TTK4115

Lecture 3/4

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This lecture

1. Canonical Forms

2. Realizations

3. Discretization

4. Controllability

    Controllability Gramians

    Eigenvector tests

    Controllability in practice

    Controllability indices

Topic

Canonical Forms

Canonical Forms

- Using the change of basis $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$ we can change a system into infinitely many similar forms.
- Some of these forms are more useful than others.
- Some of these are called *canonical*.

Equivalence/Similarity transform

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned}$$

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\bar{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{T}\bar{\mathbf{x}} + \mathbf{D}\mathbf{u} \end{aligned}$$

Canonical Forms

Canonical Forms[1]

- Jordan Form
- Modal Form
- Companion form
- Controllable form
- Observable form

[1] This list is not exhaustive.

Canonical Forms

Jordan Form

The Jordan form is the most convenient to use when solving the system. We have seen that this form is very practical for finding solutions to LTI systems:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

Diagonal matrix

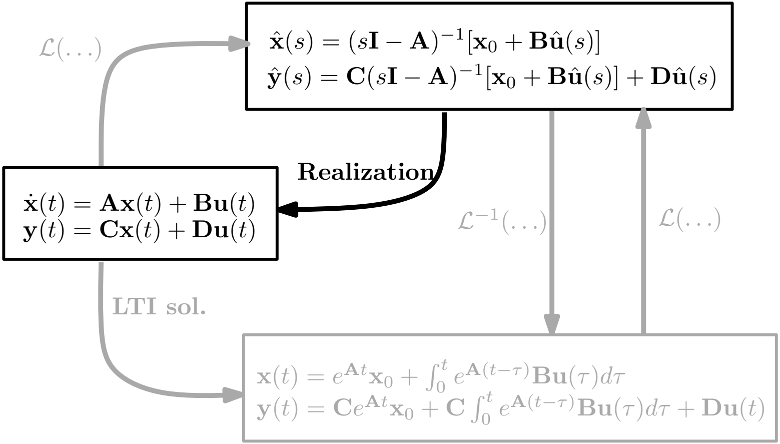
$$\mathbf{A} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \Rightarrow e^{\mathbf{A}t} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{bmatrix}$$

Jordan Block

$$\mathbf{A} = \mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \quad \Rightarrow e^{\mathbf{A}t} = \begin{bmatrix} e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 & \frac{1}{3!}e^{t\lambda}t^3 \\ 0 & e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 \\ 0 & 0 & e^{t\lambda} & e^{t\lambda}t \\ 0 & 0 & 0 & e^{t\lambda} \end{bmatrix}$$

Topic

LTI systems overview



Realizations

The final piece in the diagram

Realizations

Key points

Realization

- We have seen that a transformation $\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}$ can change the state equation..
- but the transfer function remains the same.
- When we realize, we start with a transfer function $\mathbf{H}(s)$..
- and generate a state-space $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$
- that yields $\mathbf{H}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

Note

There are infinitely many state-spaces we could realize to!

Note

We usually go for a *canonical* form.

Realizations

Conditions

Proper

A transfer function must be proper to have a realization:

$$h(s) = \frac{n(s)}{d(s)} \quad \Rightarrow \deg d(s) \geq n(s)$$

$$|h_p(j\infty)| < \infty, \quad |h_{sp}(j\infty)| = 0$$

Rational

A transfer function must be rational to have a realization.

- The degrees of the numerator and denominator must be finite.
- All lumped LTI systems are rational.

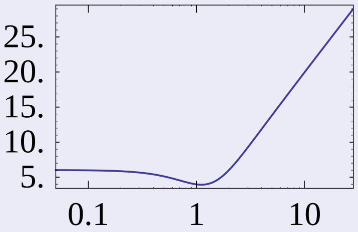
Proper transfer functions

We must have a proper transfer function for realization.

Example

$$h(s) = \frac{2 + 2s + s^2}{1 + s}$$

$|h(i\omega)|$



Question:

Is this a proper transfer function?

Signals are amplified

at infinite frequencies.. no device can do this

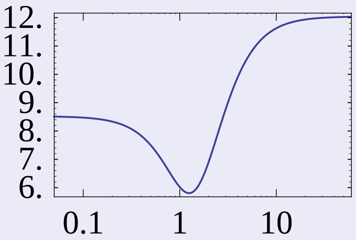
Proper transfer functions

Proper transfer functions behave nicely at high frequencies

Example

$$h(s) = \frac{4\left(2 + 2s + s^2\right)}{3 + 4s + s^2}$$

$|h(i\omega)|$



Question:

Is this a proper transfer function?

Answer

Yes, *but not strictly proper*. The transferfunction is finite at infinite frequencies:

$$\lim_{\omega \to \infty} h(j\omega) = h_\infty \neq 0$$

Quiz

Are these transfer functions realizable?

