INTERNAL WAVES

2. CONTINUOUS STRATIFICATION

Bibliography

- Gerkema- Zimmerman (2008). *An introduction to internal waves*
  - *http://stockage.univ-brest.fr/~gula/Ondes/gerkema.pdf*
- Gill (1982) *: Atmosphère-Ocean Dynamics*
- Kundu-Cohen (1987). *Fluid Mechanics. Third edition*
- Cushman-Roisin*. Introduction to geophysical fluid Dynamics*

- **1.2 : Internal waves with continuous stratification**
  - Equations
  - Method of vertical modes
  - Method of characteristics

Local Static Stability

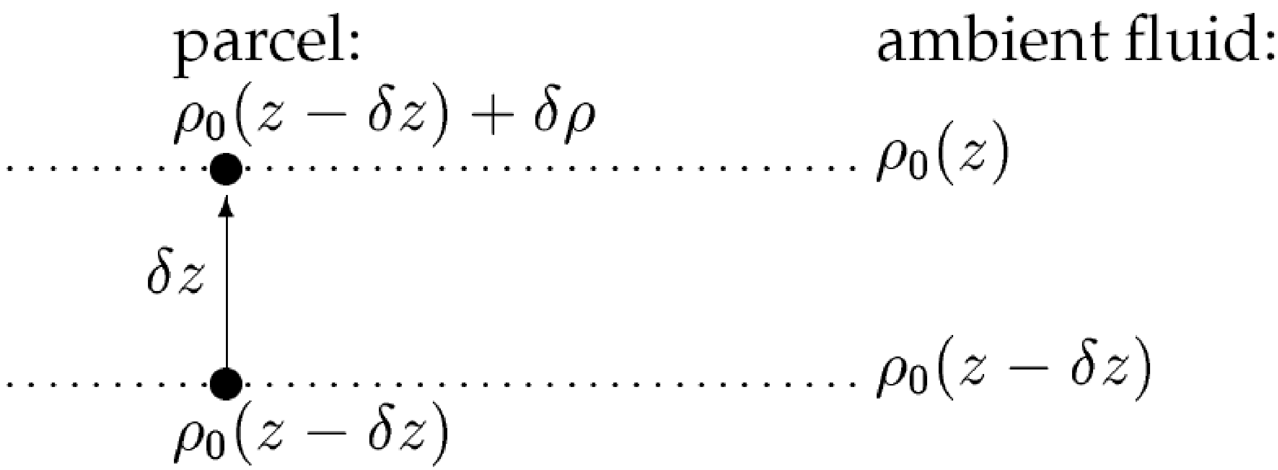
- Stratification:

$$\rho = \rho_0(z)$$

- A fluid is
  - **stably stratified** if a displaced parcel tends to return to its original position,
  - **unstably stratified** if it tends to move further away from its original position
  - **neutrally stratified** if it tends to stay where it is.

Local Static Stability

- Let's move a parcel:



- Buoyancy force:

$$\rho_0(z)\ddot{\delta z} = g\left(\rho_0(z) - \rho_0(z-\delta z) - \delta\rho\right)$$

Local Static Stability

- With background density variation:

$$\rho_0(z) - \rho_0(z - \delta z) = \frac{d\rho_0}{dz}\delta z$$

- And parcel density variation?
  - *From thermodynamics, if entropy and salinity are conserved during displacement:*

$$\delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_{\eta,S}\delta p = c_s^{-2}\delta p$$

Where c_s is the speed of sound

Local Static Stability

- So we get:

$$\rho_0(z)\ddot{\delta z} = g\left(\frac{d\rho_0}{dz}\delta z + \frac{\rho_0 g \delta z}{c_s^2}\right)$$

Local Static Stability

- Simple Harmonic oscillator:

$$\underbrace{\ddot{\delta z} - \frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)}_{N^2}\delta z = 0$$

- Brunt-Vaisala frequency: N^2

- Solutions: $e^{\pm \mathrm{i} N t}$ $\quad\ddot{\delta z} + N^2 \delta z = 0$

- Stable if $N^2 > 0$

Local Static Stability

- Simple Harmonic oscillator:

$$\ddot{\delta z} + N^2 \delta z = 0$$

- Solutions: $e^{\pm \mathrm{i} N t}$

- Stable if $N^2 > 0$

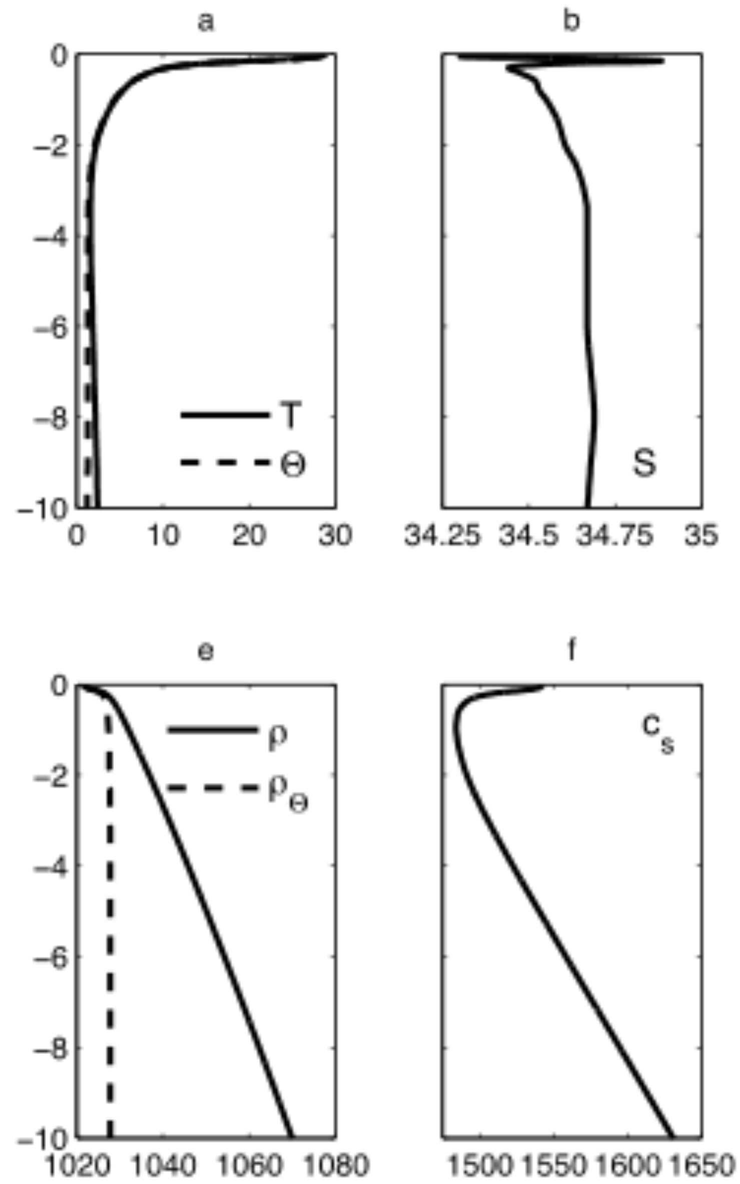
The parcel oscillates vertically at frequency *N* about its equilibrium position.

Local Static Stability

- Brunt-Vaisala frequency:

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)$$

The effect of compressibility is often neglected in the upper ocean but it is not true in general.



Vertical profiles of T, S, in-situ and potential density, and speed of sound in the Mindanao Trench.

Local Static Stability

- How do you connect density and stability?

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2}\right)$$

- It is convenient to define **the generalized potential density,** corresponding to the density that the parcel would attain if moved from z to a reference level z_r, under conservation of its entropy and salinity.

- We compute it by vertically integrating:

$$\delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_{\eta,S}\delta p = c_s^{-2}\delta p$$

Local Static Stability

- Which gives the generalized potential density:

$$\rho_r(z_r, z) = \rho_0(z) + g\int_{z_r}^{z} \frac{\rho_0(z')}{c_s^2(z, z')} dz'$$

- Such that:

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_r}{dz}(z_r, z) + \frac{g^2}{\rho_0}\int_{z_r}^{z} \rho_0(z')\frac{\partial c_s^{-2}}{\partial z}(z, z')dz'$$

Local Static Stability

- In practice we use potential density (z_r=0):

$$\rho_\Theta(z) = \rho_0(z) + g\int_0^z \frac{\rho_0(z')}{c_s^2(z,z')}dz'$$

Which is the density that the parcel would acquire if adiabatically brought to the surface.

- Such that:

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_\Theta}{dz} + \frac{g^2}{\rho_0}\int_0^z \rho_0(z')\frac{\partial c_s^{-2}}{\partial z}(z,z')dz'$$

Equations for a stratified flow

- Navier-Stokes Equations:

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \vec{\mathcal{F}} + \vec{\mathcal{D}}$$

Momentum equations

$$\frac{D\rho}{Dt} + \rho\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{DP}{Dt}$$

Thermodynamic ‘energy’ equation
(no diabatic effects)

Equations for a stratified flow

- Approximations for the momentum equation:

No forcings/dissipation

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + g\vec{k} = -\frac{\vec{\nabla} P}{\rho} + \cancel{\vec{\mathcal{F}} + \vec{D}}$$

Equations for a stratified flow

- Boussinesq Approximation:

Density perturbations small compared to mean background value:

$$\rho - \rho^* << \rho^*$$

Linearize all terms involving a product with density, except the gravity term which is already linear:

$$\rho \vec{u} \rightarrow \rho^* \vec{u}$$

$$\rho g \rightarrow \rho g$$

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- Traditional Approximation:

= neglect horizontal Coriolis term

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

$$\frac{D\vec{u}}{Dt} + f\vec{k} \times \vec{u} + \frac{\rho}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} P}{\rho^*}$$

Equations for a stratified flow

- We think as internal waves as a perturbation of a (known) static background state that has only vertical dependences

$$P = p_0(z) + p'(t, \vec{x})$$
$$\rho = \rho_0(z) + \rho'(t, \vec{x})$$

- And linearize momentum equations:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} + \frac{\rho'}{\rho^*} g\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Equations for a stratified flow

- For the thermodynamic equation: $$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{DP}{Dt}$$
- We can write: $$\frac{D\rho}{Dt} = \frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z}$$
- And: $$\frac{1}{c_s^2}\frac{DP}{Dt} = \frac{1}{c_s^2}\frac{Dp'}{Dt} + \frac{1}{c_s^2} w\frac{\partial p_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$
- So : $$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

Equations for a stratified flow

- We linearize:

$$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{Dp'}{Dt} - w\frac{\rho_0 g}{c_s^2}$$

$$\Downarrow$$

$$\frac{\partial\rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

-

Equations for a stratified flow

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\frac{\partial p'}{\partial t} - w\frac{\rho_0 g}{c_s^2}$$

- We can show that the pressure time tendencies are small using the following scalings and relations:

$$\boxed{U \ll C \ll c_{sf} \ll c_s\,.}$$

$$U \sim O(10^{-1})\,; \quad C \sim O(1)\,; \quad c_{sf} \sim O(10^1\,, 10^2)\,; \quad c_s \sim O(10^3)\,\mathrm{m\,s^{-1}}\,.$$

- Which denote the particle velocity and phase speed of internal waves, the phase speed of surface waves, and the speed of sound in seawater, respectively.

Equations for a stratified flow

- We get:

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = -w\frac{\rho_0 g}{c_s^2}$$

- So we can write:

$$\frac{\partial \rho'}{\partial t} + \left[\frac{\partial \rho_0}{\partial z} + \frac{\rho_0 g}{c_s^2}\right] w = 0$$

- Which gives

$$-\frac{g}{\rho^*}\frac{\partial \rho'}{\partial t} + N^2 w = 0$$

- With $N^2 = -\frac{g}{\rho^*}\left(\frac{\partial \rho_0}{\partial z} + \frac{\rho_0 g}{c_s^2}\right)$

Equations for a stratified flow

- And finally introducing buoyancy: $$b = -g\frac{\rho'}{\rho^*}$$
-

$$\frac{\partial b}{\partial t} + N^2 w = 0$$

Equations for a stratified flow

- For the mass equation:

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0$$

- We rewrite:

$$\rho\nabla\cdot\vec{u} = -\frac{D\rho}{Dt} = -\frac{1}{c_s^2}\frac{Dp'}{Dt} + w\frac{\rho_0 g}{c_s^2}$$

-

Equations for a stratified flow

- We look at the scales of the different terms:

$$\overbrace{\rho\frac{\partial u}{\partial x}}^{\rho_* U/L} + \overbrace{\rho\frac{\partial v}{\partial y}}^{\rho_* U/L} + \overbrace{\rho\frac{\partial w}{\partial z}}^{\rho_* W/H} = -\overbrace{\frac{1}{c_s^2}\frac{\partial p'}{\partial t}}^{P/(Tc_s^2)} - \overbrace{\frac{u}{c_s^2}\frac{\partial p'}{\partial x}}^{UP/(Lc_s^2)} - \overbrace{\frac{v}{c_s^2}\frac{\partial p'}{\partial y}}^{UP/(Lc_s^2)} - \overbrace{\frac{w}{c_s^2}\frac{\partial p'}{\partial z}}^{WP/(Hc_s^2)} + \overbrace{w\frac{\rho_0 g}{c_s^2}}^{\rho_* W g/c_s^2}$$

- We assume again a separation in time scales:

$$\boxed{U \ll C \ll c_{sf} \ll c_s\,.}$$

$$U \sim O(10^{-1})\,; \quad C \sim O(1)\,; \quad c_{sf} \sim O(10^1\,,10^2)\,; \quad c_s \sim O(10^3)\,\mathrm{m\,s^{-1}}\,.$$

- If we remove small terms we simply get: $\vec{\nabla}\cdot\vec{u} = 0$

Equations for a stratified flow

- Linearized Equations are:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} - b\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

$$\frac{\partial b}{\partial t} + N^2 w = 0$$

Thermodynamic ‘energy’ equation
(no diabatic effects)

- **Lesson 1 :**
  - Introduction about Ocean waves
  - Surface waves
  - Internal Waves (Introduction)
- **Lesson 2 :**
  - Internal Waves in the 2-layer model
  - Internal Waves with a continuous stratification (equations)
- **Lesson 3 :**
  - Internal Waves with a continuous stratification (solutions)
- **Lesson 4 :**
  - Generation of internal waves
  - Dissipation and interaction of internal waves
  - Activities (home): Numerical simulation of internal waves
- **Lesson 5 :**
  - Long waves
    - Rossby waves [Rossby, Poincare, etc.]
    - Coastal trapped waves [Kelvin waves, etc.]
    - Equatorial waves [Rossby, Kelvin, Yanai]

Presentations and material will be available at :

jgula.fr/Ondes/

Equations for a stratified flow

- Linearized Equations are:

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} - b\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation
(no source/sink)

$$\frac{\partial b}{\partial t} + N^2 w = 0$$

Thermodynamic ‘energy’
equation
(no diabatic effects)

Activity (for next session)

- Starting from linearized Equations :

$$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} - b\vec{k} = -\frac{\vec{\nabla} p'}{\rho^*}$$

$$\vec{\nabla} \cdot \vec{u} = 0 \qquad \frac{\partial b}{\partial t} + N^2 w = 0$$

- Activity:
- Write an equation for w alone.

