Laboratory Exercise 3: Numerical integration

OBJECTIVES: To implement the trapzoidal rule scheme for numerical integration and its extension to higher order schemes, and to examine their convergence.

Work through the exercises on this sheet. Try to complete as much as possible. If you finish early, work through the optional material at the end.

1 Trapezoidal rule

Write an python function `trap(n)` that will use the trapezoidal rule to integrate a function $f(x)$ between $x = a$ and $x = b$ using n intervals. Use another python function to define $f(x) = \sin(x)$ and take $a = 0$, $b = 5\pi$. The function `trap` should use a `for`-loop to loop over the endpoints x_k of the intervals, evaluating $f(x_k)$, and accumulate a running total of the relevant sum.

What is the exact value of $\int_0^{5\pi} \sin(x)\, dx$?

Either write a python program that imports your functions, or include your functions in a python program that will evaluate the trapezoidal integration for $n = 4$.

How accurate is the answer? Try successively doubling the number of intervals: does the result converge to the right answer?

Write a procedure that evaluates `trap(n)` for a range of different values of n, and plots the error versus n on a log-log plot. Can you deduce the rate of convergence from the plot, and does it agree with the theoretical rate?

2 Simpson's rule

Write a new function `simp(n)` that uses Simpson's rule with n intervals to evaluate the same integral. Do it by evaluating `trap(n)` and `trap(n/2)` and taking an appropriate combination of the results to eliminate the dominant error.

Either create a python program, or import your function at the command prompt >>> to evaluate `simp(4)`.

How accurate is the answer? Try successively doubling the number of intervals: does the result converge to the right answer?

Write a procedure that evaluates `simp(n)` for a range of different values of n, and plots the error versus n on a log-log plot. Can you deduce the rate of convergence from the plot, and does it agree with the theoretical rate? What happens to the error when the number of intervals is greater than about 100? Why?

3 Optional - higher order Romberg integration

One way of thinking about Romberg integration is as follows. The estimate of $\int_a^b f(x)\,dx$ given by the trapezoidal rule should be a smoothly varying function of $h = (b-a)/N$, or, indeed, of h^2, since the error terms are all even powers of h. Therefore, we can evaluate the trapezoidal rule estimate for several different values of h, fit a polynomial in h^2 through those estimates, then use that polynomial fit to extrapolate to the value for $h^2 = 0$. Simpson's rule, for example, effectively fits a linear function of h^2.

Write a function `romberg(n)` that will go to the next higher order estimate of the integral by evaluating `trap(n)`, `trap(n/2)`, and `trap(n/4)`, and then fit a quadratic in h^2 through those values. Use the general Lagrange interpolation formula given in the notes to construct the fit.

Repeat the exercises of sections 1 and 2 to estimate the rate of convergence of this method.

Can you see how to extend it to even higher order?