

Numerical Modelling

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the anatomy of an ocean model

#3 Discretization

Useful references

Courses:

<https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-950-atmospheric-and-oceanic-modeling-spring-2004/lecture-notes/>

GENERAL

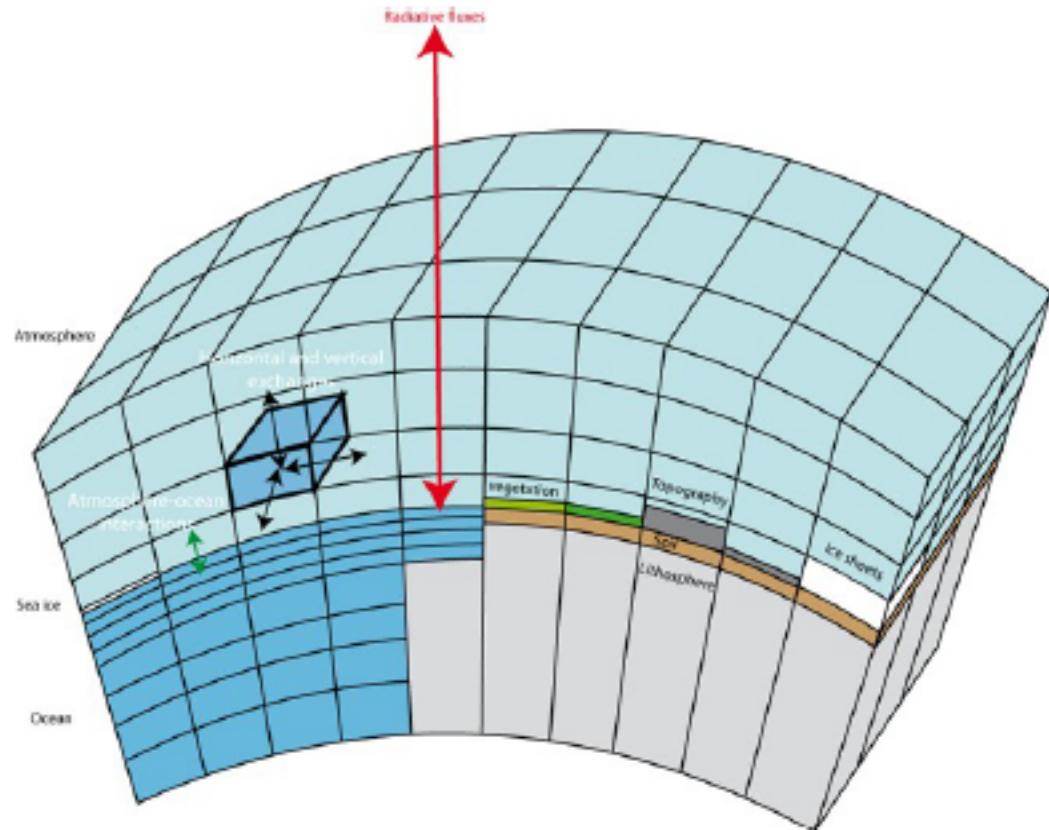
- Griffies et al., 2000, Developments in ocean climate modelling, Ocean Modelling. <http://jgula.fr/ModNum/Griffiesetal00.pdf>

ROMS:

- Shchepetkin, A., and J. McWilliams, 2005: The Regional Oceanic Modeling System (ROMS): A split-explicit, free-surface, topography-following- coordinate ocean model. Ocean Modell. <http://jgula.fr/ModNum/ShchepetkinMcWilliams05.pdf>
- <https://www.myroms.org/wiki/>

Discretization

The ocean is divided into boxes : discretization

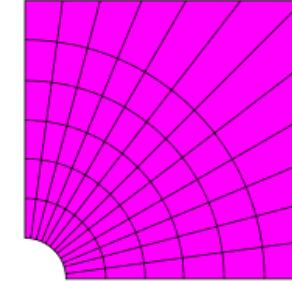


Example of a finite difference grid

Discretization

Structured grids

Identified by regular connectivity
= can be addressed by (i, j, k)

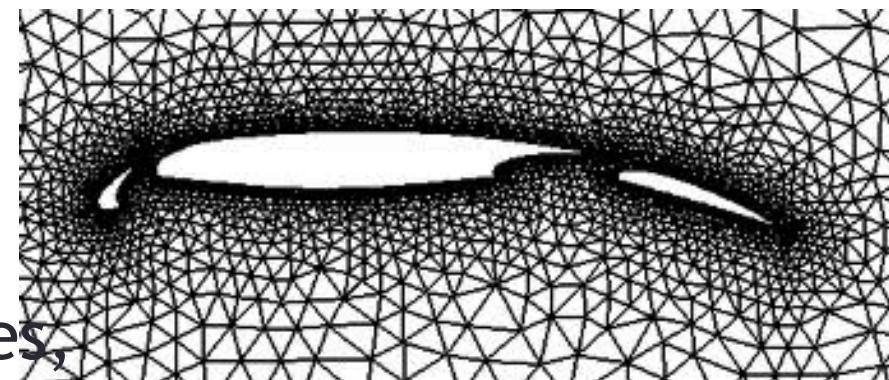


← ROMS

Unstructured grids

The domain is tiled using more general geometrical shapes (triangles, ...) pieced together to optimally fit details of the geometry.

- ✓ Good for tidal modeling, engineering applications.
- ✓ Problems:
geostrophic balance accuracy, conservation and positivity properties,



...

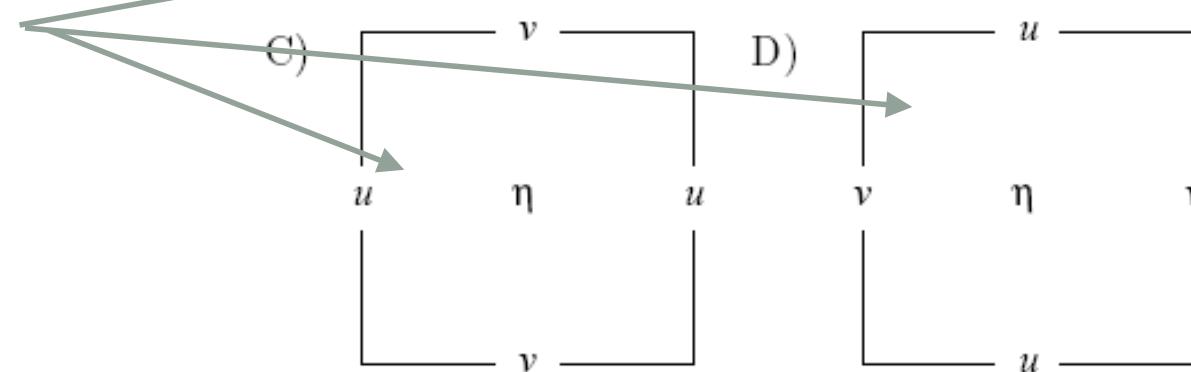
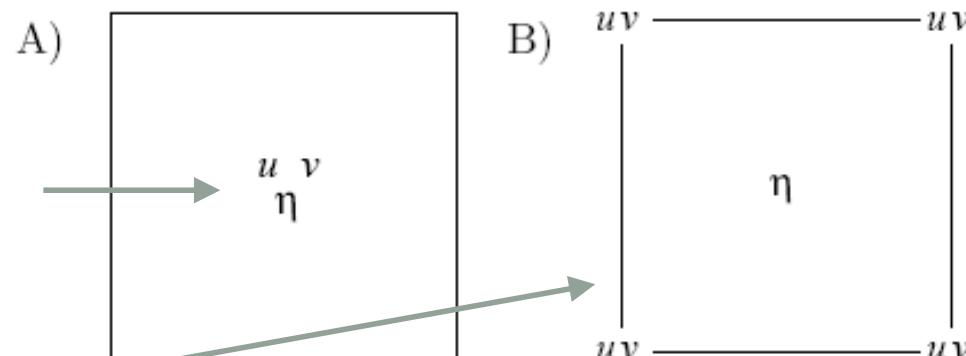
Horizontal discretization

Different types of Horizontal Grids (Arakawa Grids):

Non-staggered
(= collocated variables)

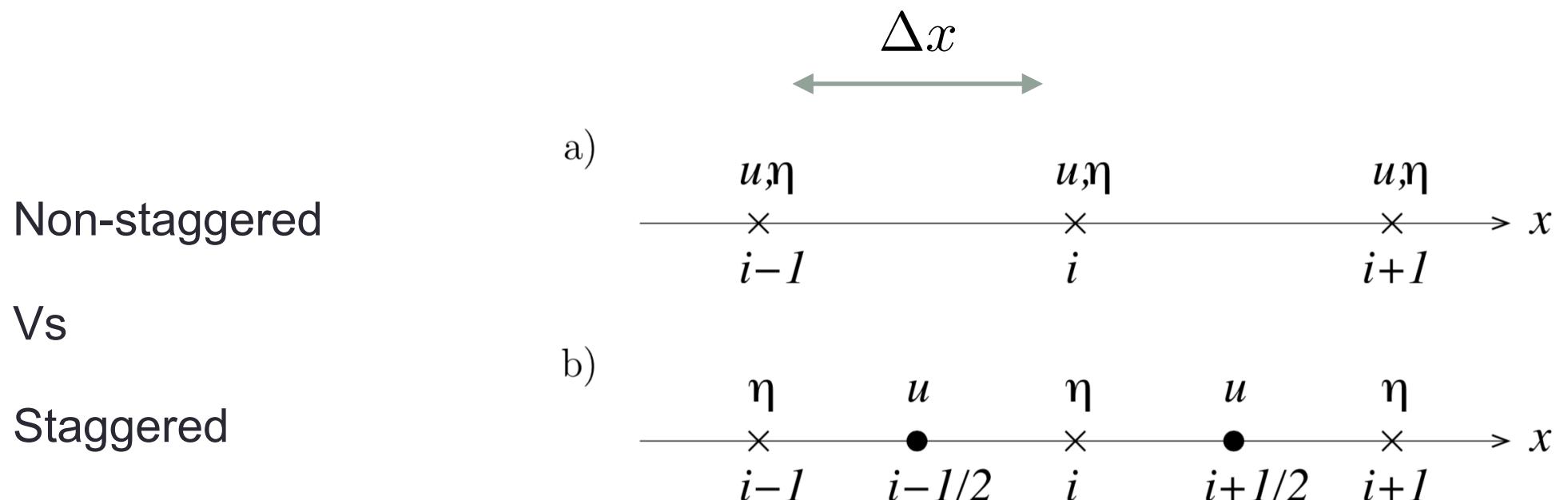
Or

Staggered



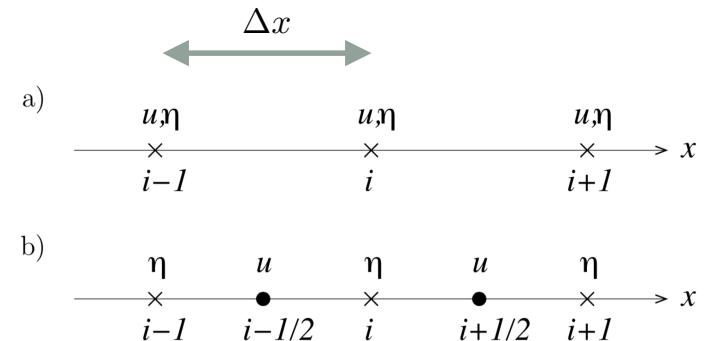
Horizontal discretization

Staggered Vs unstaggered : the 1D problem



Horizontal discretization

Staggered Vs unstaggered : the 1D problem



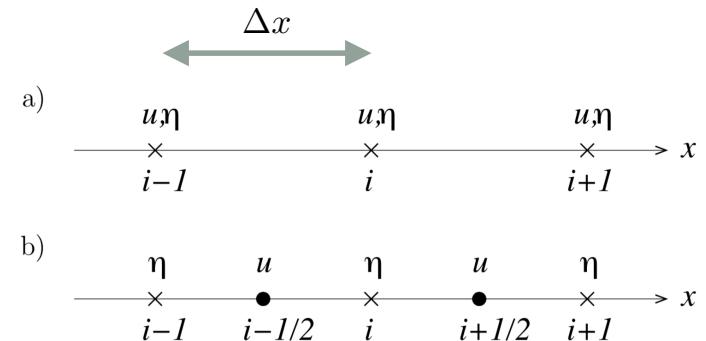
1. The advection equation $\partial_t \theta + c \partial_x \theta = 0$

Solutions of the continuous equations are non-dispersive waves

$$\theta(x, t) = \theta_o e^{i(kx - \omega t)} \text{ with dispersion relation } \omega = ck$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem



1. The advection equation $\partial_t \theta + c \partial_x \theta = 0$

Solutions of the **continuous equations** are non-dispersive waves

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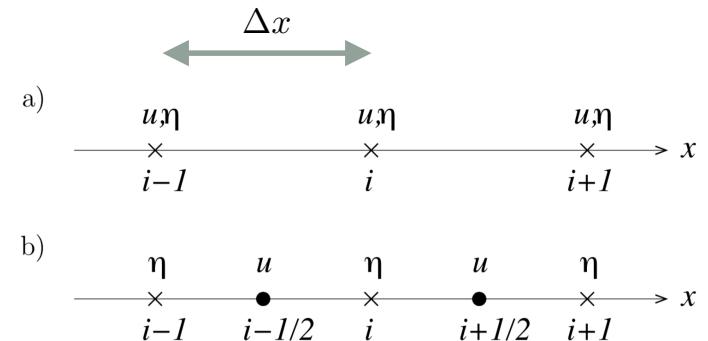
Discretized equations with the centered second order derivative are:

$$d_t \theta + \frac{c}{\Delta x} \delta_i \bar{\theta}^i = 0$$

$$d_t \theta_i + \frac{c}{2\Delta x} (\theta_{i+1} - \theta_{i-1}) = 0$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem



1. The advection equation $\partial_t \theta + c \partial_x \theta = 0$

Substituting in our solution:

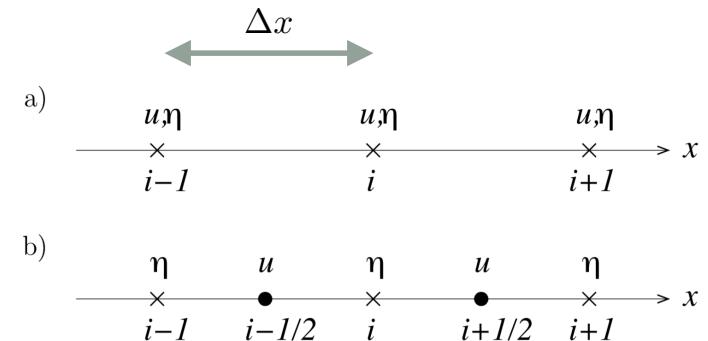
$$\theta_i(x, t) = \theta_0 e^{i(kx - \omega t)}$$

$$\theta_{i-1}(x, t) = \theta_0 e^{i(k(x - \Delta x) - \omega t)}$$

$$\theta_{i+1}(x, t) = \theta_0 e^{i(k(x + \Delta x) - \omega t)}$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem



1. The advection equation $\partial_t \theta + c \partial_x \theta = 0$

Substituting in our solution gives:

$$\begin{aligned} -i\omega &= -\frac{c}{2\Delta x} (e^{ik\Delta x} - e^{-ik\Delta x}) \\ &= -\frac{ci}{\Delta x} \sin k\Delta x \end{aligned}$$

Now the solution is **dispersive!!!**

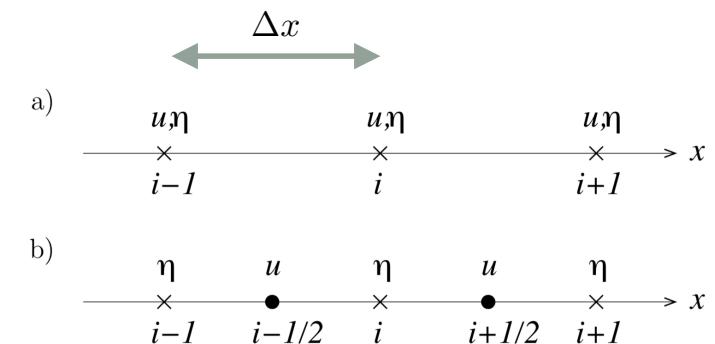
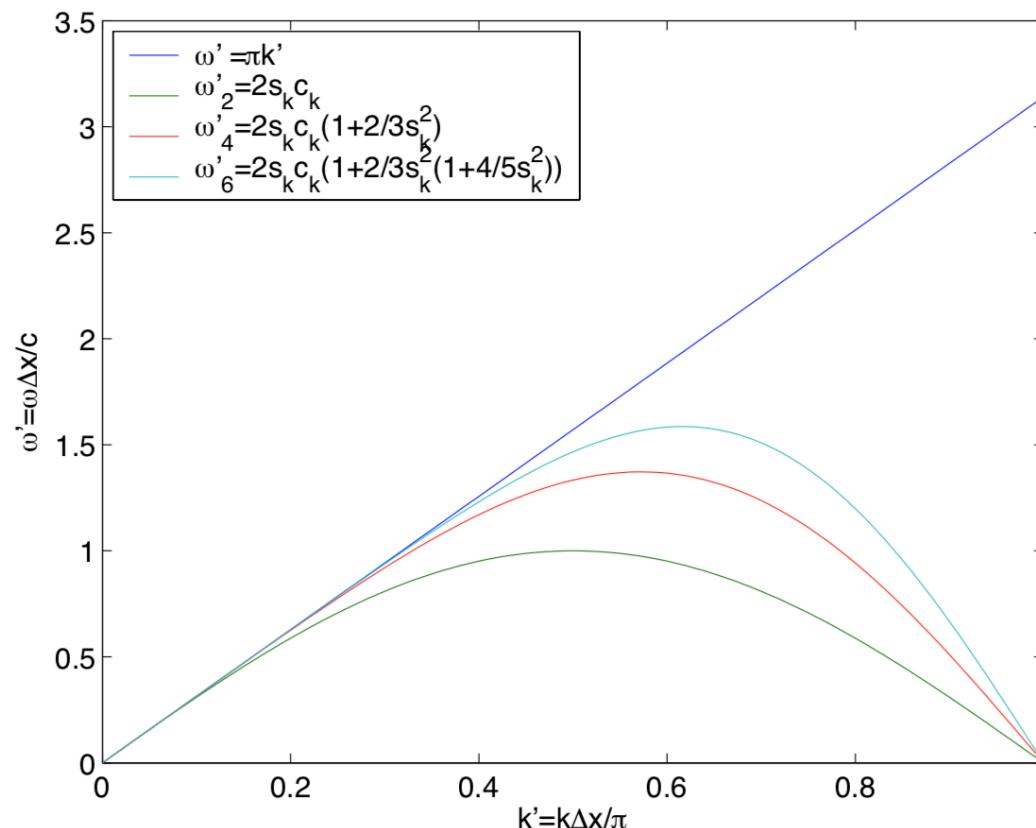
Even it will converge to the non-dispersive solution in the limit of small Δx

$$\omega = \frac{c}{\Delta x} \sin k\Delta x \stackrel{\Delta x \rightarrow 0}{=} ck$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

1. The advection equation $\partial_t \theta + c \partial_x \theta = 0$



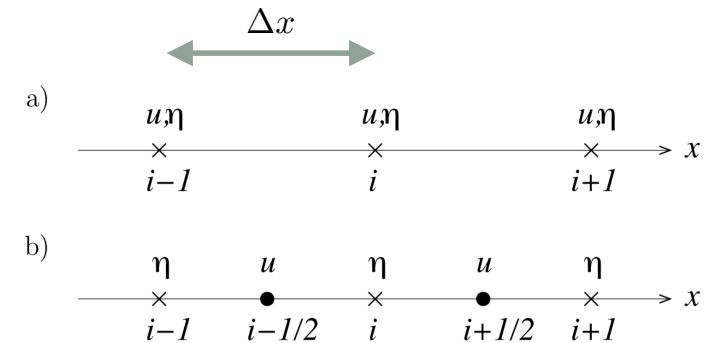
Dispersion relations for constant flow advection using second, fourth, and sixth order spatial differences.

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$\begin{aligned}\partial_t u &= -g \partial_x \eta \\ \partial_t \eta &= -H \partial_x u\end{aligned} \longrightarrow \partial_{tt} \eta = g H \partial_{xx} \eta$$



Solutions of the continuous equations are non-dispersive waves

$$\eta = \eta_o e^{i(kx - \omega t)} \quad \text{with dispersion relation} \quad \omega = \pm \sqrt{gHk}$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Gravity waves

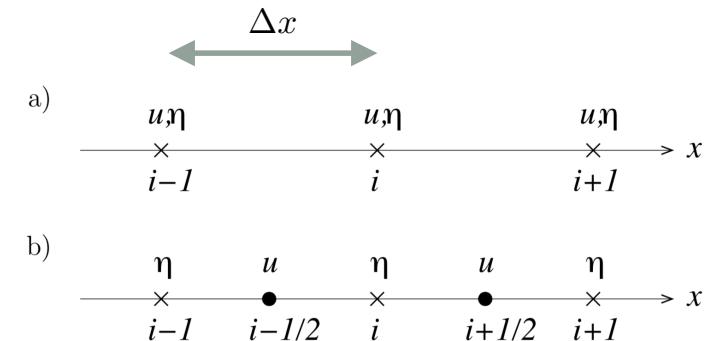
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Solutions of the continuous equations are non-dispersive waves

$$\eta = \eta_o e^{i(kx - \omega t)} \quad \text{with dispersion relation} \quad \omega = \pm \sqrt{gHk}$$

Discretized equations with the centered second order derivative on the **unstaggered grid** are:

$$\longrightarrow \partial_{tt} \eta = \frac{gH}{\Delta x^2} \delta_{ii} \bar{\eta}^{ii} \quad \text{with} \quad \delta_{ii} \bar{\eta}^{ii} = \frac{1}{4} (\eta_{i-2} - 2\eta_i + \eta_{i+2})$$



Horizontal discretization

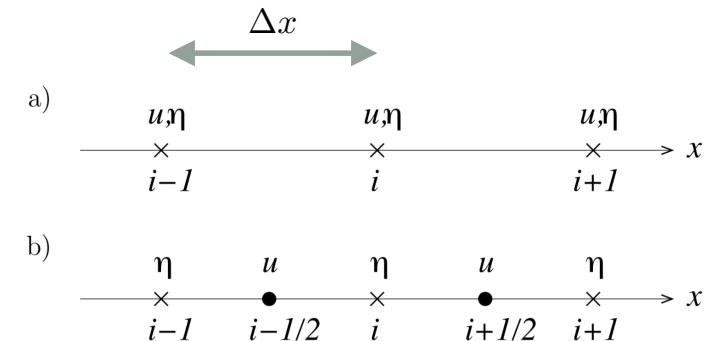
Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$\begin{aligned}\partial_t u &= -g \partial_x \eta \\ \partial_t \eta &= -H \partial_x u\end{aligned} \longrightarrow \partial_{tt} \eta = g H \partial_{xx} \eta$$

Substituting in our solution on the unstaggered grid gives :

$$\begin{aligned}-\omega^2 &= \frac{gH}{4\Delta x^2} (e^{-i2k\Delta x} - 2 + e^{i2k\Delta x}) \\ &= \frac{gH}{4\Delta x^2} (2 \cos 2k\Delta x - 2) \\ &= -\frac{4gH}{\Delta x^2} \sin^2 \frac{k\Delta x}{2} \cos^2 \frac{k\Delta x}{2}\end{aligned}$$



Horizontal discretization

Staggered Vs unstaggered : the 1D problem

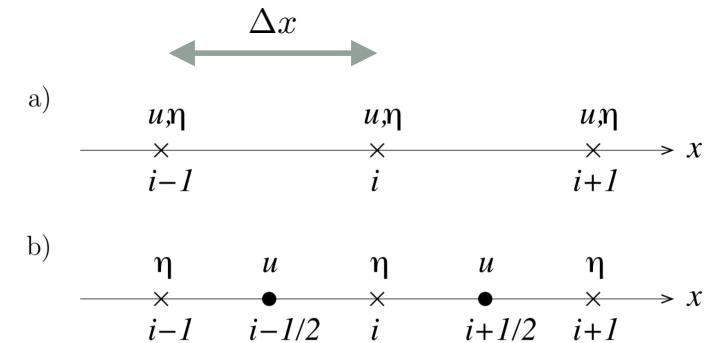
2. Gravity waves

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- Question:
 - What is the dispersion relation on the staggered grid?



Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Gravity waves

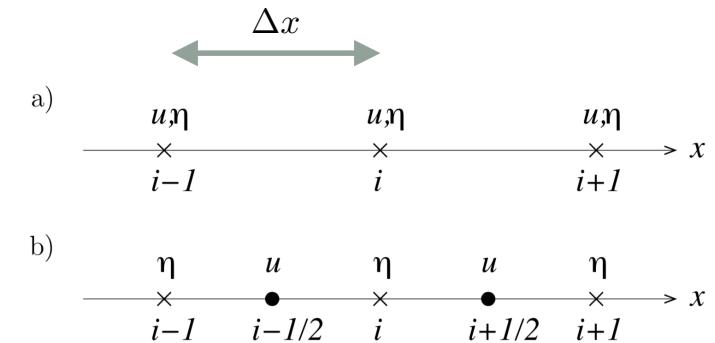
$$\begin{aligned}\partial_t u &= -g \partial_x \eta \\ \partial_t \eta &= -H \partial_x u\end{aligned} \longrightarrow \partial_{tt} \eta = g H \partial_{xx} \eta$$

Discretized equations with the centered second order derivative on the **staggered grid** are:

$$\begin{aligned}\partial_t u &= -\frac{g}{\Delta x} \delta_i \eta \\ \partial_t \eta &= -\frac{H}{\Delta x} \delta_i u\end{aligned}$$

This can be written as a system:

$$\begin{pmatrix} \partial_t & \frac{g}{\Delta x} \delta_i \\ \frac{H}{\Delta x} \delta_i & \partial_t \end{pmatrix} \begin{pmatrix} u \\ \eta \end{pmatrix} = 0 \quad \begin{pmatrix} -i\omega & \frac{2ig}{\Delta x} \sin \frac{k\Delta x}{2} \\ \frac{2iH}{\Delta x} \sin \frac{k\Delta x}{2} & -i\omega \end{pmatrix} \begin{pmatrix} u \\ \eta \end{pmatrix} = 0$$

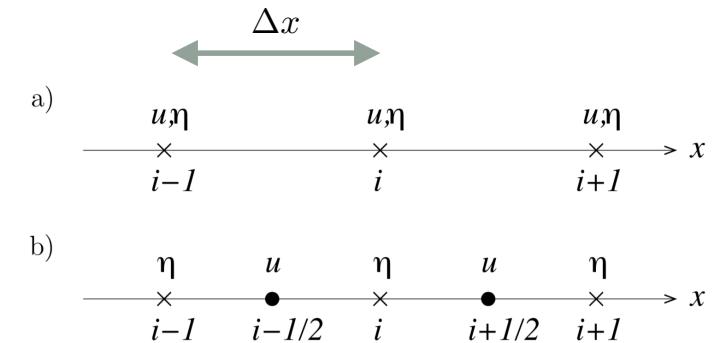


Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Gravity waves

$$\begin{aligned}\partial_t u &= -g \partial_x \eta \\ \partial_t \eta &= -H \partial_x u\end{aligned} \longrightarrow \partial_{tt} \eta = g H \partial_{xx} \eta$$

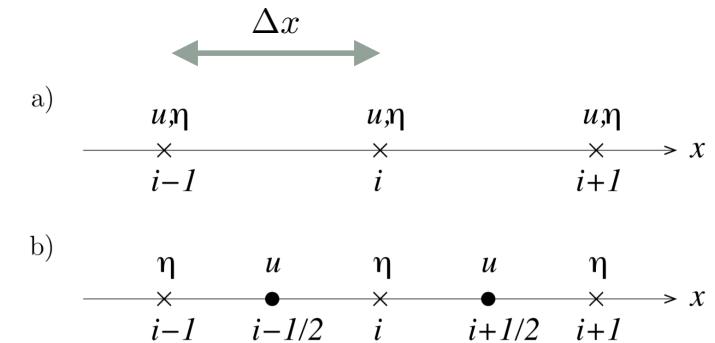


Substituting in our solution on the staggered grid gives :

$$\omega^2 - \frac{4gH}{\Delta x^2} \sin^2 \frac{k\Delta x}{2} = 0$$

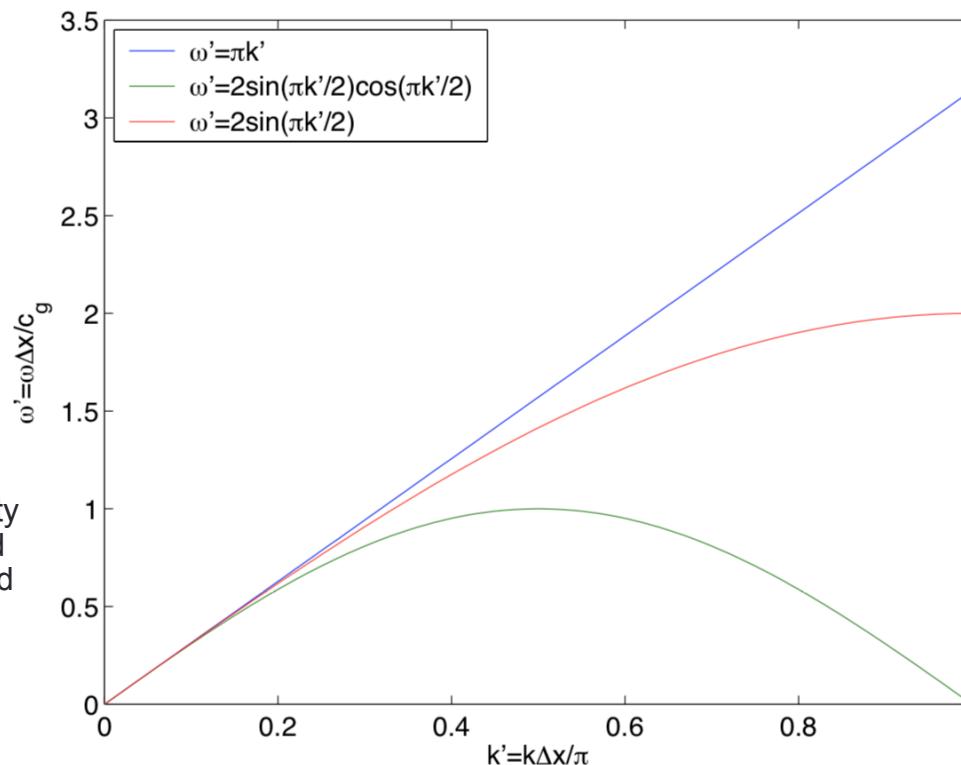
Horizontal discretization

Staggered Vs unstaggered : the 1D problem



2. Gravity waves

$$\begin{aligned} \partial_t u &= -g \partial_x \eta \\ \partial_t \eta &= -H \partial_x u \longrightarrow \partial_{tt} \eta = g H \partial_{xx} \eta \end{aligned}$$



When compared to the continuum we see that the numerical modes are still dispersive on the staggered grid, but:

there is no false extrema, unlike the non-staggered grid,

the group speed is of the correct sign everywhere, even if reduced.

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

$$\begin{aligned}\partial_t u - fv + g\partial_x \eta &= 0 \\ \partial_t v + fu &= 0 \\ \partial_t \eta + H\partial_x u &= 0\end{aligned}$$

Solutions of the continuous equations are waves following the dispersion relation:

$$\left| \begin{pmatrix} -i\omega & -f & gik \\ f & -i\omega & 0 \\ Hik & 0 & -i\omega \end{pmatrix} \right| = 0 \Rightarrow \begin{cases} \omega = 0 \\ \omega^2 = f^2 + gHk^2 \end{cases}$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

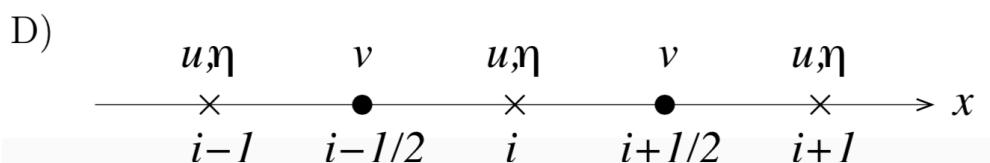
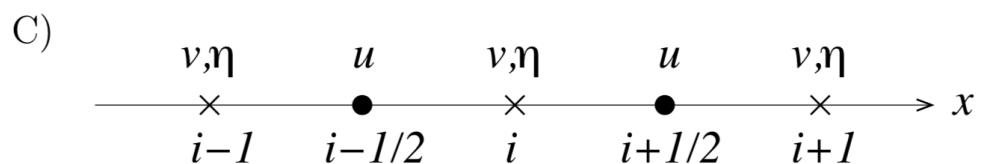
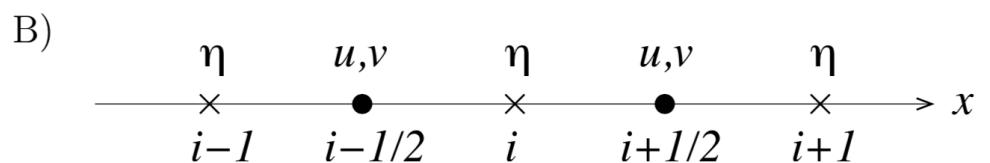
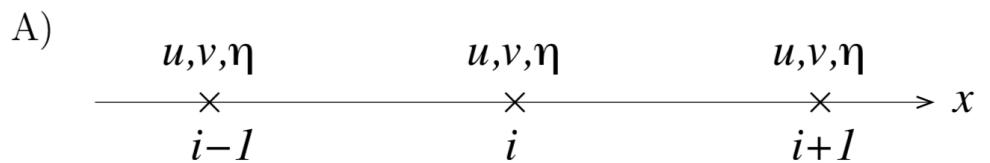
2. Inertia-Gravity waves

$$\partial_t u - fv + g \partial_x \eta = 0$$

$$\partial_t v + fu = 0$$

$$\partial_t \eta + H \partial_x u = 0$$

Now, 4 different grids are possible:



Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

- A-grid model

$$\begin{aligned}\partial_t u - f v + \frac{g}{\Delta x} \delta_i \bar{\eta}^i &= 0 \\ \partial_t v + f u &= 0 \\ \partial_t \eta + \frac{H}{\Delta x} \delta_i \bar{u}^i &= 0\end{aligned}$$

- B-grid model

$$\begin{aligned}\partial_t u - f v + \frac{g}{\Delta x} \delta_i \eta &= 0 \\ \partial_t v + f u &= 0 \\ \partial_t \eta + \frac{H}{\Delta x} \delta_i u &= 0\end{aligned}$$

- C-grid model

$$\begin{aligned}\partial_t u - f \bar{v}^i + \frac{g}{\Delta x} \delta_i \eta &= 0 \\ \partial_t v + f \bar{u}^i &= 0 \\ \partial_t \eta + \frac{H}{\Delta x} \delta_i u &= 0\end{aligned}$$

- D-grid model

$$\begin{aligned}\partial_t u - f \bar{v}^i + \frac{g}{\Delta x} \delta_i \bar{\eta}^i &= 0 \\ \partial_t v + f \bar{u}^i &= 0 \\ \partial_t \eta + \frac{H}{\Delta x} \delta_i \bar{u}^i &= 0\end{aligned}$$

Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves

The corresponding dispersion relations are :

$$s_k = \sin \frac{k\Delta x}{2} \quad c_k = \cos \frac{k\Delta x}{2}$$

$$L_d = \sqrt{gH}/f$$

A: $\frac{\omega^2}{f^2} = 1 + \frac{4L_d^2}{\Delta x^2} s_k^2 c_k^2$

B: $\frac{\omega^2}{f^2} = 1 + \frac{4L_d^2}{\Delta x^2} s_k^2$

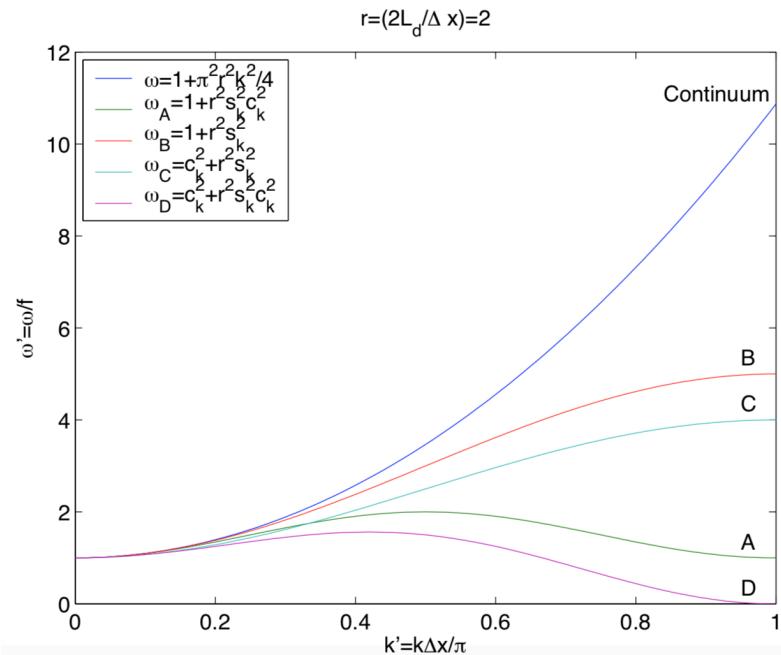
C: $\frac{\omega^2}{f^2} = c_k^2 + \frac{4L_d^2}{\Delta x^2} s_k^2$

D: $\frac{\omega^2}{f^2} = c_k^2 + \frac{4L_d^2}{\Delta x^2} s_k^2 c_k^2$

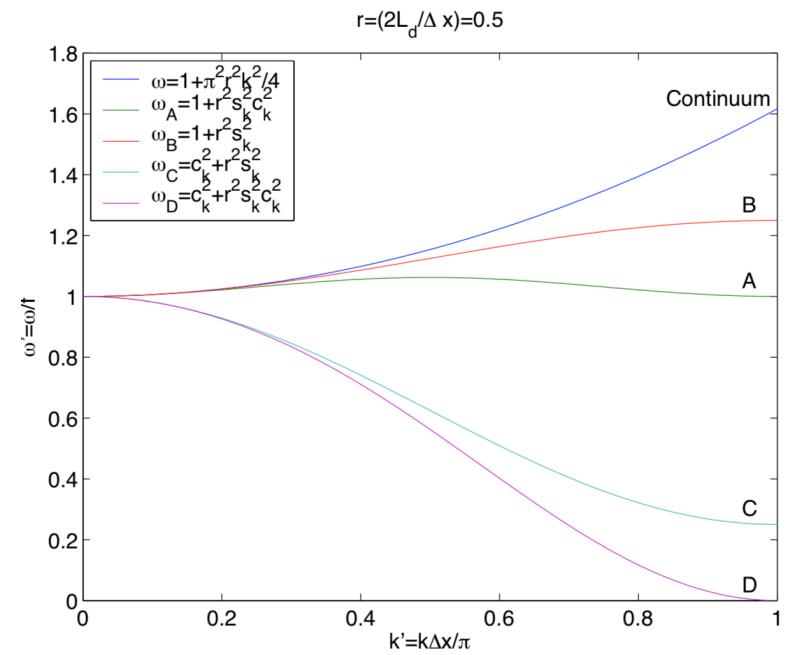
Horizontal discretization

Staggered Vs unstaggered : the 1D problem

2. Inertia-Gravity waves



deformation radius is resolved

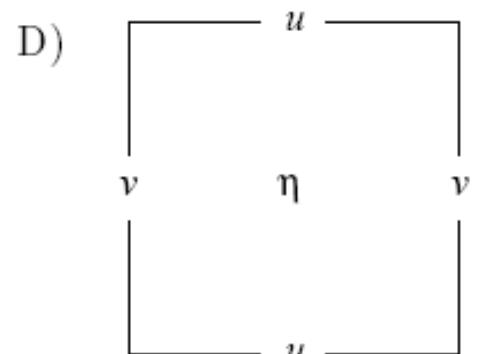
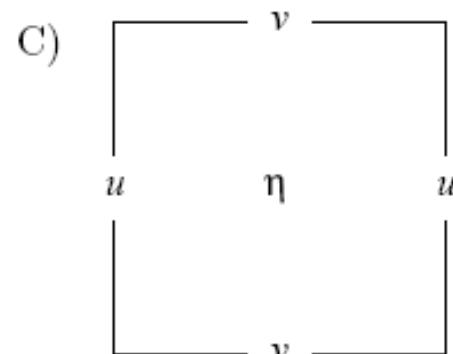
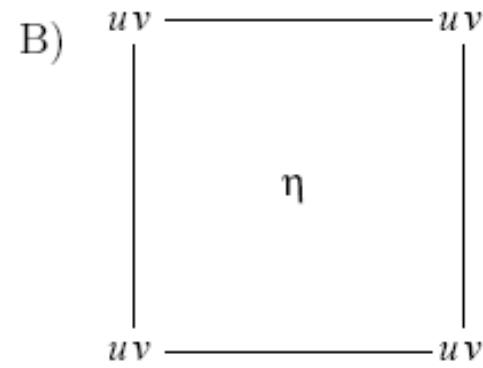
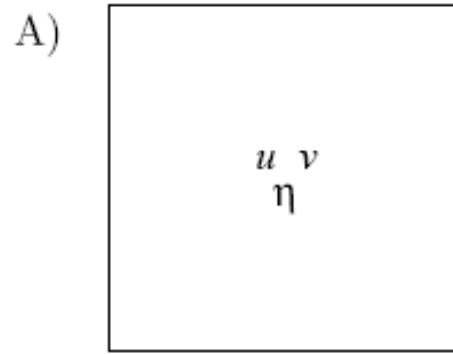


deformation radius is not resolved

Staggering variables in the form of the B grid is most likely to avoid computational modes when solving one-dimensional shallow water equations.

Horizontal discretization

Horizontal Arakawa Grids:



Linear shallow water equation:

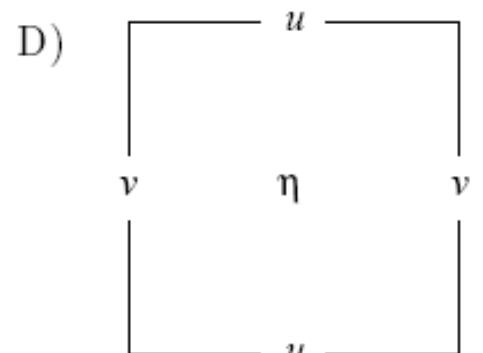
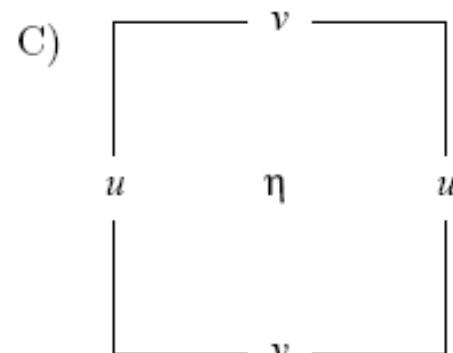
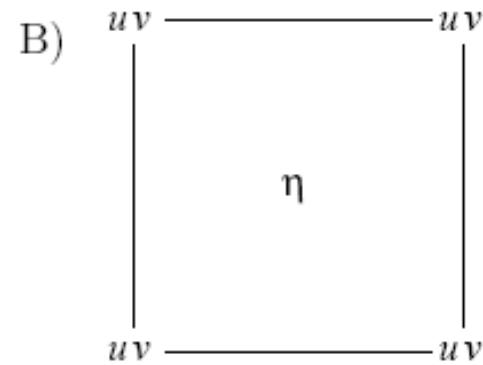
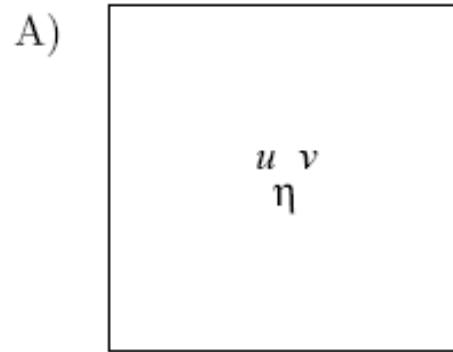
$$\partial_t u - fv + \frac{g}{\Delta x} \delta_i \eta = 0$$

$$\partial_t v + fu + \frac{g}{\Delta y} \delta_j \eta = 0$$

$$\partial_t \eta + \frac{H}{\Delta x} \delta_i u + \frac{H}{\Delta y} \delta_y \eta = 0$$

Horizontal discretization

Horizontal Arakawa Grids:



Linear shallow water equation:

$$\partial_t u - fv + \frac{g}{\Delta x} \delta_i \eta = 0$$

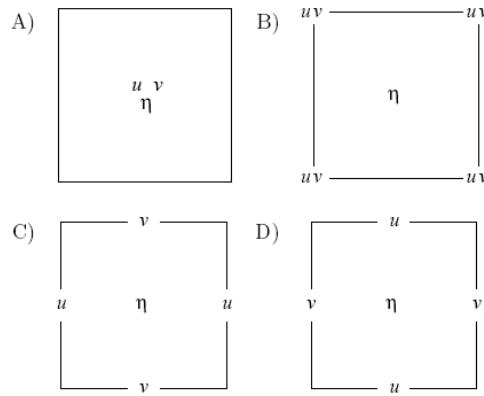
$$\partial_t v + fu + \frac{g}{\Delta y} \delta_j \eta = 0$$

$$\partial_t \eta + \frac{H}{\Delta x} \delta_i u + \frac{H}{\Delta y} \delta_y \eta = 0$$

- Question:

- Which grid minimises the number of averaging between points when solving linear SW equations in 2d?

Horizontal discretization



- A grid:

$$\begin{aligned}\partial_t u - fv + \frac{g}{\Delta x} \delta_i \bar{\eta}^i &= 0 \\ \partial_t v + fu + \frac{g}{\Delta y} \delta_j \bar{\eta}^j &= 0 \\ \partial_t \eta + \frac{H}{\Delta x} \delta_i \bar{u}^i + \frac{H}{\Delta y} \delta_j \bar{v}^j &= 0\end{aligned}$$

- B grid:

$$\begin{aligned}\partial_t u - fv + \frac{g}{\Delta x} \delta_i \bar{\eta}^j &= 0 \\ \partial_t v + fu + \frac{g}{\Delta y} \delta_j \bar{\eta}^i &= 0 \\ \partial_t \eta + \frac{H}{\Delta x} \delta_i \bar{u}^j + \frac{H}{\Delta y} \delta_j \bar{v}^i &= 0\end{aligned}$$

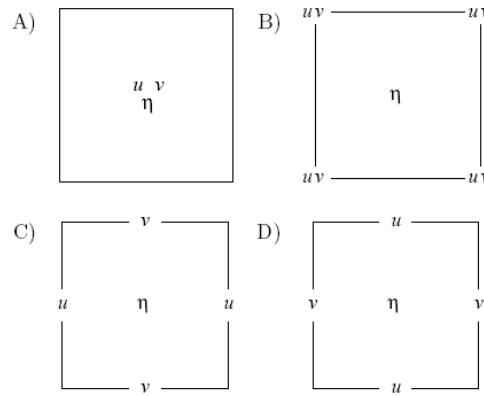
- C grid:

$$\begin{aligned}\partial_t u - f\bar{v}^{ij} + \frac{g}{\Delta x} \delta_i \eta &= 0 \\ \partial_t v + f\bar{u}^{ij} + \frac{g}{\Delta y} \delta_j \eta &= 0 \\ \partial_t \eta + \frac{H}{\Delta x} \delta_i u + \frac{H}{\Delta y} \delta_j v &= 0\end{aligned}$$

- D grid:

$$\begin{aligned}\partial_t u - f\bar{v}^{ij} + \frac{g}{\Delta x} \delta_i \bar{\eta}^{ij} &= 0 \\ \partial_t v + f\bar{u}^{ij} + \frac{g}{\Delta y} \delta_j \bar{\eta}^{ij} &= 0 \\ \partial_t \eta + \frac{H}{\Delta x} \delta_i \bar{u}^{ij} + \frac{H}{\Delta y} \delta_j \bar{v}^{ij} &= 0\end{aligned}$$

Horizontal discretization



Response of each operator:

$$\begin{aligned} R(\delta_i \phi) &= 2i \sin \frac{k \Delta x}{2} = 2i s_k \\ R(\delta_j \phi) &= 2i \sin \frac{l \Delta y}{2} = 2i s_l \\ R(\bar{\phi}^i) &= \cos \frac{k \Delta x}{2} = c_k \\ R(\bar{\phi}^j) &= \cos \frac{l \Delta y}{2} = c_l \end{aligned}$$

Dispersion relations:

- A grid:

$$\omega^2 = f^2 + \frac{4gH}{\Delta x^2} s_k^2 c_k^2 + \frac{4gH}{\Delta y^2} s_l^2 c_l^2$$

$$\text{or } \left(\frac{\omega}{f}\right)^2 = 1 + r_x^2 s_k^2 c_k^2 + r_y^2 s_l^2 c_l^2$$

- B grid:

$$\omega^2 = f^2 + \frac{4gH}{\Delta x^2} s_k^2 c_l^2 + \frac{4gH}{\Delta y^2} s_l^2 c_k^2$$

$$\text{or } \left(\frac{\omega}{f}\right)^2 = 1 + r_x^2 s_k^2 c_l^2 + r_y^2 s_l^2 c_k^2$$

- C grid:

$$\omega^2 = f^2 c_k^2 c_l^2 + \frac{4gH}{\Delta x^2} s_k^2 + \frac{4gH}{\Delta y^2} s_l^2$$

$$\text{or } \left(\frac{\omega}{f}\right)^2 = c_k^2 c_l^2 + r_x^2 s_k^2 + r_y^2 s_l^2$$

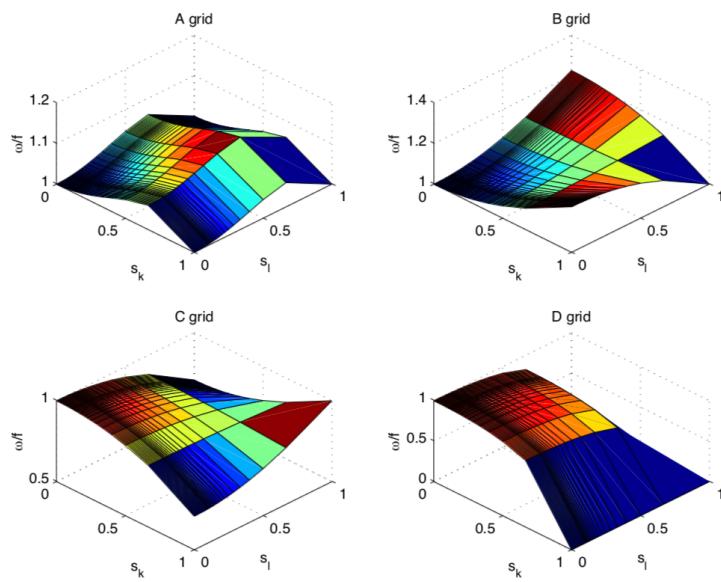
- D grid:

$$\omega^2 = f^2 c_k^2 c_l^2 + \frac{4gH}{\Delta x^2} s_k^2 c_k^2 c_l^2 + \frac{4gH}{\Delta y^2} s_l^2 c_k^2 c_l^2$$

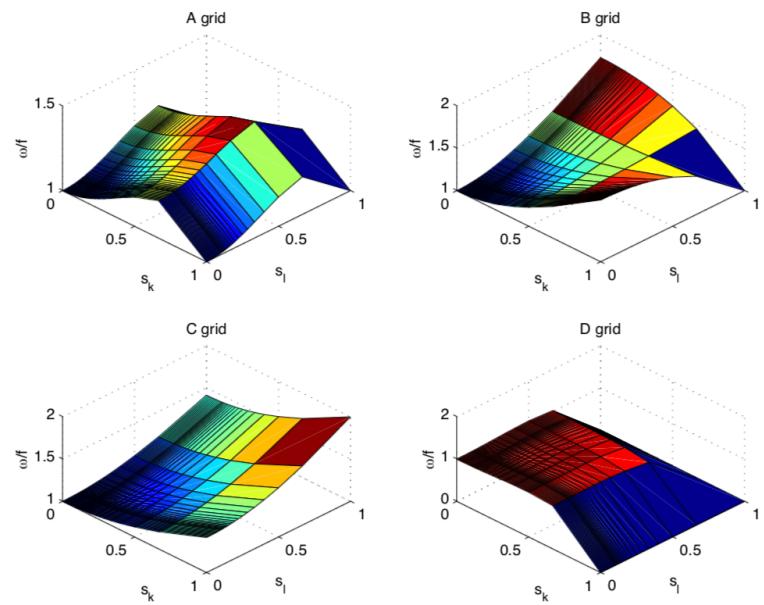
$$\text{or } \left(\frac{\omega}{f}\right)^2 = (1 + r_x^2 s_k^2 + r_y^2 s_l^2) c_k^2 c_l^2$$

Horizontal discretization

Coarse resolution:



High resolution:



D is always bad.

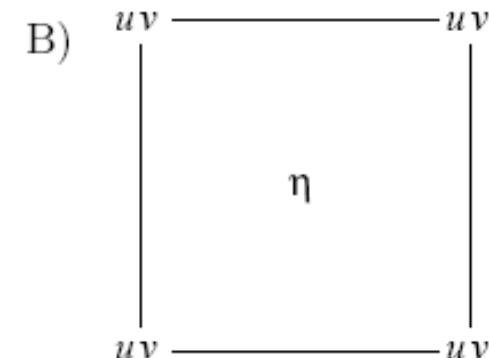
B underestimates frequency for short two-dimensional waves

C is the only grid with monotonically increasing frequency (i.e. right sign of group velocity) at high res.

