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Denote pressure gradient force in the vertical as $PGZ = -c_p\theta_e^t\frac{\partial \pi'}{\partial z}$

Denote convergence effect on π' as CON_h and CON_v, for horizontal and vertical respectively, $CON_h = -\frac{R}{c_v}\pi^t(\nabla\cdot\vec{u}_h^{\tau+\Delta\tau})$, $CON_v = -\frac{R}{c_v}\pi^t\frac{\partial w}{\partial z}$

Denote advection of π as ADV_h and ADV_v. $ADV_h = -\vec{u}_h^{\tau+\Delta\tau}\nabla\pi_h^t$

$$ADV_v = -w\cdot\frac{\partial \pi^t}{\partial z}$$

Then equations (6) and (7) can be rewritten as

$$\begin{aligned} w^{\tau+\Delta\tau} &= w^\tau + \Delta\tau\cdot PGZ(\alpha\pi'^{\tau+\Delta\tau} + \beta\pi'^{\tau}) + \Delta\tau R_w^t \\ &= w^\tau + \alpha\cdot\Delta\tau\cdot PGZ^{\tau+\Delta\tau} + \beta\Delta\tau\cdot PGZ^\tau + \Delta\tau R_w^t \qquad (8) \end{aligned}$$

$$\begin{aligned} \pi'^{\tau+\Delta\tau} &= \pi'^{\tau} + \Delta\tau\cdot CON_h + \Delta\tau\cdot CON_v(\alpha w^{\tau+\Delta\tau} + \beta w^\tau) \\ &\quad + \Delta\tau\cdot ADV_h + \Delta\tau\cdot ADV_v(\alpha w^{\tau+\Delta\tau} + \beta w^\tau) + \Delta\tau R_\pi^t \\ &= \pi'^{\tau} + \Delta\tau\cdot CON_h + \underline{\alpha\Delta\tau\cdot CON_v^{\tau+\Delta\tau}} + \beta\Delta\tau\cdot CON_v^\tau \\ &\quad + \Delta\tau\cdot ADV_h + \underline{\alpha\Delta\tau\cdot ADV_v^{\tau+\Delta\tau}} + \beta\Delta\tau\cdot ADV_v^\tau + \Delta\tau R_\pi^t \qquad (9) \end{aligned}$$

We need to substitute (9) into (8)'s $PGZ^{\tau+\Delta\tau}$ term

$$PGZ^{\tau+\Delta\tau} = -c_p\theta_p^t\frac{\partial \pi'^{\tau+\Delta\tau}}{\partial z} \qquad (\text{denote } +c_p\theta_p^t = CTH)$$

$$\begin{aligned} &= -CTH\cdot\frac{\partial}{\partial z}\Big[\pi'^{\tau} + \Delta\tau\cdot CON_h + \alpha\Delta\tau\cdot CON_v^{\tau+\Delta\tau} + \beta\Delta\tau\cdot CON_v^\tau \\ &\qquad + \Delta\tau\cdot ADV_h + \alpha\Delta\tau\cdot ADV_v^{\tau+\Delta\tau} + \beta\Delta\tau\cdot ADV_v^\tau + \Delta\tau R_\pi^t\Big] \\ &= -CTH\cdot\alpha\Delta\tau\frac{\partial}{\partial z}CON_v^{\tau+\Delta\tau} - CTH\cdot\alpha\Delta\tau\cdot\frac{\partial}{\partial z}ADV_v^{\tau+\Delta\tau} \\ &\quad \underbrace{-CTH\frac{\partial}{\partial z}\left[\pi'^{\tau} + \Delta\tau\cdot CON_h + \beta\Delta\tau\cdot CON_v^\tau + \Delta\tau\cdot ADV_h + \beta\Delta\tau ADV_v^\tau + \Delta\tau R_\pi^t\right]}_{\cancel{PGZ_{tail}}\ PGZ_{other}} \end{aligned}$$

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$$-CTH\cdot\alpha\Delta\tau\cdot\frac{\partial}{\partial z}CON_v^{\tau+\Delta\tau} = -CTH\cdot\alpha\cdot\Delta\tau\cdot\frac{\partial}{\partial z}\left(-\frac{R}{C_v}\pi^t\frac{\partial w}{\partial z}^{\tau+\Delta\tau}\right)$$

$$= CTH\cdot\alpha\cdot\Delta\tau\cdot\frac{R}{C_v}\cdot\frac{\partial}{\partial z}\left(\pi^t\frac{\partial w}{\partial z}^{\tau+\Delta\tau}\right)$$

(denote $CTHRC = CTH\frac{R}{C_v}$) $= CTHRC\cdot\alpha\cdot\Delta\tau\cdot\frac{\partial}{\partial z}\left(\pi^t\frac{\partial w}{\partial z}^{\tau+\Delta\tau}\right)$

$$\frac{\partial}{\partial z}\left(\pi^t\frac{\partial w}{\partial z}^{\tau+\Delta\tau}\right) = \frac{\partial}{\partial z}\left(\pi_k^t\frac{w_{k+1}^{\tau+\Delta\tau}-w_k^{\tau+\Delta\tau}}{\Delta z_k}\right)$$

$$= \frac{1}{\Delta z_{k-\frac{1}{2}}}\left[\pi_k^t\frac{w_{k+1}^{\tau+\Delta\tau}-w_k^{\tau+\Delta\tau}}{\Delta z_k} - \pi_{k-1}^t\frac{w_k^{\tau+\Delta\tau}-w_{k-1}^{\tau+\Delta\tau}}{\Delta z_{k-1}}\right]$$

