II.4 Convective instability

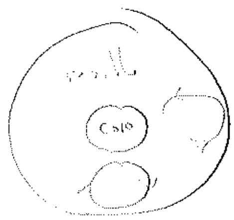
II.4.1 Rayleigh-Benard instability

Instability caused by a vertical gradient of T (cold on top of warm).

= atmosphere heated from below by the ground

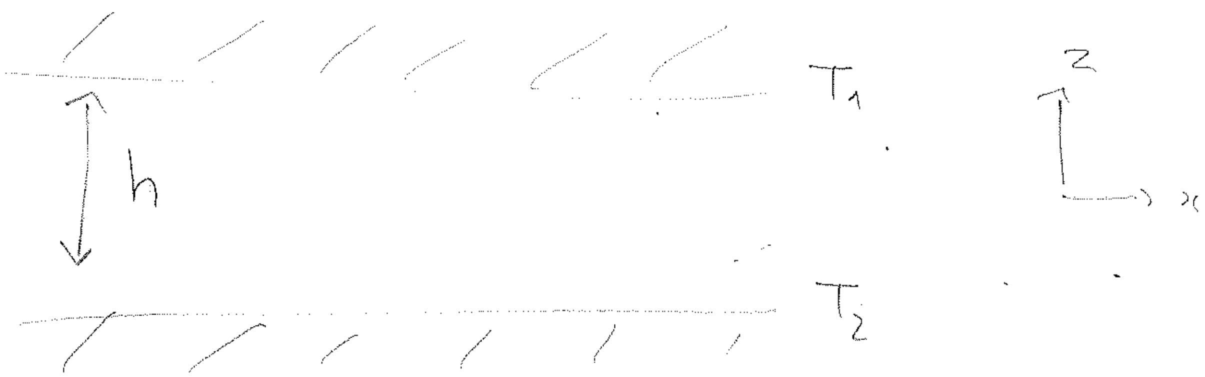
= mantle convection!

= oil in a pan



- the flow is destabilized by buoyancy forces.
- in competition with viscosity and diffusivity.

ex: Rayleigh-Benard (theoretical analysis by Rayleigh)



② Some adimensional numbers!

physical parameters are
$$\left(\text{heat equation } \frac{\partial T}{\partial t} = k\nabla^2 T\right)$$
ν = kinematic viscosity
k = thermal diffusivity
h the height of fluid.

- acceleration due to buoyancy?

Linear equation of state for $\rho(T)$, with $\rho(T_0) = \rho_0$

$\rightarrow$ $\boxed{\rho = \rho_0 (1 - \alpha(T - T_0))}$ $\alpha = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$ the thermal expansion coef.

So the acceleration is $\boxed{g\frac{\Delta \rho}{\rho_0} = g\alpha\Delta T}$

With these 4 parameters, we can define 2 numbers:

(1) The Prandtl number $\boxed{P_r = \frac{\nu}{k} = \frac{\text{viscosity}}{\text{diffusivity}}}$

[metals $P_r \ll 1$ $\rightarrow$ heat diffuses quickly = conduction
oil $P_r \gg 1$ $\rightarrow$ heat diffuses slowly = convection

(2) The Rayleigh number $\boxed{R_a = \frac{g\alpha\Delta T h^3}{\nu k}}$

experimentally : fluid is at rest if $R_a < R_{a_c}$ a critical value. if $R_a > R_{a_c}$, the flow become unstable (bifurcations $\rightarrow$ full turbulence)

(2)

- if a fluid parcel is accelerated over the height h during a time τ_0 by buoyancy forces

$$g\alpha\Delta T \sim \frac{h}{\tau_0^2}$$

- effect of viscosity (smoothing velocity gradients) corresponds to a time-scale $\tau_\nu \sim \frac{h^2}{\nu}$

- effect of diffusivity $\rightarrow$ $\tau_T \sim \frac{h^2}{k}$

So the Rayleigh number corresponds to

$$Ra = \frac{\tau_\nu \tau_T}{\tau_0^2}$$

- if Ra is small, time for diffusion of momentum and T is large compared to time to accelerate.

