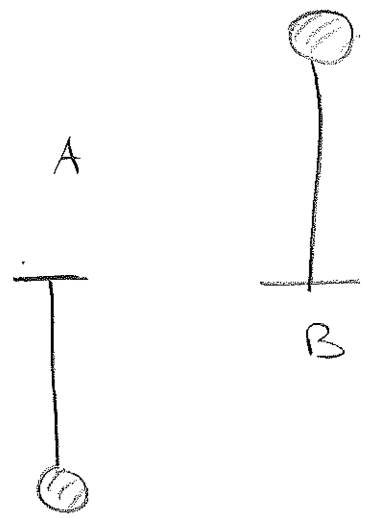
# II. Instability



## II.1 The concept of stability

ex for a pendulum there are 2 position of equilibrium:



To know which one will be observed in practice, we need to perform a stability analysis.
We need to move slightly the pendulum and see if it comes back to its equilibrium position or not. Obviously A is stable, B is not.

In hydrodynamics, we will define a background flow $U(\vec{x},t)$ in equilibrium (sol of the equations) and look at temporal evolution of small perturbations

$$u'(\vec{x},t) = u(\vec{x},t) - U(\vec{x},t)$$

- Definition: Stability in the sense of Lyapunov

  a flow is stable if

  $$\forall \varepsilon>0, \exists \delta \, \|\vec{u}(\vec{x},0) - \vec{U}(\vec{x},0)\| < \delta \Rightarrow \|\vec{u}(\vec{x},t) - \vec{U}(\vec{x},t)\| < \varepsilon$$

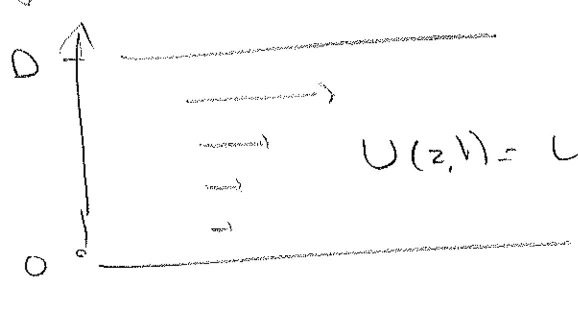
which means that if the solution is initially close to equilibrium, it will stay close at a subsequent time.

- the flow is asymptotically stable if it is stable and if $\exists \delta \, \|\vec{u}(\vec{x},0) - \vec{U}(\vec{x},0)\| < \delta$, then $\lim_{t\to\infty} \|\vec{u}(\vec{x},t)\| \to 0$

- In the following we will only look at the linear stability of steady flows.

The general method is to consider

a background flow $\vec{U}(\vec{x},t)$




$$U(z,t) = U_0 \frac{z}{D}$$

- we introduce a small perturbation $\vec{u}'(\vec{x},t)$

such that $\|\vec{u}'(\vec{x},t)\| \ll \|\vec{U}(\vec{x},t)\|$

we linearize the equations (by neglecting $u'^2, u'v'$, etc.)

- We obtain a system of homogeneous partial differential equations (with coefficients are only functions of space)

Such that we can write solutions in the form $p(\vec{x},t) = \tilde{p}'(\vec{x})e^{st}$ where $s = \sigma + i\omega$

the objective is to find the eigenvalues $s$ and the eigenmodes $\hat{p}(x)$ of the system.

- We need to find a complete basis for eigenmodes.

for example Fourier modes: (if we have spatial symmetries)

