Numerical Modelling

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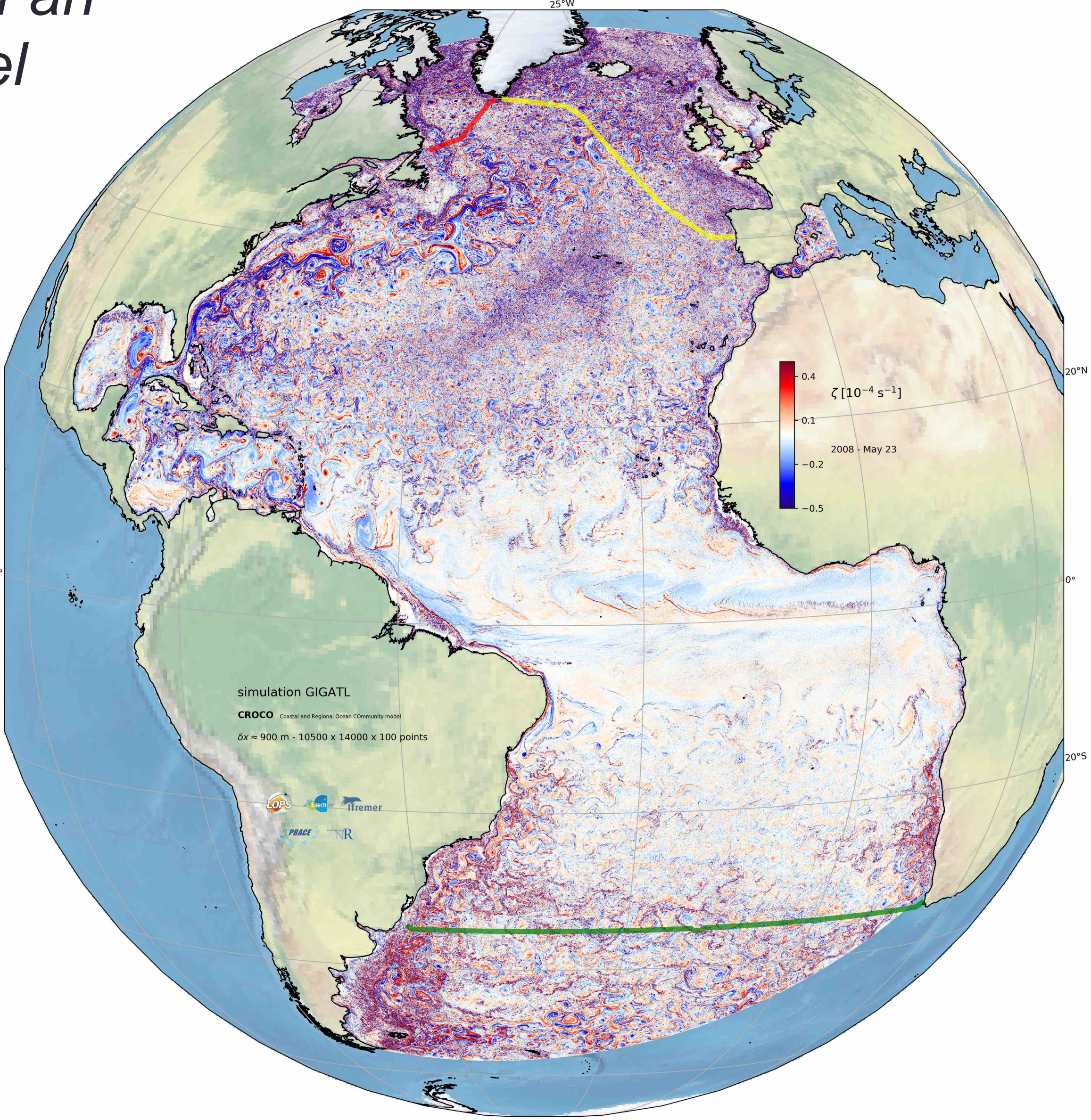
the anatomy of an ocean model



- **Lesson 1 :**
  - Introduction
  - *Activity 1 [run an ocean model]*
- **Lesson 2 :** [D109]
  - Equations of motions
  - Subgrid-scale parameterization
  - *Activity 2 [Dynamics of an ocean gyre]*
- **Lesson 3 :** [D109]
  - Horizontal Discretization
  - Numerical schemes
  - *Activity 3 [Impacts of numerics]*
- **Lesson 4 :** [D109]
  - Vertical coordinates
  - Model parameterizations
  - *Activity 4 [Impact of topography]*
- **Lesson 5 :** [D109]
  - Boundary Forcings
  - Presentation of the model CROCO
  - *Activity 4 [Design a realistic simulation]*
- **Lesson 6 :** [D109]
  - Diagnostics and validation
  - *Activity 5 [Analyze a realistic simulation]*
- **Lesson 7 :** [D109]
  - *Work on your projet*

Presentations and material will be available at :

jgula.fr/ModNum/

Useful references

Extensive courses:

- MIT: https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-950-atmospheric-and-oceanic-modeling-spring-2004/lecture-notes/
- Princeton: https://stephengriffies.github.io/assets/pdfs/GFM_lectures.pdf

Overview on ocean modelling and current challenges:

- Griffies et al., 2000, Developments in ocean climate modelling, Ocean Modelling.http://jgula.fr/ModNum/Griffiesetal00.pdf
- Griffies, 2006, "Some Ocean Model Fundamentals", In "Ocean Weather Forecasting: An Integrated View of Oceanography", 2006, Springer Netherlands. http://jgula.fr/ModNum/Griffies_Chapter.pdf
- Fox-Kemper et al, 19, “Challenges and Prospects in Ocean Circulation Models" http://jgula.fr/ModNum/FoxKemperetal19.pdf

ROMS/CROCO:

- https://www.myroms.org/wiki/
- Shchepetkin, A., and J. McWilliams, 2005: The Regional Oceanic Modeling System (ROMS): A split-explicit, free-surface, topography-following- coordinate ocean model. Ocean Modell. http://jgula.fr/ModNum/ShchepetkinMcWilliams05.pdf

#1
Which Equations?

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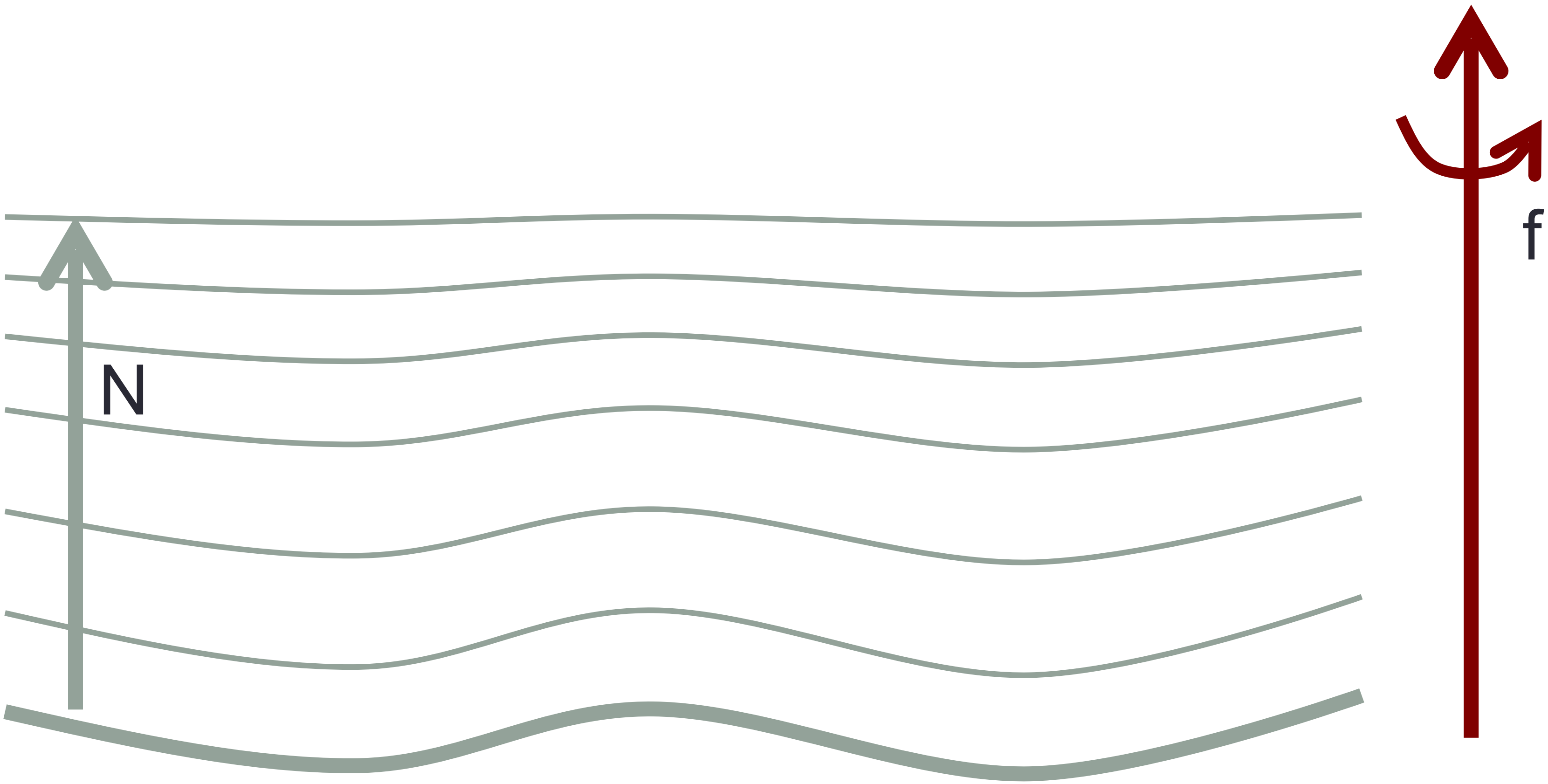
Which Equations?

