Solutions to Sample Exam on Linear Programming for MSOR 411

Name: ______________________________

1. Farmer Smith has S square meters of land on which to plant up to n different crops. Each square meter of land devoted to crop i costs c_i dollars to plant and yields k_i kilograms of the crop that then sells for p_i dollars per kilogram. Each kilogram of crop i to be produced requires q_i liters of water, of which there is a total of Q liters available. Formulate an appropriate linear program. (20 points)

   Let x_i = the number of (meters)^2 of land used to plant crop i ($i = 1, \ldots, n$). You then want to

$$
\begin{array}{lllll}
\max & \sum_{i=1}^{n} (p_i k_i - c_i) x_i & & & \\
\text{s.t.} & \sum_{i=1}^{n} q_i k_i x_i & \leq & Q & \text{(water)} \\
 & \sum_{i=1}^{n} x_i & \leq & S & \text{(land)} \\
 & \text{all } x_i & \geq & 0 &
\end{array}
$$

2. Consider the following constraint for a particular problem:

> It is necessary to determine the location of a hospital in such a way that the total driving distance to a particular blood bank does not exceed three miles. (Assume a retangular-shaped city of vertical and horizontal streets.)

To formulate this constraint, an OR student created the following two variables:

x = the number of miles from the blood bank to the hospital in the East-West direction. (A positive value for x indicates going East from the blood bank and a negative value indicates going West.)

y = the number of miles from the blood bank to the hospital in the North-South direction. (A positive value for y indicates going North from the blood bank and a negative value indicates going South.)

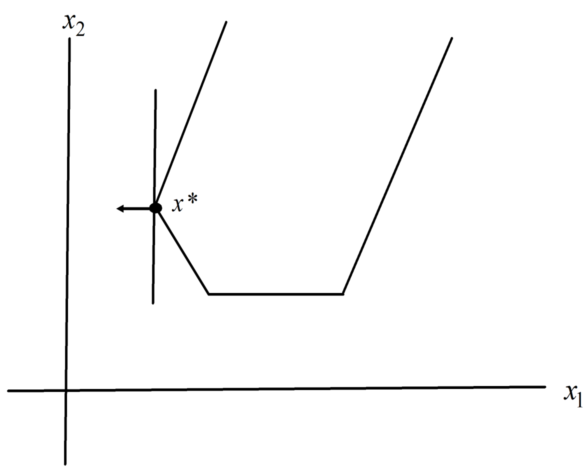
(a) With the foregoing variables x and y, formulate the desired constraint using absolute values. (5 points)

$$|x| + |y| \leq 3$$

(b) Rewrite your constraint in part (a) in such a way that you have linear programming constraints. (Hint: You will need more than one constraint.) (10 points)

$$\begin{array}{rcrcl} x & + & y & \leq & 3 \\ x & - & y & \leq & 3 \\ -x & + & y & \leq & 3 \\ -x & - & y & \leq & 3 \end{array}$$

3. Draw an example of a linear programming problem with two variables that has an optimal solution but whose feasible region is unbounded. Indicate the feasible region, the optimal solution, and the direction of improvement for the objective function. What happens to the LP if the objective function improves in the opposite direction? Explain. (15 points)



If the objective function improves in the opposite direction, then the LP will be unbounded.

4. (a) Is the vector $\mathbf{x} = (1, 0, 2/5, 0, 7/5, 2/5)$ a basic feasible solution (bfs) for the following LP? Why or why not? Explain. (5 points)

$$\begin{array}{lccccccccccccc}
\text{maximize} & 2x_1 & & & + & 4x_3 & + & 3x_4 & & & & & & \\
\text{subject to} & x_1 & + & x_2 & + & 5x_3 & + & x_4 & & & & & = & 3 \\
 & x_1 & & & - & x_3 & - & 4x_4 & + & x_5 & & & = & 2 \\
 & x_1 & & & + & 4x_3 & + & 2x_4 & & & + & x_6 & = & 3 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}$$

The vector $\mathbf{x} = (1, 0, 2/5, 0, 7/5, 2/5)$ cannot be a bfs for the given LP. This is because a bfs for this LP must have 3 basic variables (one for each of the 3 equality constraints) while the given vector has 4 basic variables (x_1, x_3, x_5 and x_6).

