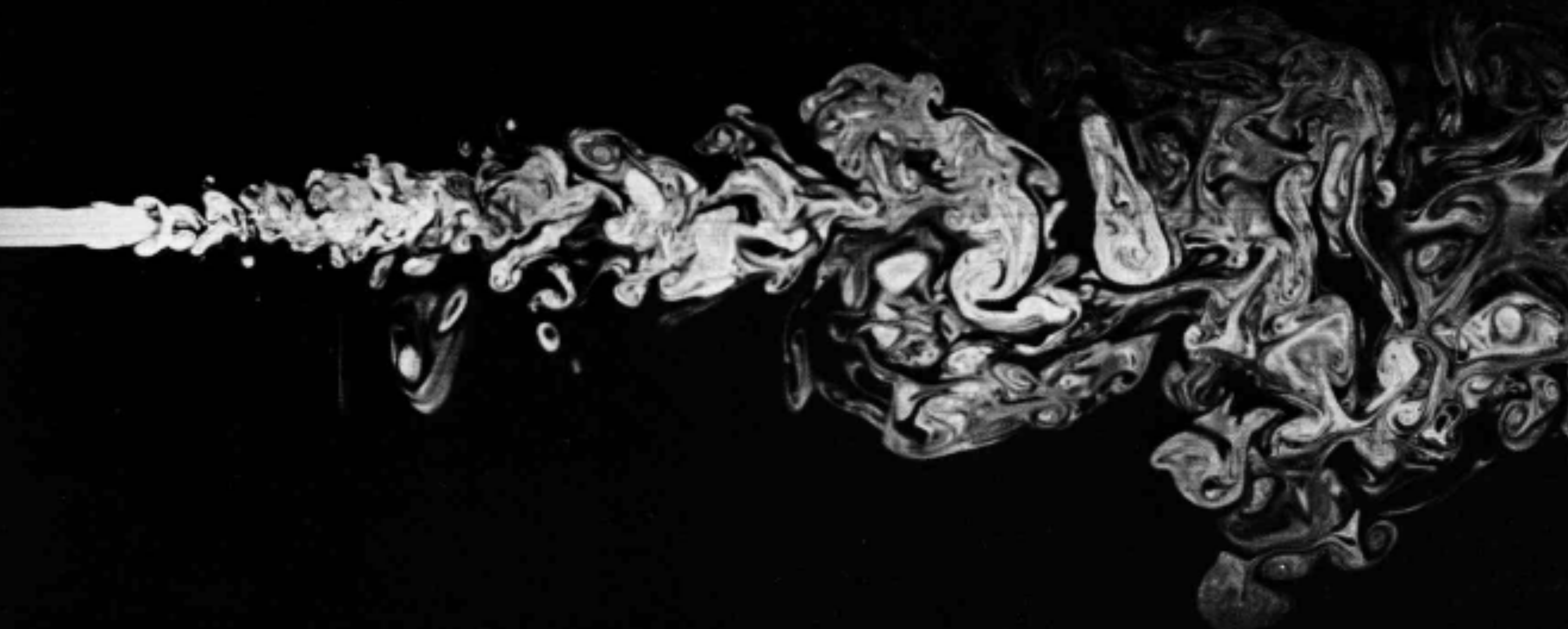
TURBULENCE

Jonathan GULA
gula@univ-brest.fr



- **Lesson 1 :** [D109]
  - Introduction
  - Properties of turbulence
- **Lesson 2 :** [D109]
  - 3D turbulence: The Kolmogorov theory
  - 2D turbulence
- **Lesson 3 :**[D109]
  - 2D turbulence (activity)
  - Geostrophic turbulence
  - Surface QG turbulence
- **Lesson 4 :**[D109]
  - Ocean turbulence (activity)
  - Turbulent diffusion
- **Lesson 5 :**[D109]
  - Stratified turbulence

Presentations and material will be available at :

jgula.fr/Turb/

Numerical Modelling

Jonathan GULA
gula@univ-brest.fr

Evaluation

- The evaluation will be based on a report, which consists in a numerical activity and summary of research articles

- Written Report will be **due Mar. 11**

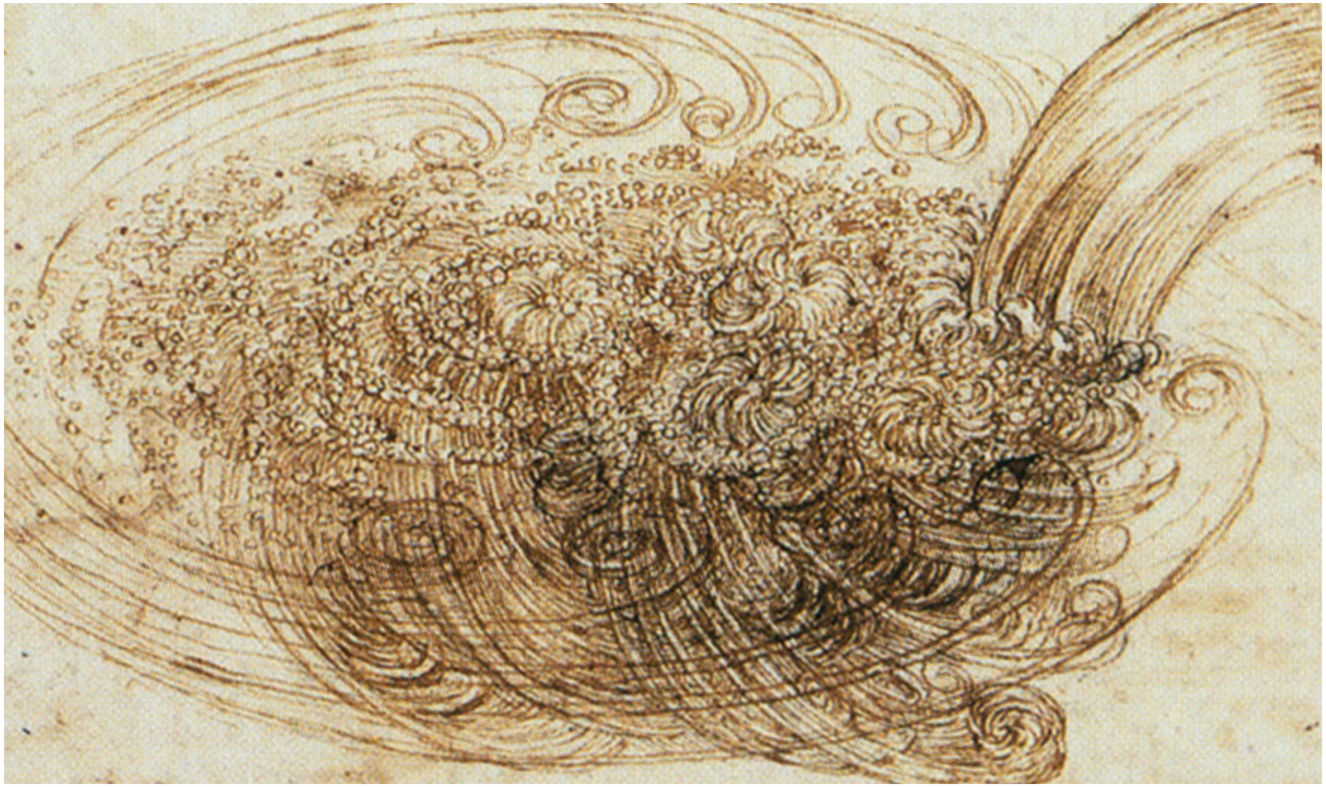
TURBULENCE

INTRODUCTION

References:

- Vallis G.K., Atmospheric and Oceanic Fluid Dynamics.
- MIT online course: *https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-820-turbulence-in-the-ocean-and-atmosphere-spring-2007/lecture-notes/*
- LaCasce J.H., Turbulence in the Atmosphere and Ocean.

What is turbulence?



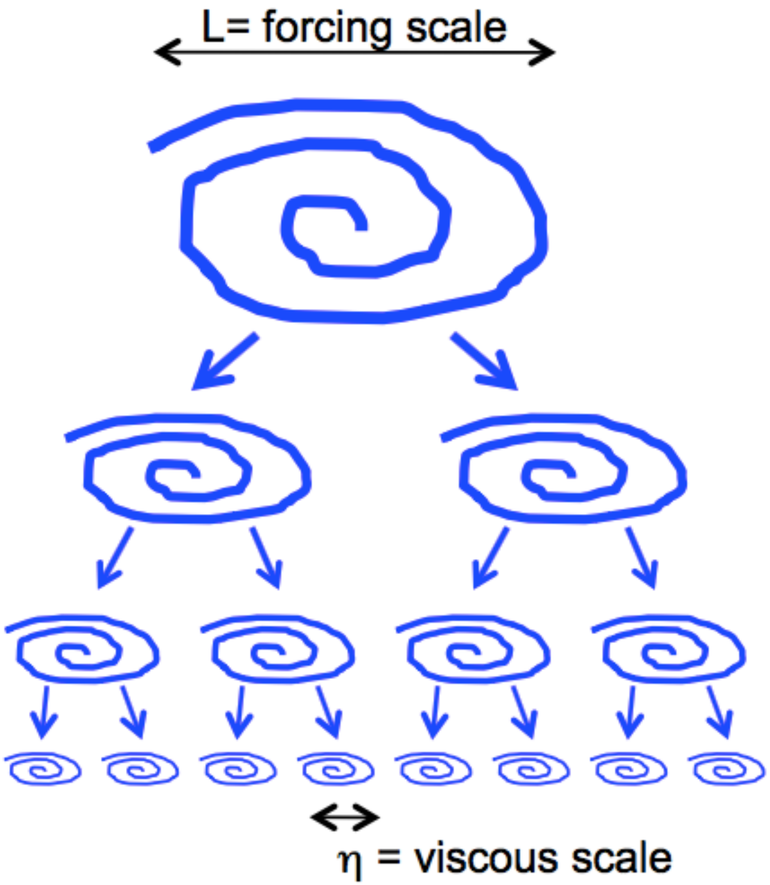
turbolenza by da Vinci [1507]

"…the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large."

What is turbulence?

No formal definition of turbulence. One of the best is by L.F. Richardson, in 1922:

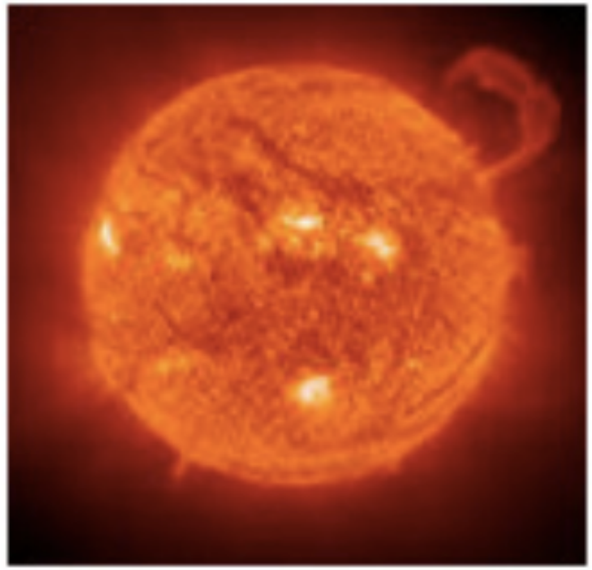
Big whorls have little whorls,
which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity.

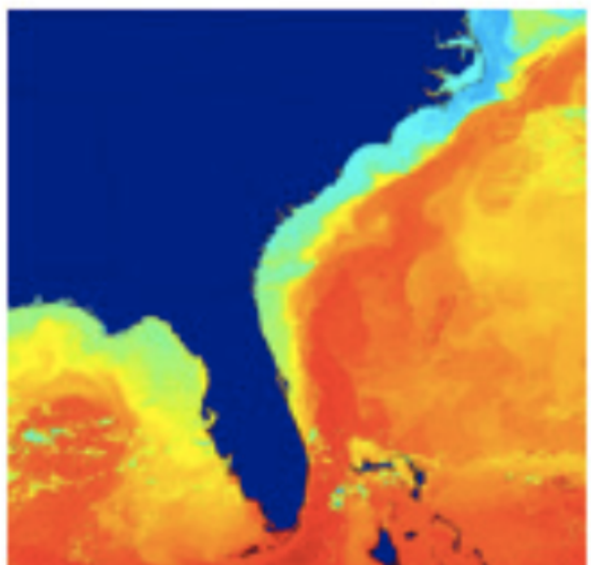


Turbulence in nature

(a) water flow from a faucet, (b) water from a garden hose, (c) flow past a curved wall, and (d) and (e) whitewater rapids whose turbulent fluctuations are so intense that air is entrained by the flow and produces small bubbles that diffusely reflect light and cause the water to appear white.

Turbulence in nature





Examples of turbulent flows at the surface of the Sun, in the Earth's atmosphere, in the Gulf Stream at the ocean surface, and in a volcanic eruption.

Turbulence in nature

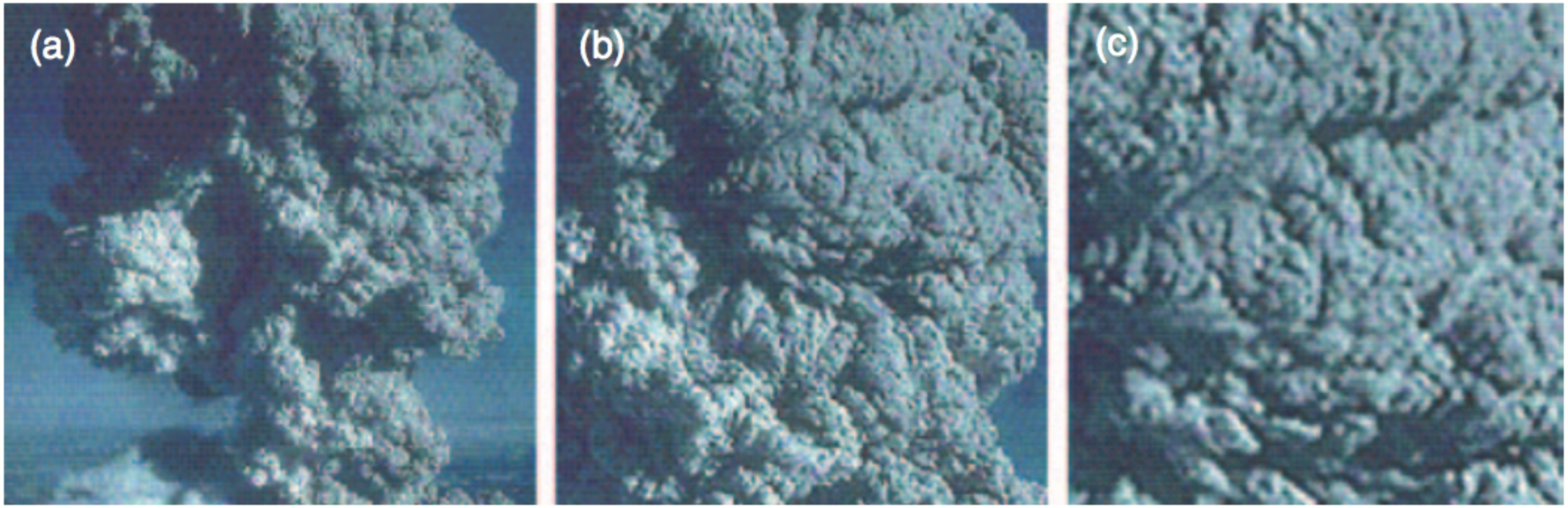
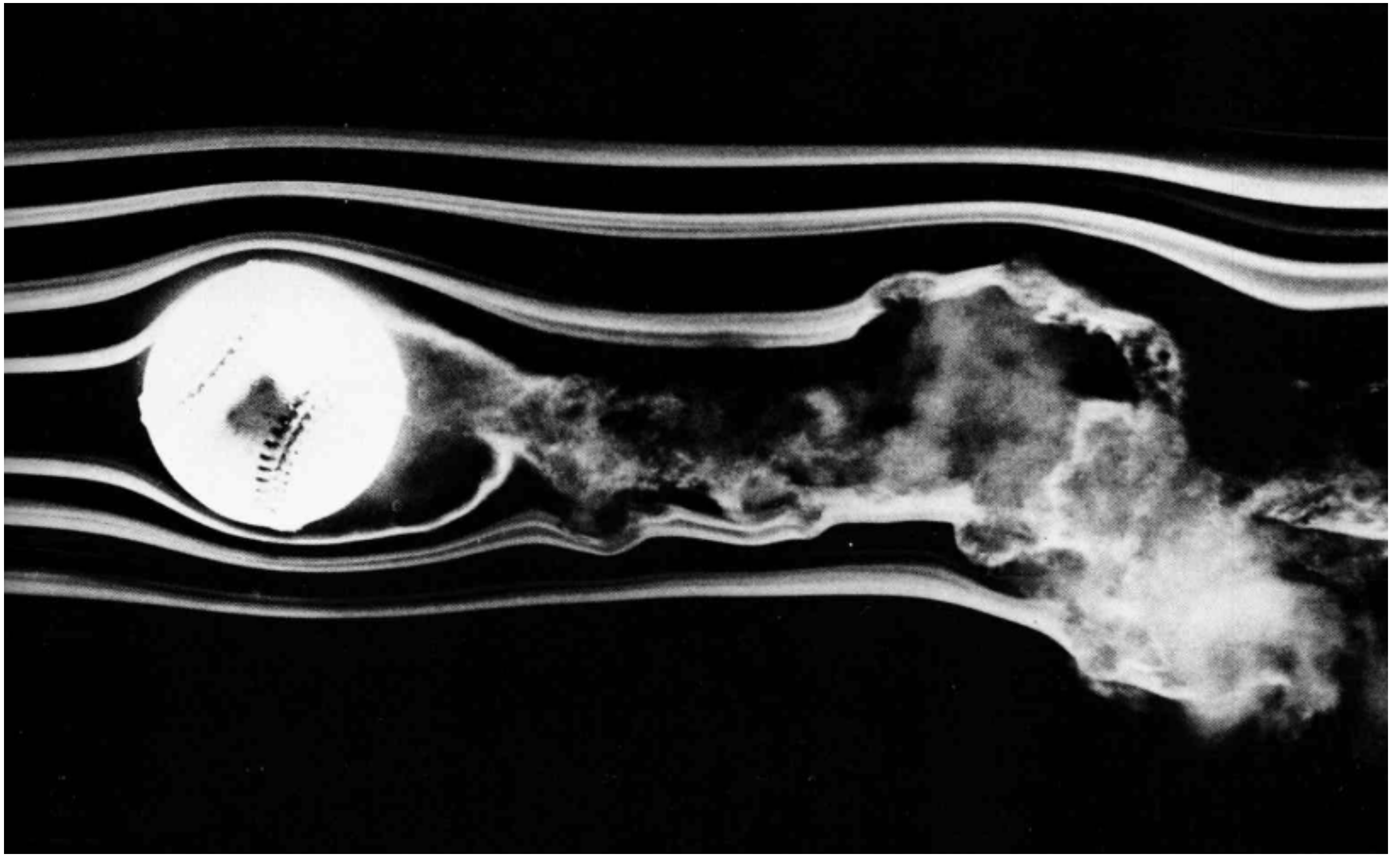


