Numerical Modelling

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the anatomy of an ocean model

- **Lesson 1 :**
  - Introduction
  - Equations of motions
  - *Activity 1 [run an ocean model]*

- **Lesson 2 :** [B 012]
  - Horizontal Discretization
  - *Activity 2 [Dynamics of an ocean gyre]*

- **Lesson 3 :** [D109]
  - Numerical schemes
  - *Activity 3 [Impacts of numerics]*

- **Lesson 4 :** [D109]
  - Vertical coordinates
  - Model parameterizations
  - *Activity 4 [Impact of topography]*

- **Lesson 5 :** [D109]
  - Boundary Forcings
  - Presentation of the model CROCO
  - *Activity 4 [Design a realistic simulation]*

- **Lesson 6 :** [D109]
  - Diagnostics and validation
  - *Activity 5 [Analyze a realistic simulation]*

- **Lesson 7 :** [D109]
  - *Work on your projet*

Presentations and material will be available at :

jgula.fr/ModNum/

Useful references

Extensive courses:

- MIT: https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-950-atmospheric-and-oceanic-modeling-spring-2004/lecture-notes/
- Princeton: https://stephengriffies.github.io/assets/pdfs/GFM_lectures.pdf

Overview on ocean modelling and current challenges:

- Griffies et al., 2000, Developments in ocean climate modelling, Ocean Modelling.http://jgula.fr/ModNum/Griffiesetal00.pdf
- Griffies, 2006, "Some Ocean Model Fundamentals", In "Ocean Weather Forecasting: An Integrated View of Oceanography", 2006, Springer Netherlands. http://jgula.fr/ModNum/Griffies_Chapter.pdf
- Fox-Kemper et al, 19, “Challenges and Prospects in Ocean Circulation Models" http://jgula.fr/ModNum/FoxKemperetal19.pdf

ROMS/CROCO:

- https://www.myroms.org/wiki/
- Shchepetkin, A., and J. McWilliams, 2005: The Regional Oceanic Modeling System (ROMS): A split-explicit, free-surface, topography-following- coordinate ocean model. Ocean Modell. http://jgula.fr/ModNum/ShchepetkinMcWilliams05.pdf

#3 Discretization

Master's degree 2nd year Marine Physics

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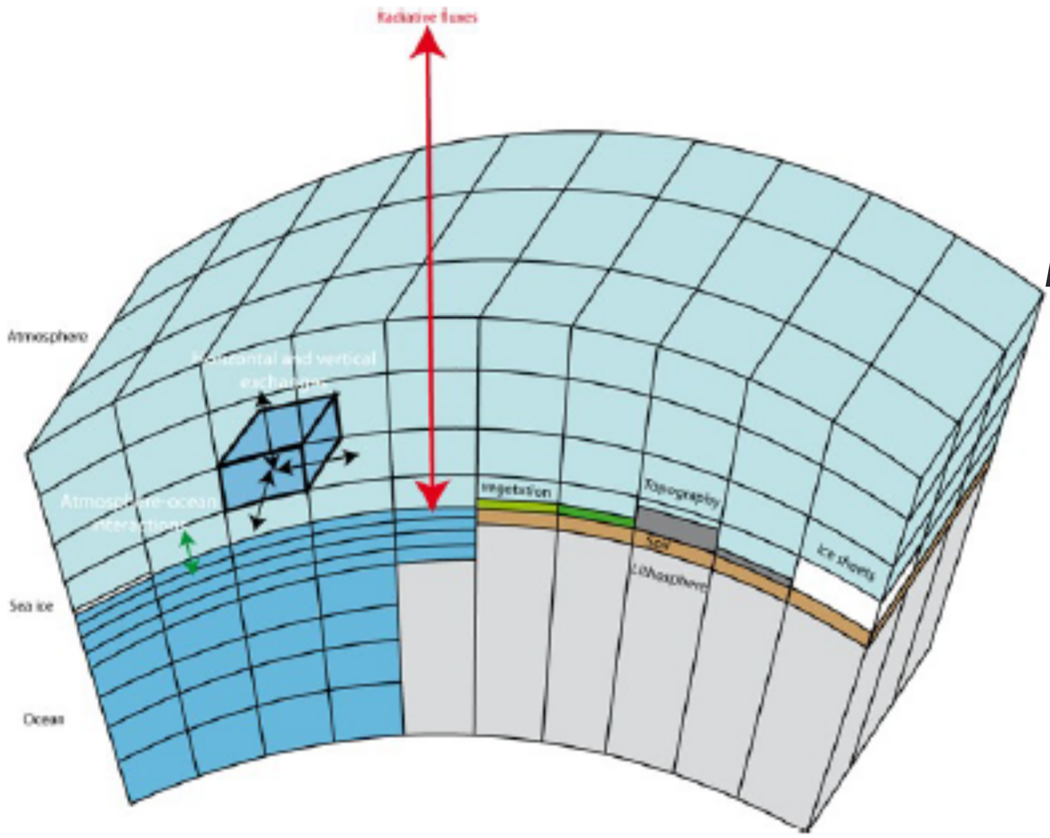
Discretization

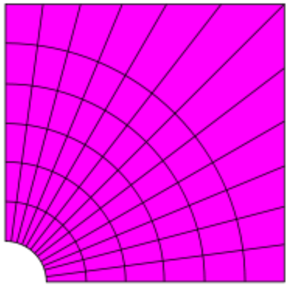
The ocean is divided into boxes : **discretization**



Example of a finite difference grid

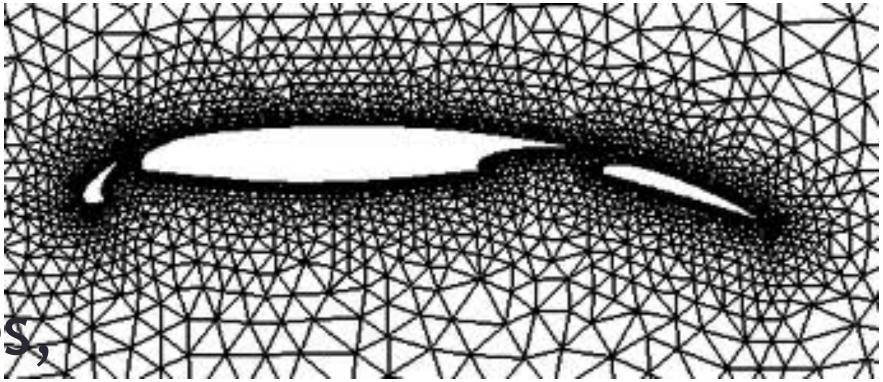
Discretization

Structured grids
Identified by regular connectivity
= can be adressed by (i,j,k)



ROMS

Unstructured grids
The domain is tiled using more general geometrical shapes (triangles, …) pieced together to optimally fit details of the geometry.



- ✓ Good for tidal modeling, engineering applications.
- ✓ Problems:
  geostrophic balance accuracy,
  conservation and positivity properties,
  …

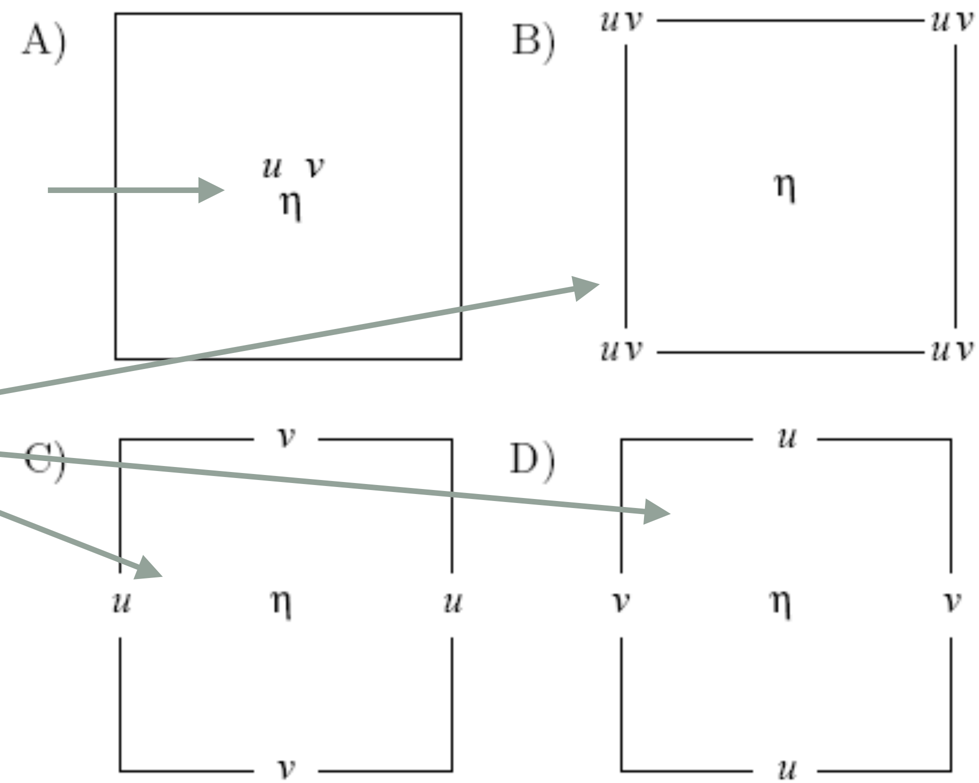
Horizontal discretization

Different types of Horizontal Grids (Arakawa Grids):

Non-staggered
(= collocated variables)

Or

Staggered



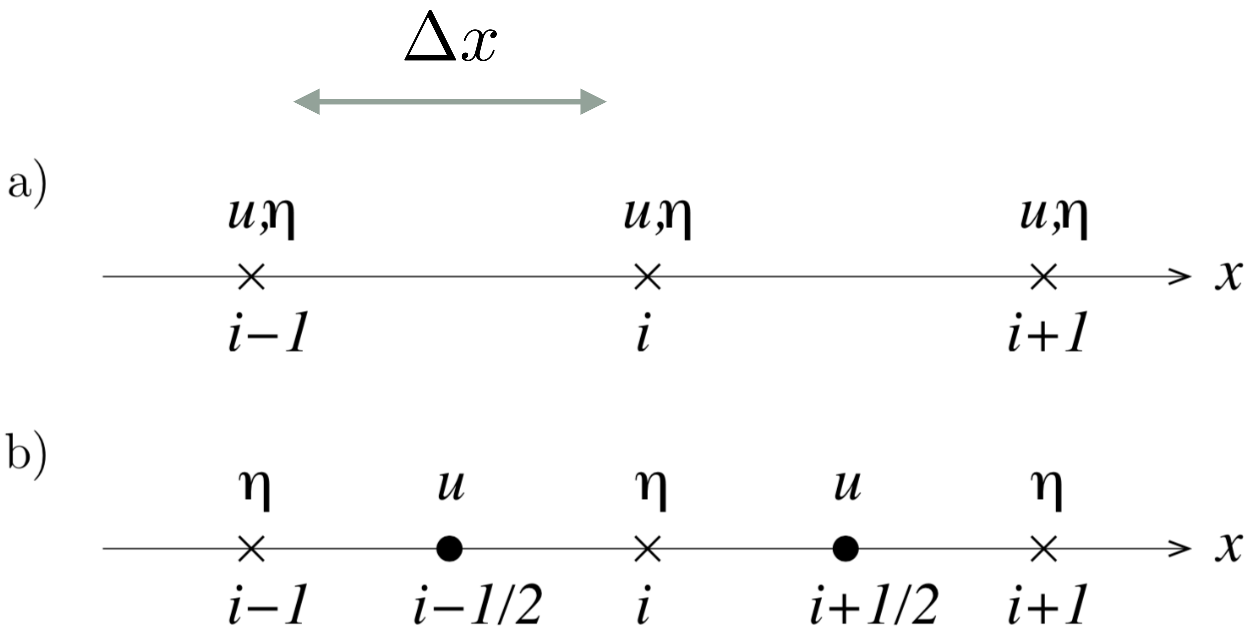
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

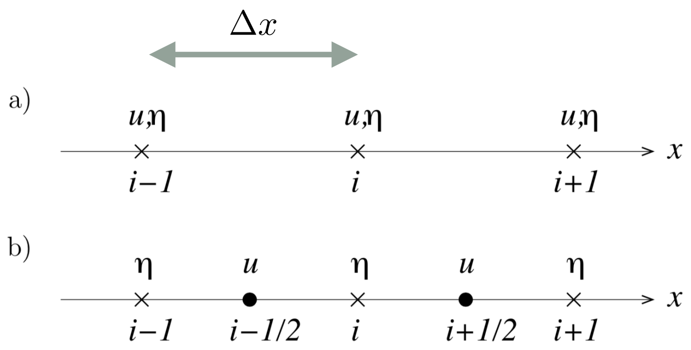
Non-staggered

Vs

Staggered



Horizontal discretization



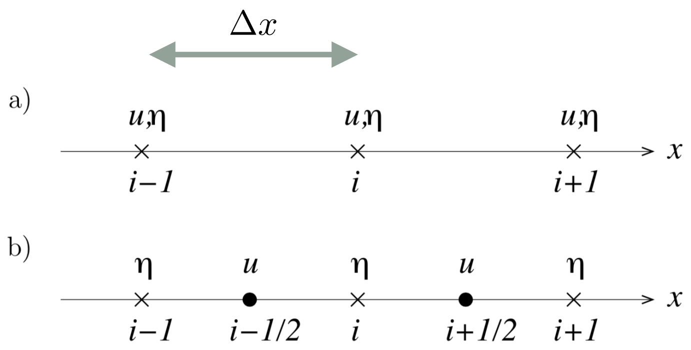
Staggered Vs unstaggered : the 1D problem

1. The advection equation $\partial_t\theta + c\partial_x\theta = 0$

Solutions of the continuous equations are non-dispersive waves

$\theta(x,t) = \theta_o e^{i(kx-\omega t)}$ *with dispersion relation* $\omega = ck$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

1. The advection equation $\partial_t \theta + c \partial_x \theta = 0$

*Solutions of the **continuous equations** are non-dispersive waves*

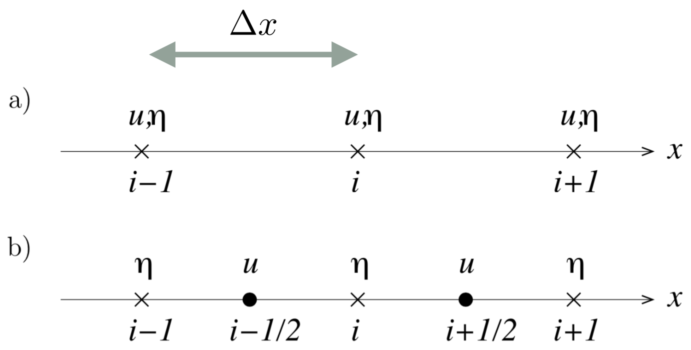
$\theta(x,t) = \theta_o e^{i(kx - \omega t)}$ *with dispersion relation* $\omega = ck$

Discretized equations with the centered second order derivative are:

$$d_t \theta + \frac{c}{\Delta x} \delta_i \overline{\theta}^i = 0$$

$$d_t \theta_i + \frac{c}{2\Delta x} \left(\theta_{i+1} - \theta_{i-1} \right) = 0$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

1. The advection equation $\partial_t\theta + c\partial_x\theta = 0$

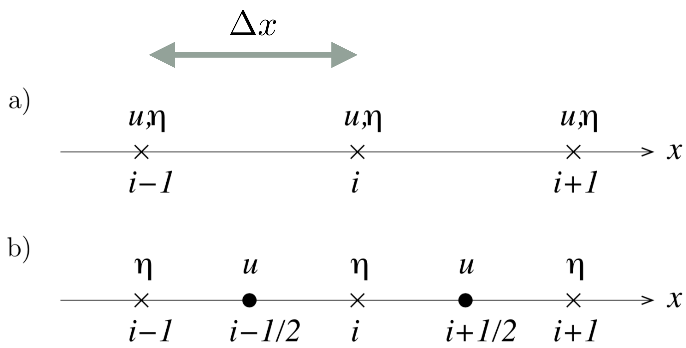
Substituting in our solution:

$$\theta_i(x,t) = \theta_0 e^{i(kx-\omega t)}$$

$$\theta_{i-1}(x,t) = \theta_0 e^{i(k(x-\Delta x)-\omega t)}$$

$$\theta_{i+1}(x,t) = \theta_0 e^{i(k(x+\Delta x)-\omega t)}$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

1. The advection equation $\partial_t \theta + c\partial_x \theta = 0$

Substituting in our solution gives:

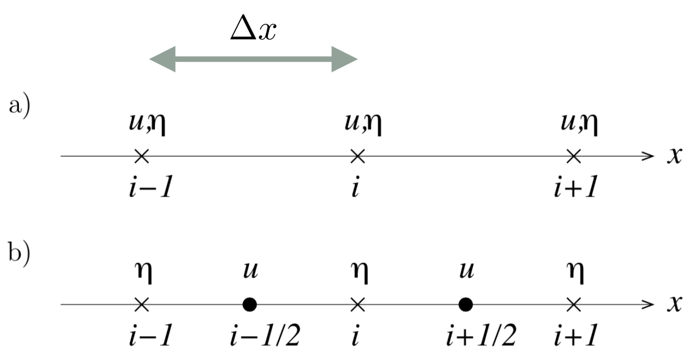
$$\begin{aligned} -i\omega &= -\frac{c}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) \\ &= -\frac{ci}{\Delta x}\sin k\Delta x \end{aligned}$$