Horizontal discretization

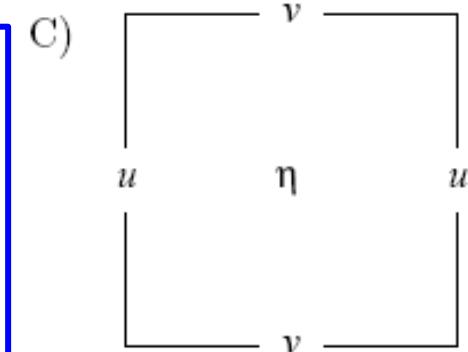
- B grid is preferred at coarse resolution, when Coriolis is important:

- Superior for poorly resolved inertia-gravity waves.
- Good for Rossby waves: collocation of velocity points.
- Bad for gravity waves: computational checkerboard mode



- C grid is preferred at fine resolution, when Coriolis is less important

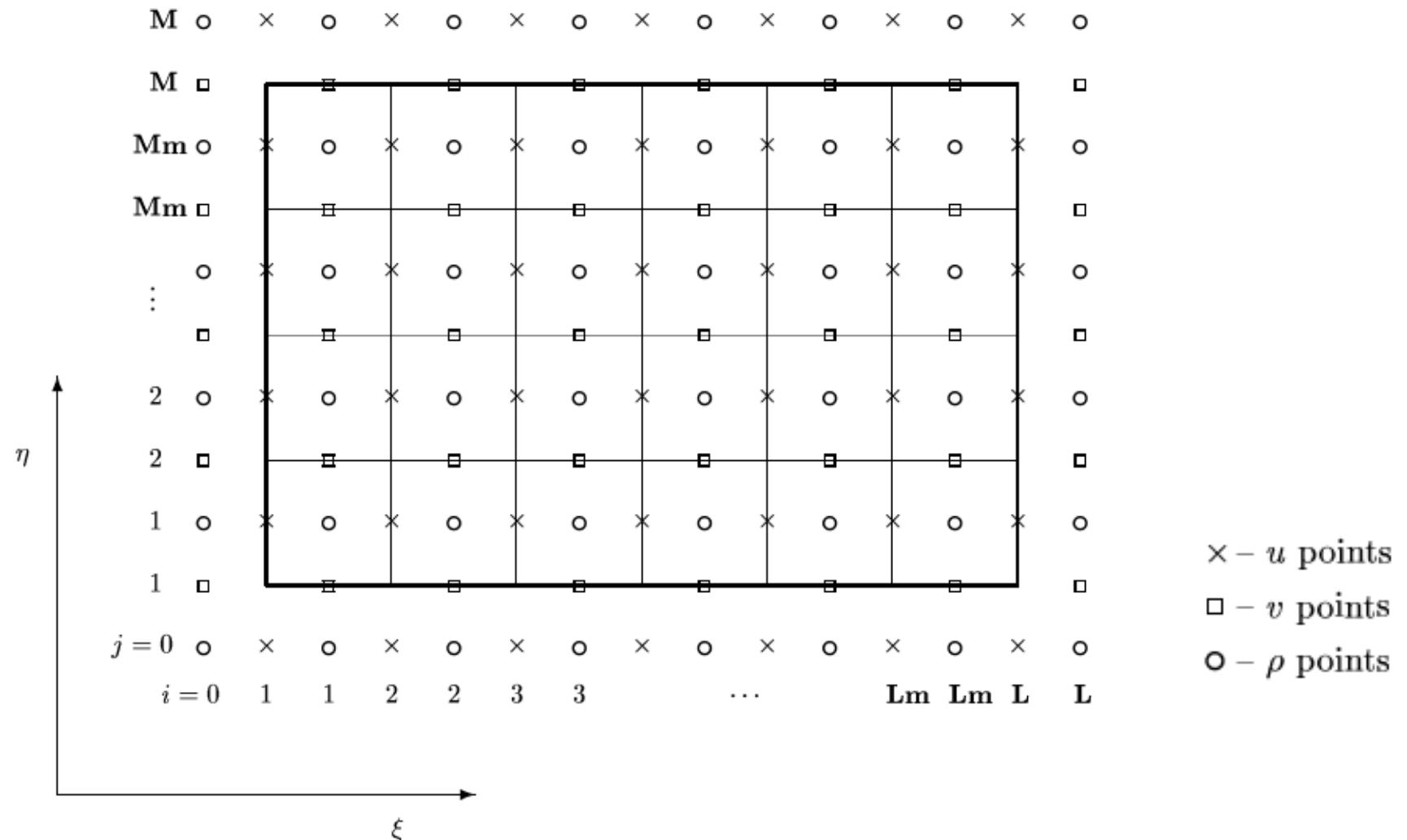
- Superior for gravity waves.
- Good for well resolved inertia-gravity waves.
- Bad for poorly resolved waves: Rossby waves (computational checkerboard mode) and inertia-gravity waves due to averaging the Coriolis force.



ROMS

Horizontal discretization

ROMS: Arakawa C-grid



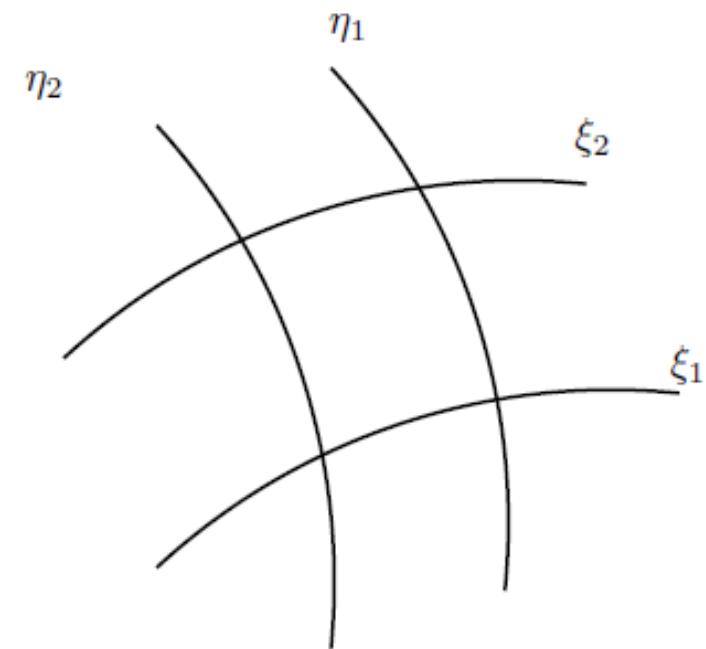
Horizontal curvilinear grid

- **ROMS:** is formulated in general horizontal curvilinear coordinates:

$$(ds)_\xi = \left(\frac{1}{m} \right) d\xi$$

$$(ds)_\eta = \left(\frac{1}{n} \right) d\eta$$

m, n : scale factors relating the differential distances to the physical arc lengths



$$\vec{v} \cdot \hat{\xi} = u$$

$$\vec{v} \cdot \hat{\eta} = v$$

Horizontal curvilinear grid

- **ROMS:** is formulated in general horizontal curvilinear coordinates:

$$(ds)_\xi = \left(\frac{1}{m} \right) d\xi$$

$$(ds)_\eta = \left(\frac{1}{n} \right) d\eta$$

With classical formulas for div, grad, curl and lap in curvilinear coordinates:

$$\nabla \phi = \hat{\xi} m \frac{\partial \phi}{\partial \xi} + \hat{\eta} n \frac{\partial \phi}{\partial \eta}$$

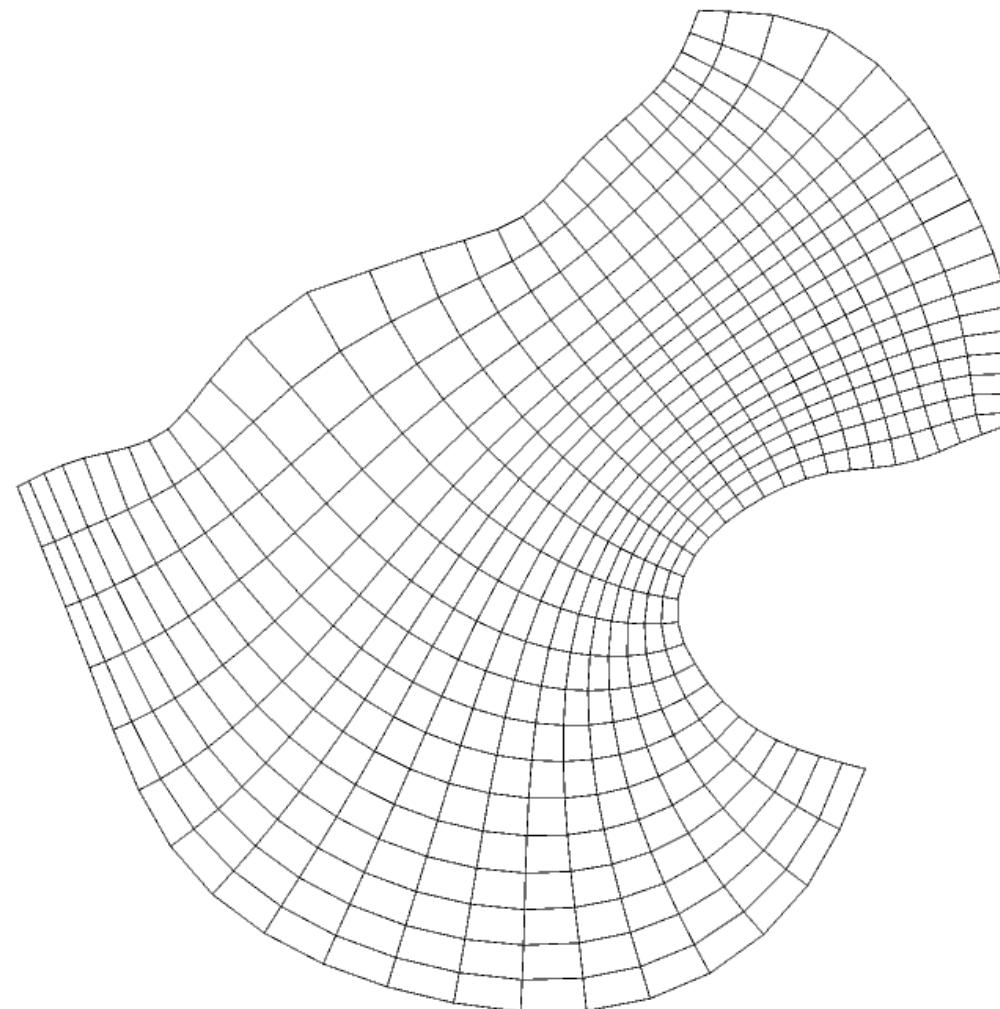
$$\nabla \cdot \vec{a} = mn \left[\frac{\partial}{\partial \xi} \left(\frac{a}{n} \right) + \frac{\partial}{\partial \eta} \left(\frac{b}{m} \right) \right]$$

$$\nabla \times \vec{a} = mn \begin{vmatrix} \hat{\xi}_1 & \hat{\xi}_2 & \hat{k} \\ \frac{\partial}{\partial \xi} & \frac{\partial}{\partial \eta} & \frac{\partial}{\partial z} \\ \frac{a}{m} & \frac{b}{n} & c \end{vmatrix}$$

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = mn \left[\frac{\partial}{\partial \xi} \left(\frac{m}{n} \frac{\partial \phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{n}{m} \frac{\partial \phi}{\partial \eta} \right) \right]$$

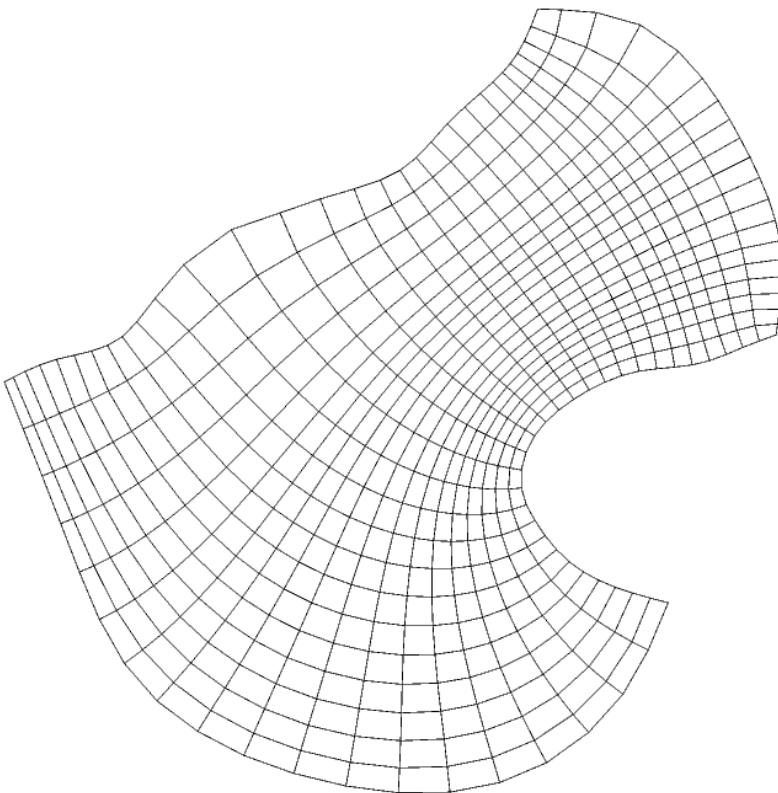
Horizontal curvilinear grid

- This is a possible grid:



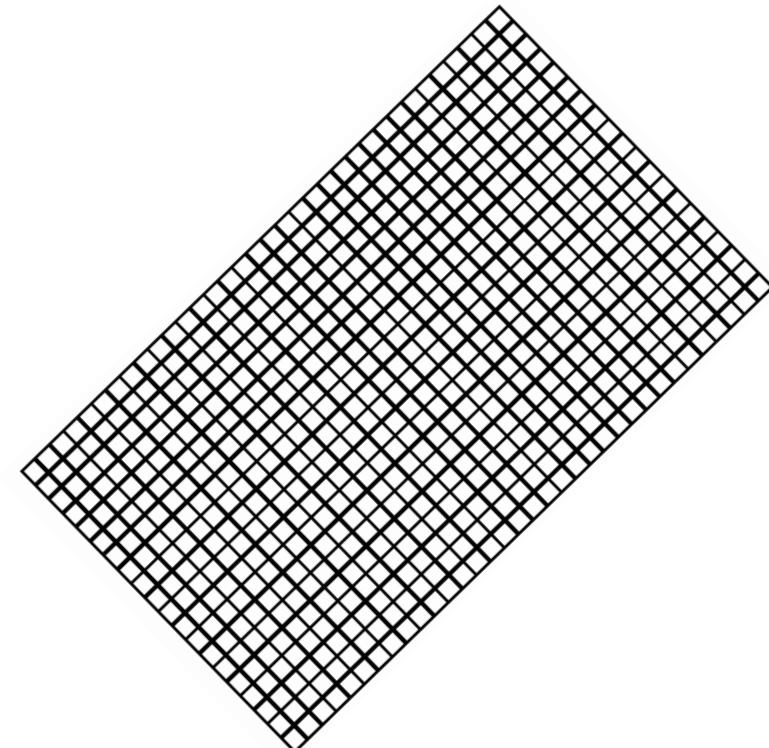
Horizontal curvilinear grid

- This is a possible grid:



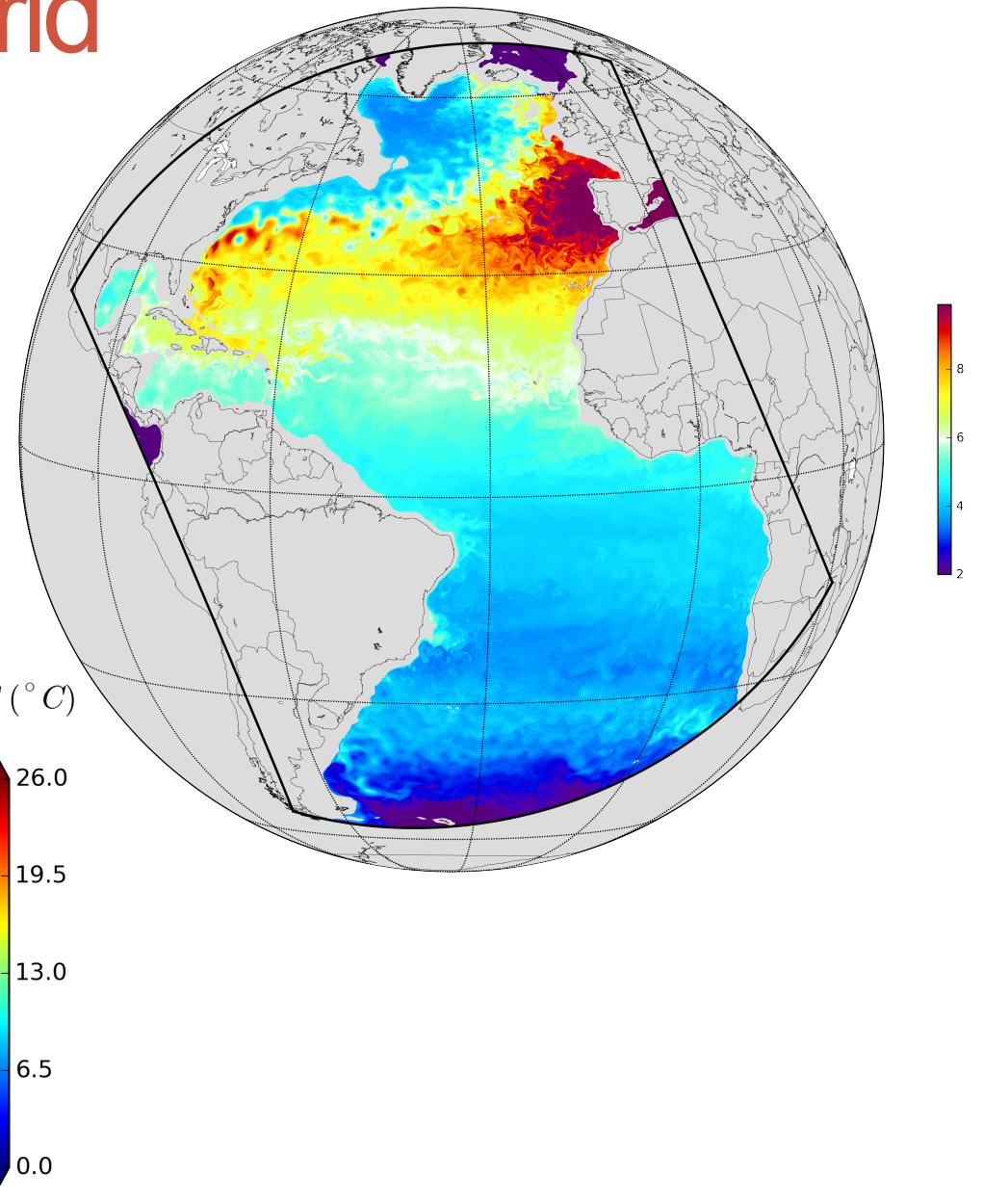
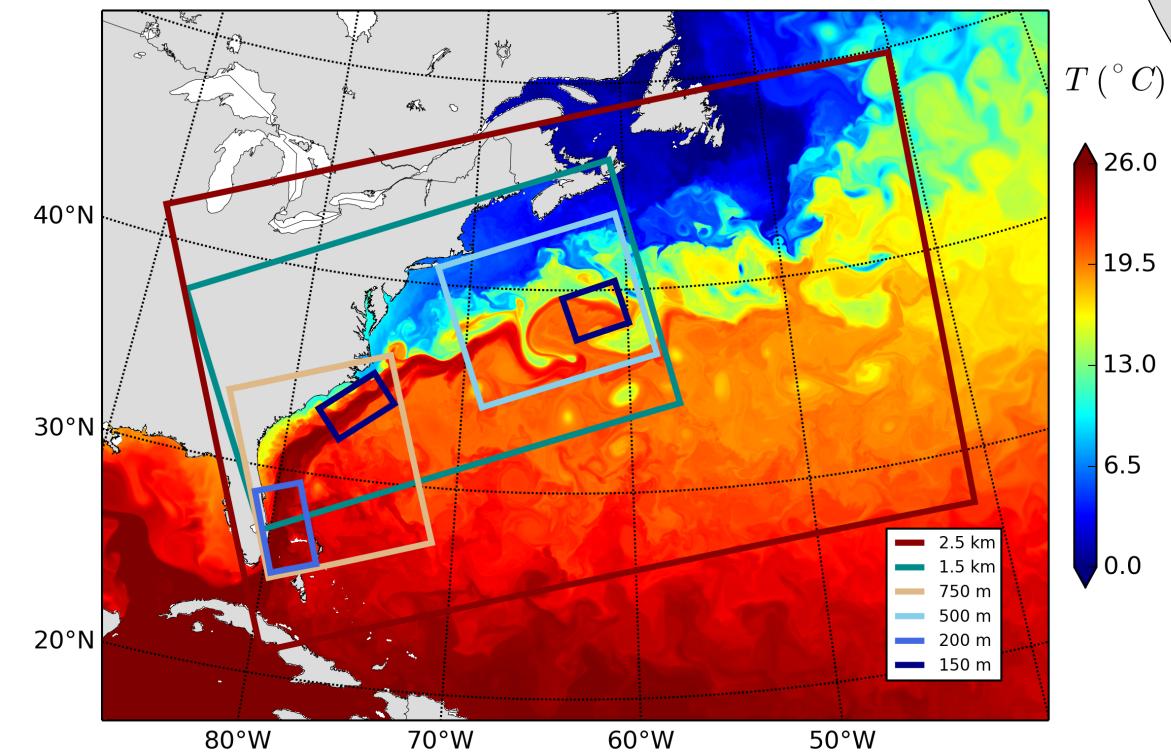
In practice variations in dx and dy should be minimized to minimize errors and optimize computation time.

So avoir extreme distortions and be as close as rectangular grids as possible (+ use land masks)



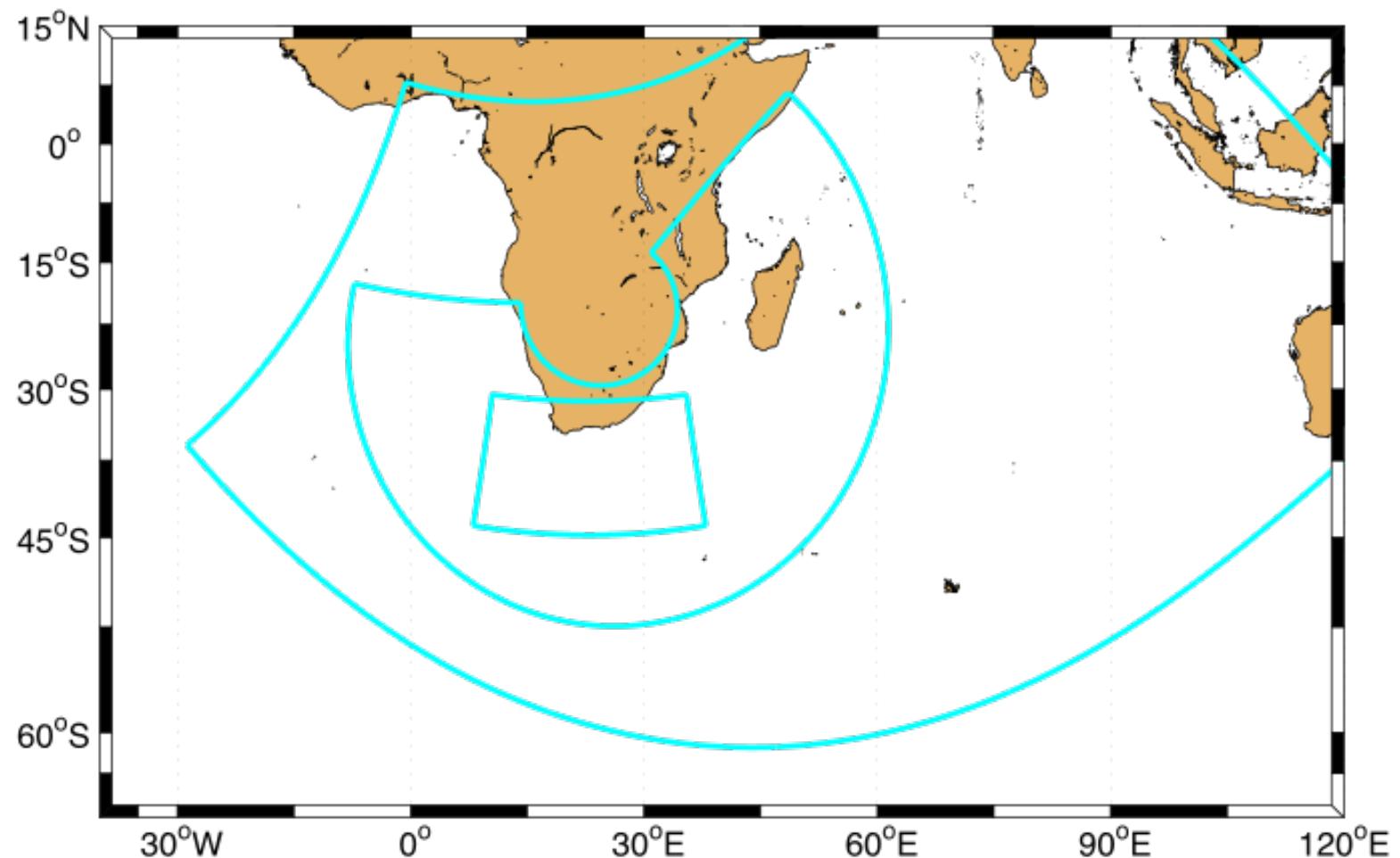
Horizontal curvilinear grid

- Example of realistic domains:



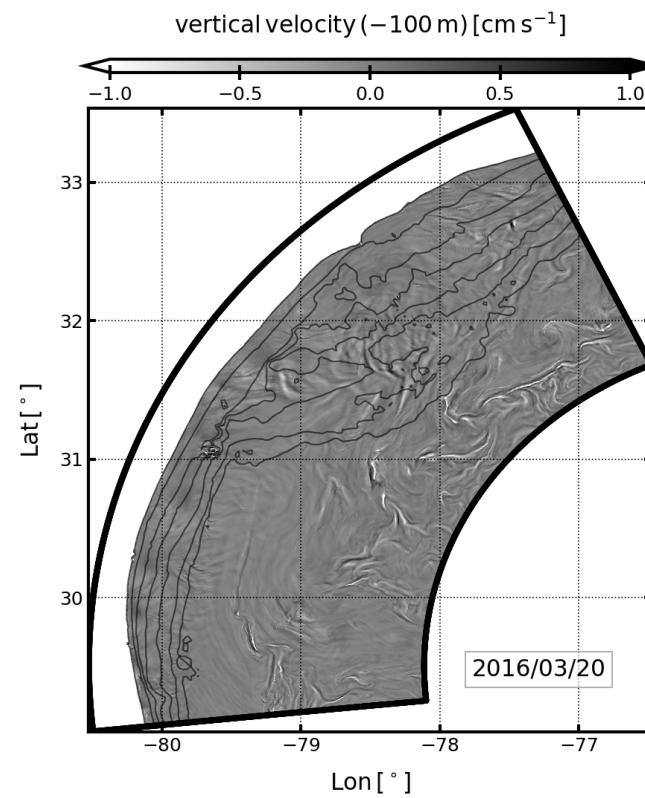
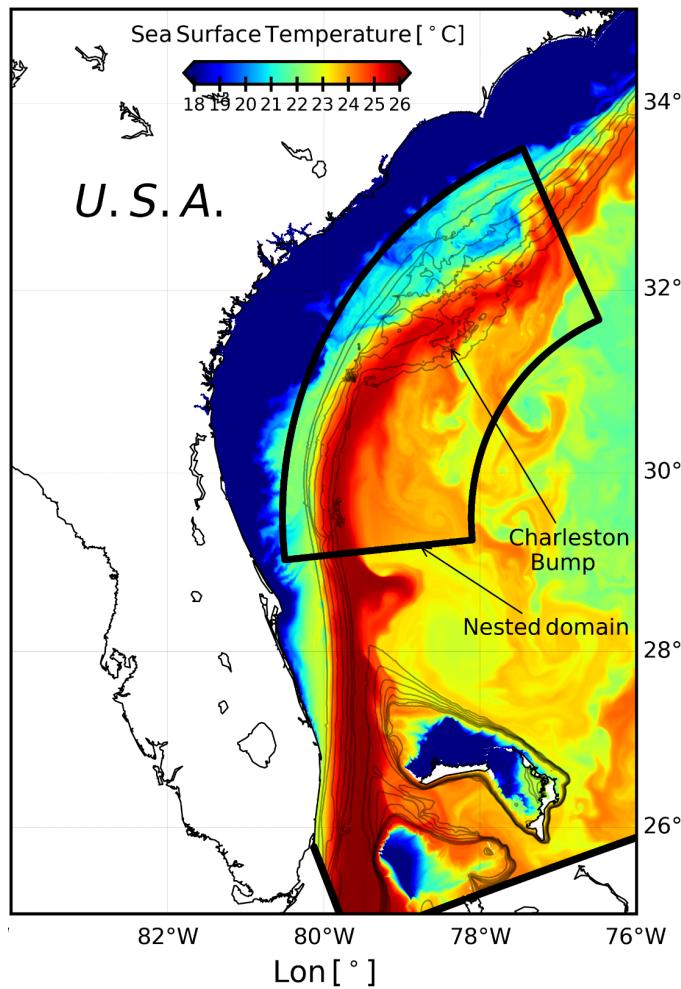
Horizontal curvilinear grid

