1 Barotropic vorticity eqution

- Momentum equations:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \vec{u}.\vec{\nabla}\vec{u} + f\vec{k} \times \vec{u} = -\frac{\vec{\nabla}P}{\rho_0} + \vec{\mathcal{F}}} \tag{1}$$

where $\vec{\mathcal{F}}$ includes all non-conservative forces.

$$\begin{aligned}
\frac{\partial u}{\partial t} + u.\frac{\partial u}{\partial x} + v.\frac{\partial u}{\partial y} - fv &= -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \mathcal{F}_x \\
\frac{\partial v}{\partial t} + u.\frac{\partial v}{\partial x} + v.\frac{\partial v}{\partial y} + fu &= -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \mathcal{F}_y
\end{aligned}$$

- We define the vertical averaged quantities as:

$$\overline{u} = \frac{1}{H}\int_{-h}^{\zeta} u\, dz \tag{2}$$

where $H = \int_{-h}^{\zeta} dz = \zeta(i,j,t) + h(i,j)$ is the total depth of the water column, with $\zeta(x,y,t)$ the free-surface height and $h(x,y)$ the depth of the topography.

We integrate the momentum equations in the vertical:

$$\begin{aligned}
\int_{-h}^{\zeta} \frac{\partial u}{\partial t}\, dz + \int_{-h}^{\zeta} (u.\frac{\partial u}{\partial x} + v.\frac{\partial u}{\partial y})\, dz - \int_{-h}^{\zeta} fv\, dz &= -\frac{1}{\rho_0}\int_{-h}^{\zeta} \frac{\partial P}{\partial x}\, dz + \int_{-h}^{\zeta} F_x\, dz \\
\underbrace{\int_{-h}^{\zeta} \frac{\partial v}{\partial t}\, dz}_{1} + \underbrace{\int_{-h}^{\zeta} (u.\frac{\partial v}{\partial x} + v.\frac{\partial v}{\partial y})\, dz}_{2} + \underbrace{\int_{-h}^{\zeta} fu\, dz}_{3} &= \underbrace{-\frac{1}{\rho_0}\int_{-h}^{\zeta} \frac{\partial P}{\partial y}\, dz}_{4} + \underbrace{\int_{-h}^{\zeta} F_y\, dz}_{5}
\end{aligned}$$

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(1) - We can write the rate of change:

$$\frac{\partial}{\partial t}\left[\int_{-h(x,y)}^{\zeta(x,y,t)} u(x,y,z,t)\; dz\right] \;=\; \int_{-h}^{\zeta}\frac{\partial u}{\partial t}\; dz + u(x,y,\zeta,t)\frac{\partial \zeta}{\partial t}$$

which gives

$$\begin{aligned}\int_{-h}^{\zeta}\frac{\partial u}{\partial t}\; dz &= \frac{\partial H\overline{u}}{\partial t} - u(\zeta).\frac{\partial \zeta}{\partial t}\\ \int_{-h}^{\zeta}\frac{\partial v}{\partial t}\; dz &= \frac{\partial H\overline{v}}{\partial t} - v(\zeta).\frac{\partial \zeta}{\partial t}\end{aligned}$$

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(2) - The advection terms are:

$$\begin{aligned}\int_{-h}^{\zeta}\left(u.\frac{\partial u}{\partial x} + v.\frac{\partial u}{\partial y} + w.\frac{\partial u}{\partial z}\right) dz &= \int_{-h}^{\zeta}\left(\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z}\right) dz = \overline{\mathcal{A}_x}\\ \int_{-h}^{\zeta}\left(u.\frac{\partial v}{\partial x} + v.\frac{\partial v}{\partial y} + w.\frac{\partial v}{\partial z}\right) dz &= \int_{-h}^{\zeta}\left(\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z}\right) dz = \overline{\mathcal{A}_y}\end{aligned}$$

