymbol{p}} p^x\tilde{\psi}(\boldsymbol{p},t)\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^y\tilde{\zeta}(\boldsymbol{q},t)\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} + \sum_{\boldsymbol{p}} p^y\tilde{\psi}(\boldsymbol{p},t)\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^x\tilde{\zeta}(\boldsymbol{q},t)\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}}.$$

- We multiply by $\exp(-i\boldsymbol{k}\cdot\boldsymbol{x})$

- And use the fact that the Fourier modes are orthogonal;

$$\int \mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}}\,\mathrm{d}A = \frac{1}{L^2}\delta(\boldsymbol{p}+\boldsymbol{q}).$$

Triads interactions

- And get:

$$\frac{\partial}{\partial t}\tilde{\psi}(\boldsymbol{k},t) = \sum_{\boldsymbol{p},\boldsymbol{q}} A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\tilde{\psi}(\boldsymbol{p},t)\tilde{\psi}(\boldsymbol{q},t) + \tilde{F}(\boldsymbol{k}) - \nu k^4 \tilde{\psi}(\mathbf{k},t),$$

where $A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = (q^2/k^2)(p^x q^y - p^y q^x)\delta(\boldsymbol{p}+\boldsymbol{q}-\mathbf{k})$

- Only vectors with $\mathbf{p}+\mathbf{q}-\mathbf{k}=0$ have a non-zero contribution = **Triads interactions**

Triads interactions

- Triads interactions can be local or non-local:

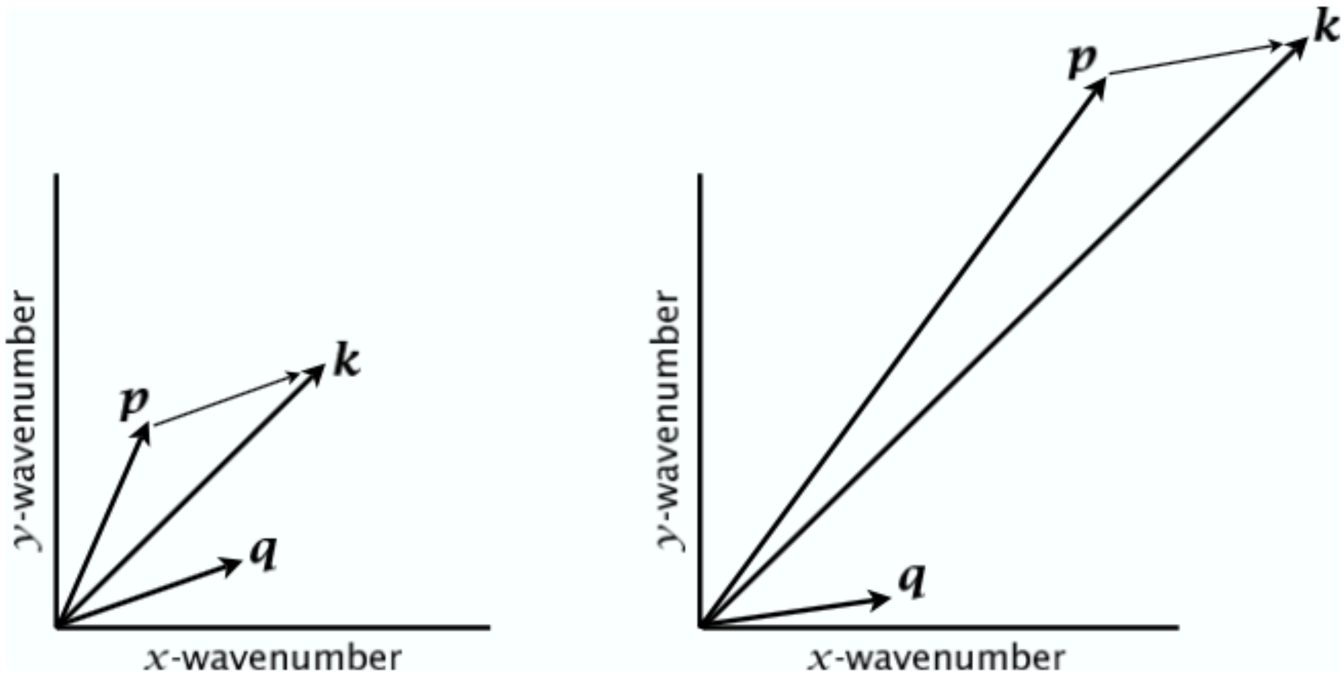


Fig. 8.1 Two interacting triads, each with $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$. On the left, a local triad with $k \sim p \sim q$. On the right, a nonlocal triad with $k \sim p \gg q$.

