Université de Bretagne Occidentale

APPLIED MATHEMATICS - I

“Perturbation Methods”

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Chapter 1 - PERTURBATION METHODS FOR ALGEBRAIC EQUATIONS

I. Introduction: Notions of simplification

To study realistic problems in mechanics several complementary approaches are possible.

- At first, theory provides us with equations: Algebraic Equations, Ordinary Differential Equations (ODE) or Partial Differential Equations (PDE). But these equations are usually extremely hard to solve mathematically.

- To process these equations, we can also use the numerical approach, which allows to integrate the equations and to predict the state of a system. However, this approach has its limitations:
    1) We do not know generally how to explain the results
    2) We do not know if they are correct (numerical approximations)
    3) We do not know the main physical mechanisms that determine the solution

- As a third approach, we usually need to consider simplified cases to better understand the realistic case:
    1) To illustrate the origin of the form of the solution
    2) To verify the numerical solutions in simple cases
    3) To identify the physical processes at work and their importance

This third approach requires a simplification of the original mathematical problem to find analytical solutions that will allow us to better understand the nature of the solutions (sensitivity to parameters, role of different physical terms).

The general simplification procedure will be:

1. Identifying the relatively small terms in the equations.
2. Neglecting these terms and solving the simplified system.
3. Checking the consistency of the approximation achieved: the approximate solution is used to estimate the neglected terms and verify that they are really small.

This common-sense method is based on the idea that small changes in the equations lead to small changes in the solution of these equations. Obviously, there are many counter-examples (exceptions) of this.

A few examples of simplifications in algebraic equations :

Example 1 :

$$\begin{cases} x + 10y = 21 \\ 5x + y = 7 \end{cases}$$

The coefficient in front of x is small compared to that of y in the first equation and we are tempted to neglect x in this equation. If we make this approximation we obtain
$x_0 = 0.98$ and $y_0 = 2.1$
With this approximation the ratio between the term neglected and the term used is
$(1.x_0) / (10.y_0) = 0.05$ and is small. The solution of the simplified system is therefore consistent. Moreover, the approximate solution is very close to the exact solution $x = 1$ and $y = 2$.

Example 2 :

$$\begin{cases} 0.01x + y = 0.1 \\ x + 101y = 11 \end{cases}$$

Just as before the coefficient in front of x is small compared to that of y in the first equation. By neglecting it one obtains the approximate solution:

$$x \sim 0.9 \text{ et } y \sim 0.1.$$

In this case $0.01x / y = 0.09$ and the approximation seems consistent.
However, the exact solution is $x = -90$, $y = 1$.
So, where is the mistake?

We write the system as $\begin{cases} \varepsilon x + y = 0.1 \\ x + 101y = 11 \end{cases}$ (here $\varepsilon = 10^{-2}$).

The simplification is to take $\varepsilon = 0$. However, this approximation is valid only if $x(\varepsilon) \sim x(0)$ et $y(\varepsilon) \sim y(0)$.

In fact, the exact solution is $x = \dfrac{0.9}{1-101\varepsilon}$ and $y = \dfrac{0.1-11\varepsilon}{1-101\varepsilon}$ with $\varepsilon \neq \dfrac{1}{101}$.

With this solution and with $\varepsilon = 10^{-2}$ we obtain $\left|\dfrac{\varepsilon x}{y}\right| = 0.9 \approx 1$. This shows that our approximation is really very bad.

We distinguish the « apparent consistency» from the « authentic consistency» which is achieved when the neglected term is really small.

Another approach is to think geometrically. actually, the original system is equivalent to find the intersection of two lines almost parallel. This explains the large uncertainty in the solution, which is the intersection point. This results in a great sensitivity to the value of ε which is related to the slope of the two lines.

We will speak here of "bad conditioned problem": small changes in its formulation lead to large changes in the solution.
In fact, here the problem has a singularity: indeed, the system moves from one case with a unique solution to one case without solution. The system considered in this example is very close to this singularity.