Equations for a stratified flow

- Finally we get an equation for *w* alone:

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0$$

Solutions of the equation

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0$$

- We suppose waves to be sinusoidal in time:

$$w = \hat{w} e^{-\mathrm{i}\omega t}$$

- And get:

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

Solutions of the equation

Two methods to solve the equation:

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

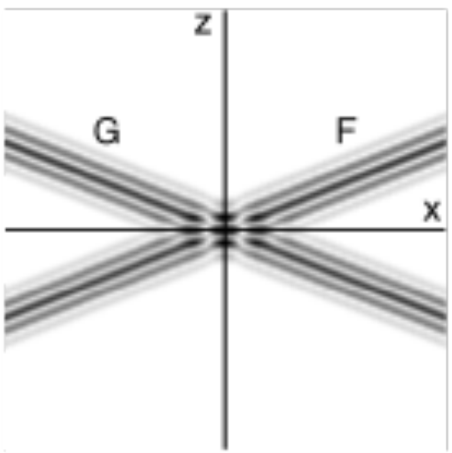
1. The method of characteristics
2. The method of modes

Solutions of the equation

1. Method of characteristics

- N = cste
- Arbitrary boundary conditions
- Solution of the form:

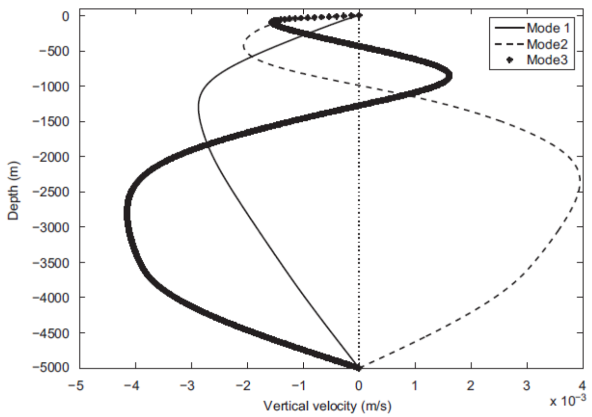
$$\hat{w} = F(\mu_+ x - z) + G(\mu_- x - z)$$



2. Method of Modes

- N = N(z)
- Horizontal surface/bottom
- Solution of the form:

$$w = W(z)e^{-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}ly}$$



Solutions: Method of characteristics

$$(N^2-\omega^2)\nabla_h^2\hat{w}-(\omega^2-f^2)\hat{w}_{zz}=0$$

- For simplicity we take $\dfrac{\partial .}{\partial y}=0$
- So that: $$(N^2-\omega^2)\hat{w}_{xx}-(\omega^2-f^2)\hat{w}_{zz}=0$$
-

Solutions: Method of characteristics

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

- For simplicity we take $$\frac{\partial .}{\partial y} = 0$$
- So that: $$(N^2 - \omega^2)\hat{w}_{xx} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

- Activity: Solve the equation for the case of constant N

  1. Find the general solution

Solutions: Method of characteristics

- Equation coefficients are all constant (if *N=cste*)

- The solution is :

$$\hat{w} = F(\mu_+ x - z) + G(\mu_- x - z)$$

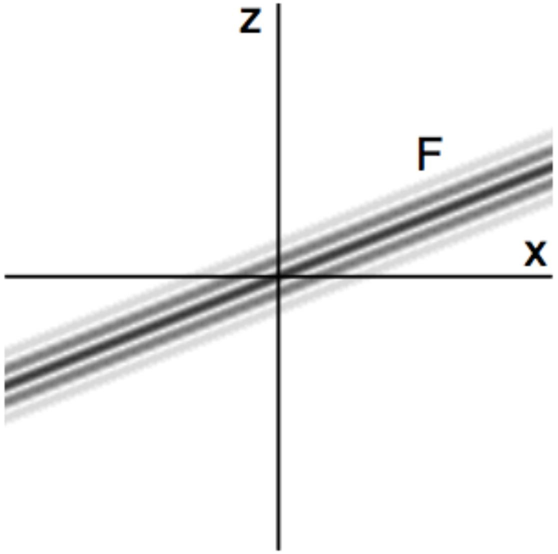
where *F* and *G* are arbitrary functions and

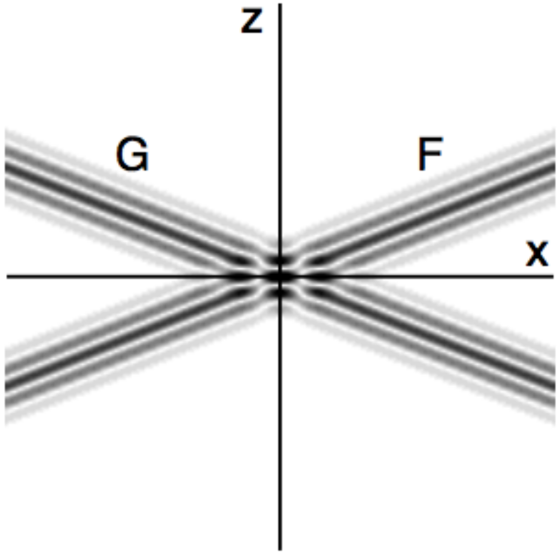
$$\mu_\pm = \pm \left(\frac{\omega^2 - f^2}{N^2 - \omega^2} \right)^{1/2}$$

Solutions: Method of characteristics

- For example we use: $$F(\xi) = \exp(-\xi^2)\exp(ik\xi)$$
- The solution is then, with $$\xi_{\pm} = \mu_{\pm}x - z$$

$$w = \exp(-\xi_+^2)\cos(k\xi_+ - \omega t) + \exp(-\xi_-^2)\cos(k\xi_- - \omega t)$$





Solutions: Method of characteristics

- Energy propagates along the lines: $$\mu_{\pm}x - z = cste$$

which are the characteristic coordinates, and which are diagonals in the x, z-plane

Solutions: Method of characteristics

$$(N^2 - \omega^2)\hat{w}_{xx} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

Activity:

- Assuming a solution of the form: $\hat{w} = w_0 e^{\mathrm{i}(kx+mz)}$
- Write the dispersion relation $\omega(k, m)$

Solutions: Method of characteristics

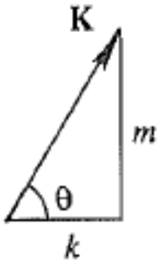
- The dispersion relation is:

$$\boxed{\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2 + m^2}}$$

Solutions: Method of characteristics

- The dispersion relation:

$$\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2 + m^2}$$



can be simplified by using polar coordinates:

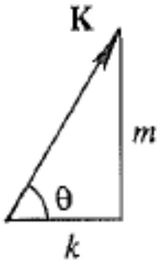
$$\vec{k} = (k, m) = \kappa(\cos\theta, \sin\theta)\,; \qquad \kappa = (k^2 + m^2)^{1/2}\,,$$

Such that:

Solutions: Method of characteristics

- The dispersion relation:

$$\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2 + m^2}$$



can be simplified by using polar coordinates:

$$\vec{k} = (k, m) = \kappa(\cos\theta, \sin\theta)\,; \quad \kappa = (k^2 + m^2)^{1/2}\,,$$

Such that:

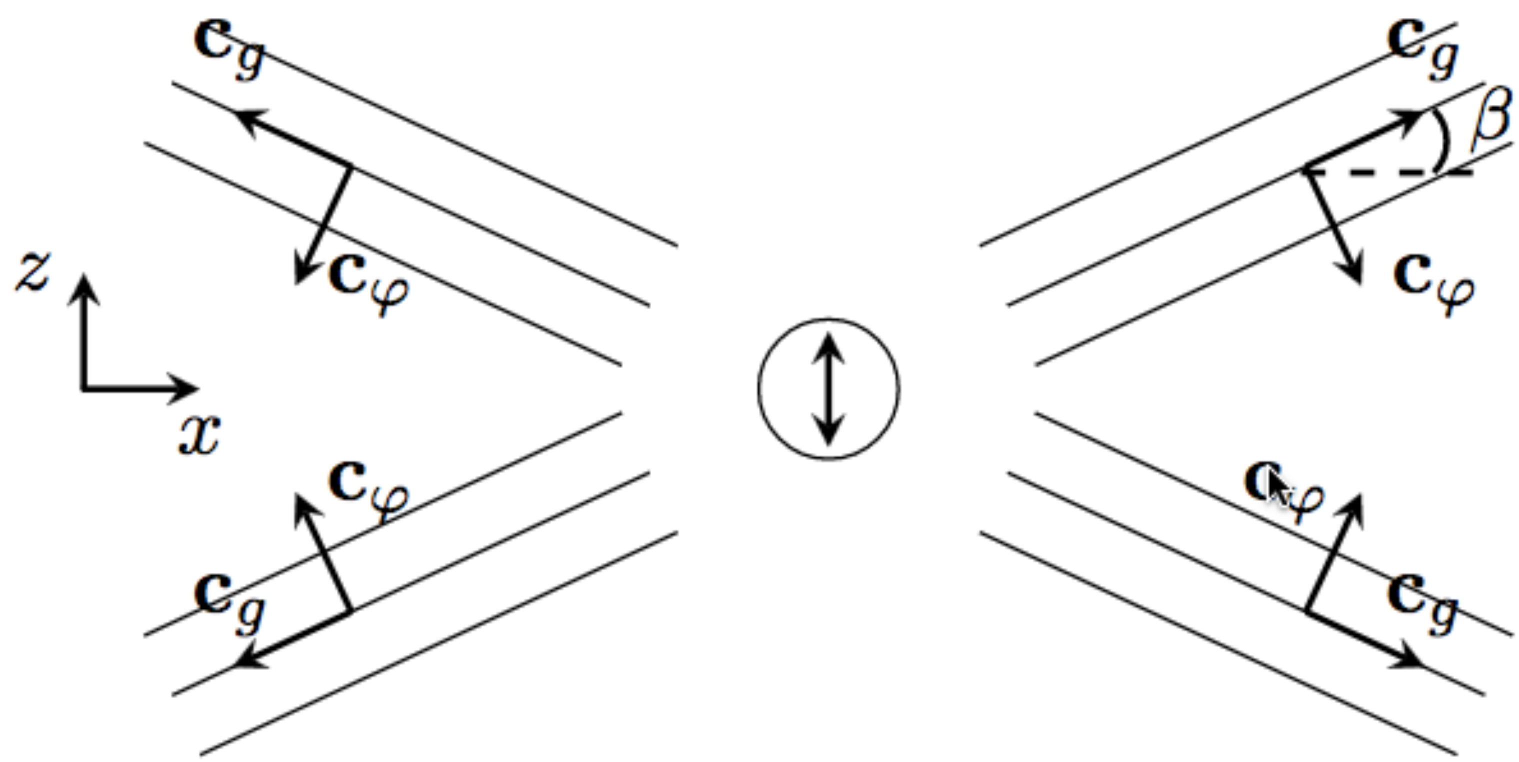
$$\boxed{\omega^2 = N^2\cos^2\theta + f^2\sin^2\theta}$$

The wave frequency depends only on the direction θ of propagation and of the local N and f.

if the frequency is imposed (e.g., by tidal forcing), all waves propagate at fixed angles from the horizontal.

Solutions: Method of characteristics

- The four possible configurations of internal wave beams created by an oscillating body at fixed frequency:



A famous experiment

J. Fluid Mech. (1967), *vol.* 28, *part* 1, *pp.* 1–16
Printed in Great Britain

A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density stratified liquid

By D. E. MOWBRAY AND B. S. H. RARITY†
Department of the Mechanics of Fluids, University of Manchester

L'expérience de MOWBRAY et RARITY (1967) conçue pour vérifier la relation de dispersion est à ce stade particulièrement intéressante pour bien comprendre les conséquences de résultats inhabituels. Il s'agit d'observer dans une cuve rectangulaire les ondes générées par un barreau oscillant verticalement à une fréquence donnée ω. La cuve est remplie d'eau salée de manière à obtenir une stratification linéaire (l'eau plus salée et donc plus dense se situe par conséquent en bas) et les ondes sont observées par ombroscopie, en utilisant le fait que l'indice optique dépend de la concentration en sel. Si des mouvements sont forcés à la pulsation ω_f dans un fluide de gradient de masse volumique constant, toute onde de vecteur d'onde faisant un angle $\beta = \arcsin(\omega_f / N)$ avec la verticale est théoriquement excitée. Comme les rayons sont donnés par les directions de la vitesse de groupe, on voit donc apparaître *une croix* (*cf.* figure 3), avec quatre branches, chacune faisant un angle β avec l'horizontale. On est donc bien loin de l'onde circulaire que donnerait une émission acoustique localisée par exemple.

Gostiaux, Dauxois, BUP868

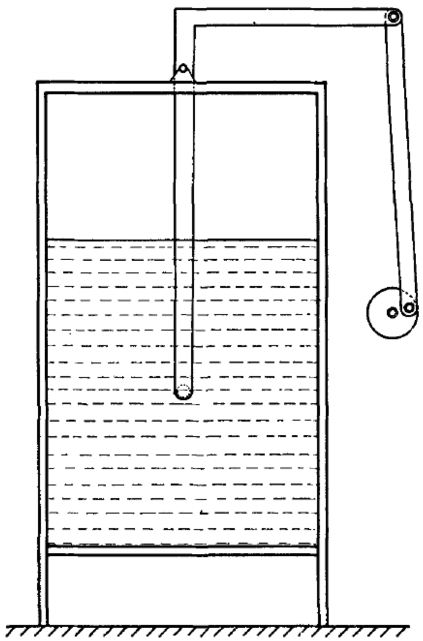


FIGURE 4. A sketch of the apparatus.

Journal of Fluid Mechanics, Vol. 28, *part* 1 — *Plate* 1

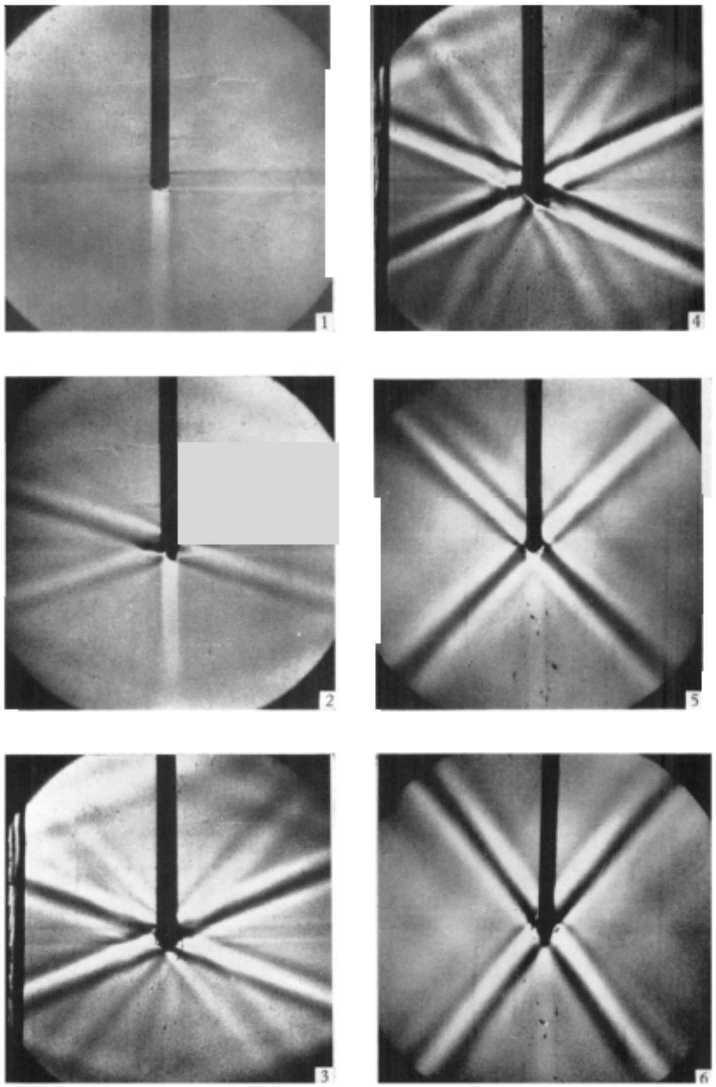


PLATE 1. (1) The image of the undisturbed fluid. (2) $\omega/\omega_0 = 0{\cdot}318$. (3) $\omega/\omega_0 = 0{\cdot}366$. (4) $\omega/\omega_0 = 0{\cdot}419$. (5) $\omega/\omega_0 = 0{\cdot}615$. (6) $\omega/\omega_0 = 0{\cdot}699$.

