Numerical Modelling

Jonathan GULA
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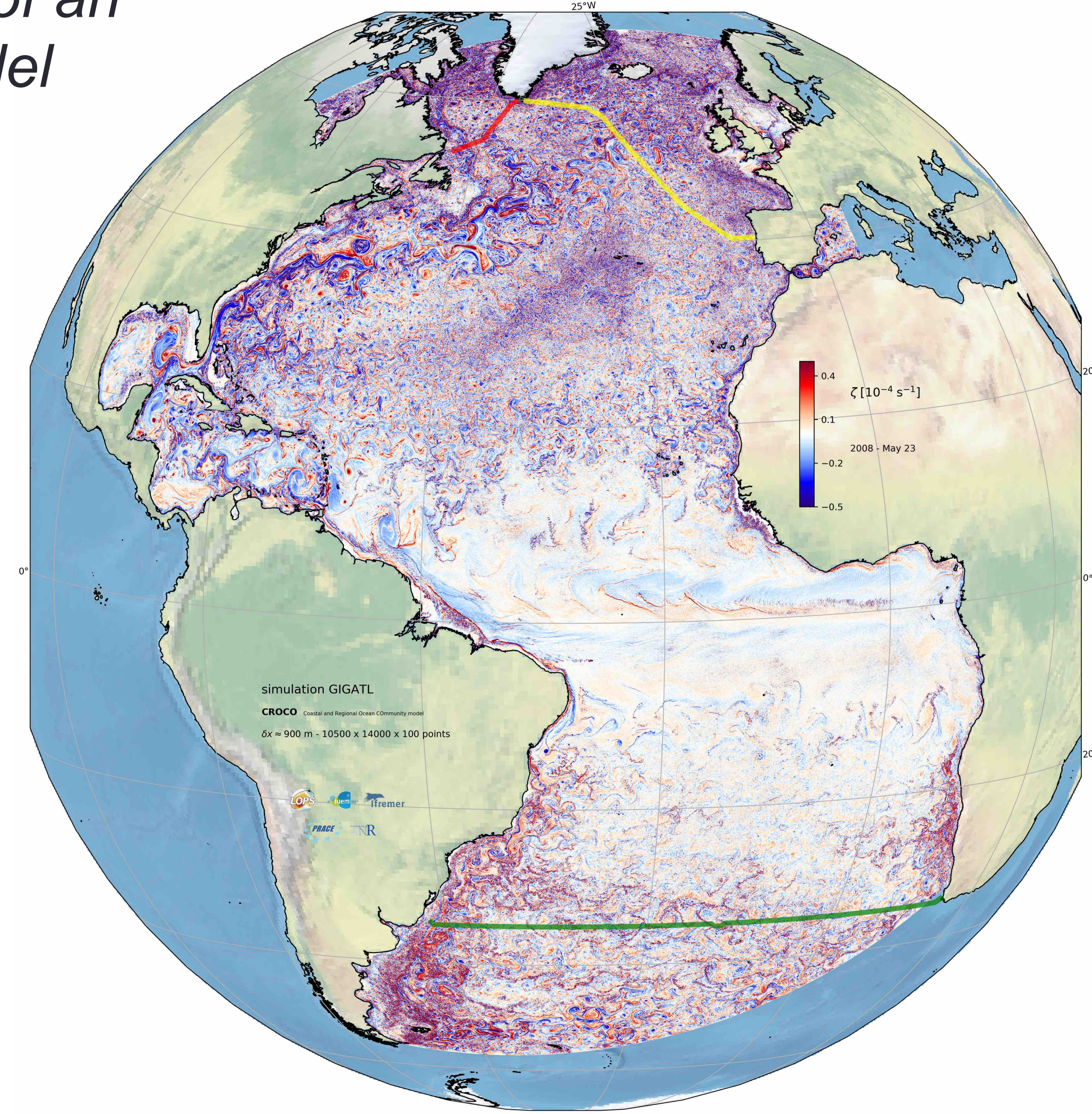
the anatomy of an ocean model



- **Lesson 1 :** [D109]
  - Introduction
  - Equations of motions
  - *Activity 1 [run an ocean model]*
- **Lesson 2 :** [D109]
  - Subgrid-scale parameterization
  - Dynamics of the ocean gyre
  - *Activity 2 [Dynamics of an ocean gyre]*
- **Lesson 3 :** [D109]
  - Horizontal Discretization
  - Dynamics of the ocean gyre (suite)
  - *Activity 2 [Dynamics of an ocean gyre]*
- **Lesson 4 :** [D109]
  - Numerical schemes
  - *Activity 3 [Impacts of numerics]*
- **Lesson 5 :** [D109]
  - Vertical coordinates
  - *Activity 3 [Impact of topography]*
- **Lesson 6 :** [D109]
  - Boundary Forcings
  - Presentation of the model CROCO
  - *Activity 5 [Design a realistic simulation]*
- **Lesson 7 :** [D109]
  - Diagnostics and validation
  - *Activity 6 [Analyze a realistic simulation]*
- **Lesson 8 :** [D109]
  - *Project*

Presentations and material will be available at :

jgula.fr/ModNum/

Useful references

Extensive courses:

- MIT: https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-950-atmospheric-and-oceanic-modeling-spring-2004/lecture-notes/
- Princeton: https://stephengriffies.github.io/assets/pdfs/GFM_lectures.pdf

Overview on ocean modelling and current challenges:

- Griffies et al., 2000, Developments in ocean climate modelling, Ocean Modelling.http://jgula.fr/ModNum/Griffiesetal00.pdf
- Griffies, 2006, "Some Ocean Model Fundamentals", In "Ocean Weather Forecasting: An Integrated View of Oceanography", 2006, Springer Netherlands. http://jgula.fr/ModNum/Griffies_Chapter.pdf
- Fox-Kemper et al, 19, “Challenges and Prospects in Ocean Circulation Models" http://jgula.fr/ModNum/FoxKemperetal19.pdf

ROMS/CROCO:

- https://www.myroms.org/wiki/
- Shchepetkin, A., and J. McWilliams, 2005: The Regional Oceanic Modeling System (ROMS): A split-explicit, free-surface, topography-following- coordinate ocean model. Ocean Modell. http://jgula.fr/ModNum/ShchepetkinMcWilliams05.pdf

#2
Subgrid-scale parameterization

Master's degree 2nd year Marine Physics

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Incompressible Navier-Stokes Equations:

- Dissipation of energy/momentum in the NS equations:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho} + \boxed{\nu \vec{\nabla}^2 \vec{u}}$$

Viscosity

Type of models

Navier Stokes

- DNS = Direct Numerical Simulation
- LES = Large Eddy Simulation
- RANS = Reynolds Averaged Navier Stokes

CFD

Process studies

PE

- PE = Primitive Equations models

SW

- SW = Shallow-Water models

Ocean Circulation Models

SQG

- SQG = Surface Quasi-Geostrophic models

QG

- QG = Quasi-Geostrophic models
- Etc.

Idealized models

Incompressible Navier-Stokes Equations:

$$\frac{\partial \vec{u}}{\partial t} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \boxed{\nu \vec{\nabla}^2 \vec{u}}$$

Non-linear terms

Viscosity

- Importance of NL terms and viscosity = Reynolds Number

$$Re = \frac{UL}{\nu}$$

Where *U* is a typical velocity of the flow and *L* is a typical length describing the flow.

Incompressible Navier-Stokes Equations:

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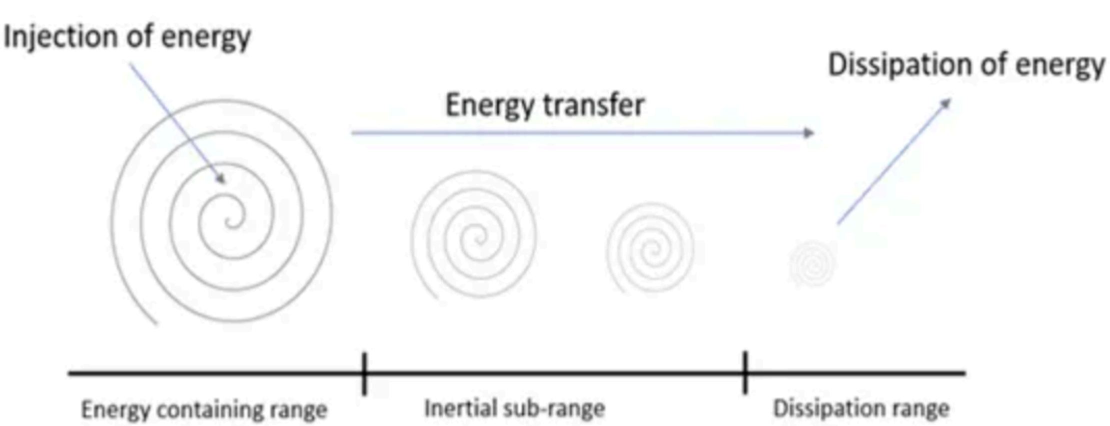
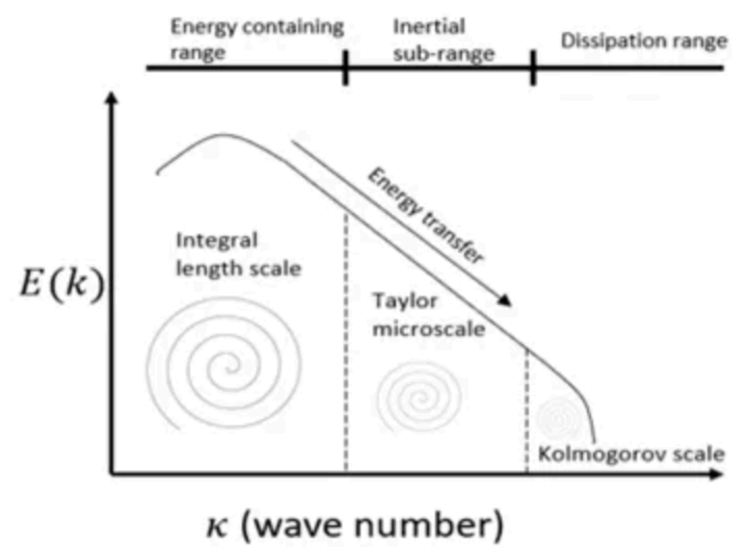
Non-linear terms