II. Perturbation methods for algebraic equations

1. Methodology

Perturbation methods are used to solve problems which contain a small parameter, usually denoted by ε, by iterative methods. These methods are so powerful that sometimes we artificially introduce a parameter ε in a problem to use them and then return to the case ε = 1. But in general, in the context of problems in continuum mechanics, this small parameter appears after a scaling of the equations.
In this chapter, we will only deal with one kind of perturbation method which uses power series. There exist other methods (parametric differentiation, successive approximation) which are described in the references cited at the end of the chapter.

The general approach is as follows:

1. Reveal a small parameter ε in front of a term of the equations. ε is called perturbation parameter.

2. Assume that the solution of the problem can be expressed as a power series of ε (perturbation series) and calculate the successive coefficients of this series (usually the first terms suffice).
3. Gather the different terms of the series to find the desired solution for the appropriate values of ε.

The underlying idea is to replace the solution of a complicated problem by solving a multitude of simpler problems. It is hoped that if the problem depends on a small perturbation parameter, then the solution of the problem as a perturbation series converges to the solution of the problem. Attention, of course this is not always the case! But in some cases, even when the series diverges, the sum of the first terms will give a satisfactory result.

However, for some problems depending on a parameter ε, a simple expansion in powers of ε may fail. This is the case when the problem has a singularity at ε = 0, that is to say that the problem changes character at this point (if ε is in front of the highest degree term of an algebraic equation for instance). In that case, if some solutions are missing then the problem may be singular: rescale the variable such that there exists a dominant equilibrium between two terms. The scaling maybe written $x = y/\delta$ or $x = \frac{y}{\varepsilon^\alpha}$ where δ or α has to be determined.

2. Illustration of regular and singular problems with algebraic equations

a) A regular problem:

Let us search approximate solutions of the 3rd degree algebraic equation:

$$x^3 - x + \varepsilon = 0 \quad \text{in which} \quad \varepsilon \ll 1$$

We search x as : $x = x_0 + \varepsilon\ x_1 + \varepsilon^2 x_2 + \quad \cdots\ldots$

Then $x^3 = x_0^3 + \varepsilon\ (3x_0^2 x_1) + \varepsilon^2 (3x_0 x_1^2 + 3x_0^2 x_2) + O(\varepsilon^3)$

Then $(x_0^3 - x_0) + \varepsilon\ (3x_0^2 x_1 - x_1 + 1) + \varepsilon^2 (3x_0 x_1^2 + 3x_0^2 x_2 - x_2) + O(\varepsilon^3) = 0$

$$\begin{cases} x_0^3 - x_0 = 0 \\ 3x_0^2 x_1 - x_1 + 1 = 0 \\ 3x_0 x_1^2 + 3x_0^2 x_2 - x_2 = 0 \end{cases} \qquad \begin{cases} x_0 (x_0^2 - 1) = 0 \\ x_1 (3x_0^2 - 1) = -1 \\ x_2 (3x_0^2 - 1) = -3x_0 x_1^2 \end{cases}$$

$$\begin{cases} x_0 = 0, +1, -1 \\ x_1 = -1/(3x_0^2 - 1) \\ x_2 = -3x_0 x_1^2 / (3x_0^2 - 1) \end{cases}$$

We thus obtain approximations as $\varepsilon \ll 1$ for the 3 roots :

$$\begin{cases} x = \varepsilon + O(\varepsilon^2) \\ x = 1 - \dfrac{1}{2}\varepsilon + O(\varepsilon^2) \\ x = -1 - \dfrac{1}{2}\varepsilon + O(\varepsilon^2) \end{cases}$$

***b*) A singular problem:**

Let us search the approximate solutions of the 3rd degree algebraic equation:

$$\varepsilon x^3 - x + 1 = 0 \quad \text{as} \quad \varepsilon \ll 1$$

Let us search x as : $x = x_0 + \varepsilon\ x_1 + \varepsilon^2 x_2 + \quad \cdots\ldots$