Triads interactions

- Starting with only 2 Fourier modes you can fill the entire spectrum due to scale interactions

$$\frac{\partial}{\partial t}\tilde{\psi}(\boldsymbol{k},t) = \sum_{\boldsymbol{p},\boldsymbol{q}} A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\tilde{\psi}(\boldsymbol{p},t)\tilde{\psi}(\boldsymbol{q},t) + \tilde{F}(\boldsymbol{k}) - \nu k^4\tilde{\psi}(\mathbf{k},t),$$

- Forcing is fixed at a certain scale(s) and dissipation acts on each Fourier mode with a coefficient that increases with wavenumber and therefore that preferentially affects small scales.

The closure problem

- *In a turbulent flow it may be virtually impossible to predict the detailed motion of each eddy, the **statistical properties** — time averages for example — might not be changing and we might like to predict such averages.*
- We can decompose the variables into mean and fluctuating components: $u = \overline{u} + u'$

Where $\overline{u} = \frac{1}{T}\int_0^T u\,dt$ or $\overline{u} = \frac{1}{N}\sum_{N\to\infty} u$

With by definition $\overline{u'} = 0$

- And try to find a closed equation for the mean part

Activity 3

- Let's try with a simple equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + uu + ru = 0$$

- Write the evolution equation for $\overline{u}$.

The closure problem

- Let's try with a simple equation: $$\frac{\mathrm{d}u}{\mathrm{d}t} + uu + ru = 0$$

- We average and get:

$$\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} + \overline{uu} + r\overline{u} = 0$$ with $$\overline{uu} = \overline{u}\,\overline{u} + \boxed{\overline{u'u'}} \neq \overline{u}\,\overline{u}.$$

unknown

- We can go the the next order, but there are always new unknowns :

$$\frac{1}{2}\frac{\mathrm{d}\overline{u^2}}{\mathrm{d}t} + \boxed{\overline{uuu}} + r\overline{u^2} = 0$$

unknown

The closure problem

- So we need a method to find close the hierarchy an make some assumptions…

- For example: $$\overline{uuuu} = \alpha \overline{uu}\,\overline{uu} + \beta \overline{uuu}$$

The closure problem

- For the NS equation:

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j \frac{\partial u_i}{\partial x_j}} + f\mathbf{k} \times \overline{u_i} + \frac{\overline{\rho}}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x_j} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_i \partial x_j}$$

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + f\mathbf{k} \times \overline{u_i} + \frac{\overline{\rho}}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x_j} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_i \partial x_j} \boxed{- \frac{\partial \overline{u_i' u_j'}}{\partial x_j}}$$

Advection for the averaged flow

Reynolds stress
= effect of subgrid-scale turbulence

The closure problem

For the NS equation:

- The number of unknowns is larger than the number of equations
- When we start deriving equations for unknowns, more variables appear.

