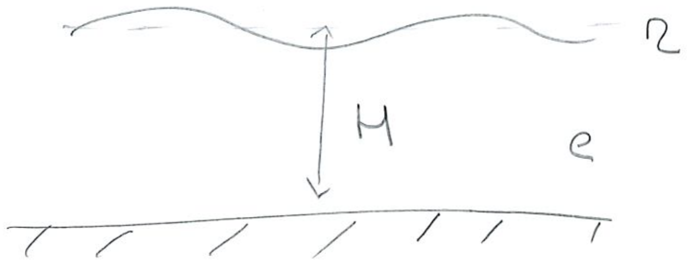
Internal waves in the 2-layer model.

Rappel shallow-water :



Prem NS : $$\frac{D\vec{u}}{Dt} + f\vec{k}\wedge\vec{u} = \frac{\vec{\nabla}p}{\rho} + \vec{g} \quad \left(\begin{array}{c}\text{no}\\\text{viscosity}\end{array}\right)$$

écoulement quasi 2d $\left\}\begin{array}{l} w = 0 \quad , u_z, v_z = 0 \\ \rho = cste \end{array}\right.$

$$\left\{\begin{array}{l} u_t + uu_x + vu_y - fv = \frac{p_x}{\rho} \\ v_t + uv_x + vv_y + fu = \frac{p_y}{\rho} \end{array}\right.$$

2nd pression hydrostatique $p(z) = \rho g(\eta - z)$

$$\Rightarrow \left\{\begin{array}{l} p_x = \rho g \eta_x \\ p_y = \rho g \eta_y \end{array}\right.$$

- conservation de la masse : $\vec{\nabla}\cdot\vec{u} = 0$ $\left[\begin{array}{l} \frac{D\eta}{Dt} = w|_\eta \\ \vec{u}\cdot\vec{\nabla}H + w|_{-H} = 0 \end{array}\right.$

$$\int_{-H}^{\eta} (u_x + v_y + w_z)\,dz = [(H+\eta)u]_x + [(H+\eta)v]_y + u|_\eta \eta_x - v|_\eta \eta_y + w|_\eta$$

$$- u|_{-H} H_x - v|_{-H} H_y - w|_{-H}$$

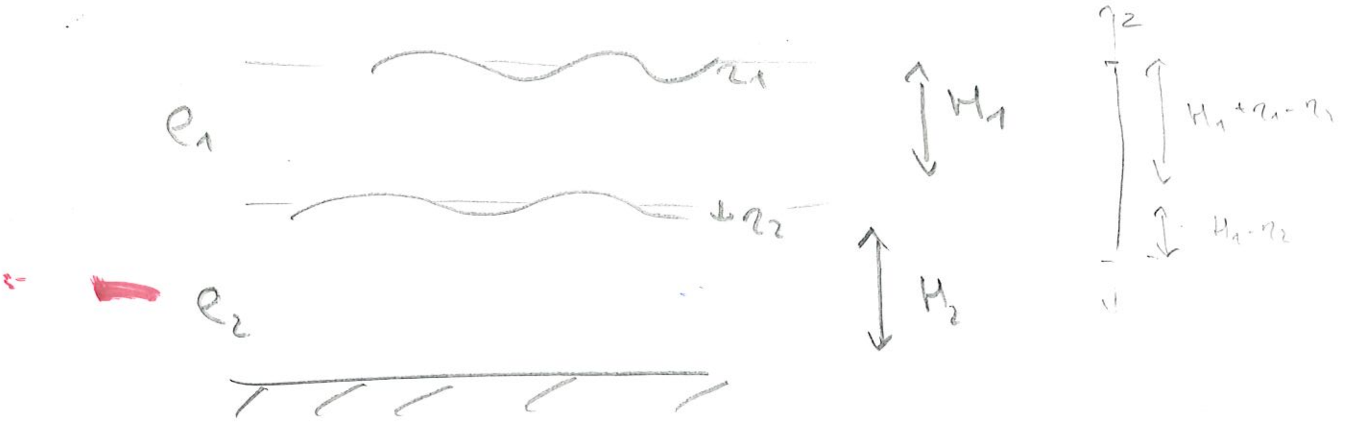
$$= \eta_t + \vec{\nabla}\cdot[(H+\eta)\vec{u}] = 0$$

$\frac{\partial\eta}{\partial t}$

$$\Longrightarrow \begin{cases} u_t + u u_x + v u_y - f v = -g \eta_x \\ v_t + u v_x + v v_y + f u = -g \eta_y \\ \eta_t + \left((H+\eta) u \right)_x + \left((H+\eta) v \right)_y = 0 \end{cases}$$

<u>with 2 layers</u> :

Marée interne en milieu bicouche.



