Université de Bretagne Occidentale

APPLIED MATHEMATICS - I

“Perturbation Methods”

Master 1 – Marine Science

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Chapter 1 - PERTURBATION METHODS FOR ALGEBRAIC EQUATIONS

I. Introduction: Notions of simplification

To study realistic problems in mechanics several complementary approaches are possible.

- At first, theory provides us with equations: Algebraic Equations, Ordinary Differential Equations (ODE) or Partial Differential Equations (PDE). But these equations are usually extremely hard to solve mathematically.

- To process these equations, we can also use the numerical approach, which allows to integrate the equations and to predict the state of a system. However, this approach has its limitations:
  1) We do not know generally how to explain the results
  2) We do not know if they are correct (numerical approximations)
  3) We do not know the main physical mechanisms that determine the solution

- As a third approach, we usually need to consider simplified cases to better understand the realistic case:
  1) To illustrate the origin of the form of the solution
  2) To verify the numerical solutions in simple cases
  3) To identify the physical processes at work and their importance

This third approach requires a simplification of the original mathematical problem to find analytical solutions that will allow us to better understand the nature of the solutions (sensitivity to parameters, role of different physical terms).

The general simplification procedure will be:

1. Identifying the relatively small terms in the equations.
2. Neglecting these terms and solving the simplified system.
3. Checking the consistency of the approximation achieved: the approximate solution is used to estimate the neglected terms and verify that they are really small.

This common-sense method is based on the idea that small changes in the equations lead to small changes in the solution of these equations. Obviously, there are many counter-examples (exceptions) of this.

A few examples of simplifications in algebraic equations :

Example 1 :

$$\begin{cases} x + 10y = 21 \\ 5x + y = 7 \end{cases}$$

The coefficient in front of x is small compared to that of y in the first equation and we are tempted to neglect x in this equation. If we make this approximation we obtain

$x_0 = 0.98$ and $y_0 = 2.1$

With this approximation the ratio between the term neglected and the term used is

$(1.x_0) / (10.y_0) = 0.05$ and is small. The solution of the simplified system is therefore consistent. Moreover, the approximate solution is very close to the exact solution $x = 1$ and $y = 2$.

Example 2 :

$$\begin{cases} 0.01x + y = 0.1 \\ x + 101y = 11 \end{cases}$$

Just as before the coefficient in front of x is small compared to that of y in the first equation. By neglecting it one obtains the approximate solution:

$$x \sim 0.9 \text{ et } y \sim 0.1.$$

In this case $0.01x / y = 0.09$ and the approximation seems consistent.

However, the exact solution is $x = -90$, $y = 1$.

So, where is the mistake?

We write the system as $\begin{cases} \varepsilon x + y = 0.1 \\ x + 101y = 11 \end{cases}$ (here $\varepsilon = 10^{-2}$).

The simplification is to take $\varepsilon = 0$. However, this approximation is valid only if $x(\varepsilon) \sim x(0)$ et $y(\varepsilon) \sim y(0)$.

In fact, the exact solution is $x = \dfrac{0.9}{1-101\varepsilon}$ and $y = \dfrac{0.1-11\varepsilon}{1-101\varepsilon}$ with $\varepsilon \neq \dfrac{1}{101}$.

With this solution and with $\varepsilon = 10^{-2}$ we obtain $\left|\dfrac{\varepsilon x}{y}\right| = 0.9 \approx 1$. This shows that our approximation is really very bad.

We distinguish the « apparent consistency» from the « authentic consistency» which is achieved when the neglected term is really small.

Another approach is to think geometrically. actually, the original system is equivalent to find the intersection of two lines almost parallel. This explains the large uncertainty in the solution, which is the intersection point. This results in a great sensitivity to the value of ε which is related to the slope of the two lines.

We will speak here of "bad conditioned problem": small changes in its formulation lead to large changes in the solution.
In fact, here the problem has a singularity: indeed, the system moves from one case with a unique solution to one case without solution. The system considered in this example is very close to this singularity.

II. Perturbation methods for algebraic equations

1. Methodology

Perturbation methods are used to solve problems which contain a small parameter, usually denoted by ε, by iterative methods. These methods are so powerful that sometimes we artificially introduce a parameter ε in a problem to use them and then return to the case ε = 1. But in general, in the context of problems in continuum mechanics, this small parameter appears after a scaling of the equations.
In this chapter, we will only deal with one kind of perturbation method which uses power series. There exist other methods (parametric differentiation, successive approximation) which are described in the references cited at the end of the chapter.

The general approach is as follows:

1. Reveal a small parameter ε in front of a term of the equations. ε is called perturbation parameter.

2. Assume that the solution of the problem can be expressed as a power series of ε (perturbation series) and calculate the successive coefficients of this series (usually the first terms suffice).
3. Gather the different terms of the series to find the desired solution for the appropriate values of ε.

The underlying idea is to replace the solution of a complicated problem by solving a multitude of simpler problems. It is hoped that if the problem depends on a small perturbation parameter, then the solution of the problem as a perturbation series converges to the solution of the problem. Attention, of course this is not always the case! But in some cases, even when the series diverges, the sum of the first terms will give a satisfactory result.

However, for some problems depending on a parameter ε, a simple expansion in powers of ε may fail. This is the case when the problem has a singularity at ε = 0, that is to say that the problem changes character at this point (if ε is in front of the highest degree term of an algebraic equation for instance). In that case, if some solutions are missing then the problem may be singular: rescale the variable such that there exists a dominant equilibrium between two terms. The scaling maybe written $x = y/\delta$ or $x = \frac{y}{\varepsilon^\alpha}$ where δ or α has to be determined.

2. Illustration of regular and singular problems with algebraic equations

***a*) A regular problem:**

Let us search approximate solutions of the 3rd degree algebraic equation:

$$x^3 - x + \varepsilon = 0 \quad \text{in which} \quad \varepsilon \ll 1$$

We search x as : $x = x_0 + \varepsilon\ x_1 + \varepsilon^2 x_2 + \quad \cdots\ldots$

Then $x^3 = x_0^3 + \varepsilon\ (3x_0^2 x_1) + \varepsilon^2(3x_0 x_1^2 + 3x_0^2 x_2) + O(\varepsilon^3)$

Then $(x_0^3 - x_0) + \varepsilon\ (3x_0^2 x_1 - x_1 + 1) + \varepsilon^2(3x_0 x_1^2 + 3x_0^2 x_2 - x_2) + O(\varepsilon^3) = 0$

$$\begin{cases} x_0^3 - x_0 = 0 \\ 3x_0^2 x_1 - x_1 + 1 = 0 \\ 3x_0 x_1^2 + 3x_0^2 x_2 - x_2 = 0 \end{cases} \qquad \begin{cases} x_0(x_0^2 - 1) = 0 \\ x_1(3x_0^2 - 1) = -1 \\ x_2(3x_0^2 - 1) = -3x_0 x_1^2 \end{cases}$$

$$\begin{cases} x_0 = 0, +1, -1 \\ x_1 = -1/(3x_0^2 - 1) \\ x_2 = -3x_0 x_1^2 / (3x_0^2 - 1) \end{cases}$$

We thus obtain approximations as $\varepsilon \ll 1$ for the 3 roots :

$$\begin{cases} x = \varepsilon + O(\varepsilon^2) \\ x = 1 - \dfrac{1}{2}\varepsilon + O(\varepsilon^2) \\ x = -1 - \dfrac{1}{2}\varepsilon + O(\varepsilon^2) \end{cases}$$

***b*) A singular problem:**

Let us search the approximate solutions of the 3rd degree algebraic equation:

$$\varepsilon x^3 - x + 1 = 0 \quad \text{as} \quad \varepsilon \ll 1$$

Let us search x as : $x = x_0 + \varepsilon\ x_1 + \varepsilon^2 x_2 + \quad \cdots\ldots$

We get : $\begin{cases} -x_0 + 1 = 0 \\ x_0^3 - x_1 = 0 \\ 3x_0^2 x_1 - x_2 = 0 \end{cases}$ and thus : $\begin{cases} x_0 = 1 \\ x_1 = 1 \\ x_2 = 3 \end{cases}$

We have obtained only 1 approximate root: $x = 1 + \varepsilon\ + O(\varepsilon^2)$

How to get the other two roots?
And why did we miss them?