- Example of realistic domains (with gentle bendings):



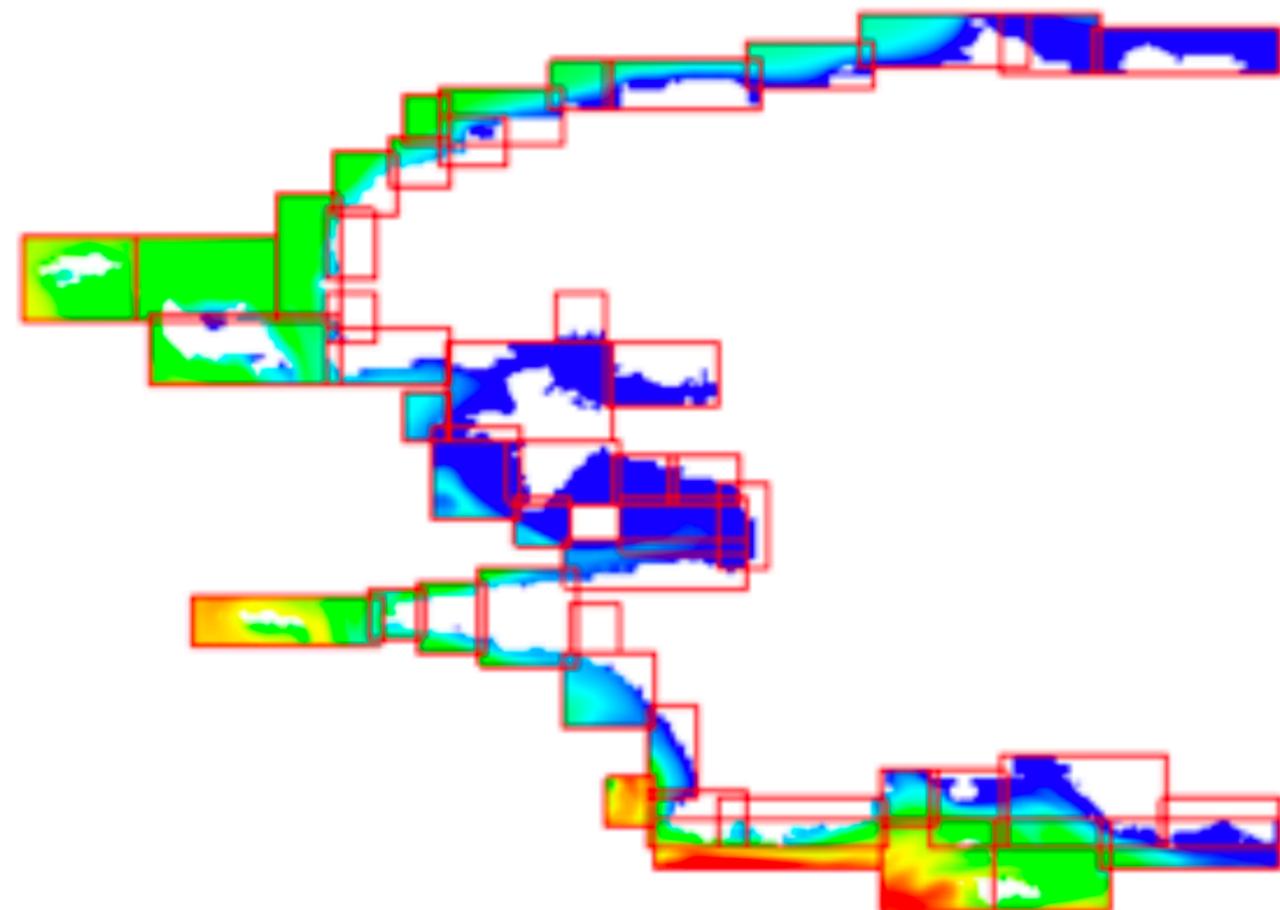
Horizontal curvilinear grid

- Example of realistic domains (with gentle bendings):



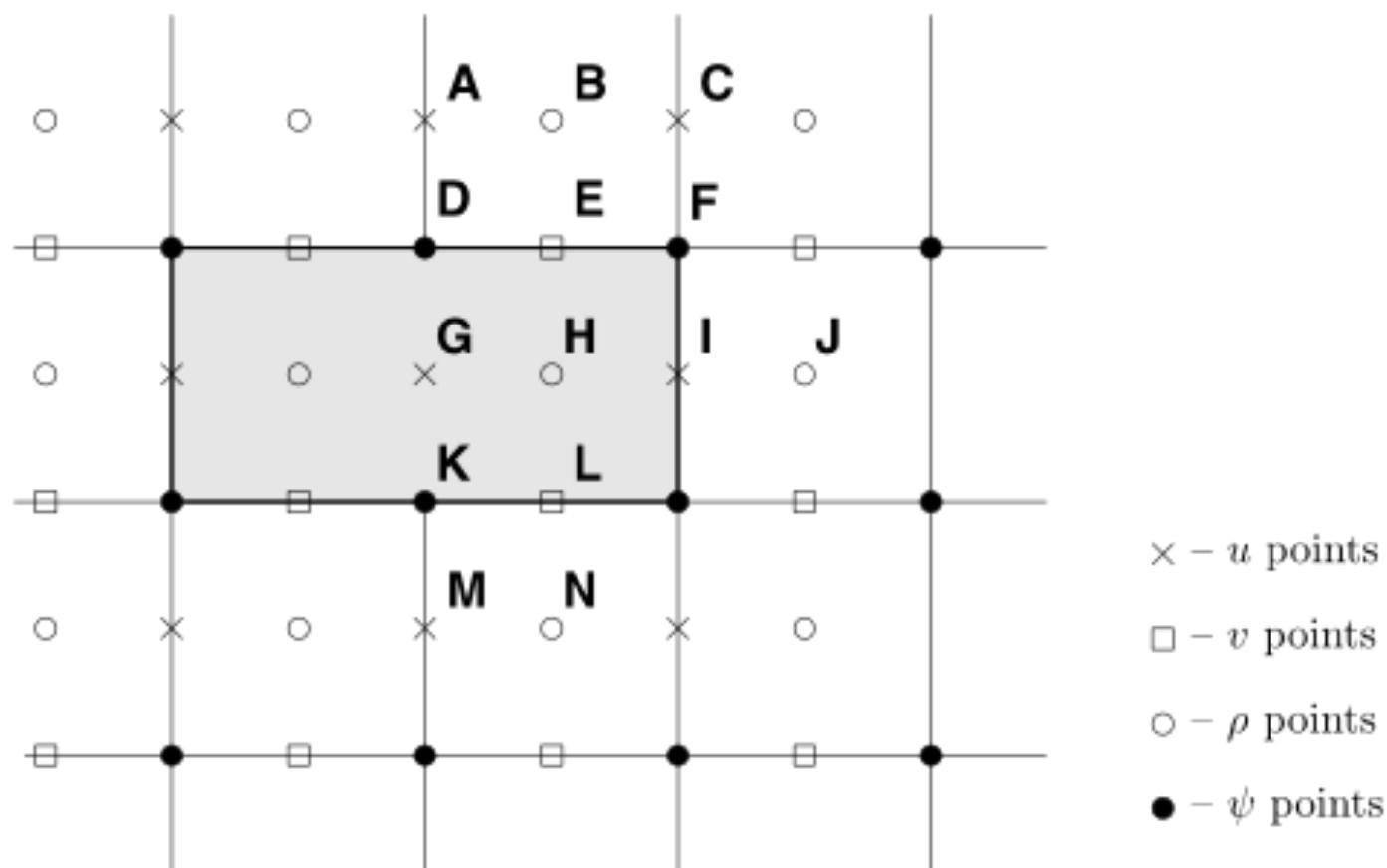
Horizontal curvilinear grid

- Another method = massive multigrain



Land/sea Mask

Variables within the masked region are set to zero by multiplying by the mask for either the u, v or rho points :



Land/sea Mask

Variables within the masked region are set to zero by multiplying by the mask for either the u, v or rho points :



Land/sea Mask

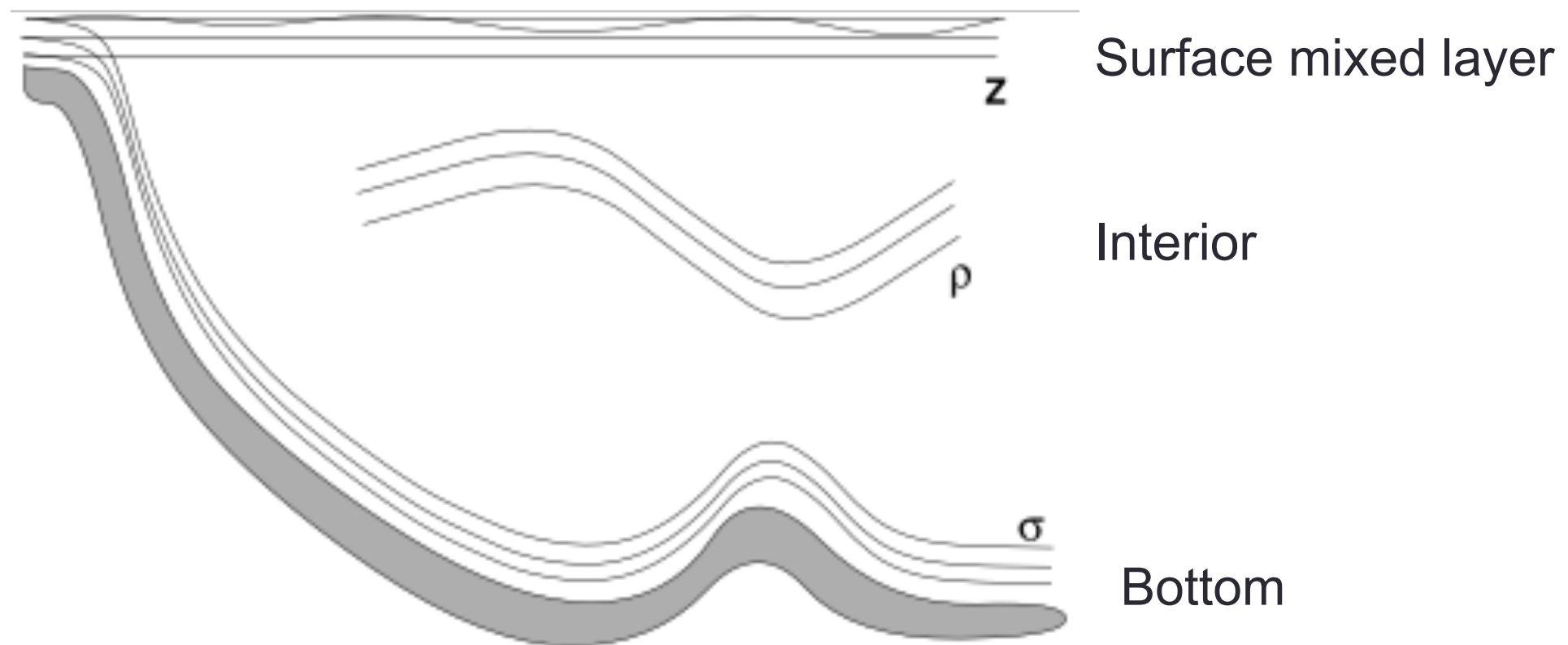
See the code routines:

```
#ifdef MASKING
# define SWITCH *
#else
# define SWITCH !
#endif

!#####
    do k=1,N
        do i=IstrU,Iend
            u(i,j,k,nnew)=(DC(i,k)-DC(i,0)) SWITCH umask(i,j)
```

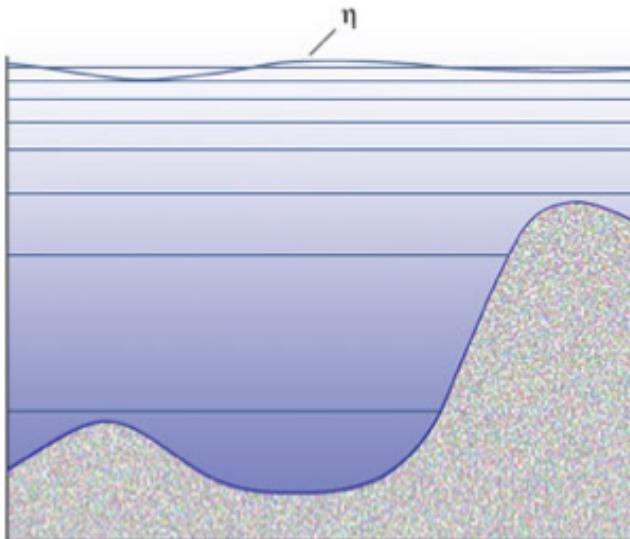
Vertical discretization

Depending on the depth, motions in the ocean will be mostly aligned along **geopotentials, isopycnal or topography**.



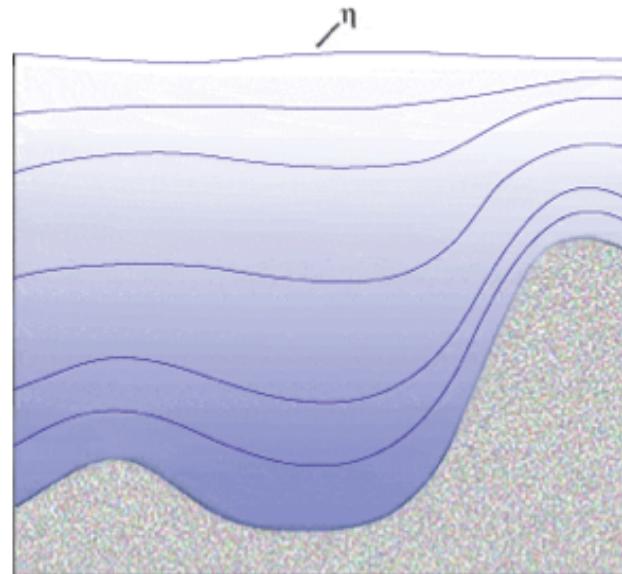
Vertical discretization

Several choices are possible to define the vertical coordinates system:



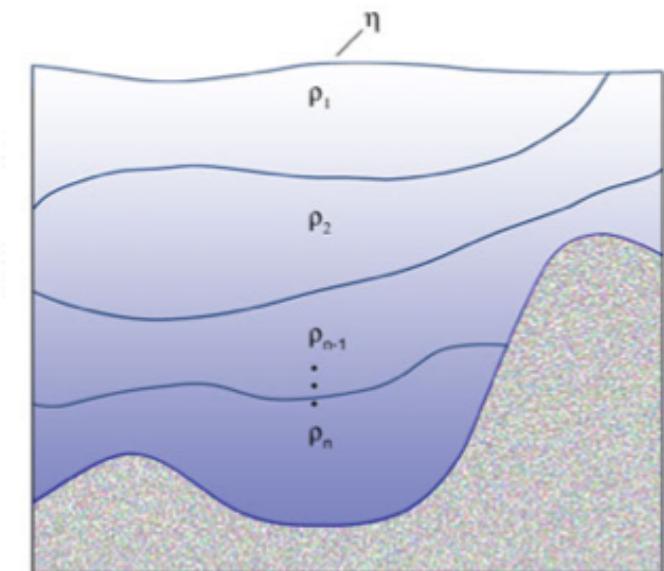
z-coordinates

Vertical coordinate is height (or depth)



sigma-coordinates

The vertical coordinate follows the bathymetry



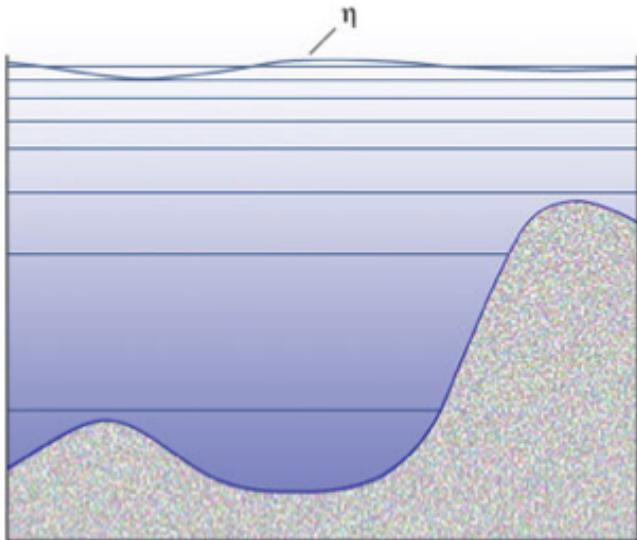
isopycnal-coordinates

The vertical coordinate is the potential density

The vertical coordinate is the major difference between models.

Vertical discretization

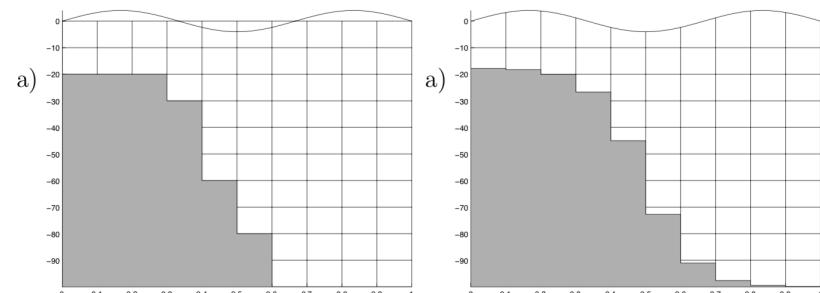
z-coordinates



PROS

- Natural in the upper ocean and for mixed-layer processes
- Ideal to compute horizontal (pressure) gradients
- Easier to implement and use

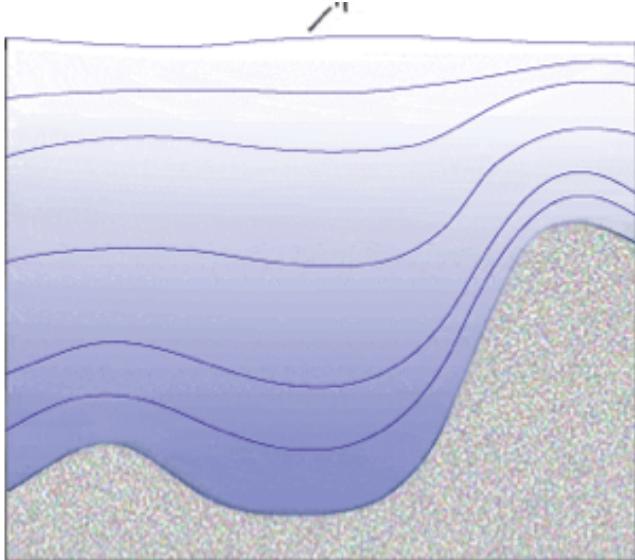
CONS



- The representation of tracer advection and diffusion along inclined density surfaces in the ocean interior is cumbersome.
- Representation of bottom topography is difficult.
- Need for bottom and lateral conditions
- Representation and parameterization of the BBL is unnatural.

Vertical discretization

sigma-coordinates



PROS

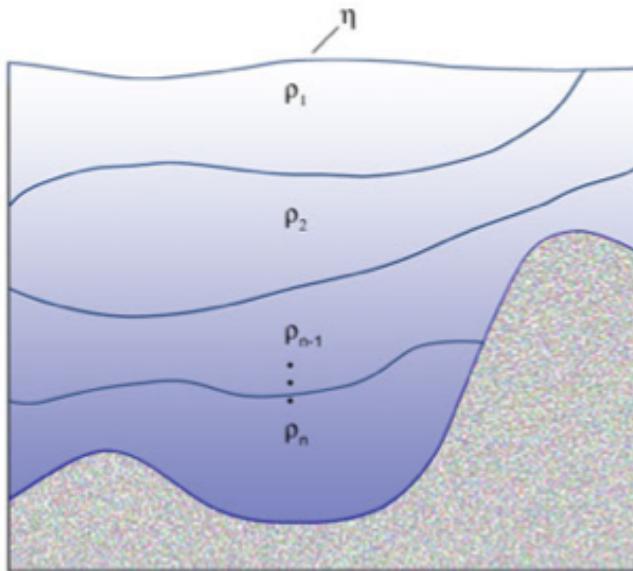
- Representation of bottom topography is natural (only bottom boundary condition)
- Representation and parameterization of the BBL is natural (more vertical res. in BBL)

CONS

- The representation of tracer advection and diffusion along inclined density surfaces in the ocean interior is cumbersome.
- Pressure gradient errors can be a problem

Vertical discretization

isopycnal-coordinates



PROS

- The representation of tracer advection and diffusion along inclined density surfaces in the ocean interior is natural.
- Water mass characteristics are preserved over long time scales

CONS

- Representing the effects of a realistic (non-linear) equation of state is cumbersome.
- Inappropriate for representing the surface mixed layer or BBL which are mostly unstratified.
- Non-hydrostatic effects/dynamics are not possible.
- Vertical and horizontal resolution are tightly connected in regions where isopycnals outcrop. This can lead to inadequate horizontal resolution in regions such as the ACC.

Vertical discretization

- Equations for a generalized coordinate system:

Consider a general vertical coordinate, r , which is assumed to be a monotonic function of height, z .

Any variable can be written in the new coordinate system: $A = A(x, y, z(x, y, r, t), t)$

Vertical derivatives can be written

$$\frac{\partial A}{\partial r} = \frac{\partial z}{\partial r} \frac{\partial A}{\partial z} \quad \frac{\partial A}{\partial z} = \frac{\partial r}{\partial z} \frac{\partial A}{\partial r}$$

And other derivatives (for $s = x, y, t$) can be written using the chain rule:

$$\left. \frac{\partial A}{\partial s} \right|_z = \left. \frac{\partial A}{\partial s} \right|_\sigma - \left. \frac{\partial A}{\partial z} \right. \left. \frac{\partial z}{\partial s} \right|_\sigma$$

So $\left. \frac{\partial A}{\partial s} \right|_z = \left. \frac{\partial A}{\partial s} \right|_\sigma - \left. \frac{\partial r}{\partial z} \right. \left. \frac{\partial A}{\partial r} \right. \left. \frac{\partial z}{\partial s} \right|_\sigma$

Vertical discretization

- Equations for a generalized coordinate system:

Using the chain rule, you can write the horizontal gradient:

$$\nabla_z A = \nabla_r A - \frac{\partial A}{\partial r} \frac{\partial r}{\partial z} \nabla_r z$$

And the vertical velocity can be written:

$$\begin{aligned} w = D_t z &= \left. \frac{\partial z}{\partial t} \right|_r + \left. \frac{\partial z}{\partial x} \right|_r D_t x + \left. \frac{\partial z}{\partial y} \right|_r D_t y + \left. \frac{\partial z}{\partial r} \right|_r D_t r \\ &= \left. \frac{\partial z}{\partial t} \right|_r + \vec{v} \cdot \nabla_r z + \dot{r} \frac{\partial z}{\partial r} \end{aligned}$$

Vertical discretization

- Equations for a generalized coordinate system:

And the total derivative becomes:

$$\begin{aligned} D_t A &= \left. \frac{\partial A}{\partial t} \right|_z + \vec{v} \cdot \nabla_z A + w \frac{\partial A}{\partial z} \\ &= \left. \frac{\partial A}{\partial t} \right|_r + \vec{v} \cdot \nabla_r A + \left(w - \left. \frac{\partial z}{\partial t} \right|_r - \vec{v} \cdot \nabla_r z \right) \frac{\partial r}{\partial z} \frac{\partial A}{\partial r} \\ &= \left. \frac{\partial A}{\partial t} \right|_r + \vec{v} \cdot \nabla_r A + \dot{r} \frac{\partial A}{\partial r} \end{aligned}$$

And the horizontal pressure gradient:

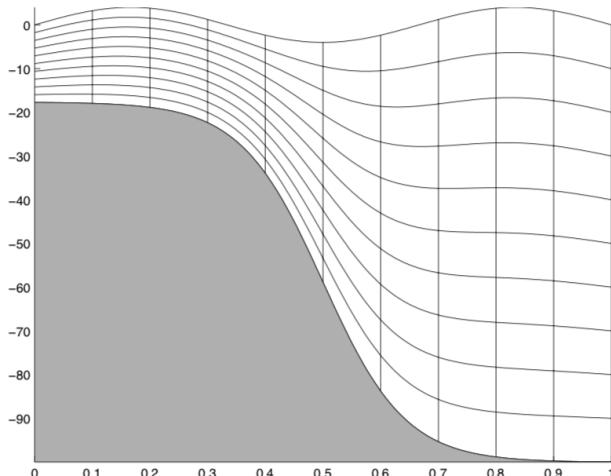
$$\begin{aligned} \nabla_z p &= \nabla_r p - \frac{\partial p}{\partial z} \nabla_r z \\ &= \nabla_r p + \rho \nabla_r g z \end{aligned}$$

Vertical grid : σ generalized coordinate

- Sigma-coordinate system:

Sigma-coordinates are terrain-following coordinates defined as:

$$\sigma = \frac{z}{H_z(x, y)}$$



$$\frac{\partial \sigma}{\partial z} = \frac{1}{H_z}$$

Vertical grid : σ generalized coordinate

- Horizontal and vertical derivatives can be written:

$$\left(\frac{\partial}{\partial x} \right)_z = \left(\frac{\partial}{\partial x} \right)_\sigma - \left(\frac{1}{H_z} \right) \left(\frac{\partial z}{\partial x} \right)_\sigma \frac{\partial}{\partial \sigma}$$
$$\left(\frac{\partial}{\partial y} \right)_z = \left(\frac{\partial}{\partial y} \right)_\sigma - \left(\frac{1}{H_z} \right) \left(\frac{\partial z}{\partial y} \right)_\sigma \frac{\partial}{\partial \sigma}$$
$$\frac{\partial}{\partial z} = \left(\frac{\partial s}{\partial z} \right) \frac{\partial}{\partial \sigma} = \frac{1}{H_z} \frac{\partial}{\partial \sigma}$$

Vertical grid : σ generalized coordinate

- Example: (Hydrostatic) Potential vorticity (PV) is defined as

$$q = \left(f \vec{k} + \vec{\nabla} \times \vec{u} \right) \cdot \vec{\nabla} b$$

$$q = \left[f + \frac{\partial v}{\partial x} \Big|_z - \frac{\partial u}{\partial y} \Big|_z \right] \frac{\partial \rho}{\partial z} - \frac{\partial v}{\partial z} \frac{\partial \rho}{\partial x} \Big|_z + \frac{\partial u}{\partial z} \frac{\partial \rho}{\partial y} \Big|_z$$

- Question: How do you compute PV in sigma-coordinates?

