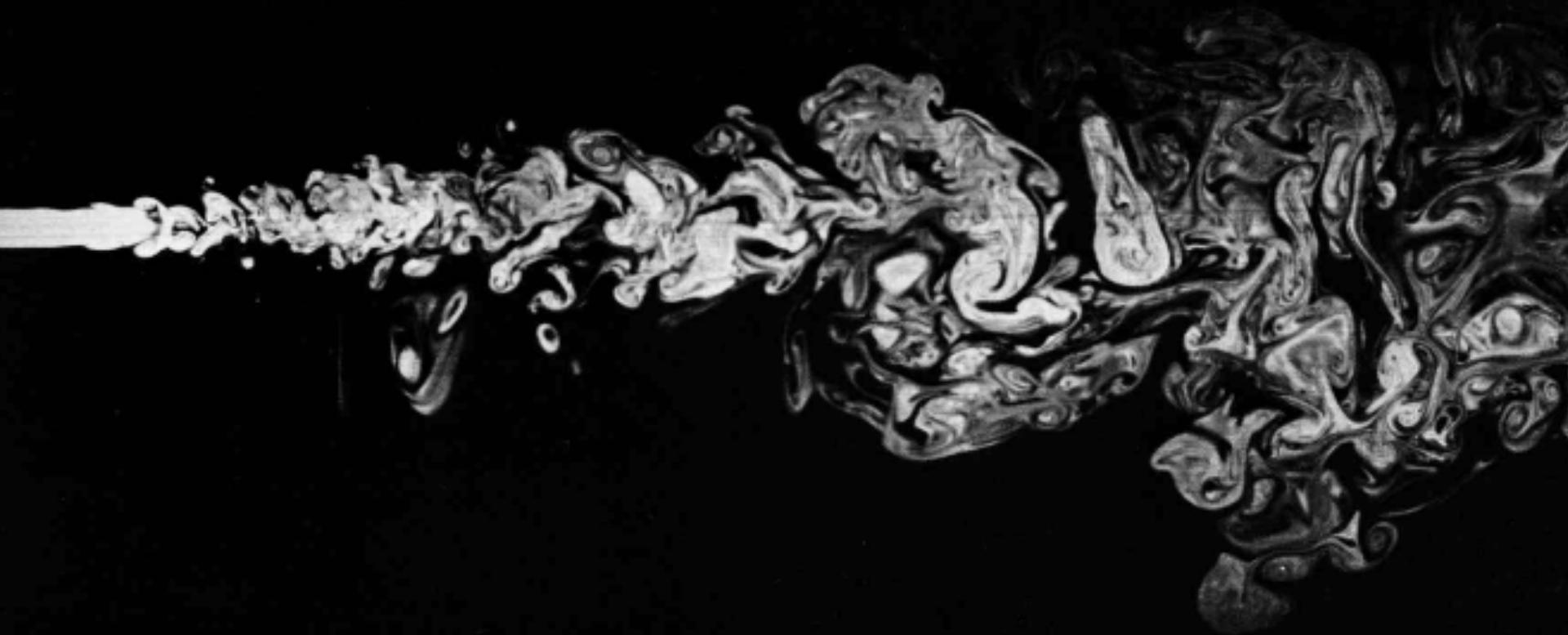


# TURBULENCE

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- **Lesson 1 : [D109]**
  - Introduction
  - Properties of turbulence
- **Lesson 2 : [D109]**
  - 3D turbulence: The Kolmogorov theory
  - 2D turbulence
- **Lesson 3 :[D109]**
  - 2D turbulence (activity)
  - Geostrophic turbulence
  - Surface QG turbulence

- **Lesson 4 :[D109]**
  - Ocean turbulence (activity)
  - Turbulent diffusion
- **Lesson 5 :[D109]**
  - Stratified turbulence

Presentations and material will be available at :

**[jgula.fr/Turb/](http://jgula.fr/Turb/)**

# Numerical Modelling

Jonathan GULA  
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## Evaluation

- The evaluation will be based on a report, which consists in a numerical activity and summary of research articles
- Written Report will be **due Mar. 11**

# References:

- Vallis G.K., Atmospheric and Oceanic Fluid Dynamics.
- MIT online course: <https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-820-turbulence-in-the-ocean-and-atmosphere-spring-2007/lecture-notes/>
- LaCasce J.H., Turbulence in the Atmosphere and Ocean.

# TURBULENCE

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## INTRODUCTION

- **Lesson 1 - Introduction:**

- *What is turbulence?*
- *Why do we care about turbulence?*
- *Where does it come from?*
- Properties of turbulence:
  - Unpredictability
  - Scale interactions
- The closure problem

# What is turbulence?

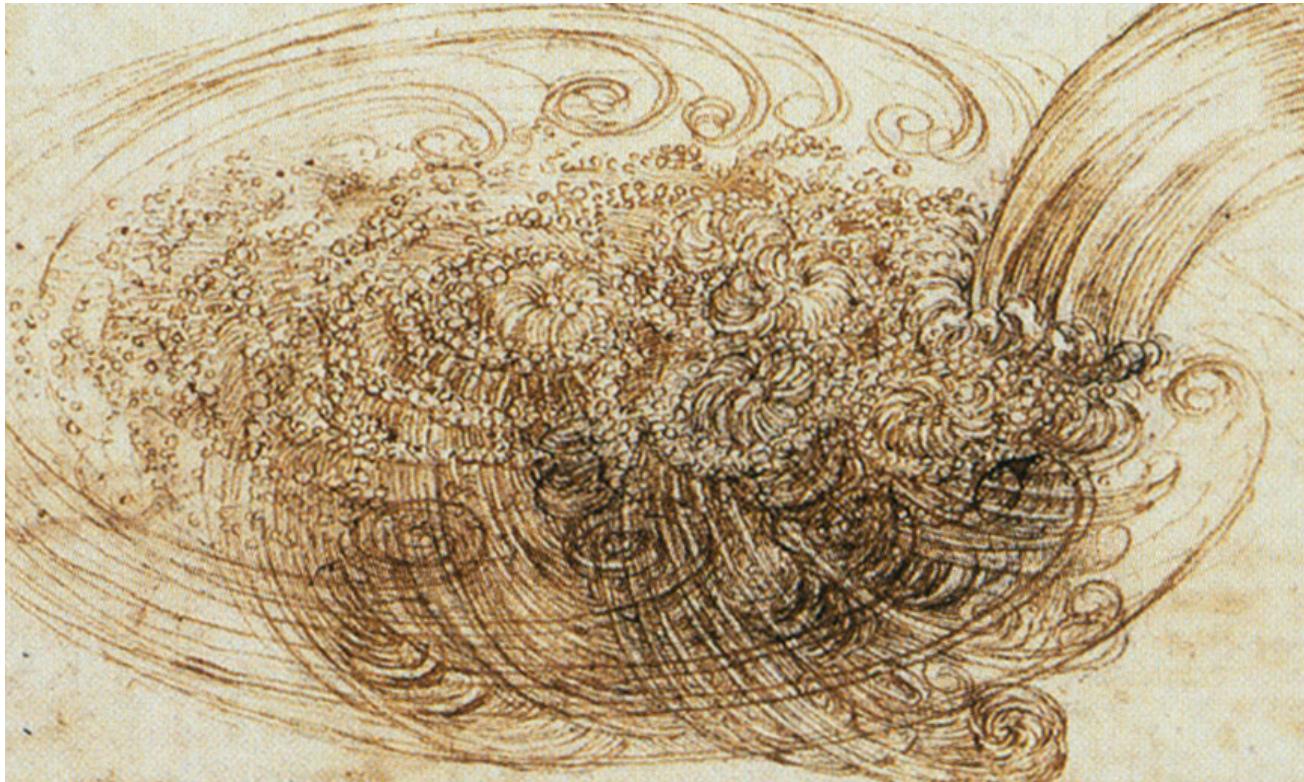
**Werner Heisenberg  
(1901 - 1976)**



“When I meet God, I’m going to ask him two questions: why relativity? And why turbulence?”

I really believe he’ll have an answer for the first.”

# What is turbulence?



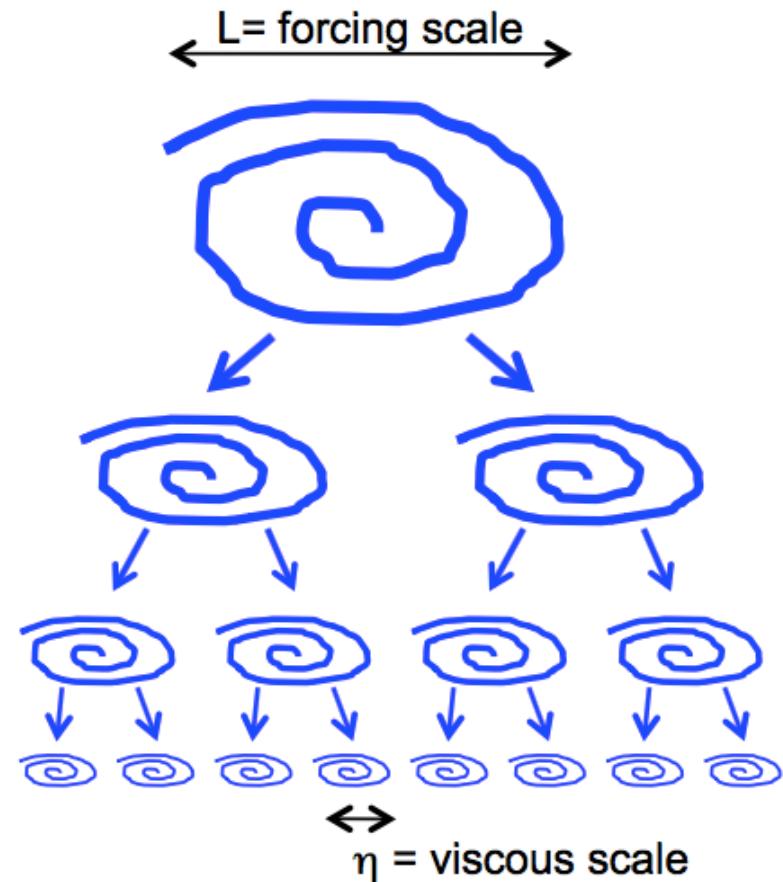
**turbolenza** by da Vinci [1507]

*“...the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large.”*

# What is turbulence?

No formal definition of turbulence. One description as a poem from by L.F. Richardson, in 1922:

*Big whirls have little whirls,  
which feed on their velocity;  
And little whirls have lesser whirls,  
And so on to viscosity.*

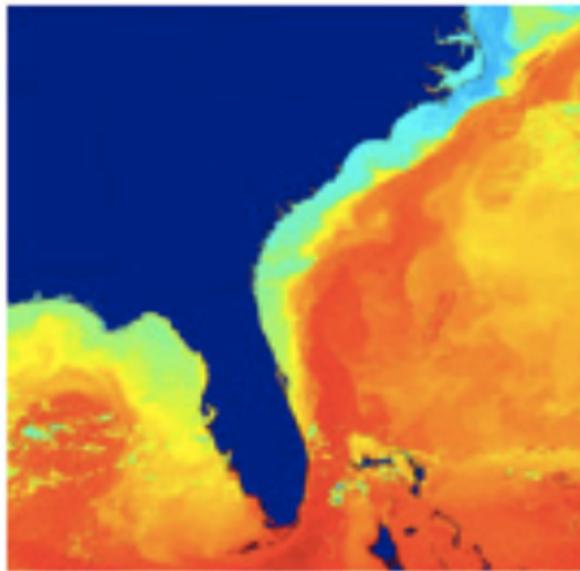
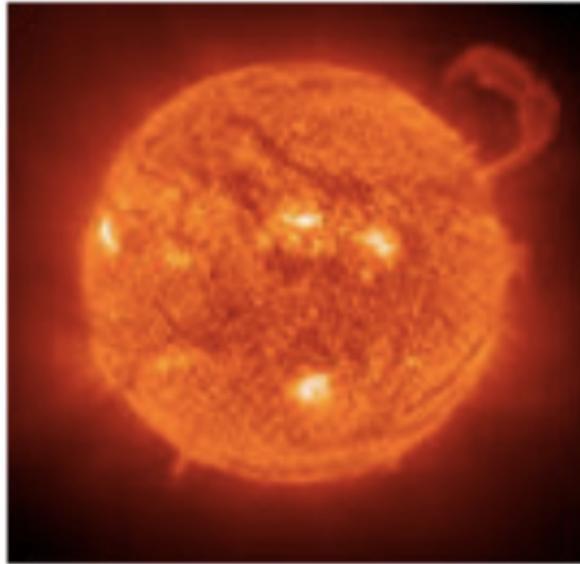


# Turbulence in nature



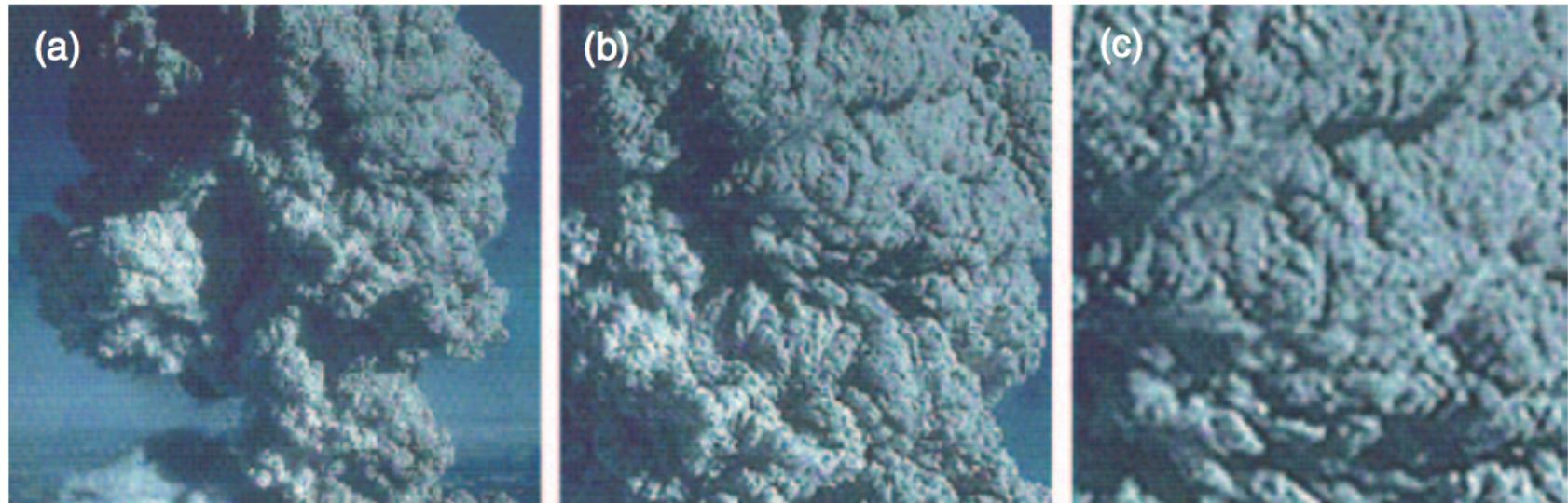
(a) water flow from a faucet, (b) water from a garden hose, (c) flow past a curved wall, and (d) and (e) whitewater rapids whose turbulent fluctuations are so intense that air is entrained by the flow and produces small bubbles that diffusely reflect light and cause the water to appear white.

# Turbulence in nature



Examples of turbulent flows at the surface of the Sun, in the Earth's atmosphere, in the Gulf Stream at the ocean surface, and in a volcanic eruption.

# Turbulence in nature



**Figure 2. Scale-Independence in Turbulent Flows**

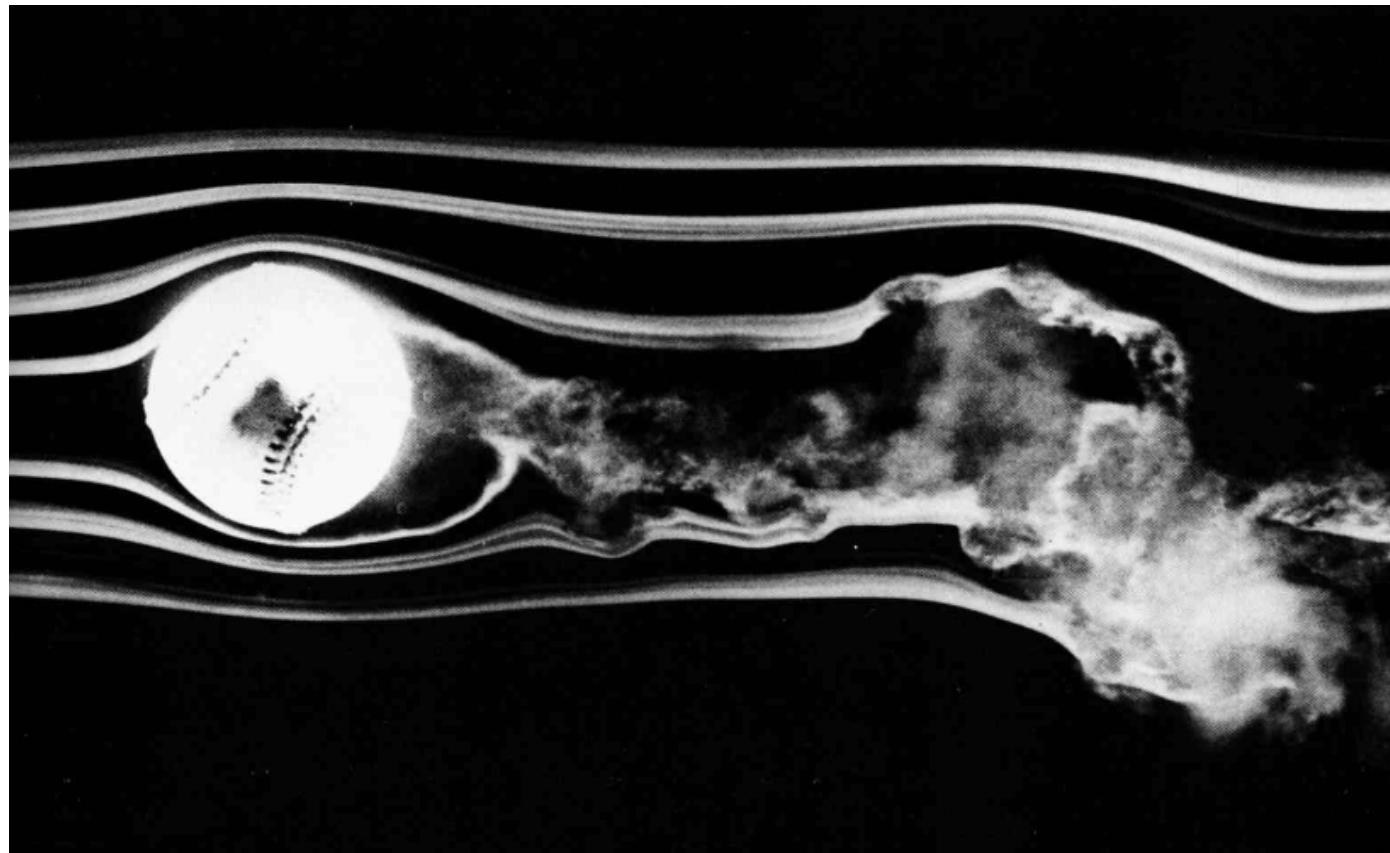
The turbulent structure of the pyroclastic volcanic eruption of Mt. St. Helens shown in (a) is expanded by a factor of 2 in (b) and by another factor of 2 in (c). The characteristic scale of the plume is approximately 5 km. Note that the expanded images reveal the increasingly finer scale structure of the turbulent flow. The feature of scale independence, namely, that spatial images or temporal signals look the same (statistically) under increasing magnification is called self-similarity.

