Internal Waves in the Ocean

Master 2 — Physique de l’Océan et du Climat

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Outline

1. A general introduction to ocean waves
2. What are internal waves ? Why do we study internal waves ?
3. Internal waves in the two-layer shallow-water model
4. **Internal waves in the continuously-stratified model**
5. Generation of internal waves
6. Propagation of internal waves
7. Dissipation of internal waves and impacts

Local Static Stability

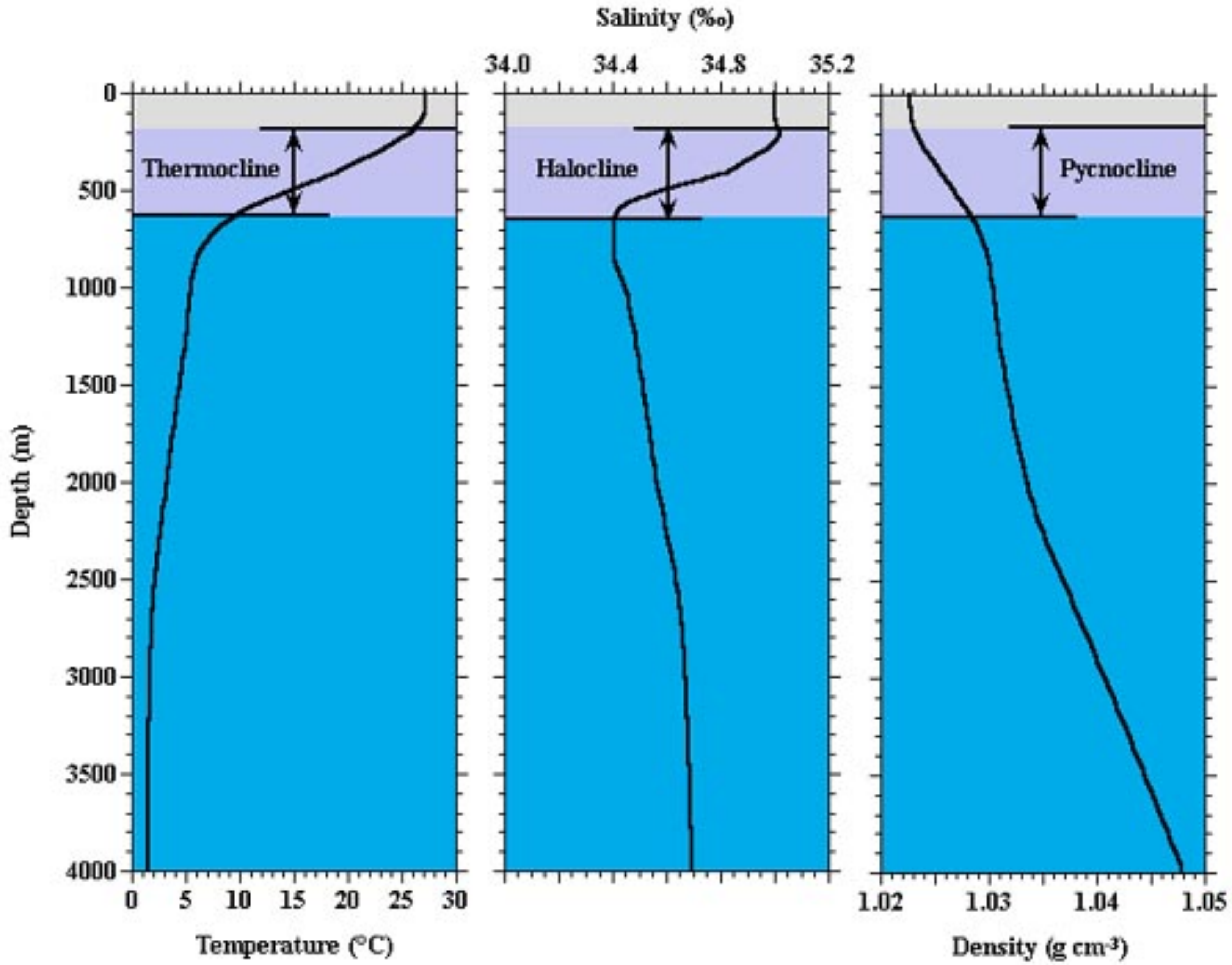
- Stratification:

$$\rho = \rho_0(z)$$

- A fluid is
  - **stably stratified** if a displaced parcel tends to return to its original position,
  - **unstably stratified** if it tends to move further away from its original position
  - **neutrally stratified** if it tends to stay where it is.

Local Static Stability

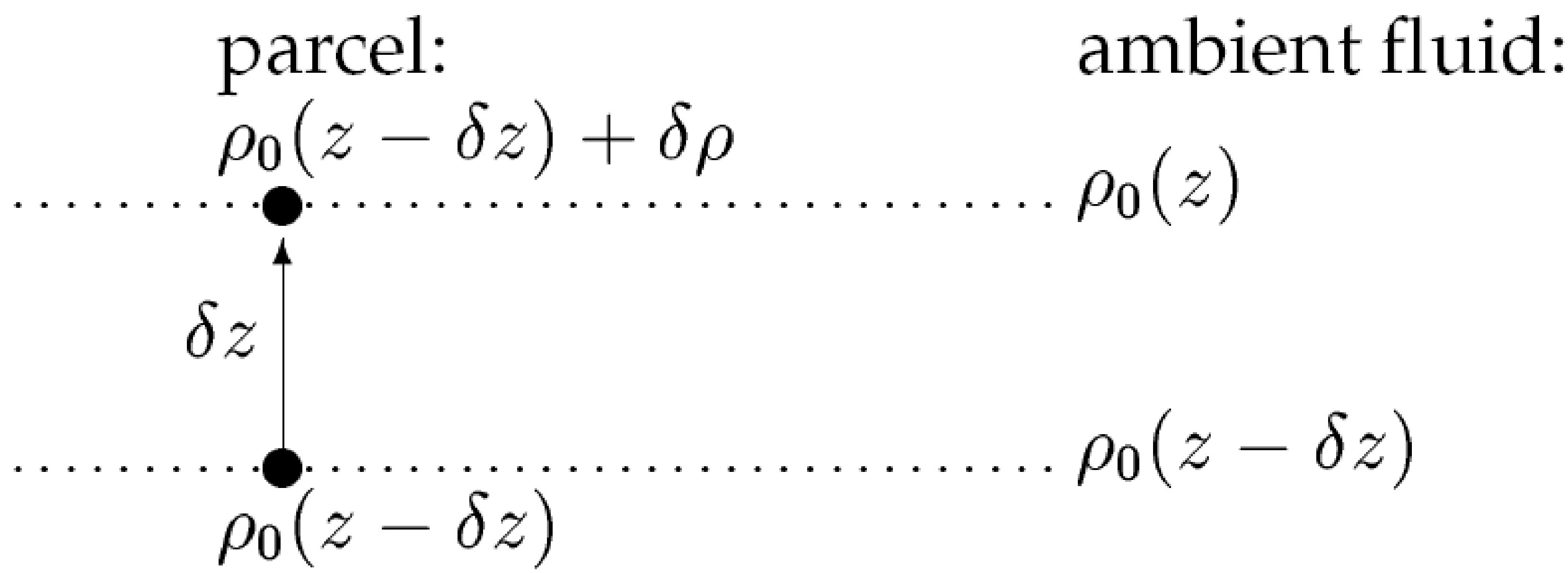
Stratification of the ocean:



with Equation of state : $\rho = \rho(T, S, p)$

Local Static Stability

- Let’s move a parcel:



- Buoyancy force:

$$\rho_0(z)\ddot{\delta z} = g\left(\rho_0(z) - \rho_0(z-\delta z) - \delta\rho\right)$$

Local Static Stability

- With background density variation:

$$\rho_0(z) - \rho_0(z - \delta z) = \frac{d\rho_0}{dz}\delta z$$

- And parcel density variation?
  - *From thermodynamics, if entropy and salinity are conserved during displacement:*

$$\delta\rho = \left(\frac{\partial \rho}{\partial p}\right)_{\eta,S} \delta p = c_s^{-2}\delta p$$

Where c_s is the speed of sound

Local Static Stability

- So we get:

$$\rho_0(z)\ddot{\delta z} = g\left(\frac{d\rho_0}{dz}\delta z + \frac{\rho_0 g \delta z}{c_s^2}\right)$$

Local Static Stability

- Simple Harmonic oscillator:

$$\ddot{\delta z} \underbrace{- \frac{g}{\rho_0} \left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2} \right)}_{N^2} \delta z = 0$$

- Brunt-Vaisala frequency: N^2

- Solutions: $e^{\pm \mathrm{i} N t}$

$$\ddot{\delta z} + N^2 \delta z = 0$$

- Stable if $N^2 > 0$

Local Static Stability

- Simple Harmonic oscillator:

$$\ddot{\delta z} + N^2 \delta z = 0$$

- Solutions: $e^{\pm \mathrm{i} N t}$

- Stable if $N^2 > 0$

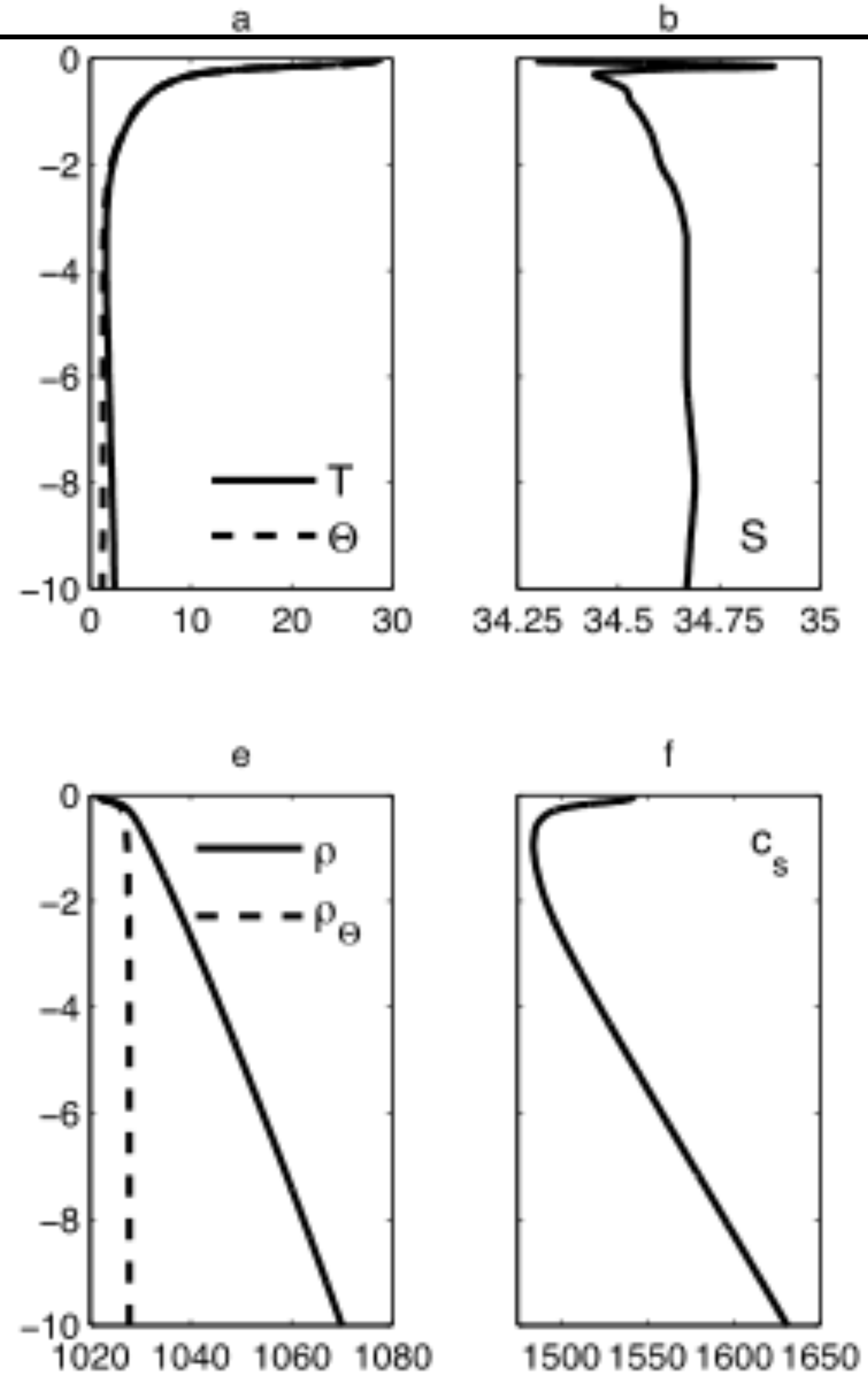
The parcel oscillates vertically at frequency *N* about its equilibrium position.

