Introduction to Computational Contact Mechanics

Part I. Basics

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WEMESURF short course on contact mechanics and tribology
Paris, France, 21-24 June 2010

Preface

- To whom the course is aimed?
- Developpers and users.
- What is the aim?
- Accurate contact modeling, correct interpretation, etc.
- FEM - Finite Element Method, FEA - Finite Element Analysis

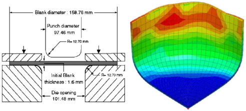
Outline

- Mathematical foundation
  - contact geometry;
  - optimization methods.
- Inside the Finite Element programme
  - contact detection;
  - contact discretization;
  - account of contact.
- Contact problem resolution with FEM: guide for engineer
  - master-slave approach;
  - boundary conditions;
  - spurious cases.
- Content
  - demonstrative;
  - simple;
  - general.

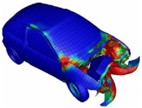
Plan

Finite Element Method

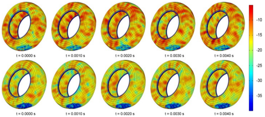
- FEM a powerfull and multi-purpose tool for linear and non-linear dynamic and static continuum mechanical and multi-physic problems



[Rousselier et al.]



[Klyavin et al.]



[Brinkmeiera et al.]

Finite Element Method

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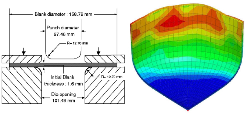

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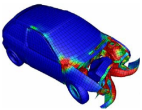

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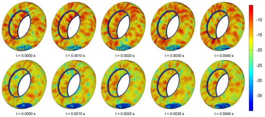


[Brinkmeiera et al.]

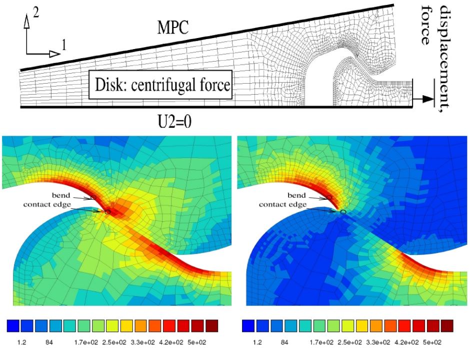
- Contact problems are not continuum and they require:
  - new rigorous mathematical basis: geometry, optimization, non-smooth analysis;
  - particular treatment of the finite element algorithms;
  - smart use of the Finite Element Analysis (FEA).

Finite Element Method

Finite element analysis of contact problems

- assembled components;

Disk-blade contact



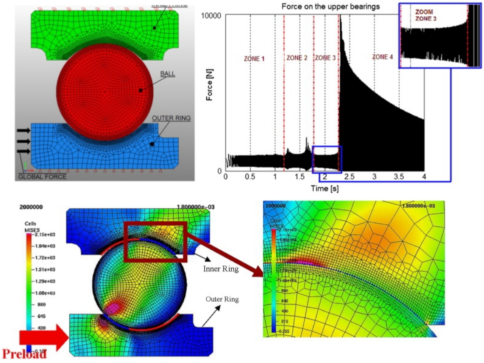
T.Dick, G.Cailletaud
Centre des Matériaux

Finite Element Method

Finite element analysis of contact problems

- assembled components;
- bearings;

Rolling bearing



F.Massi et al.
LaMCoS, INSA-Lyon et al.

Finite Element Method

Finite element analysis of contact problems

- assembled components;
- bearings;
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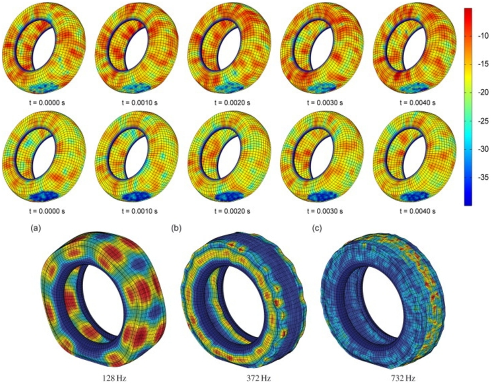
Tire rolling noise simulation



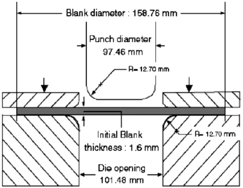
M.Brinkmeiera, U.Nackenhorst et al.
University of Hannover et al.

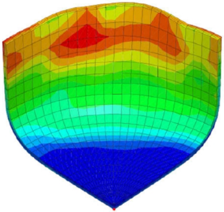
Finite Element Method

Finite element analysis of contact problems

- assembled components;
- bearings;
- rolling contact;
- forming processes;

Deep drawing





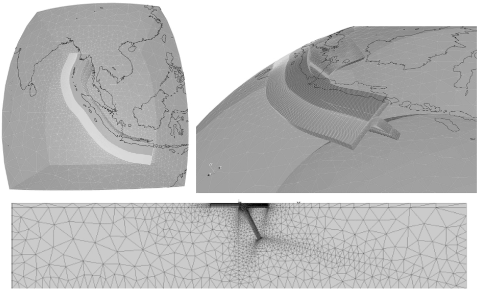
G.Rousselier et al.
Centre des Matériaux et al.

Finite Element Method

Finite element analysis of contact problems

- assembled components;
- bearings;
- rolling contact;
- forming processes;
- geomechanical contact;

Post seismic relaxation



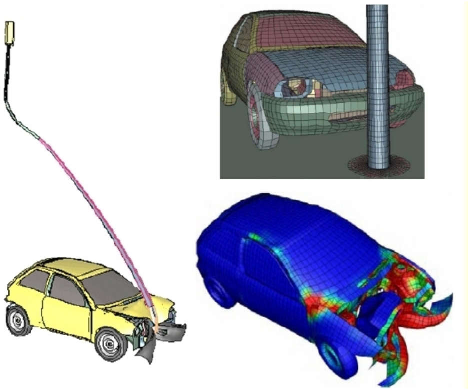
J.D.Garaud, L.Fleitout, G.Cailletaud
Centre des Matériaux, ENS

Finite Element Method

Finite element analysis of contact problems

- assembled components;
- bearings;
- rolling contact;
- forming processes;
- geomechanical contact;
- crash tests;

Crash-test



O.Klyavin, A.Michailov, A.Borovkov
St Petersbourg State University

Finite Element Method

Finite element analysis of contact problems

- assembled components;
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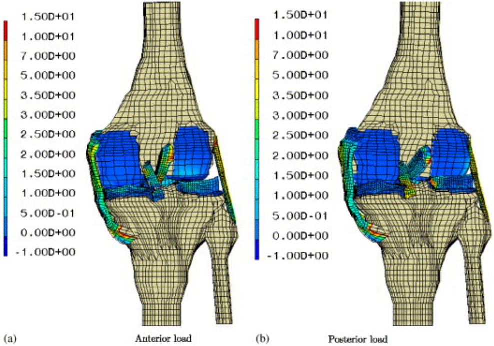
- crash tests;
- human joints;

Knee simulation



E.Peña, B.Calvoa et al.
University of Zaragoza

Finite Element Method

Finite element analysis of contact problems

- assembled components;
- bearings;
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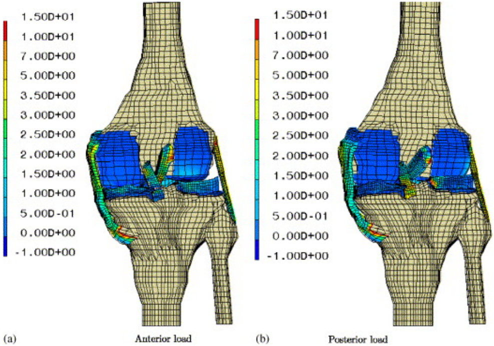
- forming processes;
- geomechanical contact;
- crash tests;
- human joints;
- and many others.

Knee simulation



E.Peña, B.Calvoa et al.
University of Zaragoza

They trust in the FEA

Leading international industrial companies

AIRBUS BOEING GM

HUMMER Ford

DAIMLER

Schlumberger CATERPILLAR®

SIEMENS RENAULT Ferrari

LG SAMSUNG ELECTRONICS eDF CORNING Discovering Beyond Imagination

Mechanical problem

From a real life problem to an engineering problem

Need to determine:

- the **problematic:**
  - strength/life-time/fracture;
  - vibration/buckling;
  - thermo-electro-mechanical.
- relevant **geometry;**
- relevant **loads:**
  - static/quasi-static/dynamic;
  - mechanical/thermic;
  - volume/surface.
- relevant **material:**
  - rigid/elastic/plastic/visco-plastic;
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- relevant **scale:**
  - macro/meso/micro.

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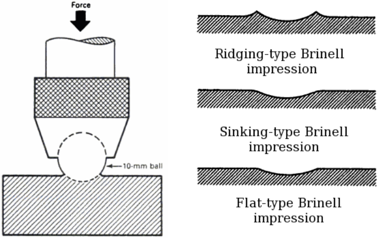
For example: Brinell hardness test



Scheme of the Brinell hardness test and different types of impression

[Harry Chandler, Hardness testing]

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Need to determine:

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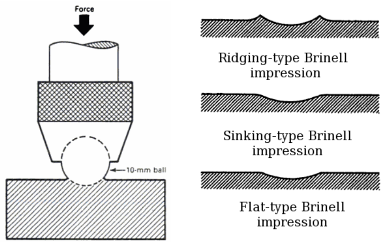

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FE model:

- **analysis type:**
  - stress-strain state;
  - eigen values;
  - coupled physics.

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- **microstructure:**
  - RVE/microstructure/–.

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FE model:

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[a] rigid in FEA: much more harder than another solid, special boundary conditions or geometrical representation.

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- **microstructure:**
  - homogeneous > RVE.

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Mechanical problem

From engineering problem to a finite element model

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- problematic
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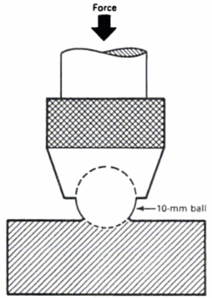

FE model:

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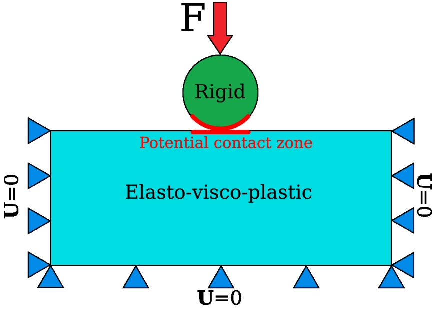

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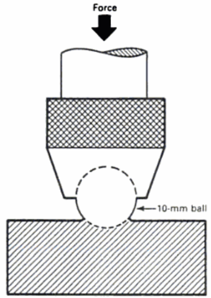

FE model:

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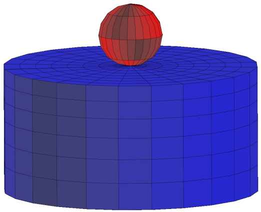

Finite element mesh :)

Symmetry and plane problems

How to solve problems faster

Main ideas:

- symmetry
  - geometry AND loading;
- 3D to 2D:
  - axisymmetry/plane strain/plane stress;
- 3D to smaller 3D:
  - half/quarter/sector;

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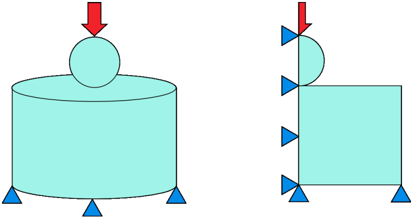
Axisymmetry



Axisymmetry of geometry
axisymmetry of load

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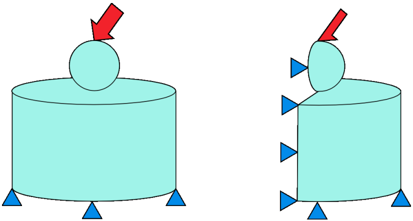
Mirror symmetry



Axisymmetry of geometry
mirror symmetry of load

Symmetry and plane problems

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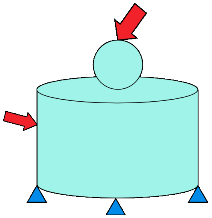
No symmetry



Axisymmetry of geometry
no symmetry of load

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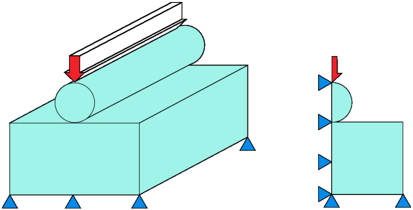
Plain strain



Mirror symmetry of geometry
mirror symmetry of load
very long structure or fixed edges

Symmetry and plane problems

How to solve problems faster

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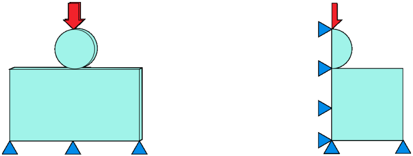
Plain stress

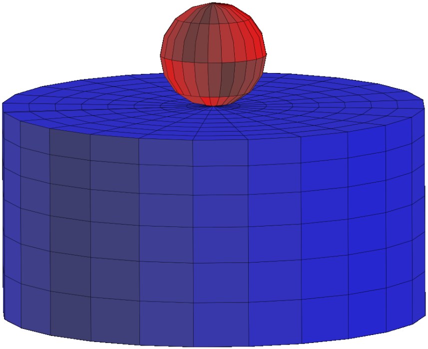


Mirror symmetry of geometry
mirror symmetry of load
very thin structure

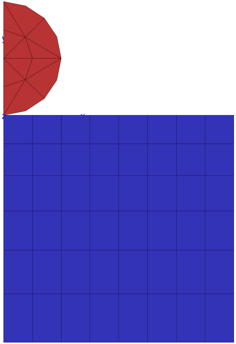
Symmetry and plane problems

How to solve problems faster

Full 3D mesh



Axisymmetric 2D mesh



Finite element mesh

Basics

In general the finite element mesh should

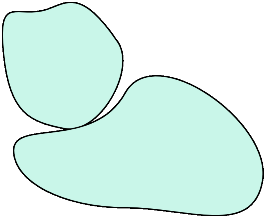
- fulfil the required solution precision;
- correctly represent relevant geometry;
- not be enormous;
- be fine where strain is large;
- be rough where strain is small;
- avoid too oblong elements.

In contact problems the finite element mesh should

- not allow corners at master surface;
- be very fine and precise on both contacting surfaces;
- use carefully quadratic elements.

Boundary conditions

- Two solids Ω_1 and Ω_2.
- Volumetric forces $\mathbf{F}_v$: e.g. inertia $m\ddot{\mathbf{u}}$.
- Neumann (static, force) boundary conditions: distributed and concentrated loads.
- Dirichlet (kinematic, displacement) boundary conditions: displacements.
- In each solid we fulfil $div(\sigma) + \mathbf{F}_v = 0$



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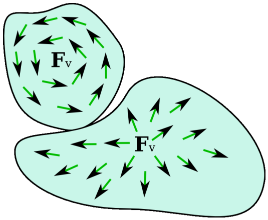

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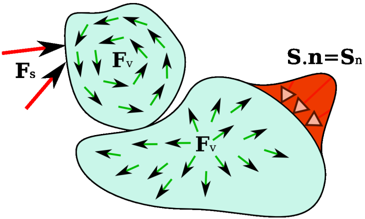

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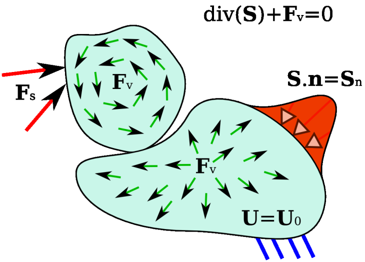

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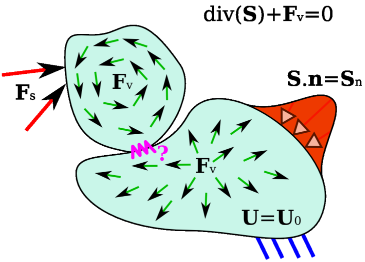
- What are the contact boundary conditions?



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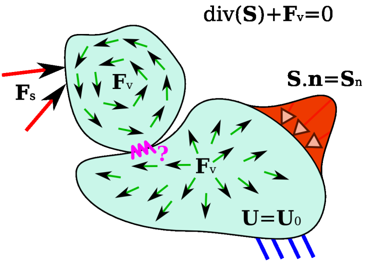
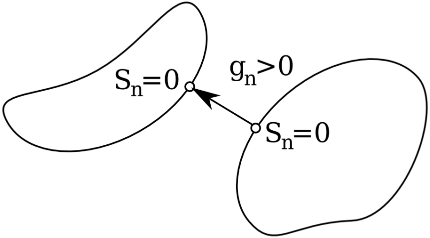
- Dirichlet (kinematic, displacement) boundary conditions: displacements.
- In each solid we fulfil $div(\sigma) + \mathbf{F}_v = 0$
- What are the contact boundary conditions?