We can develop them as:

$$\begin{aligned}\int_{-h}^{\zeta}\frac{\partial uu}{\partial x}\; dz &= \frac{\partial H\overline{uu}}{\partial x} - u^2(\zeta).\frac{\partial \zeta}{\partial x} - u^2(-h).\frac{\partial h}{\partial x}\\ \int_{-h}^{\zeta}\frac{\partial uv}{\partial y}\; dz &= \frac{\partial H\overline{uv}}{\partial y} - u(\zeta)v(\zeta)\frac{\partial \zeta}{\partial y} - u(-h)v(-h)\frac{\partial h}{\partial y}\\ \int_{-h}^{\zeta}\frac{\partial uw}{\partial z}\; dz &= u|_{\zeta}\left(\frac{\partial \zeta}{\partial t} + u|_{\zeta}\frac{\partial \zeta}{\partial x} + v|_{\zeta}\frac{\partial \zeta}{\partial y}\right) - u|_{-h}\left(u|_{-h}\frac{\partial -h}{\partial x} + v|_{-h}\frac{\partial -h}{\partial y}\right)\\ \int_{-h}^{\zeta}\frac{\partial uv}{\partial x}\; dz &= \frac{\partial H\overline{uv}}{\partial x} - u(\zeta)v(\zeta)\frac{\partial \zeta}{\partial x} - u(-h)v(-h)\frac{\partial h}{\partial x}\\ \int_{-h}^{\zeta}\frac{\partial vv}{\partial y}\; dz &= \frac{\partial H\overline{vv}}{\partial y} - v^2(\zeta).\frac{\partial \zeta}{\partial y} - v^2(-h).\frac{\partial h}{\partial y}\\ \int_{-h}^{\zeta}\frac{\partial vw}{\partial z}\; dz &= v|_{\zeta}\left(\frac{\partial \zeta}{\partial t} + u|_{\zeta}\frac{\partial \zeta}{\partial x} + v|_{\zeta}\frac{\partial \zeta}{\partial y}\right) - v|_{-h}\left(u|_{-h}\frac{\partial -h}{\partial x} + v|_{-h}\frac{\partial -h}{\partial y}\right)\end{aligned}$$

using boundary conditions:

$$
\begin{aligned}
\omega|_\zeta &= \frac{\partial\zeta}{\partial t} + u|_\zeta \frac{\partial\zeta}{\partial x} + v|_\zeta \frac{\partial\zeta}{\partial y} \\
\omega|_{-h} &= u|_{-h} \frac{\partial -h}{\partial x} + v|_{-h} \frac{\partial -h}{\partial y}
\end{aligned}
$$

and write:

$$
\begin{aligned}
\overline{\mathcal{A}_x} &= \frac{\partial H\overline{uu}}{\partial x} + \frac{\partial H\overline{uv}}{\partial y} + u|_\zeta \cdot \frac{\partial\zeta}{\partial t} \\
\overline{\mathcal{A}_y} &= \frac{\partial H\overline{vv}}{\partial y} + \frac{\partial H\overline{uv}}{\partial x} + v|_\zeta \cdot \frac{\partial\zeta}{\partial t}
\end{aligned}
$$

Note that the $u|_\zeta \cdot \frac{\partial\zeta}{\partial t}$ and $v|_\zeta \cdot \frac{\partial\zeta}{\partial t}$ terms will disappear when added to the term (1).

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(3) - Coriolis terms are:

$$
\begin{aligned}
-\int_{-h}^{\zeta} fv\, dz &= -fH\overline{v} \\
\int_{-h}^{\zeta} fu\, dz &= fH\overline{u}
\end{aligned}
$$

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(4) Pressure terms are-

$$
\begin{aligned}
-\frac{1}{\rho_0}\int_{-h}^{\zeta} \frac{\partial P}{\partial x}\, dz &= -\frac{1}{\rho_0}\frac{\partial H\overline{P}}{\partial x} + \frac{1}{\rho_0}P(\zeta).\frac{\partial\zeta}{\partial x} + \frac{1}{\rho_0}P(-h)\frac{\partial h}{\partial x} \\
-\frac{1}{\rho_0}\int_{-h}^{\zeta} \frac{\partial P}{\partial y}\, dz &= -\frac{1}{\rho_0}\frac{\partial H\overline{P}}{\partial y} + \frac{1}{\rho_0}P(\zeta).\frac{\partial\zeta}{\partial y} + \frac{1}{\rho_0}P(-h)\frac{\partial h}{\partial y}
\end{aligned}
$$