* pression dans chaque couche :

$$P_1 = \rho_1 g \left(\eta_1 - z\right)$$

$$P_2 = \rho_1 g \left(H_1 + \eta_1 - \eta_2\right) + \rho_2 g \left(-z - (H_1 - \eta_2)\right)$$

$$= \rho_1 g \left(H_1+\eta_1-\eta_2\right) + \rho_2 g \left(-z - H_1 + \eta_2\right)$$

* gradient de pression :

$$\frac{1}{\rho_1}\frac{\partial P_1}{\partial x} = \frac{P_{1x}}{\rho_1} = g\,\eta_{1x} \; ; \; \frac{1}{\rho_2}\frac{\partial P_2}{\partial x} = \frac{P_{2x}}{\rho_2} = \frac{\rho_1}{\rho_2} g \left(\eta_{1x} - \eta_{2x}\right) + g\,\eta_{2x}$$

$$= \frac{\rho_1}{\rho_2} g\,\eta_{1x} + \left(1 - \frac{\rho_1}{\rho_2}\right) g\,\eta_{2x}$$

on définit la gravité réduite :

$$g' = \left(1 - \frac{\rho_1}{\rho_2}\right) g \implies \frac{P_{2x}}{\rho_2} = g'\eta_{2x} + (g - g')\,\eta_{1x}$$

en 1re approximation $\boxed{g \gg g'}$

exemple pour océan côtier : $\rho = \rho_0 (1 - \alpha T)$

$$g' \approx \frac{\rho_2 - \rho_1}{\rho_0} g = \alpha (T_1 - T_2) g \quad \begin{cases} \alpha \sim 2.10^{-4}\ K^{-1} \\ \Delta T \sim 10\ K \end{cases}$$

$$= 2.10^{-3} g$$

$$\frac{P_{2x}}{\rho_2} = g' \eta_{2x} + g \eta_{1x} \qquad \frac{P_{1x}}{\rho_1} = g \eta_{1x}$$

* Equation du mouvement linéarisée :

$$(1) \begin{cases} u_{1t} - f v_1 + g \eta_{1x} = 0 \\ v_{1t} + f u_1 + g \eta_{1y} = 0 \end{cases} \qquad \left[\begin{array}{l} (\eta_1 - \eta_2)_t + H_1 u_{1x} + H_1 v_{1y} = 0 \\ \eta_{2t} + H_2 u_{2x} + H_2 v_{2y} = 0 \end{array} \right.$$

$$(2) \begin{cases} u_{2t} - f v_2 + g \eta_{1x} + g' \eta_{2x} = 0 \\ v_{2t} + f u_2 + g \eta_{1y} + g' \eta_{2y} = 0 \end{cases}$$

on cherche des solutions sous la forme :

$$A = A_0 e^{i(\omega t - kx - ly)}$$

c-à-d
$$\begin{cases} A_{tt} = -\omega^2 A \\ A_{xx} = -k^2 A \\ A_{yy} = -l^2 A \end{cases}$$

on dérive les équations du mouvement par t (2)

$$\begin{cases} u_{1tt} - \beta v_{1t} + g\eta_{1xt} = u_{1tt} - \beta[-\beta u_1 - g\eta_{1y}] + g\eta_{1xt} \\ v_{1tt} + \beta u_{1t} + g\eta_{1yt} = v_{1tt} + \beta[\beta v_1 - g\eta_{1x}] + g\eta_{1yt} = 0 \end{cases}$$

$$u_{1tt} + \beta^2 u_1 = -g\beta\eta_{1y} - g\eta_{1xt}$$

$$v_{1tt} + \beta^2 v_1 = g\beta\eta_{1x} - g\eta_{1yt}$$

$$\begin{cases} u_1 = \dfrac{-g\beta\eta_{1y} - g\eta_{1xt}}{\beta^2 - \omega^2} \\ v_1 = \dfrac{g\beta\eta_{1x} - g\eta_{1yt}}{\beta^2 - \omega^2} \end{cases}$$

$$\begin{cases} u_2 = \dfrac{-g\beta\eta_{2y} - g\eta_{1xt} - g'\eta_{2xt}}{\beta^2 - \omega^2} \\ v_2 = \dfrac{g\beta\eta_{2x} - g\eta_{1yt} - g'\eta_{2yt}}{\beta^2 - \omega^2} \end{cases}$$

on remplace dans l'équation de continuité :

$$\eta_{1t} - \eta_{2t} + \frac{H_1}{\beta^2 - \omega^2}(\cancel{-g\beta\eta_{1xy}} - g\eta_{1xxt}) + \frac{H_1}{\beta^2 - \omega^2}(\cancel{g\beta\eta_{1xy}} - g\eta_{1yyt}) = 0$$

$$\eta_{2t} + \frac{H_2}{\beta^2 - \omega^2}(\cancel{-g\beta\eta_{2xy}} - g\eta_{1xxt} - g'\eta_{2xxt}) + \frac{H_2}{\beta^2 - \omega^2}(\cancel{g\beta\eta_{2xy}} - g\eta_{1yyt} - g'\eta_{2yyt}) = 0$$

...