Account of contact

Signorini conditions

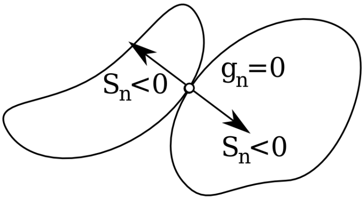


- Signorini conditions of nonpenetration $g_n > 0$ and non-adhesion $\sigma_n \leq 0$

$$\boxed{g_n\sigma_n = 0,\ g_n \geq 0,\ \boldsymbol{\sigma} \leq 0, \quad \sigma_n = \boldsymbol{\sigma} \cdot \mathbf{n}}$$

- Contact boundary conditions $\sim$ unknown Neumann (force) boundary conditions depending on the geometry.

Account of contact
Signorini conditions



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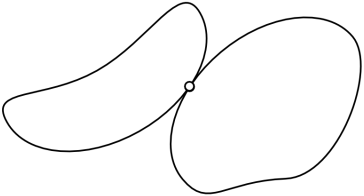
Account of contact

Coulomb's friction



- Coulomb's friction conditions

$$\boxed{|\dot{\mathbf{g}}_t|\,(|\boldsymbol{\sigma}_t| + \mu\sigma_n) = 0;\;\; |\boldsymbol{\sigma}_t| \le -\mu\sigma_n;\;\; \dot{\mathbf{g}}_t = |\dot{\mathbf{g}}_t|\frac{\boldsymbol{\sigma}_t}{|\boldsymbol{\sigma}_t|},\quad \boldsymbol{\sigma}_t = \boldsymbol{\sigma}\cdot\mathbf{t}}$$

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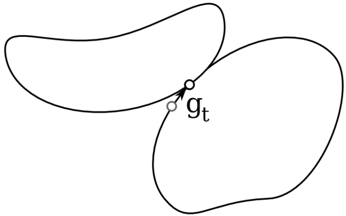
Account of contact

Coulomb's friction



- Coulomb's friction conditions

$$\boxed{|\dot{\mathbf{g}}_t|\left(|\boldsymbol{\sigma}_t| + \mu\sigma_n\right) = 0;\ \ |\boldsymbol{\sigma}_t| \le -\mu\sigma_n;\ \ \dot{\mathbf{g}}_t = |\dot{\mathbf{g}}_t|\frac{\boldsymbol{\sigma}_t}{|\boldsymbol{\sigma}_t|},\quad \boldsymbol{\sigma}_t = \boldsymbol{\sigma}\cdot\mathbf{t}}$$

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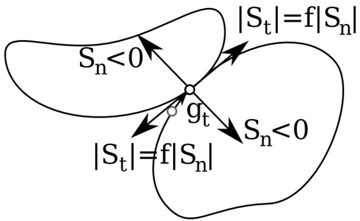
Account of contact

Coulomb's friction



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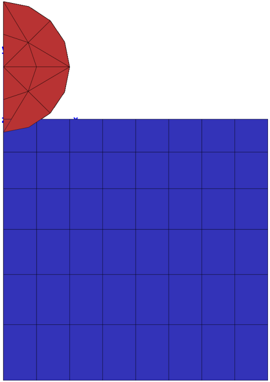
- Contact boundary conditions $\sim$ unknown Neumann (force) boundary conditions depending on the geometry.

Contact detection

- Two FE meshes penetrate each other
- What penetrates?
  - Nodes, lines, elements?
- Into what it penetrates?
  - Into elements, under surface?
- How do we detect such penetration?



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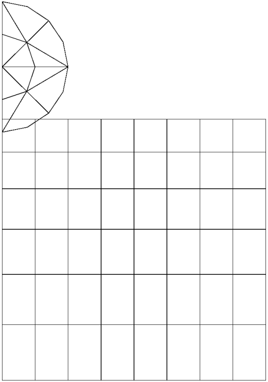

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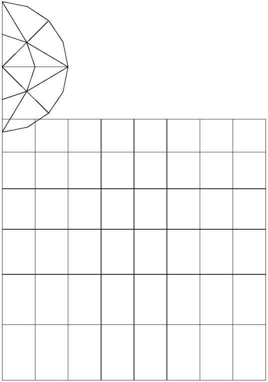

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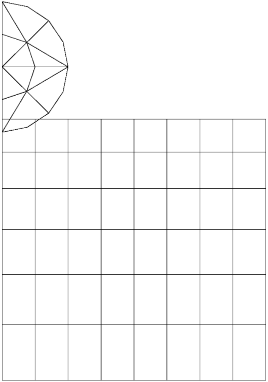

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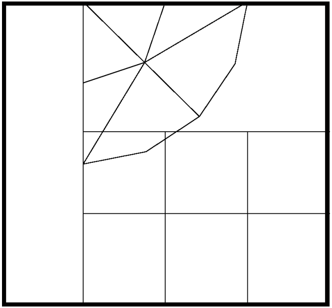

Zoom on penetration

Contact detection

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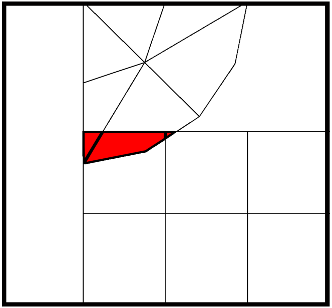
- How do we detect such penetration?



Volume intersection

Contact detection

- Two FE meshes penetrate each other
- What penetrates?
  - Nodes, lines, elements?
- Into what it penetrates?
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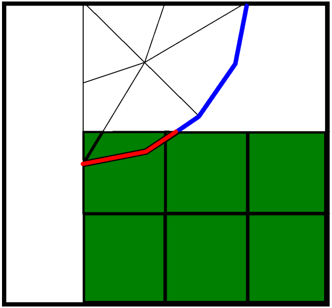
- How do we detect such penetration?



Surface-in-volume 1

Contact detection

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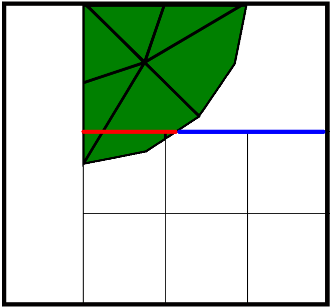
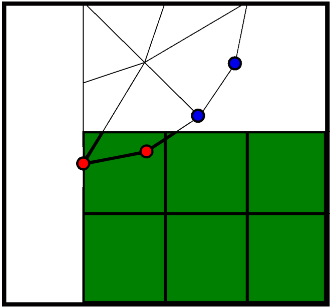


Surface-in-volume 2

Contact detection

- Two FE meshes penetrate each other
- What penetrates?
  - Nodes, lines, elements?
- Into what it penetrates?
  - Into elements, under surface?
- How do we detect such penetration?



Nodes-in-volume 1

Contact detection

- Two FE meshes penetrate each other
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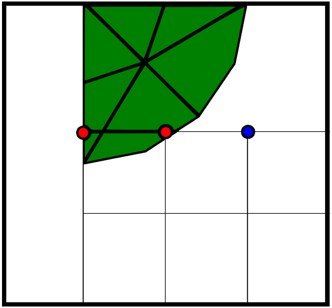

Nodes-in-volume 2

Contact detection

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  - Nodes, lines, elements?
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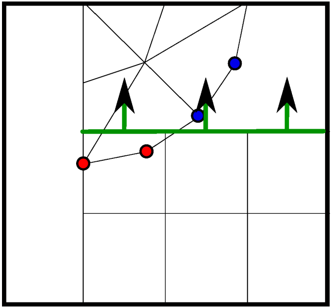

Nodes-to-surface 1

Contact detection

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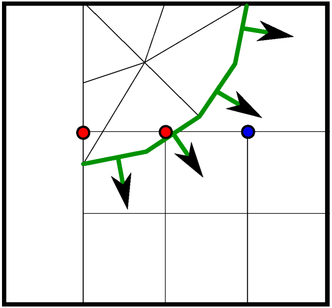
- How do we detect such penetration?

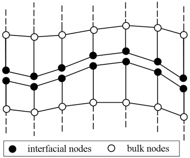


Nodes-to-surface 2

Contact discretization

What is elementary contact contrubutor?

- Local contacting unit?
- Node-to-node
- Node-to-segment/Node-to-edge/Node-to-surface
- Gauss point to surface



Node-to-node discretization: small deformation/small sliding

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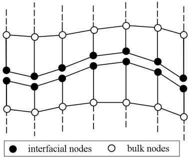

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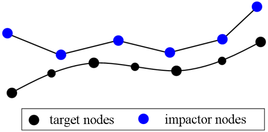

Large deformation/large sliding

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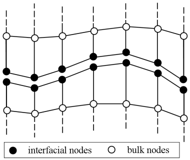
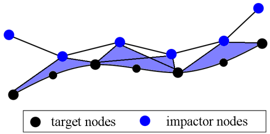


Node-to-node discretization: small deformation/small sliding



Node-to-segment discretization large deformation/large sliding

Contact discretization

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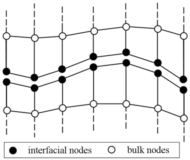
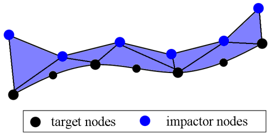


Node-to-node discretization: small deformation/small sliding



Contact domain method: large deformation/large sliding

Contact discretization

What is elementary contact contrubutor?

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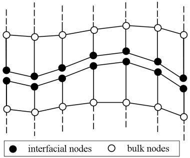
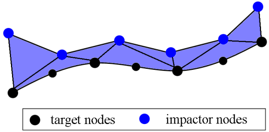
Conception of the **contact element**



Node-to-node discretization: small deformation/small sliding



Contact domain method: large deformation/large sliding

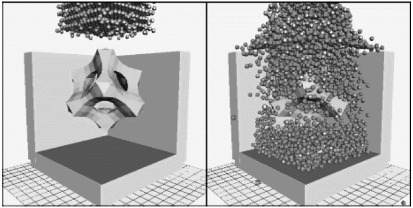
Plan

Spatial search and local detection

Spatial search

Detection of contacting solids: multibody systems

- Discrete Element Method
- Molecular dynamics

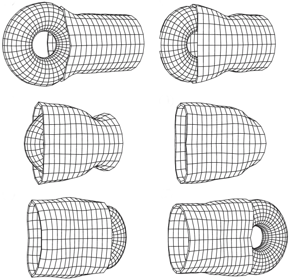


Example of DEM application [Williams,O'Connor,1999]

Local contact detection

Detection of contacting nodes and surfaces of two discretized solids

- Finite Element Method
- Smoothed Particle Hydrodynamics



Torus–cylinder impact problem [B. Yang, T.A. Laursen, 2006]

Spatial search and local detection

Spatial search

Detection of contacting solids: multibody systems

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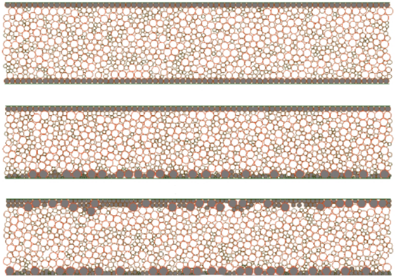

3rd body layer modeling
[V.-D. Nguyen, J. Fortin et al., 2009]

Local contact detection

Detection of contacting nodes and surfaces of two discretized solids

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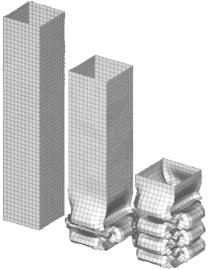
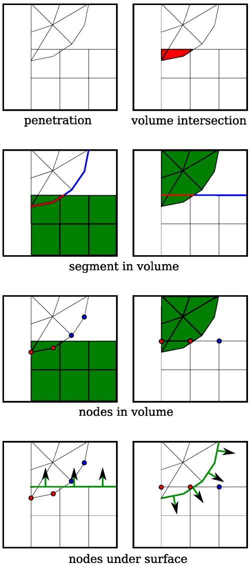


Buckling with self-contact
[T. Belytschko, W.K. Liu, B. Moran, 2000]

Local contact detection

Basic ideas

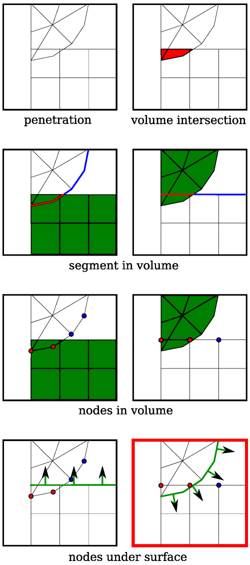
- How to detect contact?
- What penetrates and where?
- What/where approach



Local contact detection

Basic ideas

- How to detect contact?
- What penetrates and where?
- What/where approach, for example,
    - What? **Slave** nodes
    - Where? Under **master** surface



Local contact detection

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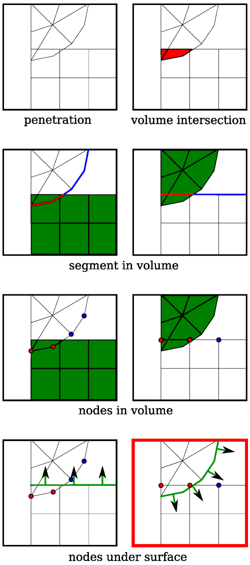

Local contact detection

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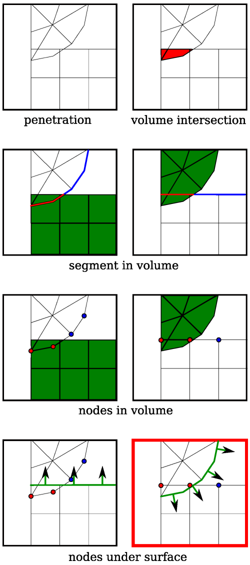
New paradigm

BUT! We need to detect contact before any penetration occures!



Local contact detection

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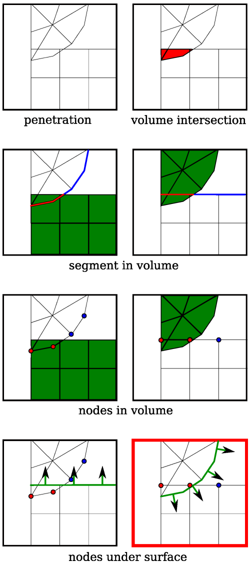

New paradigm

BUT! We need to detect contact before any penetration occures!

Contact detection idea

In general the detection consists in checking if slave nodes are close enough to the master surface

Local contact detection
Basic ideas

- How to detect contact?
- What penetrates and where?
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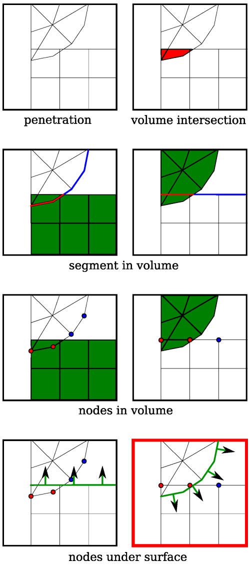
New paradigm

BUT! We need to detect contact before any penetration occures!

Contact detection idea

In general the detection consists in checking if slave nodes are close enough[a] to the master surface

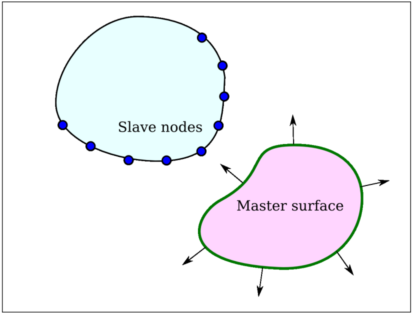
[a] What does it mean close enough?



Maximal detection distance

Maximal detection distance concept $d_{\max}$

If a slave node is closer than $d_{\max}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.



Solid with slave nodes and solid with a master surface.