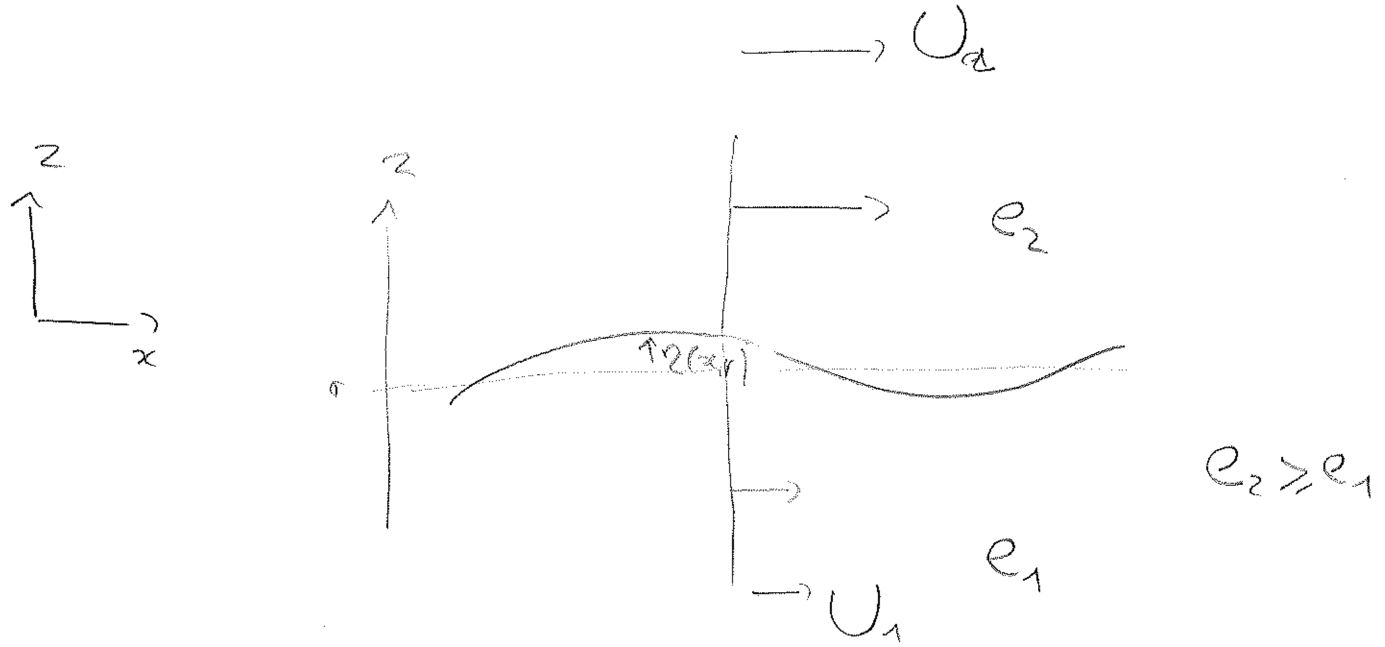
$$\hat{p}(\vec{x}) = \iiint \tilde{p}(k,l,m,s)\, e^{i(kx+ly+mz)}\, dk\, dl\, dm$$

So we can get a dispersion relation $F(k,l,m,s) = 0$

- How do we know if the model is stable?

$$\text{if } s = \sigma + i\omega \text{ so } \begin{cases} \text{if } \sigma = Re(s) > 0 \Rightarrow \text{unstable mode} \\ \text{if } \sigma = 0 \Rightarrow \text{neutral mode} \\ \text{if } \sigma < 0 \Rightarrow \text{asymptotically stable} \end{cases}$$

# II.2 The Kelvin-Helmholtz instab.




2 fluids with different densities and a vertical velocity shear.

We assume 2 incompressible fluid layers
The flow is irrotational in each layer
(with the exception of the interface)

We can write ~~a~~ potentials

$$\vec{u}_1 = \vec{\nabla}\phi_1 \quad \Rightarrow \quad \nabla^2\phi_1 = 0$$
$$\vec{u}_2 = \vec{\nabla}\phi_2 \qquad\qquad \nabla^2\phi_2 = 0$$

$$\vec{u}_1 = \vec{u}_1' + \vec{U}_1 \qquad \lim_{|z|\to\infty} \vec{\nabla}\phi_1 = \vec{U}_1$$
$$\vec{u}_2 = \vec{u}_2' + \vec{U}_2 \qquad \lim_{|z|\to\infty} \vec{\nabla}\phi_2 = \vec{U}_2$$

at the interface, which is a material surface, $z = \zeta$

$$\phi_{iz} = \frac{D_i}{Dt}\zeta$$

we write non hydrostatic Euler equations in the x-z plane:

(5)

$$\begin{cases} u_{it} + u_i u_{ix} + w_i u_{iz} = -\dfrac{P_{ix}}{\rho_i} \\ w_{it} + u_i w_{ix} + w_i w_{iz} + g = -\dfrac{P_{iz}}{\rho_i} \\ u_{ix} + w_{iz} = 0 \end{cases}$$

with dynamical condition at the interface $P_1|_{z=\eta} = P_2|_{z=\eta}$

and the kinematic condition ——— material interface:

$$\frac{D\eta}{Dt} = \eta_t + u_i \eta_x = w_i|_{z=\eta}$$

- There is a stationary solution. $\begin{cases} u_i = U_i = cst \\ w_i = 0, \ \eta = 0 \\ P_{iz} = -g\rho_i \end{cases}$