The reason is that : as $\varepsilon \longrightarrow 0$ the two other roots tend to infinity so that εx^3 keeps a finite value and thus can not be neglected.

We thus have to rescale the problem:
So we search $y(\varepsilon)$ such that $x(\varepsilon) = \dfrac{y(\varepsilon)}{\delta(\varepsilon)}$ and $y(\varepsilon) = O(1)$ as $\varepsilon \longrightarrow 0$

We get : $$\underbrace{\frac{\varepsilon}{\delta^3}}_{(1)} y^3 - \underbrace{\frac{1}{\delta}}_{(2)} y + \underbrace{1}_{(3)} = 0$$

Our first task is to determine $\delta(\varepsilon)$ such that this equation contains 2 terms of the same order as $\varepsilon \longrightarrow 0$ (what we call a « dominant equilibrium »). If it is not the case, we get only trial or impossible solution.

This analysis gives 3 cases:

- if $(1) \sim (2)$ then $\delta \sim \sqrt{\varepsilon}$ which give $\frac{\varepsilon}{\delta^3} \gg 1$ (OK as $\varepsilon \ll 1$)
- if $(1) \sim (3)$ then $\delta \sim \varepsilon^{1/3}$ which give $1 \ll \varepsilon^{-\frac{1}{3}}$ (IMPOSSIBLE as $\varepsilon \ll 1$)
- if $(2) \sim (3)$ then $\delta \sim 1$ which give $1 \gg \varepsilon$ (OK as $\varepsilon \ll 1$)

The first case leads to $\delta \ll \sqrt{\varepsilon}$ and the scaled equation $y^3 - y + \delta = 0$

With $y = y_0 + \delta\ y_1 + O(\delta^2)$ we get : $y = \pm 1 - \dfrac{1}{2}\delta + O(\delta^2)$

And back to x: $x = \dfrac{y}{\delta} = \dfrac{y}{\sqrt{\varepsilon}} = \pm\dfrac{1}{\sqrt{\varepsilon}} - \dfrac{1}{2} + O(\sqrt{\varepsilon})$

(we notice that the third case would lead to the regular solution found before)

The approximation of the three roots as $\varepsilon \ll 1$ are thus:

$$\begin{cases} 1 + \varepsilon + O(\varepsilon^2) \\ \dfrac{1}{\sqrt{\varepsilon}} - \dfrac{1}{2} + O(\sqrt{\varepsilon}) \\ -\dfrac{1}{\sqrt{\varepsilon}} - \dfrac{1}{2} + O(\sqrt{\varepsilon}) \end{cases}$$

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Chapter 2 - ASYMPTOTIC EXPANSIONS

A few notations:

Asymptotically smaller

$f(x) \ll g(x) \;\; (x \to x_0)$: $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$ other notation : $f(x) = o\big(g(x)\big)$

Asymptotically equal

$f(x) \sim g(x) \;\; (x \to x_0)$: $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$

Asymptotically bounded

$f(x) = O\big(g(x)\big) \quad (x \to x_0)$: $\exists M \quad |f| \le M|g| \quad \forall x$ sufficiently close to x_0

I. Some reminders about power series

Definition : A power series is a series that can be written as : $\sum_{n=0}^{\infty} A_n (x-x_0)^n = \lim_{N\to\infty} \sum_{n=0}^{N} A_n (x-x_0)^n$. For x values at which the limit exists the series is convergent.

Series convergence (*Ratio test*): A series $S = \sum_{n=0}^{\infty} s_n$ converges if the absolute value of its consecutive terms ration $\left|\frac{s_{n+1}}{s_n}\right|$ converges towards a finite limit $\rho < 1$ as n tends towards infinity. It diverges if $\rho > 1$. No conclusion is possible if $\rho = 1$.

Power series convergence: as a consequence, for a power series $\rho = \lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| |x - x_0|$ and if $L = \lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right|$ exists, then the series converges for $|x - x_0| < \frac{1}{L}$

When L exists, the distance $R = \frac{1}{L}$ is called convergence radius of the series.

Convergent power series and Taylor expansions: If $\sum_{n=0}^{\infty} A_n (x-x_0)^n$ is convergent in a neighborhood of x_0, then it represents a function $f(x)$ infinitely derivable with continuous derivatives.

One shows that :

$$f^{(k)}(x_0) = k!\,A_k \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} A_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n .$$

This is the Taylor expansion of $f(x)$ around x_0. Such a function is called a regular function (analytical function).

Series remainder: If a power series is convergent for all x, then necessarily its remainder has to tend towards 0:

$$f(x) = \sum_{n=0}^{\infty} A_n (x-x_0)^n = A_0 + A_1(x-x_0) + \cdots + A_N (x-x_0)^N + \underbrace{A_{N+1}(x-x_0)^{N+1} + \cdots}_{R_N(x)}$$

Remainder: $R_N(x) = \sum_{n=N+1}^{\infty} A_n (x-x_0)^n \xrightarrow[N\to\infty]{} 0 \quad \forall x$

II. Asymptotic expansions

Definition : $\sum_{n=0}^{\infty} A_n (x-x_0)^n$ is an asymptotic expansion of the function $f(x)$ as $x \to x_0$ if : $\quad \forall N, \quad f(x) - \sum_{n=0}^{N} A_n (x-x_0)^n << (x-x_0)^N, \quad \text{as} \quad (x \to x_0).$

We then write : $f(x) \sim \sum_{n=0}^{+\infty} A_n (x-x_0)^n$, as $(x \to x_0)$

Remark 1 : This is equivalent to :

$$\forall N, \quad \forall \varepsilon, \quad \exists \delta, \quad \text{as } |x - x_0| < \delta \Rightarrow \left| f(x) - \sum_{n=0}^{N} A_n (x-x_0)^n \right| < \varepsilon |x-x_0|^N$$

Or also equivalently :

$$\forall N, \quad f(x) - \sum_{n=0}^{N} A_n (x-x_0)^n = o\left((x-x_0)^N\right), \quad \text{as} \quad (x \to x_0)$$

Remark 2 : A convergent series $\sum_{n=0}^{\infty} A_n (x-x_0)^n$ over an interval containing x_0 is always asymptotic to its limit $f(x)$. But be careful, lots of asymptotic expansions are not convergent series !

We note $\varepsilon_N(x) = f(x) - \sum_{n=0}^{N} A_n (x-x_0)^n = f(x) - S_N(x)$

- *For a convergent power series* :

$$f(x) = \sum_{n=0}^{\infty} A_n (x-x_0)^n = \underbrace{A_0 + A_1(x-x_0) + \cdots + A_N(x-x_0)^N}_{S_N(x)} + \underbrace{A_{N+1}(x-x_0)^{N+1} + \cdots}_{R_N(x)}$$

necessarily : $\varepsilon_N(x) = R_N(x) = \sum_{n=N+1}^{\infty} A_n (x-x_0)^n \xrightarrow[N\to\infty]{} 0 \quad \forall x$

- *For an asymptotic expansion* :

We impose that $\varepsilon_N(x) \ll (x-x_0)^N$, as $(x \to x_0) \forall N$

Convergence is an absolute property
Asymptoticity is a relative property

Remark 3 : Non-integer powers are possible in asymptotic expansions : $\sum_{n=0}^{\infty} A_n (x-x_0)^{\alpha n}, \alpha > 0$

Asymptoticity close to infinity : if $x_0 = \infty$ we write :
$f(x) \sim \sum_{n=0}^{+\infty} A_n x^{-\alpha n}$, as $(x \to \infty)$ with $\varepsilon_N(x) << x^{-\alpha N}$ as $x \to \infty$ $\forall N$

Generalized asymptotic expansions :
If $(\phi_n(x))_n$ are functions, we say that $f(x) \sim \sum_{n=0}^{+\infty} A_n \phi_n(x)$, as $(x \to x_0)$ if

$\forall N, f(x) \sim \sum_{n=0}^{N} A_n \phi_n(x) \ll \phi_N(x)$, as $(x \to x_0)$

An example of asymptotic expansion / divergent series close to infinity.