# Turbulence in nature



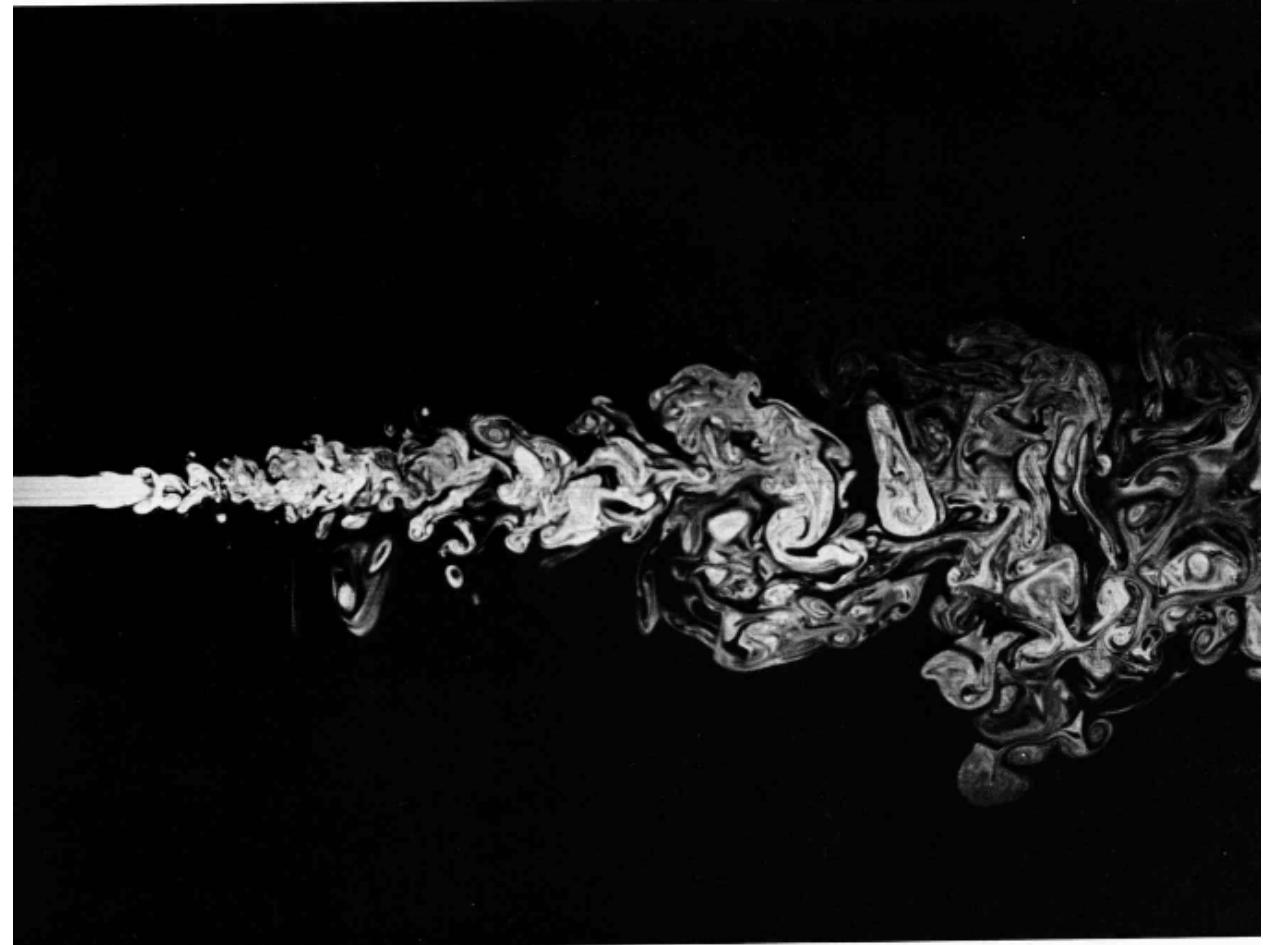
A panoramic Hubble image of the Carina nebula showing the turbulent effects of the stellar winds and ionizing radiation from massive stars on the molecular cloud out of which the stars were born. The width of the image is 16 pc. [<https://www.nature.com/articles/s41550-020-01277-w>]

# Turbulence in the lab

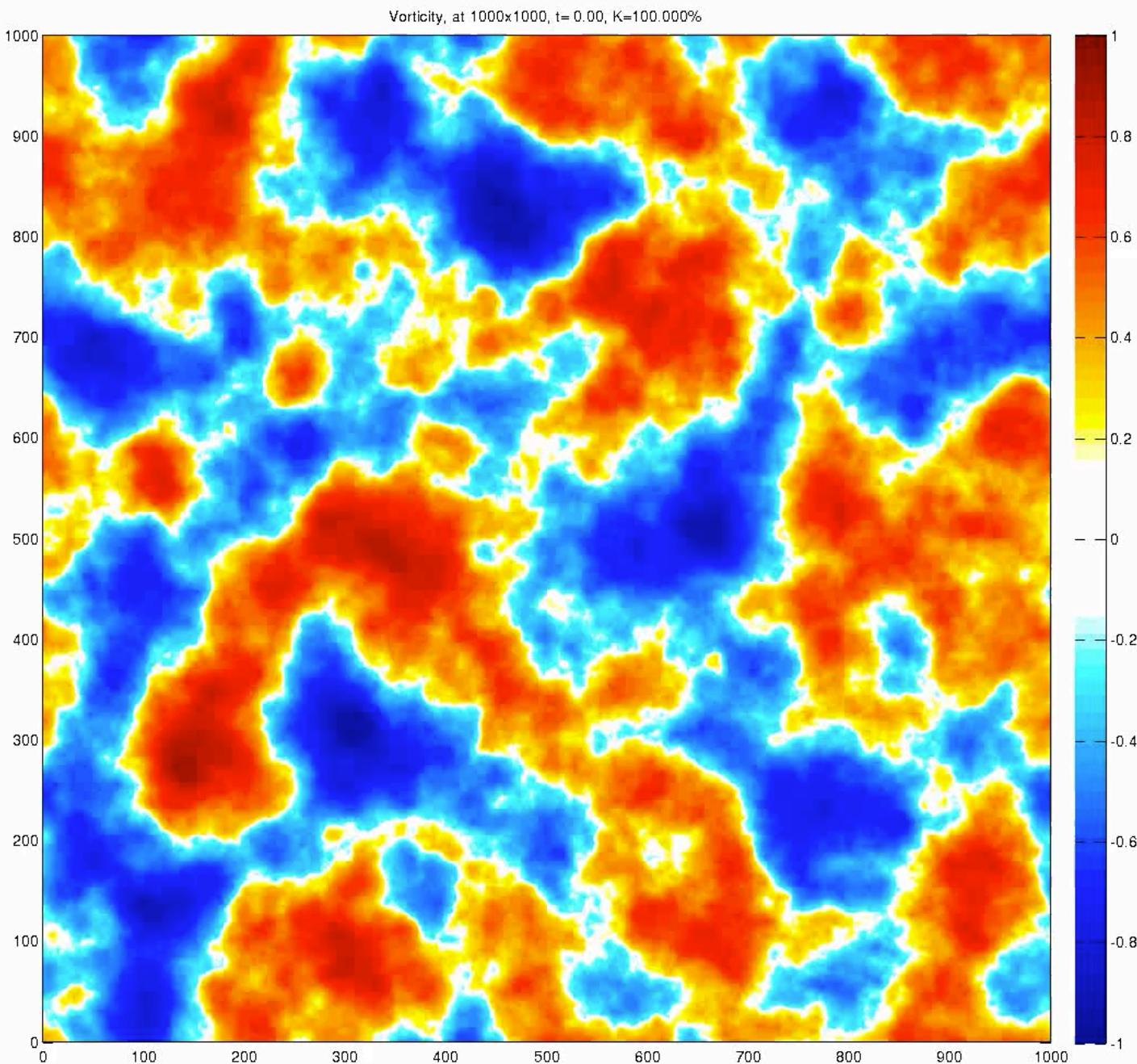


Spinning Baseball [Van Dyke, 82]

# Turbulence in the lab

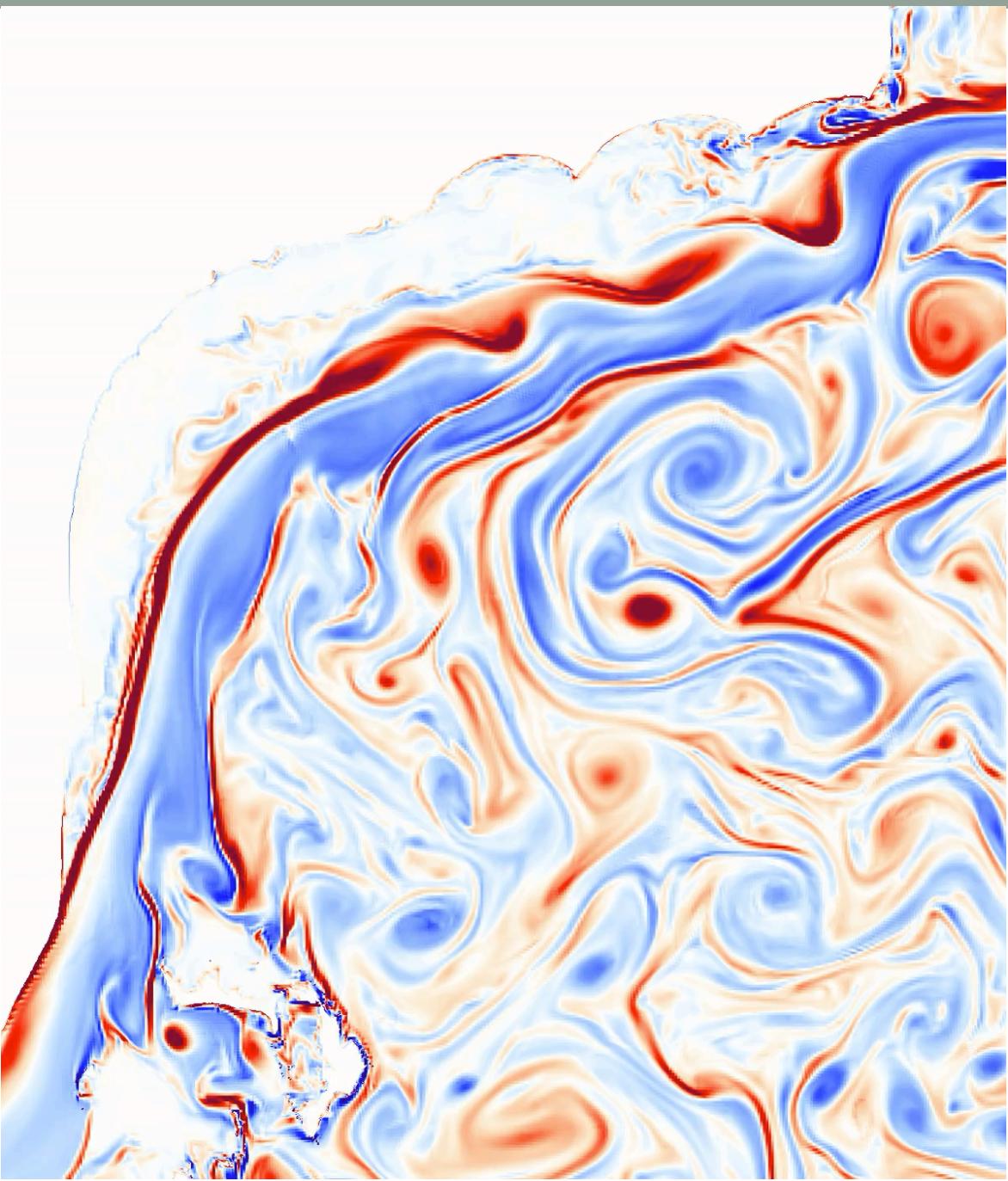


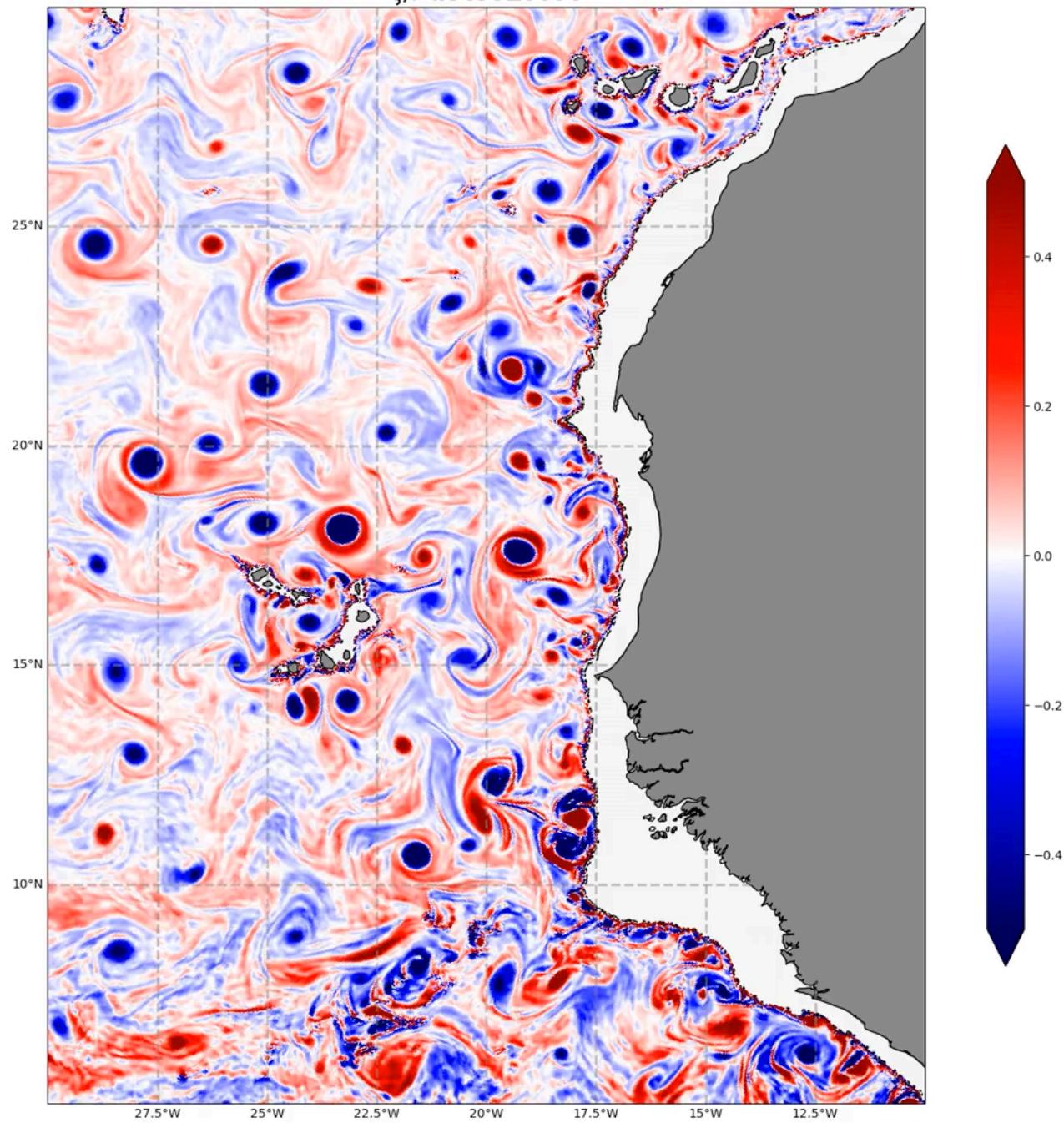
Turbulent water jet ( $Re = 2300$ ) [Van Dyke, 82]



# Introduction

# Turbulence in a realistic ocean model



$\zeta/f$  at iso27.60



Entry: V0074

## The beauty of turbulent convection: a particle tracking endeavor

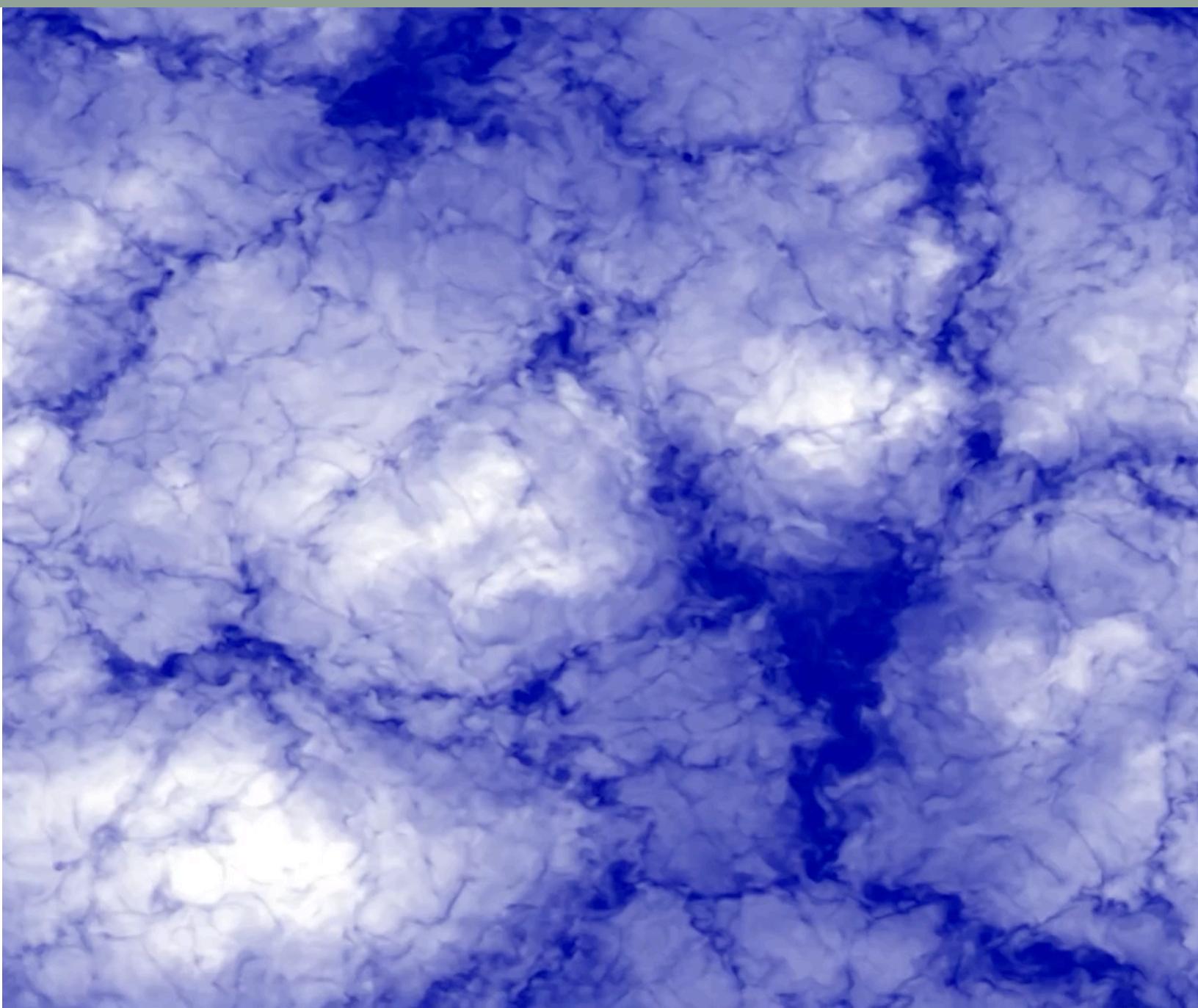
Philipp Godbersen, Johannes Bosbach, Daniel Schanz, Andreas Schröder

Institute of Aerodynamics and Flow Technology  
German Aerospace Center (DLR)



# Introduction

# Atmospheric Convection



# Why do we care about turbulence?

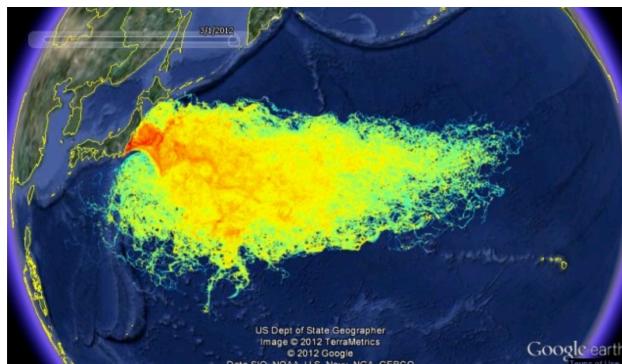
- Weather prediction



- Aeronautics



- Dispersion of pollutants



And many more ...

- **Lesson 1 - Introduction:**

- *What is turbulence?*
- *Why do we care about turbulence?*
- ***Where does it come from?***
- Properties of turbulence:
  - Unpredictability
  - Scale interactions
- The closure problem

# Navier-Stokes Equations

The Navier-Stokes equations probably contain all of turbulence.

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Momentum equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0$$

Mass conservation  
(no source/sink)

*Kinematic viscosity:*

$$\nu_{water} \approx 10^{-6} \text{m}^2 \text{s}^{-1}$$

$$\nu_{air} \approx 1.5 \times 10^{-5} \text{m}^2 \text{s}^{-1}$$

# Navier-Stokes Equations

## Millennium Prize problems:

Since understanding the Navier–Stokes equations is considered to be the first step to understanding the elusive phenomenon of [turbulence](#), the [Clay Mathematics Institute](#) in May 2000 made this problem one of its seven [Millennium Prize problems](#) in mathematics. It offered a [US \\$1,000,000](#) prize to the first person providing a solution for a specific statement of the problem:<sup>[1]</sup>

*Prove or give a counter-example of the following statement:*

In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

See <https://www.jgula.fr/Turb/Articles/Robert.pdf>

# Navier-Stokes Equations

+ Boussinesq approximation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{\mathcal{F}}$$

Momentum equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation  
(no source/sink)

# Navier-Stokes Equations

+ Boussinesq approximation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} + f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{F}$$

Time  
variation

Advection  
(inertia)

Rotation

Gravity

Pressure  
gradient

Dissipation  
(viscosity)

Forcings

$$\vec{\nabla} \cdot \vec{u} = 0$$

Mass conservation  
(no source/sink)

# Navier-Stokes Equations

+ Boussinesq approximation

- Turbulence arises from the non-linear terms

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} - f \vec{k} \times \vec{u} + g \vec{k} = -\frac{\vec{\nabla} P}{\rho_0} + \nu \vec{\nabla}^2 \vec{u} + \vec{F}$$

Advection  
(inertia)

- Only one here = advection (*quadratic nonlinearity*)
- Turbulence results from the nonlinear nature of advection, which enables interaction between motions on different spatial scales.