Local Static Stability

- Brunt-Vaisala frequency:

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)$$

The effect of compressibility is often neglected in the upper ocean but it is not true in general.



Vertical profiles of T, S, in-situ and potential density, and speed of sound in the Mindanao Trench.

Local Static Stability

- How do you connect density and stability?

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)$$

- It is convenient to define **the generalized potential density,** corresponding to the density that the parcel would attain if moved from z to a reference level z_r, under conservation of its entropy and salinity.

- We compute it by vertically integrating:

$$\delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_{\eta,S} \delta p = c_s^{-2}\delta p$$

Local Static Stability

- Which gives the generalized potential density:

$$\rho_r(z_r, z) = \rho_0(z) + g \int_{z_r}^{z} \frac{\rho_0(z')}{c_s^2(z, z')} dz'$$

- Such that:

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho_r}{dz}(z_r, z) + \frac{g^2}{\rho_0} \int_{z_r}^{z} \rho_0(z') \frac{\partial c_s^{-2}}{\partial z}(z, z') dz'$$

- and for $z_r = z$:

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho_r}{dz}(z, z)$$

Local Static Stability

- In practice we use potential density, where z_r is fixed. For example at the surface $(z_r = 0)$:

$$\rho_\Theta(z) = \rho_0(z) + g\int_0^z \frac{\rho_0(z')}{c_s^2(z,z')}dz'$$

Which is the density that the parcel would acquire if adiabatically brought to the surface.

- Such that:

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_\Theta}{dz} + \frac{g^2}{\rho_0}\int_0^z \rho_0(z')\frac{\partial c_s^{-2}}{\partial z}(z,z')dz'$$

Equations for a stratified flow

- Navier-Stokes Equations:

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}} + \vec{\mathcal{D}}$$

Momentum equations

$$\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation (no source/sink)

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2} \frac{DP}{Dt}$$

Thermodynamic ‘energy’ equation
(no diabatic effects)

Equations for a stratified flow

- Approximations for the momentum equation:

No forcings/dissipation

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \cancel{\vec{\mathcal{F}} + \vec{D}}$$

Equations for a stratified flow

- Boussinesq Approximation:

Density perturbations small compared to mean background value:

$$\rho - \rho^* << \rho^*$$

Linearize all terms involving a product with density, except the gravity term which is already linear:

$$\rho\vec{u} \to \rho^*\vec{u}$$

$$\rho g \to \rho g$$

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- Traditional Approximation:

= neglect horizontal Coriolis term

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

$$\frac{D\vec{u}}{Dt} + f\vec{k} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- We think as internal waves as a perturbation of a (known) static background state that has only vertical dependences

$$P = p_0(z) + p'(t, \vec{x})$$

$$\rho = \rho_0(z) + \rho'(t, \vec{x})$$

- And linearize momentum equations:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} + \frac{\rho'}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Equations for a stratified flow

- For the thermodynamic equation:

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{DP}{Dt}$$

- We can write:

$$\frac{D\rho}{Dt} = \frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z}$$

- And:

$$\frac{1}{c_s^2}\frac{DP}{Dt} = \frac{1}{c_s^2}\frac{Dp'}{Dt} + \frac{1}{c_s^2}w\frac{\partial p_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

- So :

$$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

Equations for a stratified flow

- We linearize:

$$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

$$\Downarrow$$

$$\frac{\partial\rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

-

Equations for a stratified flow

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

- We can show that the pressure time tendencies are small using the following scalings and relations:

$$\boxed{U \ll C \ll c_{sf} \ll c_s\,.}$$

$$U \sim O(10^{-1})\,; \quad C \sim O(1)\,; \quad c_{sf} \sim O(10^1\,, 10^2)\,; \quad c_s \sim O(10^3)\,\mathrm{m\,s^{-1}}\,.$$

- Which denote the particle velocity and phase speed of internal waves, the phase speed of surface waves, and the speed of sound in seawater, respectively.

Equations for a stratified flow

- We get: $$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = -w\frac{\rho_0 g}{c_s^2}$$
- So we can write: $$\frac{\partial \rho'}{\partial t} + \left[\frac{\partial \rho_0}{\partial z} + \frac{\rho_0 g}{c_s^2}\right] w = 0$$
- Which gives $$-\frac{g}{\rho^*}\frac{\partial \rho'}{\partial t} + N^2 w = 0$$
- With $$N^2 = -\frac{g}{\rho^*}\left(\frac{\partial \rho_0}{\partial z} + \frac{\rho_0 g}{c_s^2}\right)$$

Equations for a stratified flow

- And finally introducing buoyancy: $$b = -g\frac{\rho'}{\rho^*}$$
-

$$\frac{\partial b}{\partial t} + N^2 w = 0$$

Equations for a stratified flow

- For the mass equation: $$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0$$
- We rewrite:
- $$\rho\nabla\cdot\vec{u} = -\frac{D\rho}{Dt} = -\frac{1}{c_s^2}\frac{Dp'}{Dt} + w\frac{\rho_0 g}{c_s^2}$$

Equations for a stratified flow

- We look at the scales of the different terms:

$$\overbrace{\rho\frac{\partial u}{\partial x}}^{\rho_* U/L} + \overbrace{\rho\frac{\partial v}{\partial y}}^{\rho_* U/L} + \overbrace{\rho\frac{\partial w}{\partial z}}^{\rho_* W/H} = -\overbrace{\frac{1}{c_s^2}\frac{\partial p'}{\partial t}}^{P/(Tc_s^2)} - \overbrace{\frac{u}{c_s^2}\frac{\partial p'}{\partial x}}^{UP/(Lc_s^2)} - \overbrace{\frac{v}{c_s^2}\frac{\partial p'}{\partial y}}^{UP/(Lc_s^2)} - \overbrace{\frac{w}{c_s^2}\frac{\partial p'}{\partial z}}^{WP/(Hc_s^2)} + \overbrace{w\frac{\rho_0 g}{c_s^2}}^{\rho_* Wg/c_s^2}$$

- We assume again a separation in time scales:

$$\boxed{U \ll C \ll c_{sf} \ll c_s\,.}$$

$$U \sim O(10^{-1})\,; \quad C \sim O(1)\,; \quad c_{sf} \sim O(10^1\,,10^2)\,; \quad c_s \sim O(10^3)\,\mathrm{m\,s^{-1}}\,.$$

- If we remove small terms we simply get: $\vec{\nabla} \cdot \vec{u} = 0$

Equations for a stratified flow

- Linearized Equations are:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} - b\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation (no source/sink)

$$\frac{\partial b}{\partial t} + N^2 w = 0$$

Thermodynamic ‘energy’ equation
(no diabatic effects)

4. The continuously-stratified model

We have a set of 5 equations for 5 variables:

$$u_t - fv = -\frac{p_x}{\rho^*} \quad (1)$$

$$v_t + fu = -\frac{p_y}{\rho^*} \quad (2)$$

$$w_t - b = -\frac{p_z}{\rho^*} \quad (3)$$

$$u_x + v_y + w_z = 0 \quad (4)$$

$$b_t + N^2 w = 0 \quad (5)$$

- Activity:

- Write an equation for w alone.

Equations for a stratified flow

- Finally we get an equation for *w* alone:

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0$$

Solutions of the equation

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0$$

- We suppose waves to be sinusoidal in time:

$$w = \hat{w} e^{-\mathrm{i}\omega t}$$

- And get:

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

Solutions of the equation

Two methods to solve the equation:

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

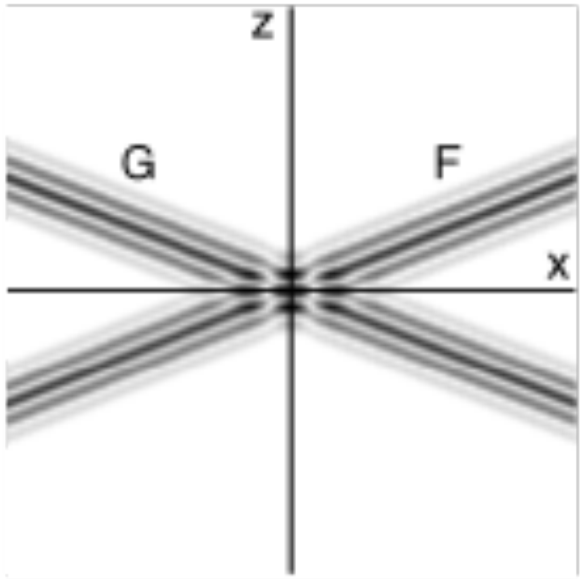
1. The method of characteristics
2. The method of modes

Solutions of the equation

1. Method of characteristics

- N = cste
- Arbitrary boundary conditions
- Solution of the form:

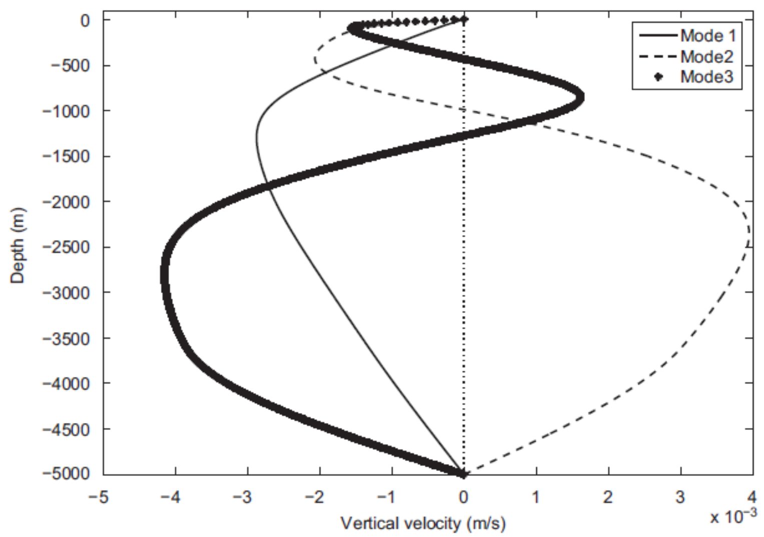
$$\hat{w} = F(\mu_+ x - z) + G(\mu_- x - z)$$



2. Method of Modes

- N = N(z)
- Horizontal surface/bottom
- Solution of the form:

$$w = W(z)e^{-\mathrm{i}\omega t+\mathrm{i}kx+\mathrm{i}ly}$$



4. The continuously-stratified model

Method of characteristics

Assumptions:

- stratification is constant, $N = cst$
- the domain is infinite, that is, there is no boundary effect
- waves are sinusoidal in time and propagate in the $x - z$ plane (no variation in the y direction): $w(x, y, z, t) = \hat{w}(x, z)\exp(i\omega t)$

The equation for w becomes: $$\boxed{\hat{w}_{xx} - \frac{\omega^2 - f^2}{N^2 - \omega^2}\hat{w}_{zz} = 0} \quad (10)$$

The method of characteristics consists in changing the variables x, z into ξ_+, ξ_- :

$$\xi_\pm = \mu_\pm x - z \text{ with } \mu_\pm = \pm\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}.$$

NB: we assume that $f < \omega < N$ so that $\mu_\pm$ exists.