Search an asymptotic expansion of the exponential integral: $Ei(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$ as $(x \to \infty)$

With successive integrations by parts:

$$\begin{aligned} Ei(x) &= \left[\frac{-e^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2}\,dt \\ &= \frac{e^{-x}}{x} + \left[\frac{e^{-t}}{t^2}\right]_x^\infty + 2\int_x^\infty \frac{e^{-t}}{t^3}\,dt \\ &\vdots \\ &= \underbrace{e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \cdots + (-1)^{N-1}\frac{(N-1)!}{x^N}\right)}_{S_N(x)} + \underbrace{(-1)^N N!\int_x^\infty \frac{e^{-t}}{t^{N+1}}\,dt}_{\varepsilon_N(x)} \end{aligned}$$

For all x , $S_N(x)$ diverges as $N \to \infty$.

$(\lim_{N\to\infty} (-1)^{N-1}\frac{(N-1)!}{x^N} \neq 0)$

One can show that : $|\varepsilon_N(x)| < \frac{N!}{x^{N+1}}\int_x^\infty e^{-t}dt = \frac{N!}{x^{N+1}}e^{-x} \xrightarrow[x\to\infty]{} 0 \quad \forall N$

and better : $\left|\frac{\varepsilon_N(x)}{e^{-x}(-1)^{N-1}(N-1)!\,x^{-N}}\right| < \frac{\frac{N!}{x^{N+1}}e^{-x}}{(N-1)!\,e^{-x}x^{-N}} = \frac{N}{x} \xrightarrow[x\to\infty]{} 0$

Thus $Ei(x) \sim e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{3!}{x^4} + \cdots\right)$ as $(x \to \infty)$ whereas the power series $\sum_{n=1}^{\infty} e^{-x}(-1)^{n-1}\frac{(n-1)!}{x^n}$ is divergent.

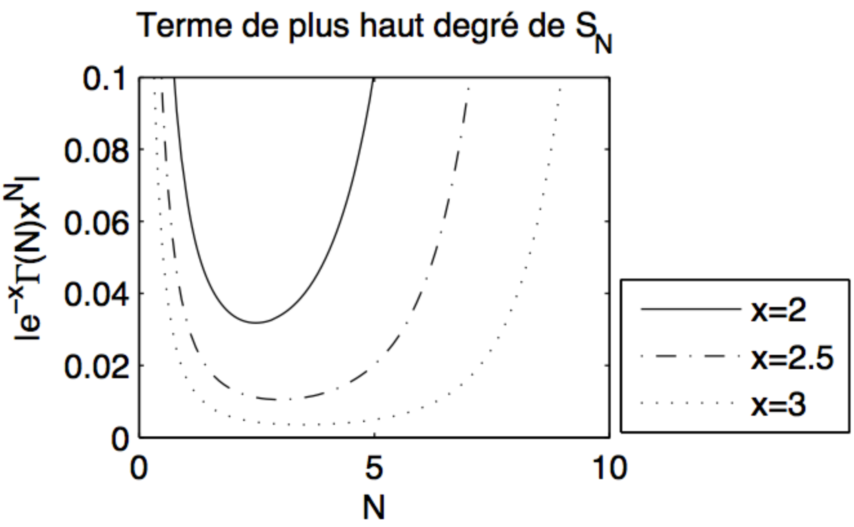

Figure 2 : *absolute value of the term of highest degree of* $S_N(x)$ *i-e* $\left|e^{-x}(-1)^{N-1}(N-1)!x^{-N}\right|$ *as a function of N for different values of x.*

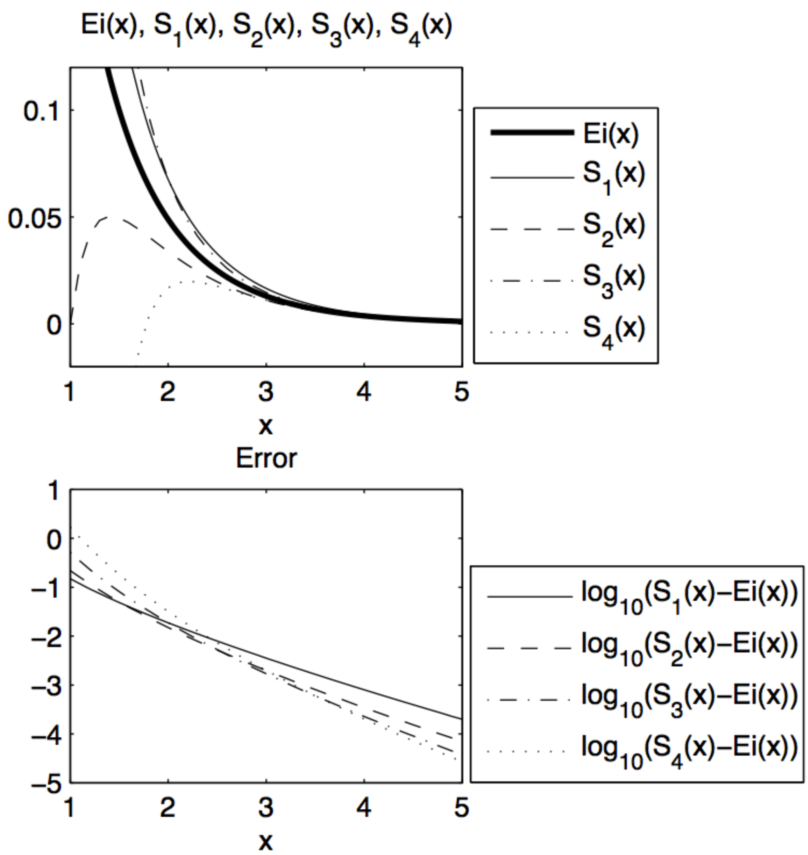

Figure 3 : : *(up) Ei(x) and the 4 first partial sums* $S_N(x)$*. (down). log (base 10) of the difference between these partial sums and exact solution.*

Remark : On figure 1 to plot the values of $\left|e^{-x}(-1)^{N-1}(N-1)!\ x^{-N}\right|$ for real values of $N > 0$ we used the Gamma function defined by :

$$\boxed{\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt \quad \text{with} \quad \operatorname{Re}(z) > 0}$$

This function is a prolongation (analytical) of the discrete factorial function.

We recall that $\boxed{\Gamma(z+1) = z\Gamma(z)}$ $\boxed{\Gamma(1) = 1}$

In particular, if n is integer: $\boxed{\Gamma(n+1) = n!}$

Properties of asymptotic expansions:

- If $f(x) \sim \sum_{n=0}^{\infty} A_n x^n$, $(x \to 0)$ then $f(x) + C \cdot e^{-1/x}$ has the same asymptotic expansion close to 0.

- **Integration / Differentiation:** One can integrate an asymptotic expansion but not always differentiate it : if f and g have the same asymptotic expansion, it is not always true for f' and g'.

- **Asymptotic expansions of integrals :**
  Let $I(x) = \int_a^b f(t,x)dt$ close to x_0.
  If $f(t,x) \sim f_0(t)$ as $(x \to x_0)$ uniformly as $a \le t \le b$, then the first term of the asymptotic expansion of $I(x)$ is :
  $$I(x) \sim \int_a^b f_0(t)dt \quad \text{as } (x \to x_0)$$
  If $f(t,x) \sim \sum_{n=0}^{+\infty} f_n(t)(x-x_0)^{\alpha n}$ as $(x \to x_0)$ $\alpha > 0$ and for all $a \le t \le b$ then
  $$f(t,x) \sim \sum_{n=0}^{+\infty} (x-x_0)^{\alpha n} \int_a^b f_n(t)dt \quad \text{as } (x \to x_0)$$

- **Unicity of the asymptotic expansion :**
  If $f(x) \sim \sum_{n=0}^{+\infty} A_n (x-x_0)^{\alpha n}$ as $(x \to x_0)$ with $\alpha > 0$, then the coefficients A_n are unique and can be calculated as follows :
  $$a_0 = \lim_{x \to x_0} f(x)$$
  $$a_1 = \lim_{x \to x_0} \frac{f(x) - A_0}{(x-x_0)^{\alpha}}$$
  $$a_2 = \lim_{x \to x_0} \frac{f(x) - (A_0 + A_1(x-x_0)^{\alpha})}{(x-x_0)^{\alpha 2}}$$
  $$etc \cdots$$
  The only condition is that these limits exist.