# Navier-Stokes Equations

+ Boussinesq approximation

- x-momentum equation:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

# Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

$$\frac{U}{T} \quad \frac{U^2}{L} \quad fU \quad \frac{P}{\rho_0 L} \quad \frac{\nu U}{L^2} \quad F$$

# Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

$\frac{U}{T}$	$\frac{U^2}{L}$	$fU$	$\frac{P}{\rho_0 L}$	$\frac{\nu U}{L^2}$	$F$
$\frac{L^2}{\nu T}$	$\frac{UL}{\nu}$	$\frac{fL^2}{\nu}$	$\frac{PL}{\rho_0 \nu U}$	1	$\frac{FL^2}{\nu U}$

# Equations scalings

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

$$\frac{UL}{\nu}$$

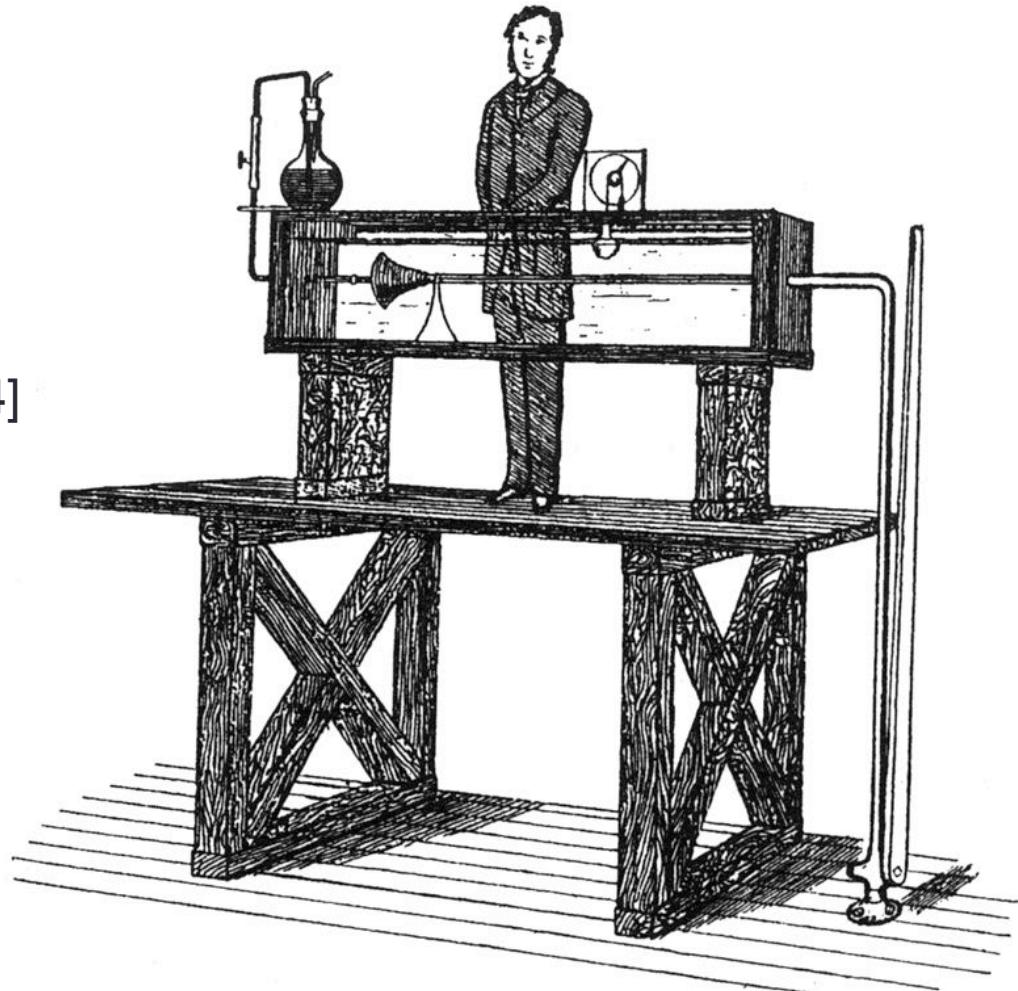
= Reynolds Number

the ratio of the non-linear terms to the viscous terms

# Reynolds Number

$$Re = \frac{UL}{\nu}$$

Osborne Reynolds experiment [1884]



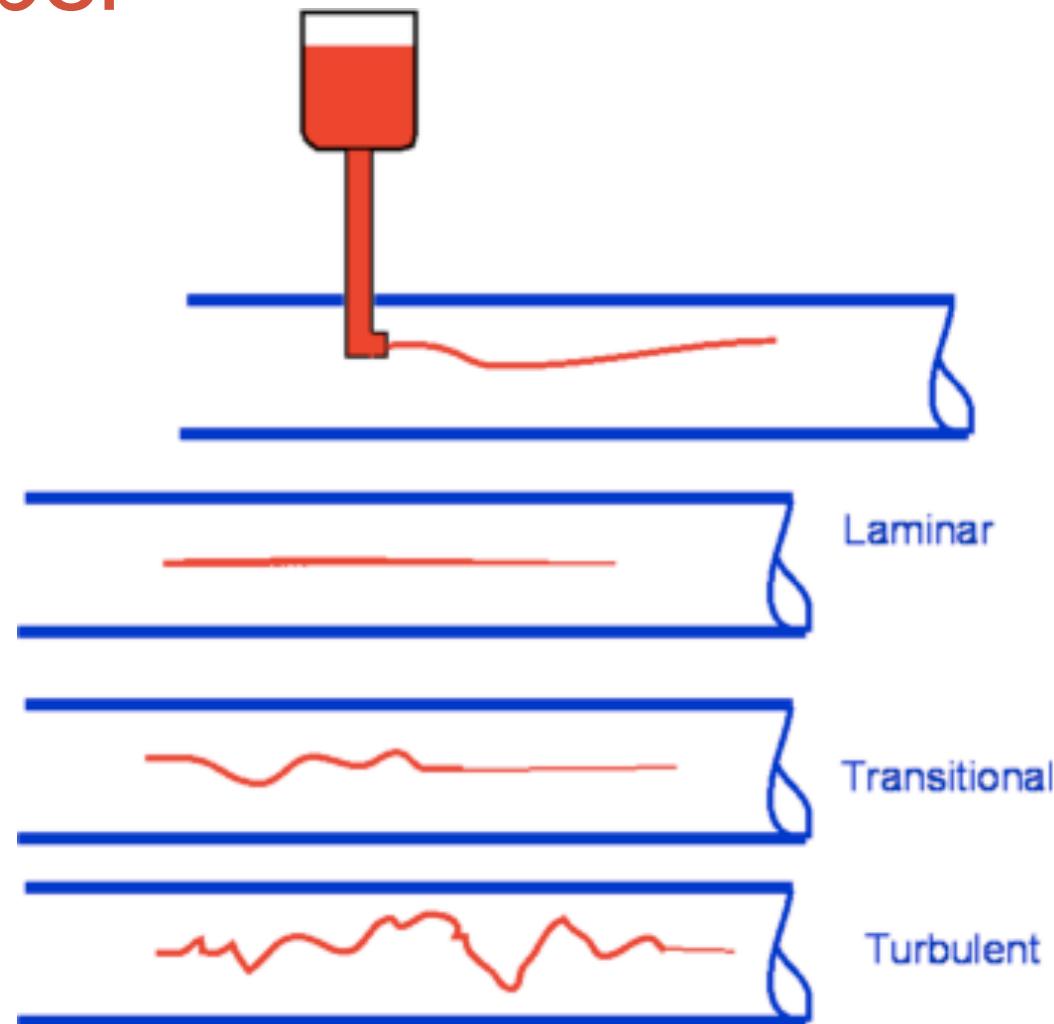
*An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous*

# Reynolds Number

$$Re = \frac{UL}{\nu}$$

$Re < 2000$  = laminar

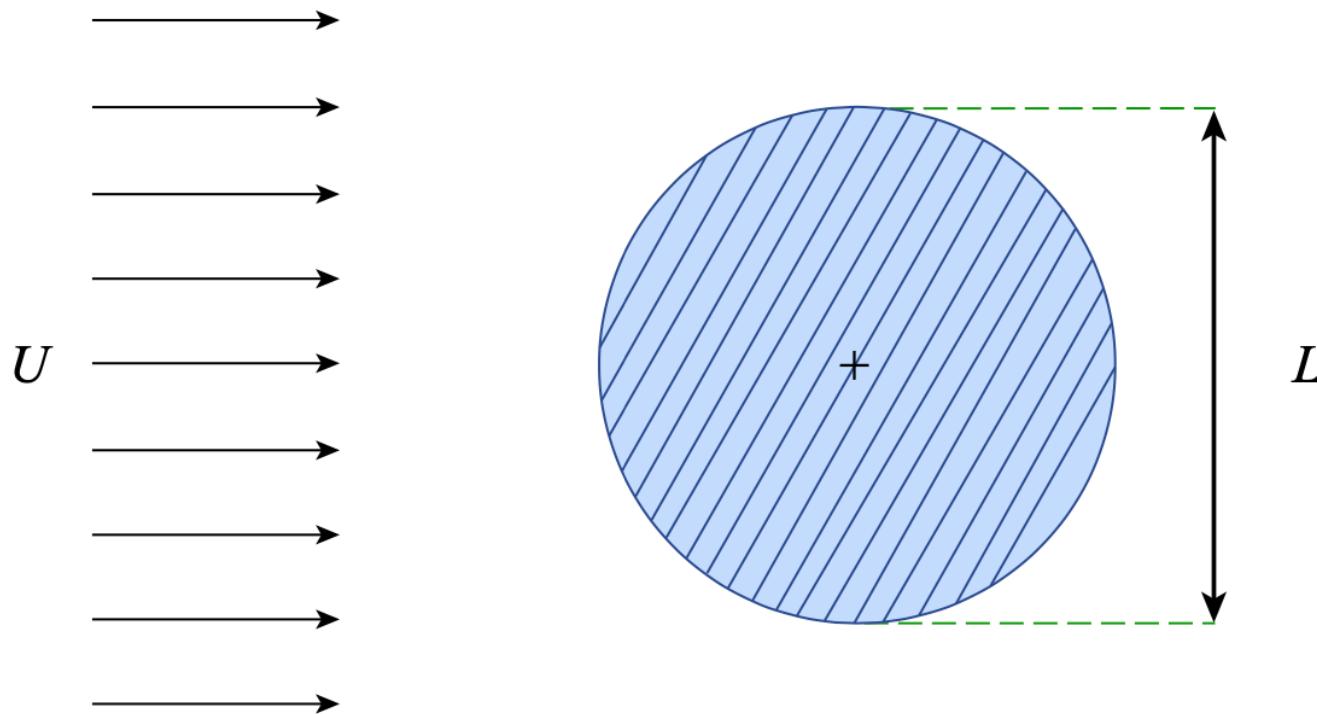
$Re > 4000$  = turbulent



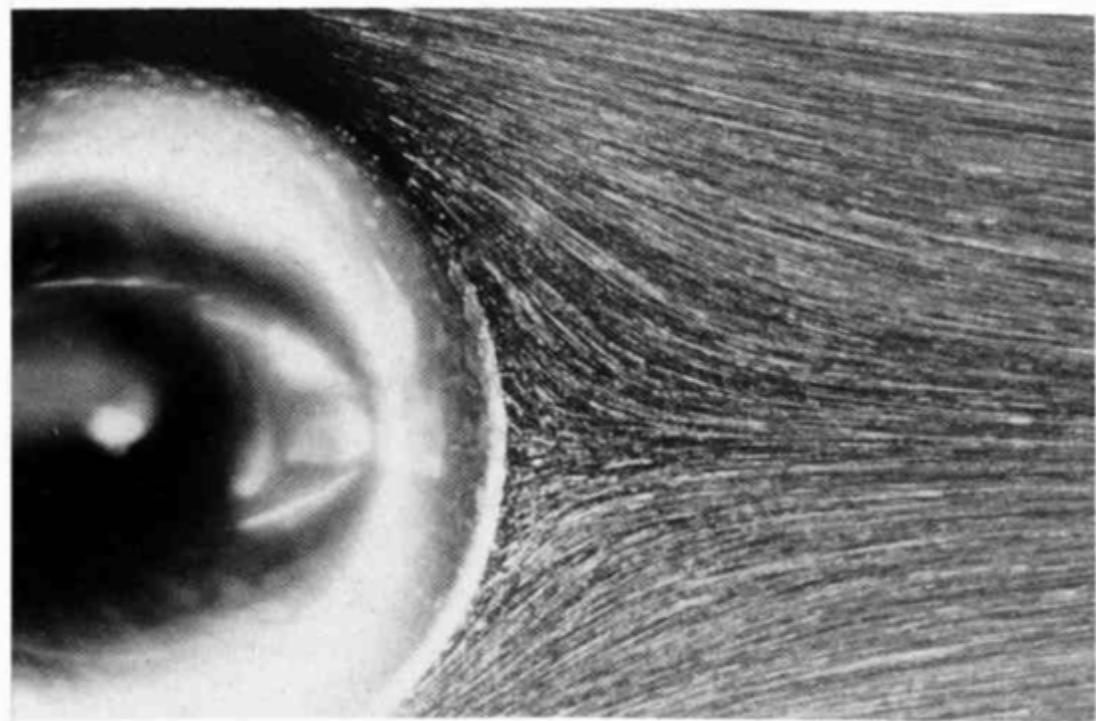
Reynolds experiment

# Transition to turbulence

Uniform flow with velocity  $U$ , incident on a cylinder of diameter  $L$

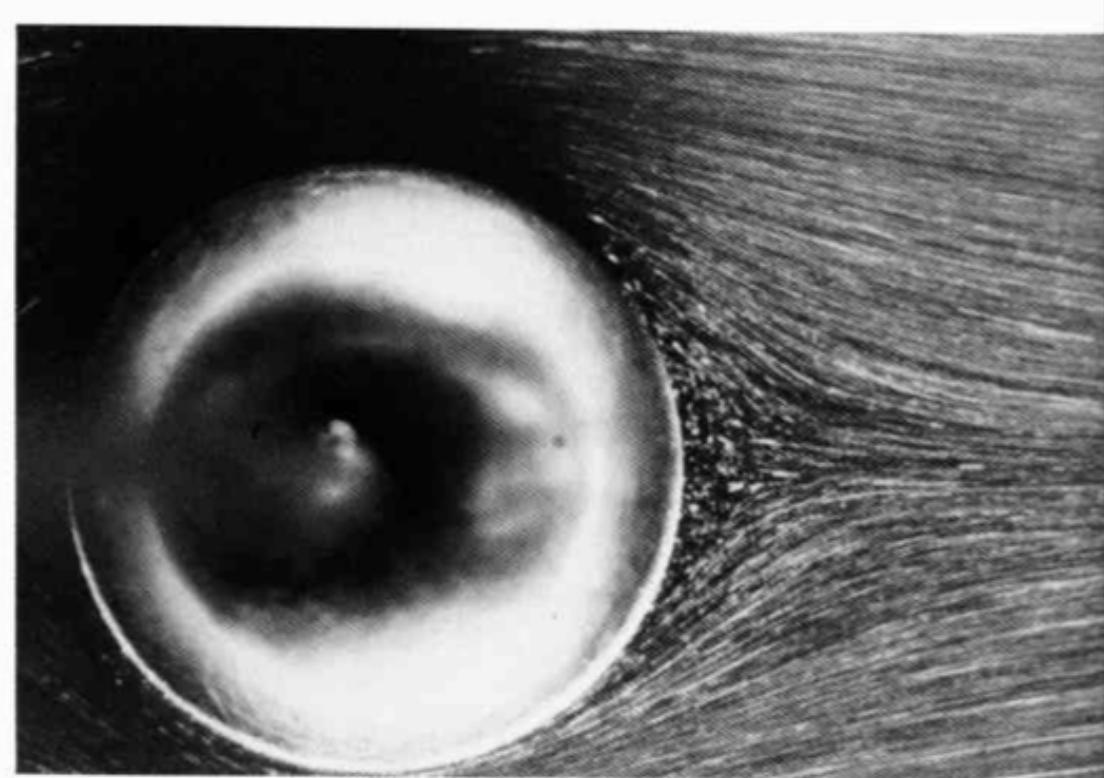


# Transition to turbulence



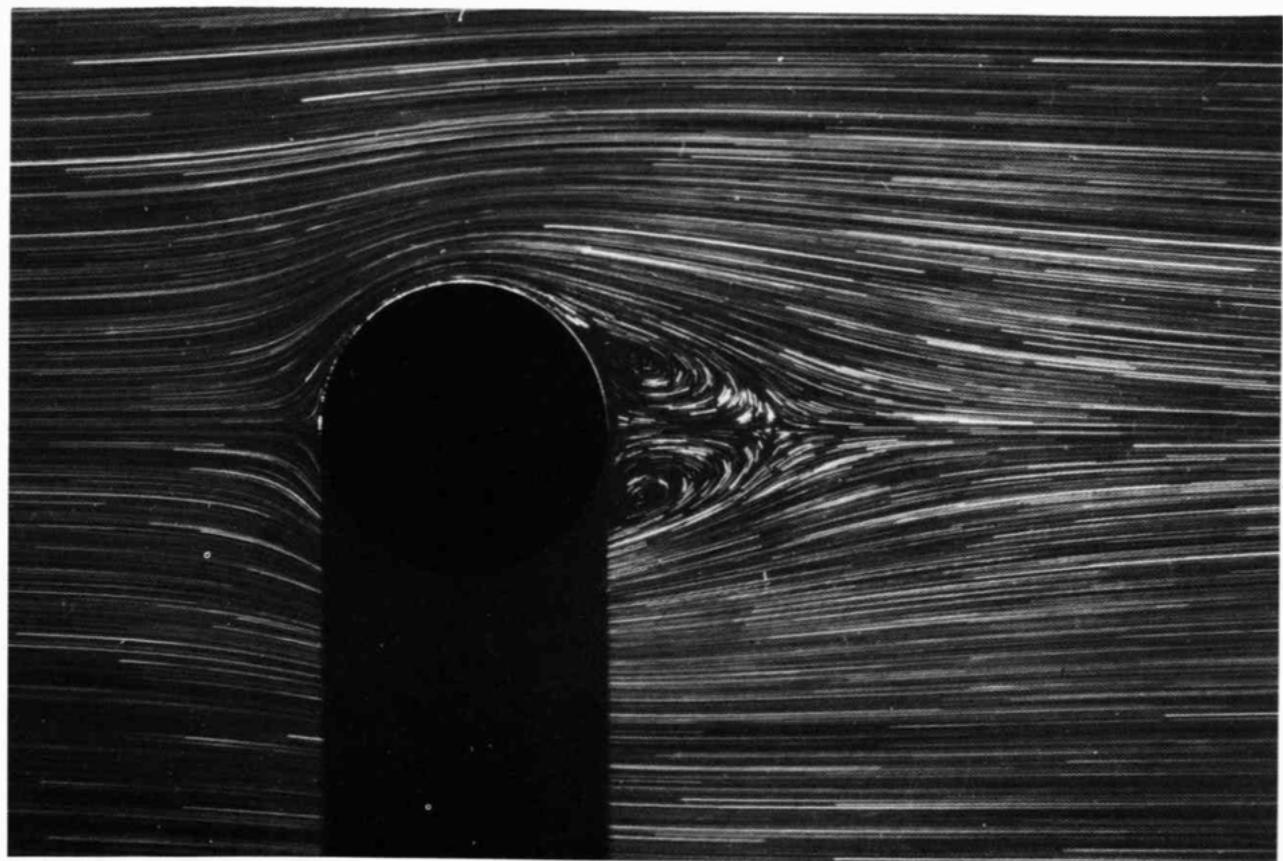
49. Sphere at  $R=25.5$ . Although it is not obvious, the flow is believed just to have separated at the rear at this Reynolds number, in contrast to the unseparated flow of figure 27. Aluminum dust is illuminated in water. *Taneda 1956b*

# Transition to turbulence



50. Sphere at  $R=26.8$ . At this slightly higher speed the flow has clearly separated over the rear of the sphere, to form a thin standing vortex ring. Aluminum dust is illuminated in water. *Taneda 1956b*

# Transition to turbulence

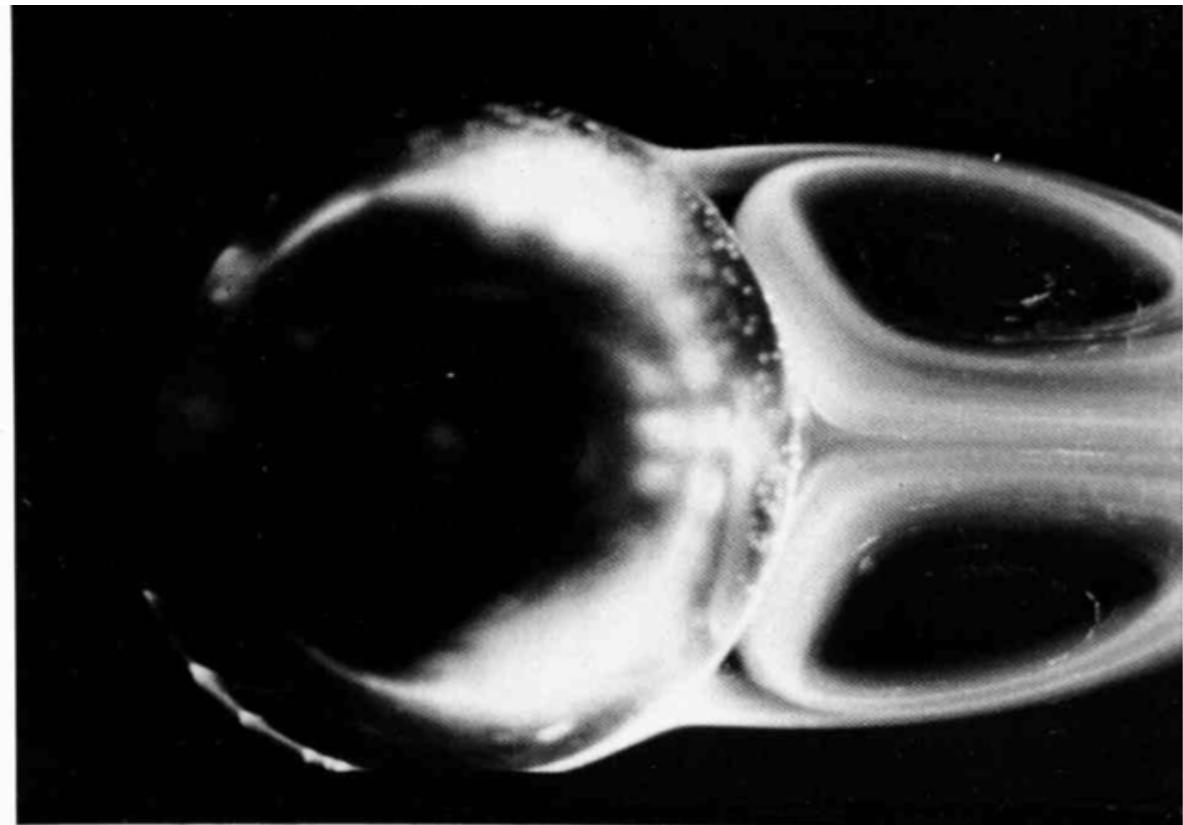


51. Sphere at  $R=56.5$ . As in figure 8, the sphere is falling steadily down the axis of a tube filled with oil, but here so large that the influence of the walls is negligible. Magne-

sium cuttings are illuminated by a sheet of light, which casts the shadow of the sphere. Archives de l'Académie des Sciences de Paris. Payard & Coutanceau 1974

# Transition to turbulence

52. Sphere at  $R=104$ . At this Reynolds number the recirculating wake extends a full diameter downstream, but is perfectly steady, as for the circle in figure 44. Visualization is by a thin coating of condensed milk on the sphere, which gradually melts and is carried into the stream of water. *Taneda 1956b*