Method of characteristics

Now we use $\hat{w}(x,z) = \bar{w}(\xi_+, \xi_-)$ with the properties:

$$\frac{\partial \hat{w}}{\partial x} = \frac{\partial \bar{w}}{\partial \xi_+}\frac{\partial \xi_+}{\partial x} + \frac{\partial \bar{w}}{\partial \xi_-}\frac{\partial \xi_-}{\partial x}\text{ , with }\frac{\partial \xi_+}{\partial x} = \mu_+ \text{ and } \frac{\partial \xi_-}{\partial x} = \mu_- \text{ .}$$

After some reorganising,

$$\frac{\partial^2 \hat{w}}{\partial x^2} = \mu_+^2\left(\frac{\partial^2 \bar{w}}{\partial \xi_+^2} + \frac{\partial^2 \bar{w}}{\partial \xi_-^2} - 2\frac{\partial^2 \bar{w}}{\partial \xi_+ \partial \xi_-}\right) \text{ and}$$

$$\frac{\partial^2 \hat{w}}{\partial z^2} = \frac{\partial^2 \bar{w}}{\partial \xi_+^2} + \frac{\partial^2 \bar{w}}{\partial \xi_-^2} + 2\frac{\partial^2 \bar{w}}{\partial \xi_+ \partial \xi_-}.$$

Method of characteristics

The equation for $\hat{w}$ reduces to:

$$\hat{w}_{xx} - \mu_+^2 \hat{w}_{zz} = \ldots = -4\mu_+^2 \boxed{\frac{\partial^2 \bar{w}}{\partial \xi_+ \partial \xi_-} = 0}.$$ That is the new PDE to solve.

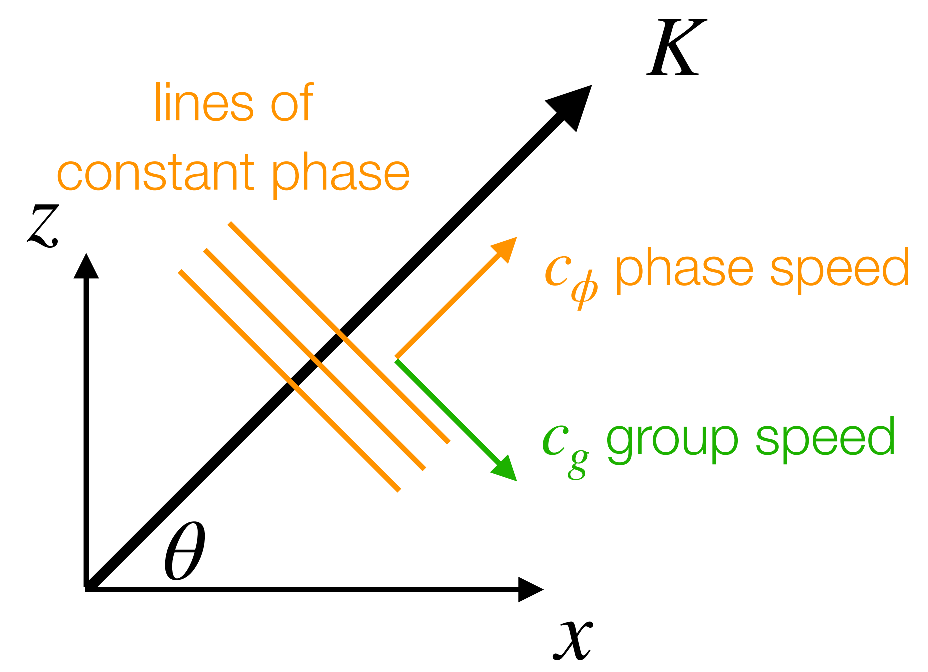
Any function of the form $\bar{w}(\xi_+, \xi_-) = F(\xi_+) + G(\xi_-)$ is a solution of the PDE, with F and G arbitrary functions.

The general solution is thus $\boxed{\hat{w}(x, z) = F(\mu_+ x - z) + G(\mu_- x - z)}$.

Method of characteristics

To find the dispersion relation, we set $w(x, y, z, t) = \hat{w}(x, y, z)\exp(i\omega t)$ with $\hat{w} = w_0 \exp(i(kx + mz))$.

From Equation (10), $\boxed{\omega^2 = \dfrac{N^2k^2 + f^2m^2}{k^2 + m^2}}$.



Using polar coordinates, the wave vector is
$\overrightarrow{K} = (k, m) = K(\cos\theta, \sin\theta)$,

we have $\boxed{\omega^2 = N^2\cos^2\theta + f^2\sin^2\theta}$

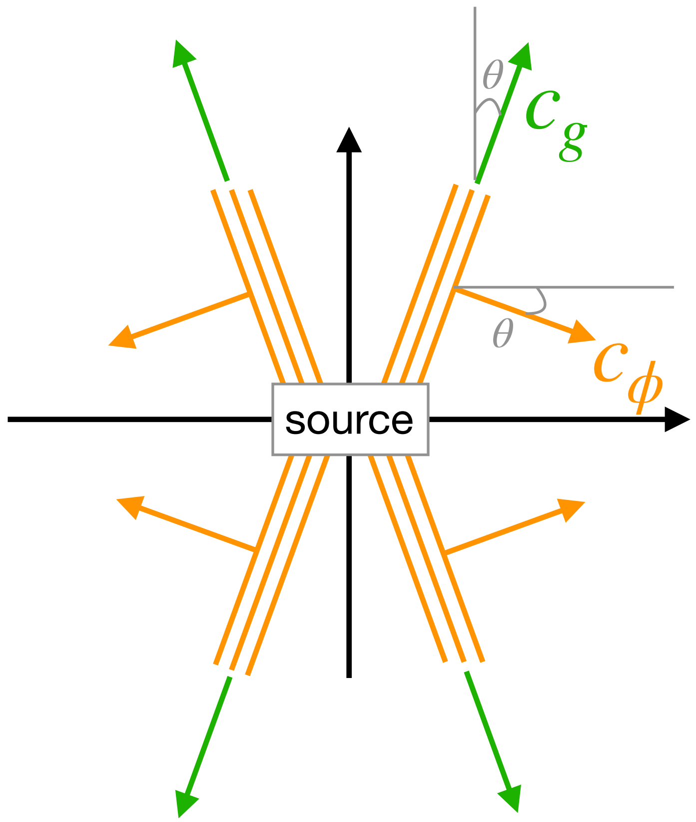
→ The wave frequency depends only on the stratification, the rotation and the direction of the wave. Conversely, the direction of the wave can be inferred from the stratification, rotation and the frequency of the wave.

Method of characteristics

The two extreme cases are:

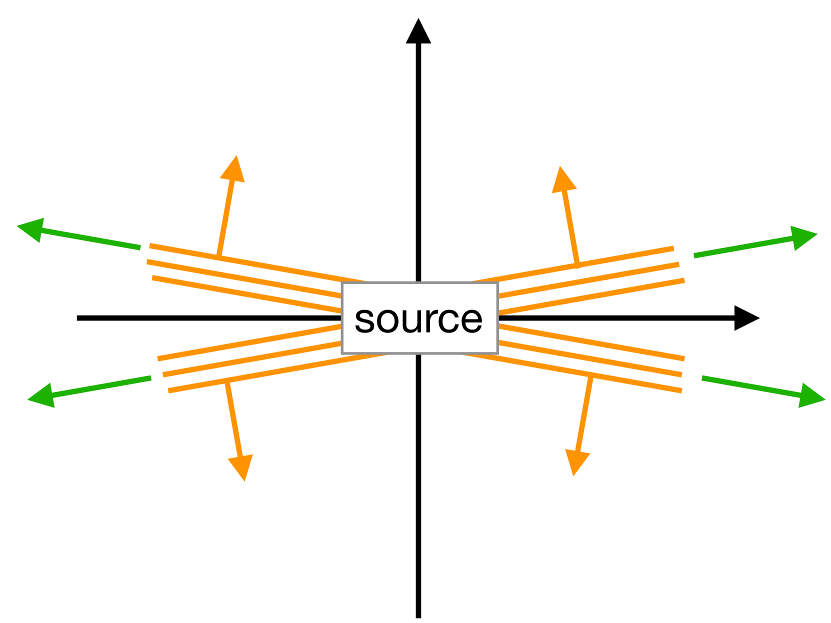
- $\omega \rightarrow N \Leftrightarrow \theta \rightarrow 0$