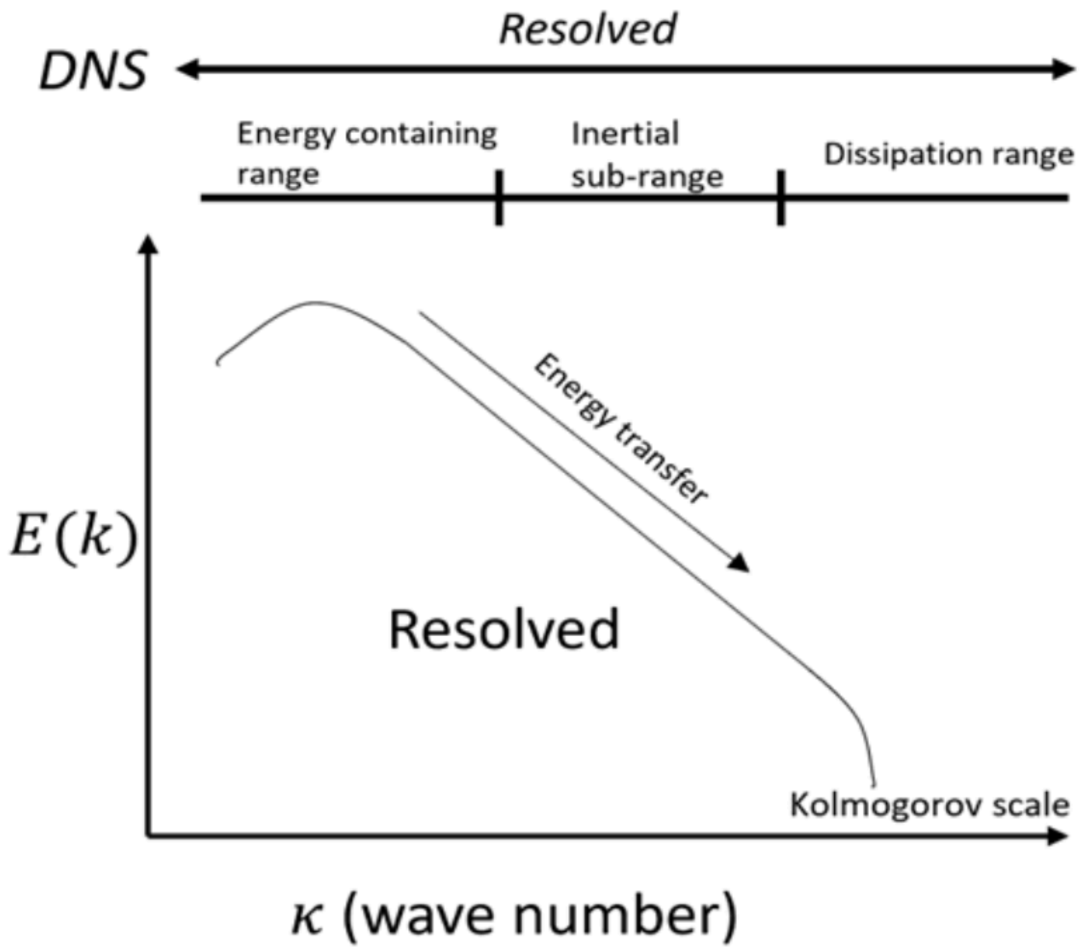
Viscosity





Direct numerical simulation (DNS)

DNS resolves the entire range of turbulent length scales down to the smallest dissipative scales (Kolmogorov scale):



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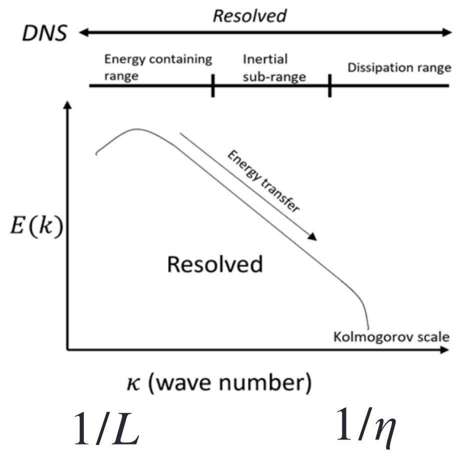

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \approx \left(\frac{\nu^3 L}{U^3}\right)^{1/4} = Re^{-3/4} L$$

where ν is the kinematic viscosity

And ϵ the rate of kinetic energy dissipation

Direct numerical simulation (DNS)

The number of floating-point operations required to complete the simulation is proportional to the number of mesh points:

$$N_x = \frac{L}{\eta} = Re^{3/4}$$

and the number of time steps: $$\frac{T}{\Delta t} = \frac{TU}{\eta} = \frac{TU}{L} Re^{3/4}$$

So the total cost scales as Re^3

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- How long would it take to run one-year of DNS for the global ocean on an exascale computer (10^{18} FLOPs/s)?

Direct numerical simulation (DNS)

What is Re for the ocean?

Given $U \sim 0.1 \mathrm{m\,s}^{-1}$, $L \sim 10^7 \mathrm{m}$, $H \sim 10^3 \mathrm{m}$, and $\nu \approx 10^{-6} \mathrm{m}^2 \mathrm{s}^{-1}$,

we have: $Re = \dfrac{UL}{\nu}$

The number of iterations (for 1 year) is thus: $\sim \dfrac{TUH}{L^2} Re^3$

Given about ~ 1000 FLOPs per iteration

So Total FLOPs ~

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PE

SW

SQG

QG

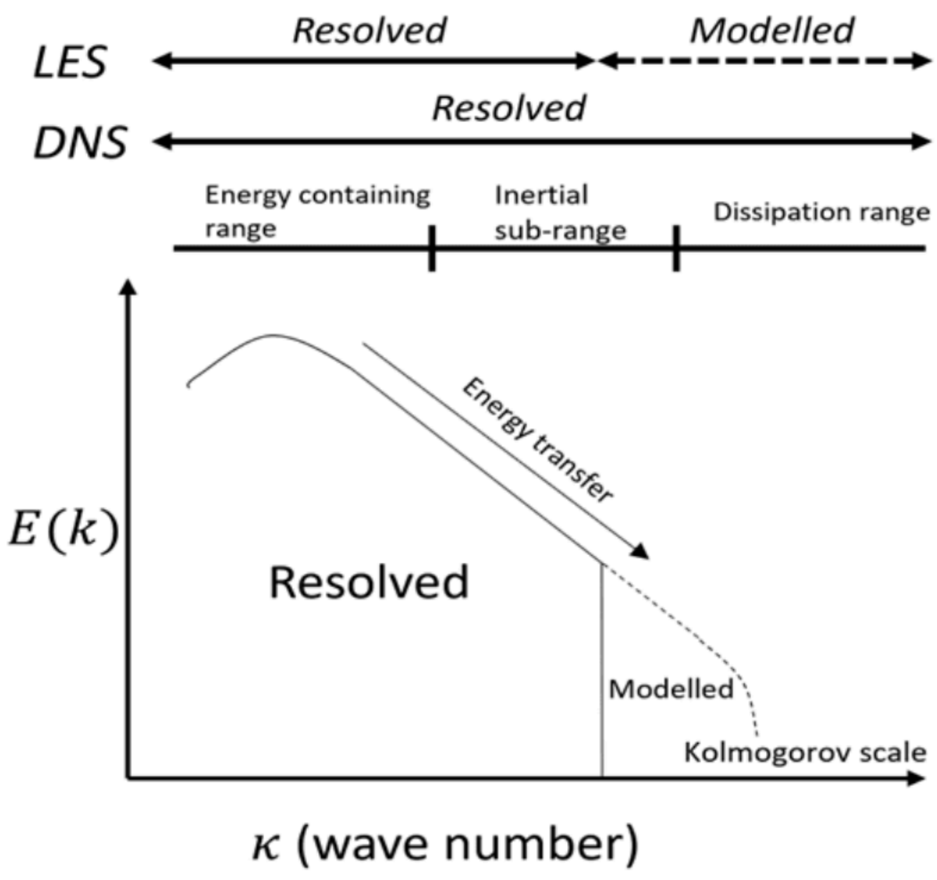
Ocean Circulation Models

Idealized models

Large Eddy Simulation

LES was first proposed by Joseph Smagorinsky in 1963 in his paper on atmospheric turbulence and weather prediction (Smagorinsky, Joseph (March 1963). "General Circulation Experiments with the Primitive Equations". *Monthly Weather Review*. **91** (3): 99–164).

Main idea: Only the large eddies in a turbulent flow are resolved, whereas the smaller more universal eddies are modelled.



Large Eddy Simulation

The flow is decomposed using a spatial filter:

$$\tilde{\phi}(\mathbf{x}, t) = \int_{\Omega} G(\mathbf{x} - \mathbf{r}; \Delta)\, \phi(\mathbf{r}, t)\, \mathrm{d}\mathbf{r}$$

$$\phi(\mathbf{x}, t) = \tilde{\phi}(\mathbf{x}, t) + \phi^{\mathrm{sgs}}(\mathbf{x}, t)$$

So the momentum equation becomes:

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}^{\mathrm{sgs}}}{\partial x_j}$$

With an additional term (**the subgrid-scale stress tensor**), which can be modelled using various schemes.

