16 Nonlinear equations, aliasing, nonlinear instability

The linear advection equation can model the advection of a quantity like moisture or potential temperature by the wind, but the equations for the wind itself are nonlinear.

16.1 Burgers equation

We will consider the one-dimensional nonlinear advection equation (Burgers equation) as a prototype:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{1}$$

The nonlinearity allows more interesting dynamics but can cause numerical problems both through truncation error and instability.

16.2 Exact solution

Let $0 \leq x \leq 1$ with periodic boundary conditions and initial condition $u(x,0) = F(x)$. The exact solution is given implicitly [Ex] by

$$u(x,t) = F(x - ut). \tag{2}$$

An expression for the gradient $\partial u/\partial x$ can be found by differentiating (2):

$$\frac{\partial u}{\partial x} = \frac{F'(x-ut)}{1 + tF'(x-ut)}. \tag{3}$$

The nonlinearity will generate large gradients as $1 + tF'(x - ut)$ becomes small, even if no large gradients exist in the initial conditions. For a numerical solution, the truncation error will inevitably become large as $\partial u/\partial x$ and higher derivatives become large.

16.3 Conservation properties

Burgers equation has two important conserved quantities: 'momentum' and 'energy'.

Let $M = \int_0^1 u\,dx$. Then

$$\begin{aligned}\frac{dM}{dt} = \int_0^1 \frac{\partial u}{\partial t}\,dx &= -\int_0^1 u\frac{\partial u}{\partial x}\,dx \\ &= -\int_0^1 \frac{1}{2}\frac{\partial}{\partial x}\left(u^2\right)\,dx = -\frac{1}{2}\left[u^2\right]_0^1 = 0.\end{aligned} \tag{4}$$

Similarly, let $E = \int_0^1 u^2\,dx$. Then

$$\begin{aligned}\frac{dE}{dt} = \int_0^1 u\frac{\partial u}{\partial t}\,dx &= -\int_0^1 u^2\frac{\partial u}{\partial x}\,dx \\ &= -\int_0^1 \frac{1}{3}\frac{\partial}{\partial x}\left(u^3\right)\,dx = -\frac{1}{3}\left[u^3\right]_0^1 = 0.\end{aligned} \tag{5}$$

Thus, momentum and energy are conserved. (This derivation holds up to the point when the first shock forms. See the book by LeVeque for a clear discussion of what happens after a shock forms.)

It is often considered a good idea for the numerical solution of a differential equation to have conservation properties analogous to those of the original equation. Consider using a centred difference approximation for the space derivative on a grid with constant Δx. An obvious discrete analogue of the momentum is

$$M_{\mathrm{num}} = \sum_{j=0}^{N} u_j \Delta x. \tag{6}$$

Then

$$\begin{aligned} \frac{dM_{\mathrm{num}}}{dt} = \Delta x \sum_{j=0}^{N} \frac{\partial u_j}{\partial t} &= -\Delta x \sum_{j=0}^{N} u_j \frac{(u_{j+1} - u_{j-1})}{2\Delta x} \\ &= -\frac{1}{2} \sum_{j=0}^{N} u_j \left(u_{j+1} - u_{j-1}\right) = 0. \end{aligned} \tag{7}$$

All terms cancel. (Recall that the periodic boundary condition implies $u_0 = u_N$ and $u_1 = u_{N+1}$.)

An obvious discrete analogue of the energy is

$$E_{\mathrm{num}} = \frac{1}{2} \sum_{j=0}^{N} u_j^2 \Delta x. \tag{8}$$

Then

$$\begin{aligned} \frac{dE_{\mathrm{num}}}{dt} = \Delta x \sum_{j=0}^{N} u_j \frac{\partial u_j}{\partial t} &= -\Delta x \sum_{j=0}^{N} u_j^2 \frac{(u_{j+1} - u_{j-1})}{2\Delta x} \\ &= -\frac{1}{2} \sum_{j=0}^{N} u_j^2 \left(u_{j+1} - u_{j-1}\right). \end{aligned} \tag{9}$$

In this case the terms do not cancel, so the numerical analogue of energy is not necessarily conserved.

In fact it is possible to construct a spatial discretization that conserves numerical analogues of both momentum and energy:

$$\left(u \frac{\partial u}{\partial x}\right)_j = \frac{1}{3} \left(u_{j-1} + u_j + u_{j+1}\right) \frac{(u_{j+1} - u_{j-1})}{2\Delta x}. \tag{10}$$

Exercise: check this.

Note these conservation properties hold when the equation is discretized in space but is left continuous in time. When the equation is discretized in time too, small errors in conservation can be introduced.

