DISCRETIZATION EFFECTS IN THE FUNDAMENTAL MATRIX COMPUTATION

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ABSTRACT

A polyhedron represents the solution set of an approximate system modeling the epipolar constraints. We introduce a new robust approach for the computation of the fundamental matrix taking into account the intrinsic errors involved in the discretization process. The problem is modeled as an approximate equation system and reduced to a linear programming form. This approach is able to compute the solution set instead of trying to compute only a single vertex of the solution polyhedron as in previous approaches. Outliers are considered as sample point matches whose errors are much bigger that the expected uncertainty ϵ. We suggest ways to deal with outliers and present an analysis with experiments in synthetic images.

Index Terms— fundamental matrix, discretization effects

1. INTRODUCTION

An *image* is a continuous two-dimensional function of brightness corresponding to the projection of a three-dimensional scene onto a plane. The spatial discretization of the continuous image samples a finite number of picture cells (pixels) to represent the image as a discrete structure. Each *pixel* in this structure is associated with the average irradiance over a small sampling area in the image plane. The discrete image is a rectangular array of pixels.

The reconstruction of a three-dimensional scene from a pair of discrete images requires, among other things, the estimation of the *fundamental matrix* [1]. The fundamental matrix encapsulates the projective geometry between two images, depending only on the internal parameters of the camera and on the relative pose. If a scene point P is projected as p in one image and p' in the other, then the image points satisfy the equation $p'^T F p = 0$, where F is a 3×3 matrix of rank 2.

A set of point correspondences between the two images is given to compute the fundamental matrix. Each point match gives rise to one linear equation in the entries of F known as the *epipolar constraint* [2]. More specifically, let p be the point $(x, y, 1)^T$ and p' be $(x', y', 1)^T$, the equation corresponding to the pair of points p and p' is

$$\begin{aligned} x'xf_{11} + x'yf_{12} + x'f_{13} &+ \\ y'xf_{21} + y'yf_{22} + y'f_{23} &+ \\ xf_{31} + yf_{32} + f_{33} &= 0. \end{aligned}$$

The set of point matches gives rise to a homogeneous system of linear algebraic equations. Given a sufficient number of matches, a solution for this system is feasible and the unknown matrix F is computed by the eight-point algorithm [3] or by the RANSAC method [4].

There are two major challenges to this approach for the computation of the fundamental matrix: incorrect point correspondences (outliers) and the lack of precision in the pixel coordinates of points used in the epipolar constraint. The wide-baseline stereo correspondence problem [5] concerns finding point matches between two images of the same scene. This is an ill-posed problem where difficulties arise from textureless regions, depth discontinuities, and image noise. These issues lead to incorrect matches between image points associated with different points in the scene. Even assuming the correspondence problem is solved so that outliers are completely avoided, the discrete nature of images inserts some amount of error in the pixel coordinates. Stereo correspondence techniques that achieve sub-pixel precision are not able to overcome this discretization error. Since real models of computation are finite, the discrete structure of images is unavoidable and, consequently, these discretization errors are intrinsically embedded in image point coordinates.

If any of the coefficients and right-hand constants in a system of equations are not known exactly, then the system is called *approximate*. The study of the effects of uncertainties in the coefficients and constants on the solution of an approximate system is called *approximate equation analysis* [6].

In this paper, we introduce a new robust approach for the computation of the fundamental matrix. Our approach considers explicitly the discretization errors embedded in the coordinates of image points. The problem is modeled as an approximate equation system and reduced to a linear programming form [7]. A linear programming module is used in the algorithm that works as a search in solution space that visits each admissible orthant. The algorithm computes eight intervals of uncertainty that contains admissible solutions for the elements of the fundamental matrix. Therefore, this approach is able to compute the polyhedron representing the solution set while previous approaches only find a single vertex of this polyhedron. This vertex is at the center of a ball with some positive radius intercepting the most number of faces of the polyhedron, where each face corresponds to a point match. We propose methods to deal with outliers (*i.e.*, incorrect matches) by considering them as sample point matches whose errors are much bigger than an expected uncertainty ϵ.

This paper is organized as follows. In Section 2, a solution set characterization is performed. We present an approximate algorithm for the fundamental matrix computation in Section 3. Our experimental results are dicussed in Section 4. Finally, Section 5 highlights our results and future work.

2. SOLUTION SET CHARACTERIZATION

Since the discrete coordinates of points in an image are not exact, there is an error associated with each spatial measurement. Due to these errors, the epipolar constraint used to compute the fundamental matrix becomes an approximate equation. The solution set for a homogeneous approximate system is a polyhedron that contains the ray representing the exact solution, since the fundamental matrix can only be determined up to a scale.