We get : $\begin{cases} -x_0 + 1 = 0 \\ x_0^3 - x_1 = 0 \\ 3x_0^2 x_1 - x_2 = 0 \end{cases}$ and thus : $\begin{cases} x_0 = 1 \\ x_1 = 1 \\ x_2 = 3 \end{cases}$

We have obtained only 1 approximate root: $\quad x = 1 + \varepsilon\ + O(\varepsilon^2)$

How to get the other two roots?
And why did we miss them?

The reason is that : as $\varepsilon \longrightarrow 0$ the two other roots tend to infinity so that εx^3 keeps a finite value and thus can not be neglected.

We thus have to rescale the problem:
So we search $y(\varepsilon)$ such that $x(\varepsilon) = \dfrac{y(\varepsilon)}{\delta(\varepsilon)}$ and $y(\varepsilon) = O(1)$ as $\varepsilon \longrightarrow 0$

We get : $\quad \underbrace{\dfrac{\varepsilon}{\delta^3}}_{(1)} y^3 - \underbrace{\dfrac{1}{\delta}}_{(2)} y + \underbrace{1}_{(3)} = 0$

Our first task is to determine $\delta(\varepsilon)$ such that this equation contains 2 terms of the same order as $\varepsilon \longrightarrow 0$ (what we call a « dominant equilibrium »). If it is not the case, we get only trial or impossible solution.

This analysis gives 3 cases:

- if $(1) \sim (2)$ then $\delta \sim \sqrt{\varepsilon}$ which give $\frac{\varepsilon}{\delta^3} \gg 1$ (OK as $\varepsilon \ll 1$)
- if $(1) \sim (3)$ then $\delta \sim \varepsilon^{1/3}$ which give $1 \ll \varepsilon^{-\frac{1}{3}}$ (IMPOSSIBLE as $\varepsilon \ll 1$)
- if $(2) \sim (3)$ then $\delta \sim 1$ which give $1 \gg \varepsilon$ (OK as $\varepsilon \ll 1$)

The first case leads to $\delta \ll \sqrt{\varepsilon}$ and the scaled equation $y^3 - y + \delta = 0$

With $y = y_0 + \delta\ y_1 + O(\delta^2)$ we get : $y = \pm 1 - \frac{1}{2}\delta + O(\delta^2)$

And back to x: $x = \frac{y}{\delta} = \frac{y}{\sqrt{\varepsilon}} = \pm\frac{1}{\sqrt{\varepsilon}} - \frac{1}{2} + O(\sqrt{\varepsilon})$

(we notice that the third case would lead to the regular solution found before)

The approximation of the three roots as $\varepsilon \ll 1$ are thus:

$$\begin{cases} 1 + \varepsilon + O(\varepsilon^2) \\ \frac{1}{\sqrt{\varepsilon}} - \frac{1}{2} + O(\sqrt{\varepsilon}) \\ -\frac{1}{\sqrt{\varepsilon}} - \frac{1}{2} + O(\sqrt{\varepsilon}) \end{cases}$$

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Chapter 2 - ASYMPTOTIC EXPANSIONS

A few notations:

Asymptotically smaller

$f(x) \ll g(x) \;\; (x \to x_0)$: $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 0$ other notation : $f(x) = o\big(g(x)\big)$

Asymptotically equal

$f(x) \sim g(x) \;\; (x \to x_0)$: $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 1$

Asymptotically bounded

$f(x) = O\big(g(x)\big) \quad (x \to x_0)$: $\exists M \quad |f| \le M|g| \quad \forall x$ sufficiently close to x_0

I. Some reminders about power series

Definition : A power series is a series that can be written as : $\sum_{n=0}^{\infty} A_n (x - x_0)^n = \lim\limits_{N \to \infty} \sum_{n=0}^{N} A_n (x - x_0)^n$. For x values at which the limit exists the series is convergent.

