Exercise 1

$$x^2 + \varepsilon x - 1 = 0$$

two first non zero terms of an asymtotic expansion of the roots.

This problem is regular when $\varepsilon = 0$ we get

$x^2 - 1 = 0$ which gives two roots 1 and -1.

So we can let $x \sim x_0 + x_1 \varepsilon + x_2 \varepsilon^2 + \cdots$

then $x^2 \sim x_0^2 + \varepsilon(2x_0x_1) + \varepsilon^2(x_1^2 + 2x_0x_2) + \cdots$

Then $x_0^2 + \varepsilon(2x_0x_1) + \varepsilon^2(x_1^2 + 2x_0x_2)$

$$+ \varepsilon x_0 + \varepsilon^2 x_1 - 1 + O(\varepsilon^3) = 0$$

thus $\begin{cases} x_0^2 \quad 1 = 0 \qquad (\text{equation for } \varepsilon = 0) \\ 2x_0x_1 + x_0 = 0 \\ x_1^2 + 2x_0x_2 + x_1 = 0 \end{cases}$

thus $\begin{cases} x_0 = +1 \quad \text{or} \quad -1 \\ x_1 = -\frac{1}{2} \\ x_2 = \frac{-x_1 - x_1^2}{2x_0} \end{cases}$ thus $\begin{cases} x \sim 1 - \frac{1}{2}\varepsilon + O(\varepsilon^2) \\ x \sim -1 - \frac{1}{2}\varepsilon + O(\varepsilon^2) \end{cases}$

Remark: with our calculus we can also give x_2 values

if $x_0 = 1$ and $x_1 = -\frac{1}{2}$ then $x_2 = \frac{1}{8}$

if $x_0 = -1$ and $x_1 = -\frac{1}{2}$ then $x_2 = -\frac{1}{8}$

and thus
$$\begin{cases} x \sim 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + O(\varepsilon^3) \\ x \sim -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3) \end{cases}$$

Exercise 2

$$y^5 + y^3 + y^2 - y = \varepsilon$$

expansion of the smallest root (in absolute value)

if $\varepsilon = 0$ then $y^5 + y^3 + y^2 - y = 0$

thus $y = 0$ is the smallest root in absolute value.

Thus we are searching for

$$y \sim y_1 \varepsilon + y_2 \varepsilon^2 + y_3 \varepsilon^3 + O(\varepsilon^4)$$

$$y^5 \sim O(\varepsilon^5)$$

$$y^3 \sim y_1^3 \varepsilon^3 + O(\varepsilon^4)$$

$$y^2 \sim y_1^2 \varepsilon^2 + 2 y_1 y_2 \varepsilon^3 + O(\varepsilon^4)$$

inserting into the equation gives:

$$y_1^3 \varepsilon^3 + y_1^2 \varepsilon^2 + 2 y_1 y_2 \varepsilon^3 - y_1 \varepsilon - y_2 \varepsilon^2 - y_3 \varepsilon^3 + O(\varepsilon^4) \sim \varepsilon$$

thus $\begin{cases} -y_1 = 1 \\ y_1^2 - y_2 = 0 \\ y_1 + 2 y_1 y_2 - y_3 = 0 \end{cases}$ thus $\begin{cases} y_1 = -1 \\ y_2 = y_1^2 = 1 \\ y_3 = y_1 + 2 y_1 y_2 = -3 \end{cases}$

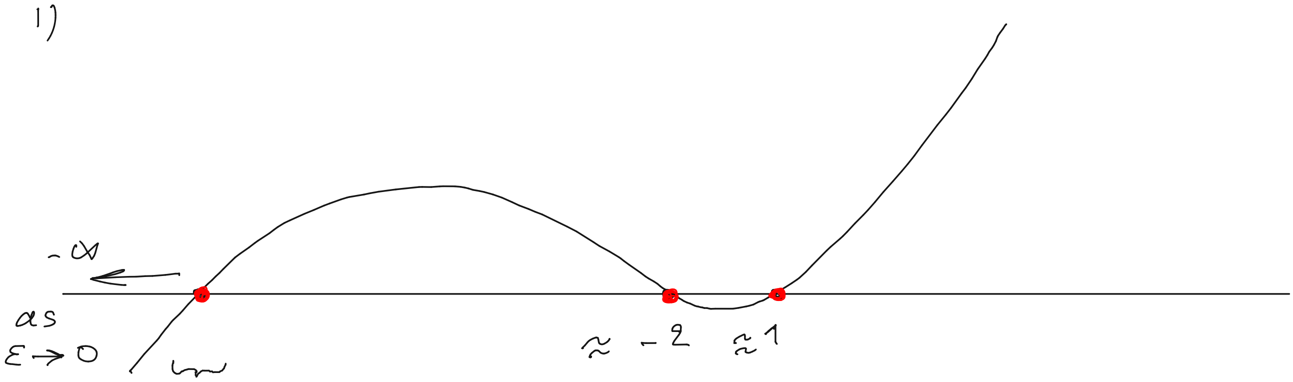
thus $\boxed{y \sim -\varepsilon + \varepsilon^2 - 3\varepsilon^3 + O(\varepsilon^4)}$

