CSE 515: Statistical Methods in Computer Science

Homework Assignment 2

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1 Kalman Filters

1. We can calculate $P(\mathbf{X}_1)$ as follows:

$$
\begin{aligned}
P(\mathbf{X}_1) &= \int_x \sum_i^k P(\mathbf{X}_1|S_0 = i, X_0 = x)P(S_0 = i)P(X_0 = x)dx \\
&= \sum_i^k P(S_0 = i) \int_x P(\mathbf{X}_1|S_0 = i, X_0 = x)P(X_0 = x)dx
\end{aligned}
$$

   Note that in the last step formula, $P(\mathbf{X}_1|S_0 = i, X_0 = x)$ and $P(X_0 = x)$ are both Gaussians, and their product is also Gaussian. Secondly, the integral of a Gaussian remains a Gaussian, so that $P(\mathbf{X}_1)$ is a mixure of Gaussian with the proceeding sum over all switches.

2. First, we calculate $P(\mathbf{X}_{t+1}|e_{1:t})$:

$$
\begin{aligned}
P(\mathbf{X}_{t+1}|e_{1:t}) &= \int_x \sum_i P(\mathbf{X}_{t+1}|X_t = x, S_t = i)P(X_t = x, S_t = i|e_{1:t})dx \\
&= \int_x \sum_i P(\mathbf{X}_{t+1}|X_t = x, S_t = i)P(X_t = x|S_t = i, e_{1:t})P(S_t = i|e_{1:t})dx \\
&= \sum_i P(S_t = i|e_{1:t}) \int_x P(\mathbf{X}_{t+1}|X_t = x, S_t = i)P(X_t = x|S_t = i, e_{1:t})dx \\
&= \sum_i P(S_t = i|e_{1:t}) \int_x P(\mathbf{X}_{t+1}|X_t = x, S_t = i)P(X_t = x|e_{1:t})dx
\end{aligned}
$$

   The last step is because $\mathbf{X}_t$ does not depends on S_t. Since $P(\mathbf{X}_t|e_{1:t})$ is m-mixture of Gaussian, and $P(\mathbf{X}_{t+1}|X_t = x, S_t = i)$ is also Gaussian, their product followed by an integration is also an km-mixture, i.e., $P(\mathbf{X}_{t+1}|e_{1:t})$ is km-mixture of Gaussian.

   Given $P(\mathbf{X}_{t+1}|e_{1:t})$, according to Beysian's lemma we have:

$$P(\mathbf{X}_{t+1}|e_{1:t+1}) = \alpha P(e_{t+1}|\mathbf{X}_{t+1})P(\mathbf{X}_{t+1}|e_{1:t})$$

   so that $P(\mathbf{X}_{t+1}|e_{1:t+1})$ is also an km-mixture of Gaussian.

3. From the last question we know that the km-mixture of Gaussian is weighted by $P(S_t = i|e_{1:t})$, i.e., the switch states given previous observations.

2 Graph and Independence Relations

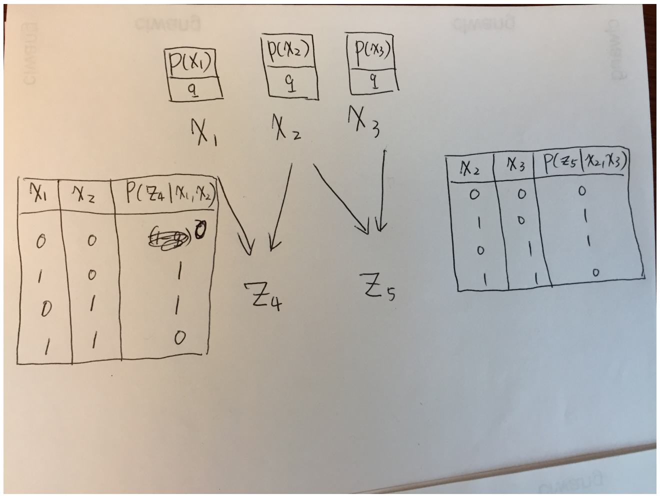
1. Given $Z_5 = 0$, we have the following distribution table.

	$X_2 = 0$	$X_2 = 1$
$X_3 = 0$	$\frac{(1-q)^2}{q^2+(1-q)^2}$	0
$X_3 = 1$	0	$\frac{q^2}{q^2+(1-q)^2}$

Given $Z_5 = 1$, we have the following distribution table.

