Continuum–Discrete Duality as a Frame-Dependent Pattern

A Spherical Volume–Boundary Gauge with BTWM Spectral Measurement Operators

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February 5, 2026

Abstract

We formalize a model in which *continuity* and *discreteness* are not primitive ontological opposites, but *frame-induced appearances* of the same patterned background. The model uses a spherical observation gauge: continuous content is represented by volumetric accumulation inside a ball, while discrete events emerge as boundary detections on the sphere or as boundary-layer limits. To connect this to the BTWM (Base–Topological Waveframe Model) viewpoint, we introduce a frame-induced self-adjoint measurement operator on a Hilbert space. Its spectrum parameterizes effective resolution, while a frame cost functional produces stability points (efficient anchors). The continuum/discrete split becomes a thresholded projection under measurement constraints rather than an intrinsic property of the background field.

Keywords: continuum/discrete duality; spherical gauge; boundary-layer limit; BTWM; spectral operator; frame cost; observation-induced patterns

Contents

1 Motivation and Scope (OSF preprint note)

This manuscript is an OSF-ready conceptual–formal framework note. It is *not* a finalized journal proof claim. Instead, it supplies a reusable mathematical representation layer: a way to encode continuum/discrete paradoxes as a single background pattern observed through different frames, with explicit gauge, cost, and spectral components.

2 Background: States, Frames, and Phase Paths

Definition 1 (Frame-indexed representation triple)**.** *Let Ω be a possibility/state space. A representation is specified by a triple $(x, \Phi, \mathcal{O})$ where $x \in \Omega$ is a state, Φ denotes a phase-path/interpretation label, and $\mathcal{O}$ is an observation/measurement frame.*

We do not assume the continuum/discrete split as primitive on Ω; the split is induced by the frame and its constraints.

3 Spherical Observation Gauge

Fix a center $c \in \Omega$ and resolution parameter $\mathrm{Res} > 0$. Define

$$B_{\mathrm{Res}}(c) = \{x : \|x - c\| \leq \mathrm{Res}\}, \qquad S_{\mathrm{Res}}(c) = \partial B_{\mathrm{Res}}(c).$$

Definition 2 (Bulk and boundary observables)**.** *Let $\rho(x)$ be a bulk density and $\sigma(x)$ a boundary sensitivity. Define*

$$V(\mathrm{Res}) := \int_{B_{\mathrm{Res}}(c)} \rho(x) \mathrm{d}V, \qquad A(\mathrm{Res}) := \int_{S_{\mathrm{Res}}(c)} \sigma(x) \mathrm{d}S.$$

Remark 1. *In this gauge: continuity corresponds to volumetric accumulation $V(\mathrm{Res})$, while discreteness corresponds to boundary detection $A(\mathrm{Res})$.*

4 Core Duality: Discreteness as Scale Response of Continuity

Assumption 1 (Regularity)**.** *Assume ρ is locally integrable and sufficiently regular so that differentiation under the integral sign and standard coarea/surface formulas apply.*

Proposition 1 (Boundary–volume derivative identity)**.** *Under the regularity assumption,*

$$\frac{d}{d\,\mathrm{Res}} \left(\int_{B_{\mathrm{Res}}(c)} \rho(x) \mathrm{d}V \right) = \int_{S_{\mathrm{Res}}(c)} \rho(x) \mathrm{d}S.$$

Definition 3 (Boundary layer)**.** *Define the shell*

$$\mathrm{Shell}_{\mathrm{Res},\varepsilon} := \{x : \mathrm{Res} - \varepsilon \leq \|x - c\| \leq \mathrm{Res}\}.$$

Proposition 2 (Boundary-layer limit)**.** *If ρ is regular enough, then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathrm{Shell}_{\mathrm{Res},\varepsilon}} \rho(x) \mathrm{d}V = \int_{S_{\mathrm{Res}}(c)} \rho(x) \mathrm{d}S.$$

5 Frame Cost and Stability Points (Energy Anchors)

Definition 4 (Frame cost functional)**.** *A frame $\mathcal{O}$ induces a cost functional $\mathcal{E}_{\mathcal{O}} : \Omega \to \mathbb{R}_{\geq 0}$, interpretable as energy/effort/complexity required to maintain/report a representation.*

Definition 5 (Stability set)**.** *The stability set under $\mathcal{O}$ is*

$$S(\mathcal{O}) := \arg\min_{x\in\Omega} \mathcal{E}_{\mathcal{O}}(x).$$

Assumption 2 (Resolution–budget coupling)**.** *Effective resolution* Res *and boundary thickness ε depend on the frame and a measurement budget:*

$$\mathrm{Res} = \mathrm{Res}(\mathcal{O}, \mathit{budget}), \qquad \varepsilon = \varepsilon(\mathcal{O}, \mathit{budget}).$$