- We will linearize the equations and look at small perturbations for this background state:

$$\begin{cases} u'_{it} + U_i u'_{ix} = -\dfrac{P'_{ix}}{\rho_i} & (1) \\ w'_{it} + U'_i w_{ix} = -\dfrac{P'_{iz}}{\rho_i} & (2) \\ u'_{ix} + w'_{iz} = 0 & (3) \end{cases} \qquad \begin{cases} u_i = U_i + u'_i \\ \quad |u'_i| \ll |U_i| \\ p_i = P_i + p'_i \\ \quad |p'_i| \ll |P_i| \end{cases}$$

→ forget the primes..

$$\frac{\partial}{\partial t}(u_{ix} + w_{iz}) = \frac{\partial}{\partial x} u_{it} + \frac{\partial}{\partial z} w_{it}$$

$$= \frac{\partial}{\partial x}\left(-U_i u_{ix} - \frac{p_{ix}}{\rho_i}\right) + \frac{\partial}{\partial z}\left(-U_i w_{ix} - \frac{p_{iz}}{\rho_i}\right)$$

$$= -\frac{p_{ixx}}{\rho_i} - \frac{p_{izz}}{\rho_i} - U_i \frac{\partial}{\partial x}\underbrace{(u_{ix} + w_{iz})}_{=0}$$

$$= -\frac{\nabla^2 p_i}{\rho_i} \implies \nabla^2 p_i = 0$$

laplace equation

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we look for solution in the form:

$$p_i(x,z,t) = \tilde{p}_i(z)\, e^{i(kx - st)} \qquad \left(\begin{array}{c} s \text{ is a complex} \\ \omega + i\sigma \end{array}\right)$$

$$\iff \frac{d^2\tilde{p}_i}{dz^2} - k^2 \tilde{p}_i = 0$$

$$\iff \tilde{p}_i(z) = \bar{p}_i\, e^{\pm kz}$$

with limit conditions : $p_1(z) \xrightarrow[z \to -\infty]{} 0$

$p_2(z) \xrightarrow[z \to +\infty]{} 0$

So
$$\begin{cases} p_1(x,z,t) = \bar{p}_1 e^{kz} e^{i(kx - st)} \\ p_2(x,z,t) = \bar{p}_2 e^{-kz} e^{i(kx - st)} \end{cases}$$

vertical velocity is also of the form

$$w_i(x,z,t) = \tilde{w}_i(z)\, e^{i(kx-st)}$$

with equation (2) $\quad w_{it} + U_i w_{ix} = -\dfrac{p_{iz}}{\varrho_i}$

$$\Longleftrightarrow \quad -is\, w_i + ikU_i w_i = -(-1)^{i+1}\frac{k p_i}{\varrho_i}$$

$$\Longleftrightarrow \quad w_i = \frac{(-1)^i k p_i}{i\varrho_i (kU_i - s)}$$

$$w_1 = \frac{+ik\bar{p}_1 e^{kz} e^{i(kx-st)}}{\varrho_1 (kU_1 - s)} \qquad w_2 = \frac{-ik\bar{p}_2 e^{-kz} e^{i(kx-st)}}{\varrho_2 (kU_2 - s)}$$

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The kinematic condition is $\quad \eta_t + U_i \eta_x = w_i|_{z=\eta}$

linearized as $\quad \eta + U_i \eta_x = w_i|_{z=0}$

with $\quad \eta(x,t) = \bar{\eta}\, e^{i(kx-st)}$

So
$$\begin{cases} \bar{\eta}\,(-is + ikU_1) = \dfrac{ik\bar{p}_1}{\varrho_1 (kU_1 - s)} \\[2ex] \bar{\eta}\,(-is + ikU_2) = \dfrac{-ik\bar{p}_2}{\varrho_2 (kU_2 - s)} \end{cases}$$

$$\bar{p}_1 = \frac{\bar{\eta}}{k}\varrho_1 (s - kU_1)^2 \qquad \bar{p}_2 = \frac{-\bar{\eta}}{k}\varrho_2 (s - kU_2)^2$$