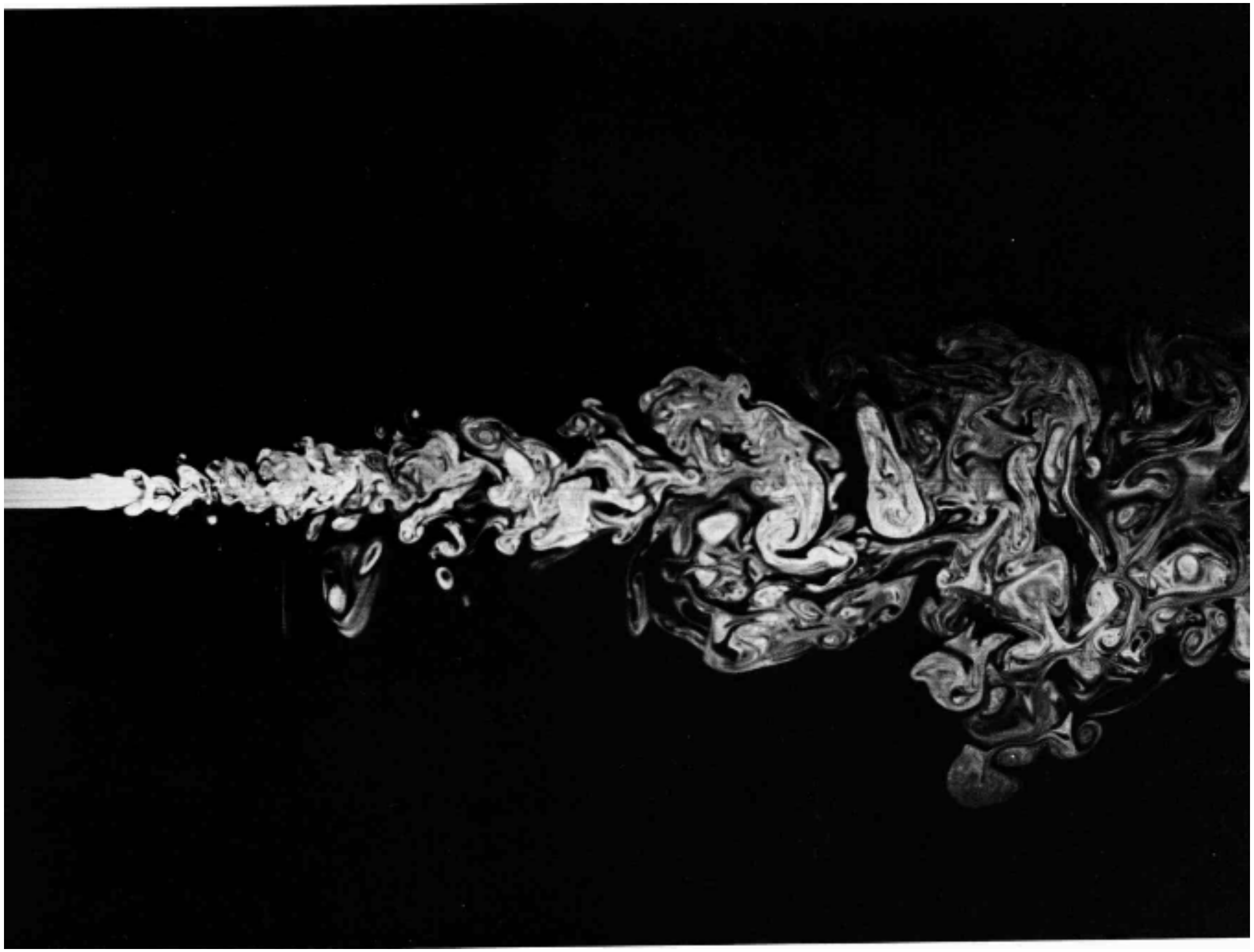
Figure 2. Scale-Independence in Turbulent Flows
The turbulent structure of the pyroclastic volcanic eruption of Mt. St. Helens shown in (a) is expanded by a factor of 2 in (b) and by another factor of 2 in (c). The characteristic scale of the plume is approximately 5 km. Note that the expanded images reveal the increasingly finer scale structure of the turbulent flow. The feature of scale independence, namely, that spatial images or temporal signals look the same (statistically) under increasing magnification is called self-similarity.

Turbulence in the lab



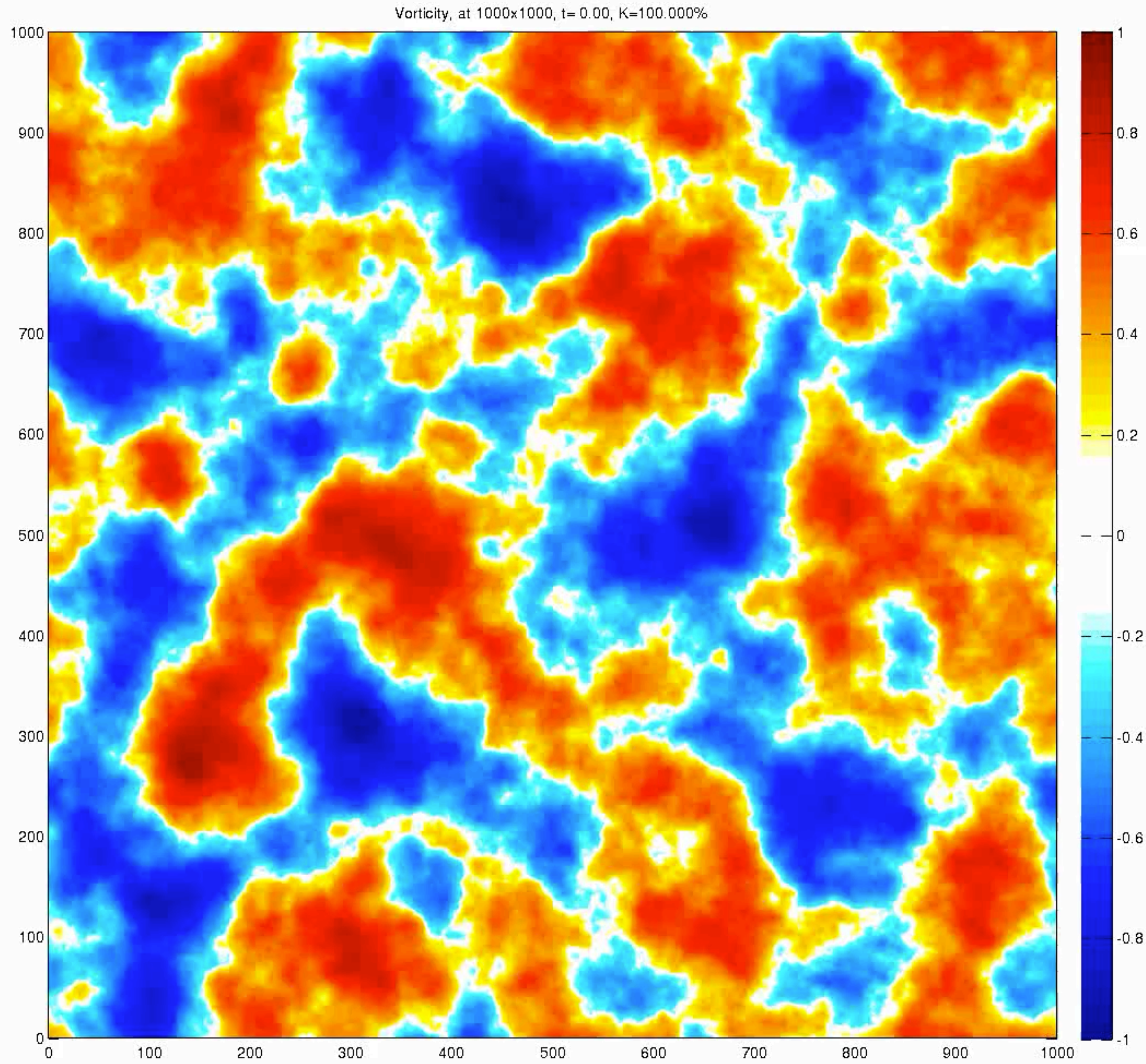
Spinning Baseball [Van Dyke, 82]

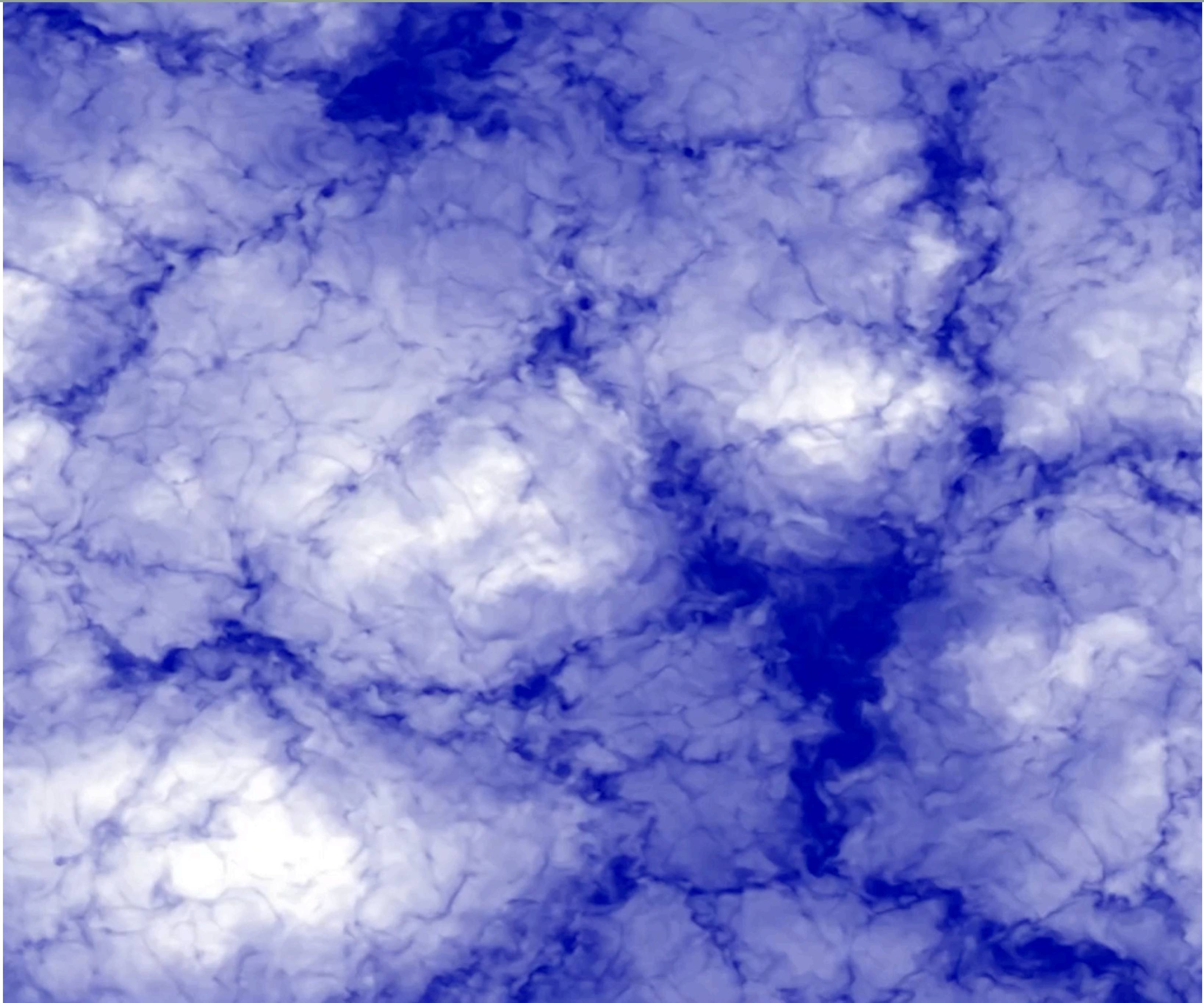
Turbulence in the lab



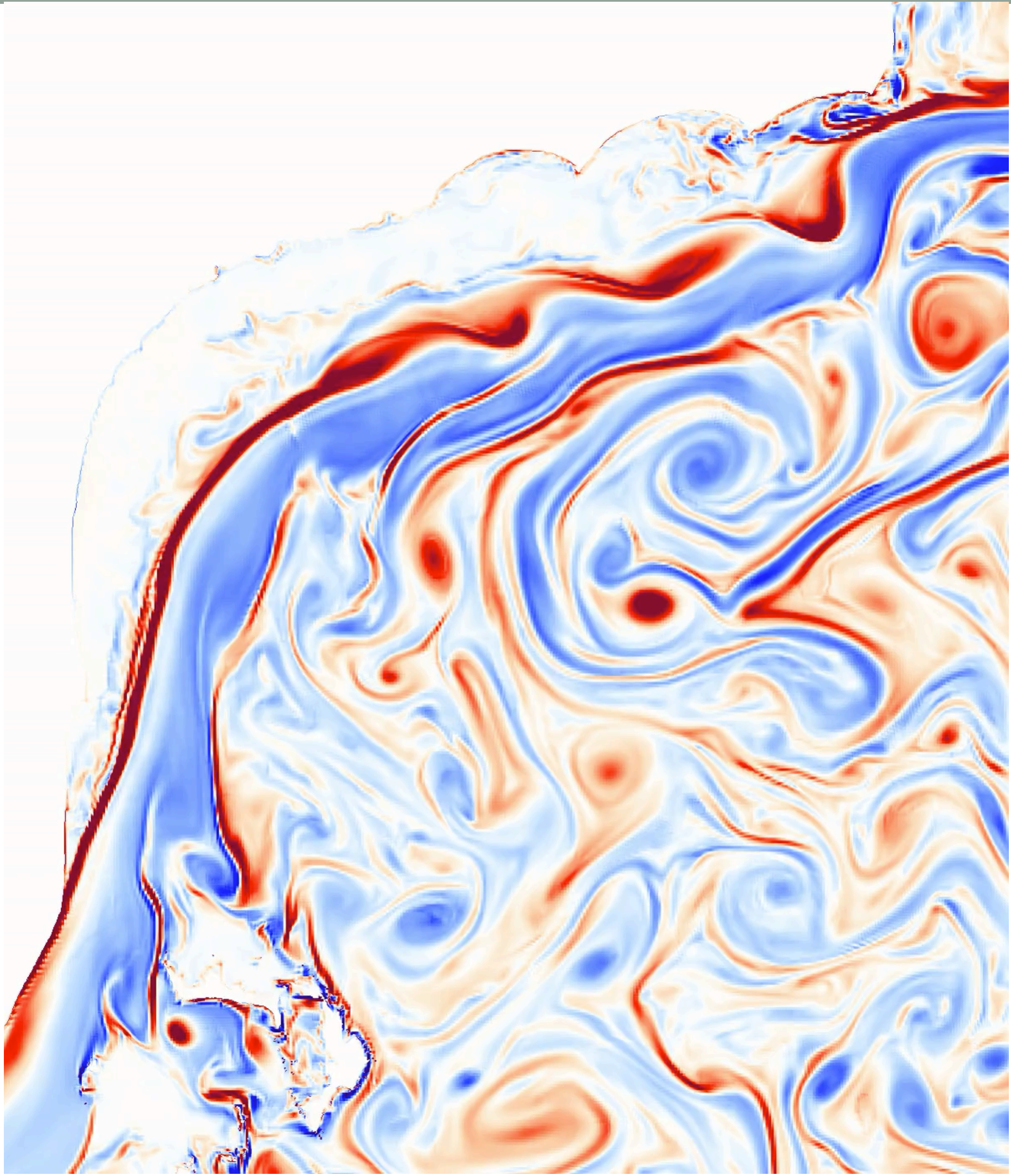
Turbulent water jet (Re = 2300) [Van Dyke, 82]

2d free-decay turbulence [G. Roullet]





Turbulence in a realistic ocean model

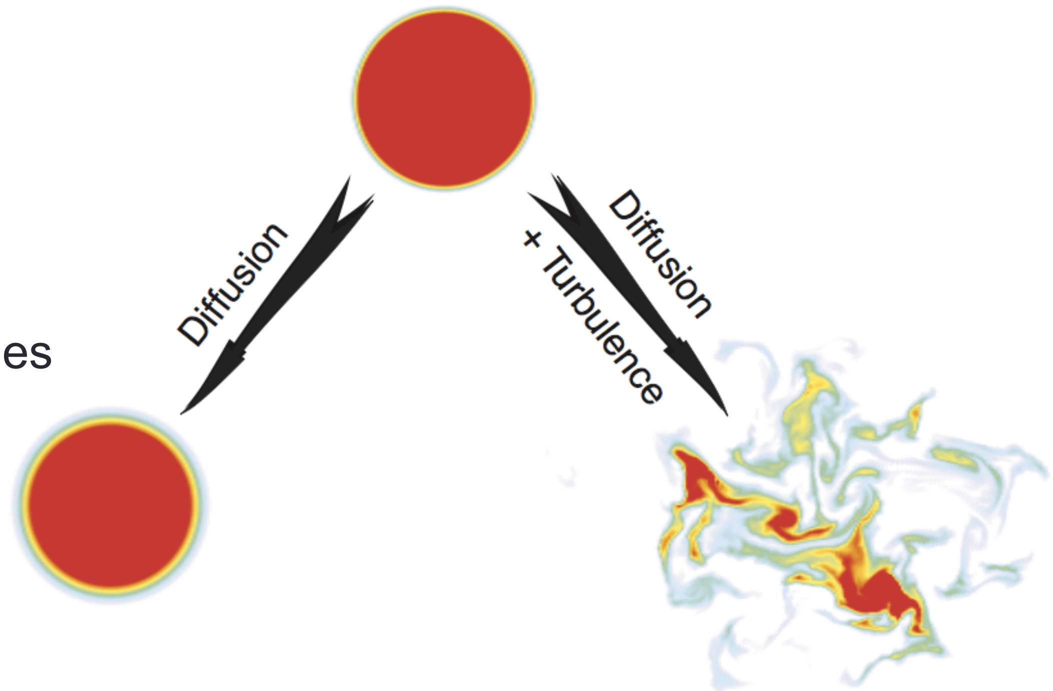


Role of turbulence

- Momentum transfer

- Scalar mixing

= homogenization of fluid properties by random molecular motions



*A comparison of **mixing enhanced by turbulence** with **mixing due to molecular processes alone**, as revealed by a numerical solution of the equations of motion.*

The initial state includes a circular region of dyed fluid in a white background. Two possible evolutions are shown: one in which the fluid is motionless (save for random molecular motions), and one in which the fluid is in a state of fully developed, two-dimensional turbulence. The mixed region (yellow/green) expands much more rapidly in the turbulent case.

- Introduction
  - *What is turbulence?*
- Properties of turbulence
  - *Where does it come from?*
  - *What does it do?*
-

Properties of turbulence

- **Broadband spectrum in space and time**
  - *Turbulent flows are characterized by structures on a broad range of spatial and temporal scales, even given smooth or periodic initial conditions and forcing.*
-

Properties of turbulence

- **Broadband spectrum in space and time**
  - *Turbulent flows are characterized by structures on a broad range of spatial and temporal scales, even given smooth or periodic initial conditions and forcing.*
- **Dominated by nonlinearity**
  - *A field of non-interacting linear internal waves with many different frequencies and wavenumbers can also have a large range of length scales, but it is not turbulent. Why not? In a turbulent flow the different scales interact, through the nonlinear terms in the equations of motion. And these nonlinear interactions are responsible for the presence of structure on many different scales.*
-

Properties of turbulence

- **Broadband spectrum in space and time**
  - *Turbulent flows are characterized by structures on a broad range of spatial and temporal scales, even given smooth or periodic initial conditions and forcing.*
- **Dominated by nonlinearity**
  - *A field of non-interacting linear internal waves with many different frequencies and wavenumbers can also have a large range of length scales, but it is not turbulent. Why not? In a turbulent flow the different scales interact, through the nonlinear terms in the equations of motion. And these nonlinear interactions are responsible for the presence of structure on many different scales.*
- **Unpredictable in space and time**
  - *Turbulent flows are predictable for only short times and short distances. Even though we know the equations that govern the evolution of the fluid, we cannot make predictions about the details of the flow due to its sensitive dependence on initial and boundary conditions. Predictability, however, can be recovered in a statistical sense. The sensitive dependence on initial and boundary conditions is a fundamental property of chaotic systems.*

Properties of turbulence

- **Broadband spectrum in space and time**
  - *Turbulent flows are characterized by structures on a broad range of spatial and temporal scales, even given smooth or periodic initial conditions and forcing.*
- **Dominated by nonlinearity**
  - *A field of non-interacting linear internal waves with many different frequencies and wavenumbers can also have a large range of length scales, but it is not turbulent. Why not? In a turbulent flow the different scales interact, through the nonlinear terms in the equations of motion. And these nonlinear interactions are responsible for the presence of structure on many different scales.*
- **Unpredictable in space and time**
  - *Turbulent flows are predictable for only short times and short distances. Even though we know the equations that govern the evolution of the fluid, we cannot make predictions about the details of the flow due to its sensitive dependence on initial and boundary conditions. Predictability, however, can be recovered in a statistical sense. The sensitive dependence on initial and boundary conditions is a fundamental property of chaotic systems.*
- **Time irreversible**
  - *Turbulent motions are not time reversible. As time goes on, turbulent motions tend to forget their initial conditions and reach some equilibrated state. Turbulence mixes stuff up, it does not unmix it.*

Navier-Stokes Equations

The Navier-Stokes equations probably contain all of turbulence.

