Solutions to Homework #5

1.

Nonbasic Variables	Basic Variables	
$x_1 = 0$, $x_2 = 0$	$x_3 = 5/2$, $x_4 = 1$	bfs
$x_1 = 0$, $x_3 = 0$	$x_2 = 5/3$, $x_4 = 13/3$	bfs
$x_1 = 0$, $x_4 = 0$	$x_2 = -1/2$, $x_3 = 13/4$	–
$x_2 = 0$, $x_3 = 0$	$x_1 = 5$, $x_4 = 6$	bfs
$x_2 = 0$, $x_4 = 0$	$x_1 = -1$, $x_3 = 3$	–
$x_3 = 0$, $x_4 = 0$	$x_1 = -13$, $x_2 = 6$	–

2.

(a) $x = (0,0,3,0,2)$ is a bfs because $x_B = (x_3, x_5) = (3,2) = B^{-1}b = Ib$.

(b) $x = (1,0,-1,0,0)$ is not a bfs because $x_3 = -1 < 0$.

(c) $x = (0,1,1,0,0)$ is a bfs because $x_B = (x_2, x_3) = (1,1) = B^{-1}b$ since:

$$B^{-1}b = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = x_B.$$

(d) $x = (1/2,1,0,0,0)$ is not a bfs because this vector does not satisfy the second constraint.

3. The vector $\mathbf{x} = (2,3,1,0,1,0,4)$ is not a bfs for the LP because this vector has 5 positive components while the LP has only 4 equality constraints.

4. The bfs in which x_3 and x_4 are basic is not optimal because:

$$c_N - c_B B^{-1} N = (-1,2) - (1,-1) \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = (-5/2, -3/2).$$

The bfs in which x_2 and x_4 are basic is not optimal because:

$$c_N - c_B B^{-1} N = (-1,1) - (2,-1) \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = (-2,1).$$

The bfs in which x_1 and x_4 are basic is optimal because:

$$c_N - c_B B^{-1} N = (2,1) - (-1,-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} = (6,5) \geq (0,0).$$

5. The test for optimality for a maximization problem is $\mathbf{c_N} - \mathbf{c_B} B^{-1} N \leq \mathbf{0}$.

6. The objective function would not improve because **cd** = (2, –3, –1, 4, 1) (0, 1, 3, 1, 2) = 0, which is not < 0.

7. For this problem you have:

$$c_N - c_B B^{-1} N = (-4,-1,2) - (1,1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 3 & 2 \\ -1 & 4 & 3 \end{bmatrix} = (-6,-8,-3).$$

   (a) By the First-Come-First-Served rule, $j^* = 1$, so,

$$d_B = -B^{-1}N_{*1} = -I\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} d_4 \\ d_5 \end{bmatrix} \text{ and } d_N = I_{*1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix}.$$

   (b) By Bland's rule, $j^* = 3$ (because N_{*3} corresponds to column 1 of the original A matrix, which is the left-most column), so,

$$d_B = -B^{-1}N_{*3} = -I\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} d_4 \\ d_5 \end{bmatrix} \text{ and } d_N = I_{*3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix}.$$

   (c) By the Steepest Descent rule, $j^* = 2$, so,

$$d_B = -B^{-1}N_{*2} = -I\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} d_4 \\ d_5 \end{bmatrix} \text{ and } d_N = I_{*2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix}.$$

8. For this problem, you have that $\mathbf{x_B} = (x_3, x_2) = (1, 2)$ and $\mathbf{d_B} = (d_3, d_2) = (-1/3, -2/3)$. Because both components of $\mathbf{d_B}$ are < 0, so,

$$t^* = \min\left\{-\frac{(x_B)_3}{(d_B)_3}, -\frac{(x_B)_2}{(d_B)_2}\right\} = \min\left\{-\frac{1}{-1/3}, -\frac{2}{-2/3}\right\} = 3.$$

   (a) By Bland's rule, $k^* = 2$ (because B_{*2} corresponds to column 2 of the original A matrix, which is the left-most column of the basic columns).

   (b) By the First-Come-First-Served rule, $k^* = 1$.

9. For this problem, you have:

$$c_N - c_B B^{-1} N = (-7,6,4) - (0,0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -7 & 2 & 1 \\ -8 & 9 & 6 \end{bmatrix} = (-7,6,4).$$

   Thus, by any rule, $j^* = 1$ and so

$$d_B = -B^{-1}N_{*1} = -I\begin{bmatrix} -7 \\ -8 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} d_4 \\ d_5 \end{bmatrix} \text{ and } d_N = I_{*1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

   You also have that $t^* = \infty$ because $d_B \geq \mathbf{0}$.