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(5) and finally-

$$\begin{aligned}
\int_{-h}^{\zeta} F_x \, dz &= \int_{-h}^{\zeta} \left(\frac{\partial}{\partial z} \left(K_{Mv} \frac{\partial u}{\partial z} \right) + \mathcal{D}_x \right) \, dz \\
\int_{-h}^{\zeta} F_y \, dz &= \int_{-h}^{\zeta} \left(\frac{\partial}{\partial z} \left(K_{Mv} \frac{\partial v}{\partial z} \right) + \mathcal{D}_y \right) \, dz
\end{aligned}$$

where

$$\begin{aligned}
\int_{-h}^{\zeta} \frac{\partial}{\partial z} \left(K_{Mv} \frac{\partial u}{\partial z} \right) \, dz &= K_{Mv} \frac{\partial u}{\partial z}\bigg|_{\zeta} - K_{Mv} \frac{\partial u}{\partial z}\bigg|_{-h} = \frac{1}{\rho_0} \left(\tau_x^{wind} - \tau_x^{bot} \right) \\
\int_{-h}^{\zeta} \frac{\partial}{\partial z} \left(K_{Mv} \frac{\partial v}{\partial z} \right) \, dz &= K_{Mv} \frac{\partial v}{\partial z}\bigg|_{\zeta} - K_{Mv} \frac{\partial v}{\partial z}\bigg|_{-h} = \frac{1}{\rho_0} \left(\tau_y^{wind} - \tau_y^{bot} \right)
\end{aligned}$$

and

$$\begin{aligned}
\int_{-h}^{\zeta} \mathcal{D}_x \, dz &= \overline{\mathcal{D}_x} \\
\int_{-h}^{\zeta} \mathcal{D}_y \, dz &= \overline{\mathcal{D}_y}
\end{aligned}$$

So we get finally the depth-averaged momentum equations.

$$\begin{aligned}
\frac{\partial H\overline{u}}{\partial t} &+ \frac{\partial H\overline{uu}}{\partial x} + \frac{\partial H\overline{uv}}{\partial y} - fH\overline{v} = \\
&- \frac{1}{\rho_0} \frac{\partial H\overline{P}}{\partial x} + \frac{1}{\rho_0} P(\zeta) . \frac{\partial \zeta}{\partial x} + \frac{1}{\rho_0} P(-h) \frac{\partial h}{\partial x} + \tau_x^{wind} - \tau_x^{bot} + \overline{\mathcal{D}_x} \\
\underbrace{\frac{\partial H\overline{v}}{\partial t}}_{1} &+ \underbrace{\frac{\partial H\overline{vv}}{\partial y} + \frac{\partial H\overline{uv}}{\partial x}}_{2} + \underbrace{fH\overline{u}}_{4} = \\
&- \underbrace{\frac{1}{\rho_0} \frac{\partial H\overline{P}}{\partial y} + \frac{1}{\rho_0} P(\zeta) . \frac{\partial \zeta}{\partial y} + \frac{1}{\rho_0} P(-h) \frac{\partial h}{\partial y}}_{5} + \underbrace{\tau_y^{wind} - \tau_y^{bot} + \overline{\mathcal{D}_y}}_{6}
\end{aligned}$$

Now we cross differentiates them to get the barotropic vorticity equations.