6. Conclusion

The predictions of the small amplitude theory of the phase configurations of internal waves in a stratified fluid have been tested and confirmed.

https://www.jgula.fr/Ondes/MowbrayRarity67.pdf

Solutions: Method of characteristics

- The group velocity:

$$\vec{c}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial m}\right)$$

- Can be written using the new coordinates:

$$\omega(k(\kappa,\theta), m(\kappa,\theta)) = \bar{\omega}(\kappa,\theta)$$

- Such that

$$\frac{\partial \bar{\omega}}{\partial \kappa} = \frac{\partial \omega}{\partial k}\frac{\partial k}{\partial \kappa} + \frac{\partial \omega}{\partial m}\frac{\partial m}{\partial \kappa} = \vec{c}_g \cdot \vec{k}/\kappa$$

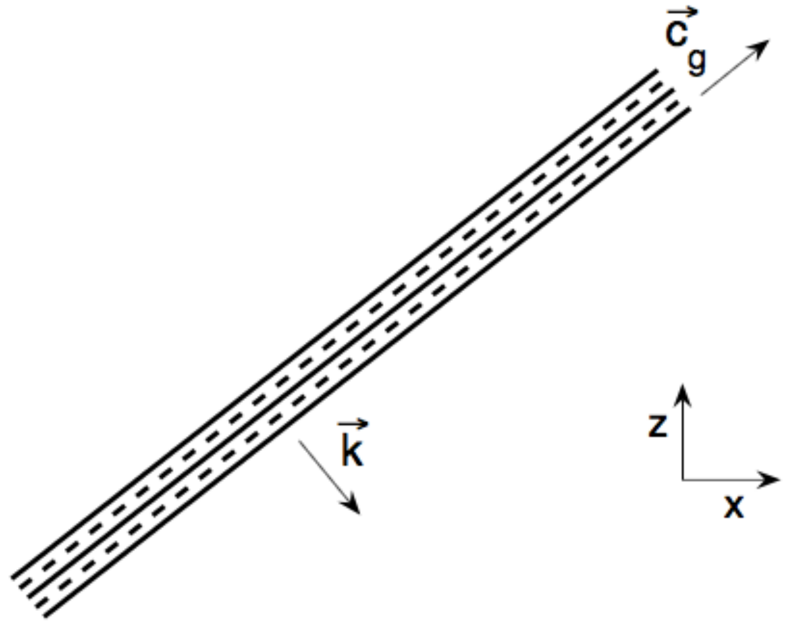
- ω does not depend on κ so:

$$\vec{c}_g \cdot \vec{k} = 0$$

$$\boxed{\vec{c}_g \perp \vec{k}}$$

The group velocity is always perpendicular to the phase speed and makes an angle θ with the vertical.

Solutions: Method of characteristics



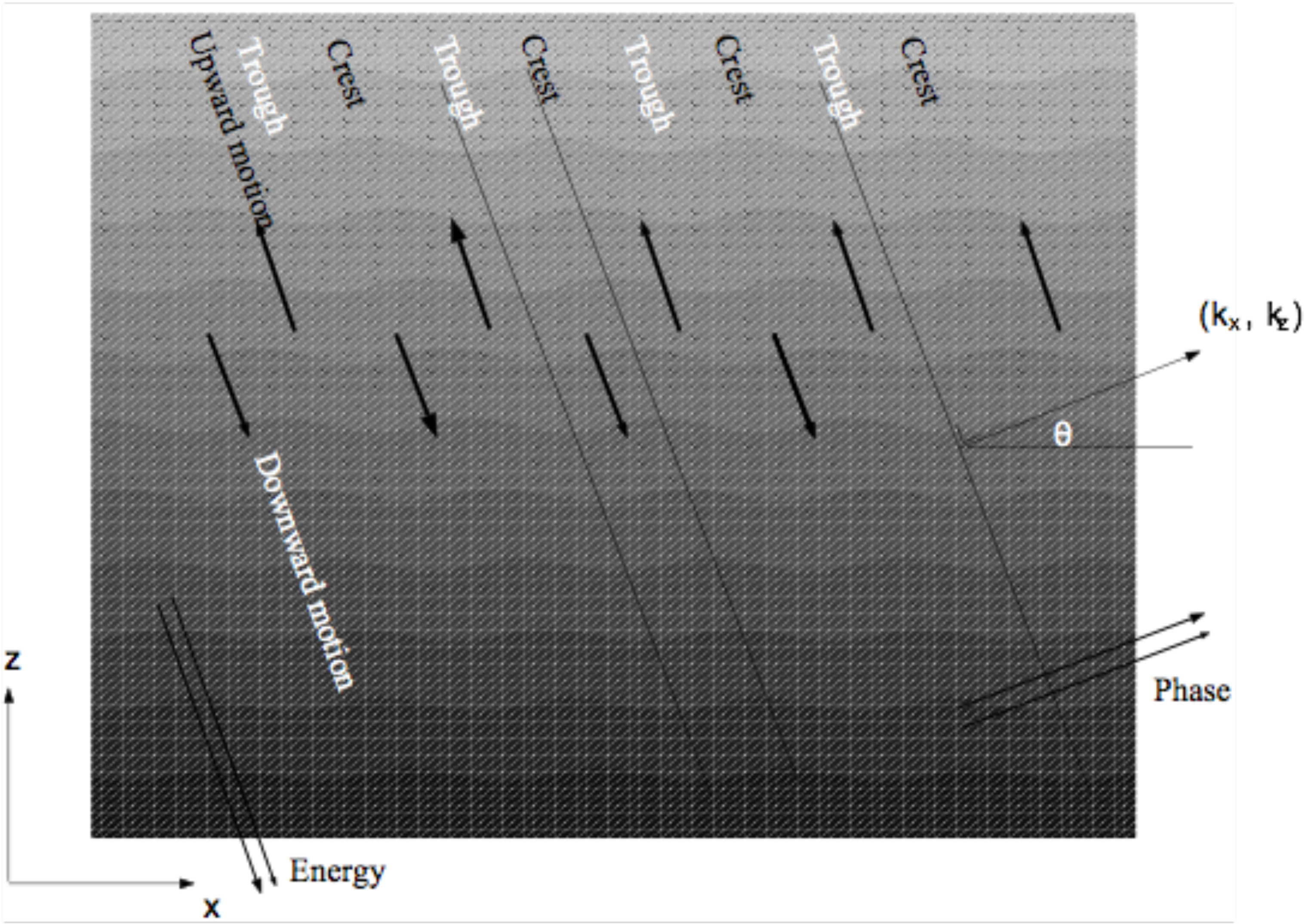
$$\tan\theta = \pm\left(\frac{N^2-\omega^2}{\omega^2-f^2}\right)^{1/2}$$

The group velocity is always perpendicular to the phase speed and makes an angle θ with the vertical.

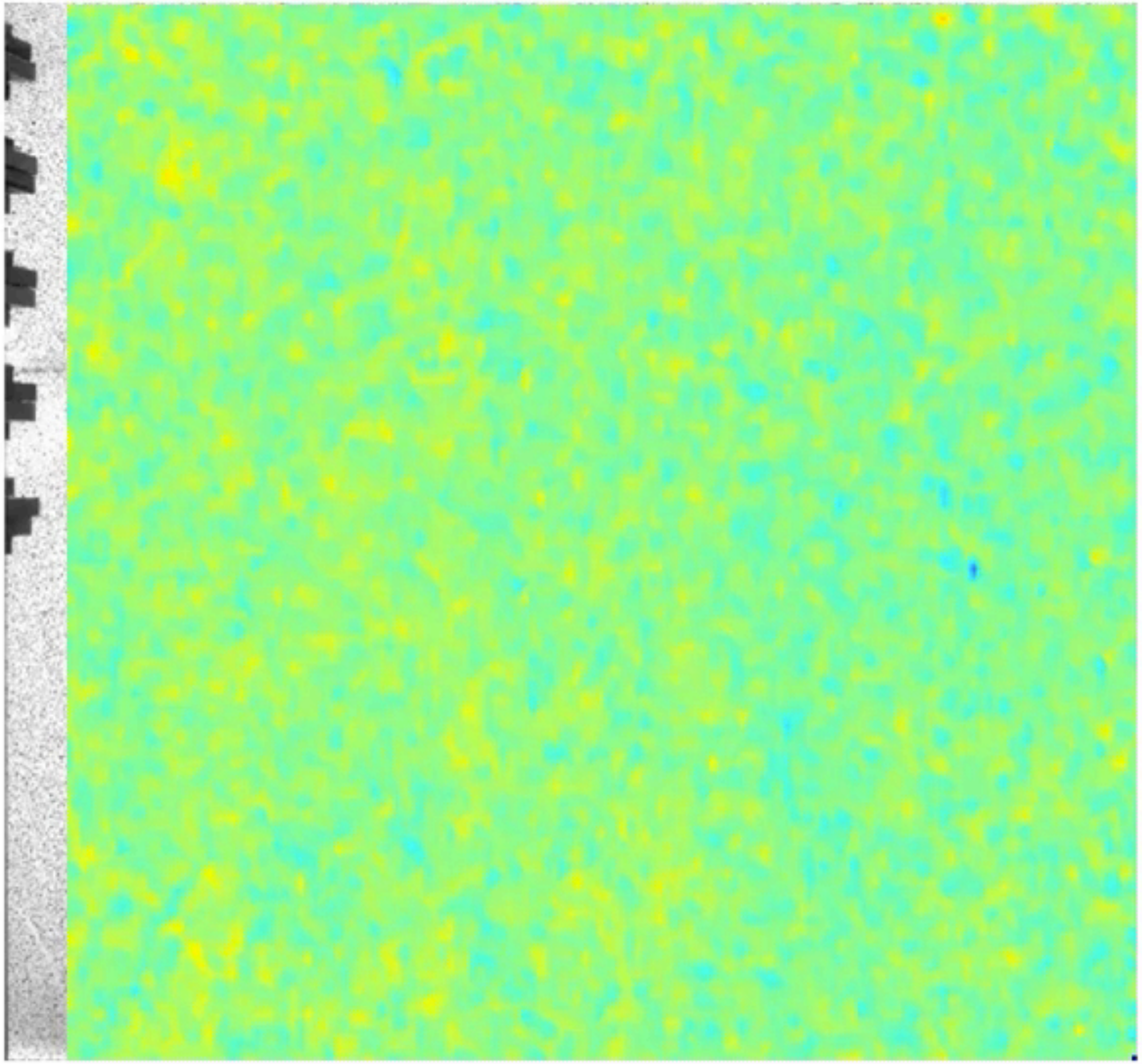
Solutions: Method of characteristics

- Using the continuity equation we can see that

$$\vec{u} \perp \vec{k}$$

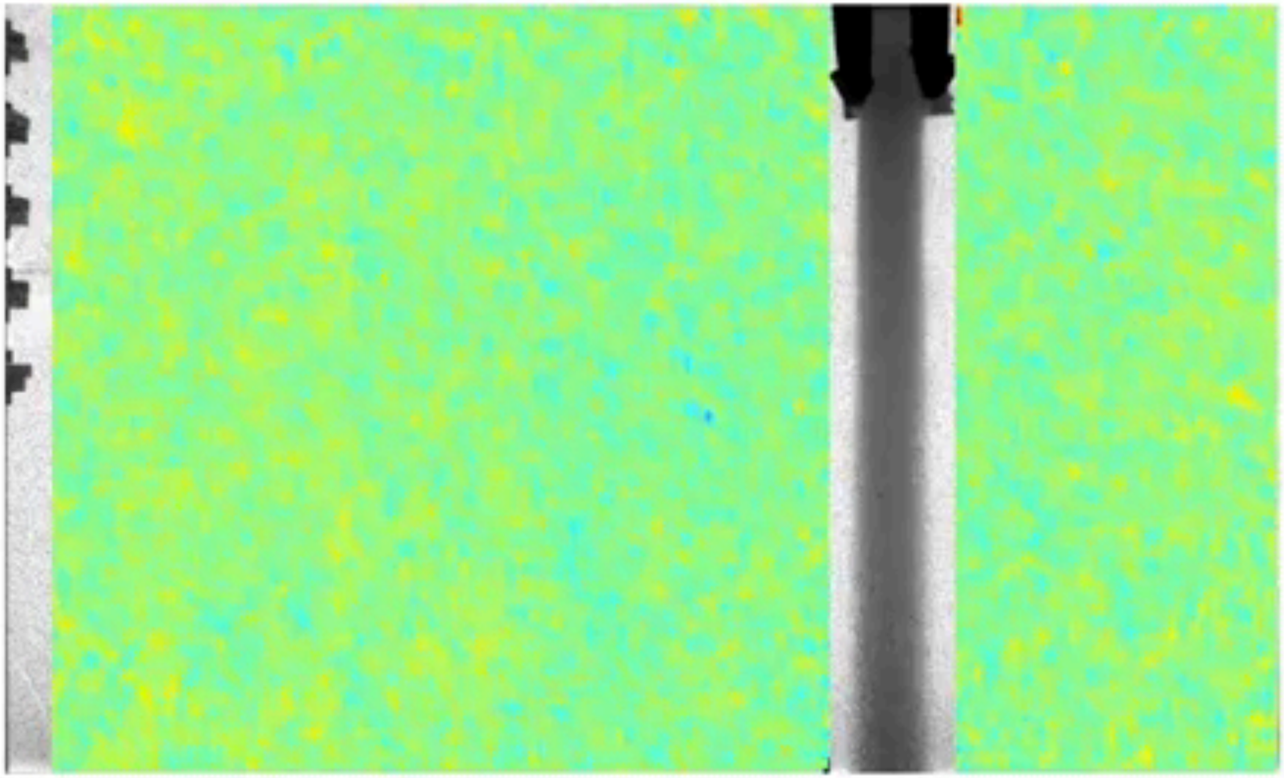


Solutions: Method of characteristics



[Animation E. Horne]

Solutions: Method of characteristics



[Animation E. Horne]

Solutions: Method of characteristics

- Solutions exist only in a range of allowable internal-wave frequencies:

$$(\mathrm{I}) \quad N \leq \omega \leq |f| \quad \text{or} \quad (\mathrm{II}) \quad |f| \leq \omega \leq N$$