Now the solution is **dispersive!!!**

Even if it will converge to the non-dispersive solution in the limit of small Δx

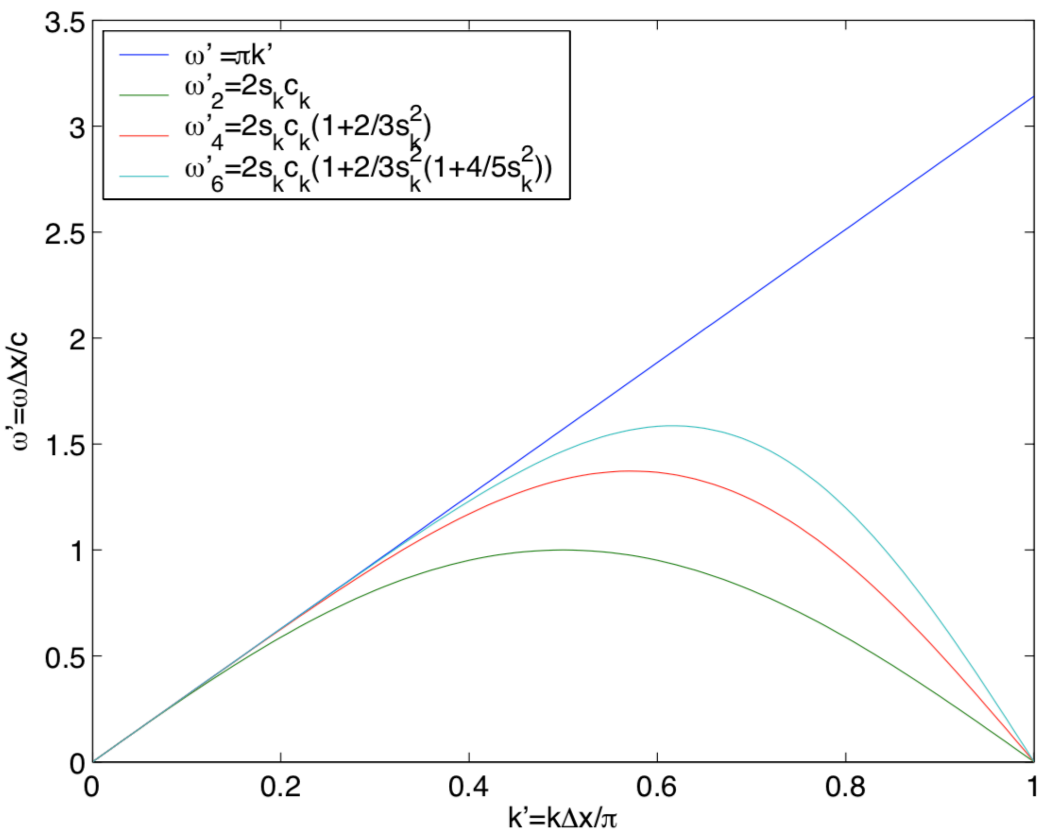
$$\omega = \frac{c}{\Delta x}\sin k\Delta x \overset{\Delta x \to 0}{=} ck$$

Horizontal discretization



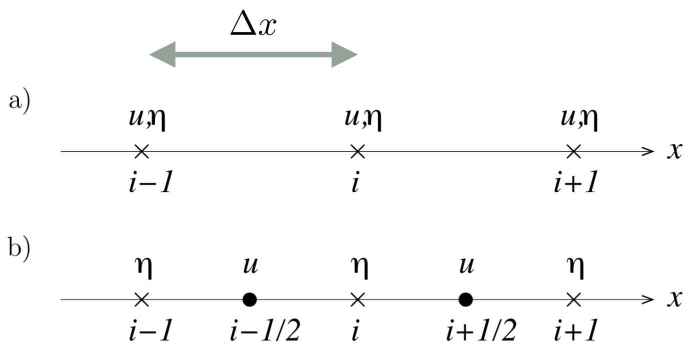
Staggered Vs unstaggered : the 1D problem

1. The advection equation $\partial_t \theta + c\partial_x \theta = 0$



Dispersion relations for constant flow advection using second, fourth, and sixth order spatial differences.

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

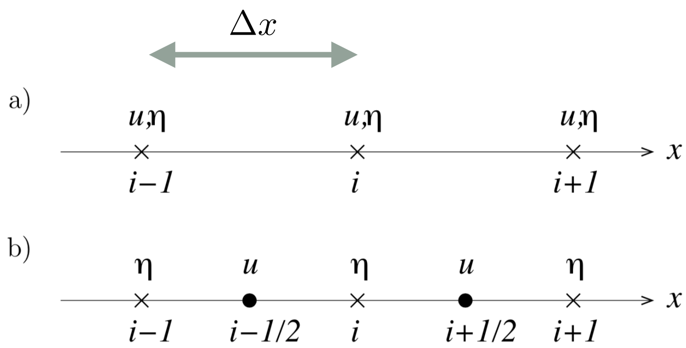
2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Solutions of the continuous equations are non-dispersive waves

$\eta = \eta_o e^{i(kx-\omega t)}$ *with dispersion relation* $\omega = \pm\sqrt{gH}k$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$
\begin{aligned}
\partial_t u &= -g\partial_x \eta \\
\partial_t \eta &= -H\partial_x u
\end{aligned}
\longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta
$$

Solutions of the continuous equations are non-dispersive waves

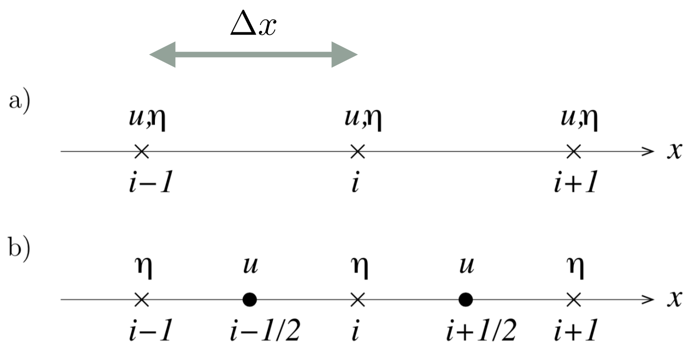
$\eta = \eta_o e^{i(kx-\omega t)}$ *with dispersion relation* $\omega = \pm\sqrt{gH}k$

Discretized equations with the centered second order derivative on the **unstaggered grid** are:

$$
\begin{aligned}
\partial_t u &= -\frac{g}{\Delta x}\delta_i \overline{\eta}^i \\
\partial_t \eta &= -\frac{H}{\Delta x}\delta_i \overline{u}^i
\end{aligned}
$$

$$\longrightarrow \partial_{tt}\eta = \frac{gH}{\Delta x^2}\delta_{ii}\overline{\eta}^{ii}$$ with $$\delta_{ii}\overline{\eta}^{ii} = \frac{1}{4}\left(\eta_{i-2} - 2\eta_i + \eta_{i+2}\right)$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

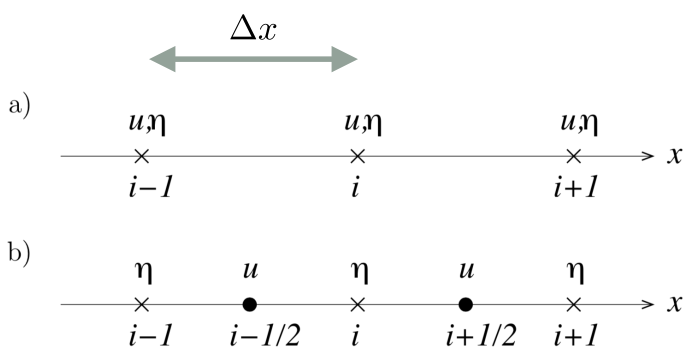
2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Substituting in our solution on the unstaggered grid gives :

$$\begin{aligned} -\omega^2 &= \frac{gH}{4\Delta x^2}\left(e^{-i2k\Delta x} - 2 + e^{i2k\Delta x}\right) \\ &= \frac{gH}{4\Delta x^2}\left(2\cos 2k\Delta x - 2\right) \\ &= -\frac{4gH}{\Delta x^2}\sin^2\frac{k\Delta x}{2}\cos^2\frac{k\Delta x}{2} \end{aligned}$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$\begin{aligned}\partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u\end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

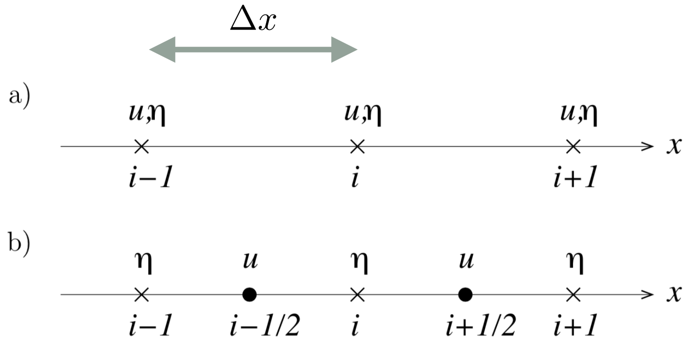
Substituting in our solution on the unstaggered grid gives :

$$\begin{aligned}-\omega^2 &= \frac{gH}{4\Delta x^2}\left(e^{-i2k\Delta x} - 2 + e^{i2k\Delta x}\right) \\ &= \frac{gH}{4\Delta x^2}\left(2\cos 2k\Delta x - 2\right) \\ &= -\frac{4gH}{\Delta x^2}\sin^2\frac{k\Delta x}{2}\cos^2\frac{k\Delta x}{2}\end{aligned}$$

- Question:
  - What is the dispersion relation on the staggered grid?

Horizontal discretization

Staggered Vs unstaggered : the 1D problem



2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

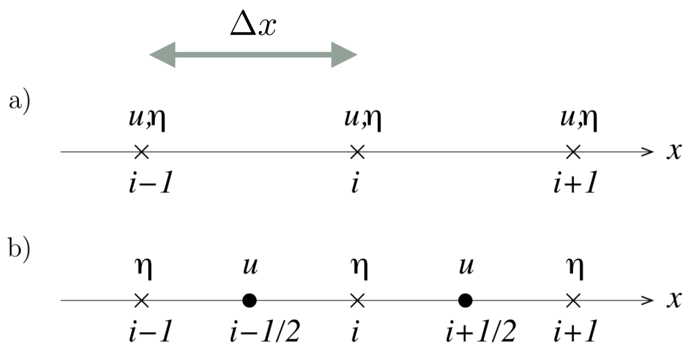
Discretized equations with the centered second order derivative on the **staggered grid** are:

$$\begin{aligned} \partial_t u &= -\frac{g}{\Delta x}\delta_i \eta \\ \partial_t \eta &= -\frac{H}{\Delta x}\delta_i u \end{aligned}$$

This can be written as a system:

$$\begin{pmatrix} \partial_t & \frac{g}{\Delta x}\delta_i \\ \frac{H}{\Delta x}\delta_i & \partial_t \end{pmatrix} \begin{pmatrix} u \\ \eta \end{pmatrix} = 0 \qquad \begin{pmatrix} -i\omega & \frac{2ig}{\Delta x}\sin\frac{k\Delta x}{2} \\ \frac{2iH}{\Delta x}\sin\frac{k\Delta x}{2} & -i\omega \end{pmatrix} \begin{pmatrix} u \\ \eta \end{pmatrix} = 0$$

Horizontal discretization



Staggered Vs unstaggered : the 1D problem

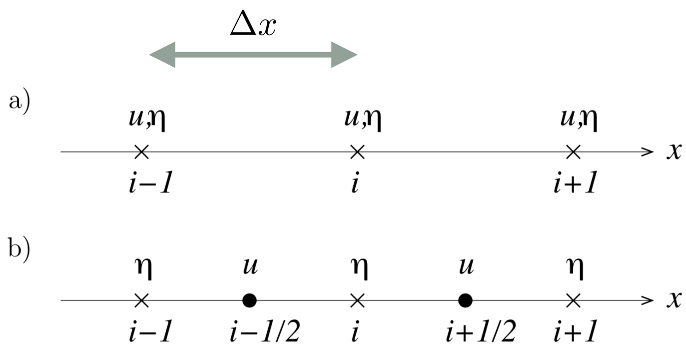
2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$

Substituting in our solution on the staggered grid gives :

$$\omega^2 - \frac{4gH}{\Delta x^2}\sin^2\frac{k\Delta x}{2} = 0$$

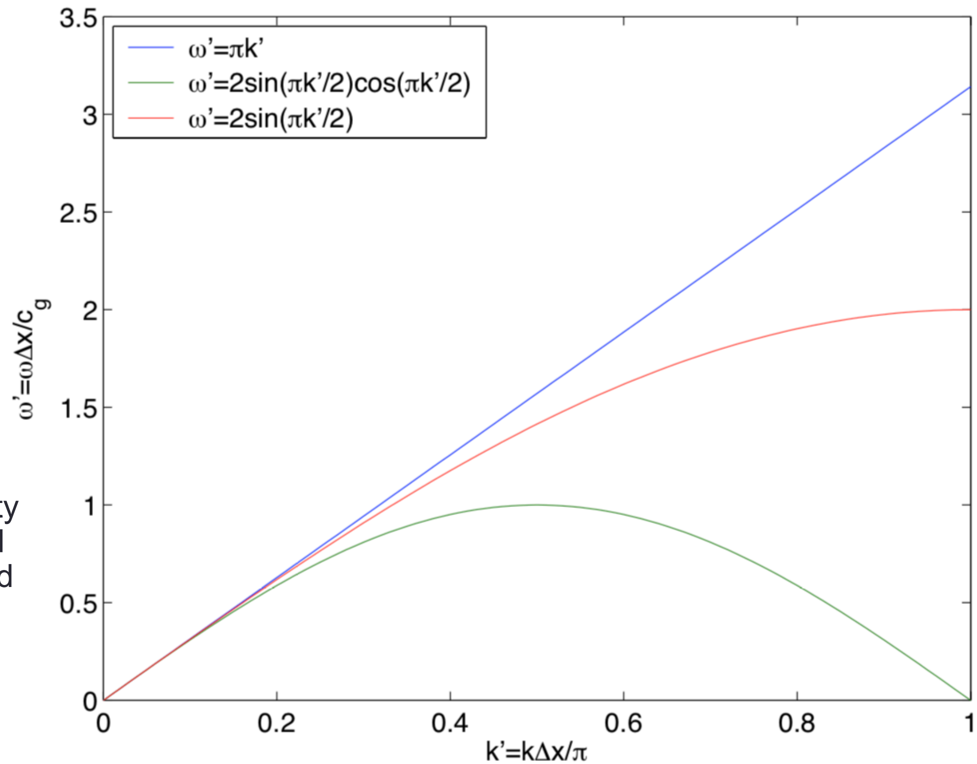
Horizontal discretization



Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$\begin{aligned} \partial_t u &= -g\partial_x \eta \\ \partial_t \eta &= -H\partial_x u \end{aligned} \longrightarrow \partial_{tt}\eta = gH\partial_{xx}\eta$$



Dispersion of numerical gravity wave for the unstaggered grid (green) and the staggered grid (red). The continuum (= k) is plotted for comparison (blue).

When compared to the continuum we see that the numerical modes are still dispersive on the staggered grid, but:

there is no false extrema, unlike the non-staggered grid,

the group speed is of the correct sign everywhere, even if reduced.

