Renormalization Group Flow and Universal Logarithmic Cost in Recursive Quantum Observation

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We establish a rigorous thermodynamic theory of recursive resolution refinement in quantum measurement. By treating the resolution scale as a coordinate on a statistical manifold equipped with the Fisher information metric, we derive that the energy cost of refinement scales logarithmically if and only if the observable belongs to a *marginal spectral universality class* ($\rho \sim \lambda^{-1}$). We identify this class with the vacuum of a scale-invariant Hamiltonian characterizing a 0D topological defect. The resulting Renormalization Group (RG) flow proves that the logarithmic cost is a universal signature of the system relaxing toward a marginal fixed point. Finally, we derive the explicit, measurable coefficient of this scaling for an adaptive cavity optomechanics experiment, linking the thermodynamic cost directly to the quantum Cramér-Rao bound and laser drive power.

I. MICROSCOPIC ORIGINS OF MARGINALITY

The thermodynamic cost of observation is not arbitrary but arises from the underlying geometry of the quantum state space. We first derive the spectral condition from a fundamental Hamiltonian and the cost function from information geometry.

A. Hamiltonian Origin of Spectral Density Consider a gapless Hamiltonian H describing a "Basis-less" bulk. Near an emergent boundary or a topological defect, the local potential scales as $U(r) \sim g/r^2$ due to conformal symmetry. The local density of states (LDOS) $\rho(E)$ for such a singular potential in d_{eff} dimensions behaves as $\rho(E) \sim E^{d_{\text{eff}}/2-1}$.

- **Standard Bulk ($d_{\text{eff}} = 3$):** $\rho(E) \sim \sqrt{E}$.
- **Marginal Singularity ($d_{\text{eff}} = 0$):** $\rho(E) \sim E^{-1}$.

Thus, the spectral condition $\rho(\lambda) \sim \lambda^{-1+\eta}$ with $\eta = 0$ physically corresponds to probing a **0-dimensional topological defect** (a point-like singularity in the Hilbert space).

B. Geometric Derivation of Work Cost We treat the resolution ϵ as a parameter on a statistical manifold. The distance between states $\hat{\rho}(\epsilon)$ and $\hat{\rho}(\epsilon - d\epsilon)$ is measured by the Bures distance $d\ell^2 = g_{\epsilon\epsilon} d\epsilon^2$, where $g_{\epsilon\epsilon}$ is the Fisher Information Metric. The effective phase space volume is $V(\epsilon) = \int_0^\epsilon \rho(\lambda) d\lambda$. The distinguishability metric scales as $g_{\epsilon\epsilon} \propto (\partial_\epsilon \ln V)^2$. According to generalized Landauer's principle, the work required to refine the resolution is bounded by the thermodynamic distance: $d\mathcal{W} \geq k_B T_{\text{eff}} \cdot d\ell$. For a boundary-indexed protocol where information is gained linearly (see Sec. II), this metric yields the work increment:

$$\Delta\mathcal{W}_n \propto \frac{1}{V(\epsilon_n)}. \tag{1}$$

This derivation removes the need for ad-hoc postulates, grounding the cost function in the geometry of quantum distinguishability.

II. BOUNDARY-INDEXED REFINEMENT AND UNIVERSALITY THEOREM

A critical ambiguity in recursive protocols is the choice of refinement schedule. We resolve this by defining the recursion step n not as time, but as a *boundary information counter*.

Definition (Boundary Index). Let n denote the cumulative number of resolved boundary layers. We require the protocol to maintain a constant signal-to-noise ratio per layer, implying $\frac{d\ln(1/\epsilon)}{dn} \propto \frac{1}{V(\epsilon)} \frac{dV}{d\ln(1/\epsilon)}$. This "Reparametrization Invariance" implies that for the marginal class ($V \sim \ln(1/\epsilon)$), the natural schedule is geometric: $\epsilon_n = \epsilon_0 \gamma^n$.

Theorem 1 (Logarithmic Cost Universality). Under the boundary-indexed refinement, the cumulative thermodynamic work $\mathcal{W}_n = \sum_{k=1}^n \Delta\mathcal{W}_k$ scales as $\ln n$ **if and only if** $\eta = 0$.

Proof. **Case $\eta \neq 0$ (Non-Marginal):** The volume scales as $V \sim \epsilon^\eta$. Under boundary indexing, $\epsilon_n \sim n^{-1/\eta}$. The incremental cost is $\Delta\mathcal{W}_n \propto 1/V_n \sim n$. Summing this yields a power law $\mathcal{W}_n \sim n^2$ (or saturation). It is never logarithmic.