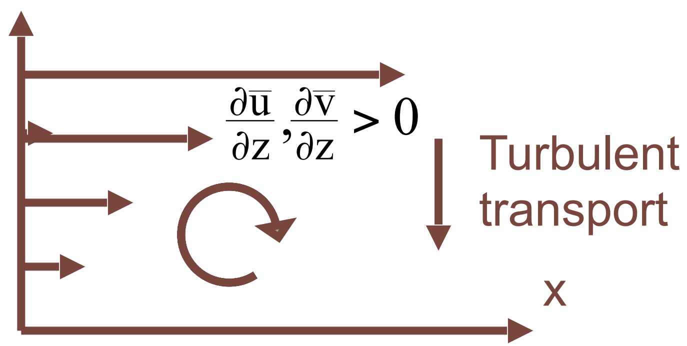
Progn. Eq. for:	Moment	Equation	No. Eqs.	No. Unknowns
$\overline{U_i}$	First	$\frac{\partial \overline{U_i}}{\partial t} = \ldots - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$	3	6
$\overline{u_i' u_j'}$	First	$\frac{\partial \overline{u_i' u_j'}}{\partial t} = \ldots - \frac{\partial \overline{u_k' u_i' u_j'}}{\partial x_k}$	6	10
$\overline{u_i' u_j' u_k'}$	First	$\frac{\partial \overline{u_i' u_j' u_k'}}{\partial t} = \ldots - \frac{\partial \overline{u_k' u_i' u_j' u_m'}}{\partial x_m}$	10	15

The closure problem

In PE models the equations are closed by parameterizing the Reynolds stresses as:

$$\overline{u'w'} = -K_{Mv}\frac{\partial u}{\partial z}$$

$$\overline{v'w'} = -K_{Mv}\frac{\partial v}{\partial z}$$



But nobody has been able to close the system, in any useful way, without introducing physical assumptions not directly deducible from the equations of motion themselves.

The Search for Universal Properties and the Kolmogorov Scaling Laws

The foundation of many theories of turbulence is the spectral theory of Kolmogorov.

This theory does not close the equations, but provides a prediction for the energy spectrum of a turbulent flow *(how much energy is present at a particular spatial scale)* by suggesting a relationship between the energy spectrum (a second order quantity in velocity) and the spectral energy flux (a third order quantity).

Activity 4

- Staring from Incompressible NS equations:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

- Write the energy equation (starting from NS equations below) integrated over a closed (or periodic) domain.

$$E = \iiint \frac{1}{2} |\vec{u}^2| \, dV \qquad \frac{d}{dt} E =$$

- *You can use:*

$$u \cdot \vec{\nabla} \vec{u} = (\vec{\nabla} \times \vec{u}) \times \vec{u} + \frac{1}{2} \vec{\nabla} (\|\vec{u}\|^2)$$

Conservation laws

- Equation for kinetic energy

$$\frac{\partial}{\partial t}\frac{1}{2}|\vec{u}^2| + \nabla \cdot (\vec{u}\frac{1}{2}|\vec{u}^2|) = -\nabla \cdot [\vec{u}(\frac{p}{\rho_0} + gz)] + \vec{u} \cdot \mathcal{F} + \nu \vec{u} \cdot \nabla^2 \vec{u}$$

- Energy integrated over a closed (or periodic) domain

$$E = \iiint \frac{1}{2}|\vec{u}^2| \, dV \qquad \frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F} \, dV + \nu \iiint \vec{u} \cdot \nabla^2 \vec{u} \, dV$$

- the inertial terms in the momentum equation conserve energy (redistribute in the domain)

Conservation laws

$$\frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F}\, dV + \nu \iiint \vec{u} \cdot \nabla^2 \vec{u}\, dV$$

- Can be rewritten using identities:

$$\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u}) = -\nabla \times \vec{\omega}$$

$$\vec{u} \cdot \nabla^2 \vec{u} = -\vec{u} \cdot (\nabla \times \omega) = -\vec{\omega} \cdot (\nabla \times \vec{u}) + \nabla \cdot (\vec{\omega} \times \vec{u})$$

$$\nu \iiint \vec{u} \cdot \nabla^2 \vec{u}\, dV = -\nu \iiint \vec{\omega} \cdot (\nabla \times \vec{u})\, dV = -\nu \iiint |\vec{\omega}|^2\, dV$$

forcing dissipation

$$\boxed{\frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F}\, dV - \nu \iiint |\vec{\omega}|^2\, dV}$$

Conservation laws

forcing dissipation

$$\frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F} \, dV - \nu \iiint |\vec{\omega}|^2 \, dV$$

- Forcing puts energy in the system and dissipation removes it.
- Dissipation is proportional to the integral of the squared vorticity, also known as the ***enstrophy***

Conservation laws

forcing dissipation

$$\boxed{\frac{d}{dt}E = \iiint \vec{u} \cdot \mathcal{F} \, dV \; - \nu \iiint |\vec{\omega}|^2 \, dV}$$

- How is the energy transferred from the forcing scales to the dissipative scales?

