FLUIDS 2

Jonathan GULA
gula@univ-brest.fr

FLUIDS 2

Course aims:

The primary aim of this course is to introduce some key aspects of how fluids respond to perturbations and to explore the mathematical descriptions of these phenomena

Online Material: https://www.jgula.fr/Fluid/

FLUIDS 2

Course structure:

The course contains 3 major parts:

<u>1: Waves (~6 sessions including 2 TPs)</u>

- General features and methods of describing waves, applications to surface gravity waves, capillary waves, internal gravity waves and sound waves

<u>2: Instabilities (~3 sessions including 1 TP)</u>

- Shear instabilities, convective instabilities, sources of energy

<u>3: Chaos and turbulence (1 session)</u>

- Introduction to dynamical chaos, the transition to turbulence, statistical descriptions of turbulence

FLUIDS 2

Logistics

You will have about 29h of classes with me, including courses and 3 TP sessions, and 18h of TD with Olivier Arzel.

You will have to write a report based on the first 2 TPs that will count towards your final mark.

The final evaluation will take the form of an exam.

FLUIDS 2

Reference material

A number of books cover different aspects of the course - I will give extra recommendations as we go along, but some general suggestions:

- *Elementary Fluid Dynamics by Acheson gives an introduction to almost all of the topics that we will see (we will go a bit further than the coverage in this book).*

- *Fluid Mechanics by Kundu and Cohen also covers most of the material, and is a bit more advanced*

- *The books Physical Fluid Dynamics by Tritton and Physical Hydrodynamics / Hydrodynamique Physique by Guyon both cover the material on instabilities*

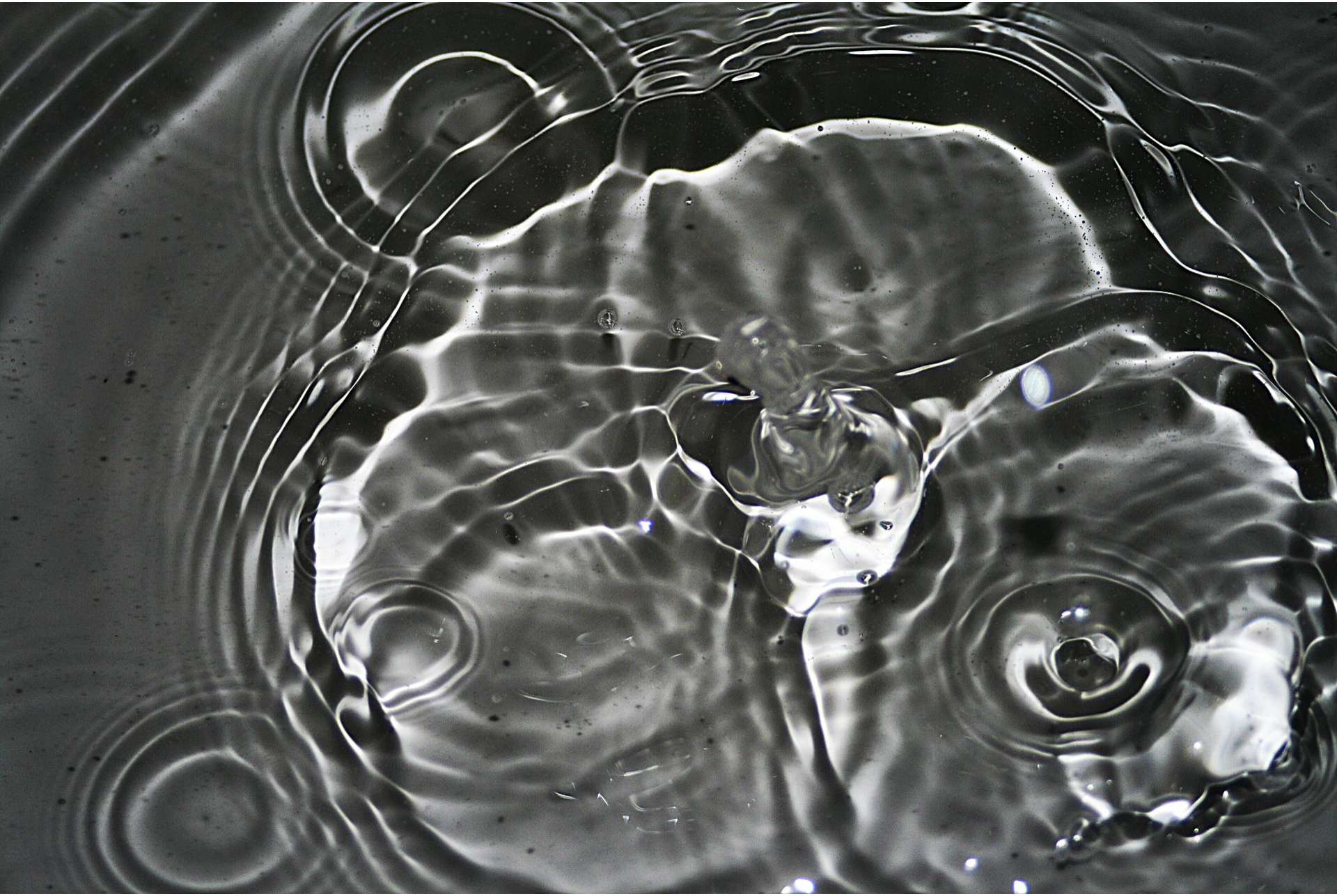
WAVES

WAVES

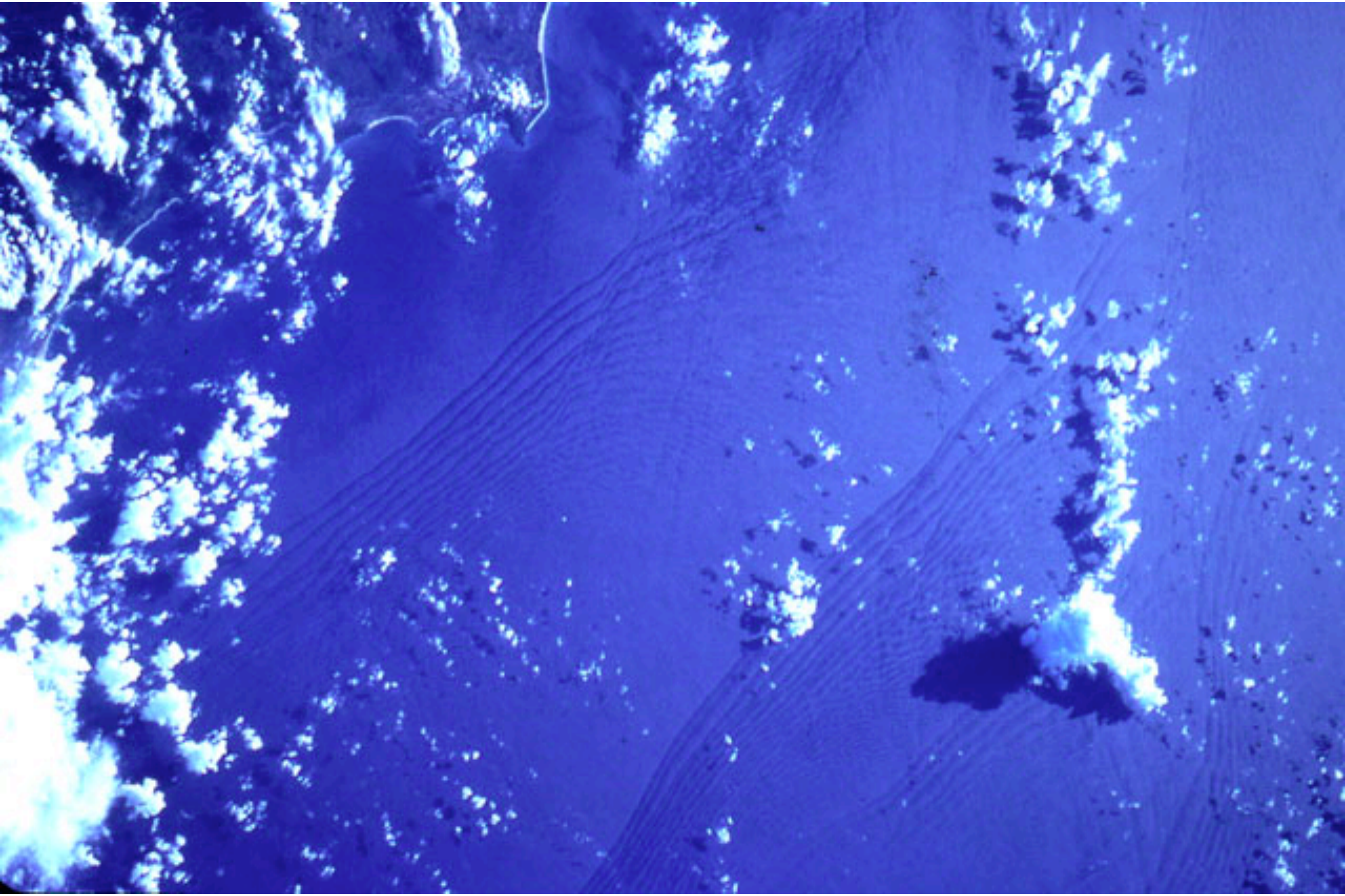
WAVES

WAVES

WAVES



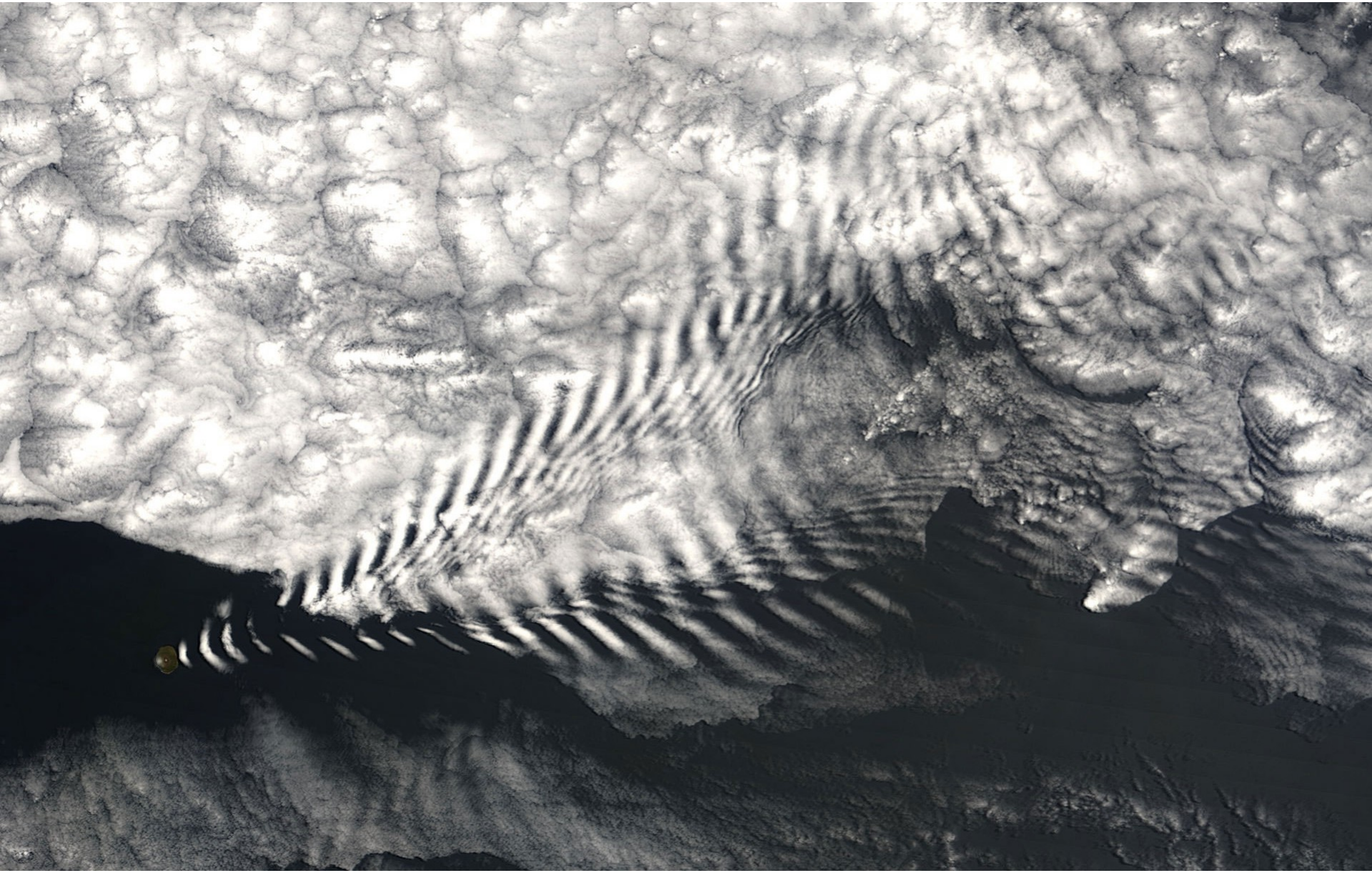
WAVES



WAVES

ENEOS
蘭 泉 洞
Ryusendo Cave
51Km
三 王 岩
Sannoiwa Rocks
20Km
浄 土 ヶ 浜
Jodogahama Beach
4Km
シートピアなあど
Seatopia Naado
1Km

