Laboratory Exercise 7: Numerical treatment of gravity waves

OBJECTIVES: To compare the dispersion properties of staggered and unstaggered meshes for solving the linearized shallow water equations

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}, \tag{1}$$

$$\frac{\partial \phi}{\partial t} = -\Phi \frac{\partial u}{\partial x}. \tag{2}$$

Staggered mesh

Browse the procedure `linswe.py` (supplied) and make sure you understand how it works. It solves the linearized, one-dimensional, shallow water equations on a periodic domain $0 \leq x \leq 1$, taking $\Phi = 1$. A CTCS discretization is used on an unstaggered grid of $N = 40$ points and the timestep is $\Delta t = 0.01$.

The initial condition is chosen to be

$$\phi(x, 0) = \begin{cases} \frac{1}{2}\left\{1 + \cos\left[4\pi\left(x - 1/2\right)\right]\right\} & \text{if } 1/4 \leq x \leq 3/4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$u(x, 0) = \phi(x, 0).$$

For the exact equations, this initial data should give a purely right-propagating disturbance, propagating at speed 1.

Run the procedure for a total of 1 time unit, so that the true solution is same as the intial condition. Plot the true solution and the numerical solution. What kind of errors does the numerical solution show?

Adapt the procedure so that it will use a CTCS scheme on a staggered mesh. Run your new procedure for a total of 1 time unit and again plot the true solution and numerical solution. Is it more accurate than the unstaggered mesh solution?

Using the theory given in lectures, work out the maximum stable timestep for the CTCS scheme on the unstaggered mesh and on the staggered mesh. Experiment with timesteps just above and just below the theoretical limits to confirm that they hold in practice.

Optional - Numerical group velocity

For the unstaggered grid scheme of Part 1, modify the initial condition by multiplying it by $(-1)^j$ where j is the grid index. This will give a packet of two-grid-length waves. How does the wave packet propagate when you run the code? Can you explain the behaviour in terms of the numerical dispersion relation?

Optional - implicit time stepping

Write a procedure to solve the linearized shallow water equations using a two-time-level trapezoidal implict timestepping scheme and a centred space scheme on a staggered mesh. You might find it helpful to base your code on the skeleton procedure `impswe.py` (supplied).

The scheme to be implemented is

$$\begin{aligned}
\frac{u_{j+1/2}^{(n+1)} - u_{j+1/2}^{(n)}}{\Delta t} + \frac{1}{2}\left(\frac{\phi_{j+1}^{(n+1)} - \phi_j^{(n+1)}}{\Delta x} + \frac{\phi_{j+1}^{(n)} - \phi_j^{(n)}}{\Delta x}\right) &= 0 \qquad (3)\\
\frac{\phi_j^{(n+1)} - \phi_j^{(n)}}{\Delta t} + \frac{1}{2}\Phi\left(\frac{u_{j+1/2}^{(n+1)} - u_{j-1/2}^{(n+1)}}{\Delta x} + \frac{u_{j+1/2}^{(n)} - u_{j-1/2}^{(n)}}{\Delta x}\right) &= 0, \qquad (4)
\end{aligned}$$

(Compare with the three-time-level scheme given in lecture notes.)

Eliminate $u_{j+1/2}^{(n+1)}$ and $u_{j-1/2}^{(n+1)}$ to obtain the discrete Helmholtz equation

$$\phi_j^{(n+1)} - \frac{\Delta t^2 \Phi}{4\Delta x^2}\left(\phi_{j+1}^{(n+1)} - 2\phi_j^{(n+1)} + \phi_{j-1}^{(n+1)}\right) = \phi_j^{(n)} - \frac{\Phi \Delta t}{\Delta x}\left(u_{j+1/2}^{(n)} - u_{j-1/2}^{(n)}\right) + \frac{\Delta t^2 \Phi}{4\Delta x^2}\left(\phi_{j+1}^{(n)} - 2\phi_j^{(n)} + \phi_{j-1}^{(n)}\right).$$

At each time step, your procedure will need to calculate the terms on the right hand side, and then feed them into the function into your python tridiagonal solver, which solves the tridiagonal linear system.

Experiment by running your procedure with different sized timesteps. Is the implicit scheme more stable than the centred-in-time scheme? What happens to the accuracy of the numerical solution as the timestep becomes very large?