Cascade of energy

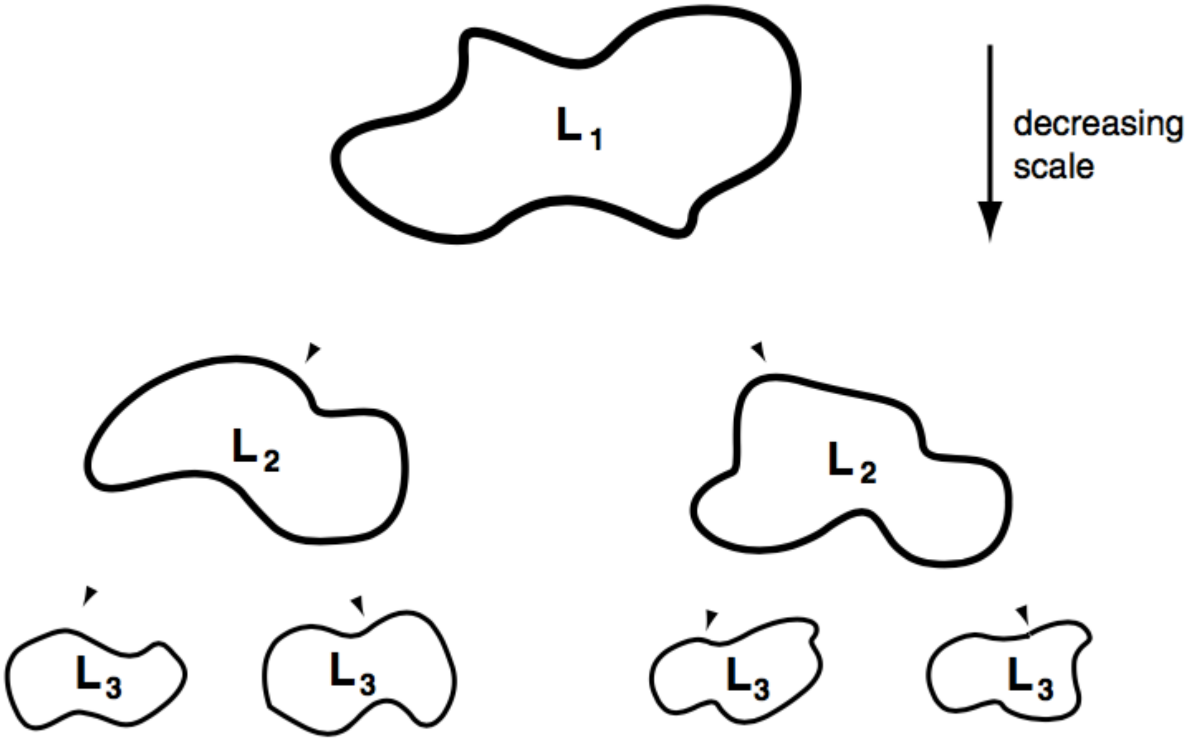


Fig. 8.2 Schema of a 'cascade' of energy to smaller scales: eddies at a large scale break up into smaller scale eddies, thereby transferring energy to smaller scales. If the transfer occurs between eddies of similar sizes (i.e., if it is spectrally local) the transfer is said to be a cascade. The eddies in reality are embedded within each other.

Kolmogorov's inertial range

- *Thus forcing puts energy into the system and dissipation removes it. We assume the forcing happens at much larger scales than the dissipation, which happens on molecular scales, and that there is a range of scales in between where neither forcing or dissipation are important.*

- Kolmogorov proposed a theory in 1941 for this transfer, which has become known as the *inertial range*, with a few assumptions:
  - Turbulence is *isotropic*—the same in all directions.
  - Turbulence is *homogeneous*—the same at all locations in space.
  - Triad interactions are *local*.

