CT216 Project Analysis

Aim: Polar Codes can be achieve Shannon's Channel Capacity bound.

- For prove that we have to show,
- For any Binary input channel Γ and any $0<a<b<1$ we have ,

$$\lim_{n\to\infty} \frac{1}{2^n} \left| \bar{S} \in \{-,+\}^n : I\left(\Gamma^{\bar{S}}\right) \in [0,a) \right| = 1 - \mathrm{I}_{\Gamma}$$

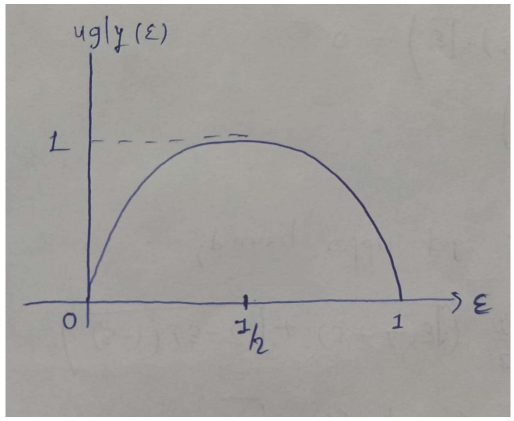
$$\lim_{n\to\infty} \frac{1}{2^n} \left| \bar{S} \in \{-,+\}^n : I\left(\Gamma^{\bar{S}}\right) \in [a,b] \right| = 0$$

$$\lim_{n\to\infty} \frac{1}{2^n} \left| \bar{S} \in \{-,+\}^n : I\left(\Gamma^{\bar{S}}\right) \in [0,a) \right| = \mathrm{I}_{\Gamma}$$

- Hence the fraction of good channels approaches the capacity of the channel.
- We show this theorem on BEC.

- We define ugliness of BEC as,

ugly(ε) = $\sqrt{4\varepsilon(1-\varepsilon)}$



- Channels with $\varepsilon = 0$ *and* $\varepsilon = 1$ are **communication friendly** channels.
- Ugly(0) = ugly(1) = 0.
- Channels with $\varepsilon = \frac{1}{2}$ is maximum ugly.
- Let find for $\varepsilon = \frac{1}{2}$ maximum ugliness,

Type equation here.

$\Gamma^+ : \mathrm{BEC}(\epsilon^+ = \epsilon^2)$

$$\begin{aligned} \text{ugly}(\epsilon^+) &= \sqrt{4\epsilon^+(1-\epsilon^+)} \\ &= \sqrt{4\epsilon^2(1-\epsilon^2)} \\ &= \sqrt{4\varepsilon(1-\varepsilon)}\,\sqrt{\varepsilon(1-\varepsilon)} \\ &= \text{ugly}(\varepsilon)\sqrt{\varepsilon(1-\varepsilon)} \end{aligned}$$

$\Gamma^- : \mathrm{BEC}(\epsilon^- = 2\epsilon - \epsilon^2)$

$$\begin{aligned} \text{ugly}(\epsilon^+) &= \sqrt{4\epsilon^-(1-\epsilon^-)} \\ &= \sqrt{4(2\epsilon - \epsilon^2)(1-2\epsilon+\epsilon^2)} \\ &= \sqrt{4\epsilon(1-\varepsilon)(1-2\epsilon+\epsilon^2)} \\ &= \text{ugly}(\varepsilon)\sqrt{(2-\varepsilon)(1-\varepsilon)} \end{aligned}$$

$$\frac{1}{2}*(\text{ugly}(\epsilon^+) + \text{ugly}(\epsilon^-)) = \text{ugly}(\varepsilon) * \frac{1}{2}\left(\sqrt{\varepsilon(1-\varepsilon)} + \sqrt{(2-\varepsilon)(1-\varepsilon)}\,\right) \qquad \text{........ (1)}$$

$$\text{f}(\varepsilon) = \sqrt{\epsilon(1+\varepsilon)} \qquad \Rightarrow \qquad f'(\varepsilon) = \frac{1}{2} * \sqrt{\varepsilon + \varepsilon^2}^{-\frac{1}{2}}(1+2\varepsilon) = \frac{1+2\varepsilon}{\sqrt{4\varepsilon(1+\varepsilon)}}$$

g(ε) = $\sqrt{(2-\varepsilon)(1-\varepsilon)}$ => g'(ε) = $\frac{1}{2} * \frac{-2\varepsilon-3}{(1-\varepsilon)(2-\varepsilon)^{\frac{1}{2}}}$