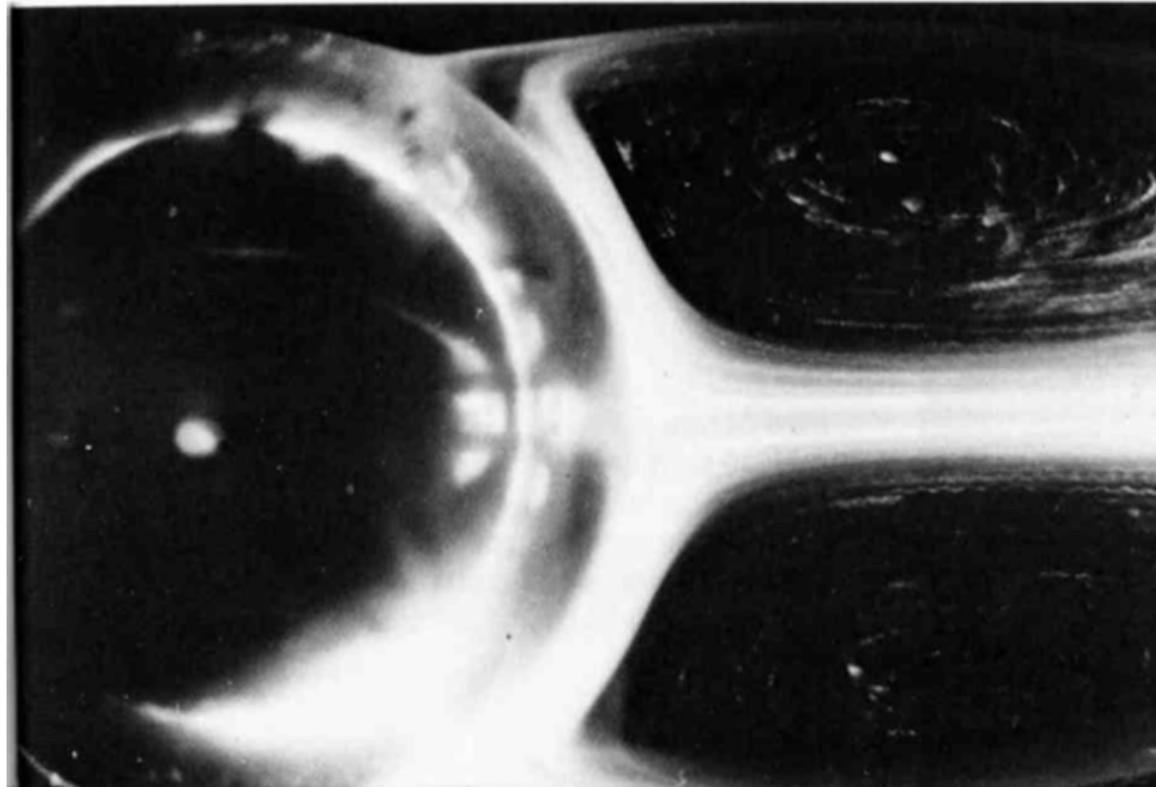


# Transition to turbulence

53. Sphere at  $R=118$ . The wake grows more slowly in axisymmetric than plane flow. These photographs have shown that the length of the recirculating region is proportional to the logarithm of the Reynolds number, whereas it grows linearly with Reynolds number for a cylinder. Aluminum dust shows the flow of water. Taneda 1956b

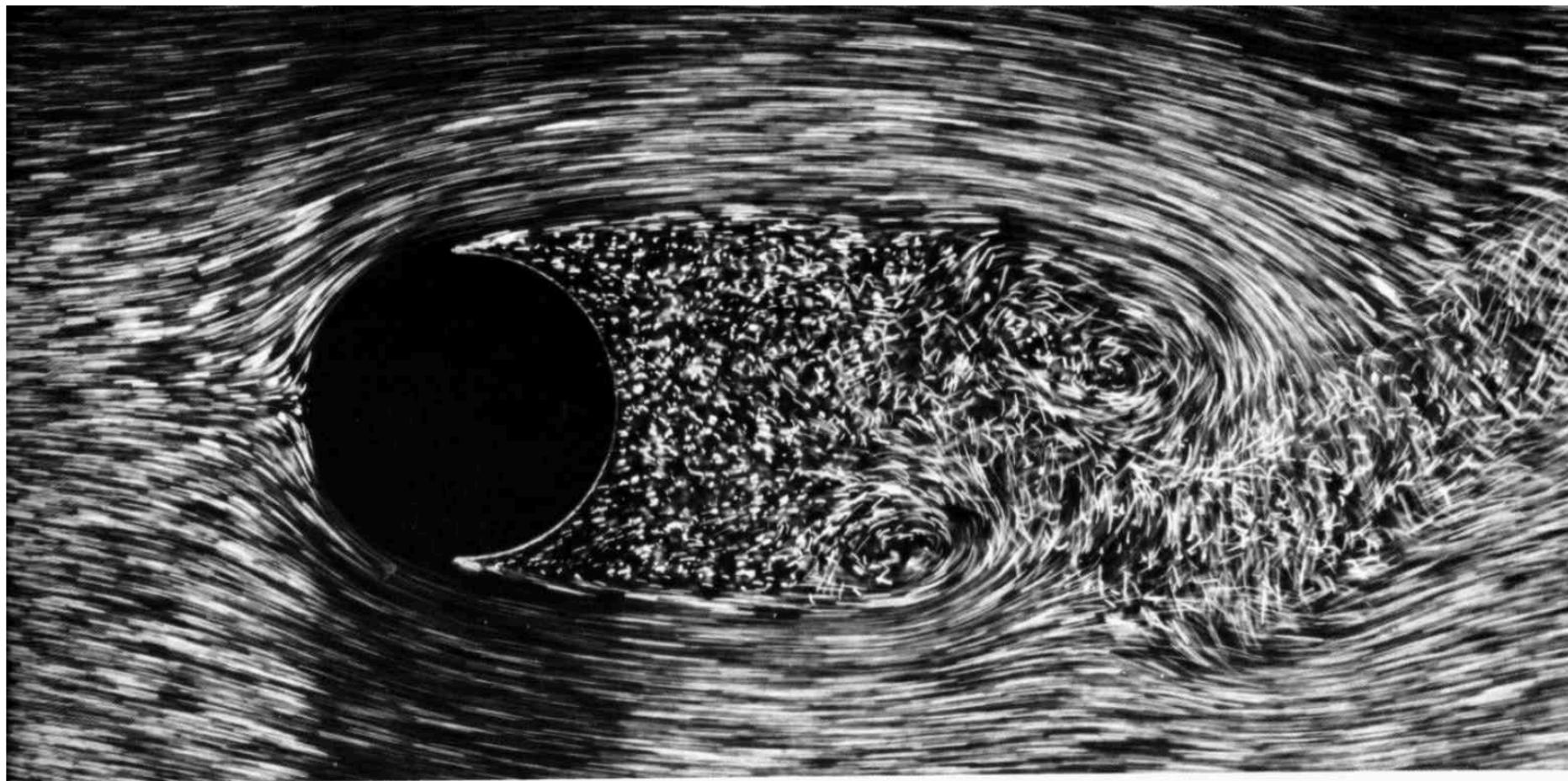


# Transition to turbulence



**54. Sphere at  $R=202$ .** The rear of the recirculating region behind a sphere begins to oscillate slowly at a Reynolds number of about 130, but the flow is still perfectly laminar at this higher speed. Visualization is by condensed milk in water. *Taneda 1956b*

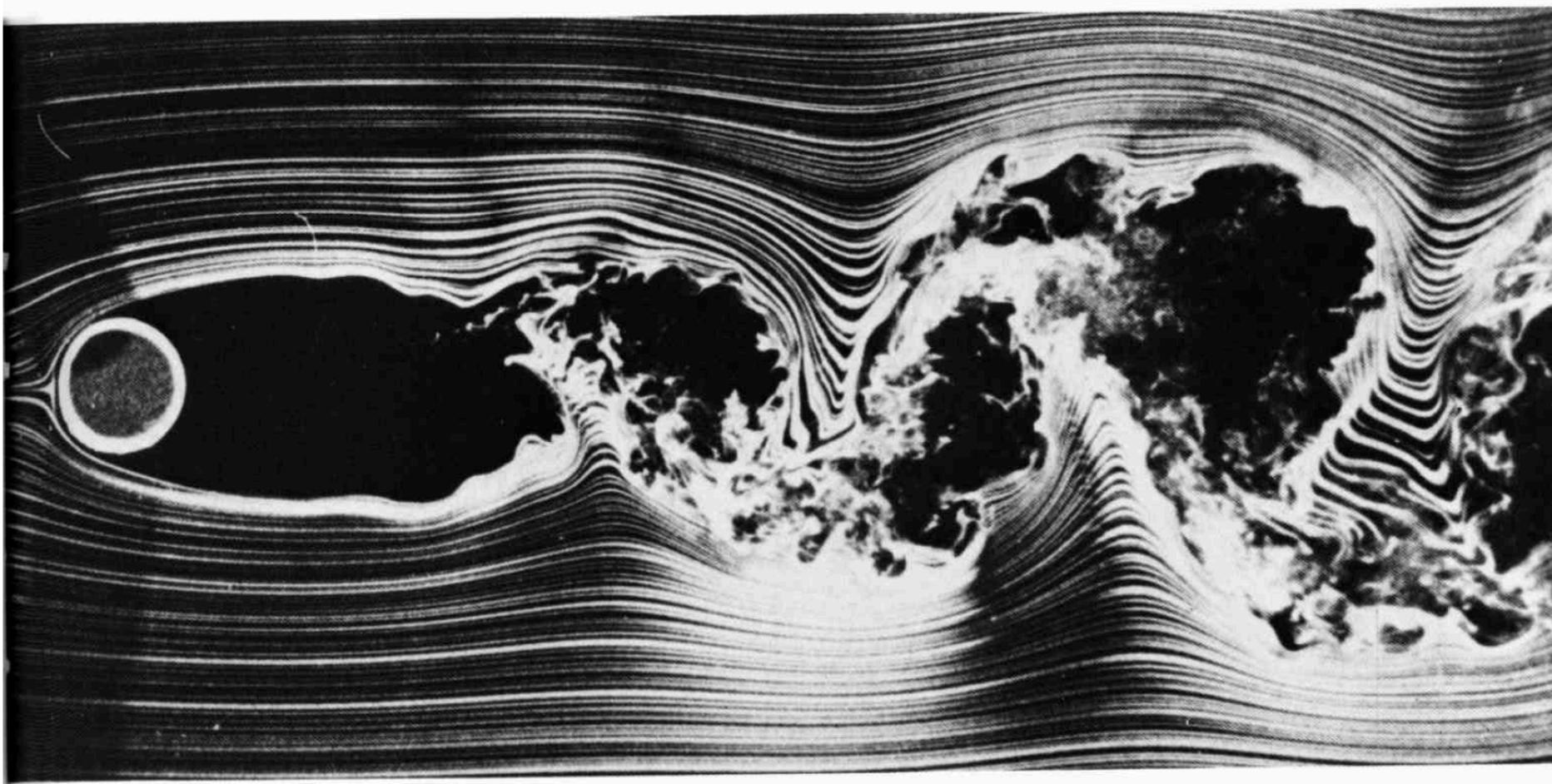
# Transition to turbulence



**47. Circular cylinder at  $R=2000$ .** At this Reynolds number one may properly speak of a boundary layer. It is laminar over the front, separates, and breaks up into a turbulent wake. The separation points, moving forward as

the Reynolds number is increased, have now attained their upstream limit, ahead of maximum thickness. Visualization is by air bubbles in water. ONERA photograph, Werlé & Gallon 1972

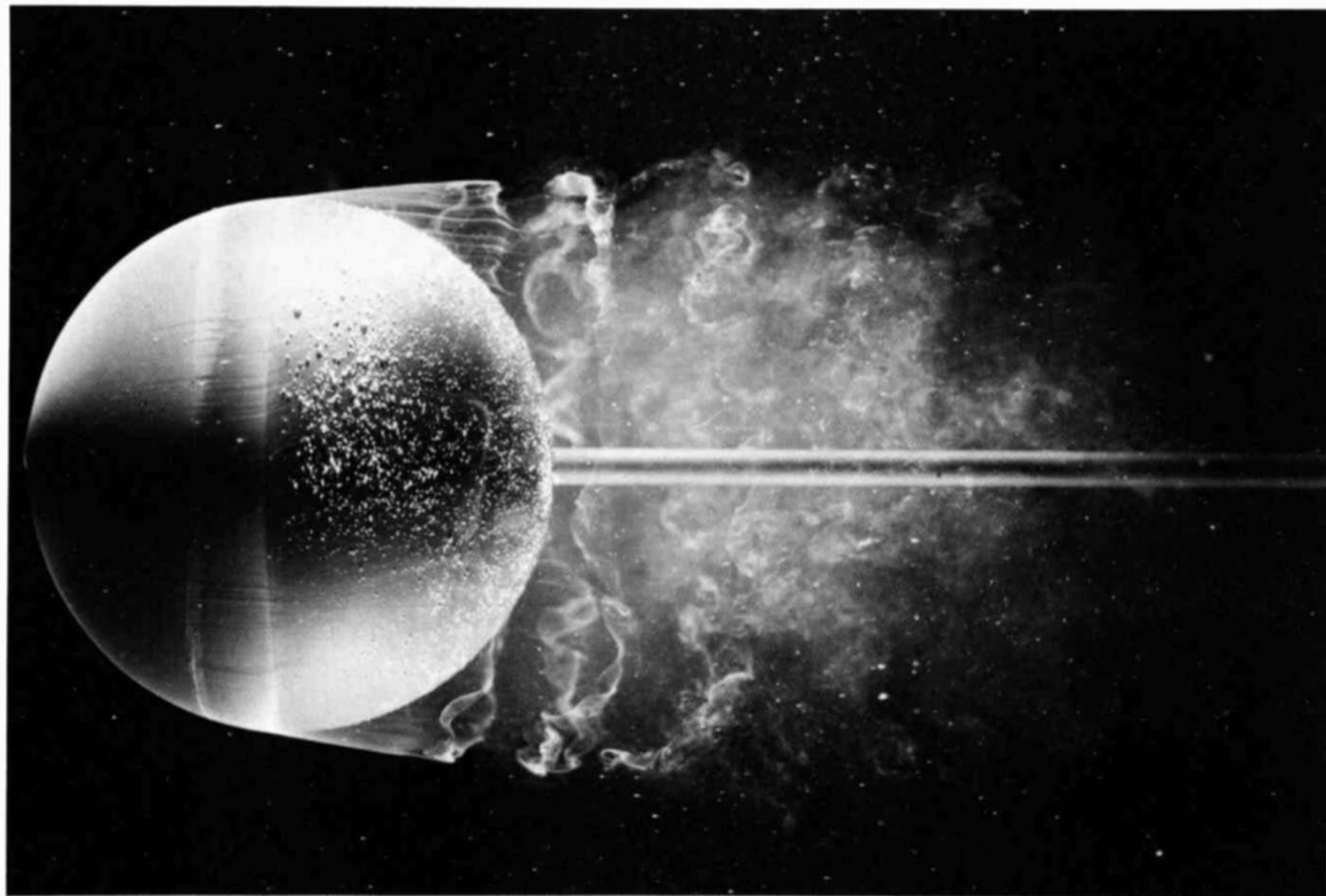
# Transition to turbulence



48. Circular cylinder at  $R=10,000$ . At five times the speed of the photograph at the top of the page, the flow pattern is scarcely changed. The drag coefficient consequently remains almost constant in the range of Reynolds

number spanned by these two photographs. It drops later when, as in figure 57, the boundary layer becomes turbulent at separation. *Photograph by Thomas Corke and Hassan Nagib*

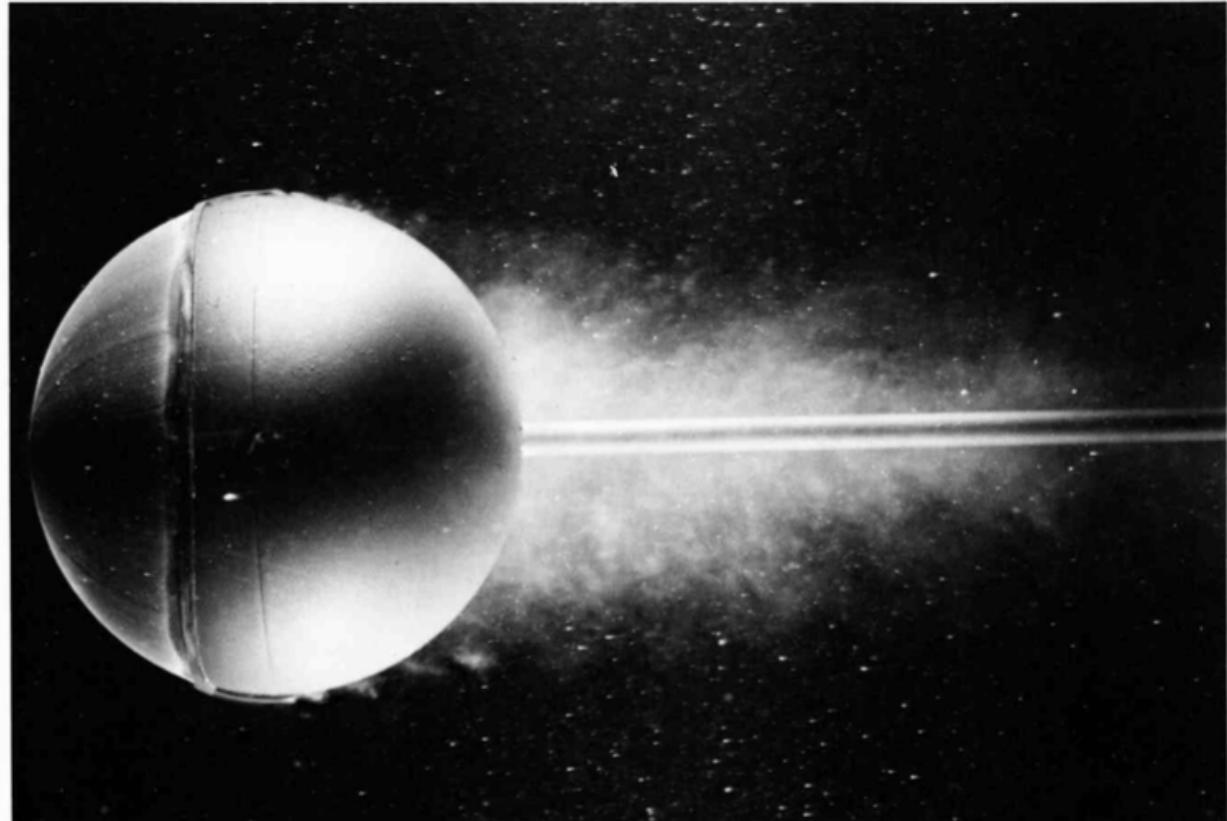
# Transition to turbulence



55. Instantaneous flow past a sphere at  $R=15,000$ . Dye in water shows a laminar boundary layer separating ahead of the equator and remaining laminar for almost one

radius. It then becomes unstable and quickly turns turbulent. ONERA photograph, Werlé 1980

# Transition to turbulence



57. Instantaneous flow past a sphere at  $R=30,000$  with a trip wire. A classical experiment of Prandtl and Wieselsberger is repeated here, using air bubbles in water. A wire hoop ahead of the equator trips the boundary layer. It becomes turbulent, so that it separates farther

rearward than if it were laminar (opposite page). The drag is thereby dramatically reduced, in a way that occurs naturally on a smooth sphere only at a Reynolds number ten times as great. ONERA photograph, Werlé 1980

# Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} u - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- **Dissipative time scale** = *time scale that is required for molecular friction to bring the motion at scale L to rest*

# Equations scalings

- Scalings: (with U,L,T,etc. typical values)

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- **Dissipative time scale** = *time scale that is required for molecular friction to bring the motion at scale L to rest*

$$\frac{L^2}{\nu T} = \frac{T_\nu}{T} \qquad T_\nu = \frac{L^2}{\nu}$$

*Kinematic viscosity:*

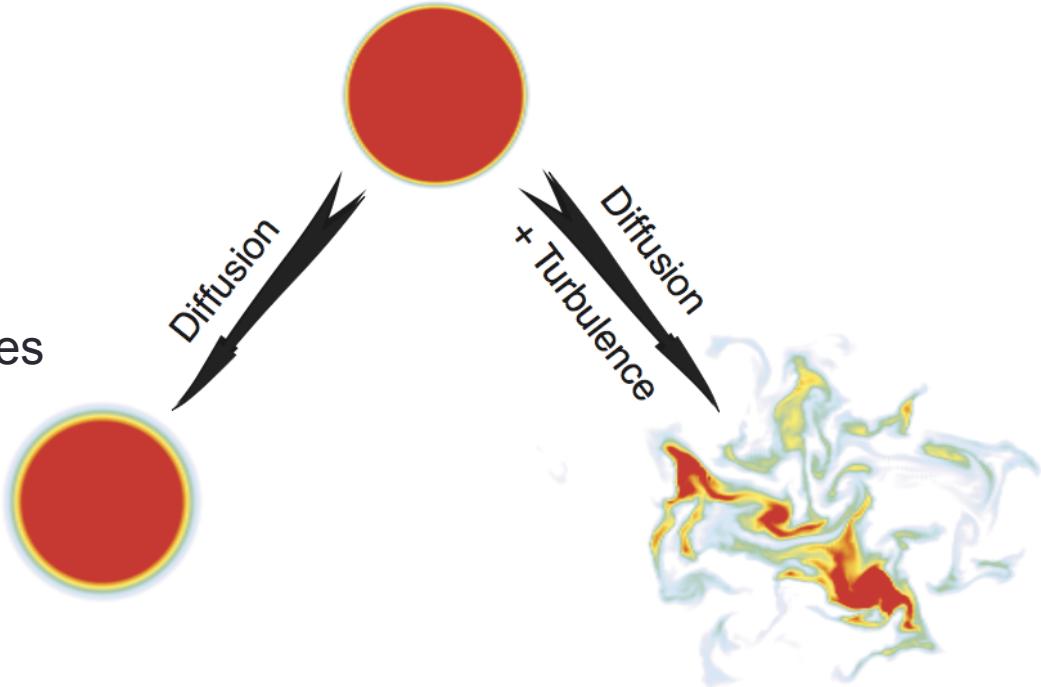
## Activity 1:

- Reynolds number for a mesoscale vortex in the ocean?
- Reynolds number for a storm system in the atmosphere?
- How long would it take for molecular dissipation to halt a storm system?
- How long would it take for molecular dissipation to halt a stirred coffee?