Maximal detection distance

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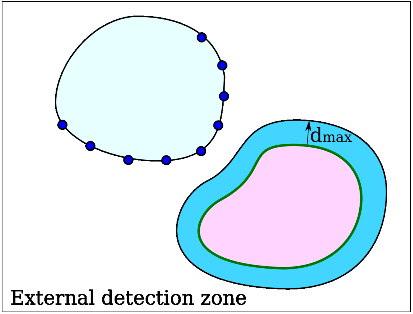

External detection zone dist $< d_{\max}$ and $g_n \geq 0$

Maximal detection distance

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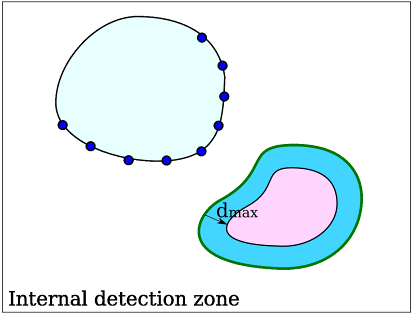
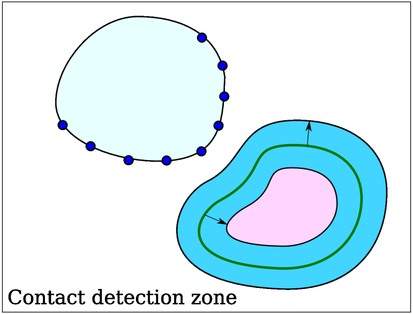


Internal detection zone dist $< d_{\max}$ and $g_n < 0$

Maximal detection distance

Maximal detection distance concept $d_{\max}$

If a slave node is closer than $d_{\max}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.



Contact detection zone dist $< d_{\max}$

Maximal detection distance

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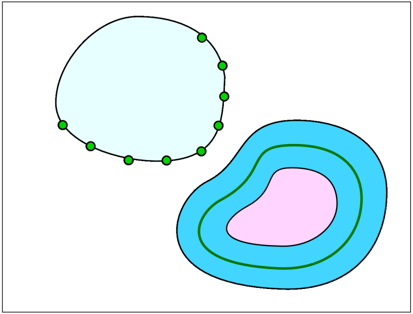

Verficiation if slave nodes are in the detection zone

Maximal detection distance

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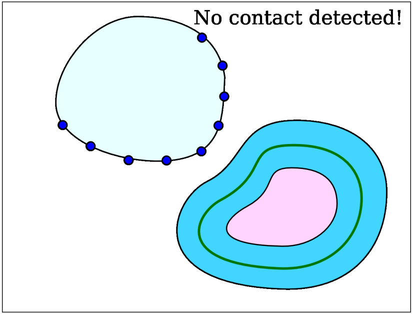

No slave nodes in the detection zone

Maximal detection distance

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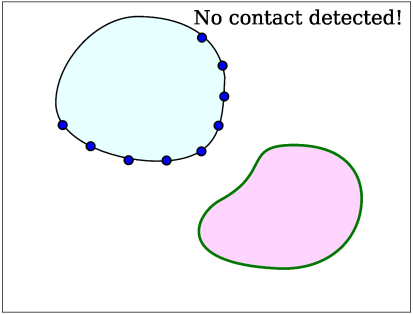

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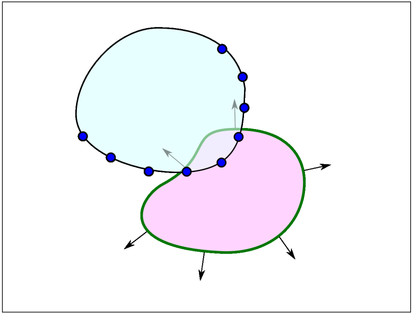

Solid with slave nodes and solid with a master surface.

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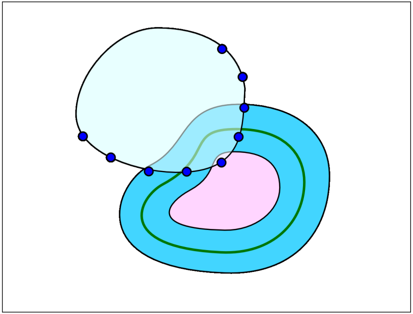

Contact detection zone dist $< d_{\max}$

Maximal detection distance

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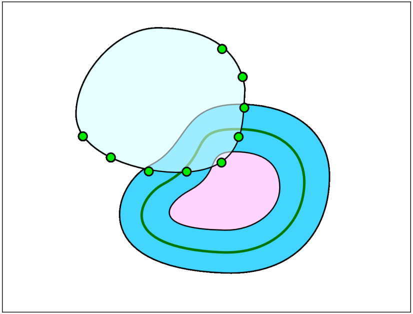
If a slave node is closer than $d_{\max}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.



Verficiation if slave nodes are in the detection zone

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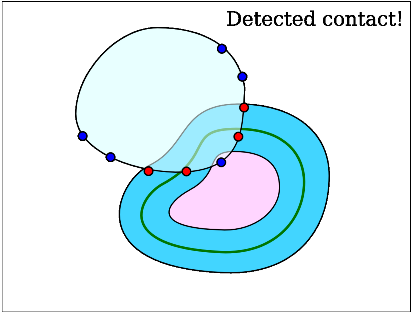

Some slave nodes are in the detection zone. One node is missed!

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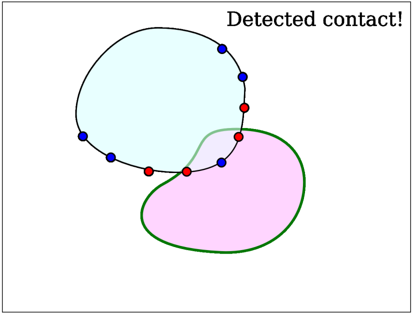

Create contact elements with detected slave nodes.

Maximal detection distance

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How to choose $d_{\max}$?

Dangerous solution

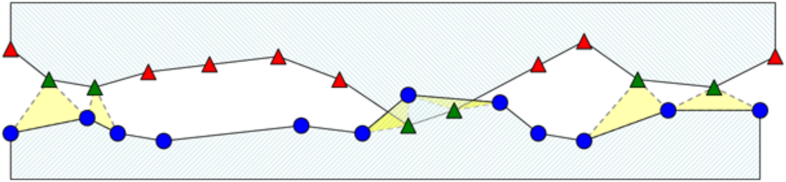
- Make the user responsible for this choice

Practical solutions: choose accordingly to

- master surface mesh;
- boundary conditions (maximal variation of displacement of contacting surface nodes during one iteration/increment)
- use both criterions.

Discretized master surface

- Master surface contsist of many elementary master surfaces.
- The aim is to find for each slave node the associated master surface.



Triangles - slave nodes and circles - master nodes which are connected by master surfaces.

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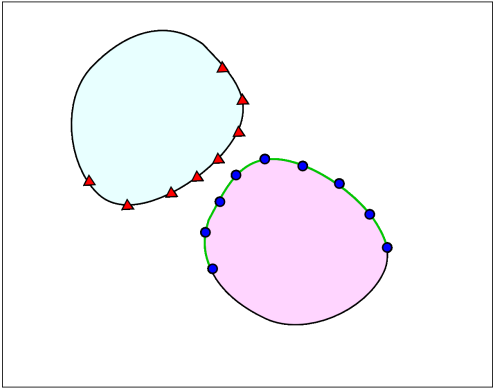

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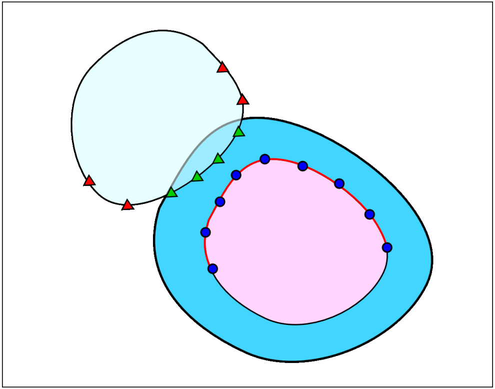

Contact detection zone.
What does it consist of?

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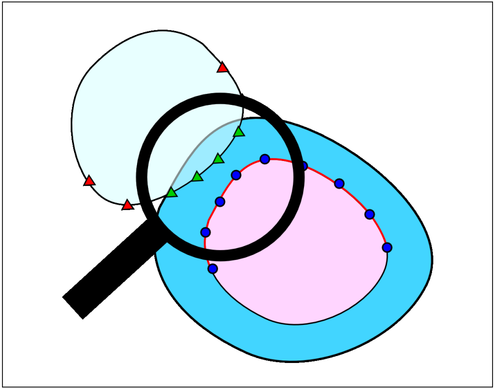

Zoom on the master contact surface and the associated detection zone.

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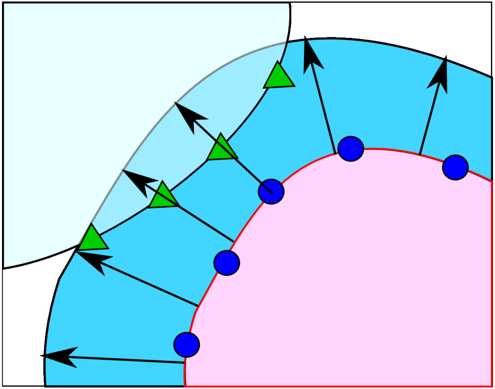

Contact detection zone is nothing but the region, where each point is closer to the master surface than $d_{\max}$

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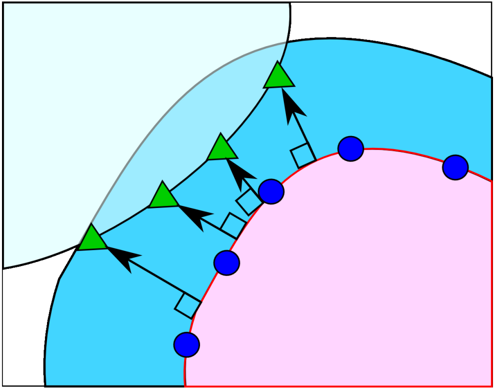

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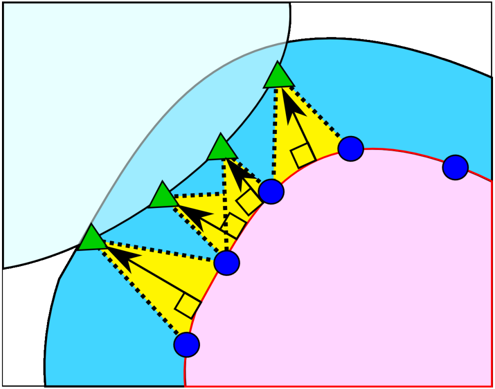

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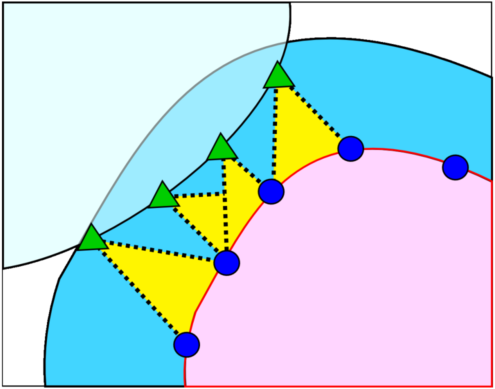

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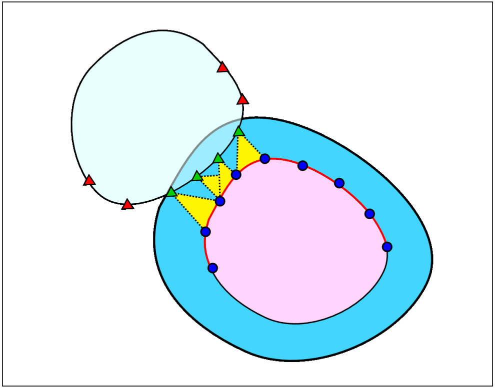

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  - In general case, nonlinear minimization problem:

$$\boxed{\min_{\xi\in[0;1]} F(\mathbf{r}_s, \xi) \Rightarrow \xi^* : \forall \xi \in [0;1], |\mathbf{r}_s - \rho(\xi^*)| \leq |\mathbf{r}_s - \rho(\xi)|}$$

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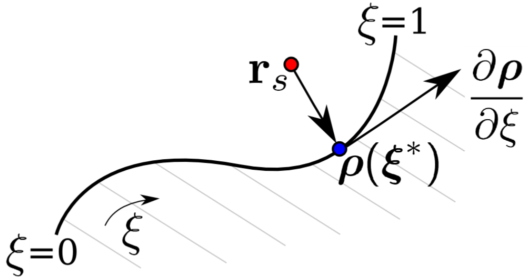

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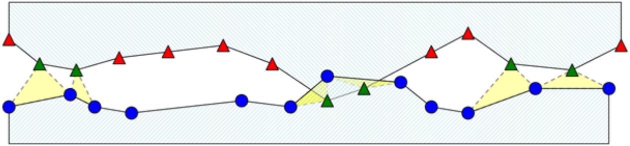

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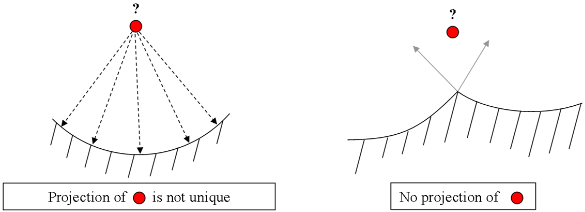
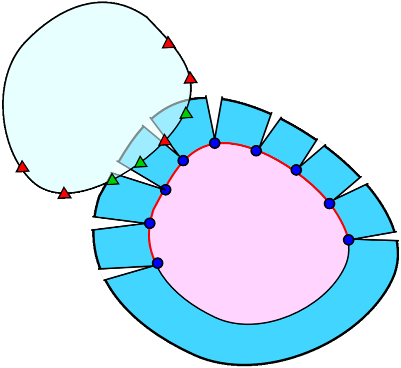


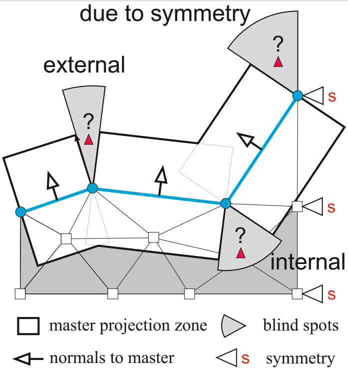
For more details see Contact geometry section

Blind spots

Blind spot problem

- Blind spots – gaps in the detection zone.
- Do not miss slave nodes situated in blind spots.
- Types of blind spots: external, internal, due to symmetry.



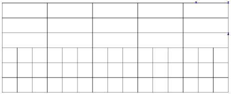


Who is master, who is slave?

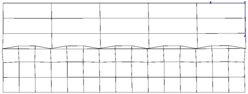
Social inequality in contact problems

Master-slave definition

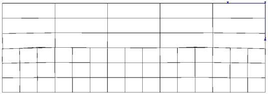
- The choice of master and slave surfaces is not random.
- Incorrect choice leads to meaningless solutions.



Initial FE mesh



Incorrect master-slave choice ☹



Correct master-slave choice ☺

Contact detection in global algorithm

At the beginning of each increment (loading step)

- \+ fast;
- \+ good convergence;
- \+ stable;
- − lack of accuracy.

At the beginning of each iteration (convergence step)

- − slow;
- − infinite looping;
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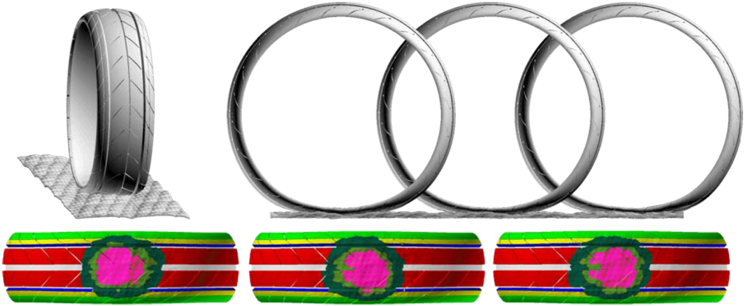

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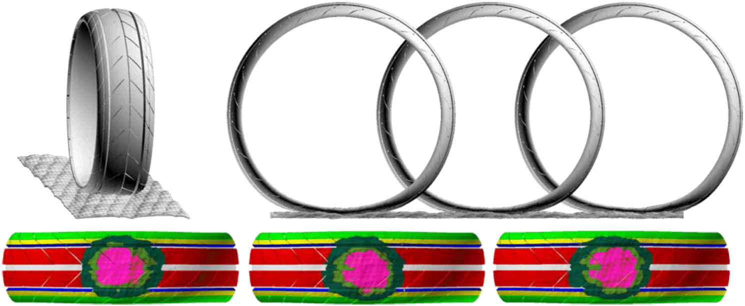


Contact detection techniques

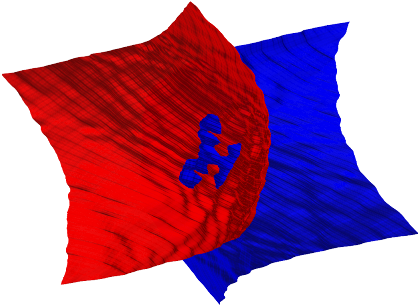
- **All-to-all detection**
  - Direct all-to-all detection
    - Slave nodes to master segments.
  - Indirect all-to-all detection
    - Slave nodes to master nodes.
    - Slave node to attached master segments.
- **Advantages:**
  - simple implementation.
- **Drawbacks:**
  - time consumming $O(N^2)$;
  - blind spots/passing by nodes.