Ingredients : rotation + stratification



Which Equations?

- Momentum equations (3d) $$\frac{D\vec{u}}{Dt} = \ldots$$
- Conservation of mass $$\frac{D\rho}{Dt} = \mathcal{S}_\rho$$
- Conservation of heat $$\frac{DT}{Dt} = \mathcal{S}_T$$
- Conservation of salinity $$\frac{DS}{Dt} = \mathcal{S}_S$$
- Equation of state : $$\rho = \rho(T, S, p)$$

[7 equations for the 7 variables: u,v,w,p,T,S,ρ]

Which Equations?

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[7 equations for the 7 variables: u,v,w,p,T,S,ρ]

Equations for momentum/mass?

- Navier-Stokes Equations (NS)
- Non-hydrostatic Primitive Equations (NH)
- Hydrostatic Primitive Equations (PE)
- Shallow-water (SW)
- Quasi-geostrophic (QG)
- 2D Euler equations
- Etc.

Type of models

Navier Stokes

- DNS = Direct Numerical Simulation
- LES = Large Eddy Simulation

CFD

Process studies

PE

- PE = Primitive Equations models

Ocean Circulation Models

SW

- SW = Shallow-Water models

SQG

- SQG = Surface Quasi-Geostrophic models

QG

- QG = Quasi-Geostrophic models
- Etc.

Idealized models

Equations for momentum/mass?

Navier-Stokes Equations:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}}$$

Momentum equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation
(no source/sink)

Equations for momentum/mass?

Navier-Stokes Equations:

$$\boxed{\frac{\partial \vec{u}}{\partial t}} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + \boxed{2\vec{\Omega} \times \vec{u}} + \boxed{g\vec{k}} = \boxed{-\frac{\vec{\nabla} P}{\rho}} + \boxed{\vec{\mathcal{F}}}$$

Time variation | Advection (inertia) | Rotation | Gravity | Pressure gradient | Forcings + Dissipation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation
(no source/sink)

Equations for momentum/mass?

Navier-Stokes Equations:

Linearized momentum equations

+ continuity equation

+ adiabatic motion :

= Acoustic modes (sound waves)

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} P$$

$$\frac{\partial P}{\partial t} = -\rho_0 c_s^2 \vec{\nabla} P \cdot \vec{u}$$

$$\partial_{tt} P = c_s^2 \nabla^2 P$$

With $c_s \approx 1500\,\mathrm{m\,s^{-1}}$ in water, a model requires a very small time-step to solve these equations.

Equations for momentum/mass?

Boussinesq Approximation:

Density perturbations small compared to mean background value:

$$\rho = \rho_0 + \rho' \qquad\qquad \rho' << \rho_0$$

Linearize all terms involving a product with density,

except the gravity term which is already linear:

$$\rho\vec{u} \quad \rightarrow \quad \rho_0\vec{u}$$

$$\rho g \quad \rightarrow \quad \rho g$$

Equations for momentum/mass?

Boussinesq Approximation :

[+ incompressibility or adiabatic]

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

$$\Big\downarrow \quad \text{Continuity equation}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

Equations for momentum/mass?

Non hydrostatic boussinesq (NH):

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho_0} g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \frac{\vec{\mathcal{F}}}{\rho_0} + \frac{\vec{\mathcal{D}}}{\rho_0}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

Easier to solve than Navier-Stokes, but still requires to invert a 3d elliptic equation for P (computationally expansive)

Equations for momentum/mass?

Hydrostatic balance:

The vertical component of the Boussinesq momentum equations is

$$\frac{\partial w}{\partial t} + \vec{u} \cdot \vec{\nabla} w + 2\Omega \cos\phi v + \boxed{\frac{\rho}{\rho_0} g = -\frac{\partial_z P}{\rho_0}} + \frac{\mathcal{F}_w}{\rho_0} + \frac{\mathcal{D}_w}{\rho_0}$$

For long horizontal motions (L >> H) the dominant balance is

H ~3000 m
L ~3000 km

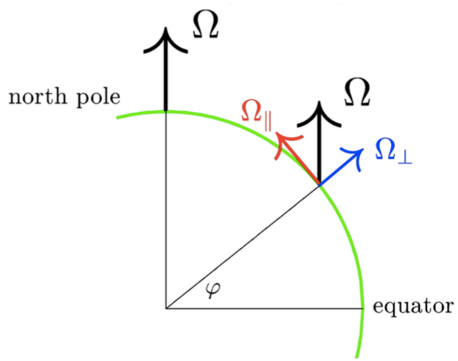
$$\frac{\partial P}{\partial z} = -\rho g$$

Such that pressure is just a vertical integral: $P = \int_z^\eta g \rho dz$

Equations for momentum/mass?

Traditional approximation

= neglect horizontal Coriolis term



$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} + \boxed{2\vec{\Omega}\times\vec{u}} + \frac{\rho}{\rho_0}g\vec{k} = -\frac{\vec{\nabla}P}{\rho_0} + \vec{\mathcal{F}} + \vec{\mathcal{D}}$$

$$\downarrow$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} + \boxed{f\vec{k}\times\vec{u}} + \frac{\rho}{\rho_0}g\vec{k} = -\frac{\vec{\nabla}P}{\rho_0} + \vec{\mathcal{F}} + \vec{\mathcal{D}}$$

Equations for momentum/mass?

Hydrostatic Primitive Equations (PE)

- 2d momentum with Boussinesq approximation:

$$\begin{aligned}
\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla}_H u + w \frac{\partial u}{\partial z} - fv &= -\frac{\partial_x P}{\rho_0} + \mathcal{F}_u + \mathcal{D}_u \\
\frac{\partial v}{\partial t} + \vec{u} \cdot \vec{\nabla}_H v + w \frac{\partial v}{\partial z} + fu &= -\frac{\partial_y P}{\rho_0} + \mathcal{F}_v + \mathcal{D}_v
\end{aligned}$$

- Hydrostatic:

$$\frac{\partial P}{\partial z} = -\rho g$$

- Continuity equation for an incompressible fluid: $\vec{\nabla} \cdot \vec{u} = 0$

Equations for PE Ocean Models

Hydrostatic Primitive Equations (PE)

- 2d momentum with Boussinesq approximation:

$$\begin{aligned}
\frac{\partial u}{\partial t} + \vec{u}\cdot\vec{\nabla}_H u + w\frac{\partial u}{\partial z} - fv &= -\frac{\partial_x P}{\rho_0} + \mathcal{F}_u + \mathcal{D}_u \\
\frac{\partial v}{\partial t} + \vec{u}\cdot\vec{\nabla}_H v + w\frac{\partial v}{\partial z} + fu &= -\frac{\partial_y P}{\rho_0} + \mathcal{F}_v + \mathcal{D}_v
\end{aligned}$$

- Hydrostatic:

$$\frac{\partial P}{\partial z} = -\rho g$$

- Continuity equation for an incompressible fluid: $\vec{\nabla}\cdot\vec{u} = 0$
- Conservation of heat and salinity $\frac{DT}{Dt} = \mathcal{S}_T \qquad \frac{DS}{Dt} = \mathcal{S}_S$
- Equation of state : $\rho = \rho(T, S, z)$

Equations for PE Ocean Models

Hydrostatic Primitive Equations (PE)

- 4 prognostics equations for **u, v, T, S**
- 3 diagnostics equations for **w, ρ, P**

Equations for PE Ocean Models

Hydrostatic Primitive Equations (PE)

- 2d momentum with Boussinesq approximation:

?

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla}_H u + w \frac{\partial u}{\partial z} - fv &= -\frac{\partial_x P}{\rho_0} + \mathcal{F}_u + \mathcal{D}_u \\
\frac{\partial v}{\partial t} + \vec{u} \cdot \vec{\nabla}_H v + w \frac{\partial v}{\partial z} + fu &= -\frac{\partial_y P}{\rho_0} + \mathcal{F}_v + \mathcal{D}_v
\end{aligned}
$$

- Hydrostatic: $$\frac{\partial P}{\partial z} = -\rho g$$
- Continuity equation for an incompressible fluid: $$\vec{\nabla} \cdot \vec{u} = 0$$
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Equations for PE Ocean Models

Hydrostatic Primitive Equations (PE)

- 4 prognostics equations for **u, v, T, S**
- 3 diagnostics equations for **w, ρ, P**

+ Forcings (wind, heat flux)

+ sub-grid scale parameterizations (bottom drag, mixing, etc.)

