A Robust Fourth Order Partial Differential Equation For Image Restoration

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***Abstract*—A hybrid fourth order partial differential equation (PDE) of nonlinear diffusion and bilateral filtering based image restoration is presented in this paper. The proposed model combines nonlinear diffusion fourth order PDEs with bilateral filtering and inherits their advantages, which not only preserves edge and robust to noise but also avoids the staircase effects. Experiment results show the effectiveness of the proposed model and demonstrate its superiority to the existing models.**

Keywords- Fourth order partial differential equation (PDE); nonlinear diffusion; image restoration; bilateral filtering.

I. INTRODUCTION

Image restoration is historically one of the oldest concerns in image processing and is still an active research topic for many civilian and military applications, such as traffic image, satellite image, medical image, astronomical image [1]. It is well known that images are slightly blurred and noised during formation, recording, and transmission processes. So image restoration aims at deblurring and denoising the observed images.

A commonly used model is the following. Let $\Omega \subset R^2$ is a bounded domain, $u:\Omega \to R$ be an original image describing the real scene, and u_0 be the observed image of the same scene. The relationship between the observed image u_0 and the real image u is expressed by the following linear equation

$$u_0 = Au + n, \tag{1}$$

where $A: L^2(\Omega) \to L^2(\Omega)$ is a blur operator (linear and bounded) and n is the additive noise. In many applications, additive Gaussian white noise is often assumed.

The goal of image restoration is to recover u as well as possible from the observed image u_0 .The deblurring and denoising problem is ill conditioned since degraded u_0 and the real images u are typically related through a blur operator A . A widely used way to overcome ill-posed minimization problem is to add a regularization term to the energy. This idea was first introduced in 1977 by Tikhonov and Arsenin [2].

In [3], Rudin, Osher and Fatemi proposed the following *total variation* (TV) based image deblurring and denoising model:

$$\frac{\partial u}{\partial t} = \lambda \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + (A^* u_0 - A^* A u), \tag{3}$$

where $\nabla \cdot$ denotes the divergence. Eq. (3) is usually called TV model. In order to regularize the parabolic term, Marquina and Osher [4] proposed another model:

$$\frac{\partial u}{\partial t} = |\nabla u|\, \lambda \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u|\,(A^* u_0 - A^* A u)\,. \tag{4}$$

which is called the *improved total variation* (ITV) model [5]. The TV model, together with its improved version, has been extensively studied [6,7, 8]. It was designed with explicit goal of preserving sharp edges in images while removing noise and other unwanted fine scale detail. However, the TV model is a second order model. This feature guaranties its ability to reconstruct images with discontinuities, but is responsible for the staircase effect [10, 11].

Most recently, Prasath and Singh [9] proposed the following adaptive model:

$$\frac{\partial u}{\partial t} = R(|\nabla u|) + (A^* u_0 - A^* A u) \tag{5}$$

where

$$R(|\nabla u|) = \begin{cases} 2a\nabla \cdot (\lambda(x)|\nabla u|), \, for \, |\nabla u| \le M, \\ 2b\nabla \cdot (\lambda(x)|\nabla u|) + c\nabla \cdot (\lambda(x)\frac{\nabla u}{|\nabla u|}) + d, \, for \, |\nabla u| > M, \end{cases} \tag{6}$$

where $\lambda(x)$ is a continuous function on Ω, $a, b, c, d > 0$, and M is the threshold specifying the relative edge strength. The threshold is dependent on the image content and noise, which has to be appropriately tuned for every new case. This

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threshold lets in noise and at high values penalizes weak edges when fixed at low values.

For $|\nabla u| \le M$, model (6) belongs to a class of anisotropic diffusion equations first proposed by Perona-Malik [19] in the form of

$$\frac{\partial u}{\partial t} = \lambda \nabla \cdot (g(|\nabla u|)\nabla u) + A^*(u_0 - Au), \tag{7}$$

where $g(\cdot)$ is a diffusion coefficient function; For $|\nabla u| > M$, model (6) belongs to a class of equations combining anisotropic diffusion equations with TV flow. Model (6) produces excellent results when compared with stand-alone anisotropic diffusion or TV model. But it still has some staircase effects which inherit from anisotropic diffusion equation and TV model.

