Numerical Analysis Using Finite Difference Method

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Introduction

This method is used to find the solution to differential equations including partial differential equations. This means that given a function's derivatives, we can obtain the original function numerically using this method. This method actually stems from the very first idea we learned when dealing with differentiation where we use it to find the derivative of a function. Given a sufficiently smooth function $f(x)$, the derivative of f is defined as

$$f'(x) = \frac{f(x+h) - f(x)}{h}.$$

For instance, if the point is 2 in the curve we can pick point $f(x+h) = f(3)$. In this, we can calculate the gradient as y_x where a change in y is $f(3) - f(2)$ and change in x is h. In this case, the gradient from this calculation is a very bad estimate, however, this estimate gets better as h gets smaller and smaller. This is the basis for the finite difference method.

The three types of difference methods

With this in mind, we can also say that there are two other possibilities. The first one we just went through in the introduction is called the forward difference method. The others are the following:

1. **Backward difference method**: In this case, we use the value behind the chosen point to find the value of $f'(x)$

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

2. **Central difference method**: This is the interesting one amongst the three because it can easily help us find the second derivative of $f(x)$. In this case, we use the 2 points on the adjacent sides of our chosen point $f(x)$ to find $f'(x)$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Second derivative using central difference method: In order to do this, we need the point between $f(x+h)$ and $f(x)$ as well as the point between $f(x-h)$ and $f(x)$ and find the gradient for them using the central difference method, i.e.,

$$
\begin{aligned}
f'(x+\frac{1}{2}h) &= \frac{f(x+h)-f(x)}{h} \\
f'(x-\frac{1}{2}h) &= \frac{f(x)-f(x-h)}{h}
\end{aligned}
$$

Then apply the method again to these 2 gradients to find $f''(x)$:

$$
\begin{aligned}
f''(x) &= \frac{f'(x+\frac{1}{2}h)-f'(x-\frac{1}{2}h)}{h} \\
&= \frac{1}{h}\Big[\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}\Big] \\
&= \frac{1}{h^2}\Big[f(x+h)-f(x)-f(x)+f(x-h)\Big] \\
&= \frac{1}{h^2}\Big[f(x+h)+f(x-h)-2f(x)\Big]
\end{aligned}
$$

In this form, we have a very powerful equation that can help us find the second derivative of a function.

Integration using central difference method: By simply rearranging these equations to make $f(x)$ the subject of the formula, we can derive the original function numerically using the derivatives. This means that we can use the central difference method to find the solution to a differential equation.

Algorithm 1 Finite Difference Method

1: Set the boundaries and conditions which are the maximum iterations(N), the step size(h), the lower limit(LL), and the upper limit(UL).
2: Define the function $f(x)$.
3: Define the second-order derivative for the function, denoted as $f''(x)$.
4: Get the values of $f(UL)$ and $f(LL)$ and use them to calculate the gradient $\frac{f(UL)-f(LL)}{UL-LL}$.
5: Create a list of elements within the range of $[LL, UL]$ in intervals of h.
6: For each element x in the list obtained in step 5, determine the value of y by multiplying x with the gradient obtained in step 4.
7: For each point (x, y) obtained in step 6, use the formula $f(x) = f(x+h) + f(x-h) - f''(x)h^2$ substituting $f''(x)$ with the defined differential equation to calculate the approximate values of $f(x)$ for the solution to the differential equation.
8: Plot these approximate values alongside the graph of the exact function $f(x)$ to validate the accuracy of the code.