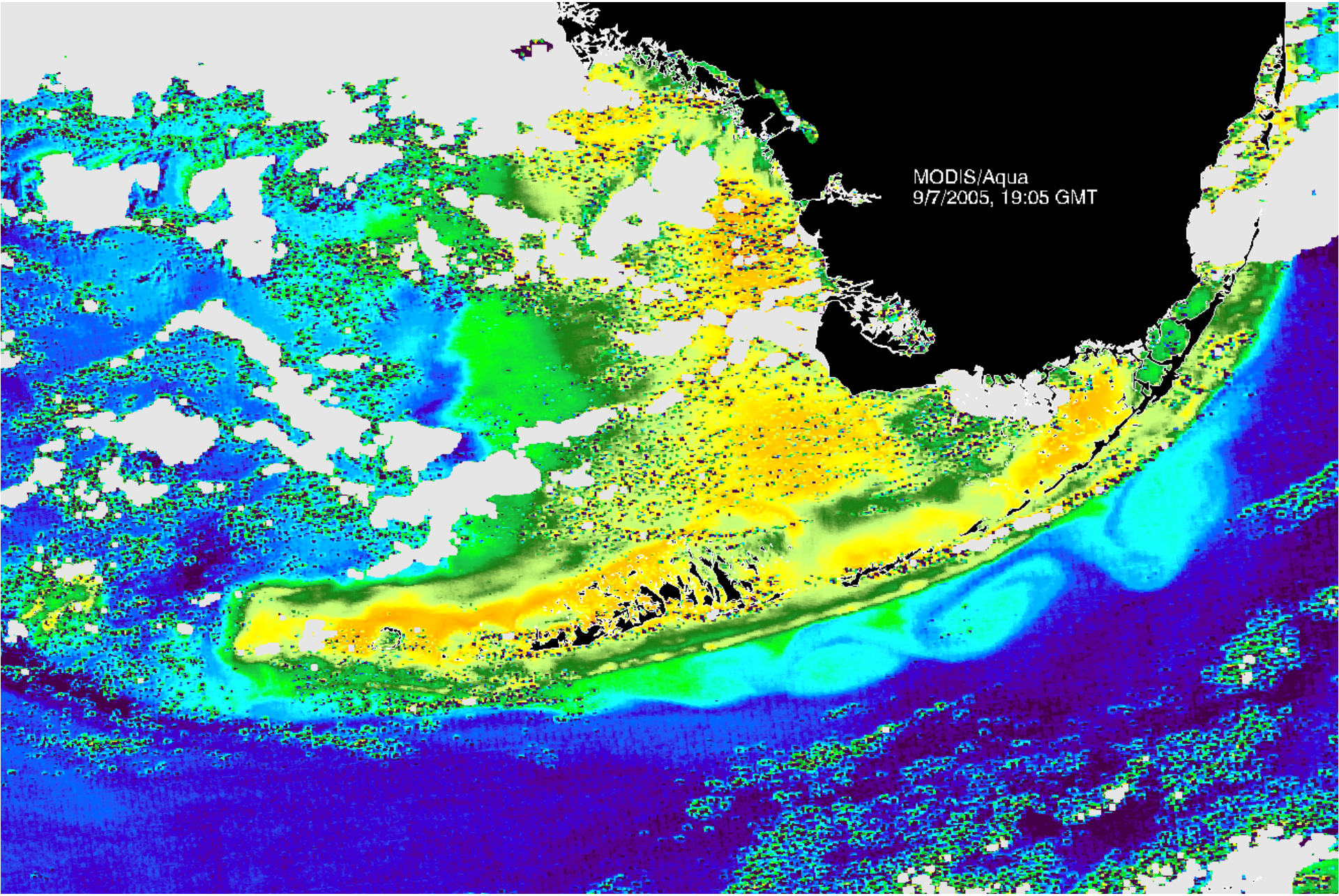
WAVES



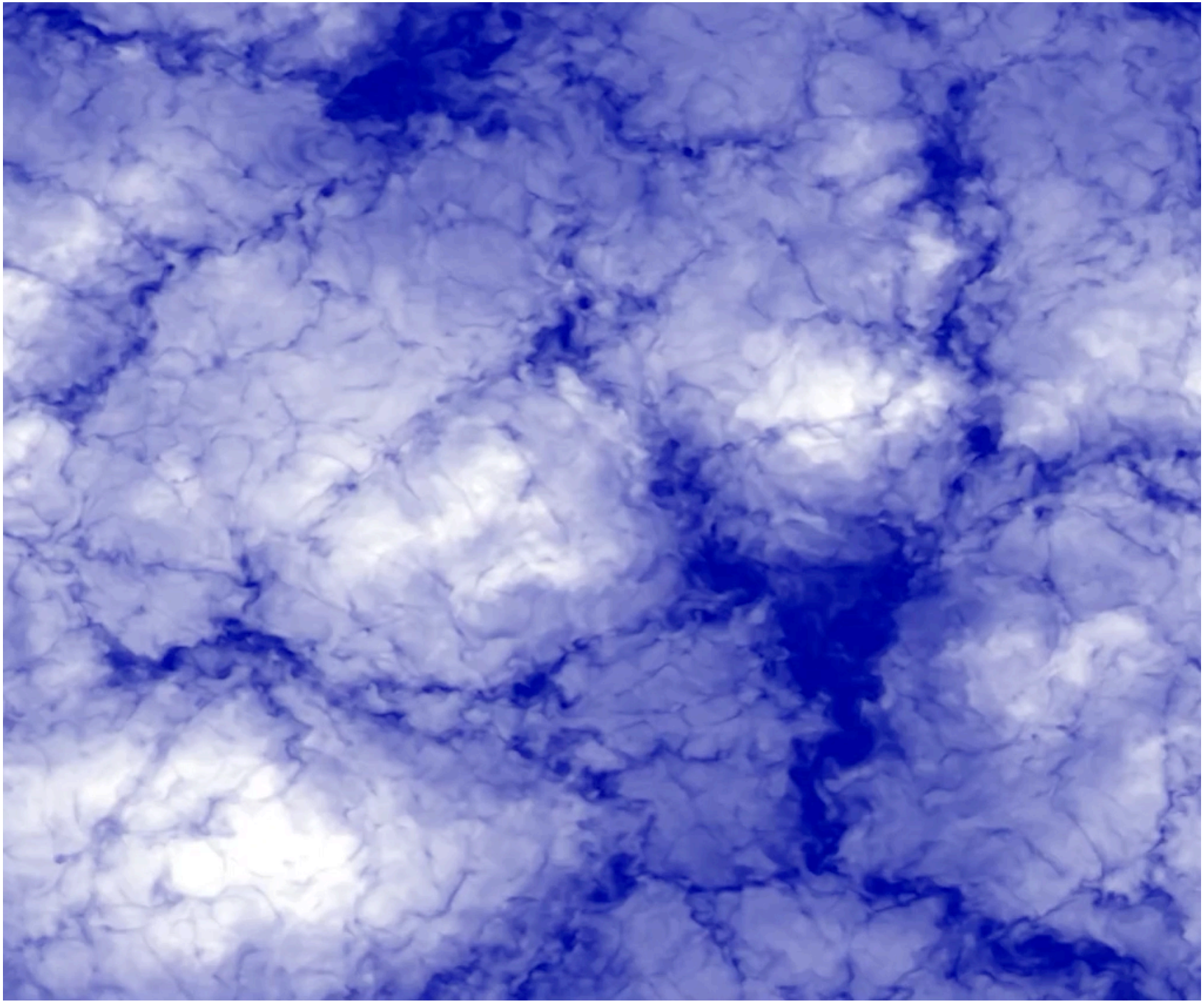
WAVES

Instabilities

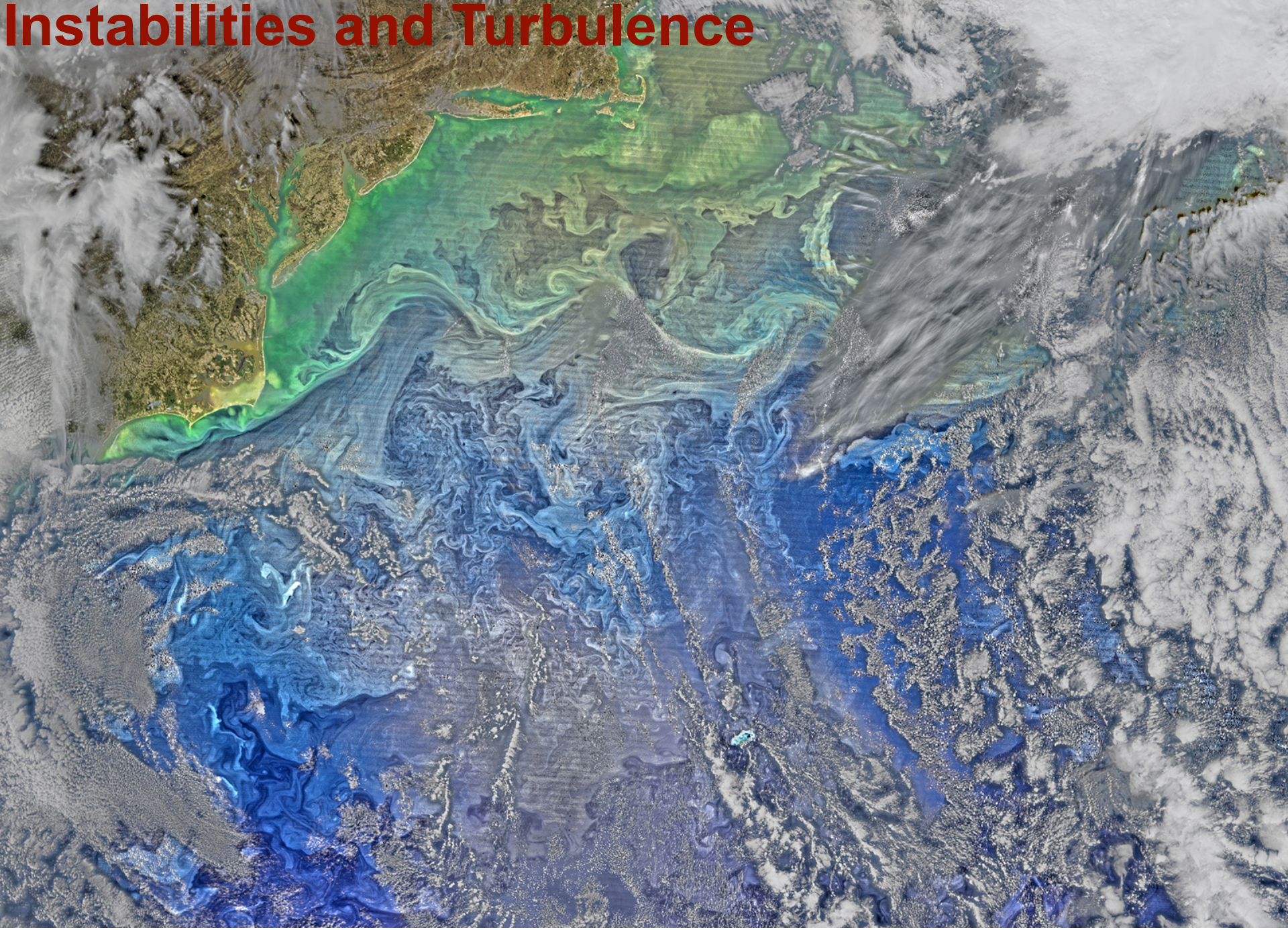
Instabilities

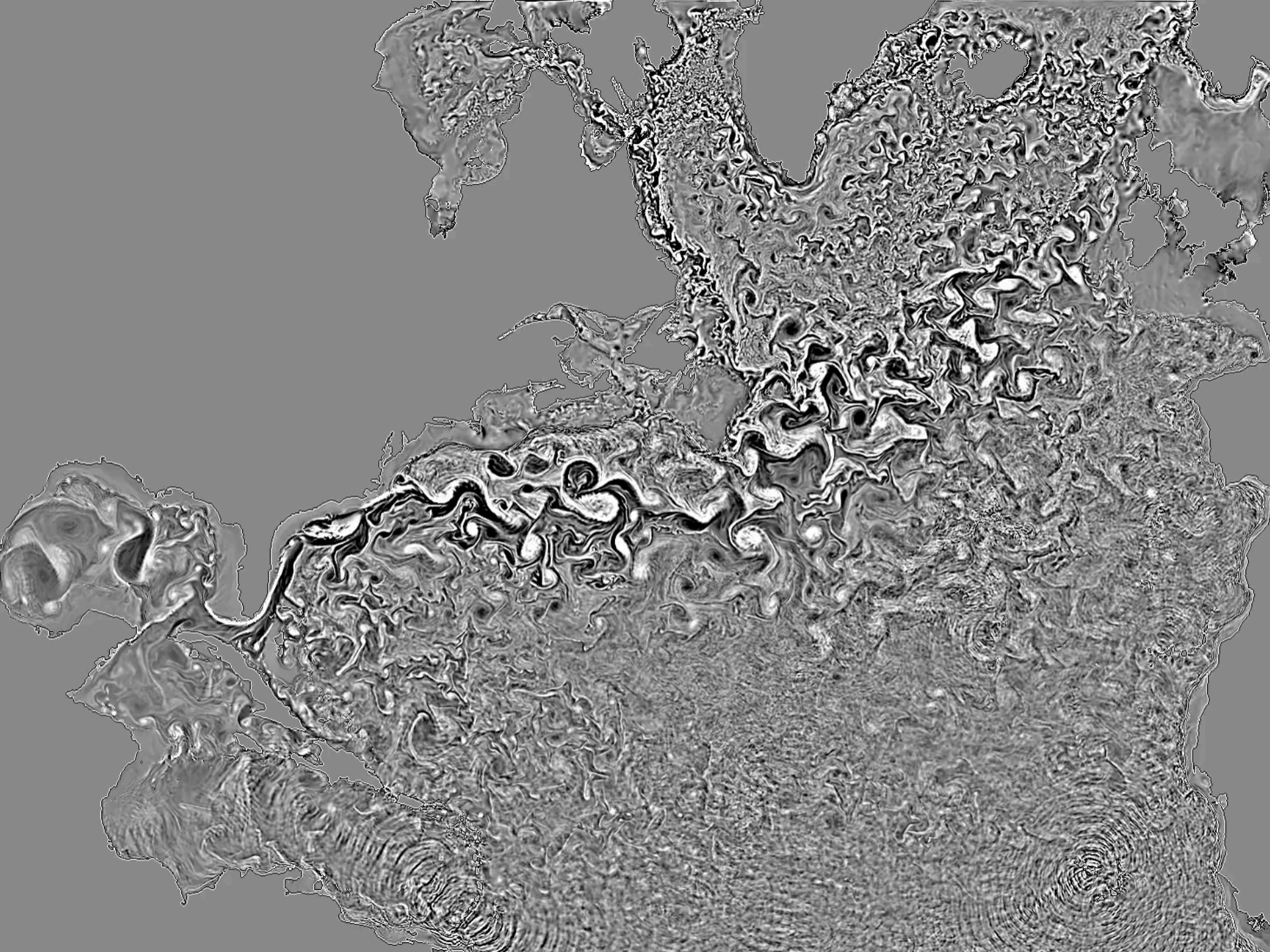


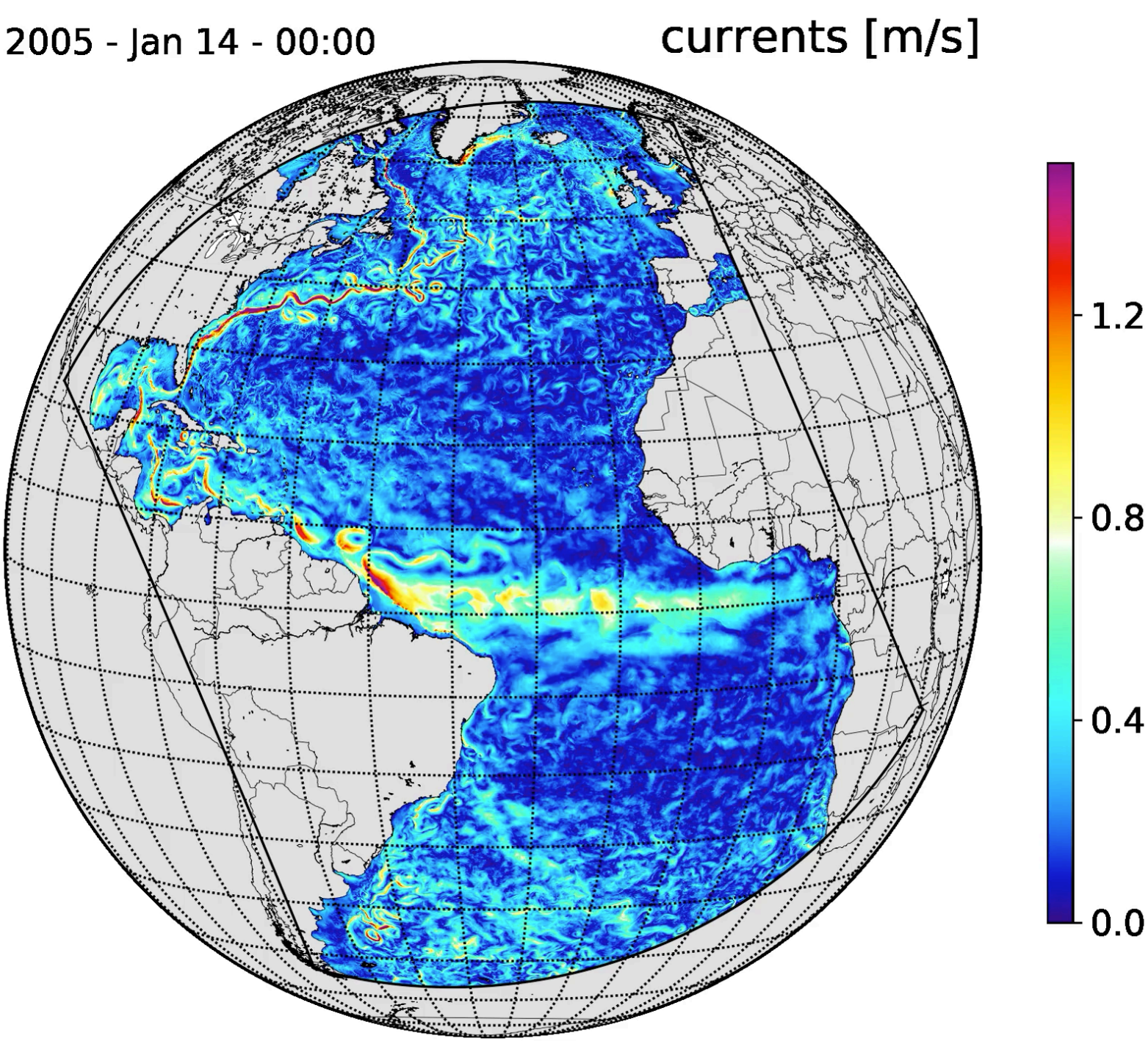
Instabilities and Turbulence



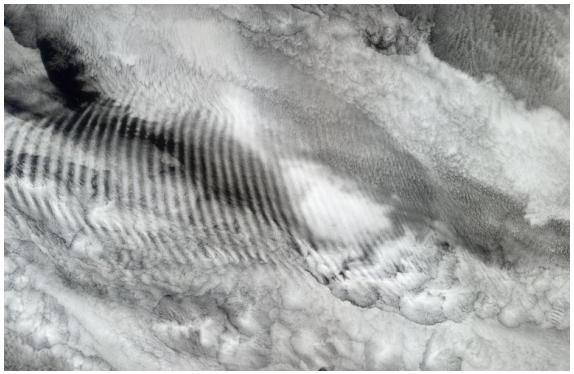
Instabilities and Turbulence

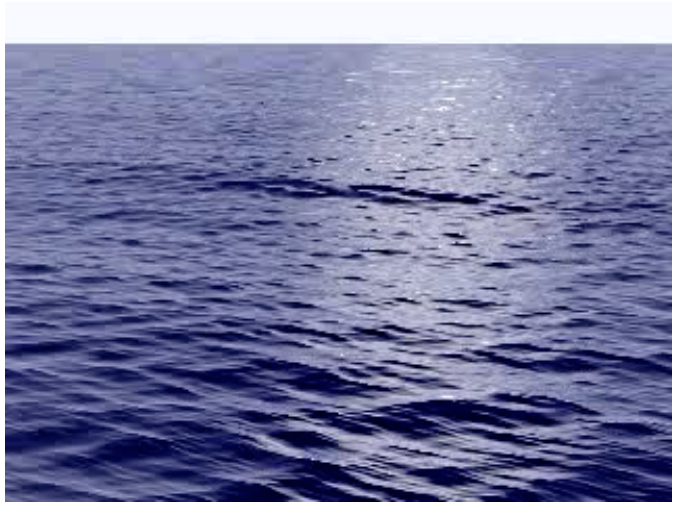




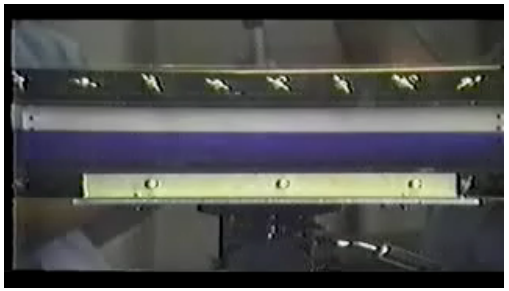

2005 - Jan 14 - 00:00
currents [m/s]
1.2
0.8
0.4
0.0

FLUIDS 2

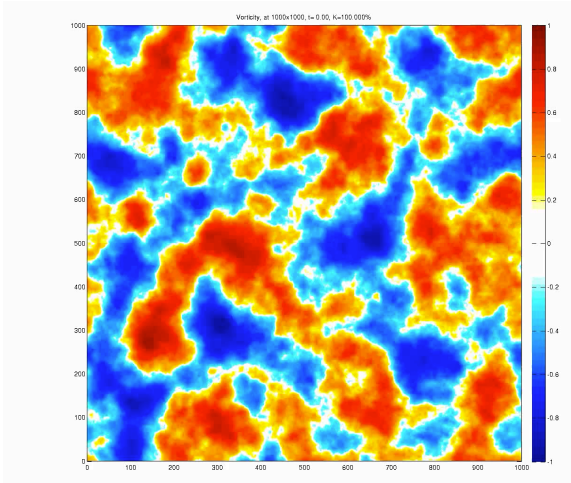




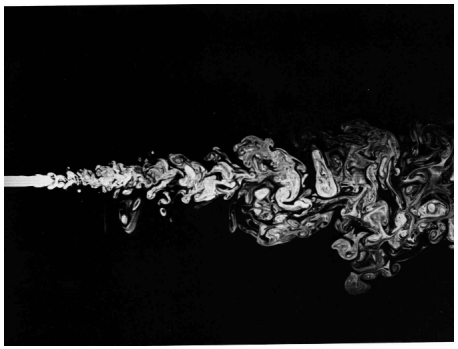
1. Waves



2. Instabilities



3. Chaos / Turbulence



I. WAVES

Jonathan GULA
gula@univ-brest.fr

I. WAVES

I.1. Introduction

I.2. General properties of Waves

I.3. Different type of ocean waves

I.3.1 Surface Gravity Waves

I.3.2 Internal Waves

I.3.3 Acoustic Waves

I.4. Ray Theory

Bibliography

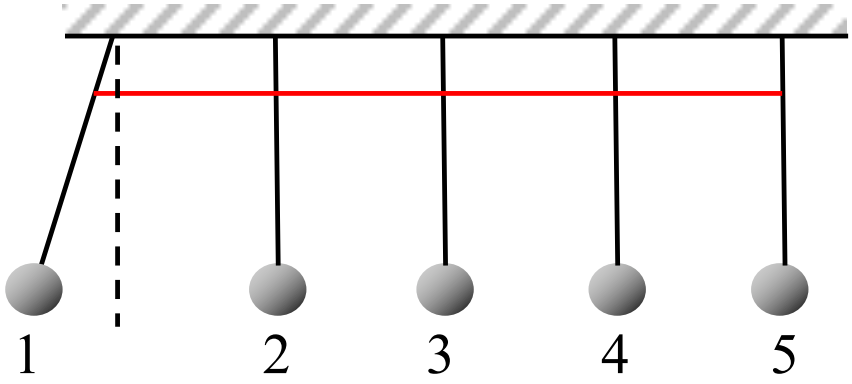
- Leblond-Mysak (1977) : *Waves in the ocean*
- Whitham (1974) : *Linear and nonlinear waves*
- Cushman-Roisin. *Introduction to geophysical fluid Dynamics*
- Ardhuin (2024) *Waves in Geosciences*
- Gerkema- Zimmerman (2008). *An introduction to internal waves*
  - *http://stockage.univ-brest.fr/~gula/Ondes/gerkema.pdf*

Definition of a wave:

- *A wave is a recognizable signal that is transferred from one part of the medium to another with a recognizable velocity of propagation. The signal may be any feature of the disturbance, such as a maximum or an abrupt change in some quantity, provided that it can be clearly recognized and its location at any time can be determined*. [Whitham: « Linear and nonlinear waves »]
-

Definition of a wave:

- *Restoring force and a continuous medium to transport oscillation*

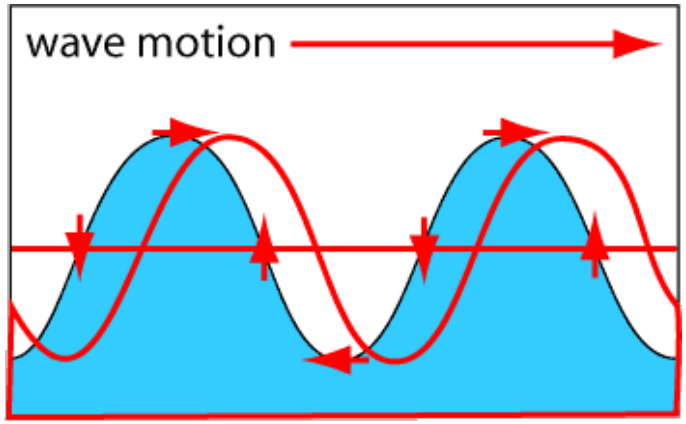


Definition of a wave:

- *Restoring force and a continuous medium to transport oscillation*

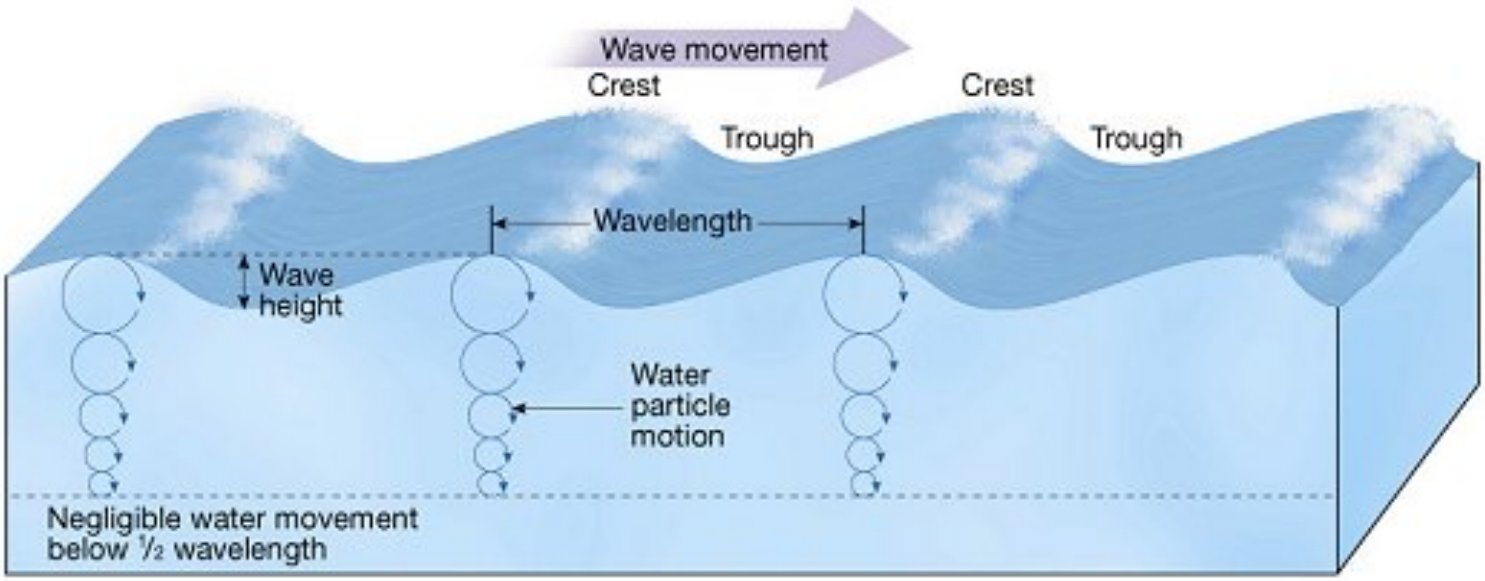
Ocean Waves

- A wave results when fluid is displaced from a position of equilibrium.
- The restoration of the fluid to equilibrium will produce a movement of the fluid back and forth, called a wave orbit.

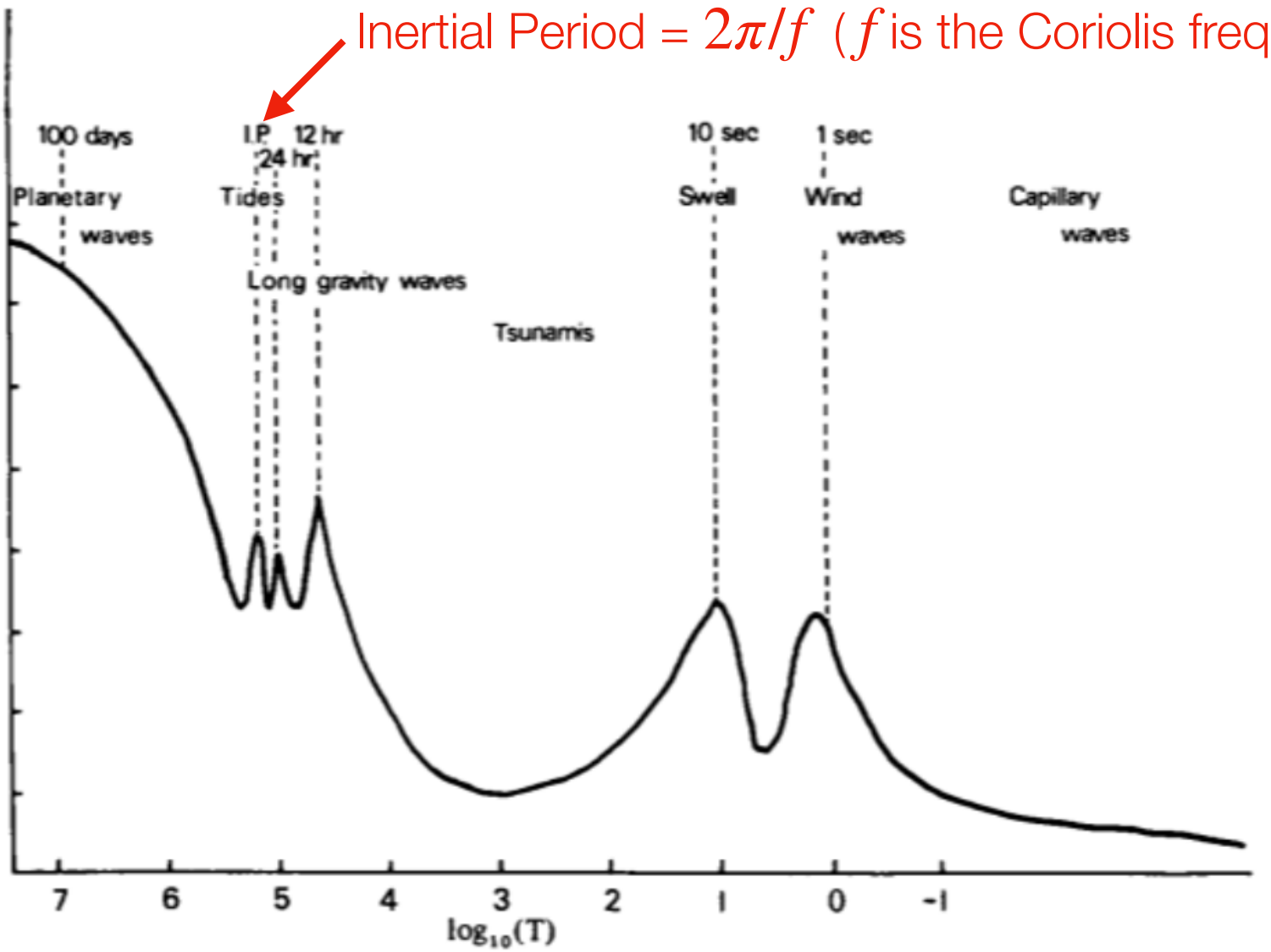


Ocean Waves

- (Linear) Waves propagates energy but do not transport water



I.1. Introduction



Schematic energy spectrum of ocean variability [Leblond & Mysak]

Ocean Waves

Different type of waves are classified on the basis of:

- Disturbing force
- Restoring force
- Properties: wavelength, frequency, amplitude, …
- Freely propagating vs forced wave

I.1. Introduction

Type of waves	Disturbing force	Restoring force	Horizontal wavelength	Period
Acoustic wave				
Capillary wave				
Surface gravity wave				
Rossby wave				
Kelvin wave				
Internal wave				

I.1. Introduction

Type of waves	Disturbing force	Restoring force	Horizontal wavelength	Period
Acoustic wave	Any motion	Pressure (compressibility)	1 mm - 10 km	< 1 s
Capillary wave				
Surface gravity wave				
Rossby wave				
Kelvin wave				
Internal wave				

I.1. Introduction

Type of waves	Disturbing force	Restoring force	Horizontal wavelength	Period
Acoustic wave	Any motion	Pressure (compressibility)	1 mm - 10 km	< 1 s
Capillary wave	Surface motion	Surface tension	1 mm - 1 cm	< 1 s
Surface gravity wave				
Rossby wave				
Kelvin wave				
Internal wave				

I.1. Introduction

Type of waves	Disturbing force	Restoring force	Horizontal wavelength	Period
Acoustic wave	Any motion	Pressure (compressibility)	1 mm - 10 km	< 1 s
Capillary wave	Surface motion	Surface tension	1 mm - 1 cm	< 1 s
Surface gravity wave	Wind, boat, earthquake, …	Gravity	1 cm - 100 km	1 s - 1 day
Rossby wave				
Kelvin wave				
Internal wave				

I.1. Introduction

Type of waves	Disturbing force	Restoring force	Horizontal wavelength	Period
Acoustic wave	Any motion	Pressure (compressibility)	1 mm - 10 km	< 1 s
Capillary wave	Surface motion	Surface tension	1 mm - 1 cm	< 1 s
Surface gravity wave	Wind, boat, earthquake, …	Gravity	1 cm - 100 km	1 s - 1 day
Rossby wave	Atmospheric forcing (wind, heat flux)	Potential vorticity (variation of Coriolis with latitude)	100 km - 1000 km	Days - months
Kelvin wave				
Internal wave				

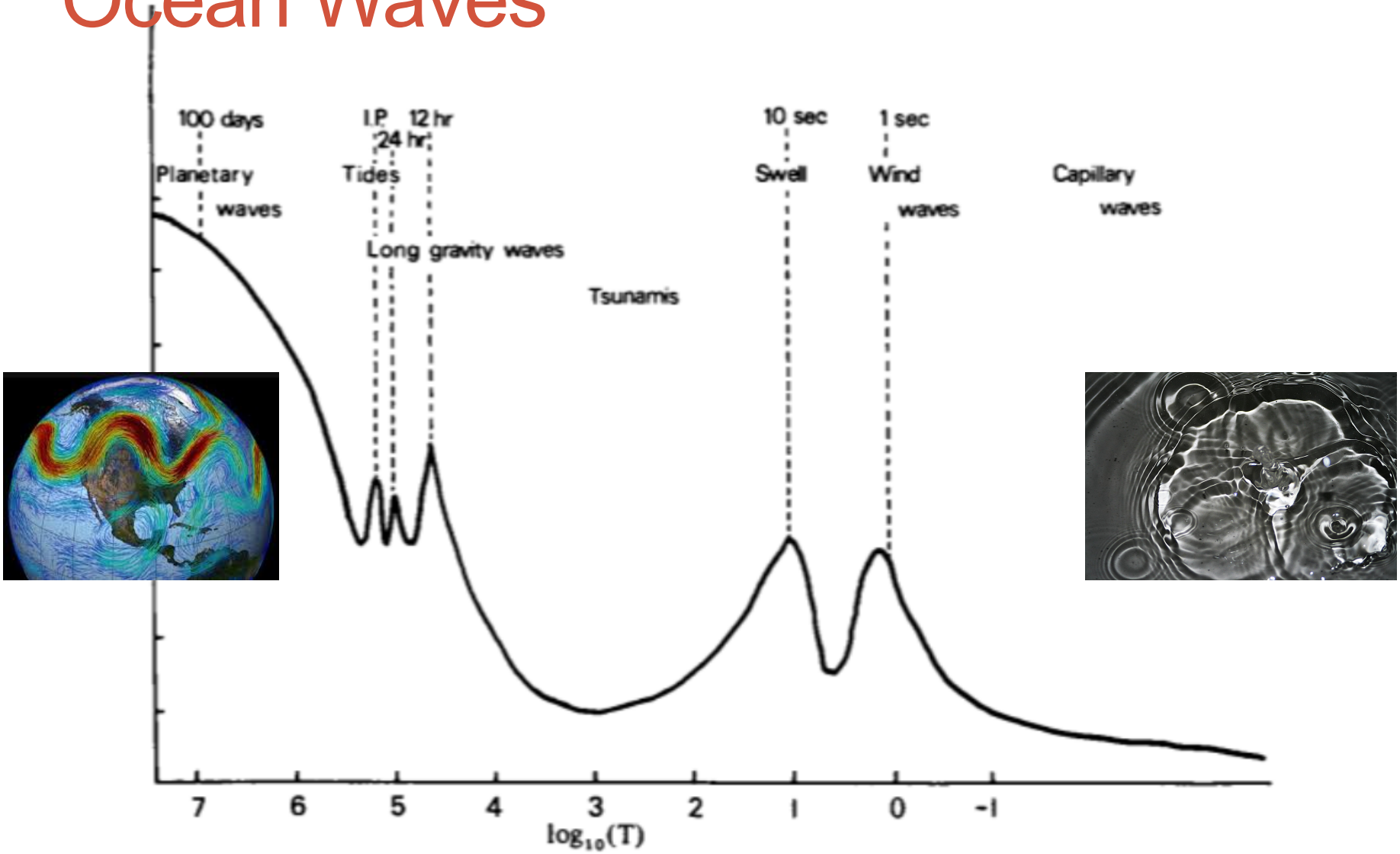
I.1. Introduction

Type of waves	Disturbing force	Restoring force	Horizontal wavelength	Period
Acoustic wave	Any motion	Pressure (compressibility)	1 mm - 10 km	< 1 s
Capillary wave	Surface motion	Surface tension	1 mm - 1 cm	< 1 s
Surface gravity wave	Wind, boat, earthquake, …	Gravity	1 cm - 100 km	1 s - 1 day
Rossby wave	Atmospheric forcing (wind, heat flux)	Potential vorticity (variation of Coriolis with latitude)	100 km - 1000 km	Days - months
Kelvin wave	Atmospheric forcing	Pressure gradient + Coriolis	10 km - 100 km	Days - months
Internal wave				