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Momentum equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation
(no source/sink)

Kinematic viscosity:

$$\nu_{water} \approx 10^{-6} \mathrm{m}^2 \mathrm{s}^{-1}$$

$$\nu_{air} \approx 1.5 \times 10^{-5} \mathrm{m}^2 \mathrm{s}^{-1}$$

Navier-Stokes Equations

Millennium Prize problems:

Since understanding the Navier–Stokes equations is considered to be the first step to understanding the elusive phenomenon of turbulence, the Clay Mathematics Institute in May 2000 made this problem one of its seven Millennium Prize problems in mathematics. It offered a US $1,000,000 prize to the first person providing a solution for a specific statement of the problem:[1]

Prove or give a counter-example of the following statement:

In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

See jgula.fr/Articles/Robert.pdf

Navier-Stokes Equations

+ Boussinesq approximation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

Navier-Stokes Equations

+ Boussinesq approximation

$$\boxed{\frac{\partial \vec{u}}{\partial t}} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + \boxed{f \vec{k} \times \vec{u}} + \boxed{g \vec{k}} = \boxed{-\frac{\vec{\nabla} P}{\rho_0}} + \boxed{\nu \vec{\nabla}^2 \vec{u}} + \boxed{\vec{\mathcal{F}}}$$

Time variation | Advection (inertia) | Rotation | Gravity | Pressure gradient | Dissipation (viscosity) | Forcings

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

Navier-Stokes Equations

+ Boussinesq approximation

- Turbulence arises from the non-linear terms

$$\frac{\partial \vec{u}}{\partial t} + \boxed{\vec{u} \cdot \vec{\nabla} \vec{u}} + f\vec{k} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Advection
(inertia)

- Only one here = advection *(quadratic nonlinearity)*
- Turbulence results from the nonlinear nature of advection, which enables interaction between motions on different spatial scales.

Navier-Stokes Equations

+ Boussinesq approximation

- x-momentum equation:

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad fU \qquad \frac{P}{\rho_0 L} \qquad \frac{\nu U}{L^2} \qquad F$$

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad fU \qquad \frac{P}{\rho_0 L} \qquad \frac{\nu U}{L^2} \qquad F$$

$$\frac{L^2}{\nu T} \qquad \boxed{\frac{UL}{\nu}} \qquad \frac{fL^2}{\nu} \qquad \frac{PL}{\rho_0 \nu U} \qquad 1 \qquad \frac{FL^2}{\nu U}$$

Equations scalings

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

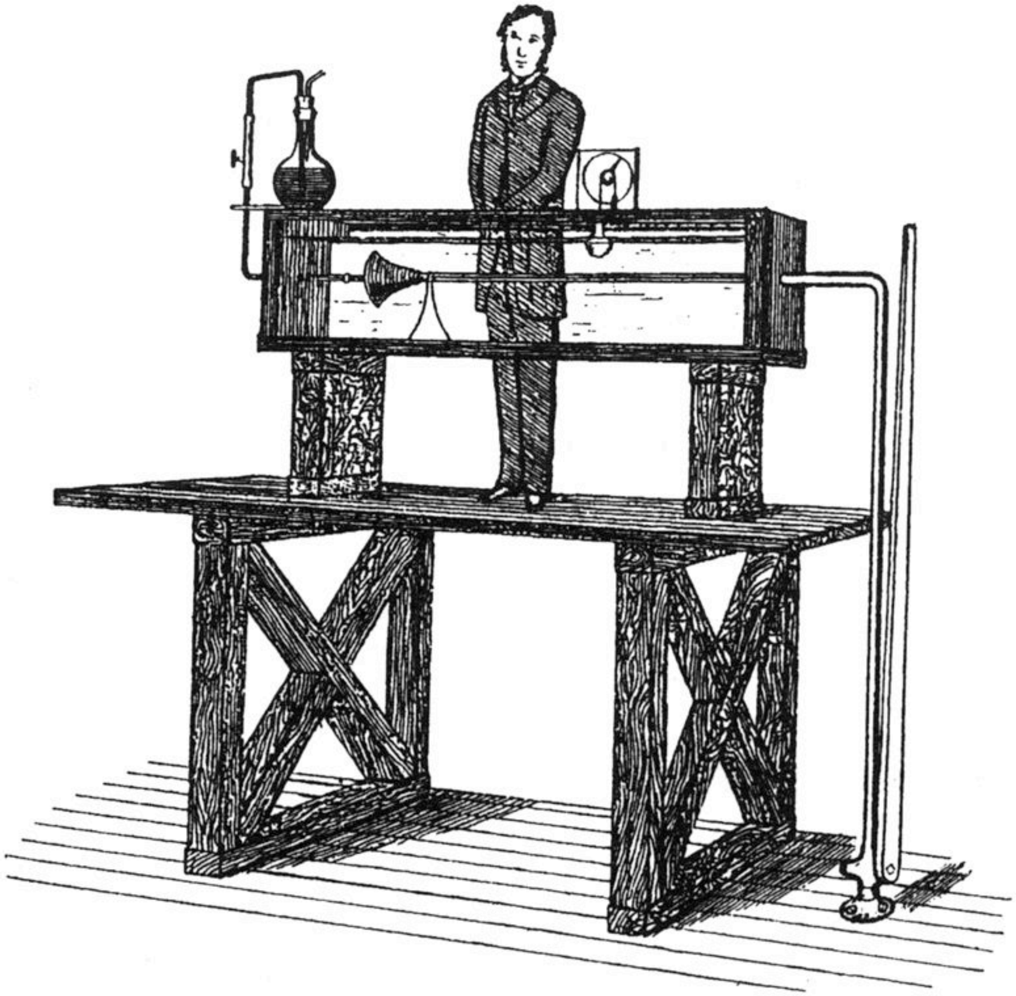
$$\boxed{\frac{UL}{\nu}}$$

= Reynolds Number

the ratio of the non-linear terms to the viscous terms

Reynolds Number

$$Re = \frac{UL}{\nu}$$



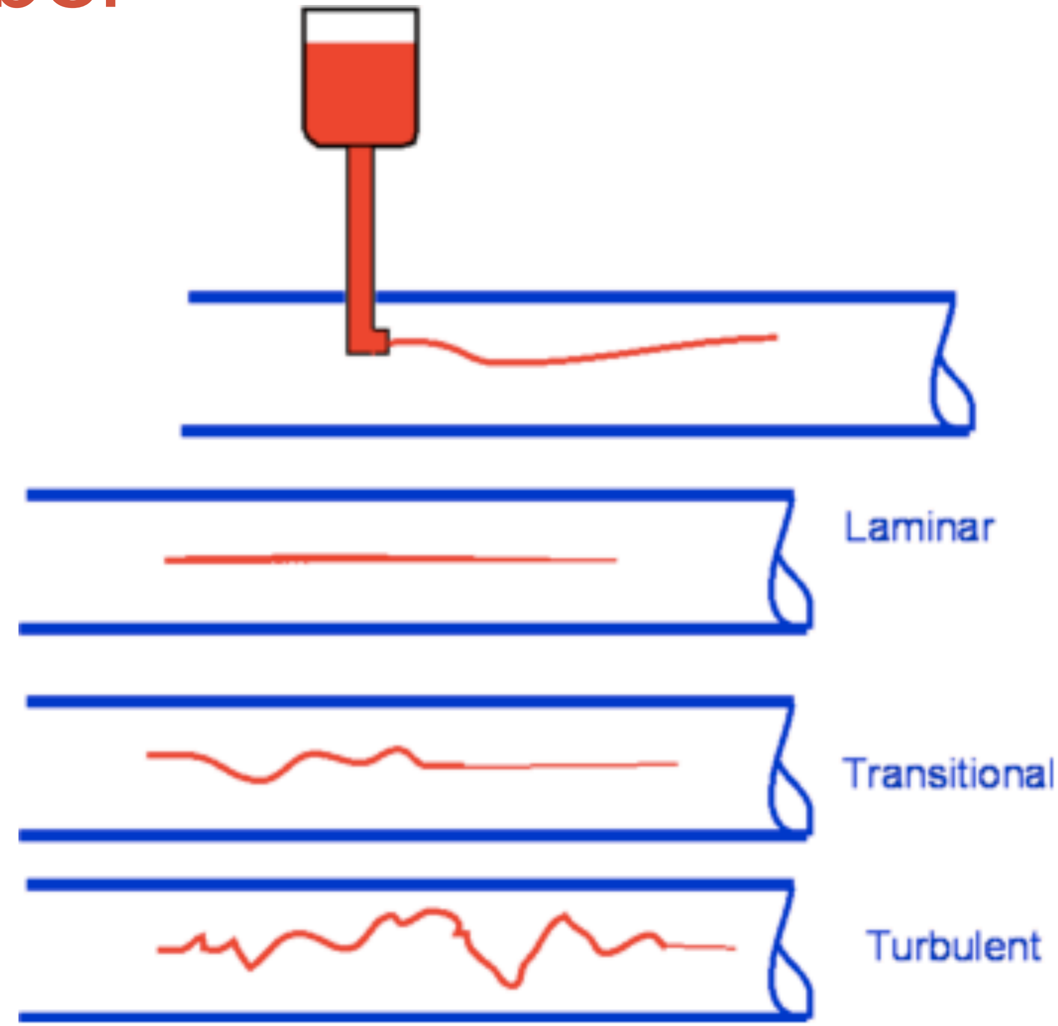
Osborne Reynolds experiment [1884]

Reynolds Number

$$Re = \frac{UL}{\nu}$$

Re < 2000 = laminar

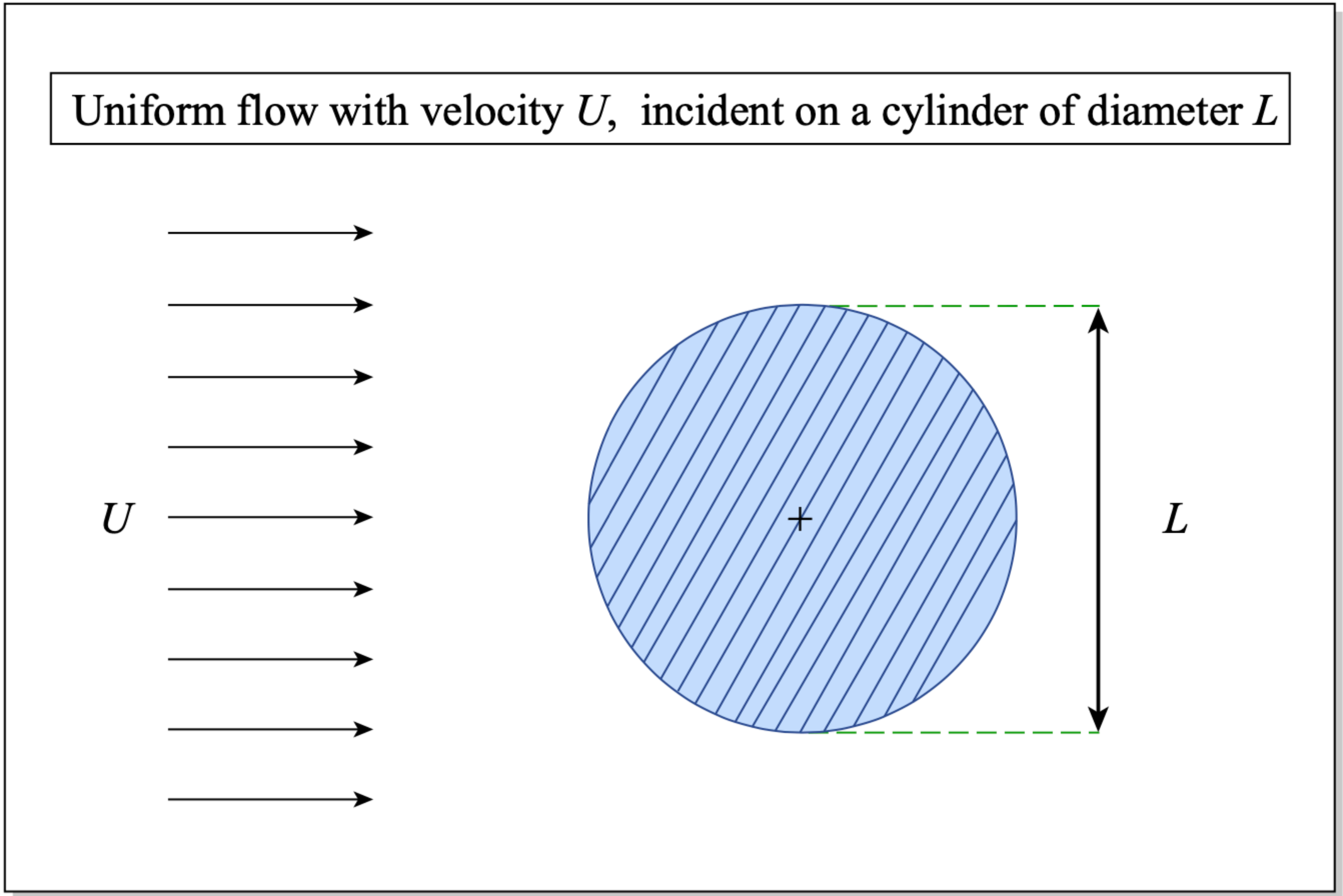
Re > 4000 = turbulent



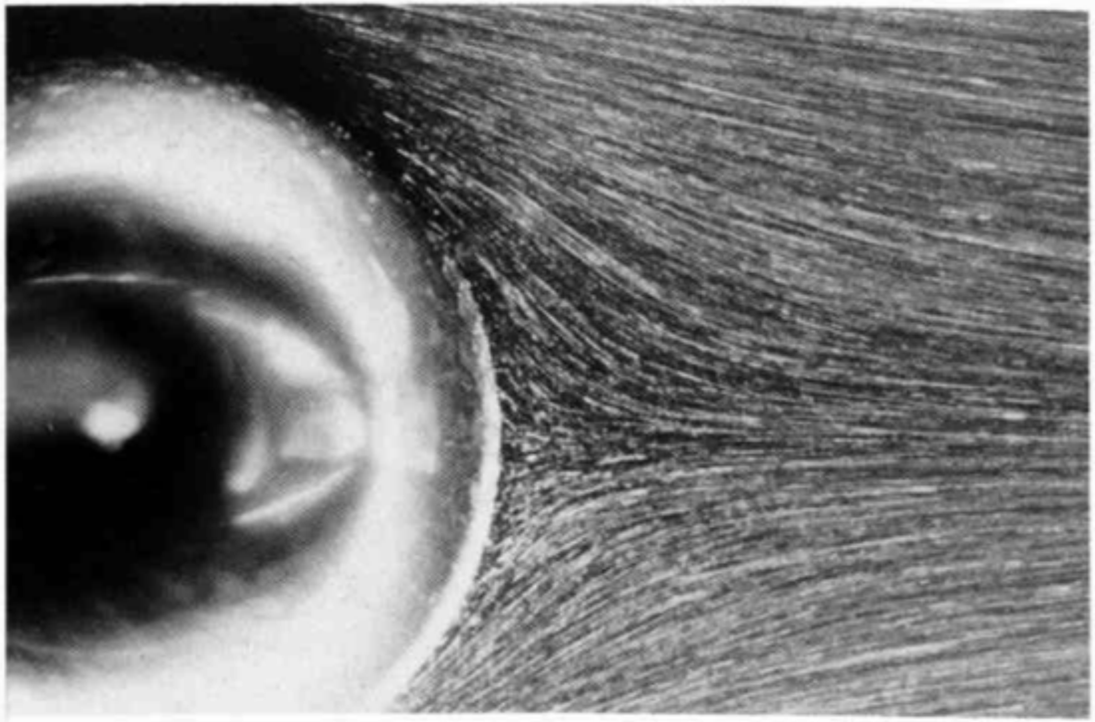
Reynolds experiment

http://www-mdp.eng.cam.ac.uk/web/library/enginfo/aerothermal_dvd_only/aero/fprops/pipeflow/node8.html