16.4 Nonlinear wave interactions

Consider a function made up of just two wavenumbers k_1 and k_2.

$$u = A\cos k_1 x + B\cos k_2 x. \tag{11}$$

Then

$$\begin{aligned}
u\frac{\partial u}{\partial x} &= -\left(A\cos k_1 x + B\cos k_2 x\right)\left(k_1 A\sin k_1 x + k_2 B\sin k_2 x\right) \\
&= -k_1 A^2 \cos k_1 x \sin k_1 x - k_1 AB\cos k_2 x \sin k_1 x \\
&\quad -k_2 AB\cos k_1 x \sin k_2 x - k_2 B^2 \cos k_2 x \sin k_2 x \\
&= -\frac{1}{2}k_1 A^2 \sin 2k_1 x - \frac{1}{2}k_2 B^2 \sin 2k_2 x \\
&\quad -\frac{1}{2}AB\left[(k_1+k_2)\sin(k_1+k_2)x + (k_1-k_2)\sin(k_1-k_2)x\right].
\end{aligned} \tag{12}$$

(We have used the formula $\sin a \cos b = [\sin(a+b) + \sin(a-b)]/2$). Thus, the $u\partial u/\partial x$ term in Burgers equation will tend to generate a superposition of waves with new wavenumbers, some higher than the wavenumbers in the original function. In general a quadratic nonlinearity will produce new wavenumbers that are sums and differences of the original wavenumbers. The transfer of energy to shorter scales is associated with the generation of large gradients. It is characteristic of fronts, shocks, and three-dimensional turbulence (but to a much lesser degree in two-dimensional turbulence—in that case there is a transfer of enstrophy to small scales).

16.5 Aliasing

The shortest wavelength that can be resolved on a grid with constant spacing Δx is $2\Delta x$. Waves with wavelength shorter than this are misrepresented and appear as waves with wavelength greater than or equal to $2\Delta x$. This effect is called *aliasing*.

Suppose Burgers equation is to be integrated on a regular grid with spacing Δx. If waves with wavelength $\lambda < 4\Delta x$ exist initially then wavelengths shorter than $2\Delta x$ will be produced by the $u\partial u/\partial x$ term. But these cannot be resolved. Instead they are aliased into other wavelengths greater than $2\Delta x$. For example, consider two waves with wavelengths $\lambda_1 = 2\Delta x$, $\lambda_2 = 4\Delta x$, i.e. $k_1 = 2\pi/2\Delta x$, $k_2 = 2\pi/4\Delta x$. The nonlinear term will produce wavenumbers $2k_1 = 2\pi/\Delta x$, $2k_2 = 2\pi/2\Delta x$, $k_1 + k_2 = 6\pi/4\Delta x$, (and others), i.e. wavelengths Δx, $2\Delta x$, $4\Delta x/3$.

The $\lambda = \Delta x$ wave appears as a constant $\lambda = \infty$. The $\lambda = 4\Delta x/3$ wave appears as a $\lambda = 4\Delta x$ wave.

16.6 Nonlinear instability

At any timestep, nonlinear interactions might produce energy at shorter wavelengths. If this energy is aliased back into the original wavelengths then more energy might be produced at the next time step. This gives rise to a feedback loop in which energy that should have escaped to smaller scales remains near the grid scale and amplifies.

Nonlinear instability can be reduced or avoided by

- Filtering the data to remove small scales, or using a scale-selective dissipation;

- Using a finite-difference scheme that conserves energy;
- Using numerical schemes specifically designed to handle small scales appropriately, such as TVD schemes (see next section).

17 Finite volume methods

Finite volume methods predict volume integrals of variables over small cells, or volumes (hence the name 'finite volume'), based on fluxes in and out of the cell.

17.0.1 Conservation laws

Many of the partial differential equations used in numerical weather and climate prediction can be written in *conservation* or *flux* form, i.e.

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{u}) = 0. \tag{13}$$

For example, the 1D linear advection equation can be written in flux form as

$$\frac{\partial \phi}{\partial t} + \frac{\partial (c\phi)}{\partial x} = 0, \tag{14}$$

giving the flux $F(\phi) = c\phi$ and the 1D Burgers equation can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \tag{15}$$

giving the flux $F(u) = \frac{1}{2}u^2$.

Consider integrating equation 13 over cell, or volume, i:

$$\int_{V_i} \frac{\partial \mathbf{u}}{\partial t}\, dV + \int_{V_i} \nabla \cdot \mathbf{F}(\mathbf{u})\, dV = 0. \tag{16}$$

Integrating the first term gives us the volume average of $\mathbf{u}$ and applying the divergence theorem to the second term gives

$$V_i \frac{d\bar{\mathbf{u}}}{dt} + \oint_{S_i} \mathbf{F}(\mathbf{u}) \cdot \mathbf{n}\, dS = 0. \tag{17}$$

For the 1D linear advection equation this becomes

$$\Delta x \frac{d\phi_i}{dt} + \int_{x_{i-1/2}}^{x_{i+1/2}} F(\phi)\, dx = 0, \tag{18}$$

or

$$\frac{d\phi_i}{dt} = -\frac{F_{x_{i+1/2}}(\phi) - F_{x_{i-1/2}}(\phi)}{\Delta x}, \tag{19}$$

where $F_{x_{i+1/2}}(\phi) = F(\phi_{x_{i+1/2}})$ is the flux on the interface at $x_{i+1/2}$. The process of finding the value of ϕ on the interfaces is called *reconstruction*. Assuming that ϕ is constant within each cell, we have two possible values for $\phi_{x_{i+1/2}}$, that calculated using the value on the left, and that calculated using the value on the right, i.e.:

$$
\begin{aligned}
\phi_{x_{i+1/2},L} &= \phi_i, \\
\phi_{x_{i+1/2},R} &= \phi_{i+1}.
\end{aligned}
$$

