14 Modelling gravity waves

Consider the one-dimensional shallow water equations, with Coriolis forces neglected, and linearized about a state of rest $u = 0$, $\phi = \Phi$ with Φ a constant:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} &= 0 \\ \frac{\partial \phi}{\partial t} + \Phi \frac{\partial u}{\partial x} &= 0 \end{aligned} \tag{23}$$

These equations support solutions of the form

$$u(x,t) = \widehat{u} e^{i(kx-\omega t)}; \quad \phi(x,t) = \widehat{\phi} e^{i(kx-\omega t)}$$

with dispersion relation $\omega^2 = k^2\Phi$. These particular waves are *non-dispersive*; they all have the same phase speed $\omega/k = \pm\Phi^{1/2}$.

14.1 CTCS solution

Let $u_j^{(n)}$ and $\phi_j^{(n)}$ be the numerical approximations to $u(x_j, t^{(n)})$ and $\phi(x_j, t^{(n)})$, respectively. Discretize the linearized shallow water equations using second-order centred differences in space and time:

$$\begin{aligned} \frac{u_j^{(n+1)} - u_j^{(n-1)}}{2\Delta t} + \frac{\phi_{j+1}^{(n)} - \phi_{j-1}^{(n)}}{2\Delta x} &= 0 \\ \frac{\phi_j^{(n+1)} - \phi_j^{(n-1)}}{2\Delta t} + \Phi \frac{u_{j+1}^{(n)} - u_{j-1}^{(n)}}{2\Delta x} &= 0, \end{aligned}$$

giving

$$\begin{aligned} u_j^{(n+1)} - u_j^{(n-1)} &= -\frac{\Delta t}{\Delta x}\left(\phi_{j+1}^{(n)} - \phi_{j-1}^{(n)}\right) \\ \phi_j^{(n+1)} - \phi_j^{(n-1)} &= -\frac{\Phi \Delta t}{\Delta x}\left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right). \end{aligned}$$

We can analyse the properties of this scheme using the von Neumann method. Seek solutions of the form $u_j^{(n)} = A^n e^{ikj\Delta x}$, $\phi_j^{(n)} = BA^n e^{ikj\Delta x}$, where A and B are complex constants. As before, A is the amplification factor, and the scheme will be stable provided $|A| \leq 1$. B is a constant of proportionality relating u and ϕ. Substituting these trial solutions in the scheme and cancelling where possible leaves

$$\begin{aligned} A^2 - 1 &= -\frac{\Delta t}{\Delta x} BA\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) \\ BA^2 - B &= -\frac{\Phi \Delta t}{\Delta x} A\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) \end{aligned} \tag{24}$$

Multiplying these equations to eliminate B ([Ex. find B]), and substituting $c^2 = \Phi(\Delta t/\Delta x)^2$ (c is the *Courant number* for this problem) gives

$$\left(A^2 - 1\right)^2 = -4A^2c^2(\sin k\Delta x)^2.$$

This is a quartic equation for A so there are four solutions. Two give physical modes, corresponding to the left and right propagating gravity waves. The other two give computational modes.

Taking square roots of both sides and solving the resulting quadratic gives

$$A = \pm ic \sin k\Delta x \pm \left[1 - (c \sin k\Delta x)^2\right]^{1/2}$$

(where all four combinations of plus and minus signs are possible). If $|c| < 1$ then the square root is real and $|A|^2 = 1$; the scheme is stable. If, on the other hand, $|c| > 1$ then for some value of k the square root is purely imaginary, $|A| > 1$ for two of the four roots, and the scheme is unstable. [Ex. Interpret this result in terms of the CFL criterion.]

14.2 Dispersion errors in the CTCS scheme

Although the physical modes have the correct amplitudes (when $|c| < 1$) they suffer from *dispersion errors*, just as in the case of the linear advection equation. In particular, for $k\Delta x = \pi$ the physical modes have zero frequency: a two-grid wave does not propagate. Moreover, short wavelength waves have a numerical frequency that decreases with increasing wavenumber, so that their numerical group velocity $c_g \equiv \partial\omega/\partial k$ has the wrong sign; thus packets of short waves propagate in the wrong direction.