- Write the expression of q using only derivatives along sigma coordinates:

$$\frac{\partial}{\partial x} \Big|_{\sigma} \quad \frac{\partial}{\partial y} \Big|_{\sigma} \quad \frac{\partial}{\partial \sigma}$$

Vertical grid : isopycnal coordinates

- For isopycnal coordinates, the vertical coordinate is: $r = \rho$
- Potential vorticity (PV) is easily defined as :

$$q = \frac{1}{h} \left[f + \left. \frac{\partial v}{\partial x} \right|_{\rho} - \left. \frac{\partial u}{\partial y} \right|_{\rho} \right]$$

with $h = \frac{\partial z}{\partial \rho}$

Vertical grid : σ generalized coordinate

ROMS: Generalized σ -Coordinate

Stretching &
condensing of
vertical resolution:

$$z(x, y, \sigma, t) = \zeta(x, y, t) + [\zeta(x, y, t) + h(x, y)] S(x, y, \sigma),$$
$$S(x, y, \sigma) = \frac{h_c \sigma + h(x, y) C(\sigma)}{h_c + h(x, y)}$$

$$S(x, y, \sigma) = \begin{cases} 0, & \text{if } \sigma = 0, \\ -1, & \text{if } \sigma = -1, \end{cases} \quad C(\sigma) = \begin{cases} 0, & \text{at the free-surface;} \\ -1, & \text{at the ocean bottom.} \end{cases}$$

Surface refinement function:

$$C(\sigma) = \frac{1 - \cosh(\theta_S \sigma)}{\cosh(\theta_S) - 1}, \quad \text{for } \theta_S > 0, \quad C(\sigma) = -\sigma^2, \quad \text{for } \theta_S \leq 0$$

Bottom refinement function:

$$C(\sigma) = \frac{\exp(\theta_B C(\sigma)) - 1}{1 - \exp(-\theta_B)}, \quad \text{for } \theta_B > 0$$

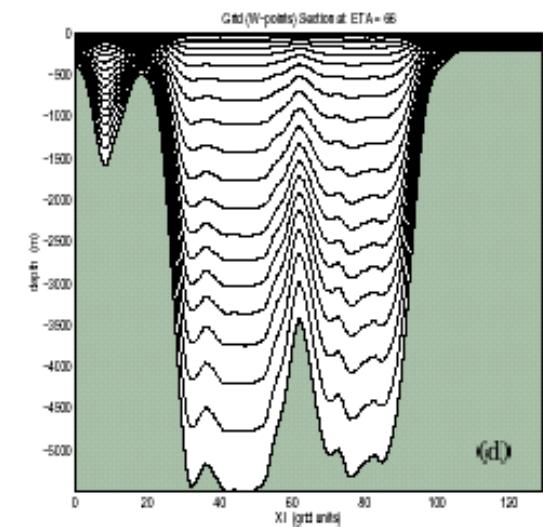
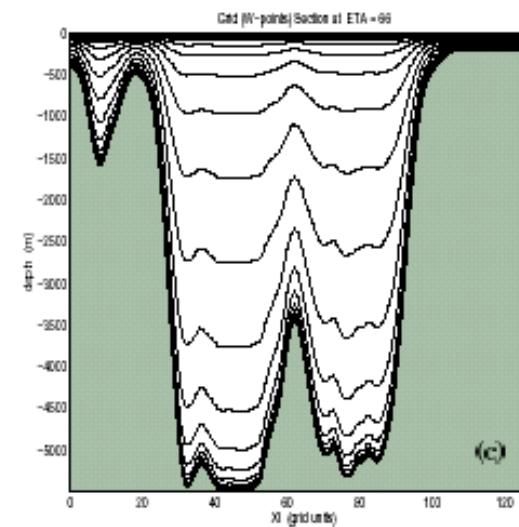
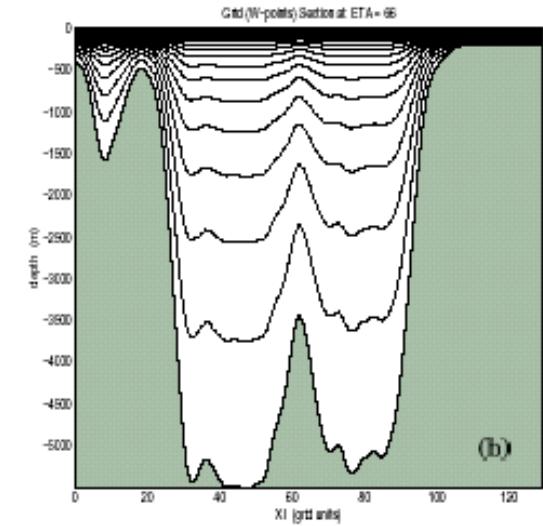
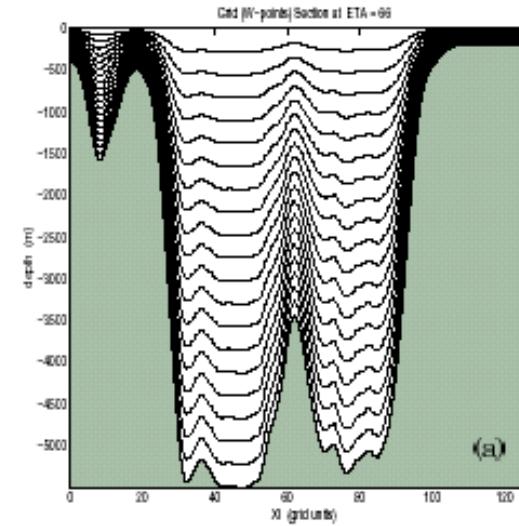
Vertical grid : σ generalized coordinate

ROMS: Generalized σ -Coordinate

Stretching & condensing of vertical resolution

θ and b : surface and bottom parameters:

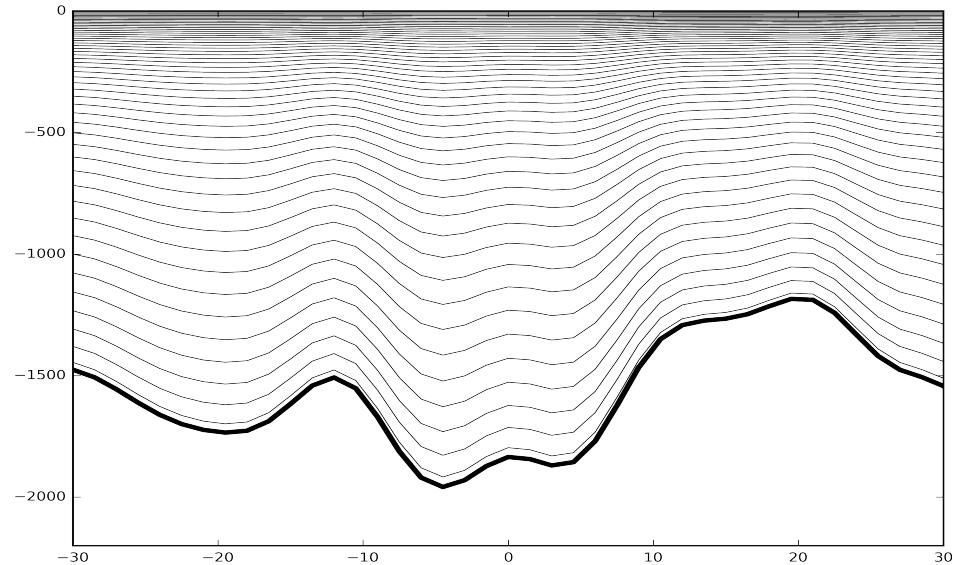
- (a) $\theta=0, b=0$
- (b) $\theta=8, b=0$
- (c) $\theta=8, b=1$
- (d) $\theta=5, b=0.4$



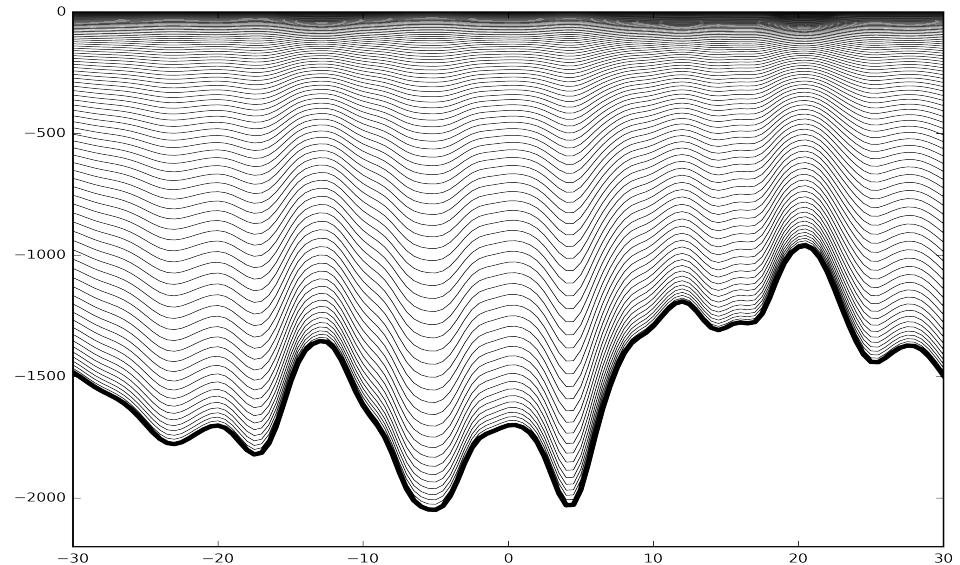
•!! Take care of the Pressure Gradient Error on steep slopes !!

Vertical grid : σ generalized coordinate

- 50 vertical levels
 $\theta=7$, $b=2$, $h_c=300$ m

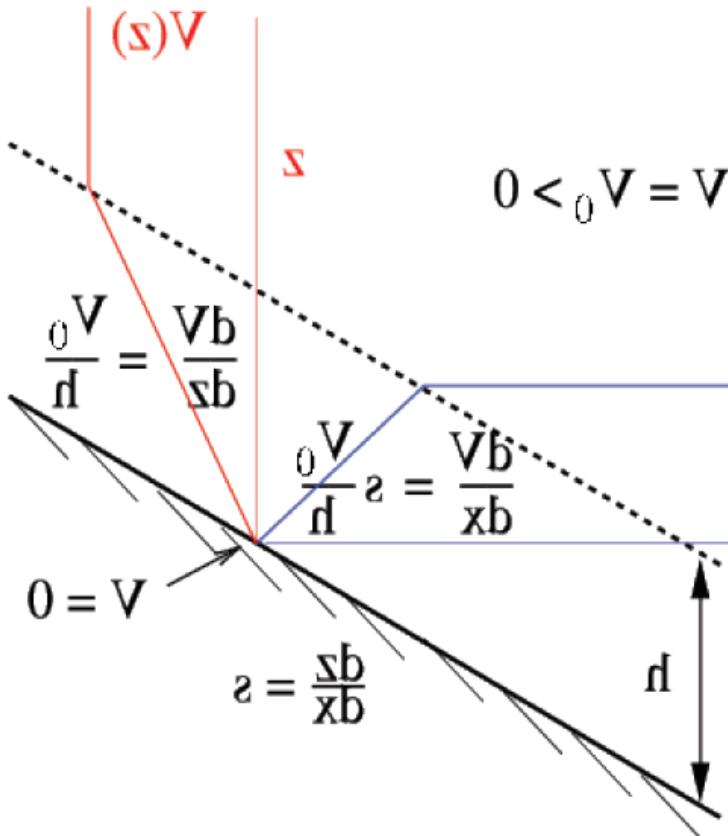


- 80 vertical levels
 $\theta=6$, $b=4$, $h_c=300$ m

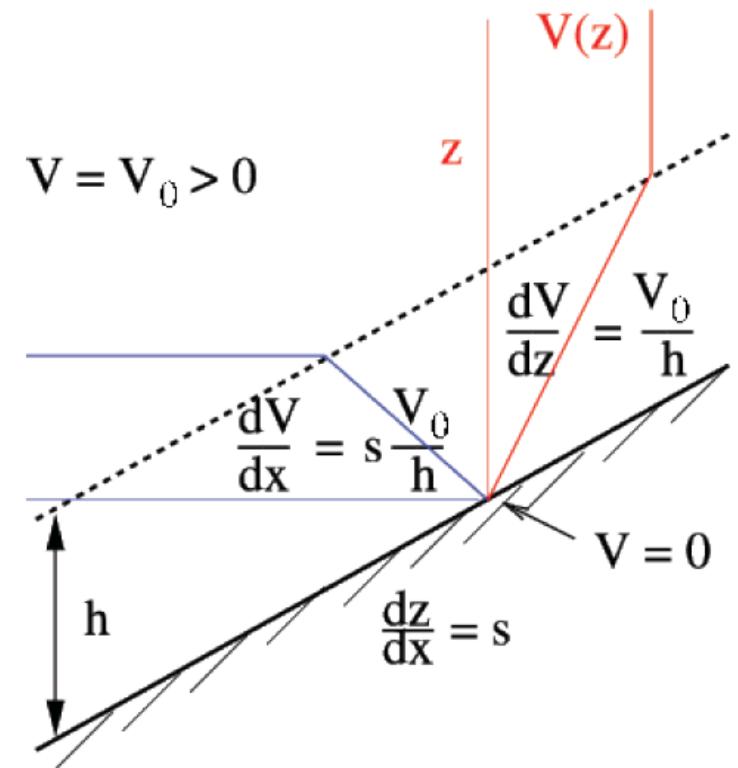


Ex: Topographic vorticity generation

Generation of vertical vorticity within the bottom boundary layer:



Current flowing with the coast on its left in the Northern hemisphere
(opposite to Kelvin wave propagation)
= **Positive vorticity generation**



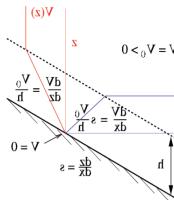
Current flowing with the coast on its right in the Northern hemisphere
(same than Kelvin wave propagation)
= **Negative vorticity generation**

Ex: Topographic vorticity generation

Positive vorticity generation :

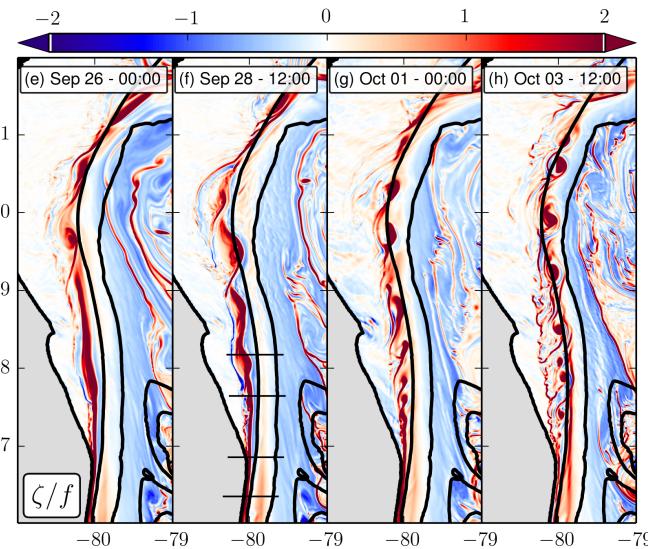
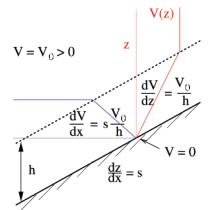
- Horizontal shear instability
- Formation of submesoscale cyclones

e.g.: *Gulf Stream along the slope*

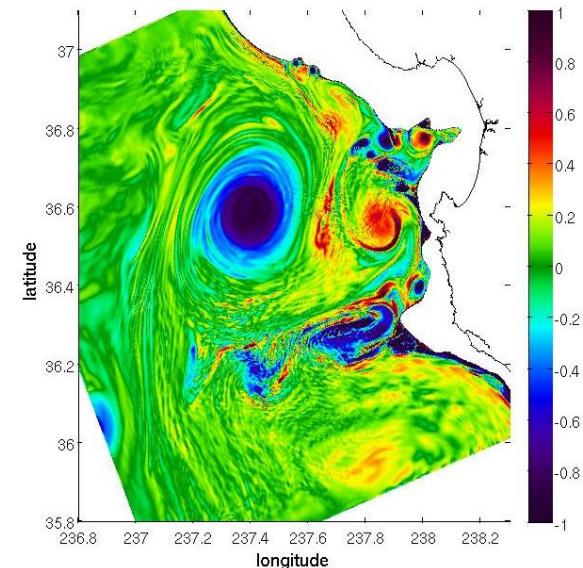


Negative vorticity generation:

- Centrifugal instability
- Small-scale turbulence, Mixing and dissipation
- Formation of submesoscale anticyclones

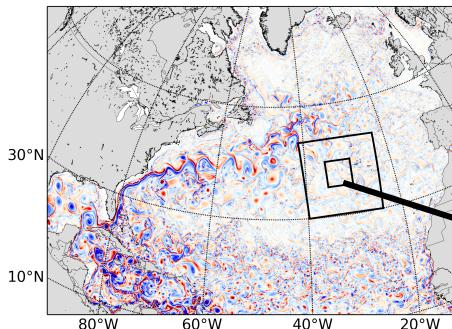


[Gula et al., GRL, 2015]

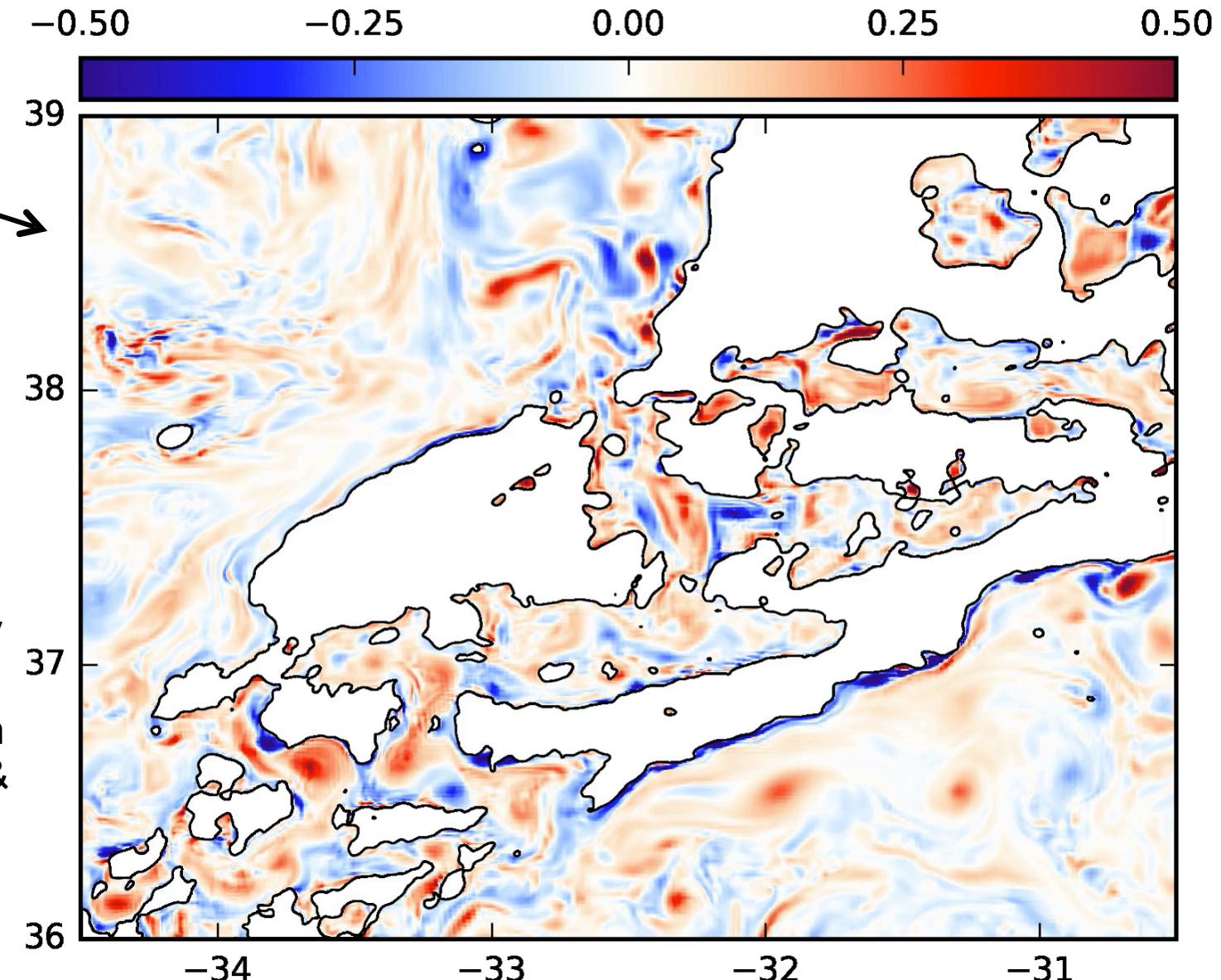


[Molemaker et al., JPO, 2015]

Ex: Topographic vorticity generation



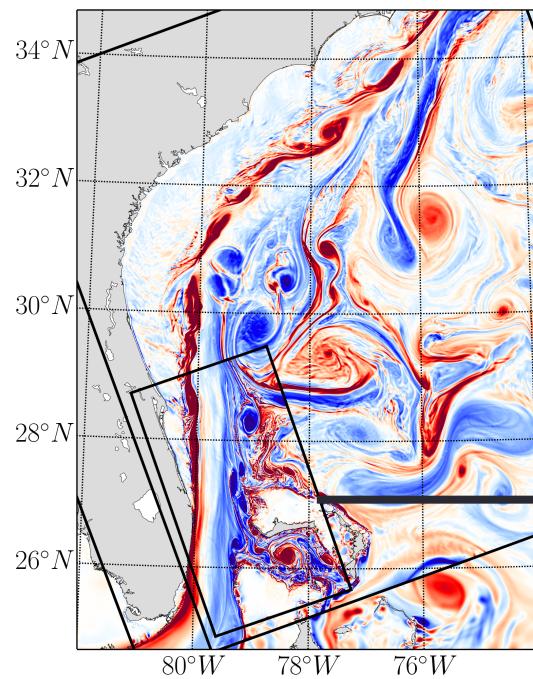
$\Delta x = 500 \text{ m}$
relative vorticity at -1800 m



- Dispersion of hydrothermal effluents and larvae from the Lucky Strike vent field
- High levels of dissipation and mixing [St Laurent & Thunherr, Nat., 2006]