Exercise 3

$$\varepsilon x^3 + x^2 + x - 2 = 0$$

first two terms (non null) of an asymptotic expansion of all roots

1)



1 singular root   2 regular roots

$\hookrightarrow$ as $\varepsilon \to 0$ εx^3 may be large so that it can not be neglected

2) Regular roots:

if $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

then $x^2 = x_0^2 + \varepsilon(2x_0x_1) + \varepsilon^2(x_1^2 + 2x_0x_2) + \dots$

and $x^3 = x_0^3 + \varepsilon(3x_0^2x_1) + \varepsilon^2(3x_0x_1^2 + 3x_0^2x_2) + \dots$

So $\varepsilon x^3 + x^2 + x - 2 = 0$ gives:

$$\varepsilon x_0^3 + \varepsilon^2(3x_0^2x_1) + x_0^2 + \varepsilon(2x_0x_1) + \varepsilon^2(x_1^2 + 2x_0x_2)$$

$$+ x_0 + \varepsilon x_1 + \varepsilon^2 x_2 - 2 = 0$$

at order ε^0: $x_0^2 + x_0 - 2 = 0$ (same as $\varepsilon = 0$ in the equation)

at order ε^1: $x_0^3 + 2x_0x_1 + x_1 = 0$

at order ε^2: $3x_0^2x_1 + x_1^2 + 2x_0x_2 + x_2 = 0$

Thus $(x_0 - 1)(x_0 + 2) = 0$ $\quad x_0 = 1$ or $x_0 = -2$

thus $x_1 = \dfrac{-x_0^3}{1 + 2x_0}$ $\quad x_1 = -\dfrac{1}{3}$ or $x_1 = -\dfrac{8}{3}$

x_0 and x_1 are non null thus it is sufficient -

the two regular roots are:

$$\boxed{\begin{array}{l} x \sim 1 - \dfrac{1}{3}\varepsilon \\ x \sim -2 - \dfrac{8}{3}\varepsilon \end{array}}$$

Remark: With our calculus we can calculate the third terms

$$x_2 = \frac{-x_1^2 - 3x_0^2 x_1}{1 + 2x_0}$$

then for $x_0 = 1$ and $x_1 = -\frac{1}{3}$: $\quad x_2 = \frac{8}{27}$

and for $x_0 = -2$ and $x_1 = -\frac{8}{3}$: $\quad x_2 = \frac{-224}{27}$

3) for the third root it becomes large as $\varepsilon \to 0$ so we try: $x = \frac{y}{\delta(\varepsilon)}$ with $y \sim O(1)$

In the equation it gives:

$$\varepsilon\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^2 + \frac{y}{\delta} - 2 = 0$$

$$\Rightarrow \underbrace{\frac{\varepsilon}{\delta^3} y^3}_{①} + \underbrace{\frac{1}{\delta^2} y^2}_{②} + \underbrace{\frac{1}{\delta} y}_{③} - \underbrace{2}_{④} = 0$$

Let us find a dominant equilibrium as $\varepsilon \to 0$

if ① $\sim$ ② then $\frac{\varepsilon}{\delta^3} \sim \frac{1}{\delta^2}$ i.e $\varepsilon \sim \delta$

then ① $\sim$ ② $\sim \frac{1}{\delta^2} \gg$ ③ and ④ Possible

if ① $\sim$ ③ then $\frac{\varepsilon}{\delta^3} \sim \frac{1}{\delta}$ i.e $\varepsilon \sim \delta^2$

then ① $\sim$ ③ $\sim \frac{1}{\delta} \ll$ ② Not dominant if ① ~ ④ then $\varepsilon \sim \delta^3$

but then ① ~ ④ ~ 1 << ③ Not dominant

In all other cases $\delta \sim 1$ so it gives $x = y$ and we find the two regular roots.