Series convergence (*Ratio test*): A series $S = \sum_{n=0}^{\infty} s_n$ converges if the absolute value of its consecutive terms ration $\left|\frac{s_{n+1}}{s_n}\right|$ converges towards a finite limit $\rho < 1$ as n tends towards infinity. It diverges if $\rho > 1$. No conclusion is possible if $\rho = 1$.

Power series convergence: as a consequence, for a power series $\rho = \lim\limits_{n \to \infty} \left|\frac{A_{n+1}}{A_n}\right| |x - x_0|$ and if $L = \lim\limits_{n \to \infty} \left|\frac{A_{n+1}}{A_n}\right|$ exists, then the series converges for $|x - x_0| < \frac{1}{L}$

When L exists, the distance $R = \frac{1}{L}$ is called convergence radius of the series.

Convergent power series and Taylor expansions: If $\sum_{n=0}^{\infty} A_n (x-x_0)^n$ is convergent in a neighborhood of x_0, then it represents a function $f(x)$ infinitely derivable with continuous derivatives.

One shows that :

$$f^{(k)}(x_0) = k!\, A_k \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} A_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n .$$

This is the Taylor expansion of $f(x)$ around x_0. Such a function is called a regular function (analytical function).

Series remainder: If a power series is convergent for all x, then necessarily its remainder has to tend towards 0:

$$f(x) = \sum_{n=0}^{\infty} A_n (x-x_0)^n = A_0 + A_1 (x-x_0) + \cdots + A_N (x-x_0)^N + \underbrace{A_{N+1}(x-x_0)^{N+1} + \cdots}_{R_N(x)}$$

$$\text{Remainder:} \quad R_N(x) = \sum_{n=N+1}^{\infty} A_n (x-x_0)^n \xrightarrow[N\to\infty]{} 0 \quad \forall x$$

II. Asymptotic expansions

Definition : $\sum_{n=0}^{\infty} A_n (x-x_0)^n$ is an asymptotic expansion of the function $f(x)$ as $x \to x_0$ if :

$$\forall N, \quad f(x) - \sum_{n=0}^{N} A_n (x-x_0)^n << (x-x_0)^N, \quad \text{as} \quad (x \to x_0).$$

We then write : $f(x) \sim \sum_{n=0}^{+\infty} A_n (x-x_0)^n$, as $(x \to x_0)$

Remark 1 : This is equivalent to :

$$\forall N, \quad \forall \varepsilon, \quad \exists \delta, \quad \text{as } |x - x_0| < \delta \Rightarrow \left| f(x) - \sum_{n=0}^{N} A_n (x-x_0)^n \right| < \varepsilon |x-x_0|^N$$

Or also equivalently :

$$\forall N, \quad f(x) - \sum_{n=0}^{N} A_n (x-x_0)^n = o\left((x-x_0)^N\right), \quad \text{as} \quad (x \to x_0)$$

Remark 2 : A convergent series $\sum_{n=0}^{\infty} A_n (x-x_0)^n$ over an interval containing x_0 is always asymptotic to its limit $f(x)$. But be careful, lots of asymptotic expansions are not convergent series !

We note $\varepsilon_N(x) = f(x) - \sum_{n=0}^{N} A_n (x-x_0)^n = f(x) - S_N(x)$

- *For a convergent power series* :

$$f(x) = \sum_{n=0}^{\infty} A_n (x-x_0)^n = \underbrace{A_0 + A_1(x-x_0) + \cdots + A_N(x-x_0)^N}_{S_N(x)} + \underbrace{A_{N+1}(x-x_0)^{N+1} + \cdots}_{R_N(x)}$$

necessarily : $\varepsilon_N(x) = R_N(x) = \sum_{n=N+1}^{\infty} A_n (x-x_0)^n \xrightarrow[N\to\infty]{} 0 \quad \forall x$