An historical one is the Smagorinsky SGS model:

$$\tau_{ij}^{\mathrm{sgs}} - \frac{1}{3}\tau_{kk}^{\mathrm{sgs}}\delta_{ij} = -2\nu_t \tilde{S}_{ij} \text{ with } \tilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) \text{ and } \nu_t = (C_s\Delta)^2\left(2\tilde{S}_{ij}\tilde{S}_{ij}\right)^{1/2}$$

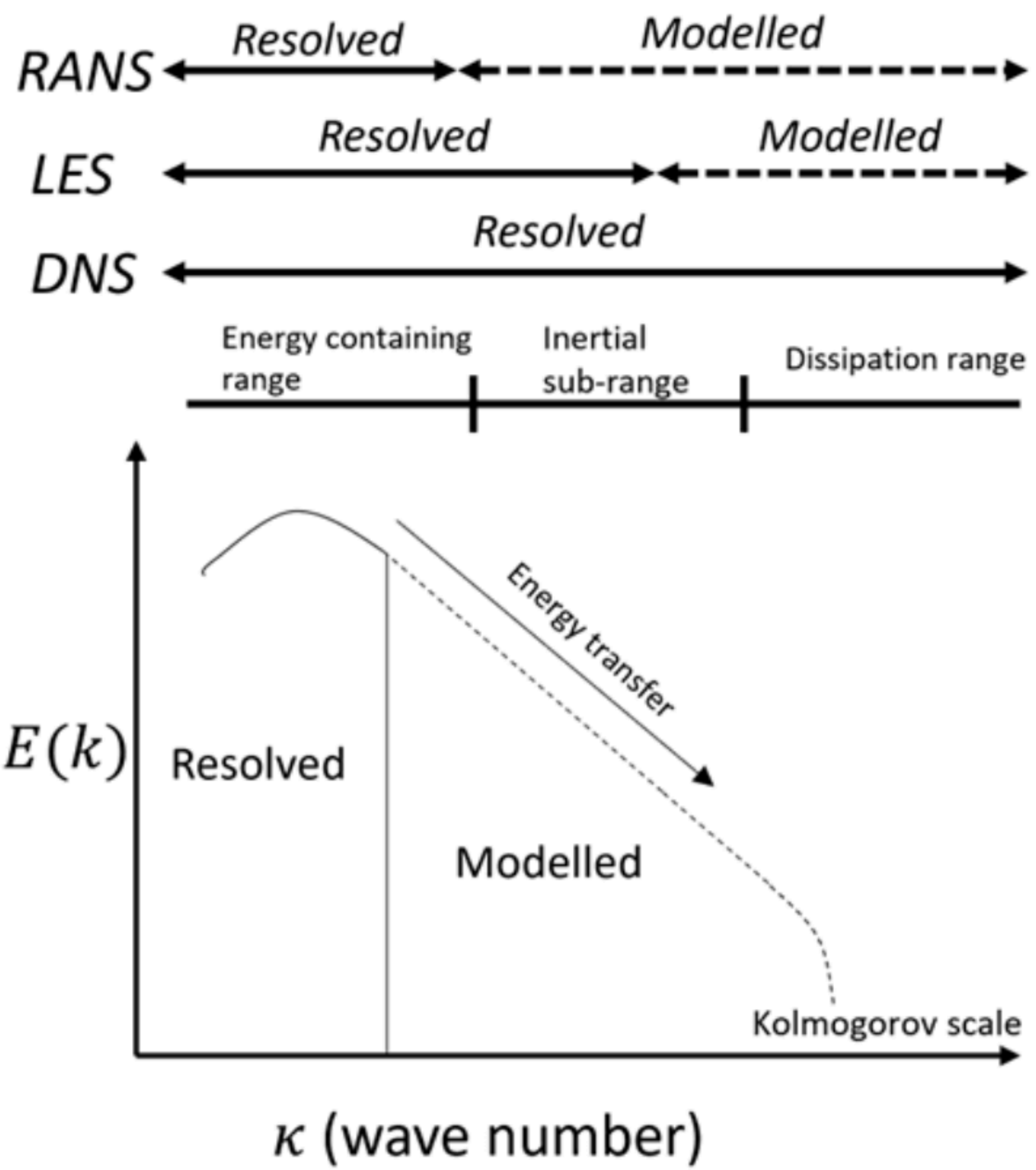
Reynolds averaging

RANS was first proposed by Osborne Reynolds.

These are the averaged Navier-Stokes equations that are obtained after employing Reynolds averaging.

The basic principle behind RANS equations is Reynolds decomposition, where a quantity is decomposed into a mean and fluctuating quantity.

Here, the mean quantity is resolved using the numerical solution, and any interactions with the fluctuating quantity are modelled.



Reynolds averaging

The idea is based on separation of mean and turbulent component: $u = \overline{u} + u'$

Traditionally using time-averaging: $$\overline{u} = \frac{1}{T}\int_0^T u \; dt$$

(but this can also be used as a spatial average or an ensemble average as long as the flow is statistically homogeneous in the averaging direction)

$$\overline{u} = \frac{1}{N}\Sigma_0^N \, u \quad \text{or} \quad \overline{u} = \frac{1}{X}\int_0^X u \; dx$$

The important property is that we have: $$\overline{u'} = 0$$

Reynolds averaging

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \nu \vec{\nabla}^2 \vec{u}$$

- Activity:

Adapt the momentum equation:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + f\mathbf{k} \times u_i + \frac{\rho}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_i \partial x_j}$$

For the mean velocity: $\dfrac{\partial \overline{u_i}}{\partial t}$ = ?

Reynolds averaging

So we resolve only the equations for the mean variables:

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j \frac{\partial u_i}{\partial x_j}} + f\mathbf{k} \times \overline{u_i} + \frac{\overline{\rho}}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x_j} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_i \partial x_j}$$

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + f\mathbf{k} \times \overline{u_i} + \frac{\overline{\rho}}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x_j} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_i \partial x_j} \boxed{- \frac{\partial \overline{u_i' u_j'}}{\partial x_j}}$$

Advection for the averaged flow

Reynolds stress
= effect of subgrid-scale turbulence

Turbulence closure

The Closure Problem :

- The number of unknowns is larger than the number of equations
- When we start deriving equations for unknowns, more variables appear.

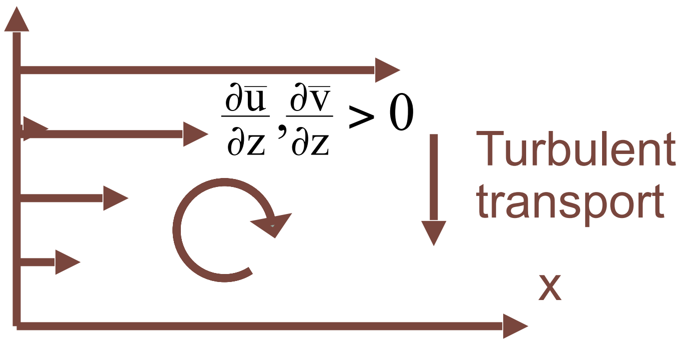
Progn. Eq. for:	Moment	Equation	No. Eqs.	No. Unknowns
$\overline{U_i}$	First	$\frac{\partial \overline{U_i}}{\partial t} = \ldots - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$	3	6
$\overline{u_i' u_j'}$	First	$\frac{\partial \overline{u_i' u_j'}}{\partial t} = \ldots - \frac{\partial \overline{u_k' u_i' u_j'}}{\partial x_k}$	6	10
$\overline{u_i' u_j' u_k'}$	First	$\frac{\partial \overline{u_i' u_j' u_k'}}{\partial t} = \ldots - \frac{\partial \overline{u_k' u_i' u_j' u_m'}}{\partial x_m}$	10	15

Turbulence closure

- In PE models the equations are closed by parameterizing the Reynolds stresses as:

$$\overline{u'w'} = -K_{Mv}\frac{\partial u}{\partial z}$$

$$\overline{v'w'} = -K_{Mv}\frac{\partial v}{\partial z}$$



Turbulence closure

In CROCO:

$$
\begin{aligned}
\mathcal{F}_u &= \frac{\partial}{\partial z}\left(K_{Mv}\frac{\partial u}{\partial z}\right) + \nabla_h(K_{Mh}.\nabla_h u) \\
\mathcal{F}_v &= \frac{\partial}{\partial z}\left(K_{Mv}\frac{\partial v}{\partial z}\right) + \nabla_h(K_{Mh}.\nabla_h v) \\
\mathcal{S}_T &= \frac{\partial}{\partial z}\left(K_{Tv}\frac{\partial T}{\partial z}\right) + \nabla_h(K_{Th}.\nabla_h T) \\
\mathcal{S}_S &= \frac{\partial}{\partial z}\left(K_{Sv}\frac{\partial S}{\partial z}\right) + \nabla_h(K_{Sh}.\nabla_h S)
\end{aligned}
$$

Vertical mixing

Horizontal diffusion
(here as a Laplacian)

Turbulence closure

Horizontal diffusion: $$K_{Mh}, K_{Th}, K_{Sh}$$

- Explicit diffusion:
  - options: **UV_VIS2, UV_VIS4, TS_DIF2, TS_DIF4** -> K values in croco.in
  - options: **UV_VIS_SMAGO**
  - see: https://croco-ocean.gitlabpages.inria.fr/croco_doc/model/model.parametrizations.diffusion.html

- and/or Implicit (comes from the advective scheme):
  - options **UV_HADV_UP3, UV_HADV_UP5, UV_HADV_WENO5, UV_HADV_TVD TS_HADV_UP3, TS_HADV_RSUP3, TS_HADV_UP5, TS_HADV_RSUP5, TS_HADV_WENO5**
  - see: https://croco-ocean.gitlabpages.inria.fr/croco_doc/model/model.numerics.advec.html#numerical-details-on-advection-schemes

Turbulence closure

Vertical Mixing: $$K_{Mv}, K_{Tv}, K_{Sv}$$

- Local Turbulence closure: GLS, k-kl (MY2.5), k-ε, κ-ω, etc. *[e.g. Warner et al, 2005, Ocean Modelling]*
  - *For example:* RANS : k-ϵ model (Burchard et al., 1998)

$$\begin{cases} \dfrac{\mathrm{d}k}{\mathrm{d}t} = P - \epsilon & \text{TKE} \\[2ex] \dfrac{\mathrm{d}\epsilon}{\mathrm{d}t} = \dfrac{\epsilon}{k}(c_{\epsilon 1}P - c_{\epsilon 2}\epsilon) & \text{dissipation} \end{cases}$$