Conclusion: dominant balance involving the first term is obtained with $\varepsilon \sim \delta$

4) Let's substitute $x \sim \frac{y}{\delta}$ in the equation with $\delta \sim \varepsilon$

It gives: $\delta\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^2 + \frac{y}{\delta} - 2 = 0$

thus $y^3 + y^2 + \delta y - 2\delta^2 = 0$

this equation is regular in y for $\delta \to 0$ $(\delta \sim \varepsilon)$

if $y \sim y_0 + \delta y_1 + \delta^2 y_2 + \dots$ is injected we get

$$y_0^3 + 3\delta y_0^2 y_1 + 3\delta^2(y_0^2 y_2 + y_0 y_1^2)$$
$$+ y_0^2 + 2\delta y_0 y_1 + \delta^2(2 y_0 y_2 + y_1^2)$$
$$+ \delta y_0 + \delta^2 y_1 - 2\delta^2 + \dots = 0$$

Thus
$$\begin{cases} y_0^3 + y_0^2 = 0 \\ 3 y_0^2 y_1 + 2 y_0 y_1 + y_0 = 0 \\ 3 y_0^2 y_2 + 3 y_0 y_1^2 + 2 y_0 y_2 + y_1^2 + y_1 - 2 = 0 \end{cases}$$

Thus $\quad y_0^2(y_0+1)=0 \quad \Rightarrow \quad y_0=-1 \quad$ or $\quad y_0=0$

the case $y_0=0$ corresponds to the regular case.

then, for $y_0=-1$ we get

$$y_1 = \frac{-1}{2+3y_0} = 1$$

thus <u>the singular root</u> is $y \sim -1+\delta$

and with $\delta=\varepsilon$ and $x=\frac{y}{\delta}$ we get

$$\boxed{x \sim -\frac{1}{\varepsilon}+1}$$

Remark: we can calculate the third term

with $y_0=-1$ and $y_1=1$ we get

$$y_2 = \frac{2-y_1-y_1^2-3y_0y_1^2}{3y_0^2+2y_0} = 3$$

then $\quad x \sim -\frac{1}{\varepsilon}+1+3\varepsilon$

TD 1 - EX. 4 & 5 SOLUTIONS

Exercice 4

$$\varepsilon^3 x^2 + \varepsilon x + 1 = 0 \qquad (1) \qquad \varepsilon \ll 1$$

1. In $\mathbb{C}$ a 2^{nd} order polynomial has 2 roots

2. With $\varepsilon = 0$ (1) $\Rightarrow 1 = 0$ with no solution. Thus there are no regular root. The equation is singular and all the roots are singular.

Remark: we expect that these roots become large as $\varepsilon \to 0$ so that they compensate the factors ε^2 and/or ε in the equation.

3. We thus need to rescale x to find these roots: $x = \dfrac{y}{\delta}$ where $\delta(\varepsilon) \ll 1$ when $\varepsilon \ll 1$ and $y \sim O(1)$.

Then (1) $\Rightarrow$ $$\underset{①}{\frac{\varepsilon^3}{\delta^2} y^2} + \underset{②}{\frac{\varepsilon}{\delta} y} + \underset{③}{1} = 0 \qquad (2)$$

Let's find δ such that there is a dominant balance

* Balance between ① and ② $\Rightarrow \dfrac{\varepsilon^3}{\delta^2} \sim \dfrac{\varepsilon}{\delta} \Rightarrow \delta \sim \varepsilon^2$

  then ① $\sim$ ② $\sim \dfrac{\varepsilon}{\delta} \sim \dfrac{\varepsilon}{\varepsilon^2} \sim \dfrac{1}{\varepsilon}$ and ③ ~ 1

  $\dfrac{1}{\varepsilon} \gg 1$ as $\varepsilon \ll 1$ thus ① $\sim$ ② $\gg$ ③ Balance is possible