- *For an asymptotic expansion* :

We impose that $\varepsilon_N(x) \ll (x-x_0)^N$, as $(x \to x_0) \forall N$

Convergence is an absolute property
Asymptoticity is a relative property

Remark 3 : Non-integer powers are possible in asymptotic expansions : $\sum_{n=0}^{\infty} A_n (x-x_0)^{\alpha n}$, $\alpha > 0$

Asymptoticity close to infinity : if $x_0 = \infty$ we write :
$f(x) \sim \sum_{n=0}^{+\infty} A_n x^{-\alpha n}$, as $(x \to \infty)$ with $\varepsilon_N(x) << x^{-\alpha N}$ as $x \to \infty$ $\forall N$

Generalized asymptotic expansions :
If $(\phi_n(x))_n$ are functions, we say that $f(x) \sim \sum_{n=0}^{+\infty} A_n \phi_n(x)$, as $(x \to x_0)$ if

$\forall N, f(x) \sim \sum_{n=0}^{N} A_n \phi_n(x) \ll \phi_N(x)$, as $(x \to x_0)$

An example of asymptotic expansion / divergent series close to infinity.

Search an asymptotic expansion of the exponential integral: $Ei(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt$ as $(x \to \infty)$

With successive integrations by parts:

$$\begin{aligned} Ei(x) &= \left[\frac{-e^{-t}}{t} \right]_x^{\infty} - \int_x^{\infty} \frac{e^{-t}}{t^2} dt \\ &= \frac{e^{-x}}{x} + \left[\frac{e^{-t}}{t^2} \right]_x^{\infty} + 2\int_x^{\infty} \frac{e^{-t}}{t^3} dt \\ &\vdots \\ &= \underbrace{e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \cdots + (-1)^{N-1} \frac{(N-1)!}{x^N} \right)}_{S_N(x)} + \underbrace{(-1)^N N! \int_x^{\infty} \frac{e^{-t}}{t^{N+1}} dt}_{\varepsilon_N(x)} \end{aligned}$$

For all x , $S_N(x)$ diverges as $N \to \infty$.
$(\lim\limits_{N\to\infty} (-1)^{N-1} \frac{(N-1)!}{x^N} \neq 0)$

One can show that : $\left|\varepsilon_N(x)\right| < \frac{N!}{x^{N+1}} \int_x^{\infty} e^{-t} dt = \frac{N!}{x^{N+1}} e^{-x} \xrightarrow[x\to\infty]{} 0 \quad \forall N$

and better : $\left| \frac{\varepsilon_N(x)}{e^{-x}(-1)^{N-1}(N-1)!\,x^{-N}} \right| < \frac{\frac{N!}{x^{N+1}} e^{-x}}{(N-1)!\,e^{-x} x^{-N}} = \frac{N}{x} \xrightarrow[x\to\infty]{} 0$

Thus $Ei(x) \sim e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{3!}{x^4} + \cdots \right)$ as $(x \to \infty)$ whereas the power series $\sum_{n=1}^{\infty} e^{-x} (-1)^{n-1} \frac{(n-1)!}{x^n}$ is divergent.