# Role of turbulence

- Momentum transfer
- Scalar mixing  
= homogenization of fluid properties by random molecular motions



*A comparison of **mixing enhanced by turbulence** with **mixing due to molecular processes alone**, as revealed by a numerical solution of the equations of motion.*

*The initial state includes a circular region of dyed fluid in a white background. Two possible evolutions are shown: one in which the fluid is motionless (save for random molecular motions), and one in which the fluid is in a state of fully developed, two-dimensional turbulence. The mixed region (yellow/green) expands much more rapidly in the turbulent case.*

# Properties of turbulence

- No formal definition for turbulence.
- It can be defined as the attempt to bring together our understanding of the laws that govern fluid dynamics (the Navier-Stokes equations) with the irregular nature of real flows.
- A useful approach is to list what properties must be present to consider a flow turbulent.

# Properties of turbulence

- **Broadband spectrum in space and time**
  - *Turbulent flows are characterized by structures on a broad range of spatial and temporal scales, even given smooth or periodic initial conditions and forcing.*
  -

# Properties of turbulence

- **Broadband spectrum in space and time**
- **Dominated by nonlinearity**
  - *A field of non-interacting linear internal waves with many different frequencies and wavenumbers can also have a large range of length scales, but it is not turbulent. Why not? In a turbulent flow the different scales interact, through the nonlinear terms in the equations of motion. And these nonlinear interactions are responsible for the presence of structure on many different scales.*
  -

# Properties of turbulence

- **Broadband spectrum in space and time**
- **Dominated by nonlinearity**
- **Unpredictable in space and time**
  - *Turbulent flows are predictable for only short times and short distances. Even though we know the equations that govern the evolution of the fluid, we cannot make predictions about the details of the flow due to its sensitive dependence on initial and boundary conditions. Predictability, however, can be recovered in a statistical sense. The sensitive dependence on initial and boundary conditions is a fundamental property of chaotic systems.*
  -

# Properties of turbulence

- **Broadband spectrum in space and time**
- **Dominated by nonlinearity**
- **Unpredictable in space and time**
- **Time irreversible**
  - *Turbulent motions are not time reversible. As time goes on, turbulent motions tend to forget their initial conditions and reach some equilibrated state. Turbulence mixes stuff up, it does not unmix it.*

# Non-linearities and scale interactions

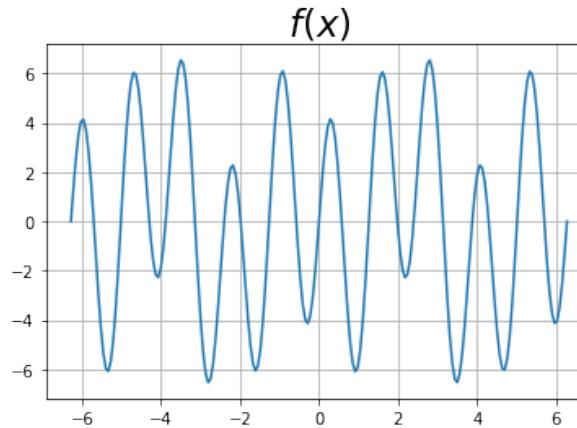
- Fourier decomposition:
- We can extract the component at a single frequency/wavenumber by Fourier transforming :
- 

$$f(x) = \sum_{k_x} \tilde{f}(k_x) e^{ik_x x}$$

$$\tilde{f}(k_x) = \int f(x) e^{-ik_x x} dx$$

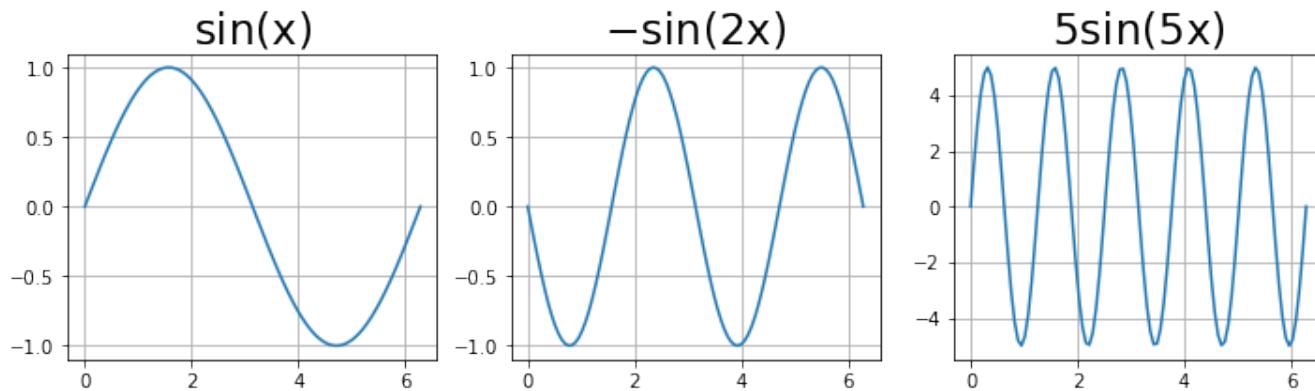
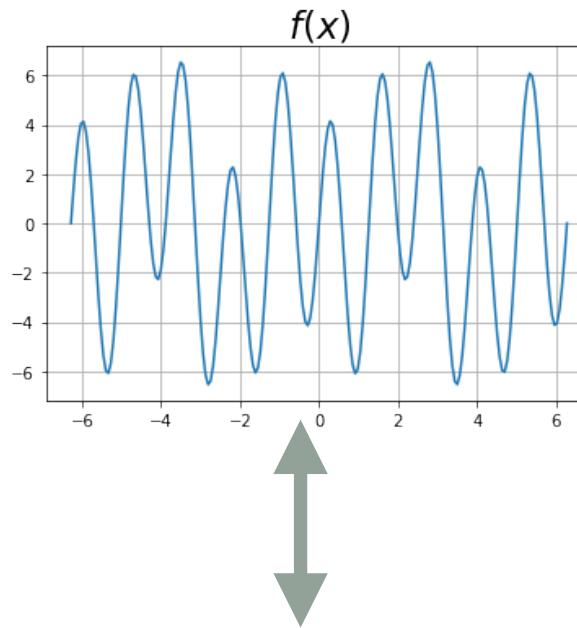
# Non-linearities and scale interactions

- For example



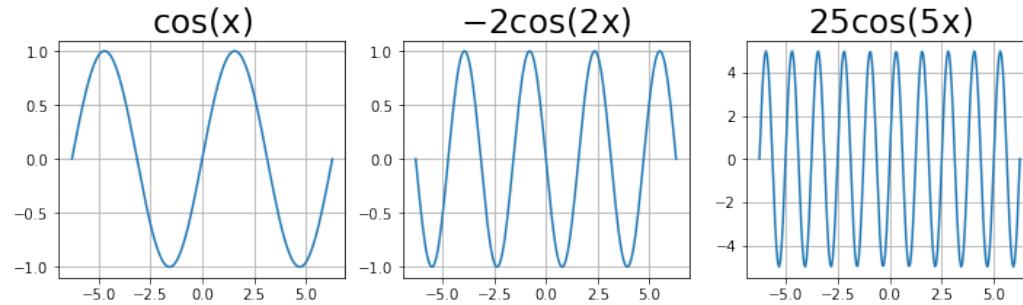
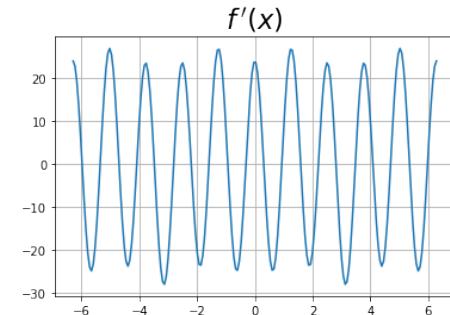
## Non-linearities and scale interactions

- For example



# Non-linearities and scale interactions

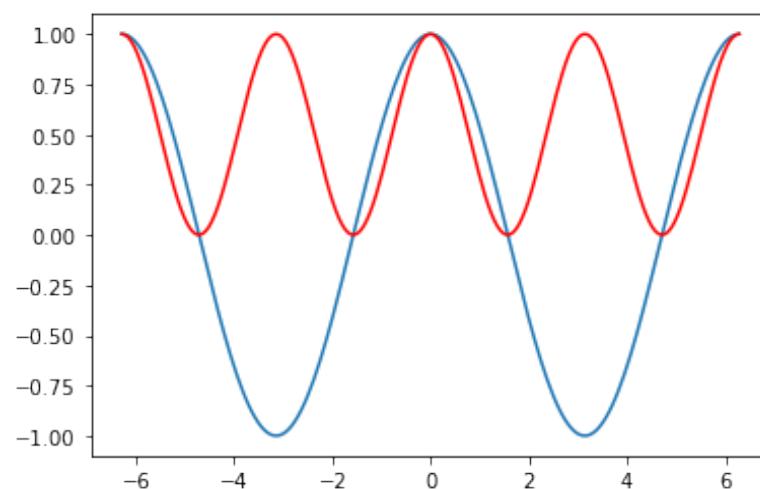
- Linear combinations / derivate of the function still have the same components:



- So for a **linear system of equations**, if you have forcings/initial condition at a given scale (e.g.  $k_x$ ), **the solution will have the same frequency/wavenumbers**.

# Non-linearities and scale interactions

- But the product of two components can give rise to a new scale in the flow.
- For example:  $f^2(x) = \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$



# Triads interactions

- The nonlinear term in the equations of motion leads to interactions among different length scales
- *Let's write an equation for an incompressible 2 dimensional flow (for simplicity)*

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \zeta, \quad \zeta = \nabla^2 \psi.$$

With  $u, v = -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}$  and  $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

# Triads interactions

- And work in Fourier space:

$$\psi(x, y, t) = \sum_{\mathbf{k}} \tilde{\psi}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \xi(x, y, t) = \sum_{\mathbf{k}} \tilde{\xi}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}},$$

where  $\mathbf{k} = \mathbf{i}k^x + \mathbf{j}k^y$ ,  $\tilde{\xi} = -k^2 \tilde{\psi}$  where  $k^2 = k^{x^2} + k^{y^2}$

# Triads interactions

- The equation (without forcings and dissipation for now) is:

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{\mathbf{k}} \tilde{\xi}(\mathbf{k}, t) e^{i \mathbf{k} \cdot \mathbf{x}} &= - \sum_{\mathbf{p}} p^x \tilde{\psi}(\mathbf{p}, t) e^{i \mathbf{p} \cdot \mathbf{x}} \times \sum_{\mathbf{q}} q^y \tilde{\xi}(\mathbf{q}, t) e^{i \mathbf{q} \cdot \mathbf{x}} \\ &\quad + \sum_{\mathbf{p}} p^y \tilde{\psi}(\mathbf{p}, t) e^{i \mathbf{p} \cdot \mathbf{x}} \times \sum_{\mathbf{q}} q^x \tilde{\xi}(\mathbf{q}, t) e^{i \mathbf{q} \cdot \mathbf{x}}. \end{aligned}$$

- We multiply by

$$\exp(-i \mathbf{k} \cdot \mathbf{x})$$

- And use the fact that the Fourier modes are orthogonal;

$$\int e^{i \mathbf{p} \cdot \mathbf{x}} e^{i \mathbf{q} \cdot \mathbf{x}} dA = \frac{1}{L^2} \delta(\mathbf{p} + \mathbf{q}).$$

where  $\delta(\mathbf{p} + \mathbf{q})$  equals unity if  $\mathbf{p} = -\mathbf{q}$  and is zero otherwise.

# Triads interactions

- And get:

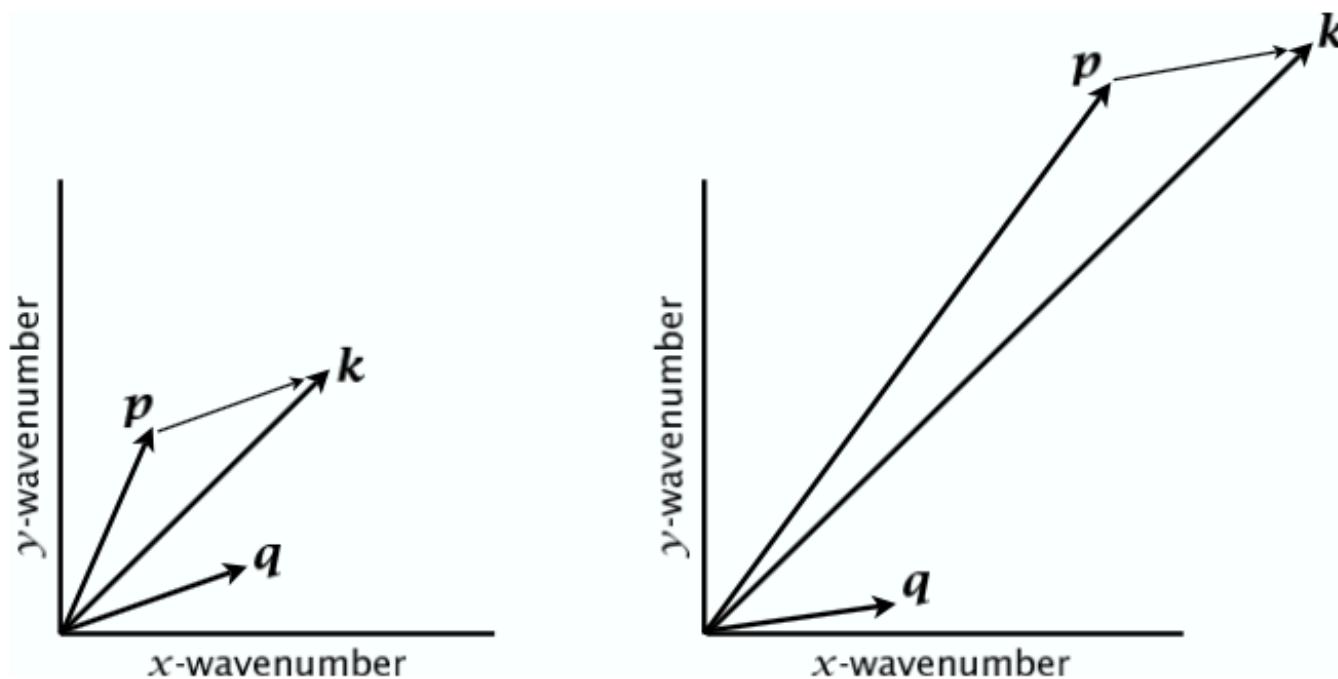
$$\frac{\partial}{\partial t} \tilde{\psi}(\mathbf{k}, t) = \sum_{\mathbf{p}, \mathbf{q}} A(\mathbf{k}, \mathbf{p}, \mathbf{q}) \tilde{\psi}(\mathbf{p}, t) \tilde{\psi}(\mathbf{q}, t) + \tilde{F}(\mathbf{k}) - \nu k^4 \tilde{\psi}(\mathbf{k}, t),$$

where  $A(\mathbf{k}, \mathbf{p}, \mathbf{q}) = (q^2/k^2)(p^x q^y - p^y q^x) \delta(\mathbf{p} + \mathbf{q} - \mathbf{k})$

- Only vectors with  $\mathbf{p} + \mathbf{q} - \mathbf{k} = 0$  have a non-zero contribution = **Triads interactions**

# Triads interactions

- Triads interactions can be local or non-local:



**Fig. 8.1** Two interacting triads, each with  $k = p + q$ . On the left, a local triad with  $k \sim p \sim q$ . On the right, a nonlocal triad with  $k \sim p \gg q$ .

# Triads interactions

- Starting with only 2 Fourier modes you can fill the entire spectrum due to scale interactions

$$\frac{\partial}{\partial t} \tilde{\psi}(\mathbf{k}, t) = \sum_{\mathbf{p}, \mathbf{q}} A(\mathbf{k}, \mathbf{p}, \mathbf{q}) \tilde{\psi}(\mathbf{p}, t) \tilde{\psi}(\mathbf{q}, t) + \tilde{F}(\mathbf{k}) - \nu k^4 \tilde{\psi}(\mathbf{k}, t),$$

- Forcing is fixed at a certain scale(s) and dissipation acts on each Fourier mode with a coefficient that increases with wavenumber and therefore that preferentially affects small scales.

# Chaotic behavior:

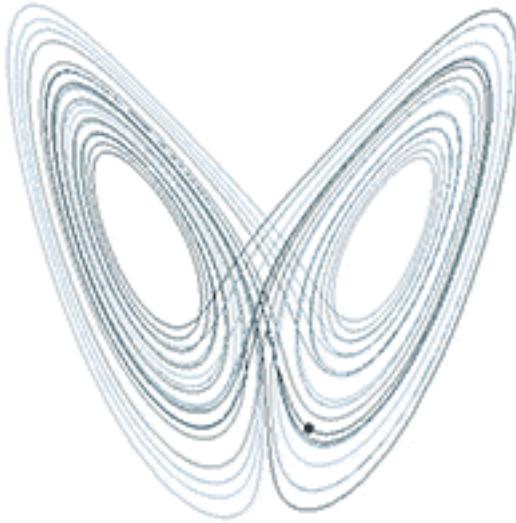
- **Chaos:** When the present determines the future, but the approximate present does not approximately determine the future [E. Lorenz]



*A double rod pendulum showing  
chaotic behavior* [[https://en.wikipedia.org/  
wiki/Chaos\\_theory](https://en.wikipedia.org/wiki/Chaos_theory)]

## Chaotic behavior:

- **Chaos:** When the present determines the future, but the approximate present does not approximately determine the future [E. Lorenz]



*Solutions in the Lorenz attractor  
(originally a model for atmospheric convection)*

$$\frac{dx}{dt} = \sigma(y - x),$$

$$\frac{dy}{dt} = x(\rho - z) - y,$$

$$\frac{dz}{dt} = xy - \beta z.$$

## A chaotic example:

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \mathcal{F}^x$$

- Let's play around with a simpler version containing a non-linear term (with a Reynolds number  $r$ ), a source and sink:

$$\frac{du}{dt} + ru^2 = -u + 1$$

- We will see how irregular and apparently stochastic solutions appear in from deterministic equations.

# Properties of turbulence: unpredictability

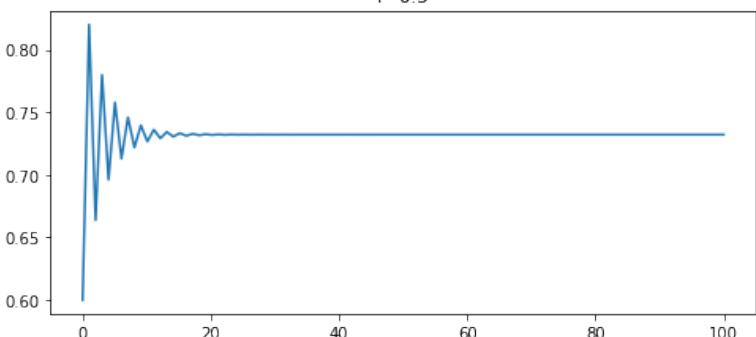
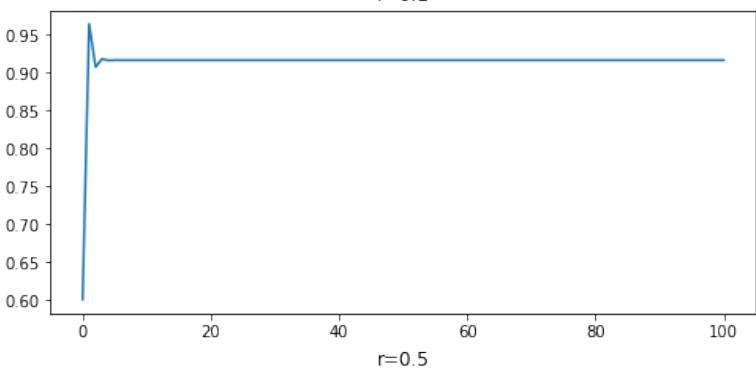
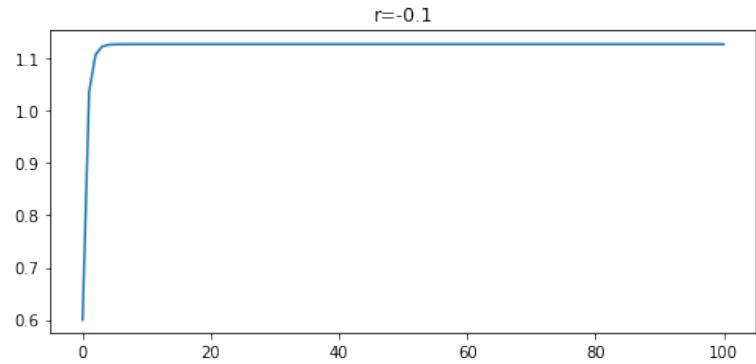
$$\frac{du}{dt} + ru^2 = -u + 1$$

