Laboratory Exercise 9: Finite Elements and Linear Advection

OBJECTIVES: To experiment with an finite element scheme for integrating the linear advection equation

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$$

on a periodic domain $0 \leq x \leq 1$. We will consider two different initial conditions:

$$\phi(x,0) = \begin{cases} \frac{1}{2}\left\{1 + \cos\left[4\pi\left(x - 1/2\right)\right]\right\} & \text{if } 1/4 \leq x \leq 3/4 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$\phi(x,0) = \begin{cases} 1 & \text{if } 1/4 \leq x \leq 3/4 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

1 Hat functions as trial functions

Write a python procedure to integrate the linear advection equation using the finite element method in the notes. First use first order euler to advance the time step, then the leap-frog scheme. Take $u = 1.0$ to be a constant, use $N = 40$ equally spaced gridpoints, and take $\Delta t = 0.01$. You might find it helpful to follow the structure of `diffuse.py` or `advect.py` and use the same tricks for coping with the periodic boundary condition and setting up the grid. You will need to use the python sparse package, but this time you will need to account for a matrix slightly more complicated than tri-diagonal.

Run your procedure for a total of 1 time unit, so that the true solution should have gone once around the domain and back to its starting position. Plot the true solution and the finite element solution on the same set of axes. What kind of errors do you get for the initial condition (1) and the inititial condition (2)?

Compare the finite element method to the FTBS scheme you coded up for linear advection. Use different time steps. What happens to function as it propagates in time?

Increase the number of grid points to $N = 150$ and run your procedure for a few steps, plotting the solution at each step. Is it more accurate than before? Why?

2 Optional - Quadratic expansion (trial) functions

Write a new procedure (or modify the one you've just created) to solve the linear advection equation using quadratic trial functions. To do this replace the hat functions with piecewise quadratic functions of the form $q(x) = c_1 + c_2x + c_3x^2$. This is more complicated because the three coefficients cannot be uniquely determined by 2 points. The best way to handle this is to use 3 points by extending the piecewise quadratic across an interval of $2\Delta x$.