I.1. Introduction

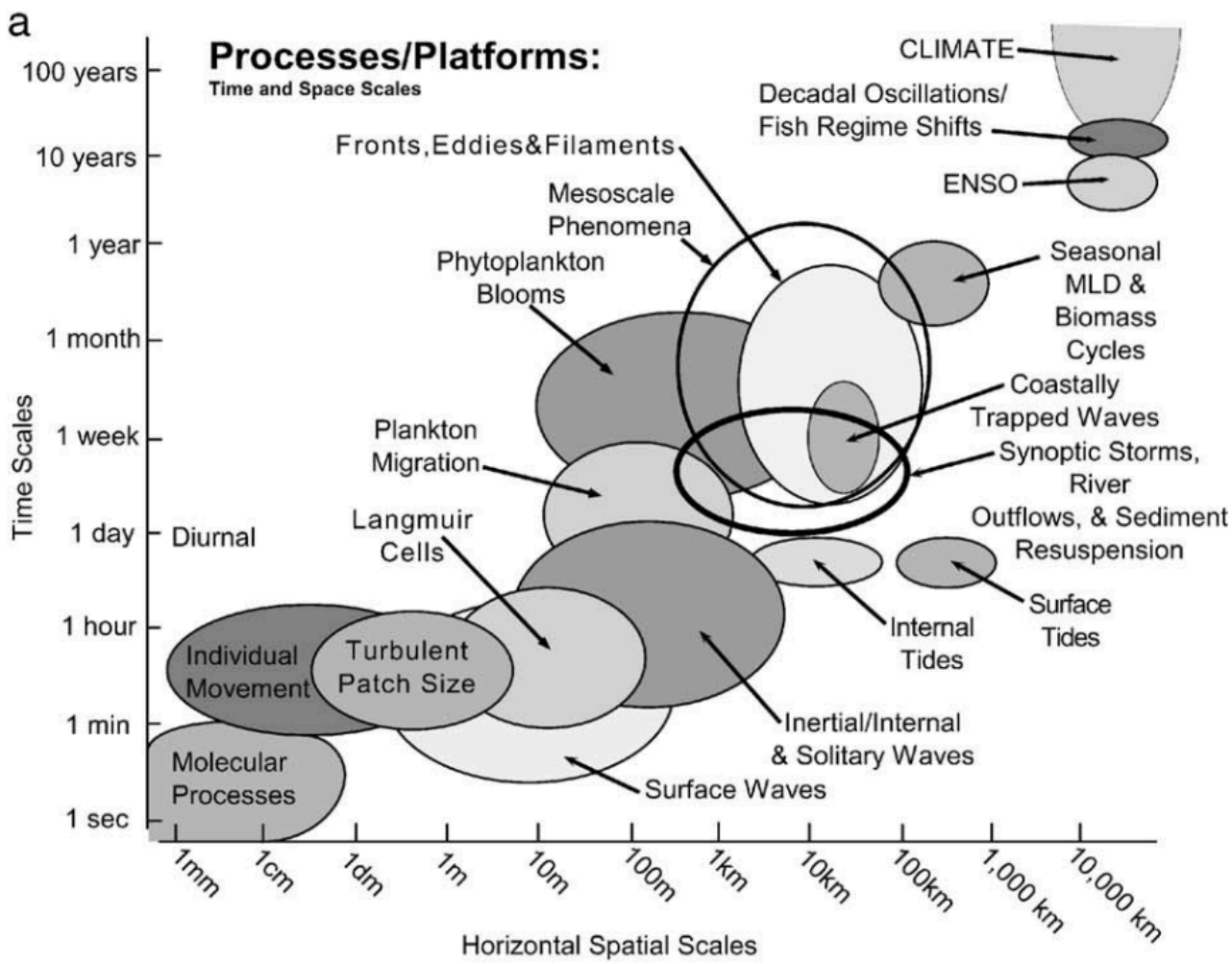
Type of waves	Disturbing force	Restoring force	Horizontal wavelength	Period
Acoustic wave	Any motion	Pressure (compressibility)	1 mm - 10 km	< 1 s
Capillary wave	Surface motion	Surface tension	1 mm - 1 cm	< 1 s
Surface gravity wave	Wind, boat, earthquake, …	Gravity	1 cm - 100 km	1 s - 1 day
Rossby wave	Atmospheric forcing (wind, heat flux)	Potential vorticity (variation of Coriolis with latitude)	100 km - 1000 km	Days - months
Kelvin wave	Atmospheric forcing	Pressure gradient + Coriolis	10 km - 100 km	Days - months
Internal wave	Tides, winds, current instability, …	Gravity (stratification) + Coriolis	1 m - 100 km	10 s - 1 day

Ocean Waves



Schematic energy spectrum of ocean variability [Leblond & Mysak]

Ocean Waves


a
Processes/Platforms:
Time and Space Scales
Time Scales
100 years
10 years
1 year
1 month
1 week
1 day
1 hour
1 min
1 sec
Diurnal
CLIMATE
Decadal Oscillations/
Fish Regime Shifts
ENSO
Fronts,Eddies&Filaments
Mesoscale
Phenomena
Phytoplankton
Blooms
Seasonal
MLD &
Biomass
Cycles
Coastally
Trapped Waves
Synoptic Storms,
River
Outflows, & Sediment
Resuspension
Plankton
Migration
Langmuir
Cells
Surface
Tides
Internal
Tides
Individual
Movement
Turbulent
Patch Size
Inertial/Internal
& Solitary Waves
Molecular
Processes
Surface Waves
1mm
1cm
1dm
1m
10m
100m
1km
10km
100km
1,000 km
10,000 km
Horizontal Spatial Scales

I. WAVES

I.1. Introduction

I.2. General properties of Waves

I.3. Different type of ocean waves

I.3.1 Surface Gravity Waves

I.3.2 Internal Waves

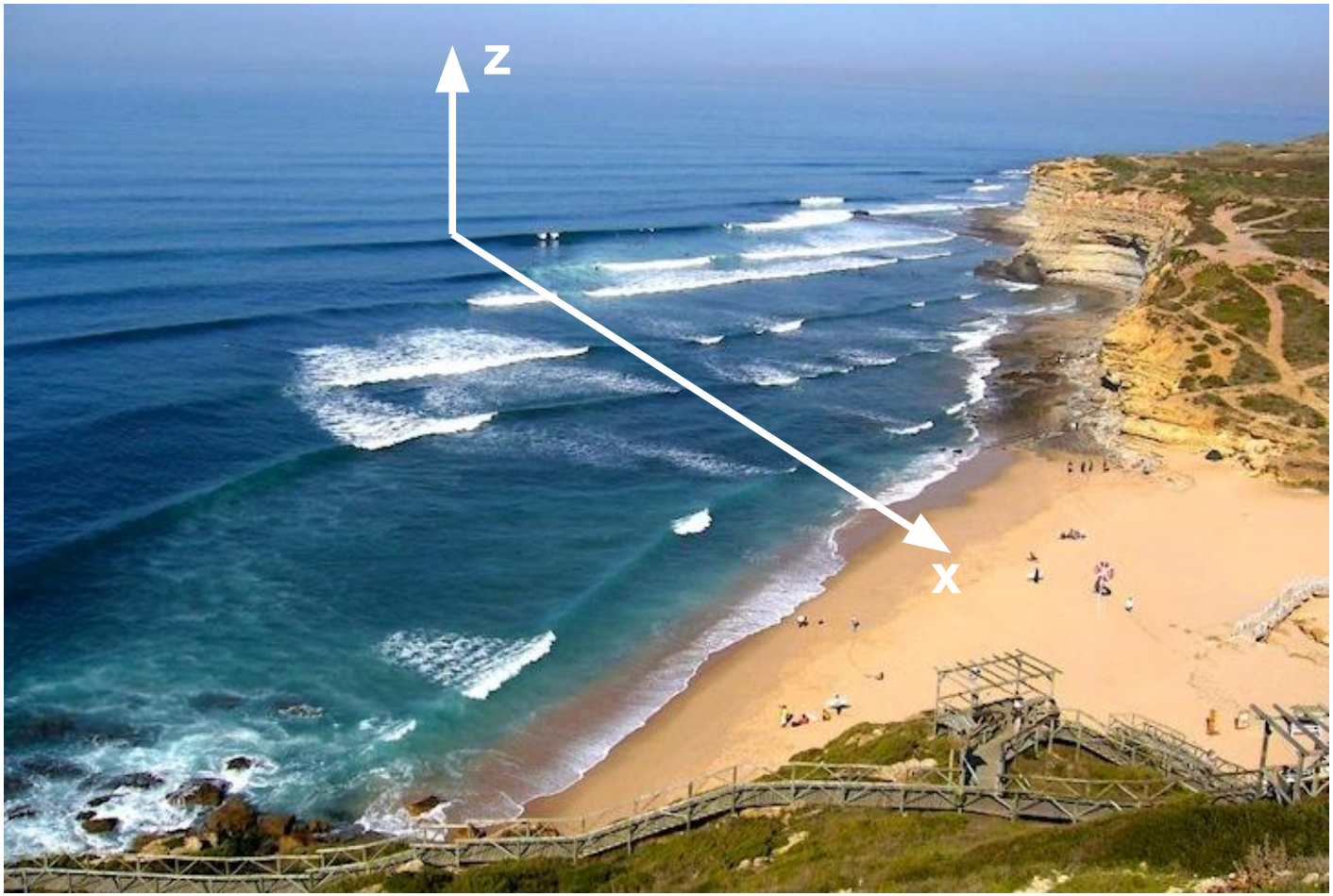
I.3.3 Acoustic Waves

I.4. Ray Theory

I.2.1 Simple Wave example

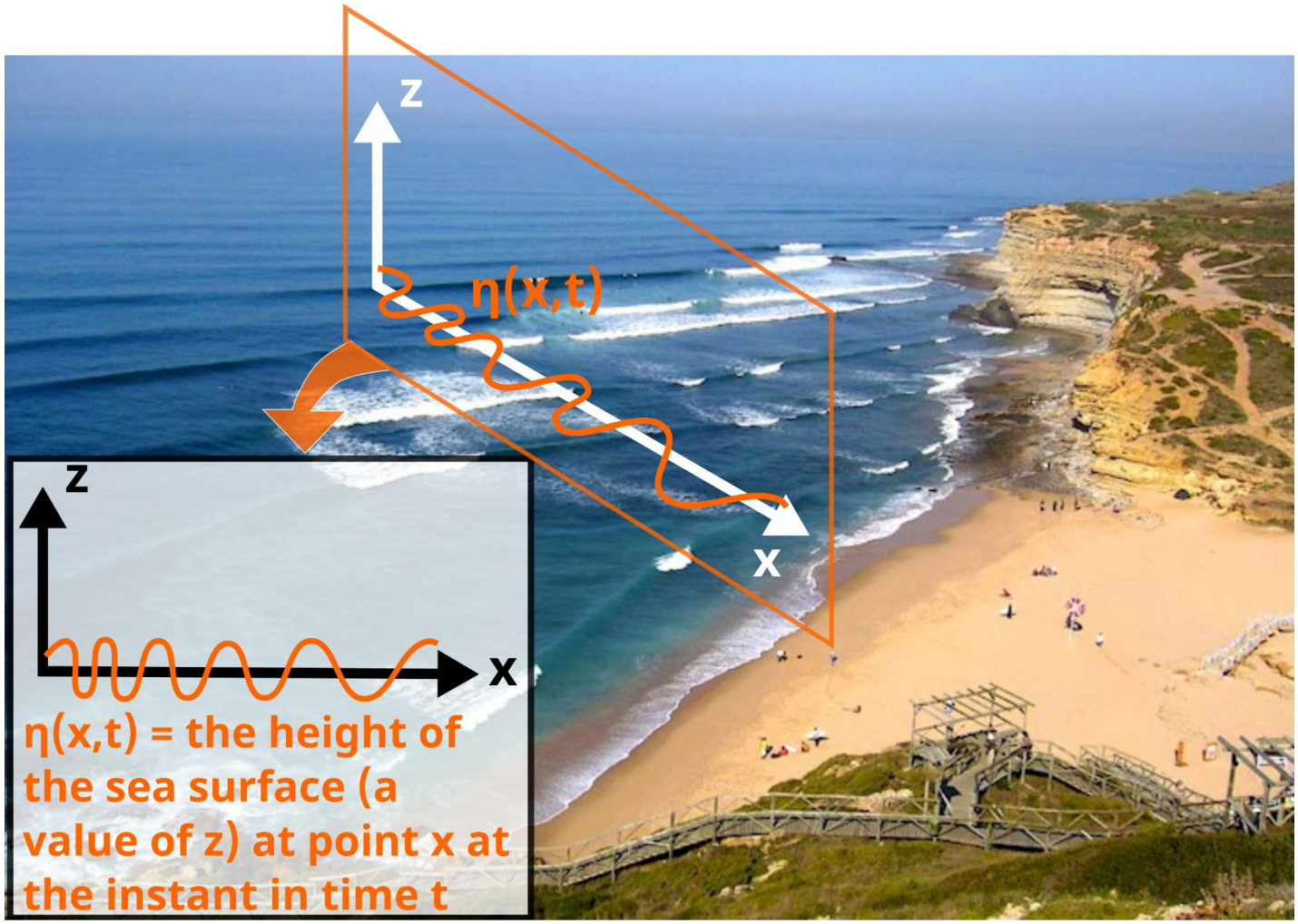
1D waves: describing a single wave

The sea surface height (η) depends on both the position in space at which we observe it, and the time at which we observe it.



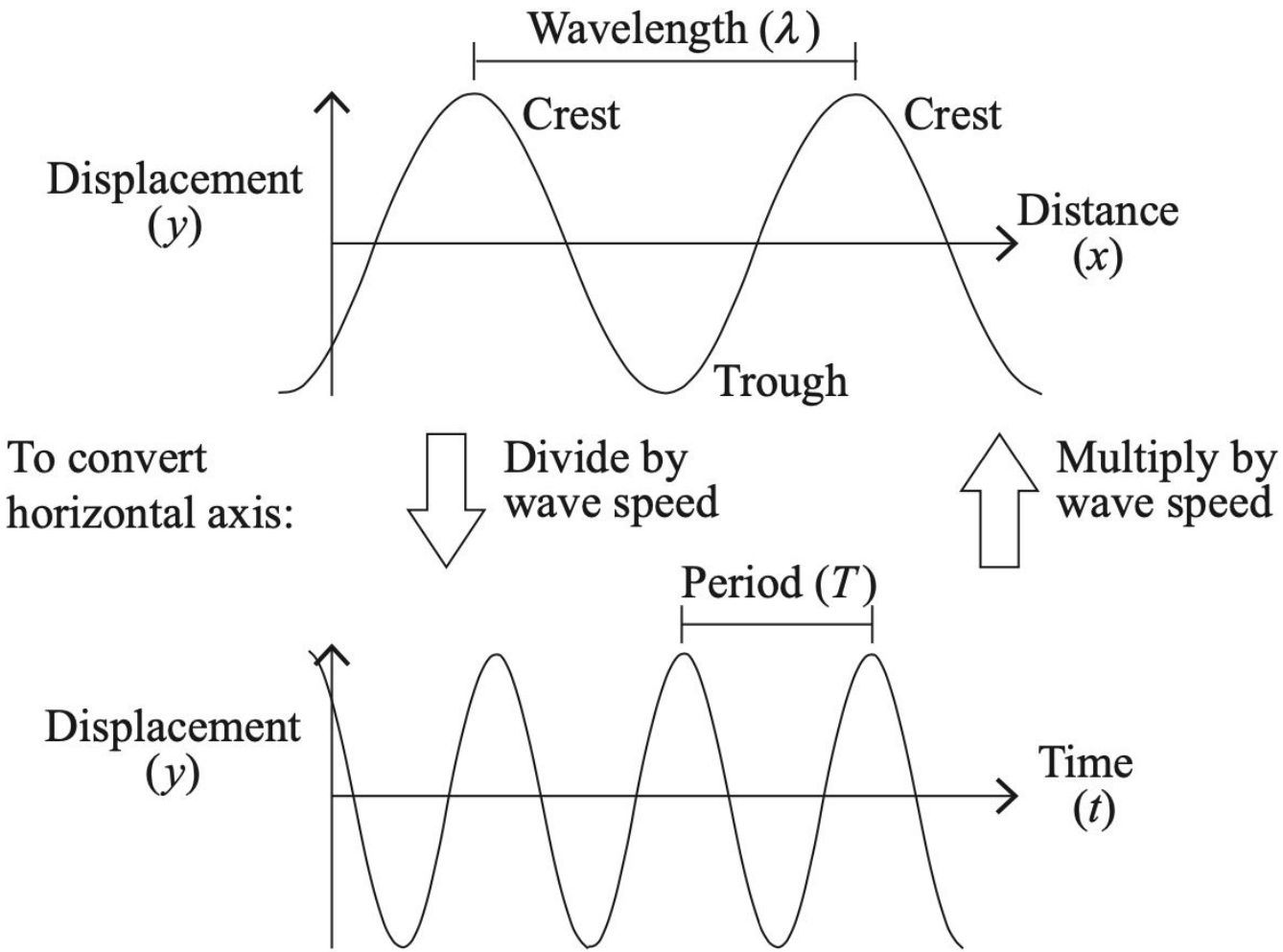
1D waves: describing a single wave

The sea surface height (η) depends on both the position in space at which we observe it, and the time at which we observe it.



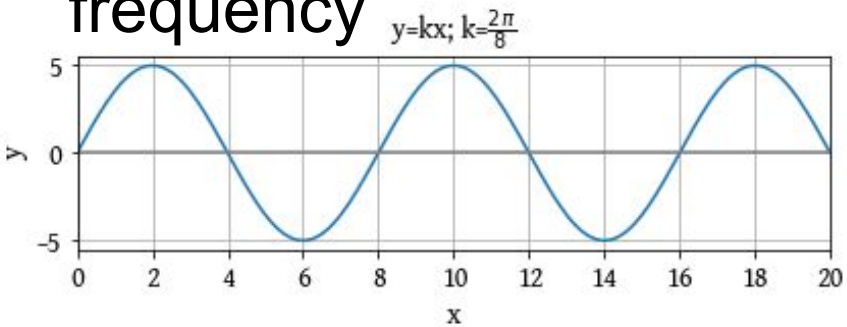
I.2.1 Simple Wave example

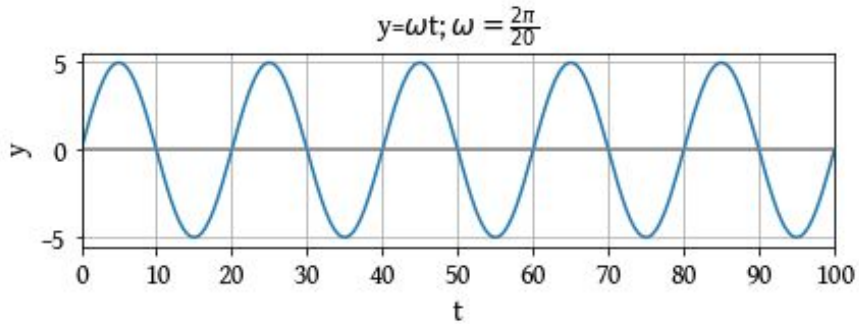
1D waves: describing a single wave



I.2.1 Simple Wave example

1D waves: phase, wavenumber and angular frequency





The phase tells us which part of a wave we are looking at at any given time : this could be a crest, a trough, or something in between. It has units of radians and can vary between 0 and 2π

The wavenumber tells us how much the phase advances in a given amount of distance

$$k = \frac{2\pi}{\lambda} = \left.\frac{\Delta\phi}{\Delta x}\right|_t$$

The angular frequency tells us how much the phase advances in a given amount of time

$$\omega = \frac{2\pi}{T} = \left.\frac{\Delta\phi}{\Delta t}\right|_x$$

I.2.1 Simple Wave example

1D waves: describing a single wave

The **wavefunction** describes the displacement at every point in space and time: $\eta = f(x, t)$

The total change in phase at a given point in space and time is given by the combination of the variations due to the individual changes in phase with space and time:

$$\Delta\phi = \Delta\phi\big|_t + \Delta\phi\big|_x$$
$$= k\Delta x + \omega\Delta t$$

The functional form of the wavefunction is thus: $\eta(x, t) = f(\underline{kx \pm \omega t})$

The displacement is determined by:

f: a function which describes the shape of the wave

and the phase of the wave

Recalling that speed = distance / time, we can define the phase speed of the wave:

$$c_p = \frac{\lambda}{T} = \frac{\frac{2\pi}{k}}{\frac{2\pi}{\omega}} = \frac{\omega}{k}$$

I.2.1 Simple Wave example

- **See notes:**
  - *Simple Wave example*

- Mathematically two main classes of waves:

Hyperbolic waves and **dispersive waves**

Hyperbolic waves

1. Hyperbolic waves are formulated in terms of hyperbolic partial differential equations, for example:

$$\eta_t + c\nabla\eta = 0 \qquad \eta_{tt} - c^2\nabla^2\eta = 0$$

Hyperbolic waves

1. Hyperbolic waves are formulated in terms of hyperbolic partial differential equations, for example:

$$\eta_t + c\nabla\eta = 0 \qquad\qquad \eta_{tt} - c^2\nabla^2\eta = 0$$

- With general solutions in the form:

$$\eta = f(x - ct) \qquad\qquad \eta = f(x - ct) + g(x + ct)$$

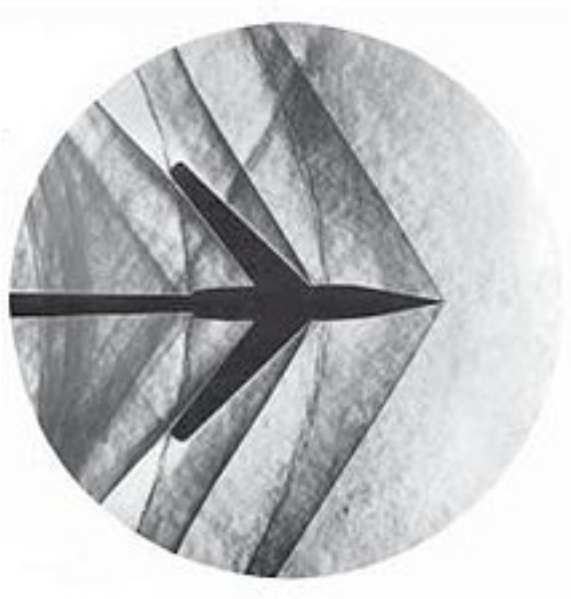
- Very frequent in acoustics, elasticity, electromagnetism, etc.

Hyperbolic waves

1. Examples of **Hyperbolic waves**

- Flood wave, tidal bores

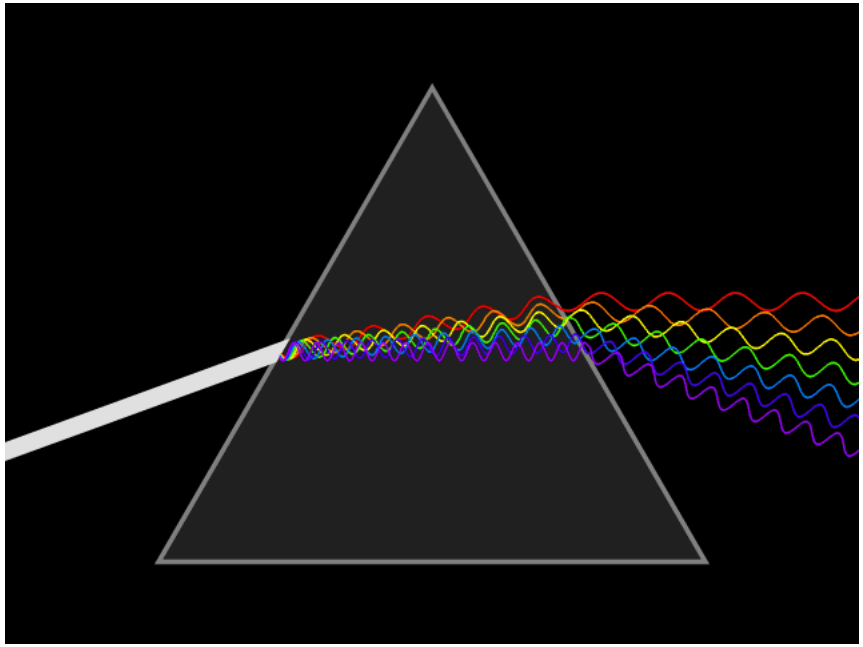
- Shock wave



Dispersive waves

- **Dispersive waves** come from a variety of PDEs and are characterised by their dispersion relation $\omega = f(k, \text{environment})$, with ω their frequency and k their wavenumber. They are observed as the superposition of waves where the Fourier components (corresponding to specific ω) propagate at different speeds.

  The wave (phase) speed is $c_\phi = \dfrac{\omega}{k}$ and the group speed is $c_g = \dfrac{\partial \omega}{\partial k}$.



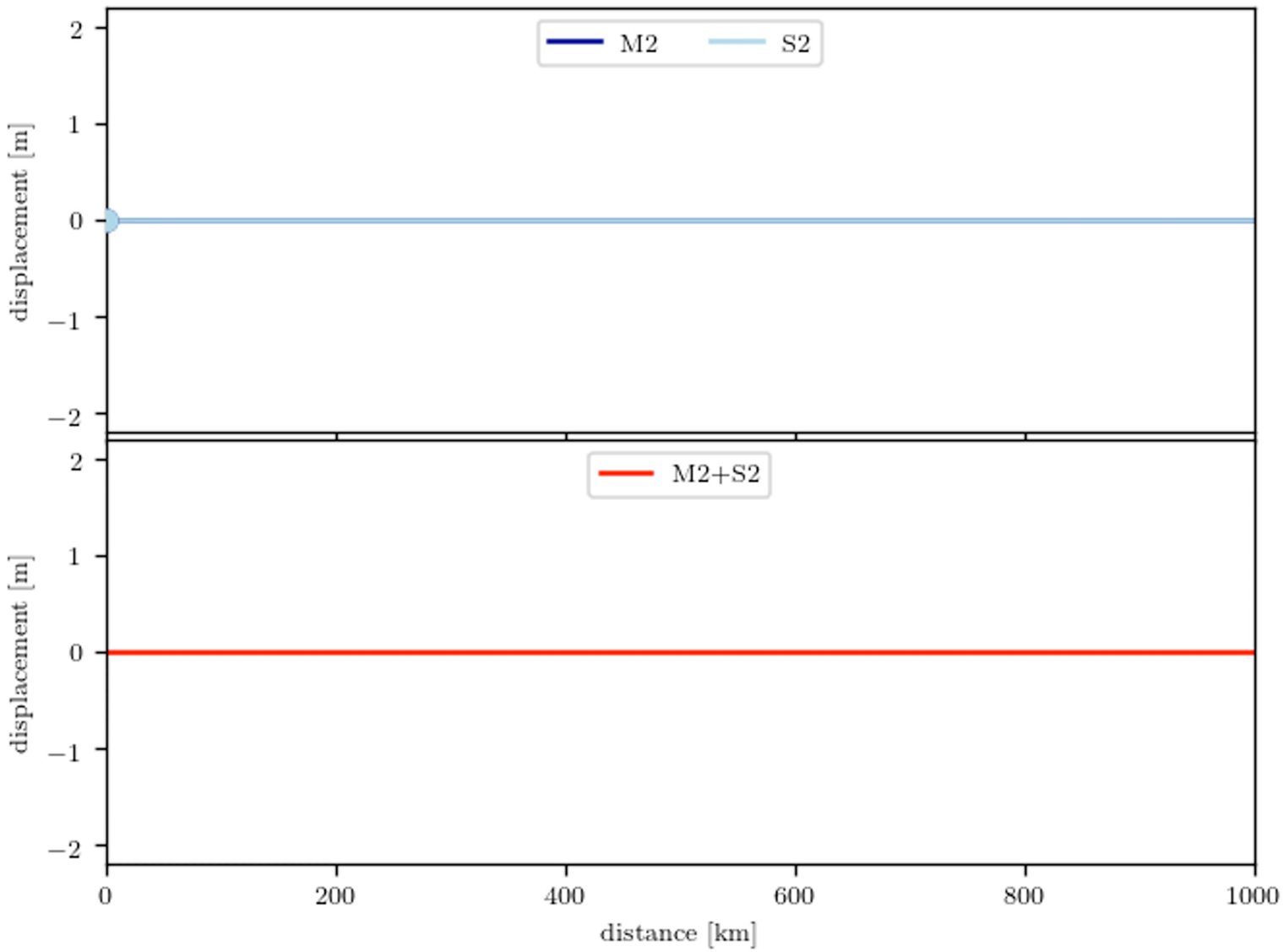
Dispersive waves

- **Dispersive waves** in the ocean: superposition of tidal waves,

  "M2 wave" (M=Moon): $\eta_{M2} = A_{M2} \exp\left(i(\omega_{M2} t - k_{M2} x)\right)$

  "S2 wave" (S=Sun): $\eta_{S2} = A_{S2} \exp\left(i(\omega_{S2} t - k_{S2} x)\right)$

I.2. General properties



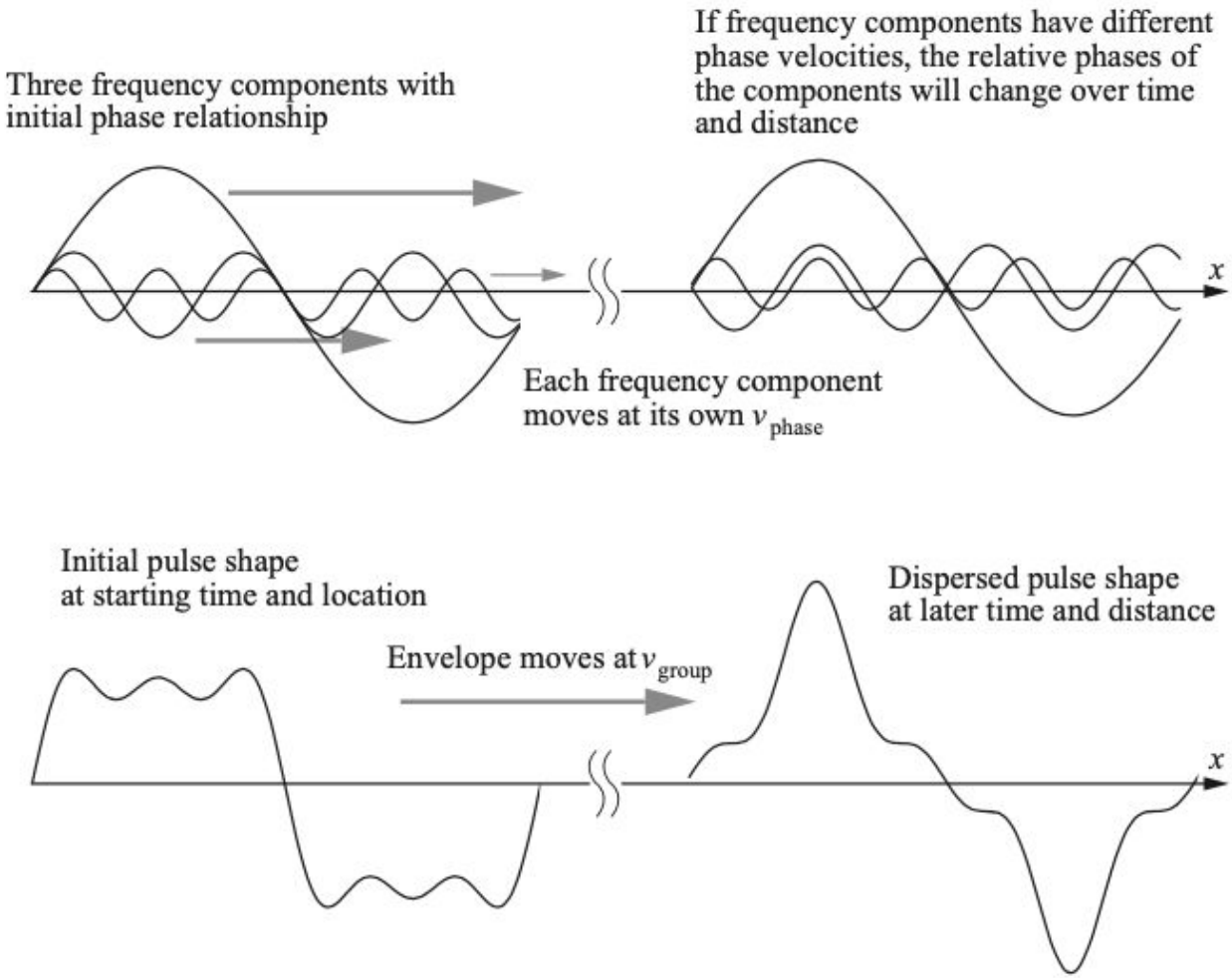
The dispersion relation

If ω, the angular frequency is a function of k, the wavenumber, then waves with different wavelengths propagate at different frequencies. The relationship between ω and k: $\omega(k)$ is called the dispersion relation.