The dynamical condition (linearized) gives: $p_2|_{z=0} - p_1|_{z=0} = g(\rho_2-\rho_1)\eta$

$$\Longleftrightarrow \qquad p_1|_{z=\eta} = p_1|_{z=0} - g\rho_1\eta$$
$$p_2|_{z=\eta} = p_2|_{z=0} - g\rho_2\eta$$

$$\bar{p}_2 - \bar{p}_1 = g(\rho_2-\rho_1)\bar{\eta}$$

$$-\frac{\bar{\eta}}{k}\rho_2(s-kU_2)^2 - \frac{\bar{\eta}}{k}\rho_1(s-kU_1)^2 = g(\rho_2-\rho_1)\bar{\eta}$$

$\Longleftrightarrow$ we obtain a dispersion relation:

$$(\rho_1+\rho_2)s^2 - (2\rho_2kU_2 + 2\rho_1kU_1)s + \rho_2k^2U_2^2 + \rho_1k^2U_1^2 + gk(\rho_2-\rho_1) = 0$$

$$(\rho_1+\rho_2)s^2 - 2(\rho_2U_2+\rho_1U_1)ks + (\rho_2U_2^2+\rho_1U_1^2)k^2 - kg\Delta\rho = 0$$
$$\Delta\rho = \rho_1-\rho_2$$

$$\Longleftrightarrow \quad \Delta = 4k^2(\rho_2U_2+\rho_1U_1)^2 - 4(\rho_1+\rho_2)\left[-kg\Delta\rho + (\rho_2U_2^2+\rho_1U_1^2)k^2\right]$$

$$\Longleftrightarrow \quad s = \frac{2(\rho_2U_2+\rho_1U_1)k \pm \Delta^{1/2}}{2(\rho_1+\rho_2)}$$

$$\Longleftrightarrow \quad s = \frac{\rho_2U_2+\rho_1U_1}{\rho_1+\rho_2}k \pm \left(\frac{k^2(\rho_2U_2+\rho_1U_1)^2}{(\rho_1+\rho_2)^2} - \frac{(\rho_2U_2^2+\rho_1U_1^2)k^2}{\rho_1+\rho_2} + \frac{kg\Delta\rho}{\rho_1+\rho_2}\right)^{1/2}$$

$$s = \frac{\rho_2U_2+\rho_1U_1}{\rho_1+\rho_2}k \pm \left(-k^2\frac{\rho_1\rho_2}{(\rho_1+\rho_2)^2}(U_1-U_2)^2 + \frac{kg\Delta\rho}{\rho_1+\rho_2}\right)^{1/2}$$

The flow is linearly stable if $\mathrm{Im}(s) = 0$

$$\Longleftrightarrow \quad \frac{kg\Delta\rho}{\rho_1+\rho_2} - k^2 \frac{\rho_1\rho_2}{(\rho_1+\rho_2)^2}(U_1-U_2)^2 \geq 0$$

$$\Longleftrightarrow \quad \boxed{kg(\rho_2^2-\rho_1^2) \geqslant k^2 \rho_1\rho_2(U_1-U_2)^2}$$

is a necessary condition for the mode $k$ stability

[ note that in 3d we get

$$\sqrt{k^2+p^2}\, g(\rho_2^2-\rho_1^2) \geqslant k^2 \rho_1\rho_1 (U_1-U_2)^2$$

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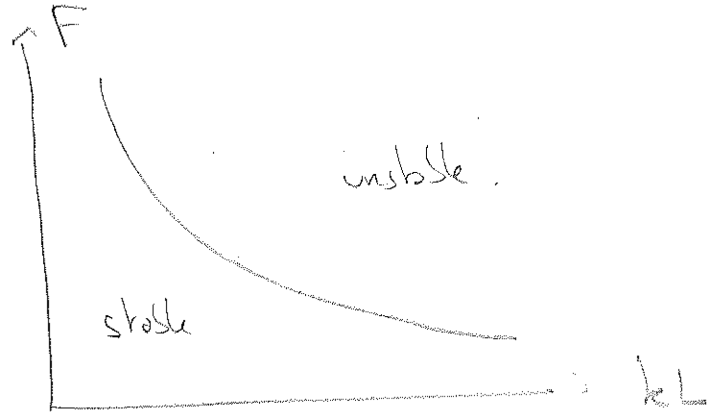
we can plot the stability curve using the reduced gravity $g' = g\dfrac{\rho_1-\rho_2}{\rho_1+\rho_2}$