- Take first derivative equal to 0 for find maximum value,

  f'(ε) + g'(ε) = 0

  =>$(1+2\varepsilon)\sqrt{(1-\varepsilon)(2-\varepsilon)}$ - $(2\varepsilon+3)\sqrt{\varepsilon(1+\varepsilon)} = 0$

  After deriving this we get $\varepsilon = \frac{1}{2}$

- After putting $\varepsilon = \frac{1}{2}$ in eq (1) we get upper bound,

$\frac{1}{2}$ *(ugly(ϵ^{+}) + ugly(ϵ^{-})) = ugly(ε) $* \frac{1}{2}(\sqrt{\varepsilon(1-\varepsilon)} + \sqrt{(2-\varepsilon)(1-\varepsilon)}) \leq$ ugly(ε) $* \frac{\sqrt{3}}{2}$

- We need to show,

$$\lim_{n\to\infty} \frac{1}{2^n} \left| \bar{S} \in \{-,+\}^n : I\left(\Gamma^{\bar{S}}\right) \in \{\delta, 1-\delta\} \right| = 0$$

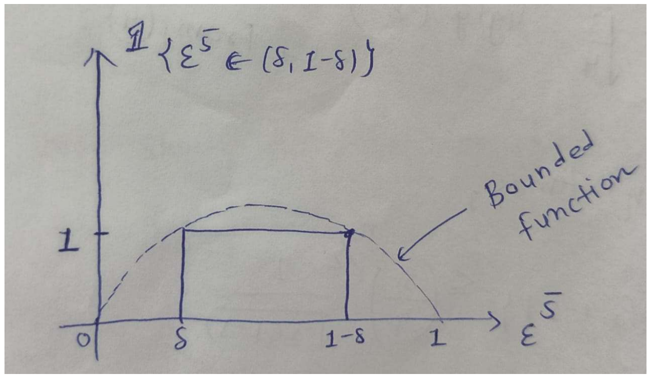
$$\lim_{n\to\infty} \frac{1}{2^n} \left| \bar{S} \in \{-,+\}^n : \left(1-\varepsilon^{\bar{S}}\right) \in \{\delta, 1-\delta\} \right| = 0$$

$$\lim_{n\to\infty} \frac{1}{2^n} \left| \bar{S} \in \{-,+\}^n : \varepsilon^{\bar{S}} \in \{\delta, 1-\delta\} \right| = 0$$

- This is equivalent to showing,

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{\bar{S} \in \{-,+\}^n} \mathbf{1}_{\left\{\varepsilon^{\bar{S}} \in (\delta, 1-\delta)\right\}} = 0$$

- We are looking for every channel which $\varepsilon^{\bar{S}}$ is between $(\delta, 1-\delta)$. (intermediate channels)
- Working with the sets in Mathematics it is not very useful to work. We move our problems set counting number of a set towards **Indicator Function**.
- We have to bound this function.



- $f(\varepsilon^{\bar{S}}) = a\sqrt{\varepsilon^{\bar{S}}(1-\varepsilon^{\bar{S}})}$
  $on\ f\ equal\ to\ 1,\quad a\sqrt{\delta(1-\delta)} = 1 \quad => \quad a = \frac{1}{\sqrt{\delta(1-\delta)}}$
- $\mathbf{1}_{\{\varepsilon^{\bar{S}} \in (\delta,1-\delta)\}} \leq f(\varepsilon) \leq a\sqrt{\varepsilon^{\bar{S}}(1-\varepsilon^{\bar{S}})}$

  $$\mathbf{1}_{\{\varepsilon^{\bar{S}} \in (\delta,1-\delta)\}} \leq \frac{\sqrt{\varepsilon^{S}(1-\varepsilon^{S})}}{\sqrt{\delta(1-\delta)}}$$