We consider a projective camera model with an unit focal length. The first camera is centered at the origin of the three-dimensional space. The principal axis is coincident with the z-axis facing the positive half-space $Z > 0$. The second camera is related to the fist one according to a rigid motion. Let P be a three-dimensional point $(X, Y, Z, 1)^T$, the point P is projected onto points p and p' with pixel coordinates $(x, y, 1)^T = (\frac{X}{Z}, \frac{Y}{Z}, 1)^T$ in the first image and $(x', y', 1)^T$ in the second image, respectively.

The discretization process inserts errors into the coordinates of the points p and p'. Hence, the discrete pixel coordinates of p and p' are defined as $(x_\diamond, y_\diamond, 1)^T = (x + e_x, y + e_y, 1)^T$ and $(x'_\diamond, y'_\diamond, 1)^T = (x' + e_{x'}, y' + e_{y'}, 1)^T$, respectively, where e_x, e_y, $e_{x'}$, $e_{y'}$ are independent spatial errors whose absolute values are less than a constant uncertainty $\epsilon > 0$ (see Fig. 1).

Once a point is projected onto an image, the exact coordinates can only be defined as belonging to intervals centered at the pixels with radius equal to the uncertainty ϵ. Hence, the epipolar constraint corresponding to the pair of points p and

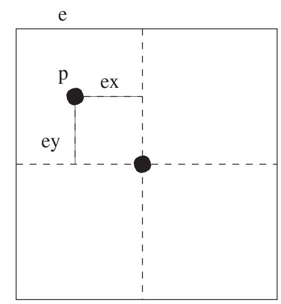

Fig. 1. Discretization process.

p' is an approximate equation[1]

$$\begin{aligned} \widehat{x'}\widehat{x}f_{11} + \widehat{x'}\widehat{y}f_{12} + \widehat{x'}f_{13} + \\ \widehat{y'}\widehat{x}f_{21} + \widehat{y'}\widehat{y}f_{22} + \widehat{y'}f_{23} + \\ \widehat{x}f_{31} + \widehat{y}f_{32} + f_{33} = 0, \end{aligned}$$

where $\widehat{x} = [x_\diamond - \epsilon, x_\diamond + \epsilon]$, $\widehat{y} = [y_\diamond - \epsilon, y_\diamond + \epsilon]$, $\widehat{x'} = [x'_\diamond - \epsilon, x'_\diamond + \epsilon]$, and $\widehat{y'} = [y'_\diamond - \epsilon, y'_\diamond + \epsilon]$.

According to interval arithmetic [8], the product of two intervals $\widehat{i} = [\underline{i}, \overline{i}]$ and $\widehat{j} = [\underline{j}, \overline{j}]$ is an interval defined as $\widehat{k} = [min\{\underline{i}\underline{j}, \underline{i}\overline{j}, \overline{i}\underline{j}, \overline{i}\overline{j}\}, max\{\underline{i}\underline{j}, \underline{i}\overline{j}, \overline{i}\underline{j}, \overline{i}\overline{j}\}]$. However, if $\underline{i} \geq 0$ and $\underline{j} \geq 0$, the product interval $\widehat{k}$ becomes $[\underline{i}\underline{j}, \overline{i}\overline{j}]$. Therefore, making these nonnegative assumptions ($x_\diamond \geq \epsilon, y_\diamond \geq \epsilon, x'_\diamond \geq \epsilon$, and $y'_\diamond \geq \epsilon$) without loss of generality, the approximate epipolar constraint becomes

$$\begin{array}{rl} [(x'_\diamond - \epsilon)(x_\diamond - \epsilon), (x'_\diamond + \epsilon)(x_\diamond + \epsilon)] & f_{11} + \\ + \, [(x'_\diamond - \epsilon)(y_\diamond - \epsilon), (x'_\diamond + \epsilon)(y_\diamond + \epsilon)] & f_{12} + \\ + \, [x'_\diamond - \epsilon, x'_\diamond + \epsilon] & f_{13} + \\ + \, [(y'_\diamond - \epsilon)(x_\diamond - \epsilon), (y'_\diamond + \epsilon)(x_\diamond + \epsilon)] & f_{21} + \\ + \, [(y'_\diamond - \epsilon)(y_\diamond - \epsilon), (y'_\diamond + \epsilon)(y_\diamond + \epsilon)] & f_{22} + \\ + \, [y'_\diamond - \epsilon, y'_\diamond + \epsilon] & f_{23} + \\ + \, [x_\diamond - \epsilon, x_\diamond + \epsilon] & f_{31} + \\ + \, [y_\diamond - \epsilon, y_\diamond + \epsilon] & f_{32} + \\ + & f_{33} = 0. \end{array}$$