  rapidly oscillating waves propagate almost vertically

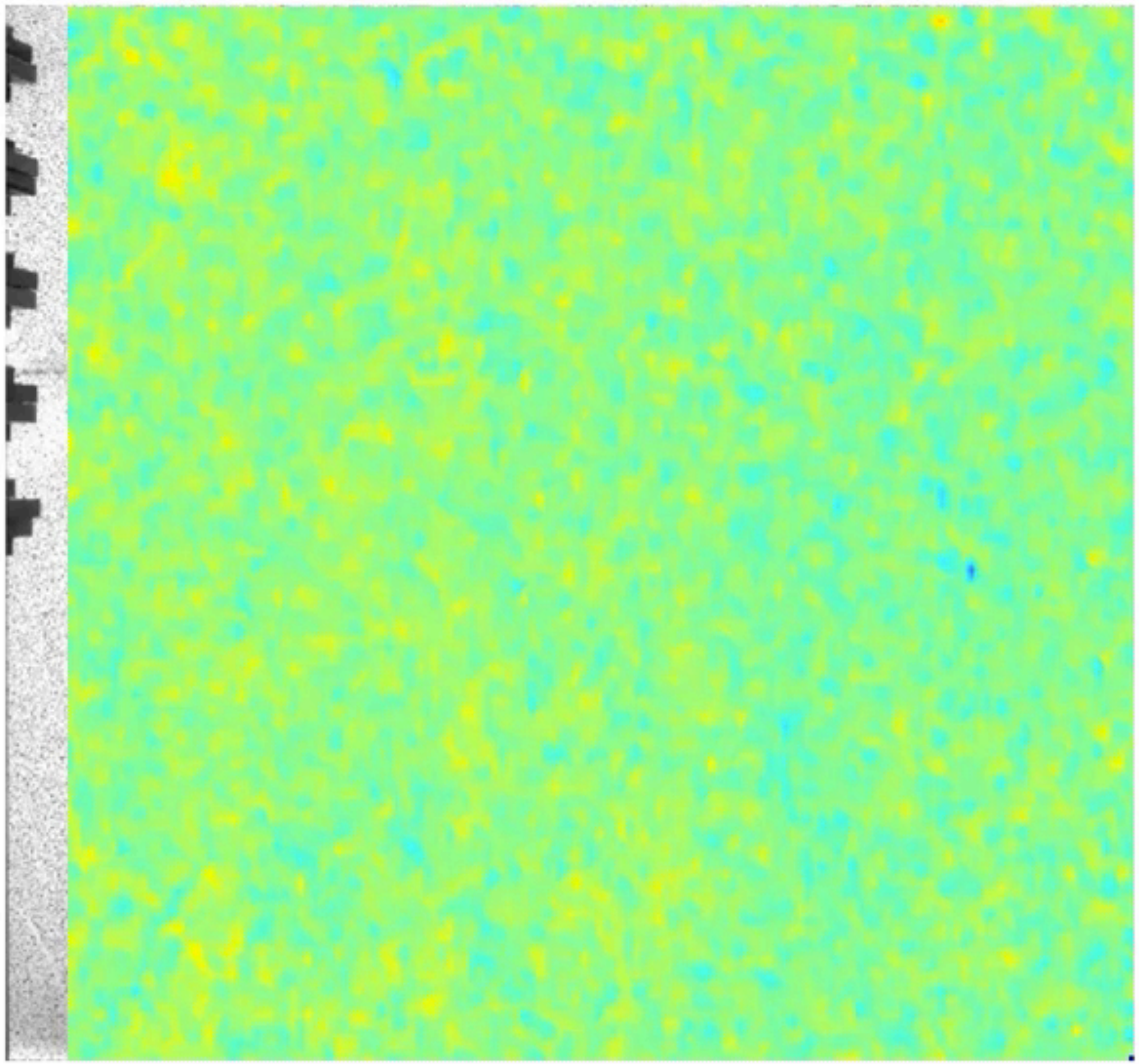


- $\omega \rightarrow f \Leftrightarrow \theta \rightarrow \dfrac{\pi}{2}$

  "near-inertial" waves propagate almost horizontally

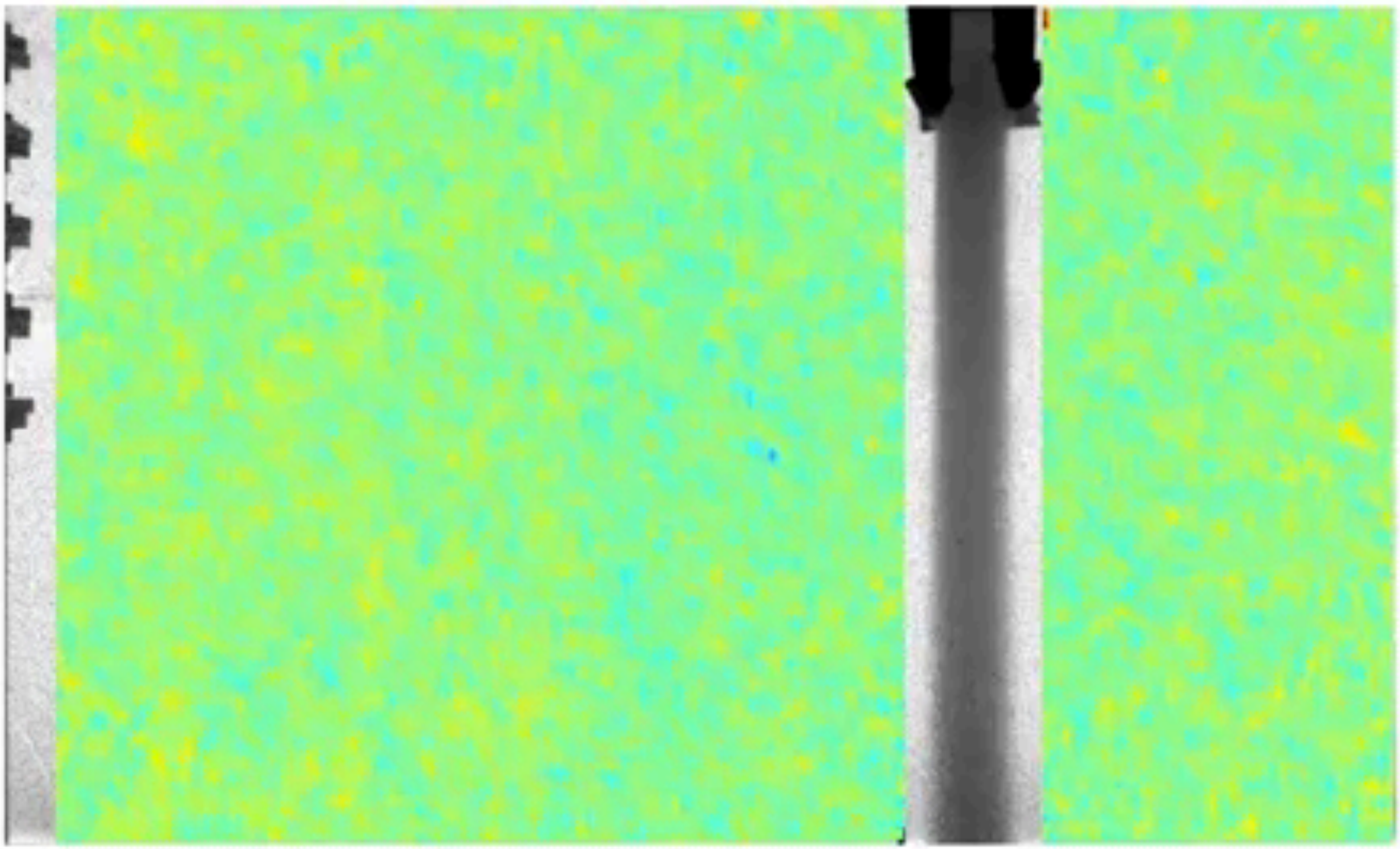


4. The continuously-stratified model



[(c) E. Horne]

4. The continuously-stratified model

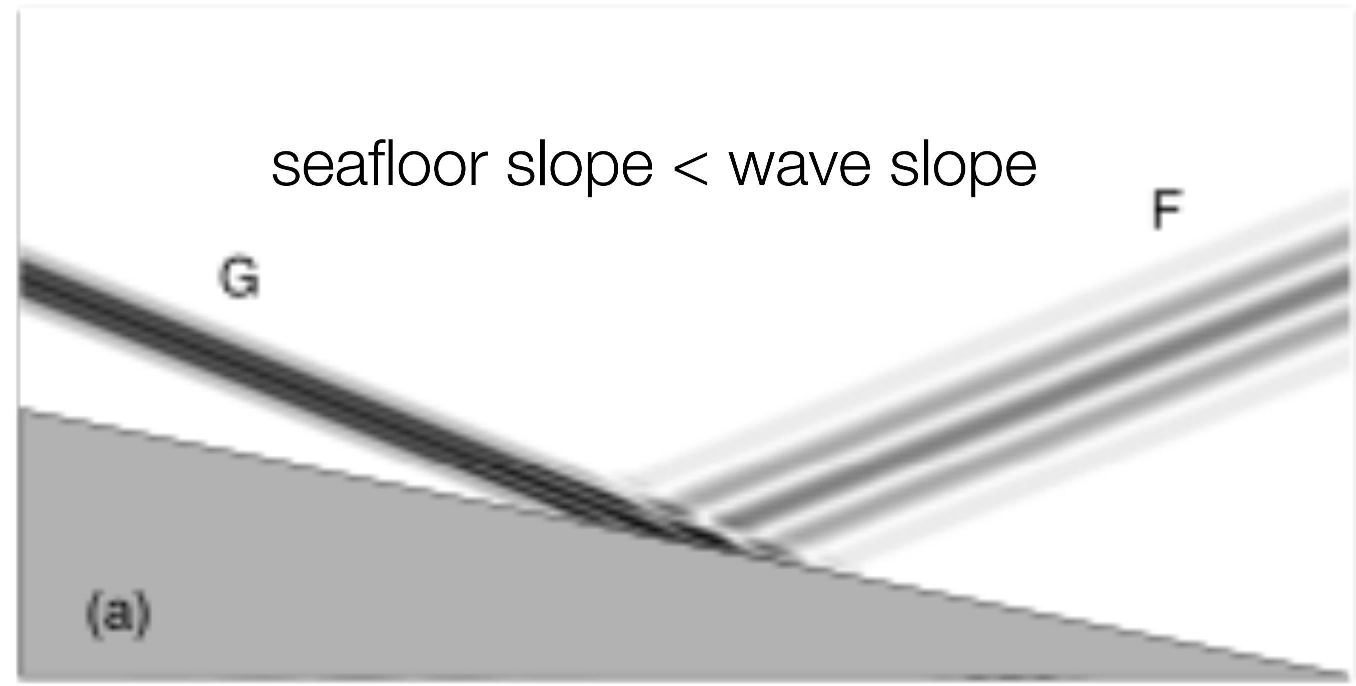


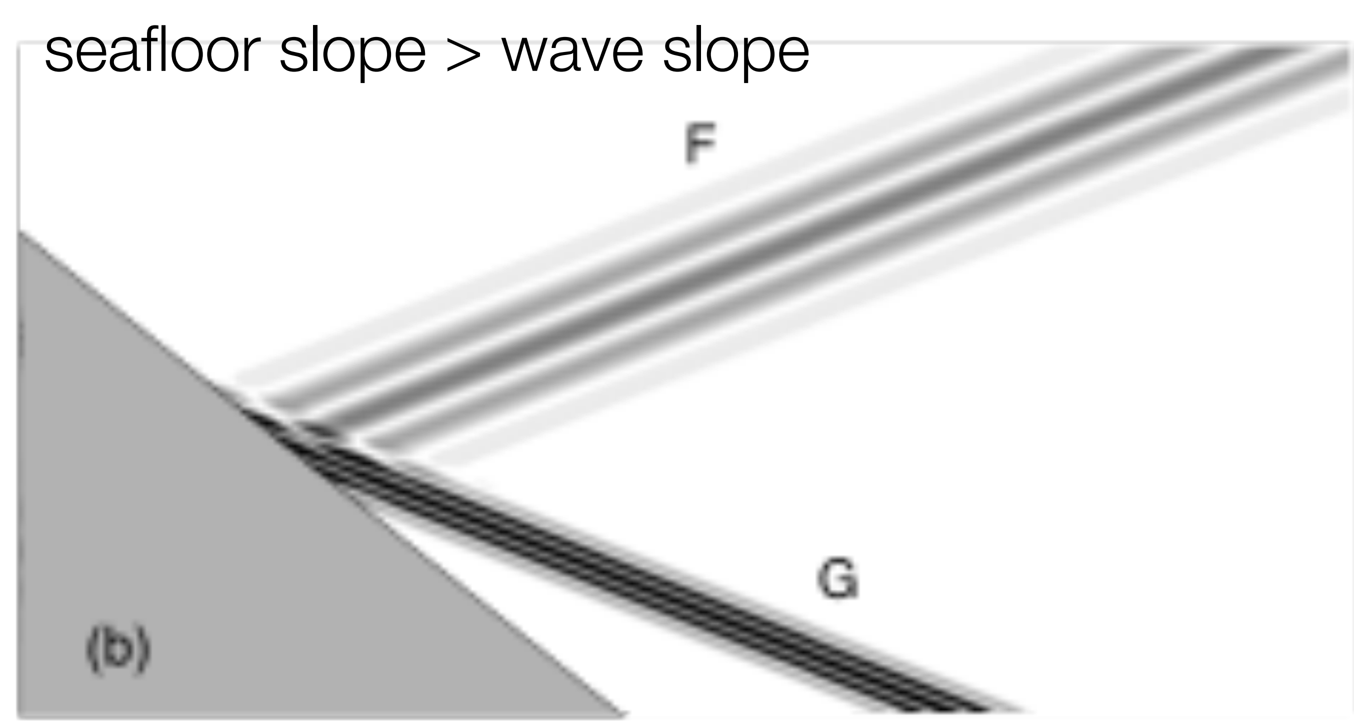
[(c) E. Horne]