6 BTWM Link: Spectral Measurement Operators as Frame Realizers

Axiom 1 (Frame induces a self-adjoint operator (BTWM waveframe operator))**.** *For each observation frame $\mathcal{O}$, there exists a (densely defined) self-adjoint operator $\mathsf{A}_{\mathcal{O}}$ on a Hilbert space $\mathcal{H}$. Its spectrum* $\mathrm{Spec}(\mathsf{A}_{\mathcal{O}})$ *parameterizes the effective discrimination modes available under $\mathcal{O}$.*

Definition 6 (Spectral-to-scale mapping)**.** *Assume a monotone map $g_{\mathcal{O}} : \mathrm{Spec}(\mathsf{A}_{\mathcal{O}}) \to \mathbb{R}_{>0}$ such that*

$$\mathrm{Res} = g_{\mathcal{O}}(\lambda) \quad \textit{for some } \lambda \in \mathrm{Spec}(\mathsf{A}_{\mathcal{O}}).$$

Definition 7 (Spectral projections)**.** *Let $\mathsf{P}_{\mathcal{O}}(\Delta)$ denote the spectral projection of $\mathsf{A}_{\mathcal{O}}$ on a Borel set $\Delta \subset \mathbb{R}$. Frame-limited measurement is modeled as coarse-graining of spectral information: only restricted collections of Δ are accessible.*

7 Continuum/Discrete as a Thresholded Spectral Projection

Definition 8 (Detection gap)**.** *Let $\Delta_{\mathcal{O}} : \Omega \to \mathbb{R}_{\geq 0}$ be a frame-dependent detection gap (distinguishability, residual energy increment, or spectral mismatch).*

Definition 9 (Frame-induced split)**.** *Fix a threshold $\tau(\mathcal{O}) > 0$ and define*

$$C(\mathcal{O}) := \{x : \Delta_{\mathcal{O}}(x) < \tau(\mathcal{O})\}, \qquad D(\mathcal{O}) := \{x : \Delta_{\mathcal{O}}(x) \geq \tau(\mathcal{O})\}.$$

Theorem 1 (Core claim: same pattern, different appearance)**.** *Assume: (i) boundary-layer limit holds; (ii)* Res *is induced via λ through $g_{\mathcal{O}}$; (iii) detection is thresholded by $\tau(\mathcal{O})$ and constrained by budget via $\mathcal{E}_{\mathcal{O}}$. Then the continuum/discrete distinction is not intrinsic to Ω but induced by $(\mathcal{O}, \mathit{budget})$: bulk accumulation dominates when* Res *is coarse and $\tau(\mathcal{O})$ is high (continuous appearance), while boundary response dominates when* Res *is tuned/fine and $\tau(\mathcal{O})$ is low (discrete appearance).*

8 Why π and Irrational Structure Naturally Appear

In Euclidean n-space,

$$\mathrm{Vol}_n(B_{\mathrm{Res}}) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\,\mathrm{Res}^n, \qquad \mathrm{Area}_{n-1}(S_{\mathrm{Res}}) = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\,\mathrm{Res}^{n-1}.$$

Thus π appears as a measure conversion coefficient between bulk and boundary.

9 Toy Models

- Radial Gaussian: $\rho(x) = e^{-\|x-c\|^2/(2\sigma^2)}$ produces smooth $V(\mathrm{Res})$ and a boundary peak in $dV/d\,\mathrm{Res}$.
- Sharp interface: $\rho(x) = \mathbf{1}_{\|x-c\|\le R}$ yields saturation in V and event-like concentration at $\mathrm{Res} = R$.

10 Limitations and OSF Use

- Gauge freedom: constrain admissible ρ and $g_{\mathcal{O}}$ to avoid over-explanation.
- Instantiation requirement: for testable predictions, specify $\mathsf{A}_{\mathcal{O}}$, $\tau(\mathcal{O})$, and $\mathcal{E}_{\mathcal{O}}$ for a domain.
- Interpretation safety: use as a representational mapping layer.

11 Conclusion

We proposed a spherical volume–boundary gauge in which continuity and discreteness are dual appearances of a single patterned background. By linking resolution to a BTWM-style spectral operator and stabilizing observation via a frame cost functional, we formalized: continuum and discrete can be the same pattern under different observation frames.

Acknowledgments

Conceptual direction by Minsu Kim. Formal structuring by GPT-5.2 (ChatGPT).