With  $u_0 = 0.6$

- Fixed points:

$$\frac{d}{dt}u = -ru^2 + 1 - u = 0$$

$$u = -\frac{1}{2r} \pm \frac{\sqrt{1+4r}}{2r}$$



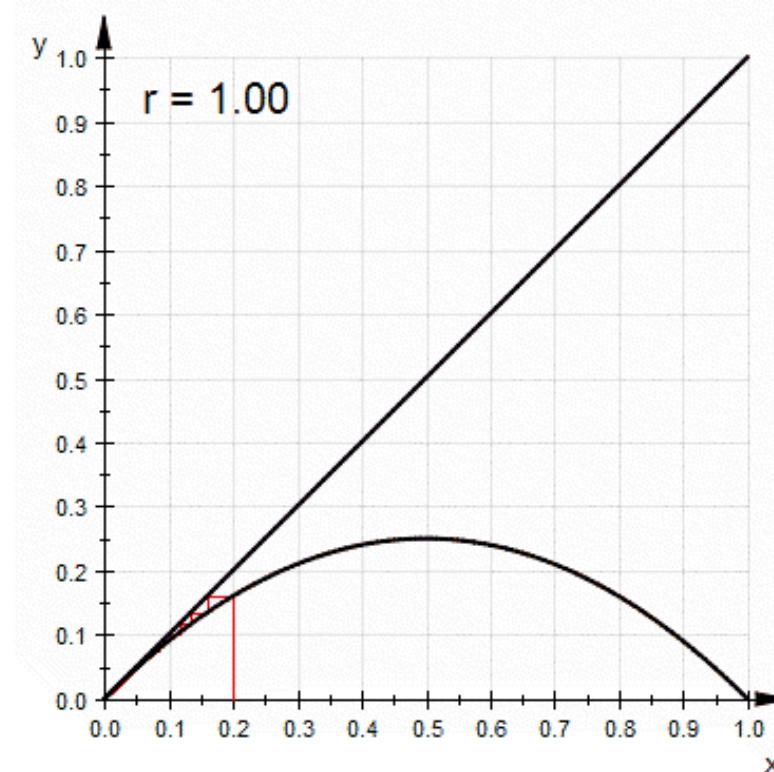
### Activity 2:

$$\frac{du}{dt} + ru^2 = -u + 1$$

- Discretize using a simple forward Euler scheme (with  $dt=1$ ) to get  $u(t+1) = F[u(t)]$
- Using the discretized equation and  $u(0) = 0.6$ , plot  $u(t)$  for  $t=[0:100]$  and  $r = [0.1, 0.5, 0.8, 1.3, 2]$ .
- Check the sensitivity to initial conditions ( $u(0)$  or  $u(0) + \text{epsilon}$ )
- Plot the pdf for  $r=2$
- Plot the power spectra of  $u$  for  $r = [0.1, 0.5, 0.8, 1.3, 2]$ ,

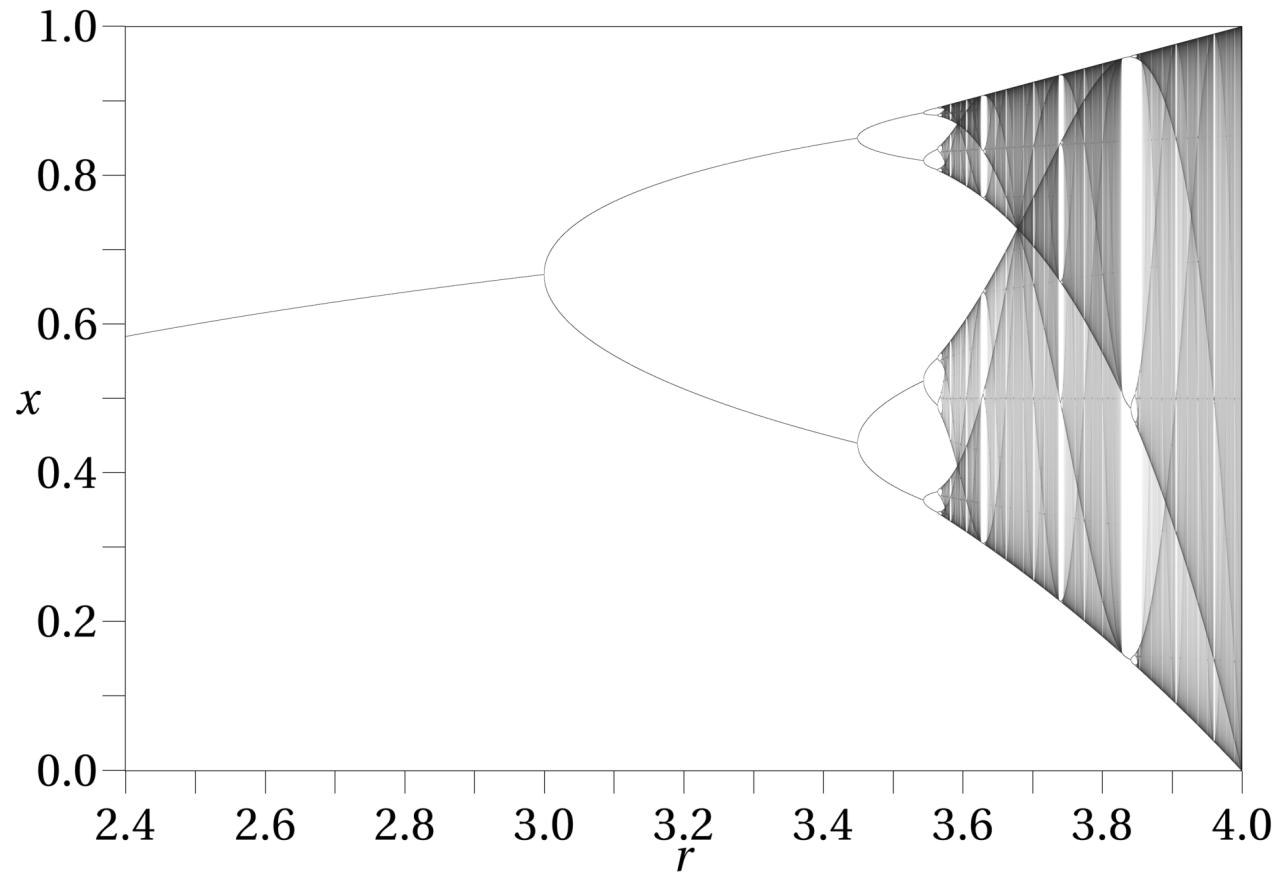
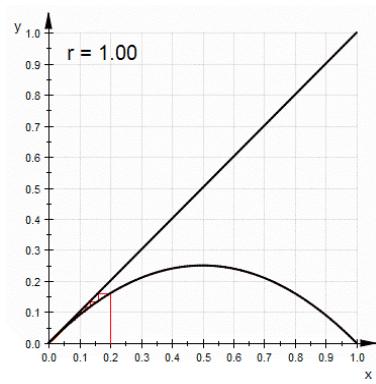
- The equation we've seen is similar to the **logistic map** by May (1976) [ijgula.fr/Turb/Articles/May76.pdf](http://ijgula.fr/Turb/Articles/May76.pdf) for the growth of population:

$$x_{n+1} = rx_n(1 - x_n)$$



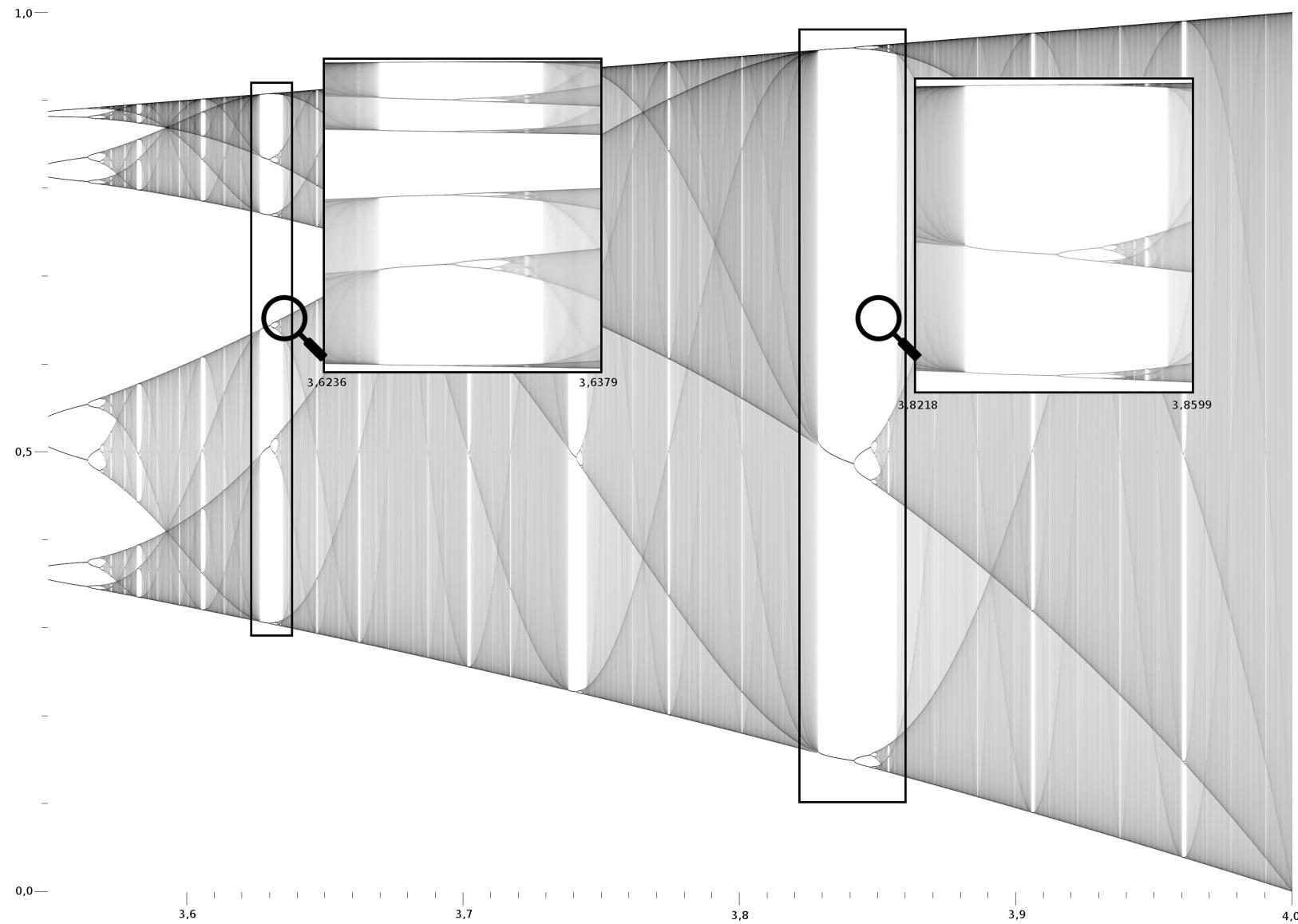
# Properties of turbulence: unpredictability

- From “laminar” to chaotic



# Properties of turbulence: unpredictability

- With islands of stability:



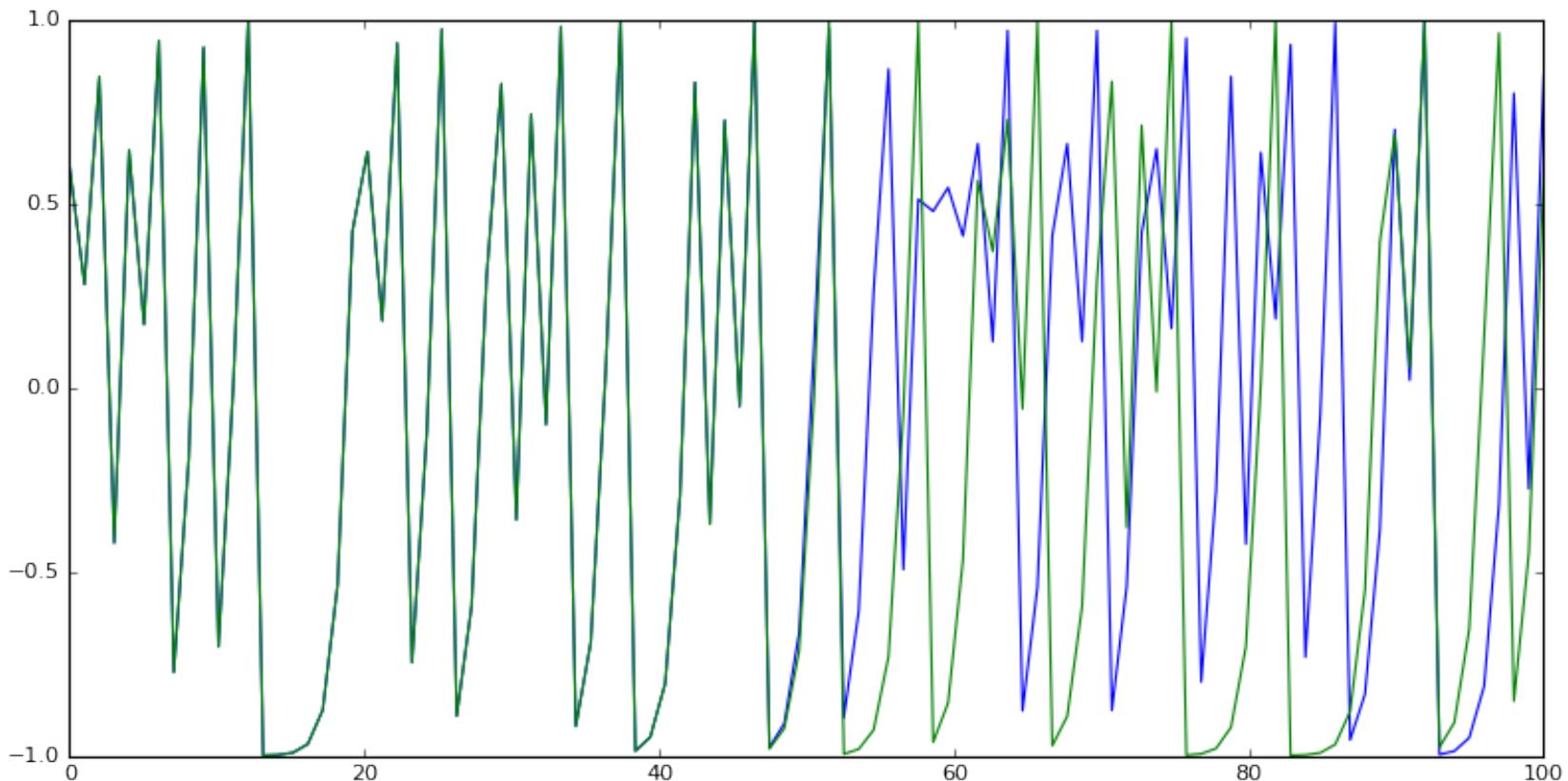
## Properties of turbulence: unpredictability

- Sensitivity to initial conditions

$$\frac{du}{dt} + ru^2 = -u + 1$$

$U_0 = 0.6$

$U_0 = 0.6 + 1e-16$



- **Sensitivity to initial conditions:**

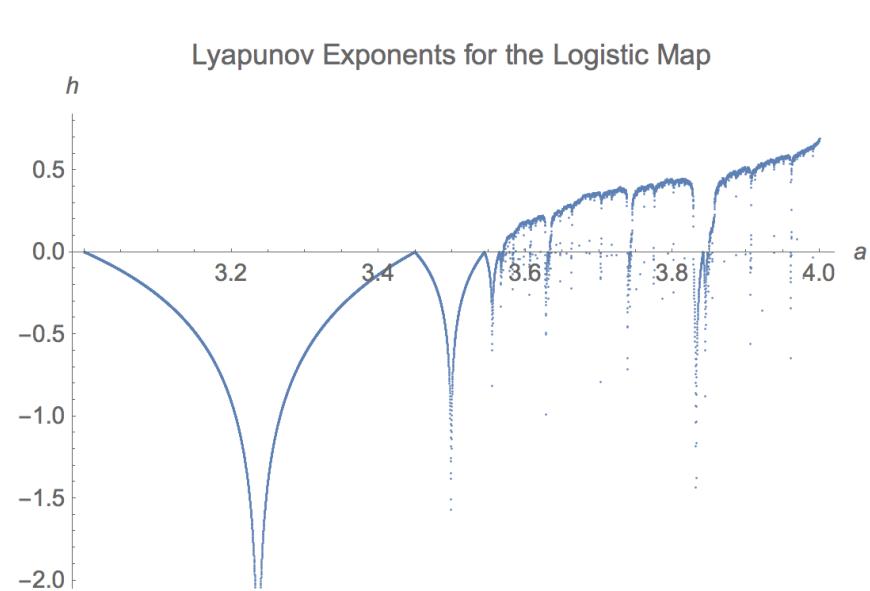
In general, for an initial difference  $x_0$  and  $x_0 + \Delta x$

We expect the difference after n iteration to grow as

Where  $\lambda(x_0)$  is the so-called **Lyapunov exponent** for the initial condition  $x_0$

$$\lambda(x_0) = \lim_{n \rightarrow \infty} \lim_{\Delta x \rightarrow 0} \frac{1}{n} \log \left| \frac{F^n(x_0 + \Delta x) - F^n(x_0)}{\Delta x} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \frac{dF^n(x_0)}{dx_0} \right|$$

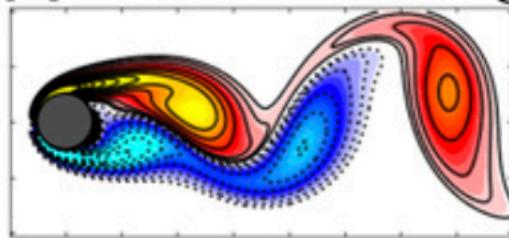
A positive Lyapunov exponent is a signature of chaos ...



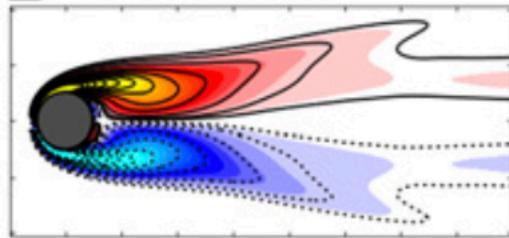
# Properties of turbulence: unpredictability

- fluid vortex shedding behind an obstacle :

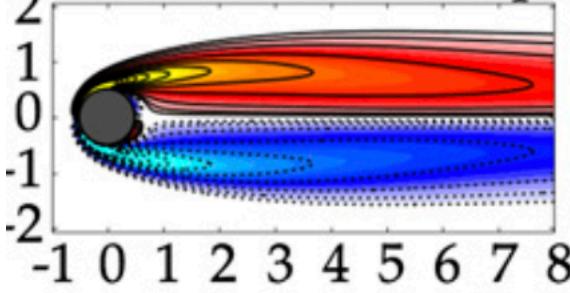
A - vortex shedding



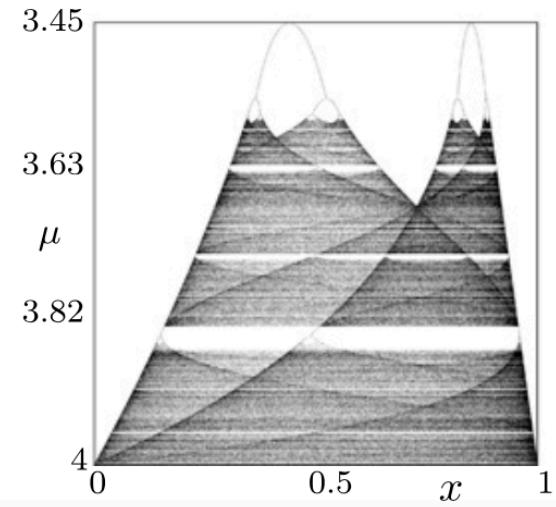
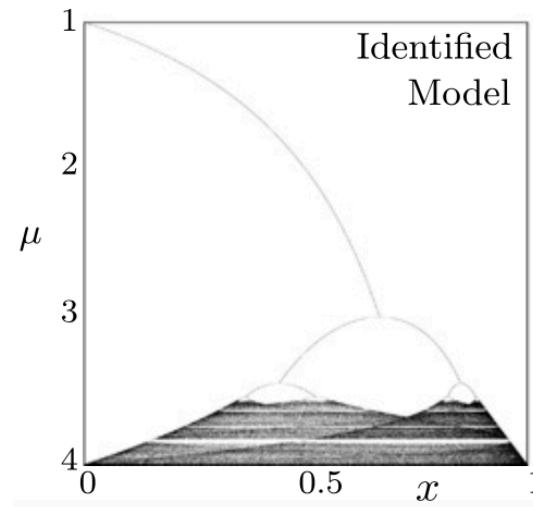
B - mean flow



C - unstable fixed pt.

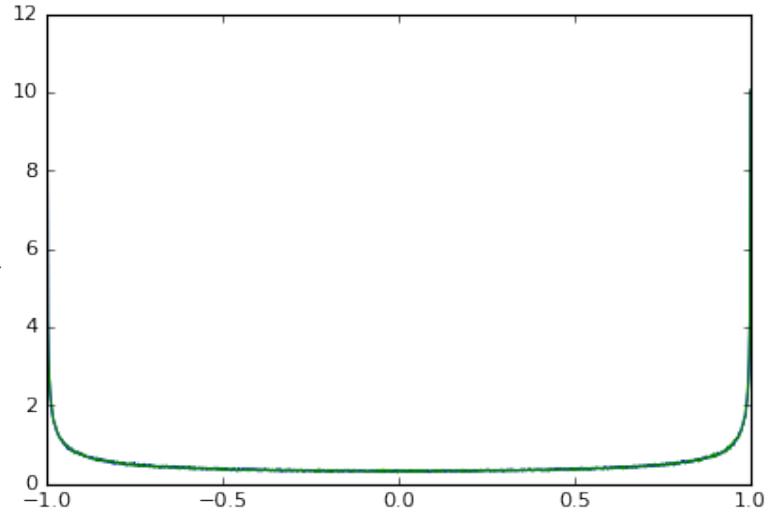
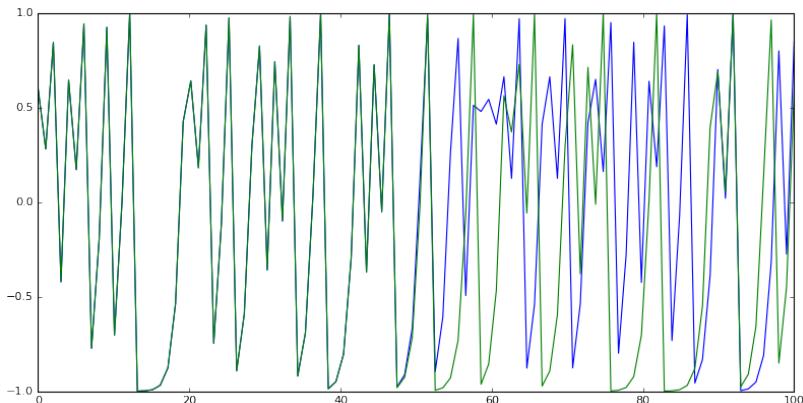


[Brunton et al, 2016]



## Properties of turbulence: unpredictability

- Hence the need for a statistical description of turbulence:
- Individual trajectories are erratic and difficult to predict. However ensemble of trajectories reflects the mean structure of the attractor and can be predicted.