#2
Subgrid-scale parameterization

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Incompressible Navier-Stokes Equations:

- Dissipation of energy/momentum in the NS equations:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \underbrace{\boxed{\nu \vec{\nabla}^2 \vec{u}}}_{\text{Viscosity}}$$

Type of models

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Ocean Circulation Models

Idealized models

Incompressible Navier-Stokes Equations:

$$\frac{\partial \vec{u}}{\partial t} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \boxed{\nu \vec{\nabla}^2 \vec{u}}$$

Non-linear terms

Viscosity

- Importance of NL terms and viscosity = Reynolds Number

$$Re = \frac{UL}{\nu}$$

Where *U* is a typical velocity of the flow and *L* is a typical length describing the flow.

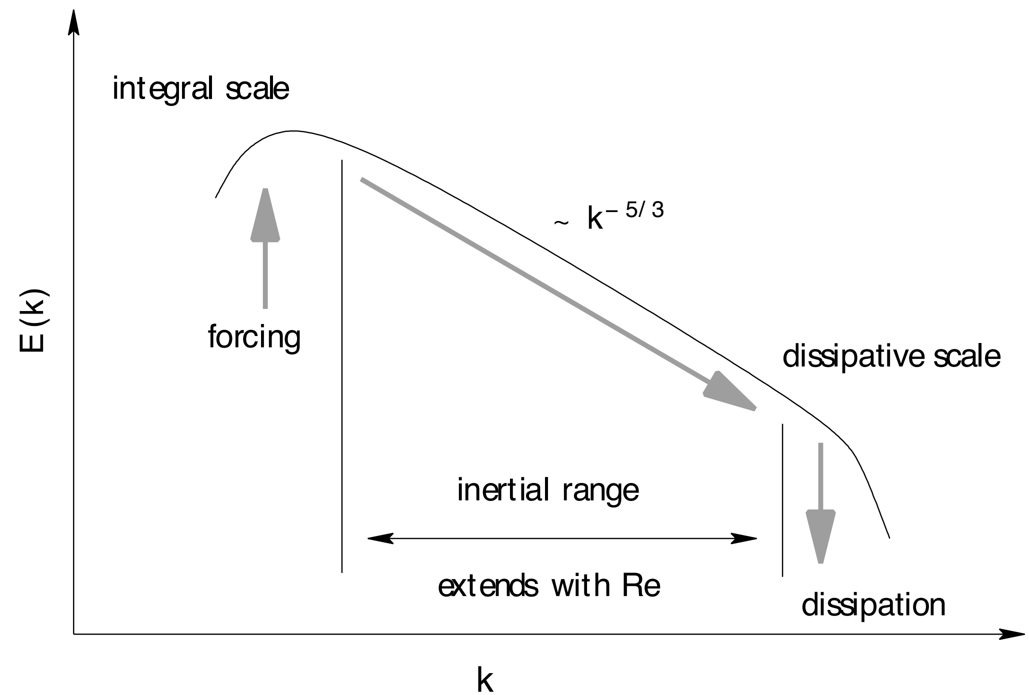
Direct numerical simulation (DNS)

DNS resolves the entire range of turbulent length scales down to the smallest dissipative scales (Kolmogorov scale):

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \approx \left(\frac{\nu^3 L}{U^3}\right)^{1/4} = Re^{-3/4} L$$

where ν is the kinematic viscosity

And ϵ the rate of kinetic energy dissipation



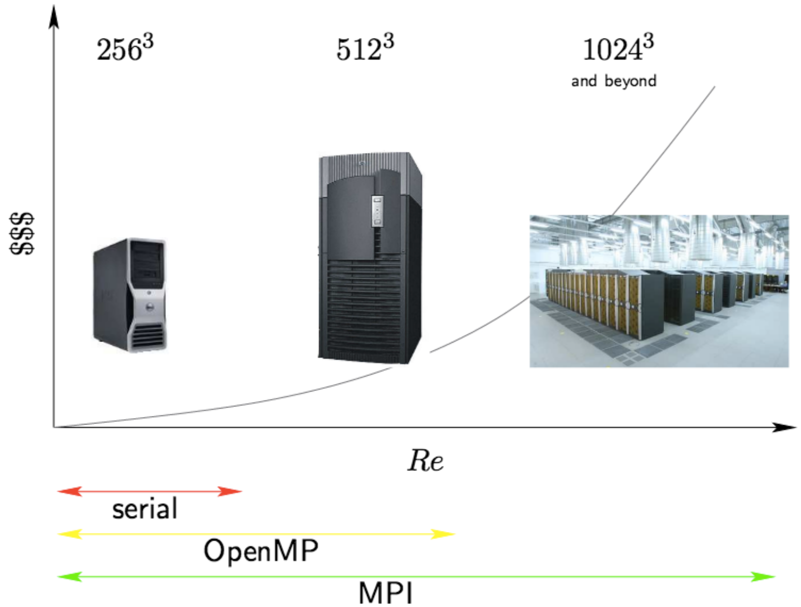
Direct numerical simulation (DNS)

The number of floating-point operations required to complete the simulation is proportional to the number of mesh points:

$$N_x = \frac{L}{\eta} = Re^{3/4}$$

and the number of time steps:

$$\frac{T}{\Delta t} = \frac{TU}{\eta} = \frac{TU}{L} Re^{3/4}$$



It is extremely expensive as the computational cost scales as *Re*³

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Ocean Circulation Models

Idealized models

Reynolds averaging

To reduce the computational cost, one need to reduce the range of time- and length-scales that are being solved for.

The idea is based on separation of mean and turbulent component:

$$u = \overline{u} + u'$$

Where

$$\overline{u} = \frac{1}{T}\int_0^T u\, dt \text{ or } \overline{u} = \frac{1}{X}\int_0^X u\, dx$$

With by definition

$$\overline{u'} = 0$$

Reynolds averaging

- Activity:

Adapt the momentum equation:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + f\mathbf{k} \times u_i + \frac{\rho}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_i \partial x_j}$$

For the mean velocity: $\frac{\partial \overline{u_i}}{\partial t}$ = ?

Reynolds averaging

So we resolve only the equations for the mean variables:

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j \frac{\partial u_i}{\partial x_j}} + f\mathbf{k} \times \overline{u_i} + \frac{\overline{\rho}}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x_j} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_i \partial x_j}$$

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + f\mathbf{k} \times \overline{u_i} + \frac{\overline{\rho}}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x_j} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_i \partial x_j} \boxed{- \frac{\partial \overline{u_i' u_j'}}{\partial x_j}}$$

Advection for the averaged flow

Reynolds stress
= effect of subgrid-scale turbulence

Turbulence closure

The Closure Problem :

- The number of unknowns is larger than the number of equations
- When we start deriving equations for unknowns, more variables appear.

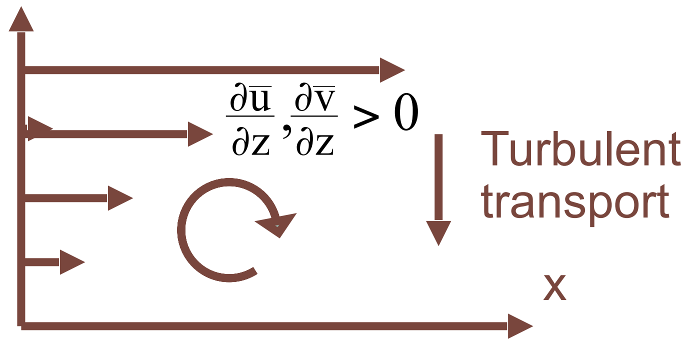
Progn. Eq. for:	Moment	Equation	No. Eqs.	No. Unknowns
$\overline{U_i}$	First	$\frac{\partial \overline{U_i}}{\partial t} = \ldots - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$	3	6
$\overline{u_i' u_j'}$	First	$\frac{\partial \overline{u_i' u_j'}}{\partial t} = \ldots - \frac{\partial \overline{u_k' u_i' u_j'}}{\partial x_k}$	6	10
$\overline{u_i' u_j' u_k'}$	First	$\frac{\partial \overline{u_i' u_j' u_k'}}{\partial t} = \ldots - \frac{\partial \overline{u_k' u_i' u_j' u_m'}}{\partial x_m}$	10	15

Turbulence closure

- In PE models the equations are closed by parameterizing the Reynolds stresses as:

$$\overline{u'w'} = -K_{Mv}\frac{\partial u}{\partial z}$$

$$\overline{v'w'} = -K_{Mv}\frac{\partial v}{\partial z}$$



Turbulence closure

In ROMS:

$$\mathcal{F}_u = \frac{\partial}{\partial z}\left(K_{Mv}\frac{\partial u}{\partial z}\right) + \nabla_h(K_{Mh}.\nabla_h u)$$

$$\mathcal{F}_v = \frac{\partial}{\partial z}\left(K_{Mv}\frac{\partial v}{\partial z}\right) + \nabla_h(K_{Mh}.\nabla_h v)$$

$$\mathcal{S}_T = \frac{\partial}{\partial z}\left(K_{Tv}\frac{\partial T}{\partial z}\right) + \nabla_h(K_{Th}.\nabla_h T)$$

$$\mathcal{S}_S = \frac{\partial}{\partial z}\left(K_{Sv}\frac{\partial S}{\partial z}\right) + \nabla_h(K_{Sh}.\nabla_h S)$$