17.0.2 Riemann problem

A conservation law with piecewise constant initial data containing a single discontinuity within the domain of interest is called a *Riemann problem*. The finite volume discretisation described in the previous section requires us to solve a set of Riemann problems to find the correct interface states. The physics of the problem is required to help us. For linear advection we know that we require the upwind value, i.e.

$$
\phi_{x_{i+1/2}} = \begin{cases} \phi_{x_{i+1/2},L} & = \phi_i \text{ for } c > 0, \\ \phi_{x_{i+1/2},R} & = \phi_{i+1} \text{ for } c < 0. \end{cases} \tag{20}
$$

.

For Burgers question the condition is more complicated. There are two possibilities, $u_L > u_R$, which results in a shock, and $u_L < u_R$ which is called *rarefaction*. Consider integrating the flux form of Burgers equation across a discontinuity located at $x = x_s(t)$, giving

$$
\frac{d}{dt}\int_a^{x_s(t)} u\,dx + \frac{d}{dt}\int_{x_s(t)}^b u\,dx = F(u_a) - F(u_b), \tag{21}
$$

where the integral has been split at the discontinuity and $u_a = u(a,t)$, $u_b = u(b,t)$. Now we apply Leibnitz's rule

$$
\int_a^{x_s(t)} \frac{\partial u}{\partial t}\,dx + u^- \frac{dx_s}{dt} + \int_{x_s(t)}^b \frac{\partial u}{\partial t}\,dx + u^+ \frac{dx_s}{dt} = F(u_a) - F(u_b), \tag{22}
$$

where u^- and u^+ are the left and right limits of $u(x,t)$ as $x \to x_s(t)$. Taking the limit as $a \to x_s(t)$ and $b \to x_s(t)$ gives

$$
S(u^- - u^+) = F(u^-) - F(u^+), \tag{23}
$$

where $S = \frac{dx_s}{dt}$ is the speed of the shock, since the integrals vanish. Substituting for the flux, we have

$$
\begin{aligned}
S &= \frac{\frac{1}{2}(u^-)^2 - \frac{1}{2}(u^+)^2}{(u^- - u^+)}, && (24) \\
&= \frac{1}{2}(u^- + u^+). && (25)
\end{aligned}
$$

We can now use this to decide on the upwind value the u should take at the interface. Combining the conditions for shock / rarefaction formation gives:

$$u_{x_{i+1/2}} = \begin{cases} u_s \text{ if } u^- > u^+, \\ u_r \text{ otherwise} \end{cases} \tag{26}$$

where u_s is the shock case, given by

$$u_s = \begin{cases} u^- \text{ if } S > 0, \\ u^+ \text{ if } S > 0, \end{cases} \tag{27}$$

and u_r is the rarefaction case, given by

$$u_r = \begin{cases} u^- \text{ if } u^- > 0, \\ u^+ \text{ if } u^+ < 0. \end{cases} \tag{28}$$

Once we have the value of u at the interface, we can compute the flux and then approximate its derivative.

17.0.3 Godunov's theorem

Godunov's theorem states that any monotonicity preserving scheme is at most first order accurate.

Monotonicity A scheme is *monotonicity preserving* is no new local extrema are created.

Total variation diminishing The *total variation* measures the grid scale oscillations that can be generated by a non monotonicity preserving scheme. It is defined by:

$$TV = \sum_{j=0}^{N} |\phi_{i+1} - \phi_i|. \tag{29}$$

A scheme is *total variation diminishing* if this value decreases in time, i.e. $TV^{n+1} \leq TV^n$.

Limiters Spurious oscillations can be removed by using a *limiter*. The idea is to approximate the flux with a weighted average of a high order approximation and a low order approximation. When the solution is smooth, the high order approximation is best but when discontinuities form the lower order approximation is preferable is it will not introduce so much oscillation. The smoothness of the function is represented by the ratio of successive gradients on the grid:

$$r_{i+\frac{1}{2}} = \frac{\phi_i - \phi_{i-1}}{\phi_{i+1} - \phi_i} \tag{30}$$

and the flux is computed as

$$F_{i+\frac{1}{2}} = \Psi_{i+\frac{1}{2}} F_H + (1 - \Psi_{i+\frac{1}{2}}) F_L \tag{31}$$

where F_H and F_L are the high order and low order fluxes respectively and $\Psi_{i+\frac{1}{2}} = \Psi(r_{i+\frac{1}{2}})$ is our limiter function. We see that $r_{i+\frac{1}{2}} \approx 1$ when the solution is smooth and $r_{i+\frac{1}{2}} < 0$ when there is a local extrema.