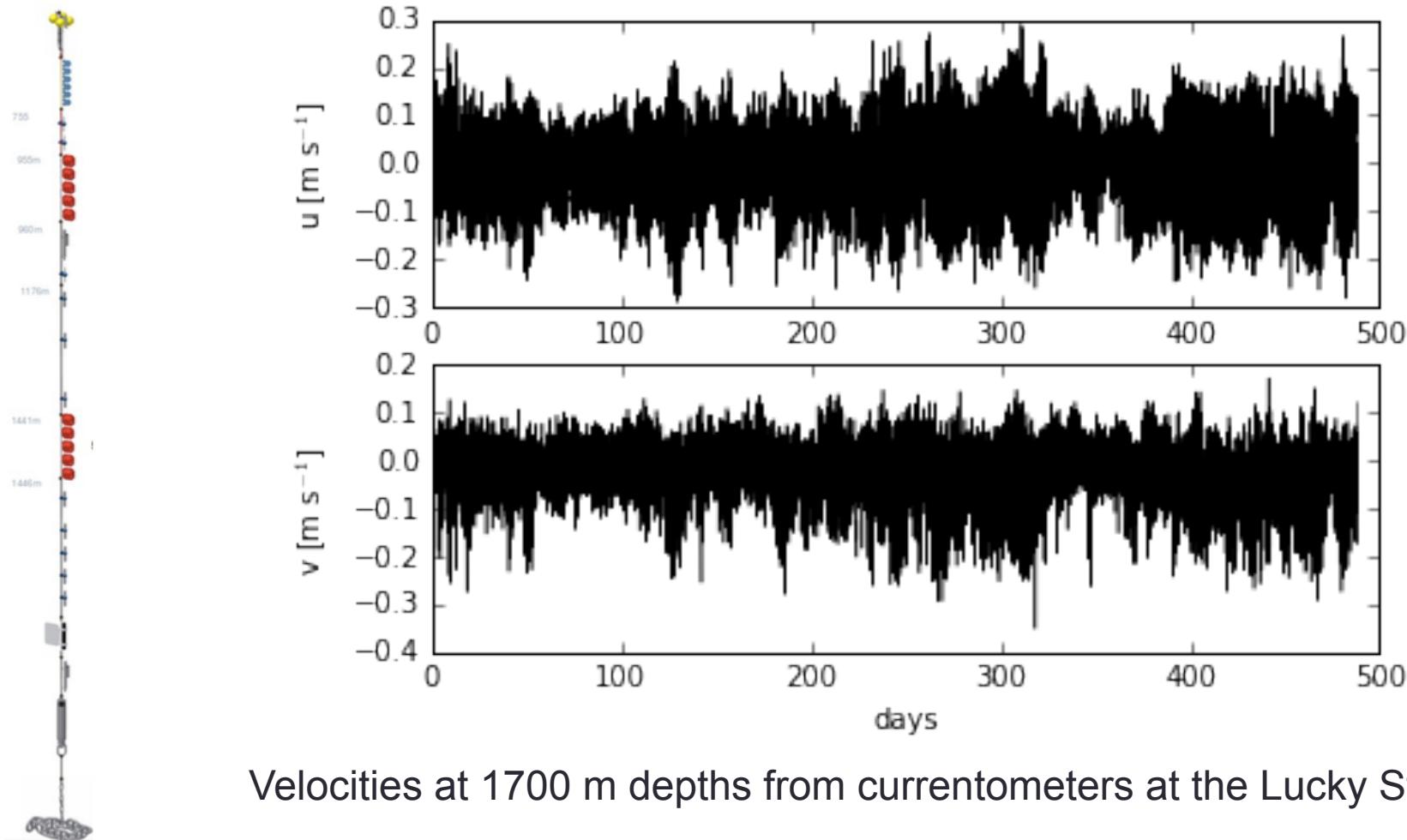


- statistical properties of the flow can be quite reproducible (pdf, mean, moments, etc.)

## Properties of turbulence: unpredictability

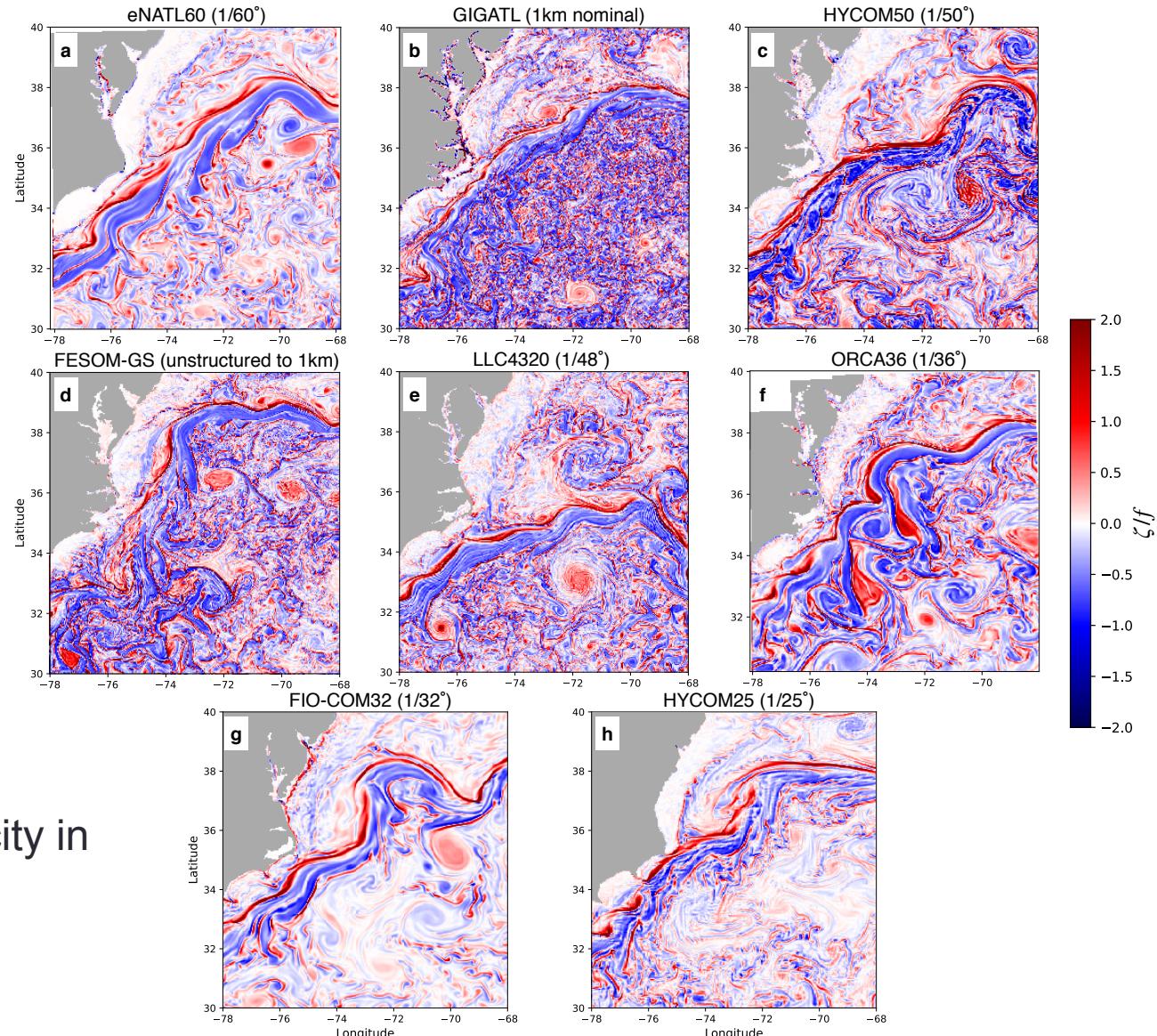
- The equation of motions are deterministic, but their solutions have many attributes of random processes.
- Individual trajectories are erratic and difficult to predict. However ensemble of trajectories reflects the mean structure of the attractor and can be predicted.
- Hence the need for a **statistical description of turbulence**.

## How to characterize properties of turbulence?



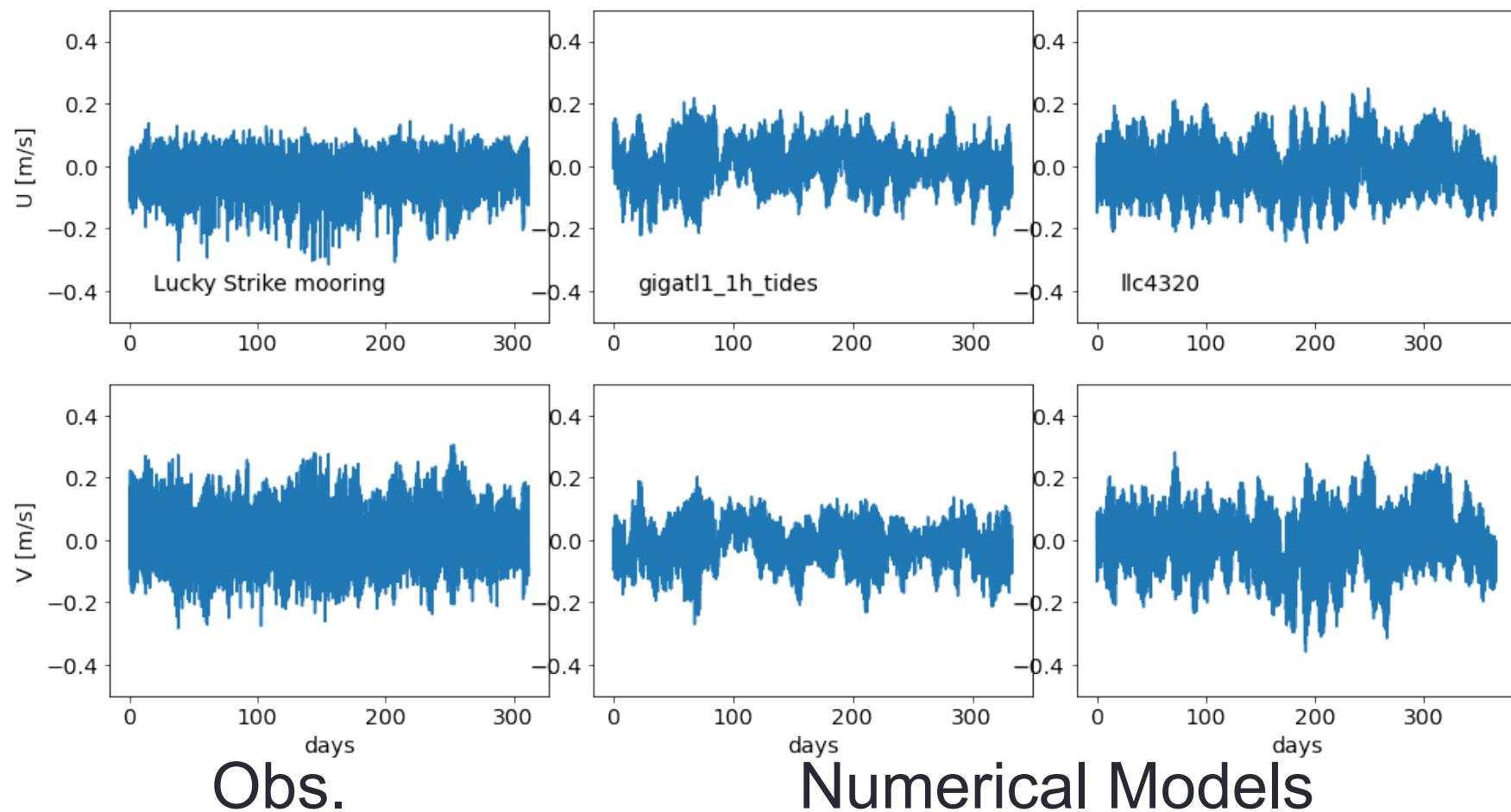
Velocities at 1700 m depths from currentometers at the Lucky Strike site

## How to characterize properties of turbulence?



Surface relative vorticity in  
the Gulf Stream

## How do we compare results from a model to observations?



Velocities at 1700 m depth at the Lucky Strike site

# How to characterize properties of turbulence?

- Turbulent flow are chaotic and not predictable in their detailed properties, but their **statistical properties are reproducible**.
- A statistical measure is an average of some kind: over the symmetry coordinates, if any are available (e.g, a time average for stationary flows); over multiple realizations (e.g, an ensemble); or over the phase space of solutions if the dynamics are homogeneous.

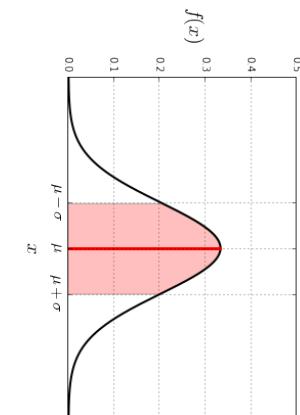
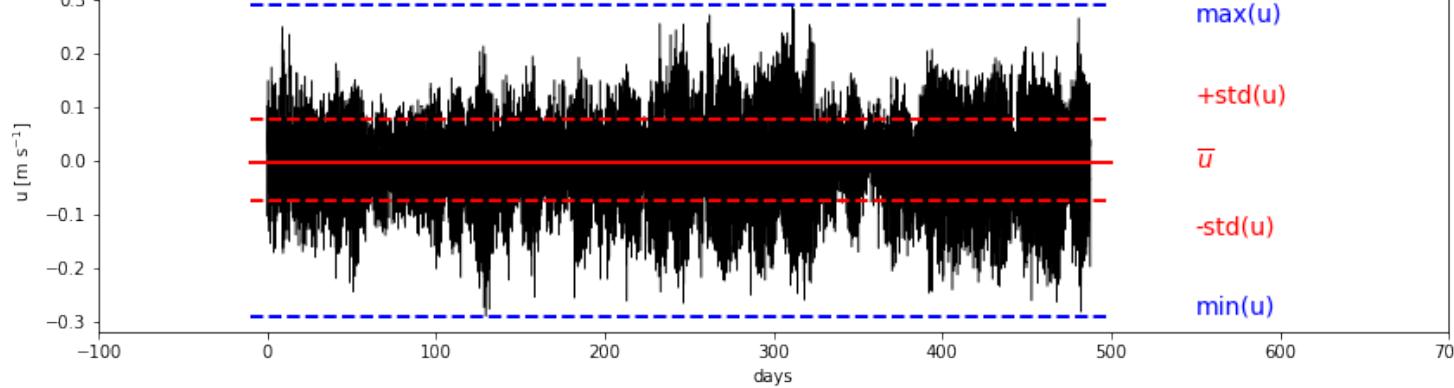
## Statistics of turbulent fields:

We look at the values distribution (= **pdf**) and its moments:

The first one is the **mean**.  $\mu = \int x f(x) dx$

The second one is the **variance** (= squared standard deviation / rms). It describes the **spread of the pdf** around the mean.

$$\sigma^2 = \int (x - \mu)^2 f(x) dx$$



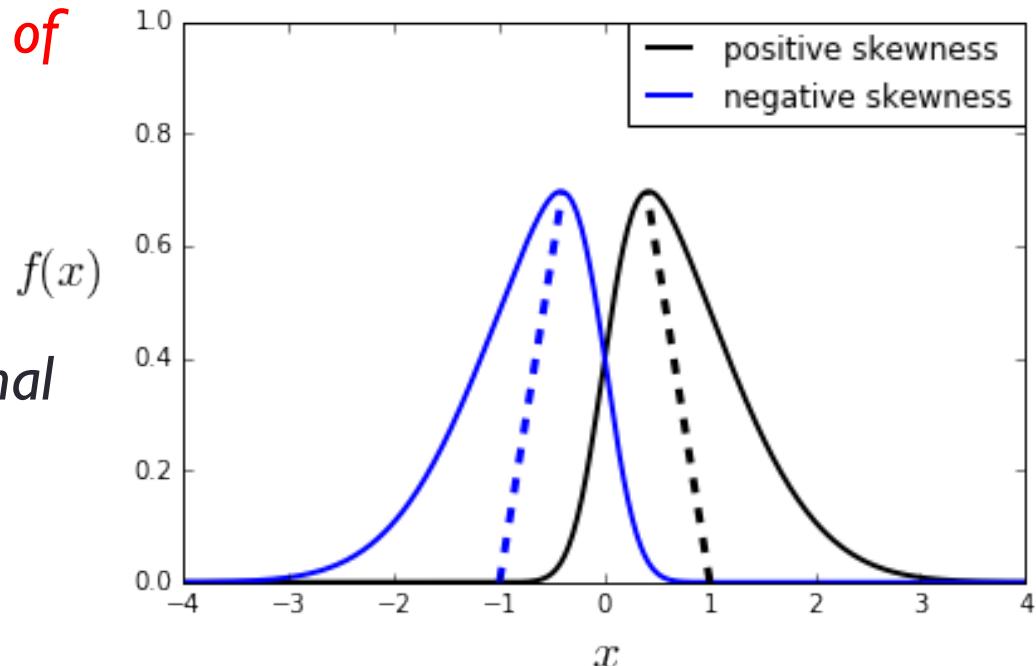
## Statistics of turbulent fields:

The third one is the **skewness**:

$$\mu_3 = \frac{1}{\sigma^3} \int (x - \mu)^3 f(x) dx$$

*It is a measure of the **asymmetry** of the **pdf** about its mean.*

*If the **pdf** is symmetric (e.g. normal distrib.), the skewness is 0*



## Statistics of turbulent fields:

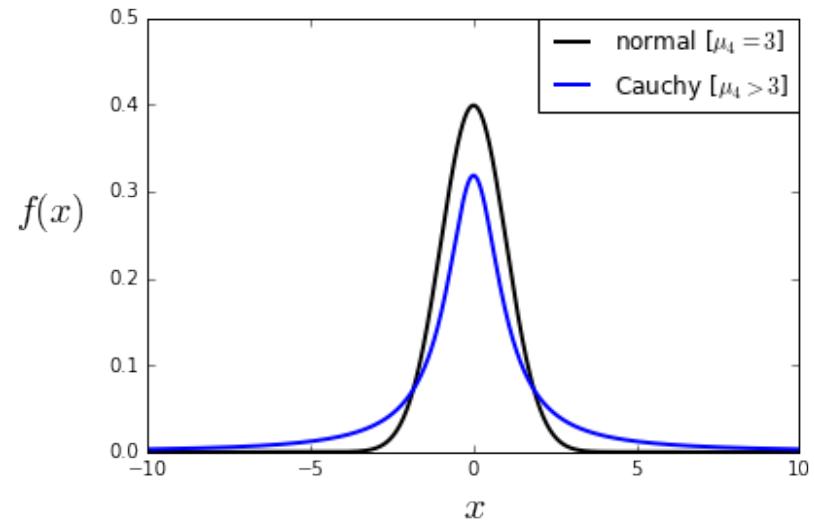
The fourth one is the **kurtosis**:

$$\mu_4 = \frac{1}{\sigma^4} \int (x - \mu)^4 f(x) dx$$

The kurtosis measures *how fat are the tails of the pdf.*

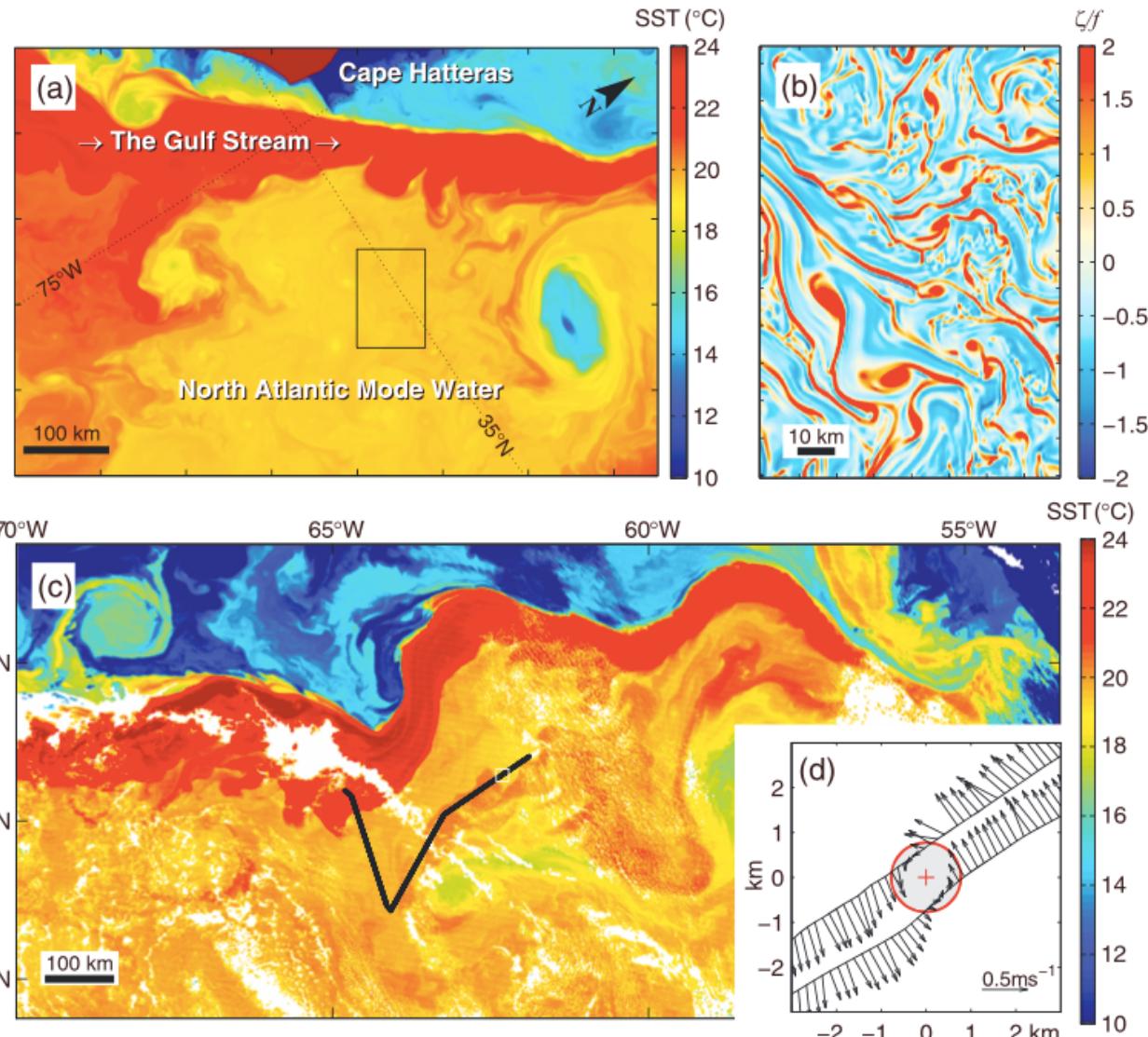
For the normal law the skewness is zero and the kurtosis is 3.