$=$ fluctuations are damped
viscous and diffusive effects dominate the flow.

- if Ra is large, it is the opposite ...

(b) Use the equations to find critical Ra

We use the NS equations

$$\frac{D\vec{u}}{Dt} = - \frac{\vec{\nabla} p}{\rho} + \vec{g} + \nu \nabla^2 \vec{u}$$

$$\frac{DT}{Dt} = k \nabla^2 T$$

$$\vec{\nabla} \cdot \vec{u} = 0 \quad \text{(we neglect compressibility)}$$

with $\rho = \rho_0 (1 - \alpha (T - T_0))$ linear EOS.

- The balanced state is

$$\boxed{\begin{aligned} \vec{U}_0 &= \vec{0} \\ \frac{1}{\rho_0} \frac{\partial P_0}{\partial z} - g &= 0 \end{aligned}}$$

$$\nabla^2 T_0 = 0$$

$$\Longrightarrow \quad T_0 = A_1 z + A_2$$

with $\begin{vmatrix} T_0(0) = T_2 \\ T_0(h) = T_1 \end{vmatrix}$

$$\Longleftrightarrow \quad T_0 = \frac{T_1 - T_2}{h} z + T_2$$

$$\boxed{T_0(z) = T_2 - \frac{\Delta T \, z}{h}}$$

- We look at small perturbations around this state:

$$\rho = \rho_0 + \rho'$$
$$P = P_0 + p'$$

Using the Boussinesq approximation $\left(\begin{array}{c}\rho' \ll \rho_0 \\ \alpha\Delta T \ll 1\end{array}\right)$ (3)

we get $$\frac{D\vec{u}'}{Dt} = -\frac{\vec{\nabla}p}{\rho_0} + \vec{g}\frac{\rho}{\rho_0} + \nu\nabla^2\vec{u}$$

$$\Leftrightarrow \frac{D\vec{u}'}{Dt} = -\frac{\vec{\nabla}p'}{\rho_0} + \vec{g}\frac{\rho'}{\rho_0} + \nu\nabla^2\vec{u}' \underbrace{-\frac{\vec{\nabla}p_0}{\rho_0} + g\frac{\rho_0}{\rho_0}}_{=0}$$

- we linearize :

$$\frac{\partial\vec{u}'}{\partial t} = -\frac{\vec{\nabla}p'}{\rho_0} + \vec{g}\frac{\rho'}{\rho_0} + \nu\nabla^2\vec{u}'$$

$$\left\{\begin{array}{ll}(1) & \dfrac{\partial\vec{u}'}{\partial t} = -\dfrac{\vec{\nabla}p'}{\rho_0} + g\alpha T\vec{k} + \nu\vec{\nabla}^2\vec{u}' \\ (2) & T_t' + w\dfrac{\partial T_0}{\partial z} = T_t' - w\dfrac{\Delta T}{h} = k\nabla^2 T' \\ (3) & \vec{\nabla}.\vec{u}' = 0\end{array}\right.$$

- we adimensionalize :

$$x \to \frac{x}{h}$$

$$t \to \frac{t}{\tau_T} = \frac{t\,k}{h^2}$$

$$u \to u\,\frac{h}{\nu}$$

$$T' \to \frac{T'}{\Delta T} \qquad p \to \frac{p\,d^2}{\rho_0\nu^2}$$

The equations become:

$$\begin{cases} \dfrac{\partial \vec{u}'}{\partial t} = -P_r \vec{\nabla} p' + Ra\, T' \vec{k} + P_r \nabla^2 \vec{u}' \\ \dfrac{\partial T'}{\partial t} - P_r w = \nabla^2 T' \\ \vec{\nabla} \cdot \vec{u} = 0 \end{cases}$$

with $P_r = \dfrac{\nu}{k}$

$$Ra = \frac{g \alpha \Delta T . h^3}{\nu k}$$

- we also define boundary conditions:

$w = 0$ at $z = 0, 1$ no normal velocity

$T' = 0$ at $z = 0, 1$ no T fluctuations at the boundary we can now reduce the system to one unknown

momentum equations:

$$(1)\quad w_t = -P_r\, p_z + R_0 T + P_r \nabla^2 w$$

$$(2)\quad u_t = -P_r\, p_x + P_r \nabla^2 u$$

$$(3)\quad v_t = -P_r\, p_y + P_r \nabla^2 v$$

$$\partial_x (2) + \partial_y (3) + \partial_z (1)$$

$$\Rightarrow \underbrace{(u_{xt} + v_{yt} + w_{zt})}_{=0} = -P_r \nabla^2 p + P_r \nabla^2 \underbrace{(u_x + v_y + w_z)}_{=0} + R_0 T_z$$

$$\Rightarrow \quad R_0 T_z = P_r \nabla^2 p$$

So $\nabla^2 (1) \Rightarrow$

$$\nabla^2 w_t = -P_r \nabla^2 p_z + R_0 \nabla^2 T + P_r \nabla^4 w$$

$$= -R_0 T_{zz} + R_0 \nabla^2 T + P_r \nabla^4 w.$$

$$\nabla^2 w_t = R_0 \nabla_h^2 T + P_r \nabla^4 w$$

$$\Rightarrow (\partial_t - P_r \nabla^2) \nabla^2 w = R_0 \nabla_h^2 T \qquad (5)$$

using $T_t - P_r w = \nabla^2 T$

$$\Leftrightarrow (\partial_t - \nabla^2) T = P_r w$$

to eliminate T in (5), we get.

$$(\partial_t - \nabla^2)(\partial_t - P_r \nabla^2)\nabla^2 w = R_a \nabla_h^2\left((\partial_t - \nabla^2) T\right).$$
$$= R_a P_r \nabla_h^2 w \quad (6)$$

with boundary conditions $T = 0 \Rightarrow \nabla_h^2 T = 0 \,\big|_{z=0,1}$

so $\nabla^2 w_t = P_r \nabla^4 w \,\big|_{z=0,1}$

- we look for solutions in the form

$$w = \hat{w}(z)\, e^{i\vec{k}\cdot\vec{x} - st}$$
with $|\vec{k}| = k$

$\Leftrightarrow$ using (6):

$$\left(s + k^2 - \frac{\partial^2}{\partial z^2}\right)\left(s + P_r k^2 - P_r \frac{\partial^2}{\partial z^2}\right)\left(-k^2 + \frac{\partial^2}{\partial z^2}\right) w$$
$$= - R_a P_r k^2 w \quad (7)$$

- we can show by multiplying (7) by w^* and integrating between $z = 0, 1$ that

s is a real if $R_a > 0$.

This is called the <u>exchange of stabilities</u>

The marginal stability corresponds to the case $s=0$

this will give us the minimum ΔT such that the flow is stable.

$$\Longrightarrow \left(k^2-\frac{\partial^2}{\partial z^2}\right)\left(k^2-\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial z^2}-k^2\right)w = -R_a k^2 w$$

$$\Longrightarrow \left(k^2-\frac{\partial^2}{\partial z^2}\right)^3 w = R_a k^2 w$$

(independent of P_r)

- if we choose free slip conditions: (not realistic but easy to solve)