(b) For the LP in part (a), is the bfs in which x_2, x_5, and x_6 are basic variables optimal? Why or why not? Explain? (5 points)

For this bfs, you have $\mathbf{x}_B = (x_2, x_5, x_6) = (3, 2, 3)$ and $\mathbf{x}_N = (x_1, x_3, x_4) = (0, 0, 0)$ and

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 5 & 1 \\ 1 & -1 & -4 \\ 1 & 4 & 2 \end{bmatrix}$$

Changing the problem from max to min, you can now test for optimality as follows:

$$\mathbf{c}_N - \mathbf{c}_B B^{-1} N = (-2, -4, -3) - (0, 0, 0) B^{-1} N = (-2, -4, -3).$$

Because the reduced costs are not all ≥ 0, this bfs is not optimal and so $j^* = 2$ (using the rule of steepest descent).

(c) For the bfs in part (b), suppose you decide to increase the nonbasic variable x_4. What is the direction of movement, that is, what are the values of $\mathbf{d} = (d_1, d_2, \ldots, d_6)$? (10 points)

Using $j^* = 3$, the direction of movement is:

$$\mathbf{d}_B = -B^{-1}N_{.j*} = -IN_{.3} = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} d_2 \\ d_5 \\ d_6 \end{bmatrix} \quad \mathbf{d}_N = I_{.j*} = I_{.3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_3 \\ d_4 \end{bmatrix}$$

Therefore, the direction of movement is $\mathbf{d} = (0, -1, 0, 1, 4, -2)$.

(d) For the bfs in part (b), suppose you decide to move in the direction $\mathbf{d} = (0, -5, 1, 0, 1, -4)$. What is t^*? Which variable leaves the basis? What is the new basis matrix? (15 points)

Because $(\mathbf{d}_B)_1 = -5 < 0$ and $(\mathbf{d}_B)_3 = -4 < 0$, you have

$$\begin{aligned} t^* &= \min\left\{-\frac{(\mathbf{x}_B)_k}{(\mathbf{d}_B)_k} : (\mathbf{d}_B)_k < 0\right\} \\ &= \min\left\{-\frac{(\mathbf{x}_B)_1}{(\mathbf{d}_B)_1}, -\frac{(\mathbf{x}_B)_3}{(\mathbf{d}_B)_3}\right\} = \min\left\{-\frac{3}{-5}, -\frac{3}{-4}\right\} = \frac{3}{5} \text{ and } k^* = 1. \end{aligned}$$

This means that $(\mathbf{x}_B)_1 = x_2$ leaves the basis. Using $j^* = 2$, you have the following new B and N matrices:

$$B = \begin{bmatrix} 5 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -4 \\ 1 & 0 & 2 \end{bmatrix}$$

5. On the way to school one day, an OR student had two pieces of paper. One contained a primal linear program and its optimal solution and the other paper contained the corresponding dual linear program and its optimal solution. At school, she discovered that she only had the second sheet with the following information:

$$\begin{array}{lrcrcl} \text{maximize} & 6u_1 & + & 3u_2 & & \\ \text{subject to} & -u_1 & + & 2u_2 & \leq & 6 \\ & u_1 & + & u_2 & \leq & 5 \\ & 3u_1 & - & 2u_2 & \leq & 2 \\ & 2u_1 & - & u_2 & \leq & 1 \\ & -2u_1 & + & 3u_2 & \leq & 7 \end{array} \qquad \mathbf{u}^* = (2,3).$$

(a) What was the original primal LP? (10 points)

The original LP is:

$$\begin{array}{lrcrcrcrcrcl} \text{minimize} & 6x_1 & + & 5x_2 & + & 2x_3 & + & 1x_4 & + & 7x_5 & & \\ \text{subject to} & -x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & - & 2x_5 & = & 6 \\ & 2x_1 & + & x_2 & - & 2x_3 & - & x_4 & + & 3x_5 & = & 3 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

(b) Using duality theory and the given optimal solution $\mathbf{u}^*$, construct an optimal primal solution without using the simplex algorithm. (Hint: Which constraints are binding?) (15 points)

At $\mathbf{u}^* = (2,3)$, constraints 2 and 4 of the dual LP are binding. Therefore, x_2 and x_4 are the basic variables and x_1, x_3, and x_5 are the nonbasic variables, so,

$$B^{-1} = [A_{.2}, A_{.4}] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$$

The values of the basic variables are:

$$B^{-1}\mathbf{b} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

It is easy to check that $\mathbf{x}^* = (0,4,0,1,0)$ is feasible for the primal, $\mathbf{u}^* = (2,3)$ is feasible for the dual, and $\mathbf{cx}^* = \mathbf{u}^*\mathbf{b} = 21$ and so both are optimal.