	$X_2 = 0$	$X_2 = 1$
$X_3 = 0$	0	0.5
$X_3 = 1$	0.5	0

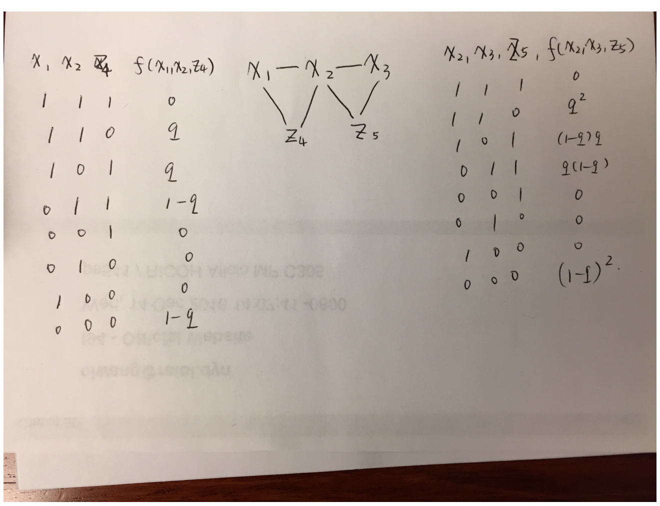
2. The directed graph representation of the dependencies is show below.



Independence relations:

- X_1, X_2 are independent if Z_4 is unknown.
- X_2, X_3 are independent if Z_5 is unknown.
- X_1, X_3 are independent if (1) X_2 is known, or (2) Z_4 is unknown, or (3) Z_5 is unknown.
- Z_4, Z_5 are independent if X_2 is known.
- Z_4, X_3 are independent if (1) X_2 is known or (2) Z_5 is unknown.
- Z_5, X_1 are independent if (1) Z_4 is unknown or (2) X_2 is known.

3. The undirected graph representation of the relation is shown below.



4. If we want $Z_5 \perp X_3$, we need $P(Z_5|X_3) = P(Z_5)$, i.e., $0.5 = 2q(1-q)$, which requires $q = 0.5$. Similarly, when $q = 0.5$, $Z_4 \perp X_1$. This information does not show up in either diagram.

3 BN2O Networks

1. Given F_i and the subset of parents $D_1, ..., D_l$, their joint distribution is shown as follows:

$$
\begin{aligned}
P(F_i = f_i^0, D_1, ..., D_l) =& \sum_{D_{l+1},...,D_k} P(F_i = f_i^0, D_1, ..., D_l, D_{l+1}, ..., D_k) \\
=& \sum_{D_{l+1},...,D_k} P(F_i = f_i^0 | D_1, ..., D_k) P(D_1, ..., D_k) \\
=& \sum_{D_{l+1},...,D_k} \left[(1-\lambda_{i,0}) \prod_{j=1}^{k} (1-\lambda_{i,j})^{d_j} \prod_{t=1}^{k} P(D_t) \right] \\
=& (1-\lambda_{i,0}) \prod_{j=1}^{l} (1-\lambda_{i,j})^{d_j} \sum_{D_{l+1},...,D_k} \prod_{t=l+1}^{k} (1-\lambda_{i,t})^{d_t} P(D_t)
\end{aligned}
$$

Let $A = \sum_{D_{l+1},...,D_k} \prod_{t=l+1}^{k} (1-\lambda_{i,t})^{d_t} P(D_t)$ and $\lambda'_{i,0} = 1 - A(1-\lambda_{i,0})$, we have the updated CPD as follows:

$$
\begin{aligned}
P(F_i = f_i^0 | D_1, ..., D_l) =& (1-\lambda'_{i,0}) \prod_{j=1}^{l} (1-\lambda_{i,j})^{d_j} \\
=& (1-\lambda_{i,0}) \sum_{D_{l+1},...,D_k} \prod_{t=l+1}^{k} (1-\lambda_{i,t})^{d_t} P(D_t) \prod_{j=1}^{l} (1-\lambda_{i,j})^{d_j}
\end{aligned}
$$

In this way, the joint distribution of $F_i, D_1, ..., D_l$ is maintained.

2. The posterior probability can be different from the original one since some dependencies may be removed due to the deletion of some parents. For example, given a network with D_1, D_2, D_3 and F_1, F_2, if D_1, D_2 are parents of F_1 and D_2, D_3 are parents of F_2. If we remove the node D_2, D_1, D_3 (or F_1, F_2) become conditionally independence, which was not the case for the original one.

   If no new conditional independence relations are introduced, the distribution remains exact.