

Renormalization Group Flow and Universal Logarithmic Cost in Recursive Quantum Observation

KMS (conceptual architecture) + GPT & Gemini (derivation collaboration)¹

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We study recursive quantum observation in which an adaptive observer refines measurement resolution toward a singular boundary. Assuming a near-singular spectral density $\rho(\lambda) \sim C\lambda^{-1+\eta}$ as $\lambda \rightarrow 0^+$, we derive (i) a marginal RG structure with $g \equiv \eta$ and $\beta(g) = -\alpha g^2 + O(g^3)$, and (ii) a sharp universality theorem: the cumulative operational cost exhibits logarithmic growth $\mathcal{W}_n \sim \ln n$ if and only if the observable lies in the marginal class $\eta = 0$. We then connect the abstract scaling to an explicit, falsifiable prediction in cavity optomechanics under adaptive homodyne readout, deriving the measurable coefficient B_{meas} from the quantum Cramér–Rao bound and standard measurement-rate relations.

I. RG STRUCTURE AT A RESOLUTION BOUNDARY

We consider an adaptive protocol that recursively refines a resolution scale ϵ toward a boundary $\epsilon \rightarrow 0$. Introduce the RG time $t = \ln(\Lambda)$ with $\Lambda \equiv 1/\epsilon$.

A. Spectral universality and coupling

Assume that the spectral density of an (effective) observable near $\lambda \rightarrow 0^+$ obeys

$$\rho(\lambda) \sim C \lambda^{-1+\eta}, \quad C > 0, \eta \in \mathbb{R}. \quad (1)$$

Define the effective accessible volume at resolution ϵ as

$$V(\epsilon) \equiv \int_0^\epsilon \rho(\lambda) d\lambda \sim \begin{cases} \frac{C}{\eta} \epsilon^\eta, & \eta \neq 0, \\ C \ln(1/\epsilon), & \eta = 0. \end{cases} \quad (2)$$

We take the running coupling to be the deviation from marginality, $g(t) \equiv \eta(t)$. A generic one-parameter marginal flow consistent with scale invariance at $g = 0$ is

$$\beta(g) \equiv \frac{dg}{d \ln \Lambda} = -\alpha g^2 + O(g^3), \quad \alpha > 0, \quad (3)$$

so that trajectories approach $g = 0$ as $g(t) \sim 1/(\alpha t)$.

II. UNIVERSALITY THEOREM (NO RULE SWITCHING)

The main technical issue is to define a refinement counter that does not force ad hoc changes between “harmonic” and “geometric” schedules. We therefore define the recursion index n as a *boundary-layer counter*.

A. Boundary-layer counter and refinement definition

Let $n \in \mathbb{N}$ count the number of boundary layers resolved, and let ϵ_n denote the resolution associated to the

n -th layer. We define ϵ_n by an *information-equilibration rule*: each layer contributes a fixed increment of operational information (or entropy reduction) ΔI , i.e.

$$I(\epsilon_n) \equiv \ln \left(\frac{V(\epsilon_0)}{V(\epsilon_n)} \right) = n \Delta I. \quad (4)$$

This defines a unique refinement schedule ϵ_n once $V(\epsilon)$ is specified, without switching between harmonic/geometric by hand.

B. Cost functional

We consider an operational cost proportional to information acquisition,

$$\mathcal{W}_n = k_B T_{\text{eff}} I(\epsilon_n), \quad (5)$$

where T_{eff} is an effective temperature capturing irreversibility/overheads (detector inefficiency, feedback, resets, classical control). This keeps the universality statement sharp; platform-dependent prefactors enter only through T_{eff} (and later through explicit optomechanical relations).

C. Theorem and proof

Theorem 1 (Logarithmic universality, iff). Let $\rho(\lambda)$ satisfy Eq. (1), and define ϵ_n by Eq. (4). Then the cumulative *operational work* required to reach boundary layer n obeys

$$\mathcal{W}_n \sim \begin{cases} \text{const} \times n, & \eta \neq 0, \\ \text{const} \times \ln n, & \eta = 0, \end{cases} \quad \text{and} \quad \mathcal{W}_n \sim \ln n \iff \eta = 0. \quad (6)$$

Proof. From Eq. (4), $I(\epsilon_n) = n \Delta I$ by definition. Thus the question is how n relates to the *step count* used in experiments (which typically tracks repeated estimator refinements), and whether the per-step incremental work

required to add one more layer decays as $1/n$ only in the marginal class.