Motivated by some recent results of related problems (see [11, 12, 18]), we propose a hybrid fourth order partial differential equation (PDE) by combining fourth order PDE and bilateral filtering for image denoising and deblurring. The proposed model extends the fourth order PDE for noise removal [11, 12] to denoising and deblurring problems. Moreover, our proposed model combines fourth order PDEs with bilateral filtering, which connects fourth order PDEs with the bilateral filtering, and inherits their advantages. Experiment results show that the proposed model can avoid the staircase effects well while achieving good tradeoff between noise removal and edge preservation.

II. PROPOSED METHOD

In the recent decade, a number of authors have presented analogous fourth order partial differential equations (PDE) for image de-noising [12-17]. The theoretical analysis in [16] show that fourth order equations have advantages over second order equations in some aspect. First, fourth order linear diffusion damps oscillations much faster than second order diffusion. Second, since second order PDE will evolve toward a piecewise constant approximation in smooth regions, unlike second order PDE, fourth order PDE will evolve toward and settle down to an piecewise smooth image if the image support is infinite. It is well known that piecewise smooth images look more natural than the piecewise constant images [11]. Therefore, the staircase effect will be reduced and the image will look more nature. Some new fourth order methods combine TV model with some other methods to preserve edges (see [16, 17,]). However, recent proposed fourth order PDEs methods only concentrated on image denoising, which didn't consider deblurring and denoising simultaneously.

Inspired by the ideas of [11, 12], we propose the follow fourth order PDE deblurring and denoising model:

$$\frac{\partial u}{\partial t} = -\lambda \nabla^2 (g(|\nabla^2 u|)\nabla^2 u) + A^*(u_0 - Au), \tag{8}$$

where $T > 0$, and ∇^2 denotes Laplacian operator. To our best knowledge, second order derivative is sensitive to the noise. While one may find when the image is very noisy, (1) will be unstable which could not distinguish correctly the "true" edges and "false" edges. Considering that we can eliminate some noise before solving model (1), we reformulate model (1) as follows:

$$\frac{\partial u}{\partial t} = -\lambda \nabla^2 (g(|\nabla^2 B_\sigma * u|)\nabla^2 u) + A^*(u_0 - Au), \tag{9}$$

where B_σ is a bilateral filtering [18], namely

$$B_\sigma(u(x)) = \frac{1}{W(x)} \int_\Omega G_{\sigma_s}(\xi, x) G_{\sigma_r}(u(\xi), u(x)) * u(\xi) d\xi, \tag{10}$$

with the normalization constant

$$W(x) = \int_\Omega G_{\sigma_s}(\xi, x) G_{\sigma_r}(u(\xi), u(x)) d\xi, \tag{11}$$

where G_{σ_s} will be a spatial Gaussian that decreases the influence of distant pixels, while G_{σ_r} will be a range Gaussian that decreases the influence of pixels ξ with intensity values that are very different form those of $u(x)$, e. g.

$$G_{\sigma_s} = \exp(-\frac{|\xi - x|^2}{2\sigma_s^2}), G_{\sigma_r} = \exp(-\frac{|u(\xi) - u(x)|^2}{2\sigma_r^2}), \tag{12}$$

where parameters σ_s and σ_r dictate the amount of filteringapplied in the domain and the range of the image, respectively.

We shall recall that the main purpose of the function g is to provide adaptive smoothing. It should not only precisely locate the position of the main edges, but it also inhibits diffusion at edges and allows it far from them. This is exactly what bilateral filtering accomplishes. It is proposed as a tool to reduce noise and preserve edges, by means of exploiting all relevant neighborhoods. It combines gray levels not only based on their gray similarity but also their geometric closeness, and prefers near values to distant values in both domain and range.

III. DISCREDITED NUMERICAL SCHEME

In this section, we construct an explicit discrete scheme to numerically solve differential equation (11). Let $u_{i,j}^n$ be the approximation to the value $u(x_1, x_2, t_n)$, for $x_1 = 0, 1, \cdots, M-1$ and $x_2 = 0, 1, \cdots, N-1$, denotes an image of size $M \times N$.