Kolmogorov's inertial range

- The details of the forcing and dissipation don't matter in the inertial range.
- The *only* important parameter in the inertial range is the rate at which energy is transferred downscale. We call this the energy flux: ϵ

- If we decompose velocities in Fourier space:

$$u(x, y, z, t) = \sum_{k^x, k^y, k^z} \tilde{u}(k^x, k^y, k^z, t) \mathrm{e}^{i(k^x x + k^y y + k^z z)}$$

- The energy can be written (Parseval theorem)

with energy spectral density

$$\hat{E} = \int E \, \mathrm{d}V = \frac{1}{2} \int (u^2 + v^2 + w^2) \, dV$$
$$= \frac{1}{2} \sum (|\tilde{u}|^2 + |\tilde{v}|^2 + |\tilde{w}|^2) \, \mathrm{d}\boldsymbol{k}$$

$$\hat{E} \equiv \int \mathcal{E}(k) \, \mathrm{d}k$$

Kolmogorov's inertial range

- If the rate of energy input per unit volume by stirring is equal to ϵ then if we are in a steady state there must be a flux of energy from large scales to small also equal to ϵ, and an energy dissipation rate, also ϵ
- So the general form of energy spectral density is

$$\mathcal{E}(k) = g(\varepsilon, k, k_0, k_\nu)$$

Forcing wavenumber

Dissipation wavenumber

- In the inertial range (Assuming locality), we don't feel forcing and dissipation so:

$$\mathcal{E}(k) = g(\varepsilon, k)$$

Activity 5

- Find the function g such that: $$\mathcal{E}(k) = g(\varepsilon, k)$$

Kolmogorov's inertial range

Dimensions and the Kolmogorov Spectrum

Quantity	Dimension
Wavenumber, k	1/L
Energy per unit mass, E	$U^2 = L^2/T^2$
Energy spectrum, $\mathcal{E}(k)$	$EL = L^3/T^2$
Energy Flux, ε	$E/T = L^2/T^3$

If $\mathcal{E} = f(\varepsilon, k)$ then the only dimensionally consistent relation for the energy spectrum is

$$\mathcal{E} = \mathcal{K}\varepsilon^{2/3}k^{-5/3}$$

where $\mathcal{K}$ is a dimensionless constant.

- The parameter K is a dimensionless constant, undetermined by the theory. It is known as Kolmogorov's constant and experimentally it is found to be approximately 1.5.

Kolmogorov's inertial range

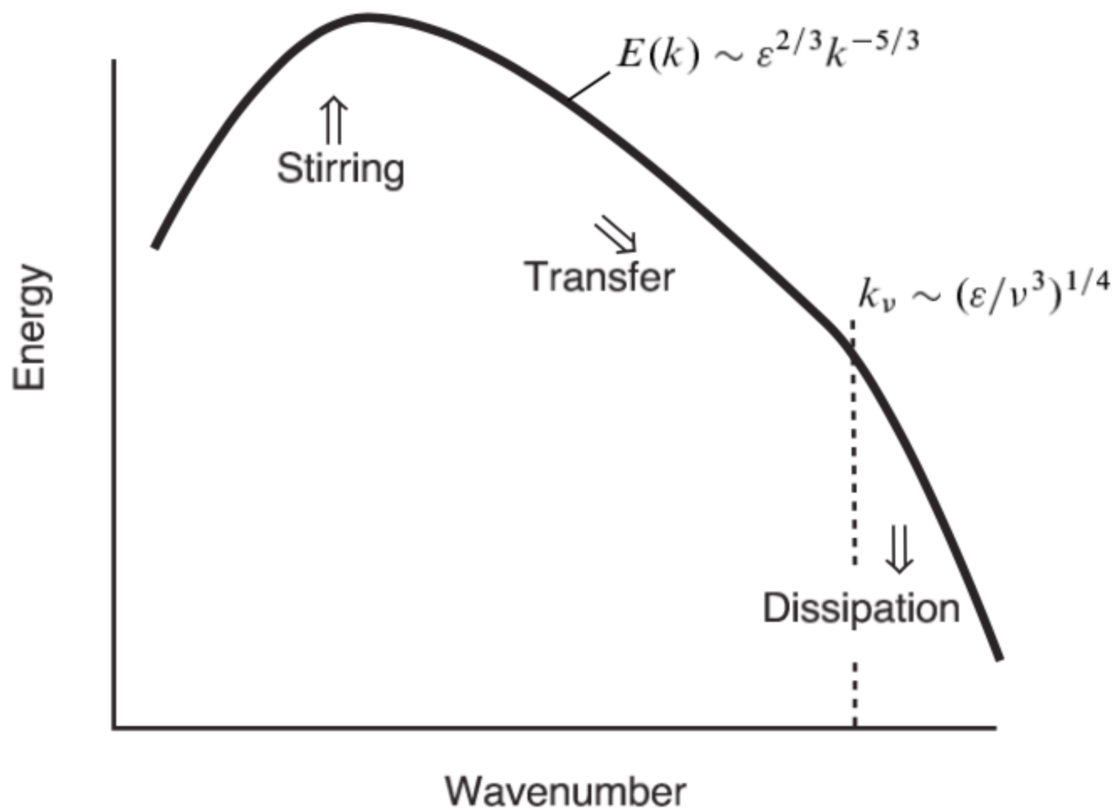
- At some small length-scale we should expect viscosity to become important and the scaling theory we have just set up will fail.
- This is given by the Kolmogorov Length scale:

$$k_\nu \sim \left(\frac{\varepsilon}{\nu^3}\right)^{1/4}, \qquad L_\nu \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

Kolmogorov's inertial range

- This is the famous '**Kolmogorov -5/3 spectrum**'. Cornerstone of turbulence theory

Figure 8.3 Schema of energy spectrum in three-dimensional turbulence, in the theory of Kolmogorov. Energy is supplied at some rate ε; it is transferred ('cascaded') to small scales, where it is ultimately dissipated by viscosity. There is no systematic energy transfer to scales larger than the forcing scale, so here the energy falls off.



Kolmogorov's inertial range

- Experimental validation

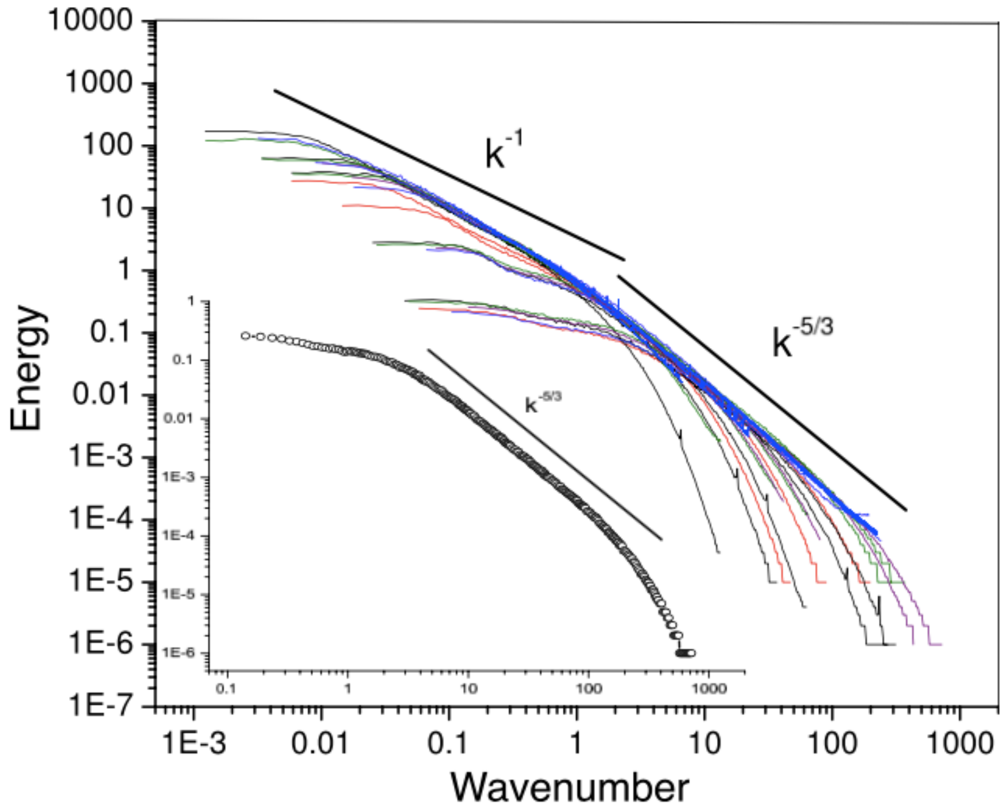