  Bound indicator function

  $\mathbf{1}_{\{\varepsilon^{\bar{S}} \in (\delta,1-\delta)\}} \leq ugly(\varepsilon) * \frac{1}{\sqrt{\delta(1-\delta)}}$ .... (3)

- We have,

  ½*(ugly(ϵ^{+}) + ugly(ϵ^{-})) = ugly(ε) $* \frac{1}{2}(\sqrt{\varepsilon(1-\varepsilon)} + \sqrt{(2-\varepsilon)(1-\varepsilon)}) \leq$ ugly(ε) $* \frac{\sqrt{3}}{2}$

  ....(4)

- $Hence,\ \frac{1}{2^n}\sum_{\bar{S} \in \{-,+\}^n} \mathbf{1}_{\{\varepsilon^{\bar{S}} \in (\delta,1-\delta)\}}$

  $= \frac{1}{2^{n-1}}\sum_{\bar{S} \in \{-,+\}^n} \frac{1}{2} * [\mathbf{1}_{\{\varepsilon^{\overline{S+}} \in (\delta,1-\delta)\}} + \mathbf{1}_{\{\varepsilon^{\overline{S-}} \in (\delta,1-\delta)\}}]$

  $\leq \frac{1}{2^{n-1}}\sum_{\bar{S} \in \{-,+\}^n} \frac{1}{2} * [\frac{ugly(\varepsilon^{\overline{S+}})}{\sqrt{\delta(1-\delta)}} + \frac{ugly(\varepsilon^{\overline{S-}})}{\sqrt{\delta(1-\delta)}}]$ (from (3))

$$\leq \frac{1}{2^{n-1}} \sum_{\bar{S}\ \epsilon\{-,+\}^n} \sqrt{\frac{3}{4}} * [\frac{1}{\sqrt{\delta(1-\delta)}} ugly(\varepsilon^{\bar{S}})] \qquad \text{(from (4))}$$

- Repeating for same step n times,

$$\frac{1}{2^n} \sum_{\bar{S}\ \epsilon\{-,+\}^n} \mathbf{1}_{\{\varepsilon^{\bar{S}}\ \epsilon\ (\delta,1-\delta)\}} \leq (\frac{3}{4})^{\frac{n}{2}} * \left[\frac{1}{\sqrt{\delta(1-\delta)}}\right]$$

- For n->∞ ,

$$\lim_{n\to\infty} (\frac{3}{4})^{\frac{n}{2}} \left[\frac{1}{\sqrt{\delta(1-\delta)}}\right] = 0$$

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Further Work

- Arbitrary BMS channel W, capacity I(W), fix a > 0.
- For any $\varepsilon > 0$, we construct codes (polar-variant) with:

$$\mathrm{R} \geq \mathrm{I(W)} - \varepsilon$$
$$\mathrm{R} \geq \mathrm{I(W)} - \frac{1}{N^{\frac{1}{\mu}}} \quad \textit{where } \mu \textit{ is scalling coefficient}$$

- For O(N log N) encoding and decoding.

Arıkan (2008) proved: $\forall \gamma > 0 \quad \dfrac{\#\{i : H(W_i) \in (\gamma, 1-\gamma)\}}{N} \to 0 \quad$ as $N \to \infty$

- We proved this using I(W) instate of H(W). Means their is no any intermediate channel for $N \to \infty$. All channels are very good or very bad. So, fraction of good channels approches to Shannon Channel Capacity.
- After 2008, many scientists try to prove this for finite N.

- For any Arbitrary BMS W.
- Desire rate(R) $\geq \mathrm{I(W)} - \frac{1}{N^{\frac{1}{\mu}}}$

- [2013 : Venkatesan Guruswami, Alexander Barg]:
  u is finite (polynomial convergence to capacity)

- [HAU'13]: u ≤ 6

- [MHU'16]: u ≤ 4.714 (u ≤ 3.639 for BEC)

- For $\mu \to 2$ can achieve Shannon capacity by Indian American Venkatesan Guruswami and Blasiok in 2019.