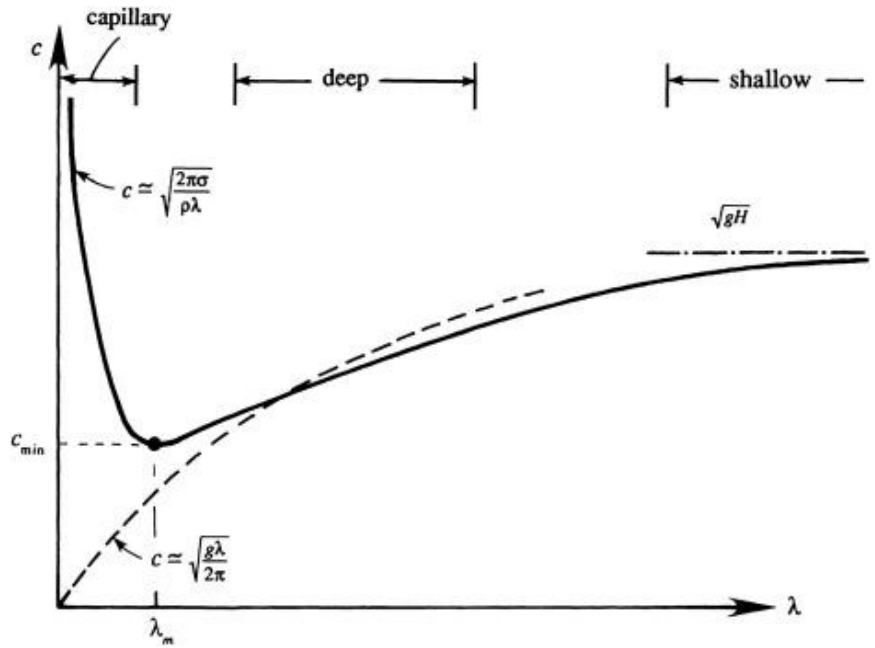
The dispersion relation is hugely important: it is different for different types of waves and asserts the physics of the wave that we are considering.

For any given type of wave, the phase speed, $\frac{\omega}{k}$, and the group speed, $\frac{\partial \omega}{\partial k}$ are not necessarily fixed values: they can vary depending on the value of ω and k that we are considering. In this case, the envelope can change shape as the wave propagates: the form of the wave changes over time

The dispersion relation



The dispersion relation



We plot the dispersion relation for a given wave as either:

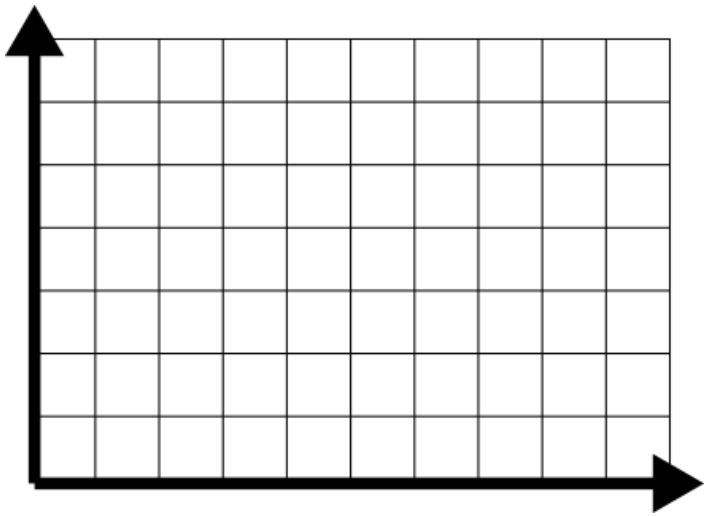
ω (y-axis) against k (x-axis)

c_p (y-axis) against λ (x-axis)

How will our figure look if…

- $\frac{\omega}{k} = \frac{\partial \omega}{\partial k}$?

What would this mean physically?



I. WAVES

I.1. Introduction

I.2. General properties of Waves

I.3. Different type of ocean waves

I.3.1 Surface Gravity Waves

I.3.2 Internal Waves

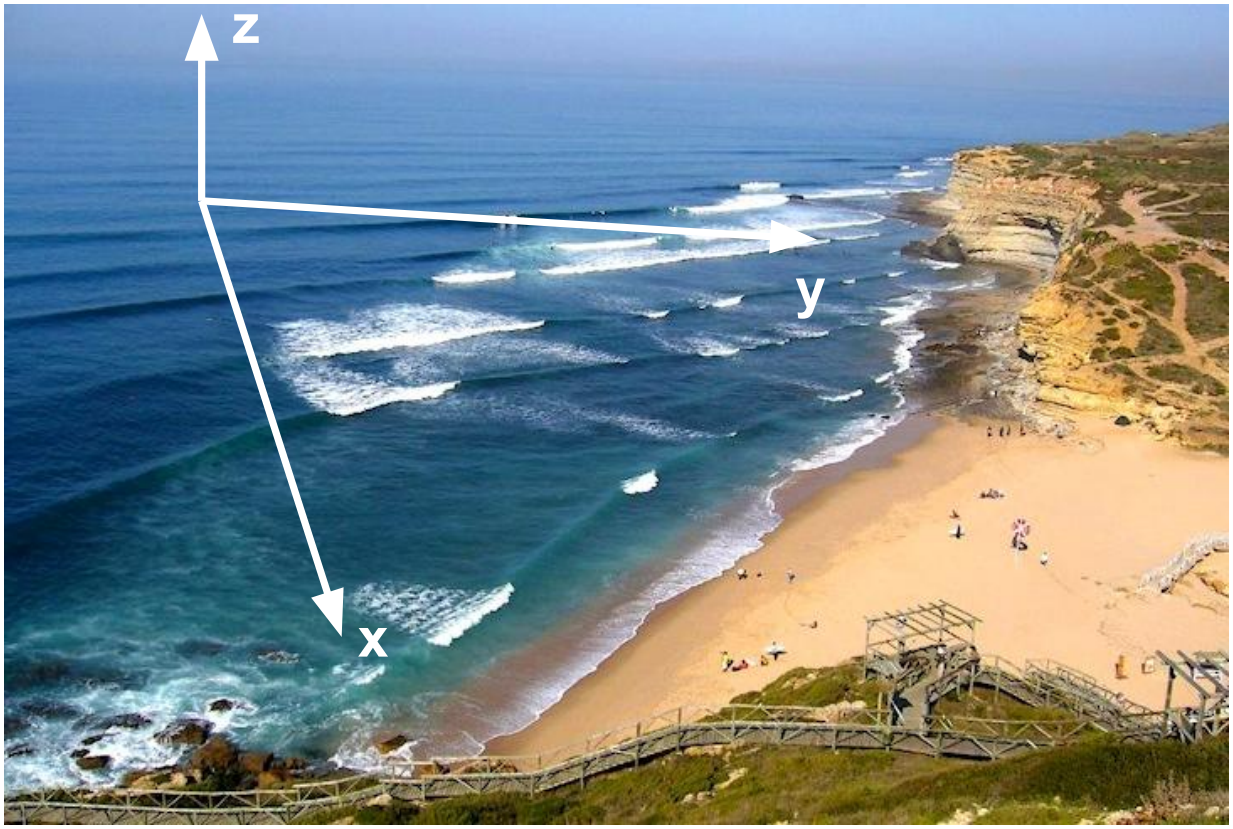
I.3.3 Acoustic Waves

I.4. Ray Theory

I.2.2 Planar Wave example

2D and 3D waves

So far we have just talked about 1D waves. Everything that we have seen so far stays exactly the same for 2D and 3D waves: we just need to generalize the wavenumber by using a vector form

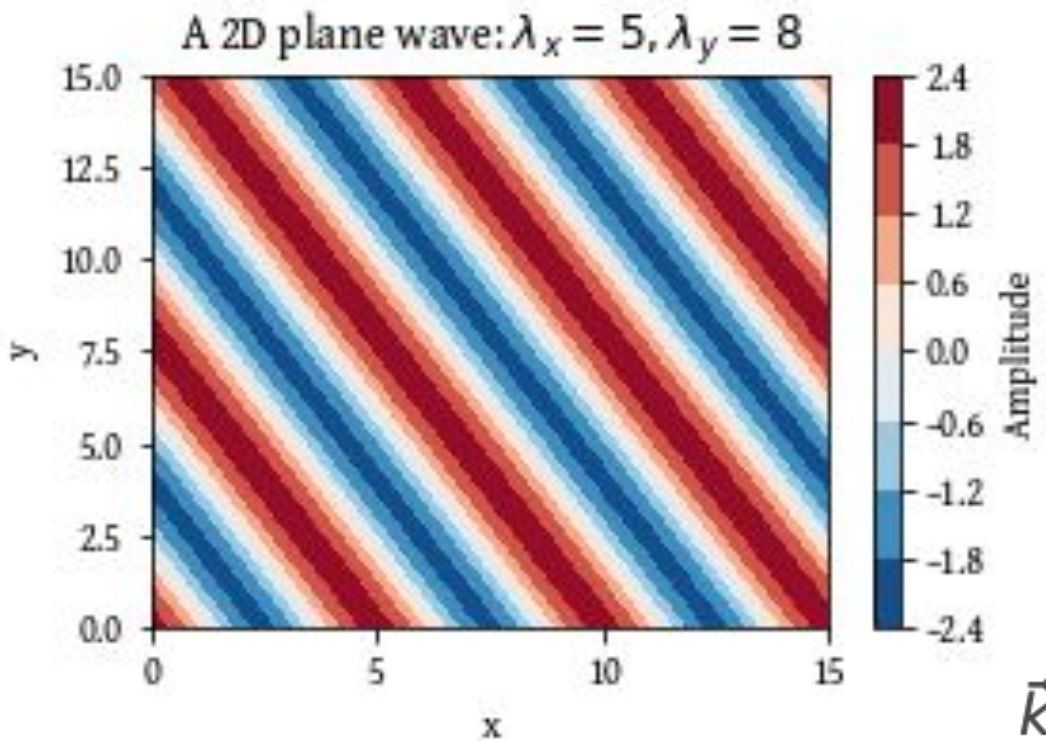


I.2. General properties

- **See notes:**
  - *Planar Wave example*

I.2. General properties

I.2.2 Planar Wave example



Wavenumbers for a 3D wave:

$$k_x = \frac{2\pi}{\lambda_x} \qquad k_y = \frac{2\pi}{\lambda_y} \qquad k_z = \frac{2\pi}{\lambda_z}$$

also sometimes written as:

$$k = \frac{2\pi}{\lambda_x} \qquad l = \frac{2\pi}{\lambda_y} \qquad m = \frac{2\pi}{\lambda_z}$$

$$\vec{k} = (k_x, k_y, k_z) \quad \text{or:} \quad \vec{k} = (k, l, m)$$

Waveform of a 3D wave: $\eta = A\cos(\vec{k}\cdot\vec{x} - \omega t)$

Wavenumber is a vector, wavelength is <u>not</u> a vector (wavelength is the distance between two troughs / two crests)

I. WAVES

I.1. Introduction

I.2. General properties of Waves

I.3. Different type of ocean waves

I.3.1 Surface Gravity Waves

I.3.2 Internal Waves

I.3.3 Acoustic Waves

I.4. Ray Theory

I.3. Different type of ocean waves

I.3.1 Surface gravity waves

	Type	Period	Wavelength	Main Forcing
Short waves	Capillary–Gravity	< 0.1 s	~cm	Wind, surface tension
	Wind waves	1–10 s	m–100 m	Local wind
	Swell	10–30 s	100–500 m	Remote wind
Long waves	Infragravity	30–300 s	100s m–km	Wave–wave interaction
	Seiche	minutes–hours	Basin-scale	Resonance
	Tsunami	10 min–hours	100–200 km	Seismic, landslides
	Storm surge	hours–days	Basin-scale	Wind + pressure
	Tide	12–24 h	Ocean-basin	Moon & Sun gravity

I.3.1 Surface gravity waves

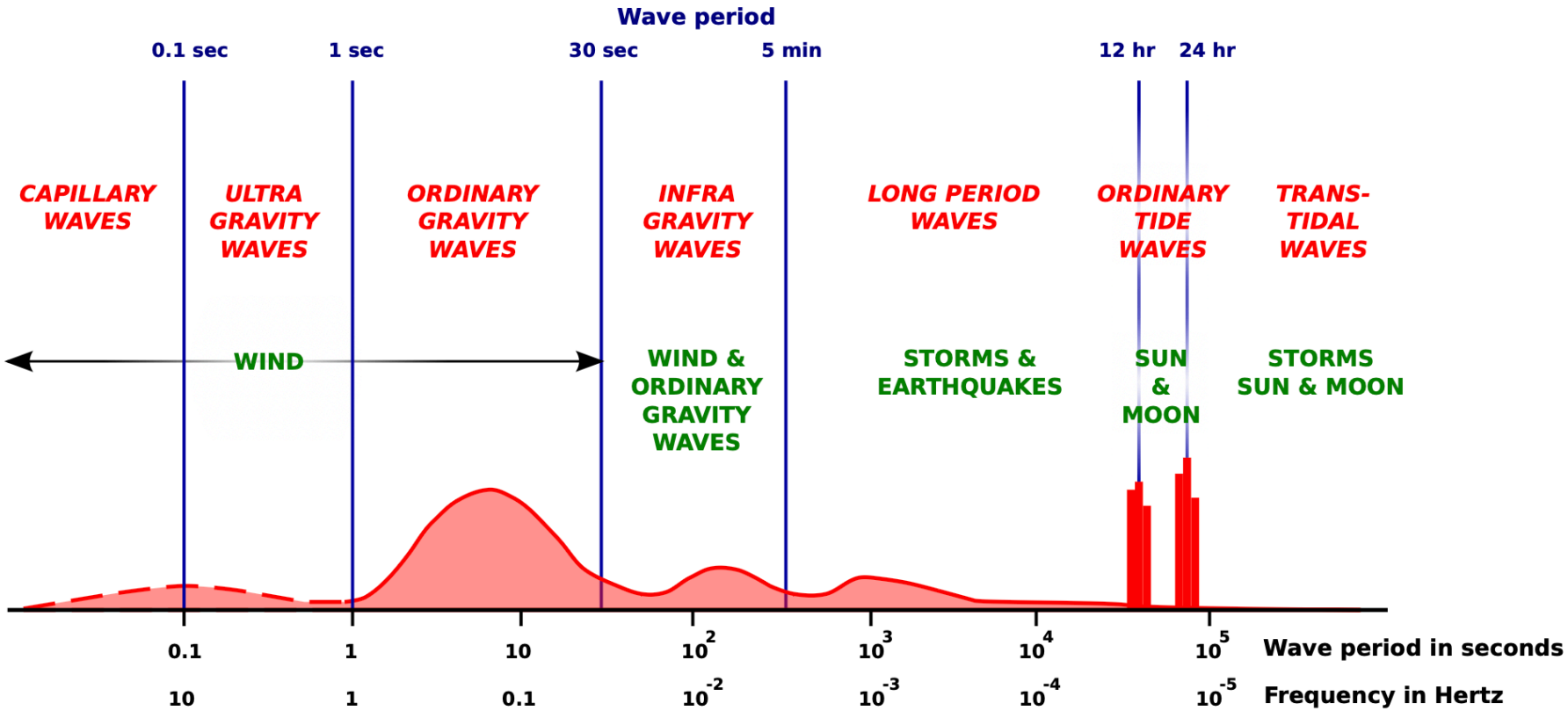


Figure 1.1: Classification of ocean surface waves with usual names in red, as a function of wave periods (x-axis), with dominant forcing mechanisms in green. Adapted from Munk (1950).

Tools for the formal analysis of the motion of a surface gravity wave

Analysis of 2D motion

We will analyse a number of different wave scenarios mathematically to describe the motion (to begin with: surface gravity waves on an infinitely deep / shallow / finite-depth ocean). The general approach that we will use is the following:

1. Identify the equations that describe the system
2. Identify our boundary conditions
3. Simplify the equations that describe the system so that we can solve them
4. Choose a trial solution for our system of equations and solve to find expressions for our variables

Often we will be trying to find expressions for x, y, u, v and/or the dispersion relation.

Before starting the mathematical description, we will review some of the ingredients that we are going to use:

- the equations of motion
- types of surface boundary conditions
- the use of linearisation to simplify an equation
- the velocity potential

Equations to describe the system

Some reminders for the equations that we will frequently be using, met previously during Fluids 1:

- Euler's equation: $$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}$$

$$\boldsymbol{g} = -\nabla\chi$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \chi\right)$$

- Conservation of mass: $$\frac{\partial \rho}{\partial t} + (\boldsymbol{u}\cdot\nabla)\rho = 0$$

Some possible surface boundary conditions

At the free surface, we can have **kinematic** and/or **dynamic** boundary conditions.

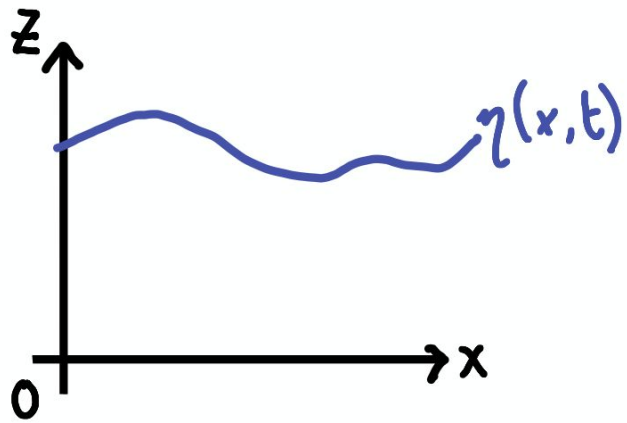
A kinematic boundary condition relates the motion of the free surface to the fluid velocities at that surface

(see next slide…)

A dynamic boundary condition evaluates the force balance at the free surface

The dynamic boundary condition is often very simple: for surface waves we will use the condition that the pressure at the fluid surface is equal to the atmospheric pressure. For very small distances (capillary waves), note that this condition will need to be modified: gravity is no longer the dominant force, and we will need to take into account the effects of surface tension.

The kinematic b.c. at the free surface (approach #1)



We begin with an equation that describes the position of the surface:

$$z_{surf} = \eta(x, t)$$

where $\eta(x, t)$ is a function that tells us the shape of the surface, and how the shape changes over time.

Now, we can rewrite this equation as: $z_{surf} - \eta(x, t) = 0$

This equation will be true for any particle lying on the free surface.

We can assign this expression to a new function, F:

$$F(z_{surf}, x, t) = z_{surf} - \eta(x, t) = 0$$

Since F=constant, it must be a conserved quantity. When we take the material derivative, this will always equal zero.

The kinematic b.c. at the free surface (approach #1)

For our function, F: $F(z_{surf}, x, t) = z_{surf} - \eta(x, t) = 0$

we will use the fact that the material derivative is zero for a conserved quantity:

$$\frac{\partial F}{\partial t} + \vec{u} \cdot \vec{\nabla} F = 0 \longrightarrow \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + w\frac{\partial F}{\partial z_{surf}} = 0$$

We can evaluate:

$$\frac{\partial F}{\partial t} = -\frac{\partial \eta}{\partial t} \qquad u\frac{\partial F}{\partial x} = -u\frac{\partial \eta}{\partial x} \qquad w\frac{\partial F}{\partial z_{surf}} = w$$

To obtain:

$$\boxed{\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w}$$

The is the kinematic boundary condition at the free surface. It says that if a particle is at the free surface at some given time, then it stays at the free surface for all other times.

Linearisation of an equation

If equations contain non-linear terms they are normally very difficult (or impossible) to solve analytically. This will be a problem that we will face in our analyses. So that we can proceed with the analyses, we will often try to make life easier for ourselves by linearising the equations.

Linearising an equation just means that we will ignore any terms that are non-linear in variables that are of small magnitude. Non-linear terms are those that consist of 2 or more terms multiplied together. **It only works when we are considering values that are small.**

Some examples: the following terms are all non-linear…

$$u^2, u\frac{\partial u}{\partial x}, v\frac{\partial u}{\partial x}, \left(\frac{\partial u}{\partial x}\right)^2, \left(\frac{\partial u}{\partial x}\right)^3, \ldots$$

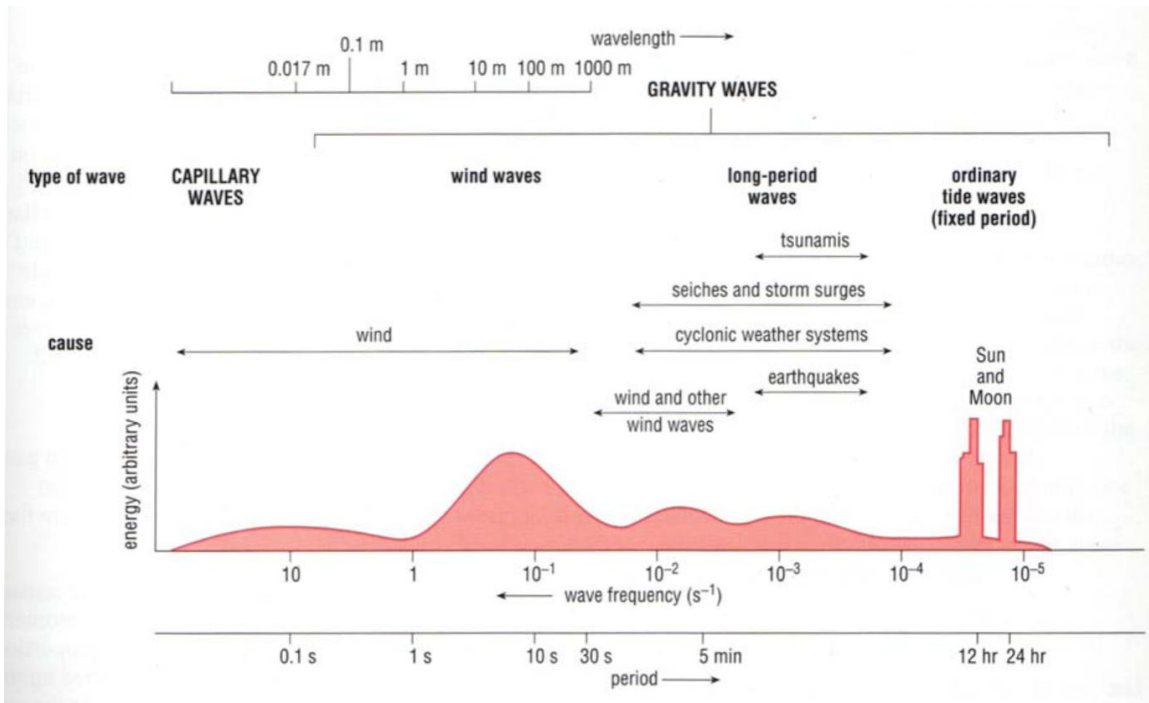
Note that the terms below are linear: they are not products of terms…

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \ldots$$

Linearisation of an equation: example

Why (and when?) are we allowed to do this? If we are considering terms that are small, then the product of two small terms will be really, really small. These product terms then contribute very little to the total, and can be ignored to obtain a solution that describes the contribution of the dominant terms in the equation.

Let's consider the kinematic boundary condition at the free surface that we just derived, with some typical values:



$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w$$

If we use:

$$\eta = A\cos(kx - \omega t)$$

with:

$$\lambda = 100\,m$$

$$t = 15\,s$$

$$A = 0.5\,m$$

we can estimate the terms on the left hand side…

Linearisation of an equation: example

The kinematic boundary condition at the free surface: $$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w$$

For a wave of form: $$\eta = A\cos(kx - \omega t)$$

We obtain: $$\frac{\partial \eta}{\partial t} = \omega A \sin(kx - \omega t)$$ max value = ωA

$$\frac{\partial \eta}{\partial x} = -kA \sin(kx - \omega t)$$ max value = kA

We haven't yet seen the expression for the speed, u, but we will soon show that it is proportional to ωA . So we can estimate the size of the two terms for our chosen values of $\lambda = 100\,m$, $t = 15\,s$ and $A = 0.5\,m$

$$\frac{\partial \eta}{\partial t} \sim \frac{2\pi}{15} \times 0.5 \sim 0.21 m/s$$

$$u\frac{\partial \eta}{\partial x} \sim \frac{2\pi}{15} \times \frac{2\pi}{100} \times 0.5^2 \sim 0.007 m/s$$

So the first term is approximately 30 times larger than the second. We can expect to still have a good estimate of w if we estimate it using only the first term

Linearisation of an equation

As an example, let's consider Euler's equations.
Which terms could we get rid of by linearising?

x-direction:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

y-direction:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

z-direction:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

mass continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

The velocity potential

During our analyses, we will often use the velocity potential, ϕ. Its definition is:

$$\vec{u} = \vec{\nabla}\phi$$

Physically, this means that the velocity potential is a scalar field, the gradient of which gives us the velocity vector. This might seem like an abstract idea, but we will only be using the velocity potential as a mathematical tool to make life easier for ourselves.