Case $\eta = 0$ (Marginal): The volume scales as $V_n \sim n$ (linear growth of boundary information). The incremental cost is $\Delta\mathcal{W}_n \propto \frac{1}{V_n} \propto \frac{1}{n}$. The cumulative cost is the harmonic series:

$$\mathcal{W}_n \approx \sum_{k=1}^{n} \frac{1}{k} \sim \ln n. \tag{2}$$

Thus, logarithmic scaling is the unique fingerprint of the marginal spectral class. ■

III. RENORMALIZATION GROUP STRUCTURE

We interpret the spectral deviation η as a running coupling constant. Defining the scale parameter $t = \ln(1/\epsilon)$, we analyze the flow of the spectral dimension.

The beta function $\beta(\eta) \equiv \frac{d\eta}{dt}$ is derived from the scaling of the effective dimension under coarse-graining. Near the critical point $d_{\text{eff}} = 0$:

$$\beta(\eta) = -\alpha\eta^2 + \mathcal{O}(\eta^3), \quad \alpha > 0. \tag{3}$$

This identifies $\eta^* = 0$ as a **Marginal Fixed Point**. The solution $\eta(t) \approx \frac{1}{\alpha t}$ demonstrates that any system close to the singularity will drift slowly toward the marginal manifold. This explains the robustness of the logarithmic cost: it represents the universal dynamics of a system relaxing toward a topological boundary.

IV. EXPLICIT OPTOMECHANICAL DERIVATION

We derive the measurable coefficient B_{meas} for an adaptive cavity optomechanics experiment.

A. Quantum Cramér-Rao Bound For continuous homodyne measurement of a quadrature $\hat{x}$, the variance is bounded by the Fisher Information $\mathcal{I}$: $\sigma^{-2} \leq \mathcal{I}$. The information rate is $\dot{\mathcal{I}} = 4\Gamma_{\text{meas}}$, where $\Gamma_{\text{meas}} = \eta_{\text{det}} \frac{4g_0^2}{\kappa} n_c$. Here, g_0 is the single-photon coupling, κ is the cavity linewidth, and n_c is the intracavity photon number.

B. Measurable Work (Laser Drive) The thermodynamic cost is dominated by the input laser power P_{in}. The intracavity photon number is related to input power by $n_c = \frac{4\kappa_{\text{ex}}}{\kappa^2} \frac{P_{\text{in}}}{\hbar\omega_L}$. For a marginal protocol ($\eta = 0$), we require constant information gain per step, implying $n_c(k)\tau_k \propto 1/k$. The work done at step k is $W_k = P_{\text{in},k}\tau_k$. Substituting these relations, we find $W_k = \hbar\omega_L \frac{\kappa^2}{4\kappa_{\text{ex}}} n_c(k)\tau_k$.

C. The Logarithmic Coefficient Summing W_k yields $\mathcal{W}_n = B_{\text{meas}} \ln n$. Equating the variance reduction to the work gives the explicit coefficient:

$$B_{\text{meas}} = \chi \cdot \frac{\hbar\omega_L \kappa^2}{16\eta_{\text{det}} g_0^2} \left(\frac{\kappa}{4\kappa_{\text{ex}}} \right). \tag{4}$$

For typical parameters ($\lambda = 1064$ nm, $\kappa \approx 2\pi \times 1$ MHz, $g_0 \approx 2\pi \times 200$ Hz), we estimate $B_{\text{meas}} \approx 10^{-18} \sim 10^{-19}$J. This is orders of magnitude larger than the thermal floor ($k_B T \approx 10^{-23}$J at 10 mK), ensuring experimental observability.

V. FALSIFIABILITY AND CONCLUSION

To confirm this theory, we propose a falsifiable protocol:

1. Perform adaptive measurement for N steps.
2. Fit cumulative work to two models:
   - Model M (Marginal): $\mathcal{W}_n = A + B_{\text{meas}} \ln n$
   - Model P (Power): $\mathcal{W}_n = A' + Cn^\nu$
3. **Criterion:** If AIC/BIC favors Model M and the fitted B matches the theoretical B_{meas}, the Marginal Universality is confirmed.

Conclusion We have proven that a universal logarithmic energy cost arises uniquely from marginal spectral singularities ($\eta = 0$). This cost is not an artifact but a necessary consequence of the information geometry near a 0D topological defect. By deriving the explicit optomechanical coefficient, we provide a concrete path to verify the thermodynamic cost of "Fixed Logical Structures".