A few examples of asymptotic expansions of integrals :

1) $I(x)=\int_0^1 \frac{\sin(xt)}{t}\,dt$ as $(x\to 0)$

We use Taylor expansion of $\sin(x)$ (infinite convergence radius) :

$$\frac{\sin(xt)}{t}=x-\frac{x^3t^2}{3!}+\frac{x^5t^4}{5!}-\cdots+(-1)^{2p+1}\frac{x^{2p+1}t^{2p}}{(2p+1)!}+\cdots$$

$$I(x)\sim x-\frac{x^3}{18}+\frac{x^5}{600} \quad \text{as } \left(x\to 0\right)$$

2) $I(x)=\int_0^\infty \frac{e^{-t}}{1+xt}\,dt$ as $(x\to 0)$

With successive integrations by parts :

$$I(x)=\left[\frac{-e^{-t}}{1+xt}\right]_0^\infty-\int_0^\infty \frac{xe^{-t}}{\left(1+xt\right)^2}\,dt$$

$$\vdots$$

$$=\underbrace{\sum_{n=0}^{N}(-1)^n n!x^n}_{S_N}+\underbrace{(-1)^{N+1}(N+1)!\int_0^\infty \frac{x^{N+1}e^{-t}}{\left(1+xt\right)^{N+2}}\,dt}_{\varepsilon_N}$$

$\left(\sum_{n=0}^{\infty}(-1)^n n!x^n\right)$ diverges, and $\quad \forall x \quad \varepsilon_N \xrightarrow[N\to\infty]{} \infty$

However : for a given N, $\varepsilon_N \xrightarrow[x\to 0]{} 0$

Thus : $I(x)\sim \sum_{n=0}^{+\infty}(-1)^n n!\,x^n$, as $(x\to 0)$

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Série SCHAUM (développements de Taylor, séries entières)

Chapter 3 - PERTURBATION METHODS FOR DIFFERENTIAL EQUATIONS

I. A regular problem: the projectile problem

Let a projectile of mass m launched vertically with a speed V. We denote by x(t) its position depending on time t. According to the gravitation law and the 2nd Newton's law:

$$\frac{d^2x}{dt^2} = -g\,\frac{R^2}{(\mathrm{x}+\mathrm{R})^2}$$

with initial conditions (IC) : $x(0) = 0$ and $\dot{x}(0) = V$

Let us nondimensionalize the problem in choosing V^2/g as scale for x: $x = \underbrace{\frac{V^2}{g}}_{\substack{odg\\ de x}}\ \underbrace{\tilde{x}}_{odg 1}$

where $\tilde{x}$ in dimensionless. It is valid scaling when $x \ll R$.
If the acceleration is of order g, then time scale T of the system is fixed.
Indeed, if $t = T\tilde{t}$ then $\frac{d^2x}{dt^2} = \frac{V^2}{gT^2}\frac{d^2\tilde{x}}{d\tilde{t}^2}$. And if we want: $\frac{V^2}{gT^2} = g$, then necessarily: $T = \frac{V}{g}$.

According with this scaling : $x = \frac{V^2}{g}\tilde{x}$ and $t = \frac{V}{g}\tilde{t}$ and we get the nondimensional problem:

$$\boxed{\ddot{\tilde{x}} = \frac{-1}{(1+\varepsilon\tilde{x})^2}, \quad \tilde{x}(0) = 0, \quad \dot{\tilde{x}}(0) = 1, \quad \text{avec} \quad \varepsilon = \frac{V^2}{gR}}$$

Given the parameter ε, we suppose that the solution $x(t, \varepsilon)$ is expandable as a power series: $x(t,\varepsilon) = \sum_{n=0}^{\infty} x_n(t)\,\varepsilon^n$. Then: $\ddot{x} = \sum_{n=0}^{\infty} \ddot{x}_n(t)\,\varepsilon^n$.

Attention : as it is common to do to alleviate the notations, we deleted the ~ over x and t. So in the following calculations x and t are nondimensional variables.

Let us consider the order 2 expansion:

$$\ddot{x} = \ddot{x}_0(t) + \varepsilon\ \ddot{x}_1(t) + \varepsilon^2\ddot{x}_2(t) + O(\varepsilon^3)$$

Using binomial expansion:

$$(1+\alpha)^n = 1 + n\alpha + n\frac{(n-1)}{2!}\alpha^2 + n\frac{(n-1)(n-2)}{3!}\alpha^3 + \cdots$$

which converges if α<1, we get :

$$\left(1+\varepsilon\, x\right)^{-2} = 1 + (-2x_0)\varepsilon \; + (-2x_1 + 3x_0^2)\varepsilon^2 + O(\varepsilon^3)$$

To verify the equation $\ddot{x} = -(1+\varepsilon x)^{-2}$ the powers of ε have to be identical from part to part of the equal sign. We get :

$$\begin{cases} \ddot{x}_0 = -1 & x_0(0) = 0 & \dot{x}_0(0) = 1 \\ \ddot{x}_1 = 2x_0 & x_1(0) = 0 & \dot{x}_1(0) = 0 \\ \ddot{x}_2 = 2x_1 - 3x_0^2 & x_2(0) = 0 & \dot{x}_2(0) = 0 \end{cases}$$

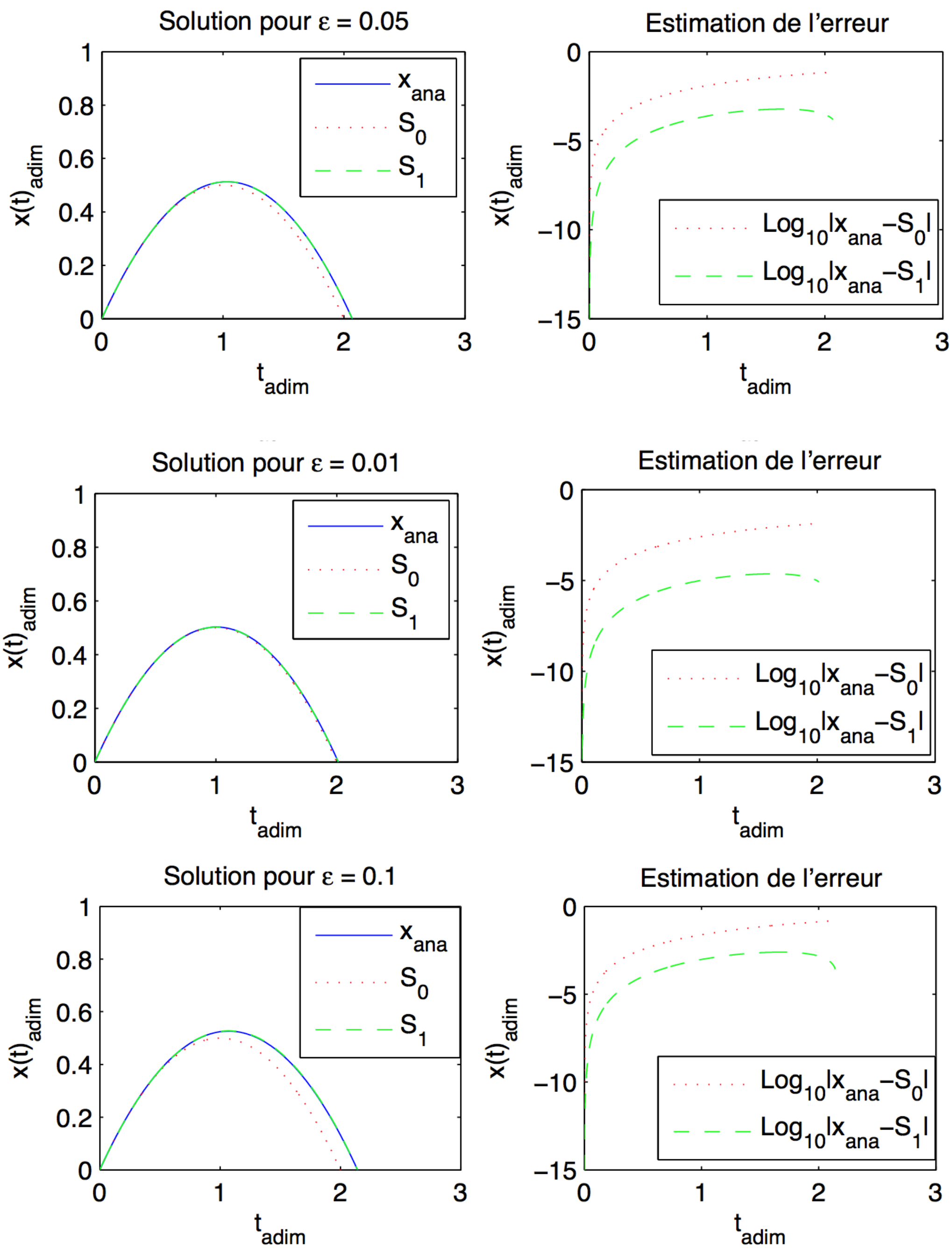
(Remark : Do not forget to develop the initial conditions !)