- $g_1(s) = \frac{1}{s}$
- $g_2(s) = s$
- $g_3(s) = \frac{1}{s+1}$
- $g_4(s) = \frac{1}{s-1}$
- $g_5(s) = \frac{s}{s+1}$
- $g_6(s) = e^{-\tau s}, \quad \tau > 0$
- $g_7(s) = \frac{1-\frac{\tau}{2}s}{1+\frac{\tau}{2}s}, \quad \tau > 0$

Improper/Proper/Strictly proper

Improper

$$H_{i.p.}(s) = k_P + k_D s + \frac{k_I}{s} = \frac{k_D s^2 + k_P s + k_I}{s} \quad \text{PID regulator}$$

Proper

$$H_p(s) = \frac{s}{Ts + 1} \quad \text{Band-limited differentiator}$$

Strictly proper

$$H_{s.p.}(s) = \frac{1}{s^2 m + ds + k} \quad \text{Msd.}$$

Realizations

Strictly proper transfer functions

Decomposition

We decompose the proper transfer function as:

$$\mathbf{G}(s) = \overbrace{\mathbf{G}_{sp}(s)}^{\text{strictly proper}} + \overbrace{\mathbf{G}_{\infty}}^{\text{constant}}$$

Question: where is **D**?

$$\begin{aligned} \hat{\mathbf{y}}(s) &= \mathbf{G}_{sp}(s)\hat{\mathbf{u}}(s) + \mathbf{G}_{\infty}\hat{\mathbf{u}}(s) \\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s) + \mathbf{D}\hat{\mathbf{u}}(s) \end{aligned}$$

Matching

The crucial next step is to select a state-space model with unknown coefficients:

$$\Sigma_r : \; \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$$

that can represent our transfer-function.

Matching

We shall use the **Controllable Canonical Form** today. This is one of many choices.

Let's pick a nice **A** for the realization

Four states $\rightarrow$ up to s^4 in the denominator of $g(s)$.

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$(s\mathbb{I} - \mathbf{A})^{-1}$

$$(s\mathbb{I}-\mathbf{A})^{-1} = \frac{1}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4} \begin{bmatrix} s^3 & -s^2\alpha_2 - s\alpha_3 - \alpha_4 & -s^2\alpha_3 - s\alpha_4 & -s^2\alpha_4 \\ s^2 & s^3 + s^2\alpha_1 & -s\alpha_3 - \alpha_4 & -s\alpha_4 \\ s & s^2 + s\alpha_1 & s^3 + s^2\alpha_1 + s\alpha_2 & -\alpha_4 \\ 1 & s + \alpha_1 & s^2 + s\alpha_1 + \alpha_2 & s^3 + s^2\alpha_1 + s\alpha \end{bmatrix}$$

Let's pick a nice **B** for the realization too

Four states $\rightarrow$ up to s^4 in the denominator of $g(s)$.

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} \frac{1}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

Realizations

Controllable form

What about $\mathbf{C}$?

Four states $\rightarrow$ up to s^4 in the denominator of $g(s)$.

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} \frac{1}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{G}_{sp}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{s^3 n_1 + s^2 n_2 + s n_3 + n_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

Example: Mass spring damper transfer-function realization

$$\frac{x(s)}{f(s)} = \frac{y(s)}{u(s)} = \frac{1}{ms^2 + sd + k} = \frac{1/m}{s^2 + s(d/m) + (k/m)}$$

Controllable canonical form, $n = 2$

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} n_1 & n_2 \end{bmatrix}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{G}_{sp}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{s\overbrace{n_1}^{0} + \overbrace{n_2}^{1/m}}{s^2 + s\underbrace{\alpha_1}_{d/m} + \underbrace{\alpha_2}_{k/m}}$$

Realization

The mass spring damper back on state-space form:

$$\mathbf{A} = \begin{bmatrix} -d/m & -k/m \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1/m \end{bmatrix}$$

Realizations

Controllable form

Controllable canonical form: p inputs, q outputs

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 \mathbb{I}_p & -\alpha_2 \mathbb{I}_p & -\alpha_3 \mathbb{I}_p & -\alpha_4 \mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 \end{bmatrix}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{G}_{sp}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{s^3\mathbf{N}_1 + s^2\mathbf{N}_2 + s\mathbf{N}_3 + \mathbf{N}_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

$d(s)$

We have to find the common denominator of $\mathbf{G}_{sp}(s)$: $d(s) = s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4$

Realizations

Example 1

Realize $\mathbf{G}(s)$ to controllable canonical form:

$$\mathbf{G}(s) = \begin{bmatrix} \frac{-10+4s}{1+2s} & \frac{3}{2+s} \\ \frac{1}{(2+s)(1+2s)} & \frac{1+s}{(2+s)^2} \end{bmatrix}$$

Find $\mathbf{G}_\infty$

$$\mathbf{D} = \mathbf{G}_\infty = \lim_{s\to\infty} \begin{bmatrix} \frac{-10+4s}{1+2s} & \frac{3}{2+s} \\ \frac{1}{(2+s)(1+2s)} & \frac{1+s}{(2+s)^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Find $\mathbf{G}_{sp} = \mathbf{G}(s) - \mathbf{G}_\infty$

$$\mathbf{G}_{sp} = \begin{bmatrix} \frac{-10+4s}{1+2s} & \frac{3}{2+s} \\ \frac{1}{(2+s)(1+2s)} & \frac{1+s}{(2+s)^2} \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{12}{1+2s} & \frac{3}{2+s} \\ \frac{1}{2+5s+2s^2} & \frac{1+s}{(2+s)^2} \end{bmatrix}$$

