INTERNAL WAVES

2. CONTINUOUS STRATIFICATION

Bibliography

- Gerkema- Zimmerman (2008). *An introduction to internal waves*
  - http://stockage.univ-brest.fr/~gula/Ondes/gerkema.pdf

- Gill (1982) *: Atmosphère-Ocean Dynamics*

- Kundu-Cohen (1987). *Fluid Mechanics. Third edition*

- Cushman-Roisin*. Introduction to geophysical fluid Dynamics*

- **1.2 : Internal waves with continuous stratification**
  - Equations
  - Method of vertical modes
  - Method of characteristics

Local Static Stability

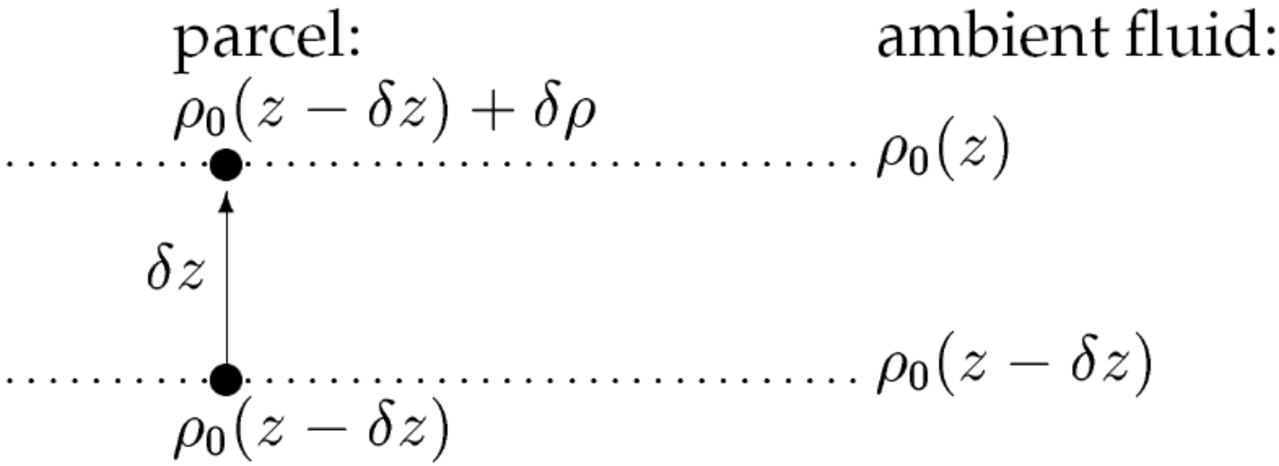
- Stratification:

$$\rho = \rho_0(z)$$

- A fluid is
  - **stably stratified** if a displaced parcel tends to return to its original position,
  - **unstably stratified** if it tends to move further away from its original position
  - **neutrally stratified** if it tends to stay where it is.

Local Static Stability

- Let’s move a parcel:



- Buoyancy force:

$$\rho_0(z)\ddot{\delta z} = g\left(\rho_0(z) - \rho_0(z - \delta z) - \delta\rho\right)$$

Local Static Stability

- With background density variation:

$$\rho_0(z) - \rho_0(z - \delta z) = \frac{d\rho_0}{dz}\delta z$$

- And parcel density variation?
  - *From thermodynamics, if entropy and salinity are conserved during displacement:*

$$\delta \rho = \left(\frac{\partial \rho}{\partial p}\right)_{\eta, S} \delta p = c_s^{-2} \delta p$$

Where c_s is the speed of sound

Local Static Stability

- So we get:

$$\rho_0(z)\ddot{\delta z} = g\left(\frac{d\rho_0}{dz}\delta z + \frac{\rho_0 g \delta z}{c_s^2}\right)$$

Local Static Stability

- Simple Harmonic oscillator:

$$\underbrace{\ddot{\delta z} - \frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)}_{N^2}\delta z = 0$$

- Brunt-Vaisala frequency: N^2

- Solutions: $e^{\pm \mathrm{i} N t}$ $\qquad \ddot{\delta z} + N^2 \delta z = 0$