* Balance between ① and ③ $\Rightarrow \dfrac{\varepsilon^3}{\delta^2} \sim 1 \Rightarrow \delta \sim \varepsilon^{3/2}$

  then ① $\sim$ ③ ~ 1 and ② $\sim \dfrac{\varepsilon}{\delta} \sim \dfrac{\varepsilon}{\varepsilon^{3/2}} = \dfrac{1}{\sqrt{\varepsilon}}$

  $1 \ll \dfrac{1}{\sqrt{\varepsilon}}$ as $\varepsilon \ll 1$ thus ① $\sim$ ③ $\ll$ ② No dominant balance

* Balance between ② and ③ $\Rightarrow \dfrac{\varepsilon}{\delta} \sim 1 \Rightarrow \delta \sim \varepsilon$

  then ② $\sim$ ③ ~ 1 and ① $\sim \dfrac{\varepsilon^3}{\delta^2} \sim \dfrac{\varepsilon^3}{\varepsilon^2} = \varepsilon$

  $1 \gg \varepsilon$ as $\varepsilon \ll 1$ thus ② $\sim$ ③ $\gg$ ① Balance is possible

We thus found 2 possible values for δ: $\delta \sim \varepsilon^2$ and $\delta \sim \varepsilon$

4.

* Let's try with $\delta = \varepsilon$: (2) $\Rightarrow \delta y^2 + \underbrace{y + 1}_{\text{dominant balance}} = 0$

We notice that this equation is singular thus we expect that 1 root (or 2) are singular. But we see also that one root is regular (because $\varepsilon = 0 \Rightarrow y + 1 = 0$ which has a solution).

So we can try with $y = y_0 + \delta y_1 + \delta^2 y_2 + O(\delta^3)$

Then $y^2 = y_0^2 + 2\delta y_0 y_1 + \delta^2(y_1^2 + 2y_1 y_2) + O(\delta^3)$

(2) $\Rightarrow \delta y_0^2 + 2\delta^2 y_0 y_1 + y_0 + \delta y_1 + \delta^2 y_2 + 1 + O(\delta^3) = 0$

$$\Rightarrow \begin{cases} \text{order } \delta^0 : & y_0 + 1 = 0 \\ \delta^1 : & y_0^2 + y_1 = 0 \\ \delta^2 : & 2y_0 y_1 + y_2 = 0 \end{cases} \Rightarrow \begin{cases} y_0 = -1 \\ y_1 = -y_0^2 = -1 \\ y_2 = -2y_0 y_1 = -2 \end{cases}$$

We find one root: $y \sim -1 - \delta - 2\delta^2$ with $x = \dfrac{y}{\delta}$ and $\delta = \varepsilon$

thus $x \underset{\varepsilon \ll 1}{\sim} -\dfrac{1}{\varepsilon} - 1 - 2\varepsilon$

* Let's try with $\delta \sim \varepsilon^2$: (2) $\Rightarrow \dfrac{\delta^{3/2}}{\delta^2} y^2 + \dfrac{\delta^{1/2}}{\delta} y + 1 = 0$

(2) $\Rightarrow \delta^{-1/2} y^2 + \delta^{-1/2} y + 1 = 0$

(2) $\Rightarrow \underbrace{y^2 + y}_{\text{dominant balance}} + \sqrt{\delta} = 0$

So we can try $y = y_0 + \sqrt{\delta} y_1 + (\sqrt{\delta})^2 y_2 + O(\sqrt{\delta}^3)$

It is easyer to work with ε instead of $\sqrt{\delta}$ because $\varepsilon = \sqrt{\delta}$

We have (2) $\Rightarrow y^2 + y + \varepsilon = 0$

and we try $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + O(\varepsilon^3)$

$y^2 = y_0^2 + 2\varepsilon y_0 y_1 + \varepsilon^2(y_1^2 + 2y_0 y_2) + O(\varepsilon^3)$

and we find : (2) $\Rightarrow y_0^2 + 2\varepsilon y_0 y_1 + \varepsilon^2(y_1^2 + 2y_0 y_2) + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon + O(\varepsilon^3) = 0$

Which gives:

$$\begin{cases} \text{at order } \varepsilon^0: & y_0^2 + y_0 = 0 \\ \varepsilon^1: & 2y_0y_1 + y_1 + 1 = 0 \\ \varepsilon^2: & y_1^2 + 2y_0y_2 + y_2 = 0 \end{cases} \Rightarrow \begin{cases} y_0 = 0 \quad \text{or} \quad -1 \\ y_1 = \dfrac{-1}{1+2y_0} \\ y_2 = \dfrac{-y_1^2}{1+2y_0} \end{cases}$$