Find common denominator $d(s)$

$$\mathbf{G}_{sp} = \frac{1}{s^3 + (9/2)s^2 + 6s + 2} \begin{bmatrix} -6(2+s)^2 & 3(1+s/2)(1+2s) \\ 1+s/2 & (1/2+s)(1+s) \end{bmatrix}$$

Realizations

Example 1

Realize $\mathbf{G}(s)$ to controllable canonical form:

$$\mathbf{G}_{sp} = \frac{1}{d(s)}\left(\begin{bmatrix} -24-24s & 3+\frac{15s}{2} \\ 1+\frac{s}{2} & \frac{1}{2}+\frac{3s}{2} \end{bmatrix} + s^2 \begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}\right)$$

$$d(s) = s^3 + (9/2)s^2 + 6s + 2$$

Find numerator matrices $\mathbf{N}_i$

$$\mathbf{G}_{sp} = \frac{1}{d(s)}\left(\overbrace{\begin{bmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{bmatrix}}^{\mathbf{N}_3} + s\overbrace{\begin{bmatrix} -24 & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}}^{\mathbf{N}_2} + s^2\overbrace{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}}^{\mathbf{N}_1}\right)$$

$$d(s) = s^3 + (9/2)s^2 + 6s + 2$$

Realizations

Example 1

Realize $\mathbf{G}(s)$ to controllable canonical form:

$$\mathbf{G}_{sp} = \frac{1}{d(s)}\left[\mathbf{N}_3 + s\mathbf{N}_2 + s^2\mathbf{N}_1\right] \quad \mathbf{G}_\infty = \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{N}_1 = \begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix} \quad \mathbf{N}_2 = \begin{bmatrix} -24 & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \quad \mathbf{N}_3 = \begin{bmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$d(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = s^3 + (9/2)s^2 + 6s + 2$$

Realize:

$$\mathbf{A} = \begin{bmatrix} -\alpha_1\mathbb{I}_p & -\alpha_2\mathbb{I}_p & -\alpha_3\mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 \end{bmatrix}$$

Topic

Analog state space model

The continuous state space model is analog[2]:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

To simulate it *as is* would require an analog computer.

Discretization

Discretization is a necessary step for computer simulation. A recursive model is sought:

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]\end{aligned}$$

Many methods are available for obtaining a discretized model. We examine the two most common methods: **Exact** and **Euler** discretization[3].

Approach

$$\text{TF} \xrightarrow{\text{Realize}} \text{CLTI} \xrightarrow{\text{Discretize}} \text{DLTI} \xrightarrow{\text{Recursion}} \text{Solution}$$

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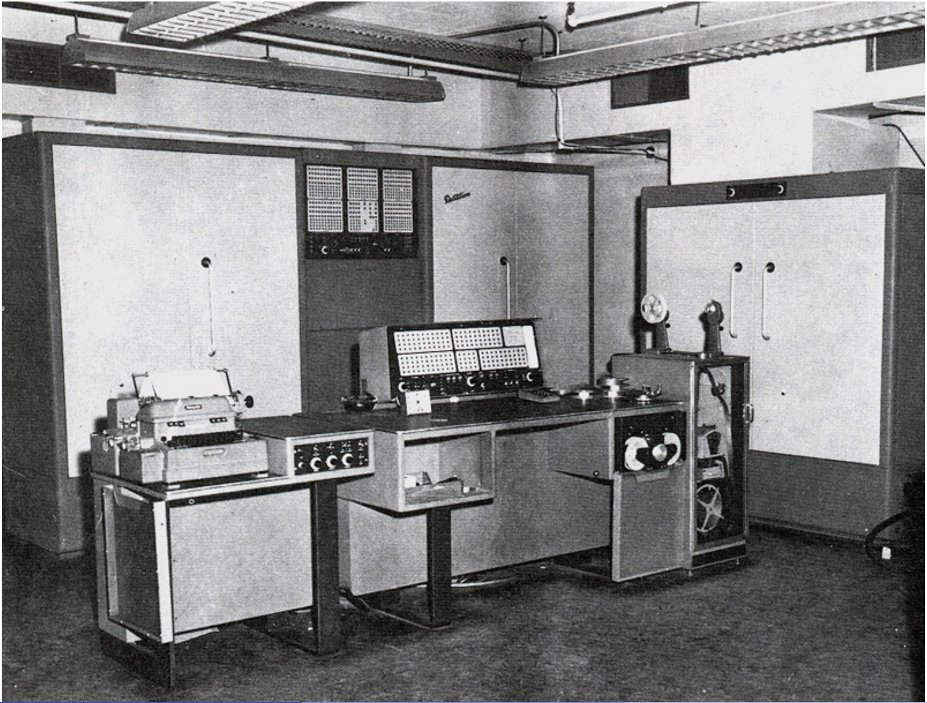
[2]Some systems are discrete by nature, such as financial systems or discrete filters. Most plants will however be continuous as they are based on a physical model.

[3]These are repectively the best and worst of the common methods.



LTI solution

The exact solution of the LTI system forms the theoretical basis of **exact** discretization.

