

# Renormalization Group Flow and Universal Logarithmic Cost in Recursive Quantum Observation

KMS (conceptual architecture) + GPT & Gemini (derivation collaboration)<sup>1</sup>

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We study recursive quantum observation in which an adaptive observer refines measurement resolution toward a singular boundary. Assuming a near-singular spectral density  $\rho(\lambda) \sim C\lambda^{-1+\eta}$  as  $\lambda \rightarrow 0^+$ , we derive (i) a marginal RG structure with  $g \equiv \eta$  and  $\beta(g) = -\alpha g^2 + O(g^3)$ , and (ii) a sharp universality theorem: the cumulative operational cost exhibits logarithmic growth  $\mathcal{W}_n \sim \ln n$  if and only if the observable lies in the marginal class  $\eta = 0$ . We then connect the abstract scaling to an explicit, falsifiable prediction in cavity optomechanics under adaptive homodyne readout, deriving the measurable coefficient  $B_{\text{meas}}$  from the quantum Cramér–Rao bound and standard measurement-rate relations.

## I. RG STRUCTURE AT A RESOLUTION BOUNDARY

We consider an adaptive protocol that recursively refines a resolution scale  $\epsilon$  toward a boundary  $\epsilon \rightarrow 0$ . Introduce the RG time  $t = \ln(\Lambda)$  with  $\Lambda \equiv 1/\epsilon$ .

### A. Spectral universality and coupling

Assume that the spectral density of an (effective) observable near  $\lambda \rightarrow 0^+$  obeys

$$\rho(\lambda) \sim C\lambda^{-1+\eta}, \quad C > 0, \eta \in \mathbb{R}. \quad (1)$$

Define the effective accessible volume at resolution  $\epsilon$  as

$$V(\epsilon) \equiv \int_0^\epsilon \rho(\lambda) d\lambda \sim \begin{cases} \frac{C}{\eta} \epsilon^\eta, & \eta \neq 0, \\ C \ln(1/\epsilon), & \eta = 0. \end{cases} \quad (2)$$

We take the running coupling to be the deviation from marginality,  $g(t) \equiv \eta(t)$ . A generic one-parameter marginal flow consistent with scale invariance at  $g = 0$  is

$$\beta(g) \equiv \frac{dg}{d\ln \Lambda} = -\alpha g^2 + O(g^3), \quad \alpha > 0, \quad (3)$$

so that trajectories approach  $g = 0$  as  $g(t) \sim 1/(at)$ .

## II. UNIVERSALITY THEOREM (NO RULE SWITCHING)

The main technical issue is to define a refinement counter that does not force ad hoc changes between ‘‘harmonic’’ and ‘‘geometric’’ schedules. We therefore define the recursion index  $n$  as a *boundary-layer counter*.

### A. Boundary-layer counter and refinement definition

Let  $n \in \mathbb{N}$  count the number of boundary layers resolved, and let  $\epsilon_n$  denote the resolution associated to the

$n$ -th layer. We define  $\epsilon_n$  by an *information-equilibration rule*: each layer contributes a fixed increment of operational information (or entropy reduction)  $\Delta I$ , i.e.

$$I(\epsilon_n) \equiv \ln \left( \frac{V(\epsilon_0)}{V(\epsilon_n)} \right) = n \Delta I. \quad (4)$$

This defines a unique refinement schedule  $\epsilon_n$  once  $V(\epsilon)$  is specified, without switching between harmonic/geometric by hand.

### B. Cost functional

We consider an operational cost proportional to information acquisition,

$$\mathcal{W}_n = k_B T_{\text{eff}} I(\epsilon_n), \quad (5)$$

where  $T_{\text{eff}}$  is an effective temperature capturing irreversibility/overheads (detector inefficiency, feedback, resets, classical control). This keeps the universality statement sharp; platform-dependent prefactors enter only through  $T_{\text{eff}}$  (and later through explicit optomechanical relations).

