10.018 Modelling Space and Systems

Week 1 Class 1 Supplementary Notes

Vector Operations, Lines and Planes

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Outline

1. Vector Operations
2. Lines and Planes
3. Summary

Vector Operations

Vector Addition

Vectors can be added or subtracted coordinate-wise.

Example:

Consider the vector $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and vector $\vec{b} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$. Find $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

Solution:

$$\vec{a} + \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} \quad , \quad \vec{a} - \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

Vector Operations

Scalar Multiplication

Vectors can be multiplied by scalars coordinate-wise.

Example:

Continuing from previous example, compute $3\vec{a} + 2\vec{b}$.

Solution:

$$3\vec{a} + 2\vec{b} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}.$$

Note: In this course, we generally take scalars to be real numbers. Of course we may sometimes work with complex numbers (we will specify them in such occasions).

Dot Product (or Scalar Product)

Definition

Consider two vectors in $\mathbb{R}^n$ denoted by $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

The dot product $\vec{u} \cdot \vec{v}$ is given by

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

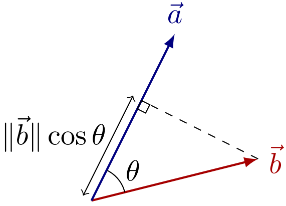
Definition

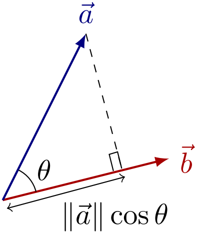
The length or norm of the vector $\vec{u}$ is

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = \sqrt{\vec{u} \cdot \vec{u}}.$$

Geometrical Interpretation

Consider two vectors $\vec{a}$ and $\vec{b}$ where the angle between them is θ.





In the left figure, we have $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta = \|\vec{a}\|(\|\vec{b}\| \cos\theta)$. This is a multiplication of norm of $\vec{a}$ with the component of $\vec{b}$ in the direction of $\vec{a}$.

Equivalently, in the right figure, we have $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta = \|\vec{b}\|(\|\vec{a}\| \cos\theta)$. This is a multiplication of norm of $\vec{b}$ with the component of $\vec{a}$ in the direction of $\vec{b}$.

Dot Product (or Scalar Product)

Example:

Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Find the angle between $\vec{a}$ and $\vec{b}$.

Solution:

Their norms are $\|\vec{a}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ and $\|\vec{b}\| = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}$. The dot product $\vec{a} \cdot \vec{b}$ is given by

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = (1)(1) + (2)(1) + (3)(3) = 12.$$

Since $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$, we find that $12 = \sqrt{14}\sqrt{11}\cos\theta$. Thus $\cos\theta \approx 0.97$ or $\theta \approx 14.8°$.

Dot Product (or Scalar Product)

We list down some properties:

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ to be some vectors in $\mathbb{R}^n$, and c is any scalar.

- Dot products are commutative.

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

- Dot products are distributive.

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$
$$(c\vec{u}) \cdot \vec{v} = c\,(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$$

Remarks:

- The vectors $\vec{u}$ and $\vec{v}$ are orthogonal (or perpendicular) if and only if $\vec{u} \cdot \vec{v} = 0$.
- Dot products are scalars.

Cross Product (or Vector Product)

Since cross product is defined only for vectors in $\mathbb{R}^3$, we define the following.

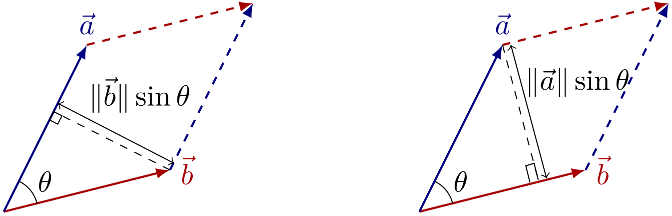
Definition

Consider the vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. The cross-product $\vec{u} \times \vec{v}$ is given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}.$$

Geometrical Interpretation

Consider two vectors $\vec{a}$ and $\vec{b}$ where the angle between them is θ.



In the left figure, we have $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta = \|\vec{a}\|(\|\vec{b}\| \sin\theta)$.
Equivalently, in the right figure, we have
$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta = \|\vec{b}\|(\|\vec{a}\| \sin\theta)$.

Magnitude of the cross product is the area of the parallelogram formed by the two vectors.

Cross Product (or Vector Product)

Example:

Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Find $\vec{a} \times \vec{b}$, $\left(\vec{a} \times \vec{b}\right) \cdot \vec{a}$ and $\left(\vec{a} \times \vec{b}\right) \cdot \vec{b}$.

Solution:

$\vec{a} \times \vec{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$, $\left(\vec{a} \times \vec{b}\right) \cdot \vec{a} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0$ and

$\left(\vec{a} \times \vec{b}\right) \cdot \vec{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0.$

Remark: We can see that $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$.