We can define the velocity potential if we have an irrotational fluid. In that case, the vorticity is zero, by definition:

$$\vec{\xi} = \vec{\nabla} \wedge \vec{u} = 0$$

According to the rules of vector calculus, we can thus write **u** as the gradient of a scalar field, since, for any scalar field, s, we have:

$$\vec{\nabla} \wedge \vec{\nabla} s = 0$$

Note: this is why irrotational flow is sometimes called potential flow

The velocity potential

Why are we doing this abstract-seeming step? The advantage is that we can reduce a vector, **u**, comprising three unknown components (u,v,w) into a single unknown scalar quantity ϕ

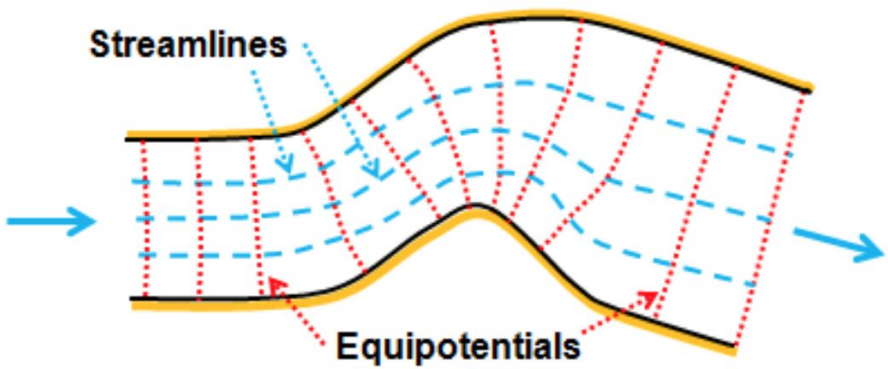
Although the physical intuition about what this quantity represents might seem complicated, mathematically this step can make things much simpler for us. We can solve our equations to find an expression for the scalar quantity ϕ, and then we can find the three components of the velocity very simply at the end of our analysis by differentiating this expression.

For information: the lines of constant velocity potential (called equipotentials) are perpendicular to the lines of constant streamfunction, ψ (called streamlines). For the velocity potential, we have:

$$u = \frac{\partial \phi}{\partial x} \quad , \quad v = \frac{\partial \phi}{\partial y}$$

and for the streamfunction:

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x}$$

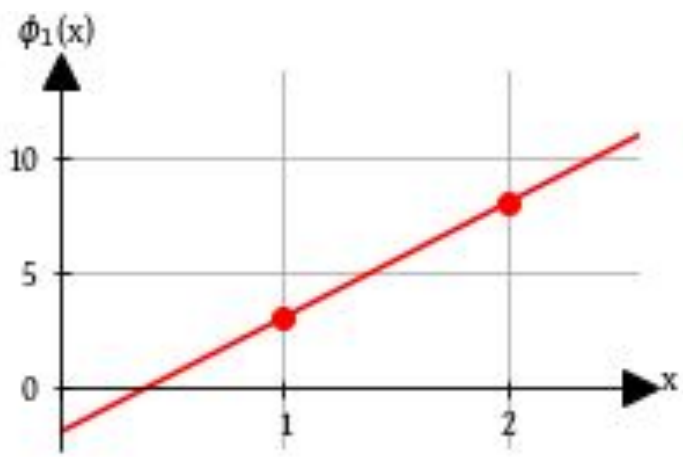


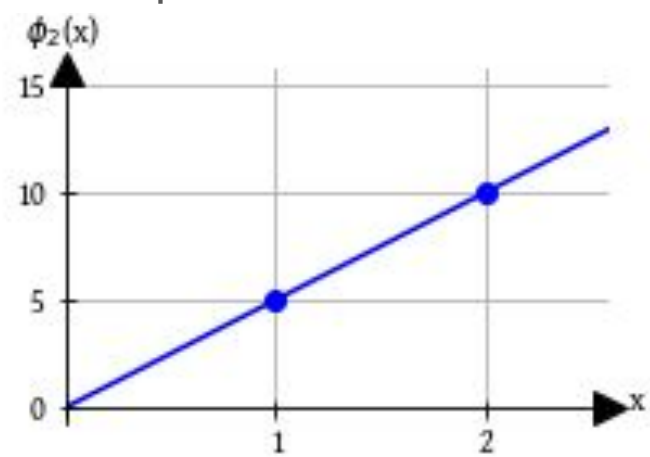
The velocity potential

The velocity potential varies over space, and can also vary over time, to give us values of (u,v,w) that change over time.

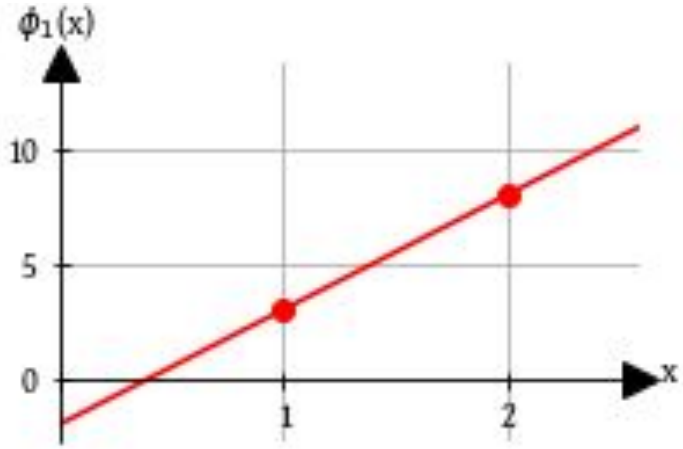
Consider first a given moment in time, t_0. Because we are only interested in the spatial gradients of ϕ (because they yield the components of **u**) the velocity potential for **u** is not unique: we could add any constant value to all points of the field, and we would still obtain the same solution for **u**.

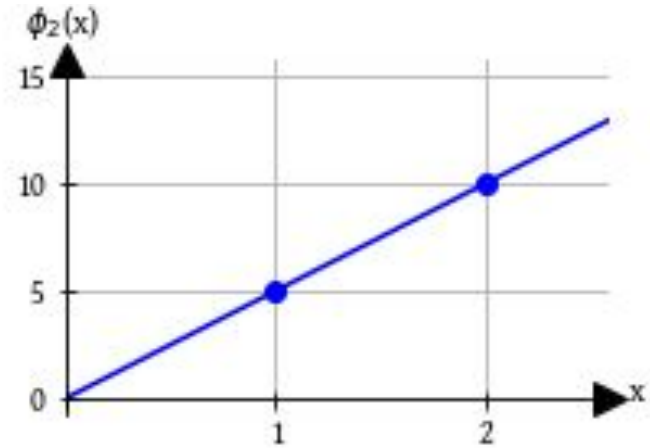
Example: imagine that we have a 1D field with u=5 m/s at all points x at time t_0. The following 2 fields of ϕ will both give us the required result:





The velocity potential





$$\phi_1(x=1, t=t_0) = 3$$

$$\phi_1(x=2, t=t_0) = 8$$

$$\left.\frac{\Delta\phi_1}{\Delta x}\right|_{t=t_0} = \frac{8-3}{2-1} = 5$$

$$\phi_2(x=1, t=t_0) = 5$$

$$\phi_2(x=2, t=t_0) = 10$$

$$\left.\frac{\Delta\phi_2}{\Delta x}\right|_{t=t_0} = \frac{10-5}{2-1} = 5$$

$$\phi_2(x, t_0) = \phi_1(x, t_0) + 2$$

If we add a constant value to the function, it has no effect on the gradient – it only changes the intercept. And **we are interested only in the gradient** of the velocity potential, which we will use to obtain velocities.

The velocity potential

We can apply this argument at every time step: for any given time step, we can add a constant value to ϕ and still obtain the same result for **u**.

But: we do not need to add the same constant value at every time step. For example, if we added 2 at time step t_0, as in the previous example, this does not in any way constrain the value that we can add at the next time step, t_1. We can still add any arbitrary value that we choose at time t_1 and obtain the correct value of **u** at that time step.

The result of this is that we can add a time series of values to ϕ. For each value of the time series, we can add that constant value to all points in the 3D field ϕ. This can sometimes be useful, because it can allow us to get rid of certain terms by including them in ϕ.

Formally, we can say that: if $\phi(x, y, z, t)$ is a solution for the velocity potential of **u**, then $\phi(x, y, z, t) + G(t)$ is also a solution for the velocity potential of **u**, where G(t) is a time series of values.

Bernoulli's equation for unsteady, incompressible flow

The Bernoulli equation with which you are probably very familiar:

$$P + \frac{1}{2}\rho v^2 + \rho g z = constant$$

makes the hypothesis that the flow is **steady** (= not dependent on time).

By manipulating the Euler equation, and substituting in our new expression for the velocity potential, we can derive Bernoulli's equation for **unsteady**, incompressible flow. It enables us to make a link between the pressure and the fluid dynamics, and we will use this equation when we define the dynamic surface boundary condition.

Bernoulli's equation for unsteady, incompressible flow

We will start with Euler's equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla}\left(\frac{p}{\rho} + \chi\right) \quad \text{where} \quad \vec{g} = -\vec{\nabla}\chi$$

and use the vector calculus identity:

$$\vec{u} \cdot \vec{\nabla}\vec{u} = (\vec{\nabla} \wedge \vec{u}) \wedge u + \vec{\nabla}\left(\frac{1}{2}\vec{u}^2\right)$$

to obtain:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{\nabla} \wedge \vec{u}) \wedge \vec{u} = -\vec{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}\vec{u}^2 + \chi\right)$$

For irrotational flow, the second term on the LHS vanishes:

$$\frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}\vec{u}^2 + \chi\right)$$

Bernoulli's equation for unsteady, incompressible flow

Using the velocity potential, $\vec{u} = \vec{\nabla}\phi$, we then obtain:

$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right) = -\vec{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}\vec{u}^2 + \chi\right)$$

and finally, integrating over ∇, we obtain Bernoulli's equation for unsteady, incompressible flow:

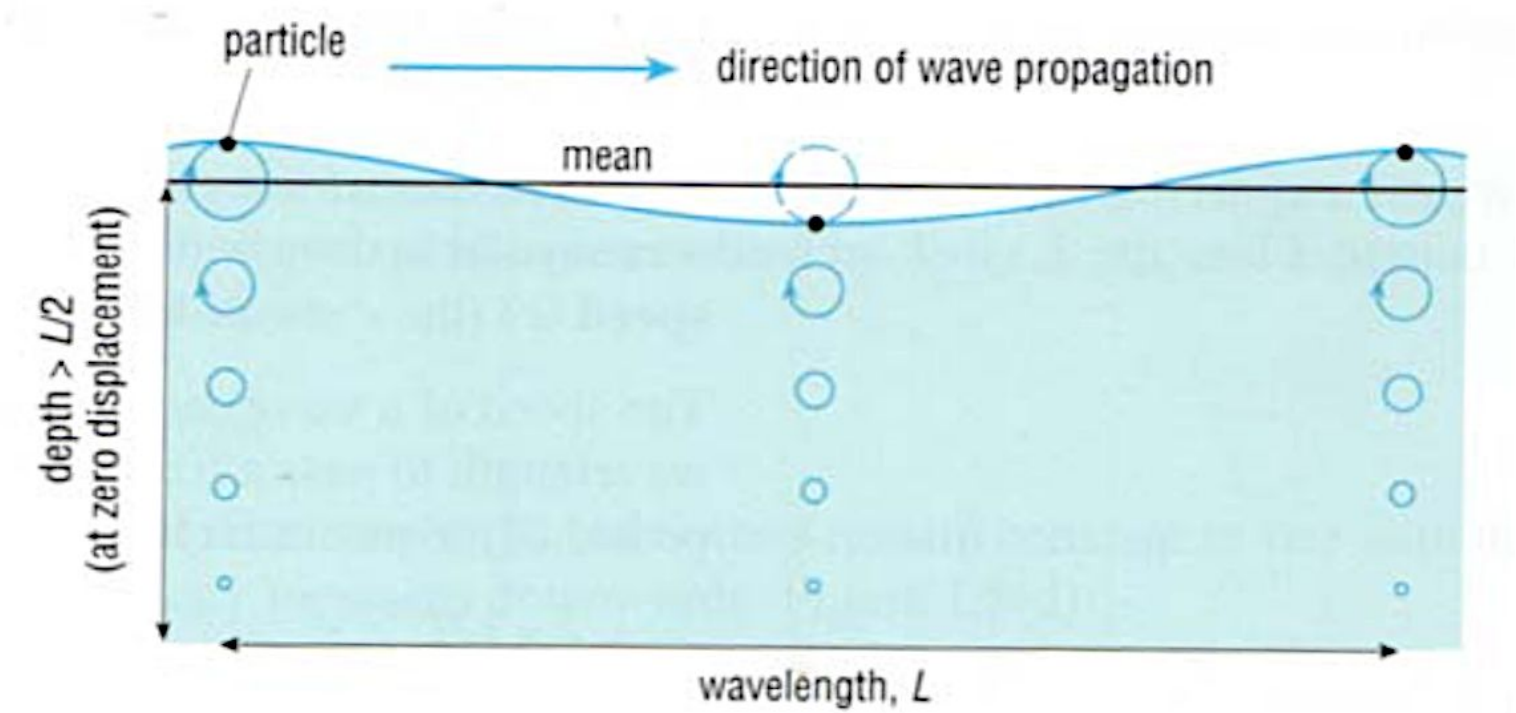
$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\vec{u}^2 + \chi = G(t)$$

where G(t) is some time-varying function that does not depend on x, y or z. If we are solving for ϕ, it is possible to get rid of G(t) by including it in ϕ, as discussed earlier for the velocity potential.

Mathematical description…

We now have all the tools we need to derive the positions, velocities and dispersion relation for surface gravity waves.

We will start by considering an infinitely deep ocean, and then progressively change the boundary conditions to look at the case of a finite-depth ocean and a shallow ocean.



I.3. Different type of ocean waves

I.3.1 Surface gravity waves

I.3.1.1 Long waves (shallow-water waves)

I.3.1.2 Short waves (deep-water waves)

I.3.1.3 The general case

I.3.1.4 Capillary waves

I.3.1.1 Long waves

- **See notes:**
  - *Derivation of SW model*
  - *Solution for shallow-water waves*

I.3.1.1 Long waves

Example of long waves:

- **Tsunamis**
- **Tidal bores**

Tsunamis

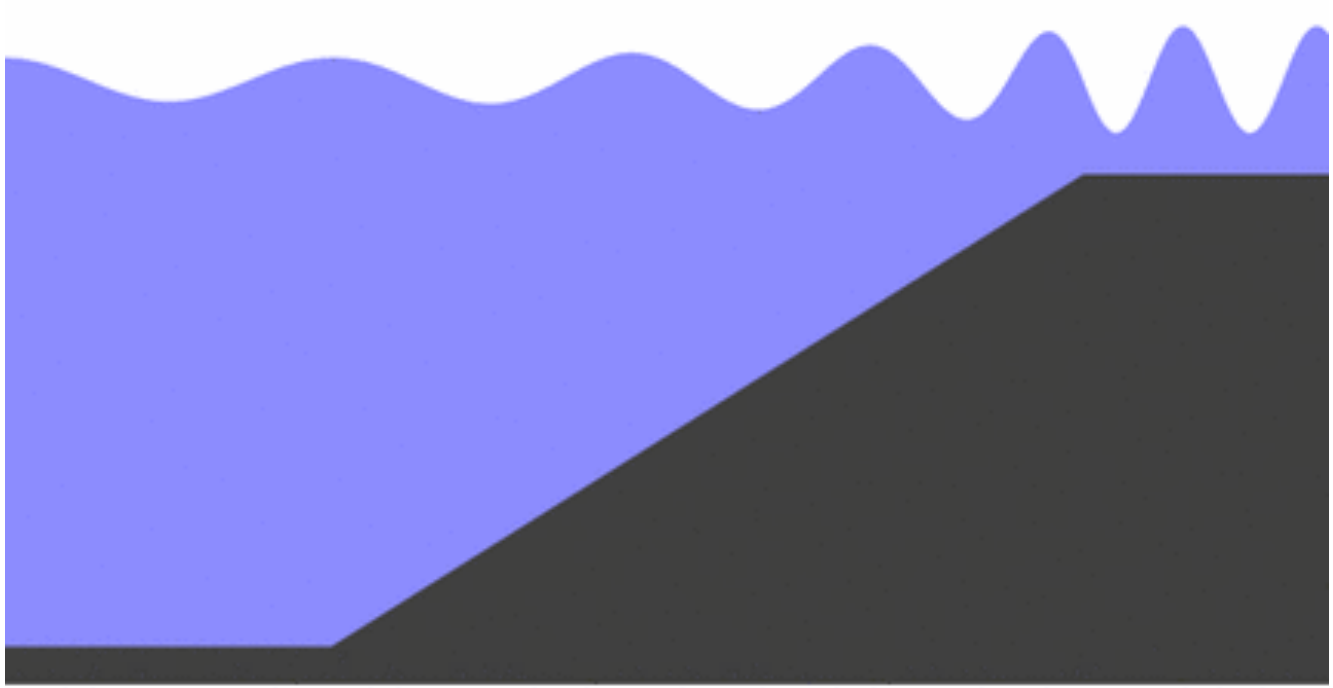
Tsunami means “harbor wave” in Japanese.

Sometimes called "tidal waves" because they "feel" like big tides (but are not due to sun/moon pull like tides).

Tsunamis

Main cause of tsunamis are earthquakes that trigger sharp movements of the seafloor.

Tsunamis



$D_1 = 5000\,\mathrm{m}$

$\lambda_1 = 200\,\mathrm{km}$

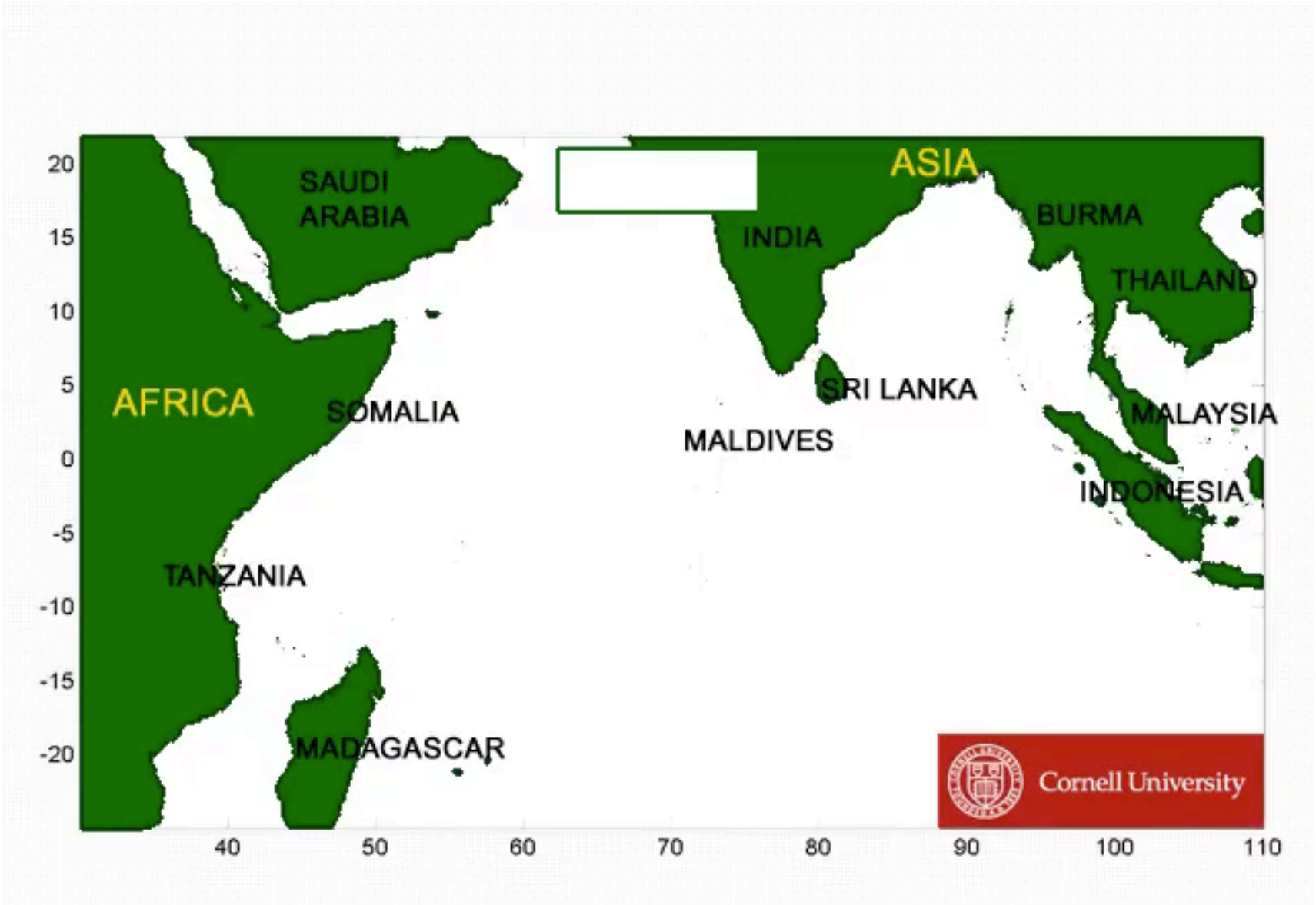
$h_1 = 1\mathrm{m}$

$c_1?\ T_1?$

$D_2 = 10\,\mathrm{m}$

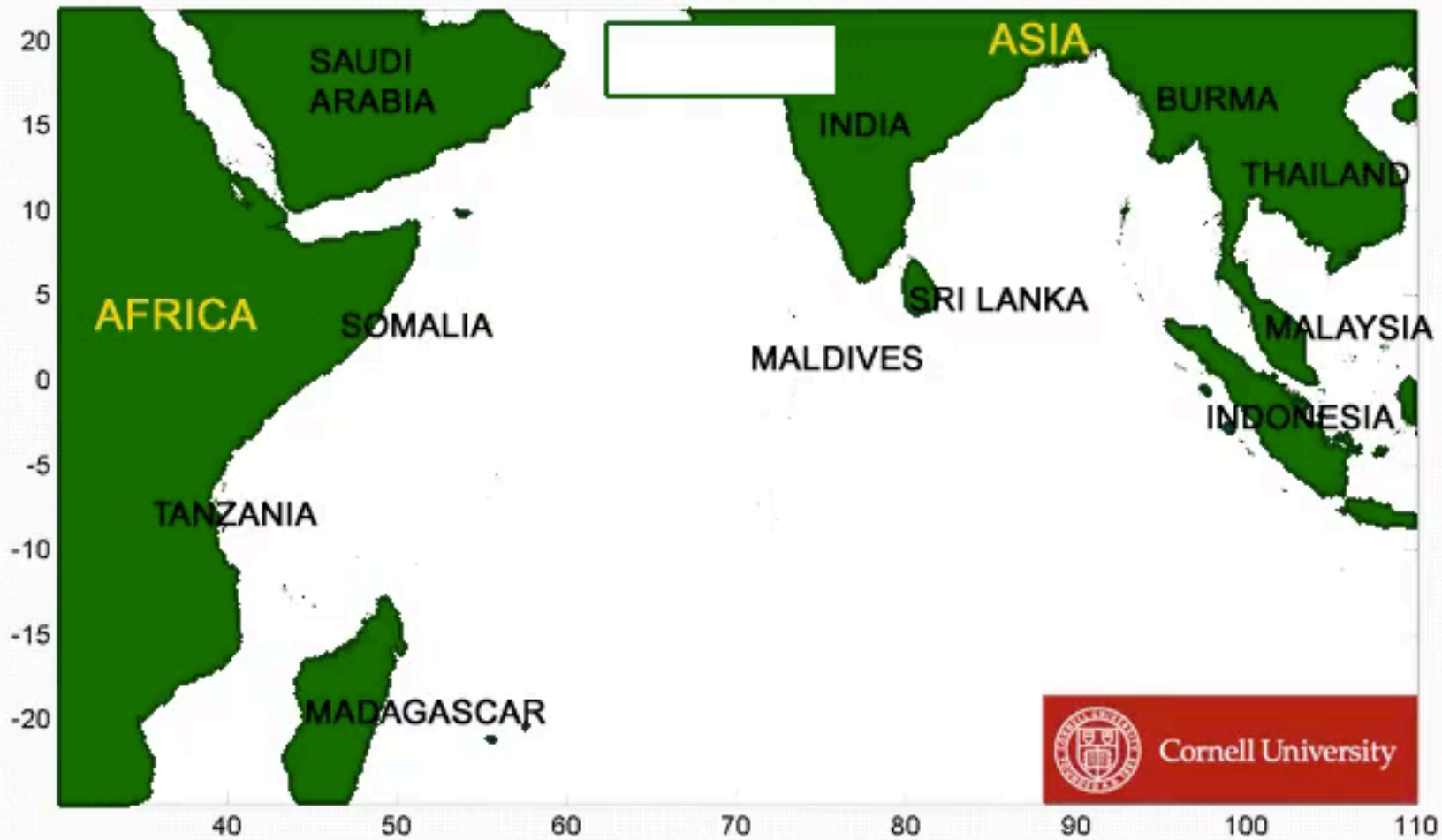
$c_2?\ T_2?\ \lambda_2?\ h_2?$

I.3.1.3 Long waves

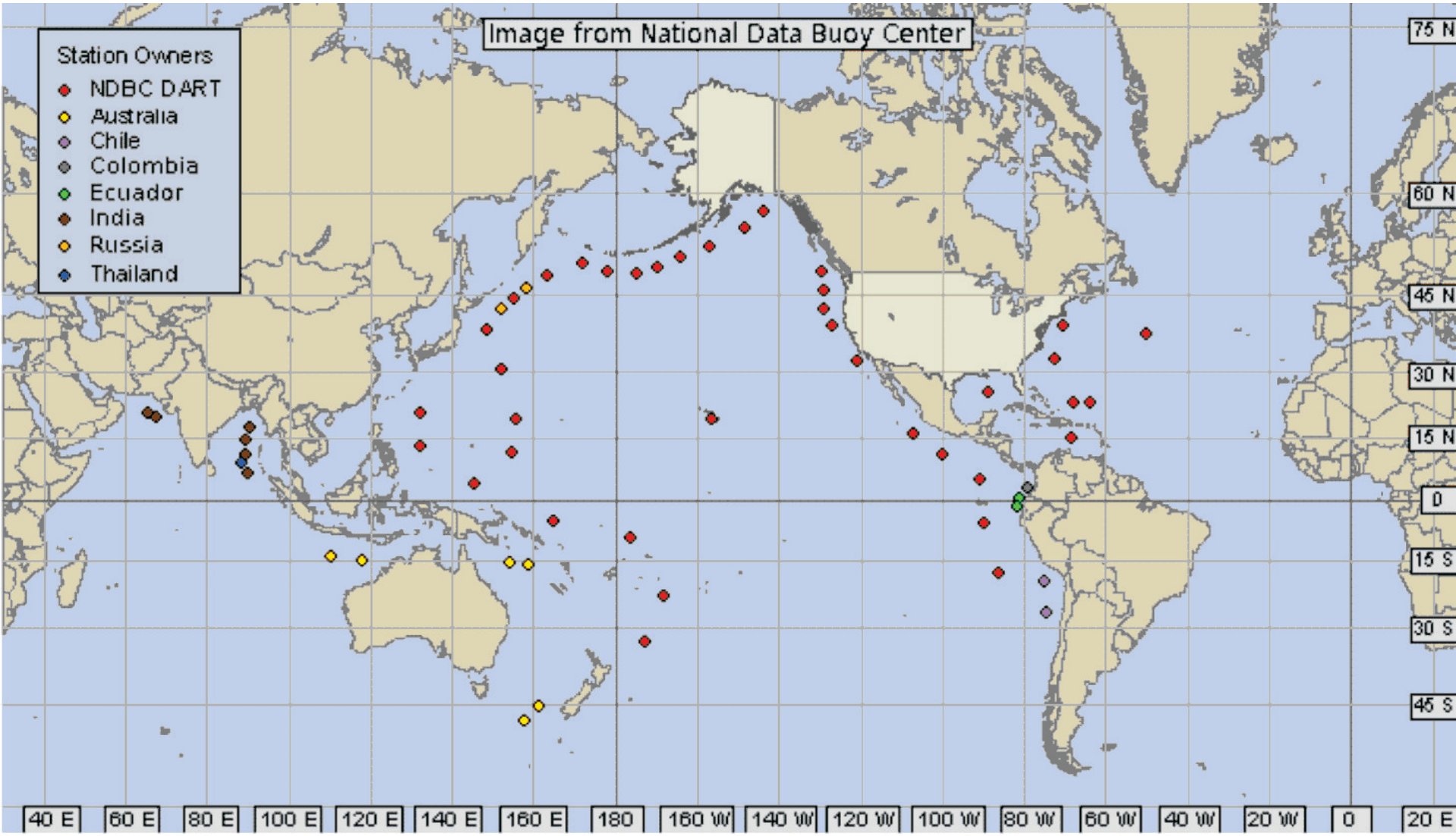


The 2004 Indian Ocean tsunami was among the deadliest natural disasters in human history, with at least 230,000 people killed or missing in 14 countries bordering the Indian Ocean.

