- Applied Mathematics -
Introduction to perturbation methods

Partical exercises : Perturbation methods for algebraic equations

Exercice 1 : Regular perturbation

We consider the roots of the quadratic equation

$$x^2 + \epsilon x - 1 = 0$$

Find the two first non zero terms of an asymptotic expansion of these roots.

Exercice 2 : Regular perturbation

Determine the first three (non vanishing) terms of an asymptotic expansion of the smallest root (in absolute value) of

$$y^5 + y^3 + y^2 - y = \epsilon$$

when $\epsilon \longrightarrow 0$. Hint: Start by determining this root when $\epsilon = 0$.

Exercice 3 : Singular perturbation

We want to determine the first two (non vanishing) terms of an asymptotic expansion of all the roots of

$$\epsilon x^3 + x^2 + x - 2 = 0$$

1. Do a graphical representation to see where the problem sits.
2. Determine the first two terms of the regular roots
3. By defining the following scaling $x = y/\delta(\epsilon)$ propose choice of $\delta(\epsilon)$ which gives a dominant balance with the ϵx^3 term at leading order.
4. Substitute $x = y/\delta(\epsilon)$ in the equation
5. Find the expansion of the last root using the equation for y.

Exercice 4 : Singular perturbation

We want to determine the first two (non vanishing) terms of an asymptotic expansion of all the roots of

$$\epsilon^3 x^2 + \epsilon x + 1 = 0$$

1. We look for roots $x \in \mathbb{C}$. How many roots do have this polynomial?
2. How many roots are regular?
3. By defining the following scaling $x = y/\delta(\epsilon)$ propose choices of $\delta(\epsilon)$
4. Substitute $x = y/\delta(\epsilon)$ in the equation
5. Find the 3 first terms of the expansion of the two roots by using the equation for y.
6. Discuss the scaling by comparing with the exact solution.

Exercice 5 : expansions with non integer powers of ϵ

Determine the first two (non vanishing) terms of an asymptotic expansion of the root of

$$(x+1)^3 - (\epsilon + \frac{27}{4})x = 0$$

close to $\frac{1}{2}$.
Hint: by expanding in ϵ^n show that this problem is pathological (this kind of expansion does not work). The trick consists here in expanding in $(\sqrt{\epsilon})^n$. Find the solution with such an expansion form.