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

$$\begin{aligned} \partial_t u - fv + g\partial_x \eta &= 0 \\ \partial_t v + fu &= 0 \\ \partial_t \eta + H\partial_x u &= 0 \end{aligned}$$

Solutions of the continuous equations are waves following the dispersion relation:

$$\left| \begin{pmatrix} -i\omega & -f & gik \\ f & -i\omega & 0 \\ Hik & 0 & -i\omega \end{pmatrix} \right| = 0 \Rightarrow \begin{cases} \omega = 0 \\ \omega^2 = f^2 + gHk^2 \end{cases}$$

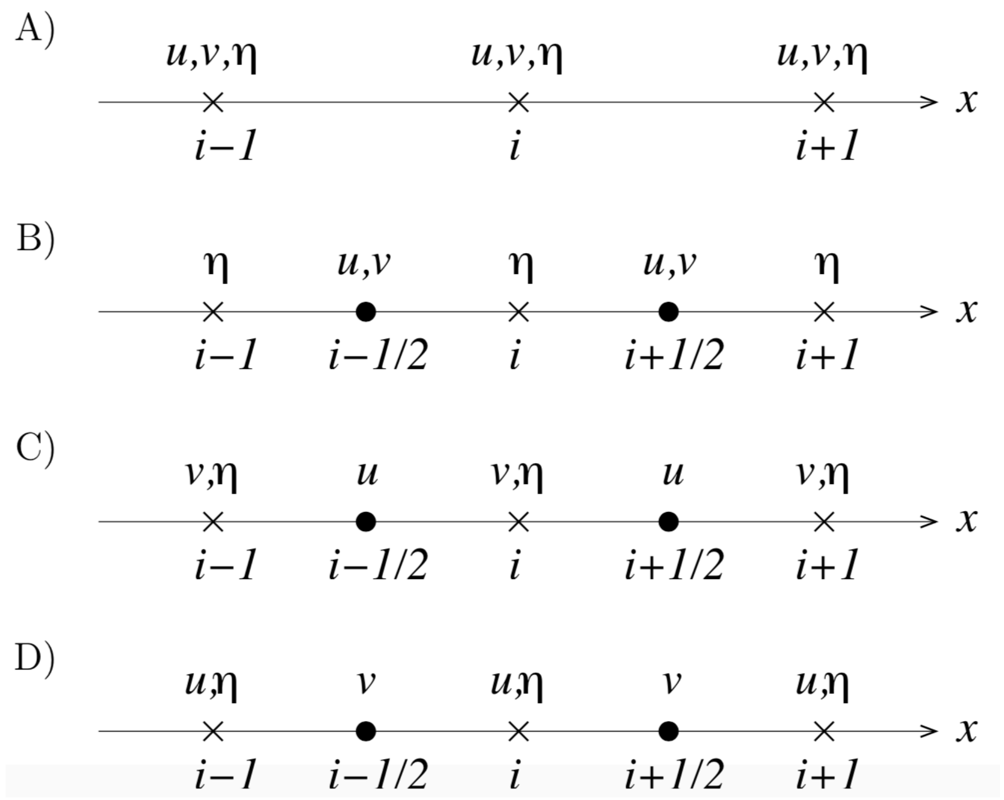
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

$$\begin{aligned} \partial_t u - fv + g\partial_x \eta &= 0 \\ \partial_t v + fu &= 0 \\ \partial_t \eta + H\partial_x u &= 0 \end{aligned}$$

Now, 4 different grids are possible:



Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

- A-grid model

$$\begin{aligned} \partial_t u - f v + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\ \partial_t v + f u &= 0 \\ \partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i &= 0 \end{aligned}$$

- B-grid model

$$\begin{aligned} \partial_t u - f v + \frac{g}{\Delta x}\delta_i \eta &= 0 \\ \partial_t v + f u &= 0 \\ \partial_t \eta + \frac{H}{\Delta x}\delta_i u &= 0 \end{aligned}$$

- C-grid model

$$\begin{aligned} \partial_t u - f \overline{v}^i + \frac{g}{\Delta x}\delta_i \eta &= 0 \\ \partial_t v + f \overline{u}^i &= 0 \\ \partial_t \eta + \frac{H}{\Delta x}\delta_i u &= 0 \end{aligned}$$

- D-grid model

$$\begin{aligned} \partial_t u - f \overline{v}^i + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\ \partial_t v + f \overline{u}^i &= 0 \\ \partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i &= 0 \end{aligned}$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

The corresponding dispersion relations are :

$$\text{A:} \quad \frac{\omega^2}{f^2} = 1 + \frac{4L_d^2}{\Delta x^2} s_k^2 c_k^2$$

$$\text{B:} \quad \frac{\omega^2}{f^2} = 1 + \frac{4L_d^2}{\Delta x^2} s_k^2$$

$$\text{C:} \quad \frac{\omega^2}{f^2} = c_k^2 + \frac{4L_d^2}{\Delta x^2} s_k^2$$

$$\text{D:} \quad \frac{\omega^2}{f^2} = c_k^2 + \frac{4L_d^2}{\Delta x^2} s_k^2 c_k^2$$

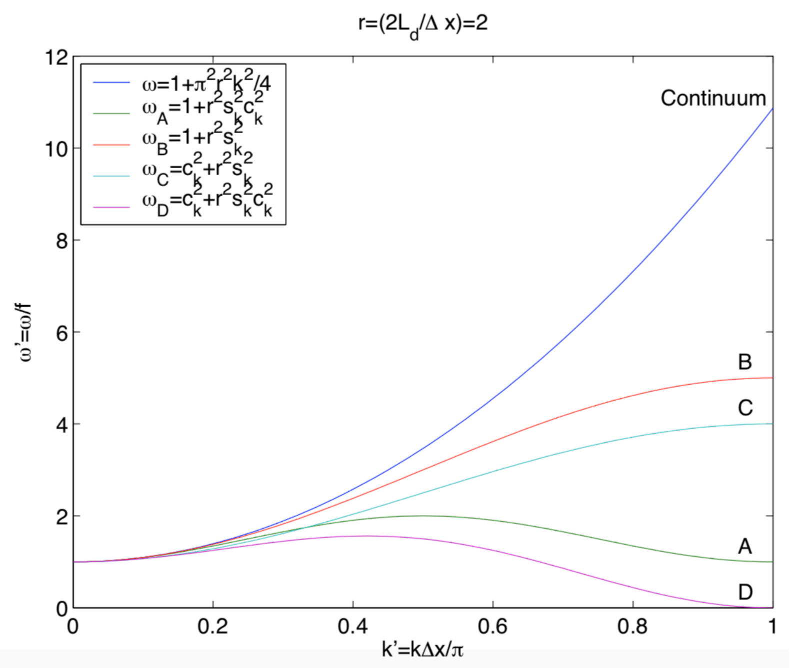
$$s_k = \sin \frac{k\Delta x}{2} \qquad c_k = \cos \frac{k\Delta x}{2}$$

$$L_d = \sqrt{gH}/f$$

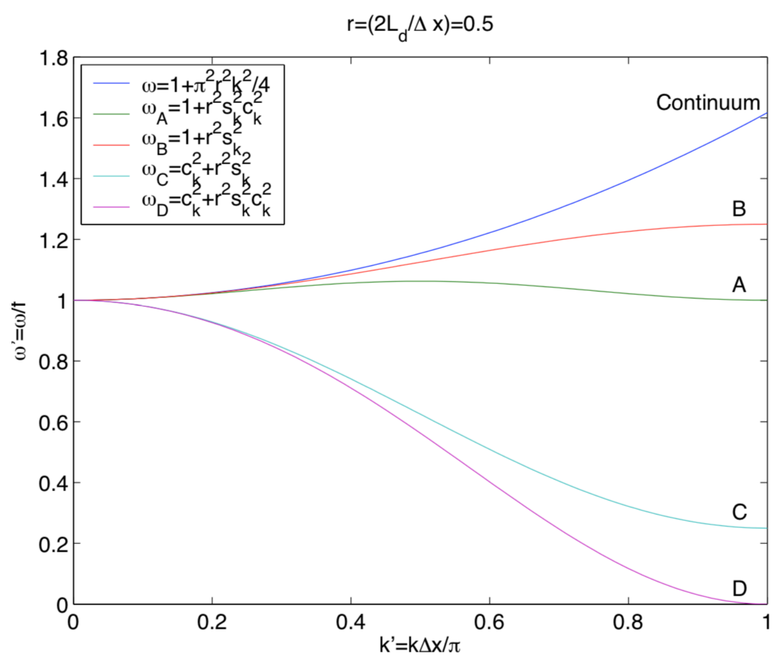
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves



deformation radius is resolved

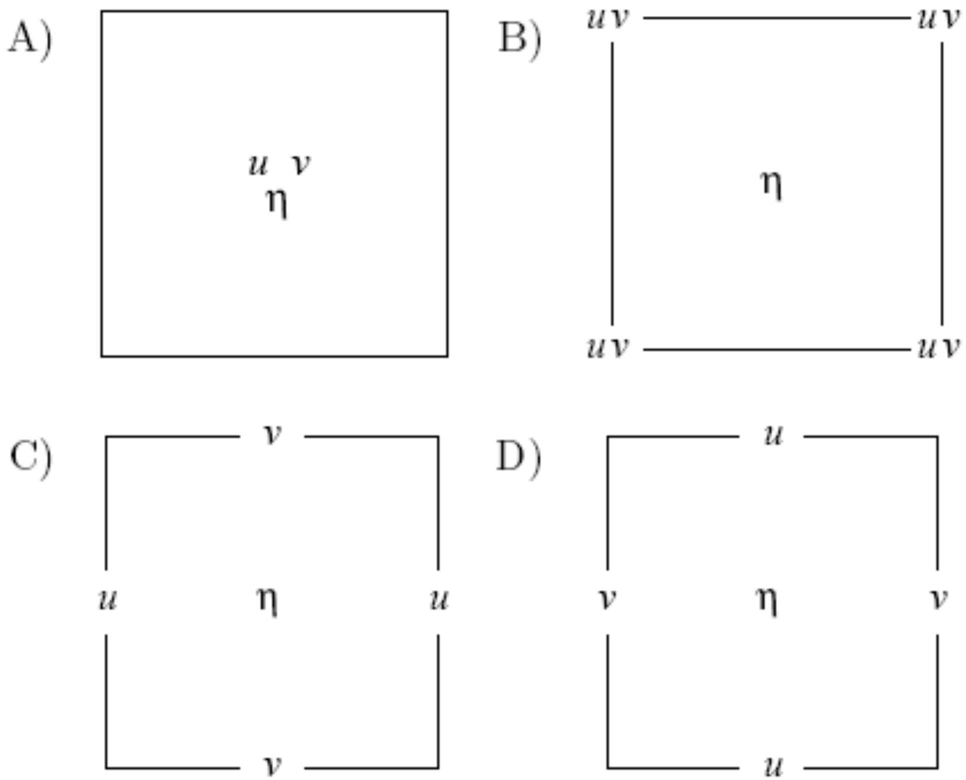


deformation radius is not resolved

Staggering variables in the form of the B grid is most likely to avoid computational modes when solving one-dimensional shallow water equations.

Horizontal discretization

Horizontal Arakawa Grids:

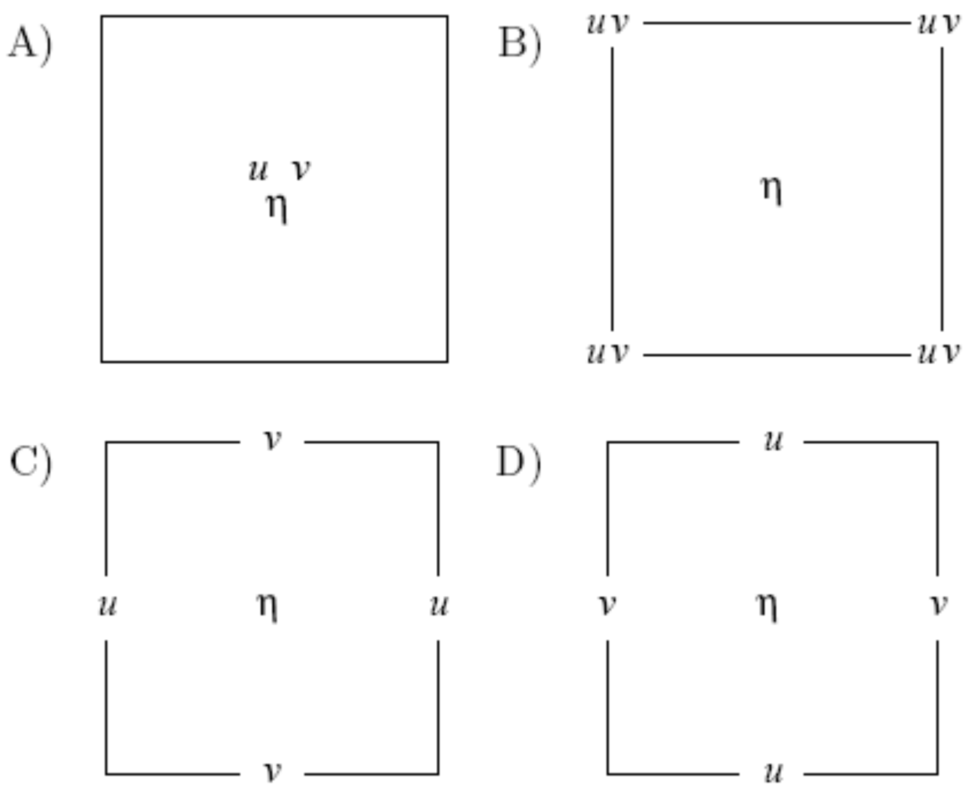


Linear shallow water equation:

$$\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f u + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_j v &= 0
\end{aligned}$$

Horizontal discretization

Horizontal Arakawa Grids:

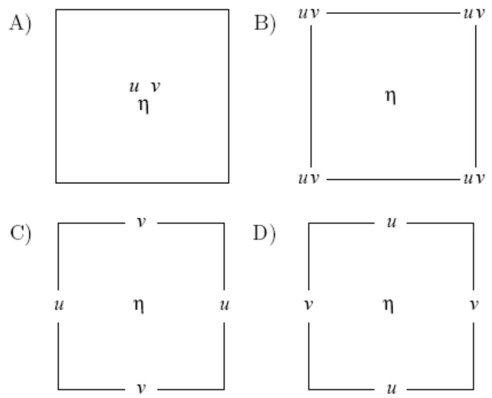


Linear shallow water equation:

$$
\begin{aligned}
\partial_t u - fv + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + fu + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_y \eta &= 0
\end{aligned}
$$

- Question:
  - Which grid minimises the number of averaging between points when solving linear SW equations in 2d?

Horizontal discretization



- A grid:

$$\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \overline{\eta}^i &= 0 \\
\partial_t v + f u + \frac{g}{\Delta y}\delta_j \overline{\eta}^j &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^i + \frac{H}{\Delta y}\delta_j \overline{v}^j &= 0
\end{aligned}$$

- B grid:

$$\begin{aligned}
\partial_t u - f v + \frac{g}{\Delta x}\delta_i \overline{\eta}^j &= 0 \\
\partial_t v + f u + \frac{g}{\Delta y}\delta_j \overline{\eta}^i &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^j + \frac{H}{\Delta y}\delta_j \overline{v}^i &= 0
\end{aligned}$$

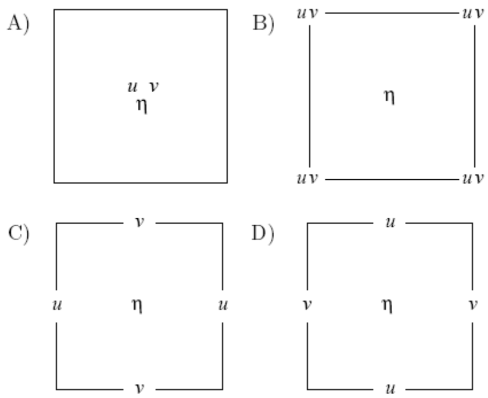
- C grid:

$$\begin{aligned}
\partial_t u - f \overline{v}^{ij} + \frac{g}{\Delta x}\delta_i \eta &= 0 \\
\partial_t v + f \overline{u}^{ij} + \frac{g}{\Delta y}\delta_j \eta &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i u + \frac{H}{\Delta y}\delta_j v &= 0
\end{aligned}$$

- D grid:

$$\begin{aligned}
\partial_t u - f \overline{v}^{ij} + \frac{g}{\Delta x}\delta_i \overline{\eta}^{ij} &= 0 \\
\partial_t v + f \overline{u}^{ij} + \frac{g}{\Delta y}\delta_j \overline{\eta}^{ij} &= 0 \\
\partial_t \eta + \frac{H}{\Delta x}\delta_i \overline{u}^{ij} + \frac{H}{\Delta y}\delta_j \overline{v}^{ij} &= 0
\end{aligned}$$

Horizontal discretization



Response of each operator:

$$
\begin{aligned}
R(\delta_i \phi) &= 2i \sin \frac{k\Delta x}{2} = 2i s_k \\
R(\delta_j \phi) &= 2i \sin \frac{l\Delta y}{2} = 2i s_l \\
R(\overline{\phi}^i) &= \cos \frac{k\Delta x}{2} = c_k \\
R(\overline{\phi}^j) &= \cos \frac{l\Delta y}{2} = c_l
\end{aligned}
$$

Dispersion relations:

- A grid:

$$
\begin{aligned}
\omega^2 &= f^2 + \frac{4gH}{\Delta x^2} s_k^2 c_k^2 + \frac{4gH}{\Delta y^2} s_l^2 c_l^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= 1 + r_x^2 s_k^2 c_k^2 + r_y^2 s_l^2 c_l^2
\end{aligned}
$$

- B grid:

$$
\begin{aligned}
\omega^2 &= f^2 + \frac{4gH}{\Delta x^2} s_k^2 c_l^2 + \frac{4gH}{\Delta y^2} s_l^2 c_k^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= 1 + r_x^2 s_k^2 c_l^2 + r_y^2 s_l^2 c_k^2
\end{aligned}
$$

- C grid:

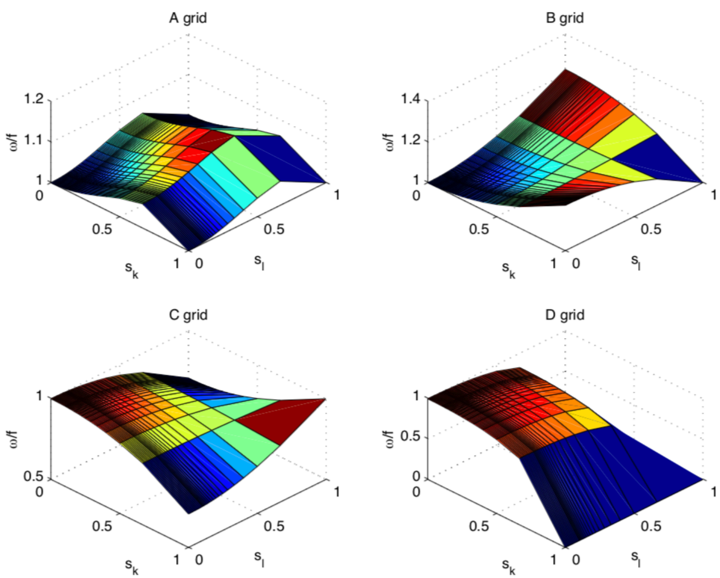
$$
\begin{aligned}
\omega^2 &= f^2 c_k^2 c_l^2 + \frac{4gH}{\Delta x^2} s_k^2 + \frac{4gH}{\Delta y^2} s_l^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= c_k^2 c_l^2 + r_x^2 s_k^2 + r_y^2 s_l^2
\end{aligned}
$$

- D grid:

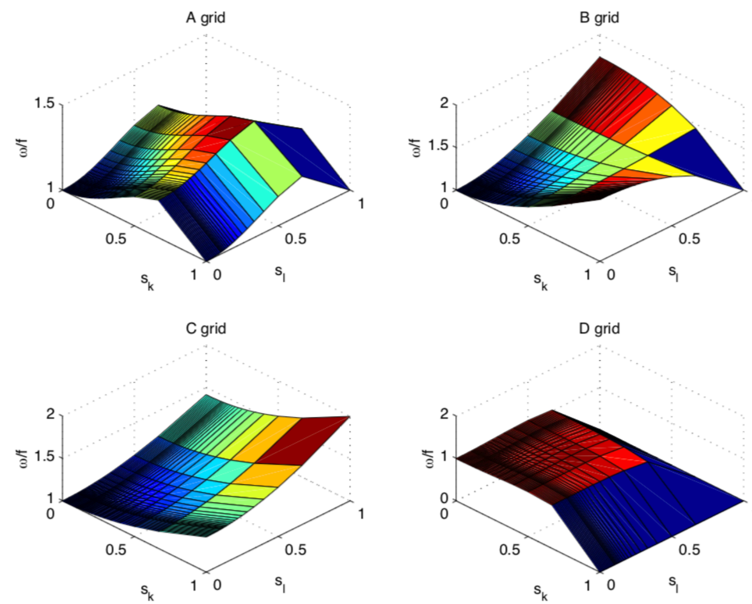
$$
\begin{aligned}
\omega^2 &= f^2 c_k^2 c_l^2 + \frac{4gH}{\Delta x^2} s_k^2 c_k^2 c_l^2 + \frac{4gH}{\Delta y^2} s_l^2 c_k^2 c_l^2 \\
\text{or} \qquad \left(\frac{\omega}{f}\right)^2 &= (1 + r_x^2 s_k^2 + r_y^2 s_l^2) c_k^2 c_l^2
\end{aligned}
$$

Horizontal discretization

Coarse resolution:



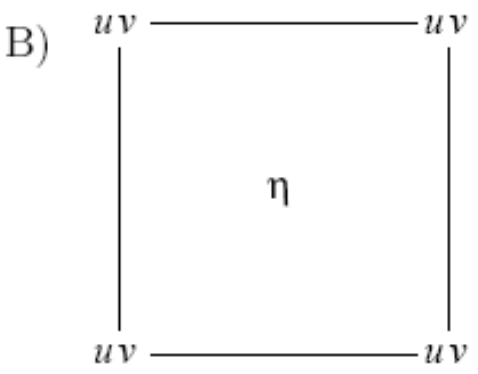
High resolution:



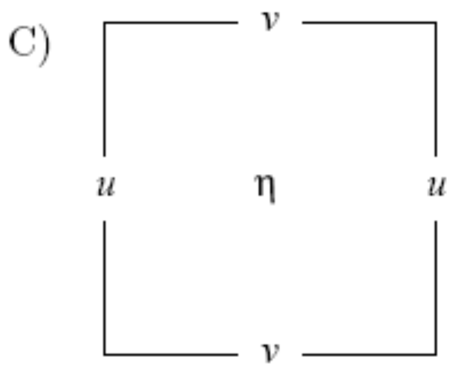
D is always bad.
B underestimates frequency for short two-dimensional waves
C is the only grid with monotonically increasing frequency (i.e. right sign of group velocity) at high res.

Horizontal discretization

- B grid is prefered at coarse resolution, when Coriolis is important:
  - Superior for poorly resolved inertia-gravity waves.
  - Good for Rossby waves: collocation of velocity points.
  - Bad for gravity waves: computational checkerboard mode



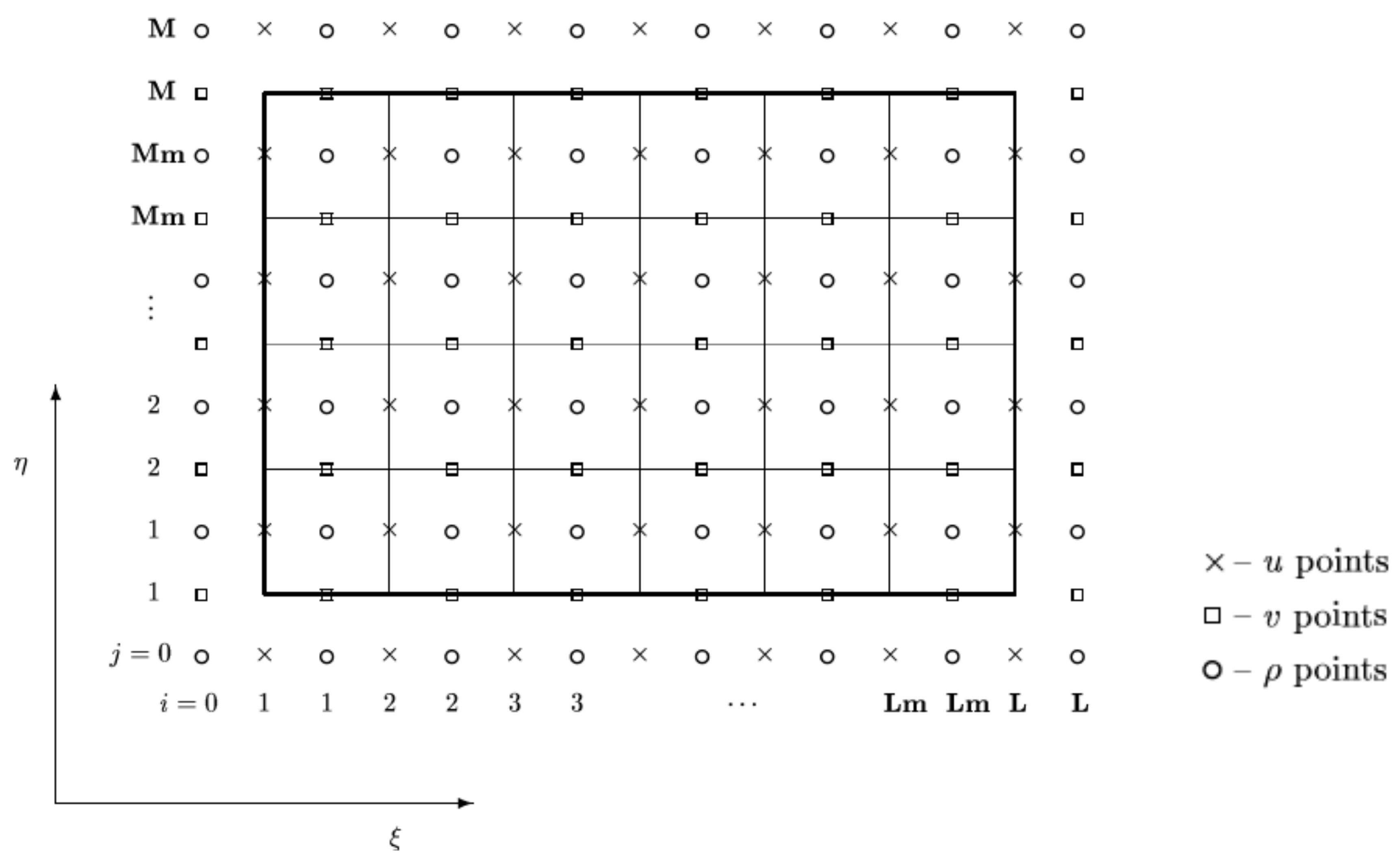
- C grid is prefered at fine resolution, when Coriolis is less important
  - Superior for gravity waves.
  - Good for well resolved inertia-gravity waves.
  - Bad for poorly resolved waves: Rossby waves (computational checkerboard mode) and inertia-gravity waves due to averaging the Coriolis force.



ROMS

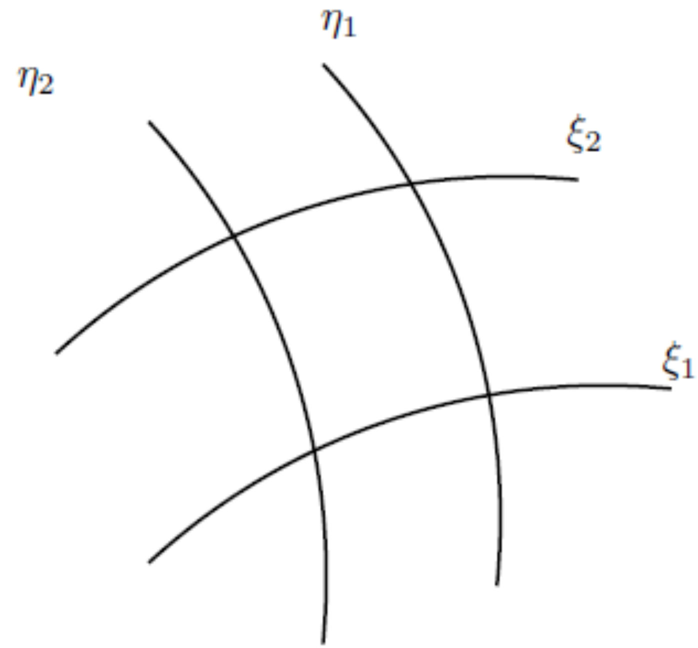
Horizontal discretization

ROMS: Arakawa C-grid



Horizontal curvilinear g

- **ROMS:** is formulated in general horizontal curvilinear coordinates:



$$(ds)_\xi = \left(\frac{1}{m}\right) d\xi$$

$$(ds)_\eta = \left(\frac{1}{n}\right) d\eta$$

$$\vec{v} \cdot \hat{\xi} = u$$

m, n: scale factors relating the differential distances to the physical arc lengths

$$\vec{v} \cdot \hat{\eta} = v$$

Horizontal curvilinear grid

- **ROMS:** is formulated in general horizontal curvilinear coordinates:

$$(ds)_\xi = \left(\frac{1}{m}\right) d\xi$$

$$(ds)_\eta = \left(\frac{1}{n}\right) d\eta$$

With classical formulas for div, grad, curl and lap in curvilinear coordinates:

$$\nabla\phi = \hat{\xi} m \frac{\partial \phi}{\partial \xi} + \hat{\eta} n \frac{\partial \phi}{\partial \eta}$$

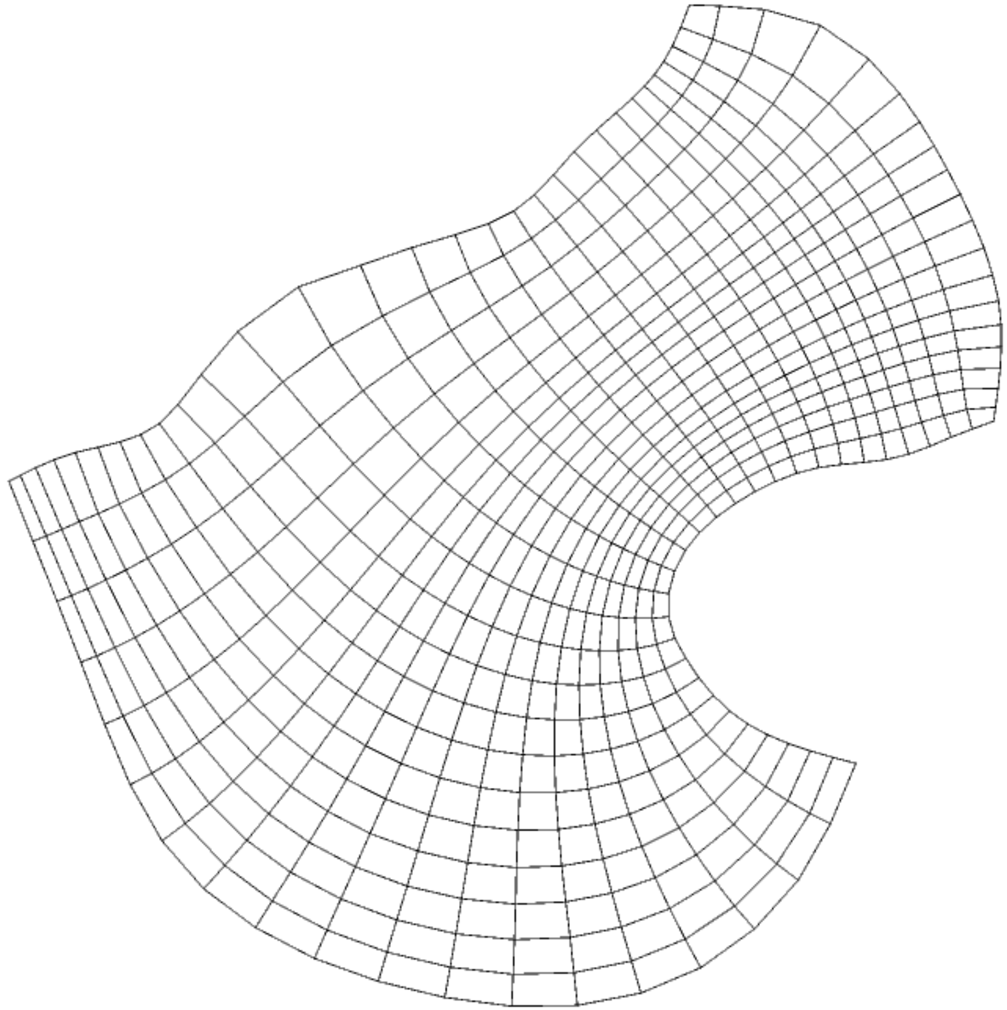
$$\nabla \cdot \vec{a} = mn \left[\frac{\partial}{\partial \xi}\left(\frac{a}{n}\right) + \frac{\partial}{\partial \eta}\left(\frac{b}{m}\right) \right]$$

$$\nabla \times \vec{a} = mn \begin{vmatrix} \frac{\hat{\xi}_1}{m} & \frac{\hat{\xi}_2}{n} & \hat{k} \\ \frac{\partial}{\partial \xi} & \frac{\partial}{\partial \eta} & \frac{\partial}{\partial z} \\ \frac{a}{m} & \frac{b}{n} & c \end{vmatrix}$$

$$\nabla^2\phi = \nabla \cdot \nabla\phi = mn \left[\frac{\partial}{\partial \xi}\left(\frac{m}{n}\frac{\partial \phi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{n}{m}\frac{\partial \phi}{\partial \eta}\right) \right]$$

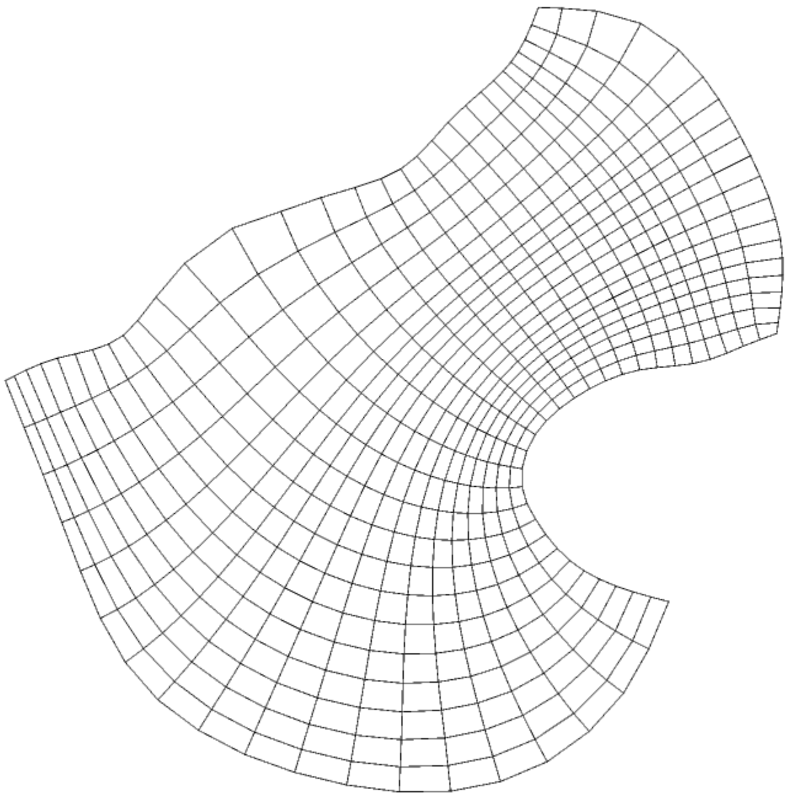
Horizontal curvilinear grid

- This is a possible grid:



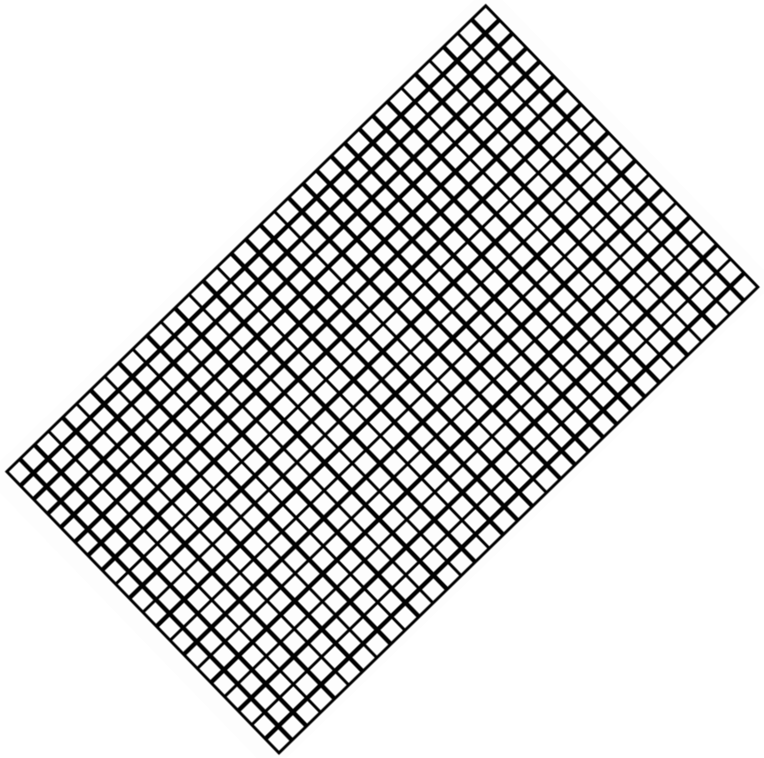
Horizontal curvilinear grid

- This is a possible grid:



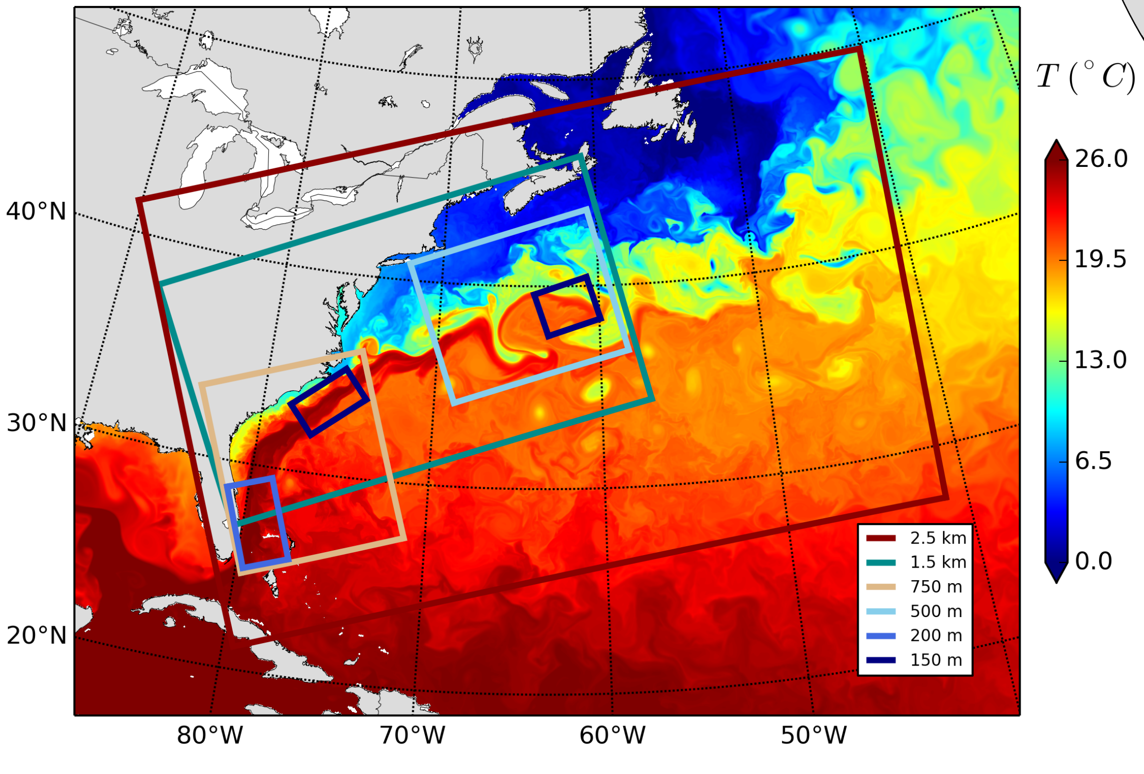
In practice variations in *dx* and *dy* should be minimized to minimize errors and optimize computation time.

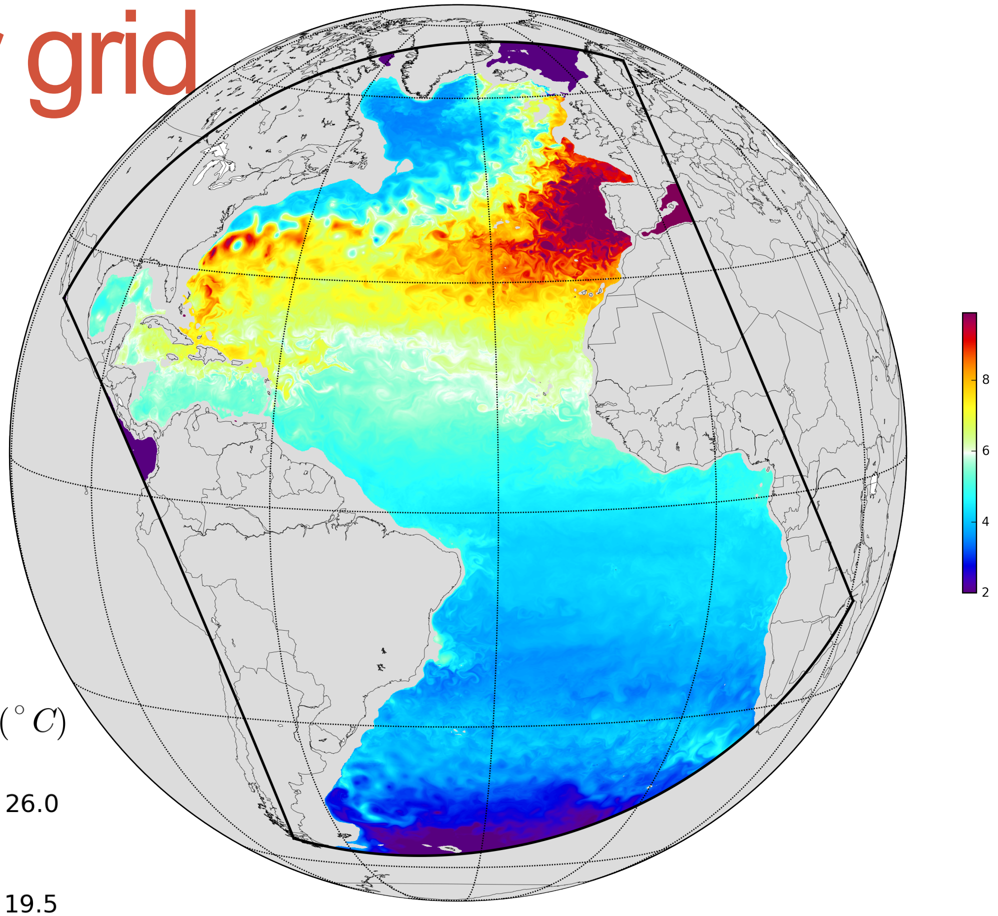
So avoir extreme distorsions and be as close as rectangular grids as possible (+ use land masks)



Horizontal curvilinear grid

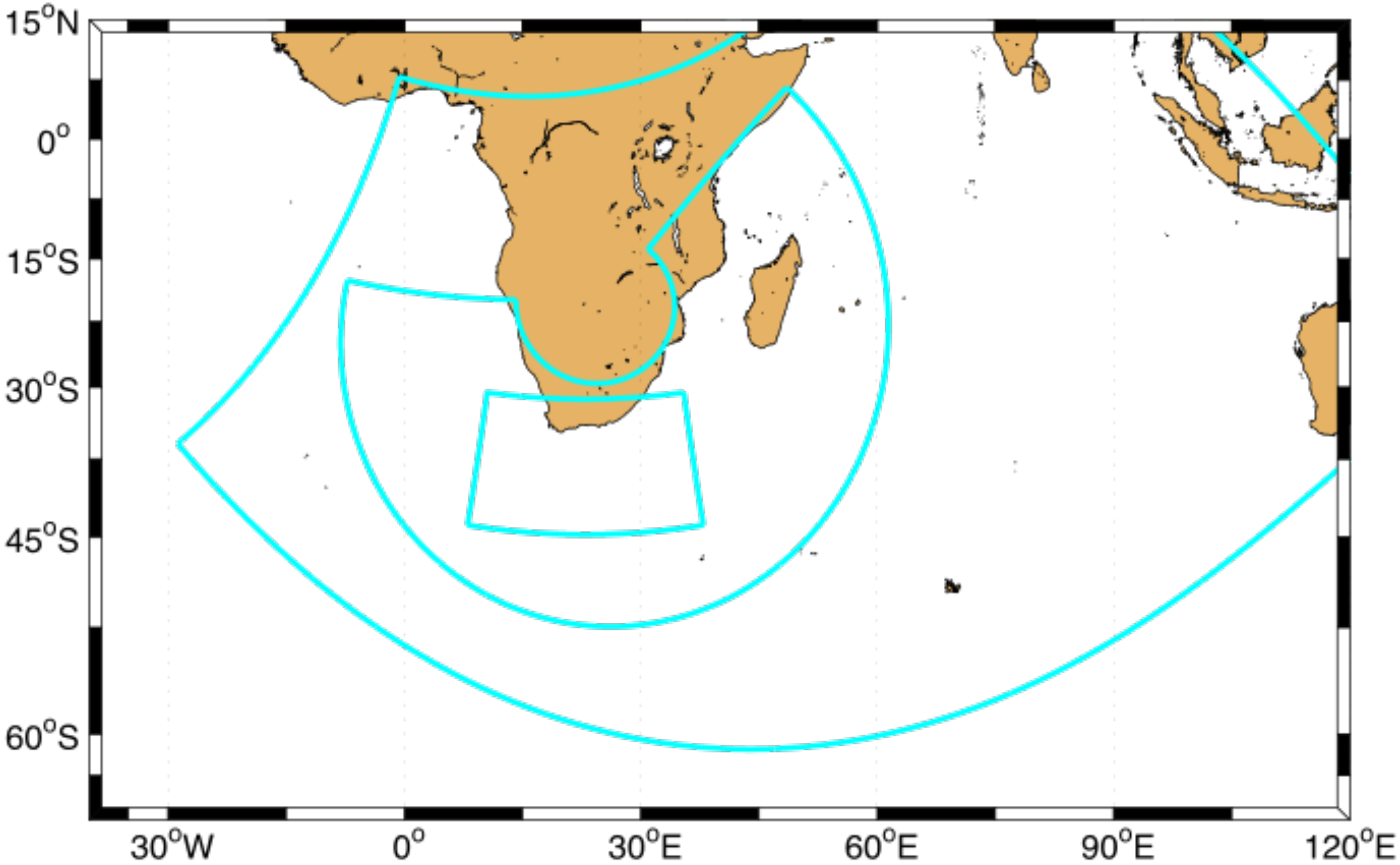
- Example of realistic domains:





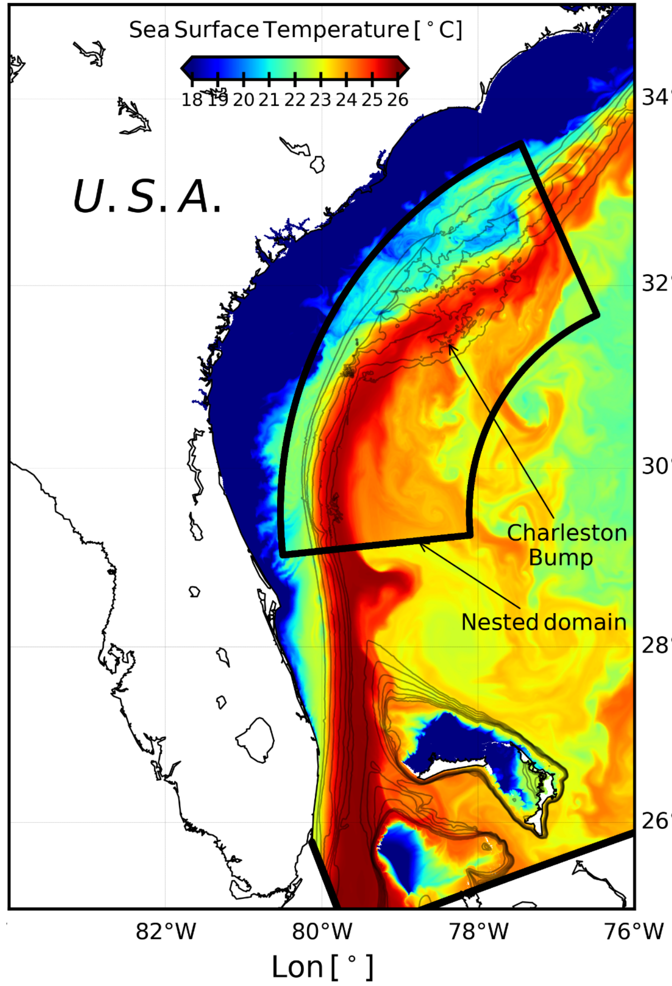
Horizontal curvilinear grid

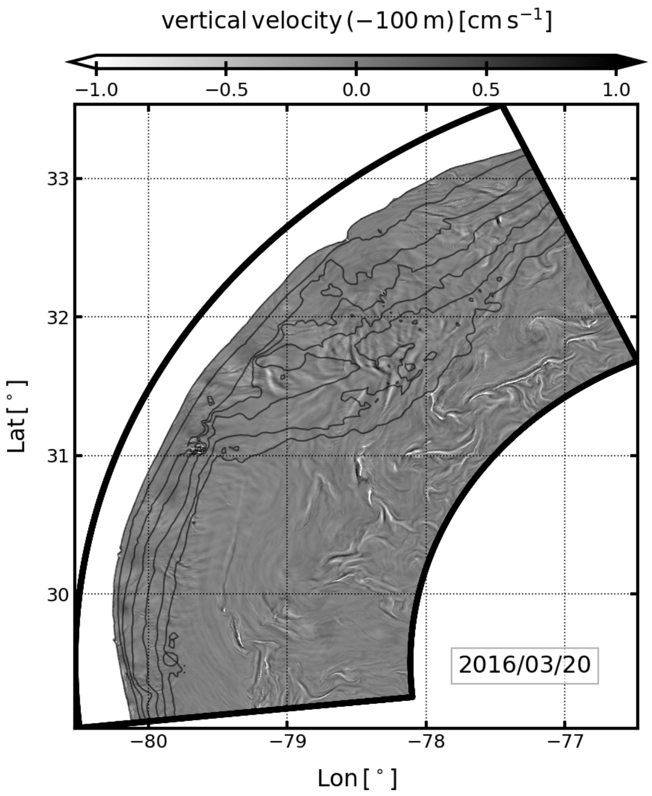
- Example of realistic domains (with gentle bendings):



Horizontal curvilinear grid

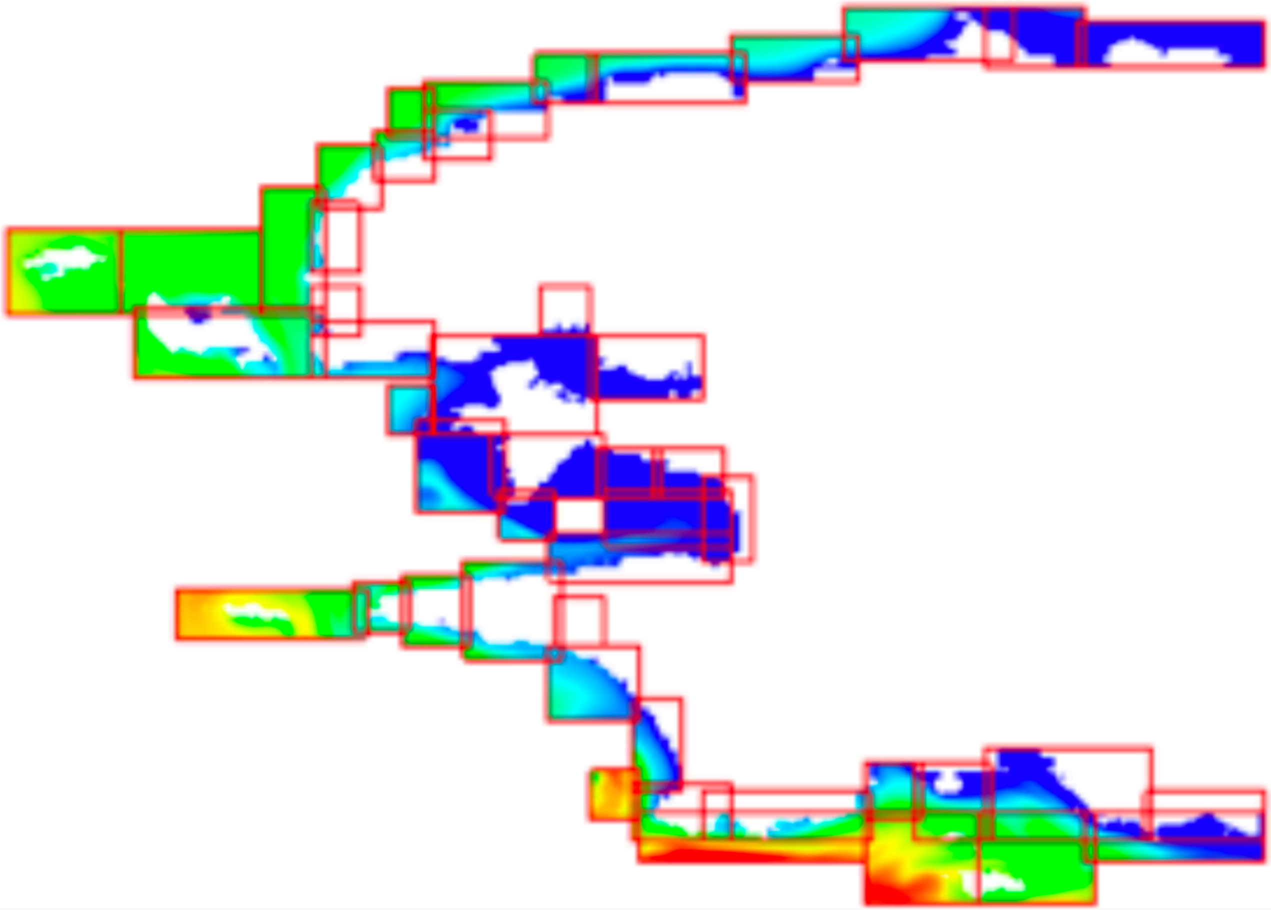
- Example of realistic domains (with gentle bendings):