I.3.1.1 Long waves

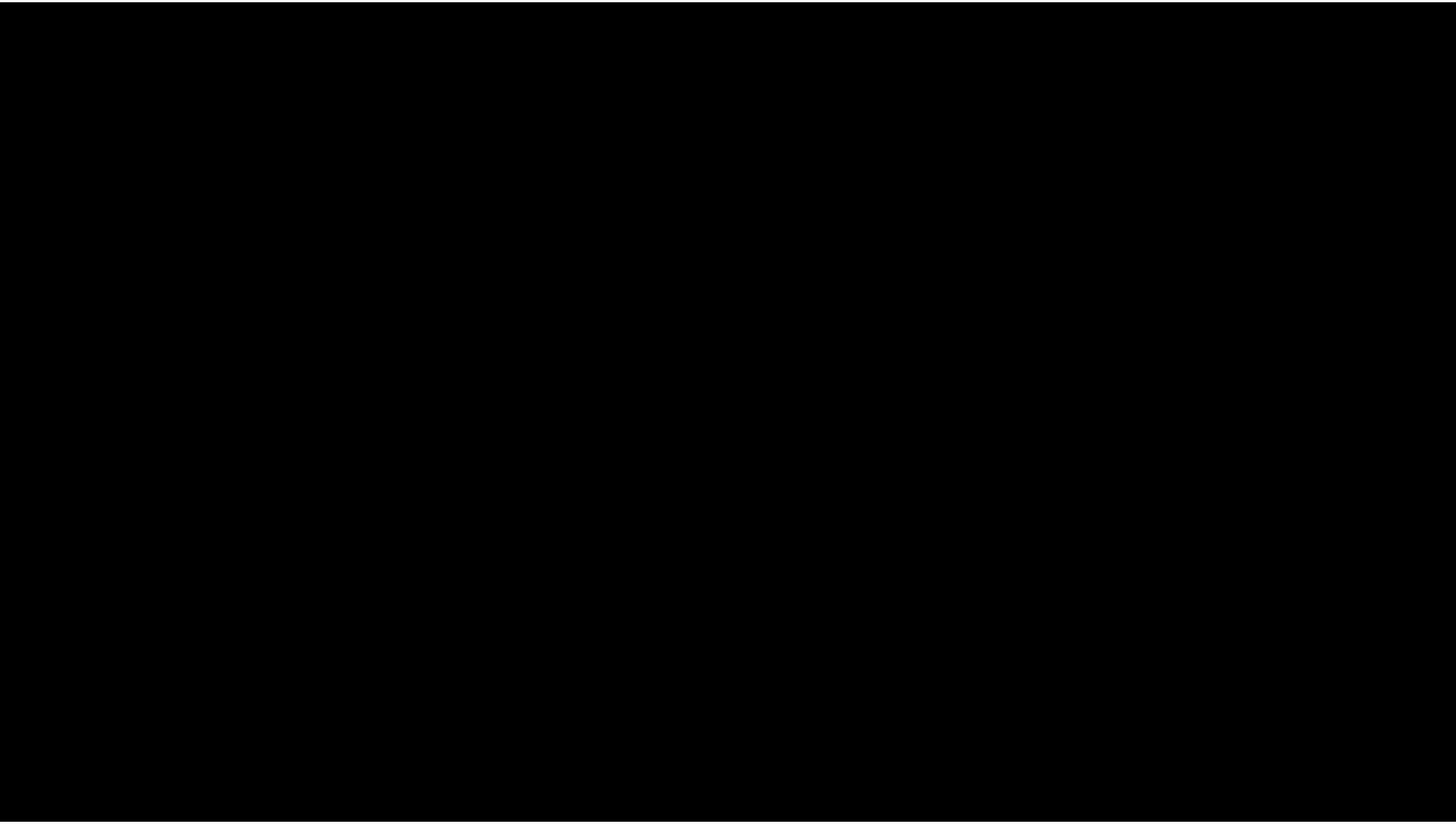


I.3.1.1 Long waves



Tsunami detection buoys have been placed around many tectonically active locations. M
Map is from NOAA Tsunami Detection website http://www.ndbc.noaa.gov/dart.shtml

I.3.1.1 Long waves

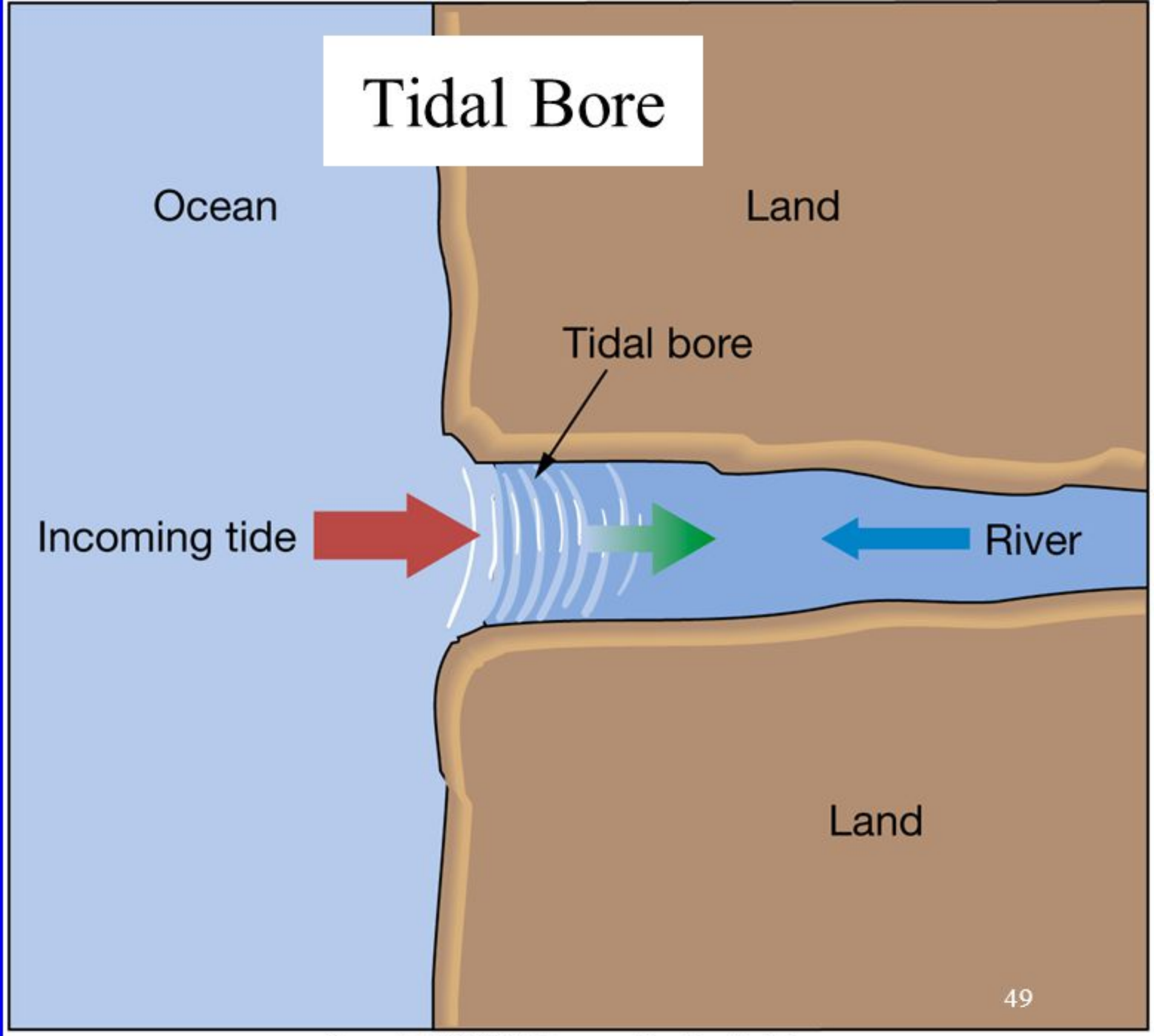


I.3.1.1 Long waves

Example of long waves:

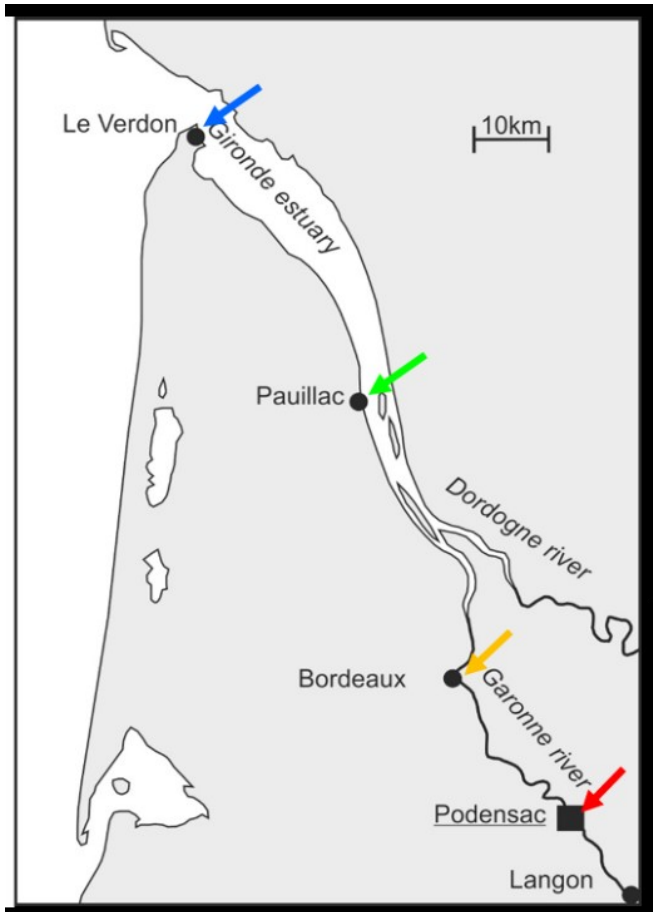
- **Tsunamis**
- **Tidal bores**

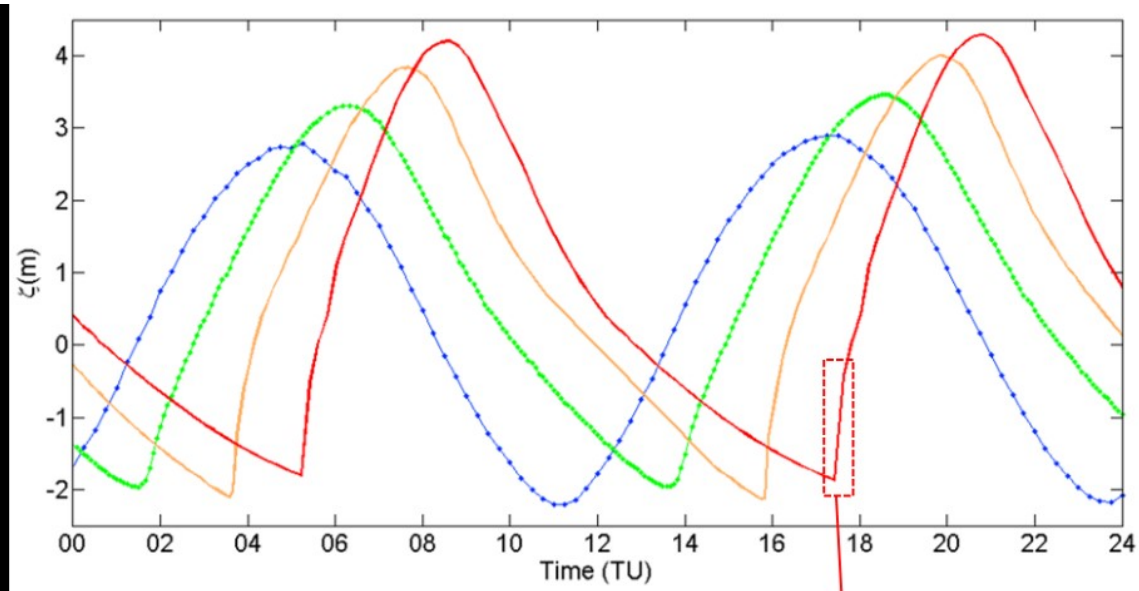
A tidal bore is a wall of water that surges upriver with the advancing high tide.

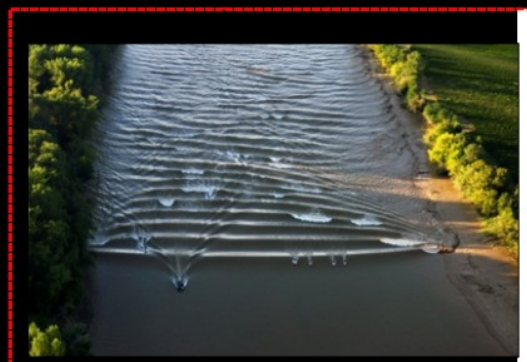


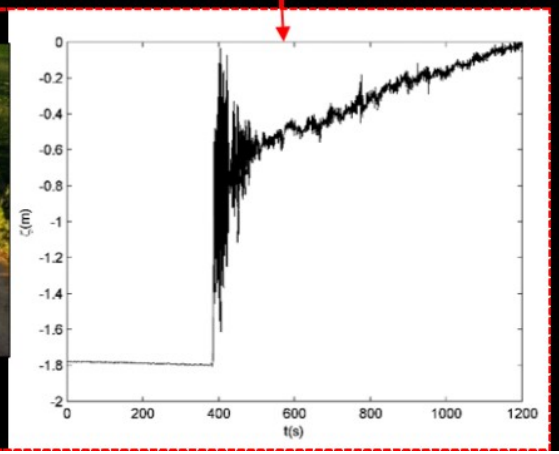
Copyright © 2005 Pearson Prentice Hall, Inc.

Tidal bores


Le Verdon
Gironde estuary
10km
Pauillac
Dordogne river
Bordeaux
Garonne river
Podensac
Langon


ζ(m)
Time (TU)



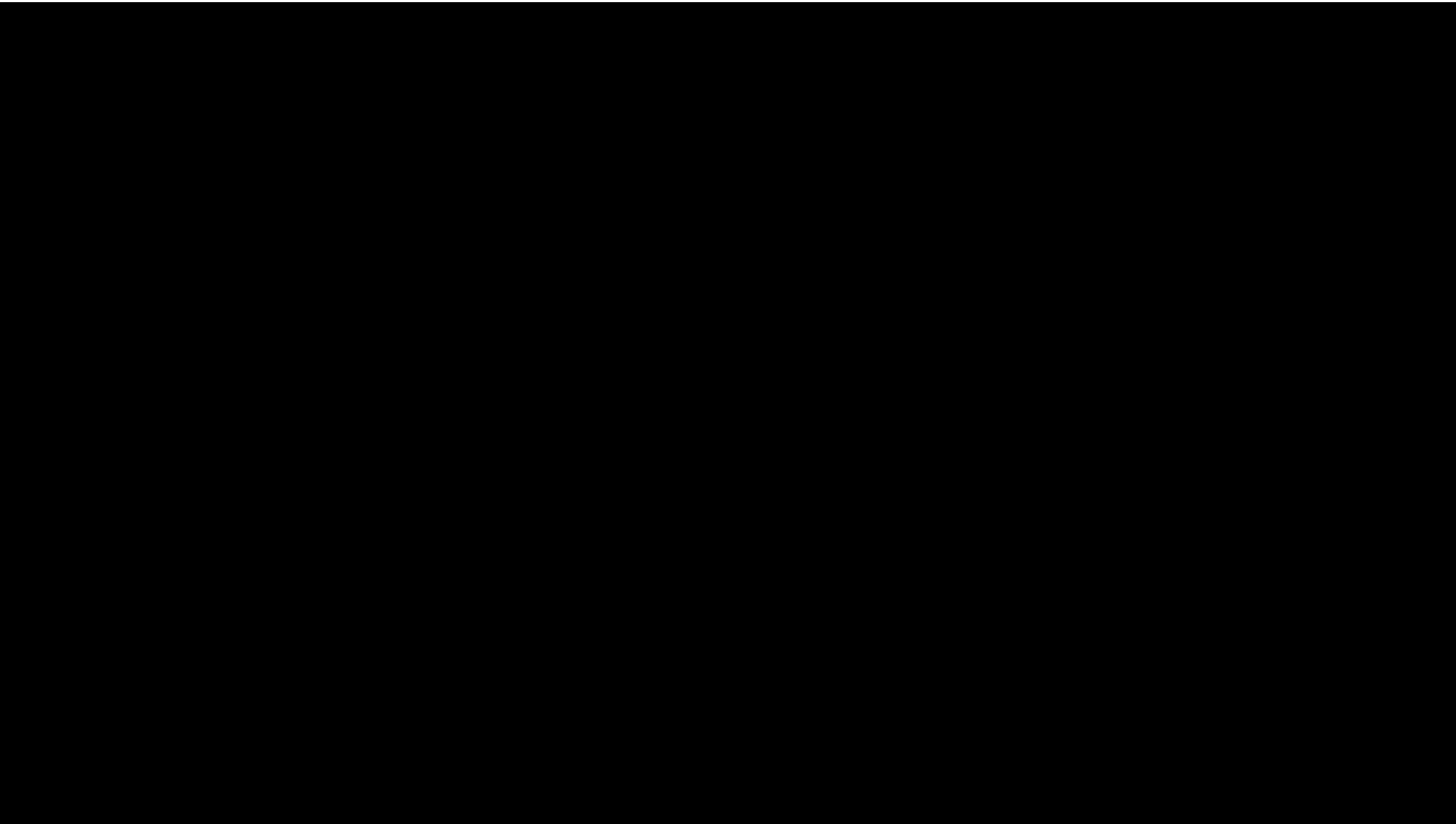

ζ(m)
t(s)

I.3.1.3 Long waves

I.3.1.3 Long waves

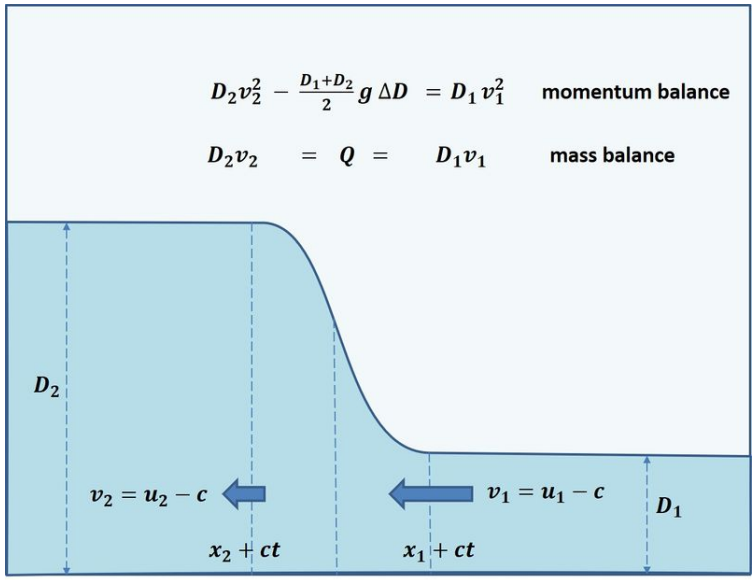
A group of surfers ride the bore tide at Alaska's Turnagain Arm in July 2014. Alaska's most famous bore tide occurs here in the lower arm of the Cook Inlet where waves can reach 6 to 10 feet (1.8 to 3 meters) tall and move at 10 to 15 mph (16 to 24 kph).

I.3.1.3 Long waves



Tidal bores

Tidal bores are non-linear and require non linear shallow water equations.

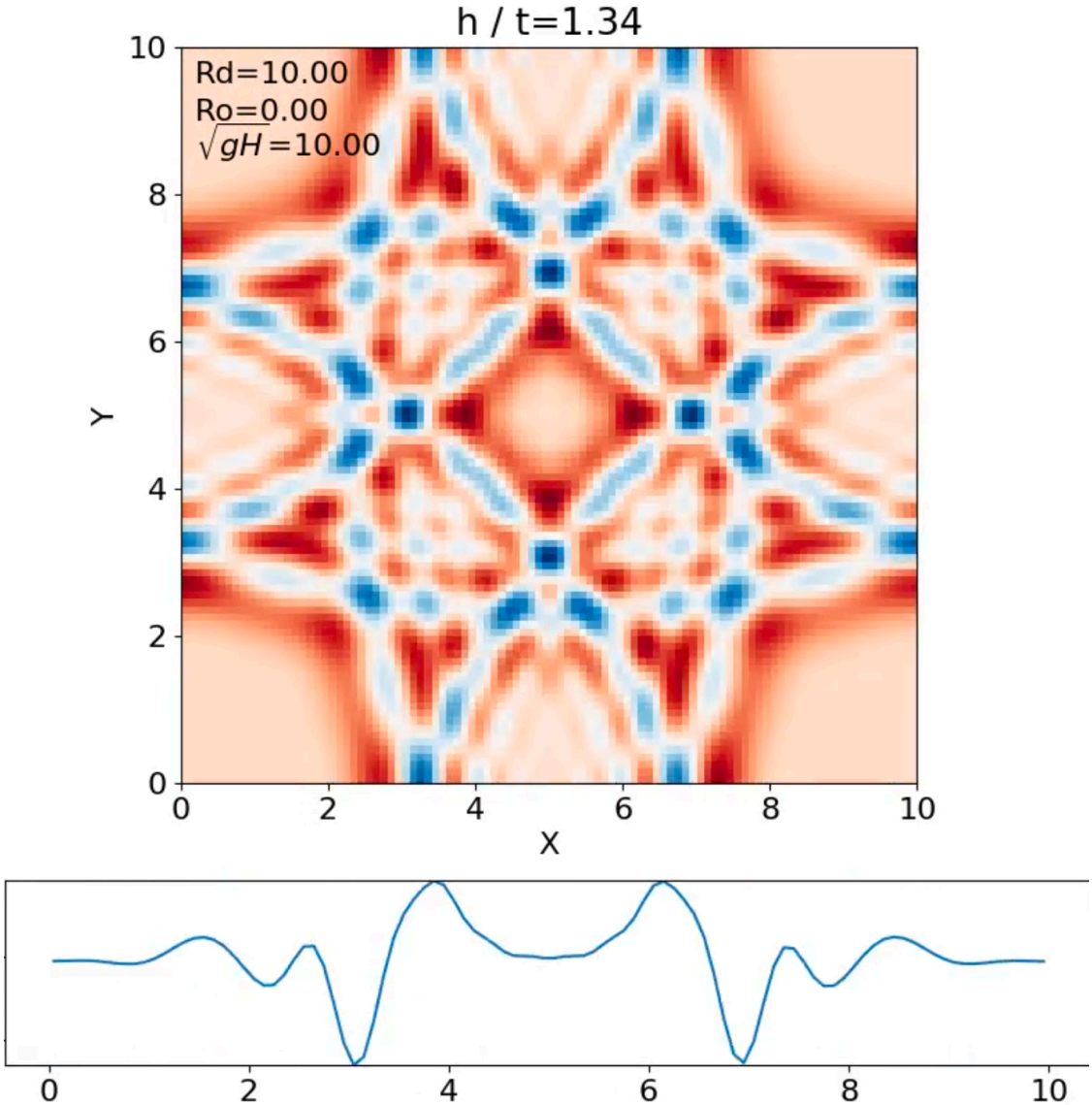


Solutions of the propagation of a bore/solution can described by a Korteweg-deVries (KdV) type equation:

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0$$

I.3.1.1 Long waves

TP1



I.3.1 Surface gravity waves

I.3.1.1 Long waves

I.3.1.2 Short waves

I.3.1.3 The general case

I.3.1.4 Capillary waves

I.3.1.2 Short waves

- **See notes:**
  - *Solution for deep-water waves*

Wind waves versus Swells

Feature	Wind Waves	Swells
Origin	Generated by local winds	Formed from decayed wind waves traveling away from storms
Wavelength	Shorter (meters to ~100 m)	Longer (100–500 m)
Period	Short (1–10 s)	Long (10–30 s)
Appearance	Steep, irregular, choppy	Smooth, uniform, regular
Direction	Aligned with local wind	Can travel across ocean basins, independent of local wind
Energy	Confined near wind source	Can transport energy thousands of km
Persistence	Disappear quickly when wind stops	Can last for days to weeks