Fig. 8.4 The energy spectrum of 3D turbulence measured in some experiments at the Princeton Superpipe facility.[5] The outer plot shows the spectra from a large-number of experiments at different Reynolds numbers, with the magnitude of their spectra appropriately rescaled. Smaller scales show a good -5/3 spectrum, whereas at larger scales the eddies feel the effects of the pipe wall and the spectra are a little shallower. The inner plot shows the spectrum in the centre of the pipe in a single experiment at $Re \approx 10^6$.

Kolmogorov's inertial range

- Experimental validation

measurements in a jet in the laboratory

-

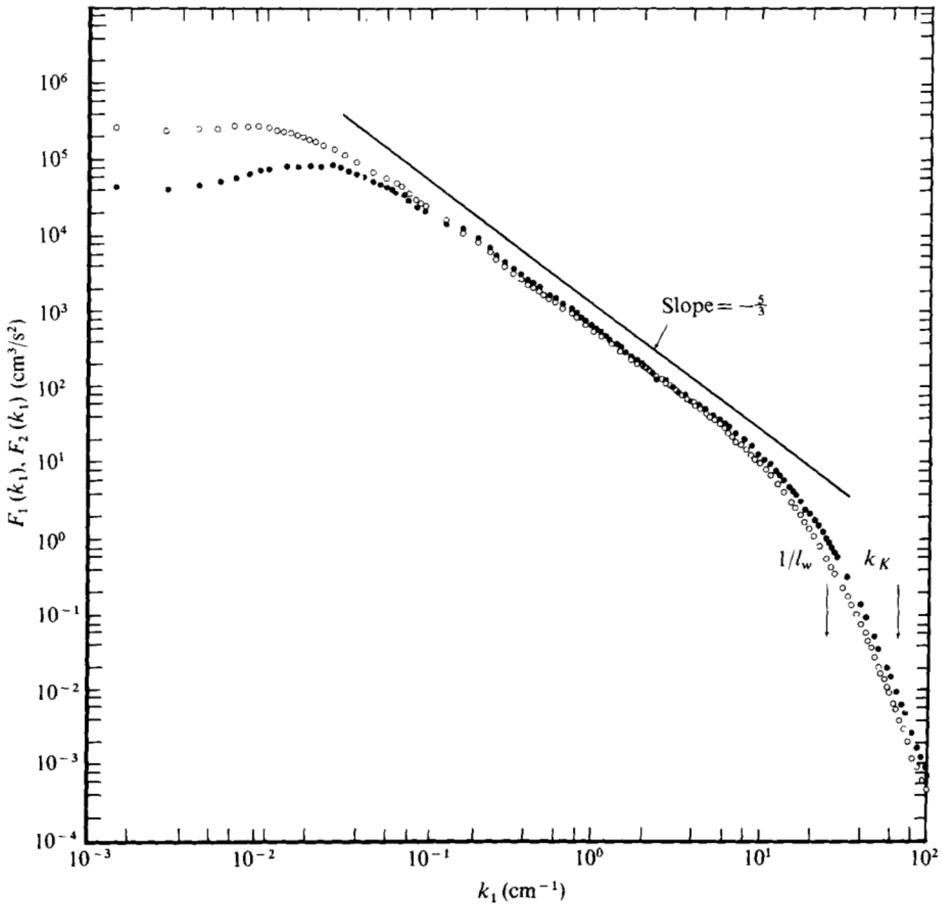


FIGURE 15. One-dimensional spectra of streamwise- and lateral-component velocity fluctuations for an axisymmetric jet; $Re = 3{\cdot}7 \times 10^5$, $x/d = 70$, $r/d = 0$. ○, $F_1(k_1)$; ●, $F_2(k_1)$.

Champagne, 78