We then solve each equation:

$$\begin{cases} x_0(t) = t - \dfrac{t^2}{2} \\ x_1(t) = \dfrac{t^3}{3} - \dfrac{t^4}{12} \\ x_2(t) = -\dfrac{t^4}{4} + \dfrac{11}{60}t^5 - \dfrac{11}{360}t^6 \end{cases}$$

The solution of the nondimensional problem is thus :

$$\boxed{x(t) = (t - \frac{t^2}{2}) + \varepsilon\;(\frac{t^3}{3} - \frac{t^4}{12}) + \varepsilon^2(-\frac{t^4}{4} + \frac{11}{60}t^5 - \frac{11}{360}t^6) + O(\varepsilon^3)}$$



In this section only the simplest perturbation method has been presented which is called regular perturbation method. However, for some problems depending on a parameter ε, a simple expansion in powers of ε may fail. This is the case when the problem has a singularity at $\varepsilon = 0$, that is to say that the problem changes character at this point (if ε is in front of the highest derivative of a differential equation for instance). This is also the case when the solution of the problem has significant variations near a boundary condition: the solution approximated by a regular perturbation method becomes incompatible with the boundary conditions (for instance in boundary layers in fluid mechanics). To address this type of problem other more subtle perturbation methods have to be developed. This is the objective of the end of this course. These methods fall within the class of singular perturbation methods and we will see one of them: the WKB method. Other methods exist as the boundary layers method, and the multiple scales methods.

II. Singular problems: Global analysis of Ordinary Differential Equations with the WKB Method.

1. Dissipative and dispersive singular problems:

Dissipative processes

Let us consider the problem:

$$\epsilon y'' - y = 0 \quad \forall x \in]0,1[\qquad y(0) = 0 \quad y(1) = 1$$

This is a singular perturbation problem: the non perturbed problem ($\varepsilon = 0$) has no solution (y can not both remain constant and satisfy the two boundary conditions). In particular the solution of the perturbed problem can not be developed as $y(x) = \sum_{n=0}^{\infty} y_n(x)\varepsilon^n$ (regular perturbation method) because then $y_0(x)$ does not exist.

For this particular problem the exact solution is known: $y(x) = \dfrac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}$

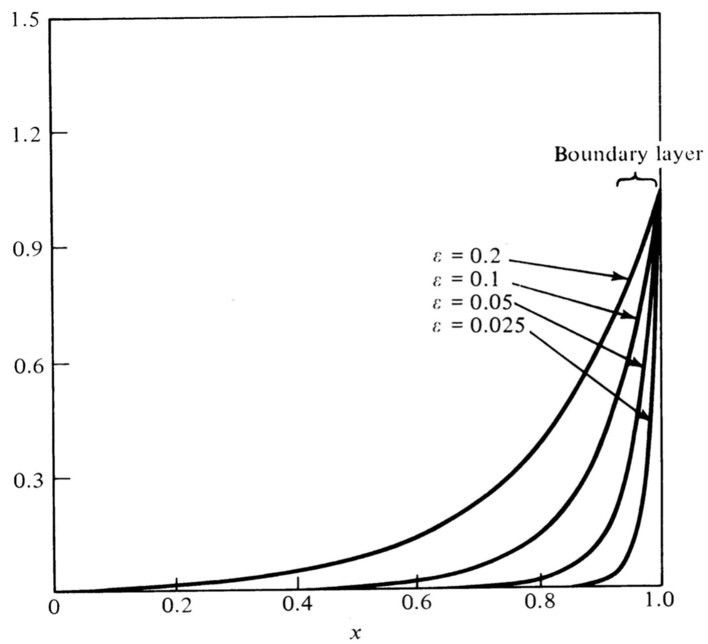


Figure 7.2 A plot of $y(x) = (e^{x/\varepsilon} - 1)/(e^{1/\varepsilon} - 1)$ $(0 \le x \le 1)$ for $\varepsilon = 0.2, 0.1, 0.05, 0.025$. When ε is small $y(x)$ varies rapidly near $x = 1$; this localized region of rapid variation is called a boundary layer. When ε is negative the boundary layer is at $x = 0$ instead of $x = 1$. This abrupt jump in the location of the boundary layer as ε changes sign reflects the singular nature of the perturbation problem.

One observes that as ε becomes very small but non null $y(x)$ is quasi constant over the interval [0,1] except over a tiny interval close to $x=1$ with a $O(\varepsilon)$ width. This interval is called boundary layer (« couche limite» in french). One says that the problem has a local « breakdown » in this boundary layer as $\varepsilon \to 0$, that is to say close to the boundary condition. In that case the solution is exponentially decreasing from this boundary condition this is why the term “dissipative” is used.

Dispersive processes:

Let us consider the problem:

$$\epsilon y'' + y = 0 \quad \forall x \in]0,1[\qquad y(0) = 0 \quad y(1) = 1$$

This is again a singular problem; however the problem has a global « breakdown » instead of local.

Indeed, the exact solution is: $y(x) = \dfrac{\sin\left(x/\sqrt{\varepsilon}\right)}{\sin\left(1/\sqrt{\varepsilon}\right)} \quad \forall \varepsilon \neq \left(n\pi\right)^{-2}$

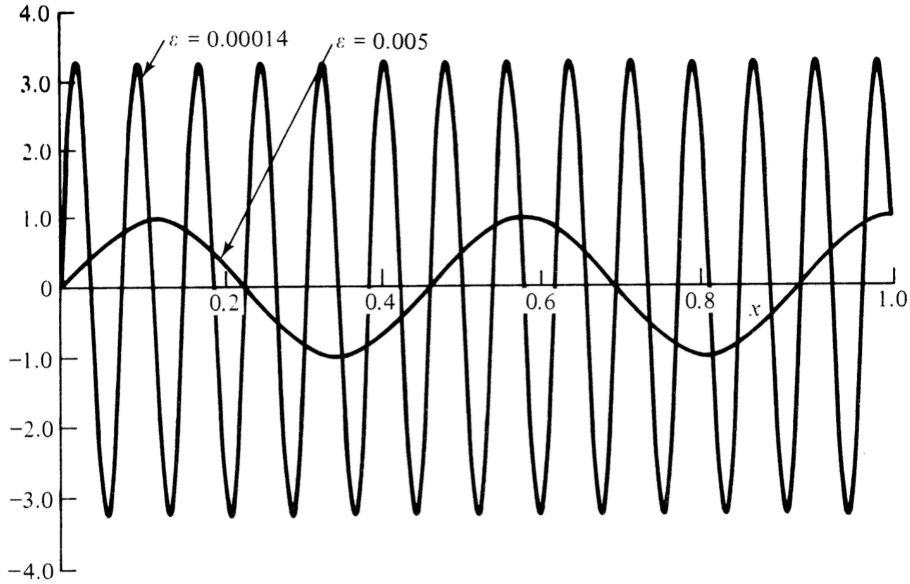


Figure 7.3 A plot of $y(x) = [\sin(x\varepsilon^{-1/2})]/[\sin(\varepsilon^{-1/2})]$ $(0 \le x \le 1)$ for $\varepsilon = 0.005$ and 0.00014. As ε gets smaller the oscillations become more violent; as $\varepsilon \to 0+$, $y(x)$ becomes discontinuous over the entire interval. The WKB approximation is a perturbative method commonly used to describe functions like $y(x)$ which exhibit rapid variation on a global scale.