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

Sampling

Time is discretized into intervals of duration T: $t = kT$. Sample index is denoted k. The state solution from one sample to the next is:

$$\mathbf{x}((k+1)T) = e^{\mathbf{A}T}\mathbf{x}(kT) + \int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Here $\mathbf{x}[k] \triangleq \mathbf{x}(t)|_{t=kT}$ serves as an initial condition
The resulting solution is evaluated at $\mathbf{x}[k+1] \triangleq \mathbf{x}(t)|_{t=(k+1)T}$.

Piecewise constant input

The input is assumed to stay approximately constant between samples:

$$\mathbf{u}[k] \simeq \mathbf{u}(t), \quad kT \leq t \leq (k+1)T$$

Sampled model

$$\mathbf{x}[k+1] = e^{\mathbf{A}T}\mathbf{x}[k] + \left(\int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}d\tau\right)\mathbf{Bu}[k]$$

Substitution rule

$$\int_{y(a)}^{y(b)} F(x)dx = \int_a^b F(y(x))\frac{dy}{dx}dx$$

Change of variable: $\alpha(\tau) \triangleq (k+1)T - \tau, \quad d\tau = -d\alpha$

Integration limits are simplified:

$$\tau_0 = kT \to \alpha_0 = T, \quad \tau_1 = (k+1)T \to \alpha_1 = 0$$

along with integrand:

$$e^{\mathbf{A}[(k+1)T-\tau]} \to e^{\mathbf{A}\alpha}$$

$$\mathbf{B}_d = \left(\int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}d\tau\right)\mathbf{B} = \left(\int_0^T e^{\mathbf{A}\alpha}d\alpha\right)\mathbf{B}$$

Discretization

Exactly discretized model

$$
\begin{aligned}
\mathbf{x}[k+1] &= \overbrace{e^{\mathbf{A}T}}^{\mathbf{A}_d}\mathbf{x}[k] + \overbrace{\left(\int_0^T e^{\mathbf{A}\alpha}d\alpha\right)\mathbf{B}}^{\mathbf{B}_d}\mathbf{u}[k] \\
\mathbf{y}[k] &= \underbrace{\mathbf{C}}_{\mathbf{C}_d}\mathbf{x}[k] + \underbrace{\mathbf{D}}_{\mathbf{D}_d}\mathbf{u}[k]
\end{aligned}
$$

Discrete time system

This model is *exact* under the assumption:

$$
\mathbf{u}[k] = \mathbf{u}(t), \quad kT \leq t \leq (k+1)T
$$

It is recursive, and very efficient in implementation:

$$
\begin{aligned}
\mathbf{x}[k+1] &= \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \\
\mathbf{y}[k] &= \mathbf{C}_d\mathbf{x}[k] + \mathbf{D}_d\mathbf{u}[k]
\end{aligned}
$$

Discretization

Euler discretization

Euler's method proceeds via the definition of the derivative[4]:

$$\dot{\mathbf{x}}[k] \approx \frac{\mathbf{x}[k+1] - \mathbf{x}[k]}{T}$$

Thus:

$$\dot{\mathbf{x}}[k] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \quad \Rightarrow \mathbf{x}[k+1] = \mathbb{I}\mathbf{x}[k] + T\mathbf{A}\mathbf{x} + T\mathbf{B}\mathbf{u}$$

Stability

Euler's method may be unstable although the underlying plant is stable. This problem gets worse with larger timesteps. A mathematical criterion for first order systems may be stated as:

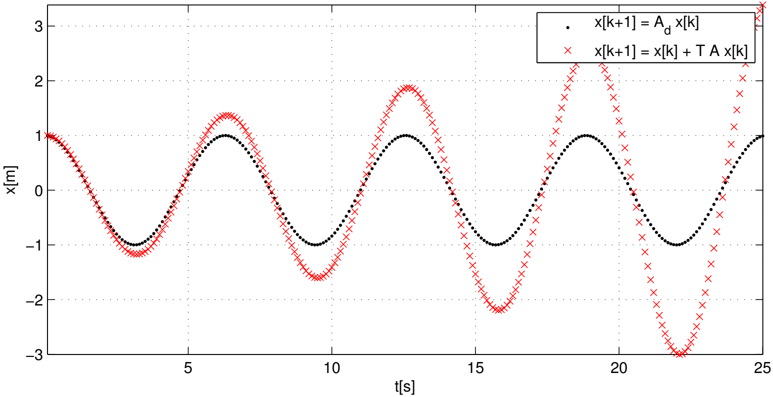
$$|1 + T\lambda| \leq 1$$

Insufficiently stable systems or large timesteps will result in a divergent solution.