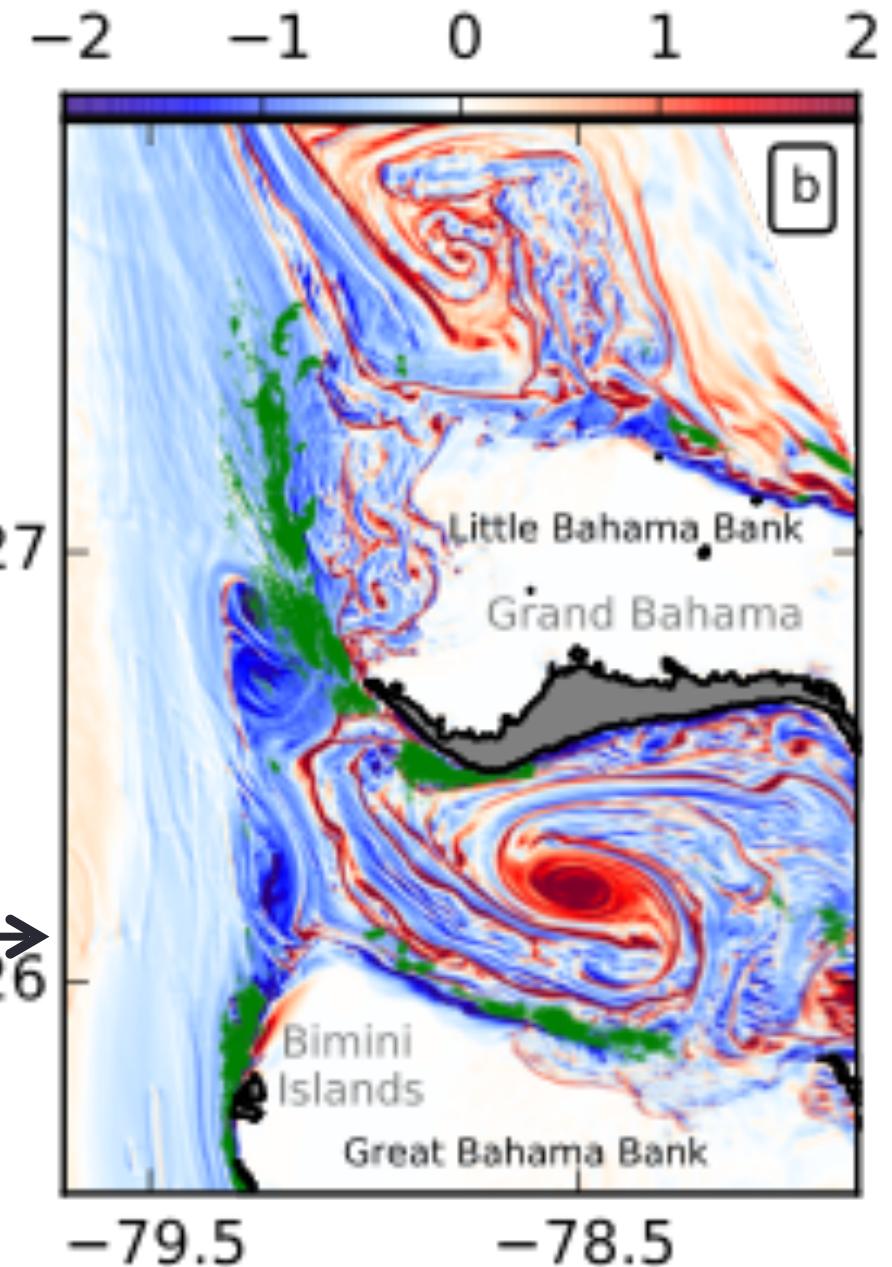
Ex: Topographic vorticity generation

Anticyclonic vorticity generation by bottom drag on the slope on Bahamas slope



$$\Delta x = 200 \text{ m}$$

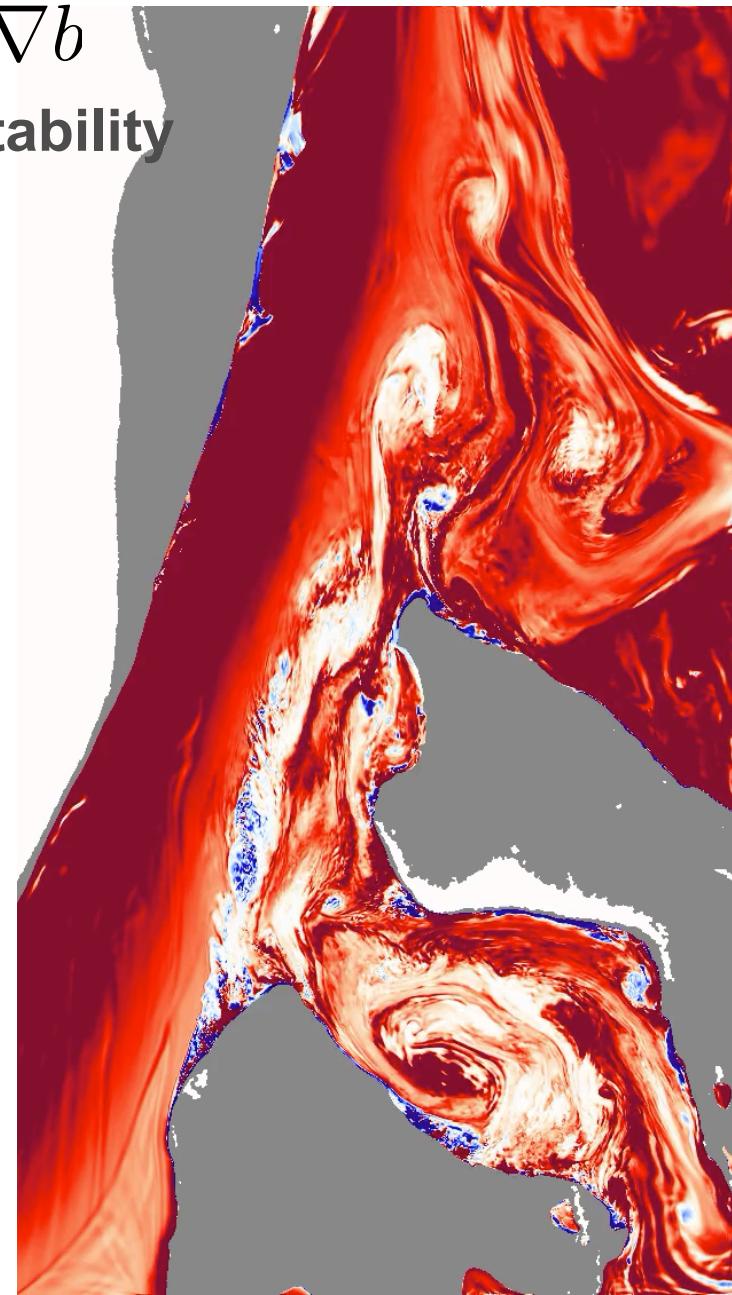
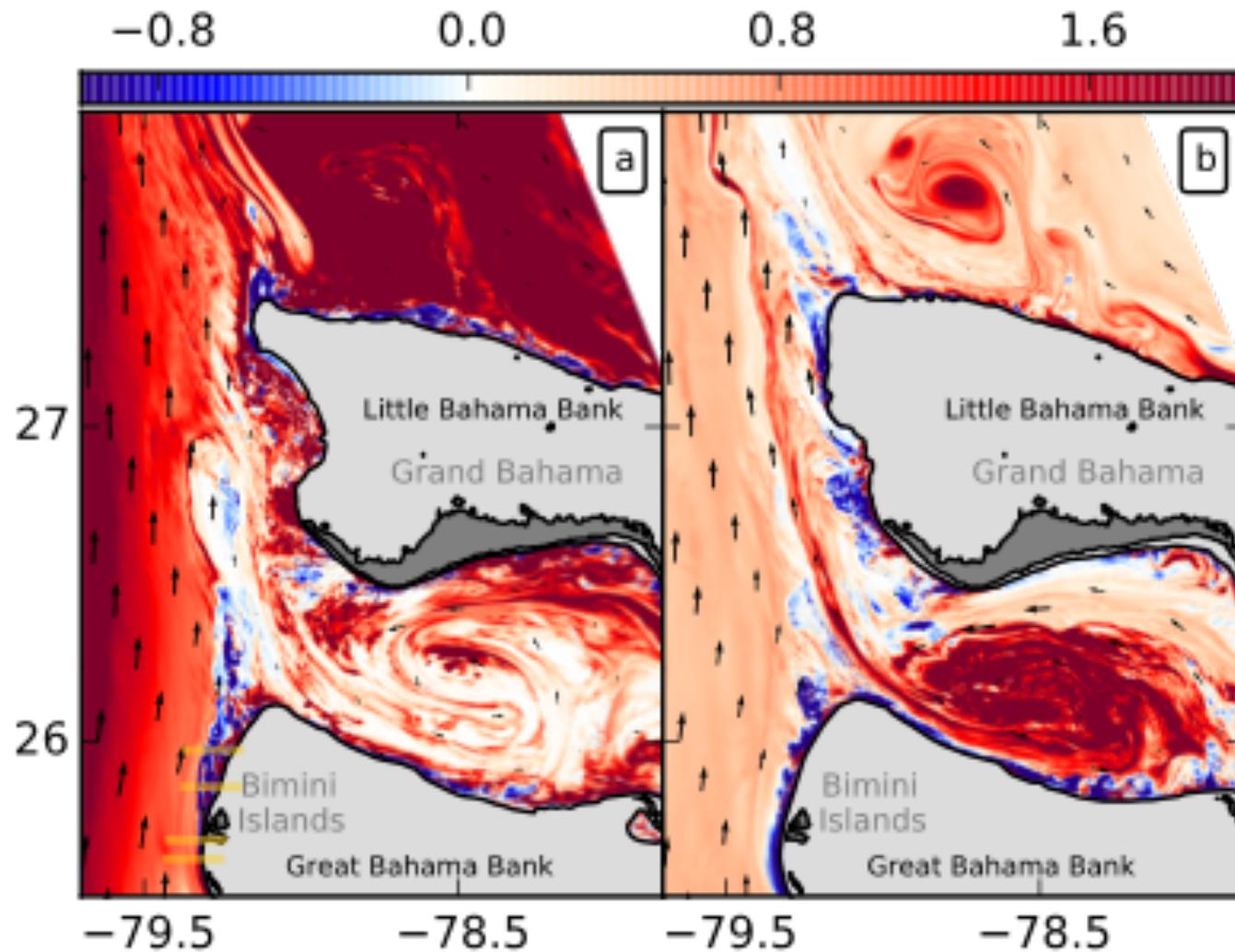
Topographic generation of submesoscale centrifugal instability and energy dissipation



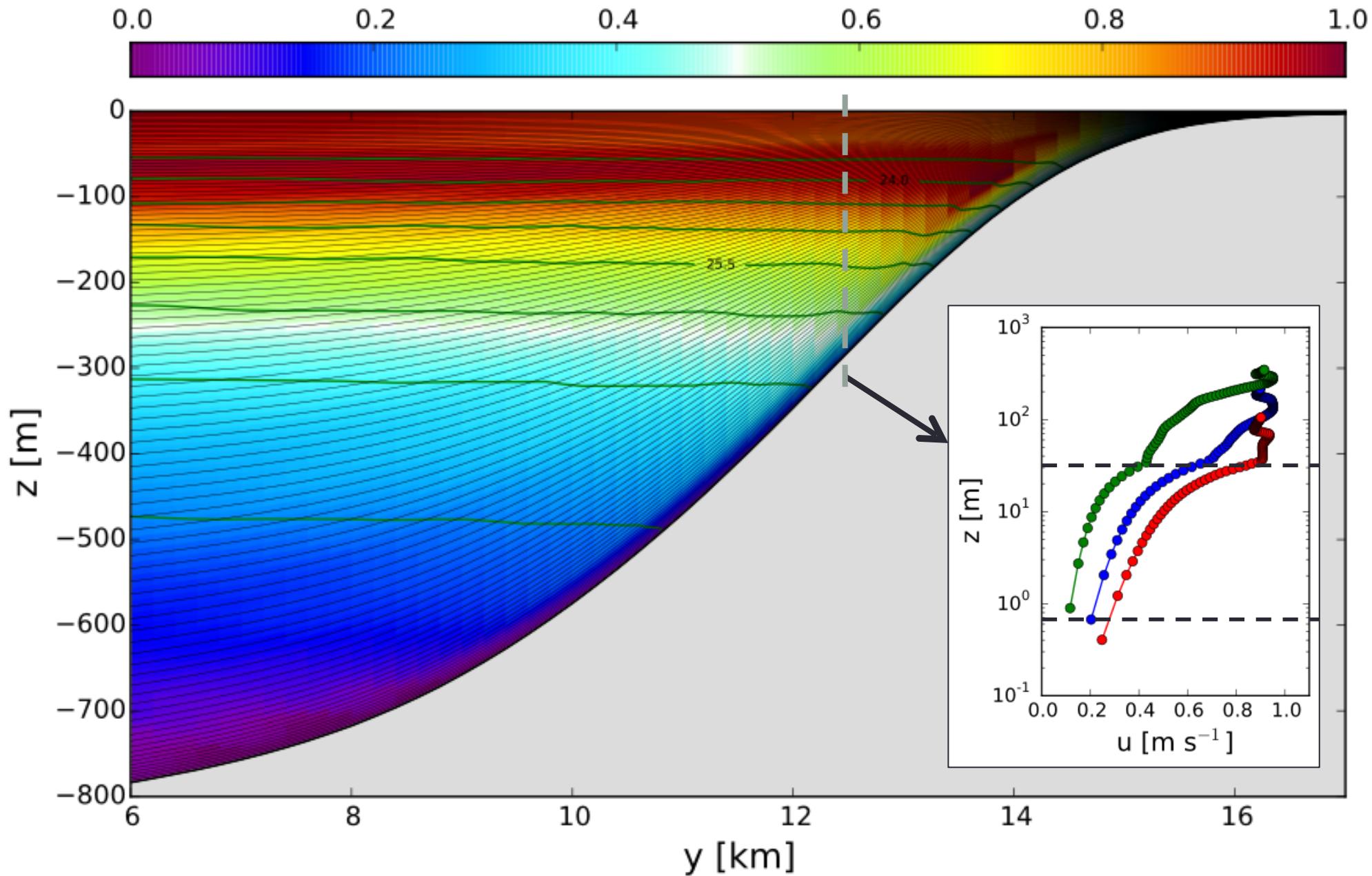
Ex: Topographic vorticity generation

$$q = (f \vec{z} + \vec{\nabla} \times \vec{u}) \cdot \vec{\nabla} b$$

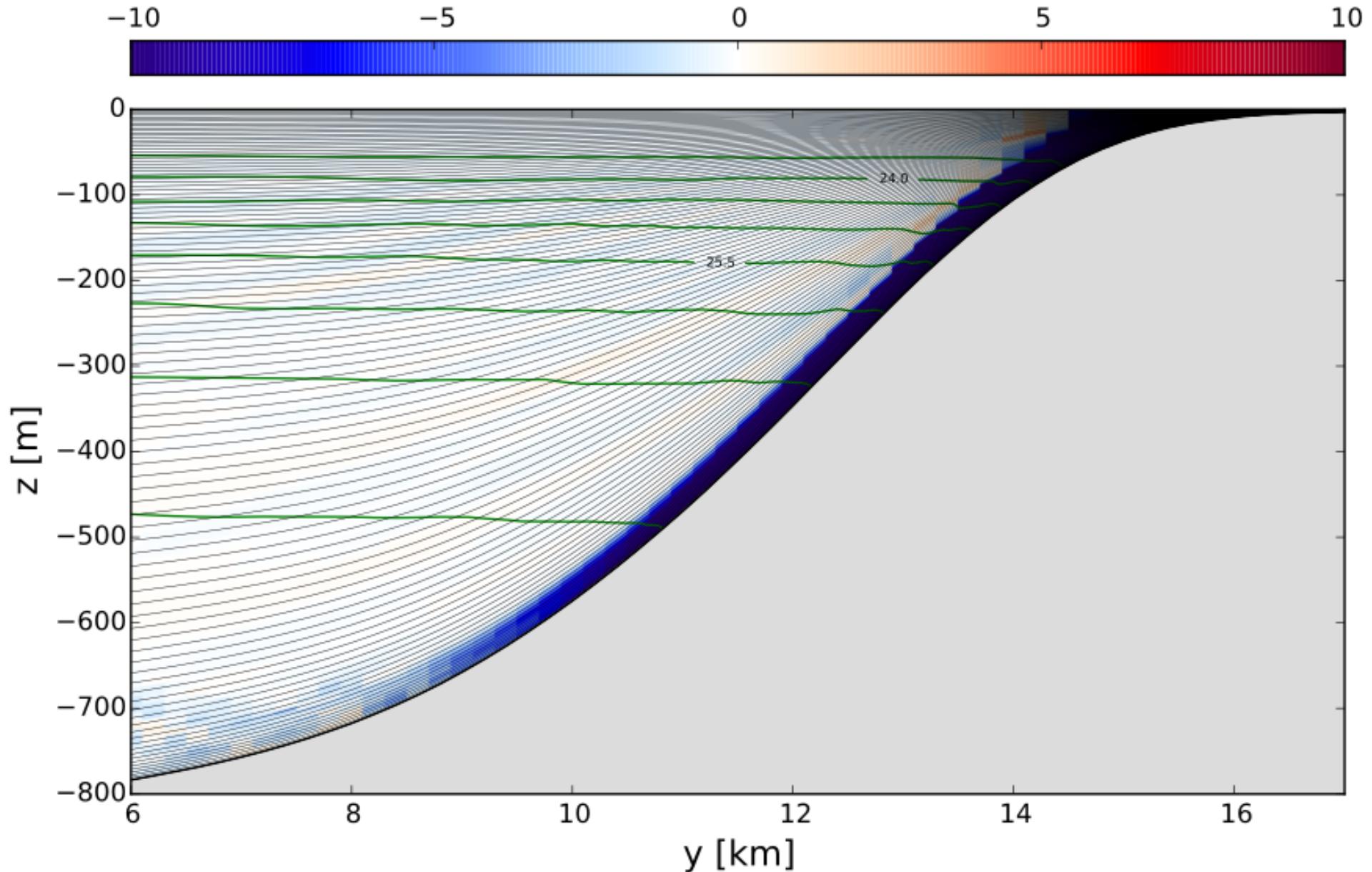
Negative PV ($f q < 0$) = centrifugal (inertial) instability



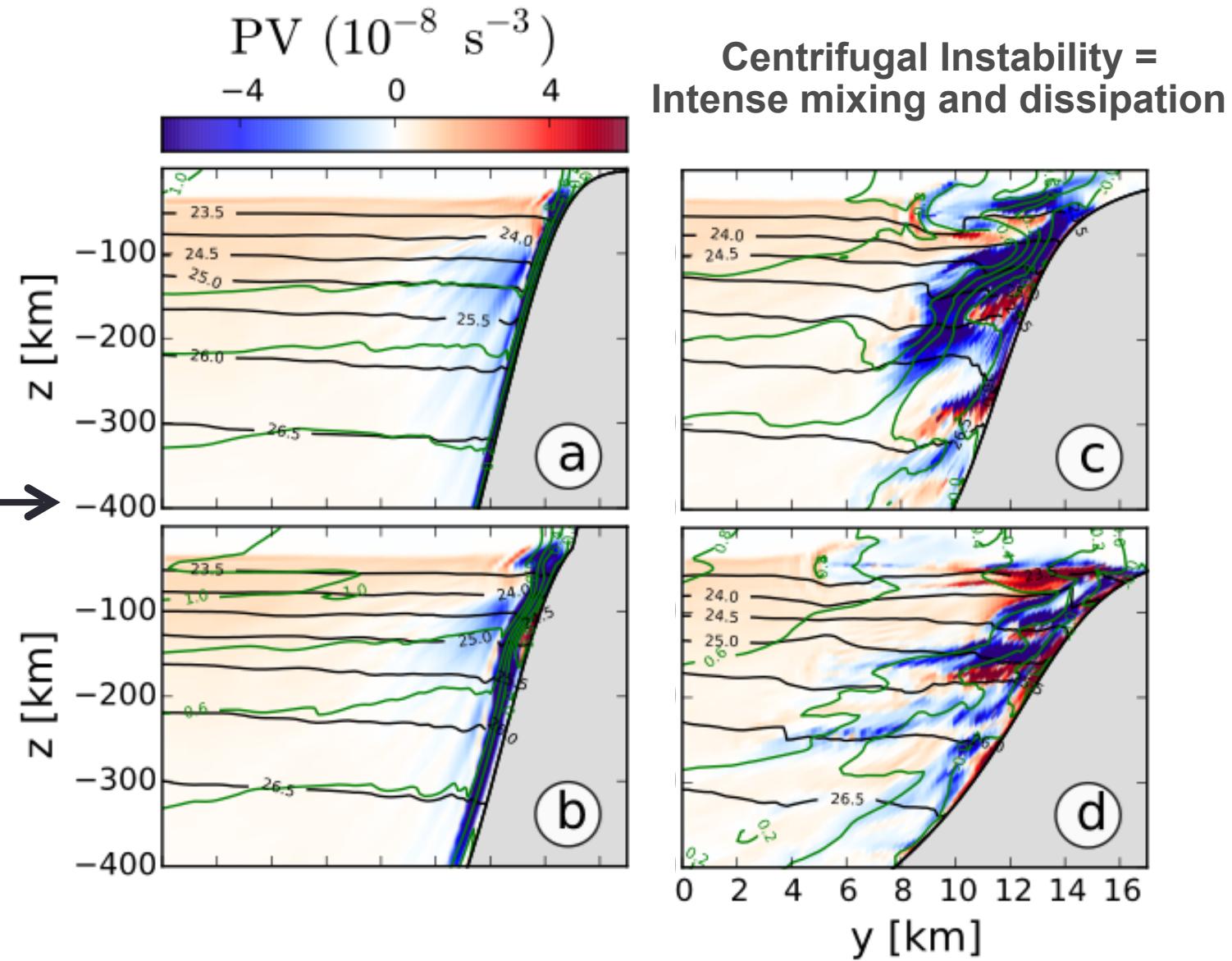
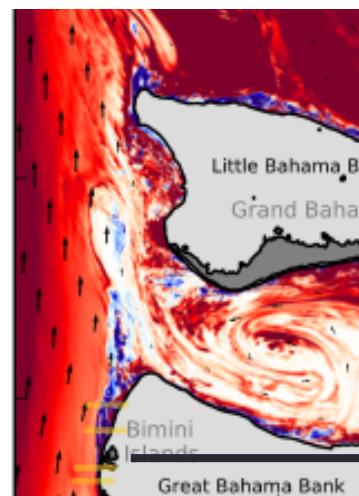
Ex: Topographic vorticity generation



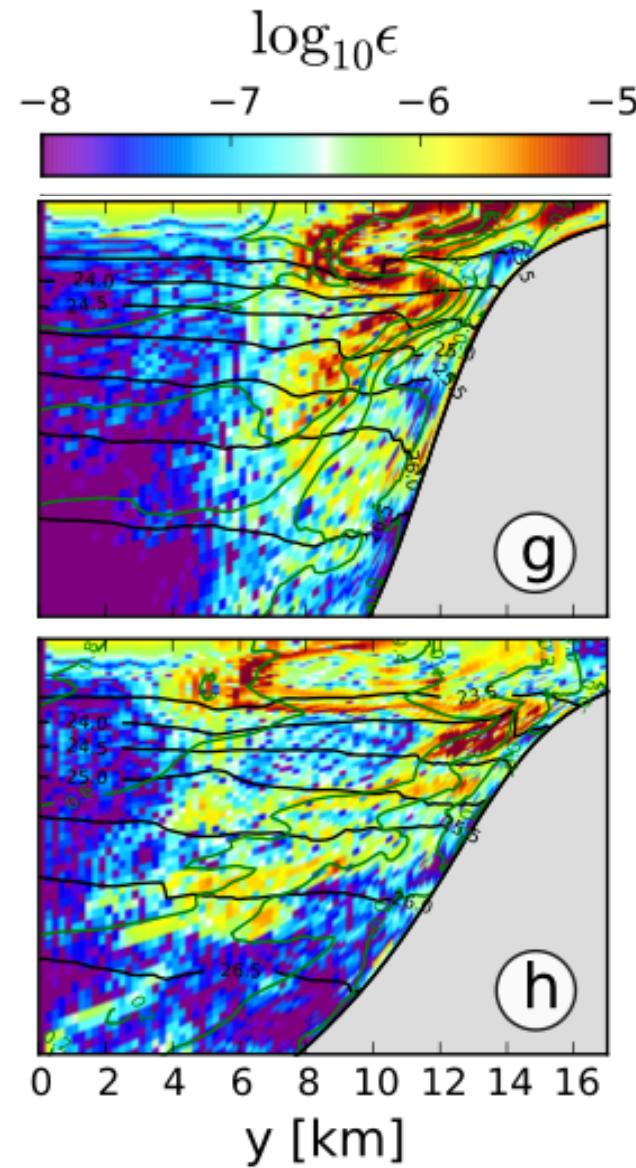
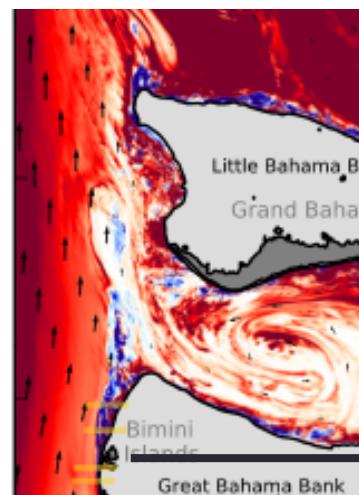
Ex: Topographic vorticity generation



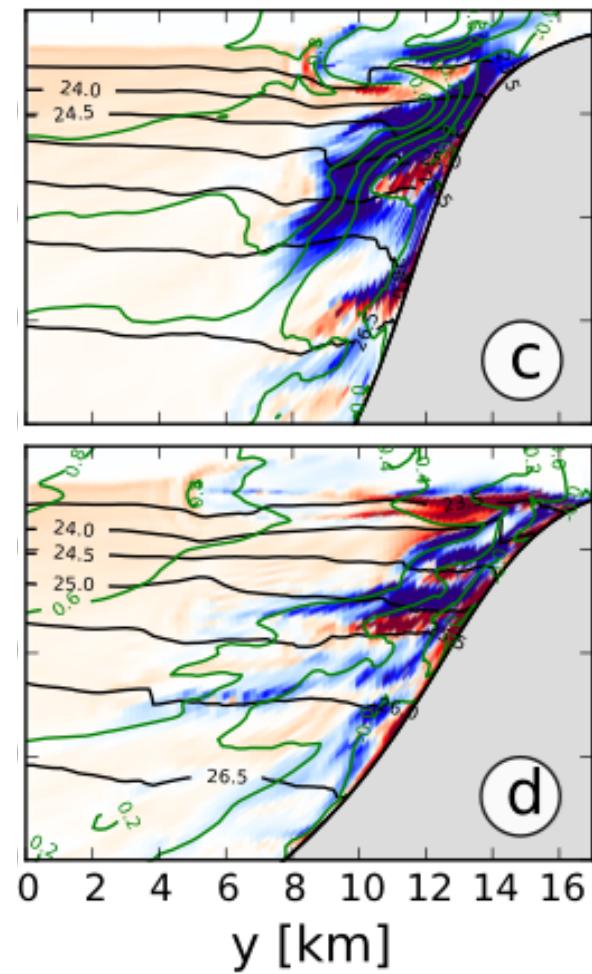
Ex: Topographic vorticity generation



Ex: Topographic vorticity generation

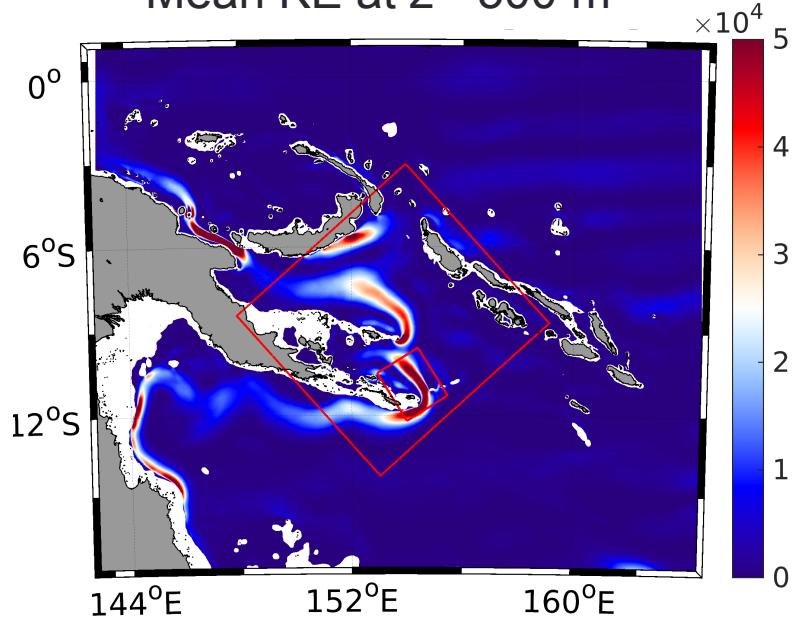


Centrifugal Instability =
Intense mixing and dissipation



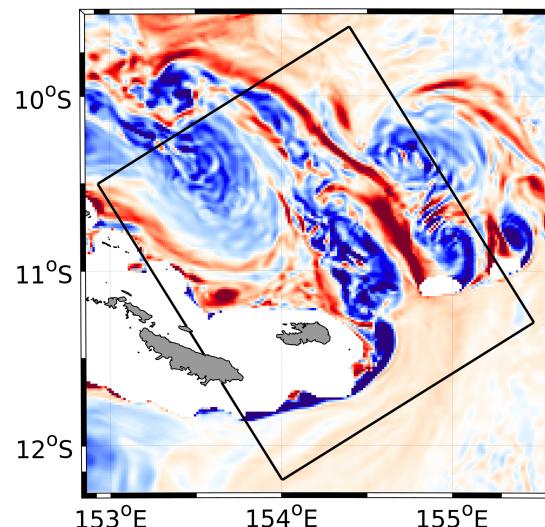
Ex: Topographic vorticity generation

Mean KE at $z=-500$ m

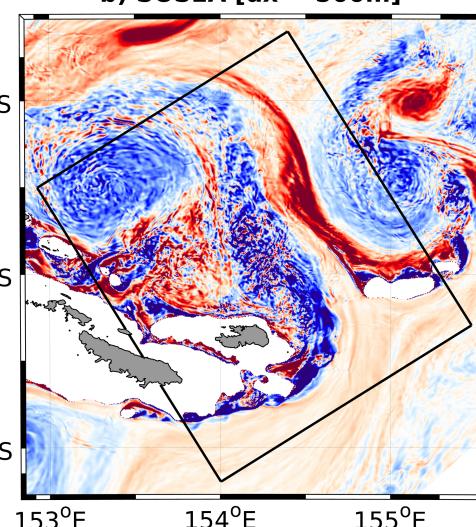


Gulf of Papua Undercurrent

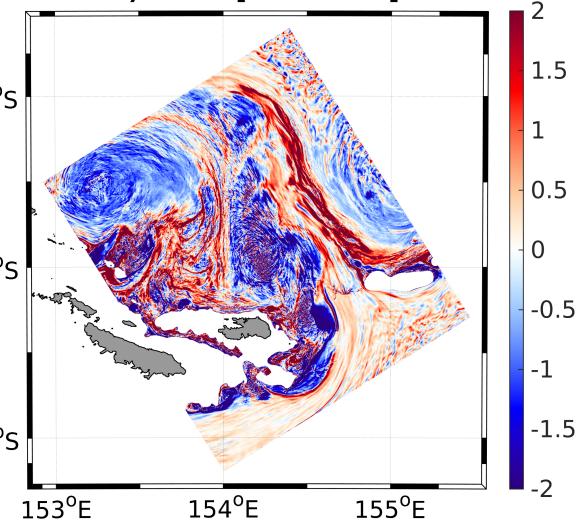
a) PAPUA [$dx = 1500\text{m}$]