Transition to turbulence

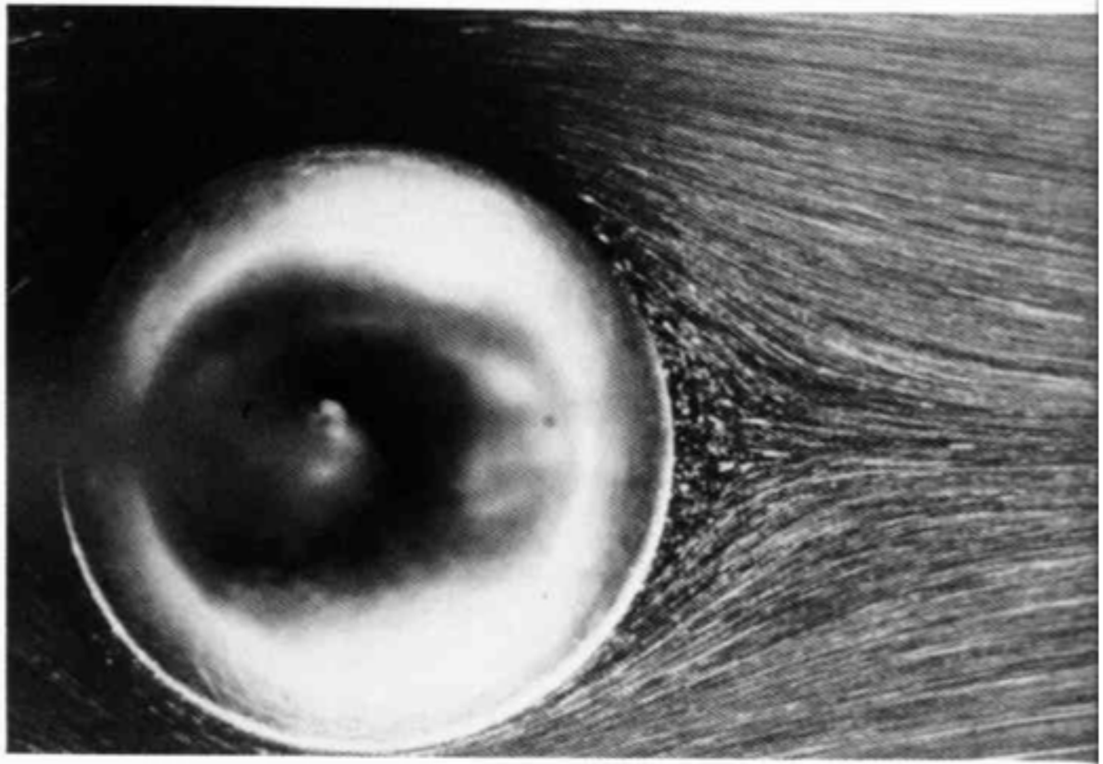


Transition to turbulence



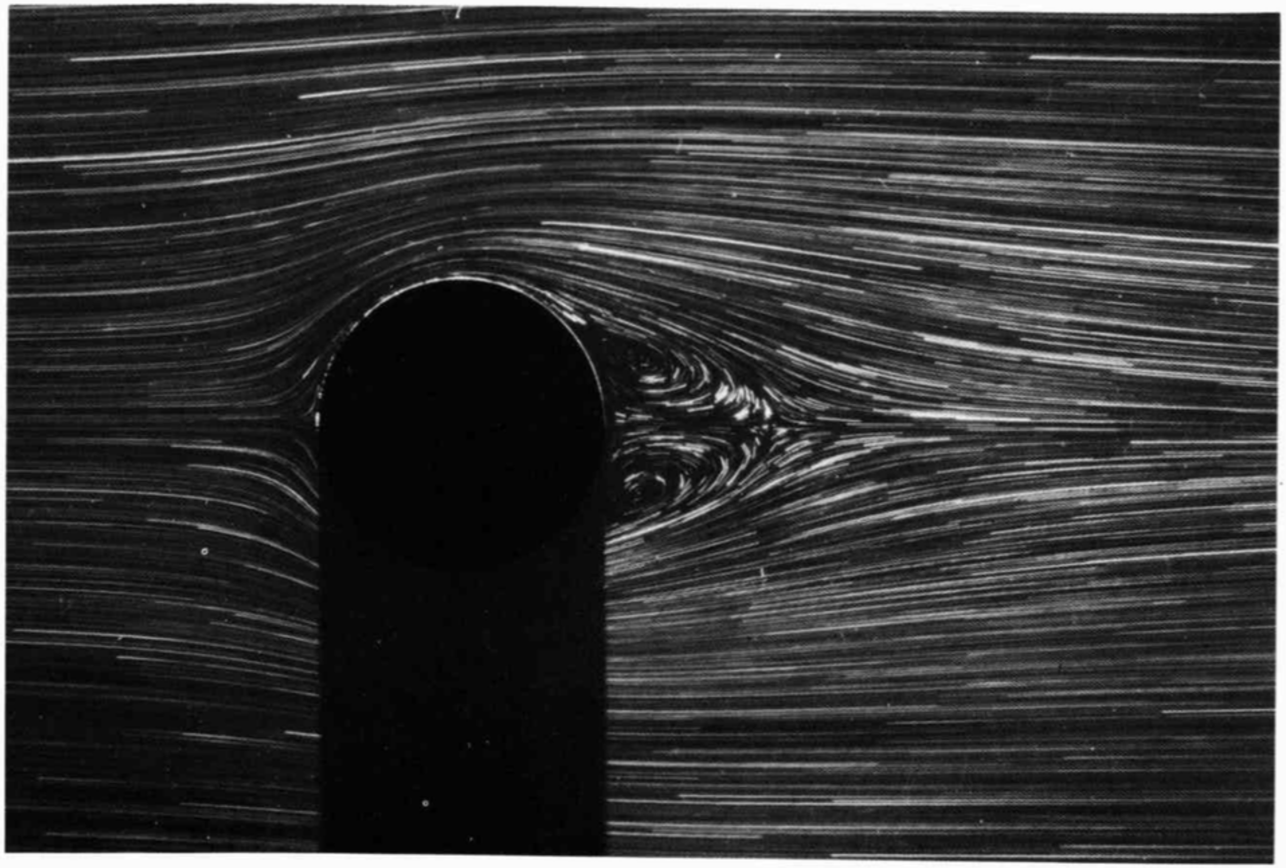
49. Sphere at R=25.5. Although it is not obvious, the flow is believed just to have separated at the rear at this Reynolds number, in contrast to the unseparated flow of figure 27. Aluminum dust is illuminated in water. *Taneda 1956b*

Transition to turbulence



50. Sphere at R**=26.8.** At this slightly higher speed the flow has clearly separated over the rear of the sphere, to form a thin standing vortex ring. Aluminum dust is illuminated in water. *Taneda* 1956b

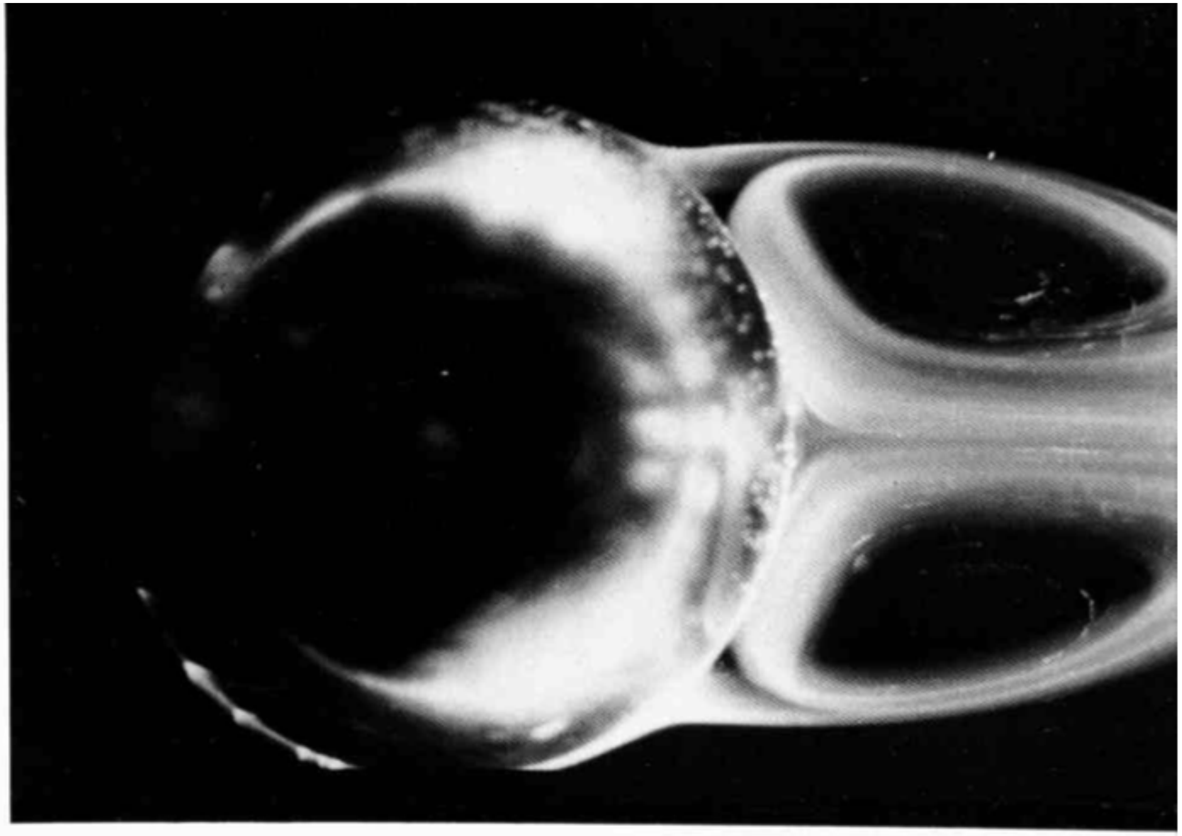
Transition to turbulence



51. Sphere at $R = 56.5$. As in figure 8, the sphere is falling steadily down the axis of a tube filled with oil, but here so large that the influence of the walls is negligible. Magnesium cuttings are illuminated by a sheet of light, which casts the shadow of the sphere. *Archives de l'Académie des Sciences de Paris. Payard & Coutanceau 1974*

Transition to turbulence

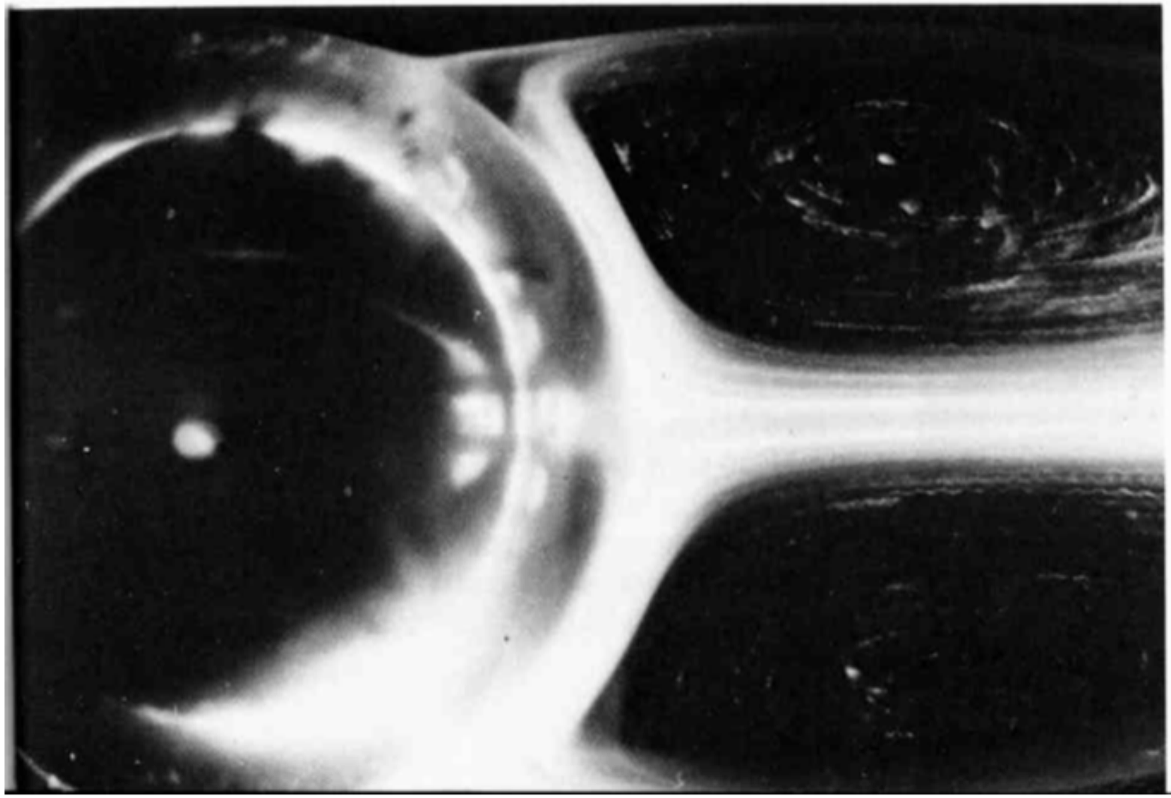
52. Sphere at R=104. At this Reynolds number the recirculating wake extends a full diameter downstream, but is perfectly steady, as for the circle in figure 44. Visualization is by a thin coating of condensed milk on the sphere, which gradually melts and is carried into the stream of water. *Taneda 1956b*



Transition to turbulence

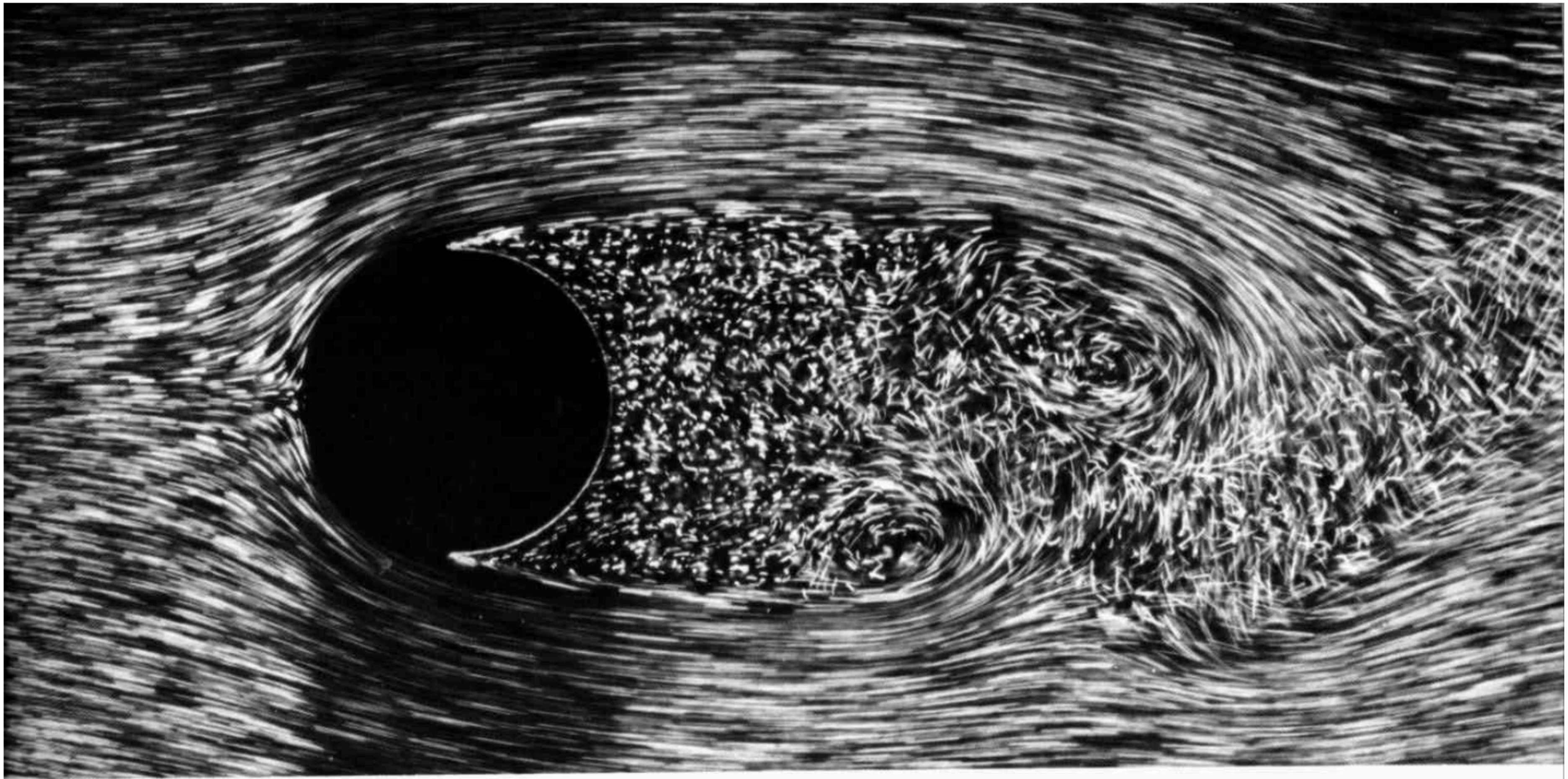
53. Sphere at R=118. The wake grows more slowly in axisymmetric than plane flow. These photographs have shown that the length of the recirculating region is proportional to the logarithm of the Reynolds number, whereas it grows linearly with Reynolds number for a cylinder. Aluminum dust shows the flow of water. *Taneda 1956b*

Transition to turbulence



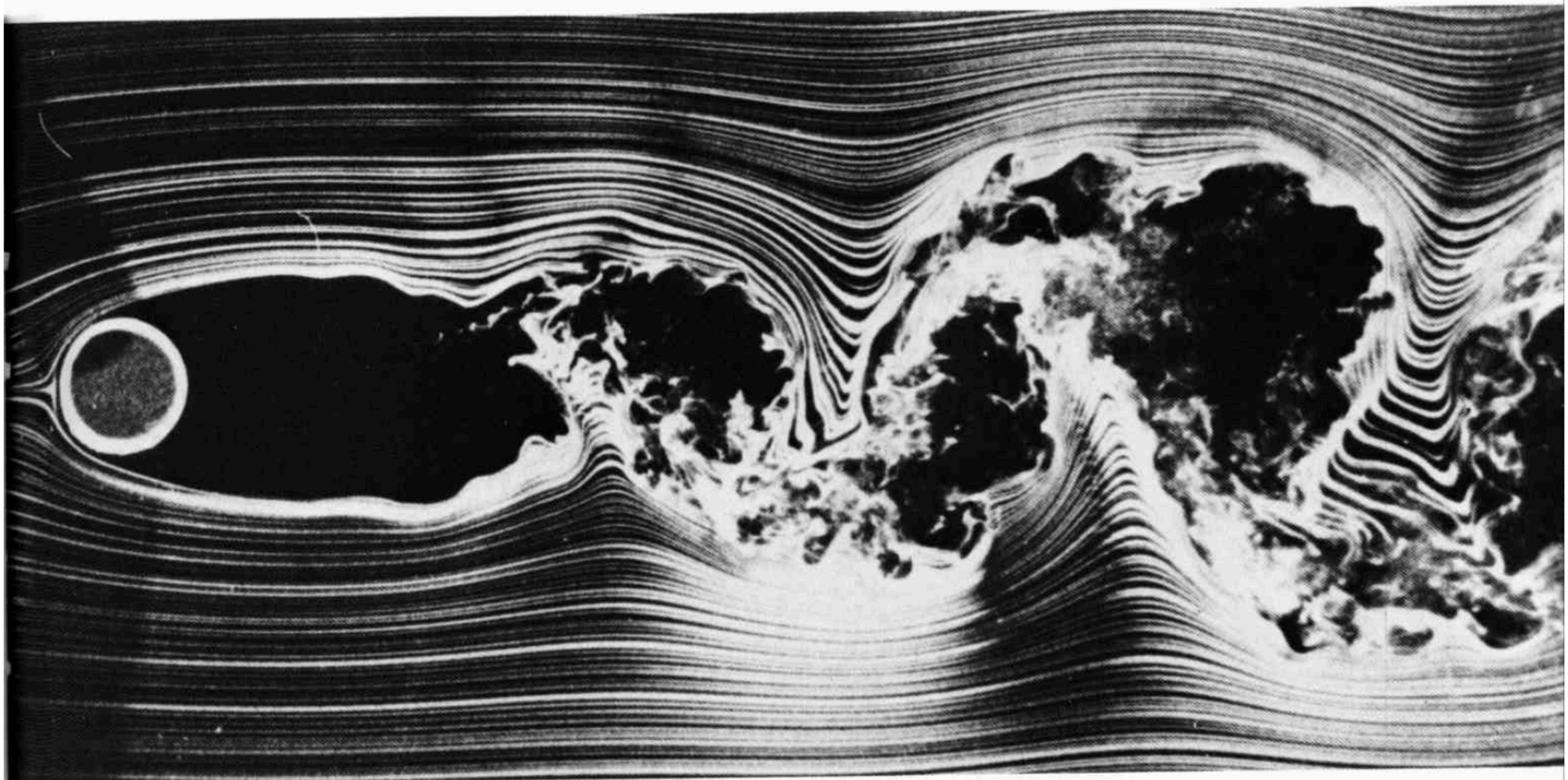
54. Sphere at $R = 202$. The rear of the recirculating region behind a sphere begins to oscillate slowly at a Reynolds number of about 130, but the flow is still perfectly laminar at this higher speed. Visualization is by condensed milk in water. *Taneda 1956b*

