Renormalization Group Flow and Universal Logarithmic Cost in Recursive Quantum Observation II: RG and Universality

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We formalize a universality principle for recursive quantum observation in which resolution is refined stepwise toward a singular boundary. Assuming a spectral density $\rho(\lambda) \sim C\lambda^{-1+\eta}$ as $\lambda \to 0^+$, we define an effective spectral volume $V(\epsilon)$ and a boundary-indexed refinement protocol. We prove a necessary and sufficient condition: the cumulative cost $\mathcal{W}_N$ exhibits logarithmic scaling $\mathcal{W}_N \sim \ln N$ if and only if $\eta = 0$ (the marginal spectral class). We also present the corresponding RG interpretation, where the marginal trajectory is characterized by a slow approach to the fixed manifold.

I. RG SETUP AND BOUNDARY-INDEXED REFINEMENT

Resolution as a flow parameter. We treat the resolution ϵ as a dynamical control variable and define the RG "time"

$$t \equiv \ln\left(\frac{\epsilon_0}{\epsilon}\right) = \ln\left(\frac{\Lambda}{\Lambda_0}\right), \qquad \Lambda \equiv \epsilon^{-1}. \tag{1}$$

A *boundary-indexed* refinement protocol counts recursion depth by an integer n such that t increases linearly:

$$t_n = n\,\delta, \qquad \epsilon_n = \epsilon_0 e^{-n\delta} \quad (\delta > 0). \tag{2}$$

This is equivalent to a geometric refinement in ϵ and corresponds to uniform steps in $\ln \Lambda$.

Spectral universality class. Assume that the observable induces a spectral density near $\lambda \to 0^+$:

$$\rho(\lambda) \sim C\,\lambda^{-1+\eta}, \qquad C > 0,\ \eta \in \mathbb{R}. \tag{3}$$

Define the *effective spectral volume* accessible up to resolution ϵ by

$$V(\epsilon) \equiv \int_0^{\epsilon} \rho(\lambda)\,\mathrm{d}\lambda. \tag{4}$$

Then

$$V(\epsilon) \sim \begin{cases} \dfrac{C}{\eta}\epsilon^{\eta}, & \eta \neq 0, \\ C\ln\left(\dfrac{\epsilon_0}{\epsilon}\right) = Ct, & \eta = 0. \end{cases} \tag{5}$$

The marginal class is $\eta = 0$.

Cost functional (boundary-limited driving). We consider protocols where the incremental cost at step n is inversely proportional to available effective volume:

$$\Delta\mathcal{W}_n \equiv \frac{w_0}{V(\epsilon_n)}, \qquad w_0 > 0, \tag{6}$$

and define the cumulative cost

$$\mathcal{W}_N \equiv \sum_{n=1}^{N} \Delta\mathcal{W}_n. \tag{7}$$

This choice isolates the scaling implication of shrinking accessible spectral mass; Sections III–IV (Part III) will connect w_0 to measurement-rate and control-work in concrete platforms.

RG interpretation (minimal). On the marginal manifold, $V(\epsilon) \sim t$ grows only linearly with RG time, producing harmonic step costs. Away from marginality, $V(\epsilon)$ changes exponentially in n under (2), yielding non-logarithmic cumulative behavior.

II. UNIVERSALITY THEOREM AND THE HARMONIC–GEOMETRIC RESOLUTION

We now state and prove the universality result in a way that keeps the refinement rule fixed throughout.

Theorem 1 (Logarithmic universality, iff condition). Let $\rho(\lambda)$ satisfy (3), and let the protocol be boundary-indexed as in (2) with cost per step (6). Then the cumulative cost satisfies

$$\mathcal{W}_N \sim \ln N \quad \Longleftrightarrow \quad \eta = 0. \tag{8}$$

Proof. Using (5) and (2), treat cases:

(i) Non-marginal case $\eta \neq 0$. From (5),

$$V(\epsilon_n) \sim \frac{C}{\eta}\epsilon_n^{\eta} = \frac{C}{\eta}\epsilon_0^{\eta} e^{-\eta n\delta}. \tag{9}$$

Hence

$$\Delta\mathcal{W}_n = \frac{w_0}{V(\epsilon_n)} \sim \frac{w_0\,\eta}{C\,\epsilon_0^{\eta}}\,e^{\eta n\delta}. \tag{10}$$

Therefore the partial sums are geometric:

$$\mathcal{W}_N \sim \sum_{n=1}^{N} e^{\eta n\delta} = \begin{cases} \mathcal{O}(e^{\eta N\delta}), & \eta > 0, \\ \mathcal{O}(1), & \eta < 0, \end{cases} \tag{11}$$

which is *never* logarithmic in N.

(ii) Marginal case $\eta = 0$. From (5) and (2), $V(\epsilon_n) \sim Ct_n = Cn\delta$. Then

$$\Delta\mathcal{W}_n = \frac{w_0}{V(\epsilon_n)} \sim \frac{w_0}{C\delta} \cdot \frac{1}{n}, \tag{12}$$

and thus

$$\mathcal{W}_N \sim \frac{w_0}{C\delta}\sum_{n=1}^{N}\frac{1}{n} = \frac{w_0}{C\delta}\left(\ln N + \gamma_E + o(1)\right), \tag{13}$$

where γ_E is Euler's constant. Hence $\mathcal{W}_N \sim \ln N$.

Combining (i) and (ii) proves (8). ■

Remark (Why harmonic vs geometric confusion disappears). If one uses a *harmonic* refinement in ϵ (e.g., $\epsilon_k = \epsilon_0/k$), the natural scaling variable for recursion is no longer k but $t = \ln(\epsilon_0/\epsilon)$, and $t_k = \ln k$. The boundary-indexed definition (2) simply chooses n to be linear in t, which is the standard RG choice. This is precisely the regime in which marginality yields harmonic step cost $1/n$ and cumulative $\ln N$.

III. (PLACEHOLDER) RG FLOW OF η

IV. (PLACEHOLDER) EXPERIMENTAL COEFFICIENT B_{meas}