- **Elaborated techniques**
  - Bounding boxes
    - Account only close regions.
  - Regular grid methods: bucket [Benson].
    - Detect only into a cell and in neighbouring cells
  - Sorting methods: heap sort, octree [Williams, O'Connor].
    - Data tree construction and tree search.
- **Advantages:**
  - relatively fast $O(N)$, $O(N\log N)$..
- **Drawbacks:**
  - blind spots/passing by nodes.

Contact detection techniques

- **All-to-all detection**
- **Elaborated techniques**

Example: two curved contacting surfaces –
1 million of slave nodes against **1 million** of master segments.



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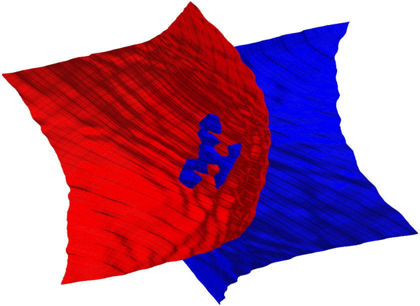

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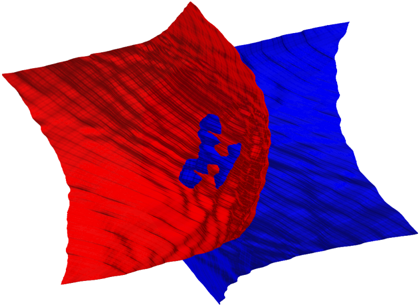

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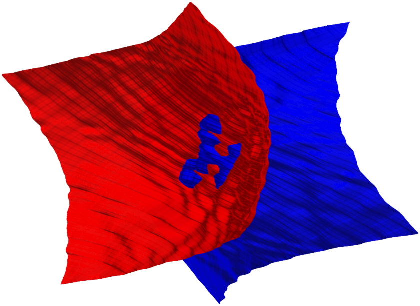
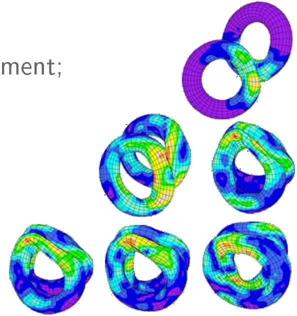


- Indirect all-to-all technique
  - **187 hours(!)**
- Bounding box+grid detection
  - **1 minute**

Summary
Contact detection

- Global search/local contact detection;
- slave-master or what-where approach;
- conception of the maximal detection distance and its choice;
- contact geometry and contact detection - closest point definition;
- existance and uniqueness of the closest point;
- from continuous and smooth to discretized C^0 surface;
- attention - blind spots;
- detect contact at the beginning of increment;
- different detection techniques;
- self-contact detection.
- contact detection is strongly connected with contact geometry and contact discretization.

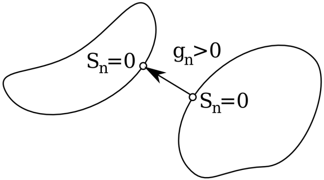


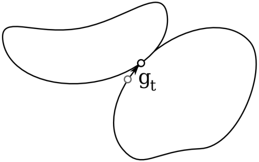
[B. Yang, T.A. Laursen, 2006]

Plan

Introduction

- Geometry is a foundation for
  - master-slave approach;
  - detection;
  - discretization method;
  - solution.
- Geometrical quantities
  - penetration or normal gap g_n - normal contact;
  - tangential sliding $\Delta \mathbf{g}_t$ - frictional effects.





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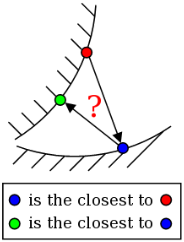

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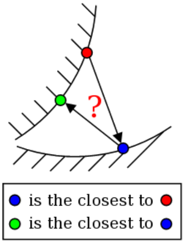

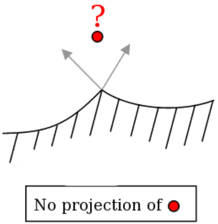

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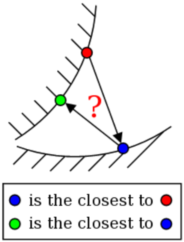
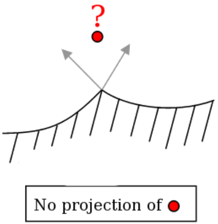
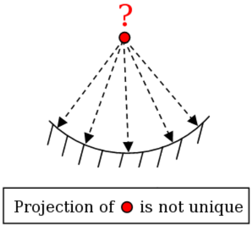






Smoothness of contact surface

Do real surfaces are smooth or not?

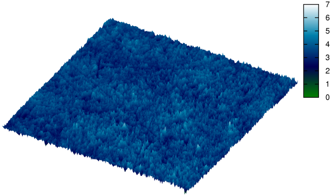
- It depends on the scale.
- The smaller scale, the higher roughness.
- Fractal surface.

Surface deforms even without contact

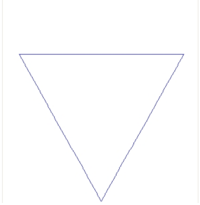
- macro deformation;
- escaping dislocations and twins;
- relaxation at nano-scale.

Finite element surface is not smooth

- convergence problems;
- unphysical oscilations;
- remedy - special smoothing techniques.



Polished metal surface
400x600 microns specimen



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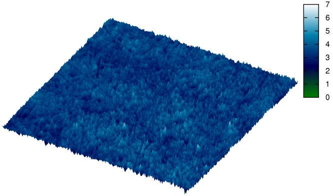

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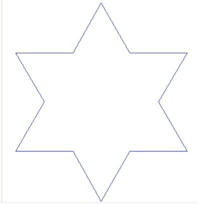

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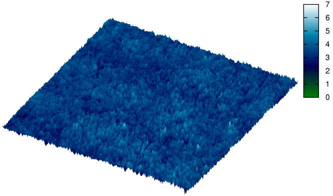

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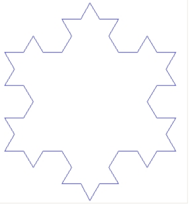


Fractal surface

Smoothness of contact surface

Do real surfaces are smooth or not?

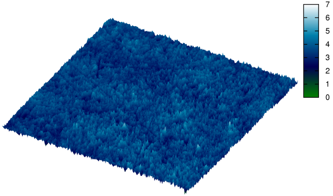
- It depends on the scale.
- The smaller scale, the higher roughness.
- Fractal surface.

Surface deforms even without contact

- macro deformation;
- escaping dislocations and twins;
- relaxation at nano-scale.

Finite element surface is not smooth

- convergence problems;
- unphysical oscilations;
- remedy - special smoothing techniques.



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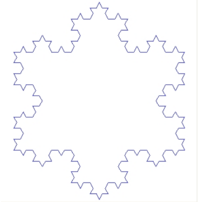

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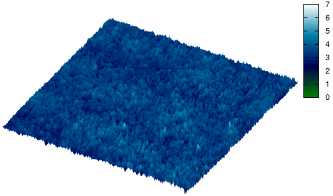

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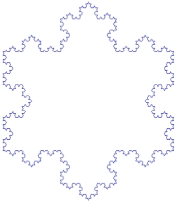

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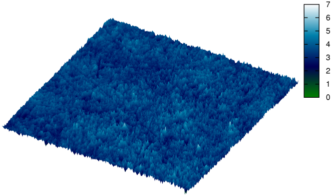

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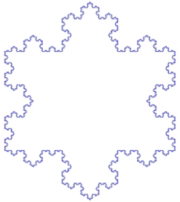

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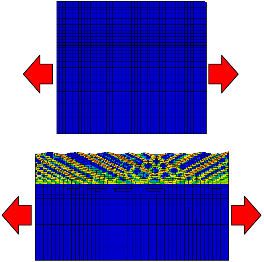
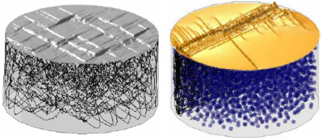


Deformation twinnings in coating [Forest, 2000]



Escaping dislocations [Fivel, 2009]

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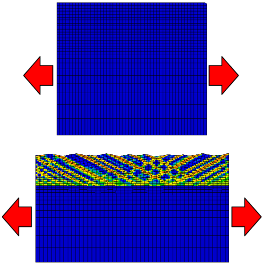

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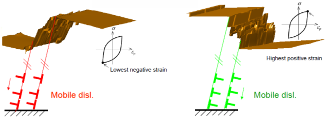

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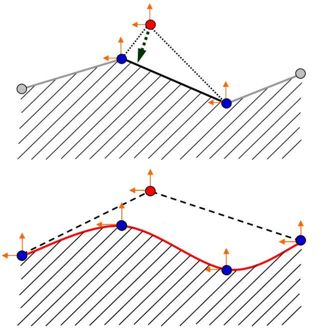

2D surface smoothing with Bezier curves

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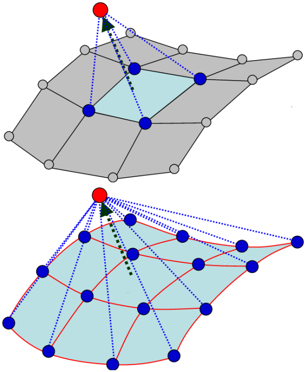


3D surface smoothing with Bezier surface

Account of geometrical quantities

First order variations

Contact in the weak form

- Geometrical quantities: normal gap $g_n = g_n(\mathbf{x})$ and tangential sliding $\dot{\mathbf{g}}_t = \dot{\mathbf{g}}_t(\mathbf{x}^0, \mathbf{x})$
- Virtual work at contact surface $\delta W^{\text{cont}} = \delta W_n + \delta W_t$
  - normal contact $W_n(\sigma_n, g_n) \Rightarrow \delta W_n(\sigma_n, g_n, \delta\sigma_n, \delta g_n)$
  - frictional contact $W_t(\boldsymbol{\sigma}_t, \dot{\mathbf{g}}_t) \Rightarrow \delta W_t(\boldsymbol{\sigma}_t, \dot{\mathbf{g}}_t, \delta\boldsymbol{\sigma}_t, \delta\dot{\mathbf{g}}_t)$
- Resulting nonlinear equation $R\left(\delta W^{\text{int}}, \delta W^{\text{ext}}, \delta W^{\text{cont}}\right) = 0$

Need of analytical expressions

- First order variation of the normal gap and tangential sliding

$$\boxed{\delta g_n = \frac{\partial g_n}{\partial \mathbf{x}} \cdot \delta \mathbf{x}} \qquad \boxed{\delta \dot{\mathbf{g}}_t = \frac{\partial \dot{\mathbf{g}}_t}{\partial \mathbf{x}} \cdot \delta \mathbf{x}}$$

Account of geometrical quantities

Second order variations

Linearization of the weak form

- Resulting nonlinear equation $R\left(\delta W^{\text{int}}, \delta W^{\text{ext}}, \delta W^{\text{cont}}\right) = 0$
- Linearization $R(\mathbf{x}) = 0 \Rightarrow R|_{\mathbf{x}_0} + \Delta R(x)|_{\mathbf{x}_0} \approx 0$
- $\Delta R(\mathbf{x}) = \Delta\delta W^{\text{int}} + \Delta\delta W^{\text{ext}} + \Delta\delta W^{\text{cont}}$
  - normal contact $\Delta\delta W_n(\sigma_n, g_n, \delta\sigma_n, \delta g_n, \Delta\delta\sigma_n, \Delta\delta g_n)$
  - frictional contact $\Delta\delta W_t(\boldsymbol{\sigma}_t, \dot{\mathbf{g}}_t, \delta\boldsymbol{\sigma}_t, \delta\dot{\mathbf{g}}_t, \Delta\delta\boldsymbol{\sigma}_t, \Delta\delta\dot{\mathbf{g}}_t)$

Need of analytical expressions

- Second order variation of the normal gap and tangential sliding

$$\boxed{\Delta\delta g_n = \Delta\mathbf{x} \cdot \frac{\partial^2 g_n}{\partial \mathbf{x}^2} \cdot \delta\mathbf{x}} \qquad \boxed{\Delta\delta\dot{\mathbf{g}}_t = \Delta\mathbf{x} \cdot \frac{\partial^2 \dot{\mathbf{g}}_t}{\partial \mathbf{x}^2} \cdot \delta\mathbf{x}}$$

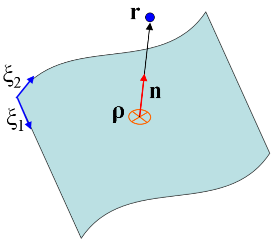
Point and surface

Master-slave approach

Master-slave

Slave node penetrates under and slides over the master surface.

- Slave node – point $\mathbf{r}_s$
- Master surface – $\rho(\xi)$
- Surface parametrization $\xi = \{\xi_1,\ \xi_2\}$
- Projection of the slave node
  $\rho(\xi_p)$
- Normal to the master surface
  $\mathbf{n}(\xi_p)$



Geometrical quantities

- Normal gap $\boxed{g_n = (\mathbf{r}_s - \rho)\cdot\mathbf{n}}$
- Tangential sliding $\boxed{\dot{\mathbf{g}}_t dt = \delta\rho(\xi)}$

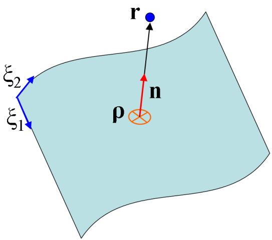
Point and surface

Master-slave approach

Master-slave

Slave node penetrates under and slides over the master surface.

- Slave node – point $\mathbf{r}_s = \mathbf{r}_s(t)$
- Master surface – $\rho(\xi) = \rho(t, \xi)$
- Surface parametrization $\xi = \{\xi_1,\ \xi_2\}$
- Projection of the slave node
  $$\rho(\xi_p) = \rho(t, \xi(\mathbf{r}_s(t)))$$
- Normal to the master surface
  $$\mathbf{n}(\xi_p) = \mathbf{n}(t, \xi(\mathbf{r}_s(t)))$$



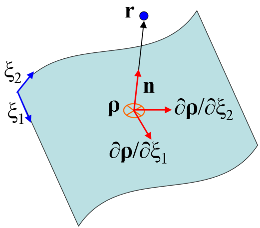
Geometrical quantities

- Normal gap $\boxed{g_n(t) = (\mathbf{r}_s(t) - \rho(t, \xi(\mathbf{r}_s(t))) \cdot \mathbf{n}(t, \xi(\mathbf{r}_s(t)))}$
- Tangential sliding $\boxed{\dot{\mathbf{g}}_t(t, \xi(\mathbf{r}_s(t)))dt = \delta\rho(t, \xi(\mathbf{r}_s(t)))}$

Continuum geometrical formulation

Covariant and contravariant bases, fundamental surface tensors

- Covariant surface basis
  $$\frac{\partial \rho}{\partial \xi} = \left\{ \frac{\partial \rho}{\partial \xi_1}; \frac{\partial \rho}{\partial \xi_2} \right\}$$
- Contravariant surface basis
  $$\frac{\partial \hat\rho}{\partial \xi} = \left\{ \frac{\partial \rho}{\partial \xi^1}; \frac{\partial \rho}{\partial \xi^2} \right\}$$
- Basis change
  $$\frac{\partial \hat\rho}{\partial \xi} = \mathbf{A}^{-1} \frac{\partial \rho}{\partial \xi}\; \frac{\partial \rho}{\partial \xi^i} = a^{ij} \frac{\partial \rho}{\partial \xi_j}$$
- 1^{st} fundamental covariant surface tensor
  $$\mathbf{A} \sim a_{ij} = \frac{\partial \rho}{\partial \xi_i} \cdot \frac{\partial \rho}{\partial \xi_j}$$
- 1^{st} fundamental contravariant surface tensor
  $$\mathbf{A}^{-1} \sim a^{ij} = \frac{\partial \rho}{\partial \xi^i} \cdot \frac{\partial \rho}{\partial \xi^j}$$
- 2^{nd} fundamental surface tensor
  $$\mathbf{H} \sim h_{ij} = \mathbf{n} \cdot \frac{\partial^2 \rho}{\partial \xi_i \partial \xi_j}$$



