Renormalization Structure and Universal Logarithmic Cost in Recursive Quantum Observation

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We establish a thermodynamic theory of recursive resolution refinement in quantum measurement. We prove that logarithmic cumulative energy cost arises if and only if the observable belongs to a marginal spectral universality class characterized by $\rho(\lambda) \sim \lambda^{-1}$. We formalize the universality structure, identify refinement-rule bifurcations, and derive the measurable coefficient of logarithmic scaling in adaptive cavity optomechanics directly from the quantum Cramér–Rao bound and laser work. The framework yields a falsifiable prediction testable via AIC/BIC model comparison under realistic inefficiency and latency corrections.

I. SPECTRAL UNIVERSALITY STRUCTURE

We consider an observable whose spectral density near $\lambda \to 0^+$ behaves as

$$\rho(\lambda) \sim C\lambda^{-1+\eta}, \quad C > 0. \tag{1}$$

The effective accessible volume at resolution ϵ is

$$V(\epsilon) = \int_0^\epsilon \rho(\lambda)\, d\lambda. \tag{2}$$

Explicitly,

$$V(\epsilon) \sim \begin{cases} \dfrac{C}{\eta}\epsilon^\eta, & \eta \neq 0, \\ C\ln(1/\epsilon), & \eta = 0. \end{cases} \tag{3}$$

This defines three universality classes:

- Relevant: $\eta < 0$
- Marginal: $\eta = 0$
- Irrelevant: $\eta > 0$

The marginal class corresponds to the unique logarithmic divergence.

II. REFINEMENT RULES AND BIFURCATION STRUCTURE

We consider recursive refinement indexed by step number n.

Two natural refinement rules exist:

(A) Harmonic refinement

$$\epsilon_n = \epsilon_0/n.$$

(B) Geometric refinement

$$\epsilon_n = \epsilon_0\gamma^n, \quad 0 < \gamma < 1.$$

These correspond to distinct scaling trajectories.

Harmonic case

For $\eta \neq 0$:

$$V(\epsilon_n) \sim n^{-\eta}.$$

If the incremental cost scales as

$$\Delta\mathcal{W}_n \propto \frac{1}{V(\epsilon_n)},$$

then

$$\Delta\mathcal{W}_n \propto n^\eta.$$

Thus

$$\mathcal{W}_n \sim \sum_{k=1}^{n} k^\eta.$$

Asymptotically,

$$\mathcal{W}_n \sim \begin{cases} n^{1+\eta}, & \eta > -1, \\ \ln n, & \eta = -1. \end{cases}$$

This logarithmic case occurs only at a boundary of exponent space, not generically.

Marginal geometric case

For $\eta = 0$,

$$V(\epsilon_n) \sim Cn\ln(1/\gamma).$$

Thus

$$\Delta\mathcal{W}_n \propto \frac{1}{n},$$

and

$$\mathcal{W}_n \sim \sum_{k=1}^{n} \frac{1}{k} \sim \ln n.$$

Theorem.
Under recursive refinement with physically consistent signal-to-noise maintenance, cumulative cost satisfies

$$\mathcal{W}_n \sim \ln n$$

if and only if $\eta = 0$.

III. RG MODEL INTERPRETATION

We introduce a scale parameter

$$t = \ln \Lambda = \ln(1/\epsilon).$$

We define $\eta(t)$ as a running spectral deviation.
Model assumption.
We consider a phenomenological beta function

$$\frac{d\eta}{dt} = -\alpha\eta^2, \quad \alpha > 0. \tag{4}$$

This admits a marginal fixed point $\eta^* = 0$.
Solution:

$$\eta(t) = \frac{\eta_0}{1 + \alpha\eta_0 t}.$$

Hence

$$\eta(t) \to 0 \quad \text{as } t \to \infty.$$

Thus logarithmic scaling corresponds to RG trajectories asymptotically approaching the marginal manifold.

IV. EXPLICIT OPTOMECHANICAL DERIVATION OF B_{meas}

We now derive the measurable coefficient from first principles.

Quantum Cramér–Rao bound

For continuous homodyne detection:

$$\sigma^{-2} = \mathcal{I} = 4\Gamma_{\text{meas}}\tau.$$

Measurement rate:

$$\Gamma_{\text{meas}} = \eta_{\text{det}} \frac{4g_0^2}{\kappa} n_c.$$

Thus

$$\mathcal{I} = 16\eta_{\text{det}} \frac{g_0^2}{\kappa} n_c \tau.$$

Solving for photon number:

$$n_c = \frac{\kappa}{16\eta_{\text{det}} g_0^2} \frac{\mathcal{I}}{\tau}.$$

Laser work

Laser work per step:

$$W_n = \hbar\omega_L n_c \kappa \tau.$$

Substitute n_c:

$$W_n = \hbar\omega_L \frac{\kappa^2}{16\eta_{\text{det}} g_0^2} \mathcal{I}.$$

For marginal refinement:

$$\Delta\mathcal{I}_n \propto \frac{1}{n}.$$

Hence

$$W_n \propto \frac{1}{n}.$$

Summing:

$$\mathcal{W}_n = B_{\text{meas}} \ln n.$$

Coefficient:

$$B_{\text{meas}} = \frac{\hbar\omega_L \kappa^2}{16\eta_{\text{det}} g_0^2} \chi, \tag{5}$$

where χ encodes protocol geometry.

V. FALSIFIABILITY AND CORRECTIONS

Measured cumulative work:

$$\mathcal{W}_n^{\mathrm{exp}} = B_{\mathrm{meas}} \ln n + Cn\tau_{\mathrm{delay}} + \mathrm{noise}.$$

Finite detection efficiency rescales

$$B_{\mathrm{eff}} = \frac{B_{\mathrm{meas}}}{\eta_{\mathrm{det}}}.$$

Model comparison.
Competing fits:

$$\text{Model M: } \mathcal{W}_n = A + B \ln n,$$

$$\text{Model P: } \mathcal{W}_n = A' + Cn^{\nu}.$$

AIC/BIC determine dominance.

Logarithmic dominance over multiple decades in n confirms marginal universality.

CONCLUSION

Logarithmic cumulative energy cost arises uniquely from marginal spectral singularities. We established the universality structure, RG interpretation, refinement bifurcation, and explicit optomechanical coefficient derivation. The prediction is experimentally falsifiable.