Thus: $\begin{cases} y_0 = 0 \\ y_1 = -1 \\ y_2 = -1 \end{cases}$ or $\begin{cases} y_0 = -1 \\ y_1 = +1 \\ y_2 = 1 \end{cases}$

We find the 2 roots: $\begin{cases} y \sim -\varepsilon - \varepsilon^2 + O(\varepsilon^3) \\ y \sim -1 + \varepsilon + \varepsilon^2 + O(\varepsilon^3) \end{cases}$

And with $x = \dfrac{y}{\delta} = \dfrac{y}{\varepsilon^2}$ we get: $\begin{cases} x \sim -\dfrac{1}{\varepsilon} - 1 + O(\varepsilon) \\ x \sim -\dfrac{1}{\varepsilon^2} + \dfrac{1}{\varepsilon} + 1 + O(\varepsilon) \end{cases}$

We see that one terme is missing in the first root, we need to go farther. However we did the job before and found for the first scaling the corresponding root with 3 first terms of the expansion.

So the 2 roots are $\begin{cases} x \sim -\dfrac{1}{\varepsilon} - 1 - 2\varepsilon + O(\varepsilon^2) \\ x \sim -\dfrac{1}{\varepsilon^2} + \dfrac{1}{\varepsilon} + 1 + O(\varepsilon) \end{cases}$

Remark: that's a good exercise to find the missing term (-2ε) by developping the equation for y_3 with $y_0 = 0$, $y_1 = 1$, $y_2 = -1$ you should find $y_3 = -2$.

Let us try:

We have: $y^2 + y + \varepsilon = 0$

and $y = y_0 + y_1 \varepsilon + y_2 \varepsilon^2 + y_3 \varepsilon^3 + O(\varepsilon^4)$

$$y^2 = y_0^2 + \varepsilon(2y_0y_1) + \varepsilon^2(y_1^2 + 2y_0y_2) + \varepsilon^3(2y_0y_3 + 2y_1y_2) + O(\varepsilon^4)$$

Thus
$$\begin{cases} y_0^2 + y_0 = 0 \\ 2y_0y_1 + y_1 + 1 = 0 \\ y_1^2 + 2y_0y_2 + y_2 = 0 \\ 2y_0y_3 + 2y_1y_2 + y_3 = 0 \end{cases}$$

with $y_0 = 0$ we get $y_1 = -1$, $y_2 = -1$ as before

and $y_3 = \dfrac{-2y_1y_2}{1 + 2y_0} = -2$ Done!

5. We can check with the exact solution for $\varepsilon > 0$

We get: $x = \dfrac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4\varepsilon^3}}{2\varepsilon^3} = \dfrac{-1 \pm \sqrt{1 - 4\varepsilon}}{2\varepsilon^2}$

We know that $\sqrt{1+\alpha} \sim 1 + \dfrac{\alpha}{2} + O(\alpha^2)$ (Taylor)

So $x \sim \dfrac{-1 \pm (1 - 2\varepsilon)}{2\varepsilon^2} \sim \begin{cases} \dfrac{1}{\varepsilon^2} \\ \text{or} \\ \dfrac{1}{\varepsilon} \end{cases}$

These are the order of magnitude of the two roots we found.

Exercice 5

$$(x+1)^3 - \left(\varepsilon + \frac{27}{4}\right)x = 0 \qquad (1)$$

Let's try with $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

then $x+1 = (x_0+1) + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

Then $(x+1)^3 = (x_0+1)^3 + 3(x_0+1)^2 \varepsilon x_1 + 3(x_0+1)\varepsilon^2 x_1^2 + 3(x_0+1)^2 \varepsilon^2 x_2 + O(\varepsilon^3)$

then (1) $\Rightarrow (x_0+1)^3 - \frac{27}{4}x_0 + \varepsilon\left(3(x_0+1)^2 x_1 - x_0 - \frac{27}{4}x_1\right) + O(\varepsilon^2) = 0$

Thus
$$\begin{cases} (x_0+1)^3 - \frac{27}{4}x_0 = 0 & (2) \\ \left(3(x_0+1)^2 - \frac{27}{4}\right)x_1 - x_0 = 0 & (3) \end{cases}$$