One observes that as ε becomes very small but non null, $y(x)$ oscillates more and more « rapidly » in x over the whole interval [0,1]. And as $\varepsilon \to 0^+$, the limit solution is discontinued over the whole interval [0,1]. This is a global « breakdown ». The « dispersive » term is used in relation to a wave solution which oscillates rapidly with small and slow changes in wavelength and slow amplitude variations in function of x.

The WKB method is a powerful method to obtain global approximate solutions of these two kinds of problems. This is naturally a singular perturbation method but it is valid only for linear ordinary differential equations which exhibit a small term ε in front of its highest degree term. This method is very useful to find approximate « global solutions » of these equations (over the whole domain corresponding to the problem) where classical perturbation methods give only local approximate solutions.

Its name, WKB method, comes from Wentzel, Kramers & Brillouin who contributed to develop the method. It is sometimes also called WKBJ for Jeffreys who also contributed.

2. The WKB method

a) General principle of the WKB method

The WKB method is to search for approximate solutions $y(x)$ as an exponential function of a power series:

$$y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right] \qquad \delta \to 0 \qquad (1)$$

We put this expression into the equation and we determine δ in function of ε and also the functions $S_0(x)$, $S_1(x)$, $S_2(x)$, ...

Dissipative and dispersive problems are all characterized by exponential like solutions with a real exponent in the dissipative case and an imaginary exponent in the dispersive case.
It is thus logical to search for a solution as an exponential approximation:

$$y(x) \sim A(x)e^{S(x)/\delta}, \quad \delta \to 0 \qquad (2)$$

δ is a scale over which the solution varies rapidly (width of the boundary layer for a dissipative problem and wavelength for a dispersive problem).
In the equation (2) the phase $S(x)$ is supposed to be non constant and slowly variable in a « breakdown » region. When $S(x)$ is constant $y(x)$ is given by the slowly variable amplitude $A(x)$.
The key idea of the WKB method is to use this particular form (2) of the solution but in practice it is more useful to express $S(x)$ and $A(x)$ as power series of δ, which gives (1).

We say that $S(x)$ is slowly variable if $S(x)$ varies as $O(1)$ over an O(1) interval.
We say that $S(x)$ is rapidly variable if $S(x)$ varies as $O(1)$ over an O(ε) interval (ε«1)

b) WKB method development

Let us consider the following equation:

$$\varepsilon^2 \ddot{y} = Q(x)y \qquad \text{with } \; Q(x) \neq 0$$

Let us search WKB approximate solutions as $\varepsilon \ll 1$, as follows :

$$y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right] \qquad (\delta \to 0)$$

Differentiating this expression we get:

$$\dot{y}(x) \sim \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n \dot{S}_n(x)\right)\exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right] \qquad (\delta \to 0)$$
$$\ddot{y}(x) \sim \left(\frac{1}{\delta^2}\left(\sum_{n=0}^{\infty}\delta^n \dot{S}_n(x)\right)^2 + \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n \ddot{S}_n(x)\right)\exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right] \qquad (\delta \to 0)$$

And replacing these expressions into the equation gives:

$$\frac{\varepsilon^2}{\delta^2}\dot{S}_0^2 + 2\frac{\varepsilon^2}{\delta}\dot{S}_0\dot{S}_1 + \frac{\varepsilon^2}{\delta}\ddot{S}_0^2 + \cdots = Q(x)$$

We then suppose that there exists a dominant equilibrium between the right hand term ($Q(x)$) and the greatest left hand term ($\frac{\varepsilon^2}{\delta^2}\dot{S}_0^2$) which gives: $\frac{\varepsilon^2}{\delta^2}\dot{S}_0^2 \sim Q(x)$, and thus $\delta = \varepsilon$

Then :

$$\dot{S}_0^2 = Q(x)$$
$$2\dot{S}_0\dot{S}_1 + \ddot{S}_0 = 0$$
$$2\dot{S}_0\dot{S}_n + \ddot{S}_{n-1} + \sum_{j=1}^{n-1}\dot{S}_j\dot{S}_{n-j} = 0 \quad \forall n \geq 2$$

Thus

$$S_0(x) = \pm\int_a^x \sqrt{Q(t)}dt$$

and $S_1(x) = -\frac{1}{2}\ln\left(\sqrt{Q(x)}\right)$

Thus, at the order 1 we get :

$$y(x) \sim C_1\left(Q(x)\right)^{-1/4} e^{\frac{1}{\varepsilon}\int_a^x \sqrt{Q(t)}dt} + C_2\left(Q(x)\right)^{-1/4} e^{\frac{-1}{\varepsilon}\int_a^x \sqrt{Q(t)}dt} \qquad (\varepsilon \to 0)$$

where a is an arbitrary integration constant.

The constants C_1 and C_2 are determined by the initial conditions.

Let us suppose that: $y(0) = A \quad \dot{y}(0) = B$.
And let $a = 0$

$$y(0) \sim (C_1 + C_2)\big(Q(0)\big)^{-\frac{1}{4}} = A$$

$$\dot{y}(0) \sim -\frac{1}{4}(C_1 + C_2)\dot{Q}(0)\big(Q(0)\big)^{-\frac{5}{4}} + \frac{1}{\epsilon}(C_1 - C_2)\big(Q(0)\big)^{\frac{1}{4}} = B$$

If $A = 0$ and $B = 1$ we get:

$$y(x) \sim \epsilon\big(Q(x)Q(0)\big)^{-\frac{1}{4}} sinh\left(\int_0^x \frac{\sqrt{Q(t)}}{\epsilon}dt\right) \qquad (\epsilon \to 0)$$

In particular, if $Q(x) = \left(1+x^2\right)^2$ we obtain:

$$y(x) \sim \frac{\varepsilon}{\sqrt{1+x^2}}\sinh\left(\frac{1}{\varepsilon}(x+\frac{x^3}{3})\right) \qquad (\varepsilon \to 0)$$

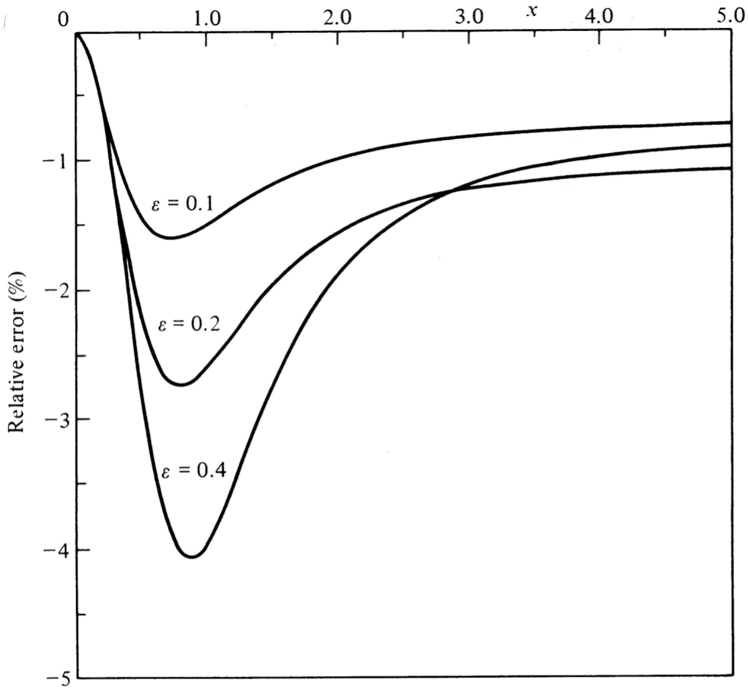


Figure 10.1 A plot of the relative error between the exact solution to the initial-value problem $\varepsilon^2 y'' = (1 + x^2)^2 y$ [$y(0) = 0$, $y'(0) = 1$] in (10.1.18) and the leading-order WKB approximation to $y(x)$, $y(x) \sim \varepsilon(1 + x^2)^{-1/2} \sinh[(x + x^2/3)/\varepsilon]$ $(\varepsilon \to 0)$, in (10.1.19) for three values of ε. The relative error is defined as (WKB approximation − exact solution)/(exact solution).