Vertical mixing

Horizontal diffusion

Turbulence closure

Vertical Mixing: $$K_{Mv}, K_{Tv}, K_{Sv}$$

- Local Turbulence closure: GLS, k-kl (MY2.5), k-ε, κ-ω, etc. *[e.g. Warner et al, 2005, Ocean Modelling]*
- Non local K-profile parameterization (KPP) *[Large et al, 1994, Rev. of Geophysics]*

Horizontal diffusion:

$$K_{Mh}, K_{Th}, K_{Sh}$$

- Explicit diffusion
- Implicit (comes from the advective scheme)

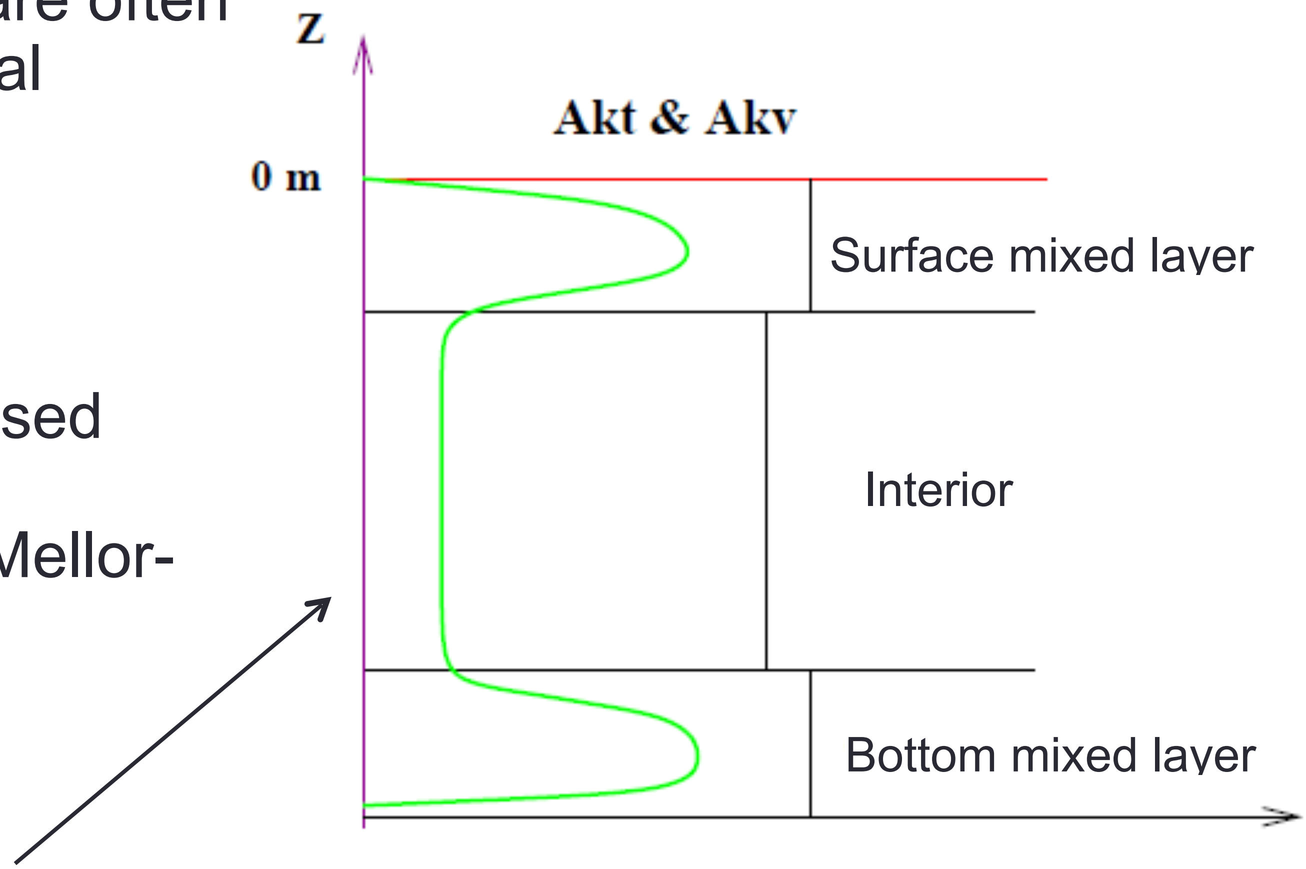
Non local K-profile parameterization

❑ Mixed layer schemes are often based on one-dimensional « column physics »

❑ Boundary layer parameterizations are based either on:

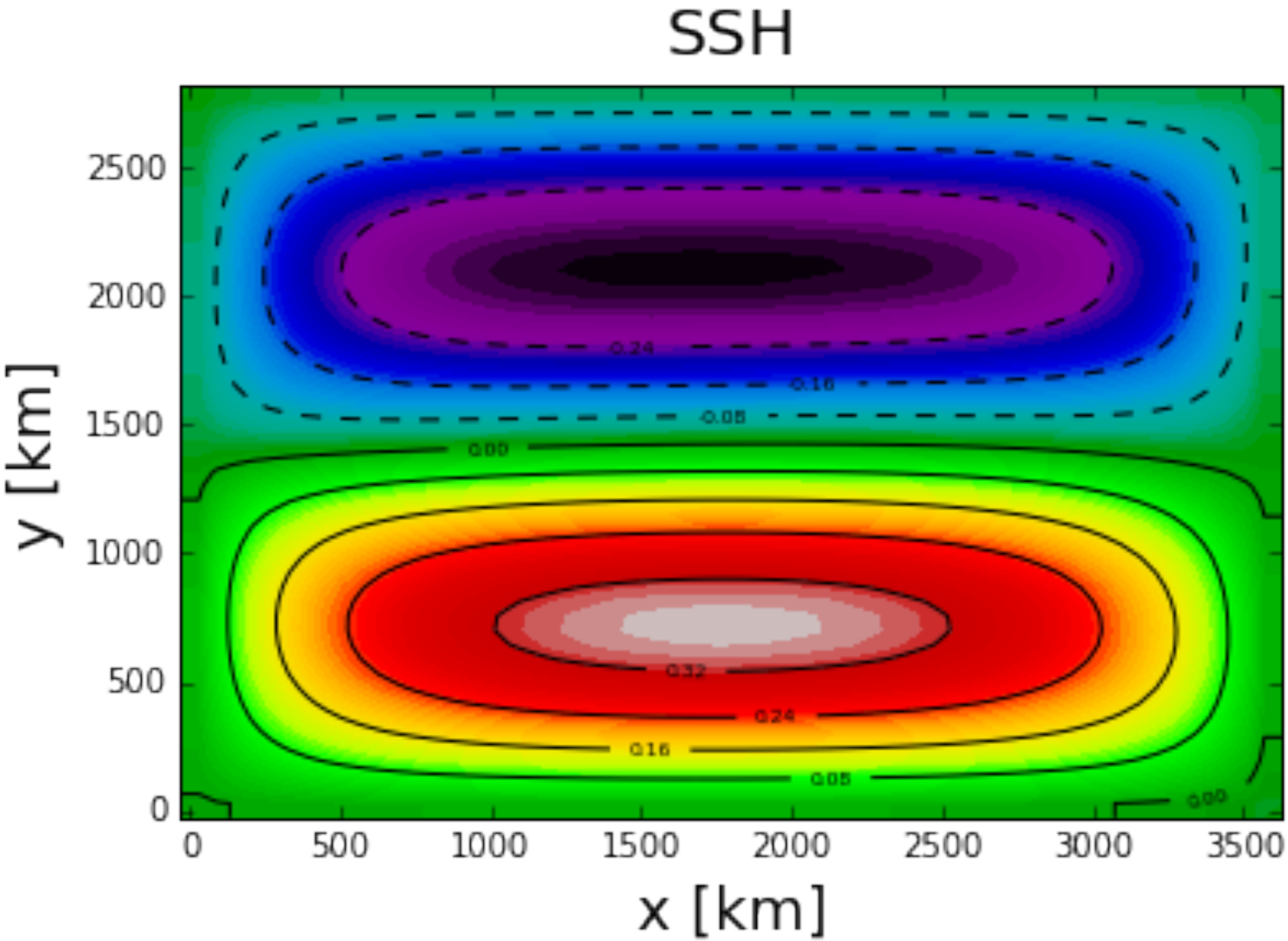
- Turbulent closure (Mellor-Yamada, TKE)
- **K profile (KPP)**

Mixing coefficient profils



Principle scheme of KPP turbulent

Activity 2 – Run an idealized ocean basin II



Momentum equations

$$\frac{\partial u}{\partial t} = -u_j \frac{\partial u}{\partial x_j} - w \frac{\partial u}{\partial z} + fv - \frac{P_x}{\rho_0} + \mathcal{V}_u + \mathcal{D}_u + \mathcal{S}_u$$

$$\underbrace{\frac{\partial v}{\partial t}}_{rate} = \underbrace{-u_j \frac{\partial v}{\partial x_j}}_{hadv} \underbrace{- w \frac{\partial v}{\partial z}}_{vadv} - \underbrace{fu}_{cor} - \underbrace{\frac{P_y}{\rho_0}}_{Prsgrd} + \underbrace{\mathcal{V}_v}_{vmix} + \underbrace{\mathcal{D}_v}_{hmix+hdiff} + \underbrace{\mathcal{S}_v}_{nudg}$$

Barotropic vorticity balance

- Barotropic vorticity: $\Omega = \frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y}$ with $\overline{u} = \int_{-h}^{\zeta} u \, dz,$

- The barotropic vorticity balance equation of the flow is obtained by integrating the momentum equations in the vertical and cross differentiating them:
  - [cf: https://www.jgula.fr/ModNum/vort_balance.pdf]

$$\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}} = - \underbrace{\vec{\nabla}.(f\vec{\overline{u}})}_{\text{planet. vort. adv.}} + \underbrace{\frac{\mathbf{J}(P_b, h)}{\rho_0}}_{\text{bot. pres. torque}} + \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}$$
$$+ \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

$$A_\Sigma = \frac{\partial^2(\overline{vv} - \overline{uu})}{\partial x \partial y} + \frac{\partial^2 \overline{uv}}{\partial x \partial x} - \frac{\partial^2 \overline{uv}}{\partial y \partial y},$$

Stommel's gyre

THE WESTWARD INTENSIFICATION OF WIND-DRIVEN OCEAN CURRENTS

Henry Stommel

(Contribution No. 408, Woods Hole Oceanographic Instiution)

Abstract--A study is made of the wind-driven circulation in a homogeneous rectangular ocean under the influence of surface wind stress, linearised bottom friction, horizontal pressure gradients caused by a variable surface height, and Coriolis force.