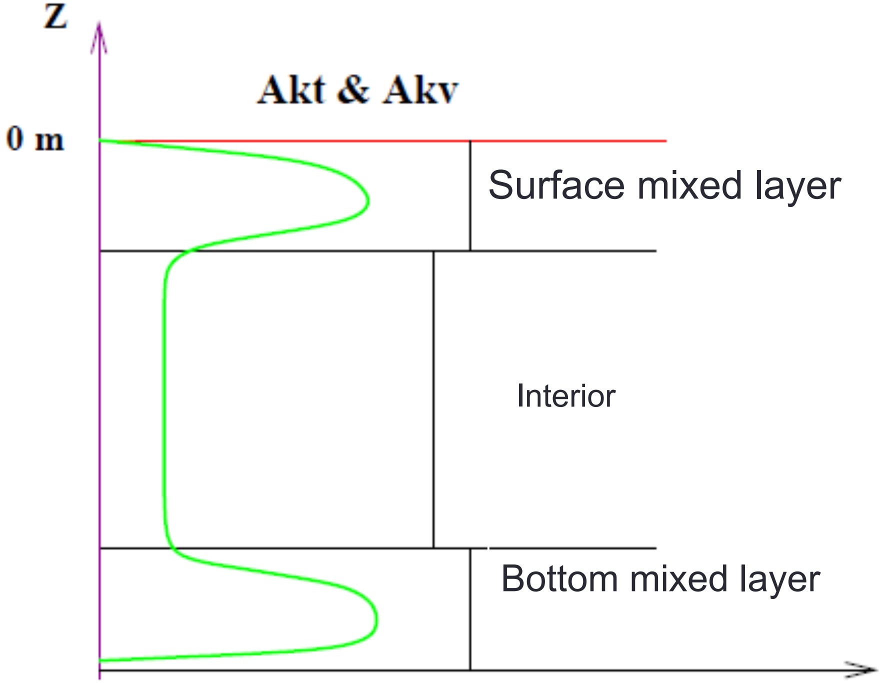
- Non local K-profile parameterization (KPP) *[Large et al, 1994, Rev. of Geophysics]*

Turbulence closure

Vertical Mixing: $$K_{Mv}, K_{Tv}, K_{Sv}$$

- Non local K-profile parameterization (KPP) *[Large et al, 1994, Rev. of Geophysics]*

Mixing coefficient profiles



Principle scheme of KPP

Turbulence closure

In CROCO:

$$\mathcal{F}_u = \frac{\partial}{\partial z}\left(K_{Mv}\frac{\partial u}{\partial z}\right) + \nabla_h(K_{Mh}.\nabla_h u)$$

$$\mathcal{F}_v = \frac{\partial}{\partial z}\left(K_{Mv}\frac{\partial v}{\partial z}\right) + \nabla_h(K_{Mh}.\nabla_h v)$$

$$\mathcal{S}_T = \frac{\partial}{\partial z}\left(K_{Tv}\frac{\partial T}{\partial z}\right) + \nabla_h(K_{Th}.\nabla_h T)$$

$$\mathcal{S}_S = \frac{\partial}{\partial z}\left(K_{Sv}\frac{\partial S}{\partial z}\right) + \nabla_h(K_{Sh}.\nabla_h S)$$

Vertical mixing

Horizontal diffusion

Boundary conditions

- Boundary conditions become:

top ($\sigma = 0$)

$$\left(\frac{K_m}{H_z}\right)\frac{\partial u}{\partial \sigma} = \tau_s^x(x,y,t)$$
$$\left(\frac{K_m}{H_z}\right)\frac{\partial v}{\partial \sigma} = \tau_s^y(x,y,t)$$

Wind stress

$$\left(\frac{K_C}{H_z}\right)\frac{\partial C}{\partial \sigma} = \frac{Q_C}{\rho_o c_P}$$

Tracer forcing (heat / P-E, etc)

$$\Omega = 0$$

and bottom ($\sigma = -1$)

$$\left(\frac{K_m}{H_z}\right)\frac{\partial u}{\partial \sigma} = \tau_b^x(x,y,t)$$
$$\left(\frac{K_m}{H_z}\right)\frac{\partial v}{\partial \sigma} = \tau_b^y(x,y,t)$$

Bottom friction

$$\left(\frac{K_C}{H_z}\right)\frac{\partial C}{\partial s} = 0$$
$$\Omega = 0.$$

Bottom Boundary conditions

1. Linear friction, with r ***friction velocities*** [m/s] $\rightarrow$ $$(\tau_b^x, \tau_b^y) = -r\ (u_b, v_b)$$

2. Quadratic friction, controled by a constant drag coefficient **Cd** $\rightarrow$ $$(\tau_b^x, \tau_b^y) = C_d \sqrt{u_b^2 + v_b^2}\ (u_b, v_b)$$

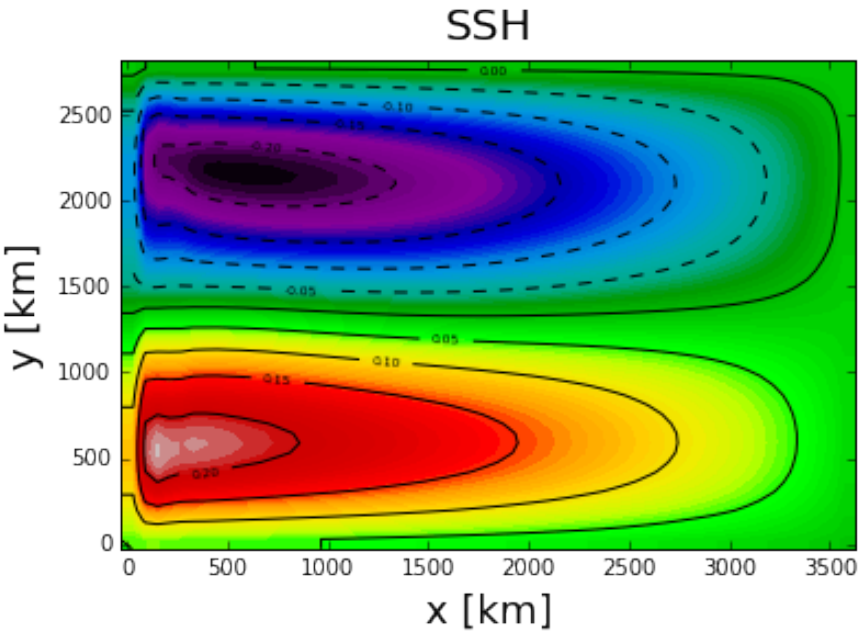
3. Quadratic friction coefficient, using variable **Cd** (Von Karman log. layer) $\rightarrow$

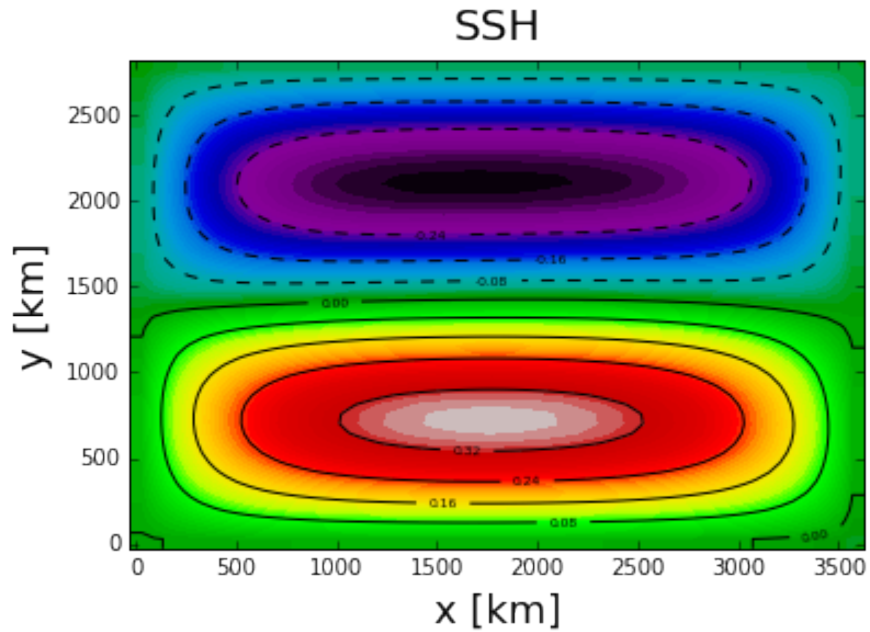
$$\begin{cases} (\tau_b^x, \tau_b^y) = C_d \sqrt{u_b^2 + v_b^2}\ (u_b, v_b) \\ C_d = \left(\frac{\kappa}{\log[\Delta z_b / z_r]}\right)^2 \text{ si } C_d^{min} < C_d < C_d^{max} \\ C_d = C_d^{min} \text{ ou } C_d^{max} \\ \kappa = 0.41 \end{cases}$$

z_r = Roughness Length

Δz_b = thickness of the first bottom level

Activity 2 – Run an idealized ocean basin and diagnose barotropic vorticity balance





$$\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}} = -\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}} + \underbrace{\frac{\mathbf{J}(P_b, h)}{\rho_0}}_{\text{bot. pres. torque}} + \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}$$

$$+ \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

Stommel's gyre

THE WESTWARD INTENSIFICATION OF WIND-DRIVEN OCEAN CURRENTS

Henry Stommel

(Contribution No. 408, Woods Hole Oceanographic Institution)

Abstract--A study is made of the wind-driven circulation in a homogeneous rectangular ocean under the influence of surface wind stress, linearised bottom friction, horizontal pressure gradients caused by a variable surface height, and Coriolis force.

An intense crowding of streamlines toward the western border of the ocean is discovered to be caused by variation of the Coriolis parameter with latitude. It is suggested that this process is the main reason for the formation of the intense currents (Gulf stream and others) observed in the actual oceans.