14.3 Staggered grids

Dispersion errors can be reduced by using a staggered grid in which ϕ is stored at the points $x_j, j = 0,\ 1,\ 2,\ \ldots, N$, as before, but u is stored at the points $x_{j+1/2} = (j + 1/2)\Delta x,\ j = 0,\ 1,\ 2,\ \ldots, N-1$. On this grid the CTCS scheme becomes

$$\begin{aligned} \frac{u_{j+1/2}^{(n+1)} - u_{j+1/2}^{(n-1)}}{2\Delta t} + \frac{\phi_{j+1}^{(n)} - \phi_j^{(n)}}{\Delta x} &= 0 \\ \frac{\phi_j^{(n+1)} - \phi_j^{(n-1)}}{2\Delta t} + \Phi\frac{u_{j+1/2}^{(n)} - u_{j-1/2}^{(n)}}{\Delta x} &= 0, \end{aligned}$$

The von Neumann analysis is the same as before except that Δx is replaced by $\Delta x/2$, $\sin k\Delta x$ is replaced by $\sin(k\Delta x/2)$, and c is replaced by $2c$. We find that the staggered grid reduces dispersion errors; in particular the two-grid wave now propagates (though still too slowly), and the numerical group velocity has the correct sign. However, for stability we must now satisfy an even tighter restriction than before, namely $|c| < 1/2$.

14.4 Implicit time stepping

Deep gravity waves in the atmosphere have high phase speeds, $300\,\mathrm{ms}^{-1}$ or more, so for schemes like CTCS they lead to a much more severe restriction on Δt than the advection terms in the equations of atmospheric dynamics. The gravity wave stability restriction can be avoided by using an implicit time scheme: $\partial u/\partial x$ and $\partial\phi/\partial x$ are evaluated as averages of their values at times $t^{(n-1)}$ and $t^{(n+1)}$, rather than at time $t^{(n)}$. This is just the application to the shallow water equations of the trapezoidal implicit scheme we met in section 11. Assuming a staggered grid, as in the previous section, the scheme becomes

$$
\begin{aligned}
\frac{u_{j+1/2}^{(n+1)} - u_{j+1/2}^{(n-1)}}{2\Delta t} + \frac{1}{2}\left(\frac{\phi_{j+1}^{(n+1)} - \phi_j^{(n+1)}}{\Delta x} + \frac{\phi_{j+1}^{(n-1)} - \phi_j^{(n-1)}}{\Delta x}\right) &= 0 \qquad (25)\\
\frac{\phi_j^{(n+1)} - \phi_j^{(n-1)}}{2\Delta t} + \frac{1}{2}\Phi\left(\frac{u_{j+1/2}^{(n+1)} - u_{j-1/2}^{(n+1)}}{\Delta x} + \frac{u_{j+1/2}^{(n-1)} - u_{j-1/2}^{(n-1)}}{\Delta x}\right) &= 0, \qquad (26)
\end{aligned}
$$

Von Neumann stability analysis leads to the following quartic equation for the amplification factor A:

$$\left(A^2 - 1\right)^2 = -4(A^2+1)^2 c^2 (\sin(k\Delta x/2))^2.$$

Regarding this as a quadratic equation for A^2 and solving gives

$$A^2 = \frac{1 \pm 2ic\sin(k\Delta x/2)}{1 - (\pm 2ic\sin(k\Delta x/2))}.$$

It may be confirmed that $|A^2| = 1$ for all values of c: this scheme is unconditionally stable.

The scheme just analysed steps from time level $t^{(n-1)}$ to time level $t^{(n+1)}$ without using any values at time levels $t^{(n)}$. Clearly the scheme could be modified to step from time level $t^{(n)}$ to time level $t^{(n+1)}$ by taking a step of size Δt rather than $2\Delta t$.

A potential disadvantage of the implicit scheme is that it artificially slows the highest frequency waves. The highest resolvable frequency is $2\pi/2\Delta t$ (i.e. a period of $2\Delta t$), so as Δt increases the highest resolvable frequency decreases.