An intense crowding of streamlines toward the western border of the ocean is discovered to be caused by variation of the Coriolis parameter with latitude. It is suggested that this process is the main reason for the formation of the intense currents (Gulf stream and others) observed in the actual oceans.

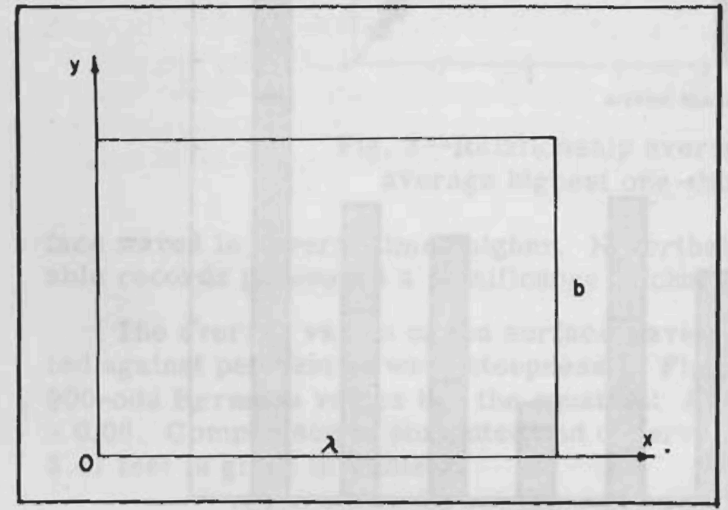


Fig. 1--Ocean basin dimensions and the coordinate system

- Momentum equations:

The steady state equations of motion, with the inertial terms omitted because they are small, are written in the form

$$0 = f(D + h)v - F \cos (\pi y/b) - Ru - g(D + h)\partial h/\partial x \qquad (1)$$

$$0 = - f(D + h)u - Rv - g(D + h)\partial h/\partial y \qquad (2)$$

- Barotropic vorticity equation:

$$v(D + h)(\partial f/\partial y) + (F\pi/b) \sin (\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Stommel's gyre

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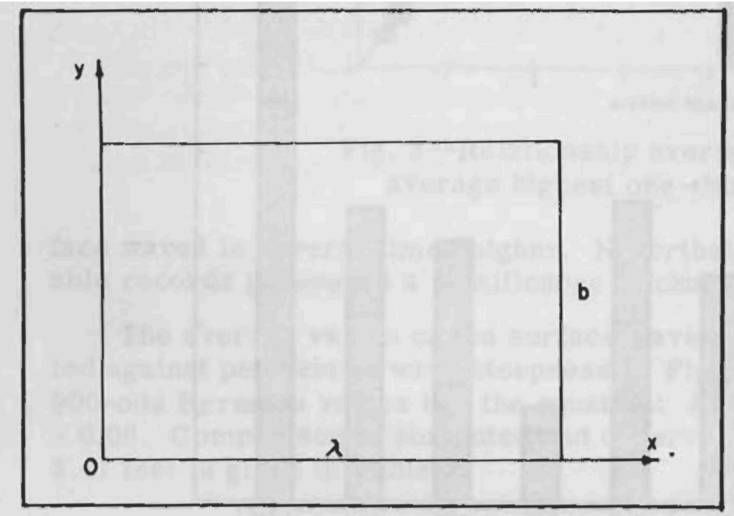


Fig. 1--Ocean basin dimensions and the coordinate system

- Momentum equations:

Coriolis Wind Drag Pressure gradient

The steady state equations of motion, with the inertial terms omitted because they are small, are written in the form

$$0 = f(D + h)v + F\cos(\pi y/b) - Ru - g(D + h)\partial h/\partial x \qquad (1)$$

$$0 = -f(D + h)u - Rv - g(D + h)\partial h/\partial y \qquad (2)$$

- Barotropic vorticity equation:

Planetary vorticity advection | Wind Curl | Drag Curl

$$v(D + h)(\partial f/\partial y) + (F\pi/b)\sin(\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Stommel's gyre

- No rotation / or constant rotation ($\dfrac{\partial f}{\partial y} = \beta = 0$)

The equation for the stream function is therefore

$$\psi = \gamma (b/\pi)^2 \sin (\pi y/b)[e^{(x - \lambda)\pi/b} + e^{-x\pi/b} - 1]$$

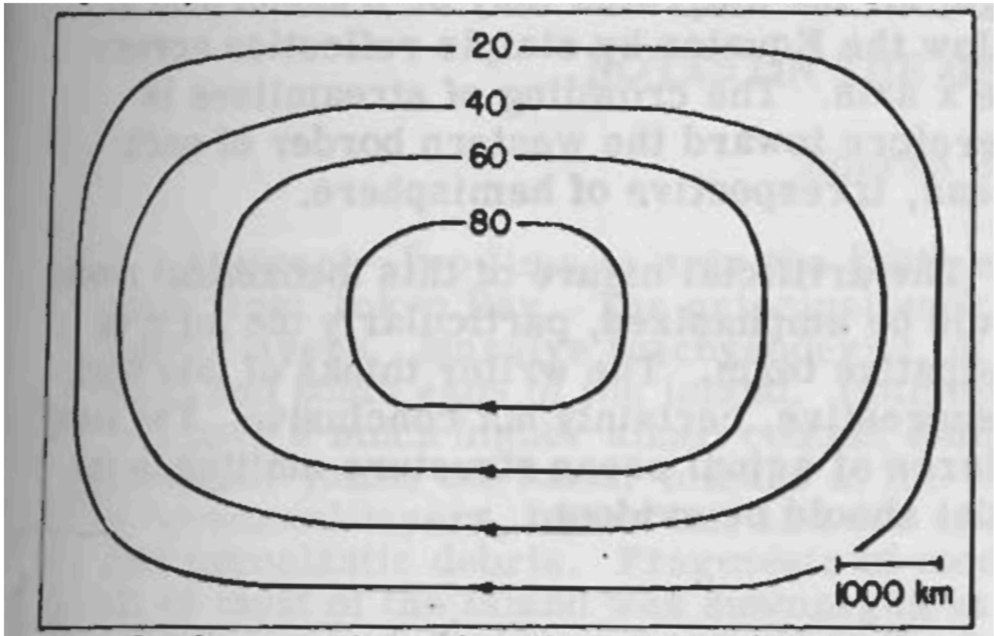


Fig. 2--Streamlines for the case of both the non-rotating and uniformly rotating oceans

Plan ary v ity
a n

Wind
Curl

Drag
Curl

$$\cancel{v(D\ldots)(\partial f/\partial y)} + (F\pi/b) \sin (\pi y/b) + R(\partial v/\partial x - \partial u/\partial y) = 0$$

Barotropic vorticity balance

- No rotation / or constant rotation ($\frac{\partial f}{\partial y} = \beta = 0$)
-

$$\ldots = - \ldots + \ldots + \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}$$

$$+ \ldots - \ldots$$

Stommel's gyre

- With latitudinal variation of Coriolis

$$\frac{\partial f}{\partial y} = \beta \neq 0$$

- **Formation of a western boundar**

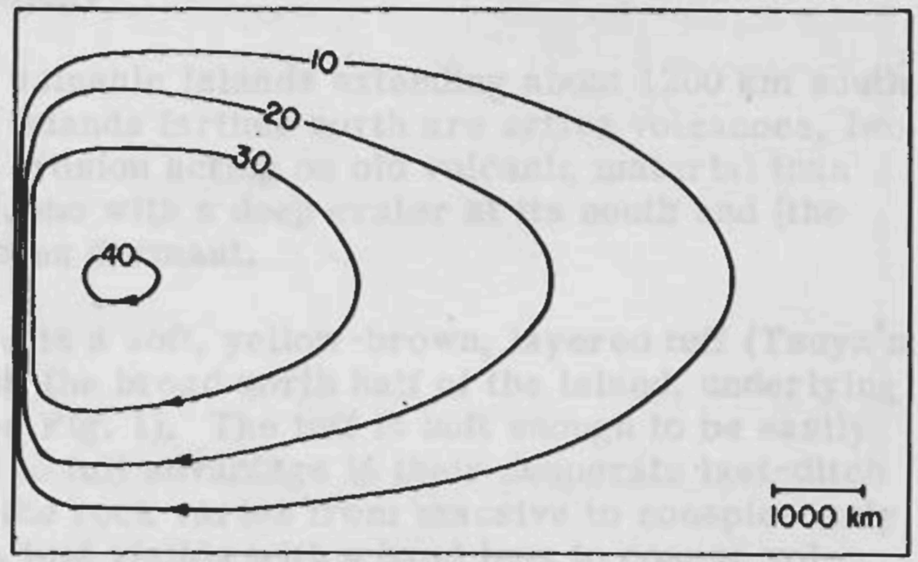


Fig. 5--Streamlines for the case where the Coriolis force is a linear function of latitude