and the Froude number $F = \dfrac{U_1-U_2}{(g'L)^{1/2}}$

so $$kL\,F^2 = \frac{(\rho_1+\rho_2)^2}{\rho_1\rho_2}$$

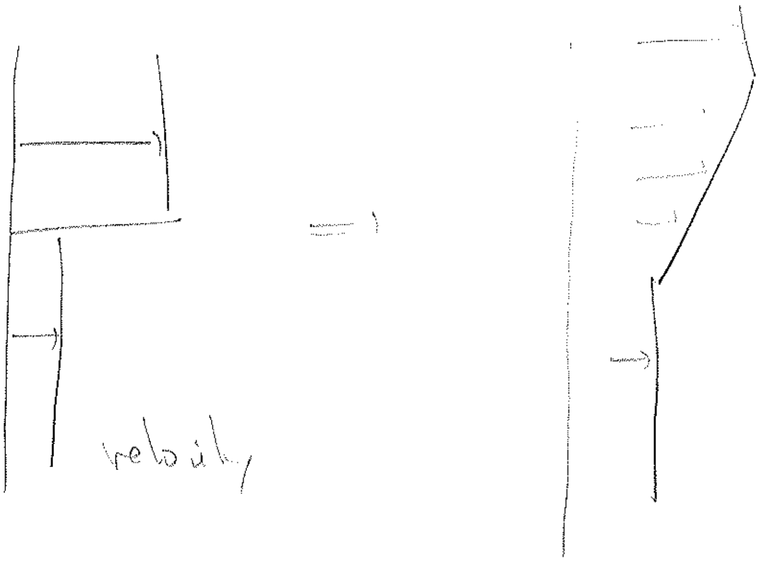



The most unstable modes are long wave numbers. For a given shear there will always be a $k$ large enough to be unstable.

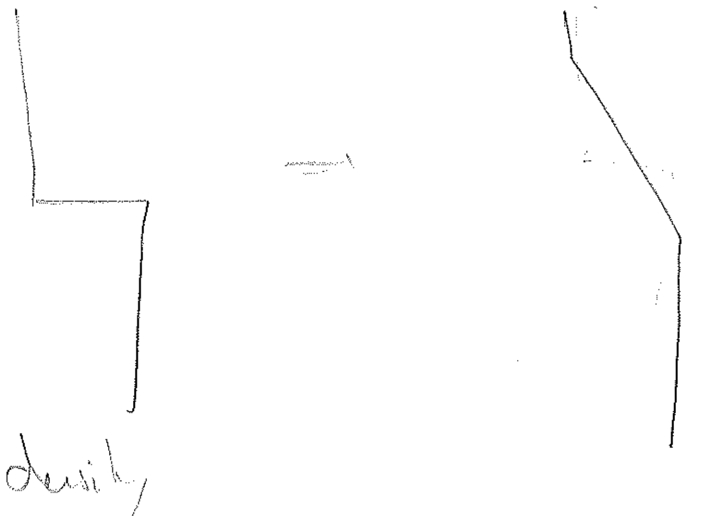
It will only be stabilised at very high wave numbers by capillary forces or viscosity.

The destabilizing effect is the shear - the stabilizing effect is the stratification.

The instability will smooth the velocity gradients - and convert kinetic energy into potential energy.



velocity



density

2 limiting cases:

(a) $U_1 = U_2 = 0 \implies$

$$s = \pm i \left( \frac{k g \Delta\rho}{\rho_1 + \rho_2} \right)^{1/2} \Rightarrow \omega^2 = g' |k|$$

always stable: short waves propagating on the interface.

(b) $\rho_1 = \rho_2 \implies s = \frac{U_1 + U_2}{2} k \pm \left( - \frac{k^2 (U_1 - U_2)^2}{4} \right)^{1/2}$

$$= \frac{U_1 + U_2}{2} k \pm i \frac{k (U_1 - U_2)}{4}$$

always unstable = vorticity sheet with growth rates increasing with $k$.

(in practice limited by diffusion)

II.2.2 KH instability with a continuous stratification

$$N = \sqrt{-\frac{g}{\rho} \frac{\partial \rho}{\partial z}} \qquad \overline{U}(z)$$

see Cushman-Roisin p 154.

we can derive integral constraints

showing that if the inequality:

Ep barrier

$$Ri = \frac{N^2}{(d\overline{U}/dz)^2} > \frac{1}{4}$$

Ek available

holds, the shear flow is stable.