4. The continuously-stratified model

Method of characteristics

When a wave impinges on a seafloor slope, its frequency ω is conserved, hence its propagating slope is conserved.



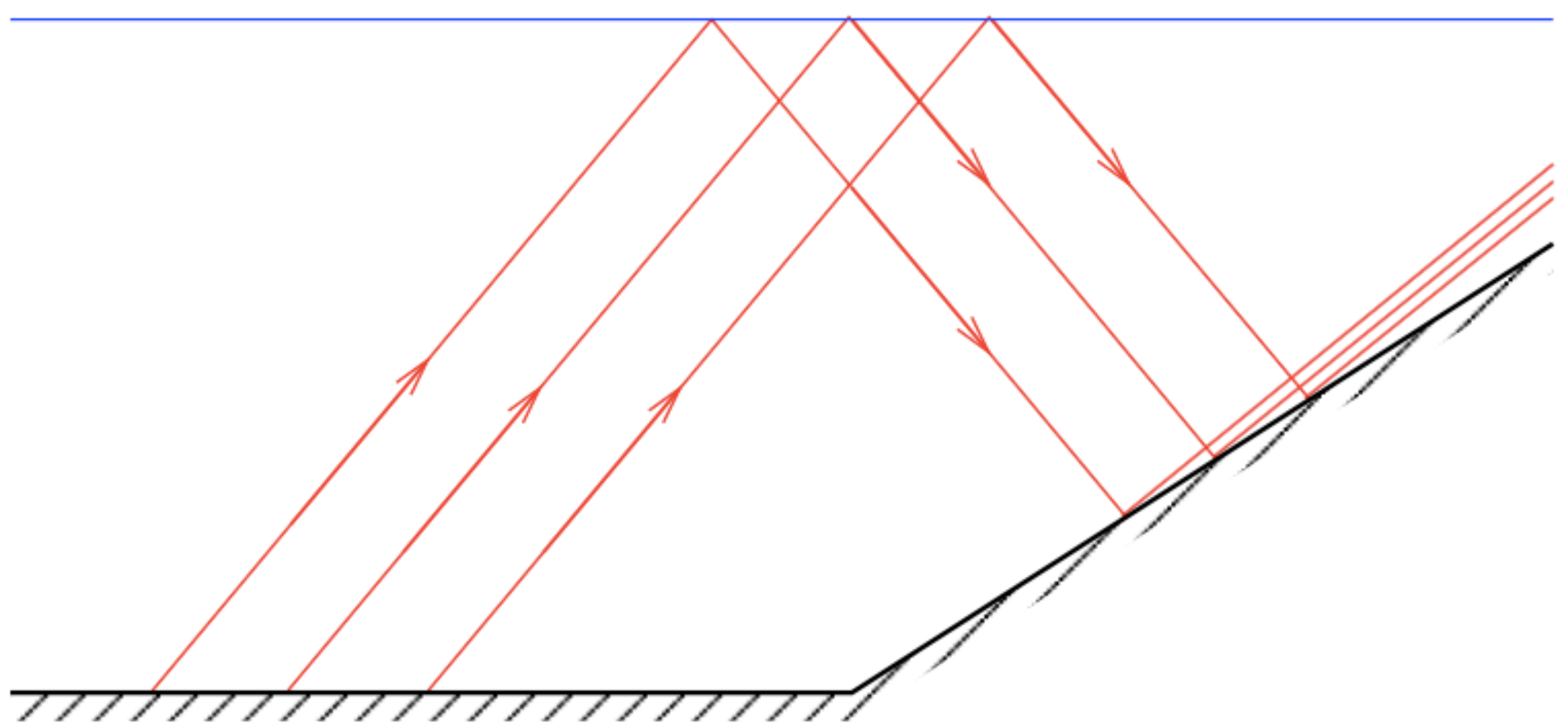


4. The continuously-stratified model

Method of characteristics

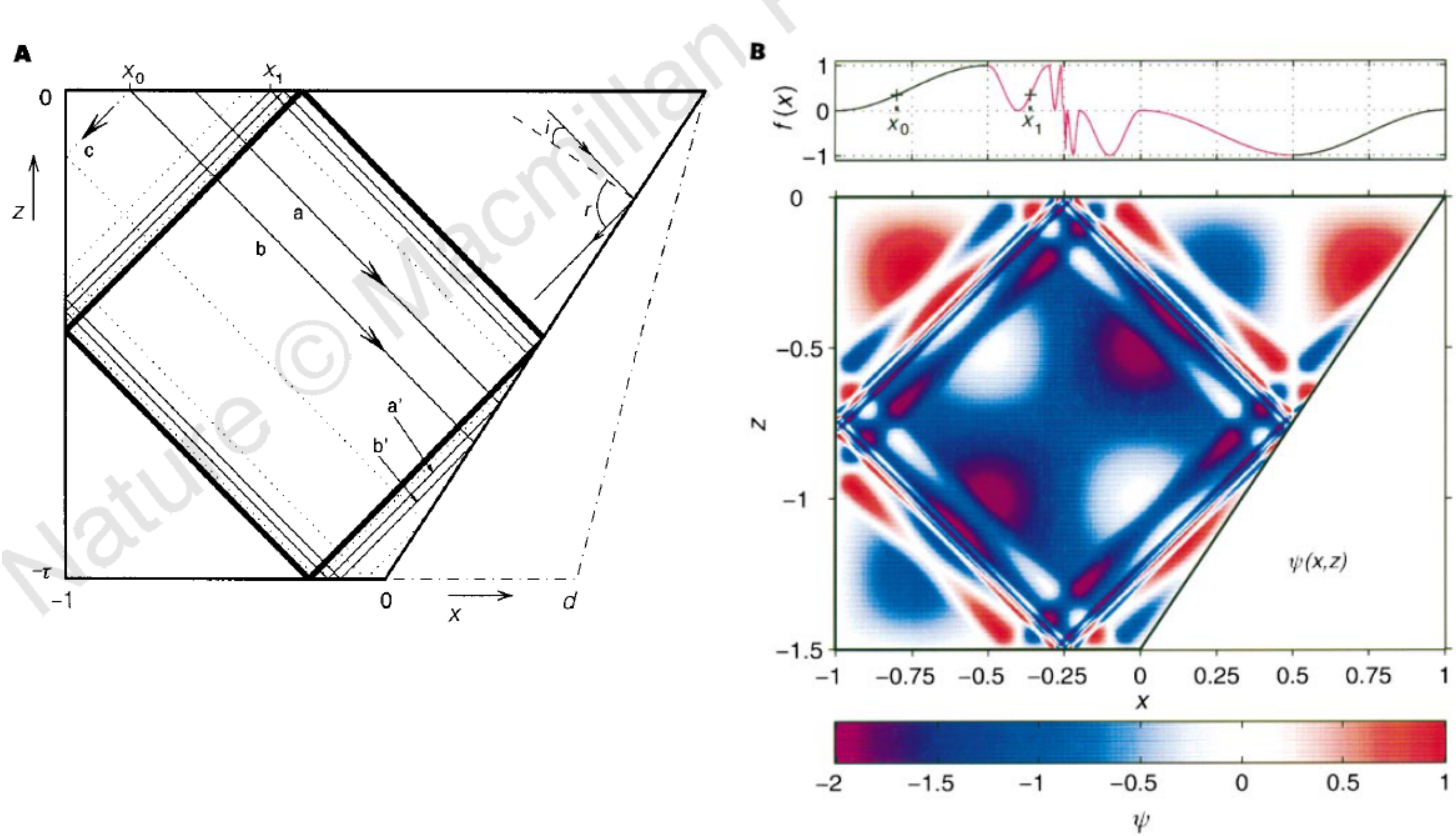
When a wave impinges on a seafloor slope, its frequency ω is conserved, hence its propagating slope is conserved.

Energy can focus in narrow bands.



Method of characteristics

Specific geometries can work as wave attractors.



Maas et al. 1997

4. The continuously-stratified model

Method of characteristics

Specific geometries can work as wave attractors.