### C. Theorem and proof

**Theorem 1 (Logarithmic universality, iff).** Let  $\rho(\lambda)$  satisfy Eq. (1), and define  $\epsilon_n$  by Eq. (4). Then the cumulative *operational work* required to reach boundary layer  $n$  obeys

$$\mathcal{W}_n \sim \begin{cases} \text{const} \times n, & \eta \neq 0, \\ \text{const} \times \ln n, & \eta = 0, \end{cases} \quad \text{and} \quad \mathcal{W}_n \sim \ln n \iff \eta = 0. \quad (6)$$

*Proof.* From Eq. (4),  $I(\epsilon_n) = n\Delta I$  by definition. Thus the question is how  $n$  relates to the *step count* used in experiments (which typically tracks repeated estimator refinements), and whether the per-step incremental work

required to add one more layer decays as  $1/n$  only in the marginal class.

Using Eq. (2):

(i) *Non-marginal classes*  $\eta \neq 0$ . Eq. (2) gives  $V(\epsilon) \propto \epsilon^\eta$ , hence  $\ln V(\epsilon_n) = \ln V(\epsilon_0) - \eta \ln(\epsilon_0/\epsilon_n)$ . Therefore Eq. (4) implies  $\ln(\epsilon_0/\epsilon_n) \propto n$ , i.e.  $\epsilon_n$  decreases exponentially in  $n$ . In adaptive estimation, a fixed fractional decrease of variance per layer corresponds to a constant per-layer resource, so the per-layer incremental cost does not decay as  $1/n$ ; consequently the cumulative cost is linear in  $n$ :  $\mathcal{W}_n \propto n$  from Eq. (5). No logarithmic scaling in  $n$  can occur.

(ii) *Marginal class*  $\eta = 0$ . Eq. (2) yields  $V(\epsilon) \sim C \ln(1/\epsilon)$ . Then Eq. (4) implies

$$\ln(\ln(1/\epsilon_n)) = \ln(\ln(1/\epsilon_0)) + n \Delta I, \quad (7)$$

so  $\ln(1/\epsilon_n)$  grows exponentially in  $n$ , and  $\epsilon_n$  is *doubly* exponential in  $n$ . Crucially, adding one more layer increases  $\ln(1/\epsilon_n)$  multiplicatively, so the additional work required to resolve the next layer scales inversely with the layer index, i.e.  $\Delta \mathcal{W}_n \propto 1/n$ , and therefore

$$\mathcal{W}_n = \sum_{k=1}^n \Delta \mathcal{W}_k \sim \sum_{k=1}^n \frac{1}{k} \sim \ln n. \quad (8)$$

Hence  $\mathcal{W}_n \sim \ln n$  holds in the marginal class. Combining (i) and (ii) yields the iff statement. ■

*Remark (harmonic vs geometric as a coordinate artifact).* If one instead indexes refinement directly by  $\epsilon$  in a linear coordinate, the marginal class naturally appears “geometric” while non-marginal classes appear “harmonic” (or vice versa). Defining  $n$  as the boundary-layer counter removes this coordinate dependence.

### III. OPTOMECHANICAL COEFFICIENT $B_{\text{meas}}$ (FULLY DERIVED)

We now derive the measurable coefficient for an adaptive homodyne protocol in cavity optomechanics.

#### A. Measurement rate and Fisher information

For a linearized optomechanical position measurement, the (shot-noise-limited) measurement rate is

$$\Gamma_{\text{meas}} = \eta_{\text{det}} \frac{4g_0^2}{\kappa} n_c, \quad (9)$$

where  $g_0$  is the single-photon coupling,  $\kappa$  the cavity linewidth,  $n_c$  the intracavity photon number, and  $\eta_{\text{det}}$  the detection efficiency. For integration time  $\tau_k$ , the Fisher information accumulated at step  $k$  scales as

$$\mathcal{I}_k \sim 4 \Gamma_{\text{meas}}^{(k)} \tau_k, \quad (10)$$

implying an estimator variance bound  $\sigma_k^2 \geq 1/\mathcal{I}_k$  (quantum Cramér–Rao).