[4] $x[k] = x(kT)$

Euler's Method vs Discretization

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \\ \mathbf{x}_e[k+1] &= \mathbf{x}_e[k] + T\mathbf{A}\mathbf{x}_e[k] + T\mathbf{B}\mathbf{u}[k] \end{aligned}$$



Topic

Example

Consider the single input system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \quad \mathbf{A} \in \mathbb{R}^{4\times 4}$$

4 steps forward

We step the system forward by infinitesimally small time steps $\Delta t = \epsilon$:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \epsilon\mathbf{A}\mathbf{x}_k + \epsilon\mathbf{b}u_k$$

Starting point is $\mathbf{x}_0$

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{x}_0 + \epsilon\mathbf{A}\mathbf{x}_0 + \epsilon\mathbf{b}u_0 \\
\mathbf{x}_2 &= \mathbf{x}_1 + \epsilon\mathbf{A}\mathbf{x}_1 + \epsilon\mathbf{b}u_1 \\
\mathbf{x}_3 &= \mathbf{x}_2 + \epsilon\mathbf{A}\mathbf{x}_2 + \epsilon\mathbf{b}u_2 \\
\mathbf{x}_4 &= \mathbf{x}_3 + \epsilon\mathbf{A}\mathbf{x}_3 + \epsilon\mathbf{b}u_3
\end{aligned}$$

Last step may be written as:

$$\mathbf{x}_4 = (\mathbb{I} + \epsilon\mathbf{A})^4\,\mathbf{x}_0 + \epsilon\,(\mathbb{I} + \epsilon\mathbf{A})^3\,\mathbf{b}u_0 + \epsilon\,(\mathbb{I} + \epsilon\mathbf{A})^2\,\mathbf{b}u_1 + \epsilon\,(\mathbb{I} + \epsilon\mathbf{A})\,\mathbf{b}u_2 + \epsilon\mathbf{b}u_3$$

The n'th step is linear in the initial condition and the sequence of inputs:

$$
\begin{aligned}
\mathbf{x}_4 &= (\mathbb{I} + \epsilon\mathbf{A})^4 \mathbf{x}_0 + \epsilon (\mathbb{I} + \epsilon\mathbf{A})^3 \mathbf{b} u_0 + \epsilon (\mathbb{I} + \epsilon\mathbf{A})^2 \mathbf{b} u_1 + \epsilon (\mathbb{I} + \epsilon\mathbf{A}) \mathbf{b} u_2 + \epsilon \mathbf{b} u_3 \\
&= (\mathbb{I} + \epsilon\mathbf{A})^4 \mathbf{x}_0 + \epsilon \begin{bmatrix} (\mathbb{I} + \epsilon\mathbf{A})^3 \mathbf{b} & (\mathbb{I} + \epsilon\mathbf{A})^2 \mathbf{b} & (\mathbb{I} + \epsilon\mathbf{A}) \mathbf{b} & \mathbf{b} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}
\end{aligned}
$$

Gather linearly dependent columns:

$$
\begin{aligned}
&\epsilon \begin{bmatrix} (\mathbb{I} + \epsilon\mathbf{A})^3 \mathbf{b} & (\mathbb{I} + \epsilon\mathbf{A})^2 \mathbf{b} & (\mathbb{I} + \epsilon\mathbf{A}) \mathbf{b} & \mathbf{b} \end{bmatrix} = \epsilon \begin{bmatrix} \mathbf{b} & \cancel{\mathbf{b}} & \cancel{\mathbf{b}} & \cancel{\mathbf{0}} \end{bmatrix} \\
&+ \epsilon^2 \begin{bmatrix} 3\mathbf{A}\mathbf{b} & \cancel{2\mathbf{A}\mathbf{b}} & \cancel{\mathbf{0}} & \cancel{\mathbf{0}} \end{bmatrix} + \epsilon^3 \begin{bmatrix} 3\mathbf{A}^2\mathbf{b} & \cancel{\mathbf{0}} & \cancel{\mathbf{0}} & \cancel{\mathbf{0}} \end{bmatrix} + \begin{bmatrix} \epsilon^4\mathbf{A}^3\mathbf{b} & \epsilon^3\mathbf{A}^2\mathbf{b} & \epsilon^2\mathbf{A}\mathbf{b} & \epsilon\mathbf{b} \end{bmatrix} \\
&\qquad = \begin{bmatrix} \epsilon^4\mathbf{A}^3\mathbf{b} & \epsilon^3\mathbf{A}^2\mathbf{b} & \epsilon^2\mathbf{A}\mathbf{b} & \epsilon\mathbf{b} \end{bmatrix} + LDC
\end{aligned}
$$

The n'th step is linear in the initial condition and the sequence of inputs

$$\mathbf{x}_4 = \overbrace{(\mathbb{I} + \epsilon\mathbf{A})^4\, \mathbf{x}_0}^{\text{zir}} + \overbrace{\left(\begin{bmatrix} \epsilon^4\mathbf{A}^3\mathbf{b} & \epsilon^3\mathbf{A}^2\mathbf{b} & \epsilon^2\mathbf{A}\mathbf{b} & \epsilon\mathbf{b} \end{bmatrix} + LDC\right)\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}}^{\text{zsr}}$$

Key idea

Iff $\begin{bmatrix} \epsilon^4\mathbf{A}^3\mathbf{b} & \epsilon^3\mathbf{A}^2\mathbf{b} & \epsilon^2\mathbf{A}\mathbf{b} & \epsilon\mathbf{b} \end{bmatrix}$ has *full rank*, we can choose $\mathbf{x}_4$ as we like with our inputs.

Controllability matrix

Iff the controllability matrix has full rank: $\text{rank}(\mathcal{C}) = n$

$$\mathcal{C} \triangleq \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{(n-1)}\mathbf{b} \end{bmatrix}$$

the state can be placed anywhere with the right sequence of inputs.
-**This is controllability**.