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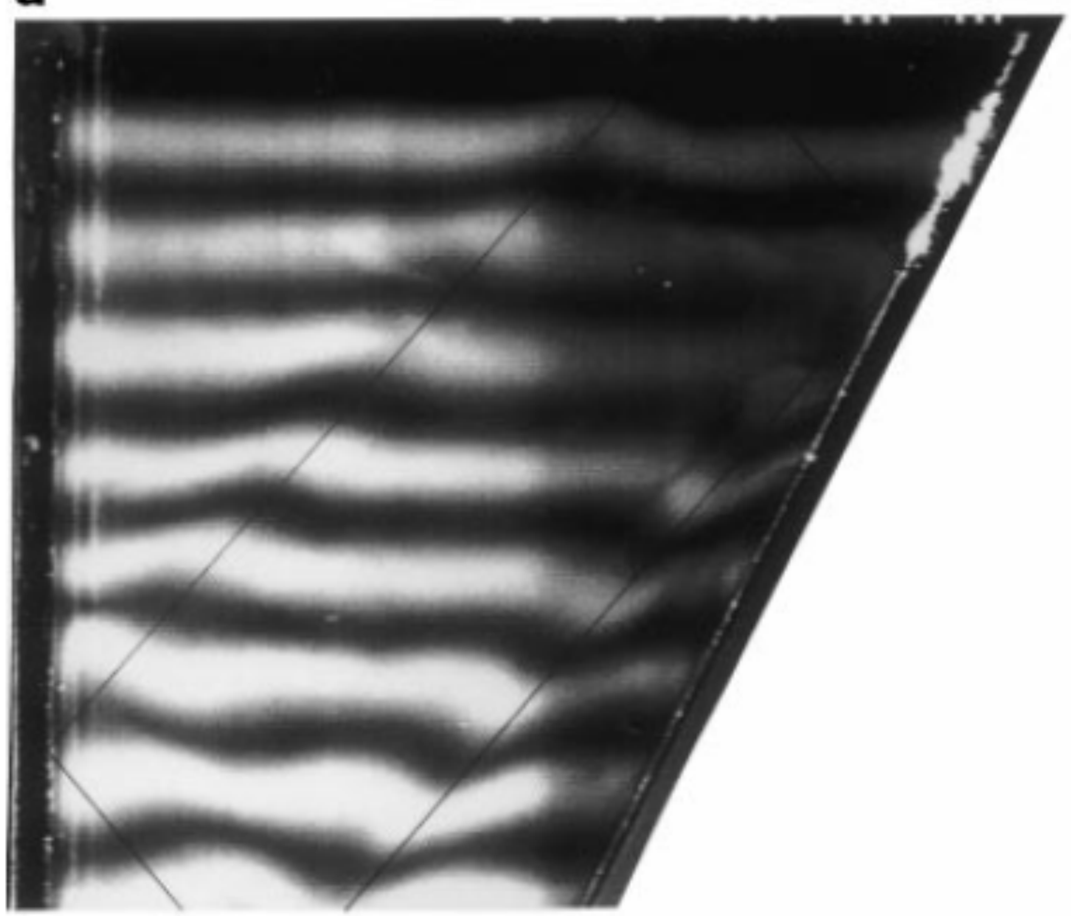

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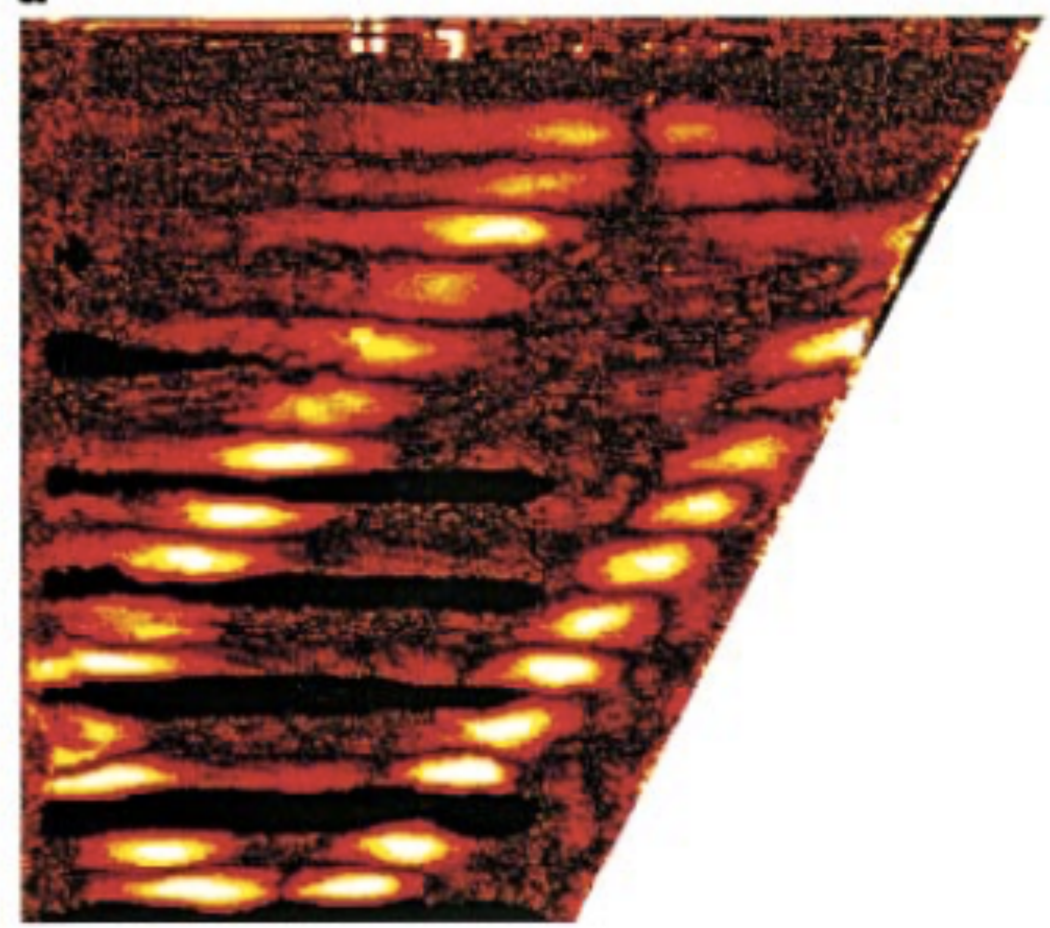


Maas et al. 1997

Method of characteristics

Main limit of the method of characteristics: no top or bottom boundary condition is specified. The ocean is supposed to be “infinite” ! What happens with a finite-depth ocean ? → **method of modes**

4. The continuously-stratified model

Method of modes

Assumptions:

- $N = N(z)$
- flat sea surface height (*rigid lid*) and flat seafloor

We start back from equation (9): $(\nabla^2 w)_{tt} + f^2 w_{zz} + N^2 \nabla_h^2 w = 0$,

and assume a solution in the form: $w = W(z)\exp(i(\omega t - kx))$, that is, the vertical structure is decoupled from the horizontal structure (+ no variation in y).

The equation for w becomes $W_{zz} + k^2 \dfrac{N^2 - \omega^2}{\omega^2 - f^2} W = 0$, which can be written as:

$$\boxed{W_{zz} + m^2 W = 0} \quad \text{with} \quad \boxed{m^2(z) = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}}$$

Method of modes

If $N(z) = cst$, the solution can be written: $W(z) = A\cos(mz) + B\sin(mz)$.

The top and bottom boundary conditions are: $w(0) = w(-H) = 0$.
They give: $m = \pm\, n\dfrac{\pi}{H}$, which translates into the dispersion relation:

$$\boxed{k_n = \pm\frac{n\pi}{H}\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}} \quad \text{or} \quad \boxed{\omega^2 = \frac{N^2 k_n^2 + \frac{n^2\pi^2}{H^2} f^2}{k_n^2 + \frac{n^2\pi^2}{H^2}}}$$

We recover the dispersion relation from the method of characteristics.

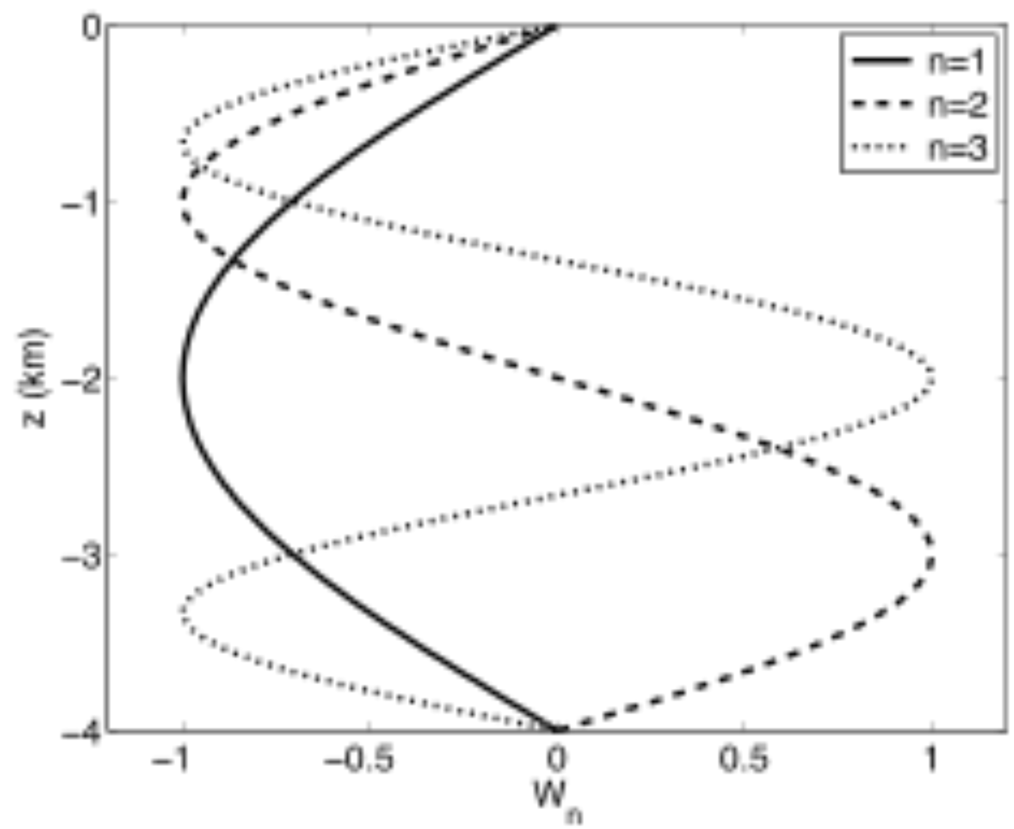
4. The continuously-stratified model

Method of modes

The general solution for w is the superposition of modes:

$$w(x,z,t) = \sum_n W_n(z) a_n \cos(k_n x - \omega t).$$

If $N = cst$, $w(x,z,t) = \sum_n a_n \sin\left(\frac{n\pi z}{H}\right) \cos(k_n x - \omega t).$



Gerkema & Zimmerman (2008)

Method of modes

From the initial set of equations, we get the vertical structure of u, v, p, b:

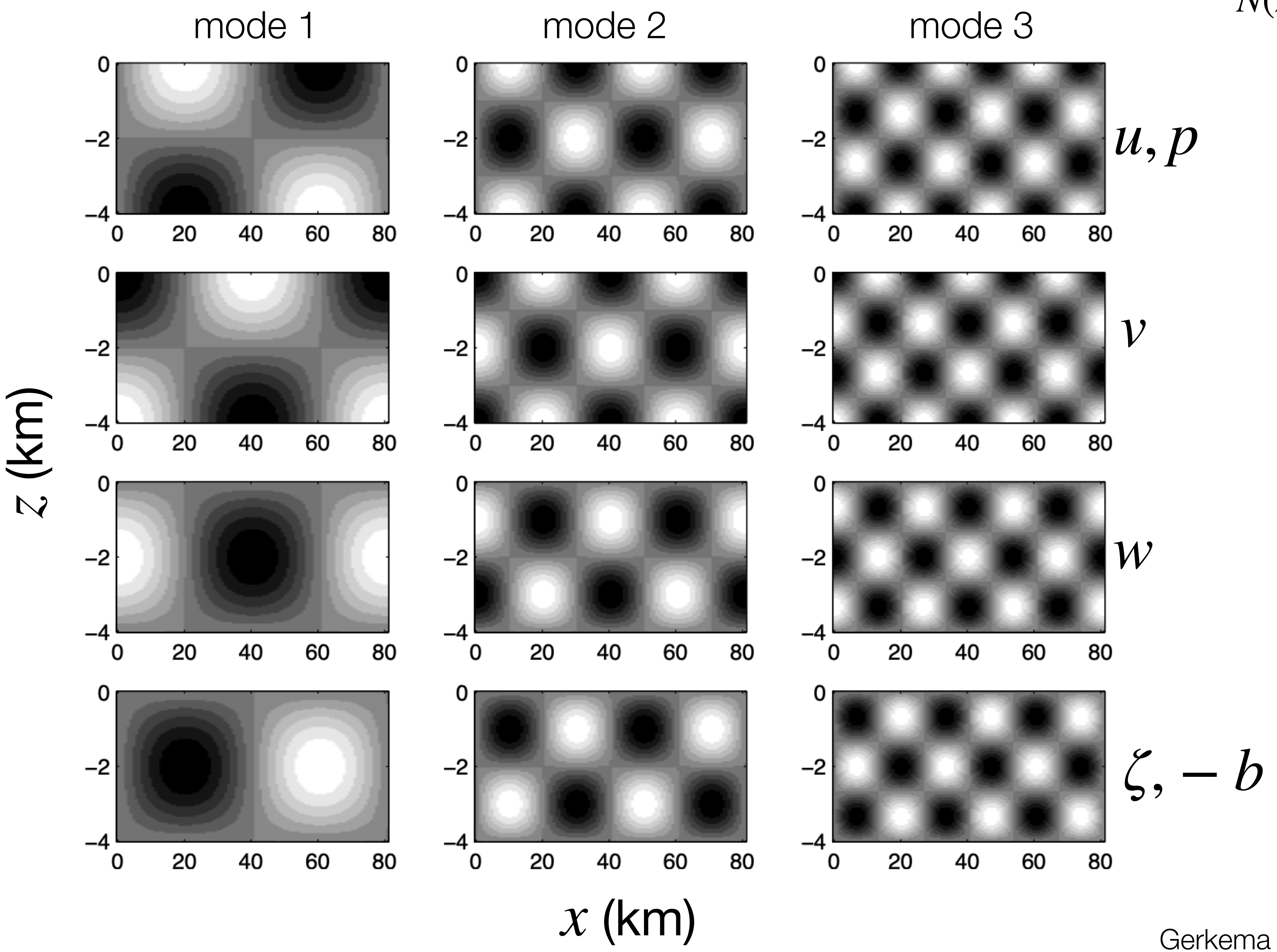
$$U(z) = \frac{i}{k}\frac{\partial W}{\partial z} \; ; \; V(z) = \frac{f}{\omega k}\frac{\partial W}{\partial z} \; ; \; P(z) = i\rho_0 \frac{\omega^2 - f^2}{\omega k^2}\frac{\partial W}{\partial z} \; ; \; B(z) = -\frac{iN^2}{\omega} W(z)$$

Gerkema & Zimmerman (2008)

4. The continuously-stratified model

Method of modes

$N(z) = cst$



ζ is the vertical displacement of isopycnals.

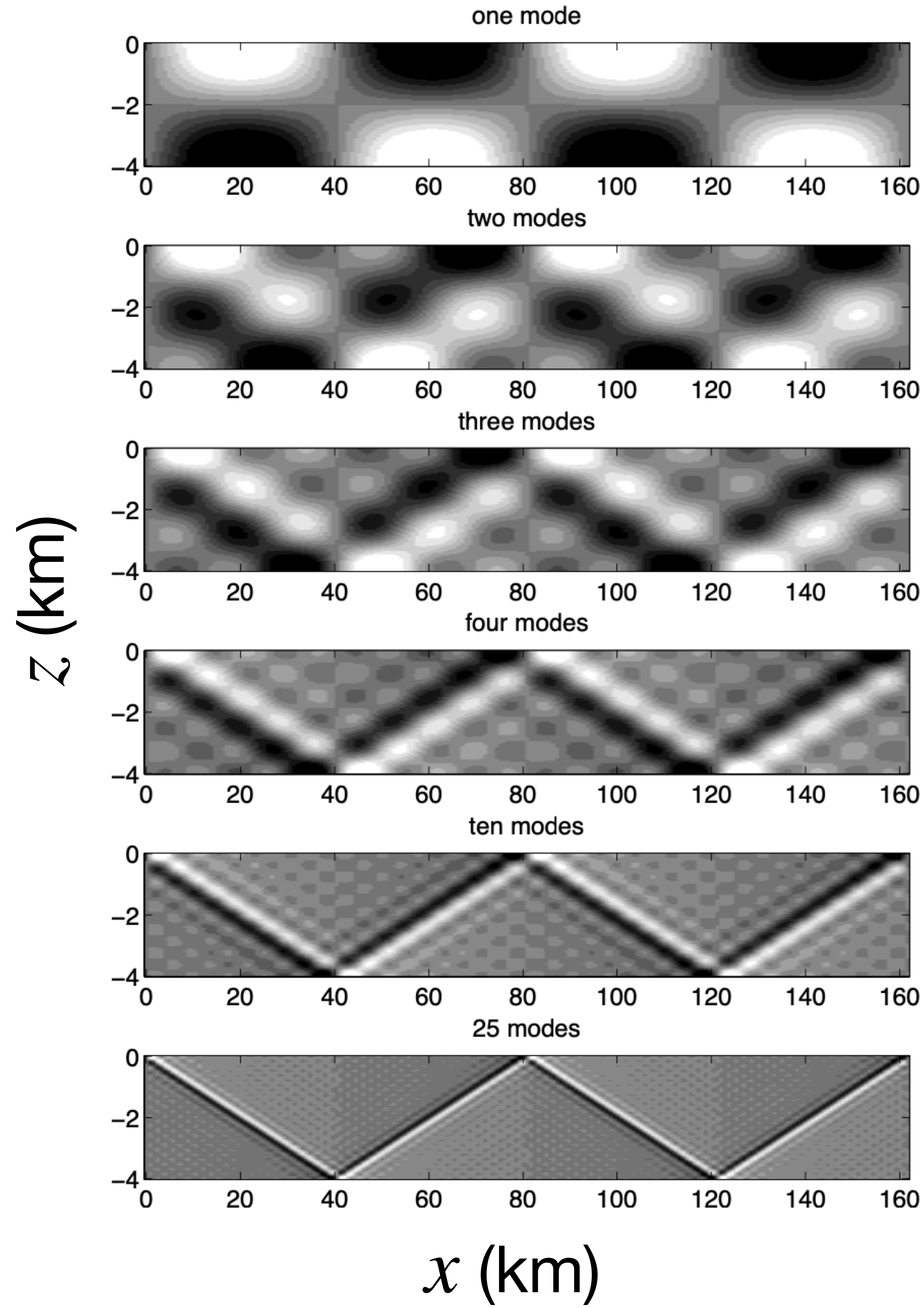
Gerkema & Zimmerman (2008)

4. The continuously-stratified model

Method of modes

The superposition of modes shows up as beams

NB: $N(z) = cst$ in the figure

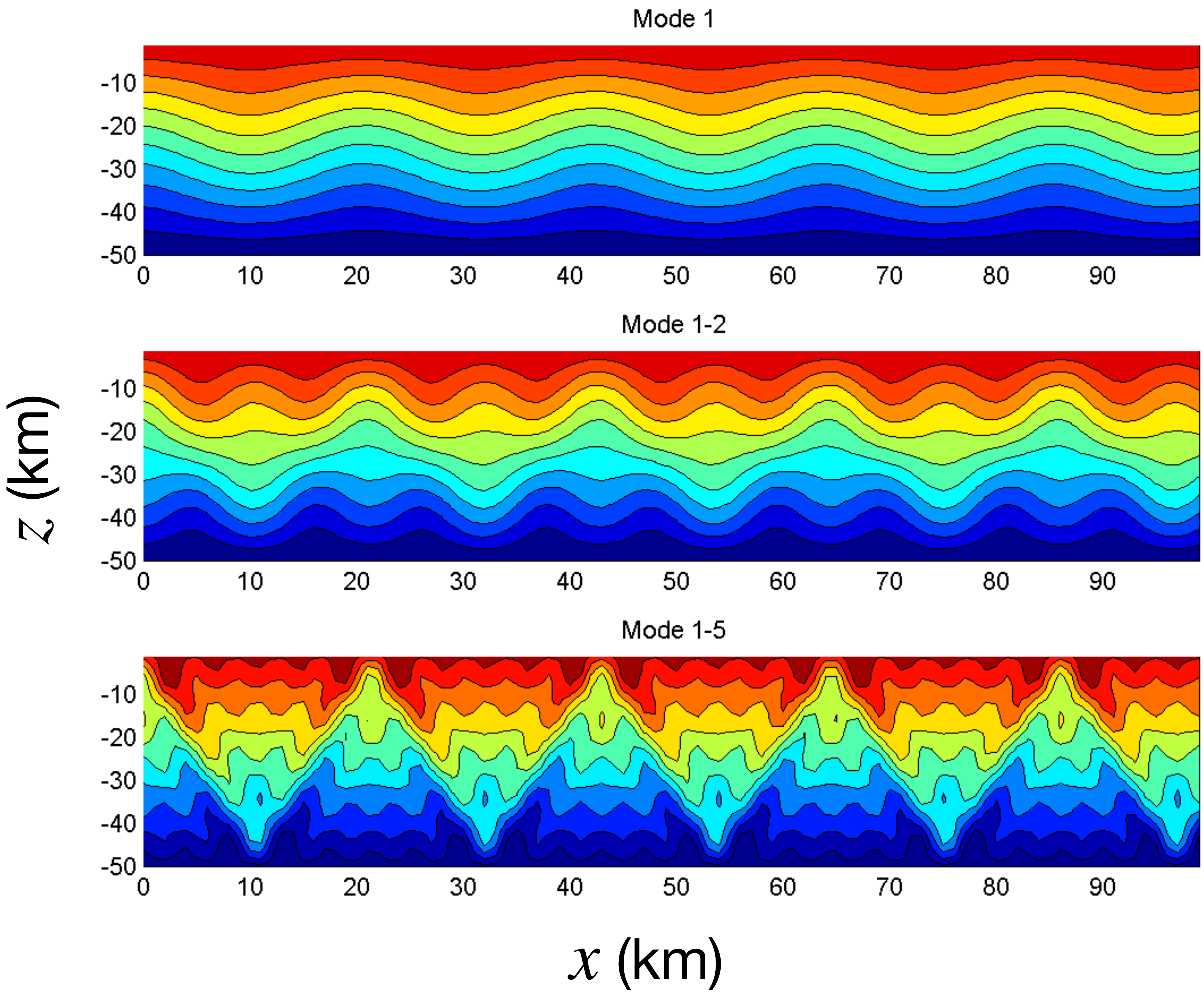


Gerkema & Zimmerman (2008)

