Laboratory Exercise 9: Finite volume methods

OBJECTIVES: To implement the finite volume methods introduced in the lecture to solve the 1D linear advection equation

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0,$$

and the 1D inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

Linear advection

Equivalent finite difference methods

Show that

1. combining a centred time discretisation with the flux given by $F_{i+1/2} = \frac{c}{2}(\phi_{i+1} + \phi_i)$ recovers the CTCS finite difference scheme.
2. combining a forward in time discretisation with the flux given by $F_{i+1/2} = c\phi_i$ recovers the FTBS finite difference scheme.

Higher order upwind schemes

Higher order upwind schemes can be constructed by using a larger stencil. the QUICK (quadratic upwind scheme), derived from Taylor series expansions, gives a third order approximation for $\phi_{i+1/2}$ using the values at three points, ϕ_{i+1}, ϕ_i and ϕ_{i-1}:

$$\phi_{i+1/2} = \frac{1}{8}(3\phi_{i+1} + 6\phi_i - \phi_{i-1}), \tag{1}$$

and similarly for $\phi_{i-1/2}$.

Implement this scheme combined with the Matsuno forward-backward timestepping (it is unstable with forward-in-time) for the linear advection problem from session 3, i.e. use a periodic domain $0 \leq x \leq 1$ and set the velocity to 1. We will consider two different initial conditions:

$$\phi(x,0) = \begin{cases} \frac{1}{2}\left\{1 + \cos\left[4\pi\left(x - 1/2\right)\right]\right\} & \text{if } 1/4 \leq x \leq 3/4 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\phi(x,0) = \begin{cases} 1 & \text{if } 1/4 \leq x \leq 3/4 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Burgers equation

Implement the finite volume upwind method described in the lecture notes to solve Burgers equation on the domain $0 \leq x \leq 1$ with a top hat initial condition given by:

$$u(x,0) = \begin{cases} 1 & 0 < x < 1/4, \\ 2 & 1/4 \leq x \leq 3/4, \\ 1 & 3/4 < x < 1. \end{cases} \tag{4}$$