Continuum geometrical formulation

First order variations

- First order variation of the normal gap δg_n

$$\boxed{\delta g_n = \mathbf{n} \cdot (\delta \mathbf{r}_s - \delta \boldsymbol{\rho})}$$

- First order variation of the surface parameter $\delta \boldsymbol{\xi}$

$$\boxed{\delta \boldsymbol{\xi} = (\mathbf{A} - g_n \mathbf{H})^{-1} \cdot \left(\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\xi}} \cdot (\delta \mathbf{r}_s - \delta \boldsymbol{\rho}) + g_n \mathbf{n} \cdot \delta \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\xi}} \right)}$$

Continuum geometrical formulation

Second order variations

- Second order variation of the normal gap δg_n

$$
\boxed{\begin{aligned}
\Delta\delta g_n = & -\mathbf{n}\cdot\left(\delta\frac{\partial\rho}{\partial\xi}\cdot\Delta\xi+\Delta\frac{\partial\rho}{\partial\xi}\cdot\delta\xi+\delta\xi\cdot\frac{\partial^2\rho}{\partial\xi^2}\cdot\Delta\xi\right)+\\
& + g_n\delta\xi\cdot\mathbf{H}\cdot\mathbf{A}^{-1}\cdot\mathbf{H}\cdot\Delta\xi+\\
& + g_n\left(\mathbf{n}\cdot\delta\frac{\partial\rho}{\partial\xi}\right)\cdot\mathbf{A}^{-1}\cdot\left(\Delta\frac{\partial\rho}{\partial\xi}\cdot\mathbf{n}\right)
\end{aligned}} \qquad (1)
$$

- Second order variation of the surface parameter $\delta\xi$

$$
\boxed{\begin{aligned}
\Delta\delta\xi = (g_n\mathbf{H}-\mathbf{A})^{-1}\cdot\Bigg[& \frac{\partial\rho}{\partial\xi}\cdot\left(\delta\frac{\partial\rho}{\partial\xi}\cdot\Delta\xi+\Delta\frac{\partial\rho}{\partial\xi}\cdot\delta\xi+\delta\xi\cdot\frac{\partial^2\rho}{\partial\xi^2}\cdot\Delta\xi\right)\\
& - g_n\mathbf{n}\cdot\left(\delta\frac{\partial^2\rho}{\partial\xi^2}\cdot\Delta\xi+\Delta\frac{\partial^2\rho}{\partial\xi^2}\cdot\delta\xi+\delta\xi\cdot\frac{\partial^3\rho}{\partial\xi^3}\cdot\Delta\xi\right)\\
+\left\{\Delta\frac{\partial\rho}{\partial\xi}\cdot\mathbf{n}+\mathbf{H}\cdot\Delta\xi\right\}\cdot & \left(\mathbf{I}\left\{\mathbf{n}\cdot(\delta\rho-\delta\mathbf{r}_s)\right\}+g_n\mathbf{A}^{-1}\cdot\frac{\partial\rho}{\partial\xi}\cdot\left\{\delta\frac{\partial\rho}{\partial\xi}+\frac{\partial^2\rho}{\partial\xi^2}\cdot\delta\xi\right\}\right)\\
+\left\{\delta\frac{\partial\rho}{\partial\xi}\cdot\mathbf{n}+\mathbf{H}\cdot\delta\xi\right\}\cdot & \left(\mathbf{I}\left\{\mathbf{n}\cdot(\Delta\rho-\Delta\mathbf{r}_s)\right\}+g_n\mathbf{A}^{-1}\cdot\frac{\partial\rho}{\partial\xi}\cdot\left\{\Delta\frac{\partial\rho}{\partial\xi}+\frac{\partial^2\rho}{\partial\xi^2}\cdot\Delta\xi\right\}\right)\Bigg]
\end{aligned}} \qquad (2)
$$

Approximation

Small penetration

Approximation of zero penetration

Normal gap is small $g_n \approx 0$.

$$\boxed{\delta g_n = \mathbf{n} \cdot (\delta \mathbf{r}_s - \delta \boldsymbol{\rho})} \qquad \boxed{\delta \xi = \mathbf{A}^{-1} \cdot \frac{\partial \boldsymbol{\rho}}{\partial \xi} \cdot (\delta \mathbf{r}_s - \delta \boldsymbol{\rho})}$$

$$\boxed{\Delta \delta g_n = -\mathbf{n} \cdot \left(\delta \frac{\partial \boldsymbol{\rho}}{\partial \xi} \cdot \Delta \xi + \Delta \frac{\partial \boldsymbol{\rho}}{\partial \xi} \cdot \delta \xi + \delta \xi \cdot \frac{\partial^2 \boldsymbol{\rho}}{\partial \xi^2} \cdot \Delta \xi \right)}$$

$$\boxed{\begin{aligned} \Delta \delta \xi = -\mathbf{A}^{-1} \cdot \Bigg[& \frac{\partial \boldsymbol{\rho}}{\partial \xi} \cdot \left(\delta \frac{\partial \boldsymbol{\rho}}{\partial \xi} \cdot \Delta \xi + \Delta \frac{\partial \boldsymbol{\rho}}{\partial \xi} \cdot \delta \xi + \delta \xi \cdot \frac{\partial^2 \boldsymbol{\rho}}{\partial \xi^2} \cdot \Delta \xi \right) + \\ & + \left\{ \Delta \frac{\partial \boldsymbol{\rho}}{\partial \xi} \cdot \mathbf{n} + \mathbf{H} \cdot \Delta \xi \right\} \cdot \left(\mathbf{I} \left\{ \mathbf{n} \cdot (\delta \boldsymbol{\rho} - \delta \mathbf{r}_s) \right\} \right) + \\ & + \left\{ \delta \frac{\partial \boldsymbol{\rho}}{\partial \xi} \cdot \mathbf{n} + \mathbf{H} \cdot \delta \xi \right\} \cdot \left(\mathbf{I} \left\{ \mathbf{n} \cdot (\Delta \boldsymbol{\rho} - \Delta \mathbf{r}_s) \right\} \right) \Bigg] \end{aligned}} \quad (3)$$

Discretized contact geometry

From continuum formulation to the Finite Element Method

Finite element method formalism

$$\boldsymbol{r} = \sum_{i=1}^{N} \phi_i(\boldsymbol{\xi})\boldsymbol{x}_i = \sum_{i=1}^{N} \phi_i(\xi_1,\xi_2)\boldsymbol{x}_i$$

$$[\boldsymbol{X}] = [\boldsymbol{X}(t)] = [\boldsymbol{x}_0(t),\ \boldsymbol{x}_1(t),\ \dots,\ \boldsymbol{x}_N(t)]^T;$$

$$[\Phi] = [\Phi(\boldsymbol{\xi})] = [0,\ \phi_1(\boldsymbol{\xi}),\ \dots,\ \phi_N(\boldsymbol{\xi})]^T;$$

$$[\Phi'_i] = \left[\frac{\partial\Phi(\boldsymbol{\xi})}{\partial\xi_i}\right] = [0,\ \phi_{1,i}\ ,\ \dots,\ \phi_{N,i}]^T;$$

$$\mathbf{r}_s = \mathbf{r}_s(t) = \boldsymbol{x}_0(t) = \boxed{[\mathbf{S}_0]^T[\boldsymbol{X}]},\ \text{where } [\mathbf{S}_0] = [1,\ 0,\ \dots\ ,\ 0]^T.$$

$$\boldsymbol{\rho} = \boldsymbol{\rho}(t,\boldsymbol{\xi}_p) = \phi_i(\boldsymbol{\xi}_p)\boldsymbol{x}_i = \left[\Phi(\boldsymbol{\xi}_p)\right]^T[\boldsymbol{X}(t)],= \boxed{[\Phi]^T[\boldsymbol{X}]}$$

$$\boldsymbol{\rho}_i = \frac{\partial\boldsymbol{\rho}}{\partial\xi_i} = \left.\frac{\partial\boldsymbol{\rho}(t,\boldsymbol{\xi})}{\partial\xi_i}\right|_{\boldsymbol{\xi}_p} = \left[\left.\frac{\partial\Phi(\boldsymbol{\xi})}{\partial\xi_i}\right|_{\boldsymbol{\xi}_p}\right]^T[\boldsymbol{X}(t)] = \boxed{[\Phi'_i]^T[\boldsymbol{X}]}$$

Discretized contact geometry

From continuum formulation to the Finite Element Method II

First variations of geometrical quantities

$$\boxed{\delta g_n = \begin{bmatrix} \mathbf{n} \\ -\phi_1 \mathbf{n} \\ \vdots \\ -\phi_N \mathbf{n} \end{bmatrix}^T \cdot \begin{bmatrix} \delta \boldsymbol{x}_0 \\ \delta \boldsymbol{x}_1 \\ \vdots \\ \delta \boldsymbol{x}_N \end{bmatrix} = \left[\nabla g_n\right]^T \cdot \delta \left[\boldsymbol{X}\right]} \tag{4}$$

$$\boxed{\delta \xi_i = c_{ij} \begin{bmatrix} \dfrac{\partial \boldsymbol{\rho}}{\partial \xi_j} \\ -\dfrac{\partial \boldsymbol{\rho}}{\partial \xi_j} \phi_1 + g_n \mathbf{n} \phi_{1,j} \\ \vdots \\ -\dfrac{\partial \boldsymbol{\rho}}{\partial \xi_j} \phi_N + g_n \mathbf{n} \phi_{N,j} \end{bmatrix}^T \cdot \begin{bmatrix} \delta \boldsymbol{x}_0 \\ \delta \boldsymbol{x}_1 \\ \vdots \\ \delta \boldsymbol{x}_N \end{bmatrix} = \left[\boldsymbol{\nabla} \xi_i\right]^T \cdot \delta \left[\boldsymbol{X}\right]} \tag{5}$$

Discretized contact geometry

From continuum formulation to the Finite Element Method II

Second order variation of the normal gap $\Delta\delta g_n$

$$\boxed{\begin{aligned}\Delta\delta g_n =&\delta\left[\boldsymbol{X}\right]^T \cdot \Big\{ -\mathbf{n}\left[\Phi_i'\right]\otimes\left[\boldsymbol{\nabla}\xi_i\right]^T - \left[\boldsymbol{\nabla}\xi_i\right]\otimes\left[\Phi_i'\right]^T\mathbf{n} \\ &-\left(h_{ij} - g_n h_{ik} a^{km} h_{mj}\right)\left[\boldsymbol{\nabla}\xi_i\right]\otimes\left[\boldsymbol{\nabla}\xi_j\right]^T \\ &+g_n a^{ij}\mathbf{n}\left[\Phi_i'\right]\otimes\left[\Phi_j'\right]^T\mathbf{n}\Big\}\cdot\Delta\left[\boldsymbol{X}\right] = \\ &=\delta\left[\boldsymbol{X}\right]^T\cdot\left[\nabla\nabla g_n\right]\cdot\Delta\left[\boldsymbol{X}\right]\end{aligned}} \qquad (6)$$

Discretized contact geometry

From continuum formulation to the Finite Element Method II

Second order variation of surface parameter $\Delta\delta\xi$

$$
\begin{aligned}
\Delta\delta\xi_i = \delta\,[X]^T \cdot \Bigg\{ -c_{ij} \Bigg[& \left[\Phi'_k\right] \frac{\partial\rho}{\partial\xi_j} \otimes [\nabla\xi_k]^T + [\nabla\xi_k] \otimes \frac{\partial\rho}{\partial\xi_j} \left[\Phi'_k\right]^T \\
& + \left(\frac{\partial\rho}{\partial\xi_j} \cdot \frac{\partial^2\rho}{\partial\xi_k\partial\xi_m} + g_n \mathrm{n} \cdot \frac{\partial^3\rho}{\partial\xi_k\partial\xi_j\partial\xi_m} \right) [\nabla\xi_k] \otimes [\nabla\xi_m]^T \\
& + g_n \left[\Phi''_{jk}\right] \mathrm{n} \otimes [\nabla\xi_k]^T + g_n [\nabla\xi_k] \otimes \mathrm{n} \left[\Phi''_{jk}\right]^T \\
& - \delta_{kj} \left([\nabla g_n] \otimes \left(\mathrm{n} \left[\Phi'_k\right]^T + h_{ks} [\nabla\xi_s]^T \right) + \left(\left[\Phi'_k\right] \mathrm{n} + h_{ks} [\nabla\xi_s] \right) \otimes [\nabla g_n]^T \right) \\
& + g_n a^{km} \left(\left[\Phi'_j\right] \frac{\partial\rho}{\partial\xi_m} \otimes \left(\mathrm{n} \left[\Phi'_k\right]^T + h_{ks} [\nabla\xi_s]^T \right) + \left(\mathrm{n} \left[\Phi'_k\right] + h_{ks} [\nabla\xi_s] \right) \otimes \frac{\partial\rho}{\partial\xi_m} \left[\Phi'_j\right]^T \right) \\
& + g_n a^{km} \left(\frac{\partial\rho}{\partial\xi_m} \cdot \frac{\partial^2\rho}{\partial\xi_j\partial\xi_l} \right) \left([\nabla\xi_l] \otimes \left(\mathrm{n} \left[\Phi'_k\right]^T + h_{ks} [\nabla\xi_s]^T \right) + \right. \\
& \left. + \left(\mathrm{n} \left[\Phi'_k\right] + h_{ks} [\nabla\xi_s] \right) \otimes [\nabla\xi_l]^T \right) \Bigg] \Bigg\} \cdot \Delta\,[X] = \\
= \delta\,[X]^T \cdot & \left[\nabla\nabla\xi_i\right] \cdot \Delta\,[X]
\end{aligned}
\tag{7}
$$

Plan

Introduction

Discretization

Discretization of the contact area into elementary units responsible for the contact stress transmission from one contacting surface to another.

Units:

- surface nodes;
- surface of elements;
- Gauss points onto surfaces;

Problem types:

- small deformation - no slip;
- large deformation - arbitrary slip.

Discretization method and assymetry:

- methods which enforce assymetry of surfaces:
  - node-to-segment.
- methods which reduce this assymetry:
  - segment-to-segment, mortar, Nitsche;
  - contact domain method.

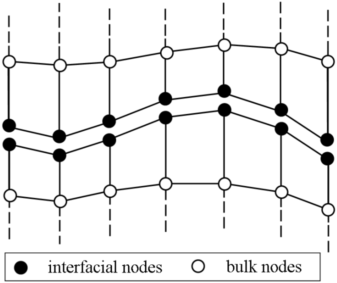
NTN – Node-to-node

Node-to-node discretization

[Francavilla & Zienkiewicz, 1975], [Oden, 1981], [Kikuchi & Oden, 1988]

Advantages:

- ☺ very simple;
- ☺ passes Taylor's test[1].



Scheme of two conforming meshes.
Pairing nodes form NTN contact elements.

[1] Taylor's patch test requires that a uniform contact stress transmittes correctly from one contacting surface to another. See section Finite Element Analysis.

NTN – Node-to-node

Node-to-node discretization

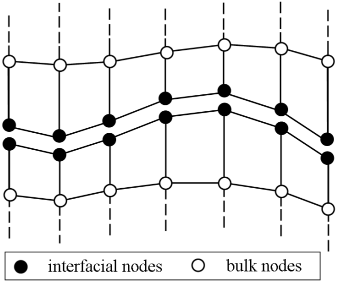
[Francavilla & Zienkiewicz, 1975], [Oden, 1981], [Kikuchi & Oden, 1988]

Advantages:

- ☺ very simple;
- ☺ passes Taylor's test[1].

Drawbacks:

- ☹ small deformation;
- ☹ small slip;
- ☹ requires conforming FE meshes.



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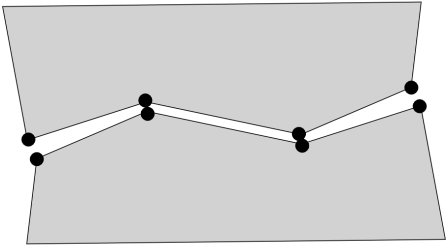
[1] Taylor's patch test requires that a uniform contact stress transmittes correctly from one contacting surface to another. See section Finite Element Analysis.