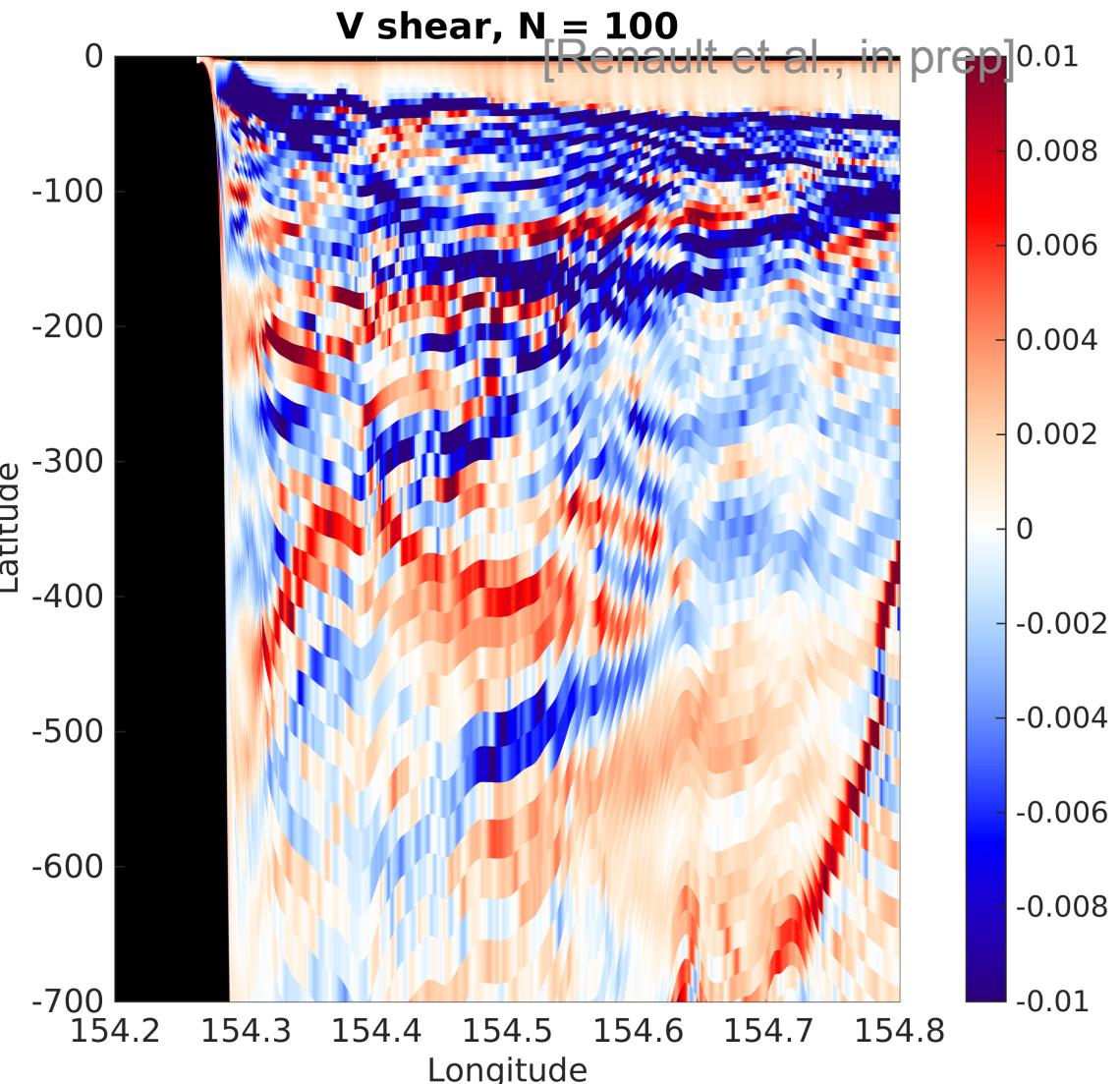
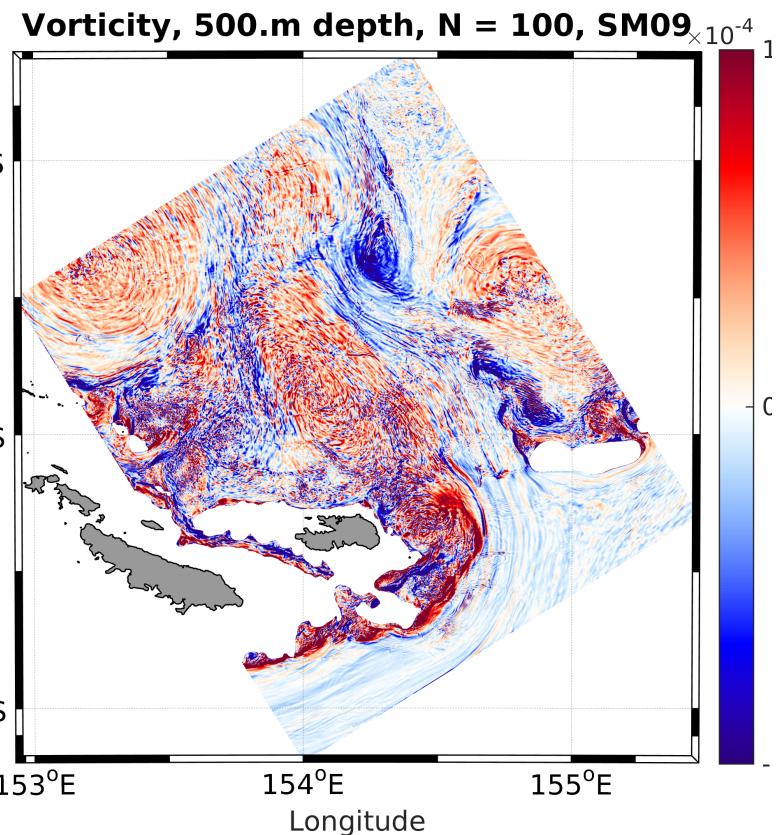
b) SOSEA [$dx = 500\text{m}$]



c) LOUIS [$dx = 150\text{m}$]

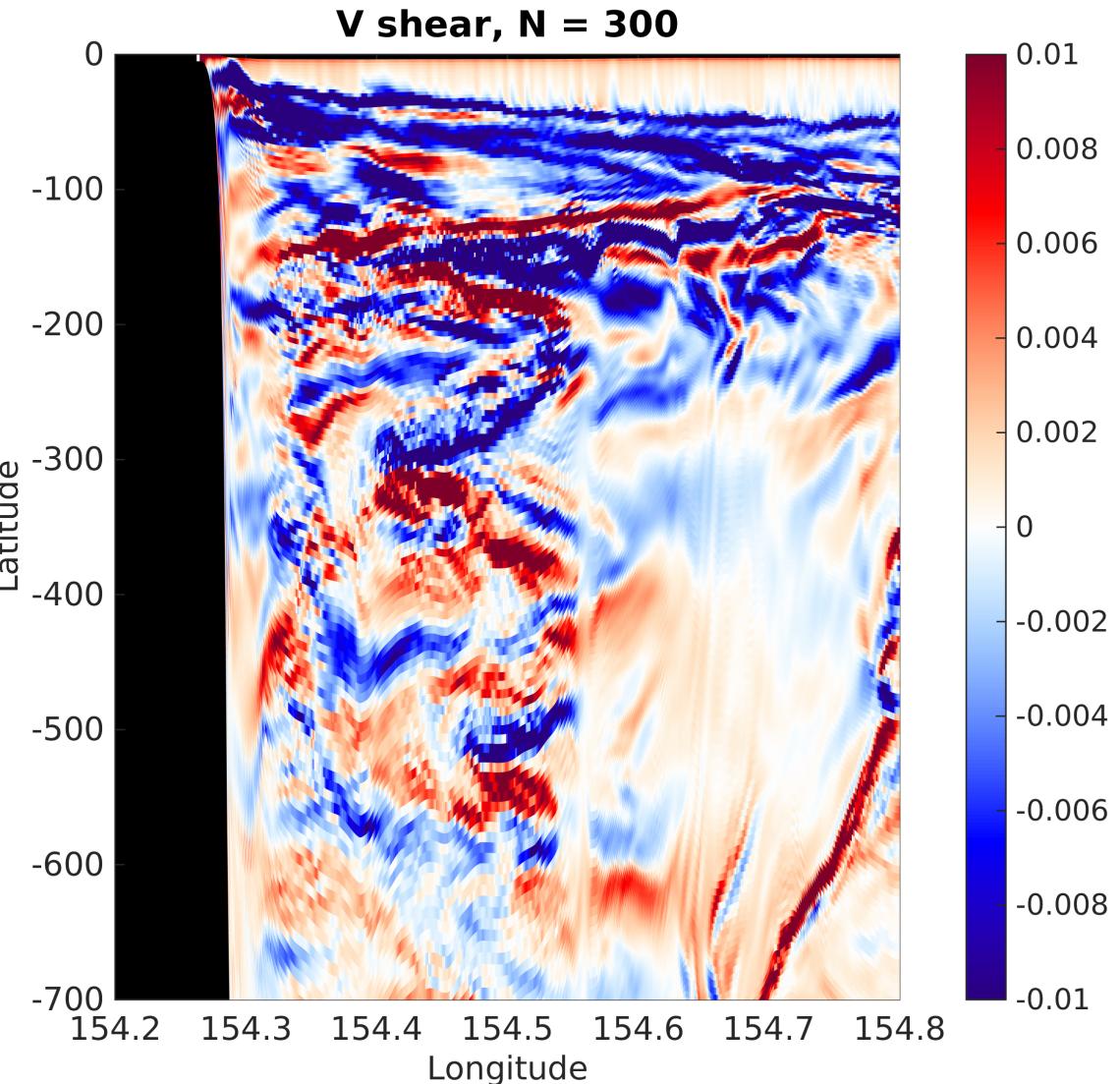
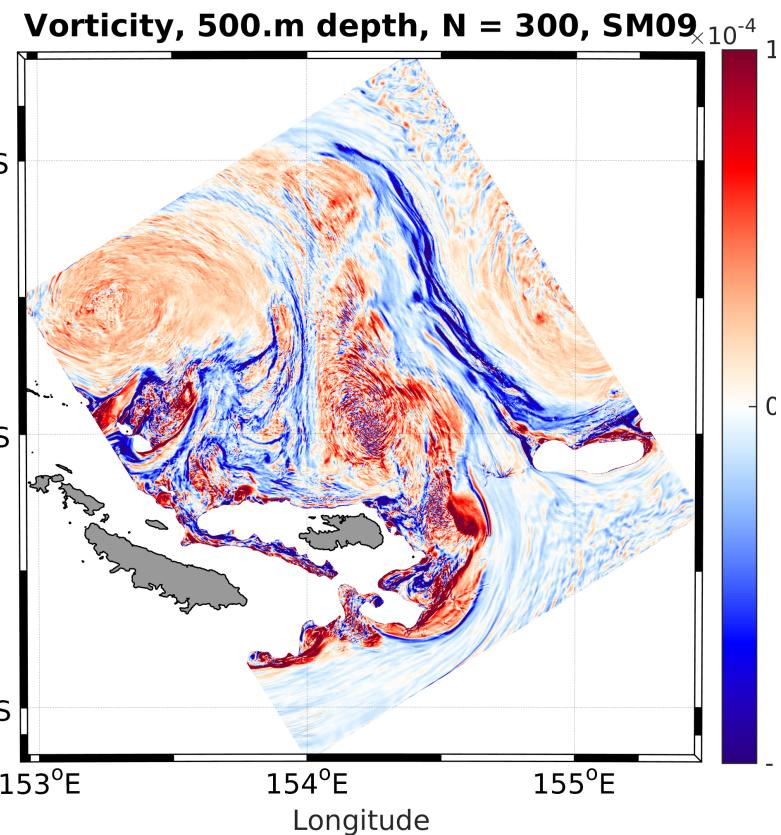


Ex: Need in Vertical resolution?



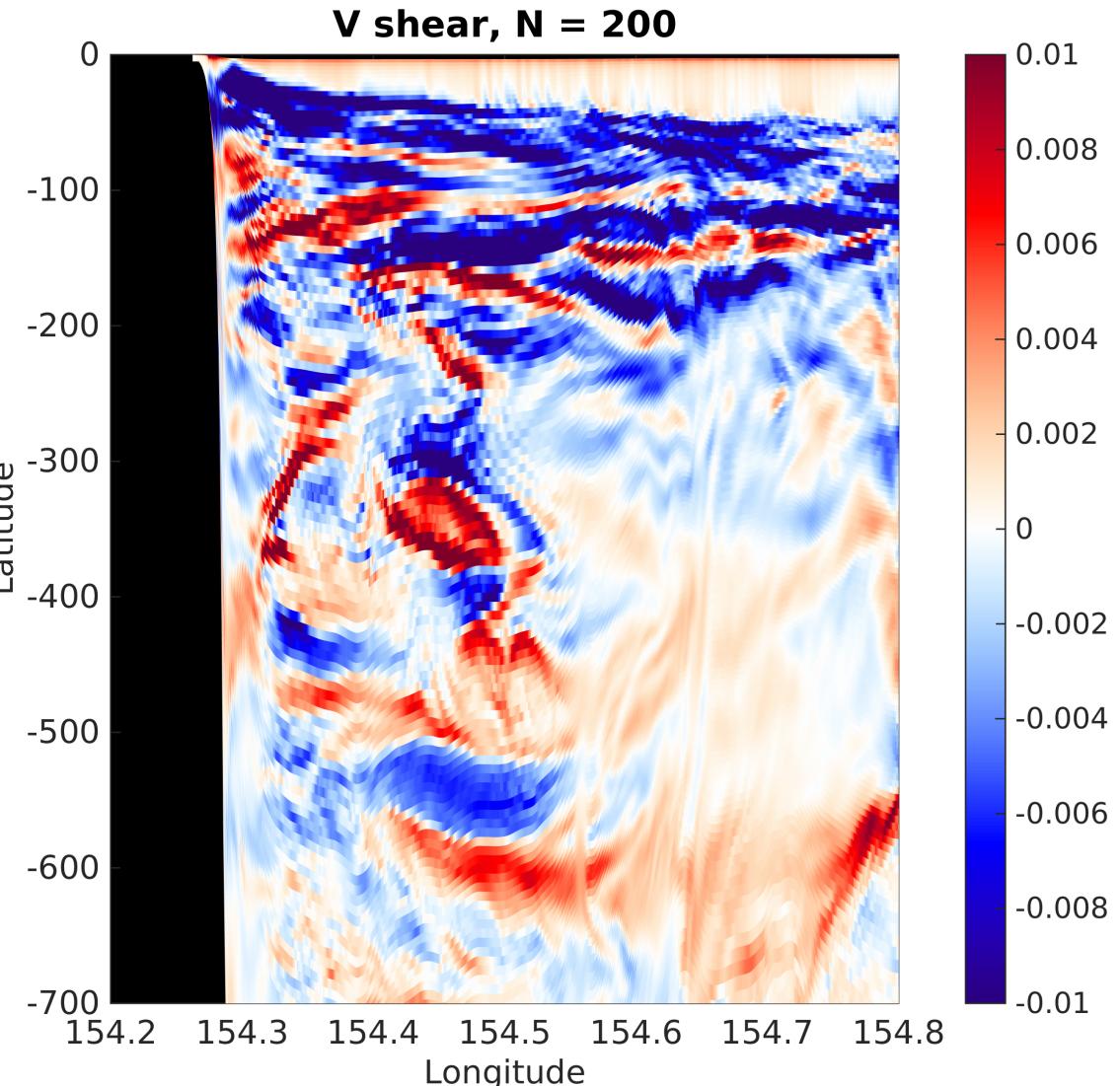
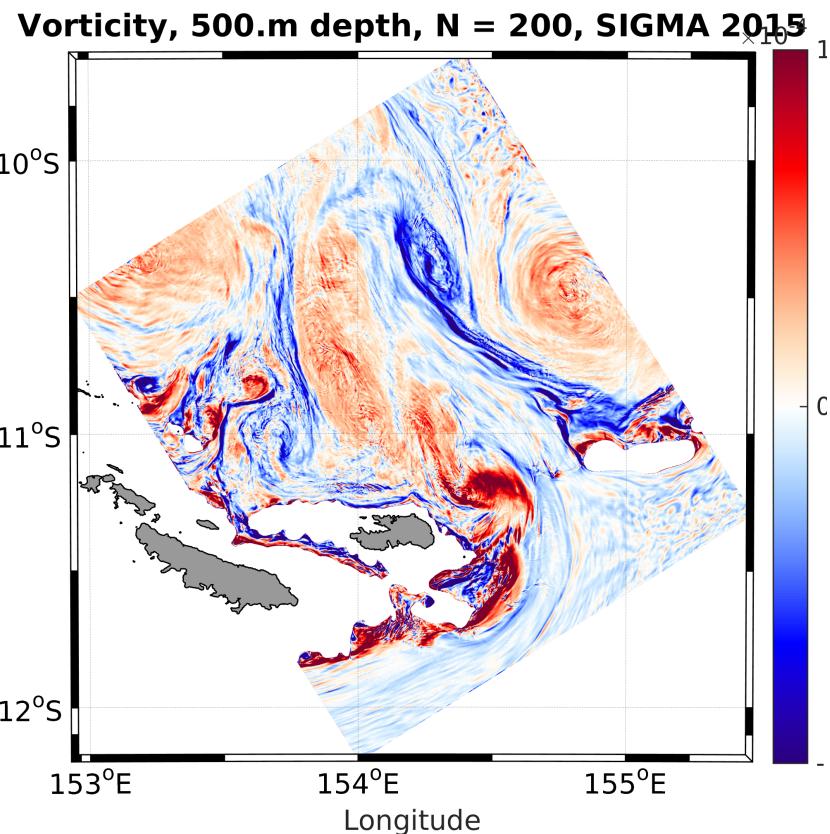
Ex: What happens when underresolving vertical scales with HR simulations (Solomon Sea)

Ex: Need in Vertical resolution?



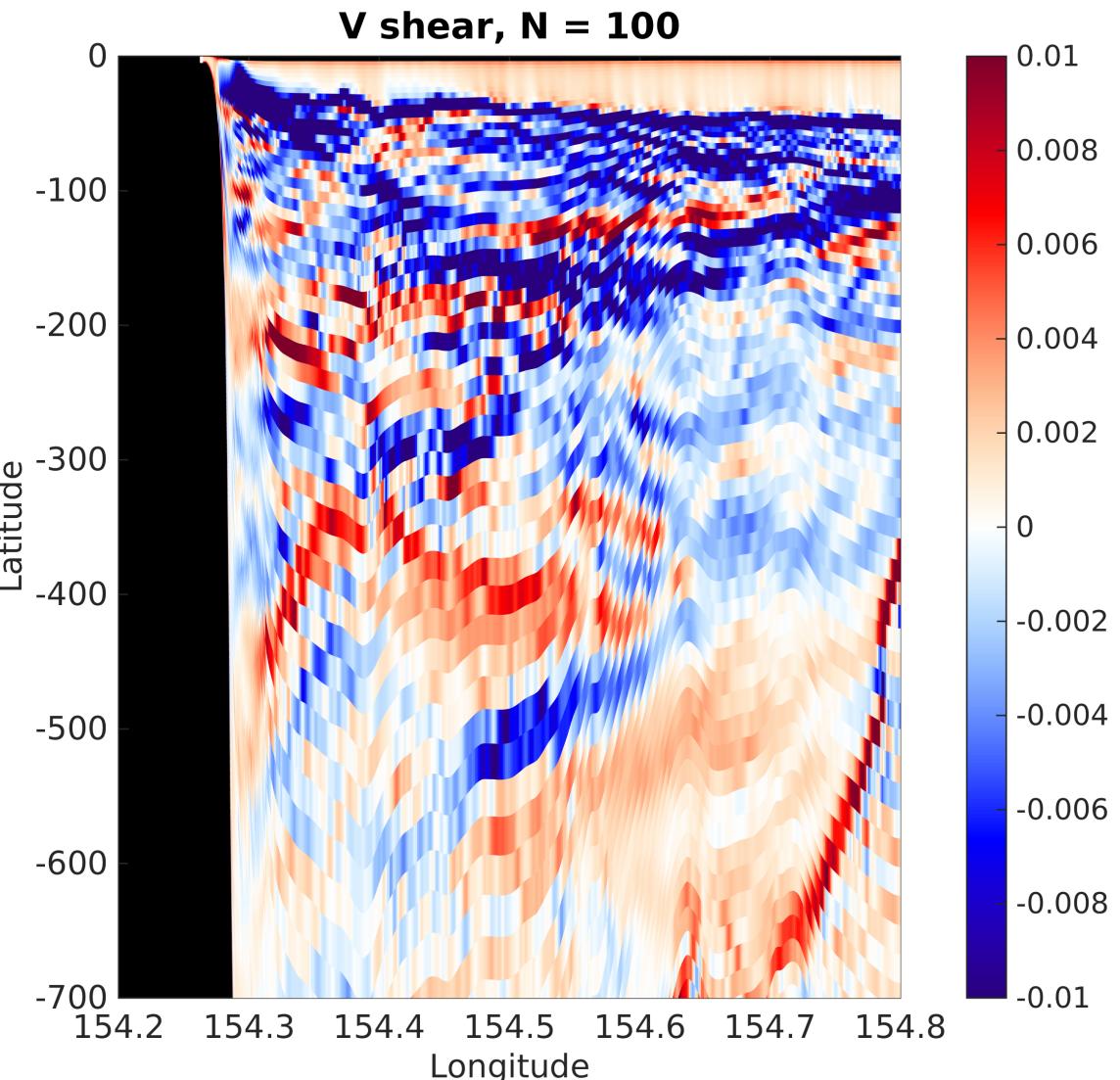
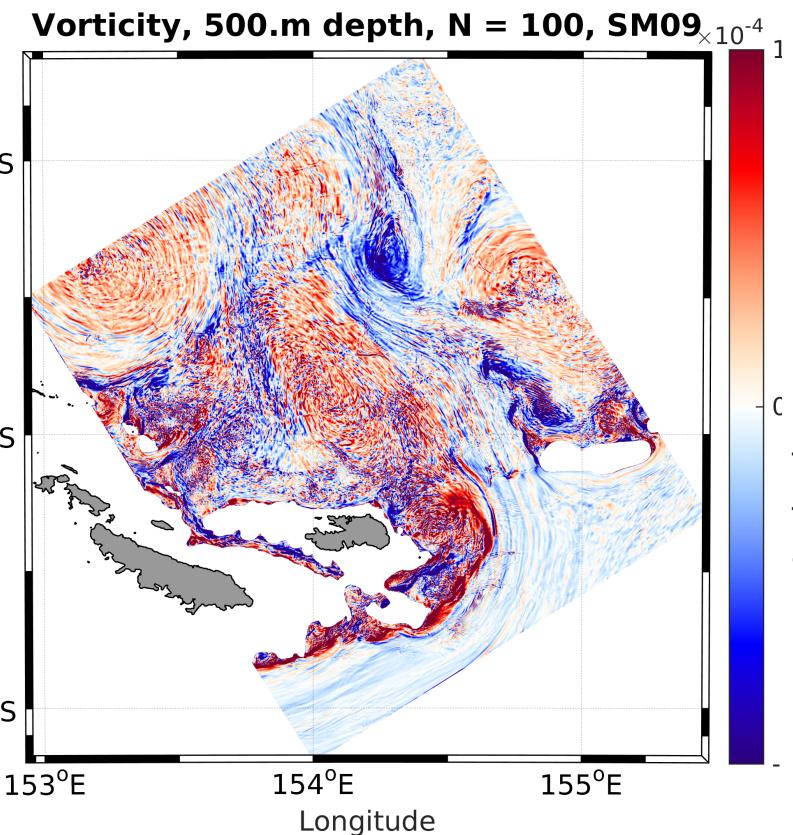
From N=100 levels to N=300 levels...

Ex: Need in Vertical resolution?



N=200 levels with a more balanced distribution

Ex: Need in Vertical resolution?