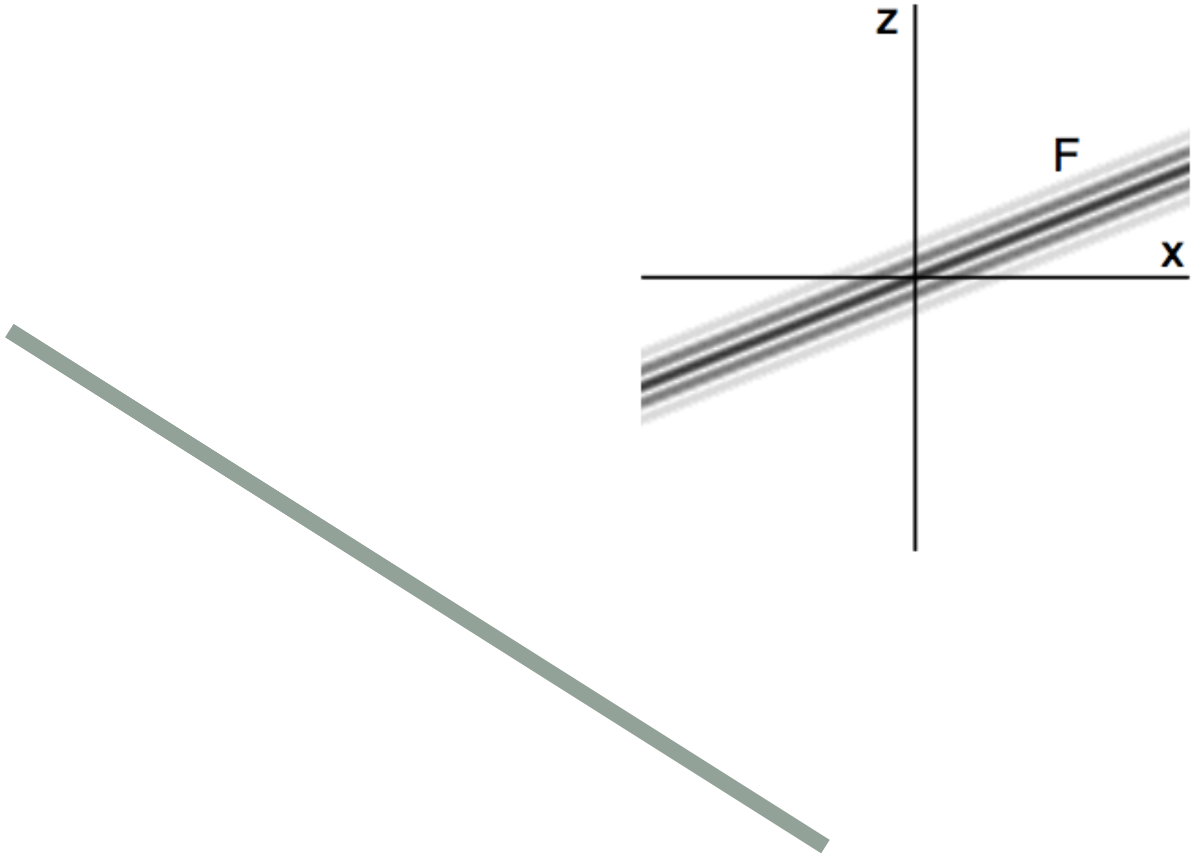
- 2 extreme cases can occur:

$\omega \rightarrow f$ k becomes vertical, and the group velocity horizontal

$\omega \rightarrow N$ k becomes horizontal and the group velocity vertical

Solutions: Method of characteristics

Wave Reflection?

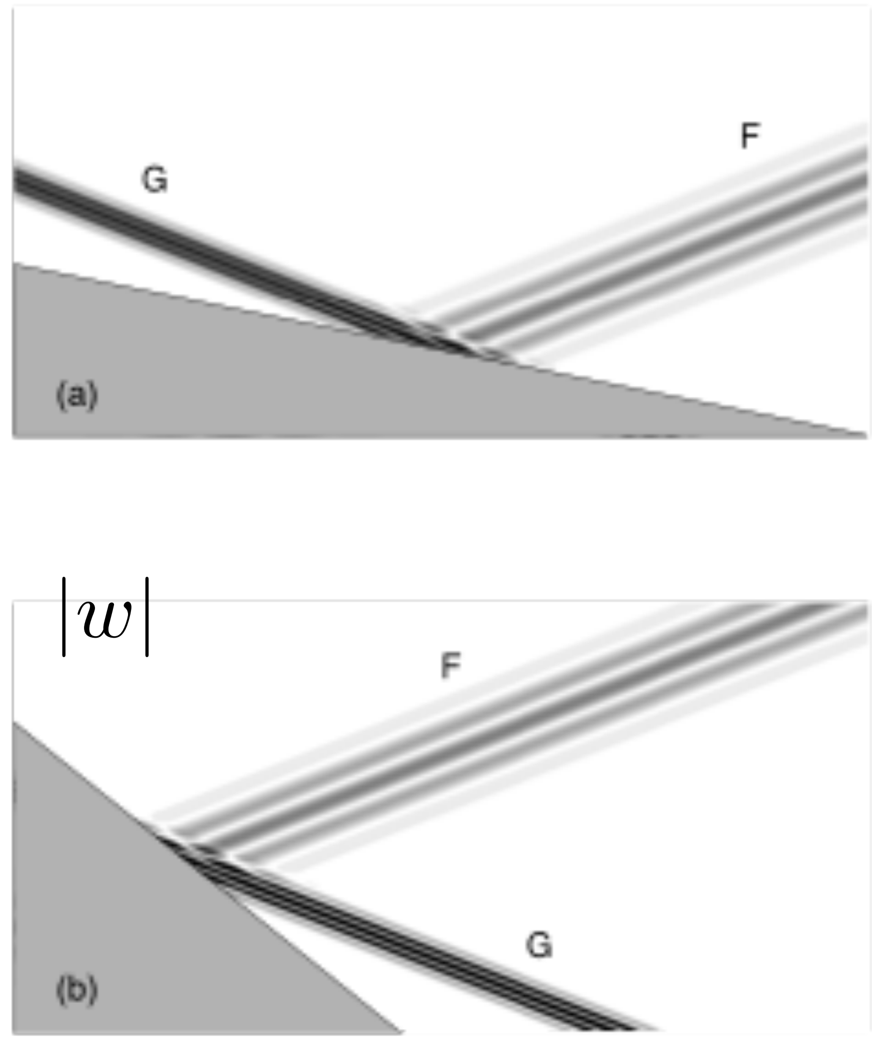


Solutions: Method of characteristics

Wave Reflection

The wave frequency is determined by the angle of propagation (θ). Vice versa, for a given frequency ω, the angle θ is fixed.

After reflection, energy must again propagate at an angle θ with the vertical, since the wave frequency has not changed.



Solutions: Method of characteristics

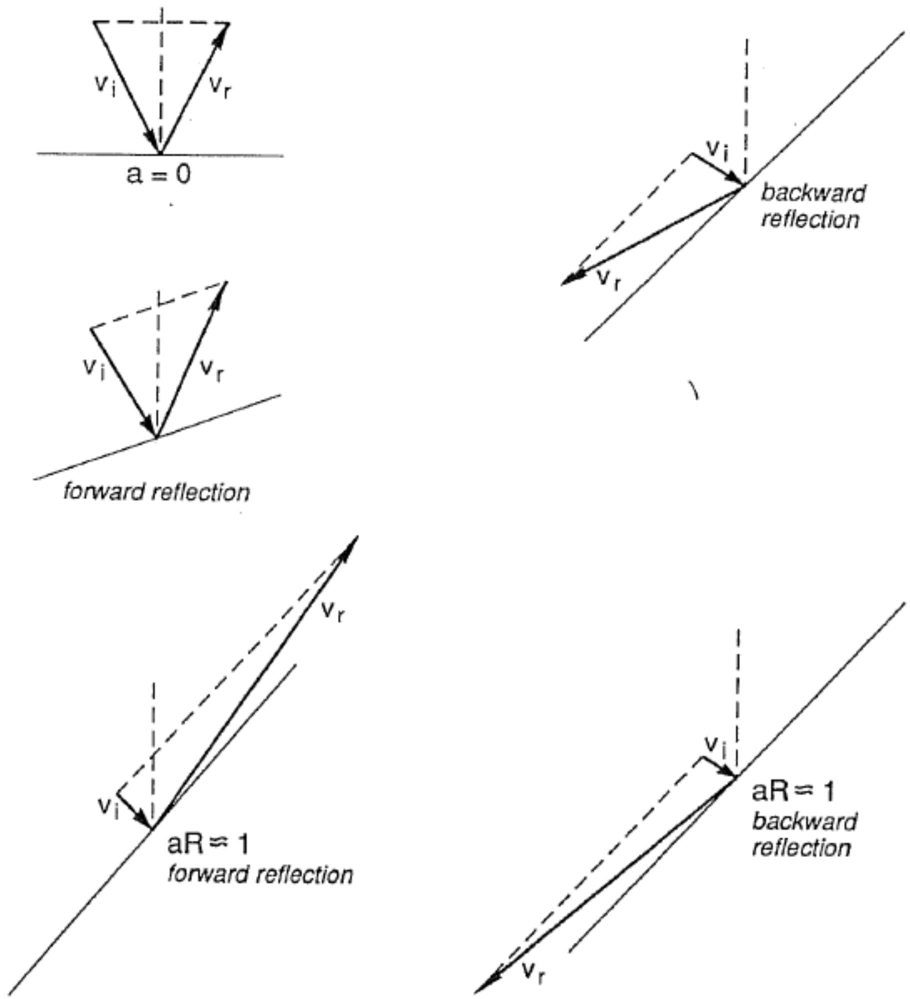
Wave Reflection

Several possible cases:

- For a flat bottom, energy is reflected with the same angle with the vertical
- Depending on the bottom slope, the reflection can be forward or backward

- The critical angle corresponds to a slope with the same angle than the ray:

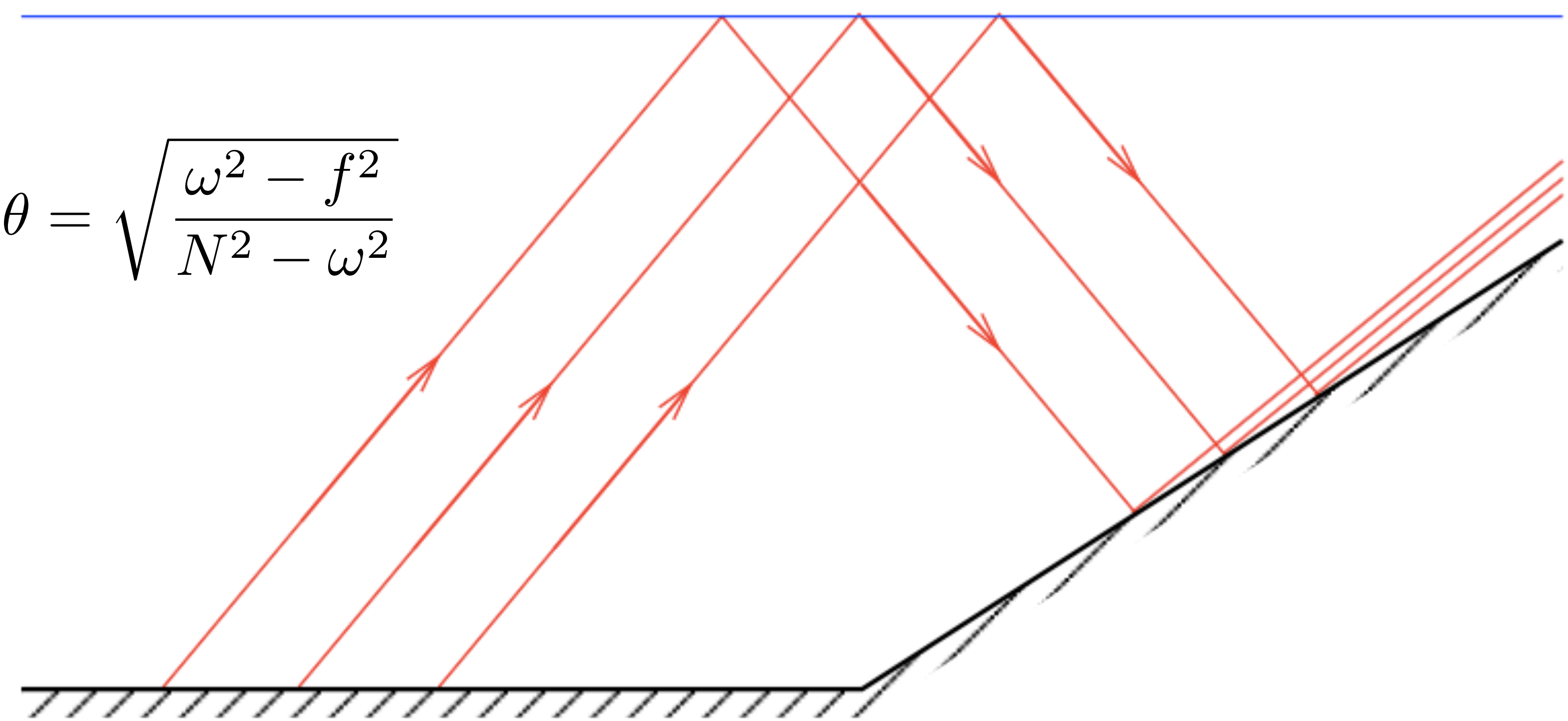
$$\tan \theta = \pm \left(\frac{N^2 - \omega^2}{\omega^2 - f^2} \right)^{1/2}$$



Reflection of internal waves

Internal wave characteristics approaching a sloping boundary. After reflection, the energy in the waves is concentrated into a narrower band (energy density increases).

$$\tan\theta = \sqrt{\frac{\omega^2 - f^2}{N^2 - \omega^2}}$$



Another famous experiement:

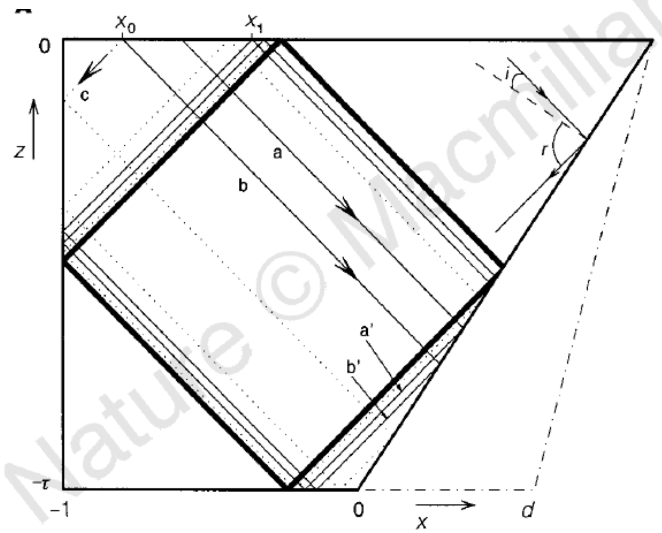
Observation of an internal wave attractor in a confined, stably stratified fluid

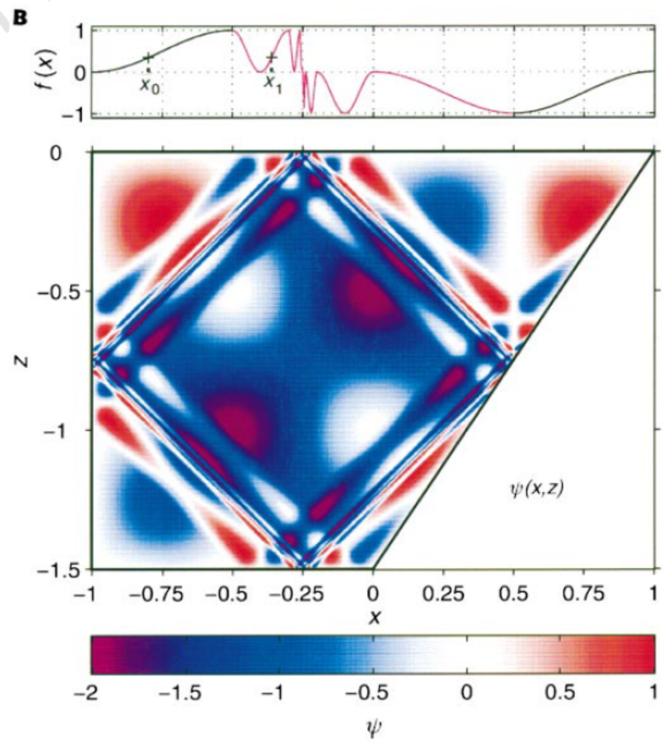
Leo R. M. Maas*, Dominique Benielli†, Joël Sommeria† & Frans-Peter A. Lam*