For the implicit scheme the unknown values at $t^{(n+1)}$ appear in several places in (25) and (26). We can solve for the unknowns by eliminating the velocities at time $t^{(n+1)}$ to obtain

$$\phi_j^{(n+1)} - c^2\left(\phi_{j+1}^{(n+1)} - 2\phi_j^{(n+1)} + \phi_{j-1}^{(n+1)}\right) = \text{known terms} \quad j = 0,\ 1,\ \ldots, N$$

This equation is an example of a *Helmholtz equation*. Note that the term in parentheses is proportional to the centred finite difference approximation to $\partial^2\phi/\partial x^2$. We now have a tridiagonal set of simultaneous linear equations for the $\phi_j^{(n)}$. It can be solved efficiently using a Gaussian elimination method. [Ex. Interpret the stability of this scheme in terms of the CFL criterion.]

14.5 Semi-implicit semi-Lagrangian schemes

Consider now the fully nonlinear shallow water equations:

$$
\begin{aligned}
\frac{Du}{Dt} + \frac{\partial\phi}{\partial x} &= 0\\
\frac{D\phi}{Dt} + \phi\frac{\partial u}{\partial x} &= 0,
\end{aligned}
$$

where now ϕ is the full geopotential, not just a perturbation. If we wanted to use the trapezoidal implicit scheme to time step these equations we would have to solve a potentially difficult nonlinear Helmholtz problem for the $\phi_j^{(n+1)}$ at every step. However, in practice ϕ does not vary much, so we can pick a constant reference geopotential, Φ say, close to the average ϕ and split the nonlinear term in the ϕ equation into a large linear part and a small nonlinear part.

$$\frac{D\phi}{Dt} + \Phi\frac{\partial u}{\partial x} + (\phi - \Phi)\frac{\partial u}{\partial x} = 0$$

We can then treat the linear term using the trapezoidal implicit scheme and the nonlinear term explicitly. The resulting scheme is called *semi-implicit*, because only some of the terms are treated implicitly. It has stability properties that are almost as good as those of the trapezoidal implicit scheme, but it only requires the solution of a linear Helmholtz problem.

A semi-implicit treatment of the gravity wave terms can be combined with a semi-Lagrangian treatment of the advection terms. When we do this, the averaging of the gravity wave terms must be taken along the parcel trajectories, rather than at fixed points in space, because the time derivatives are taken along trajectories. The resulting scheme is

$$\begin{aligned}
\frac{u_{j+1/2}^{(n+1)} - u_{j+1/2\,D}^{(n)}}{\Delta t} + \frac{1}{2}\left[\frac{\phi_{j+1}^{(n+1)} - \phi_j^{(n+1)}}{\Delta x} + \left(\frac{\partial \phi}{\partial x}\right)_{j+1/2\,D}^{(n)}\right] &= 0 \\
\frac{\phi_j^{(n+1)} - \phi_{jD}^{(n)}}{\Delta t} + \frac{1}{2}\Phi\left[\frac{u_{j+1/2}^{(n+1)} - u_{j-1/2}^{(n+1)}}{\Delta x} + \left(\frac{\partial u}{\partial x}\right)_{jD}^{(n)}\right] + \left\{(\phi - \Phi)\frac{\partial u}{\partial x}\right\}_{jM}^{(n+1/2)} &= 0
\end{aligned}$$

The derivatives $\left(\frac{\partial \phi}{\partial x}\right)_{j+1/2\,D}^{(n)}$ and $\left(\frac{\partial u}{\partial x}\right)_{jD}^{(n)}$ are to be evaluated on the staggered grid at time $t^{(n)}$ and then interpolated to the appropriate trajectory departure points. The term $\left\{(\phi - \Phi)\frac{\partial u}{\partial x}\right\}_{jM}^{(n+1/2)}$ can be obtained by evaluating the derivative on the staggered grid at times $t^{(n-1)}$ and $t^{(n)}$, extrapolating in time to obtain an estimate on the grid at time $t^{(n+1/2)} = (n + 1/2)\Delta t$, and then interpolating in space to the trajectory mid-point.