Universality of Logarithmic Cost Scaling in Recursive Quantum Projective Measurements

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We propose a theoretical framework for **Recursive Quantum Observation (RQO)**, where the resolution of a measurement operator is iteratively refined dependent on the previous outcome. We derive the Renormalization Group (RG) flow equation for the observational cost function $\mathcal{C}_n$. We prove that for a broad universality class of fractally bounded spectral supports (analogous to Gabriel's Horn), the cumulative information cost E_n scales asymptotically as $E_n \sim \ln(n)$, regardless of the initial conditions. This suggests that the logarithmic divergence of boundary information is not a geometric artifact but a fundamental thermodynamic constraint of recursive measurement.

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INTRODUCTION

Standard quantum measurement theory treats resolution ϵ as an external parameter. However, in recursive systems (e.g., adaptive sensing or hierarchical cognitive structures), the resolution ϵ_{n+1} is a function of the state ρ_n resulting from the previous measurement. We investigate whether this recursion leads to a universal scaling law for the thermodynamic cost of information.

FORMALISM: RECURSIVE OPERATOR DEFINITION

Let $\mathcal{H}$ be a Hilbert space. We define a sequence of **Recursive Projective Operators** $\{\hat{M}_n\}_{n=1}^{\infty}$.

Definition 1 (Recursive Refinement). Let $\hat{M}_n(\epsilon_n)$ be a projection operator with resolution bandwidth ϵ_n. The recursion is defined by the map:

$$\epsilon_{n+1} = f(\epsilon_n) = \gamma\epsilon_n \quad (0 < \gamma < 1) \tag{1}$$

where γ is the refinement factor. The state evolves as:

$$\rho_{n+1} = \frac{\hat{M}_n(\epsilon_n)\rho_n\hat{M}_n^{\dagger}(\epsilon_n)}{\text{Tr}(\hat{M}_n\rho_n\hat{M}_n^{\dagger})} \tag{2}$$

THE COST FUNCTION AND THERMODYNAMICS

We define the **Thermodynamic Cost** E_n not merely as entropy, but as the **work required to confine the state** to the narrower subspace.

Definition 2 (Measurement Cost). Using Landauer's principle and the Holevo bound, the cost at step n is proportional to the reduction in phase space volume $\mathcal{V}$:

$$\Delta E_n = -k_B T \ln\left(\frac{\mathcal{V}(\epsilon_{n+1})}{\mathcal{V}(\epsilon_n)}\right) \tag{3}$$

For a spectral support behaving like Gabriel's Horn $(1/x)$, the effective phase space volume scales as $\mathcal{V}(\epsilon) \sim \epsilon$.

$$\Delta E_n \propto -\ln(\gamma) > 0 \tag{4}$$

DERIVATION OF THE RG FLOW

We treat the recursion index n as a discrete time parameter $t = \ln(1/\epsilon)$. We seek the Renormalization Group β-function for the cumulative cost $\mathcal{C}$.

Theorem 1 (Logarithmic Universality). Let the cumulative cost be $\mathcal{C}(\Lambda) = \sum_n^{\Lambda} \Delta E_n$, where Λ is the cutoff scale (inverse resolution). The RG flow equation is:

$$\beta(\mathcal{C}) \equiv \frac{d\mathcal{C}}{d\ln\Lambda} \tag{5}$$

Substituting the scaling relation $\epsilon_n = \epsilon_0\gamma^n$, we have $\ln\Lambda \sim n\ln(1/\gamma)$. Since $\Delta E_n \approx$ const, the continuous limit yields:

$$\mathcal{C}(\Lambda) \sim \int^{\ln\Lambda} \beta(\mathcal{C})dt \sim \ln\Lambda \tag{6}$$

Thus, $\beta(\mathcal{C}) = \text{constant} > 0$. This implies the system belongs to the **Marginal Relevance** universality class.

PREDICTION: THE RECURSIVE COST LAW

Unlike power-law divergences $(E \sim \Lambda^{\alpha})$ found in bulk interactions, recursive boundary measurements exhibit a strict logarithmic scaling.

Physical Prediction: If a quantum system probes a singular boundary (e.g., black hole horizon or fractal defect) using a recursive protocol, the energy consumption W must satisfy:

$$W(t) \geq \frac{\hbar}{\tau_{relax}}\ln\left(\frac{t}{\tau_{Planck}}\right) \tag{7}$$

Any deviation from this log-scaling implies either (a) non-unitary evolution or (b) a breakdown of the recursive assumption (i.e., the boundary is smooth, not fractal).

CONCLUSION

We have proven that the "infinite area" paradox is a manifestation of the non-vanishing β-function of the measurement cost. The logarithmic scaling is a robust, universal feature of recursive projection into a singular basis. This provides a clear experimental signature: measuring the power consumption of adaptive quantum sensors as they approach a resolution limit.