* *Netherlands Institute for Sea Research, PO Box 59, 1790 AB Texel, The Netherlands*
† *Ecole Normale et Superieure de Lyon, Laboratoire de Physique, 46 allée d'Italie, 69364 Lyon cedex 07, France*

NATURE | VOL 388 | 7 AUGUST 1997

https://www.jgula.fr/Ondes/maasetal97.pdf





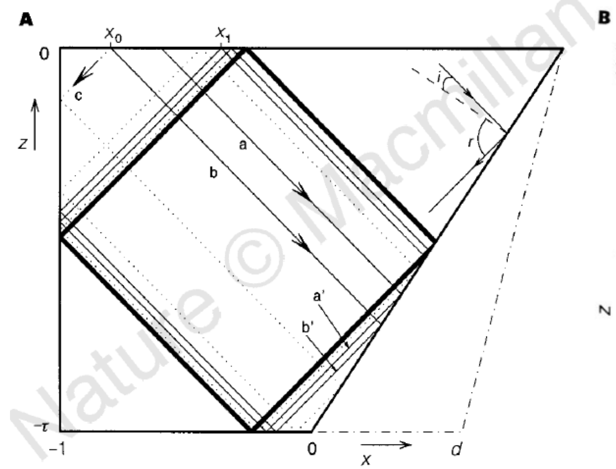
Another famous experiement:

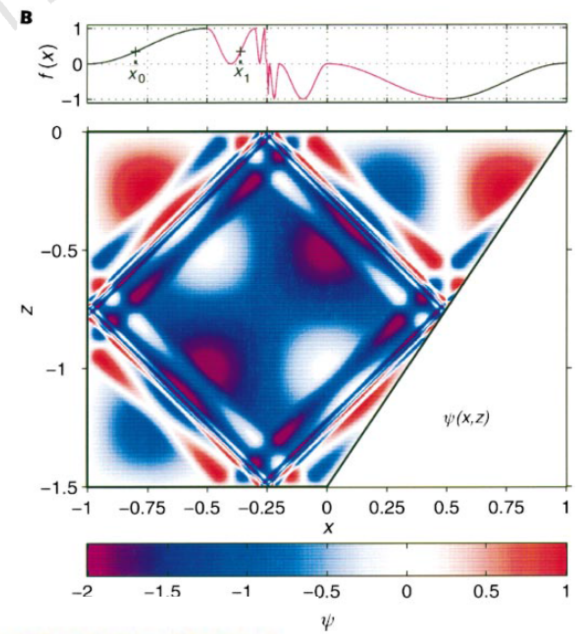
Observation of an internal wave attractor in a confined, stably stratified fluid

Leo R. M. Maas*, Dominique Benielli†, Joël Sommeria† & Frans-Peter A. Lam*

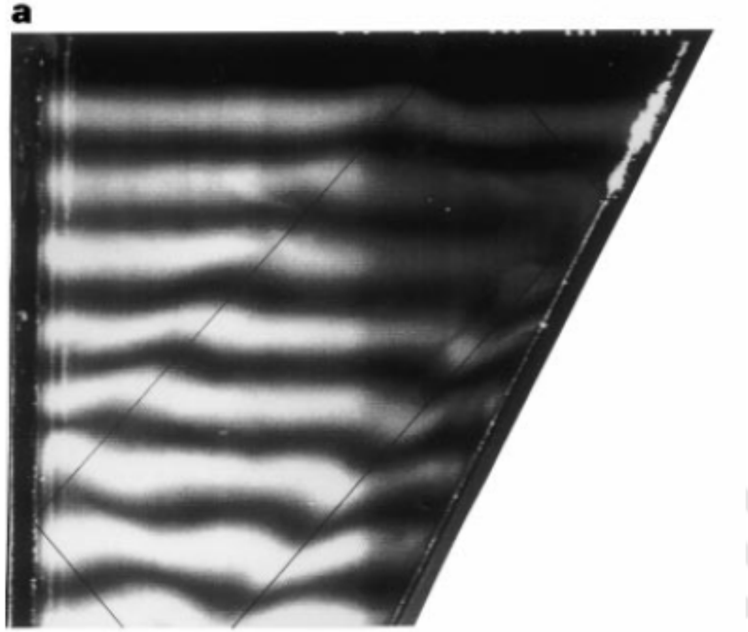
* *Netherlands Institute for Sea Research, PO Box 59, 1790 AB Texel, The Netherlands*
† *Ecole Normale et Superieure de Lyon, Laboratoire de Physique, 46 allée d'Italie, 69364 Lyon cedex 07, France*

NATURE | VOL 388 | 7 AUGUST 1997

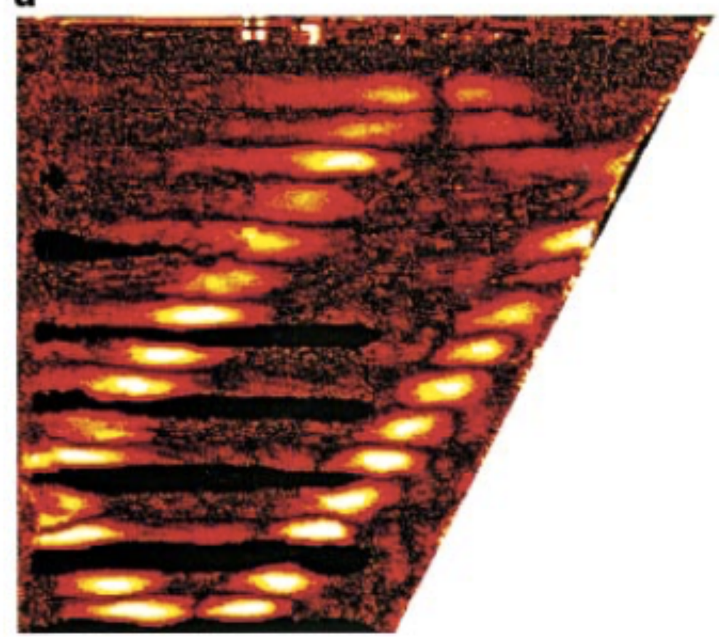




a



a

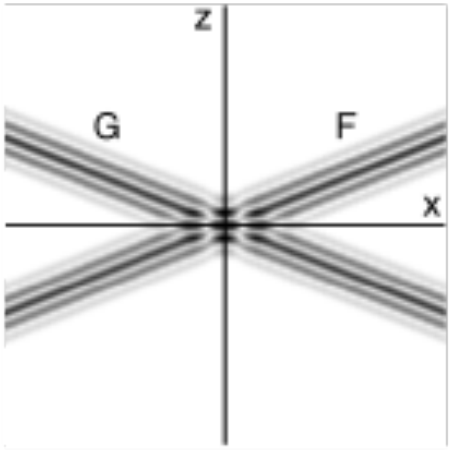


Solutions of the equation

1. Method of characteristics

- N = cste
- Arbitrary boundary conditions
- Solution of the form:

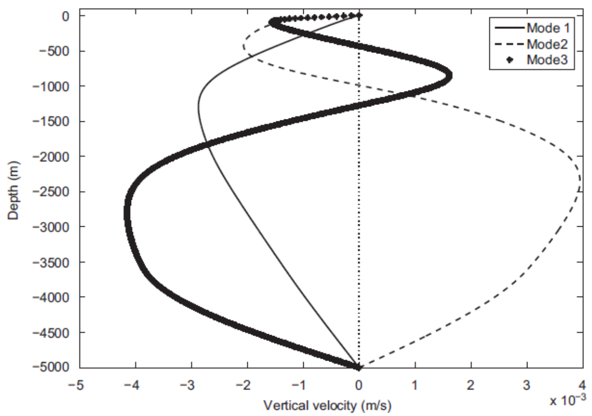
$$\hat{w} = F(\mu_+ x - z) + G(\mu_- x - z)$$



2. Method of Modes

- N = N(z)
- Horizontal surface/bottom
- Solution of the form:

$$w = W(z)e^{-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}ly}$$



Solutions: Method of modes

$$(N^2 - \omega^2)\nabla_h^2 \hat{w} - (\omega^2 - f^2)\hat{w}_{zz} = 0$$

N = N(z)

- We restrict ourselves to the (x,z) plane and look for a solution of the form:

$$w = W(z)e^{-\mathrm{i}\omega t + \mathrm{i}kx}$$

- Which gives the equation:

$$\boxed{W'' + k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2} W = 0}$$

- With boundary conditions:

$$W = 0 \quad \text{at} \quad z = 0, -H\,.$$

Solutions: Method of modes

- These equations form a Sturm-Liouville problem, which for fixed ω has an infinite number of solutions W_n (eigenfunctions, vertical modes) with corresponding eigenvalues k_n.

- The general solution will be the superposition:

$$w = \sum_n W_n(z) \left[a_n^{\pm} \exp i(k_n^{\pm} x - \omega t) \right]$$

Solutions: Method of modes

- Two types of solution depending on the sign of:

$$m^2(z) = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}$$

- Oscillatory if $m^2(z) \geq 0.$

$$\boxed{\text{(I)} \quad N(z) \leq \omega \leq |f| \quad \text{or} \quad \text{(II)} \quad |f| \leq \omega \leq N(z)\,.}$$

- Exponential decay otherwise.

Solutions: Method of modes

- Case 1 - Constant N:

$$W'' + m^2 W = 0 \qquad m^2(z) = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}$$

- Activity: Solve the equation for the case of constant N

  1. Find the dispersion relation

  2. Find the general solution

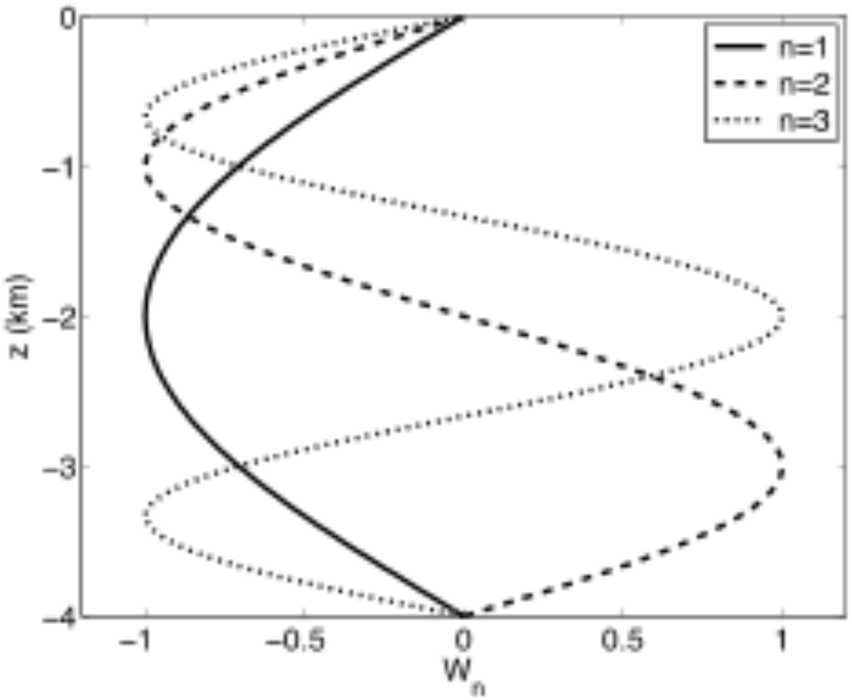
Solutions: Method of modes

- For a constant N:

$$k_n = \pm \frac{n\pi}{H}\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}, \qquad n = 1, 2, 3, \cdots .$$

$$W_n = \sin\left(\frac{n\pi z}{H}\right), \qquad n = 1, 2, 3, \cdots .$$

$$w = \sum_n a_n \sin\left(\frac{n\pi z}{H}\right) \cos(k_n x - \omega t) .$$



- Note that higher vertical modes have shorter wavelengths

Solutions: Method of modes

- We can also write:

$$\omega^2 = \frac{N^2k^2 + f^2(\frac{n\pi}{H})^2}{k^2 + (\frac{n\pi}{H})^2}.$$

  - Horizontal phase speed:

$$\begin{aligned} c &= \frac{[N^2k^2 + f^2(\frac{n\pi}{H})^2]^{1/2}}{k[k^2 + (\frac{n\pi}{H})^2]^{1/2}} \\ &= \pm\left(\frac{H\omega}{n\pi}\right)\left(\frac{N^2 - \omega^2}{\omega^2 - f^2}\right)^{1/2}. \end{aligned}$$

  - Horizontal group speed:

$$\begin{aligned} c_g &= \frac{k(\frac{n\pi}{H})^2(N^2 - f^2)}{[N^2k^2 + f^2(\frac{n\pi}{H})^2]^{1/2}\,[k^2 + (\frac{n\pi}{H})^2]^{3/2}} \\ &= \pm\left(\frac{H}{n\pi}\right)\frac{(\omega^2 - f^2)^{1/2}(N^2 - \omega^2)^{3/2}}{\omega(N^2 - f^2)}. \end{aligned}$$

- **And see that higher vertical modes propagate more slowly**

Solutions: Method of modes

- For a constant N:

$$k_n = \pm \frac{n\pi}{H} \left(\frac{\omega^2 - f^2}{N^2 - \omega^2} \right)^{1/2}, \qquad n = 1, 2, 3, \cdots.$$

$$\begin{aligned}
u &= -\sum_n a_n \frac{n\pi}{k_n H} \cos\left(\frac{n\pi z}{H}\right) \sin(k_n x - \omega t) \\
v &= \frac{f}{\omega} \sum_n a_n \frac{n\pi}{k_n H} \cos\left(\frac{n\pi z}{H}\right) \cos(k_n x - \omega t) \\
p &= -\rho_* \frac{\omega^2 - f^2}{\omega} \sum_n a_n \frac{n\pi}{k_n^2 H} \cos\left(\frac{n\pi z}{H}\right) \sin(k_n x - \omega t) \\
b &= \frac{N^2}{\omega} \sum_n a_n \sin\left(\frac{n\pi z}{H}\right) \sin(k_n x - \omega t).
\end{aligned}$$