Cross Product (or Vector Product)

We list down some properties:

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ to be some vectors in $\mathbb{R}^n$, and c is any scalar.

- Cross products are anti-commutative.

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

- Cross products are distributive.

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$
$$(c\vec{u}) \times \vec{v} = c\,(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$$

- Cross products are not associative.

$$\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$$

Outline

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2. Lines and Planes
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Lines in $\mathbb{R}^2$

Definition

The normal form of the equation of a line in $\mathbb{R}^2$ is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

where $\vec{r}_0$ is the position vector of a specific point on the line and $\vec{n} \neq \vec{0}$ is a normal vector of the line.

Definition

The general form of the equation of a line in $\mathbb{R}^2$ is

$$ax + by = c$$

where $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector of the line.

Lines in $\mathbb{R}^2$

Definition

The vector form of the equation of a line in $\mathbb{R}^2$ (or higher dimension) is

$$\vec{r} = \vec{r}_0 + t\vec{d} \quad , \quad \text{for all } t \in \mathbb{R}$$

where $\vec{r}_0$ is the position vector of a specific point on the line and $\vec{d} \neq \vec{0}$ is a direction vector of the line ('a vector parallel to the line').

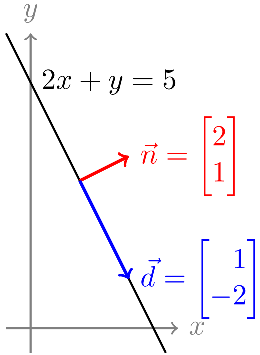
Remarks:

Normal vector and direction vector are orthogonal. i.e. $\vec{n} \cdot \vec{d} = 0$.

Lines in $\mathbb{R}^2$

Example:

Consider a line with general form $2x + y = 5$ with normal vector $\vec{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and direction vector $\vec{d} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.



The normal form of is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = 0.$$

The vector form is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Planes in $\mathbb{R}^3$

Definition

The normal form of the equation of a plane in $\mathbb{R}^3$ is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

where $\vec{r}_0$ is the position vector of a specific point on the plane and $\vec{n} \neq \vec{0}$ is a normal vector of the plane.

Planes in $\mathbb{R}^3$

Definition

The general form of the equation of a plane in $\mathbb{R}^3$ is

$$ax + by + cz = d$$

where $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a normal vector of the plane, and d is the dot product of the normal vector and a specific point on the plane.

Planes in $\mathbb{R}^3$

Definition

The vector form of the equation of a plane in $\mathbb{R}^3$ (or higher dimensions) is

$$\vec{r} = \vec{r}_0 + s\vec{u} + t\vec{v} \quad , \quad \text{for all } s, t \in \mathbb{R}$$

where $\vec{r}_0$ is the position vector of a specific point on the plane, $\vec{u}$ and $\vec{v}$ are direction vectors of the plane. ($\vec{u}$ and $\vec{v}$ are non-zero and parallel to the plane but NOT parallel to each other.)

Planes in $\mathbb{R}^3$

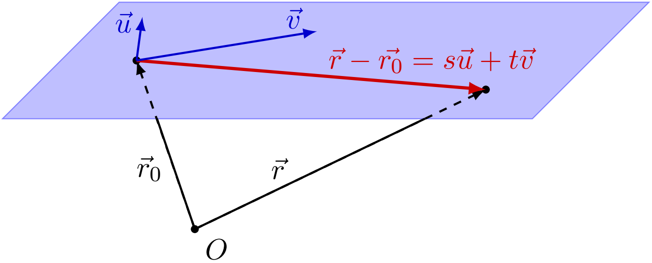


Figure: Illustration of the vector form of the equation of a plane.

You can download the **GeoGebra 3D Calculator** in App Store or Google Play to utilise the augmented reality (AR) capability for more realistic visualisation.

`https://www.geogebra.org/m/k2xdx35v`

Activity

Consider a plane with normal vector $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and contains a point $r_0 = (2, 0, 3)$.

(a) Find the general form of the equation of the plane.

(b) Show that $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are parallel to the plane.
Hence write down the vector form of the equation of the plane.

Activity: Solution

(a) Consider an arbitrary point point $r = (x, y, z)$. Using the normal form of the equation of the plane, we have

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \qquad \Rightarrow \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right) = 0.$$

Rearranging, we obtain the general form of the equation of the plane

$$x - y + z = 5.$$

Activity: Solution

(b) We compute $\vec{u} \times \vec{v}$ and get

$$\vec{u} \times \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \vec{n}.$$

Since $\vec{u} \times \vec{v}$ gives the normal vector of the plane, we conclude they are parallel to the plane. Thus the vector form of the equation of the plane is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

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Summary

We have covered:

- vector operations,
- lines and planes.

References:

Textbook	Section
Hughes-Hallet	13.3 - 13.4.
Simmons	18.2 - 18.4.

You may try the exercises therein.

Online Resources

Click on each box to access the video

dot product | cross product | lines | planes