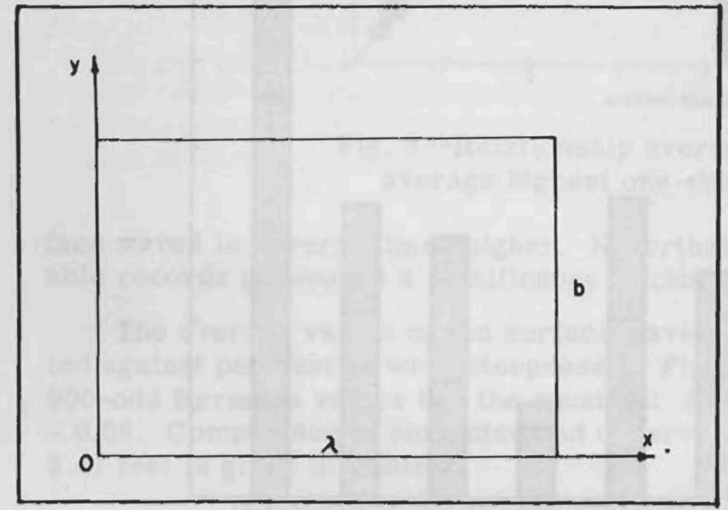


Fig. 1--Ocean basin dimensions and the coordinate system

- Momentum equations:

The steady state equations of motion, with the inertial terms omitted because they are small, are written in the form

$$0 = f(D + h)v - F\cos(\pi y/b) - Ru - g(D + h)\partial h/\partial x \qquad (1)$$

$$0 = -f(D + h)u - Rv - g(D + h)\partial h/\partial y \qquad (2)$$

- Barotropic vorticity equation:

$$v(D + h)(\partial f/\partial y) + (F\pi/b)\sin(\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Stommel's gyre

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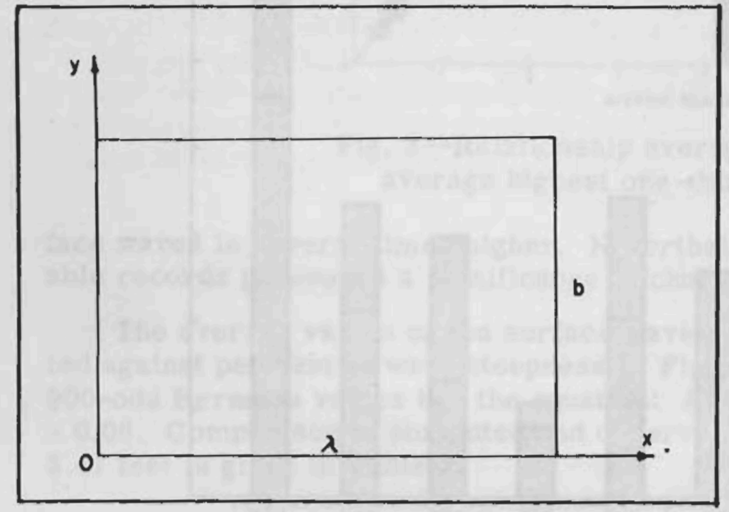


Fig. 1--Ocean basin dimensions and the coordinate system

- Momentum equations:

Coriolis Wind Drag Pressure gradient

The steady state equations of motion, with the inertial terms omitted because they are small, are written in the form

$$0 = \boxed{f(D + h)v} - \boxed{F \cos (\pi y/b)} - \boxed{Ru} - \boxed{g(D + h)\partial h/\partial x} \qquad (1)$$

$$0 = -\boxed{f(D + h)u} - \boxed{Rv} - \boxed{g(D + h)\partial h/\partial y} \quad \dots\dots\dots\dots\dots\dots \quad (2)$$

- Barotropic vorticity equation:

Planetary vorticity advection — Wind Curl — Drag Curl

$$\boxed{v(D + h)(\partial f/\partial y)} + \boxed{(F\pi/b) \sin (\pi y/b)} + \boxed{R(\partial v/\partial x - \partial u/\partial y)} = 0$$

Stommel's gyre

- No rotation / or constant rotation ($\frac{\partial f}{\partial y} = \beta = 0$)

The equation for the stream function is therefore

$$\psi = \gamma(b/\pi)^2 \sin(\pi y/b)[e^{(x - \lambda)\pi/b} + e^{-x\pi/b} - 1]$$

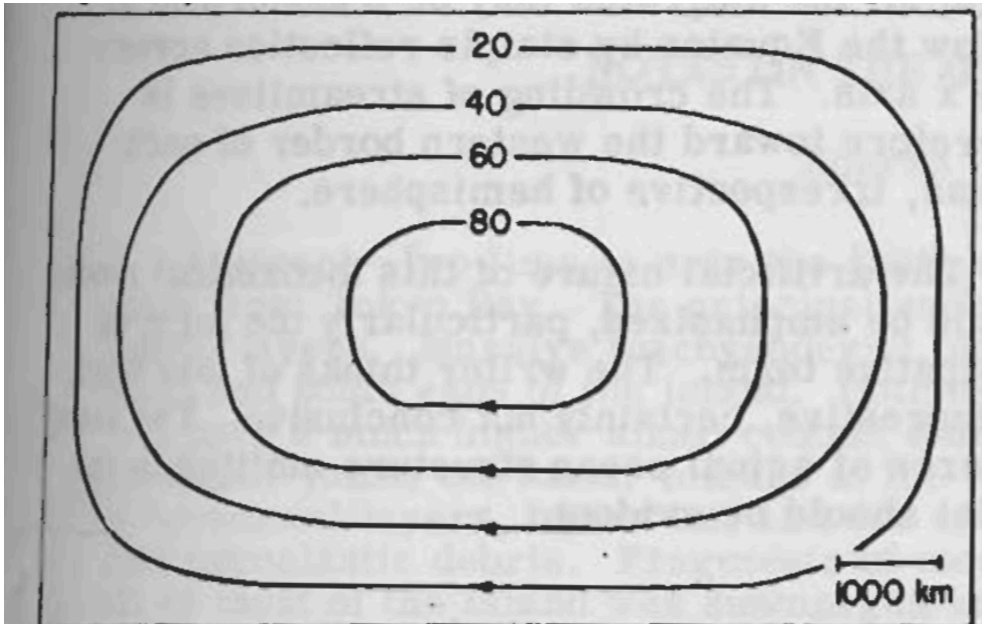


Fig. 2--Streamlines for the case of both the non-rotating and uniformly rotating oceans

Plan ary v city a n

Wind Curl

Drag Curl

$$\cancel{v(D\ldots)(\partial f/\partial y)} + (F\pi/b)\sin(\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Stommel's gyre

- With latitudinal variation of Coriolis

$$\frac{\partial f}{\partial y} = \beta \neq 0$$

- **Formation of a western boundary**

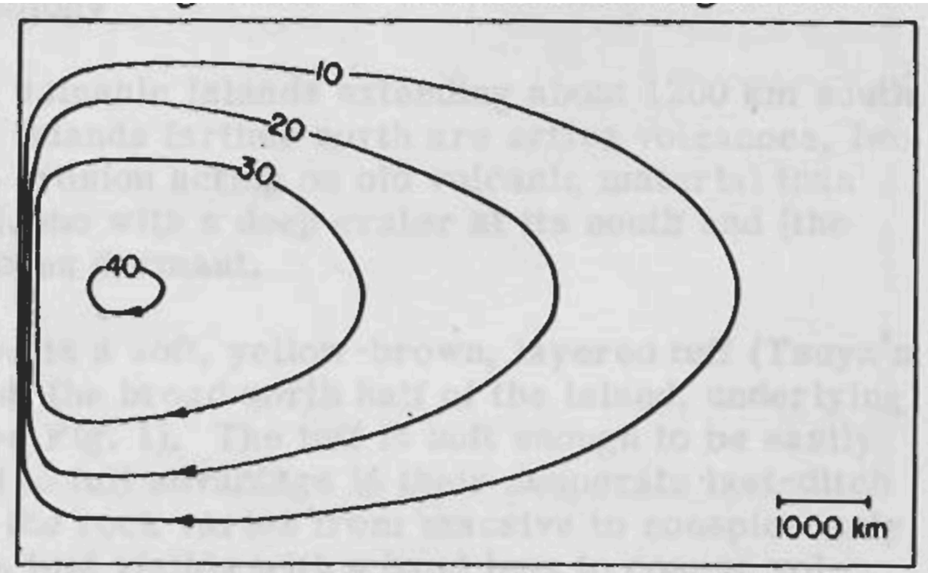


Fig. 5--Streamlines for the case where the Coriolis force is a linear function of latitude

Stommel's gyre (no beta)

- Forcings and data
  - Bottom topography + Land mask
  - Atmospheric surface boundary forcing
  - Initial oceanic conditions
  - Lateral oceanic boundary conditions

- **Flat bottom**
- **Constant wind** (sustr(i,j)=-cff1*cos(2.*pi/el*yr(i,j)))
- **Resting state**
- **Vertical walls**
-

Stommel's gyre (no beta)

- No rotation / or constant rotation (. $\frac{\partial f}{\partial y} = \beta = 0$)
- **cppdefs.h**

```
# define UV_COR
# define UV_VIS2
# define TS_DIF2

# define ANA_GRID
# define ANA_INITIAL
```

- **croco.in**

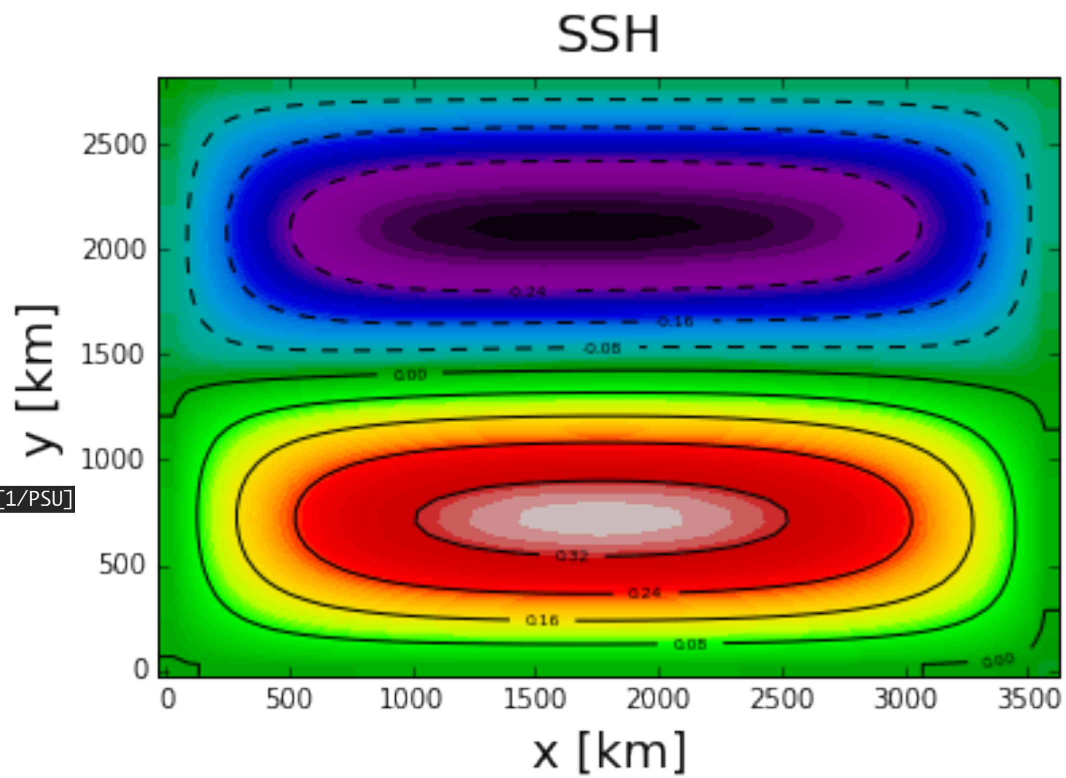
```
bottom_drag:     RDRG(m/s),      RDRG2, Zob [m],  Cdb_min, Cdb_max
                 3.e-4              0.      0.        0.      0.
gamma2:
                 1.
lin_EOS_cff:  R0 [kg/m3], T0 [Celsius], S0 [PSU], TCOEF [1/Celsius], SCOEF [1/PSU]
              30.         0.            0.        0.28               0.
lateral_visc:   VISC2     [m^2/sec ]
                 1000.  0.
tracer_diff2: TNU2          [m^2/sec]
                 1000. 0.
```