10. (a) The new basic and nonbasic variables are $\mathbf{x_B} = (x_6, x_5)$ and $\mathbf{x_N} = (x_1, x_3, x_4, x_2)$.

(b) The new matrices are:

$$B = [A_{*6}, A_{*5}] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \text{ and } N = [A_{*1}, A_{*3}, A_{*4}, A_{*2}] = \begin{bmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

(c) The new inverse is:

$$B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(d) The values of the new basic variables are:

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

(e) You have $\mathbf{x} + t^*\mathbf{d} = (0, 2, 0, 0, 3, 0) + 2\,(0, -1, 0, 0, 1, 1) = (0, 0, 0, 0, 5, 2)$, so $\mathbf{x_B} = (x_6, x_5) = (2, 5)$.

(f) The new test for optimality is:

$$c_N - c_B B^{-1} N = (1,1,0,-2) - (1,-4)\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix} = (12,-1,5,1).$$

11. The initial bfs and iteration 1 are:

$$x_B = (x_1, x_4) = (3,2) \qquad x_N = (x_2, x_3) = (0,0)$$

$$B = I, \qquad N = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

$$c_N - c_B B^{-1} N = (-2,-3) - (1,1)I\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = (-1,-5), \text{ so } j^* = 2.$$

$$d_B = -B^{-1}N_{*j^*} = -IN_{*2} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$t^* = \min\left\{-\frac{(x_B)_k}{(d_B)_k} : (d_B)_k < 0\right\} = \min\left\{-\frac{(x_B)_2}{(d_B)_2}\right\} = \frac{2}{5}, \text{ so } k^* = 2.$$

Pivoting results in the following bfs and iteration 2:

$$x_B = (x_1, x_3) = (21/5, 2/5) \qquad x_N = (x_2, x_4) = (0,0)$$

$$B^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3/5 \\ 0 & 1/5 \end{bmatrix}, \qquad N = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$$

$$c_N - c_B B^{-1} N = (-2,1) - (1,-3)\begin{bmatrix} 1 & 3/5 \\ 0 & 1/5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} = (-4,1), \text{ so } j^* = 1.$$

$$d_B = -B^{-1} N_{*j^*} = -B^{-1} N_{*1} = -\begin{bmatrix} 1 & 3/5 \\ 0 & 1/5 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 3/5 \end{bmatrix}$$

$$t^* = \min\left\{ -\frac{(x_B)_k}{(d_B)_k} : (d_B)_k < 0 \right\} = \min\left\{ -\frac{(x_B)_1}{(d_B)_1} \right\} = 21, \text{ so } k^* = 1.$$

Pivoting results in the following bfs and iteration 3:

$$x_B = (x_2, x_3) = (21,13) \qquad x_N = (x_1, x_4) = (0,0)$$

$$B^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$c_N - c_B B^{-1} N = (1,1) - (-2,-3)\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (20,13).$$

The reduced costs are all ≥ 0, so the current solution of $\mathbf{x} = (x_1, x_2, x_3, x_4) = (0, 21, 13, 0)$ with objective function value $\mathbf{cx} = -81$ is optimal.

12. The phase 1 LP is:

$$\begin{array}{llllllllll} \text{Min} & & & & & & y_1 & + y_2 & + y_3 & \\ & x_1 & & + 3x_3 & - x_4 & & + y_1 & & & = 5 \\ & 2x_1 + x_2 & & & - x_5 & & & + y_2 & & = 7 \\ & & x_2 + & x_3 & & - x_6 & & & + y_3 & = 3 \\ & x_1, & x_2, & x_3, & x_4, & x_5, x_6, & y_1, & y_2, & y_3 & \geq 0 \end{array}$$

13. The original LP is feasible because, in the optimal solution to the phase 1 LP, $y_1 = y_2 = 0$. Phase 1 has produced an initial bfs for the original LP because both y_1 and y_2 are nonbasic at the end of phase 1.

14. The original LP is not feasible because, in the optimal solution to the phase 1 LP, $y_2 = 5 > 0$.

15. The original LP is feasible because, in the optimal solution to the phase 1 LP, $y_1 = y_2 = 0$. However, phase 1 has not produced an initial bfs for the original LP because one of the y-variables, namely, y_1, is basic at the end of phase 1.