(a) Calculating the Laplacian of the image intensity functions

$$\nabla^2 u_{i,j}^n = 0.5 \left[\frac{u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j-1}^n + u_{i,j+1}^n - 4u_{i,j}^n}{h^2} \right]$$

$$+0.25\left[\frac{u_{i+1,j+1}^n+u_{i-1,j+1}^n+u_{i+1,j-1}^n+u_{i-1,j-1}^n-4u_{i,j}^n}{h^2}\right]$$

with symmetric boundary conditions

$$u_{-1,j}^n=u_{0,j}^n, u_{N+1,j}^n=u_{N,j}^n,\ j=0,1,2,\cdots,N\ , \tag{13}$$

$$u_{i,-1}^n=u_{i,0}^n, u_{i,M+1}^n=u_{i,M}^n\ ,\ i=0,1,2,\cdots,M\ , \tag{14}$$

(b) Calculating the value of the following function

$$\varphi_{i,j}^n=(A^*A(u)-A^*u_0)_{i,j}^n\ , \tag{15}$$

$$\phi_{i,j}^n=\lambda g(|\nabla^2 B_\sigma * u_{i,j}^n|)\nabla^2 u_{i,j}^n\ . \tag{16}$$

(c) Calculating the Laplacian of $\phi_{i,j}^n$ as

$$\nabla^2\phi_{i,j}^n=0.5\left[\frac{\phi_{i+1,j}^n+\phi_{i-1,j}^n+\phi_{i,j-1}^n+\phi_{i,j+1}^n-4\phi_{i,j}^n}{h^2}\right]$$

$$+0.25\left[\frac{\phi_{i+1,j+1}^n+\phi_{i-1,j+1}^n+\phi_{i+1,j-1}^n+\phi_{i-1,j-1}^n-4\phi_{i,j}^n}{h^2}\right]$$

with symmetry boundary conditions

$$\phi_{-1,j}^n=\phi_{0,j}^n, \phi_{N+1,j}^n=\phi_{N,j}^n,\ j=0,1,2,\cdots,N\ , \tag{17}$$

$$\phi_{i,-1}^n=\phi_{i,0}^n, \phi_{i,M+1}^n=\phi_{i,M}^n\ ,\ i=0,1,2,\cdots,M\ . \tag{18}$$

(d) Finally, the numerical approximation to the differential equation (11) is given as

$$u_{i,j}^{n+1}=u_{i,j}^n-\Delta t(\nabla^2\phi_{i,j}^n+\varphi_{i,j}^n)\ , \tag{19}$$

where Δt denotes time step size.

IV. EXPERIMENTAN RESULTS

To verify the effective of our proposed model (8) and (9), we compare it with ITV model [5] and model (6) [10]. The values chosen for the various parameters are $\sigma_s=5$, $\sigma_r=0.1$, $\Delta t=0.1$, $\lambda=0.5$ and $h=1$. A general linear blur is in the form of $Au(x,y)=h*u(x,y)$, $(x,y)\in\Omega$, where h is called the blur kernel function. The kernel h is determined as the PSF of the sensor, which could be unknown of the problem. Gaussian blur kernel is often used to model the thermal properties of CCD image acquisition sensors [1], defined as

$$h(x,y)=\frac{1}{2\pi\gamma^2}\exp(-\frac{x^2+y^2}{2\gamma^2})\ ,\ (x,y)\in\Omega\ , \tag{20}$$

where γ is a positive real constant.

In the following experiments, we adopt improvement in signal to noise ratio (ISNR) is used to measure the quality of the restored image [8]

$$ISNR=10\cdot\log 10(\frac{\sum_{i,j}[u(i,j)-u_0(i,j)]^2}{\sum_{i,j}[u(i,j)-u_{new}(i,j)]^2})\ , \tag{21}$$

where $u_0(\cdot)$ is the initial image (noised and blurred image) and $u_{new}(\cdot)$ is the restored image. The value of ISNR is large, the restored image is better.

Our numerical experiment is performed on the License Plate image (Fig. 1 (a)) and dynamic range in [0,255]. The noisy and blurry image is shown in Fig. 1 (b) generated by convolving Fig. 1 (a) with a 129×129 discrete approximation of a Gaussian blur kernel h defined from (20) setting $\gamma=5$ and adding Gaussian white noise with variance $\sigma = 25$. Table 1 gives a qualitative comparison for different models. For the ITV model, ISNR is 7951; For the model (6), 2.5568; For model (8) without BF, ISNR is 3.1437; For model (9) with BF, ISNR is 3.7116. From the standpoint of ISNR value, our proposed model produced the best quality. From the standpoint of perceptual view, Fig. 4 (c) shows the result obtained by using ITV model. The output of the ITV model is blurred, and some of the numbers are not at all discernible. For example, the last number "2" confused, while the number "8" can be misinterpreted as "B". The result restored by using model (6) is shown in Fig. 4 (d), which performs relatively better, but some of the letters are still confused. For comparison, our proposed model yields the best result with distinctly preserve edges, as shown in Fig. 4 (e)-(f). But model (9) with BF can get larger value of ISNR and better perceptual quality than model (8) without BF.