Barotropic vorticity balance

- No rotation / or constant rotation ($\dfrac{\partial f}{\partial y} = \beta = 0$)
-

$$\underbrace{\quad}_{\text{rate}} = \boxed{-\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \underbrace{\quad , h)}_{\text{bot. p s. torque}} \boxed{+\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}}$$

$$+ \underbrace{\quad}_{\text{hor usion.}} - \underbrace{\quad}_{\text{NL tion}}$$

Stommel's gyre (no beta)

- Forcings and data
  - Bottom topography + Land mask
  - Atmospheric surface boundary forcing
  - Initial oceanic conditions
  - Lateral oceanic boundary conditions
-

- **Flat bottom**
- **Constant wind** (sustr(i,j)=-cff1*cos(2.*pi/el*yr(i,j)))
- **Resting state**
- **Vertical walls**
-

Stommel's gyre (no beta)

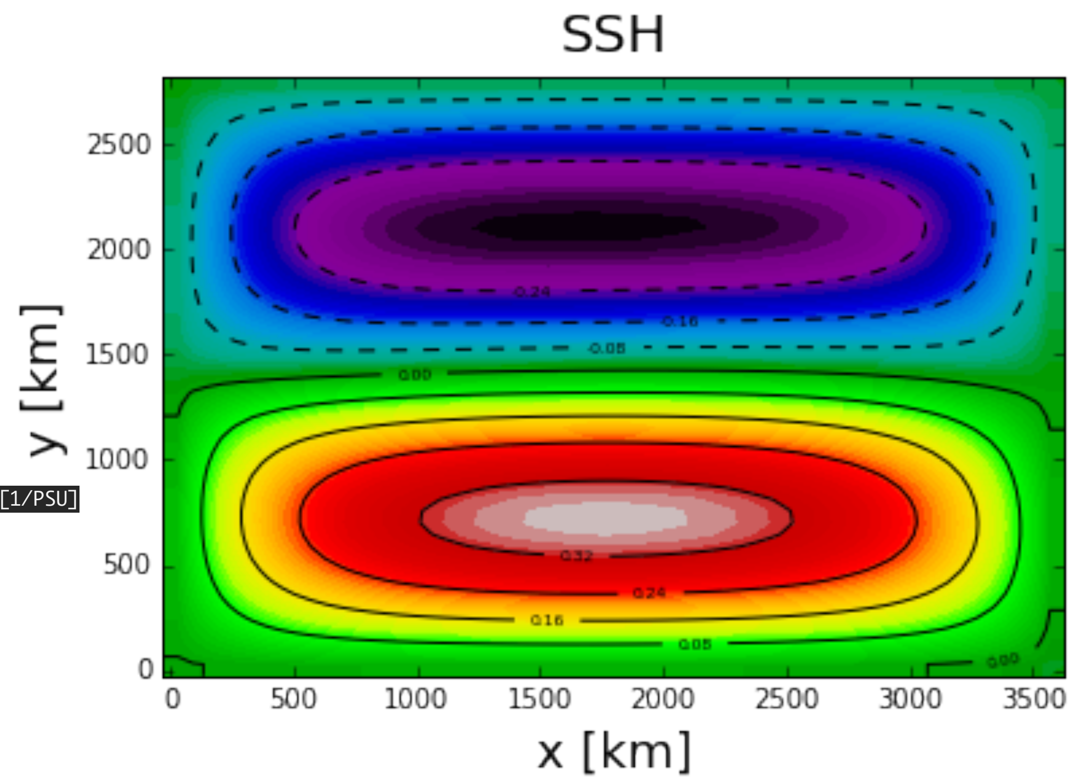
- No rotation / or constant rotation ($\frac{\partial f}{\partial y} = \beta = 0$)
- **cppdefs.h**

```
# define UV_COR
# define UV_VIS2
# define TS_DIF2

# define ANA_GRID
# define ANA_INITIAL
```

- **croco.in**

```
bottom_drag:     RDRG(m/s),      RDRG2, Zob [m],  Cdb_min, Cdb_max
                 3.e-4             0.      0.         0.       0.
gamma2:
                 1.
lin_EOS_cff:  R0 [kg/m3], T0 [Celsius], S0 [PSU], TCOEF [1/Celsius], SCOEF [1/PSU]
              30.          0.             0.        0.28                  0.
lateral_visc:   VISC2     [m^2/sec ]
                 1000.  0.
tracer_diff2: TNU2          [m^2/sec]
                 1000. 0.
```



ROMS simulation after 20 years

- **ana_grid.F**

```
f0=1.E-4
beta=0.
```

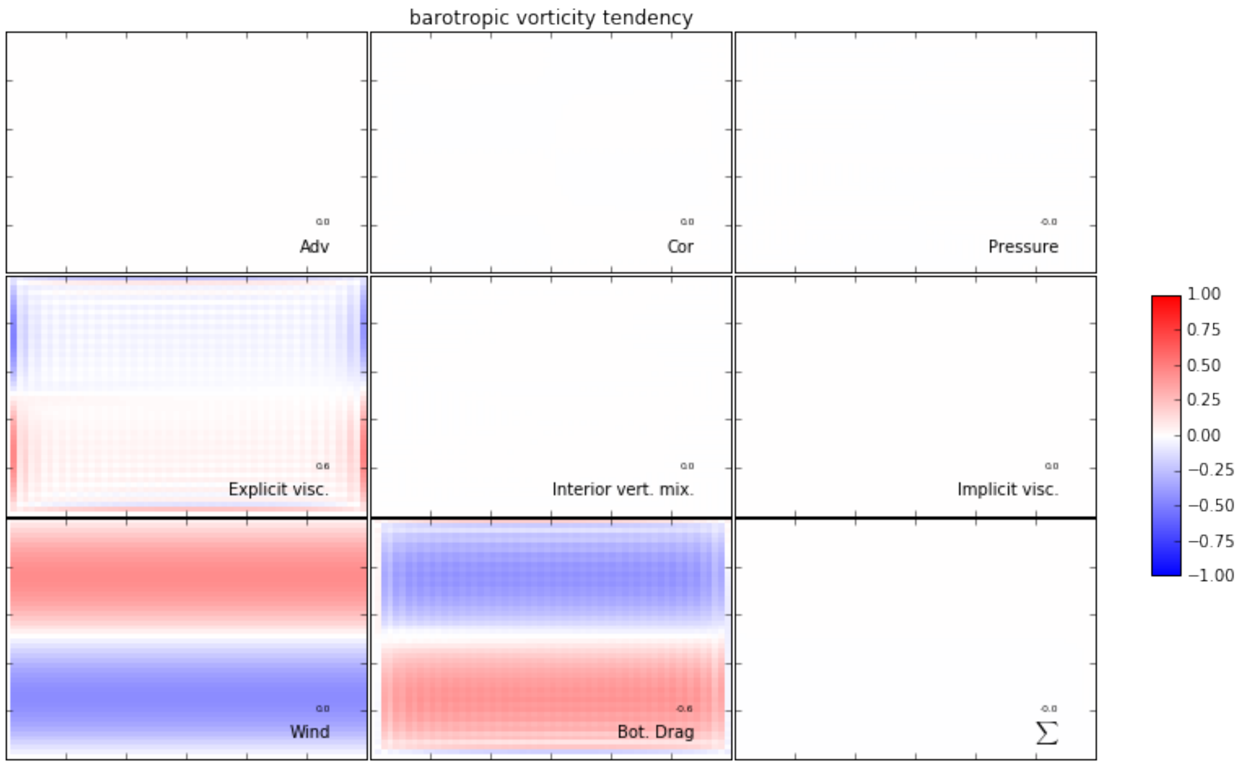
- **param.h**

```
      parameter (LLm0=60,    MMm0=50,    N=10)
```

Stommel's gyre (no beta)

- No rotation / or constant rotation

$$\underbrace{\cancel{\frac{\partial \Omega}{\partial t}}}_{\text{rate}} = -\underbrace{\cancel{\vec{\nabla}\cdot(f\vec{u})}}_{\text{planet. vort. adv.}} + \underbrace{\cancel{\frac{\mathbf{J}(P_b, h)}{\rho_0}}}_{\text{bot. pres. torque}} + \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}} + \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{\cancel{\mathcal{A}}}_{\text{NL advection}}$$