I.3.1.2 Short waves

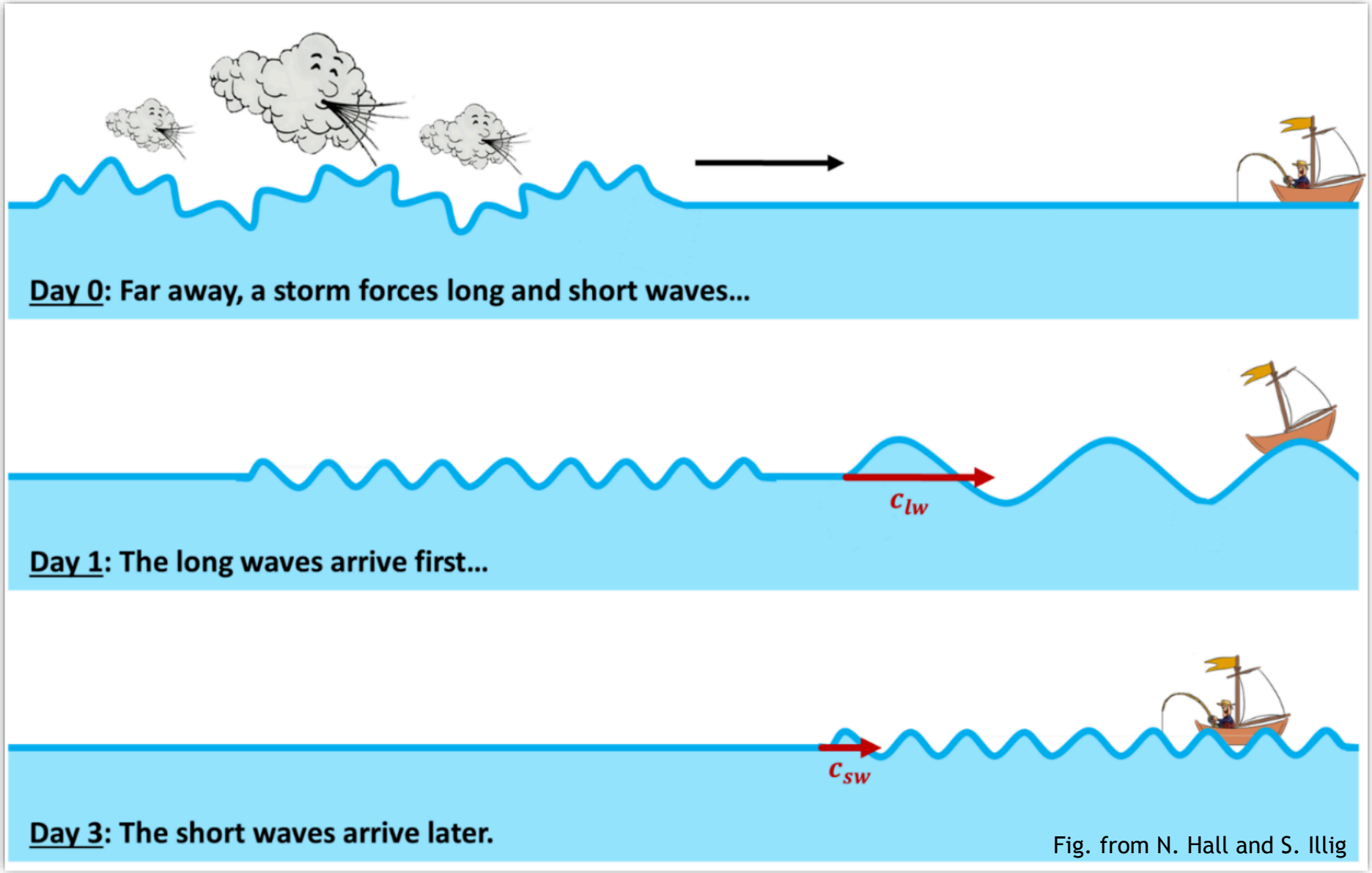


Fig. from N. Hall and S. Illig

I.3.1.2 Short waves

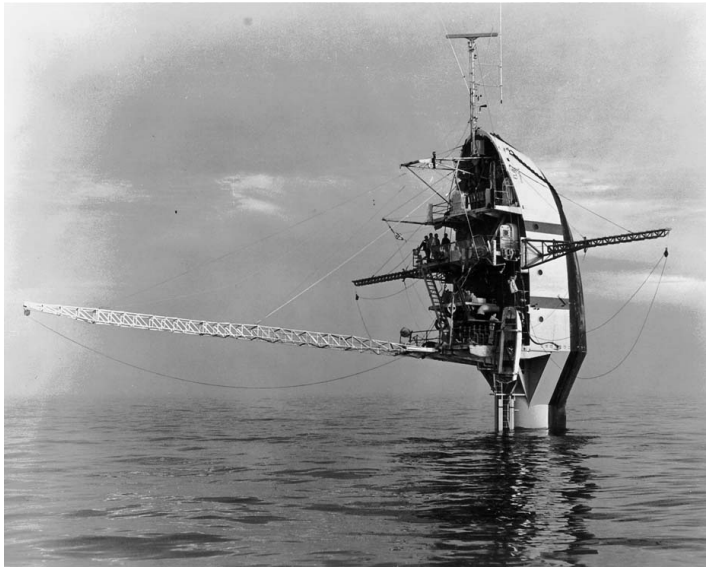
Munk's experiment in 1963: **"Waves Across the Pacific"**

His equations said that the swells hitting beaches in Mexico began some 9,000 miles away — somewhere in the southern reaches of the Indian Ocean, near Antarctica.

"Could it be?" he wrote in an autobiographical sketch. Could a storm halfway across the world produce a patch of moving water that traveled from near the South Pole, up past Australia, then past New Zealand, then across the vast expanse of the Pacific, arriving still intact — at a beach off Mexico?

He decided to find out for himself.

That is why, in 1957, Walter Munk designed a global, real-life wave-watching experiment.



https://www.youtube.com/watch?v=MX5cKoOm6Pk&pbjreload=10

I.3.1 Surface gravity waves

I.3.1.1 Long waves

I.3.1.2 Short waves

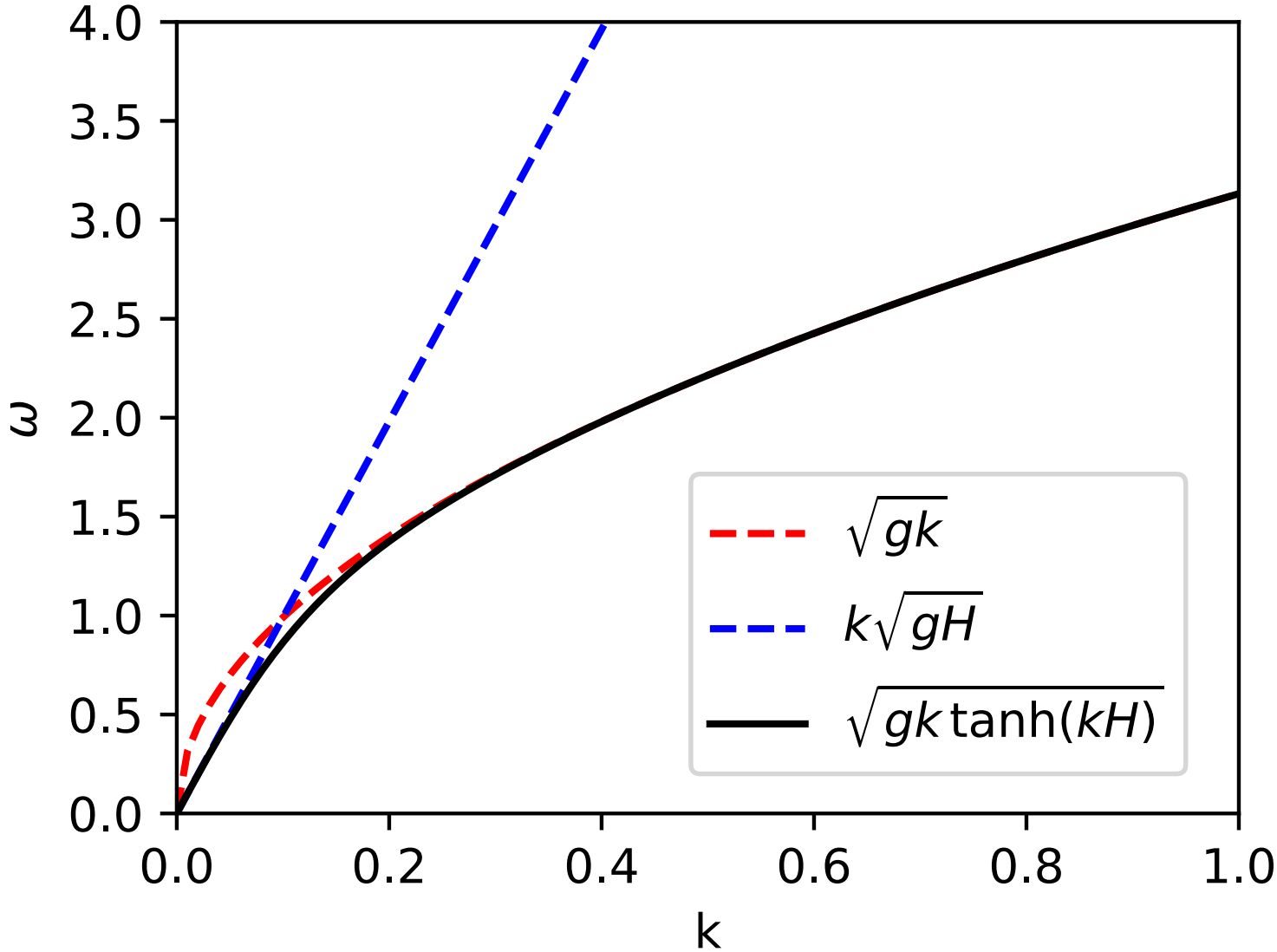
I.3.1.3 The general case

I.3.1.4 Capillary waves

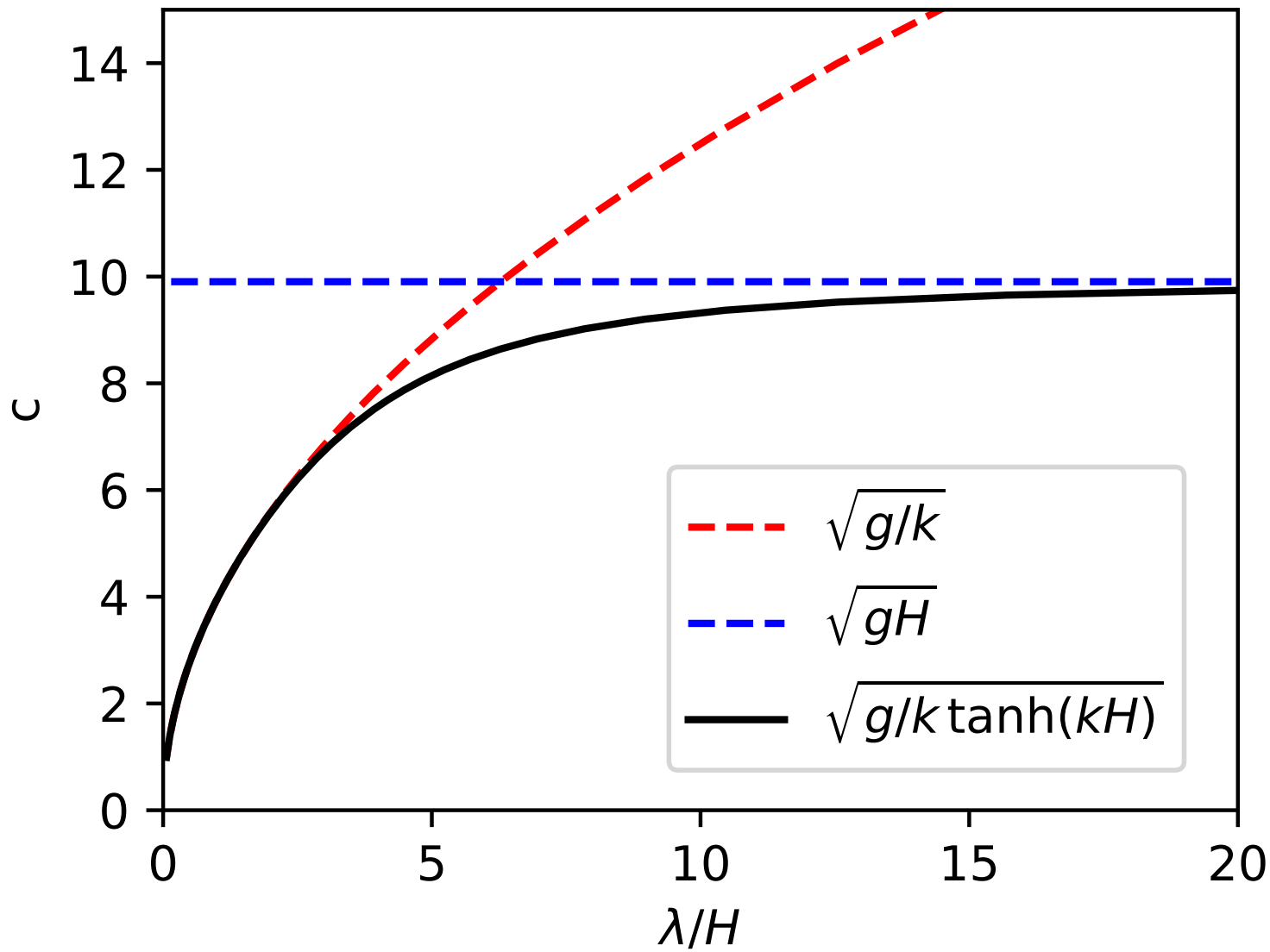
I.3.1.3 The general case

- **See notes:**
  - *Solution for the general case*

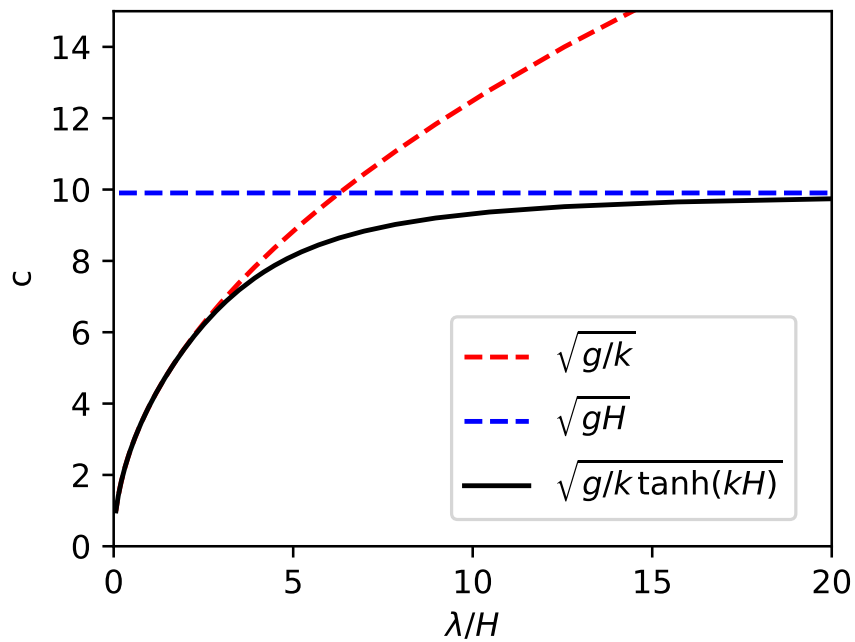
I.3.1.3 The general case

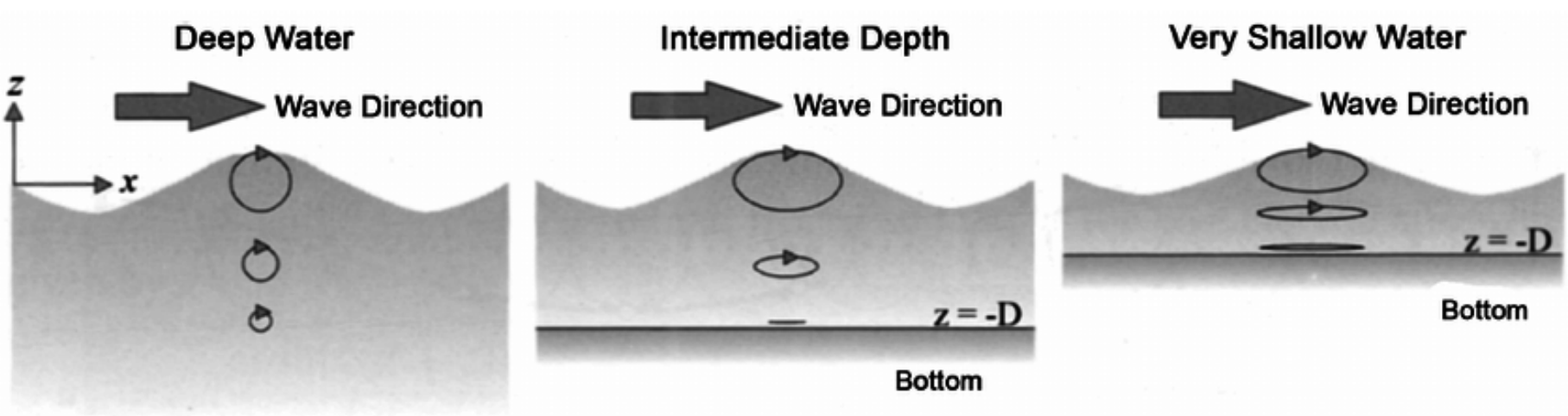


I.3.1.3 The general case

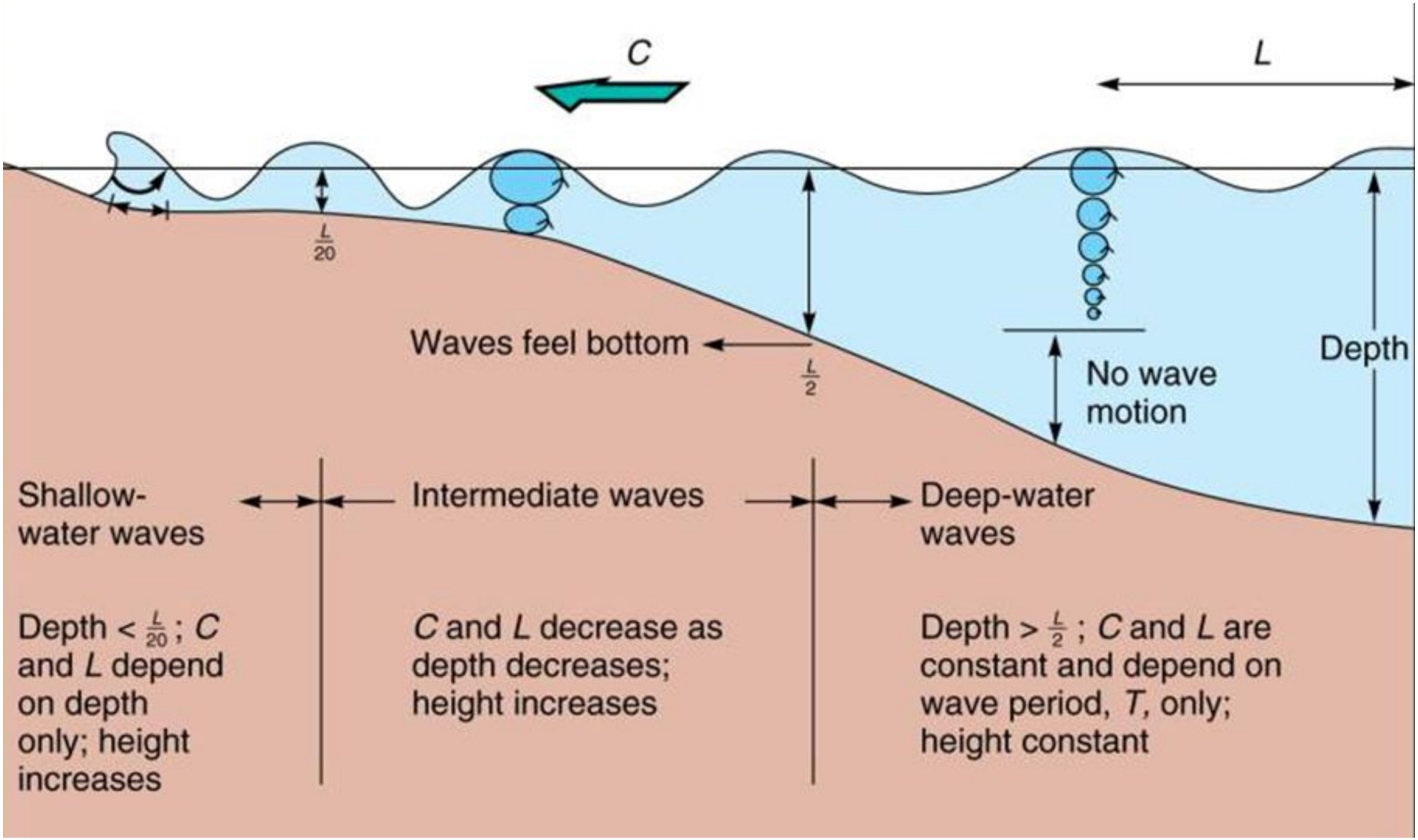


I.3.1.3 The general case





I.3.1.3 The general case



I.3.1.3 The general case

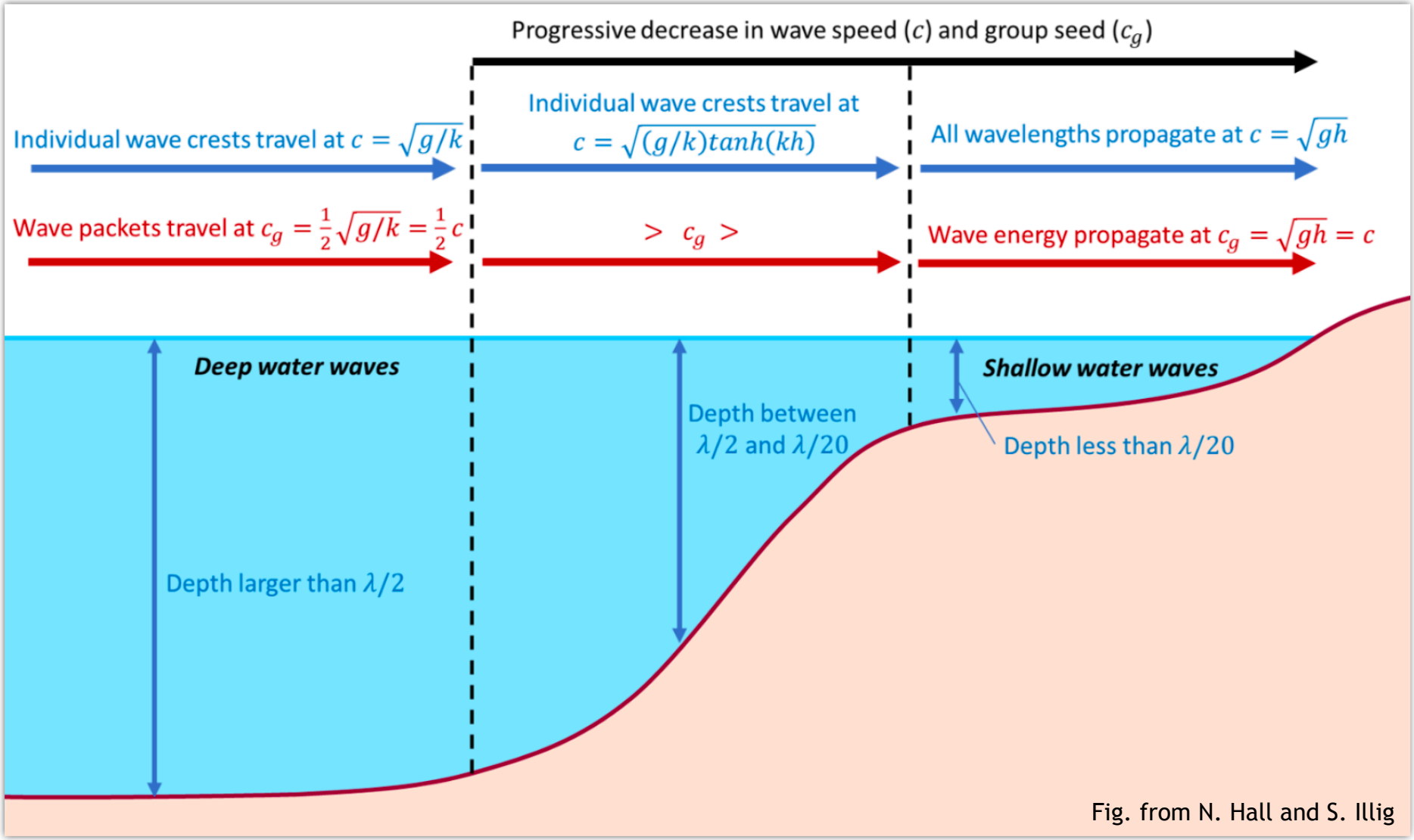


Fig. from N. Hall and S. Illig

Waves change properties approaching the coast

I.3.1.3 The general case

Water gets shallower so waves must slow down.

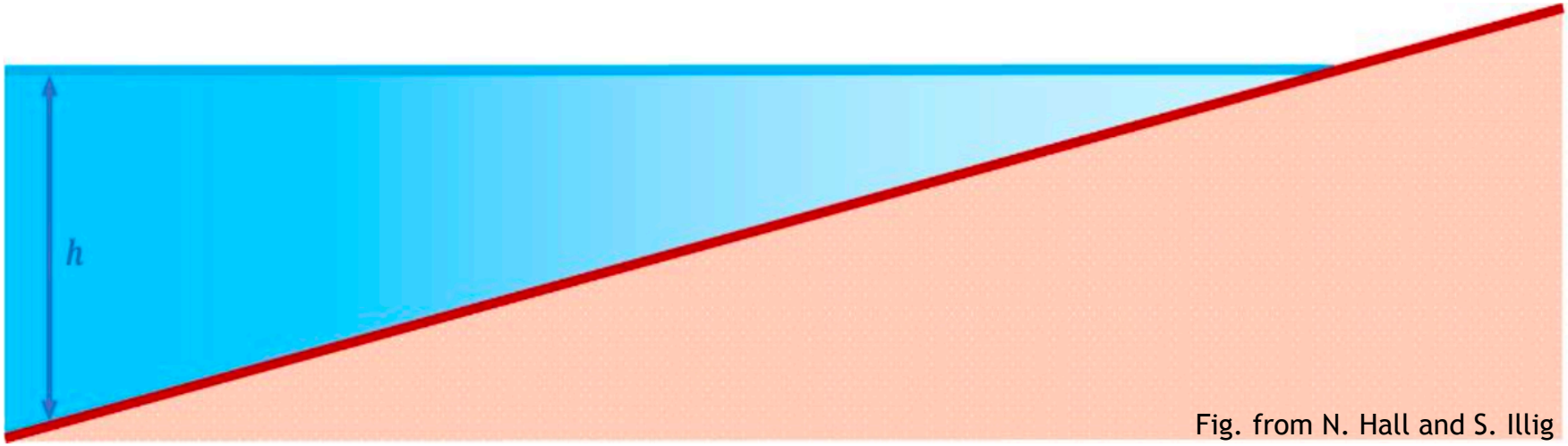


Fig. from N. Hall and S. Illig

So what happens?