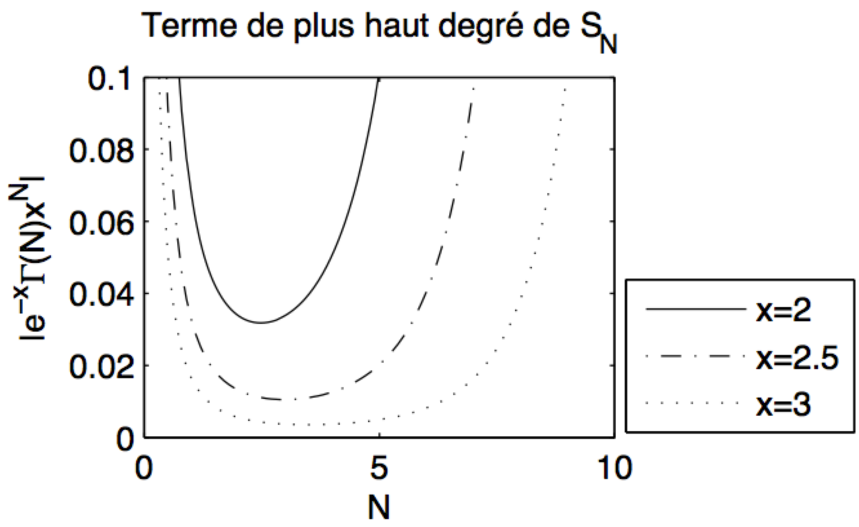

Figure 2 : *absolute value of the term of highest degree of* $S_N(x)$ *i-e* $\left|e^{-x}(-1)^{N-1}(N-1)!x^{-N}\right|$ *as a function of N for different values of x.*

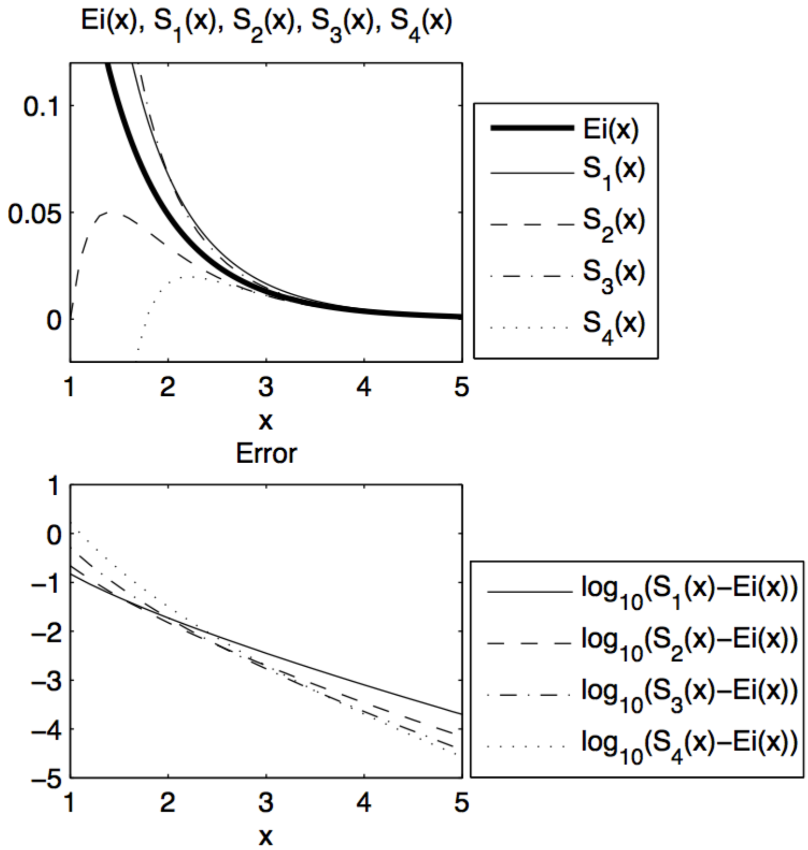

Figure 3 : : *(up) Ei(x) and the 4 first partial sums* $S_N(x)$. *(down). log (base 10) of the difference between these partial sums and exact solution.*

Remark : On figure 1 to plot the values of $\left| e^{-x}(-1)^{N-1}(N-1)!\ x^{-N} \right|$ for real values of $N > 0$ we used the Gamma function defined by :

$$\boxed{\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt \quad \text{with} \quad \mathrm{Re}(z) > 0}$$

This function is a prolongation (analytical) of the discrete factorial function.