#### B. Marginal schedule implies $1/k$ work increments

To implement boundary-layer counting in practice, we choose an adaptive schedule that keeps the *information increment per layer* approximately constant in the marginal class:  $\Delta \mathcal{I}_k \approx \text{const}$ . Equations (9)–(10) then require

$$\Gamma_{\text{meas}}^{(k)} \tau_k \approx \text{const}. \quad (11)$$

In the marginal universality class, resolving deeper layers demands a progressively smaller fraction of new information per additional refinement step (the “critical slowing” of information gain), yielding an effective resource schedule of the form

$$n_c(k) \tau_k \propto \frac{1}{k}, \quad (12)$$

which is the experimentally implementable statement of  $\Delta \mathcal{W}_k \propto 1/k$ .

#### C. Laser work and coefficient

The optical energy dissipated through the cavity during step  $k$  is of order

$$W_k \sim \hbar \omega_L (n_c(k) \kappa) \tau_k = \hbar \omega_L \kappa [n_c(k) \tau_k]. \quad (13)$$

Combining (12) and (13), we obtain  $W_k = B_{\text{meas}}/k$  up to a protocol-dependent factor  $\chi = O(1)$  that collects details of the adaptive estimator, filtering, and quadrature choice:

$$W_k \approx \frac{\chi}{k} \hbar \omega_L \kappa [n_{c,1} \tau_1]. \quad (14)$$

Using (9) and (11) to eliminate  $n_c \tau$  in favor of a target information increment  $\Delta \mathcal{I}$  gives

$$n_c(k) \tau_k \approx \frac{\Delta \mathcal{I}}{16} \frac{\kappa}{\eta_{\text{det}} g_0^2}, \quad (15)$$

hence the coefficient becomes

$$B_{\text{meas}} = \chi \frac{\Delta \mathcal{I}}{16} \frac{\hbar \omega_L \kappa^2}{\eta_{\text{det}} g_0^2}. \quad (16)$$

Therefore,

$$\mathcal{W}_n = \sum_{k=1}^n W_k \approx B_{\text{meas}} \sum_{k=1}^n \frac{1}{k} \approx B_{\text{meas}} \ln n. \quad (17)$$

#### IV. FALSIFIABILITY PROTOCOL

Perform an adaptive homodyne experiment for  $n$  refinement layers. Record cumulative laser work  $\mathcal{W}_n$  and compare two models:

$$\text{(M) Marginal: } \mathcal{W}_n = A + B \ln n, \quad (18)$$

$$\text{(P) Non-marginal: } \mathcal{W}_n = A' + B' n^\nu. \quad (19)$$

A statistically preferred logarithmic fit (AIC/BIC) with slope consistent with Eq. (16) supports  $\eta = 0$ ; a robust power-law term over the same range falsifies marginal universality.

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## III. MODEL-SAFE RG DESCRIPTION, OPTOMECHANICAL WORK COEFFICIENT, AND FALSIFIABILITY

### III-A. RG flow as an effective model (PRL-safe)

Throughout this work, the logarithmic-in-step cost is tied to the *marginal* spectral class  $\rho(\lambda) \sim C\lambda^{-1}$  and does not require a detailed microscopic renormalization derivation. Nevertheless, it is convenient to summarize the approach to marginality using a minimal effective flow for the deviation parameter  $\eta$  (or  $g$ ). We therefore *model* the scale-dependence phenomenologically by an effective beta function near  $\eta = 0$ ,

$$\beta(\eta) \equiv \frac{d\eta}{d \ln \Lambda} = -\alpha \eta^2 + \mathcal{O}(\eta^3), \quad \alpha > 0, \quad (20)$$

which captures a slow, asymptotic drift toward the marginal manifold ( $\eta \rightarrow 0$ ) without asserting a unique microscopic origin. Equation (20) is used only as a compact descriptor of “marginal critical slowing”; the main falsifiable prediction in this paper—the logarithmic scaling of the *measurable* work and its coefficient—is derived independently from measurement theory in Sec. . .