NTN – Node-to-node

Definition of the normal

Definition of the normal for the node-to-node discretization



Two FE mesh with matching nodes in the contact zone

NTN – Node-to-node

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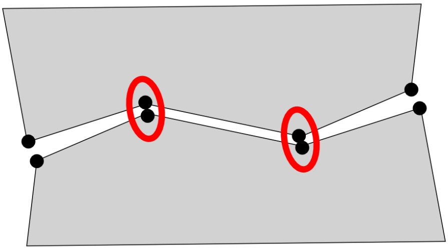

NTN contact element detection

NTN – Node-to-node

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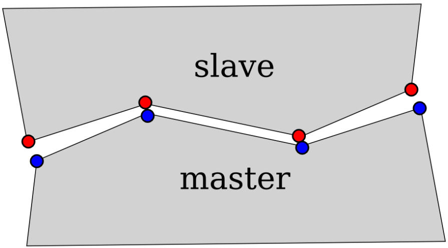

Master-slave discretization

NTN – Node-to-node

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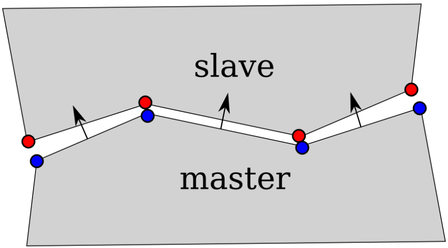

Normals on the master surface

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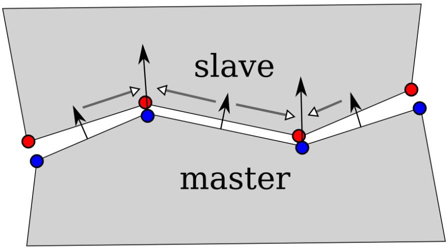

Definition of the normals at master nodes as an average of the normals of adjacent segments

NTN – Node-to-node

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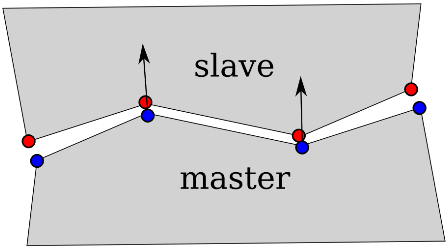


Definition of the normals at master nodes

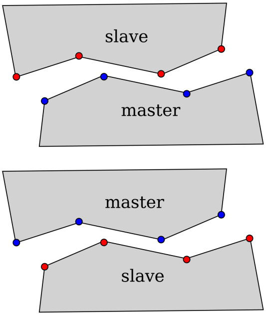
NTS – Node-to-Segment/Node-to-Surface

Node-to-segment discretization

[Hughes, 1977] [Hallquist, 1979] [Bathe & Chaudhary, 1985] [Wriggers et al., 1990]

Advantages:

- ☺ simple;
- ☺ large deformations and slip;
- ☺ mesh independent.



Scheme of two non-matching meshes

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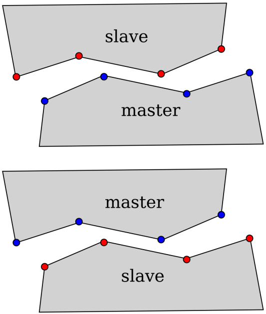

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A modified NTS algorithm passing the contact patch test

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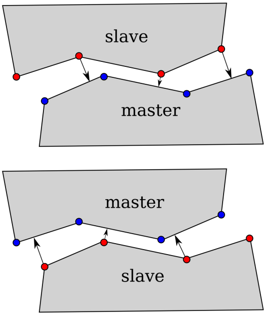

Detection of contact elements

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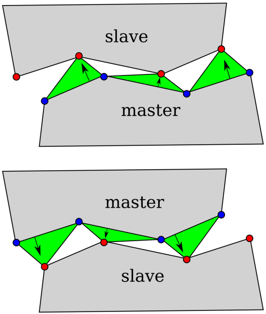

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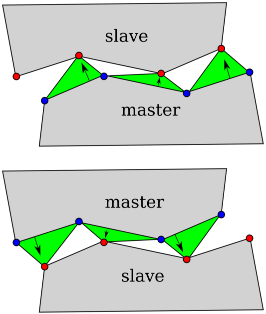

NTS contact elements

Remark: for a stable large deformation implementation NTS should be supplemented with Node-to-Vertex and Node-to-Edge.

Segment-to-segment

Segment-to-segment discretization

[Simo et al., 1985], [Zavarise & Wriggers, 1998]

Advantages:

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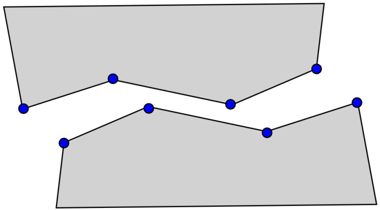

Scheme of two non-matching meshes

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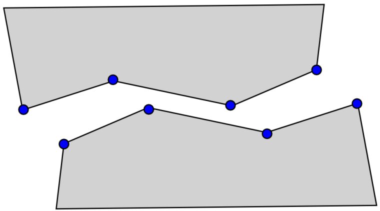
- ☹ only 2D version;
- ☹ constant contact pressure within one segment.



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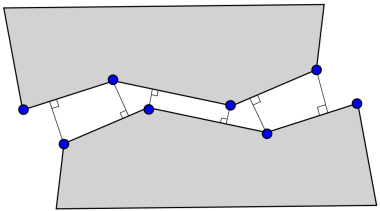

Projections of both surfaces

Segment-to-segment

Segment-to-segment discretization

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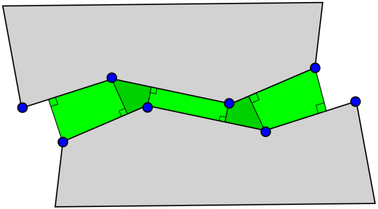

Contact sub-elements

Segment-to-segment

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[Simo et al., 1985], [Zavarise & Wriggers, 1998]

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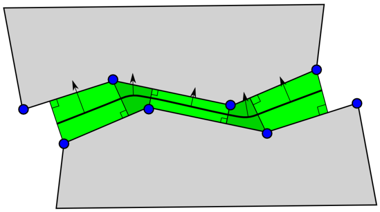

Integration line

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Mortar and Nitsche methods

Mortar and Nitsche discretizations

[Bernadi et al., 1994][Wohlmuth, 2000][Puso&Laursen, 2003][Becker&Hansbo, 1999]

Advantages:

- ☺ passes Taylor's test;
- ☺ correct contact stress distribution within contact element;
- ☺ use of any order shape functions;
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- ☺ correct contact stress distribution within contact element;
- ☺ use of any order shape functions;
- ☺ large deformations and slip;
- ☺ mesh independent.

Drawbacks:

- ☹ very complicated implementation[1];
- ☹ stability problem for curved surfaces.

[1] *"3D implementation is a nightmare, but it's feasible."*
T.A. Laursen about mortar method, ECCM, 2010

CDM – Contact Domain Method

Contact domain method for discretization

[Oliver, Hartmann et al. 2009]

Advantages:

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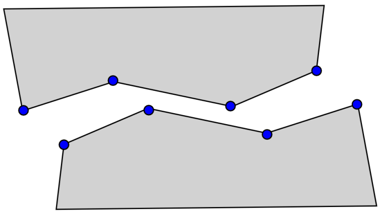

Scheme of two non-matching meshes

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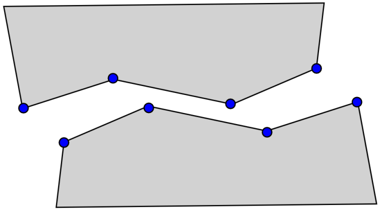

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[1] Triangluation problems for arbitrary contacting surfaces in 3D.

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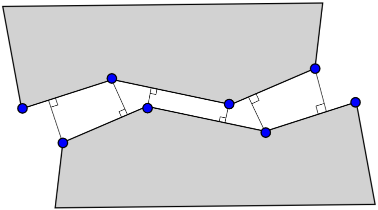

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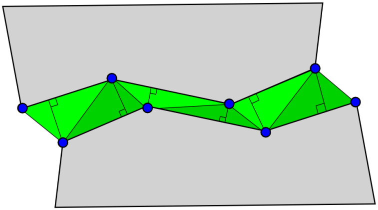

Contact domain construction

[1] Triangluation problems for arbitrary contacting surfaces in 3D.

Smoothing technique

Smothing of the master surface with

- Hermite polynoms;
- P-slines;
- Bézier curves;
- etc.

Consequences

- fulfils requirements of C^1-smoothness all along the master surface;
- nonphysical edge effects;
- complicated in 3D – requires special FE discretizations.

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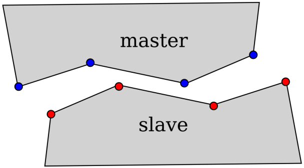

Scheme of two non-matching meshes

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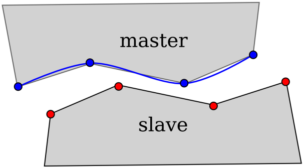

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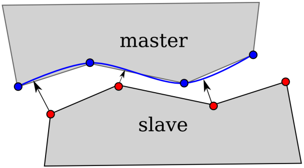

Contact detection

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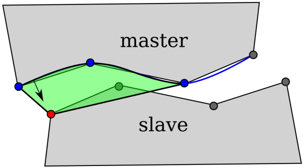

Contact element construction (edge contact element)

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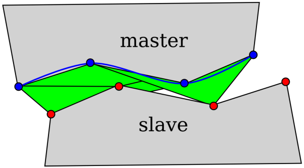
Examples of NTS contact elements smoothed with Bézier curves



Constructed smoothed contact elements

Smoothing technique

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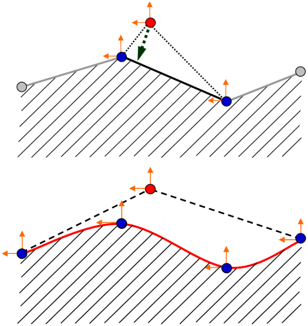

Ordinary (top) and smothed (bottom) NTS contact element in 2D

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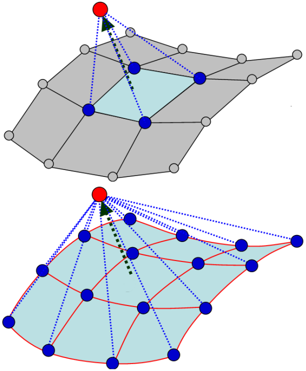

Ordinary (top) and smothed (bottom) NTS contact element in 3D

Plan

Introduction

Boundary value problem with constraints

Continous formulation of boundary value problem

- Partial differential equation

$$\boxed{\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_v = 0 \quad \text{in } \Omega^{1,2}}$$

- Neumann and Dirichlet boundary conditions

$$\boxed{\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f}_0 \quad \text{at } \Gamma_N^{1,2}, \qquad \mathbf{u} = \mathbf{u}_0 \quad \text{at } \Gamma_D^{1,2}}$$

- Contact constraints: non-penetration and non-adhesion at Γ_c – Signorini's conditions

$$\boxed{g_n \sigma_n = 0,\ g_n \geq 0,\ \boldsymbol{\sigma} \leq 0, \quad \sigma_n = \boldsymbol{\sigma} \cdot \mathbf{n}}$$

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Introduction

Weak form with contact terms

Continous formulation of the weak form for contact problems

- Weak form

$$\boxed{\int_{\Omega^{1,2}} \boldsymbol{\sigma} \cdot\cdot \delta\varepsilon d\Omega - \int_{\Omega^{1,2}} \mathbf{f}_v \cdot \delta\mathbf{u} d\Omega - \int_{\Gamma_N^{1,2}} f_0 \cdot \delta\mathbf{u} d\Gamma = 0}$$

- Contact term in the weak form, contact pressure $\sigma_n^c = \sigma \cdot \mathbf{n}$ at Γ_c

$$\boxed{\int_{\Omega^{1,2}} \boldsymbol{\sigma} \cdot\cdot \delta\varepsilon d\Omega - \int_{\Omega^{1,2}} \mathbf{f}_v \cdot \delta\mathbf{u} d\Omega - \int_{\Gamma_N^{1,2}} f_0 \cdot \delta\mathbf{u} d\Gamma - \int_{\Gamma_c} \sigma_n^c \delta g_n d\Gamma = 0}$$

- Variational inequality ($\sigma_n^c \delta g_n \geq 0$)

$$\boxed{\int_{\Omega^{1,2}} \boldsymbol{\sigma} \cdot\cdot \delta\varepsilon d\Omega \geq \int_{\Omega^{1,2}} \mathbf{f}_v \cdot \delta\mathbf{u} d\Omega + \int_{\Gamma_N^{1,2}} f_0 \cdot \delta\mathbf{u} d\Gamma}$$

- Variational equality

$$\boxed{\int_{\Omega^{1,2}} \boldsymbol{\sigma} \cdot\cdot \delta\varepsilon d\Omega - \int_{\Omega^{1,2}} \mathbf{f}_v \cdot \delta\mathbf{u} d\Omega - \int_{\Gamma_N^{1,2}} f_0 \cdot \delta\mathbf{u} d\Gamma + \mathbf{C} = 0}$$

Introduction

Weak form with contact terms

Continous formulation of the weak form for contact problems

- Weak form

$$\boxed{\int_{\Omega^{1,2}} \boldsymbol{\sigma} \cdot\cdot \delta\varepsilon d\Omega - \int_{\Omega^{1,2}} \mathbf{f}_v \cdot \delta\mathbf{u} d\Omega - \int_{\Gamma_N^{1,2}} f_0 \cdot \delta\mathbf{u} d\Gamma = 0}$$

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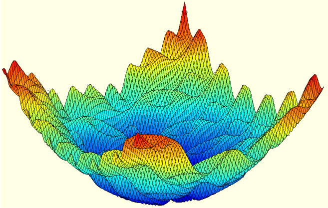
Methods for contact resolution

- Variational inequality
  [Duvaut & Lions, 1976], [Kikuchi & Oden, 1988]
- Variational equality[1]
  - **optimization methods**
    [Kikuchi & Oden, 1988], [Bertsekas, 1984], [Luenberger, 1984], [Curnier & Alart, 1991], [Wriggers, 2006]
  - **mathematical programming methods**
    [Conry & Siereg, 1971], [Klarbring, 1986]



[1] Often used with so-called **active set strategy**, which determines which contact elements are active (in contact) and which are not.