$$\eta_{1t} - \eta_{2t} + \frac{H_1}{\beta^2-\omega^2}\left[-g\eta_{1xxt} - g\eta_{1yyt}\right] = 0$$

$$\eta_{2t} + \frac{H_2}{\beta^2-\omega^2}\left[g\eta_{1xxt} - g\eta_{1yyt} - g'\eta_{2xxt} - g'\eta_{2yyt}\right] = 0$$

on simplifie le ∂t et on utilise $\begin{cases} \eta_{1xx} = \text{[redacted]}\ \eta_1 \\ \eta_{1yy} = -\ell^2 \eta_1 \end{cases}$

$$\begin{cases} \left(1 + \dfrac{gH_1}{\beta^2-\omega^2}\overbrace{\left[k^2+\ell^2\right]}^{K^2}\right)\eta_1 - \eta_2 = 0 \\[2ex] +\dfrac{gH_2}{\beta^2-\omega^2}\left[k^2+\ell^2\right]\eta_1 + \left[1 + \dfrac{g'H_2}{\beta^2-\omega^2}(k^2+\ell^2)\right]\eta_2 = 0 \end{cases}$$

déterminant nul : $\left[\beta^2-\omega^2 + gH_1K^2\right]\left[\beta^2-\omega^2 + g'H_2K^2\right]$

$$+ gH_2K^2(\beta^2-\omega^2) = 0$$

$$(\beta^2-\omega^2)^2 + (gH_2K^2 + gH_1K^2 + g'H_2K^2)(\beta^2-\omega^2) + gg'H_1H_2K^4 = 0$$

déterminant $\Delta = (gHK^2)^2 - 4gg'H_1H_2K^4$

$$= K^4(g^2H^2 - 4gg'H_1H_2)$$

solutions : $\beta^2-\omega^2 = \dfrac{-gHK^2 \pm K^2\sqrt{g^2H^2 - 4gg'H_1H_2}}{2}$

avec au premier ordre

$$\sqrt{g^2H^2 - 4gg'H_1H_2} = gH\left[1 - 2g'\frac{H_1H_2}{gH^2}\right]$$

Solutions :

$$\omega^2 - f^2 = \begin{cases} gHk^2\left(1 - \dfrac{g'}{g}\dfrac{H_1H_2}{H^2}\right) \\[2ex] \dfrac{g'H_1H_2}{H}k^2 \end{cases}$$

mode barocline : $\omega_c^2 = f^2 + \dfrac{g'H_1H_2}{H} k^2$

mode barotrope : $\omega_b^2 = \underbrace{f^2 + gHk^2}_{\text{onde inertie gravité (onde Poincaré)}} \Big[1 - \underbrace{\dfrac{g'H_1H_2}{gH^2}}_{\text{correction}}\Big]$

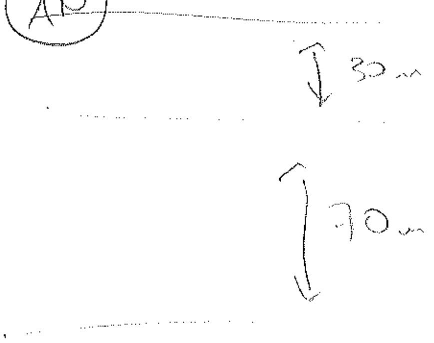
vitesse de phase $\boxed{c_c^2 = \dfrac{\omega^2}{k^2} = \dfrac{g'H_1H_2}{H\left(1 - f^2/\omega^2\right)}}$

$$\boxed{c_b^2 = gH\left[\dfrac{1 - \dfrac{g'H_1H_2}{gH^2}}{1 - f^2/\omega^2}\right]}$$

Relation entre élévation dans la thermocline et la surface libre pour le mode barocline :

$$\eta_1 = \frac{\eta_2}{1 + \dfrac{g H_1 k^2}{\beta^2 - \omega^2}}$$

AN



avec $$\beta^2 - \omega^2 = - g' \frac{H_1 H_2}{H} k^2$$

$$\Leftrightarrow \quad \eta_1 = \frac{\eta_2}{1 - \dfrac{g H}{g' H_2}} = \frac{\eta_2}{1 - \dfrac{100}{2.10^{3} \times 70}}$$

$$= - \frac{\eta_2}{713}$$

Si $\eta_2 \simeq 10\,m \longrightarrow \eta_1 \simeq 1{,}4$ cm

la trace en surface des ondes internes est faible.

Vitesse du mode barotrope :

$$\begin{cases} U_1 = (u_1^2+v_1^2)^{1/2} = -\dfrac{c}{H_1}\eta_2\left(1+\dfrac{\beta^2}{\omega^2}\right)^{1/2} \\ U_2 = (u_2^2+v_2^2)^{1/2} = \dfrac{c}{H_2}\eta_2\left(1+\dfrac{\beta^2}{\omega^2}\right)^{1/2} \end{cases}$$

on peut observer que $U_1 H_1 = -U_2 H_2$

= <u>pas de transport net !</u>

$$\begin{cases} a\eta_1 + b\eta_2 = 0 \\ c\eta_1 + d\eta_2 = 0 \end{cases}$$

$$\Downarrow$$

$$ad - cb = 0$$