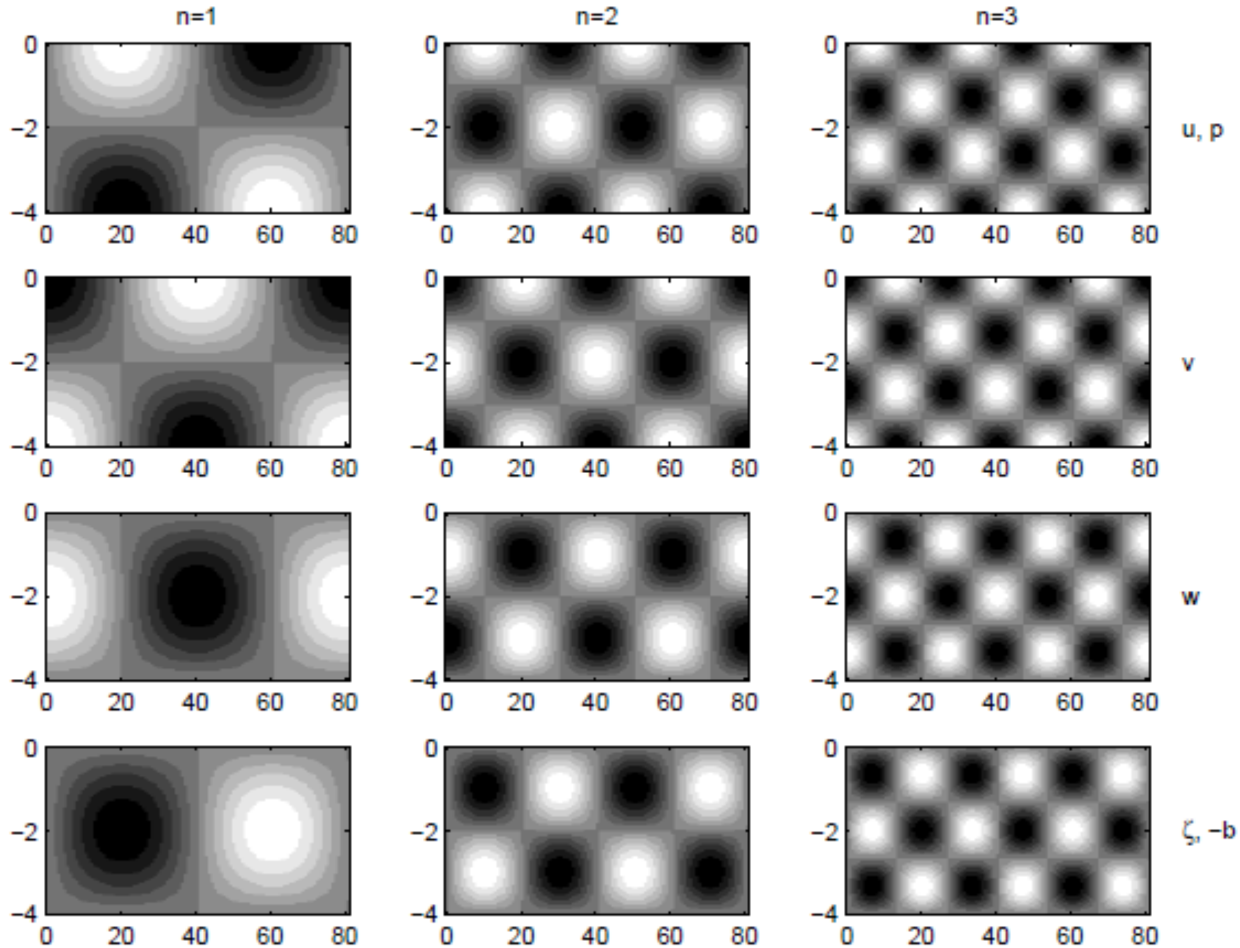


Fig. 5.4: Spatial structure of the first three modes, for constant stratification, at $t = 0$. Upper panels show the horizontal velocity component u; the second row, the tranverse velocity v; the third row, the vertical velocity w; and the lowest panels the isopycnal elevation ζ. The first and fourth rows also represent pressure p and minus buoyancy b, respectively. White denotes negative values; black, positive ones. Parameters are: $N = 1 \times 10^{-3}$, $f = 1 \times 10^{-4}$, and $\omega = 1.405 \times 10^{-4}$ (the semi-diurnal lunar tidal frequency, M_2), all in rad s^{-1}. Along the vertical is water depth, with $H = 4$ km; horizontal distances are also in km.

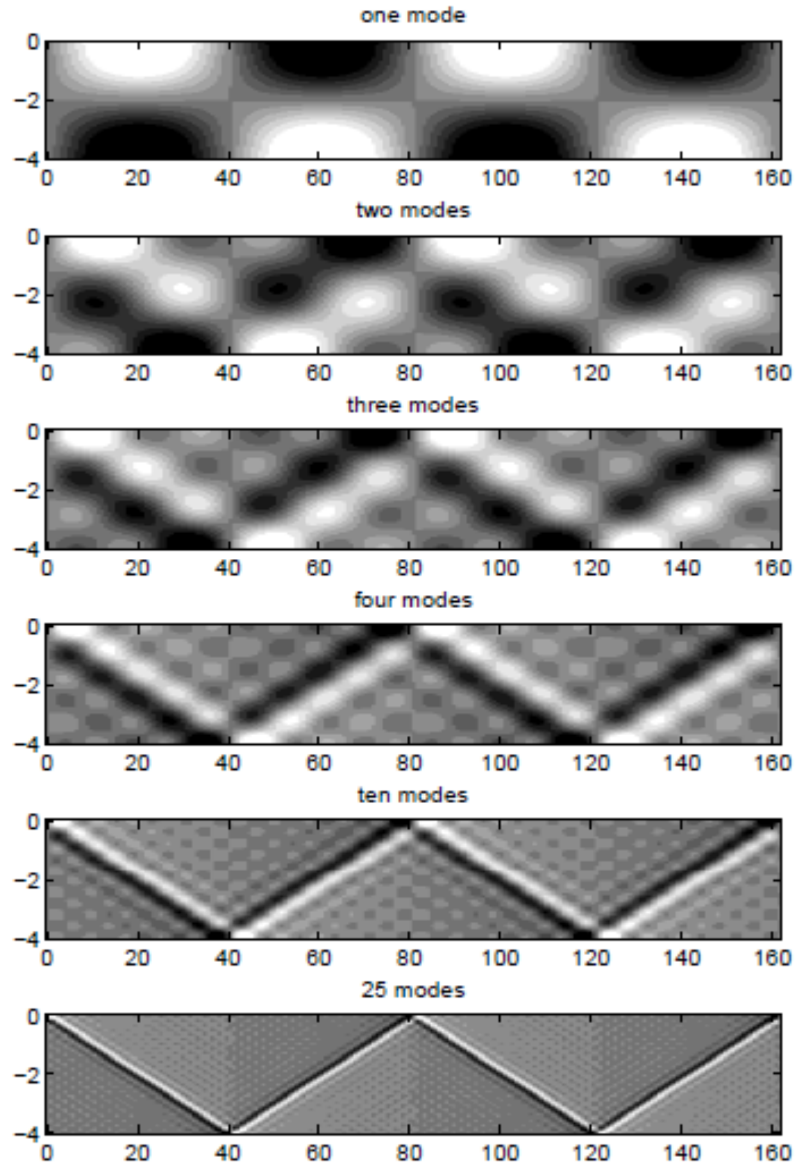
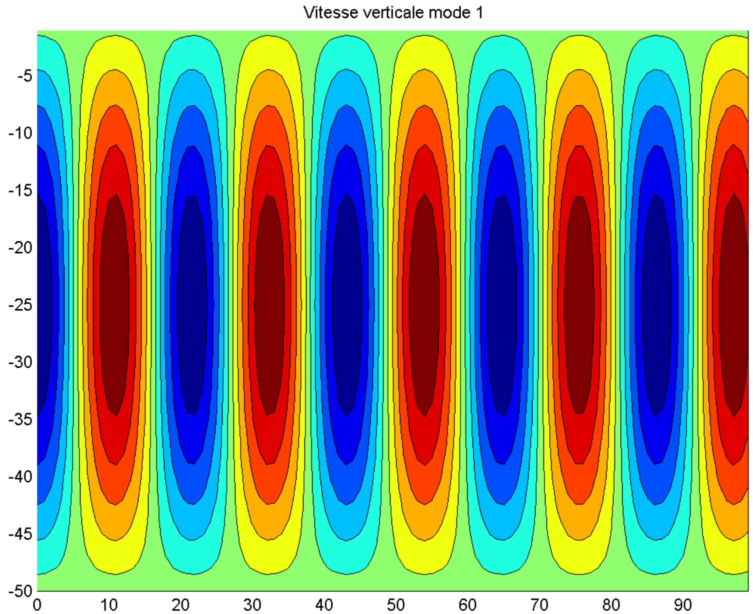


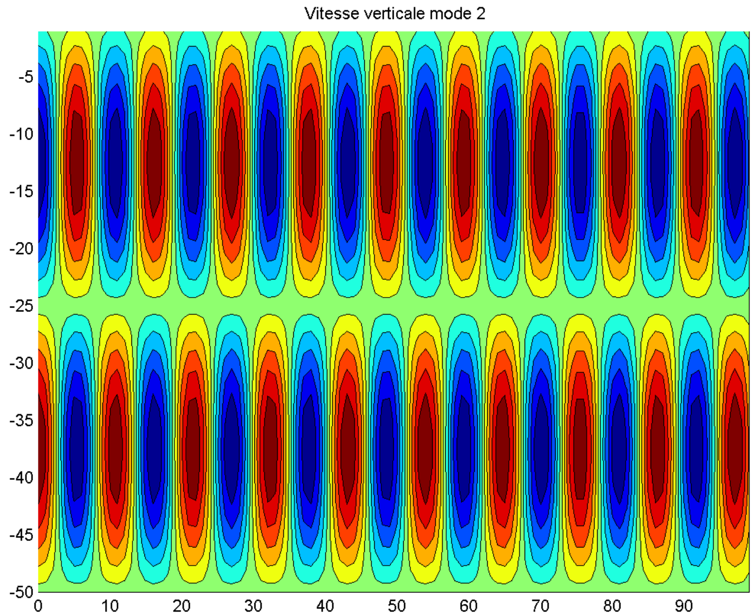
Fig. 5.6: Superpositions with an increasing number of modes, each at $t = 0$, showing the horizontal current velocity u. White denotes negative values; black, positive ones. Parameters as in Figure 5.4.

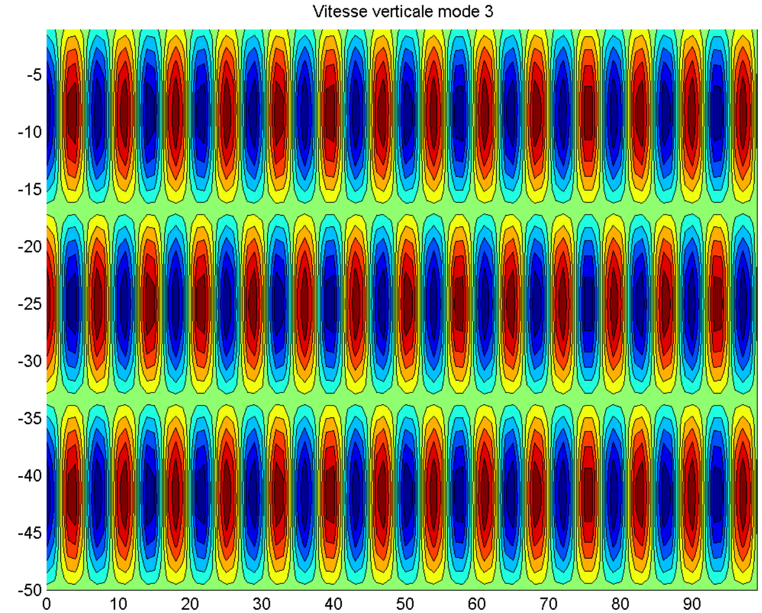
- **Numerical example:**

  - Latitude=47°N
  - omega=omega_M2=1.4 e^{-4} s^{-1}
  - H=50 m
  - Constant stratification (linear variation of temperature from 12°C at bottom to 12°+10°C at the surface (N2= 4*10^{-4} s^{-2})
  - Arbitrary coefficients a_n=1

Vertical velocity







$$k_n = \frac{n\pi}{H}\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}$$

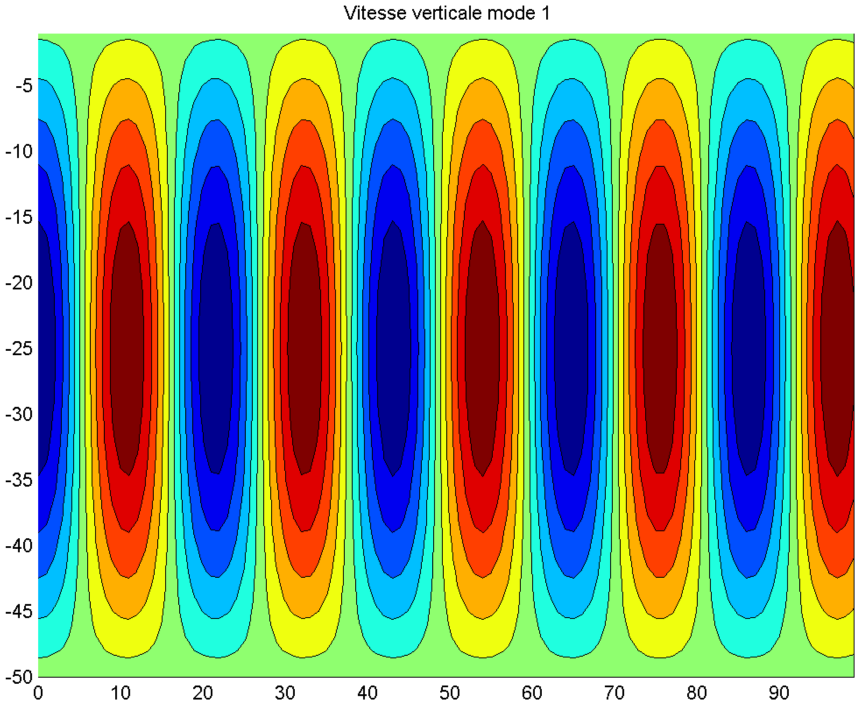
Wavelength :