Optimization methods

Function to minimize $f(\mathbf{x})$ and constraint $g_i(\mathbf{x}) \geq 0,\ i = 1, N$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Optimization methods

Function to minimize $f(\mathbf{x})$ and constraint $g_i(\mathbf{x}) \geq 0,\ i = 1, N$

- **Penalty method**

$$f_p(\mathbf{x}) = f(\mathbf{x}) + \boxed{r \langle g(\mathbf{x}) \rangle^2}$$

$$\nabla f_p(\bar{\mathbf{x}}) = \nabla f(\bar{\mathbf{x}}) + 2r\nabla \langle g(\bar{\mathbf{x}}) \rangle \nabla g(\bar{\mathbf{x}}) = 0$$

$$\bar{\mathbf{x}} \xrightarrow{r \longrightarrow \infty} \mathbf{x}^*$$

- Lagrange multipliers method
- Augmented Lagrangian method

Optimization methods

Function to minimize $f(\mathbf{x})$ and constraint $g_i(\mathbf{x}) \geq 0,\ i = 1, N$

- **Penalty method**
- **Lagrange multipliers method**

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boxed{\lambda_i g_i(\mathbf{x})}$$

$$\min_{g(\mathbf{x}) \geq 0} \{f(\mathbf{x})\} \longrightarrow \boxed{\mathbf{x}^* = \bar{\mathbf{x}}} \longleftarrow \min\{\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})\}$$

$$\nabla_{\mathbf{x},\boldsymbol{\lambda}} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}) + \lambda_i \nabla_x g_i(\mathbf{x}) \\ g_i(\mathbf{x}) \end{bmatrix} = 0,\ \lambda_i \leq 0$$

- **Augmented Lagrangian method**

Optimization methods

Function to minimize $f(\mathbf{x})$ and constraint $g_i(\mathbf{x}) \geq 0, \;\; i = 1, N$

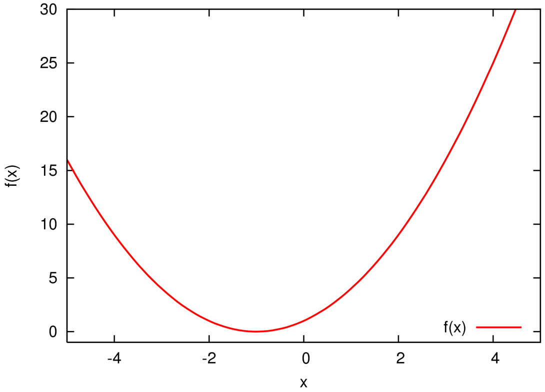
- Penalty method
- Lagrange multipliers method
- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boxed{\lambda_i g_i(\mathbf{x})} + \boxed{r \left\langle g_i(\mathbf{x}) \right\rangle^2}$$

$$\min_{g(\mathbf{x})\geq 0} \{f(\mathbf{x})\} \longrightarrow \boxed{\mathbf{x}^* = \bar{\mathbf{x}}} \longleftarrow \min\{\mathcal{L}_a(\mathbf{x}, \boldsymbol{\lambda})\}$$

$$\nabla_{\mathbf{x},\boldsymbol{\lambda}} \mathcal{L}_a = \begin{bmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}) + \lambda_i \nabla_x g_i(\mathbf{x}) + 2r\nabla \left\langle g_i(\mathbf{x}) \right\rangle \nabla g_i(\mathbf{x}) \\ g_i(\mathbf{x}) \end{bmatrix} = 0$$

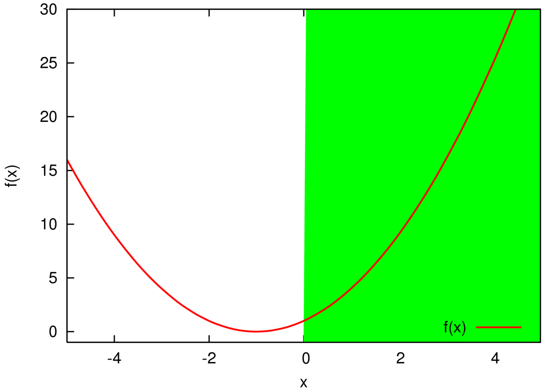
Optimization methods
Demonstration



$$\text{Function}: f(x) = x^2 + 2x + 1$$
$$\text{Constrain}: g(x) = x \geq 0$$
$$\text{Solution}: x^* = 0$$

Optimization methods

Demonstration



$$\text{Function}: \; f(x) = x^2 + 2x + 1$$

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$$\text{Solution}: \; x^* = 0$$

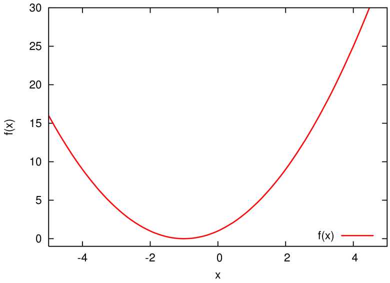
Optimization methods

Demonstration :: penalty method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Penalty method

$$f_p(x) = f(x) + \boxed{r\,\langle -g(x) \rangle^2}$$



Optimization methods

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$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

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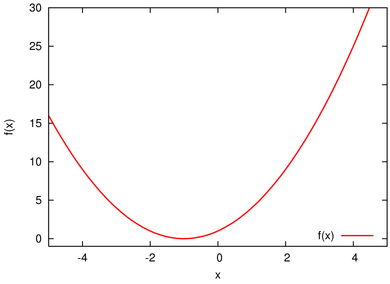
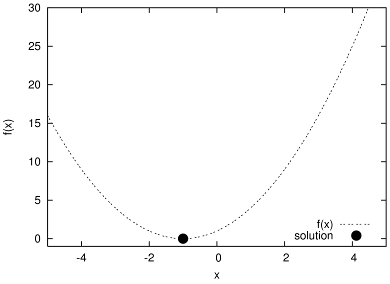




$\mathbf{r = 0}$

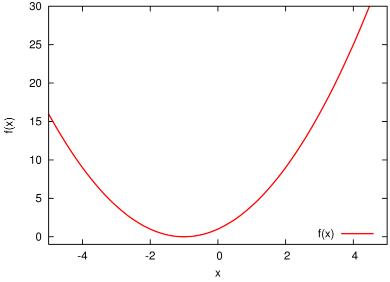
Optimization methods

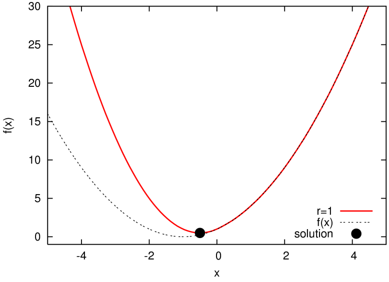
Demonstration :: penalty method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$f_p(x) = f(x) + \boxed{r\left\langle -g(x)\right\rangle^2}$$





$\mathbf{r = 1}$

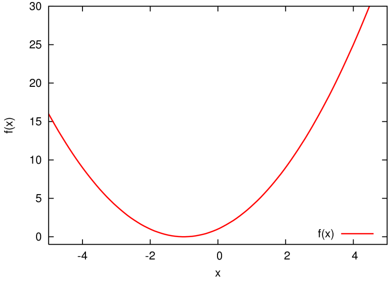
Optimization methods

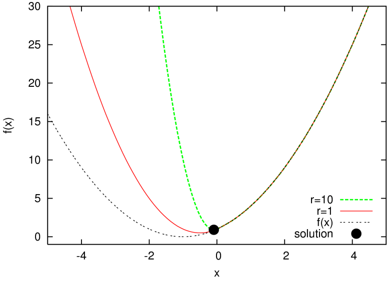
Demonstration :: penalty method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Penalty method

$$f_p(x) = f(x) + \boxed{r \left\langle -g(x) \right\rangle^2}$$





$\mathbf{r = 10}$

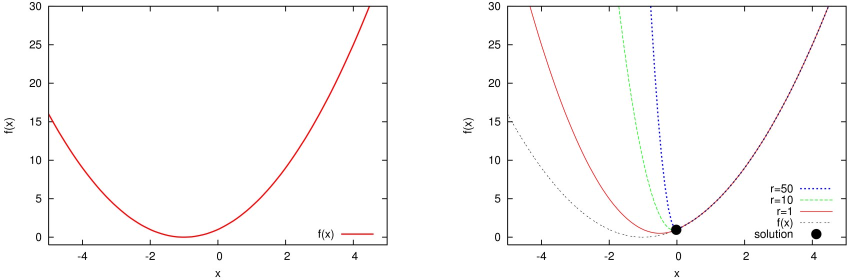
Optimization methods

Demonstration :: penalty method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$f_p(x) = f(x) + \boxed{r \left\langle -g(x) \right\rangle^2}$$



$\mathbf{r = 50}$

Optimization methods

Demonstration :: penalty method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$f_p(x) = f(x) + \boxed{r \langle -g(x) \rangle^2}$$

Advantages ☺

- simple physical interpretation;
- no additional degrees of freedom;
- smooth functional.

Drawbacks ☹

- solution is not exact:
  - too small penalty → large penetration;
  - too large penalty → ill-conditioning of the global matrix;
- user has to choose penalty r properly.

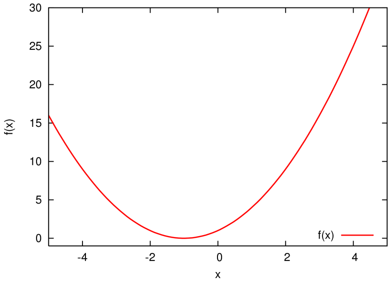
Optimization methods

Demonstration :: Lagrange multipliers method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{\lambda g(x)} \rightarrow \text{Saddle point} \rightarrow \min_x \max_\lambda \mathcal{L}(x,\lambda)$$



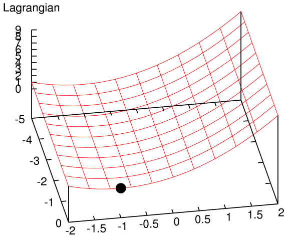
Optimization methods

Demonstration :: Lagrange multipliers method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Lagrange multipliers method

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{\lambda g(x)} \rightarrow \text{ Saddle point } \rightarrow \min_x \max_\lambda \mathcal{L}(x,\lambda)$$



Additional unknown λ

Optimization methods

Demonstration :: Lagrange multipliers method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Lagrange multipliers method

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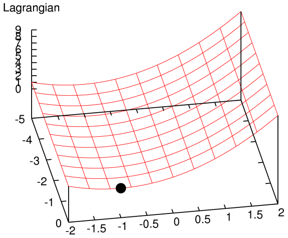
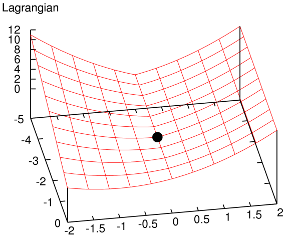


Additional unknown λ



Lagrangian

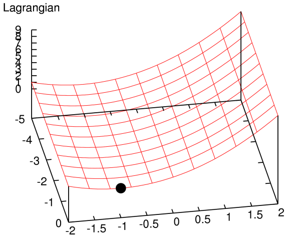
Optimization methods

Demonstration :: Lagrange multipliers method

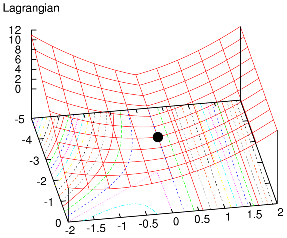
$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Lagrange multipliers method

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{\lambda g(x)} \rightarrow \text{Saddle point} \rightarrow \min_x \max_\lambda \mathcal{L}(x,\lambda)$$



Additional unknown λ



Lagrangian

Optimization methods

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Lagrange multipliers method

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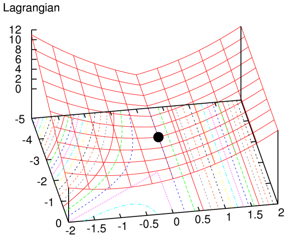
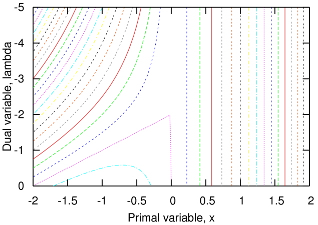


Lagrangian



Lagrangian (isolines)

Optimization methods

Demonstration :: Lagrange multipliers method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{\lambda g(x)} \to \text{Saddle point} \to \min_x \max_\lambda \mathcal{L}(x,\lambda)$$

Advantages ☺

- exact solution.

Drawbacks ☹

- Lagrangian is not smooth;
- additional degrees of freedom increase the problem.

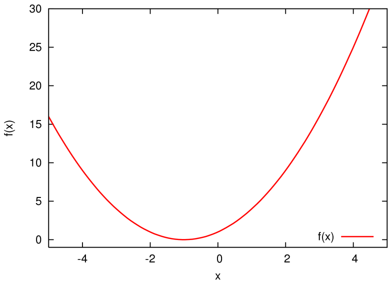
Optimization methods

Demonstration :: Augmented Lagrangian method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{r\left\langle -g(x)\right\rangle^2} + \boxed{\lambda g(x)} \to \min_x \max_\lambda \mathcal{L}(x,\lambda)$$



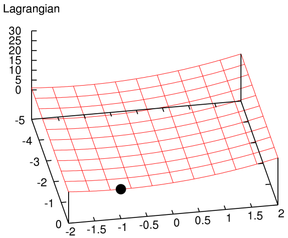
Optimization methods

Demonstration :: Augmented Lagrangian method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Augmented Lagrangian method

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{r\,\langle -g(x)\rangle^2} + \boxed{\lambda g(x)} \rightarrow \min_x \max_\lambda \mathcal{L}(x,\lambda)$$



Additional unknown λ

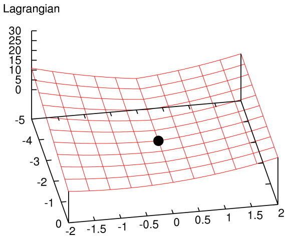
Optimization methods

Demonstration :: Augmented Lagrangian method

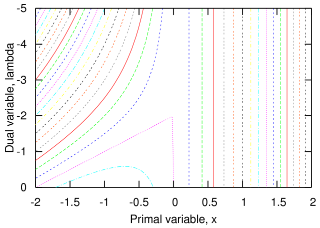
$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Augmented Lagrangian method

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{r\,\langle -g(x)\rangle^2} + \boxed{\lambda g(x)} \rightarrow \min_x \max_\lambda \mathcal{L}(x,\lambda)$$



Standard Lagrangian $r = 0$



Isolines of the standard Lagrangian $r = 0$

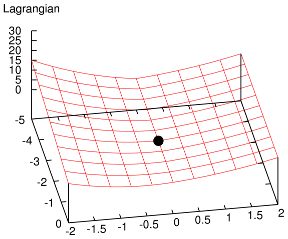
Optimization methods

Demonstration :: Augmented Lagrangian method

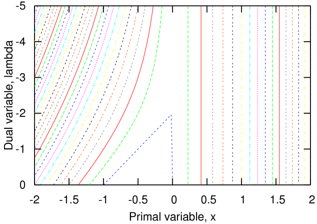
$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Augmented Lagrangian method

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{r\,\langle -g(x)\rangle^2} + \boxed{\lambda g(x)} \rightarrow \min_x \max_\lambda \mathcal{L}(x,\lambda)$$



Augmented Lagrangian $r = 1$



Isolines of the augmented Lagrangian $r = 1$

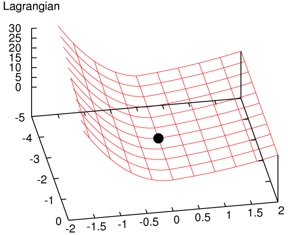
Optimization methods

Demonstration :: Augmented Lagrangian method

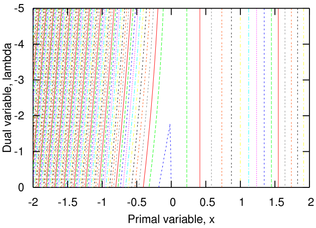
$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- ## Augmented Lagrangian method

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{r\left\langle -g(x)\right\rangle^2} + \boxed{\lambda g(x)} \to \min_x \max_\lambda \mathcal{L}(x,\lambda)$$



Augmented Lagrangian $r = 10$



Isolines of the augmented Lagrangian $r = 10$

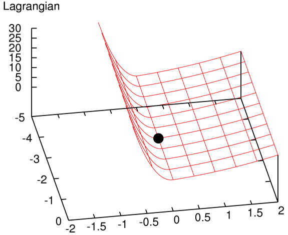
Optimization methods

Demonstration :: Augmented Lagrangian method

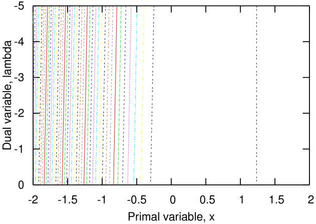
$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

Augmented Lagrangian method

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{r\left\langle -g(x)\right\rangle^2} + \boxed{\lambda g(x)} \rightarrow \min_x \max_\lambda \mathcal{L}(x,\lambda)$$



Augmented Lagrangian $r = 50$



Isolines of the augmented Lagrangian $r = 50$

Optimization methods

Demonstration :: Augmented Lagrangian method

$$f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}(x,\lambda) = f(x) + \boxed{r\,\langle -g(x)\rangle^2} + \boxed{\lambda g(x)} \to \min_x \max_\lambda \mathcal{L}(x,\lambda)$$

Advantages ☺

- exact solution;
- smoothed functional.

Drawbacks ☹

- additional degrees of freedom increase the problem.

Optimization methods

Augmented Lagrangian method + Uzawa algorithm

- Augmented Lagrangian

$$\mathcal{L}(x,\lambda) = f(x) + r\left\langle -g(x)\right\rangle^2 + \lambda g(x)$$

- Necessary conditions of the solution

$$\begin{bmatrix}\nabla_x \mathcal{L}(x,\lambda)\\ \nabla_\lambda \mathcal{L}(x,\lambda)\end{bmatrix} = 0 \begin{bmatrix}\nabla_x f(x)\\ 0\end{bmatrix} + \begin{bmatrix}-2r\left\langle -g(x)\right\rangle \nabla_x g(x)\\ 0\end{bmatrix} + \begin{bmatrix}\lambda\nabla g(x)\\ g(x)\end{bmatrix}$$

- Uzawa algorithm

$$\boxed{\lambda^{i+1} = \lambda^i - 2r\left\langle -g(x)\right\rangle}$$

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Advantages ☺

- exact solution;
- smoothed functional;
- no additional degrees of freedom.

Drawbacks ☹

☺

Plan

Introduction

- FEA requires
  - good finite element mesh
    - represents the real geometry;
    - fine enough to represent correctly stress-strain field;
    - rough enough to solve the problem in reasonable terms.
  - comprehension how close we are to the real solution;
  - careful apposition of boundary conditions.