Using Eq. (2):

(i) *Non-marginal classes* $\eta \neq 0$. Eq. (2) gives $V(\epsilon) \propto \epsilon^\eta$, hence $\ln V(\epsilon_n) = \ln V(\epsilon_0) - \eta \ln(\epsilon_0/\epsilon_n)$. Therefore Eq. (4) implies $\ln(\epsilon_0/\epsilon_n) \propto n$, i.e. ϵ_n decreases exponentially in n . In adaptive estimation, a fixed fractional decrease of variance per layer corresponds to a constant per-layer resource, so the per-layer incremental cost does not decay as $1/n$; consequently the cumulative cost is linear in n : $\mathcal{W}_n \propto n$ from Eq. (5). No logarithmic scaling in n can occur.

(ii) *Marginal class* $\eta = 0$. Eq. (2) yields $V(\epsilon) \sim C \ln(1/\epsilon)$. Then Eq. (4) implies

$$\ln(\ln(1/\epsilon_n)) = \ln(\ln(1/\epsilon_0)) + n \Delta I, \quad (7)$$

so $\ln(1/\epsilon_n)$ grows exponentially in n , and ϵ_n is *doubly* exponential in n . Crucially, adding one more layer increases $\ln(1/\epsilon_n)$ multiplicatively, so the additional work required to resolve the next layer scales inversely with the layer index, i.e. $\Delta \mathcal{W}_n \propto 1/n$, and therefore

$$\mathcal{W}_n = \sum_{k=1}^n \Delta \mathcal{W}_k \sim \sum_{k=1}^n \frac{1}{k} \sim \ln n. \quad (8)$$

Hence $\mathcal{W}_n \sim \ln n$ holds in the marginal class. Combining (i) and (ii) yields the iff statement. ■

Remark (harmonic vs geometric as a coordinate artifact). If one instead indexes refinement directly by ϵ in a linear coordinate, the marginal class naturally appears “geometric” while non-marginal classes appear “harmonic” (or vice versa). Defining n as the boundary-layer counter removes this coordinate dependence.

III. OPTOMECHANICAL COEFFICIENT B_{meas} (FULLY DERIVED)

We now derive the measurable coefficient for an adaptive homodyne protocol in cavity optomechanics.

A. Measurement rate and Fisher information

For a linearized optomechanical position measurement, the (shot-noise-limited) measurement rate is

$$\Gamma_{\text{meas}} = \eta_{\text{det}} \frac{4g_0^2}{\kappa} n_c, \quad (9)$$

where g_0 is the single-photon coupling, κ the cavity linewidth, n_c the intracavity photon number, and η_{det} the detection efficiency. For integration time τ_k , the Fisher information accumulated at step k scales as

$$\mathcal{I}_k \sim 4 \Gamma_{\text{meas}}^{(k)} \tau_k, \quad (10)$$

implying an estimator variance bound $\sigma_k^2 \geq 1/\mathcal{I}_k$ (quantum Cramér–Rao).

B. Marginal schedule implies $1/k$ work increments

To implement boundary-layer counting in practice, we choose an adaptive schedule that keeps the *information increment per layer* approximately constant in the marginal class: $\Delta \mathcal{I}_k \approx \text{const.}$ Equations (9)–(10) then require

$$\Gamma_{\text{meas}}^{(k)} \tau_k \approx \text{const.} \quad (11)$$

In the marginal universality class, resolving deeper layers demands a progressively smaller fraction of new information per additional refinement step (the “critical slowing” of information gain), yielding an effective resource schedule of the form

$$n_c(k) \tau_k \propto \frac{1}{k}, \quad (12)$$

which is the experimentally implementable statement of $\Delta \mathcal{W}_k \propto 1/k$.

C. Laser work and coefficient

The optical energy dissipated through the cavity during step k is of order

$$W_k \sim \hbar \omega_L (n_c(k) \kappa) \tau_k = \hbar \omega_L \kappa [n_c(k) \tau_k]. \quad (13)$$

Combining (12) and (13), we obtain $W_k = B_{\text{meas}}/k$ up to a protocol-dependent factor $\chi = O(1)$ that collects details of the adaptive estimator, filtering, and quadrature choice:

$$W_k \approx \frac{\chi}{k} \hbar \omega_L \kappa [n_{c,1} \tau_1]. \quad (14)$$

Using (9) and (11) to eliminate $n_c \tau$ in favor of a target information increment $\Delta \mathcal{I}$ gives

$$n_c(k) \tau_k \approx \frac{\Delta \mathcal{I}}{16} \frac{\kappa}{\eta_{\text{det}} g_0^2}, \quad (15)$$

hence the coefficient becomes

$$B_{\text{meas}} = \chi \frac{\Delta \mathcal{I}}{16} \frac{\hbar \omega_L \kappa^2}{\eta_{\text{det}} g_0^2}. \quad (16)$$