We recall that $\boxed{\Gamma(z+1) = z\Gamma(z)}$ $\boxed{\Gamma(1) = 1}$

In particular, if n is integer: $\boxed{\Gamma(n+1) = n!}$

Properties of asymptotic expansions:

- If $f(x) \sim \sum_{n=0}^{\infty} A_n x^n$, $(x \to 0)$ then $f(x) + C \cdot e^{-1/x}$ has the same asymptotic expansion close to 0.

- **Integration / Differentiation:** One can integrate an asymptotic expansion but not always differentiate it : if f and g have the same asymptotic expansion, it is not always true for f' and g'.

- **Asymptotic expansions of integrals :**
  Let $I(x) = \int_a^b f(t,x) dt$ close to x_0 .
  If $f(t,x) \sim f_0(t)$ as $(x \to x_0)$ uniformly as $a \le t \le b$, then the first term of the asymptotic expansion of $I(x)$ is :
  $$I(x) \sim \int_a^b f_0(t) dt \quad \text{as } (x \to x_0)$$
  If $f(t,x) \sim \sum_{n=0}^{+\infty} f_n(t)(x - x_0)^{\alpha n}$ as $(x \to x_0)$ $\alpha > 0$ and for all $a \le t \le b$ then
  $$f(t,x) \sim \sum_{n=0}^{+\infty} \ (x - x_0)^{\alpha n} \int_a^b f_n(t) dt \quad \text{as } (x \to x_0)$$

- **Unicity of the asymptotic expansion :**
  If $f(x) \sim \sum_{n=0}^{+\infty} A_n (x - x_0)^{\alpha n}$ as $(x \to x_0)$ with $\alpha > 0$, then the coefficients A_n are unique and can be calculated as follows :
  $$a_0 = \lim_{x \to x_0} f(x)$$
  $$a_1 = \lim_{x \to x_0} \frac{f(x) - A_0}{(x - x_0)^{\alpha}}$$
  $$a_2 = \lim_{x \to x_0} \frac{f(x) - (A_0 + A_1 (x - x_0)^{\alpha})}{(x - x_0)^{\alpha 2}}$$
  $$etc \cdots$$
  The only condition is that these limits exist.

A few examples of asymptotic expansions of integrals :

1) $I(x)=\int_0^1 \frac{\sin(xt)}{t}dt$ as $(x\to 0)$

We use Taylor expansion of $\sin(x)$ (infinite convergence radius) :

$$\frac{\sin(xt)}{t}=x-\frac{x^3t^2}{3!}+\frac{x^5t^4}{5!}-\cdots+(-1)^{2p+1}\frac{x^{2p+1}t^{2p}}{(2p+1)!}+\cdots$$

$$I(x)\sim x-\frac{x^3}{18}+\frac{x^5}{600} \quad \text{as } \left(x\to 0\right)$$

2) $I(x)=\int_0^\infty \frac{e^{-t}}{1+xt}dt$ as $(x\to 0)$

With successive integrations by parts :

$$I(x)=\left[\frac{-e^{-t}}{1+xt}\right]_0^\infty-\int_0^\infty \frac{xe^{-t}}{\left(1+xt\right)^2}dt$$

$$\vdots$$

$$=\underbrace{\sum_{n=0}^{N}(-1)^n n!x^n}_{S_N}+\underbrace{(-1)^{N+1}(N+1)!\int_0^\infty \frac{x^{N+1}e^{-t}}{\left(1+xt\right)^{N+2}}dt}_{\varepsilon_N}$$

$\left(\sum_{n=0}^{\infty}(-1)^n n!x^n\right)$ diverges, and $\quad \forall x \quad \varepsilon_N \xrightarrow[N\to\infty]{} \infty$

However : for a given N, $\varepsilon_N \xrightarrow[x\to 0]{} 0$

Thus : $I(x)\sim \sum_{n=0}^{+\infty}(-1)^n n!\,x^n$, as $(x\to 0)$

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