(1) :

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{\partial H\overline{u}}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\overline{u}\frac{\partial H}{\partial y} + H\frac{\partial \overline{u}}{\partial y}\right) \\
\frac{\partial}{\partial x}\left(\frac{\partial H\overline{v}}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\overline{v}\frac{\partial H}{\partial x} + H\frac{\partial \overline{v}}{\partial x}\right)
\end{aligned}$$

so

$$\begin{aligned}
\vec{k}.\vec{\nabla}\times(\vec{1}) &= \frac{\partial}{\partial t}\left[H\left(\frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y}\right) + \overline{v}\frac{\partial H}{\partial x} - \overline{u}\frac{\partial H}{\partial y}\right] \\
&= \frac{\partial H\omega_r}{\partial t} \\
&+ \frac{\partial}{\partial t}\left[\overline{v}\frac{\partial \zeta}{\partial x} - \overline{u}\frac{\partial \zeta}{\partial y}\right] - \frac{\partial h}{\partial x}\frac{\partial \bar{v}}{\partial t} + \frac{\partial h}{\partial y}\frac{\partial \bar{u}}{\partial t}
\end{aligned}$$

where $\omega_r = \frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y}$ is the barotropic vorticity.

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(2) - Advective non-linear terms are:

$$\begin{aligned}
\vec{k}.\vec{\nabla}\times(\vec{2}) &= \frac{\partial}{\partial x}\left(\frac{\partial H\overline{vv}}{\partial y} + \frac{\partial H\overline{uv}}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H\overline{uu}}{\partial x} + \frac{\partial H\overline{uv}}{\partial y}\right) \\
&= \frac{\partial^2 H(\overline{vv} - \overline{uu})}{\partial xy} + \frac{\partial^2 H\overline{uv}}{\partial xx} - \frac{\partial^2 H\overline{uv}}{\partial yy}
\end{aligned}$$

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(4) - The coriolis terms simplify as:

$$\begin{aligned}
\vec{k}.\vec{\nabla}\times(\vec{4}) &= \frac{\partial f}{\partial x}H\bar{u} + \frac{\partial f}{\partial y}H\bar{v} + f\left(\frac{\partial H\bar{u}}{\partial x} + \frac{\partial H\bar{v}}{\partial y}\right) \\
&= H\vec{\bar{u}}.\vec{\nabla}f + f\left(\frac{\partial H\bar{u}}{\partial x} + \frac{\partial H\bar{v}}{\partial y}\right)
\end{aligned}$$

with the use of the integral of the continuity equation:

$$\int_{-h}^{\zeta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\,dz=0$$

which gives

$$\frac{\partial H\bar{u}}{\partial x}+\frac{\partial H\bar{v}}{\partial y}=-\frac{\partial \zeta}{\partial t}$$

we get:

$$\vec{k}.\vec{\nabla}\times(\vec{4}) \quad = \quad H\vec{\bar{u}}.\vec{\nabla}f-f\frac{\partial \zeta}{\partial t}$$

Also note that we don't write $\frac{\partial f}{\partial x}=0$ because we don't necessarily use zonal-meridional coordinates.

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(5) - The pressure terms simplify as:

$$\begin{aligned}
\vec{k}.\vec{\nabla}\times(\vec{5}) &= \frac{\partial}{\partial x}\left(-\frac{1}{\rho_0}\frac{\partial H\overline{P}}{\partial y}+\frac{1}{\rho_0}P(\zeta).\frac{\partial \zeta}{\partial y}+\frac{1}{\rho_0}P(-h)\frac{\partial h}{\partial y}\right)\\
&- \frac{\partial}{\partial y}\left(-\frac{1}{\rho_0}\frac{\partial H\overline{P}}{\partial x}+\frac{1}{\rho_0}P(\zeta).\frac{\partial \zeta}{\partial x}+\frac{1}{\rho_0}P(-h)\frac{\partial h}{\partial x}\right)\\
&= \frac{1}{\rho_0}\left(\left.\frac{\partial P}{\partial x}\right|_{-h}\frac{\partial h}{\partial y}-\left.\frac{\partial P}{\partial y}\right|_{-h}\frac{\partial h}{\partial x}\right)\\
&= \frac{1}{\rho_0}J(P_b,h)
\end{aligned}$$

considering that with $\frac{\partial P(\zeta)}{\partial x}=\frac{\partial P(\zeta)}{\partial y}=0$ at the surface, and writing $P_b=P(-h)$ the pressure at the bottom.

