Laboratory Exercise 8: Burgers equation

OBJECTIVES: To examine conservation properties and nonlinear instability in numerical solutions of Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

CTCS scheme

Browse the python procedure `burgers.py` (supplied) and make sure you understand how it works. It solves Burgers equation on a periodic domain $0 \le x \le 1$. A CTCS discretization

$$\frac{u_j^{(n+1)} - u_j^{(n-1)}}{2\Delta t} + u_j^{(n)}\left(\frac{u_{j+1}^{(n)} - u_{j-1}^{(n)}}{2\Delta x}\right) = 0$$

is used on regular grid of $N = 20$ points with $\Delta t = 0.01$.

Run the procedure for a few steps and plot the numerical solution. How does it behave? What kind of errors does the numerical solution show?

Estimate the maximum Courant number for the numerical solution. On the basis of the CFL criterion, would you expect this scheme to be stable?

Conservation

The procedure includes some code to diagnose and print the "momentum" at each step. Is the momentum conserved? Add some code to diagnose and print the "energy"

$$E = \sum_j u_j^2 \Delta x$$

at each step. Rerun the procedure for a few steps. What happens to the energy?

Adapt the procedure so that it will use the momentum and energy conserving scheme

$$\frac{u_j^{(n+1)} - u_j^{(n-1)}}{2\Delta t} + \frac{1}{3}\left(u_{j-1}^{(n)} + u_j^{(n)} + u_{j+1}^{(n)}\right)\left(\frac{u_{j+1}^{(n)} - u_{j-1}^{(n)}}{2\Delta x}\right) = 0.$$

Rerun your procedure. Are the momentum and energy both now conserved? Are the problems with the original scheme reduced now?

Diffusion term

Further adapt your procedure to include a diffusion term

$$\frac{u_j^{(n+1)} - u_j^{(n-1)}}{2\Delta t} + \frac{1}{3}\left(u_{j-1}^{(n)} + u_j^{(n)} + u_{j+1}^{(n)}\right)\left(\frac{u_{j+1}^{(n)} - u_{j-1}^{(n)}}{2\Delta x}\right) = K\left(\frac{u_{j+1}^{(n-1)} - 2u_j^{(n-1)} + u_{j-1}^{(n-1)}}{\Delta x^2}\right).$$

Note the u terms in the second derivative are evaluated at time level $n-1$. (What happens if you evaluate them at time level n?)

Experiment with different values of the diffusion coefficient K. Try to find an optimum value that prevents spurious oscillations and instability without excessive smoothing.

How does the inclusion of the diffusion term affect the conservation of momentum and energy?

Exact solution

Add some code to your procedure to calculate and plot the exact solution to Burgers equation. The simplest way to do this is to note that a fluid parcel with initial velocity u will keep the same velocity and move a distance ut in time t. Thus, we can define $x_f = x + u(x,0)t$, x_f being the position at time t of the parcel initially at x, and plot $u(x_f, t) = u(x, 0)$ against x_f.

This solution will only be valid up to the point when the first shock forms. By plotting your solution at different times can you estimate the time of shock formation? Does it agree with the theory given in lectures predicting when $\partial u/\partial x$ first becomes infinite?