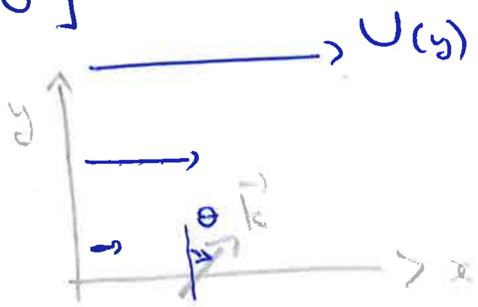
# <u>1.4C</u> Wave refraction by a horizontally sheared flow (see Leblond & Mysak, p334)

current : $\vec{U} = [U(y), 0, 0]$

wave : $\vec{k} = [k_x, k_y, 0]$




$$\begin{cases} k_x = k \sin\theta \\ k_y = k\cos\theta \end{cases}$$

intrinsic frequency : $\omega_0 = k c_0$

frequency : $\omega = k c_0 + k_x U$

in a time independent flow, $\omega$ is constant.

$$\Rightarrow \quad \omega = k c_0 + k_x U$$
$$= k c_0 + k \sin\theta \, U = \text{cste}$$

no variation in the $x$ direction : $\left(\frac{\partial U}{\partial x} = 0\right)$

$$\Rightarrow \quad \frac{dk_x}{dt} = 0 \quad \Leftrightarrow \quad k\sin\theta = \text{cste}.$$

$$\Leftrightarrow \quad \frac{\omega}{k} = c_0 + U\sin\theta$$

$$\frac{\omega}{k\sin\theta} = \frac{c_0}{\sin\theta} + U = cste$$

initial values : $c_e, U_e, \theta_e$

so
$$\boxed{\frac{c_0}{\sin\theta} + U = \frac{c_e}{\sin\theta_e} + U_e}$$

( = loi de Snell - Descartes )

ex : for long surface gravity waves :

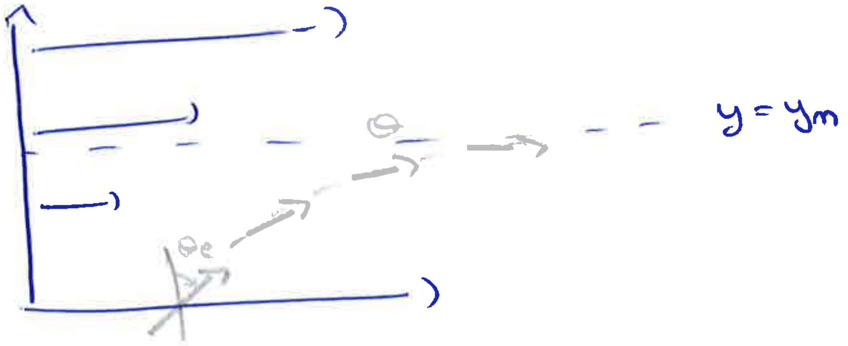
$$\begin{cases} c_0 = \sqrt{gH} \\ H = cste \\ U_e = U(y=0) = 0 \end{cases}$$

$$\Leftrightarrow \quad \frac{c_0}{\sin\theta} + U(y) = \frac{c_0}{\sin\theta_e}$$

$$\sin\theta = \frac{1}{\frac{1}{\sin\theta_e} - \frac{U}{c_0}}$$

$$\boxed{\sin\theta = \frac{\sin\theta_e}{1 - \frac{U(y)\sin\theta_e}{\sqrt{gH}}}}$$

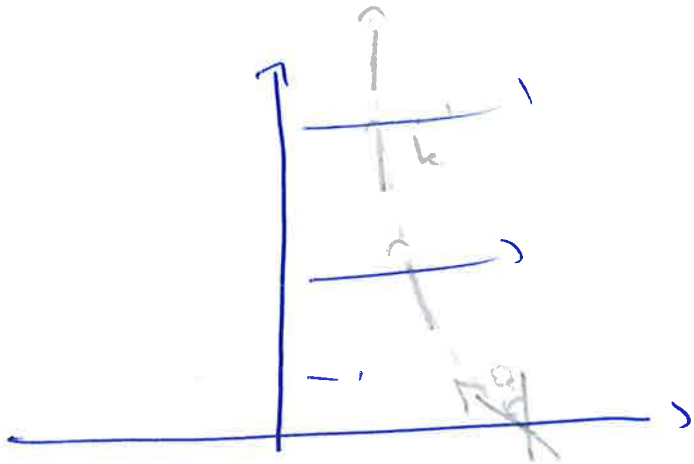
- if $0 < \Theta_e < \pi/2$ : $\Theta(M) > \Theta_e$