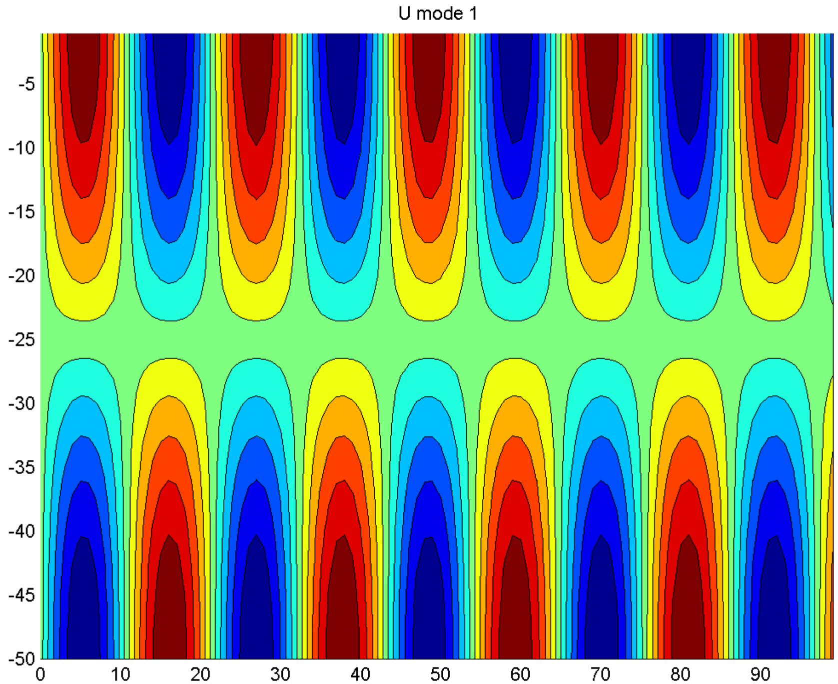
Mode 1: 21 km

Mode 2=10.5 km

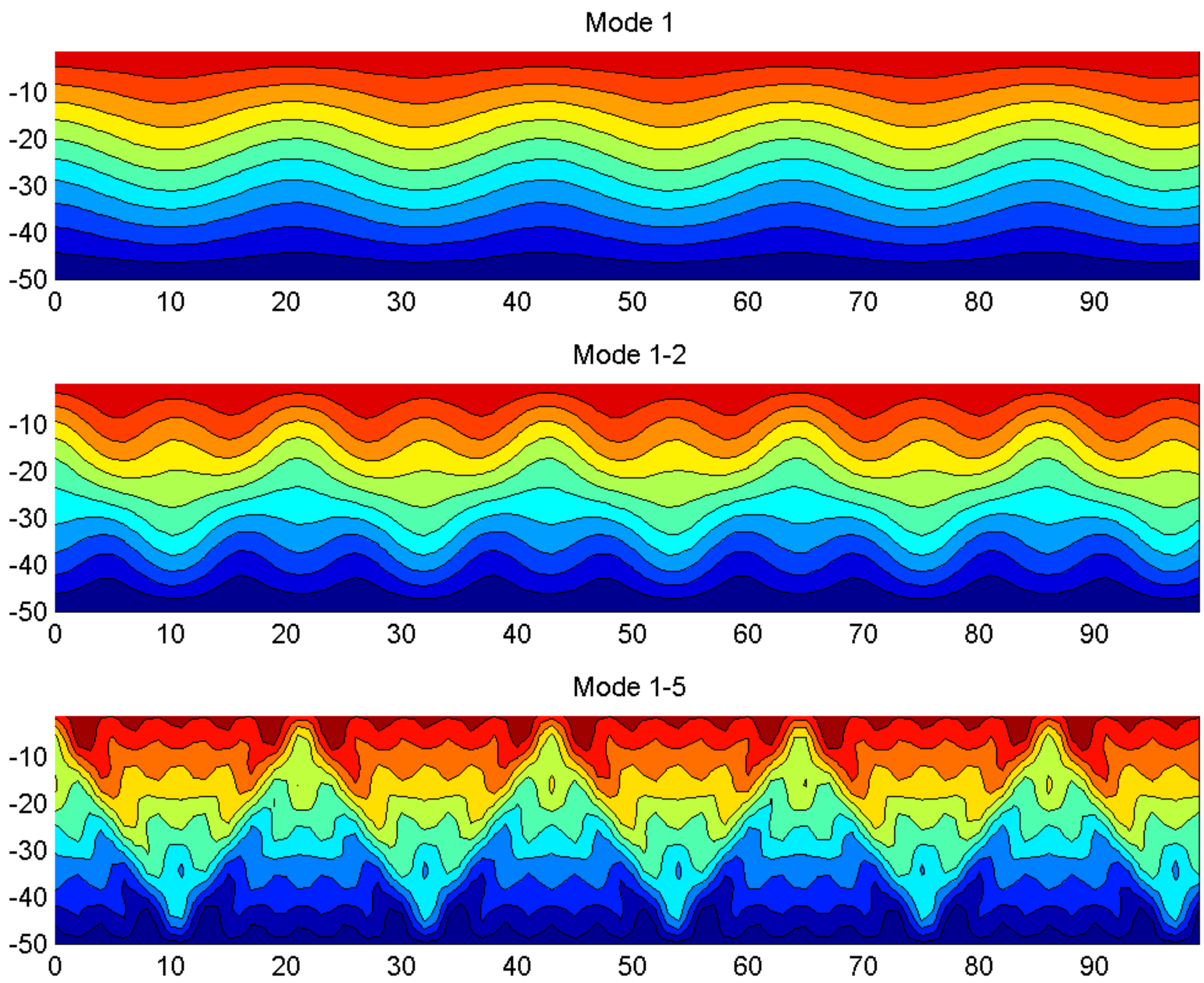
Mode 3: 7 km

- Horizontal velocity (U=iW'/kn)





Composition of modes



Solutions: Method of modes

Observations of an internal tidal beams:

Why is it curved?

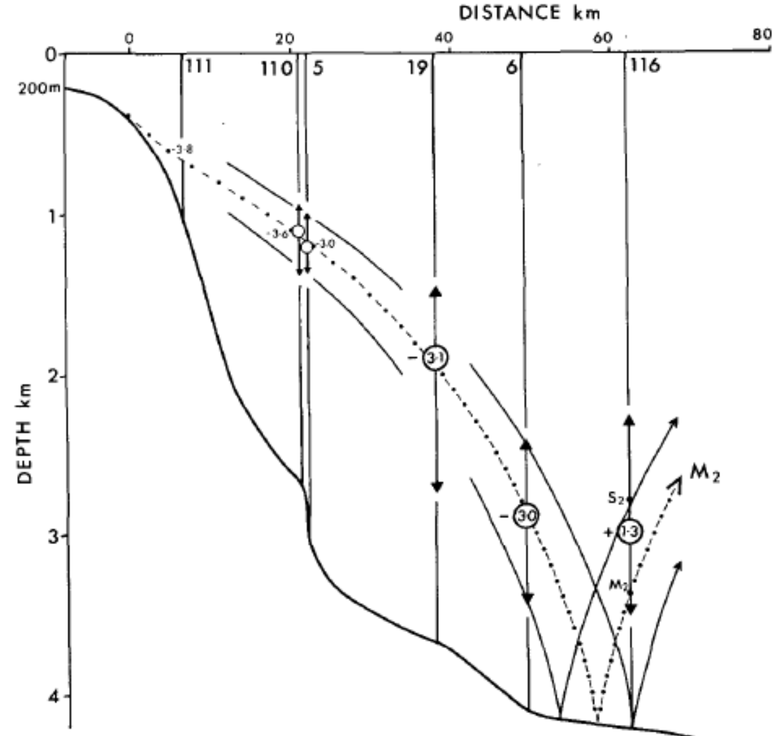
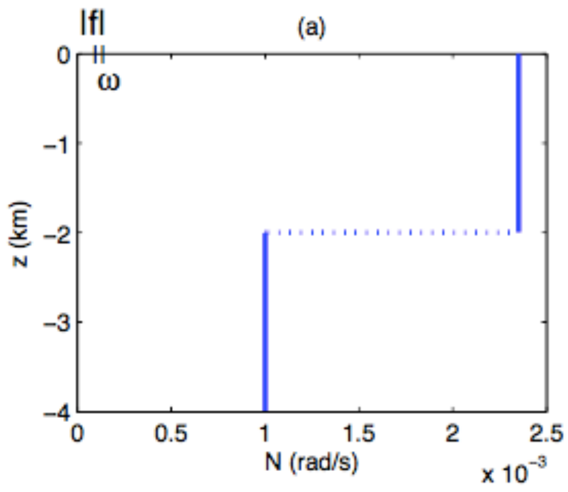


FIG. 9. Diagram showing the theoretical ray path (chained line) for a beam of internal tidal energy at the M_2 tidal frequency emanating from the critical depth (385 m) on the upper slopes and reflecting off the Biscay abyssal plain at a depth of about 4200 m, 58 km from the critical point. Also shown is a summary of the internal tidal oscillations obtained during the RRS *Challenger* cruises in 1988 (CH 31/88) and 1987 (CH 18/87). Vertical lines represent mooring and CTD station positions and are identified with numbers. CTD stations 5 and 6 and mooring 116 are from the 1988 cruise, whereas moorings 110 and 111 and CTD 19 were obtained in 1987. The depth of the maximum amplitude of the internal tidal oscillation found at each station is plotted as an open circle and the range where the amplitude is more than 70% of the maximum value is indicated by the arrows. Two further rays are shown (solid lines) passing through the 70% limits near mooring 110. The phase of the maximum upward displacement is given (within the circles) in hours with respect to HWP. A ray at the M_2 tidal frequency would intersect mooring 116 at the depth marked M_2; S_2 is the corresponding point for a ray at the S_2 tidal frequency. The topography is depicted by the bold line and is critical at 385 m; the horizontal distance scale is measured from the critical point.

Solutions: Method of modes

- For a 2-layer N:

$$N(z) = \begin{cases} N_1 & -d < z < 0 \quad \text{(upper layer)} \\ N_2 & -H < z < -d \quad \text{(lower layer)}. \end{cases}$$



Solutions: Method of modes

- For a 2-layer N:
  - We can write the solution for each layer:

$$W(z) = \begin{cases} C_1 \sin m_1 z & -d < z < 0 & \text{(upper layer)} \\ C_2 \sin m_2 (z + H) & -H < z < -d & \text{(lower layer)}, \end{cases}$$

  - The condition of continuity at the interface (for W and W') gives the dispersion relation *(which has to be solved numerically)*

$$m_2 \sin m_1 d \cos m_2 (H - d) + m_1 \cos m_1 d \sin m_2 (H - d) = 0\,.$$

Solutions: Method of modes

- For a 2-layer N:

The beam is slightly steeper in the lower layer due to the weakest stratification.

Internal reflections occur at the transition between the two layers,

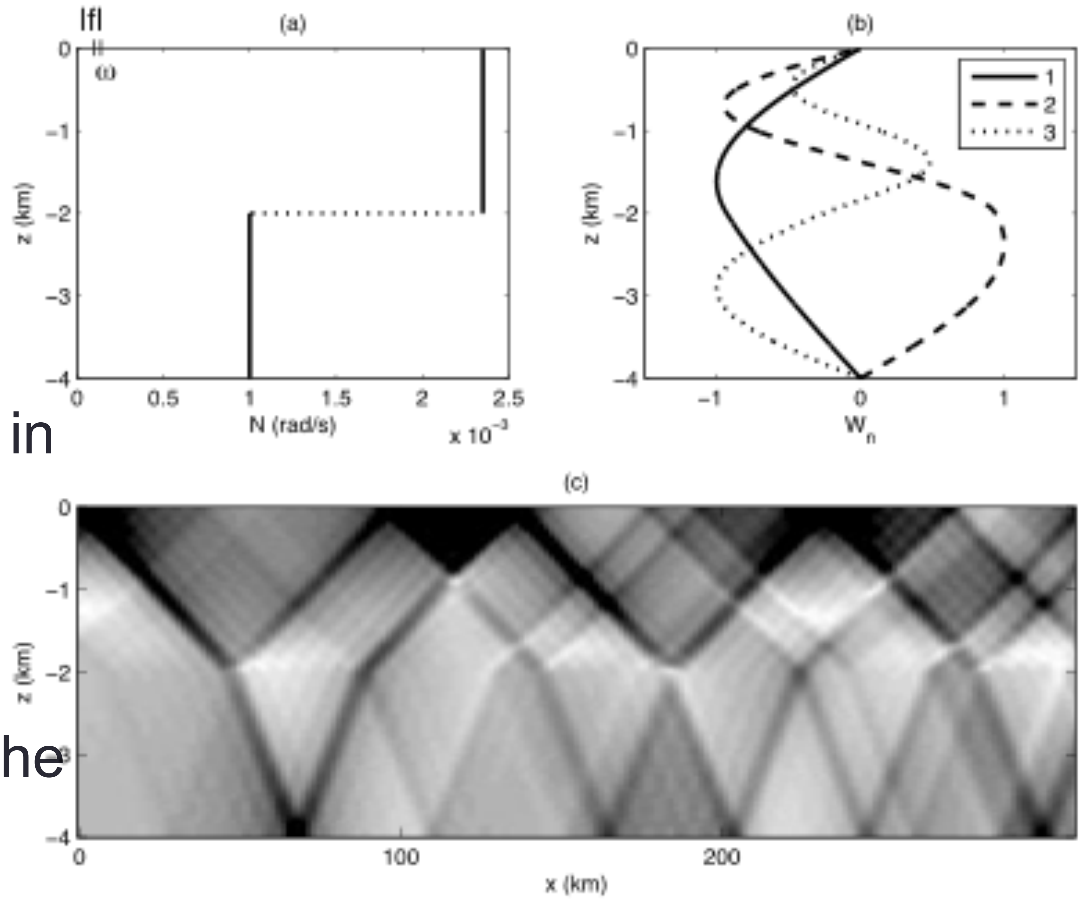
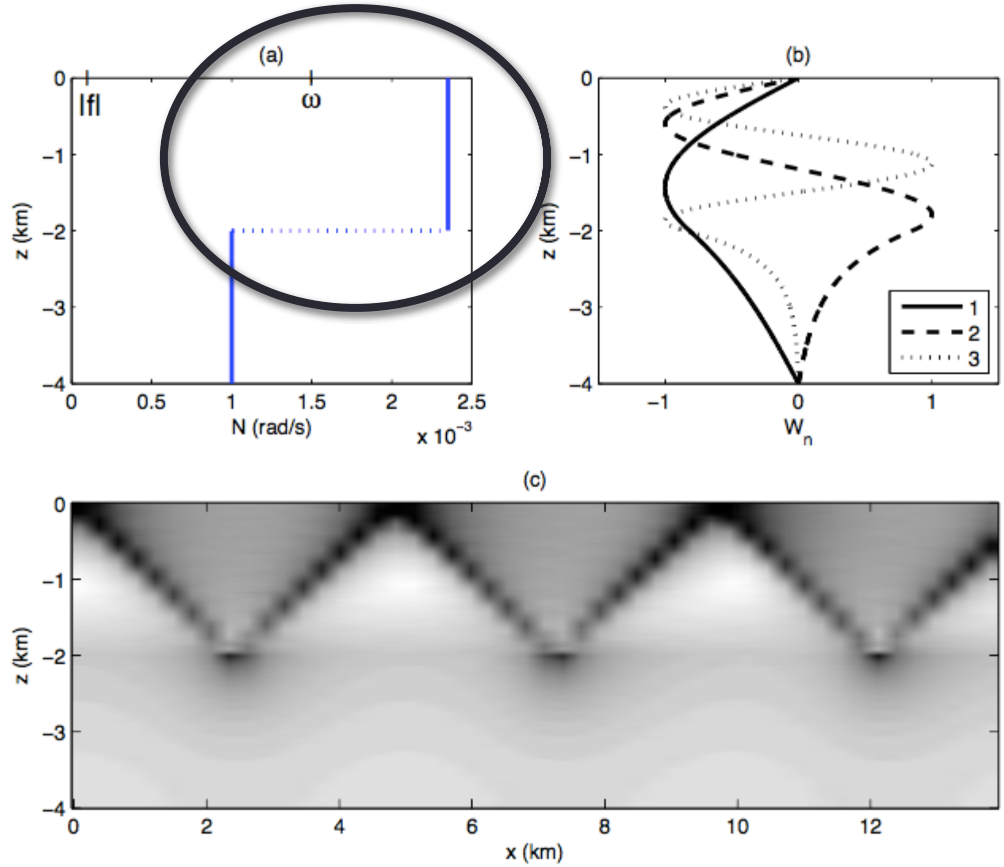


Fig. 5.7: Stratification with two layers of constant N, with $|f| < \omega < N_2 < N_1$ (**a**). Panel **b** shows the first three eigenmodes (5.30), with $C_{1,n}$ chosen such that their amplitudes are one. Modal coefficients are $a_n = 1/n$. The resulting superposition of 20 modes, representing the amplitude of u, is shown in **c**. White denotes zero; black, maximum values.

Solutions: Method of modes

- For a 2-layer N:

if: $|f| < N_2 < \omega < N_1$



High-frequency waves are trapped in the upper layer.

Fig. 5.9: Stratification with two layers of constant N; parameters are as in Figure 5.7, except for the wave frequency, which is now such that $|f| < N_2 < \omega < N_1$ (**a**). Panel **b** shows the first three eigenmodes (5.30), normalized to one. The resulting superposition of 15 modes, representing the amplitude of u, is shown in **c**. White denotes zero; black, maximum values.

Solutions: Method of modes

- For a 2-layer N:

if: $N_2 < \omega < |f| < N_1$

Low-frequency waves are trapped in the lower layer.