I.3.1.3 The general case

$$\frac{\lambda_1}{\lambda_2} = \sqrt{\frac{h_1}{h_2}}$$

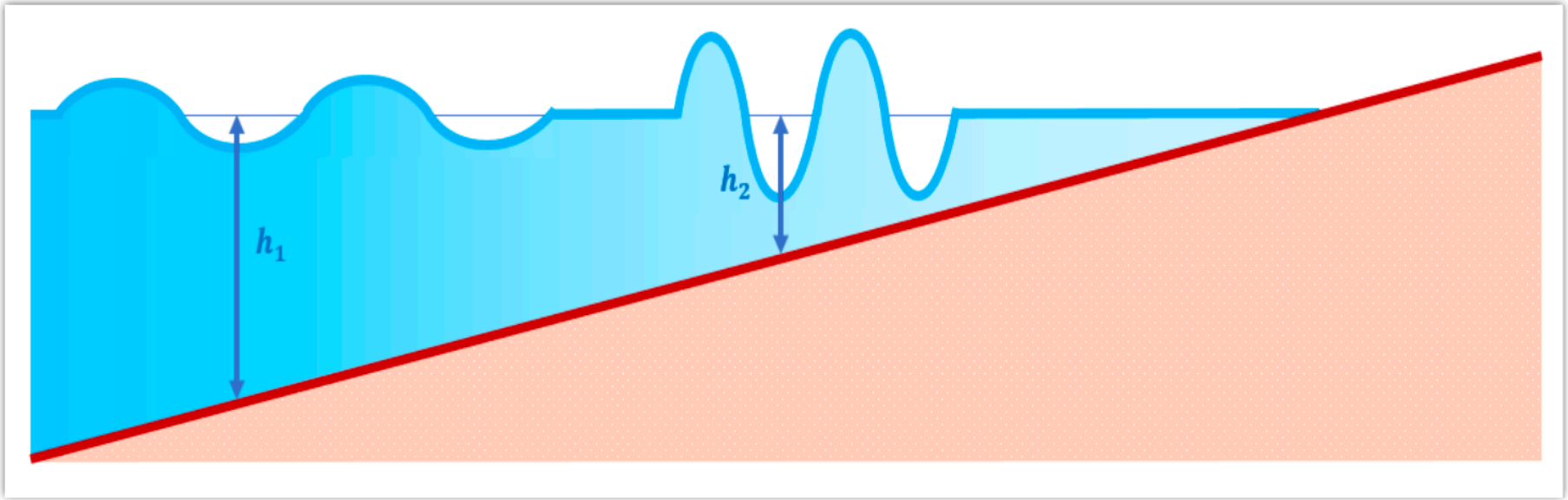
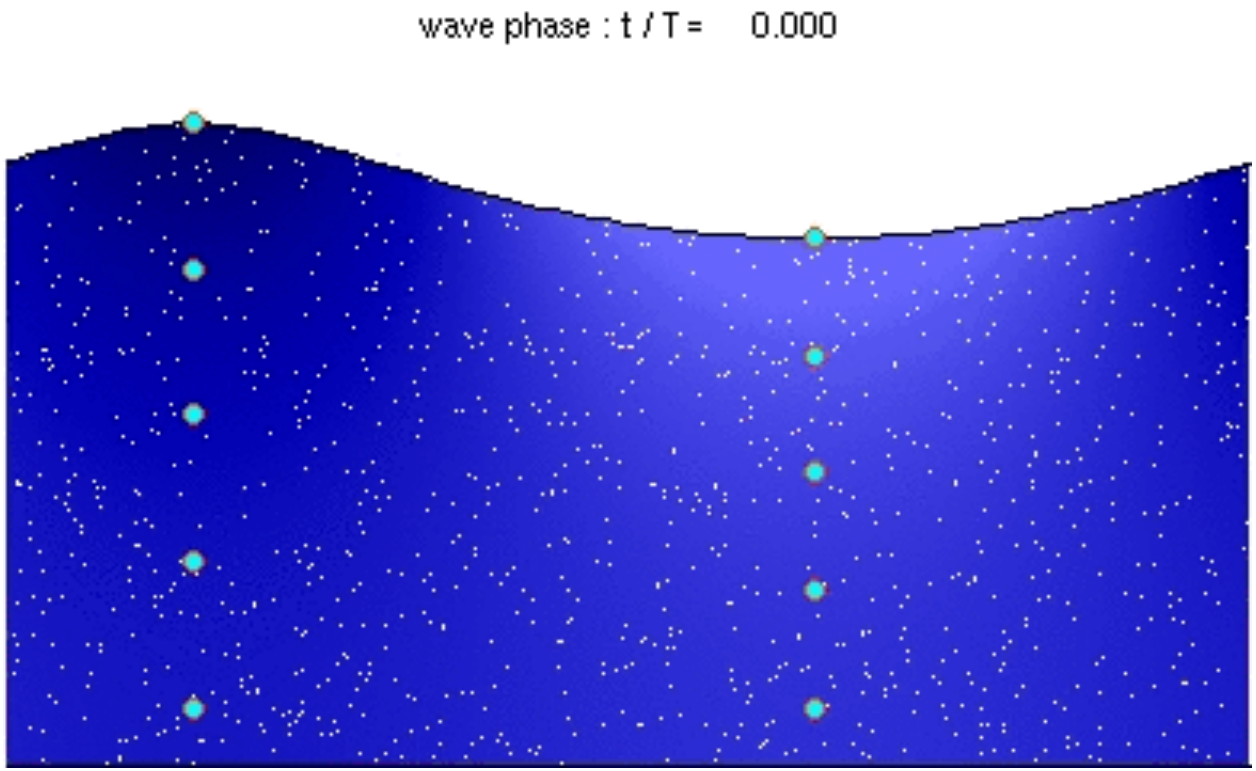


Fig. from N. Hall and S. Illig

I.3.1.3 The general case



A **nonlinear wave** also carries energy and can lead to a net transport of material (for example, Stokes drift at the ocean surface)

Animation from https://en.wikipedia.org/wiki/Wind_wave

Stokes drift

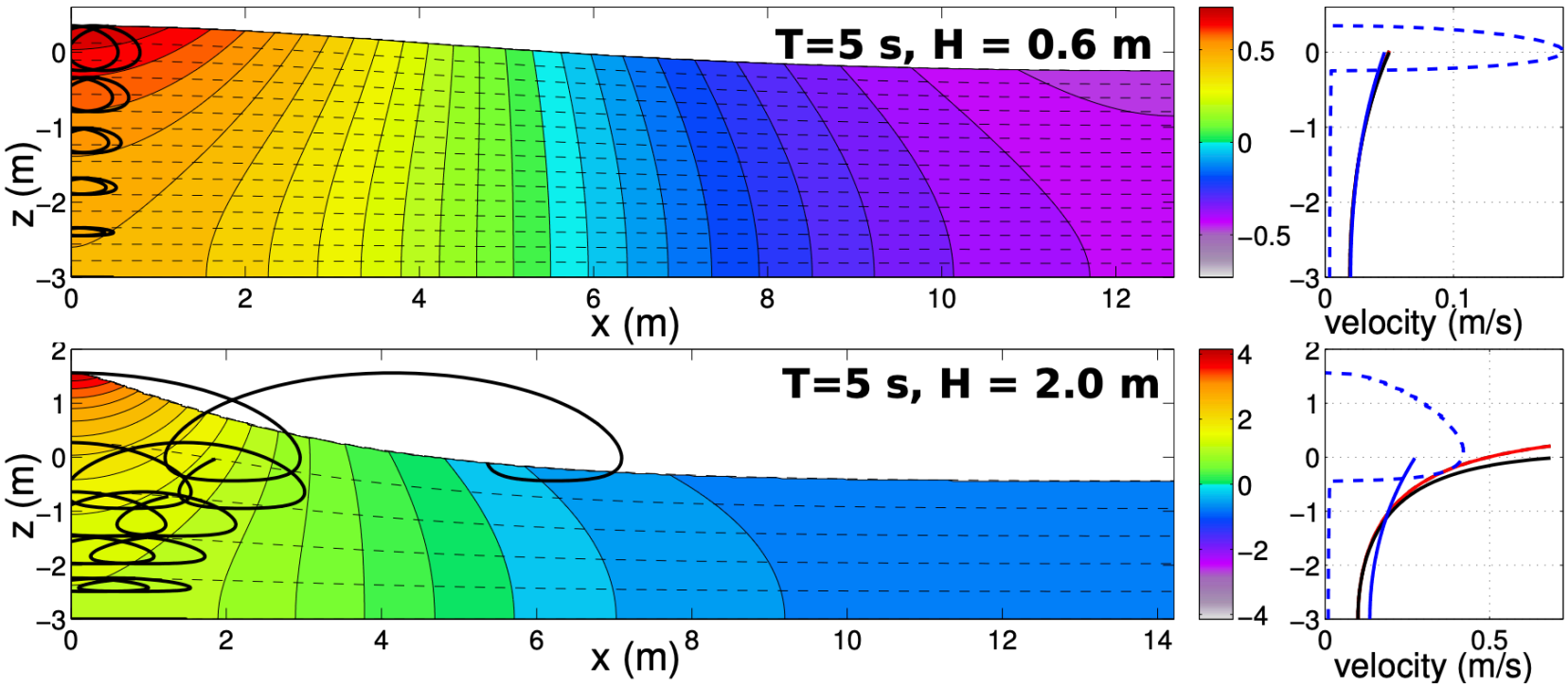
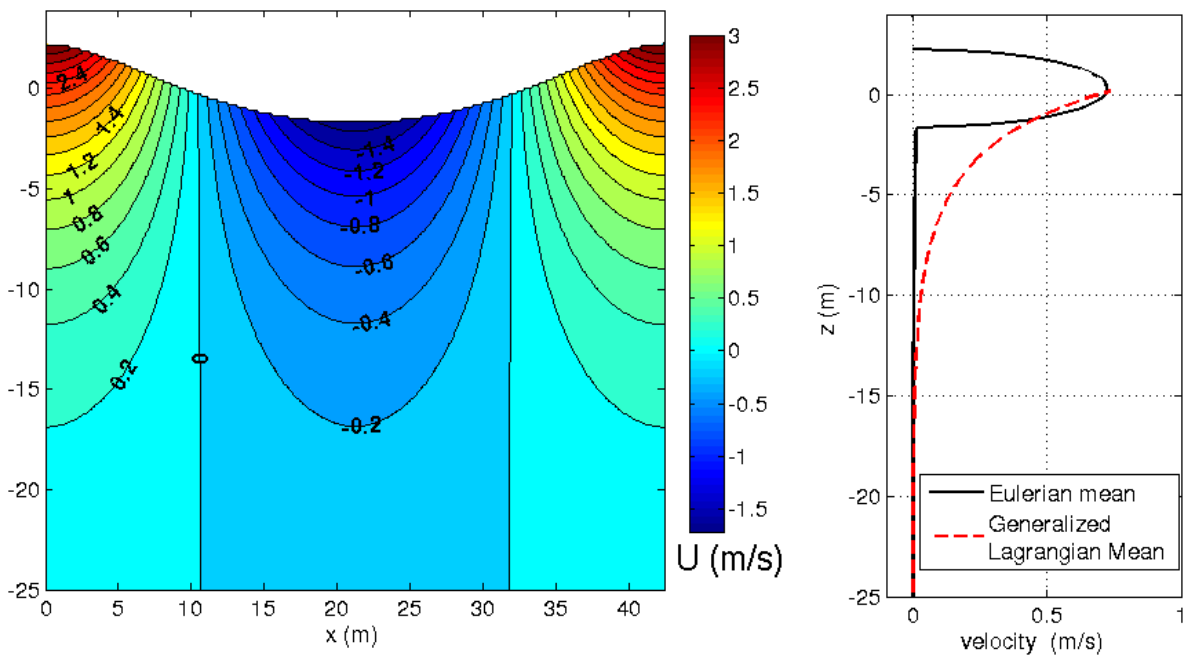


Figure 9.1: Left: Horizontal velocity field at $t = 0$ and particle trajectories integrated over 2 Eulerian periods. Solid lines are isotachs and dashed lines are streamlines in the frame of reference moving with the wave phase speed. Right: vertical profile of Eulerian mean water velocity (dashed, velocity is set to zero in the air for computing the average), Generalized Lagrangian mean (red), Lagrangian mean (black) and Lagrangian mean from the linear approximation (blue). Both top and bottom panels are computed with streamfunction theory to 80th order (Dalrymple, 1974). Non-linear terms are only significant in the bottom case.

Stokes drift

For a long wave, with surface elevation: $\eta = a\cos(\mathrm{kx} - \omega \mathrm{t})$

The velocity potential is $\phi = \dfrac{\omega}{k} a e^{kz} \sin(\mathrm{kx} - \omega \mathrm{t})$



The Stokes drift velocity (Lagrangian minus Eulerian mean) is: $\overline{u_s} = \omega k a^2 e^{2kz}$

see https://en.wikipedia.org/wiki/Stokes_drift

I.3.1 Surface gravity waves

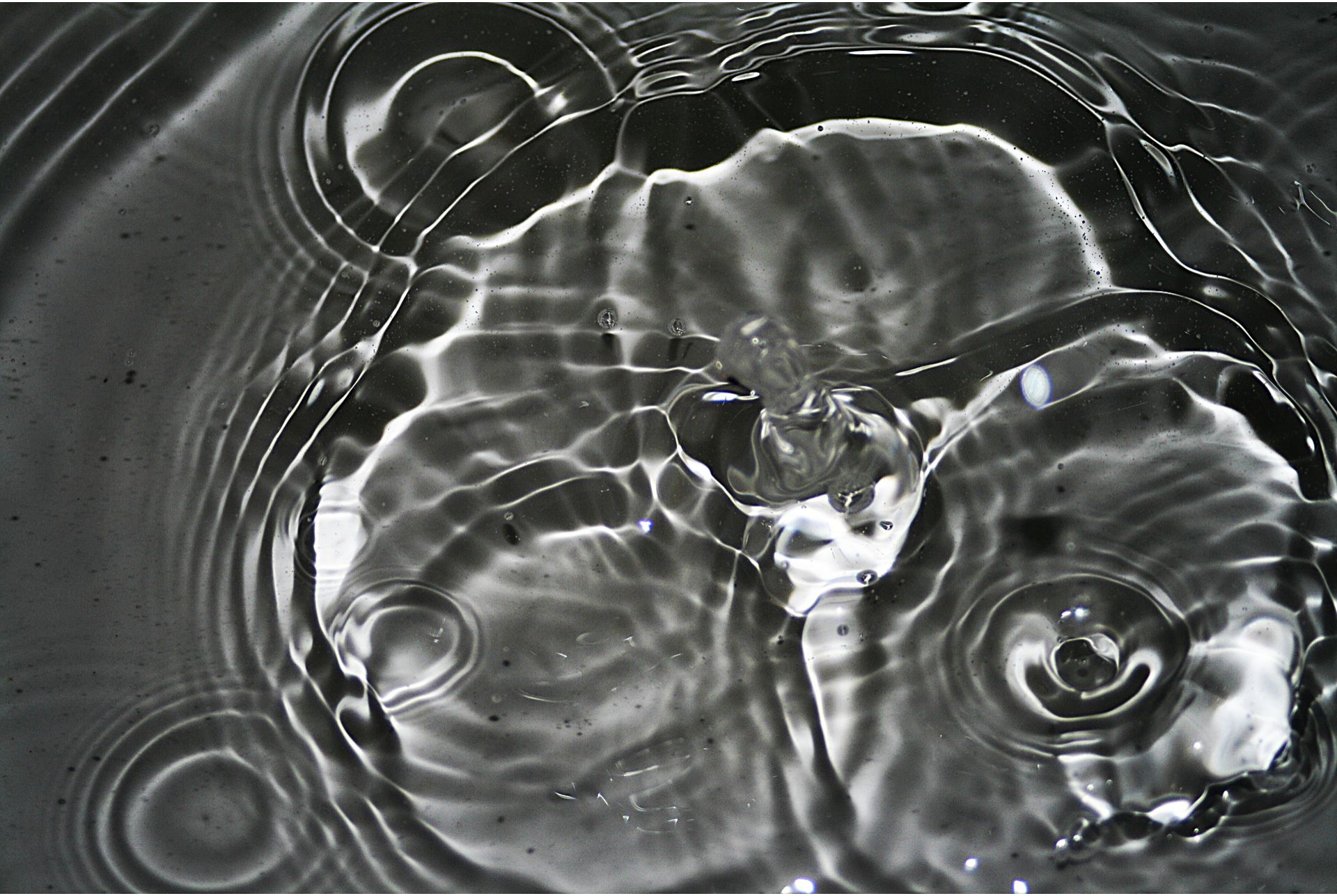
I.3.1.1 Long waves

I.3.1.2 Short waves

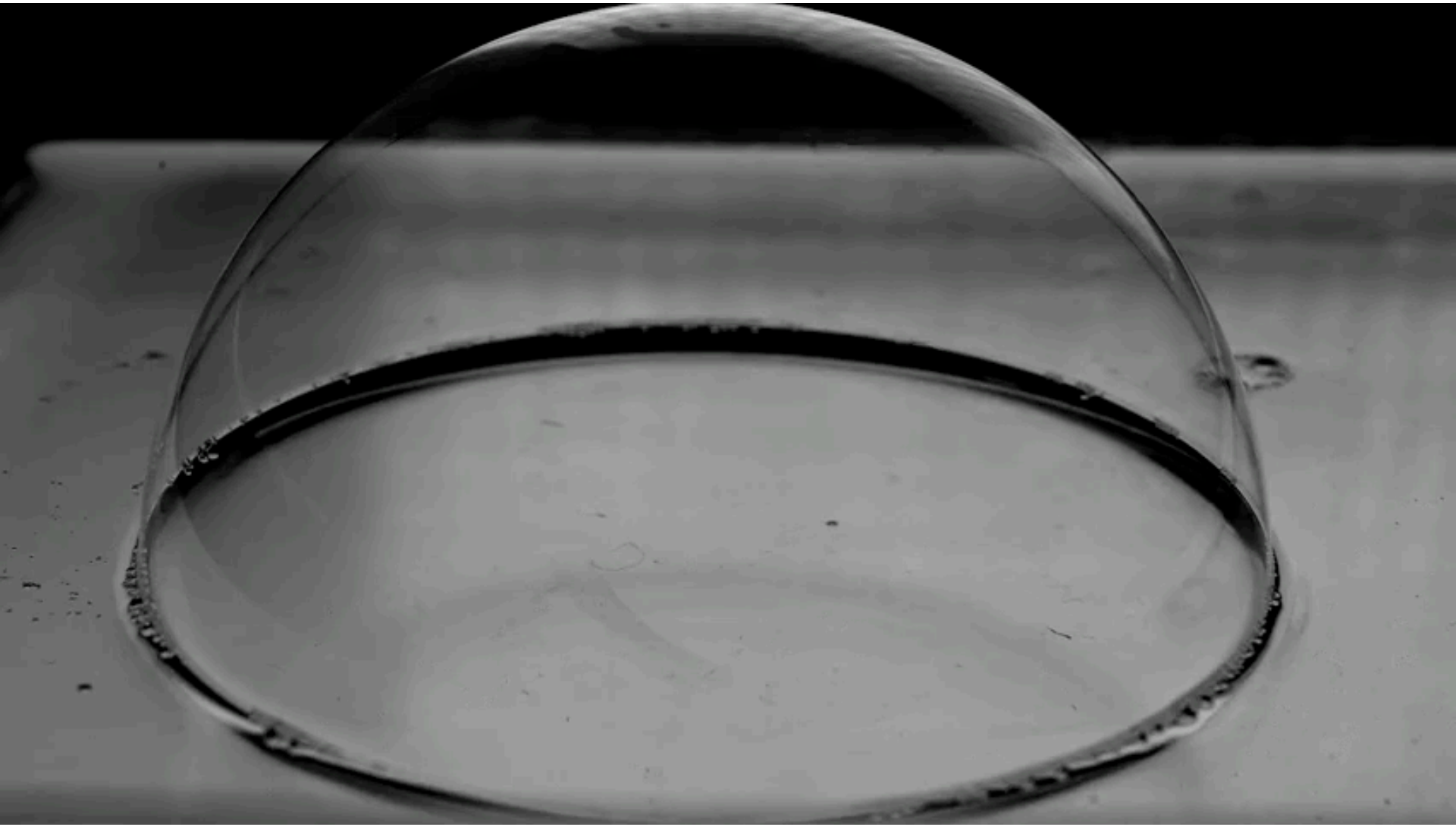
I.3.1.3 The general case

I.3.1.4 Capillary waves

I.3.1.4 Capillary waves



I.3.1.4 Capillary waves

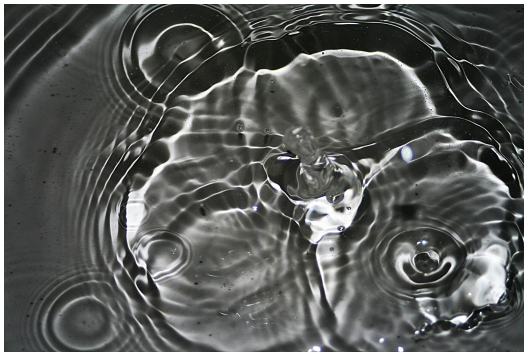


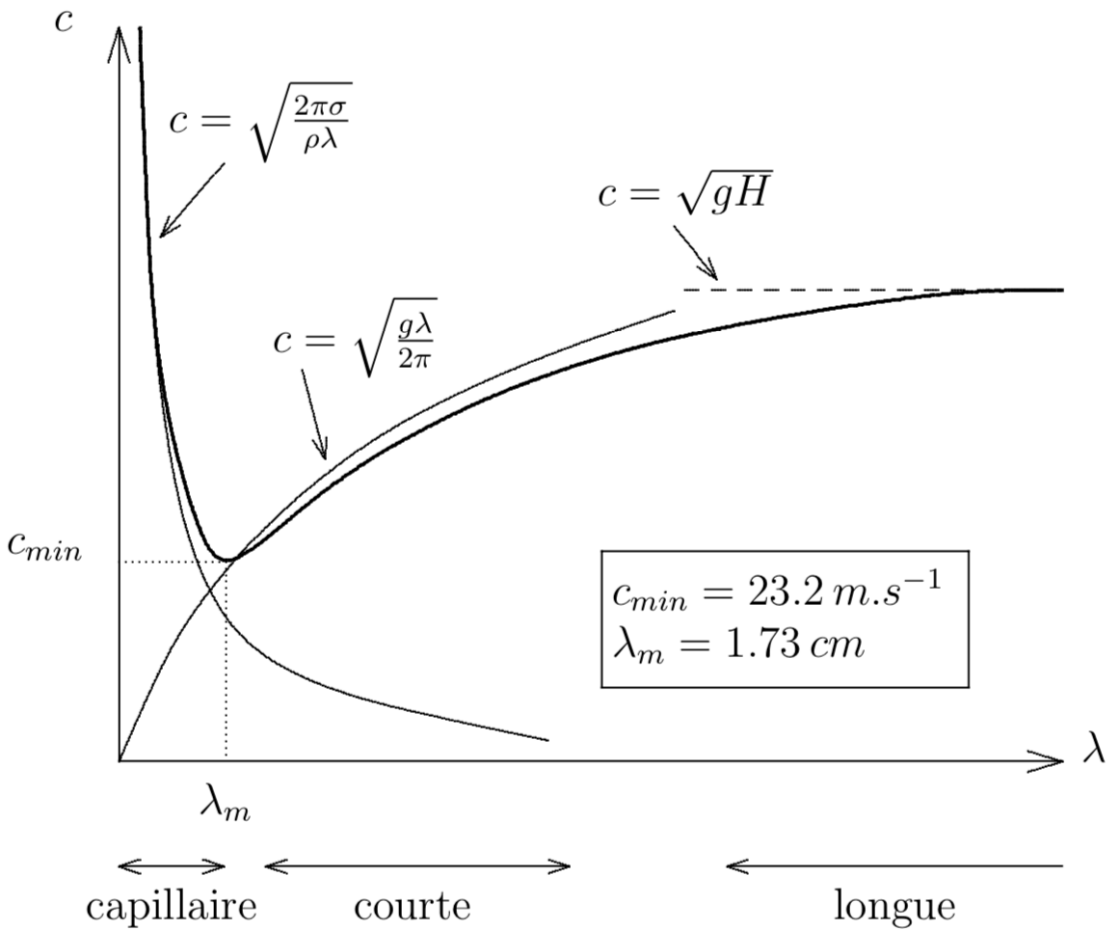
I.3.1.4 Capillary waves

- **See notes:**
  - *Solution for capillary waves*

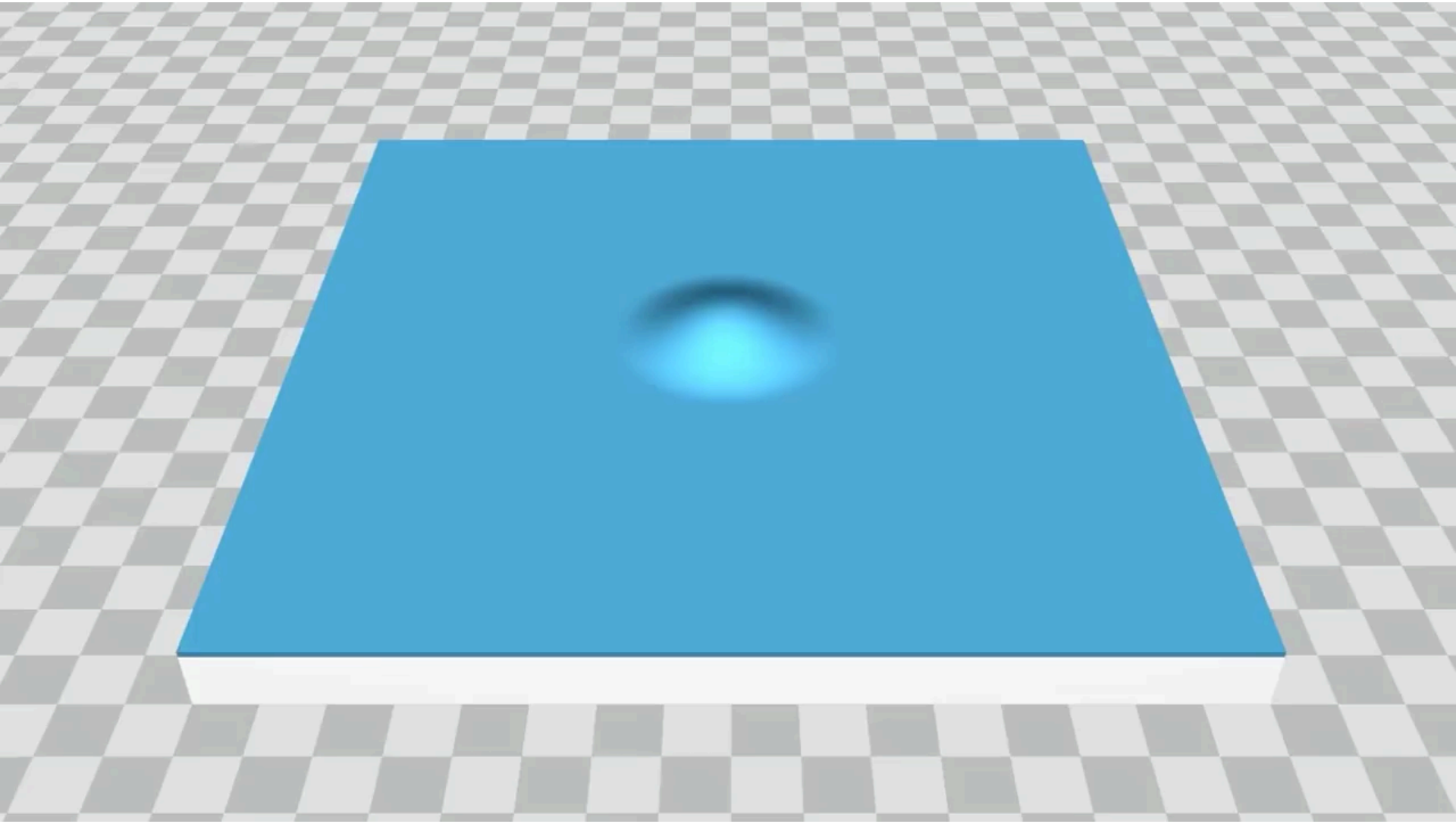
1. Surface waves

With surface tension effects:





I.3.1.4 Capillary waves



https://www.youtube.com/watch?v=BCW6_LAq6qs