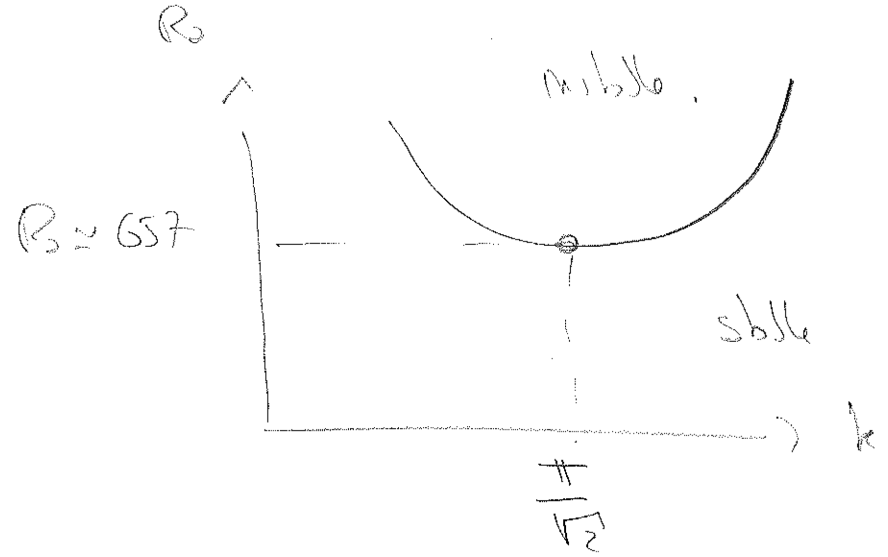
$$U_z = V_z = 0 \quad \text{at} \quad z=0,1$$

we get $w_{zz}=0$ at $z=0,1$

- solution is $w(z) = \sin(n\pi z)$

which gives marginal stability curve:

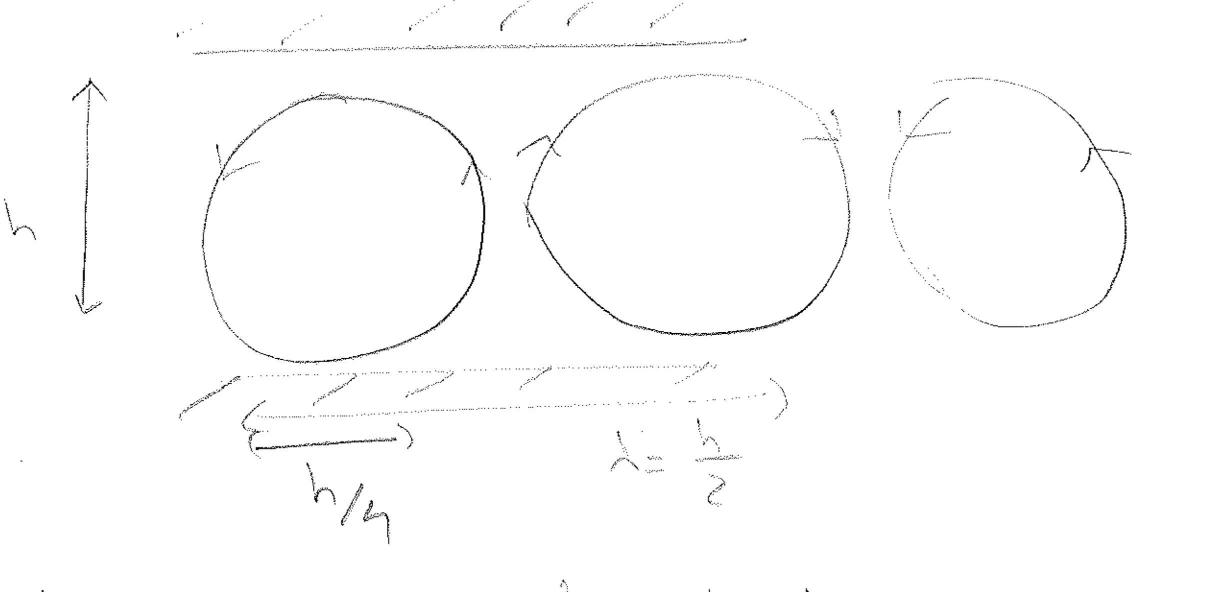
$$\boxed{R_a = \frac{(n^2\pi^2+k^2)^3}{k^2}}$$



For a given k, smaller values of Ra_c corresponds to $n=1$. and the minimum of the curve is for $k = \frac{\pi}{\sqrt{2}} \simeq 2$

The aspect ratio is close to 1.

(For larger k, the flow will be stabilized by viscous effects)



- For larger Rayleigh it will become turbulent.