Exercice : *Develop the same method but for the dispersive problem :*

$$\varepsilon\ddot{y} + y = 0 \quad \forall x \in \left]0,1\right[\qquad y(0) = 0 \qquad y(1) = 1$$

and show that we then obtain the exact solution.

c) Validity conditions of the WKB method

WKB method is clearly a singular method: if $S_0(x) \neq 0$ then the approximation $y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right]$ does not exist anymore if $\delta = 0$.

The expansions $\sum_{n=0}^{\infty}\delta^n S_n(x)$ obtained with the WKB method are generally divergent series, and we need asymptotic expansion theory. However the approximation that is obtained with such a method is generally extremely good.

The validity conditions for the order N WKB solutions are:

(1) $$\begin{cases} S_1(x) \ll \frac{1}{\delta} S_0(x) & \delta \to 0 \\ \delta S_2(x) \ll S_1(x) & \delta \to 0 \\ \vdots & \\ \delta^n S_{n+1}(x) \ll \delta^{n-1} S_n(x) & \delta \to 0 \end{cases} \quad \text{uniformly in } x$$

(2) $$\delta^N S_{N+1}(x) \ll 1 \qquad \delta \to 0$$

One can show that when these conditions are verified the relative error between $y(x)$ and the N[th] order WKB solution is::

$$\frac{y(x) - \exp\left[\frac{1}{\delta}\sum_{n=0}^{N}\delta^n S_n(x)\right]}{y(x)} \sim \delta^N S_{N+1}(x) \qquad (\delta \to 0)$$

III. Some applications of the WKB method

1. Airy equation

Approximate solution of: $\ddot{y}(x) = xy(x)$ as x becomes large.

If x is large one can let: $x = t/\varepsilon$ ($t = \varepsilon x$ is called slow variable as $\varepsilon \to 0$).

Then: $\varepsilon^2 \ddot{y}(t) = \dfrac{t}{\varepsilon} y(t)$

and then:

$$y(t) \sim C_1 \left(\frac{t}{\varepsilon}\right)^{-1/4} e^{\frac{1}{\varepsilon}\int_a^t \sqrt{\frac{u}{\varepsilon}}du} + C_2 \left(\frac{t}{\varepsilon}\right)^{-1/4} e^{\frac{-1}{\varepsilon}\int_a^t \sqrt{\frac{u}{\varepsilon}}du} \qquad (\varepsilon \to 0)$$

and finally: $y(x) \sim \dfrac{C_1}{x^{1/4}} e^{\frac{2}{3}x^{3/2}} + \dfrac{C_2}{x^{1/4}} e^{\frac{-2}{3}x^{3/2}}$ as x is large.

Reminder about Airy Functions:

Airy functions are solutions of the Airy equation $\ddot{y}(x) = xy(x)$. One knows Taylor expansions of these functions as well as expressions in terms of Bessel functions (cf. bibliographic references).

In particular we usually note $Ai(x)$ and $Bi(x)$ the two linearly independent Airy functions which are defined by their Taylor series :

$$\begin{cases} Ai(x) = 3^{-2/3} \displaystyle\sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} - 3^{-4/3} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})} \\ Bi(x) = 3^{-1/6} \displaystyle\sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} - 3^{-5/6} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})} \end{cases}$$

The following figure shows these two functions..

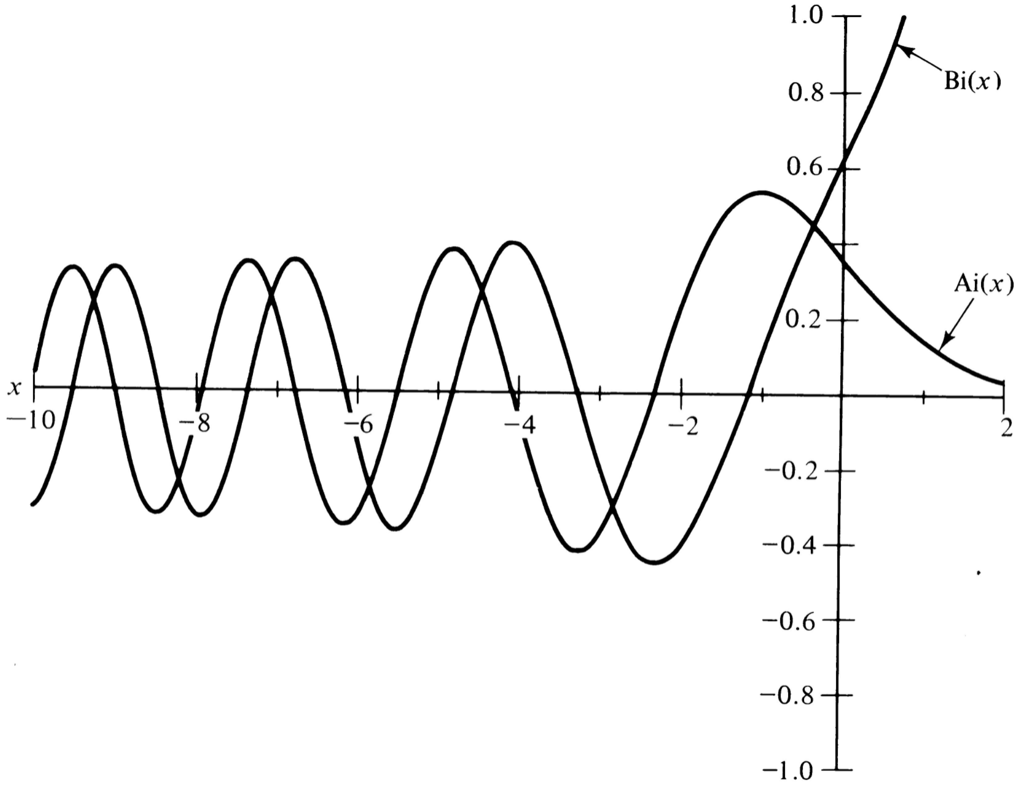


Figure 3.1 A plot of the Airy functions Ai(x) and Bi(x) for $-10 \le x \le 2$. Both functions are oscillatory for negative x; Bi(x) grows exponentially and Ai(x) decays exponentially as $x \to +\infty$.

2. Sturm-Liouville problem

$$\ddot{y}(x) + EQ(x)y(x) = 0 \quad \forall x \in [0, \pi], \qquad y(0) = y(\pi) = 0 \qquad (Q(x) > 0 \quad \forall x)$$

Such a problem has an infinite countable number of strictly positive and distinct eigenvalues$(E_1, E_2, \cdots, E_n, \cdots)$. The corresponding eigenfunctions $(y_1, y_2, \cdots, y_n, \cdots)$ are orthogonal using the scalar product:

$$\int_0^\pi y_n y_m Q(x) dx = 0 \quad m \neq n$$

And y_n can be normalized: $\int_0^\pi y_n^2 Q(x) dx = 1$

We are using here the WKB method to determine approach eigenfunctions y_n for large values of n.

We anticipate that $E_n \sim n^2 \quad (n \to \infty)$. If $\varepsilon = \dfrac{1}{E_n}$ then the problem becomes: $\varepsilon \ddot{y} + Q\, y = 0 \quad y(0) = y(\pi) = 0$ and the method will be precise as n is large ($\varepsilon \underset{n\to\infty}{\to} 0$).