Transition to turbulence



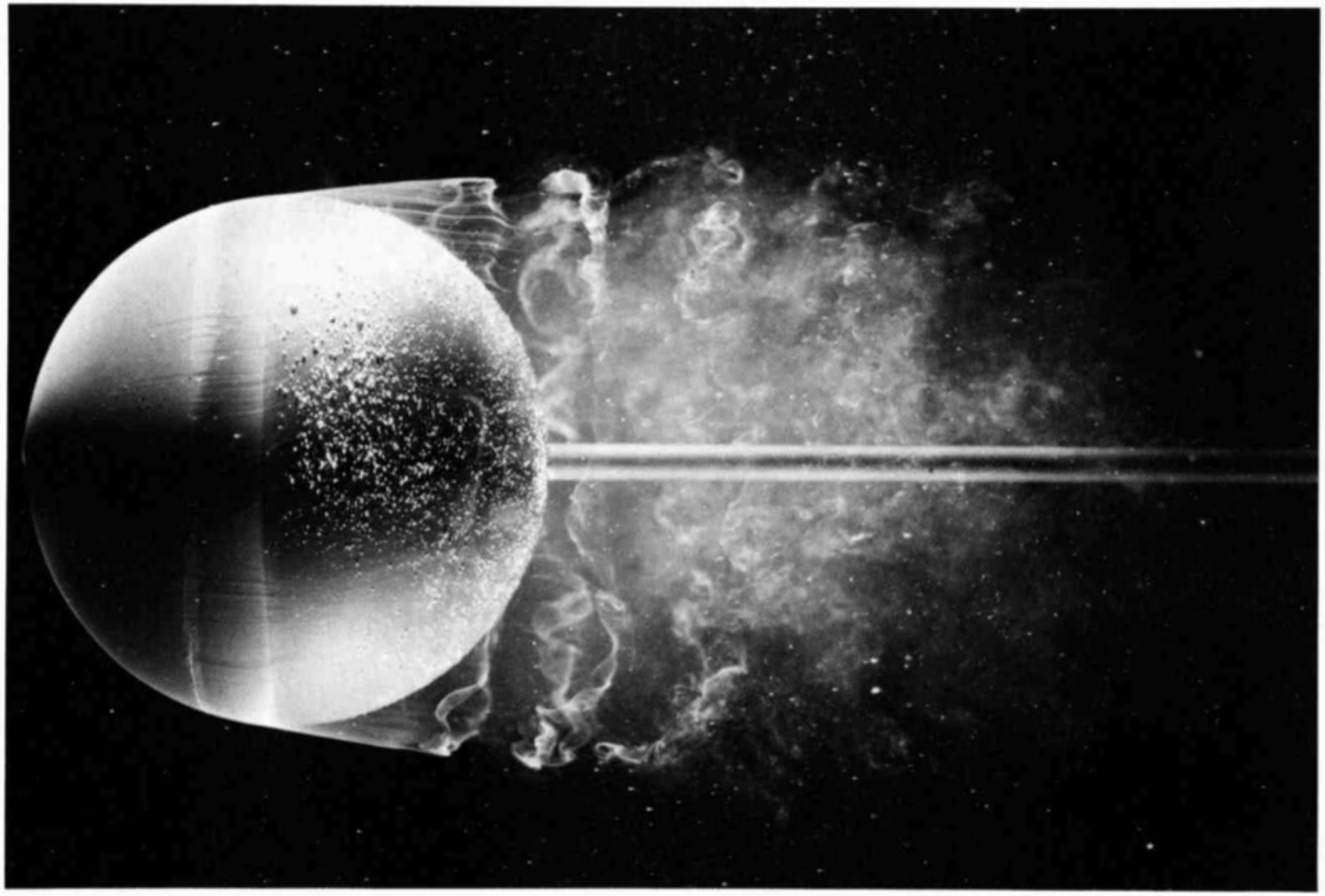
47. Circular cylinder at $R=2000$. At this Reynolds number one may properly speak of a boundary layer. It is laminar over the front, separates, and breaks up into a turbulent wake. The separation points, moving forward as the Reynolds number is increased, have now attained their upstream limit, ahead of maximum thickness. Visualization is by air bubbles in water. *ONERA photograph, Werlé & Gallon 1972*

Transition to turbulence



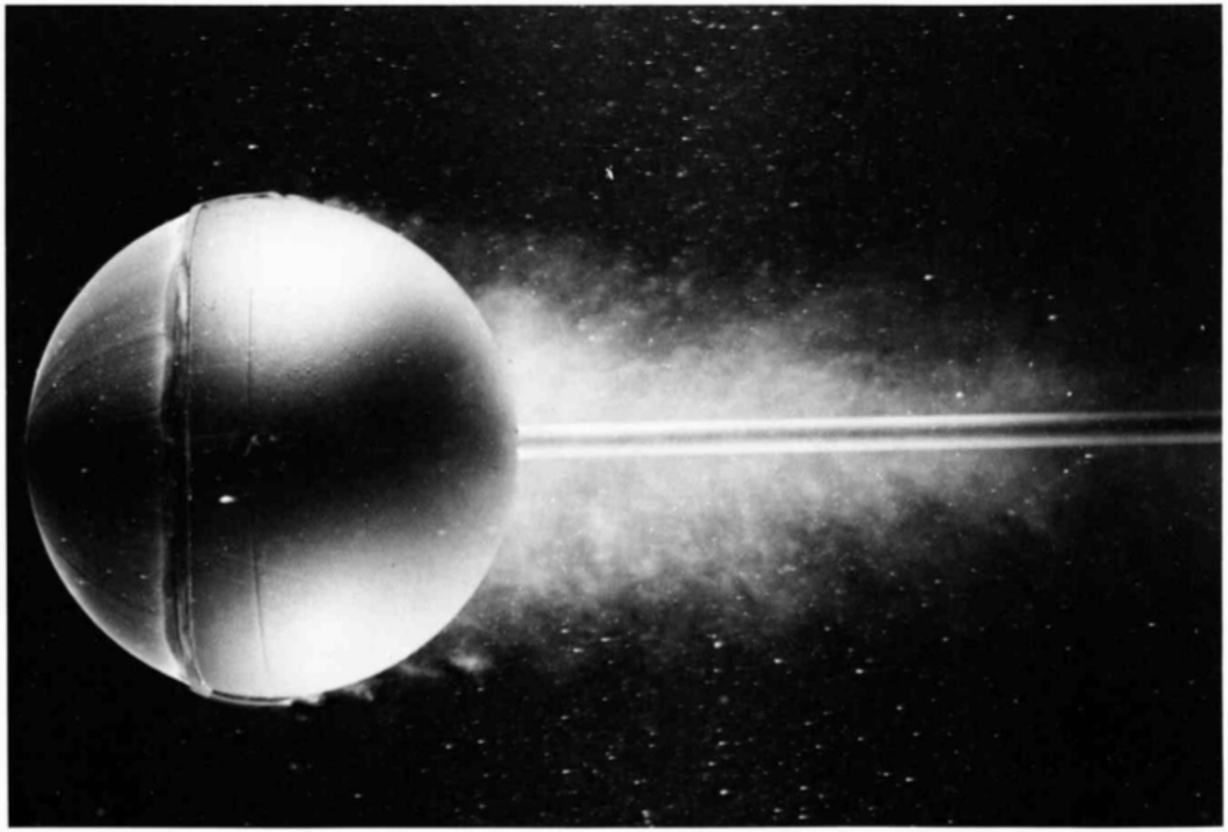
48. Circular cylinder at R=10,000. At five times the speed of the photograph at the top of the page, the flow pattern is scarcely changed. The drag coefficient consequently remains almost constant in the range of Reynolds number spanned by these two photographs. It drops later when, as in figure 57, the boundary layer becomes turbulent at separation. *Photograph by Thomas Corke and Hassan Nagib*

Transition to turbulence



55. Instantaneous flow past a sphere at R**=15,000.** Dye in water shows a laminar boundary layer separating ahead of the equator and remaining laminar for almost one radius. It then becomes unstable and quickly turns turbulent. *ONERA photograph, Werlé 1980*

Transition to turbulence



57. Instantaneous flow past a sphere at R=30,000 with a trip wire. A classical experiment of Prandtl and Wieselsberger is repeated here, using air bubbles in water. A wire hoop ahead of the equator trips the boundary layer. It becomes turbulent, so that it separates farther rearward than if it were laminar (opposite page). The drag is thereby dramatically reduced, in a way that occurs naturally on a smooth sphere only at a Reynolds number ten times as great. *ONERA photograph, Werlé 1980*

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- Dissipative time scale *= time scale that is required for molecular friction to bring the motion at scale L to rest*

Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- Dissipative time scale = *time scale that is required for molecular friction to bring the motion at scale L to rest*

$$\frac{L^2}{\nu T} = \frac{T_\nu}{T} \qquad T_\nu = \frac{L^2}{\nu}$$

Activity 1:

- Reynolds number for a mesoscale vortex in the ocean?
- Reynolds number for a storm system in the atmosphere?

- How long would it take for molecular dissipation to halt a storm system?
- How long would it take for molecular dissipation to halt a stirred coffee?

Chaotic behavior:

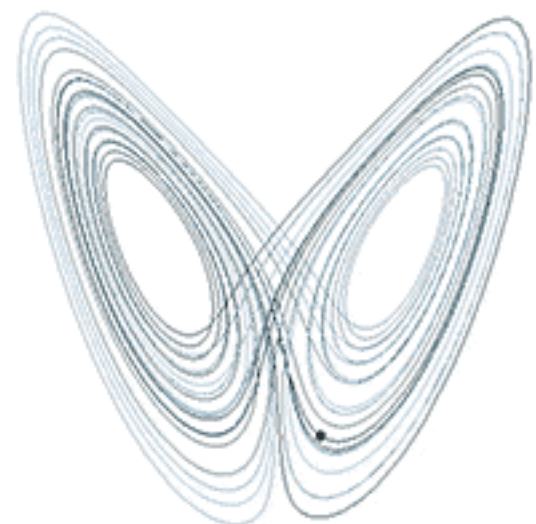
- **Chaos:** When the present determines the future, but the approximate present does not approximately determine the future *[E. Lorenz]*

A double rod pendulum showing chaotic behavior [https://en.wikipedia.org/wiki/Chaos_theory]

Chaotic behavior:

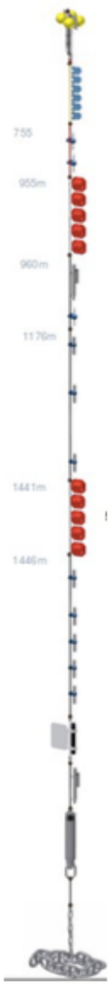
- **Chaos:** When the present determines the future, but the approximate present does not approximately determine the future *[E. Lorenz]*

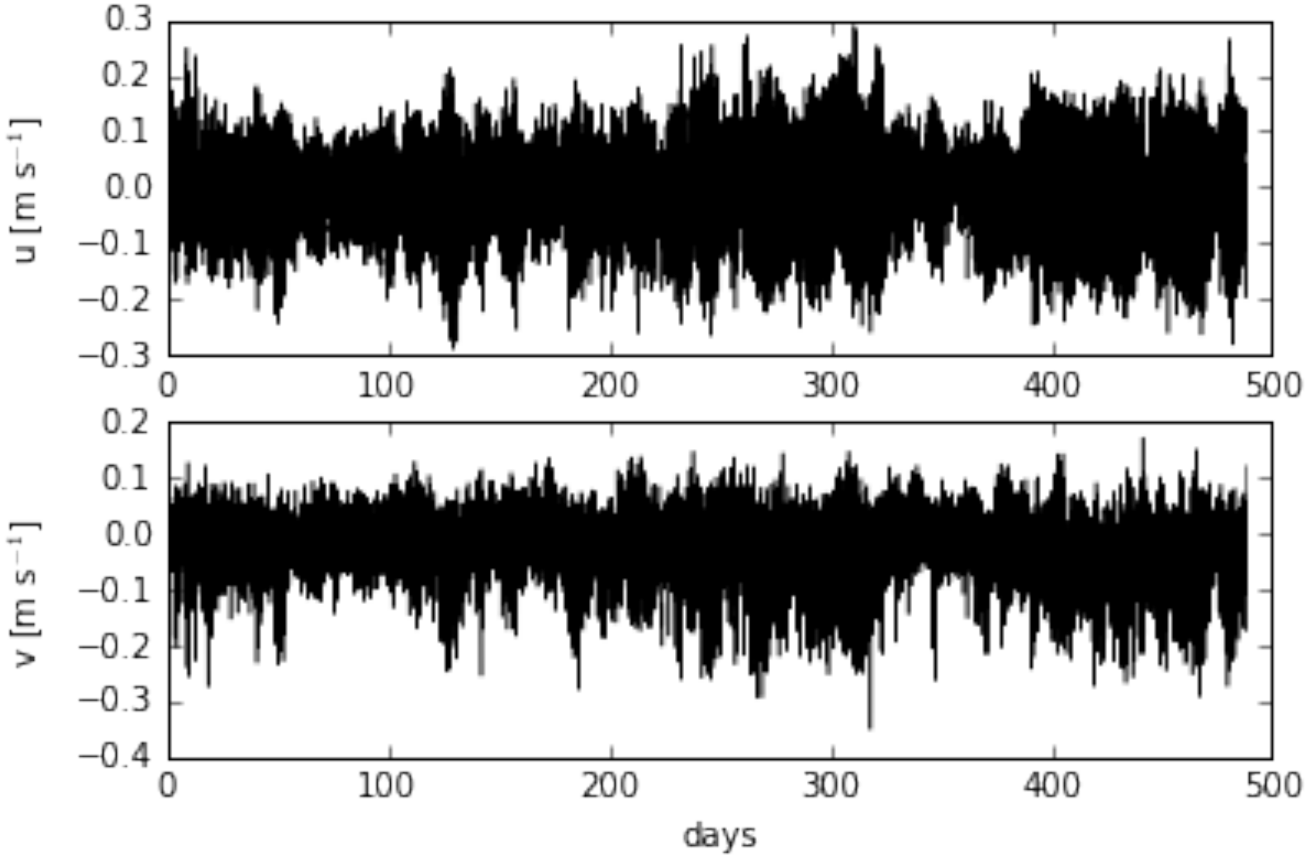
A double rod pendulum showing chaotic behavior [https://en.wikipedia.org/wiki/Chaos_theory]



Solutions in the Lorenz attractor (originally a model for atmospheric convection)

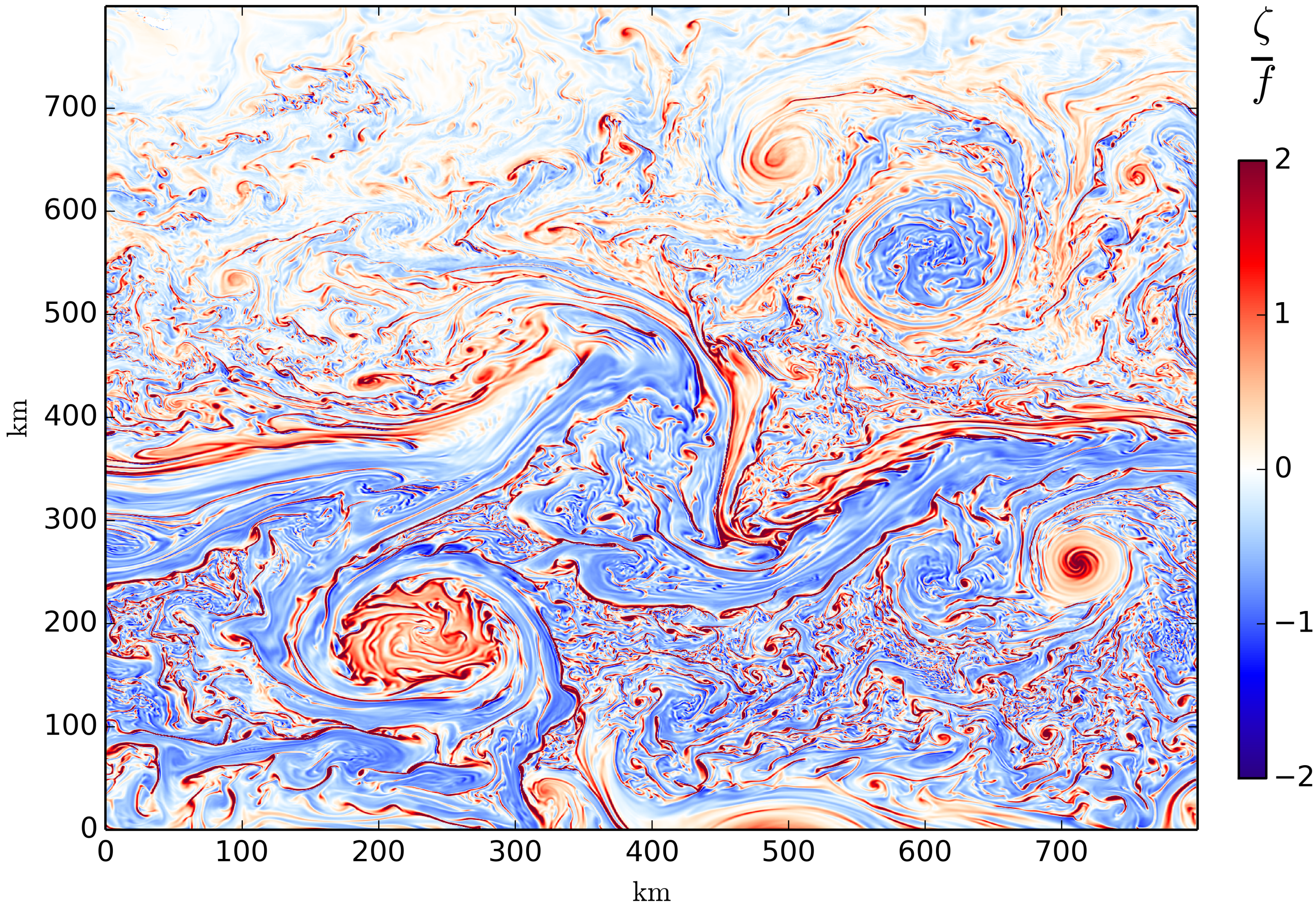
How to characterize properties of turbulence?