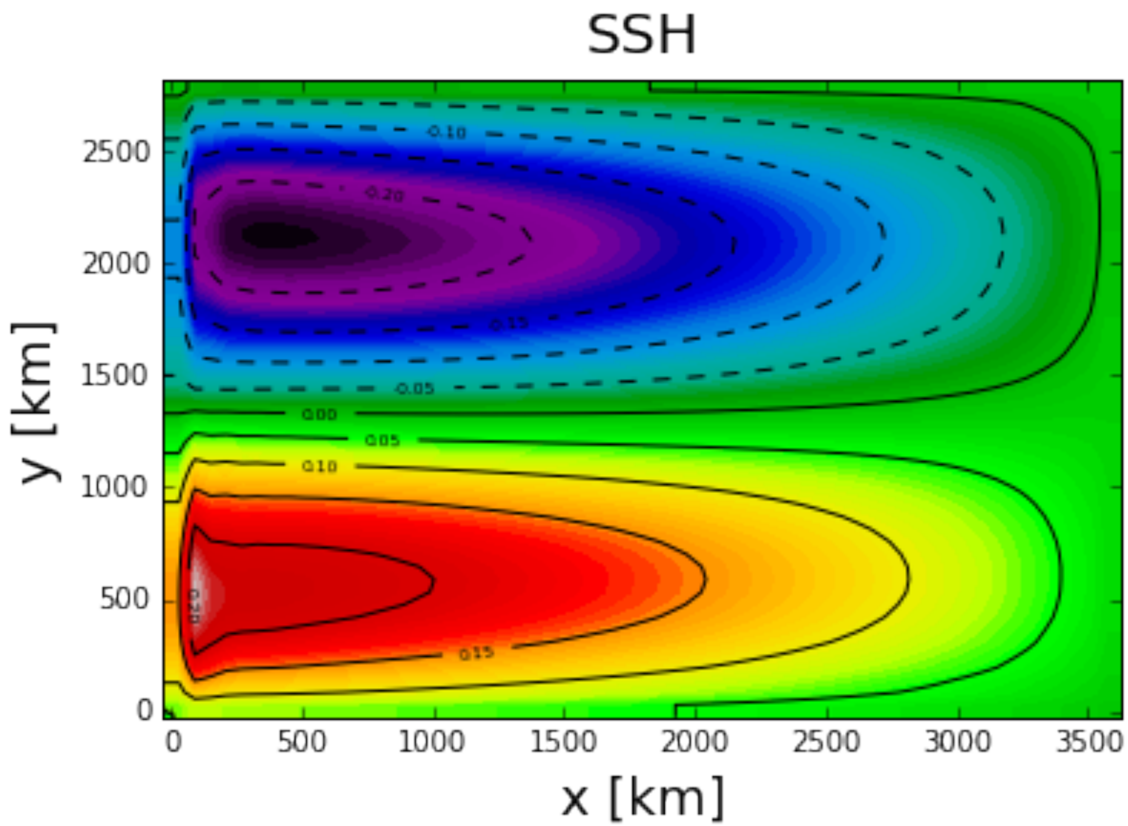
- 

Stommel's gyre (with beta)

- With latitudinal variation of Coriolis ($\frac{\partial f}{\partial y} = \beta \neq 0$)

- **ana_grid.F**

```
f0=1.E-4
beta=2.E-11
```

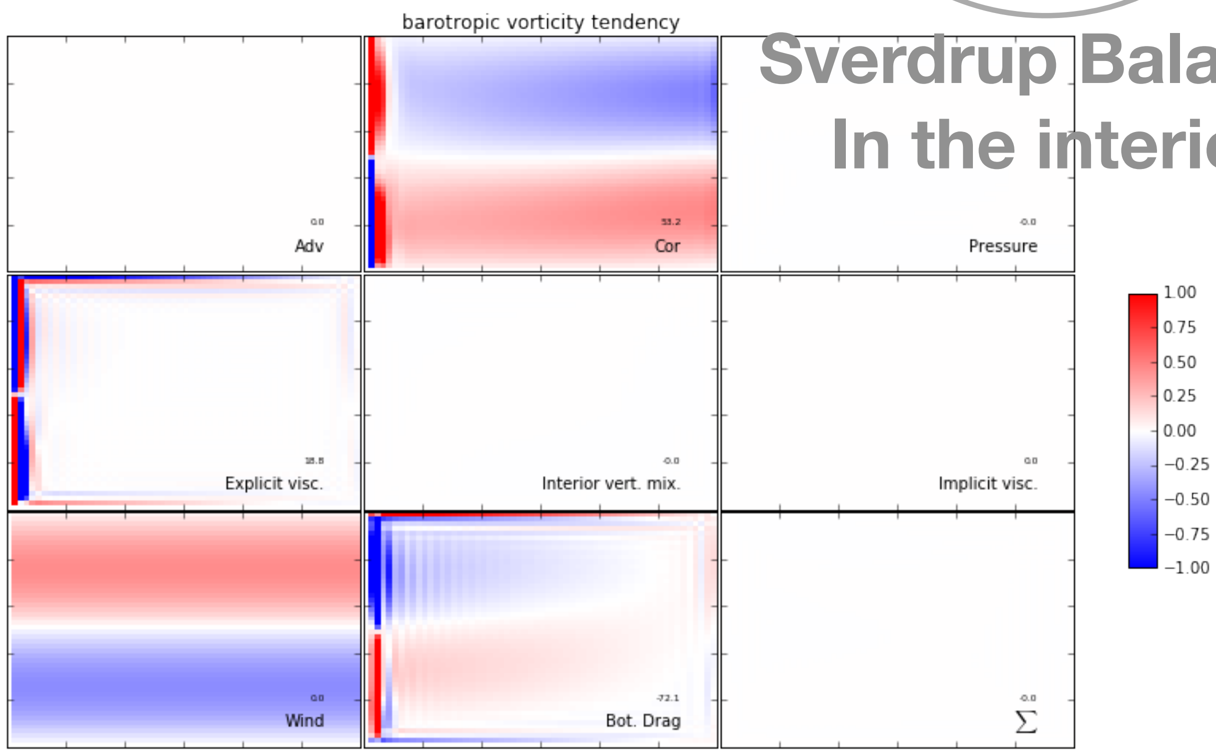


Stommel's gyre (with beta)

- With latitudinal variation of Coriolis

$$\underbrace{\cancel{\frac{\partial \Omega}{\partial t}}}_{\text{rate}} = \boxed{- \underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \underbrace{\cancel{\frac{J(P_b, h)}{\rho_0}}}_{\text{bot. pres. torque}} \boxed{+ \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}} + \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{\cancel{\mathcal{N}_\Sigma}}_{\text{NL advection}}$$

Sverdrup Balance
In the interior

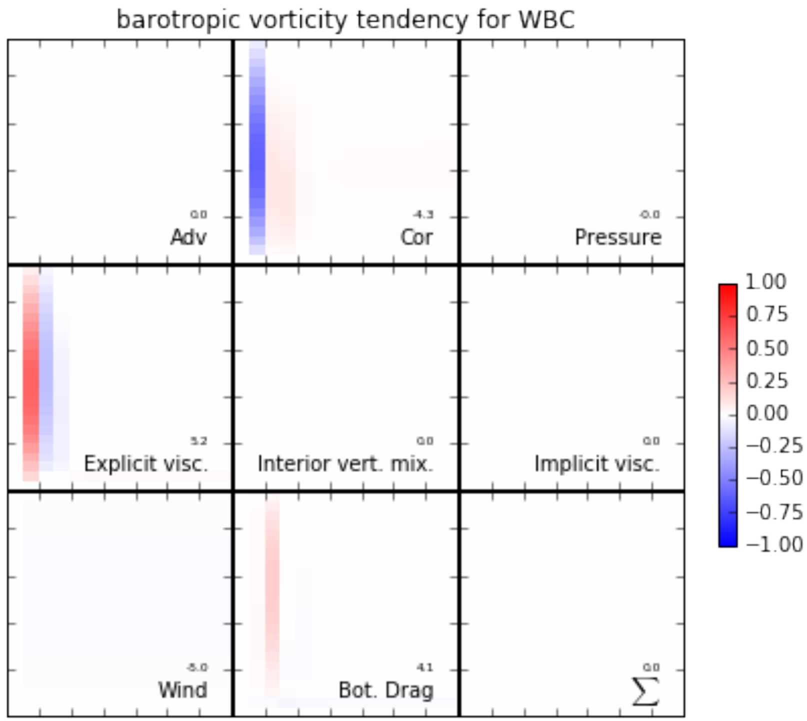
- 

Stommel's gyre (with beta)

- With latitudinal variation of Coriolis

$$\cancel{\underbrace{\frac{\partial\Omega}{\partial t}}_{\text{rate}}} = \boxed{-\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{\frac{\mathbf{J}(P_b,H)}{\rho_0}}_{\text{bot. pres. torque}}} + \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \boxed{\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}}} + \boxed{\underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}}} - \cancel{\underbrace{\mathcal{A}_\Sigma}_{\text{NL advection}}}$$

-



Zoom over the western boundary current

Munk’ gyre

JOURNAL OF METEOROLOGY

ON THE WIND-DRIVEN OCEAN CIRCULATION

By Walter H. Munk

Institute of Geophysics and Scripps Institution of Oceanography, University of California[1]

(Manuscript received 24 September 1949)

In Ekman's and Stommel's model the ocean is assumed homogeneous, a case in which the currents extend to the very bottom. Not only is this in contrast with observations, according to which the bulk of the water transport in the main ocean currents takes place in the upper thousand meters, but it also leads to mathematical complications which rendered Ekman's analysis very difficult, and forced Stommel to resort to a rather arbitrary frictional force along the bottom.