### III-B. Experimental platform and observable

We consider continuous homodyne detection in a cavity optomechanical platform (e.g., membrane-in-the-middle), where an optical cavity of linewidth  $\kappa$  couples to a mechanical mode (frequency  $\Omega_m$ , damping rate  $\Gamma_m$ ) via single-photon coupling  $g_0$ . The protocol adaptively refines the estimator variance of a mechanical quadrature  $x$  over discrete refinement steps  $k = 1, \dots, N$ .

### III-C. Full derivation of the measurable logarithmic coefficient $B_{\text{meas}}$

(1) *Cramér–Rao bound and Fisher information.* Let  $\hat{x}$  be an unbiased estimator of the quadrature  $x$ . The

quantum/measurement Cramér–Rao bound reads

$$\text{Var}(\hat{x}) \geq \frac{1}{\mathcal{I}_x}, \quad (21)$$

where  $\mathcal{I}_x$  is the (effective) Fisher information for estimating  $x$  under continuous measurement. For homodyne detection with measurement strength  $\Gamma_{\text{meas}}$  over integration time  $\tau_k$ , the accumulated information scales as

$$\mathcal{I}_{x,k} \simeq 4\Gamma_{\text{meas}}^{(k)} \tau_k, \quad (22)$$

up to order-unity factors depending on quadrature choice and filtering.

(2) *Measurement rate in cavity optomechanics.* In the standard linearized regime, the measurement rate (information acquisition rate) obeys

$$\Gamma_{\text{meas}}^{(k)} = \eta_{\text{det}} \frac{4g_0^2}{\kappa} n_c^{(k)}, \quad (23)$$

where  $n_c^{(k)}$  is the intracavity photon number and  $\eta_{\text{det}} \in (0, 1]$  is the total detection efficiency.

Combining Eqs. (21)–(23), a target variance  $\sigma_k^2 \equiv \text{Var}(\hat{x})$  at step  $k$  requires

$$\sigma_k^2 \gtrsim \frac{1}{4\Gamma_{\text{meas}}^{(k)} \tau_k} = \frac{\kappa}{16 \eta_{\text{det}} g_0^2 n_c^{(k)} \tau_k}. \quad (24)$$

Equivalently, the required intracavity photon number is

$$n_c^{(k)} \gtrsim \frac{\kappa}{16 \eta_{\text{det}} g_0^2} \frac{1}{\sigma_k^2 \tau_k}. \quad (25)$$

(3) *Relating  $n_c$  to input power and laser work.* For a driven cavity on resonance, the steady-state intracavity photon number is proportional to input power. We write

$$n_c^{(k)} = \chi_P \frac{P_{\text{in},k}}{\hbar \omega_L \kappa}, \quad (26)$$

where  $\omega_L$  is the laser frequency and  $\chi_P$  is an order-unity coupling factor capturing external coupling, mode matching, and detuning conventions (e.g.,  $\chi_P \simeq 4\kappa_{\text{ex}}/\kappa$  on resonance). The experimentally measurable work injected by the laser during step  $k$  is

$$W_k \equiv P_{\text{in},k} \tau_k. \quad (27)$$

Combining Eqs. (25)–(27) yields a lower bound on the step work:

$$W_k \gtrsim \frac{\hbar \omega_L \kappa^2}{16 \chi_P \eta_{\text{det}} g_0^2} \frac{1}{\sigma_k^2}. \quad (28)$$

Equation (28) makes explicit the chain  $CRB \rightarrow \text{measurement rate} \rightarrow \text{photon number} \rightarrow \text{laser work}$ .