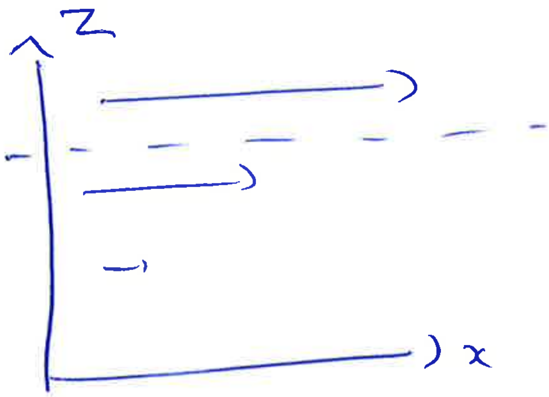

internal reflection.

when $\Theta = \frac{\pi}{2} \iff \frac{U(y_m)}{\sqrt{gH}} = \frac{1}{\sin\Theta_e} - 1$

- if $\Theta_e < 0$ : $\Theta(t) > \Theta_e$



# 1.4 D Critical layer (p 336)




- with $f=0$

$$\omega_0^2 = \frac{N^2 (k_x^2 + k_y^2)}{(k_x^2 + k_y^2 + k_z^2)}$$

- $\omega = \omega_0 + k_x U$ (is cste along a ray)

- $$c_{gz} = \frac{\partial \omega}{\partial k_z} = - \frac{N^2 (k_x^2 + k_y^2) \times 2 k_z}{k_x^2 + k_y^2 + k_z^2} \times \frac{1}{2\omega_0}$$

$$= \frac{-k_z \omega_0}{(k_x^2 + k_y^2 + k_z^2)}$$

if $\omega_0 \longrightarrow 0$ : $\begin{cases} c_{gz} \longrightarrow 0 \\ k_z \longrightarrow +\infty \end{cases}$

at the critical level $k_z$ becomes unbounded

energy is absorbed by the background

[ $\neq$ reflection when when $c_{gz} = 0$ $\begin{cases} k_z = 0 \\ \omega_0 = N \end{cases}$ ]

$$c_{gz} = -\left(1-\frac{\omega_0^2}{N^2}\right)^{1/2}\frac{\omega_0^2}{N k_x} \qquad (\text{with } k_y = 0)$$

Taylor series near $z = z_c$:

$$\omega_i(z-z_c) = \omega_i(z_c) + (z-z_c)\frac{\partial \omega_i}{\partial z} + \dots$$

$$\simeq (z-z_c)\, k_x \frac{\partial U}{\partial z}$$

$$\Longleftrightarrow \quad c_{gz} \simeq -\frac{k_x}{N}\left(\frac{\partial U}{\partial z}\right)^2 (z-z_c)^2$$

$\left(\text{with Ray equation } \frac{dz}{dt} = c_{gz}\right)$

so integrating between $z_1$ and $z_2$, we get

$$t_2 - t_1 \simeq \frac{N}{k_x\left(\frac{\partial U}{\partial z}\right)^2}\left[\frac{1}{(z_2-z_c)} - \frac{1}{(z_1-z_c)}\right]$$

becomes unbounded as $z_2 \to z_c$

a wave group never reaches
a critical level !!!

# 1.4 F Viscous dissipation

equations:
$$\vec{\nabla}\cdot\vec{u}' = 0$$
$$\begin{cases} \dfrac{\partial \rho'}{\partial t} - \dfrac{\rho_0 N^2}{g} w = \boxed{K_b \nabla^2 \rho} \\ \dfrac{\partial \vec{u}'}{\partial t} + \dfrac{\vec{\nabla} p}{\rho^*} + \dfrac{\rho}{\rho^*} g\vec{k} = \boxed{K_M \nabla^2 \vec{u}'} \end{cases}$$

viscosity

$$\boxed{\left(\frac{\partial}{\partial t} - K_B \nabla^2\right)\left(\frac{\partial}{\partial t} - K_M \nabla^2\right)\nabla^2 w + N^2 \nabla_h^2 w = 0}$$

with $w = w_0 e^{(\omega t + i\vec{k}\cdot\vec{x})}$

$$(\omega + K_B k^2)(\omega + K_M k^2) k^2 + N^2 k_h^2 = 0$$

$$\omega^2 + \omega k^2 (K_B + K_M) + K_B K_M k^4 + N^2 \frac{k_h^2}{k^2} = 0$$

(with $N = \text{cst}$)

$$\Delta = \left(k^2 (K_B + K_M)\right)^2 - 4\left(K_B K_M k^4 + N^2 \frac{k_h^2}{k^2}\right)$$

$$= \left((K_B + K_M)^2 - 4 K_B K_M\right) k^4 - 4 N^2 \frac{k_h^2}{k^2}$$

$$\omega = \frac{-k^2 (K_B + K_M) \pm i \sqrt{-\Delta}}{2}$$

$$\omega = \underbrace{-\frac{k^2}{2} (K_B + K_M)}_{<0} \pm i \frac{k_h N}{k} \sqrt{1 - \frac{1}{4} \frac{k^6 (K_B - K_M)^2}{k_h^2 N^2}}$$

waves are always damped !!

coef increase linearly with $K_B, K_M$

but quadratic with $k$

short waves are more damped than long waves

— frequency smaller than in the non-dissipative case.

— case $\Delta > 0$: no waves

viscous effects larger than $N^2$.