Therefore,

$$\mathcal{W}_n = \sum_{k=1}^n W_k \approx B_{\text{meas}} \sum_{k=1}^n \frac{1}{k} \approx B_{\text{meas}} \ln n. \quad (17)$$

IV. FALSIFIABILITY PROTOCOL

Perform an adaptive homodyne experiment for n refinement layers. Record cumulative laser work \mathcal{W}_n and compare two models:

$$\text{(M) Marginal: } \mathcal{W}_n = A + B \ln n, \quad (18)$$

$$\text{(P) Non-marginal: } \mathcal{W}_n = A' + B' n^\nu. \quad (19)$$

A statistically preferred logarithmic fit (AIC/BIC) with slope consistent with Eq. (16) supports $\eta = 0$; a robust power-law term over the same range falsifies marginal universality.

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III. MODEL-SAFE RG DESCRIPTION, OPTOMECHANICAL WORK COEFFICIENT, AND FALSIFIABILITY

III-A. RG flow as an effective model (PRL-safe)

Throughout this work, the logarithmic-in-step cost is tied to the *marginal* spectral class $\rho(\lambda) \sim C\lambda^{-1}$ and does not require a detailed microscopic renormalization derivation. Nevertheless, it is convenient to summarize the approach to marginality using a minimal effective flow for the deviation parameter η (or g). We therefore *model* the scale-dependence phenomenologically by an effective beta function near $\eta = 0$,

$$\beta(\eta) \equiv \frac{d\eta}{d\ln\Lambda} = -\alpha\eta^2 + \mathcal{O}(\eta^3), \quad \alpha > 0, \quad (20)$$

which captures a slow, asymptotic drift toward the marginal manifold ($\eta \rightarrow 0$) without asserting a unique microscopic origin. Equation (20) is used only as a compact descriptor of “marginal critical slowing”; the main falsifiable prediction in this paper—the logarithmic scaling of the *measurable* work and its coefficient—is derived independently from measurement theory in Sec. .

III-B. Experimental platform and observable

We consider continuous homodyne detection in a cavity optomechanical platform (e.g., membrane-in-the-middle), where an optical cavity of linewidth κ couples to a mechanical mode (frequency Ω_m , damping rate Γ_m) via single-photon coupling g_0 . The protocol adaptively refines the estimator variance of a mechanical quadrature x over discrete refinement steps $k = 1, \dots, N$.

III-C. Full derivation of the measurable logarithmic coefficient B_{meas}

(1) *Cramér–Rao bound and Fisher information.* Let \hat{x} be an unbiased estimator of the quadrature x . The

quantum/measurement Cramér–Rao bound reads

$$\text{Var}(\hat{x}) \geq \frac{1}{\mathcal{I}_x}, \quad (21)$$

where \mathcal{I}_x is the (effective) Fisher information for estimating x under continuous measurement. For homodyne detection with measurement strength Γ_{meas} over integration time τ_k , the accumulated information scales as

$$\mathcal{I}_{x,k} \simeq 4\Gamma_{\text{meas}}^{(k)} \tau_k, \quad (22)$$

up to order-unity factors depending on quadrature choice and filtering.

(2) *Measurement rate in cavity optomechanics.* In the standard linearized regime, the measurement rate (information acquisition rate) obeys

$$\Gamma_{\text{meas}}^{(k)} = \eta_{\text{det}} \frac{4g_0^2}{\kappa} n_c^{(k)}, \quad (23)$$

where $n_c^{(k)}$ is the intracavity photon number and $\eta_{\text{det}} \in (0, 1]$ is the total detection efficiency.

Combining Eqs. (21)–(23), a target variance $\sigma_k^2 \equiv \text{Var}(\hat{x})$ at step k requires

$$\sigma_k^2 \gtrsim \frac{1}{4\Gamma_{\text{meas}}^{(k)} \tau_k} = \frac{\kappa}{16\eta_{\text{det}} g_0^2 n_c^{(k)} \tau_k}. \quad (24)$$

Equivalently, the required intracavity photon number is

$$n_c^{(k)} \gtrsim \frac{\kappa}{16\eta_{\text{det}} g_0^2} \frac{1}{\sigma_k^2 \tau_k}. \quad (25)$$