Ex: What happens when underresolving vertical scales with HR simulations (Solomon Sea)

Vertical grid : σ generalized coordinate

- Equations become:

$$\frac{\partial u}{\partial t} - fv + \vec{v} \cdot \nabla u = -\frac{\partial \phi}{\partial x} - \left(\frac{g\rho}{\rho_o} \right) \frac{\partial z}{\partial x} - g \frac{\partial \zeta}{\partial x} + \frac{1}{H_z} \frac{\partial}{\partial \sigma} \left[\frac{K_m}{H_z} \frac{\partial u}{\partial \sigma} \right] + \mathcal{F}_u + \mathcal{D}_u$$

$$\frac{\partial v}{\partial t} + fu + \vec{v} \cdot \nabla v = -\frac{\partial \phi}{\partial y} - \left(\frac{g\rho}{\rho_o} \right) \frac{\partial z}{\partial y} - g \frac{\partial \zeta}{\partial y} + \frac{1}{H_z} \frac{\partial}{\partial \sigma} \left[\frac{K_m}{H_z} \frac{\partial v}{\partial \sigma} \right] + \mathcal{F}_v + \mathcal{D}_v$$

$$\frac{\partial C}{\partial t} + \vec{v} \cdot \nabla C = \frac{1}{H_z} \frac{\partial}{\partial \sigma} \left[\frac{K_C}{H_z} \frac{\partial C}{\partial \sigma} \right] + \mathcal{F}_T + \mathcal{D}_T$$

$$\rho = \rho(T, S, P)$$

$$\frac{\partial \phi}{\partial \sigma} = \left(\frac{-gH_z\rho}{\rho_o} \right)$$

$$\frac{\partial H_z}{\partial t} + \frac{\partial(H_z u)}{\partial x} + \frac{\partial(H_z v)}{\partial y} + \frac{\partial(H_z \Omega)}{\partial \sigma} = 0$$

Vertical grid + curvilinear grid

$$\begin{aligned}
& \frac{\partial}{\partial t} \left(\frac{H_z u}{mn} \right) + \frac{\partial}{\partial \xi} \left(\frac{H_z u^2}{n} \right) + \frac{\partial}{\partial \eta} \left(\frac{H_z u v}{m} \right) + \frac{\partial}{\partial \sigma} \left(\frac{H_z u \Omega}{mn} \right) \\
& \quad - \left\{ \left(\frac{f}{mn} \right) + v \frac{\partial}{\partial \xi} \left(\frac{1}{n} \right) - u \frac{\partial}{\partial \eta} \left(\frac{1}{m} \right) \right\} H_z v = \\
& \quad - \left(\frac{H_z}{n} \right) \left(\frac{\partial \phi}{\partial \xi} + \frac{g \rho}{\rho_o} \frac{\partial z}{\partial \xi} + g \frac{\partial \zeta}{\partial \xi} \right) + \frac{1}{mn} \frac{\partial}{\partial \sigma} \left[\frac{K_m}{H_z} \frac{\partial u}{\partial \sigma} \right] + \frac{H_z}{mn} (\mathcal{F}_u + \mathcal{D}_u)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial t} \left(\frac{H_z v}{mn} \right) + \frac{\partial}{\partial \xi} \left(\frac{H_z u v}{n} \right) + \frac{\partial}{\partial \eta} \left(\frac{H_z v^2}{m} \right) + \frac{\partial}{\partial \sigma} \left(\frac{H_z v \Omega}{mn} \right) \\
& \quad + \left\{ \left(\frac{f}{mn} \right) + v \frac{\partial}{\partial \xi} \left(\frac{1}{n} \right) - u \frac{\partial}{\partial \eta} \left(\frac{1}{m} \right) \right\} H_z u = \\
& \quad - \left(\frac{H_z}{m} \right) \left(\frac{\partial \phi}{\partial \eta} + \frac{g \rho}{\rho_o} \frac{\partial z}{\partial \eta} + g \frac{\partial \zeta}{\partial \eta} \right) + \frac{1}{mn} \frac{\partial}{\partial \sigma} \left[\frac{K_m}{H_z} \partial v \partial \sigma \right] + \frac{H_z}{mn} (\mathcal{F}_v + \mathcal{D}_v)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial t} \left(\frac{H_z C}{mn} \right) + \frac{\partial}{\partial \xi} \left(\frac{H_z u C}{n} \right) + \frac{\partial}{\partial \eta} \left(\frac{H_z v C}{m} \right) + \frac{\partial}{\partial \sigma} \left(\frac{H_z \Omega C}{mn} \right) = \\
& \quad \frac{1}{mn} \frac{\partial}{\partial s} \left[\frac{K_C}{H_z} \frac{\partial C}{\partial \sigma} \right] + \frac{H_z}{mn} (\mathcal{F}_C + \mathcal{D}_C)
\end{aligned}$$

$$\rho = \rho(T, S, P)$$

$$\frac{\partial \phi}{\partial \sigma} = - \left(\frac{g H_z \rho}{\rho_o} \right)$$

$$\frac{\partial}{\partial t} \left(\frac{H_z}{mn} \right) + \frac{\partial}{\partial \xi} \left(\frac{H_z u}{n} \right) + \frac{\partial}{\partial \eta} \left(\frac{H_z v}{m} \right) + \frac{\partial}{\partial \sigma} \left(\frac{H_z \Omega}{mn} \right) = 0.$$

Vertical grid : σ generalized coordinate

- The vertical velocity in sigma-coordinates is:

$$\Omega(x, y, \sigma, t) = \frac{1}{H_z} \left[w - \left(\frac{z + h}{\zeta + h} \right) \frac{\partial \zeta}{\partial t} - u \frac{\partial z}{\partial x} - v \frac{\partial z}{\partial y} \right]$$

- And the “true” vertical velocity:

$$w = \frac{\partial z}{\partial t} + u \frac{\partial z}{\partial x} + v \frac{\partial z}{\partial y} + \Omega H_z.$$

Vertical grid : σ generalized coordinate

- Boundary conditions become:

top ($\sigma = 0$)

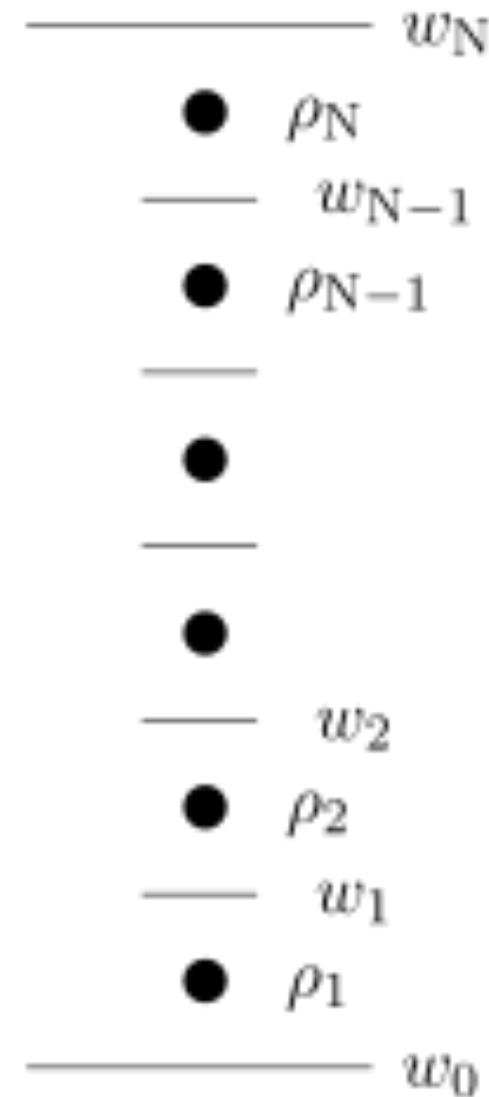
$$\begin{aligned} \left(\frac{K_m}{H_z} \right) \frac{\partial u}{\partial \sigma} &= \tau_s^x(x, y, t) \\ \left(\frac{K_m}{H_z} \right) \frac{\partial v}{\partial \sigma} &= \tau_s^y(x, y, t) \\ \left(\frac{K_C}{H_z} \right) \frac{\partial C}{\partial \sigma} &= \frac{Q_C}{\rho_o c_P} \\ \Omega &= 0 \end{aligned}$$

and bottom ($\sigma = -1$)

$$\begin{aligned} \left(\frac{K_m}{H_z} \right) \frac{\partial u}{\partial \sigma} &= \tau_b^x(x, y, t) \\ \left(\frac{K_m}{H_z} \right) \frac{\partial v}{\partial \sigma} &= \tau_b^y(x, y, t) \\ \left(\frac{K_C}{H_z} \right) \frac{\partial C}{\partial s} &= 0 \\ \Omega &= 0. \end{aligned}$$

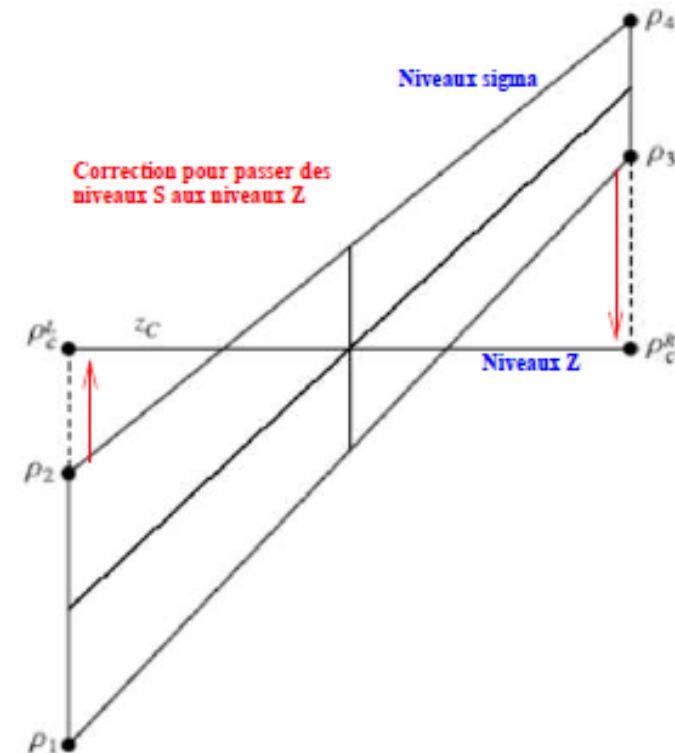
Vertical discretization

ROMS: Staggered vertical grid



Pressure Gradient force

- Truncation errors are made from calculating the baroclinic pressure gradients across sharp topographic changes such as the continental slope
- Difference between 2 large terms
- Errors can appear in the unforced flat stratification experiment



$$-\frac{1}{\rho_0} \left. \frac{\partial P}{\partial x} \right|_z = -\left. \frac{1}{\rho_0} \frac{\partial P}{\partial x} \right|_s + \frac{1}{\rho_0} \cdot \left. \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial x} \right|_s,$$

$$\epsilon \equiv \frac{\left| \left. \frac{\partial P}{\partial x} \right|_s - \left. \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial x} \right|_s \right|}{\left| \left. \frac{\partial P}{\partial x} \right|_s \right| + \left| \left. \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial x} \right|_s \right|} \ll 1,$$

Reducing PGF Truncation Errors

- Smoothing the topography using a nonlinear filter and a criterium:

$$r = \Delta h / h < 0.2$$

- Using a "density formulation"

- Using high order schemes to reduce the truncation error (4th order, McCalpin, 1994)

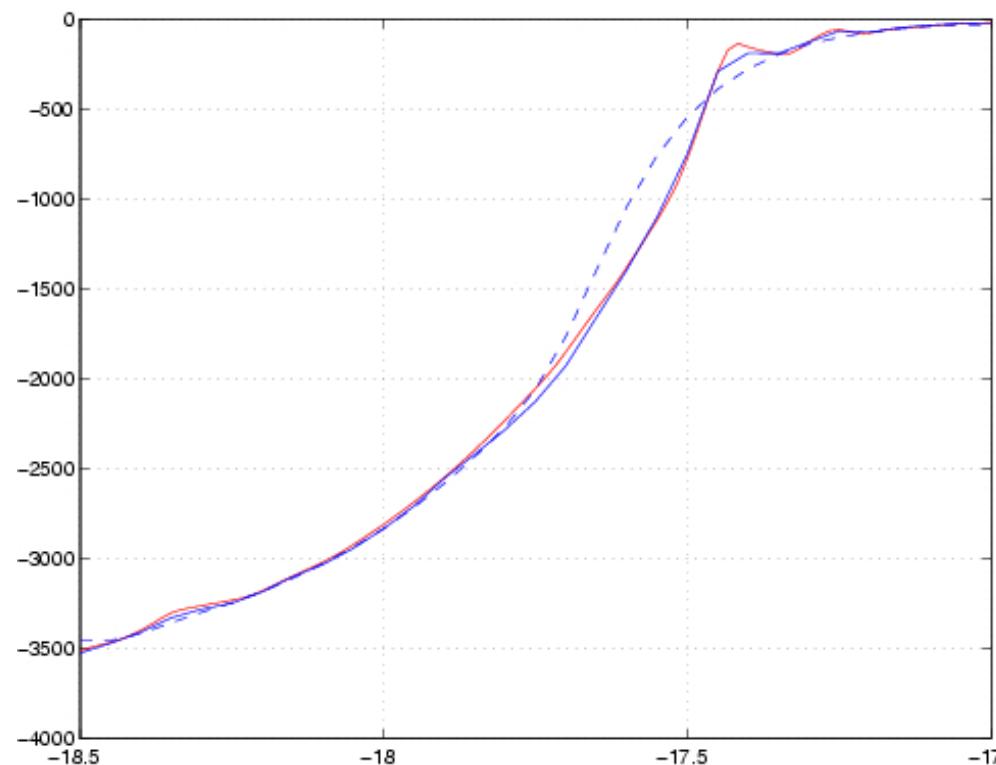
$$\begin{aligned} -\frac{1}{\rho_0} \frac{\partial P}{\partial x} \Big|_z &= -\frac{1}{\rho_0} \frac{\partial P}{\partial x} \Big|_{z=\zeta} - \frac{g}{\rho_0} \int_z^{\zeta} \frac{\partial p}{\partial x} \Big|_z dz' \\ &= -\frac{gp(\zeta)}{\rho_0} \frac{\partial \zeta}{\partial x} - \frac{g}{\rho_0} \int_z^{\zeta} \left[\frac{\partial p}{\partial x} \Big|_s - \frac{\partial p}{\partial z'} \frac{\partial z'}{\partial x} \Big|_s \right] dz', \end{aligned}$$

- Gary, 1973: subtracting a reference horizontal averaged value from density ($\rho' = \rho - \rho_a$) before computing pressure gradient
- Rewriting Equation of State: reduce passive compressibility effects on pressure gradient

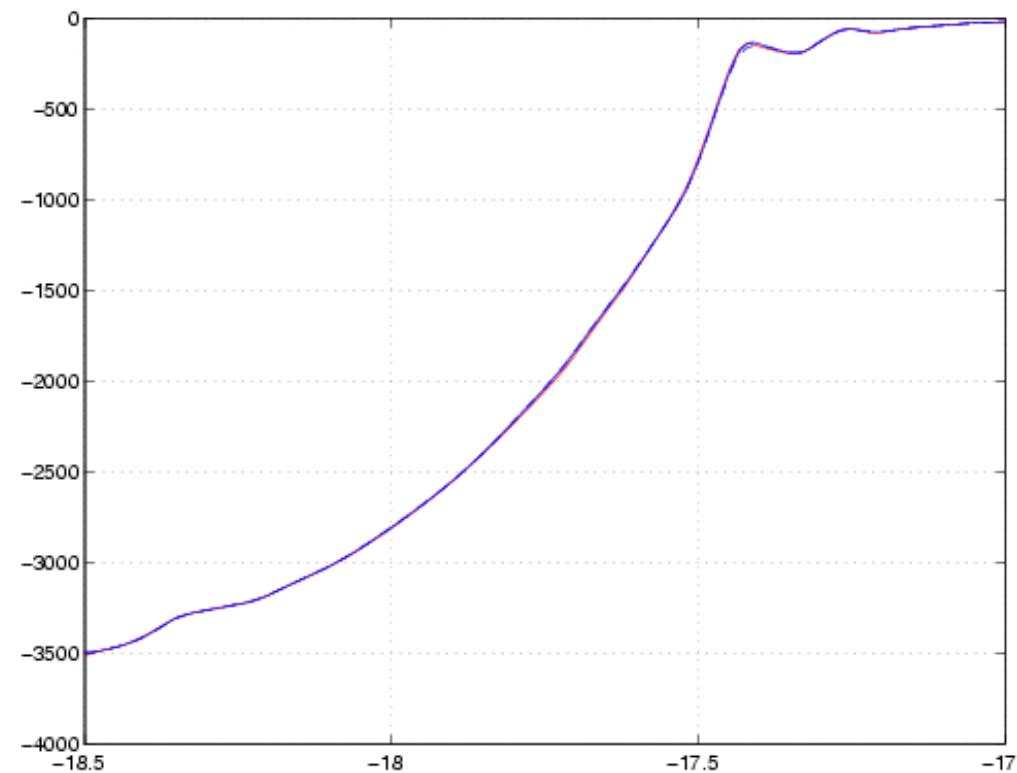
Smoothing methods

- $r = \Delta h / h$ is the slope of the logarithm of h
- One method (ROMS) consists of smoothing $\ln(h)$ until $r < r_{max}$

Res: 5 km ; $r_{max} = 0.25$

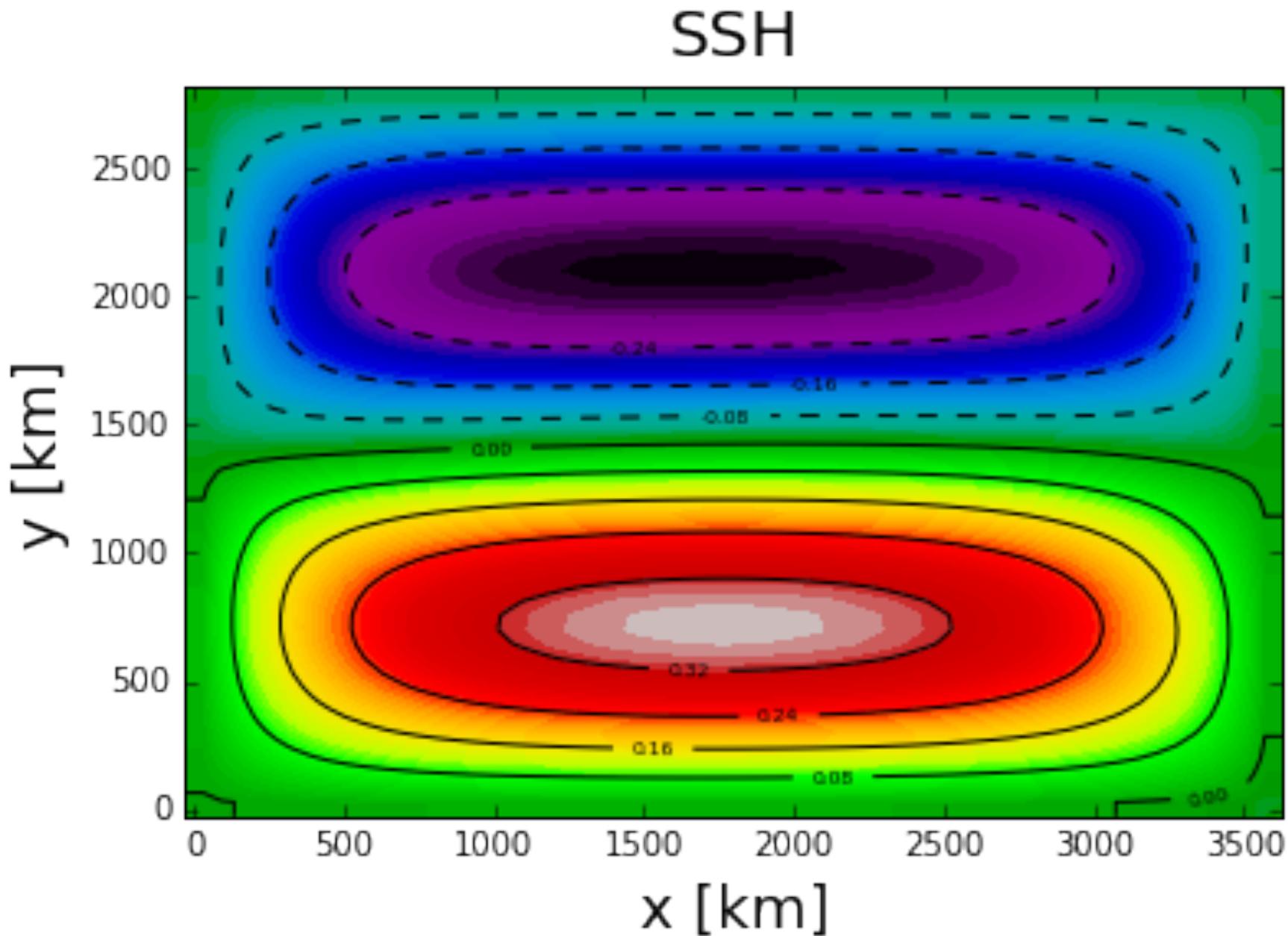


Res: 1 km ; $r_{max} = 0.25$



Activity 3 - Run an idealized ocean basin 2

Activity 2 - Run an idealized ocean basin



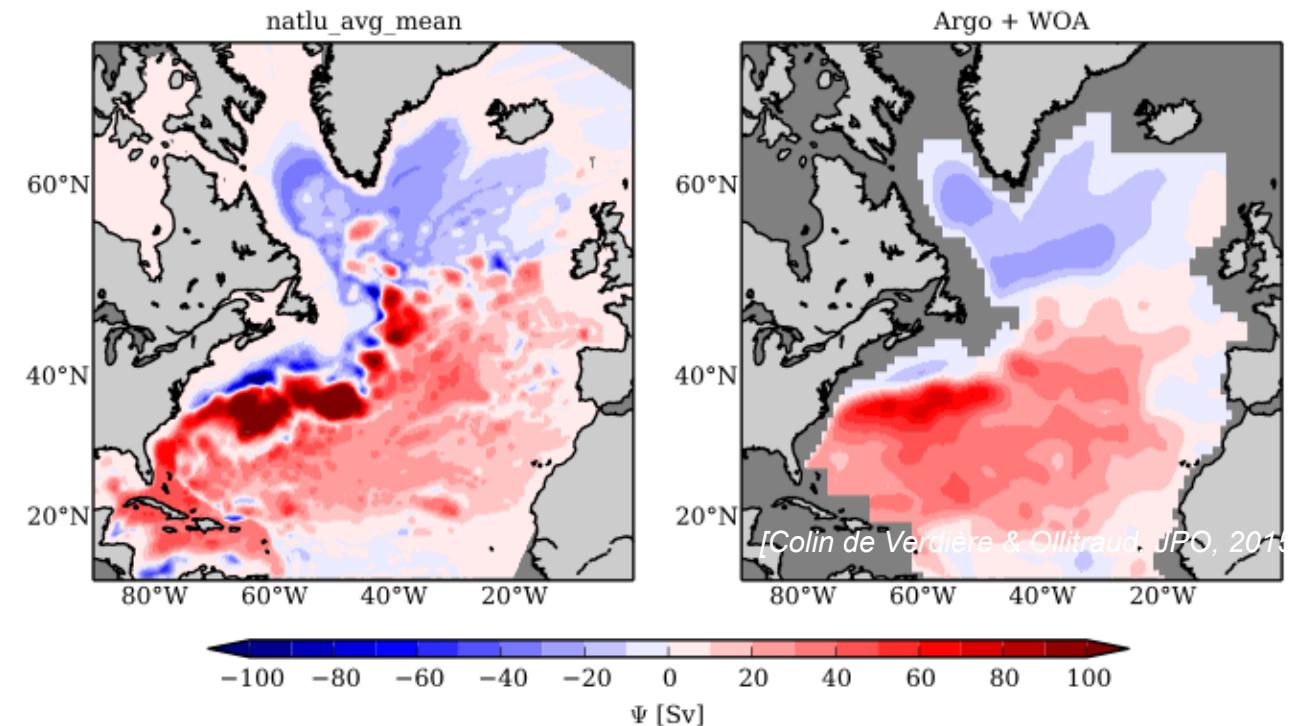
Bottom friction parametrization

1. Linear friction, with
r friction velocities [m/s] $\rightarrow (\tau_b^x, \tau_b^y) = -r (u_b, v_b)$
2. Quadratic friction, controled by a constant drag coefficient **Cd** $\rightarrow (\tau_b^x, \tau_b^y) = C_d \sqrt{u_b^2 + v_b^2} (u_b, v_b)$
3. Quadratic friction coefficient, using variable **Cd** (Von Karman log. layer) $\rightarrow \begin{cases} (\tau_b^x, \tau_b^y) = C_d \sqrt{u_b^2 + v_b^2} (u_b, v_b) \\ C_d = \left(\frac{\kappa}{\log[\Delta z_b/z_r]} \right)^2 \text{ si } C_d^{min} < C_d < C_d^{max} \\ C_d = C_d^{min} \text{ ou } C_d^{max} \\ \kappa = 0.41 \\ z_r = \text{Roughness Length} \\ \Delta z_b = \text{thickness of the first bottom level} \end{cases}$

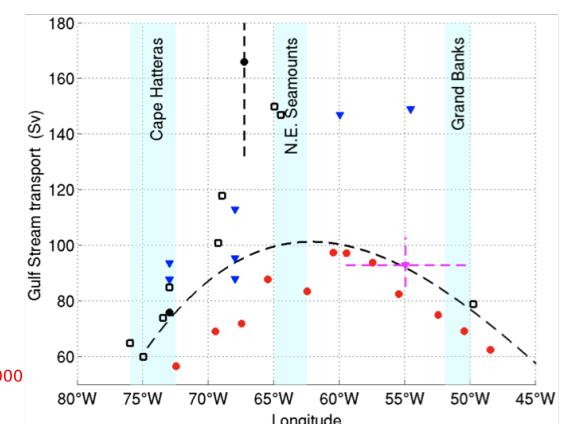
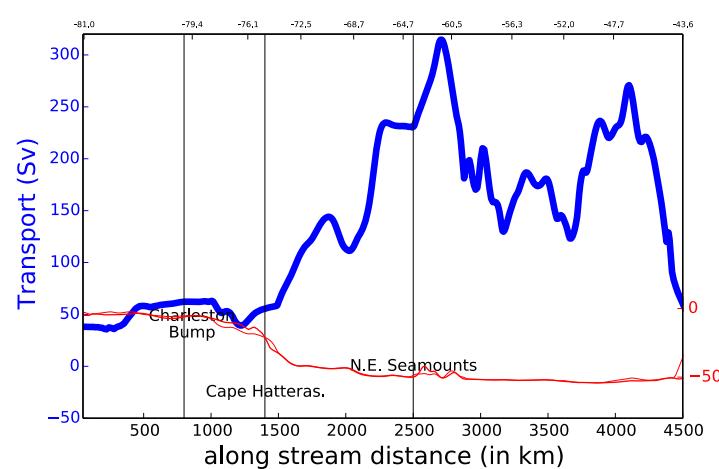
Bottom friction parametrization

Example: What happens if you take a small roughness length

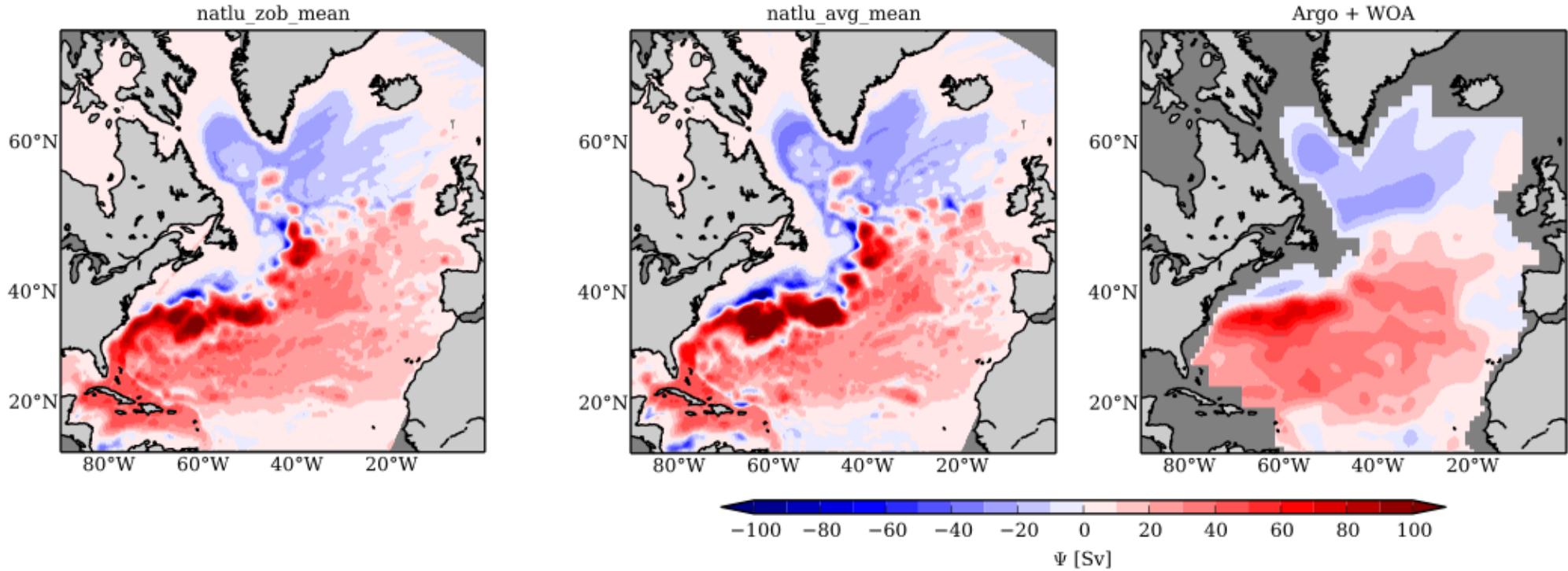
North-Atlantic ROMS-WRF Simulations



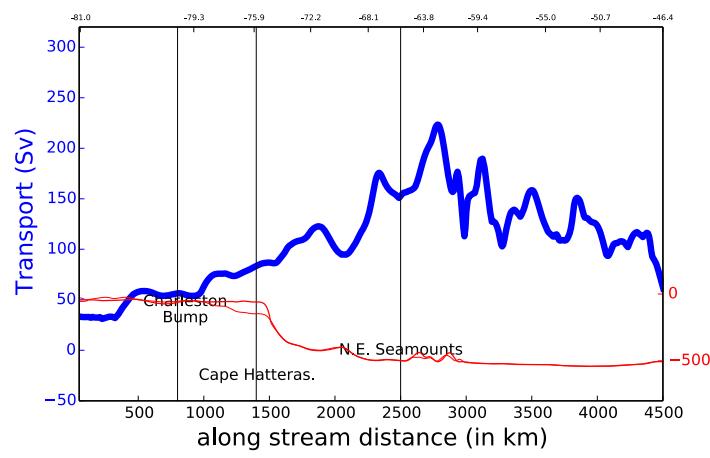
$$z_{0b} = 0.0001\text{m}$$



Bottom friction parametrization



$$z_{0b} = 0.01\text{m}$$



$$z_{0b} = 0.0001\text{m}$$