- **ana_grid.F**

```
f0=1.E-4
beta=0.
```

- **param.h**

```
      parameter (LLm0=60,    MMm0=50,    N=10)
```



CROCO simulation after 20 years

Barotropic vorticity balance

- Barotropic vorticity: $$\Omega = \frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y}$$ with $$\overline{u} = \int_{-h}^{\zeta} u \, dz,$$

- The barotropic vorticity balance equation of the flow is obtained by integrating the momentum equations in the vertical and cross differentiating them:
  - [cf: https://www.jgula.fr/ModNum/vort_balance.pdf]

Barotropic vorticity balance

- We integrate the momentum equations in the vertical, get derivatives out of integrals and get:

$$
\begin{aligned}
\frac{\partial \overline{u}}{\partial t} &+ \frac{\partial \overline{uu}}{\partial x} + \frac{\partial \overline{uv}}{\partial y} - f\overline{v} = \\
&- \frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x} + \frac{1}{\rho_0}P(\zeta).\frac{\partial \zeta}{\partial x} + \frac{1}{\rho_0}P(-h)\frac{\partial h}{\partial x} + \tau_x^{wind} - \tau_x^{bot} + \overline{\mathcal{D}_x} \\
\underbrace{\frac{\partial \overline{v}}{\partial t}}_{} &+ \underbrace{\frac{\partial \overline{vv}}{\partial y} + \frac{\partial \overline{uv}}{\partial x}}_{} + \underbrace{f\overline{u}}_{} = \\
&- \underbrace{\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial y} + \frac{1}{\rho_0}P(\zeta).\frac{\partial \zeta}{\partial y} + \frac{1}{\rho_0}P(-h)\frac{\partial h}{\partial y}}_{} + \underbrace{\tau_y^{wind} - \tau_y^{bot} + \overline{\mathcal{D}_y}}_{}
\end{aligned}
$$

Using boundary conditions:

$$
\begin{aligned}
K_{Mv}\frac{\partial u}{\partial z}\bigg|_{\zeta} - K_{Mv}\frac{\partial u}{\partial z}\bigg|_{-h} &= \frac{1}{\rho_0}\left(\tau_x^{wind} - \tau_x^{bot}\right) \\
K_{Mv}\frac{\partial v}{\partial z}\bigg|_{\zeta} - K_{Mv}\frac{\partial v}{\partial z}\bigg|_{-h} &= \frac{1}{\rho_0}\left(\tau_y^{wind} - \tau_y^{bot}\right)
\end{aligned}
$$

$$
\begin{aligned}
w|_{\zeta} &= \frac{\partial \zeta}{\partial t} + u|_{\zeta}\frac{\partial \zeta}{\partial x} + v|_{\zeta}\frac{\partial \zeta}{\partial y} \\
w|_{-h} &= u|_{-h}\frac{\partial -h}{\partial x} + v|_{-h}\frac{\partial -h}{\partial y}
\end{aligned}
$$

Barotropic vorticity balance

Then we cross differentiate them to get the barotropic vorticity equations:

-

$$\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}} = - \underbrace{\vec{\nabla}.(f\vec{\bar{u}})}_{\text{planet. vort. adv.}} + \underbrace{\frac{\mathbf{J}(P_b, h)}{\rho_0}}_{\text{bot. pres. torque}} + \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}$$
$$+ \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

Barotropic vorticity balance

Then we cross differentiate them to get the barotropic vorticity equations:

$$\underbrace{\frac{\partial\Omega}{\partial t}}_{\text{rate}} = -\underbrace{\vec{\nabla}.(f\vec{\bar{u}})}_{\text{planet. vort. adv.}} + \underbrace{\frac{\mathbf{J}(P_b, h)}{\rho_0}}_{\text{bot. pres. torque}} + \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}$$

$$+ \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

$$v(D + h)(\partial f/\partial y) + (F\pi/b)\sin(\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Barotropic vorticity balance

- Steady state ($\frac{\partial \cdot}{\partial t} = 0$); Linear; No diffusion (only bottom drag); Flat bottom

$$\cancel{\cdots} = -\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}} + \cancel{\cdots} + \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}$$

$$+\ \cancel{\cdots} - \cancel{\cdots}$$

$$v(D + h)(\partial f/\partial y) + (F\pi/b)\sin(\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Stommel's gyre (no beta)

- Steady state ($\frac{\partial \cdot}{\partial t} = 0$); Linear; No diffusion (only bottom drag); Flat bottom
- No rotation / or constant rotation ($\dfrac{\partial f}{\partial y} = \beta = 0$)

$$\underbrace{\cancel{\quad}}_{\text{rate}} = - \underbrace{\cancel{\quad}}_{\text{adv.}} + \underbrace{\cancel{\quad}}_{\text{bot. torque}} + \boxed{\underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \boxed{\underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}}$$

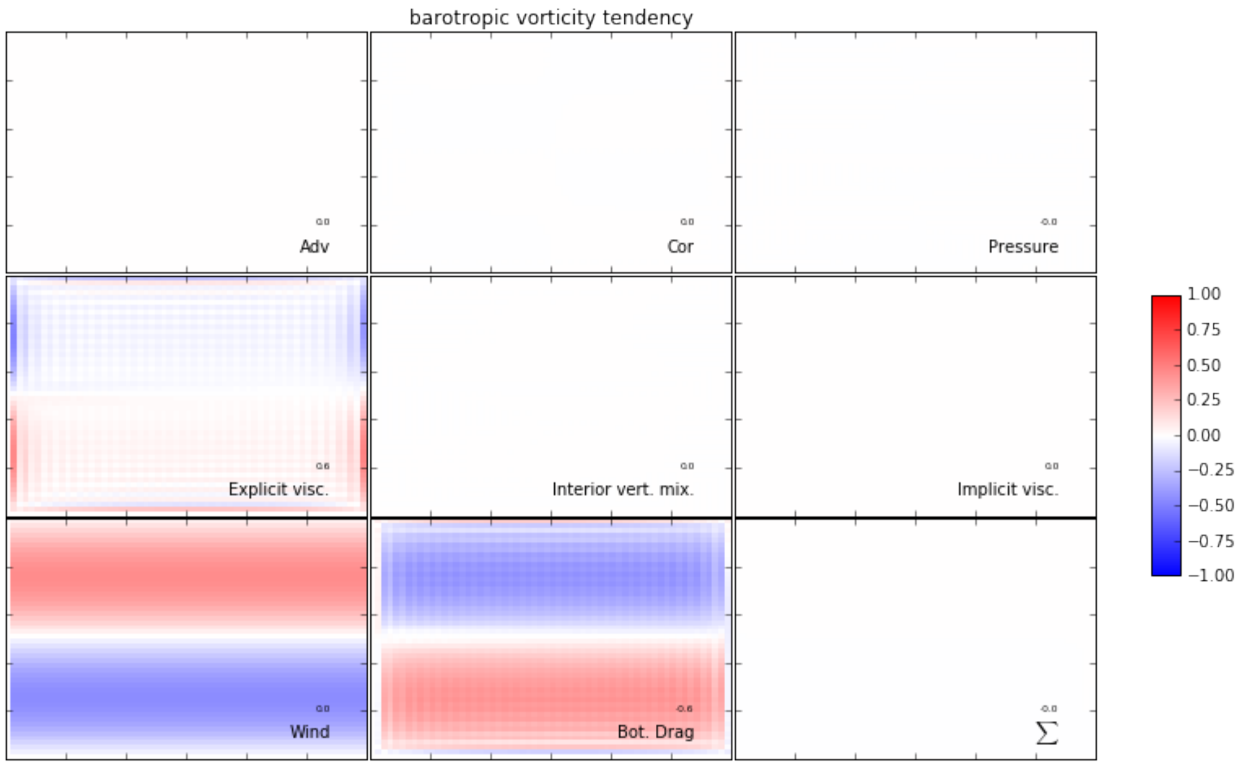
$$+ \cancel{\underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}}} - \underbrace{\cancel{\quad}}_{\text{NL}}$$

Stommel's gyre (no beta)

- No rotation / or constant rotation

$$\underbrace{\cancel{\frac{\partial \Omega}{\partial t}}}_{\text{rate}} = -\underbrace{\cancel{\vec{\nabla}\cdot(f\vec{u})}}_{\text{planet. vort. adv.}} + \underbrace{\cancel{\frac{\mathbf{J}(P_b, h)}{\rho_0}}}_{\text{bot. pres. torque}} + \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}} + \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{\cancel{\mathcal{A}}}_{\text{NL advection}}$$