To avoid these difficulties, we retain Sverdrup's integrated mass transport as the dependent variable. This device makes it possible to examine the more general case of a baroclinic ocean without having to specify the nature of the vertical distributions of density and current. In recognition of the evidence that currents essentially vanish at great depths, we shall depend on lateral friction for the dissipative forces. From Stommel we retain the rectangular boundaries, although we extend the basin to both sides of the equator and deal with the *observed* wind distribution rather than a simple sinusoidal distribution.

$$P = \int_{-h}^{z_0} p\, dz, \qquad \mathbf{M}_H = \int_{-h}^{z_0} \rho \mathbf{v}_H\, dz, \qquad \text{(2a, b)}$$

designate the integrated pressure and mass transport.

$$\nabla P + f\mathbf{k} \times \mathbf{M} - \boldsymbol{\tau} - A\nabla^2\mathbf{M} = 0. \qquad \mathbf{M} = \mathbf{k} \times \nabla\psi,$$

$$\boxed{(A\nabla^4 - \beta\, \partial/\partial x)\psi = -\operatorname{curl}_z \boldsymbol{\tau},}$$

For boundary conditions we choose

$$\psi_{\text{bdry}} = 0, \qquad (\partial\psi/\partial\nu)_{\text{bdry}} = 0, \qquad \text{(7a, b)}$$

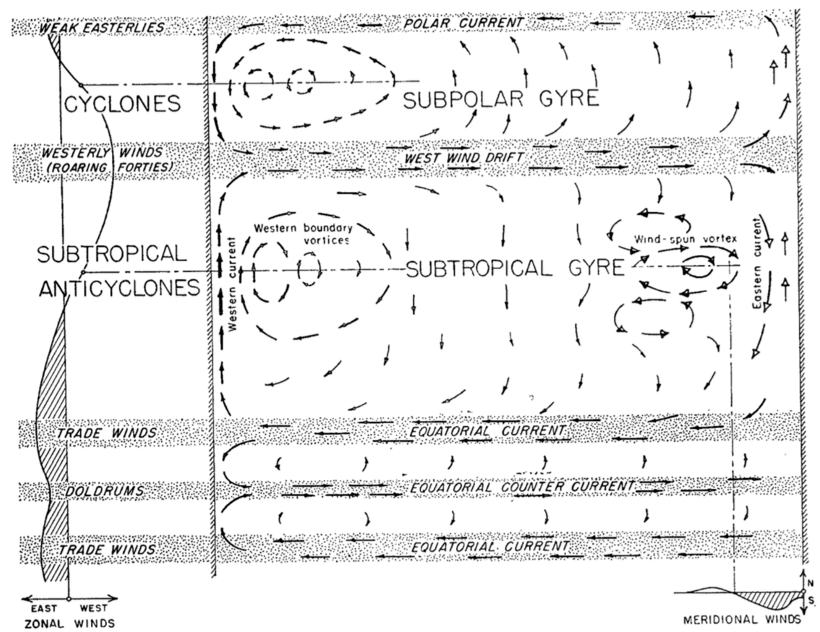


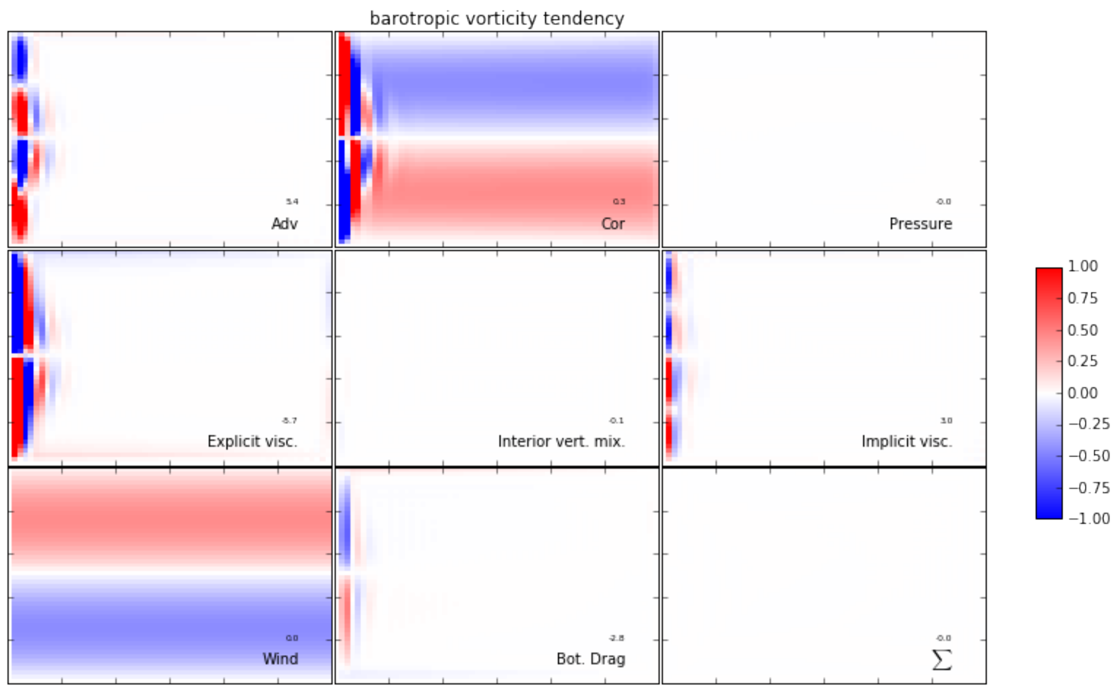
FIG. 8. Schematic presentation of circulation in a rectangular ocean resulting from zonal winds (filled arrowheads), meridional winds (open arrowheads), or both (half-filled arrowheads). The width of the arrows is an indication of the strength of the currents. The nomenclature applies to either hemisphere, but in the Southern Hemisphere the subpolar gyre is replaced largely by the Antarctic Circumpolar Current (west wind drift) flowing around the world. Geographic names of the currents in various oceans are summarized in table 3.

Gyre with beta and lateral drag

- With latitudinal variation of Coriolis

$$\cancel{\underbrace{\frac{\partial\Omega}{\partial t}}_{\text{rate}}} = \boxed{-\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \cancel{\underbrace{\frac{J(P_b,h)}{\rho_0}}_{\text{bot. pres. torque}}} \boxed{+\underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}}} - \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}} + \underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

-

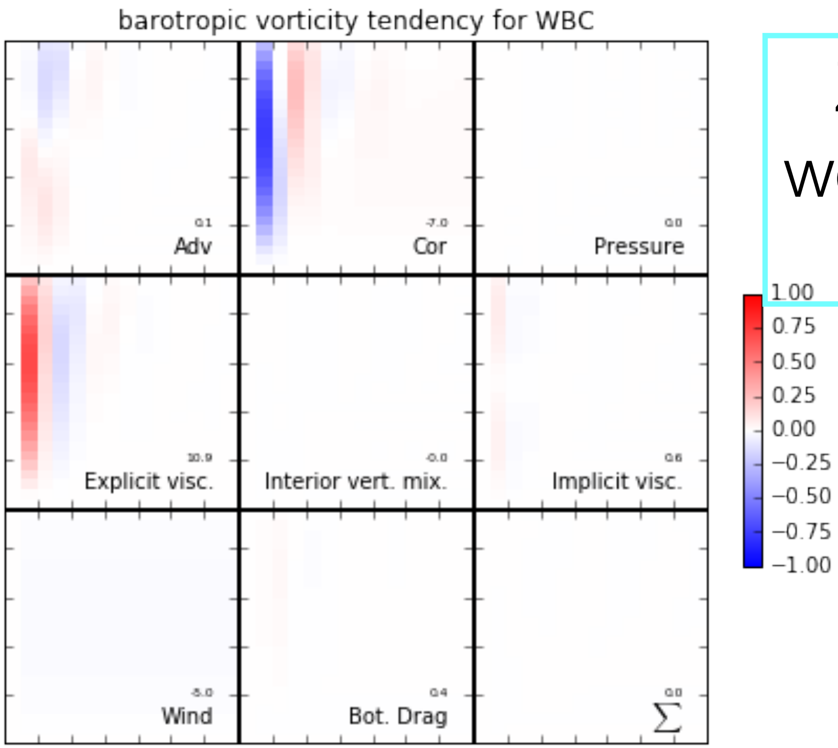


Gyre with beta and lateral drag

- With latitudinal variation of Coriolis

$$\underbrace{\frac{\partial \Omega}{\partial t}}_{\text{rate}} = \boxed{-\underbrace{\vec{\nabla}.(f\vec{u})}_{\text{planet. vort. adv.}}} + \underbrace{\cancel{\frac{J(P,h)}{\rho_0}}}_{\text{bot. pres. torque}} + \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{wind}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau}^{bot}}{\rho_0}}_{\text{bot. drag curl}} + \boxed{\underbrace{\mathcal{D}_\Sigma}_{\text{horiz.diffusion.}}} - \underbrace{A_\Sigma}_{\text{NL advection}}$$

-



Zoom over the western boundary current