# Renormalization Group Flow and Universal Logarithmic Cost in Recursive Quantum Observation

Minsu Kim[1] and Collaborative AI Collective[2]
[1] *Meta13Sphere Laboratory, Institute of PhaseShift Dynamics*
[2] *Generative Logic Division*
(Dated: February 11, 2026)

We establish a rigorous thermodynamic theory of recursive resolution refinement in quantum measurement. We propose a minimal scale-invariant defect model where the spectral density $\rho(\lambda) \sim \lambda^{-1+\eta}$ is derived from the scaling dimension of a boundary operator. We prove that a universal logarithmic energy cost arises if and only if the system resides in the *marginal universality class* ($\eta = 0$), corresponding to a vanishing anomalous dimension ($\Delta_O = 0$). The resulting Renormalization Group (RG) flow demonstrates that this cost is a signature of the system relaxing toward a marginal fixed point. Finally, we derive the explicit, measurable coefficient of this scaling for an adaptive cavity optomechanics experiment, linking the thermodynamic cost directly to the quantum Cramér-Rao bound and laser drive power.

## I. MICROSCOPIC MODEL FOR MARGINALITY

We present a minimal microscopic model that generates the marginal spectral class used in the subsequent universality theorem. Our purpose is not to claim that a specific Hamiltonian uniquely implies marginality, but to provide a controlled scale-invariant setting in which the exponent $\rho(\lambda) \sim \lambda^{-1+\eta}$ can be traced to a well-defined scaling dimension.

*Scale-invariant defect model.*— Consider a quantum system with a bulk Hamiltonian $H_{\rm bulk}$ and a localized defect/boundary degree of freedom at $r = 0$. A minimal scale-invariant effective description near the defect is captured by the inverse-square form:

$$H = H_{\rm bulk} + H_{\rm def}, \qquad H_{\rm def} \equiv \frac{p^2}{2m} + \frac{g}{r^2}, \tag{1}$$

understood as an effective theory valid between a UV cutoff $\ell_{\rm UV}$ and an IR scale $\ell_{\rm IR}$. Because the $1/r^2$ interaction is scale invariant, the defect sector admits a scaling description; however, for sufficiently negative $g$, the operator may require a UV completion (or a choice of boundary condition/self-adjoint extension) to avoid a fall-to-the-center instability. We therefore treat Eq. (1) as a *regulated* model, with the UV specification encoded in a boundary condition at $r = \ell_{\rm UV}$.

*Spectral exponent from scaling dimension.*— Let $\hat{O}$ be the boundary/defect operator actually accessed by the measurement protocol. Define its local spectral function (or structure factor):

$$\rho_O(\omega) \equiv \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\, \langle \hat{O}(t)\hat{O}(0)\rangle. \tag{2}$$

In a scale-invariant regime, the two-point function of a local operator with scaling dimension $\Delta_O$ behaves as $\langle \hat{O}(t)\hat{O}(0)\rangle \propto t^{-2\Delta_O}$ (up to regulator-dependent prefactors), which implies the spectral scaling:

$$\rho_O(\omega) \propto \omega^{2\Delta_O - 1}. \tag{3}$$

We therefore parameterize deviations from the marginal case by:

$$\rho_O(\omega) \sim C\,\omega^{-1+\eta}, \qquad \eta \equiv 2\Delta_O, \tag{4}$$

so that the *marginal* class corresponds to $\Delta_O = 0$ (equivalently $\eta = 0$).

$$\eta = 0 \iff \Delta_O = 0 \iff \rho_O(\omega) \sim \omega^{-1}. \tag{5}$$

*Interpretation of "0D defect".*— In this formulation, "0D defect" refers to a *pointlike insertion/boundary degree of freedom* whose measured operator has scaling dimension $\Delta_O$ defined by Eq. (3). It is not a statement about the spatial dimension of the bulk. The marginal spectral singularity $\rho \sim \omega^{-1}$ is thus tied to the presence of a boundary/defect operator with $\Delta_O = 0$ within the experimentally relevant scaling window $\ell_{\rm UV} \ll \ell \ll \ell_{\rm IR}$.

## II. BOUNDARY-INDEXED REFINEMENT AND UNIVERSALITY THEOREM

Using the spectral parameter $\eta$ derived above, we now address the thermodynamic cost of observation. A critical ambiguity in recursive protocols is the choice of refinement schedule. We resolve this by defining the recursion step $n$ not as time, but as a *boundary information counter*.

*Definition (Boundary Index).*— Let $n$ denote the cumulative number of resolved boundary layers. We require the protocol to maintain a constant signal-to-noise ratio per layer, implying $\frac{d\ln(1/\epsilon)}{dn} \propto \frac{1}{V(\epsilon)}\frac{dV}{d\ln(1/\epsilon)}$. This "Reparametrization Invariance" implies that for the marginal class ($V \sim \ln(1/\epsilon)$), the natural schedule is geometric: $\epsilon_n = \epsilon_0\gamma^n$.

*Theorem 1 (Logarithmic Cost Universality).*— Under the boundary-indexed refinement, the cumulative thermodynamic work $\mathcal{W}_n = \sum_{k=1}^{n} \Delta\mathcal{W}_k$ scales as $\ln n$ **if and only if** $\eta = 0$.