-

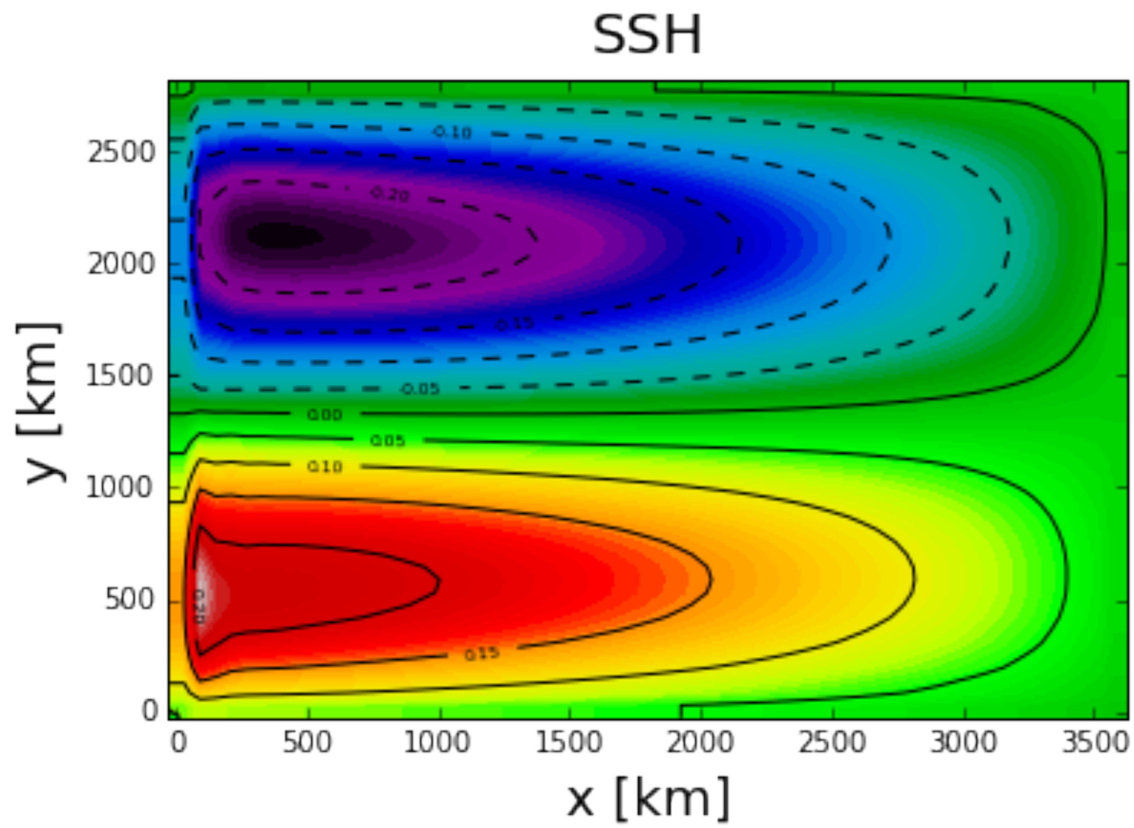


Stommel's gyre (with beta)

- With latitudinal variation of Coriolis ($\frac{\partial f}{\partial y} = \beta \neq 0$)

- **ana_grid.F**

```
f0=1.E-4
beta=2.E-11
```



Stommel's gyre (no beta)

- No rotation / or constant rotation ($\dfrac{\partial f}{\partial y} = \beta = 0$)
-

$$\cancel{\underbrace{\dots}_{\text{rate}}} = -\boxed{\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{J(\dots, h)}_{\text{bot. pres. torque}}} + \boxed{\underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \boxed{\underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}}$$

$$+ \cancel{\underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}}} - \cancel{\underbrace{\dots}_{\text{NL \dots tion}}}$$

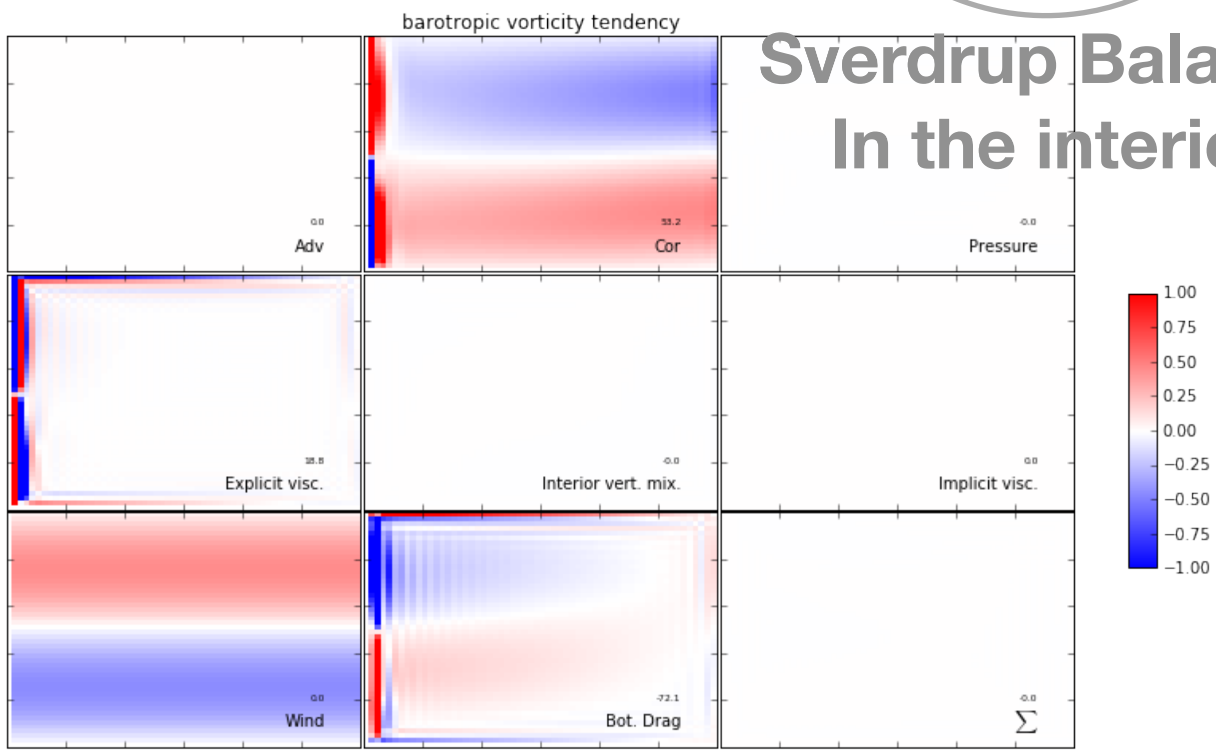
Stommel's gyre (with beta)

- With latitudinal variation of Coriolis

$$\cancel{\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}}} = \boxed{-\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{\frac{J(P_b,h)}{\rho_0}}_{\text{bot. pres. torque}}} + \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}} + \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \cancel{\underbrace{\mathcal{A}_\Sigma}_{\text{NL advection}}}$$

Sverdrup Balance
In the interior

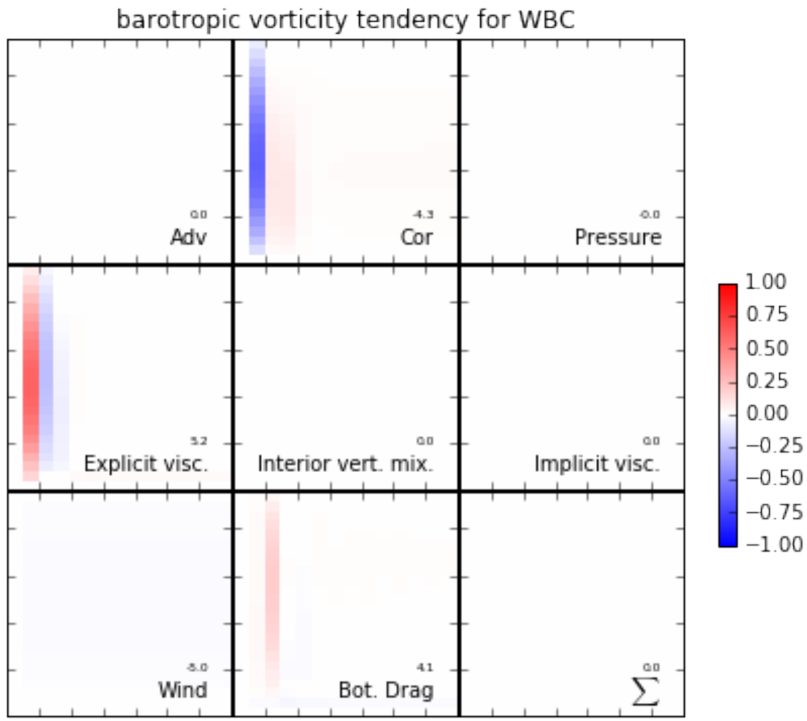
-



Stommel's gyre (with beta)

- With latitudinal variation of Coriolis

$$\cancel{\underbrace{\frac{\partial\Omega}{\partial t}}_{\text{rate}}} = \boxed{-\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{\frac{\mathbf{J}(P_b, H)}{\rho_0}}_{\text{bot. pres. torque}}} + \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}} + \boxed{\underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}}} - \cancel{\underbrace{\mathcal{A}_\Sigma}_{\text{NL advection}}}$$

- 

Zoom over the western boundary current

Munk' gyre

In Ekman's and Stommel's model the ocean is assumed homogeneous, a case in which the currents extend to the very bottom. Not only is this in contrast with observations, according to which the bulk of the water transport in the main ocean currents takes place in the upper thousand meters, but it also leads to mathematical complications which rendered Ekman's analysis very difficult, and forced Stommel to resort to a rather arbitrary frictional force along the bottom.

To avoid these difficulties, we retain Sverdrup's integrated mass transport as the dependent variable. This device makes it possible to examine the more general case of a baroclinic ocean without having to specify the nature of the vertical distributions of density and current. In recognition of the evidence that currents essentially vanish at great depths, we shall depend on lateral friction for the dissipative forces. From Stommel we retain the rectangular boundaries, although we extend the basin to both sides of the equator and deal with the *observed* wind distribution rather than a simple sinusoidal distribution.