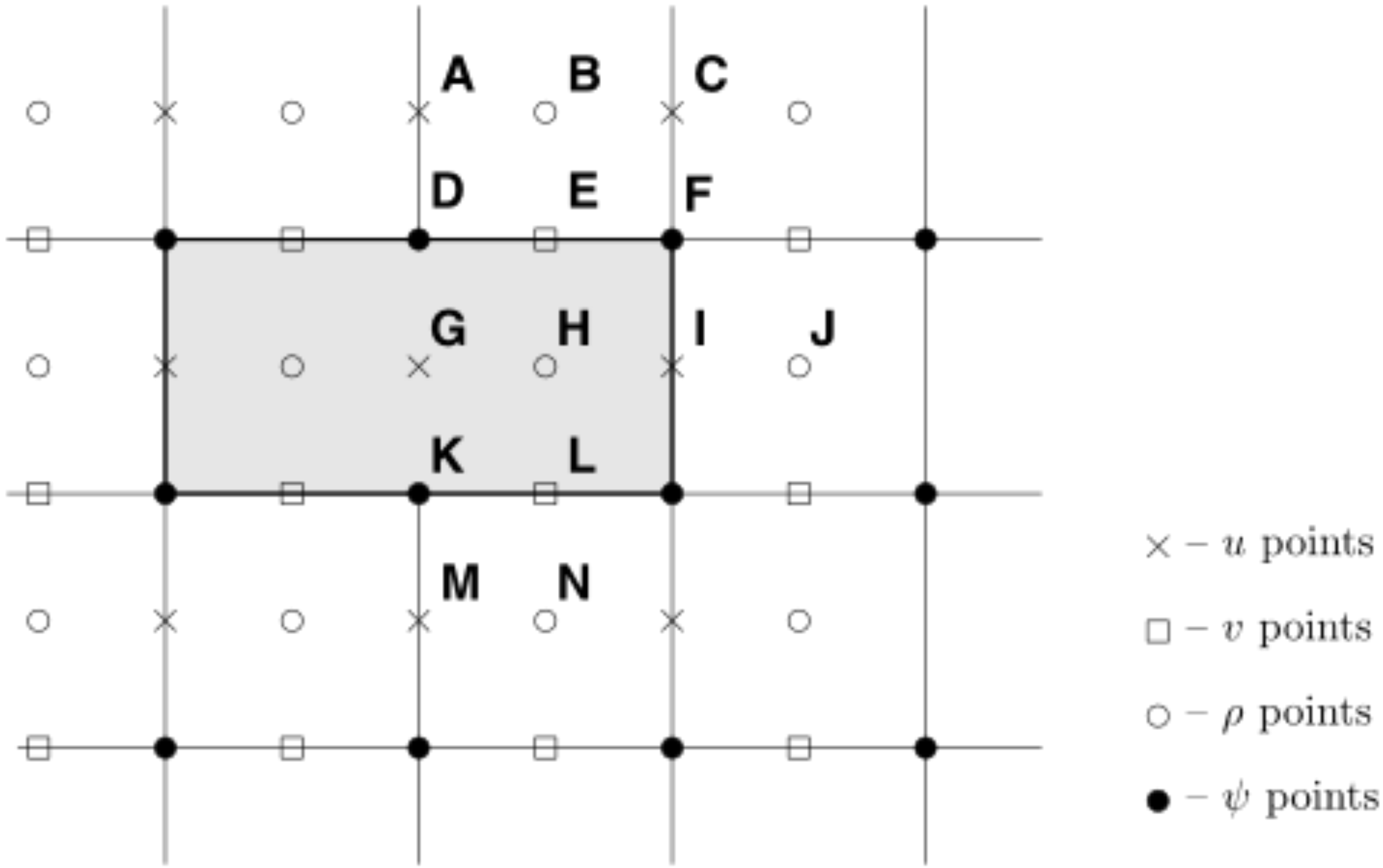
Horizontal curvilinear grid

- Another method = massive multigrain



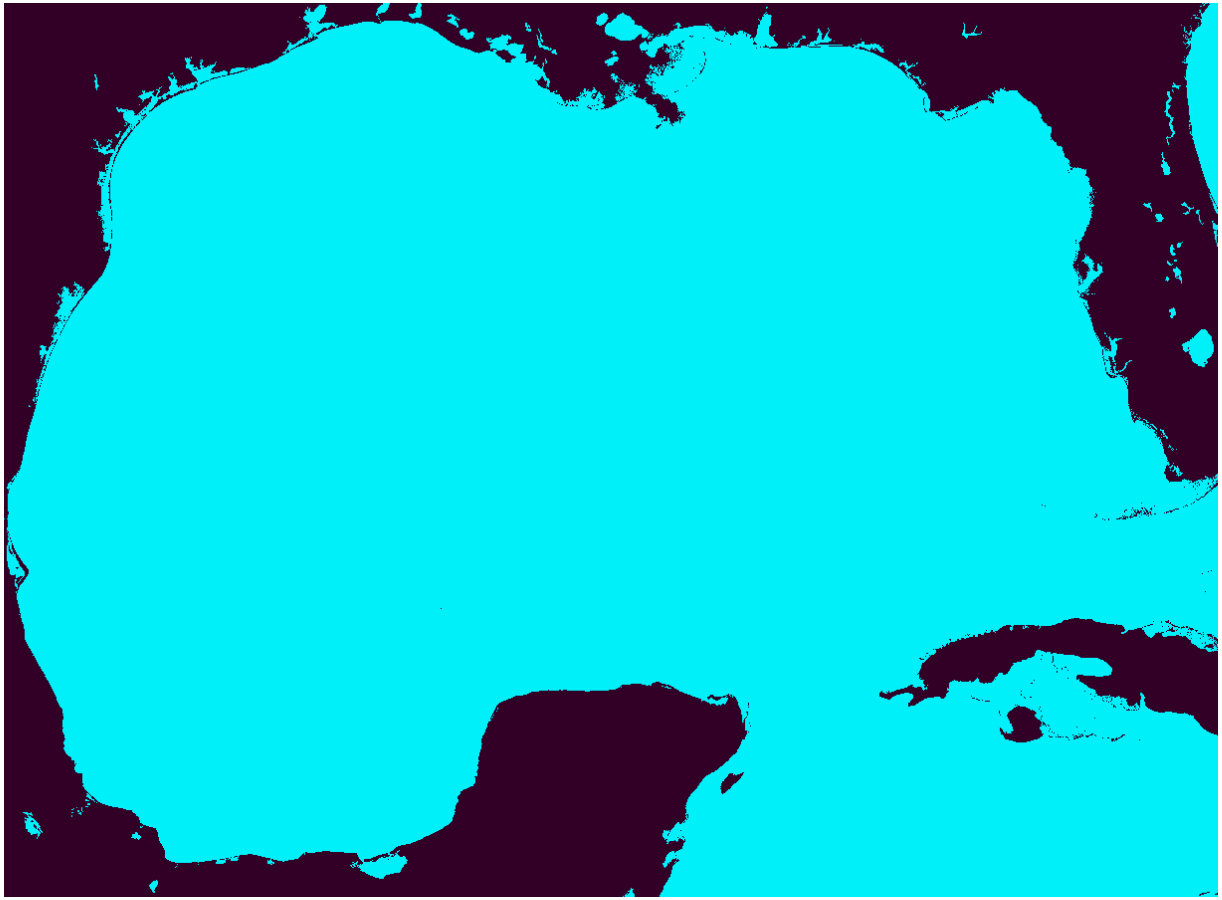
Land/sea Mask

Variables within the masked region are set to zero by multiplying by the mask for either the u, v or rho points :



Land/sea Mask

Variables within the masked region are set to zero by multiplying by the mask for either the u, v or rho points :

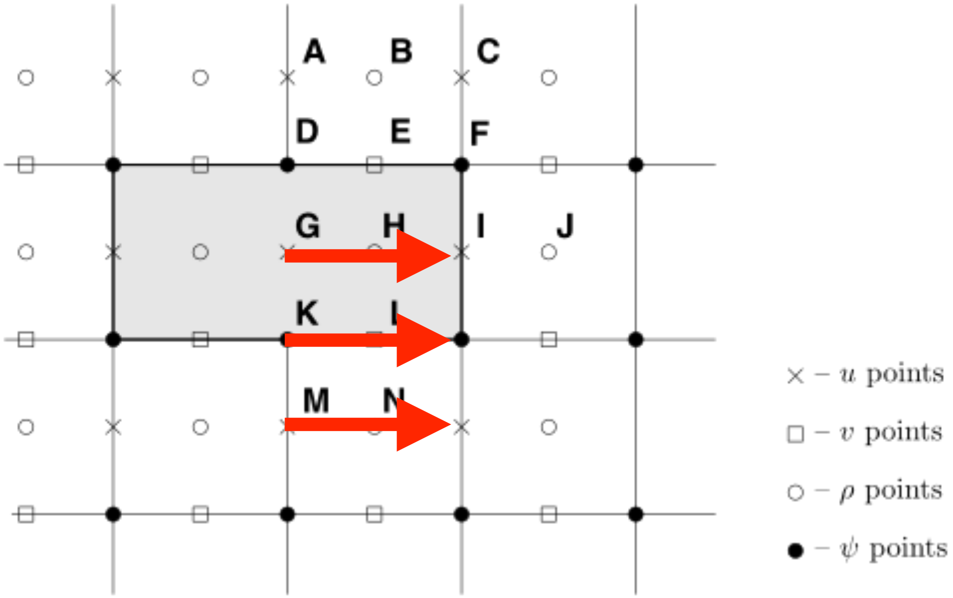


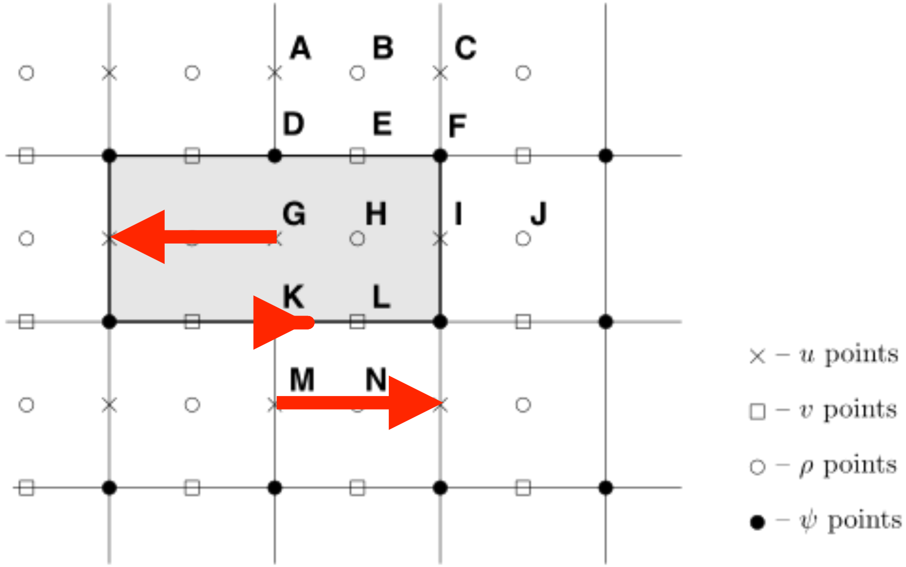
Land/sea Mask

See the code routines:

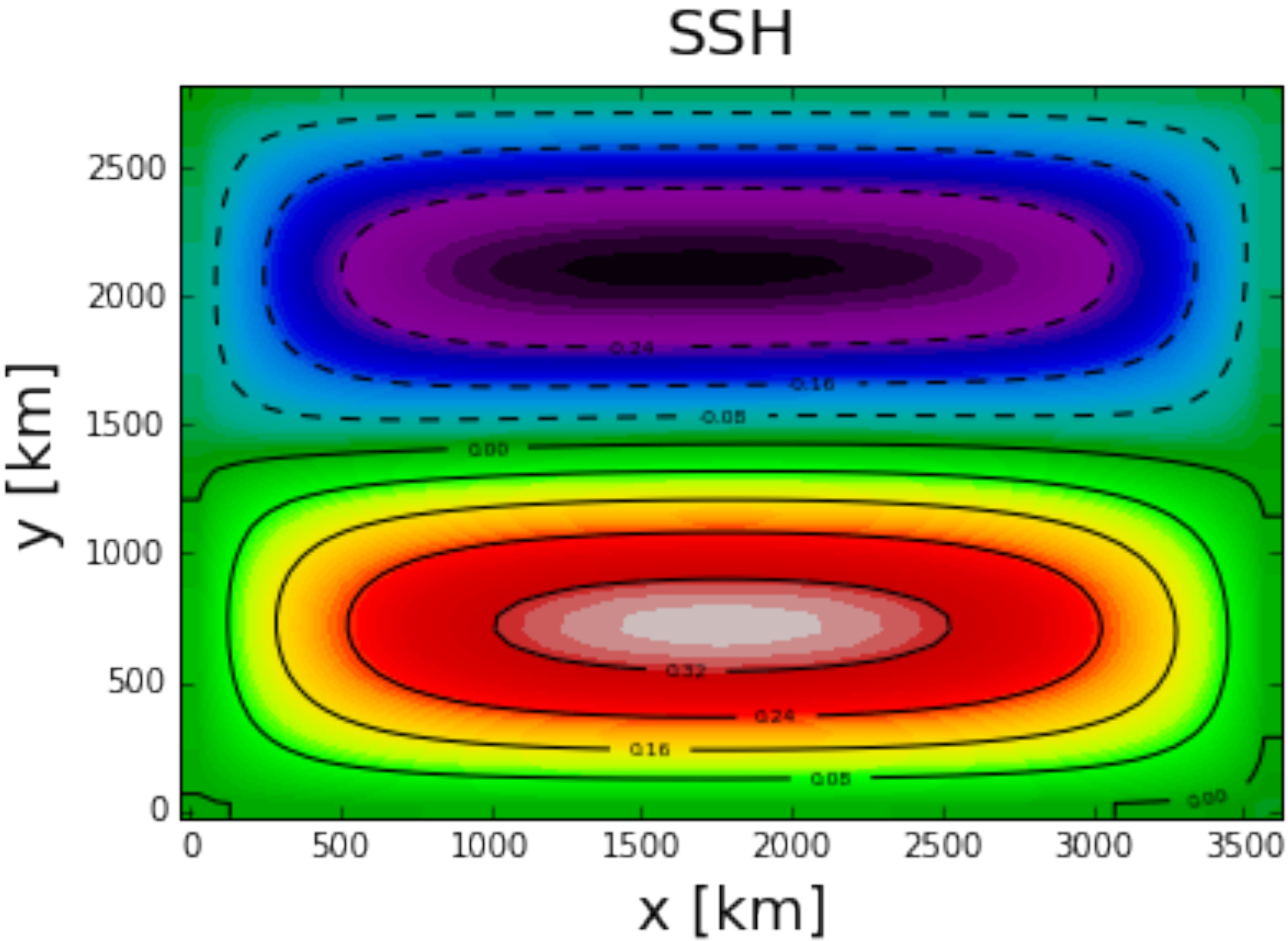
```
#ifdef MASKING
# define SWITCH *
#else
# define SWITCH !
#endif
!##########################
      do k=1,N
        do i=IstrU,Iend
          u(i,j,k,nnew)=(DC(i,k)-DC(i,0)) SWITCH umask(i,j)
```

Land/sea Mask





Activity 2 – Run an idealized ocean basin II



Momentum equations

-

$$\frac{\partial u}{\partial t} = -u_j \frac{\partial u}{\partial x_j} - w \frac{\partial u}{\partial z} + fv - \frac{P_x}{\rho_0} + \mathcal{V}_u + \mathcal{D}_u + \mathcal{S}_u$$

$$\underbrace{\frac{\partial v}{\partial t}}_{rate} = \underbrace{-u_j \frac{\partial v}{\partial x_j}}_{hadv} \underbrace{- w \frac{\partial v}{\partial z}}_{vadv} - \underbrace{fu}_{cor} - \underbrace{\frac{P_y}{\rho_0}}_{Prsgrd} + \underbrace{\mathcal{V}_v}_{vmix} + \underbrace{\mathcal{D}_v}_{hmix+hdiff} + \underbrace{\mathcal{S}_v}_{nudg}$$

Barotropic vorticity balance

- Barotropic vorticity: $$\Omega = \frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y}$$ with $$\overline{u} = \int_{-h}^{\zeta} u \, dz,$$
- The barotropic vorticity balance equation of the flow is obtained by integrating the momentum equations in the vertical and cross differentiating them:
  - [cf: https://www.jgula.fr/ModNum/vort_balance.pdf]

$$\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}} = -\underbrace{\vec{\nabla}.(f\vec{\overline{u}})}_{\text{planet. vort. adv.}} + \underbrace{\frac{\mathbf{J}(P_b, h)}{\rho_0}}_{\text{bot. pres. torque}} + \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}} + \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

$$A_\Sigma = \frac{\partial^2(\overline{vv} - \overline{uu})}{\partial x \partial y} + \frac{\partial^2 \overline{uv}}{\partial x \partial x} - \frac{\partial^2 \overline{uv}}{\partial y \partial y},$$

Stommel's gyre

THE WESTWARD INTENSIFICATION OF WIND-DRIVEN OCEAN CURRENTS

Henry Stommel

(Contribution No. 408, Woods Hole Oceanographic Instiution)

Abstract--A study is made of the wind-driven circulation in a homogeneous rectangular ocean under the influence of surface wind stress, linearised bottom friction, horizontal pressure gradients caused by a variable surface height, and Coriolis force.