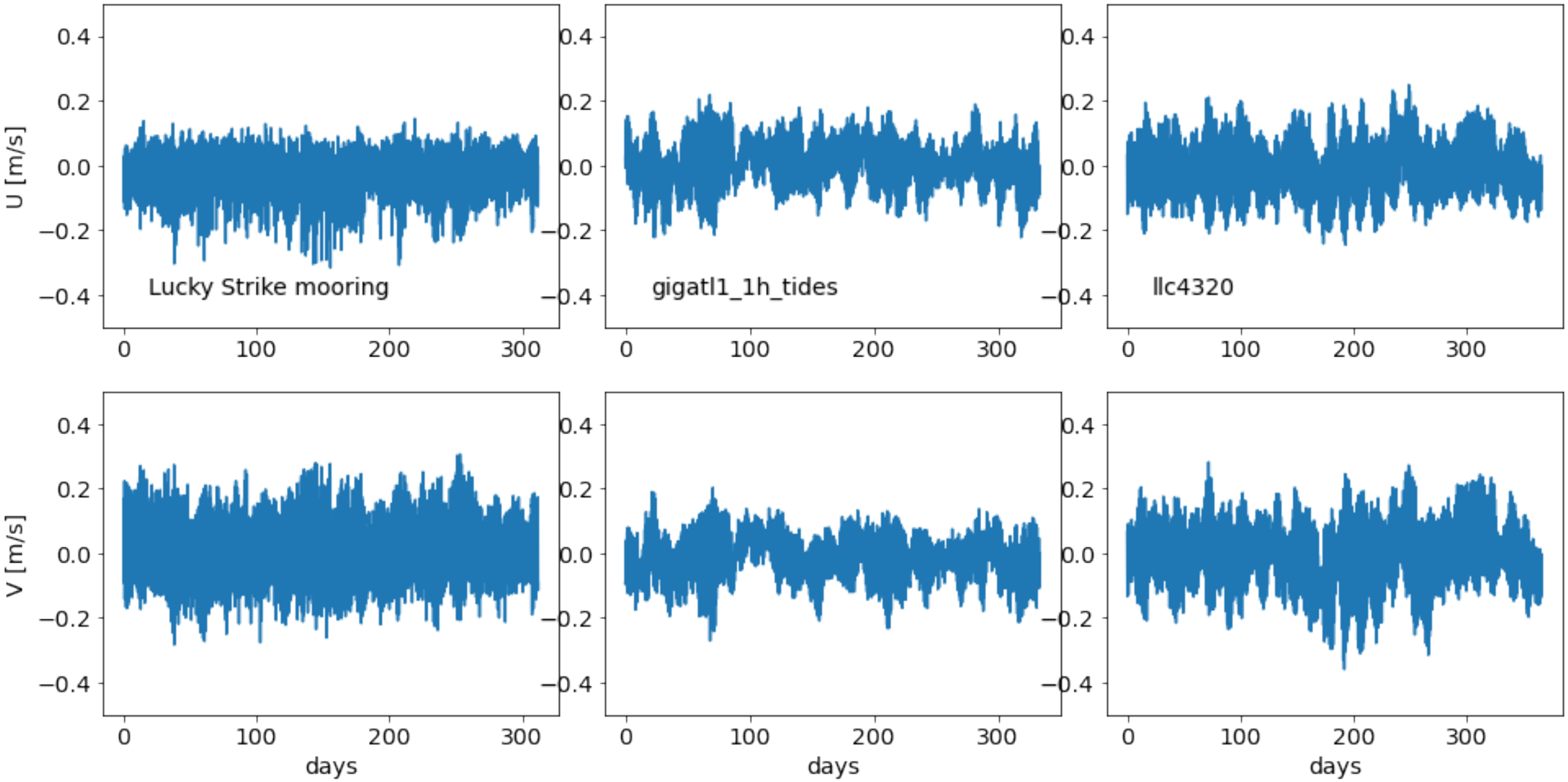
Velocities at 1700 m depths from currentometers at the Lucky Strike site

How to characterize properties of turbulence?



Surface relative vorticity in the Gulf Stream

How do we compare results from a model to observations?



Obs. Numerical Models

Velocities at 1700 m depth at the Lucky Strike site

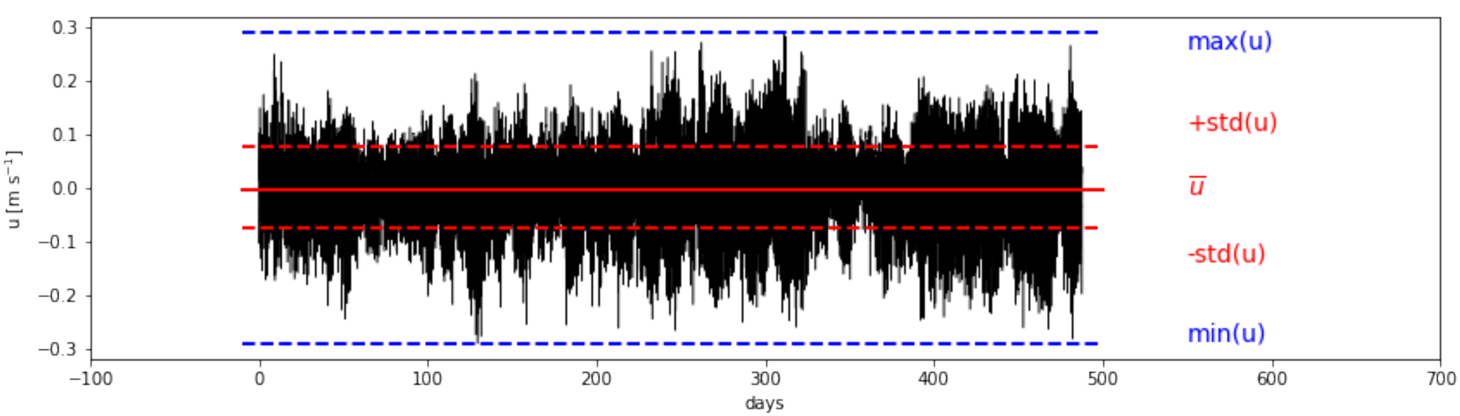
Statistics of turbulent fields:

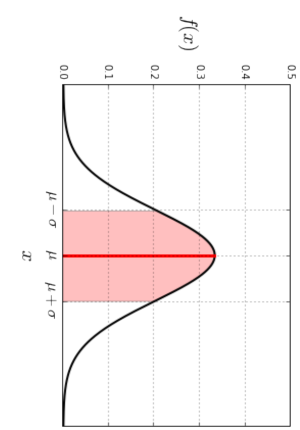
We look at the values distribution (= **pdf**) and its moments:

The first one is the **mean.** $\mu = \int x f(x) dx$

The second one is the **variance** (= squared standard deviation / rms). It describes the spread of the pdf around the mean.

$$\sigma^2 = \int (x-\mu)^2 f(x)\, dx$$





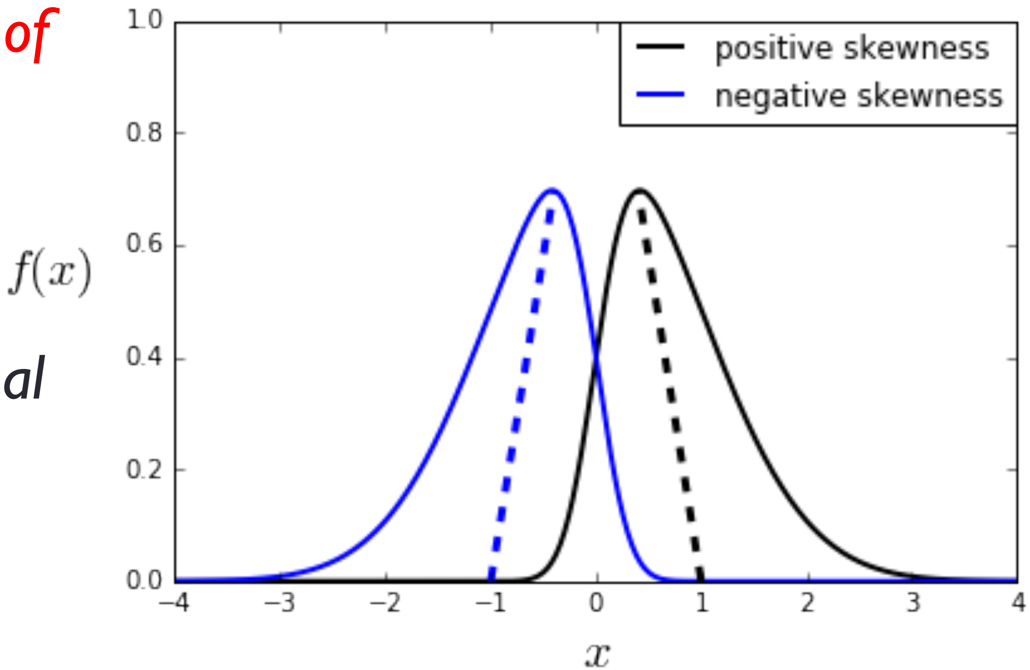
Statistics of turbulent fields:

The third one is the **skewness:**

$$\mu_3 = \frac{1}{\sigma^3} \int (x-\mu)^3 f(x)\, dx$$

It is a measure of the asymmetry of the pdf about its mean.

If the pdf is symmetric (e.g. normal distrib.), the skewness is 0



Statistics of turbulent fields:

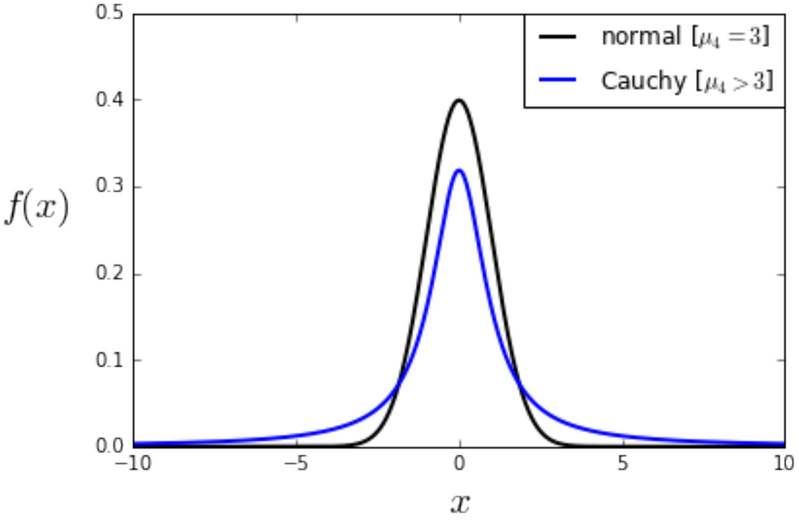
The fourth one is the **kurtosis:**

$$\mu_4 = \frac{1}{\sigma^4}\int (x-\mu)^4 f(x)\,dx$$

The kurtosis measures how fat are the tails of the pdf.

For the normal law the skewness is zero and the kurtosis is 3.

Values larger than 3 indicate more likely extreme values than the normal law.

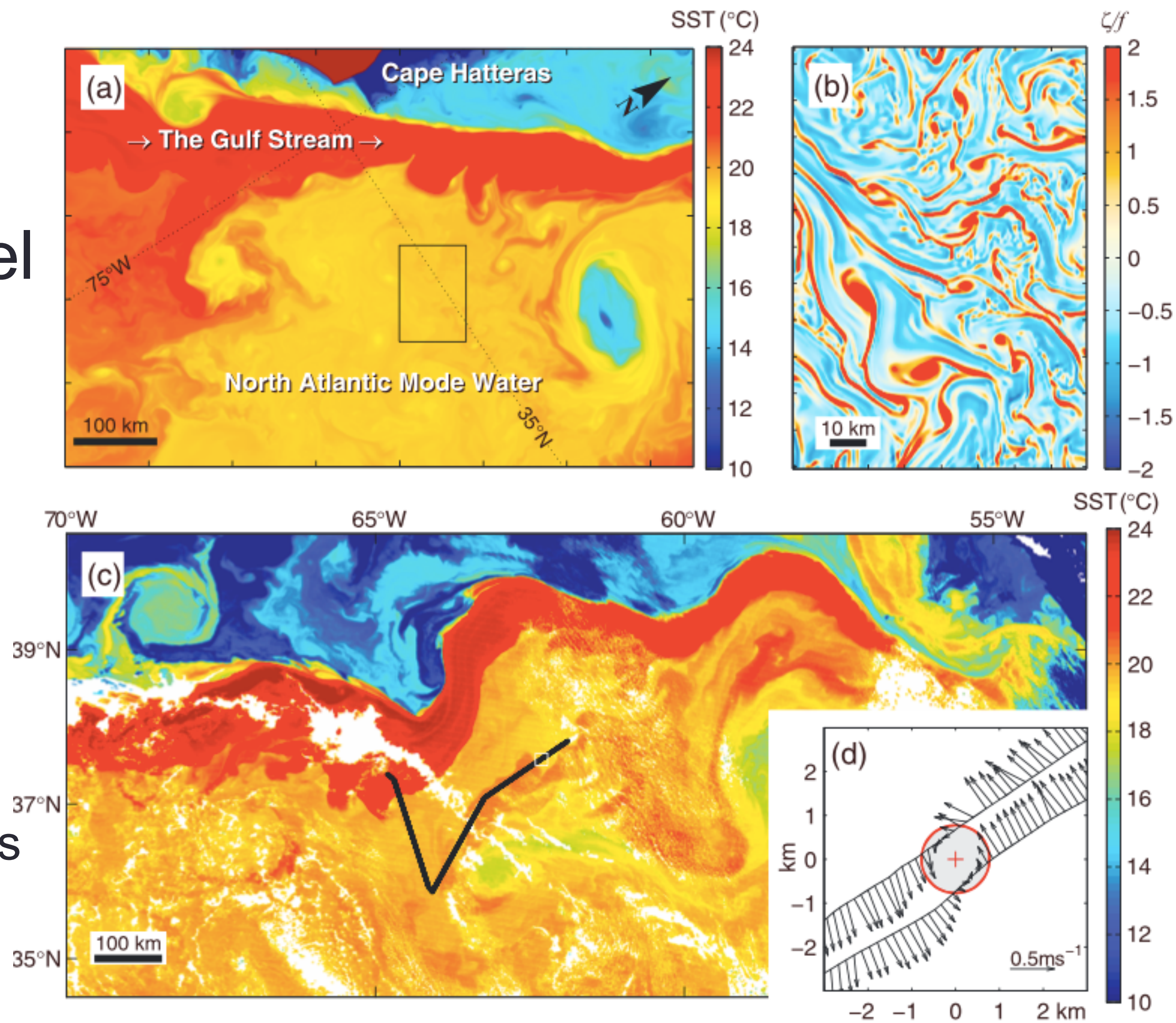


Ex: How do we compare results from a model to observations:

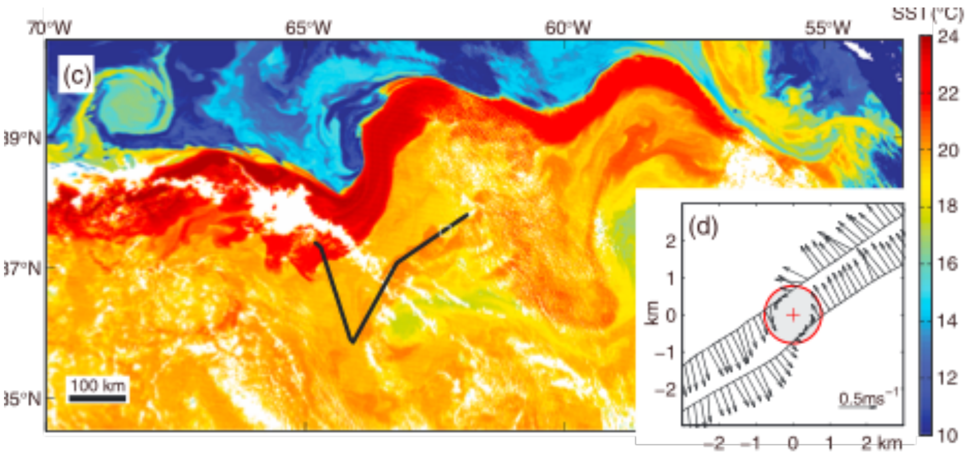
Numerical Model

Observations

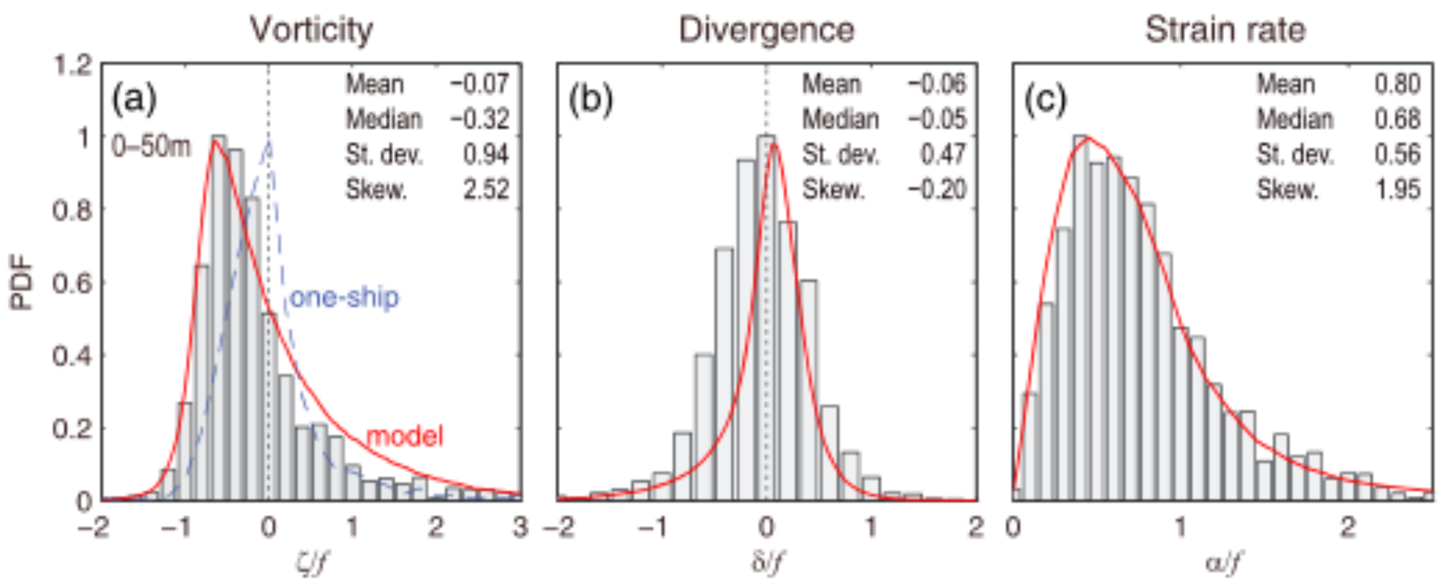
[satellite SST + velocity measurements from 2 parallel ships]



Ex: How do we compare results from a model to observations:

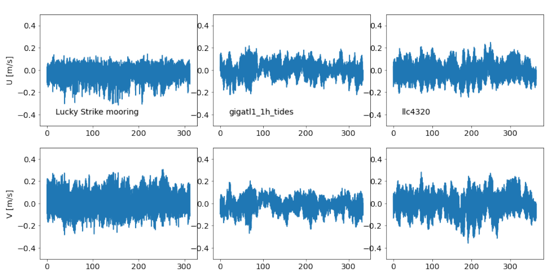


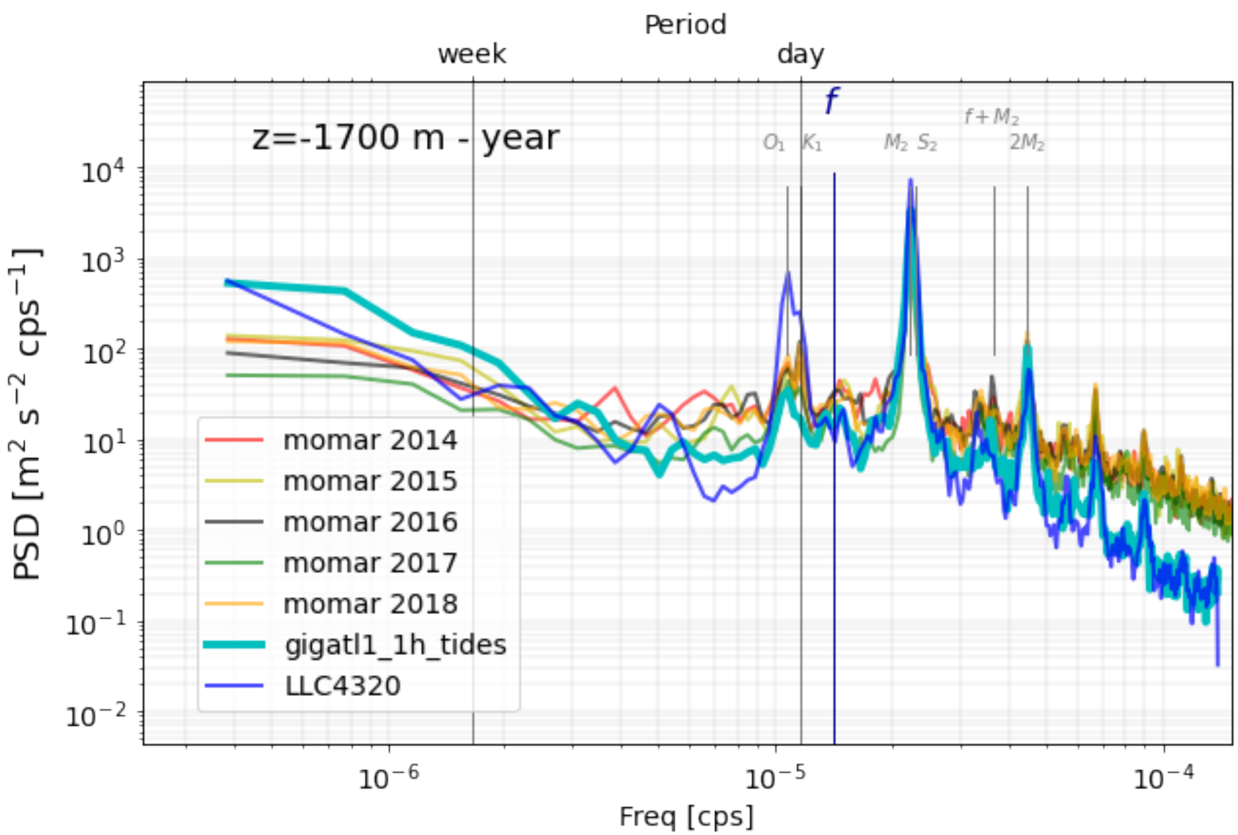
Ex: Shcherbina et al, 2015.