From the part I of this chapter, the WKB approximate solutions are :

$$\left(Q(x)\right)^{-1/4} \sin\left(\sqrt{E}\int_0^x \sqrt{Q(t)}dt\right) \quad \text{and} \quad \left(Q(x)\right)^{-1/4} \cos\left(\sqrt{E}\int_0^x \sqrt{Q(t)}dt\right)$$

$$y(0) = 0 \Rightarrow \; y(x) \sim C\left(Q(x)\right)^{-1/4} \sin\left(\sqrt{E}\int_0^x \sqrt{Q(t)}dt\right) \quad E \to \infty$$

$$y(\pi) = 0 \Rightarrow \sqrt{E_n} \sim \frac{n\pi}{\int_0^\pi \sqrt{Q(t)}dt} \quad (n \to \infty)$$

Thus
$$E_n \sim \left(\frac{n\pi}{\int_0^\pi \sqrt{Q(t)}dt}\right)^2 \quad (n \to \infty)$$

and
$$y_n(x) \sim C_n \left(Q(x)\right)^{-1/4} \sin\left(\sqrt{E_n}\int_0^x \sqrt{Q(t)}dt\right)$$

And using the normalisation $\int_0^\pi y_n^2\, Q(x)dx = 1$ we obtain C_n as:

$$\int_0^\pi C_n^2 \frac{1}{\sqrt{Q(x)}} \sin^2\left(\sqrt{E_n}\int_0^x \sqrt{Q(t)}dt\right) Q(x)dx \sim 1 \quad (n \to \infty)$$

We let: $u = \sqrt{E_n}\int_0^x \sqrt{Q(t)}dt$ and get : $C_n^2 \sim \dfrac{2}{\int_0^\pi \sqrt{Q(t)}\, dt} \quad (n \to \infty)$

Thus:

$$y_n(x) \sim \left(\int_0^\pi \frac{\sqrt{Q(t)}}{2} dt\right)^{-1/2} \left(Q(x)\right)^{-1/4} \sin\left(n\pi \frac{\int_0^x \sqrt{Q(t)}dt}{\int_0^\pi \sqrt{Q(t)}dt}\right) \quad (n \to \infty)$$

Approximation for $Q(x) = (x+\pi)^4$:

$$E_n = \frac{9n^2}{49\pi^2}$$

$$y_n(x) \sim \sqrt{\frac{6}{7\pi^3}}\frac{1}{x+\pi}\sin\frac{n(x^3+3\pi x^2+3\pi^2 x)}{7\pi^2}$$

Table 10.1 A comparison of the exact eigenvalues E_n of the Sturm-Liouville problem $y''(x) + E(x+\pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$ with the leading-order WKB prediction [see (10.1.34)] for these eigenvalues $E_n \sim 9n^2/49\pi^2$ $(n \to \infty)$

As expected, this prediction becomes more accurate as n increases. The relative error is defined as (approximate − exact)/(exact)

n	E_n(WKB)	E_n(exact)	Relative error, %
1	0.00188559	0.00174401	8.1
2	0.00754235	0.00734865	2.6
3	0.0169703	0.0167524	1.3
4	0.0301694	0.0299383	0.77
5	0.0471397	0.0469006	0.51
10	0.188559	0.188305	0.13
20	0.754235	0.753977	0.035
40	3.01694	3.01668	0.009

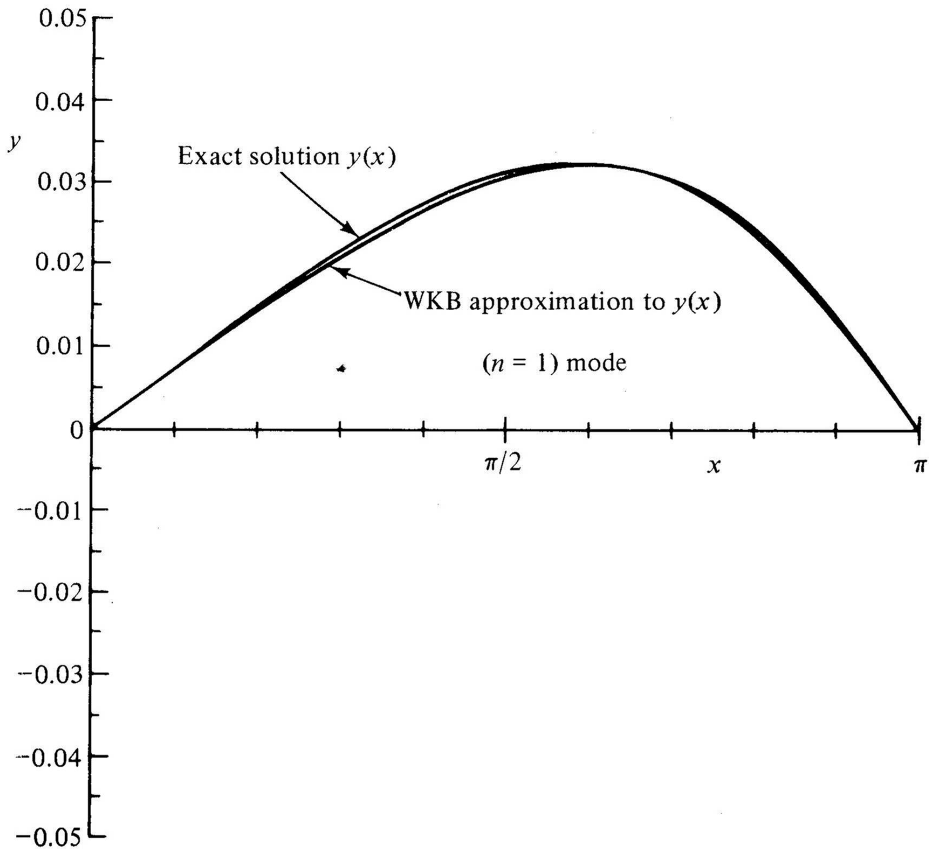


Figure 10.2 Comparison of the exact solution to $y''(x) + E_n(x + \pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$, with the WKB approximation to this solution as given in (10.1.35) for the lowest ($n = 1$) mode. Although WKB becomes exact as $n \to \infty$, this plot shows that even when $n = 1$ the WKB approximation is extraordinarily accurate.

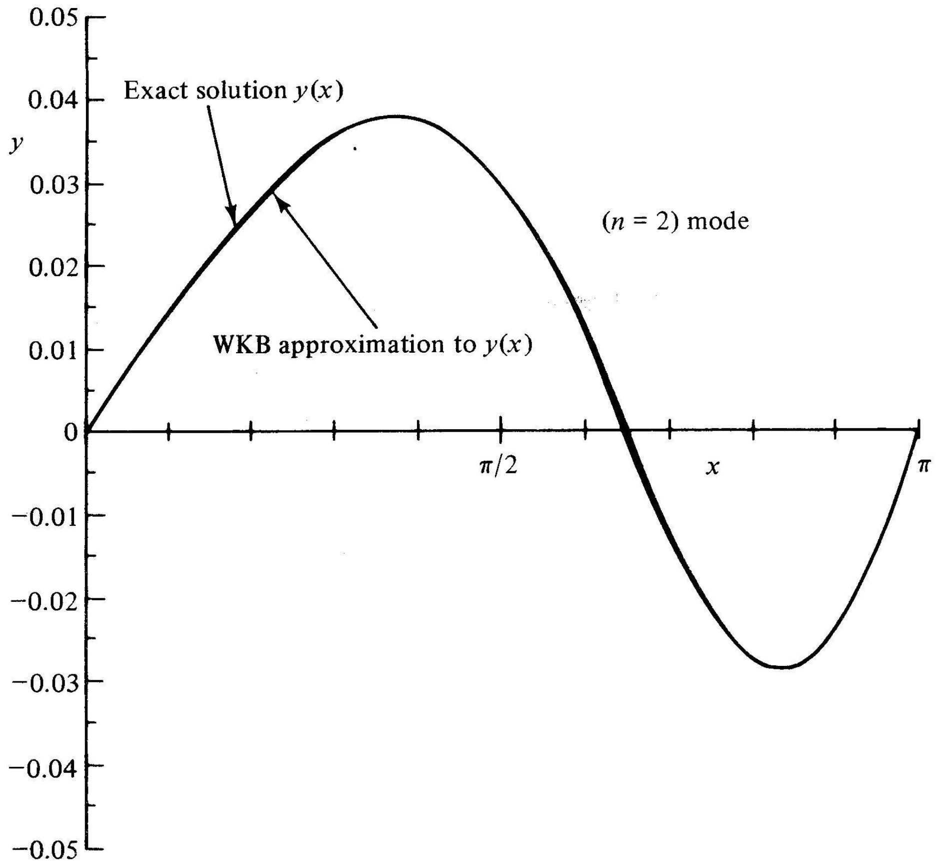


Figure 10.3 Same as in Fig. 10.2 except that $n = 2$. The exact eigenfunction and the WKB approximation are almost indistinguishable.

Bibliography (chapter 3) :

Bender & Orszag: Advance Mathematical Methodes for Scientists and Engineers - Chapter 10 (Mc Graw Hill - Advanced book program)

Morse & Feshbach: Methods of Theoretical Physics - Chapter 9 (Mc Graw Hill - International series in pure and applied mathematics)