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(6) - Forcing and dissipative terms can be written as:

$$\vec{k}.\vec{\nabla}\times(\vec{6}) \quad = \quad \frac{1}{\rho_0}\left(\frac{\partial \tau_y^{wind}}{\partial x}-\frac{\partial \tau_x^{wind}}{\partial y}\right)-\frac{1}{\rho_0}\left(\frac{\partial \tau_y^{bot}}{\partial x}-\frac{\partial \tau_x^{bot}}{\partial y}\right)+\frac{\partial \overline{\mathcal{D}_y}}{\partial x}-\frac{\partial \overline{\mathcal{D}_x}}{\partial y}$$

So we get the equation:

$$\boxed{\begin{aligned} \underbrace{H\frac{\partial \omega_r}{\partial t}}_{\text{rate}} + \underbrace{H\vec{u}.\vec{\nabla} f}_{\text{planet. vort.}} &= \underbrace{\frac{J(P_b,h)}{\rho_0}}_{\text{bot. pres. torque}} + \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau^{wind}}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla} \times \frac{\vec{\tau^{bot}}}{\rho_0}}_{\text{bot. drag curl}} \\ &+ \underbrace{F}_{\text{horiz.dissip.}} - \underbrace{A}_{\text{NL adv terms}} - \underbrace{L}_{\text{left overs}} \end{aligned}}$$

where the curl of NL advection terms is:

$$A = \frac{\partial^2 H(\overline{vv} - \overline{uu})}{\partial xy} + \frac{\partial^2 H\overline{uv}}{\partial xx} - \frac{\partial^2 H\overline{uv}}{\partial yy}$$

Horiz. dissip. is included in:

$$F = \frac{\partial \overline{\mathcal{D}_y}}{\partial x} - \frac{\partial \overline{\mathcal{D}_x}}{\partial y}$$

Note that we are using implicit diffusion so the $\mathcal{D}$ terms are numerically "included" in the advection terms.

And finally left overs are:

$$L = \frac{\partial}{\partial t}\left[\overline{v}\frac{\partial H}{\partial x} - \overline{u}\frac{\partial H}{\partial y}\right] + \omega_r\frac{\partial \zeta}{\partial t} - f\frac{\partial \zeta}{\partial t}$$

We can alternatively use the depth-integrated barotropic vorticity (instead of depth averaged):

$$\omega_\Sigma = \frac{\partial H\overline{v}}{\partial x} - \frac{\partial H\overline{u}}{\partial y} = H\omega_r + \left[\overline{v}\frac{\partial H}{\partial x} - \overline{u}\frac{\partial H}{\partial y}\right]$$

where the only difference in the equation will be the term L replaced by:

$$L_\Sigma = -f\frac{\partial \zeta}{\partial t} = f\left[\frac{\partial H\bar{u}}{\partial x} + \frac{\partial H\bar{v}}{\partial y}\right] = f\vec{\nabla}.(H\vec{u})$$

which can be interpreted as planetary vortex stretching of the total layer thickness.

So finally we can write:

$$\boxed{\begin{aligned}\underbrace{\frac{\partial \omega_\Sigma}{\partial t}}_{\text{rate}} = -\underbrace{H\vec{u}.\vec{\nabla}f}_{\text{planet. vort.}} - \underbrace{f\vec{\nabla}.(H\vec{u})}_{\text{vort. stretch}} + \underbrace{\frac{J(P_b,h)}{\rho_0}}_{\text{bot. pres. torque}} + \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau^{wind}}}{\rho_0}}_{\text{wind curl}} - \underbrace{\vec{k}.\vec{\nabla}\times\frac{\vec{\tau^{bot}}}{\rho_0}}_{\text{bot. drag curl}} \\ + \underbrace{F}_{\text{horiz.dissip.}} - \underbrace{A}_{\text{NL adv terms}}\end{aligned} \qquad (3)}$$