Topic

Controllability Gramian

Given an LTI system:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Attempt to place the system at $\mathbf{x}_1$ at $t = t_1$:

$$\mathbf{x}_1 = \mathbf{x}(t_1) = e^{\mathbf{A}t_1}\mathbf{x}_0 + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

We clearly need the proper input signal $\mathbf{u}(t)$ to do this.

If we can find such an input, the system is controllable.

Controllability Gramian: definition

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)}d\tau$$

The existence of a nonsingular Controllability Gramian is important because it guarantees that a sufficient $\mathbf{u}(t)$ exists.

Place the system at $\mathbf{x}_1$ at $t = t_1$

$$\mathbf{x}_1 = \mathbf{x}(t_1) = e^{\mathbf{A}t_1}\mathbf{x}_0 + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

We need an input $\mathbf{u}(t)$ to do this.

Educated guess:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1-t)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right]$$

Result

$$\mathbf{x}_1 = e^{\mathbf{A}t_1}\mathbf{x}_0 - \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\left(\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1-\tau)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right]\right)d\tau$$

$$= e^{\mathbf{A}t_1}\mathbf{x}_0 - \overbrace{\left(\int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1-\tau)}d\tau\right)}^{\mathbf{W}_c(t_1)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right]$$

$$= e^{\mathbf{A}t_1}\mathbf{x}_0 - \left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right] = \underline{\mathbf{x}_1}$$

Iff $\mathbf{W}_c(t)$ is invertible, the system is controllable.

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

When is $\mathbf{W}_c(t)$ invertible?

- $e^{\mathbf{A}t}$ may be expressed as a linear combination of $\{\mathbb{I}, \mathbf{A}, \ldots, \mathbf{A}^{n-1}\}$
- $e^{\mathbf{A}t}\mathbf{B}$ may be expressed as a linear combination of $\{\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\}$

Consider the effect of the vector $\mathbf{v}$

$$\mathbf{v}^\mathsf{T}\mathbf{W}_c(t)\mathbf{v} = \int_0^t \overbrace{\mathbf{v}^\mathsf{T} e^{\mathbf{A}\tau}\mathbf{B}}^{\mathbf{0}}\,\overbrace{\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{v}}^{\mathbf{0}}\, d\tau$$

Key point:

$$\overbrace{\mathbf{v}^\mathsf{T} e^{\mathbf{A}\tau}\mathbf{B}}^{\mathbf{0}} \quad \Rightarrow \overbrace{\mathbf{v}^\mathsf{T}\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}}^{\mathbf{0}}$$

The Gramian is invertible iff the controllability matrix has full rank!

Equivalent Statements on Controllability

Controllability Gramian

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

Iff $\mathbf{W}_c(t)$ is invertible, the system is controllable.

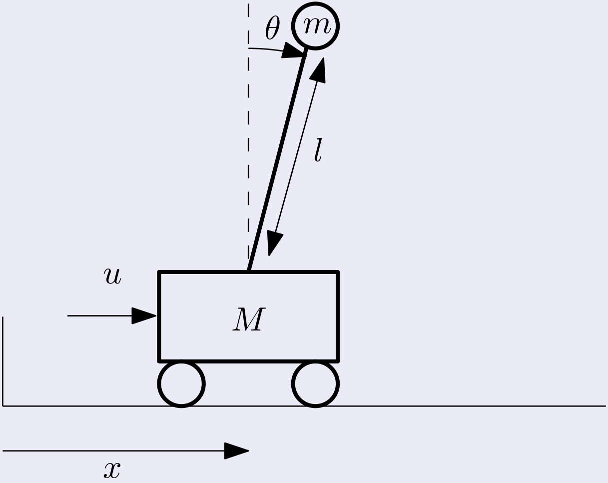
Controllability Matrix

$$\mathcal{C} = \overbrace{\begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}}^{np} \}n$$

Iff the controllability matrix has full rank: $\text{rank}(\mathcal{C}) = n$, the system is controllable.

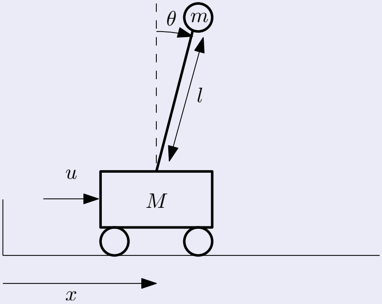
Controllability

Example



Controllability

Example



Linearized EOM

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(m+M)}{lM} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{lM} \end{bmatrix} u$$

Controllability

Linearized EOM

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(m+M)}{lM} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{lM} \end{bmatrix} u$$

Controllability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Controllability Matrix: Full row rank

$$\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{15}{2} \\ \frac{1}{2} & 0 & \frac{5}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{15}{2} & 0 \end{bmatrix}$$

Controllability Gramian: Invertible

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

Controllability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Move the cart from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}_1 = [1, 0, 0, 0]^\mathsf{T}$, $t_1 = 3s$

Let's use the Gramian:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1 - t)} \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1 \right]$$

