Master Physique Marine (M1)
Fluides II

Hydrodynamic instabilities

Homework 1

We want to use the two layer model shown in the lectures (mean flow U_1 in the upper half plane, mean flow U_2 in the lower half plane) to discuss the shear instability from an energetic point of view.

(1) First write out the linearized momentum equations satisfied by the perturbations on each side of the discontinuity and derive an equation for the kinetic energy of perturbations averaged over the volume of the fluid as :

$$\partial_t KE = (U_1 - U_2)\overline{p' \frac{dh'}{dx}}$$

where p is the pressure at the interface, h the particle displacement in the z direction of the interface, KE the total kinetic energy of pertubations and the brackets on the rhs denote the x average of quantities evaluated on the material surface. This term called « form drag » is just the x component of the pressure forces acting on the material surface. For unstable (damped) modes in the case U_1-U_2>0, conclude on the sign of form drag.

(2) Compute the pressure of an unstable wave to find out the phase lag betweeh p and h and check the above conclusion (it is easy if you take U_1=U and U_2=-U). Make a drawing of the material surface and indicate the pressure forces acting on it. Show then that the x momentum must increase (decrease) below (above) the crests. In this sense we may say that the unstable perturbations act to reduce the shear of the mean flow (which is the source of the instability in the first place).

Hint : Multiply the vector equations scalarly by the velocity vector in each layer. Show that averaging along x of terms of the form d/dx gives nothing on account of peridicitry in x. Show that the boundary terms at infinity in the z direction vanish as well. Be very careful of the terms evaluated on the material surface of discontinuity.

Homework 2 : barotropic shear instability

(1) Starting from the incompressible Navier-Stokes momentum equation (homogeneous fluid) for a mean flow $\mathbf{U_i}$ + perturbation $\mathbf{u_i}$ in 2D (i) derive the eddy kinetic energy equation for the perturbations as :

$$\frac{d}{dt}\int \frac{1}{2} u_i^2 \, dV = -\int u_i u_j \frac{\partial Ui}{\partial x_j} dV - \epsilon$$

where the dissipation term is

$$\epsilon=\nu\int\left(\frac{\partial u_i}{\partial x_j}\right)^2 dV$$

The domain of integration is a periodic channel in x and limited by 2 plane walls at y 1 and y 2 .

(2) Interpret the terms in the above energy equation in the unstable case of a simple shear flow U(y) as in lectures. Relate the shape of the perturbations to the sign of dU/dy.

Howework 3 : KH instability

We add stratification in the shear instability discussed in lectures (we keep it 2D in x and z). ρ_1 and ρ_2 are the density in layers 1 and 2 and gravity is in the negative z direction. The domain is infinite in the positive and negative z directions. To fix ideas U_1 could be the wind at some upper level and U_2 the smaller wind at lower level so that $U_1 - U_2 > 0$. Otherwise the sign of $\rho_1 - \rho_2$ is arbitrary as both signs can occur in the atmosphere and we talk of stable or unstable stratification.

(1) Obtain the condition for the flow to be unstable and discuss it in term of wavenumber k of the perturbations and a form of Richardson number (an inverse Froude number here) :

$$R_i=\frac{gL(\rho_1-\rho_2)/\rho_0}{(U_1-U_2)^2}$$

Here L is an arbitrary length scale to make R_i adimensional but define ρ_0. For which direction of the wavenumber vector do perturbations reach the largest growth rate ?

(2) Construct the stability diagram in the Ri-kL space. Indicate the marginal curve Ri(kL) of of zero grwoth rate that separates unstable and stable regions. What is the effect of the sign $\rho_1-\rho_2$? (consider also the $U_1 - U_2 = 0$ case).

Hint : To implement the continuity of pressure at the interface, use the velocity potentials and the Bernouilli theorem (instead of streamfunction as in lectures). These continuity conditions have to be implemented for both mean and perturbations.

Homework 4 : Thermal convection in the Earth

Rigid plates in the lithosphere (thickness=100 km) move around at the surface of the earth and there was much debate about the processes responsible for this continental drift. Arthur Holmes proposed thermal convection in the mantle below the lithosphere in 1931. Geophysicists did not like the idea because the mantle supported shear waves that indicated a solid material (no shear waves exist in a Newtonian fluid because there is no resistance to shear deformation). However there is a very slow post glacial rebound of the earth surface following the melting of the ice cap which started 10ky ago. This introduced the idea of a fluid behavior on very long time scales. The process of solids flowing under pressure is called creep (glaciers are made of solid crystalline ice but also flow very slowly). A viscosity $\mu=10^{21}$ Pa.s has been inferred for the upper mantle of thickness b=700 km from the study of the post glacial rebound. The rebound is still observed today (Iceland and Norvegia rise).

Thermal convection in the mantle is rather different from classical Rayleigh-Benard convection.

The mantle is heated internally by heat sources H due to radioactive decay of ^{235}U, ^{238}U, ^{232}Th, ^{40}K isotopes.

(1) Formulate the 2D linear stability problem taking stress free boundaries, an isothermal upper surface, and an adiabatic lower surface (no heat flux).

(2) Determine the basic thermal (conductive) equilibrium state. What is the temperature at 700 km depth?

(3) Obtain the coupled equations for the perturbations: stream function and temperature with their BCs. We do not ask that you solve them.

(4) The Rayleigh number for this problem is :

$$Ra=\frac{g\,\alpha\,\rho_0^2\,H\,b^5}{K\,\mu\,k}$$

Take g=10 ms -2 . Other needed properties for rocks are :
thermal expansion coefficient α=3.10^{-5} Kelvin^{-1}
specific mass of rocks in the mantle ρ_0 =4000 kg m^{-3},
thermal conductivity K=4 W m^{-1} Kelvin^{-1}
thermal diffusivity k=1 mm^2 s^{-1}
b=700 km
radioactive heating H=9 10^{-12} W kg^{-1}

Check dimensions and compute the Rayleigh number for the mantle and compare to the critical value for the onset of convection Ra_c =867.8. The associated critical wavelength is such that 2πb/λ c =1.79. This was essentially the basis of the argument given by Holmes. Now how convection cells move the rigid plates of the crust above remains a tough question.