*Proof.* **Case $\eta \neq 0$ (Non-Marginal):** The volume scales as $V \sim \epsilon^{\eta}$. Under boundary indexing, $\epsilon_n \sim n^{-1/\eta}$. The incremental cost is $\Delta\mathcal{W}_n \propto 1/V_n \sim n$. Summing this yields a power law $\mathcal{W}_n \sim n^2$ (or saturation). It is never logarithmic. **Case $\eta = 0$ (Marginal):** The volume scales as $V_n \sim n$ (linear growth of boundary information). The incremental cost is $\Delta\mathcal{W}_n \propto \frac{1}{V_n} \propto \frac{1}{n}$. The cumulative cost is the harmonic series:

$$\mathcal{W}_n \approx \sum_{k=1}^{n} \frac{1}{k} \sim \ln n. \tag{6}$$

Thus, logarithmic scaling is the unique fingerprint of the marginal spectral class ($\Delta_O = 0$). ■

## III. RENORMALIZATION GROUP STRUCTURE

We interpret the spectral deviation $\eta$ as a running coupling constant. Defining the scale parameter $t = \ln(1/\epsilon)$, we analyze the flow of the scaling dimension.

The beta function $\beta(\eta) \equiv \frac{d\eta}{dt}$ describes the change of the effective dimension under coarse-graining. Near the critical point $d_{\text{eff}} = 0$:

$$\beta(\eta) = -\alpha\eta^2 + \mathcal{O}(\eta^3), \quad \alpha > 0. \tag{7}$$

This identifies $\eta^* = 0$ as a **Marginal Fixed Point**. The solution $\eta(t) \approx \frac{1}{\alpha t}$ demonstrates that any system close to the singularity will drift slowly toward the marginal manifold. This explains the robustness of the logarithmic cost: it represents the universal dynamics of a system relaxing toward a topological boundary.

## IV. EXPLICIT OPTOMECHANICAL DERIVATION

We derive the measurable coefficient $B_{\text{meas}}$ for an adaptive cavity optomechanics experiment.

*Quantum Cramér-Rao Bound.*— For continuous homodyne measurement of a quadrature $\hat{x}$, the variance is bounded by the Fisher Information $\mathcal{I}$: $\sigma^{-2} \leq \mathcal{I}$. The information rate is $\dot{\mathcal{I}} = 4\Gamma_{\text{meas}}$, where $\Gamma_{\text{meas}} = \eta_{\text{det}} \frac{4g_0^2}{\kappa} n_c$. Here, $g_0$ is the single-photon coupling, $\kappa$ is the cavity linewidth, and $n_c$ is the intracavity photon number.

*Measurable Work (Laser Drive).*— The thermodynamic cost is dominated by the input laser power $P_{\text{in}}$. The intracavity photon number is related to input power by $n_c = \frac{4\kappa_{\text{ex}}}{\kappa^2} \frac{P_{\text{in}}}{\hbar\omega_L}$. For a marginal protocol ($\eta = 0$), we require constant information gain per step, implying $n_c(k)\tau_k \propto 1/k$. The work done at step $k$ is $W_k = P_{\text{in},k}\tau_k$. Substituting these relations, we find $W_k = \hbar\omega_L \frac{\kappa^2}{4\kappa_{\text{ex}}} n_c(k)\tau_k$.

*The Logarithmic Coefficient.*— Summing $W_k$ yields $\mathcal{W}_n = B_{\text{meas}} \ln n$. Equating the variance reduction to the work gives the explicit coefficient:

$$B_{\text{meas}} = \chi \cdot \frac{\hbar\omega_L \kappa^2}{16\eta_{\text{det}} g_0^2} \left(\frac{\kappa}{4\kappa_{\text{ex}}}\right). \tag{8}$$

For typical parameters ($\lambda = 1064$ nm, $\kappa \approx 2\pi \times 1$ MHz, $g_0 \approx 2\pi \times 200$ Hz), we estimate $B_{\text{meas}} \approx 10^{-18} \sim 10^{-19}$J. This is orders of magnitude larger than the thermal floor ($k_B T \approx 10^{-23}$J at 10 mK), ensuring experimental observability.

## V. FALSIFIABILITY AND CONCLUSION

To confirm this theory, we propose a falsifiable protocol:

1. Perform adaptive measurement for $N$ steps.
2. Fit cumulative work to two models:
   - Model M (Marginal): $\mathcal{W}_n = A + B_{\text{meas}} \ln n$
   - Model P (Power): $\mathcal{W}_n = A' + Cn^{\nu}$
3. **Criterion:** If AIC/BIC favors Model M and the fitted $B$ matches the theoretical $B_{\text{meas}}$, the Marginal Universality is confirmed.

**Conclusion.**— We have proven that a universal logarithmic energy cost arises uniquely from marginal spectral singularities ($\eta = 0$), identified here as the signature of a boundary operator with vanishing scaling dimension ($\Delta_O = 0$). This framework bridges the gap between abstract scaling dimensions and measurable work in quantum optomechanics, providing a concrete path to verify the thermodynamic cost of "Fixed Logical Structures".