4. The continuously-stratified model

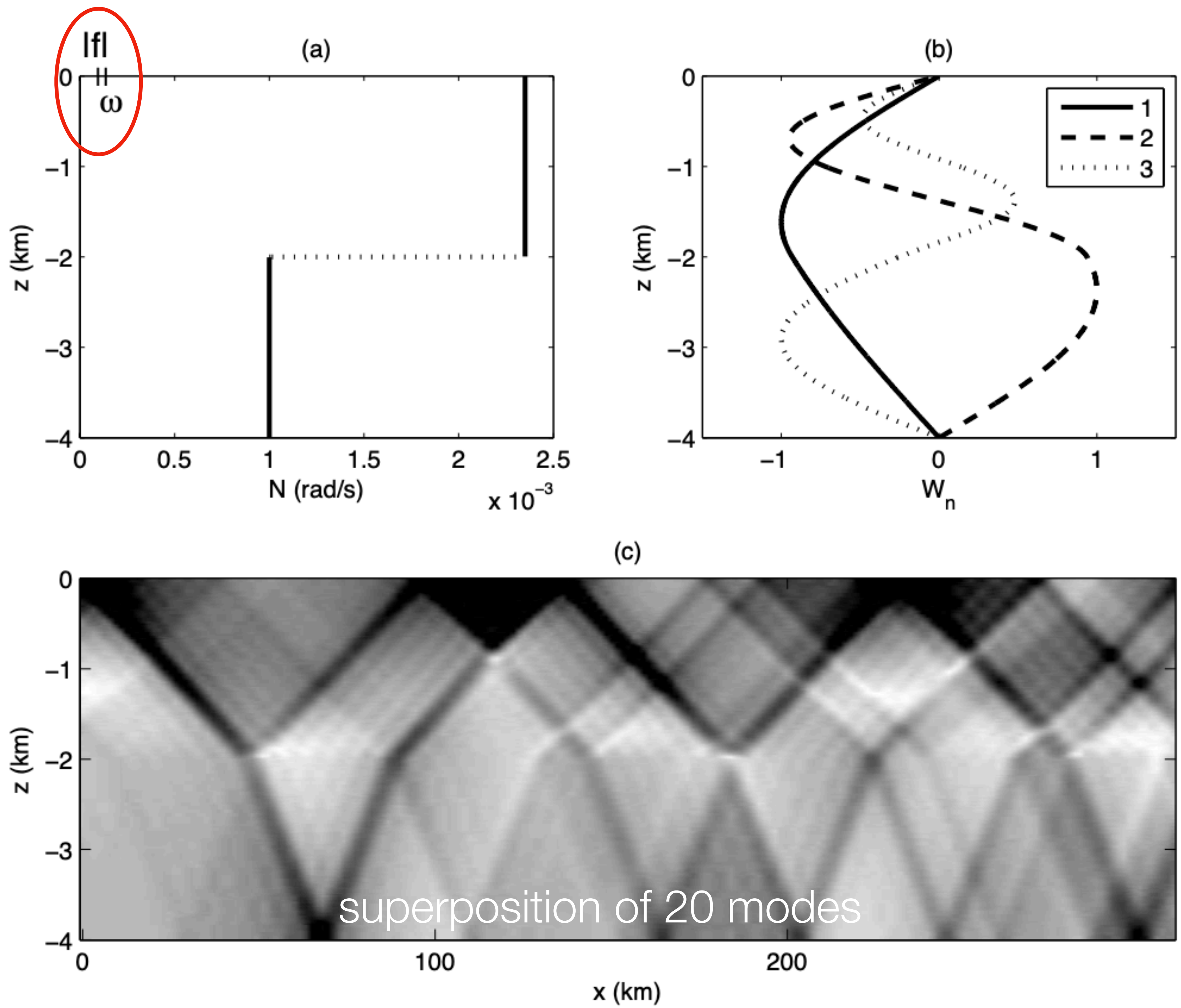
Method of modes

$N(z) = cst$



Method of modes

case of a piecewise-constant $N(z)$

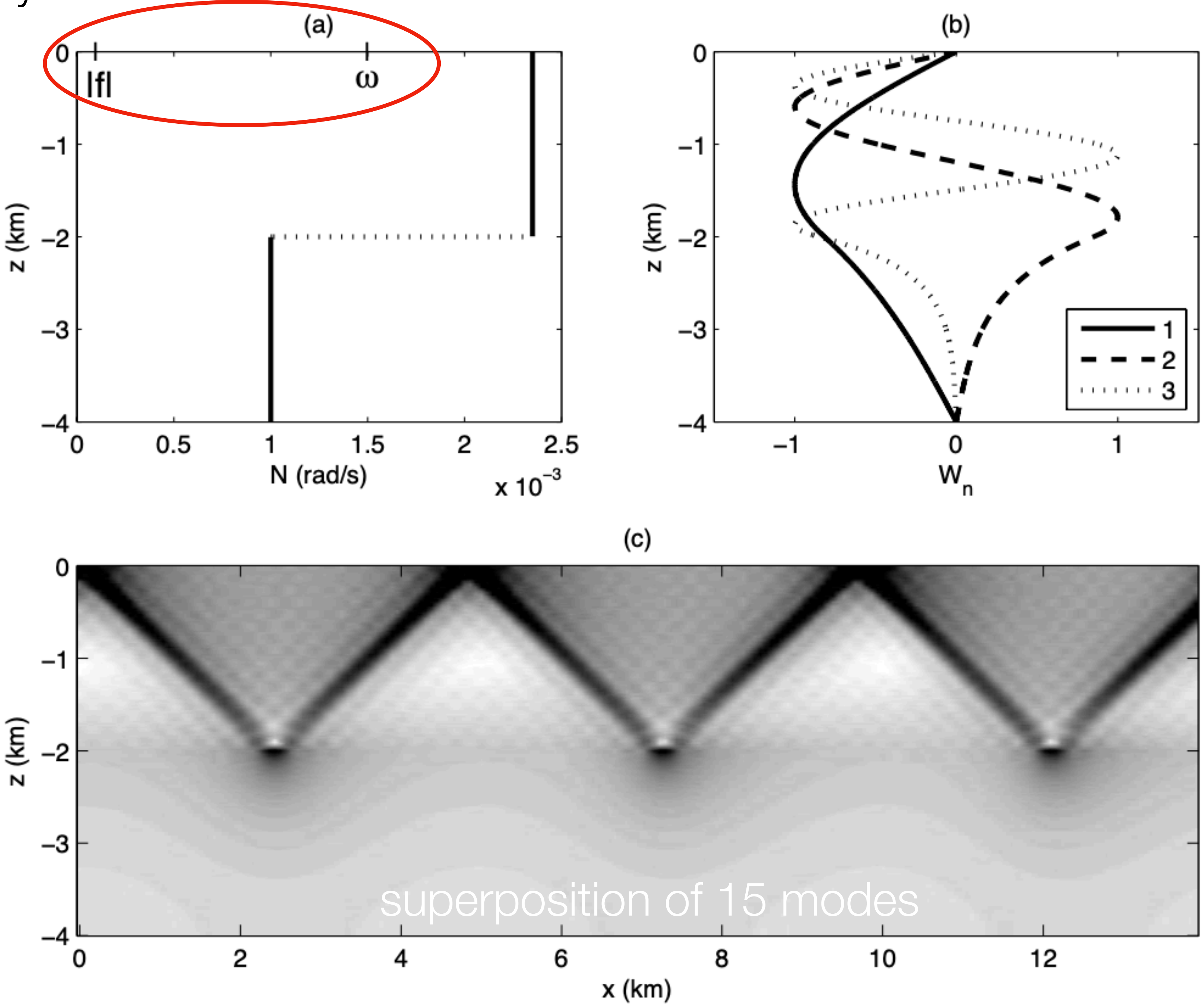


Gerkema & Zimmerman (2008)

4. The continuously-stratified model

Method of modes

case of a piecewise-constant $N(z)$, with $\omega > N$ in the bottom layer: wave trapping in the surface layer.

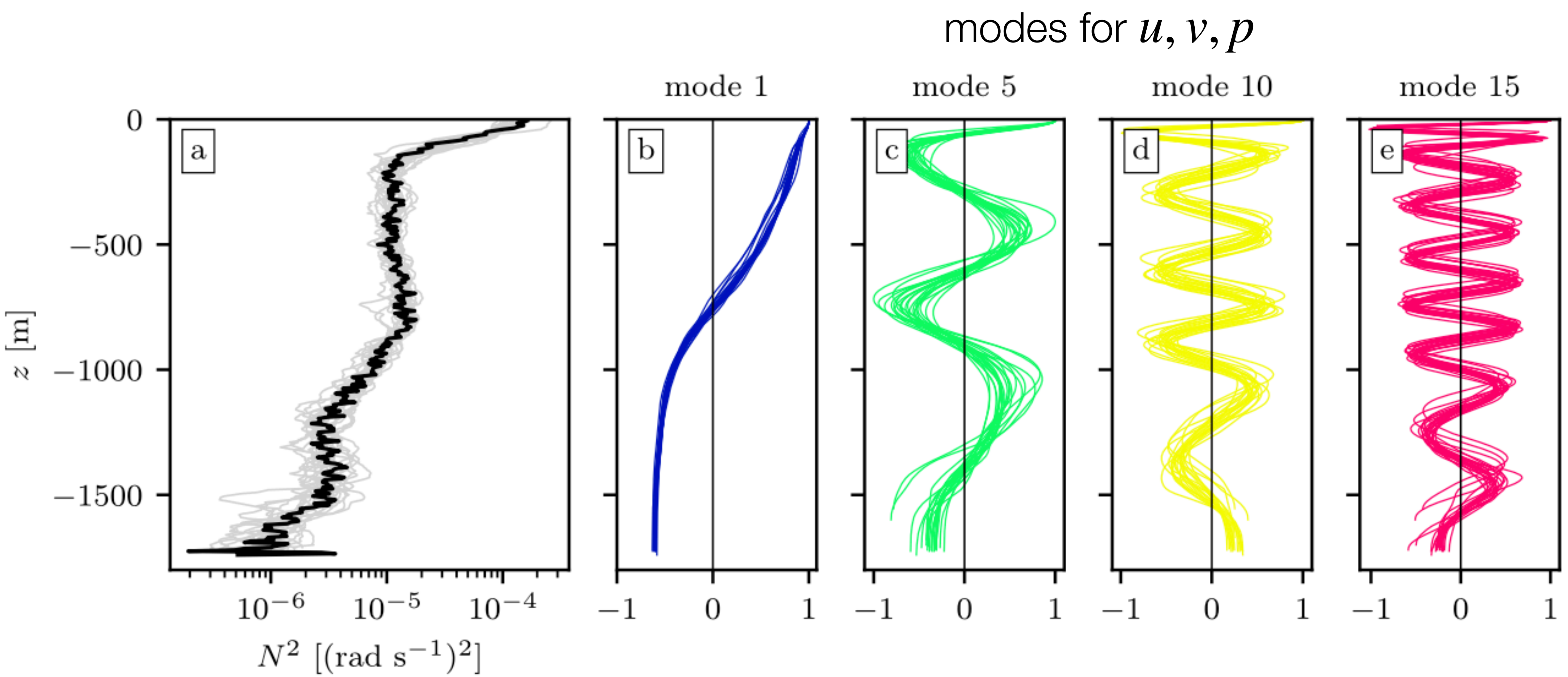


Gerkema & Zimmerman (2008)

4. The continuously-stratified model

Method of modes

Case of a real stratification $N(z)$ over the Mid-Atlantic Ridge



4. The continuously-stratified model

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- Maas et al., Nature 1997, *Observation of an internal wave attractor in a confined, stably stratified fluid*
- Vic & Ferron, JGR 2023, *Observed Structure of an Internal Tide Beam Over the Mid-Atlantic Ridge*