Statistics of turbulent fields:

Another way is to compute "**power spectral density**" (= spectra of energy or tracer variance)





Computing Power Spectra

- We have a signal *(e.g. velocity u along dimension x)*
- *The total energy is* $$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx$$
-

Computing Power Spectra

- We have a signal *(e.g. velocity u along dimension x)*
- *The total energy is* $$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx$$
- *Using the Fourier transform of the variable:*

$$\hat{u}(k) = \int_{-\infty}^{+\infty} e^{-2\pi i k x} u(x) dx$$

- *the power spectral density is* $$|\hat{u}(k)|^2$$
  - *It is the density of energy per unit wavenumber (or frequency)*

Computing Spectra

- Parseval's theorem states that:

$$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx = \int_{-\infty}^{+\infty} |\hat{u}(k)|^2 dk$$

  - *Integral of energy in the physical domain is equal to integral of spectral energy density over all wavenumbers.*

Computing Spectra

- In a finite and discrete domain:

$$\hat{u}(k) = \sum_{x=0}^{N-1} u(x) e^{-ikx\frac{2\pi}{N}}$$

$$\text{for } k = 0, \ldots, N-1$$

- And the power spectra is defined as: $$\frac{\hat{u}(k)\hat{u}^*(k)}{N}$$

Computing Spectra

A simple example in Python:

```
def myfft(u):
    nx = u.shape[0]
    k = np.fft.rfftfreq(nx,d=1)
    psd = (np.abs(np.fft.rfft((u)))**2)/nx
    return k, psd
```

A chaotic example:

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- Let's play around with a simpler version containing a non-linear term (with a Reynolds number r), a source and sink:

$$\frac{du}{dt} + ru^2 = -u + 1$$

-

Properties of turbulence: unpredictability

$$\frac{du}{dt} + ru^2 = -u + 1$$

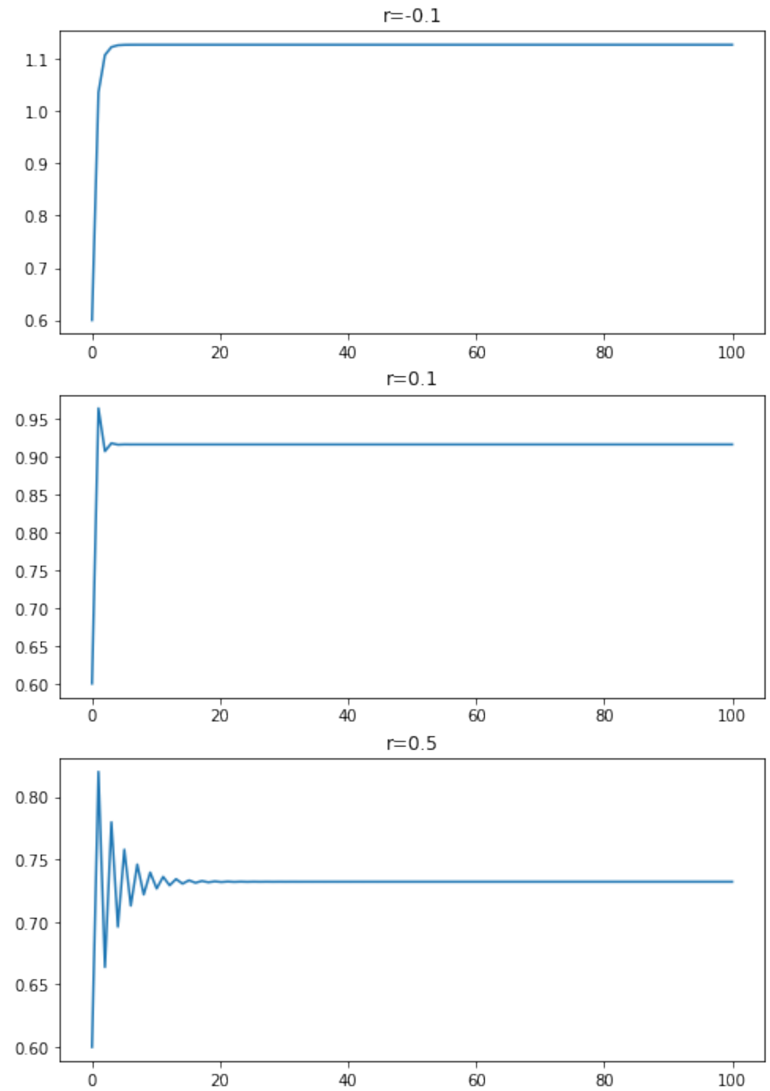
With $u_0 = 0.6$

- Fixed points:

$$\frac{d}{dt}u = -ru^2 + 1 - u = 0$$

$$u = -\frac{1}{2r} \pm \frac{\sqrt{1+4r}}{2r}$$

- Stable fixed points?



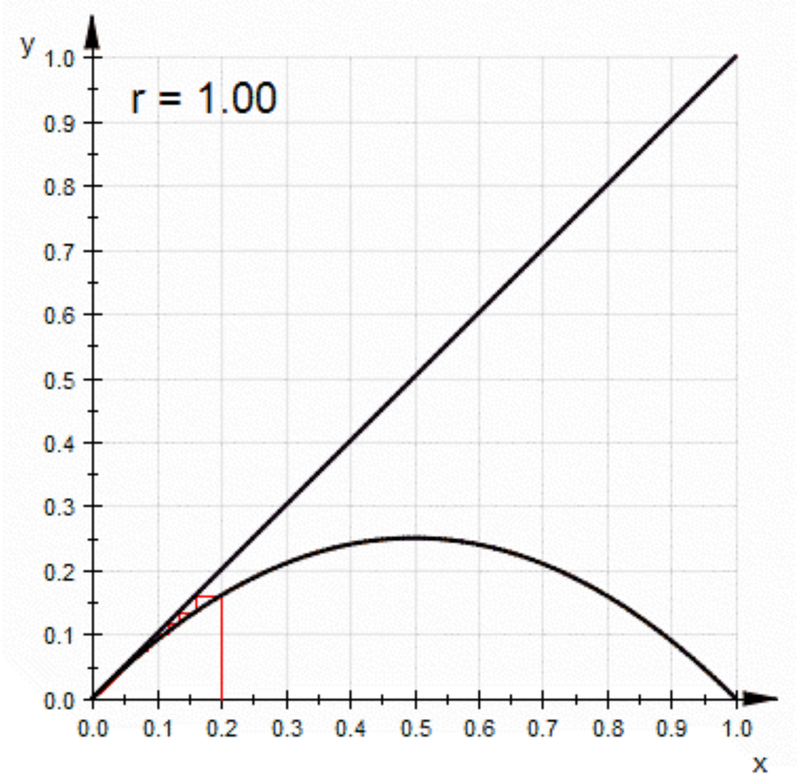
Activity 2:

$$\frac{du}{dt} + ru^2 = -u + 1$$

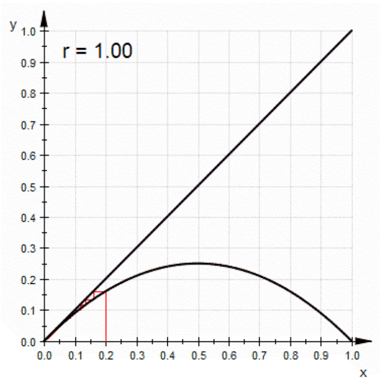
- Discretize using a simple forward Euler scheme (with dt=1) to get $u(t+1) = F[u(t)]$
- Using the discretized equation and u(0) = 0.6, plot u(t) for t=[0:100] and r = [0.1, 0.5, 0.8, 1.3, 2].
- Check the sensitivity to initial conditions (u(0) or u(0) + epsilon)
- Plot the pdf for r=2
- Plot the power spectra of u for r = [0.1, 0.5, 0.8, 1.3, 2],

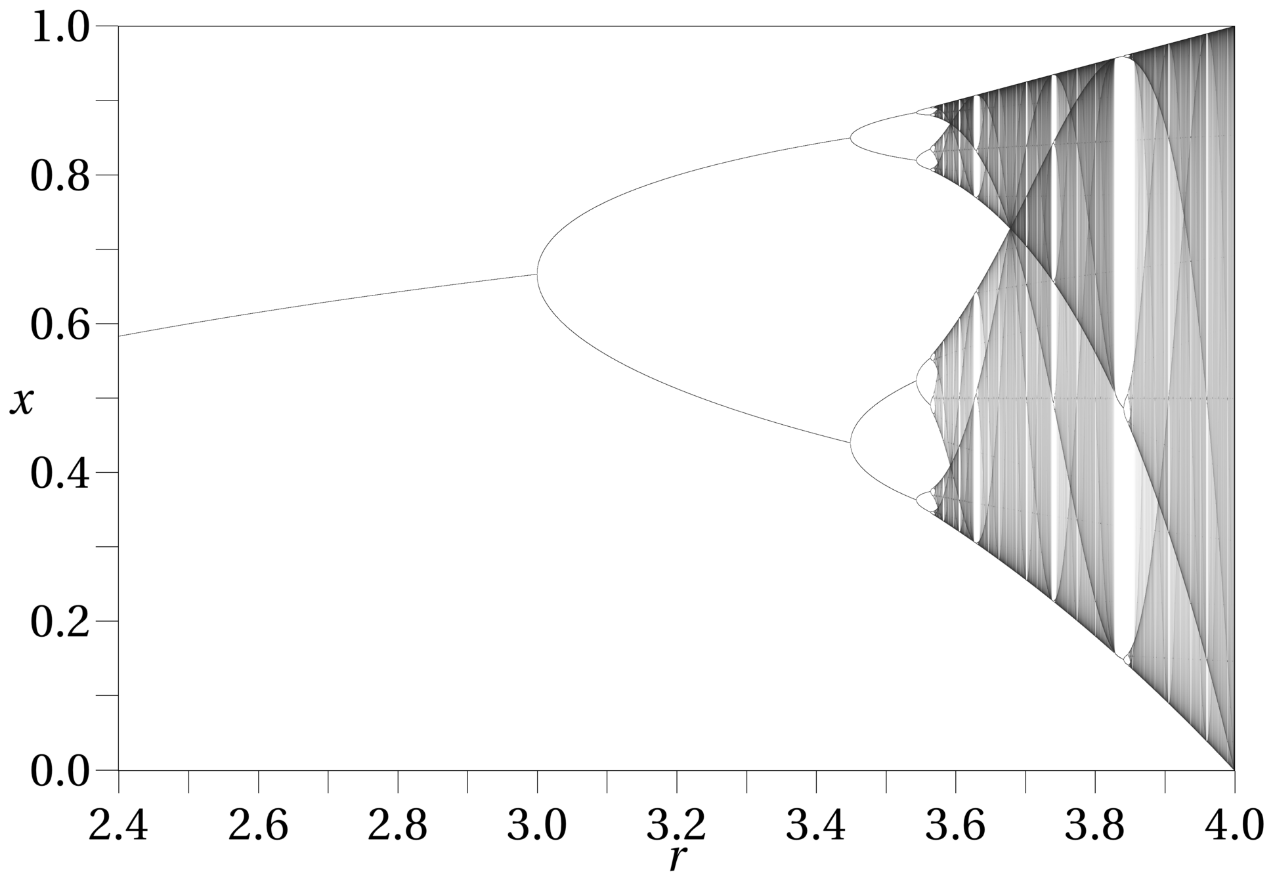
- The equation we've seen is similar to the **logistic map** by May (1976) [jgula.fr/Articles/May76.pdf] for the growth of population:

$$x_{n+1} = r x_n (1 - x_n)$$

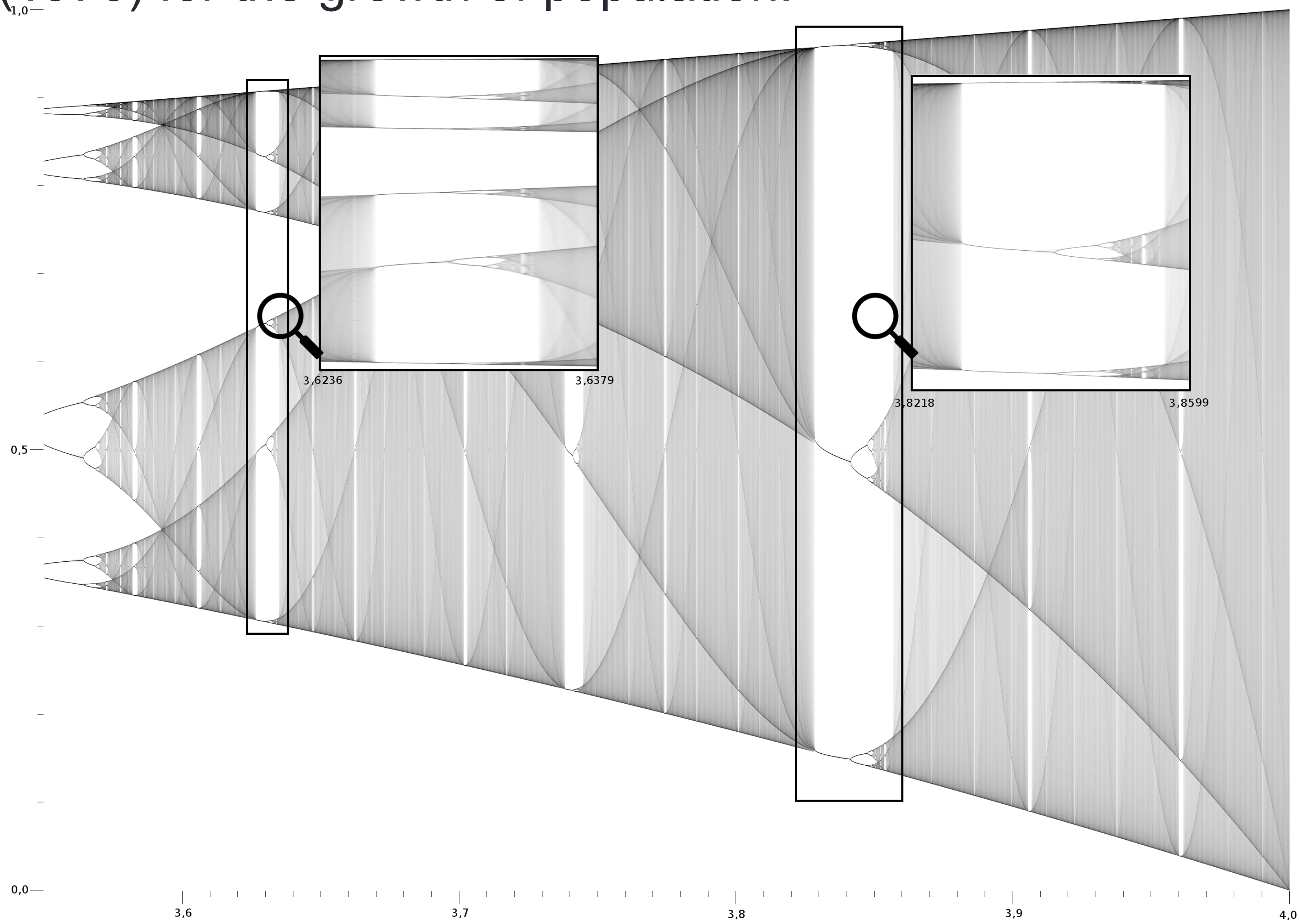


- The equation we've seen is similar to the **logistic map** by May (1976) for the growth of population:





- The equation we've seen is similar to the **logistic map** by May (1976) for the growth of population:

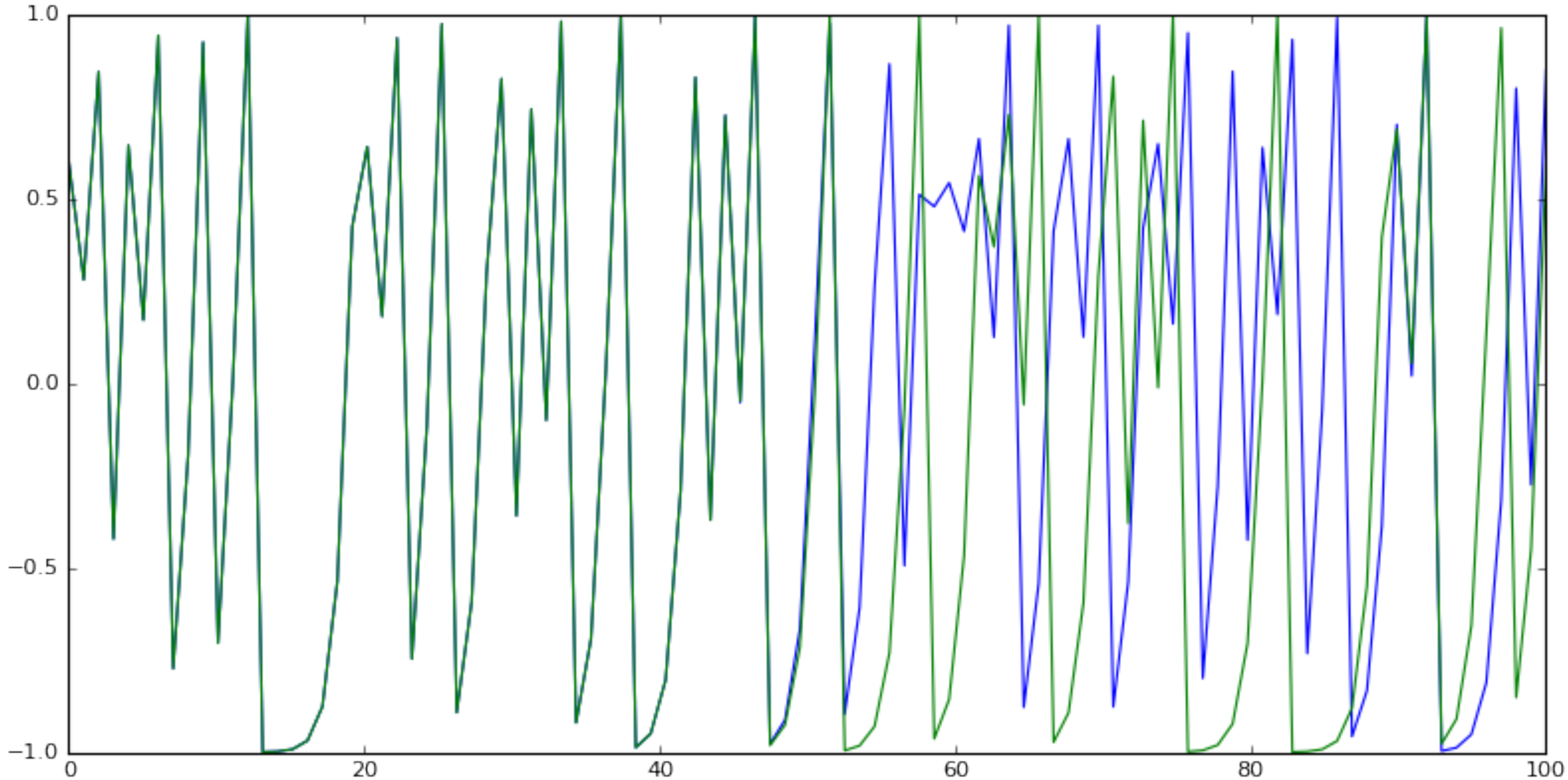


- Sensitivity to initial conditions

$$\frac{du}{dt} + ru^2 = -u + 1$$

U0 = 0.6

U0 = 0.6 + 1e-16



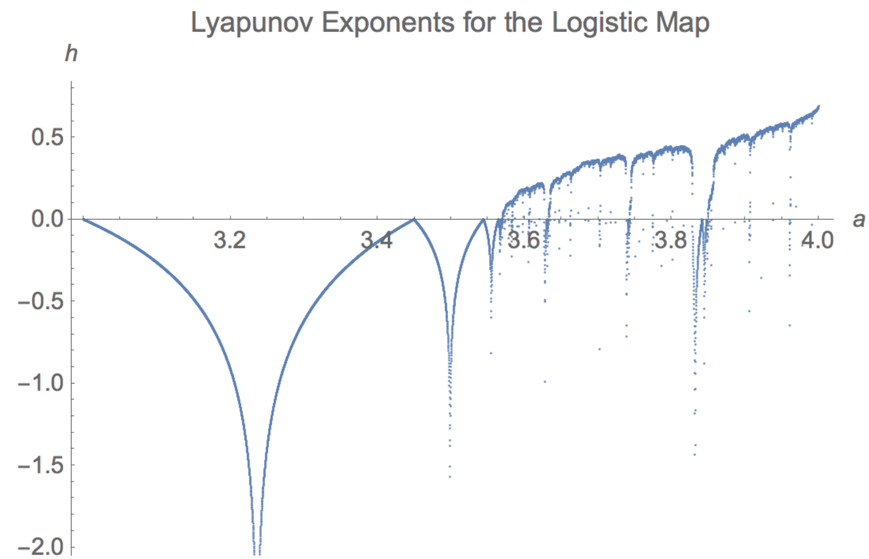
- **Sensitivity to initial conditions:**