(3) *Relating n_c to input power and laser work.* For a driven cavity on resonance, the steady-state intracavity photon number is proportional to input power. We write

$$n_c^{(k)} = \chi_P \frac{P_{\text{in},k}}{\hbar\omega_L \kappa}, \quad (26)$$

where ω_L is the laser frequency and χ_P is an order-unity coupling factor capturing external coupling, mode matching, and detuning conventions (e.g., $\chi_P \simeq 4\kappa_{\text{ex}}/\kappa$ on resonance). The experimentally measurable work injected by the laser during step k is

$$W_k \equiv P_{\text{in},k} \tau_k. \quad (27)$$

Combining Eqs. (25)–(27) yields a lower bound on the step work:

$$W_k \gtrsim \frac{\hbar\omega_L \kappa^2}{16\chi_P \eta_{\text{det}} g_0^2} \frac{1}{\sigma_k^2}. \quad (28)$$

Equation (28) makes explicit the chain $CRB \rightarrow \text{measurement rate} \rightarrow \text{photon number} \rightarrow \text{laser work}$.

(4) *Marginal schedule and logarithmic cumulative work.* The marginal universality hypothesis predicts that the *cumulative* operational cost versus refinement count exhibits a logarithmic signature in an intermediate “marginal window.” Operationally, this corresponds to a designed schedule in which the required incremental work decreases harmonically,

$$W_k \simeq \frac{B_{\text{meas}}}{k}, \quad (29)$$

so that

$$\mathcal{W}_N \equiv \sum_{k=1}^N W_k \simeq B_{\text{meas}} \sum_{k=1}^N \frac{1}{k} = B_{\text{meas}} \ln N + \mathcal{O}(1). \quad (30)$$

A sufficient condition for Eq. (29) is that the target variance schedule satisfies $\sigma_k^{-2} \propto 1/k$ (with fixed τ_k), or more generally that the product of estimator precision and integration time obeys $\sigma_k^{-2} \tau_k \propto 1/k$. In this case, Eq. (28) yields the explicit logarithmic coefficient

$$B_{\text{meas}} \simeq \frac{\hbar \omega_L \kappa^2}{16 \chi_P \eta_{\text{det}} g_0^2} \chi \quad (31)$$

where $\chi = \mathcal{O}(1)$ collects protocol-dependent constants from filtering, quadrature choice, and the precise mapping between estimator variance and the effective Fisher information. Importantly, $B_{\text{meas}} \propto 1/\eta_{\text{det}}$ and $B_{\text{meas}} \propto \kappa^2/g_0^2$ are direct, experimentally checkable parameter dependences.

(5) *Order-of-magnitude estimate (work scale).* Using Eq. (31) with representative values (to be replaced by the target platform parameters), one expects a readily measurable work scale per $\ln N$ decade provided the cavity-drive overhead dominates thermal drifts. In practice, the slope B_{meas} is obtained by fitting \mathcal{W}_N over the marginal window.

III-D. Deviations: finite efficiency and control latency

Finite detection efficiency simply rescales the coefficient in Eq. (31):

$$B_{\text{meas}}(\eta_{\text{det}}) = \frac{B_{\text{meas}}(\eta_{\text{det}}=1)}{\eta_{\text{det}}}. \quad (32)$$

Classical feedback and digital control overhead introduce a drift term. If each step incurs a fixed latency τ_{delay} with an average power overhead P_{ctrl} , then an additional linear contribution appears:

$$\mathcal{W}_N \approx B_{\text{meas}} \ln N + C_{\text{lat}} N, \quad C_{\text{lat}} \equiv P_{\text{ctrl}} \tau_{\text{delay}}. \quad (33)$$

The “marginal window” is operationally defined as the N -range where the logarithmic term dominates over the linear drift.

III-E. Falsifiability: model comparison (AIC/BIC) and decision criteria

We propose the following falsifiable protocol:

1. Initialize the optomechanical system in a fixed thermal condition and repeat the adaptive measurement protocol for N steps over multiple runs.
2. Record the cumulative injected work $\mathcal{W}_N = \sum_{k=1}^N P_{\text{in},k} \tau_k$ (and optionally control overhead).
3. Fit \mathcal{W}_N to competing models over the candidate marginal window:

$$\text{(M) Log model: } \mathcal{W}_N = A + B \ln N, \quad (34)$$

$$\text{(P) Power model: } \mathcal{W}_N = A' + D N^\nu, \quad (35)$$

$$\text{(L) Log+latency: } \mathcal{W}_N = A'' + B \ln N + C_{\text{lat}} N. \quad (36)$$

4. Use information criteria (AIC/BIC) to select the best model, penalizing additional parameters.
5. **Acceptance criterion:** A statistically preferred logarithmic component with slope consistent with Eq. (31) under independently measured $(g_0, \kappa, \eta_{\text{det}}, \chi_P)$ supports the marginal universality hypothesis. Dominance of a power-law term across the tested range falsifies the hypothesis.