6. Consider the problem of Midas Miner, who landed a space ship on an asteroid and discovered lots of gold and silver rock, for which he obtained the following information: (a) 1 pound of gold rock takes up 2 cubic feet of cargo space and brings a profit of \$3 thousand when sold, (b) 1 pound of silver rock takes up 1 cubic feet of cargo space and brings a profit of \$2 thousand when sold, and (c) the space ship can hold no more than 100 pounds and 150 cubic feet of cargo. Answer the following questions by using the attached computer output that was obtained when solving the following problem of maximizing profits while not exceeding cargo capacities:

$$\begin{array}{lrcrcr} \text{maximize} & 3x_1 & + & 2x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 100 \\ & 2x_1 & + & x_2 & \leq & 150 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

where

$$\begin{array}{lcl} x_1 & = & \text{the number of pounds of gold rock to take on the ship,} \\ x_2 & = & \text{the number of pounds of silver rock to take on the ship.} \end{array}$$

(a) Explain the meaning of the value of the slack variable associated with the weight constraint in the optimal solution. (5 points)

From the Answer Report, the value of the slack variable associated with the weight constraint is 0. This means that the current plan of $x_1^* = 50$ and $x_2^* = 50$ uses all available 100 pounds of cargo weight capacity.

(b) While preparing to load the amounts of gold and silver rock, as specified by the optimal solution, Midas heard a news report that the price of gold has just gone up by \$500 per pound. How does this information affect the optimal solution and optimal profit Midas will make? Explain. (10 points)

This information changes the objective function coefficient of gold from 3 to 3.5. As indicated in the Adjustable Cells portion of the Sensitivity Report, this increase of 0.5 is within the allowable increase of 1. Therefore, the optimal solution of $x_1^* = 50$ and $x_2^* = 50$ does not change. However, the new profit for Midas is:

$$3.5x_1^* + 2x_2^* = 3.5(50) + 2(50) = 275.$$

(c) Which would Midas prefer having (at no cost): (i) 50 additional pounds of weight capacity or (ii) 50 additional cubic feet of cargo space? Explain. (5 points)

As both increases of 50 in the rhs of the weight and volume constraints are within the allowable increase of 50 indciated in the Constraints portion of the Sensitivy Reprots, the corresponding shadow prices are valid. However, these shadow prices are both 1. Therefore, Midas is indifferent between having 50 more pounds of cargo weight capacity and 50 more $(\mathrm{tf})^3)$ of cargo space.

(d) Midas has just discovered that he has less fuel than he thought and can therefore load only 70 pounds of cargo on his ship. What effect will this have on the optimal solution and the optimal objective function value? Explain. (10 points)

This change reduces the rhs of the weight constraint by 30. As this amount is outside the allowable decrease of 255 in the Constraints portion of the Sensitivity Report, you would have to change the rhs of this constraint to 70 and resolve the problem to obtain the new optimal solution and optimal objective function value.

Microsoft Excel 12.0 Answer Report
Worksheet: [SampleExam.xls]Sheet1
Report Created: 10/1/2008 4:13:22 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
$F10$	Profit	7	250

Adjustable Cells

Cell	Name	Original Value	Final Value
$C10$	Gold	1	50
$D10$	Silver	2	50

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$C13$	Weight Used	100	\$C\$13 $\leq$ \$D\$13	Binding	0
$C14$	Volume Used	150	\$C\$14 $\leq$ \$D\$14	Binding	0

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [SampleExam.xls]Sheet1
Report Created: 10/1/2008 4:13:22 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C10$	Gold	50	0	3	1	1
$D10$	Silver	50	0	2	1	0.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$C13$	Weight Used	100	1	100	50	25
$C14$	Volume Used	150	1	150	50	50