In general, for an initial difference x_0 and $x_0 + \Delta x$

We expect the difference after n iteration to grow as $|\delta x_n| = \Delta x e^{n\lambda(x_0)}$

Where $\lambda(x_0)$ is the so-called **Lyapunov exponent** for the initial condition x_0

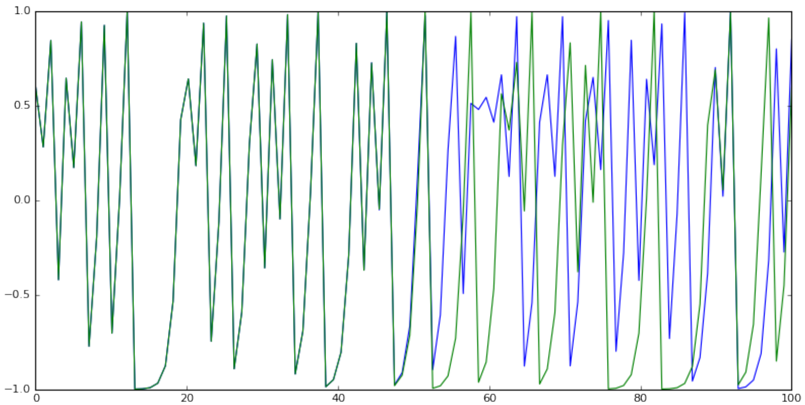
$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\Delta x \to 0} \frac{1}{n} \log \left| \frac{F^n(x_0 + \Delta x) - F^n(x_0)}{\Delta x} \right| = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{dF^n(x_0)}{dx_0} \right|$$

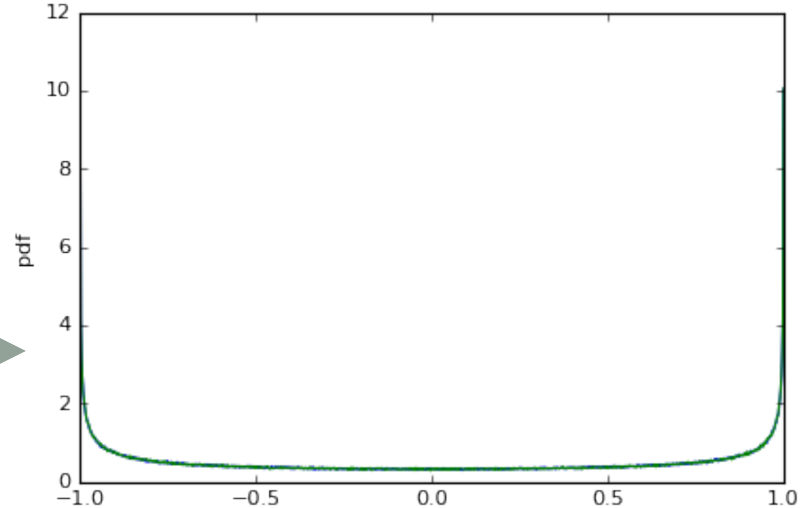


A positive Lyapunov exponent is a signature of chaos …

Properties of turbulence: unpredictability

- Hence the need for a statistical description of turbulence:

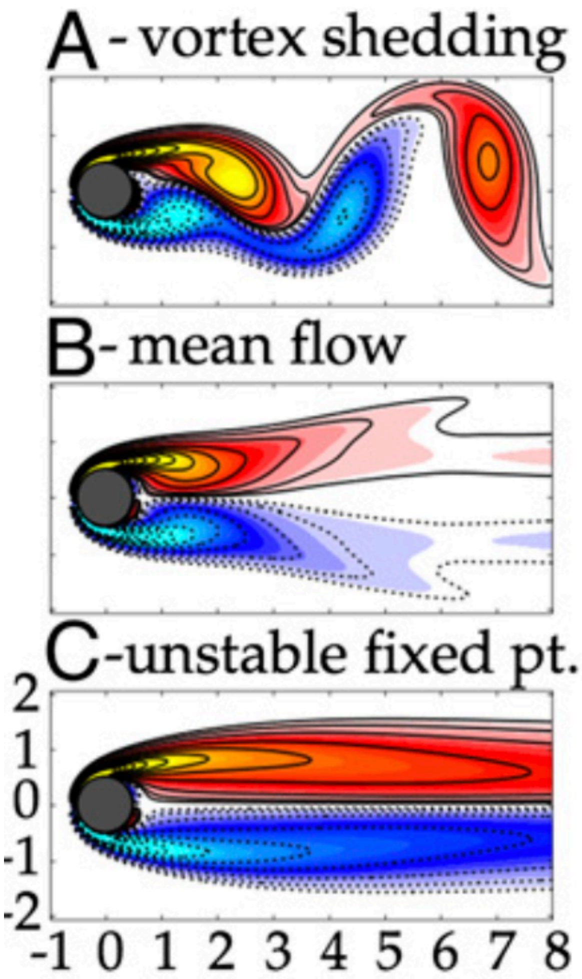




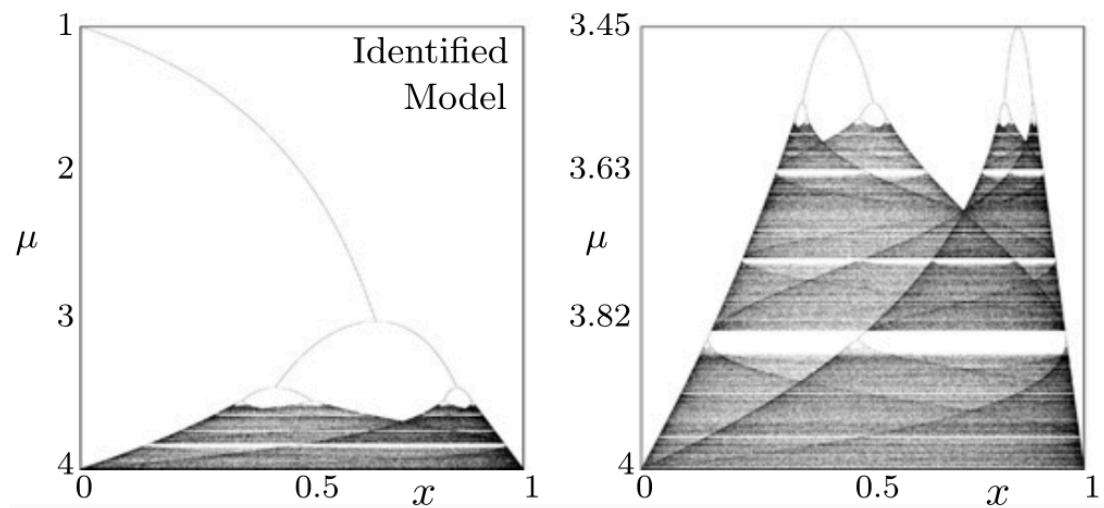
- statistical properties of the flow can be quite reproducible (pdf, mean, moments, etc.)
-

Properties of turbulence: unpredictability

- fluid vortex shedding behind an obstacle :



[Brunton et al, 2016]



-

The closure problem

- *In a turbulent flow it may be virtually impossible to predict the detailed motion of each eddy, the* ***statistical properties*** *— time averages for example — might not be changing and we might like to predict such averages.*
- We can decompose the variables into mean and fluctuating components: $$u = \overline{u} + u'$$

Where $\overline{u} = \frac{1}{T}\int_0^T u\, dt \text{ or } \overline{u} = \frac{1}{N}\sum_{N\to\infty} u$

With by definition $\overline{u'} = 0$

- And try to find a closed equation for the mean part

Activity 5

- Let's try with a simple equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + uu + ru = 0$$

- Write the evolution equation for $\overline{u}$.

The closure problem

- Let's try with a simple equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + uu + ru = 0$$

- We average and get:

$$\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} + \overline{uu} + r\overline{u} = 0$$ with $$\overline{uu} = \overline{u}\,\overline{u} + \boxed{\overline{u'u'}} \neq \overline{u}\,\overline{u}.$$

unknown

- We can go the the next order, but there are always new unknowns :

$$\frac{1}{2}\frac{\mathrm{d}\overline{u^2}}{\mathrm{d}t} + \boxed{\overline{uuu}} + r\overline{u^2} = 0$$

unknown

The closure problem

- So we need a method to find close the hierarchy an make some assumptions…
- For example: $$\overline{uuuu} = \alpha\overline{uu}\,\overline{uu} + \beta\overline{uuu}$$

The closure problem

- For the NS equation:

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j \frac{\partial u_i}{\partial x_j}} + f\mathbf{k} \times \overline{u_i} + \frac{\bar{\rho}}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x_j} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_i \partial x_j}$$

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + f\mathbf{k} \times \overline{u_i} + \frac{\bar{\rho}}{\rho_0} g\mathbf{k} = -\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x_j} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_i \partial x_j} \boxed{- \frac{\partial \overline{u_i' u_j'}}{\partial x_j}}$$

Advection for the averaged flow

Reynolds stress
= effect of subgrid-scale turbulence

The closure problem

For the NS equation:

- The number of unknowns is larger than the number of equations
- When we start deriving equations for unknowns, more variables appear.

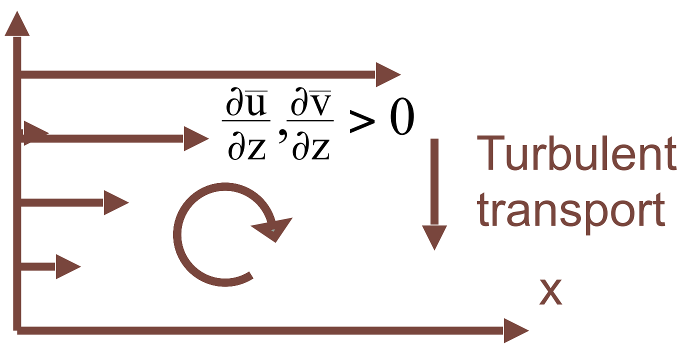
Progn. Eq. for:	Moment	Equation	No. Eqs.	No. Unknowns
$\overline{U_i}$	First	$\frac{\partial \overline{U_i}}{\partial t} = \ldots - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$	3	6
$\overline{u_i' u_j'}$	First	$\frac{\partial \overline{u_i' u_j'}}{\partial t} = \ldots - \frac{\partial \overline{u_k' u_i' u_j'}}{\partial x_k}$	6	10
$\overline{u_i' u_j' u_k'}$	First	$\frac{\partial \overline{u_i' u_j' u_k'}}{\partial t} = \ldots - \frac{\partial \overline{u_k' u_i' u_j' u_m'}}{\partial x_m}$	10	15

The closure problem

In PE models the equations are closed by parameterizing the Reynolds stresses as:

$$\overline{u'w'} = -K_{Mv}\frac{\partial u}{\partial z}$$

$$\overline{v'w'} = -K_{Mv}\frac{\partial v}{\partial z}$$



But nobody has been able to close the system, in any useful way, without introducing physical assumptions not directly deducible from the equations of motion themselves.

Triads interactions

- The nonlinear term in the equations of motion leads to interactions among different length scales
- *Let's write an equation for an incompressible 2 dimensional flow (for simplicity)*

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \zeta, \quad \zeta = \nabla^2 \psi.$$

With $u, v = -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}$ *and* $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Triads interactions

- *And work in Fourier space:*

$$\psi(x,y,t) = \sum_{\boldsymbol{k}} \tilde{\psi}(\boldsymbol{k},t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \zeta(x,y,t) = \sum_{\boldsymbol{k}} \tilde{\zeta}(\boldsymbol{k},t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $\boldsymbol{k} = \mathbf{i}k^x + \mathbf{j}k^y$, $\tilde{\zeta} = -k^2\tilde{\psi}$ where $k^2 = k^{x2} + k^{y2}$

Triads interactions

- The equation (without forcings and dissipation for now) is:

$$\frac{\partial}{\partial t}\sum_{\boldsymbol{k}}\tilde{\zeta}(\boldsymbol{k},t)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} = -\sum_{\boldsymbol{p}} p^x\tilde{\psi}(\boldsymbol{p},t)\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^y\tilde{\zeta}(\boldsymbol{q},t)\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} + \sum_{\boldsymbol{p}} p^y\tilde{\psi}(\boldsymbol{p},t)\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^x\tilde{\zeta}(\boldsymbol{q},t)\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}}.$$

- We multiply by $\exp(-i\boldsymbol{k}\cdot\boldsymbol{x})$

- And use the fact that the Fourier modes are orthogonal;

$$\int \mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}}\,\mathrm{d}A = \frac{1}{L^2}\delta(\boldsymbol{p}+\boldsymbol{q}).$$

Triads interactions

- And get:

$$\frac{\partial}{\partial t}\tilde{\psi}(\boldsymbol{k},t) = \sum_{\boldsymbol{p},\boldsymbol{q}} A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\tilde{\psi}(\boldsymbol{p},t)\tilde{\psi}(\boldsymbol{q},t) + \tilde{F}(\boldsymbol{k}) - \nu k^4\tilde{\psi}(\mathbf{k},t),$$

where $A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = (q^2/k^2)(p^x q^y - p^y q^x)\delta(\boldsymbol{p}+\boldsymbol{q}-\mathbf{k})$

- Only vectors with $\mathbf{p}+\mathbf{q}-\mathbf{k}=0$ have a non-zero contribution = **Triads interactions**

Triads interactions

- Triads interactions can be local or non-local:

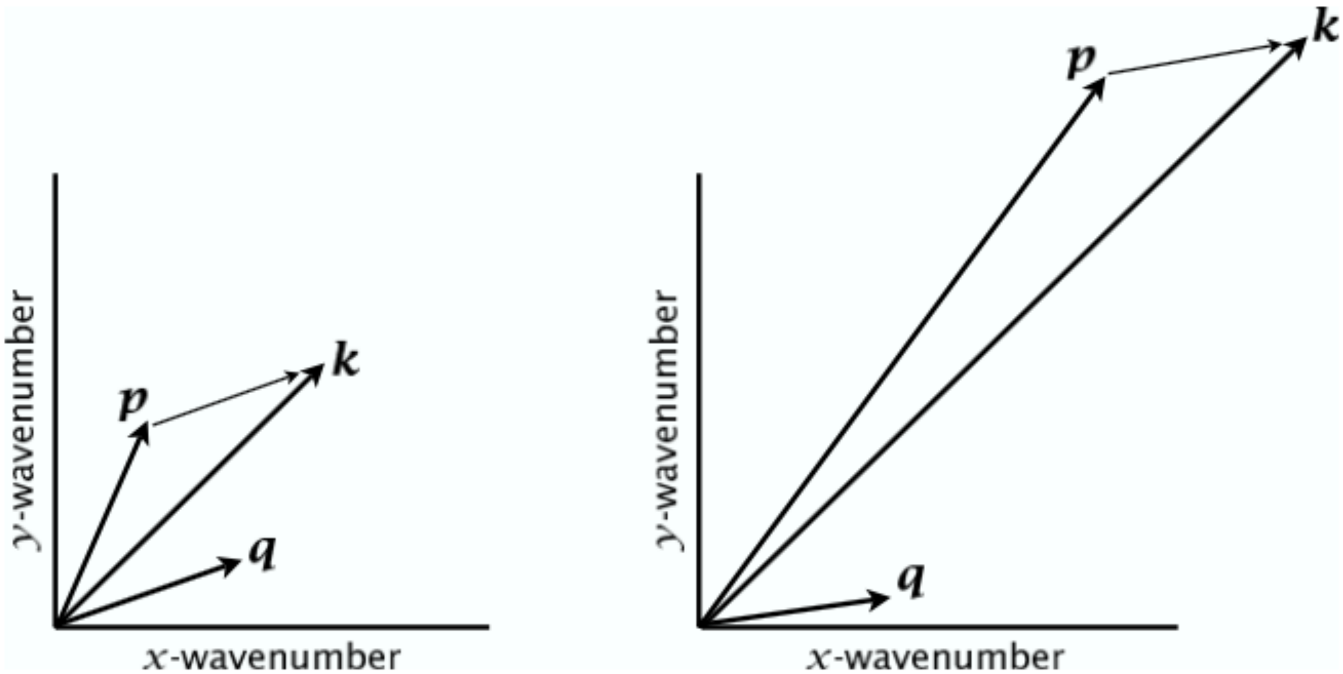


Fig. 8.1 Two interacting triads, each with $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$. On the left, a local triad with $k \sim p \sim q$. On the right, a nonlocal triad with $k \sim p \gg q$.

Triads interactions

- Starting with only 2 Fourier modes you can fill the entire spectrum due to scale interactions

$$\frac{\partial}{\partial t}\tilde{\psi}(\boldsymbol{k},t) = \sum_{\boldsymbol{p},\boldsymbol{q}} A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\tilde{\psi}(\boldsymbol{p},t)\tilde{\psi}(\boldsymbol{q},t) + \tilde{F}(\boldsymbol{k}) - \nu k^4\tilde{\psi}(\mathbf{k},t),$$

- Forcing is fixed at a certain scale(s) and dissipation acts on each Fourier mode with a coefficient that increases with wavenumber and therefore that preferentially affects small scales.