(4) *Marginal schedule and logarithmic cumulative work.* The marginal universality hypothesis predicts that the *cumulative* operational cost versus refinement count exhibits a logarithmic signature in an intermediate “marginal window.” Operationally, this corresponds to a designed schedule in which the required incremental work decreases harmonically,

$$W_k \simeq \frac{B_{\text{meas}}}{k}, \quad (29)$$

so that

$$\mathcal{W}_N \equiv \sum_{k=1}^N W_k \simeq B_{\text{meas}} \sum_{k=1}^N \frac{1}{k} = B_{\text{meas}} \ln N + \mathcal{O}(1). \quad (30)$$

A sufficient condition for Eq. (29) is that the target variance schedule satisfies  $\sigma_k^{-2} \propto 1/k$  (with fixed  $\tau_k$ ), or more generally that the product of estimator precision and integration time obeys  $\sigma_k^{-2} \tau_k \propto 1/k$ . In this case, Eq. (28) yields the explicit logarithmic coefficient

$$B_{\text{meas}} \simeq \frac{\hbar \omega_L \kappa^2}{16 \chi_P \eta_{\text{det}} g_0^2} \chi \quad (31)$$

where  $\chi = \mathcal{O}(1)$  collects protocol-dependent constants from filtering, quadrature choice, and the precise mapping between estimator variance and the effective Fisher information. Importantly,  $B_{\text{meas}} \propto 1/\eta_{\text{det}}$  and  $B_{\text{meas}} \propto \kappa^2/g_0^2$  are direct, experimentally checkable parameter dependences.

(5) *Order-of-magnitude estimate (work scale).* Using Eq. (31) with representative values (to be replaced by the target platform parameters), one expects a readily measurable work scale per  $\ln N$  decade provided the cavity-drive overhead dominates thermal drifts. In practice, the slope  $B_{\text{meas}}$  is obtained by fitting  $\mathcal{W}_N$  over the marginal window.

### III-D. Deviations: finite efficiency and control latency

Finite detection efficiency simply rescales the coefficient in Eq. (31):

$$B_{\text{meas}}(\eta_{\text{det}}) = \frac{B_{\text{meas}}(\eta_{\text{det}}=1)}{\eta_{\text{det}}}. \quad (32)$$

Classical feedback and digital control overhead introduce a drift term. If each step incurs a fixed latency  $\tau_{\text{delay}}$  with an average power overhead  $P_{\text{ctrl}}$ , then an additional linear contribution appears:

$$\mathcal{W}_N \approx B_{\text{meas}} \ln N + C_{\text{lat}} N, \quad C_{\text{lat}} \equiv P_{\text{ctrl}} \tau_{\text{delay}}. \quad (33)$$

The “marginal window” is operationally defined as the  $N$ -range where the logarithmic term dominates over the linear drift.

### III-E. Falsifiability: model comparison (AIC/BIC) and decision criteria

We propose the following falsifiable protocol:

1. Initialize the optomechanical system in a fixed thermal condition and repeat the adaptive measurement protocol for  $N$  steps over multiple runs.
2. Record the cumulative injected work  $\mathcal{W}_N = \sum_{k=1}^N P_{\text{in},k} \tau_k$  (and optionally control overhead).
3. Fit  $\mathcal{W}_N$  to competing models over the candidate marginal window:

$$\text{(M) Log model: } \mathcal{W}_N = A + B \ln N, \quad (34)$$

$$\text{(P) Power model: } \mathcal{W}_N = A' + D N^\nu, \quad (35)$$

$$\text{(L) Log+latency: } \mathcal{W}_N = A'' + B \ln N + C_{\text{lat}} N. \quad (36)$$

4. Use information criteria (AIC/BIC) to select the best model, penalizing additional parameters.
5. **Acceptance criterion:** A statistically preferred logarithmic component with slope consistent with Eq. (31) under independently measured  $(g_0, \kappa, \eta_{\text{det}}, \chi_P)$  supports the marginal universality hypothesis. Dominance of a power-law term across the tested range falsifies the hypothesis.