- Stable if $N^2 > 0$

Local Static Stability

- Simple Harmonic oscillator:

$$\ddot{\delta z} + N^2 \delta z = 0$$

- Solutions: $e^{\pm \mathrm{i} N t}$

- Stable if $N^2 > 0$

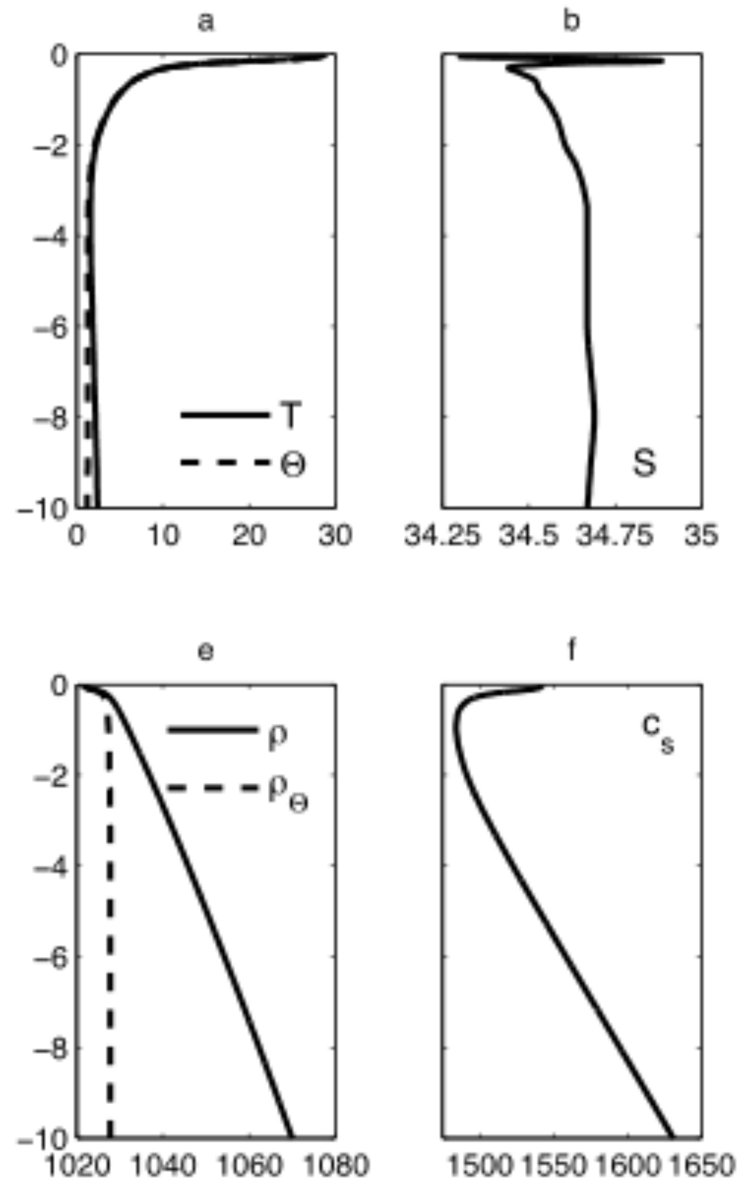
The parcel oscillates vertically at frequency *N* about its equilibrium position.

Local Static Stability

- Brunt-Vaisala frequency:

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)$$

The effect of compressibility is often neglected in the upper ocean but it is not true in general.



Vertical profiles of T, S, in-situ and potential density, and speed of sound in the Mindanao Trench.

Local Static Stability

- How do you connect density and stability?

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)$$

- It is convenient to define **the generalized potential density,** corresponding to the density that the parcel would attain if moved from z to a reference level z_r, under conservation of its entropy and salinity.

- We compute it by vertically integrating:

$$\delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_{\eta,S} \delta p = c_s^{-2}\delta p$$

Local Static Stability

- Which gives the generalized potential density:

$$\rho_r(z_r, z) = \rho_0(z) + g \int_{z_r}^{z} \frac{\rho_0(z')}{c_s^2(z, z')} dz'$$

- Such that:

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho_r}{dz}(z_r, z) + \frac{g^2}{\rho_0} \int_{z_r}^{z} \rho_0(z') \frac{\partial c_s^{-2}}{\partial z}(z, z') dz'$$

Local Static Stability

- In practice we use potential density (z_r=0):

$$\rho_\Theta(z) = \rho_0(z) + g\int_0^z \frac{\rho_0(z')}{c_s^2(z,z')}dz'$$

Which is the density that the parcel would acquire if adiabatically brought to the surface.

- Such that:

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_\Theta}{dz} + \frac{g^2}{\rho_0}\int_0^z \rho_0(z')\frac{\partial c_s^{-2}}{\partial z}(z,z')dz'$$

Equations for a stratified flow

- Navier-Stokes Equations:

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}} + \vec{\mathcal{D}}$$

Momentum equations

$$\frac{D\rho}{Dt} + \rho\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{DP}{Dt}$$

Thermodynamic ‘energy’
equation
(no diabatic effects)

Equations for a stratified flow

- Approximations for the momentum equation:

No forcings/dissipation

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \cancel{\vec{\mathcal{F}} + \vec{D}}$$

Equations for a stratified flow

- Boussinesq Approximation:

Density perturbations small compared to mean background value:

$$\rho - \rho^* << \rho^*$$

Linearize all terms involving a product with density, except the gravity term which is already linear:

$$\rho\vec{u} \rightarrow \rho^*\vec{u}$$

$$\rho g \rightarrow \rho g$$

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- Traditional Approximation:

= neglect horizontal Coriolis term

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

$$\frac{D\vec{u}}{Dt} + f\vec{k} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- We think as internal waves as a perturbation of a (known) static background state that has only vertical dependences

$$P = p_0(z) + p'(t, \vec{x})$$
$$\rho = \rho_0(z) + \rho'(t, \vec{x})$$

- And linearize momentum equations:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} + \frac{\rho'}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Equations for a stratified flow

- For the thermodynamic equation:

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{DP}{Dt}$$

- We can write:

$$\frac{D\rho}{Dt} = \frac{D\rho'}{Dt} + w\frac{\partial\rho_0}{\partial z}$$

- And:

$$\frac{1}{c_s^2}\frac{DP}{Dt} = \frac{1}{c_s^2}\frac{Dp'}{Dt} + \frac{1}{c_s^2}w\frac{\partial p_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

- So :

$$\frac{D\rho'}{Dt} + w\frac{\partial\rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

Equations for a stratified flow

- We linearize:

$$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

$$\Downarrow$$

$$\frac{\partial\rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

-

Equations for a stratified flow

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

- We can show that the pressure time tendencies are small using the following scalings and relations:

$$\boxed{U \ll C \ll c_{sf} \ll c_s\,.}$$

$$U \sim O(10^{-1})\,; \quad C \sim O(1)\,; \quad c_{sf} \sim O(10^1\,, 10^2)\,; \quad c_s \sim O(10^3)\,\mathrm{m\,s^{-1}}\,.$$

- Which denote the particle velocity and phase speed of internal waves, the phase speed of surface waves, and the speed of sound in seawater, respectively.

Equations for a stratified flow

- We get: $$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = -w\frac{\rho_0 g}{c_s^2}$$
- So we can write: $$\frac{\partial \rho'}{\partial t} + \left[\frac{\partial \rho_0}{\partial z} + \frac{\rho_0 g}{c_s^2}\right] w = 0$$
- Which gives $$-\frac{g}{\rho^*}\frac{\partial \rho'}{\partial t} + N^2 w = 0$$
- With $N^2 = -\frac{g}{\rho^*}\left(\frac{\partial \rho_0}{\partial z} + \frac{\rho_0 g}{c_s^2}\right)$

Equations for a stratified flow

- And finally introducing buoyancy: $$b = -g\frac{\rho'}{\rho^*}$$
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$$\frac{\partial b}{\partial t} + N^2 w = 0$$

Equations for a stratified flow

- For the mass equation:

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0$$

- We rewrite:

$$\rho\nabla\cdot\vec{u} = -\frac{D\rho}{Dt} = -\frac{1}{c_s^2}\frac{Dp'}{Dt} + w\frac{\rho_0 g}{c_s^2}$$

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Equations for a stratified flow

- We look at the scales of the different terms:

$$\rho\overbrace{\frac{\partial u}{\partial x}}^{\rho_* U/L}+\rho\overbrace{\frac{\partial v}{\partial y}}^{\rho_* U/L}+\rho\overbrace{\frac{\partial w}{\partial z}}^{\rho_* W/H}=-\overbrace{\frac{1}{c_s^2}\frac{\partial p'}{\partial t}}^{P/(Tc_s^2)}-\overbrace{\frac{u}{c_s^2}\frac{\partial p'}{\partial x}}^{UP/(Lc_s^2)}-\overbrace{\frac{v}{c_s^2}\frac{\partial p'}{\partial y}}^{UP/(Lc_s^2)}-\overbrace{\frac{w}{c_s^2}\frac{\partial p'}{\partial z}}^{WP/(Hc_s^2)}+\overbrace{w\frac{\rho_0 g}{c_s^2}}^{\rho_* Wg/c_s^2}$$

- We assume again a separation in time scales:

$$\boxed{U \ll C \ll c_{sf} \ll c_s\,.}$$

$$U \sim O(10^{-1})\,;\quad C\sim O(1)\,;\quad c_{sf}\sim O(10^1\,,10^2)\,;\quad c_s \sim O(10^3)\,\mathrm{m\,s^{-1}}\,.$$

- If we remove small terms we simply get: $\vec{\nabla}\cdot\vec{u}=0$

Equations for a stratified flow

- Linearized Equations are:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} - b\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

$$\frac{\partial b}{\partial t} + N^2 w = 0$$

Thermodynamic ‘energy’
equation
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Activity (for next session)

- Starting from linearized Equations :

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} - b\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

$$\vec{\nabla} \cdot \vec{u} = 0 \qquad \frac{\partial b}{\partial t} + N^2 w = 0$$

- Activity:

- Write an equation for w alone.