A fundamental matrix is represented by a ray cf in the solution space, where c is a scalar and f is a vector $(f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{f32}, f_{33})$. The vector f specifies an orthant according to the signs of f_{ij} for $i, j = 1, ..., 3$. This orthant contains a nontrivial ($c \neq 0$) exact solution f. The admissible solution conditions for the

[1]The intervals associated with the unknowns of the approximate system are not independent. However, the assumption that these intervals are independent implies a solution set which contains the original solution set that considers interval dependency.

approximate equation in an orthant has the form

$$
\begin{aligned}
(x'_\diamond x_\diamond \pm \epsilon(x'_\diamond + x_\diamond) + \epsilon^2)f_{11} &+ \\
(x'_\diamond y_\diamond \pm \epsilon(x'_\diamond + y_\diamond) + \epsilon^2)f_{12} &+ \\
(x'_\diamond \pm \epsilon)f_{13} &+ \\
(y'_\diamond x_\diamond \pm \epsilon(y'_\diamond + x_\diamond) + \epsilon^2)f_{21} &+ \\
(y'_\diamond y_\diamond \pm \epsilon(y'_\diamond + y_\diamond) + \epsilon^2)f_{22} &+ \\
(y'_\diamond \pm \epsilon)f_{23} &+ \\
(x_\diamond \pm \epsilon)f_{31} &+ \\
(y_\diamond \pm \epsilon)f_{32} &+ \\
f_{33} &\geq 0.
\end{aligned}
$$

3. AN APPROXIMATE ALGORITHM FOR THE FUNDAMENTAL MATRIX

The true intervals of uncertainty of an approximate system can be found by using the necessary and sufficient conditions for admissible solutions and applying Linear Programming (LP) techniques. If one of the unknowns is specified then LP can be used to find the restricted intervals in which the remaining unknowns lie. In order to obtain the intervals of uncertainty in the unknowns of an approximate system of equations, the approximate equation problem is reduced to a standard LP form [6].

In order to satisfy the non-negativity condition of the LP form, the variables are changed according to a particular orthant. The variables $f'_j \geq 0$, for $j = 1, ..., m$ are introduced such that $f'_j = f_j$ if $f_j \geq 0$ and $f'_j = -f_j$ otherwise.

The inequality conditions for admissible solutions are expressed in terms of equations by introducing *slack variables*. A constraint $\sum_{j=1}^{m} a_{ij} f'_j \leq b_i$ becomes $\sum_{j=1}^{m}(a_{ij} f'_j) + f'_{m+1} = b_i$ and a constraint $\sum_{j=1}^{m} a_{ij} f'_j \geq b_i$ becomes $\sum_{j=1}^{m}(a_{ij} f'_j) - f'_{m+1} = b_i$, where the slack variable f'_{m+1} is nonnegative. This way, the $2n$ slack variables are introduced to convert the $2n$ inequalities to equalities. The $2n$ equations are multiplied by -1 when $b_i < 0$ to satisfy $b_i \geq 0$, for $i = 1, ..., n$.

Since solving the LP problem leads to the minimum value of the objective function, the minimum and maximum values of the kth component of f' is found by taking the objective function as f'_k and $-f'_k$, respectively. Therefore, the m intervals of uncertainty are obtained by solving $2m$ LP problems for each orthant. The set of solutions to an LP problem for a particular orthant is a closed convex set. Hence, the solution set of an approximate system consists of the union of the convex subsets in all 2^m orthants.

An *admissible orthant* contains at least one admissible solution and an *empty orthant* contains no admissible solution. The solution set S is the union of the solution subsets in the admissible orthants. The intervals of uncertainty can be found in all 2^m orthants, but empty orthants should be avoided.

The orthant to be considered first is the one known to contain at least one admissible solution x^*. If x^* has q zero components, the search for a solution set must start in the 2^q orthants which x^* lies. If all values found for the endpoints of the orthant intervals of uncertainty are finite and none are zero, then an insular subset of the solution set exists in this particular orthant and this insular subset is the entire solution set. If any of the endpoints are not bounded, then the system is critically ill-conditioned. However, if q' intervals of uncertainty in the orthant have endpoints equal to zero or are unbounded, then the solution may extend into other $2^{q'} - 1$ adjacent orthants. An *adjacent orthant* corresponds to a sequence of q' components each of which may be either positive or negative. In this case, the solution set in the adjacent orthants must be investigated. If endpoints are found to be zero in any of the adjacent orthant intervals of uncertainty for components of f other than the q' components in the current orthant, then further orthants must be searched.