Controllability Gramian

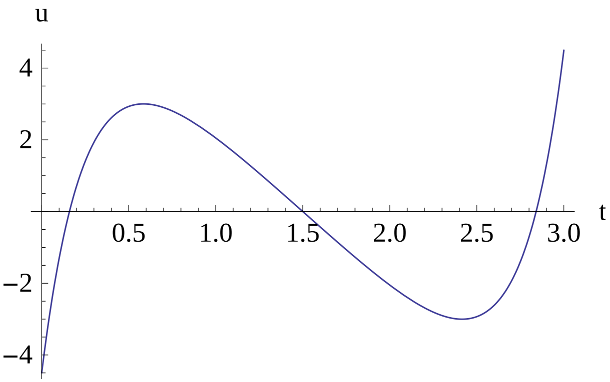
$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

Controllability

Move the cart from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}_1 = [1, 0, 0, 0]^\top$, $t_1 = 3s$

Let's use the Gramian:

$$\mathbf{u}(t) = -\mathbf{B}^\top e^{\mathbf{A}^\top (t_1 - t)} \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}t_1} \mathbf{x}_0 - \mathbf{x}_1 \right]$$

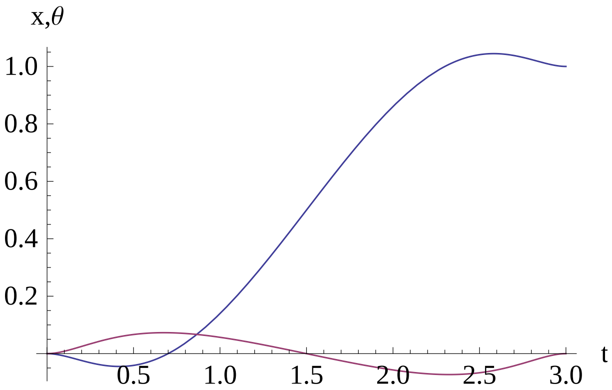


Controllability

Move the cart from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}_1 = [1, 0, 0, 0]^\mathsf{T}$, $t_1 = 3s$

Let's use the Gramian:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1 - t)} \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}t_1} \mathbf{x}_0 - \mathbf{x}_1 \right]$$



Topic

Eigenvector test

Basic idea

If the Controllability matrix has full rank, there is no vector $\mathbf{v}$ such that:

$$\mathcal{C}^{\mathsf{T}}\mathbf{v} = \begin{bmatrix} \mathbf{B}^{\mathsf{T}} \\ \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} \\ \dots \\ \mathbf{B}^{\mathsf{T}}(\mathbf{A}^{\mathsf{T}})^{(n-1)} \end{bmatrix} \mathbf{v} = \mathbf{0}, \quad \forall \mathbf{v} \neq \mathbf{0}$$

Let $\mathbf{v}$ be an eigenvector of $\mathbf{A}^{\mathsf{T}}$: $\mathbf{A}^{\mathsf{T}}\mathbf{q} = \lambda\mathbf{v}$

$$\begin{bmatrix} \mathbf{B}^{\mathsf{T}} \\ \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} \\ \dots \\ \mathbf{B}^{\mathsf{T}}(\mathbf{A}^{\mathsf{T}})^{(n-1)} \end{bmatrix} \mathbf{q} = \begin{bmatrix} \mathbf{B}^{\mathsf{T}} \\ \lambda\mathbf{B}^{\mathsf{T}} \\ \vdots \\ \lambda^{n-1}\mathbf{B}^{\mathsf{T}} \end{bmatrix} \mathbf{q}$$

Warning

Controllability is only possible if every eigenvector of $\mathbf{A}^{\mathsf{T}}$ is not in the null-space of $\mathbf{B}^{\mathsf{T}}$.

Eigenvector test

Warning

Controllability is only possible if every eigenvector of $\mathbf{A}^\mathsf{T}$ is not in the null-space of $\mathbf{B}^\mathsf{T}$.

Popov-Belevitch-Hautus - test

The PBH test gives an elegant test based on this insight.
An LTI system is controllable iff:

$$\text{rank}[\mathbf{A} - \lambda\mathbb{I} \;\; \mathbf{B}] = n, \quad \forall\lambda \in \mathbb{C}$$

Topic

Controllable systems

Three important points

- If the pair $\{\mathbf{A}, \mathbf{B}\}$ is controllable, so is $\{\mathbf{A} - \mathbf{BK}, \mathbf{B}\}$.
- If the system is controllable we can place the eigenvalues of the system exactly as desired.
- A controllable system can always be transformed to the controllable canonical form.

Controllability in practice

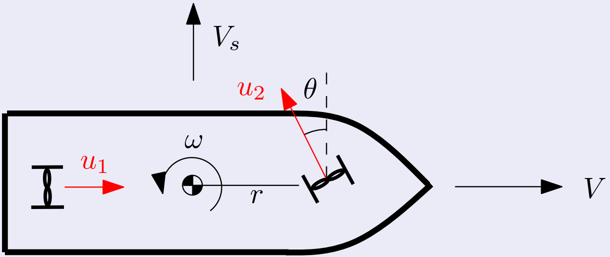
Caveats

Even if the controllability matrix has full rank, this does not mean that the system is easy to control in practice.

- The controller may require too large inputs.
- The closed loop response may be higly sensitive to modeling errors in $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$.
- The closed loop eigenvalues may have been chosen unrealistically fast.
- Fast response requires powerful actuators and an accurate model.
- The system may be ”almost uncontrollable” in practice.