We remark (see the hint) that $\frac{1}{2}$ is a double root of $(X+1)^3 - \frac{27}{4}X$

indeed $(X+1)^3 - \frac{27}{4}X = X^3 + 3X^2 + 3X + 1 - \frac{27}{4}X = X^3 + 3X^2 - \frac{15}{4}X + 1$

$$= \left(X - \frac{1}{2}\right)^2 (X - a)$$

$$= \left(X^2 - X + \frac{1}{4}\right)(X-a) = X^3 - X^2 + \frac{X}{4} - aX^2 + aX - \frac{a}{4}$$

$$= X^3 + (-1-a)X^2 + \left(\frac{1}{4}+a\right)X - \frac{a}{4} \qquad \text{with } a = -4$$

$(X+1)^3 - \frac{27}{4}x = \left(X - \frac{1}{2}\right)^2 (X+4)$ $\quad \frac{1}{2}$ double root, -4 single root

We are searching for roots close to $\frac{1}{2}$

If we choose $x_0 = \frac{1}{2}$ in (2)

then (3) $\Rightarrow \left(3\left(\frac{3}{2}\right)^2 - \frac{27}{4}\right)x_1 - \frac{1}{2} = 0 \Rightarrow 0\, x_1 - \frac{1}{2} = 0$ impossible

Thus there is no solution with the form $x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

We have to find another form. As $x_0 = \frac{1}{2}$ is a double root we can try with an expansion in powers of $\sqrt{\varepsilon}$ (see the hint)

It means $\quad x = x_0 + \sqrt{\varepsilon}\, x_1 + (\sqrt{\varepsilon})^2 x_2 + (\sqrt{\varepsilon})^3 x_3 + \cdots$

$$x = x_0 + \sqrt{\varepsilon}\, x_1 + \varepsilon\, x_2 + O(\varepsilon^{3/2})$$

Let us insert this form into the equation (1) : $(x+1)^3 - (\varepsilon + \frac{27}{4})x = 0$

We have $(x+1)^3 = (x_0+1)^3 + \sqrt{\varepsilon}\left(3(x_0+1)^2 x_1\right) + \varepsilon\left(3(x_0+1)x_1^2 + 3(x_0+1)^2 x_2\right) + O(\varepsilon^{3/2})$

and $\varepsilon x = \varepsilon x_0 + x_1 \varepsilon^{3/2} + \cdots = \varepsilon x_0 + O(\varepsilon^{3/2})$

Thus (1) $\Rightarrow (x_0+1)^3 - \frac{27}{4}x_0 + \sqrt{\varepsilon}\left(3(x_0+1)^2 x_1 - \frac{27}{4}x_1\right)$
$+ \varepsilon\left(3(x_0+1)x_1^2 + 3(x_0+1)^2 x_2 - x_0 - \frac{27}{4}x_2\right) + O(\varepsilon^{3/2}) = 0$

Thus:

at order $(\sqrt{\varepsilon})^0$: $(x_0+1)^3 - \frac{27}{4}x_0 = 0$ (4)

$(\sqrt{\varepsilon})^1$: $3(x_0+1)^2 x_1 - \frac{27}{4}x_1 = 0$ (5)

$(\sqrt{\varepsilon})^2$: $3(x_0+1)x_1^2 + 3(x_0+1)^2 x_2 - x_0 - \frac{27}{4}x_2 = 0$ (6)

We choose $x_0 = \frac{1}{2}$ as solution of (4) (always the same equation at order 0)

Then (5) $\Rightarrow 3\left(\frac{3}{2}\right)^2 \cancel{x_1} - \frac{27}{4}\cancel{x_1} = 0 \Rightarrow 0 = 0$

and (6) $\Rightarrow 3\left(\frac{3}{2}\right)x_1^2 + \cancel{3\left(\frac{3}{2}\right)^2 x_2} - \frac{1}{2} - \cancel{\frac{27}{4}x_2} = 0$

$\Rightarrow \frac{9}{2}x_1^2 = \frac{1}{2}$

$\Rightarrow x_1^2 = \frac{1}{9}$

$\Rightarrow x_1 = \pm\frac{1}{3}$

Thus the approximate solution for the roots close to $\frac{1}{2}$ as $\varepsilon \to 0$ are:

$$x \underset{\varepsilon \ll 1}{\sim} \frac{1}{2} \pm \frac{1}{3}\sqrt{\varepsilon}$$