JOURNAL OF METEOROLOGY

ON THE WIND-DRIVEN OCEAN CIRCULATION

By Walter H. Munk

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(Manuscript received 24 September 1949)

$$P = \int_{-h}^{z_0} p\,dz, \qquad \mathbf{M}_H = \int_{-h}^{z_0} \rho\,\mathbf{v}_H\,dz, \qquad (2a, b)$$

designate the integrated pressure and mass transport.

$$\nabla P + f\,\mathbf{k}\times\mathbf{M} - \boldsymbol{\tau} - A\,\nabla^2\mathbf{M} = 0. \qquad \mathbf{M} = \mathbf{k}\times\nabla\psi,$$

$$\boxed{(A\nabla^4 - \beta\,\partial/\partial x)\psi = -\,\mathrm{curl}_z\,\boldsymbol{\tau},}$$

For boundary conditions we choose

$$\psi_{\mathrm{bdry}} = 0, \qquad (\partial\psi/\partial\nu)_{\mathrm{bdry}} = 0, \qquad (7a, b)$$

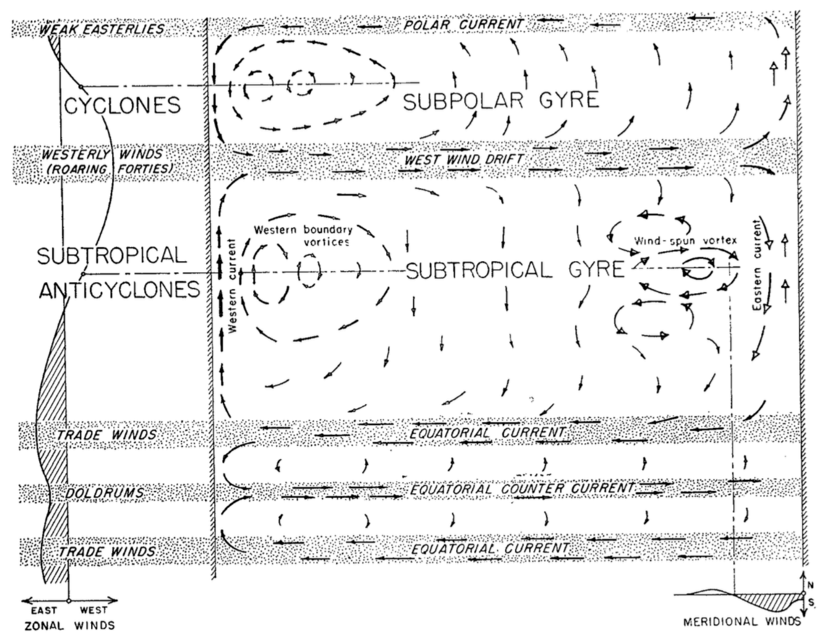


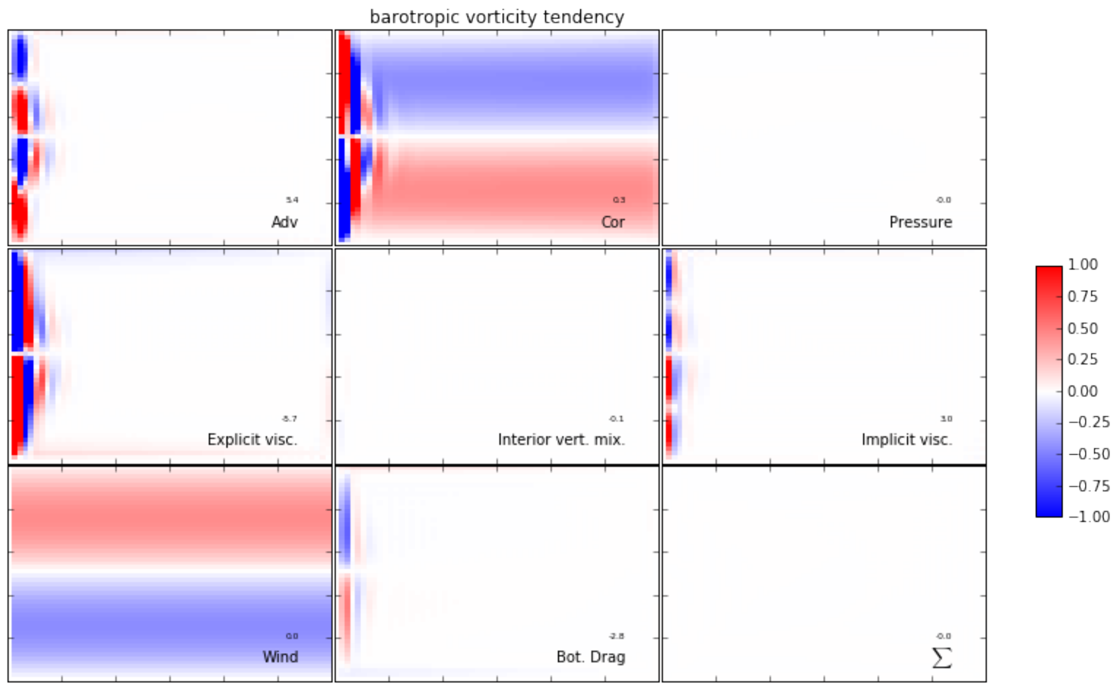
FIG. 8. Schematic presentation of circulation in a rectangular ocean resulting from zonal winds (filled arrowheads), meridional winds (open arrowheads), or both (half-filled arrowheads). The width of the arrows is an indication of the strength of the currents. The nomenclature applies to either hemisphere, but in the Southern Hemisphere the subpolar gyre is replaced largely by the Antarctic Circumpolar Current (west wind drift) flowing around the world. Geographic names of the currents in various oceans are summarized in table 3.

Gyre with beta and lateral drag

- With latitudinal variation of Coriolis

$$\cancel{\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}}} = \boxed{\underbrace{-\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{\frac{\mathbf{J}(P_b, h)}{\rho_0}}_{\text{bot. pres. torque}}} \boxed{\underbrace{+\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} \underbrace{-\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}} + \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

-

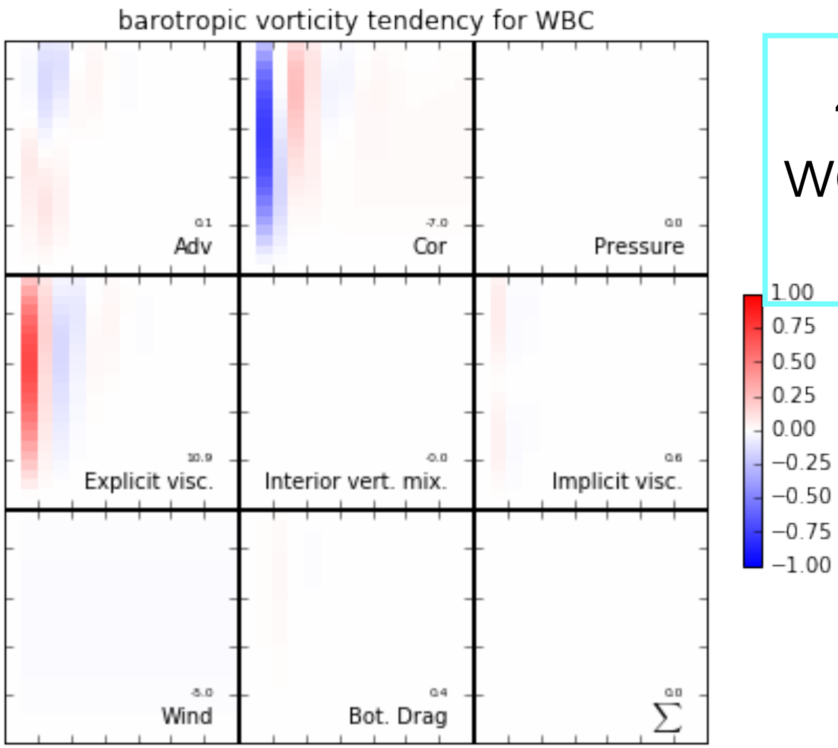


Gyre with beta and lateral drag

- With latitudinal variation of Coriolis

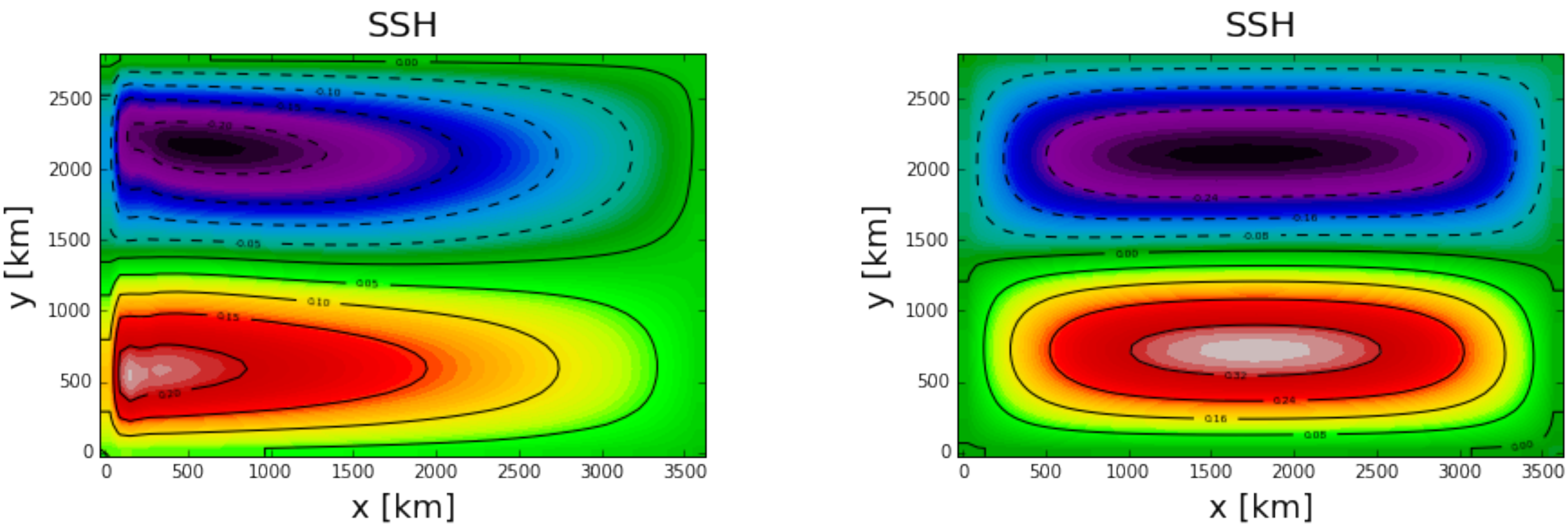
$$\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}} = \boxed{-\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{\frac{\mathbf{J}(P_b,h)}{\rho_0}}_{\text{bot. pres. torque}}} + \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}} + \boxed{\underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

-



Zoom over the western boundary current

Activity 2 – Run an idealized ocean basin and diagnose barotropic vorticity balance



$$\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}} = -\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}} + \underbrace{\frac{\mathbf{J}(P_b, h)}{\rho_0}}_{\text{bot. pres. torque}} + \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}$$

$$+ \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$