An intense crowding of streamlines toward the western border of the ocean is discovered to be caused by variation of the Coriolis parameter with latitude. It is suggested that this process is the main reason for the formation of the intense currents (Gulf stream and others) observed in the actual oceans.

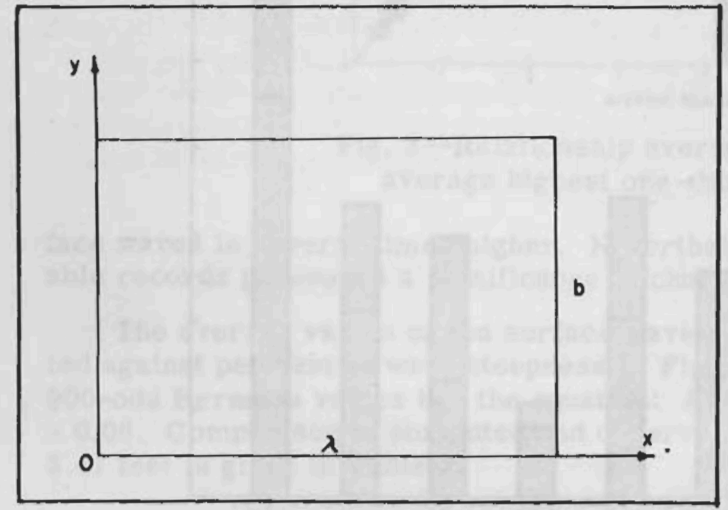


Fig. 1--Ocean basin dimensions and the coordinate system

- Momentum equations:

The steady state equations of motion, with the inertial terms omitted because they are small, are written in the form

$$0 = f(D + h)v - F\cos(\pi y/b) - Ru - g(D + h)\partial h/\partial x \qquad (1)$$

$$0 = -f(D + h)u - Rv - g(D + h)\partial h/\partial y \qquad (2)$$

- Barotropic vorticity equation:

$$v(D + h)(\partial f/\partial y) + (F\pi/b)\sin(\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Stommel's gyre

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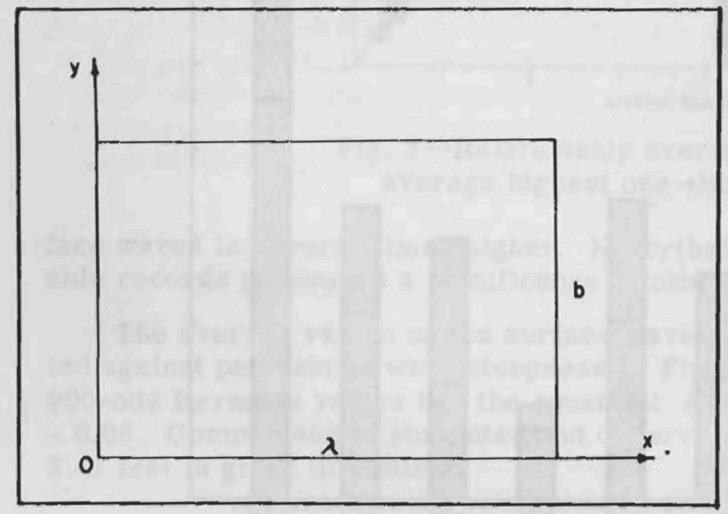


Fig. 1--Ocean basin dimensions and the coordinate system

- Momentum equations:

Coriolis   Wind   Drag   Pressure gradient

The steady state equations of motion, with the inertial terms omitted because they are small, are written in the form

$$0 = f(D + h)v + F\cos(\pi y/b) - Ru - g(D + h)\partial h/\partial x \qquad (1)$$

$$0 = -f(D + h)u - Rv - g(D + h)\partial h/\partial y \ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

- Barotropic vorticity equation:

Planetary vorticity advection   Wind Curl   Drag Curl

$$v(D + h)(\partial f/\partial y) + (F\pi/b)\sin(\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Stommel's gyre

- No rotation / or constant rotation ($\dfrac{\partial f}{\partial y} = \beta = 0$)

The equation for the stream function is therefore

$$\psi = \gamma (b/\pi)^2 \sin (\pi y/b)[e^{(x - \lambda)\pi/b} + e^{-x\pi/b} - 1]$$

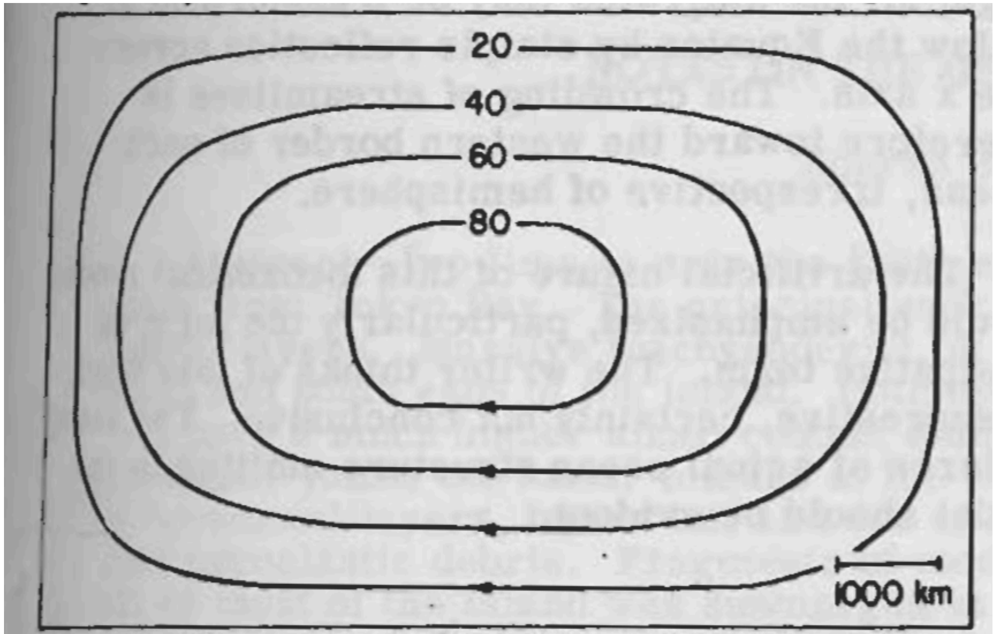


Fig. 2--Streamlines for the case of both the non-rotating and uniformly rotating oceans

Plan ary v city a n

Wind Curl

Drag Curl

$$\cancel{v(D\ldots)(\partial f/\partial y)} + (F\pi/b) \sin (\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Barotropic vorticity balance

- No rotation / or constant rotation ($\frac{\partial f}{\partial y} = \beta = 0$)
-

$$\ldots = - \ldots + \ldots + \boxed{\underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \boxed{\underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}}$$

$$+ \ldots - \ldots$$

Stommel's gyre

- With latitudinal variation of Coriolis

$$\frac{\partial f}{\partial y} = \beta \neq 0$$

- **Formation of a western boundar**

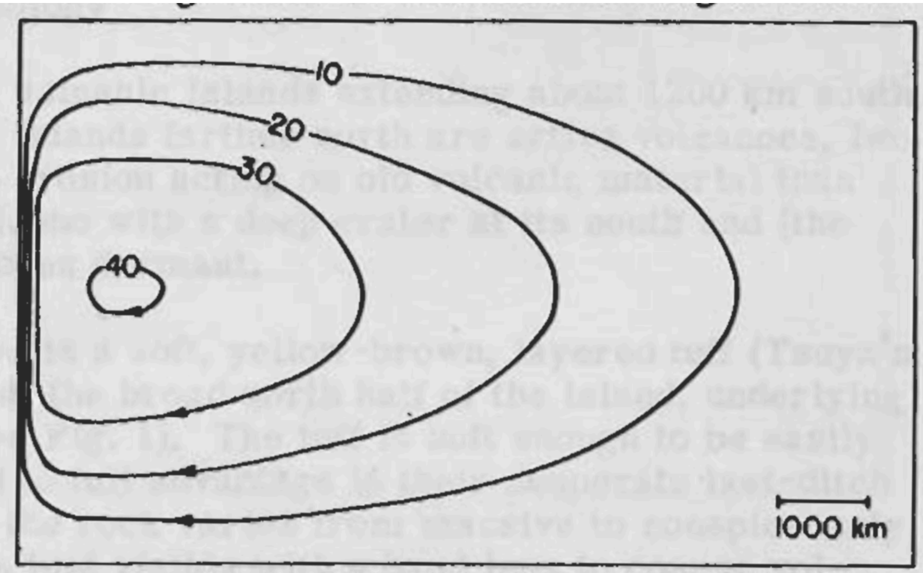


Fig. 5--Streamlines for the case where the Coriolis force is a linear function of latitude

Barotropic vorticity balance

- No rotation / or constant rotation ($\frac{\partial f}{\partial y} = \beta = 0$)
-

$$\cancel{\underbrace{\cdots}_{\text{ra\ldots e}}} = \boxed{\underbrace{-\ \vec{\nabla}.(f\vec{\bar{u}})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{\cdots, h)}_{\text{bot. p\ldots s. torque}}} + \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}$$

$$+ \cancel{\underbrace{\mathcal{D}\cdots}_{\text{hor\ldots usion.}}} - \cancel{\underbrace{\mathcal{A}\cdots}_{\text{NL\ldots tion}}}$$

Stommel's gyre (no beta)

- Forcings and data
  - Bottom topography + Land mask
  - Atmospheric surface boundary forcing
  - Initial oceanic conditions
  - Lateral oceanic boundary conditions

→

- **Flat bottom**
- **Constant wind** (sustr(i,j)=-cff1*cos(2.*pi/el*yr(i,j)))
- **Resting state**
- **Vertical walls**

Stommel's gyre (no beta)

- No rotation / or constant rotation ($\frac{\partial f}{\partial y} = \beta = 0$)
- **cppdefs.h**

```
# define UV_COR
# define UV_VIS2
# define TS_DIF2

# define ANA_GRID
# define ANA_INITIAL
```

- **croco.in**

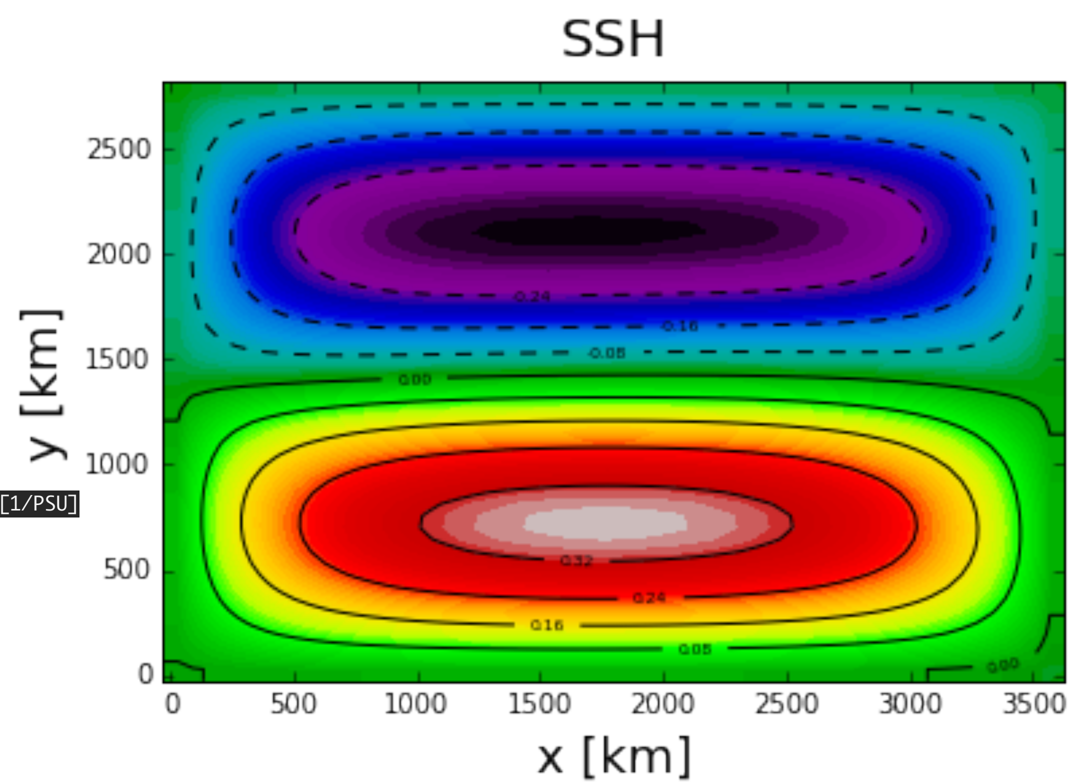
```
bottom_drag:     RDRG(m/s),      RDRG2, Zob [m],  Cdb_min, Cdb_max
                 3.e-4             0.      0.         0.       0.
gamma2:
                 1.
lin_EOS_cff:  R0 [kg/m3], T0 [Celsius], S0 [PSU], TCOEF [1/Celsius], SCOEF [1/PSU]
              30.         0.            0.        0.28                0.
lateral_visc:   VISC2     [m^2/sec ]
                 1000.  0.
tracer_diff2: TNU2          [m^2/sec]
                 1000. 0.
```

- **ana_grid.F**

```
f0=1.E-4
beta=0.
```

- **param.h**

```
      parameter (LLm0=60,    MMm0=50,    N=10)
```



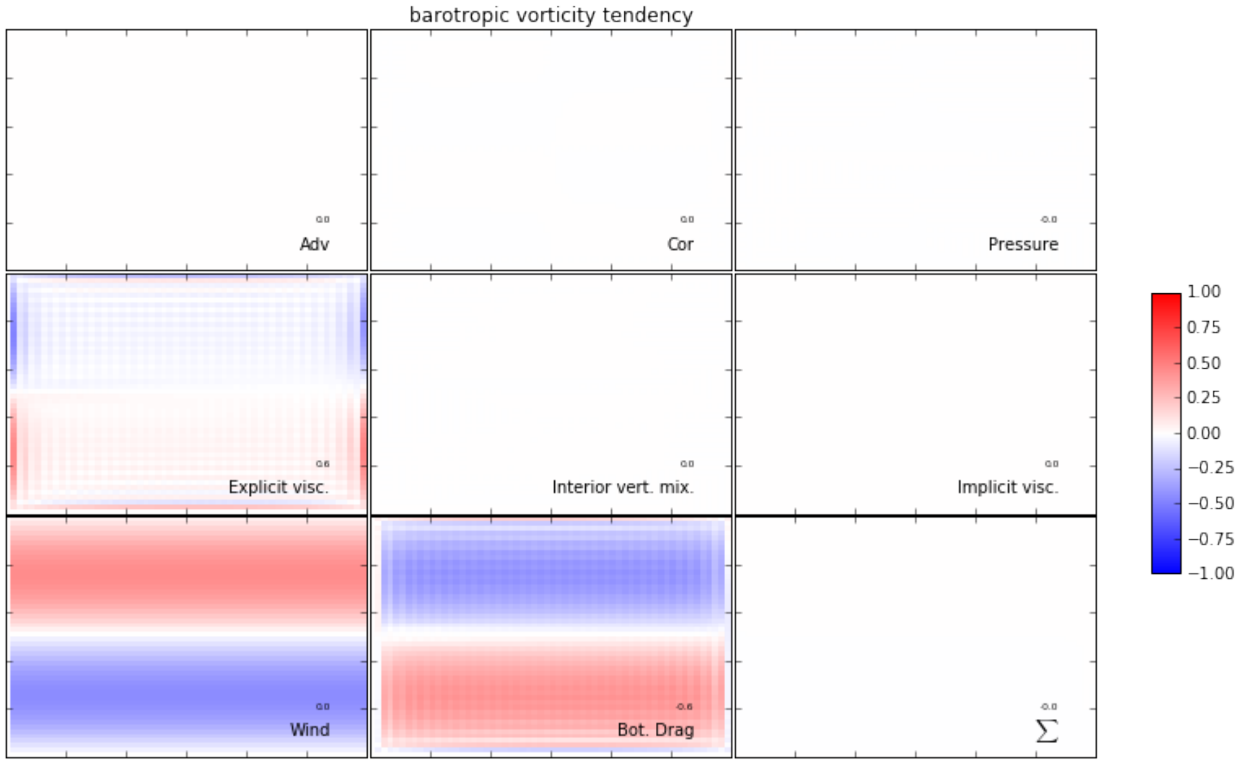
ROMS simulation after 20 years

Stommel's gyre (no beta)

- No rotation / or constant rotation

$$\underbrace{\cancel{\frac{\partial \Omega}{\partial t}}}_{\text{rate}} = -\underbrace{\cancel{\vec{\nabla}\cdot(f\vec{u})}}_{\text{planet. vort. adv.}} + \underbrace{\cancel{\frac{\mathbf{J}(P_b,h)}{\rho_0}}}_{\text{bot. pres. torque}} + \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}} + \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{\cancel{\mathcal{A}}}_{\text{NL advection}}$$