Values larger than 3 indicate more likely extreme values than the normal law.



Ex: How do we compare results from a model to observations:

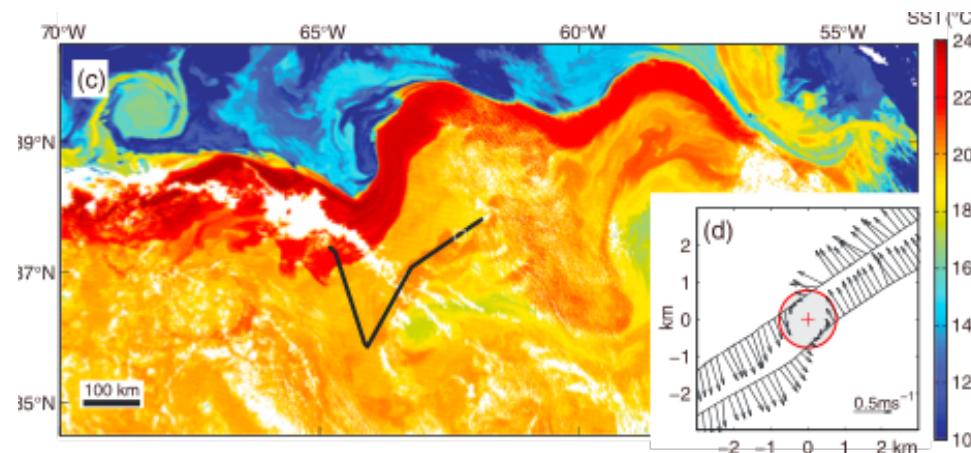
## Numerical Model



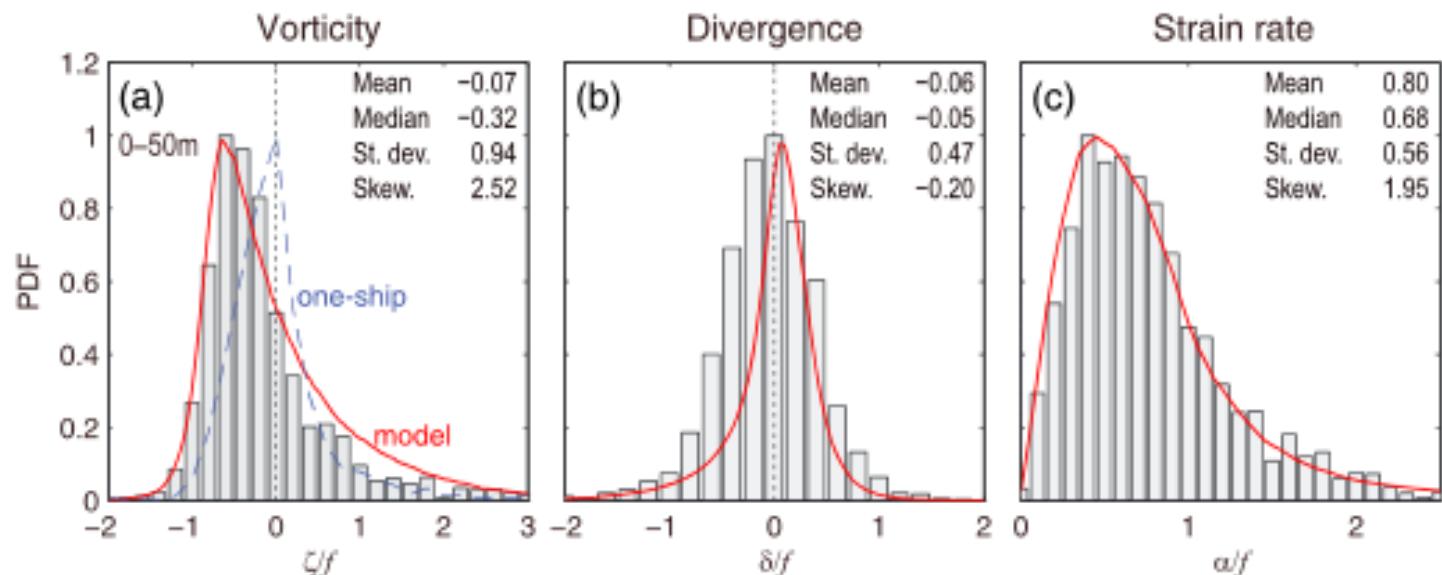
## Observations

[satellite SST +  
velocity measurements  
from 2 parallel ships]

Ex: How do we compare results from a model to observations:

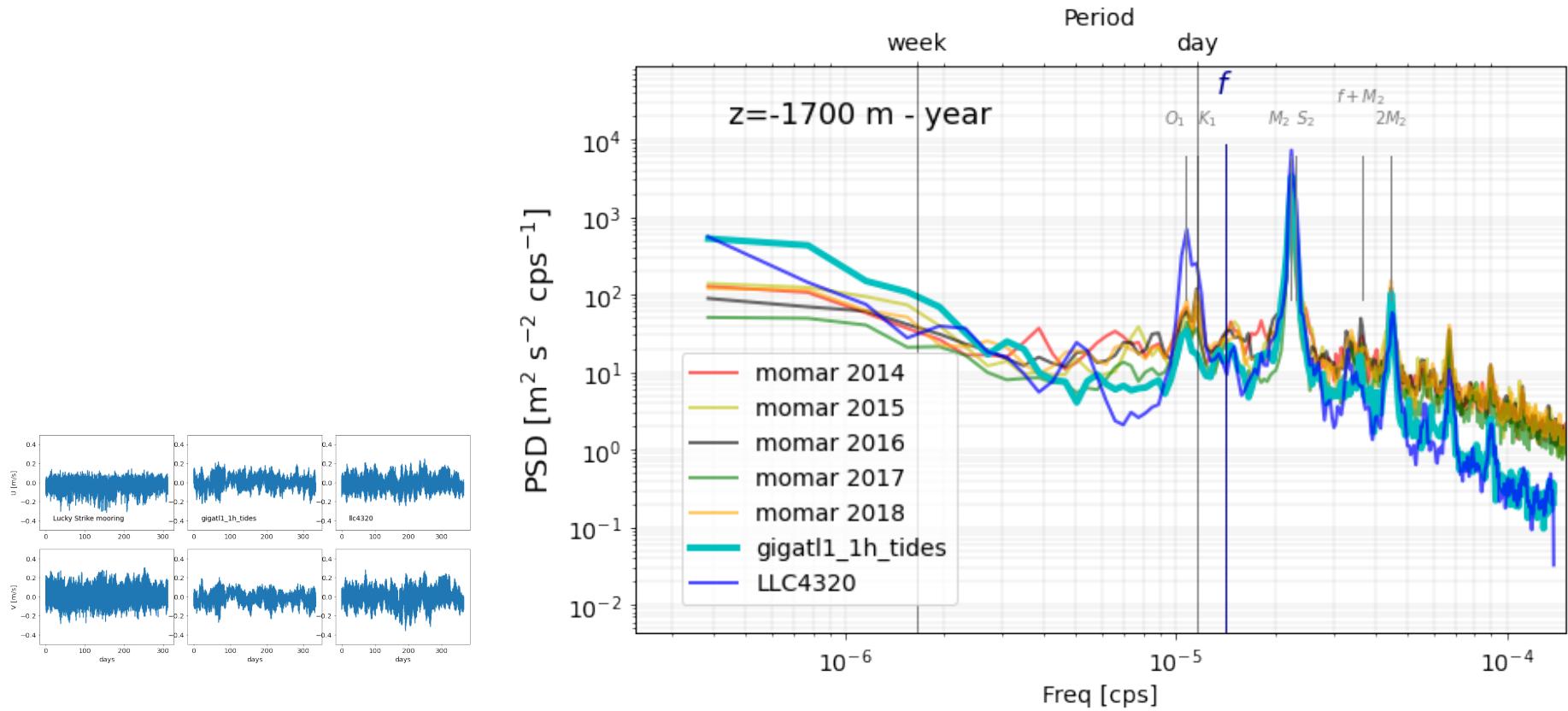


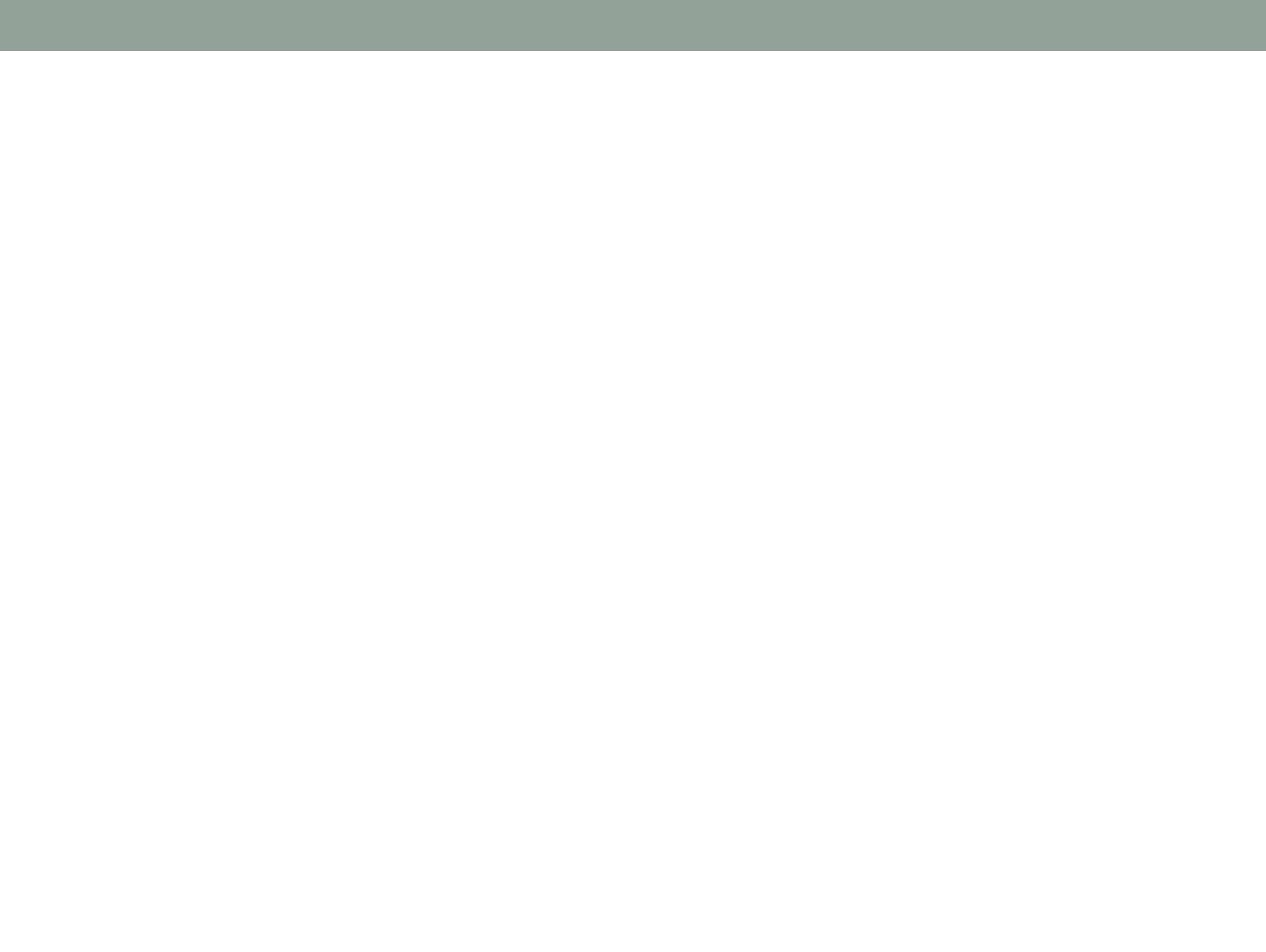
Ex: Shcherbina et al, 2015.



## Statistics of turbulent fields:

Another way is to compute “**power spectral density**” (= spectra of energy or tracer variance)





# Computing Power Spectra

- We have a signal (*e.g. velocity  $u$  along dimension  $x$* )
- *The total energy is* 
$$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx$$
-

# Computing Power Spectra

- We have a signal (e.g. *velocity  $u$  along dimension  $x$* )

- *The total energy is*

$$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx$$

- *Using the Fourier transform of the variable:*

$$\hat{u}(k) = \int_{-\infty}^{+\infty} e^{-2\pi ikx} u(x) dx$$

- *the power spectral density is*  $|\hat{u}(k)|^2$ 
  - *It is the density of energy per unit wavenumber (or frequency)*

# Computing Spectra

- Parseval's theorem states that:

$$E = \int_{-\infty}^{+\infty} |u(x)|^2 dx = \int_{-\infty}^{+\infty} |\hat{u}(k)|^2 dk$$

- *Integral of energy in the physical domain is equal to integral of spectral energy density over all wavenumbers.*

# Computing Spectra

- In a finite and discrete domain:

$$\hat{u}(k) = \sum_{x=0}^{N-1} u(x) e^{-ikx \frac{2\pi}{N}}$$

for  $k = 0, \dots, N - 1$

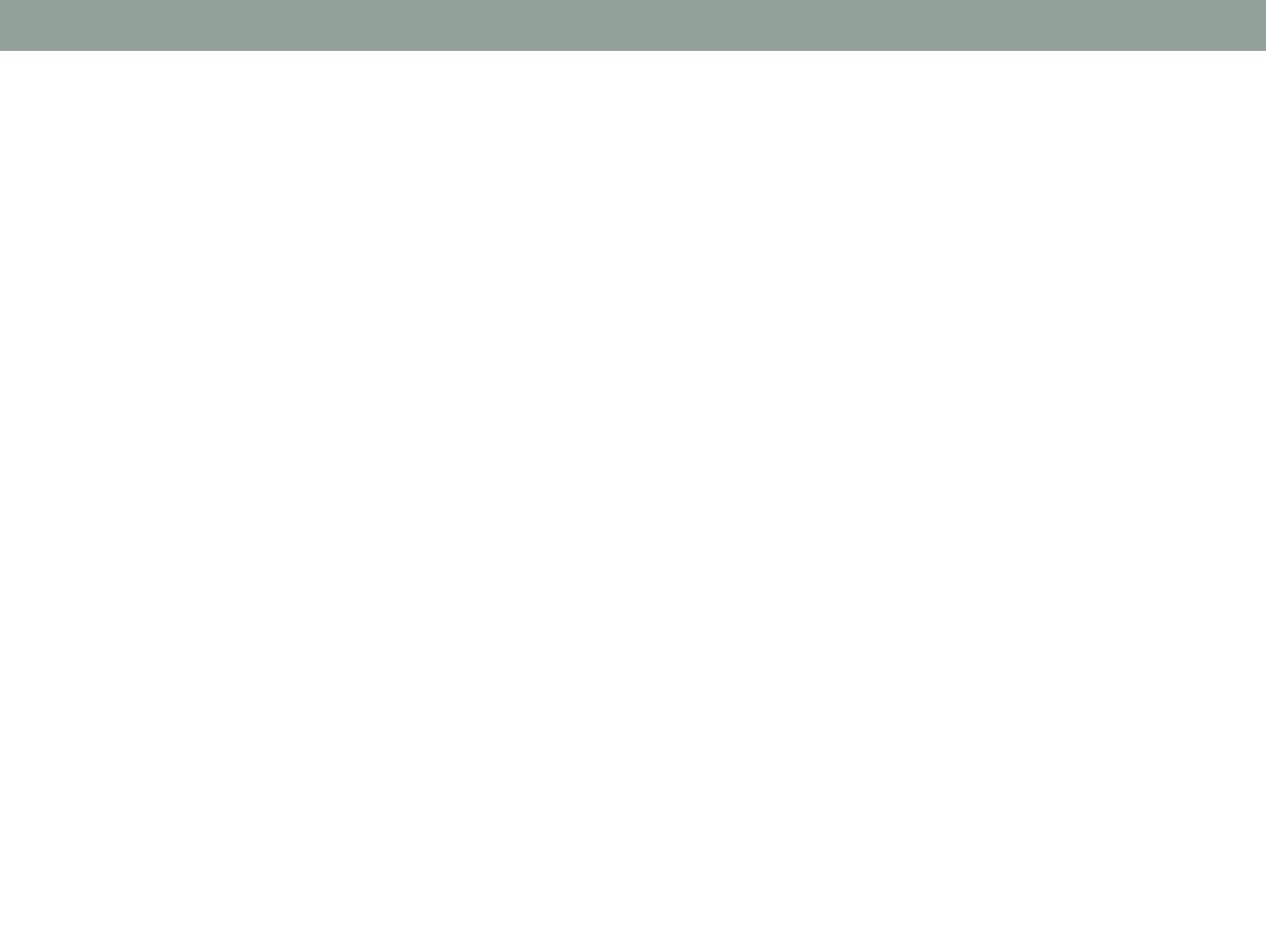
- And the power spectra is defined as:

$$\frac{\hat{u}(k)\hat{u}^*(k)}{N}$$

# Computing Spectra

A simple example in Python:

```
def myfft(u):
    nx = u.shape[0]
    k = np.fft.rfftfreq(nx,d=1)
    psd = (np.abs(np.fft.rfft((u))))**2)/nx
    return k, psd
```



# The closure problem

- *In a turbulent flow it may be virtually impossible to predict the detailed motion of each eddy, the **statistical properties** — time averages for example — might not be changing and we might like to predict such averages.*
- We can decompose the variables into mean and fluctuating components:

$$u = \bar{u} + u'$$

Where  $\bar{u} = \frac{1}{T} \int_0^T u \, dt$  or  $\bar{u} = \frac{1}{N} \sum_{N \rightarrow \infty} u$

With by definition  $\overline{u'} = 0$

- And try to find a closed equation for the mean part

## Activity 5

- Let's try with a simple equation:

$$\frac{du}{dt} + uu + ru = 0$$

- Write the evolution equation for  $\bar{u}$  (assuming the average operator can permute with the time derivative)

# The closure problem

- Let's try with a simple equation:
- We average and get:

$$\frac{du}{dt} + uu + ru = 0$$

$$\frac{d\bar{u}}{dt} + \bar{u}\bar{u} + r\bar{u} = 0 \quad \text{with} \quad \bar{u}\bar{u} = \bar{u}\bar{u} + \boxed{\bar{u}'\bar{u}'} \neq \bar{u}\bar{u}.$$

- We can go to the next order, but there are always new unknowns :

$$\frac{1}{2} \frac{d\bar{u}^2}{dt} + \boxed{\bar{u}\bar{u}\bar{u}} + r\bar{u}^2 = 0$$

unknown

# The closure problem

- So we need a method to find close the hierarchy an make some assumptions...
- For example:  $\overline{uuuu} = \alpha \overline{uu} \overline{uu} + \beta \overline{uuu}$

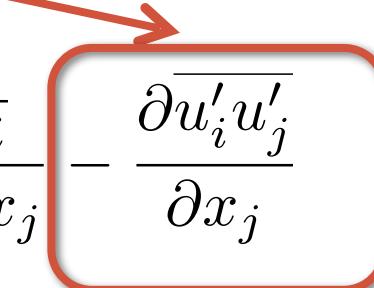
# The closure problem

- For the NS equation:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + f \mathbf{k} \times \bar{u}_i + \frac{\bar{\rho}}{\rho_0} g \mathbf{k} = - \frac{1}{\rho_0} \frac{\partial \bar{P}}{\partial x_j} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_i \partial x_j}$$



$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + f \mathbf{k} \times \bar{u}_i + \frac{\bar{\rho}}{\rho_0} g \mathbf{k} = - \frac{1}{\rho_0} \frac{\partial \bar{P}}{\partial x_j} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_i \partial x_j} - \frac{\partial \bar{u}'_i \bar{u}'_j}{\partial x_j}$$



Advection for the  
averaged flow

Reynolds stress  
= effect of subgrid-scale turbulence

# The closure problem

For the NS equation:

- The number of unknowns is larger than the number of equations
- When we start deriving equations for unknowns, more variables appear.

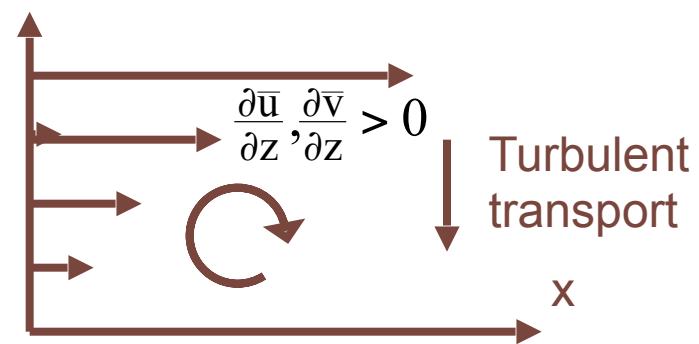
Progn. Eq. for:	Moment	Equation	No. Eqs.	No. Unknowns
$\bar{U}_i$	First	$\frac{\partial \bar{U}_i}{\partial t} = \dots - \frac{\partial u'_i u'_j}{\partial x_j}$	3	6
$\bar{u'_i u'_j}$	First	$\frac{\partial u'_i u'_j}{\partial t} = \dots - \frac{\partial u'_k u'_i u'_j}{\partial x_k}$	6	10
$\bar{u'_i u'_j u'_k}$	First	$\frac{\partial u'_i u'_j u'_k}{\partial t} = \dots - \frac{\partial u'_k u'_i u'_j u'_m}{\partial x_m}$	10	15

# The closure problem

In PE models the equations are closed by parameterizing the Reynolds stresses as:

$$\overline{u'w'} = -K_M v \frac{\partial u}{\partial z}$$

$$\overline{v'w'} = -K_M v \frac{\partial v}{\partial z}$$



But nobody has been able to close the system, in any useful way, without introducing physical assumptions not directly deducible from the equations of motion themselves.