$PZU = \frac{\pi_k^t}{\Delta z_{k-\frac{1}{2}}\Delta z_k}$

$PZL = \frac{\pi_{k-1}^t}{\Delta z_{k-\frac{1}{2}}\Delta z_{k-1}}$

$$= \frac{\pi_k^t}{\Delta z_{k-\frac{1}{2}}\Delta z_k}w_{k+1}^{\tau+\Delta\tau} - \left(\frac{\pi_k^t}{\Delta z_{k-\frac{1}{2}}\Delta z_k}+\frac{\pi_{k-1}^t}{\Delta z_{k-\frac{1}{2}}\Delta z_{k-1}}\right)w_k^{\tau+\Delta\tau} + \frac{\pi_{k-1}^t}{\Delta z_{k-\frac{1}{2}}\Delta z_{k-1}}w_{k-1}^{\tau+\Delta\tau}$$

$$= PZU\cdot w_{k+1}^{\tau+\Delta\tau} - (PZU+PZL)\,w_k^{\tau+\Delta\tau} + PZL\cdot w_{k-1}^{\tau+\Delta\tau}$$

$$-CTH\cdot\alpha\Delta\tau\cdot\frac{\partial}{\partial z}ADV_v^{\tau+\Delta\tau} = -CTH\cdot\alpha\cdot\Delta\tau\cdot\frac{\partial}{\partial z}\left(-w^{\tau+\Delta\tau}\frac{\partial\pi^t}{\partial z}\right)$$

$$= CTH\cdot\alpha\cdot\Delta\tau\left(\frac{w_{k+1}^{\tau+\Delta\tau}(\partial_z\pi^t)_{k+1} - w_{k-1}^{\tau+\Delta\tau}(\partial_z\pi^t)_{k-1}}{2\Delta z_{k-\frac{1}{2}}}\right)$$

$\frac{\partial\pi^t}{\partial z} = \partial_z\pi^t$

$DPZU = \frac{1}{2\Delta z_{k-\frac{1}{2}}}\left(\frac{\partial\pi^t}{\partial z}\right)_{k+1}$

$DPZL = \frac{1}{2\Delta z_{k-\frac{1}{2}}}\left(\frac{\partial\pi^t}{\partial z}\right)_{k-1}$

$$= CTH\cdot\alpha\cdot\Delta\tau\left(\frac{(\partial_z\pi^t)_{k+1}}{2\Delta z_{k-\frac{1}{2}}}w_{k+1}^{\tau+\Delta\tau} - \frac{(\partial_z\pi^t)_{k-1}}{2\Delta z_{k-\frac{1}{2}}}w_{k-1}^{\tau+\Delta\tau}\right)$$

$$= CTH\cdot\alpha\cdot\Delta\tau\cdot\left(DPZU\cdot w_{k+1}^{\tau+\Delta\tau} - DPZL\cdot w_{k-1}^{\tau+\Delta\tau}\right)$$

In summary,

$$PGZ^{\tau+\Delta\tau} = CTHRC\cdot\alpha\cdot\Delta\tau\cdot[PZU\cdot W_{k+1}^{\tau+\Delta\tau} - (PZU+PZL)W_k^{\tau+\Delta\tau} + PZL\cdot W_{k-1}^{\tau+\Delta\tau}]$$

$$+ CTH\cdot\alpha\cdot\Delta\tau\cdot(DPZU\cdot W_{k+1}^{\tau+\Delta\tau} - DPZL\cdot W_{k-1}^{\tau+\Delta\tau})$$

$$+ PGZ_{other}$$

$$= \alpha\cdot\Delta\tau\cdot(CTHRC\cdot PZU + CTH\cdot DPZU)\cdot W_{k+1}^{\tau+\Delta\tau}$$
$$-\alpha\cdot\Delta\tau\cdot CTHRC\cdot(PZU+PZL)\cdot W_k^{\tau+\Delta\tau}$$
$$+\alpha\cdot\Delta\tau\cdot(CTHRC\cdot PZL - CTH\cdot DPZL)W_{k-1}^{\tau+\Delta\tau}$$
$$+ PGZ_{other}$$

$$W_k^{\tau+\Delta\tau} = \underbrace{W_k^{\tau} + \beta\Delta\tau\cdot PGZ^{\tau} + \Delta\tau R_w^{t}}_{WK_{other}} + \alpha\cdot\Delta\tau\cdot PGZ^{\tau+\Delta\tau}$$

$$= \alpha^2\Delta\tau^2\cdot(CTHRC\cdot PZU + CTH\cdot DPZU)\cdot W_{k+1}^{\tau+\Delta\tau}$$
$$-\alpha^2\Delta\tau^2\cdot CTHRC\cdot(PZU+PZL)\cdot W_k^{\tau+\Delta\tau}$$
$$+\alpha^2\Delta\tau^2\cdot(CTHRC\cdot PZL - CTH\cdot DPZL)\cdot W_{k-1}^{\tau+\Delta\tau}$$
$$+\alpha\cdot\Delta\tau\cdot PGZ_{other} + WK_{other}$$

$$\underbrace{[-\alpha^2\Delta\tau^2(CTHRC\cdot PZL - CTH\cdot DPZL)]}_{A_k} W_{k-1}^{\tau+\Delta\tau}$$

$$+\underbrace{[1+\alpha^2\Delta\tau^2\cdot CTHRC\cdot(PZU+PZL)]}_{B_k} W_k^{\tau+\Delta\tau}$$

$$+\underbrace{[-\alpha^2\Delta\tau^2(CTHRC\cdot PZU + CTH\cdot DPZU)]}_{C_k}\cdot W_{k+1}^{\tau+\Delta\tau}$$

$$= \underbrace{\alpha\Delta\tau PGZ_{other} + WK_{other}}_{F_k}$$