TABLE I. ISNR VALUES OF THE RESOTRED IMAGE FOR DIFFERENT MODELS

	ITV	Model (6)	Model (8)	Model (9)
ISNR(dB)	7951	2.5568	3.1437	3.7116

V. CONCLUSIONS

In this paper, a new hybrid fourth order partial differential equation (PDE) for image restoration is proposed based on fourth order PDE and bilateral filtering. The proposed model extends the fourth order PDEs for noise removal model [11, 12] to denoising and deblurring problems. Moreover, our proposed model combines nonlinear diffusion fourth order PDEs with bilateral filtering and inherits their advantages. Numerical results indicate that the proposed model recovers well edges and reduces noise.

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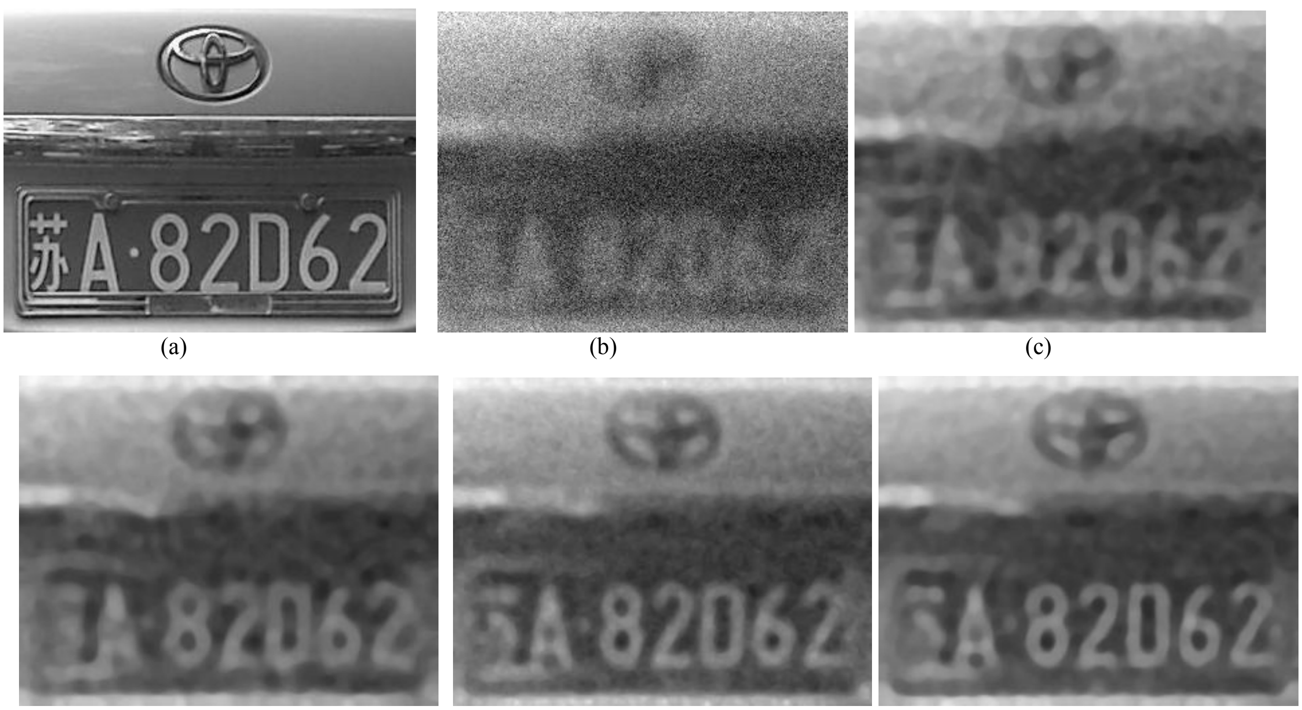

Fig. 1.**.** Comparison of different models on License Plate image. (a) Original image; (b) Noisy and blurry image; (c) ITV model; (d) Model (6); (e) Model (8); (f) Model (9)