-

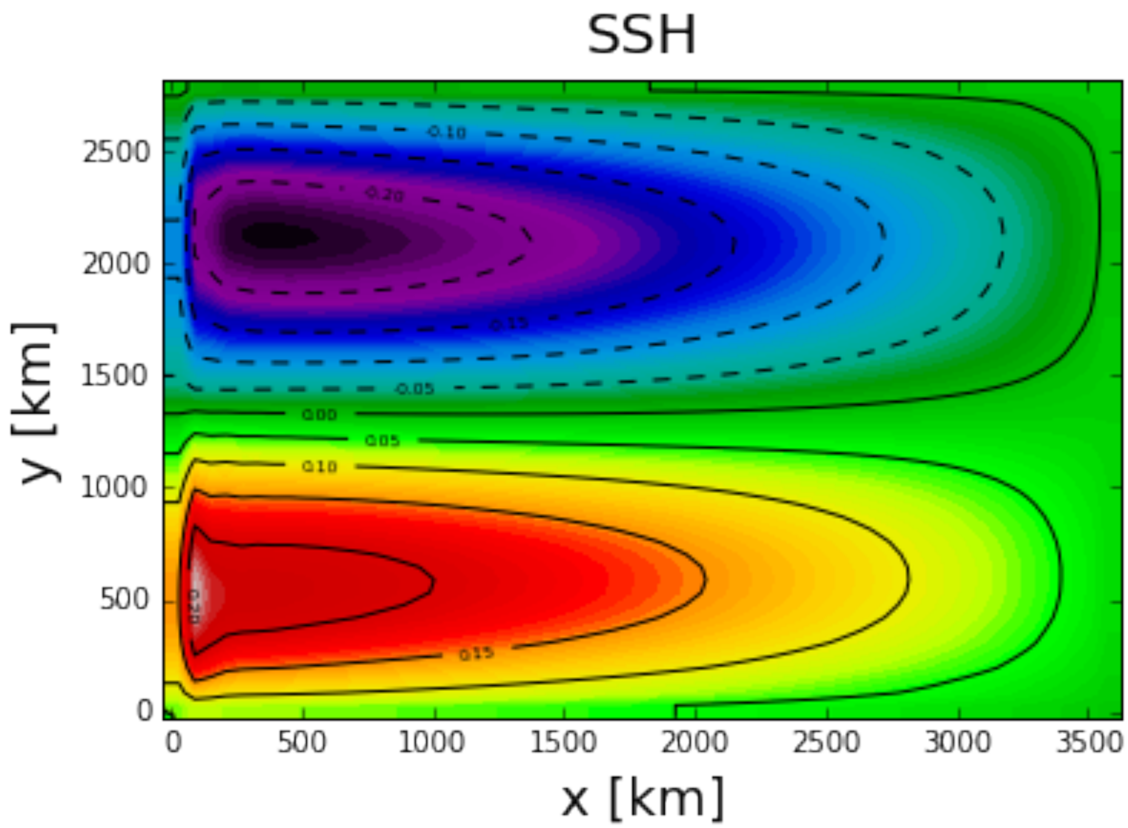


Stommel's gyre (with beta)

- With latitudinal variation of Coriolis ($\frac{\partial f}{\partial y} = \beta \neq 0$)

- **ana_grid.F**

```
f0=1.E-4
beta=2.E-11
```

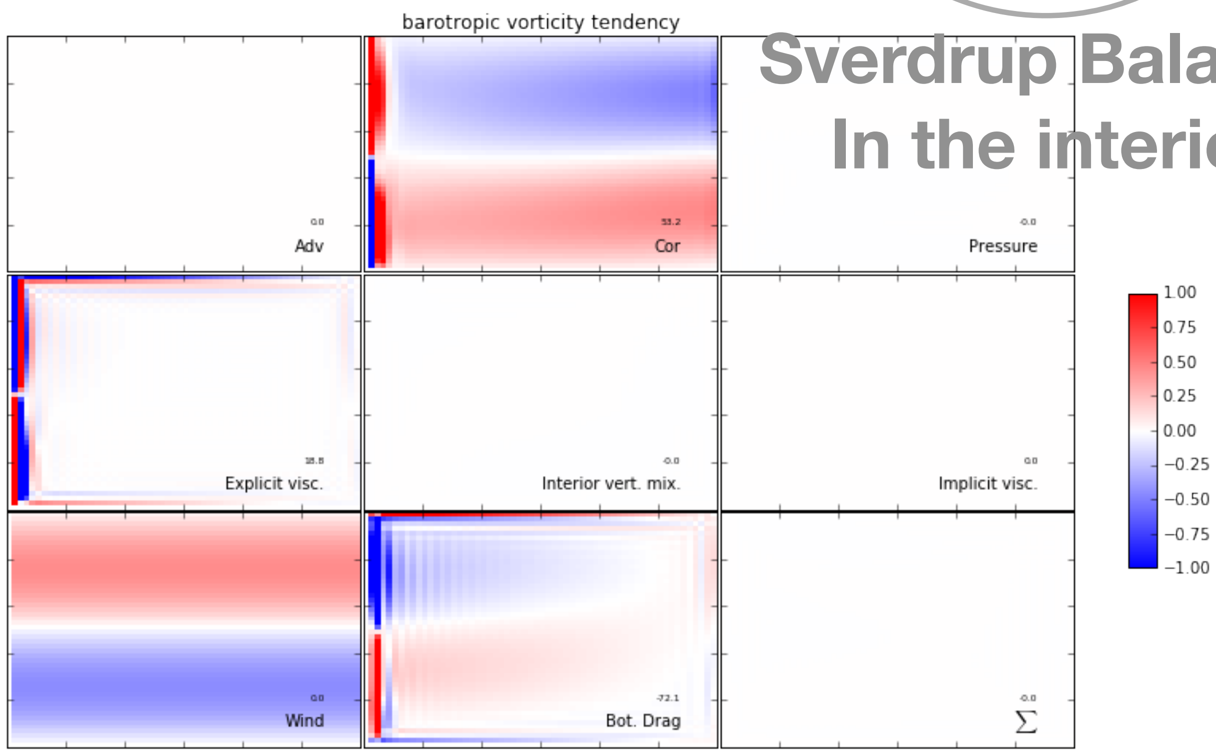


Stommel's gyre (with beta)

- With latitudinal variation of Coriolis

$$\cancel{\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}}} = \boxed{-\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{\frac{\mathbf{J}(P_b, h)}{\rho_0}}_{\text{bot. pres. torque}}} + \boxed{\underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}$$
$$+ \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \cancel{\underbrace{\mathcal{N}_A}_{\text{NL advection}}}$$

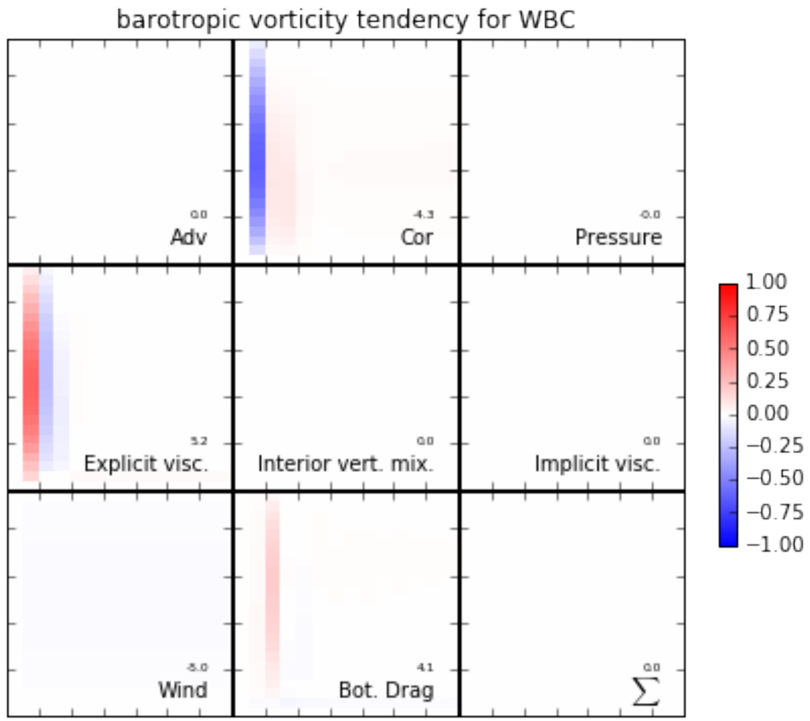
Sverdrup Balance
In the interior

- 

Stommel's gyre (with beta)

- With latitudinal variation of Coriolis

$$\underbrace{\cancel{\frac{\partial \Omega}{\partial t}}}_{\text{rate}} = \boxed{\underbrace{-\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{\frac{\mathbf{J}(P_b,H)}{\rho_0}}_{\text{bot. pres. torque}}} + \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}} + \boxed{\underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}}} - \cancel{\underbrace{\mathcal{A}_\Sigma}_{\text{NL advection}}}$$

- 

Zoom over the western boundary current

Munk' gyre

JOURNAL OF METEOROLOGY

ON THE WIND-DRIVEN OCEAN CIRCULATION

By *Walter H. Munk*

Institute of Geophysics and Scripps Institution of Oceanography, University of California[1]

(Manuscript received 24 September 1949)

In Ekman's and Stommel's model the ocean is assumed homogeneous, a case in which the currents extend to the very bottom. Not only is this in contrast with observations, according to which the bulk of the water transport in the main ocean currents takes place in the upper thousand meters, but it also leads to mathematical complications which rendered Ekman's analysis very difficult, and forced Stommel to resort to a rather arbitrary frictional force along the bottom.

To avoid these difficulties, we retain Sverdrup's integrated mass transport as the dependent variable. This device makes it possible to examine the more general case of a baroclinic ocean without having to specify the nature of the vertical distributions of density and current. In recognition of the evidence that currents essentially vanish at great depths, we shall depend on lateral friction for the dissipative forces. From Stommel we retain the rectangular boundaries, although we extend the basin to both sides of the equator and deal with the *observed* wind distribution rather than a simple sinusoidal distribution.

$$P = \int_{-h}^{z_0} p\, dz, \qquad \mathbf{M}_H = \int_{-h}^{z_0} \rho \mathbf{v}_H\, dz, \qquad (2a, b)$$

designate the integrated pressure and mass transport.

$$\nabla P + f\mathbf{k} \times \mathbf{M} - \boldsymbol{\tau} - A\nabla^2\mathbf{M} = 0. \qquad \mathbf{M} = \mathbf{k} \times \nabla\psi,$$

$$\boxed{(A\nabla^4 - \beta\, \partial/\partial x)\psi = -\operatorname{curl}_z \boldsymbol{\tau},}$$

For boundary conditions we choose

$$\psi_{\text{bdry}} = 0, \qquad (\partial\psi/\partial\nu)_{\text{bdry}} = 0, \qquad (7a, b)$$

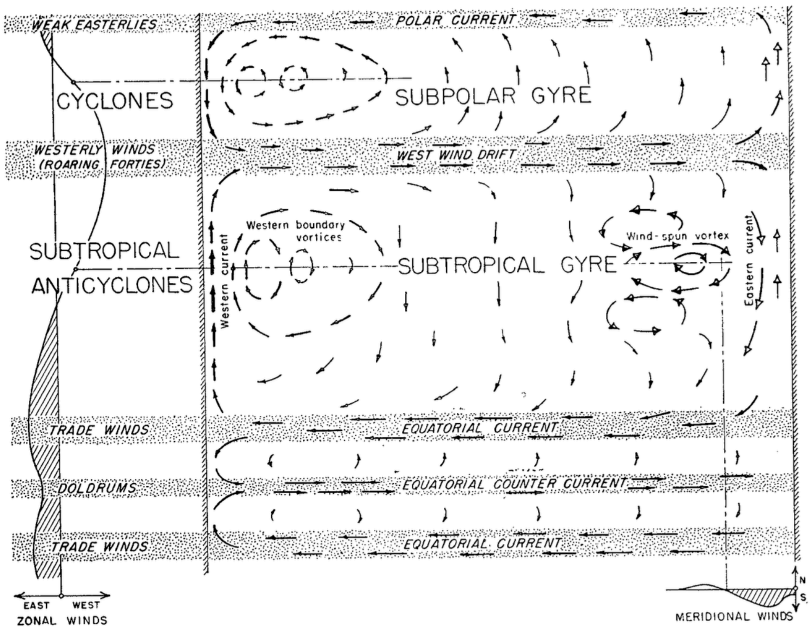


FIG. 8. Schematic presentation of circulation in a rectangular ocean resulting from zonal winds (filled arrowheads), meridional winds (open arrowheads), or both (half-filled arrowheads). The width of the arrows is an indication of the strength of the currents. The nomenclature applies to either hemisphere, but in the Southern Hemisphere the subpolar gyre is replaced largely by the Antarctic Circumpolar Current (west wind drift) flowing around the world. Geographic names of the currents in various oceans are summarized in table 3.

Gyre with beta and lateral drag

- With latitudinal variation of Coriolis

$$\underbrace{\cancel{\frac{\partial \Omega}{\partial t}}}_{\text{rate}} = \boxed{-\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \underbrace{\cancel{\frac{\mathbf{J}(P_b,h)}{\rho_0}}}_{\text{bot. pres. torque}} \boxed{+\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}} + \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

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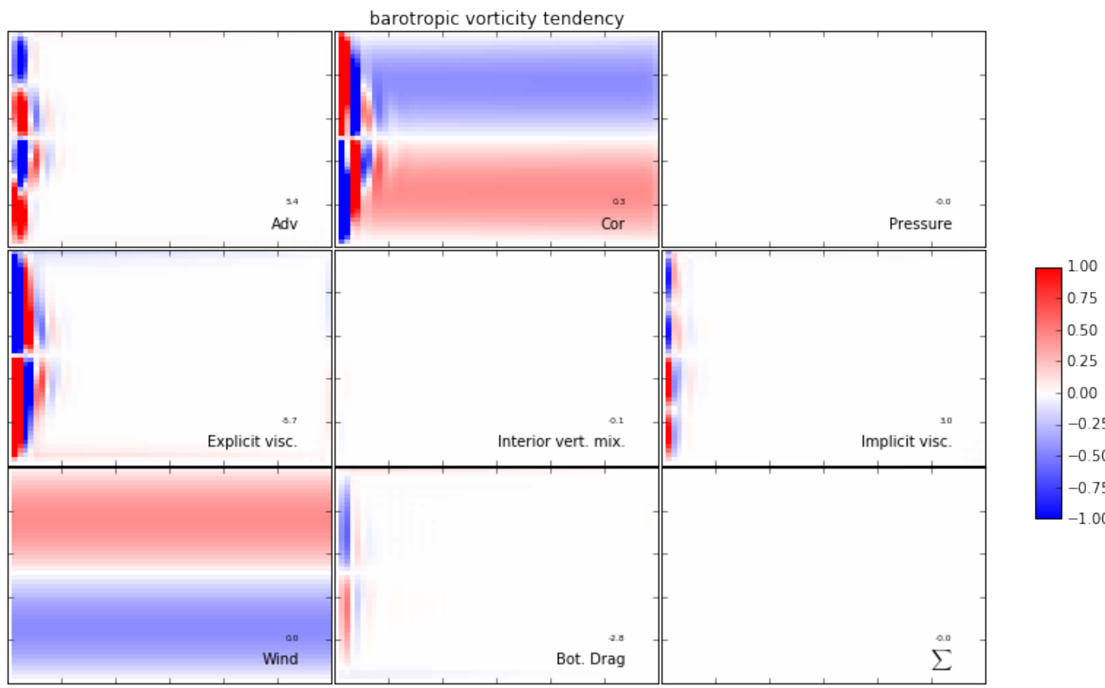

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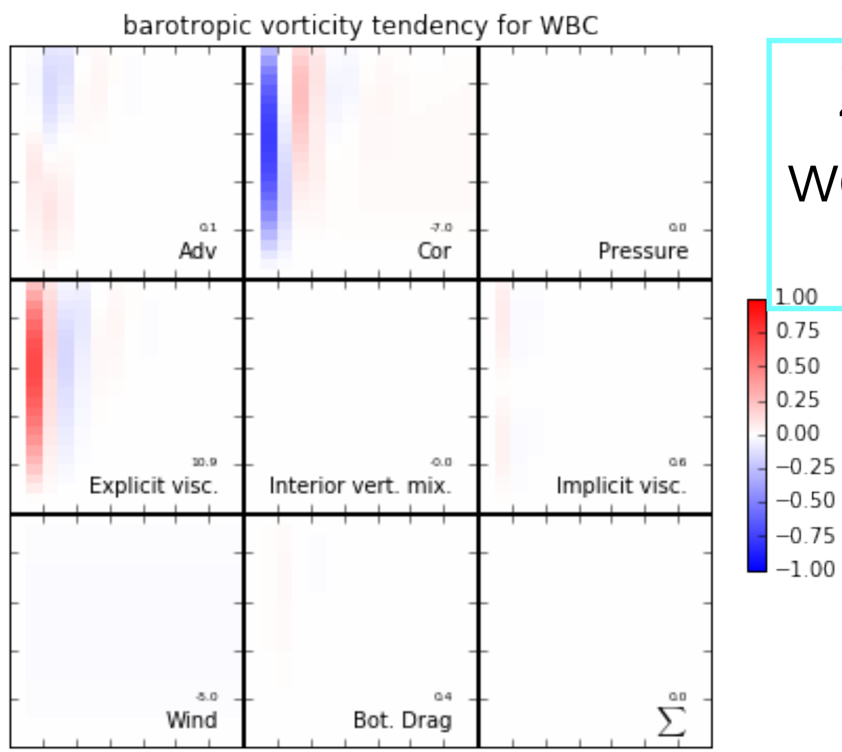

Zoom over the western boundary current