Once the intervals of uncertainty are determined, we obtain a fundamental matrix F inside these intervals. We further enforce the singularity constraint to avoid full rank matrices. To obtain a singular matrix, we use a traditional method that decomposes F into USV^T using singular value decomposition, changes the third eigenvalue in S to zero, and reconstructs the singular F with the modified S. Using this procedure, we essentially intersect the set of fundamental matrices inside the intervals of uncertainty with the set of rank 2 matrices.

3.1. Dealing with Outliers

To handle outliers, pairs of corresponding points where our error assumptions are broken, we discretize the solution space using a eight-dimensional accumulator array, where each element is associated with a fundamental matrix. For each pair of corresponding points, we compute the two inequality constraints. The elements in the accumulator associated with matrices that satisfy both constraints are incremented. Once all pairs of points are considered, the entries in the accumulator with maximum value are determined and only the pairs of points that contributed with votes to these elements are labelled as inliers. The remaining outliers are excluded from the input to our approximate algorithm.

4. EXPERIMENTAL RESULTS

We have implemented our approach to evaluate its accuracy performance when compared to state-of-art methods for the computation of the fundamental matrix. Initially, we generate a 3D scene with points in a set of sixteen 3D parametric curves. Given the extrinsic parameters of two cameras in the form of two rotation matrices R_0 and R_1 and two translation vectors T_0 and T_1, we rotate and translate the cloud of 3D points in the scene according to the respective cameras' pose.

Given the same intrinsic camera calibration matrix K for both cameras, we project each camera's cloud of 3D point into the respective image plane. Using this experimental setup, we determine a number of pairs of corresponding 2D points, actually discrete pixels, in the respective images. We also obtain the actual ground-truth fundamental matrix from the extrinsic and intrinsic calibration data.

Our experiment varies the image resolution to evaluate the accuracy of our method compared to the eight-point algorithm [3] with regards solely to the discretization effects. In other words, in this particular experiment, there are no detection errors due to noise or uncertainty since pixels' coordinates are obtained precisely. We also avoid correspondence errors that lead to outliers. Therefore, the single source of error in the computation of the fundamental matrix is the discretization effects that round real pixel coordinates to the closest integer. We vary image resolution by increasing the image's height and width. For each resolution, the focal length is adjusted accordingly to obtain similar images throughout the entire range of resolutions. In Fig. 2, we show the absolute difference error (y-axis) for all elements of the fundamental matrix as the image resolution increases (x-axis). The results of the eight-point algorithm are depicted as black dots and our approach is illustrated with red dots. We conclude that even considering only the dicretization effects in the computation, there is still significant error in the matrix obtained with the eight-point algorithm. This error decreases as the resolution increases since pixel coordinates become more and more accurate. However, the improvement seems to reach an asymptotic barrier which illustrates the need for novel methods to increase accuracy beyond this level despite the possibility of using high-resolution cameras. In addition to that, we show that our method outperforms the eight-point algorithm using only 64 point correspondences.

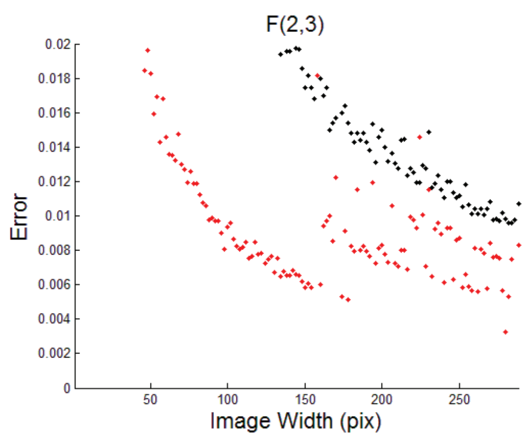

Fig. 2. A comparison between the eight-point algorithm (black dots) and our approximate algorithm (red dots) in terms of absolute error for $F_{2,3}$ as resolution increases.

5. CONCLUSIONS

We introduce a new robust approach for the computation of the fundamental matrix taking into account the intrinsic errors involved in the discretization process. The problem is modeled as an approximate equation system and reduced to a linear programming form. This approach is able to compute the solution set instead of trying to compute only a single vertex of the solution polyhedron as in previous approaches [3, 4].

Outliers are considered as sample point matches whose errors are much bigger that the expected uncertainty ϵ. We suggest ways to deal with outliers and present an outlier analysis with experiments in synthetic and real images.

The polyhedron associated with the solution set for the fundamental matrix may be used as a framework to measure the hardness involved in recovering a particular rigid motion and scene. This framework may also be used as a tool in the investigation of the relation between motion and the structure of a scene.

6. REFERENCES

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