Controllable?

Dynamic positioning

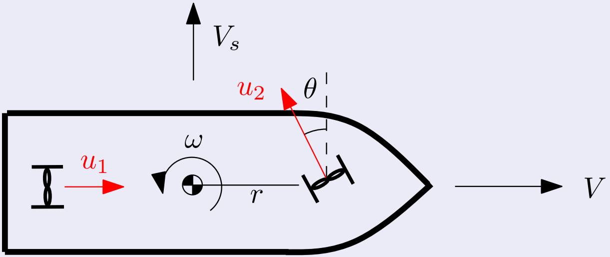


State equation

$$\begin{bmatrix} \dot{V} \\ \dot{V}_s \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -\frac{d}{m} & 0 & 0 \\ 0 & -\frac{d_s}{m} & 0 \\ 0 & 0 & -\frac{d_\omega}{J} \end{bmatrix} \begin{bmatrix} V \\ V_s \\ \omega \end{bmatrix} + \begin{bmatrix} 1/m & -\sin(\theta)/m \\ 0 & \cos(\theta)/m \\ 0 & \cos(\theta)r/J \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Controllable?

Dynamic positioning

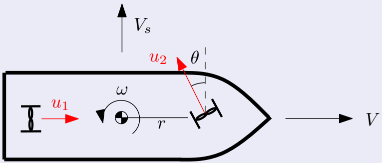


Controllability matrix

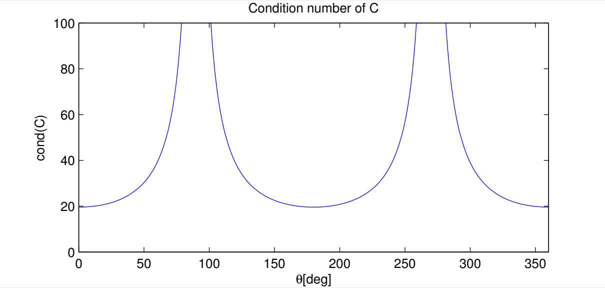
$$\mathcal{C} = \begin{bmatrix} \frac{1}{m} & -\frac{\sin(\theta)}{m} & -\frac{d}{m^2} & \frac{d\sin(\theta)}{m^2} & \frac{d^2}{m^3} & -\frac{d^2\sin(\theta)}{m^3} \\ 0 & \frac{\cos(\theta)}{m} & 0 & -\frac{\cos(\theta)d_s}{m^2} & 0 & \frac{\cos(\theta)d_s^2}{m^3} \\ 0 & \frac{r\cos(\theta)}{J} & 0 & -\frac{r\cos(\theta)d_\omega}{J^2} & 0 & \frac{r\cos(\theta)d_\omega^2}{J^3} \end{bmatrix}$$

Controllable?

Dynamic positioning

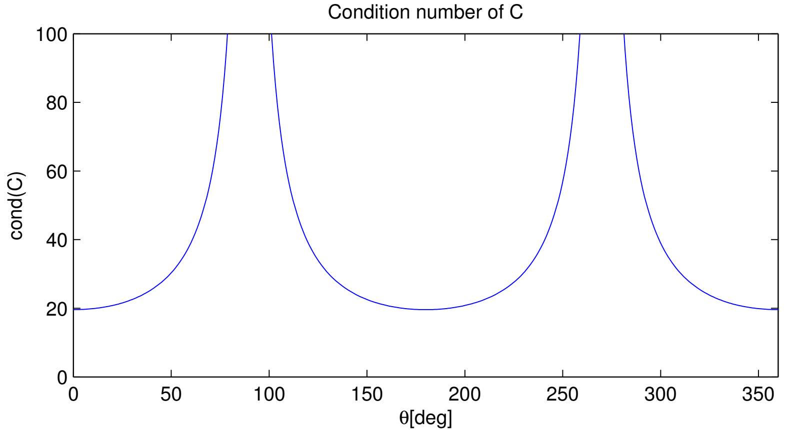


Condition number of $\mathcal{C}$



Controllable?

Condition number of $\mathcal{C}$



Topic

What's wrong with this $\mathbf{B}$?

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Answer

Linear dependency in the columns. We disregard redundant inputs.

We have p inputs. Let:

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$$

Controllability matrix

$$\mathcal{C} = \left[\begin{array}{cccc|cccc|c|cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p & \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p & \ldots & \mathbf{A}^{n-1}\mathbf{b}_1 & \mathbf{A}^{n-1}\mathbf{b}_2 & \cdots & \mathbf{A}^{n-1}\mathbf{b}_p \end{array} \right]$$

Controllability index

The controllability indices of $\mathbf{b}_p$: μ_p, are the number of linearly independent columns associated with $\mathbf{b}_p$. These indices sum to:

$$\mu_1 + \mu_2 + \cdots + \mu_p = n$$

The largest μ_i is the controllability index.

Multi-input controllability

Using these indices, we can show that it is sufficient to check the rank of:

$$\mathcal{C} = \left[\begin{array}{c|c|c|c} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A}^{n-p}\mathbf{B} \end{array} \right]$$

Final notes

Property 1

Controllability is not affected by an equivalence transformation.

Property 2

Controllability is not affected by reordering the columns of $\mathbf{B}$.