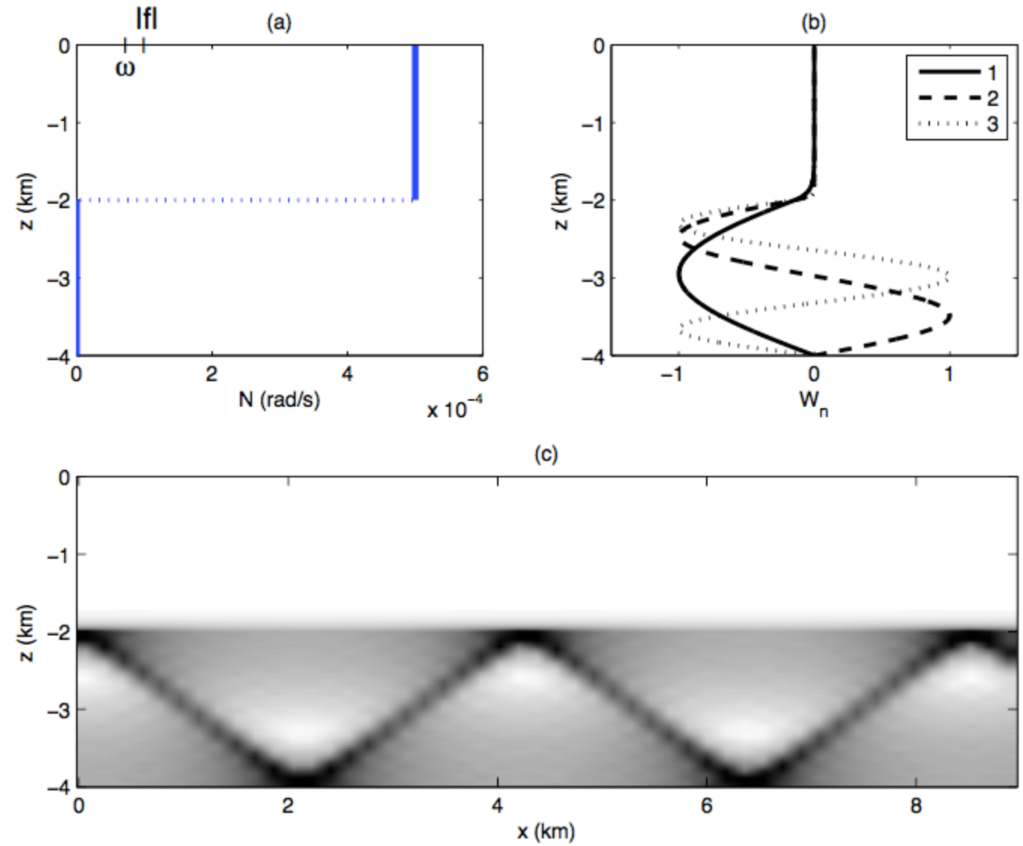


Fig. 5.10: Stratification with two layers of constant N, the lower layer being neutrally stable, $N_2 = 0$, with $N_2 < \omega < |f| < N_1$ (**a**). Panel **b** shows the first three eigenmodes (5.30), normalized to one. A superposition of 15 modes, representing the amplitude of u, is shown in **c**. White denotes zero; black, maximum values.

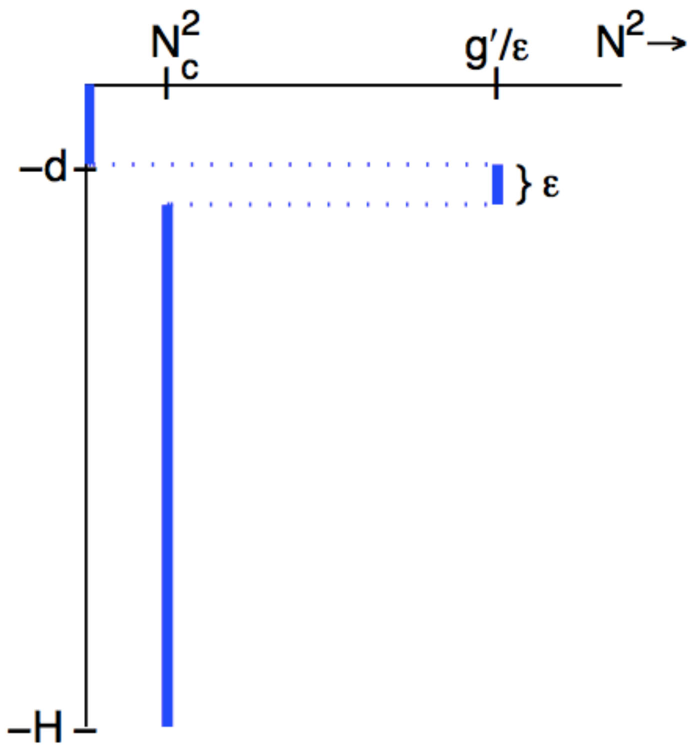
Solutions: Method of modes

- 3-layers N:

$$N^2(z) = \begin{cases} 0 & -d < z < 0 \quad \text{(mixed layer)} \\ g'/\epsilon & -d-\epsilon < z < -d \quad \text{(thermocline)} \\ N_c^2 & -H < z < -d-\epsilon \quad \text{(abyss)}\,. \end{cases}$$

We can get a more realistic stratification using 3 layers:

- a very weakly stratified upper mixed layer
- a seasonal thermocline
- a fairly weakly stratified abyssal ocean



Solutions: Method of modes

- 3-layers N:

We can get a more realistic stratification using 3 layers:

- a very weakly stratified upper mixed layer
- a seasonal thermocline
- a fairly weakly stratified abyssal ocean

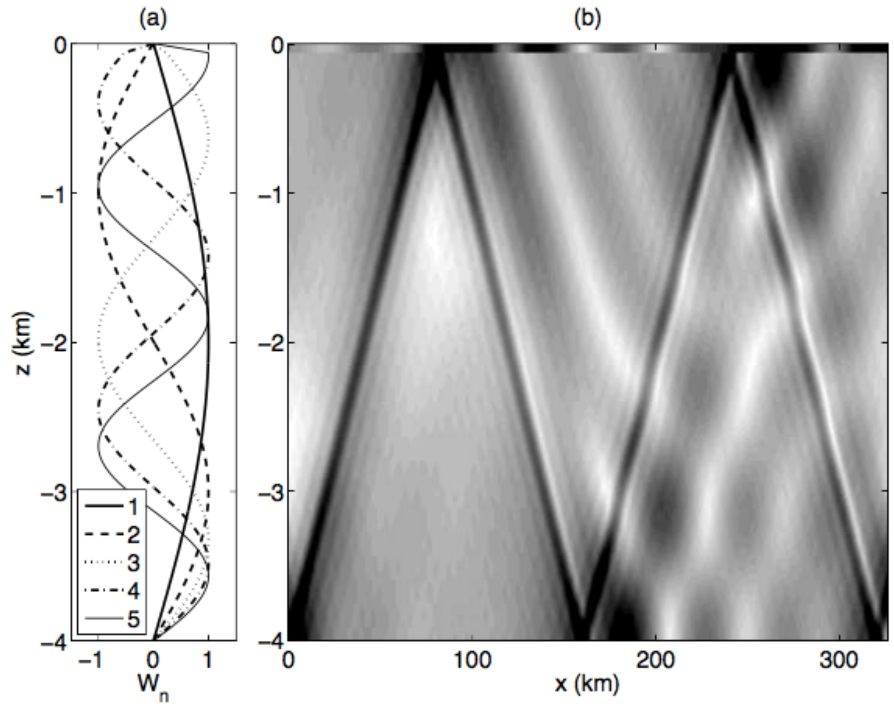


Fig. 5.12: The first five modes for the three-layer system (**a**), here with $g' = 0.005\,\mathrm{m\,s^{-2}}$. Other parameters are: $d = 60\,\mathrm{m}$ (mixed-layer depth), $N_c = 2 \times 10^{-3}$, $f = 1 \times 10^{-4}$ and $\omega = 1.4 \times 10^{-4}$, all in $\mathrm{rad\,s^{-1}}$; modal coefficients are $a_n = 1/n$. In **b**, a superposition of 25 modes, representing the amplitude of u. White denotes zero; black, maximum values.

Solutions: Method of modes

- For a linear N(z):

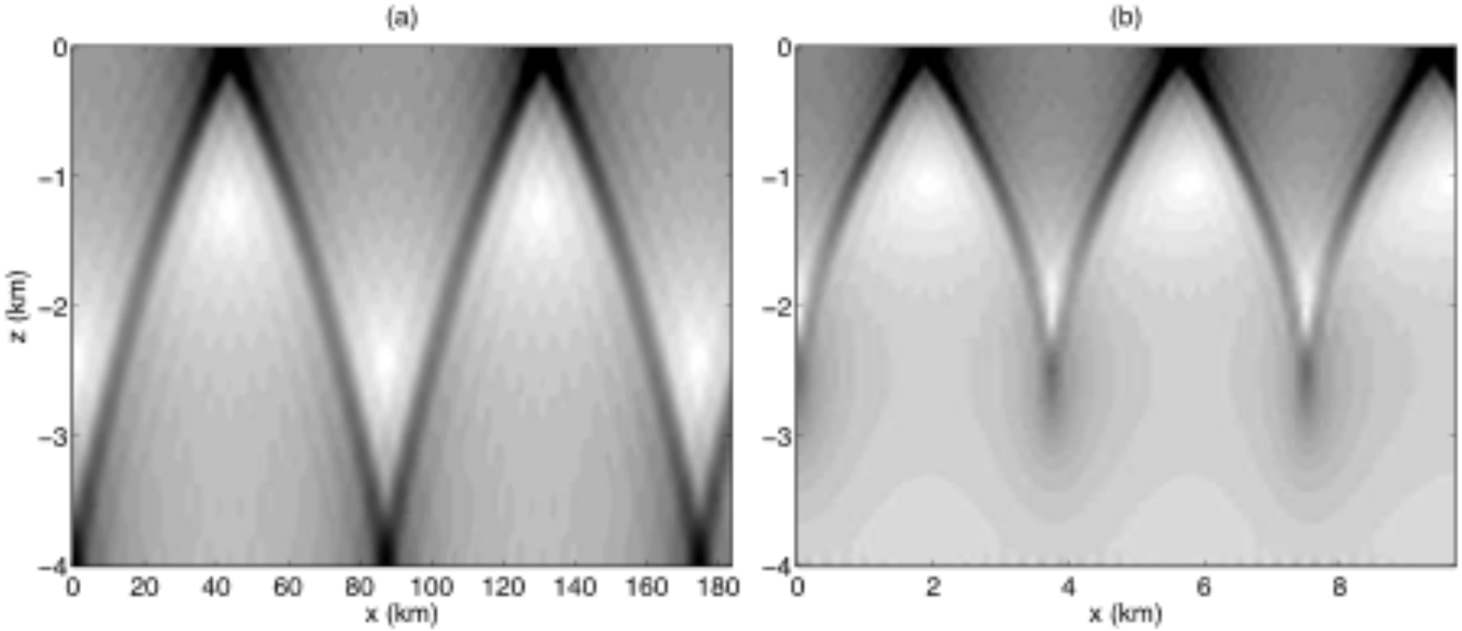
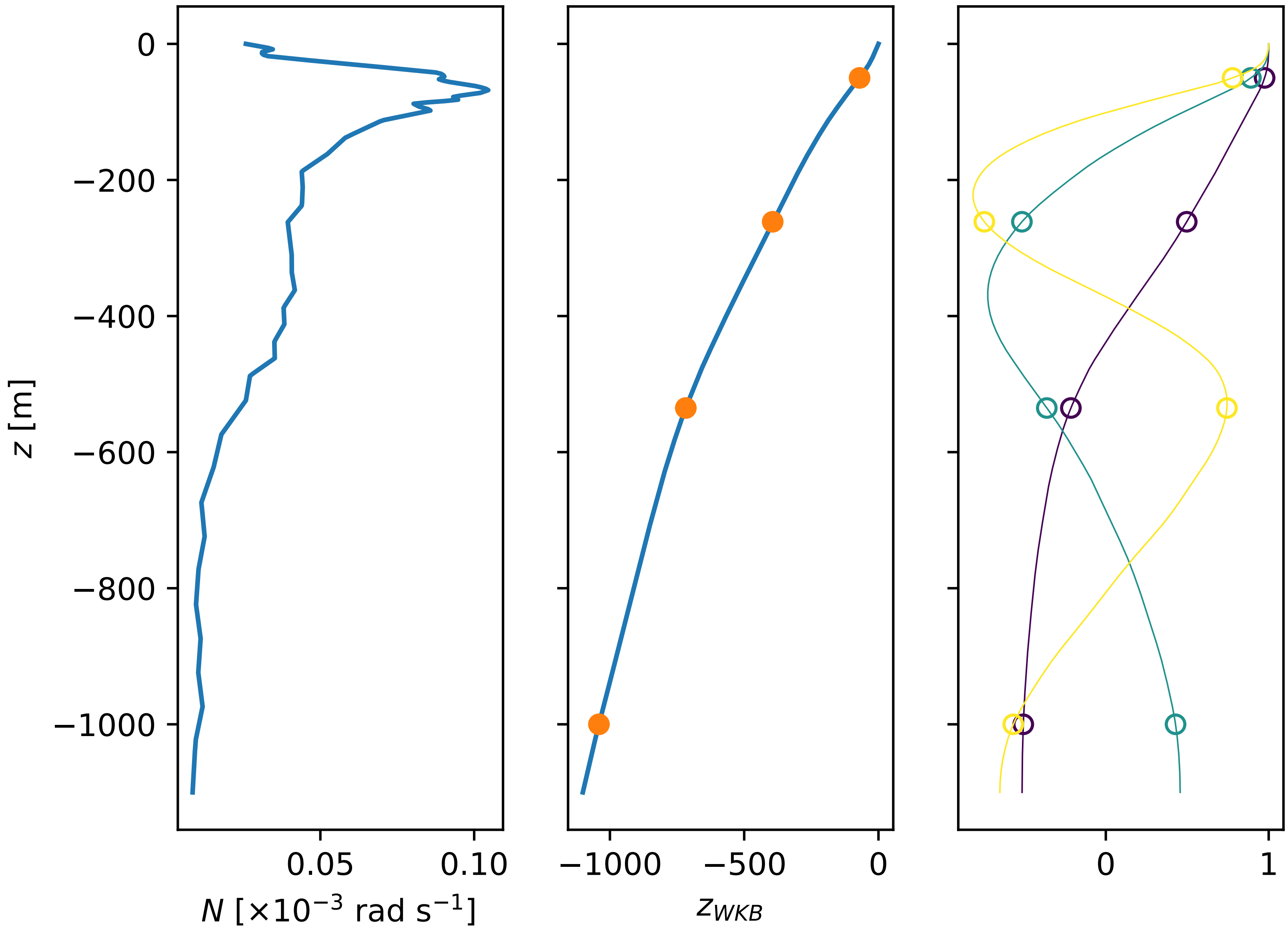


Fig. 5.16: Solution for a linearly varying $N(z)$; the amplitude of u is shown for a superposition 15 modes, with modal coefficients $a(n) = 1/n$. White denotes zero; black, maximum values. In **a**, internal-wave beams can propagate at any depth, but are refracted due to the decrease of N with depth. In **b**, a higher wave frequency is chosen, such that $|f| < N < \omega$ in the deeper part of the water column; hence, internal waves are trapped in the upper layer.

Solutions: Method of modes

In the real ocean, somewhere in the Pacific



Solutions: Method of modes

In the real ocean, somewhere in the North Atlantic