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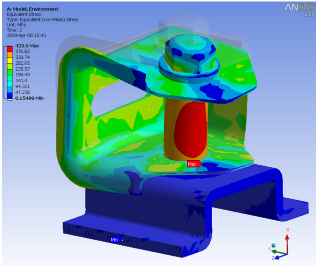
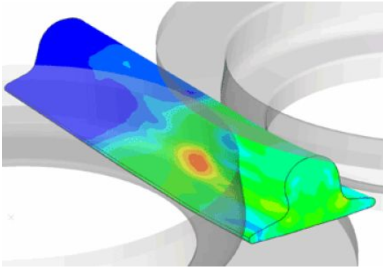


Example of contact problem solved in ANSYS (no FE mesh presented)

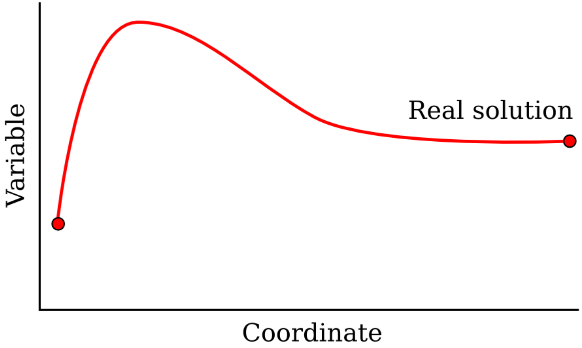


Example of contact problem solved in ABAQUS (no FE mesh presented)

Convergence by mesh

Basics

One dimensional example on mesh refinement



Convergence by mesh

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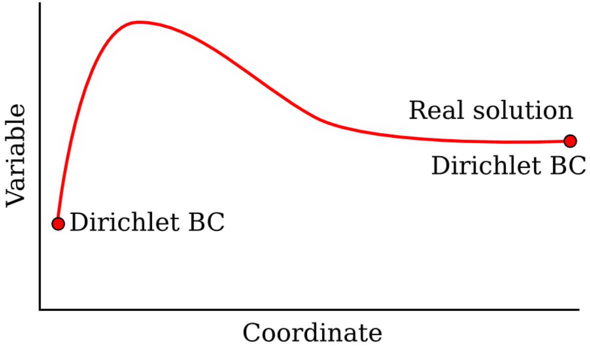
One dimensional example on mesh refinement



Convergence by mesh

Basics

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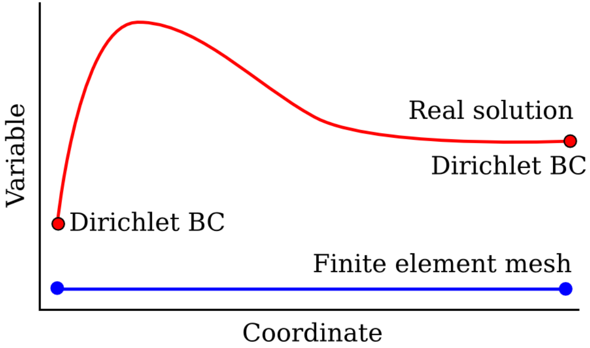

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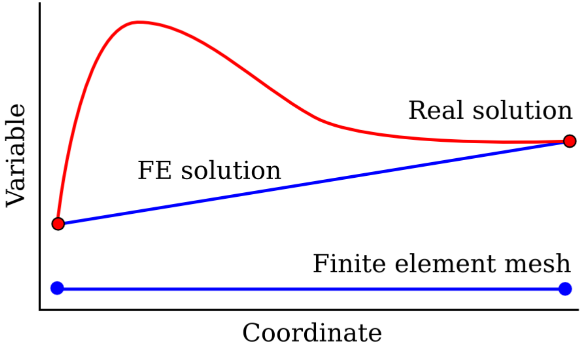

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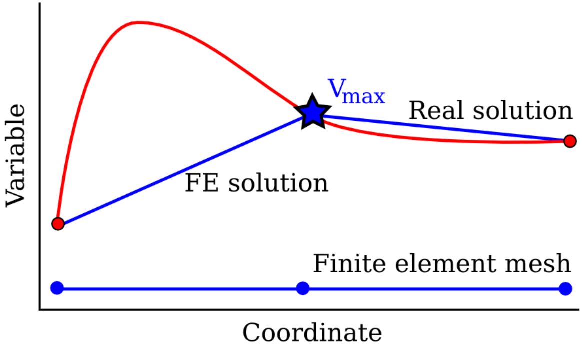

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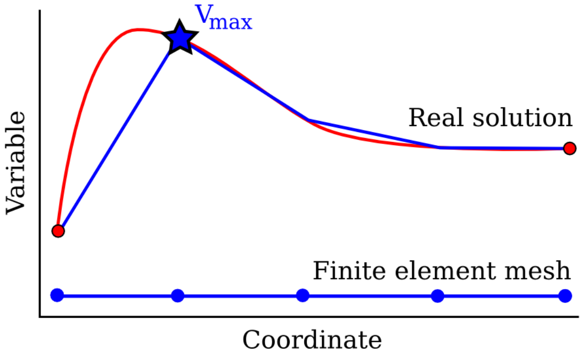

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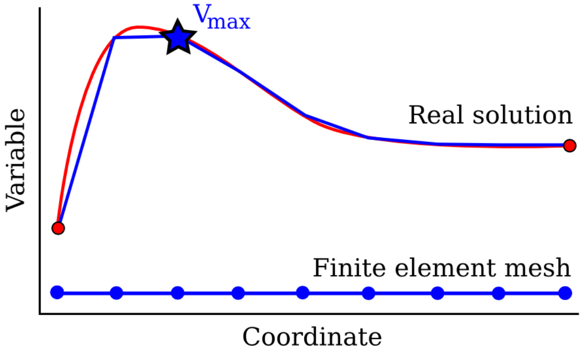

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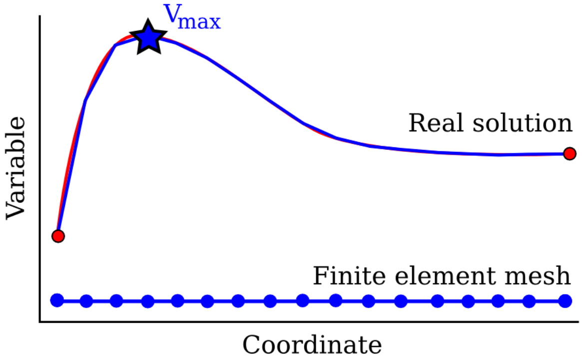

Convergence by mesh

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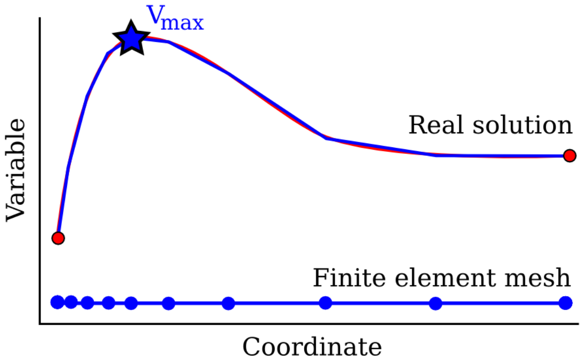
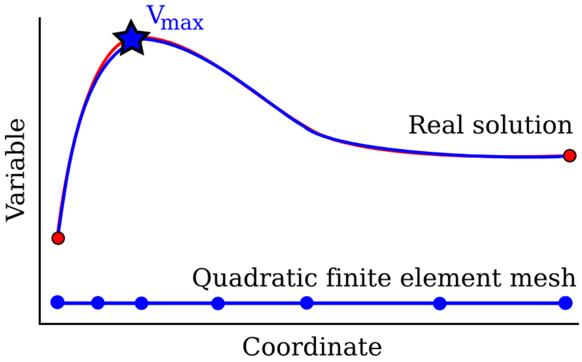


Smart meshing with linear elements

Convergence by mesh
Basics

One dimensional example on mesh refinement

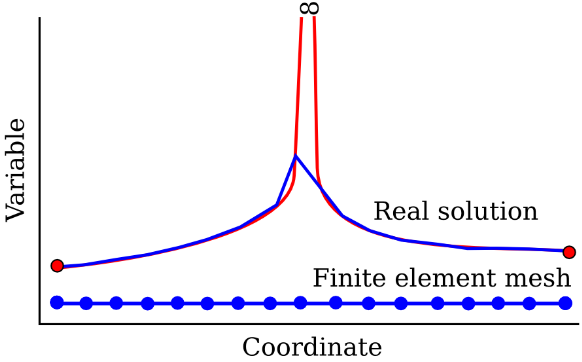


Smart meshing with quadratic elements

Convergence by mesh

Basics

One dimensional example on mesh refinement

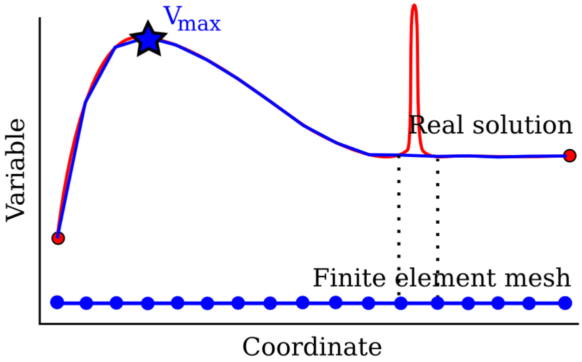


Case of singularity

Convergence by mesh

Basics

One dimensional example on mesh refinement

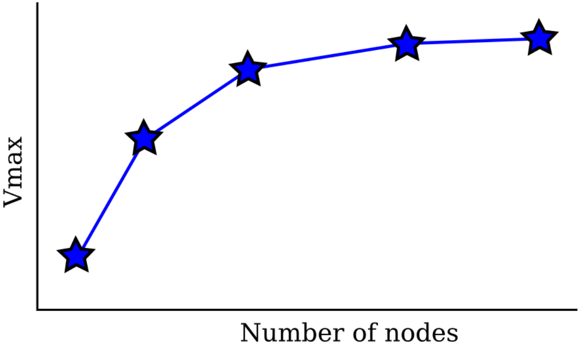


Case of hidden maximum

Convergence by mesh

Basics

One dimensional example on mesh refinement

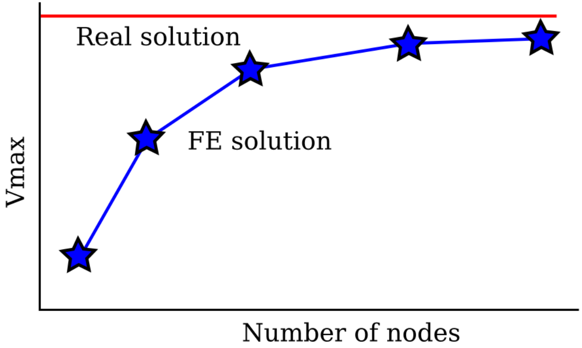


Convergence by mesh

Convergence by mesh

Basics

One dimensional example on mesh refinement



Convergence by mesh

Convergence by mesh

Basics

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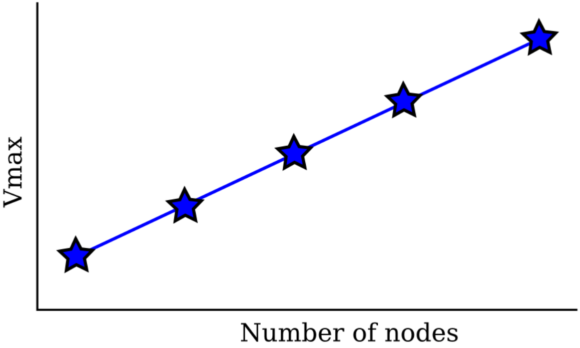
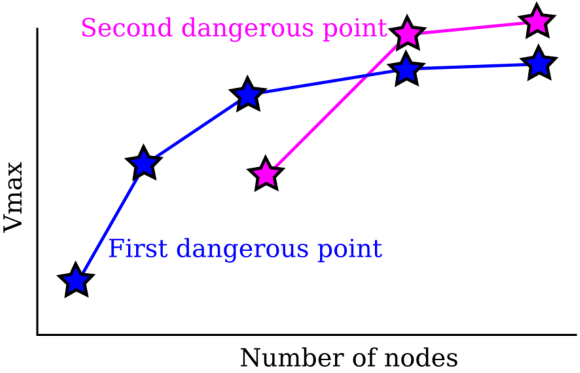


No convergence by mesh (singularity)

Convergence by mesh

Basics

One dimensional example on mesh refinement



Case of two maximums

Boundary conditions

How fast can we go?

General thinks

- Contact problems are always nonlinear
- Nonlinear problems requires slow change of boundary conditions
  - infinite looping;
  - convergence to the wrong solution.

Boundary conditions

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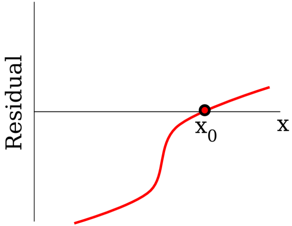
Resolution of nonlinear problem



Departure point $R(x_0, f_0) = 0$

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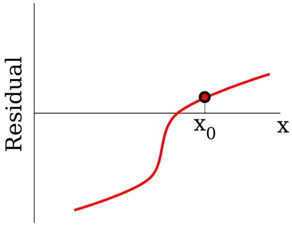

Change of boundary conditions $R(x_0, f_1) \neq 0$

Boundary conditions

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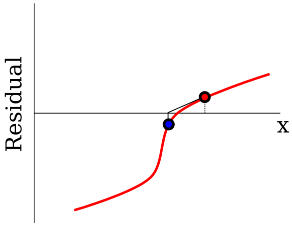

Newton-Raphson iterations

$$R(x_0, f_1) + \frac{\partial R}{\partial x}\Big|_{x_0} \delta x = 0 \rightarrow \delta x = -\frac{\partial R}{\partial x}\Big|_{x_0} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

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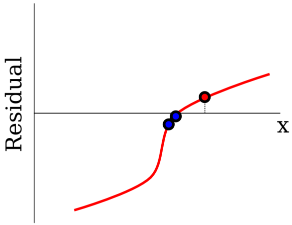

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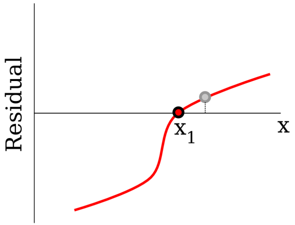

Convergence $\|x^{i+1} - x^i\| \le \varepsilon \to x_1 = x^{i+1}$

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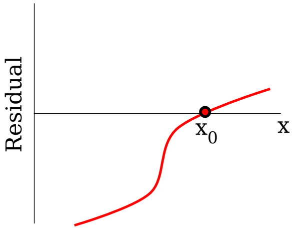
Infinite looping



Departure point $R(x_0, f_0) = 0$

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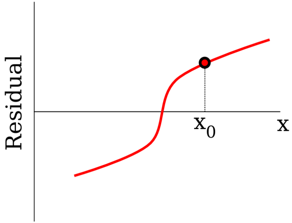

Too fast change of boundary conditions $R(x_0, f_1) \neq 0$

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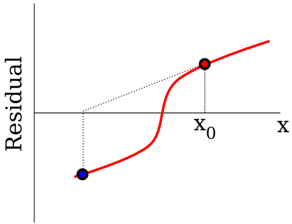

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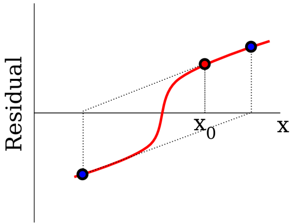

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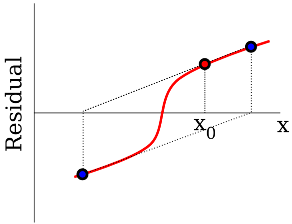

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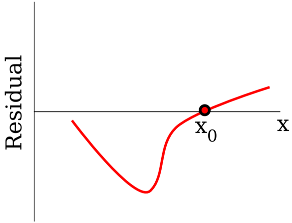

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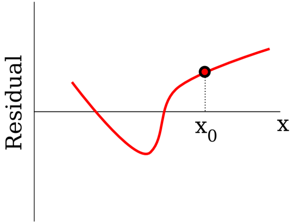

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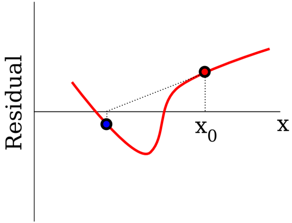

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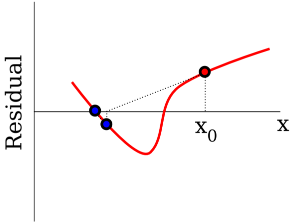

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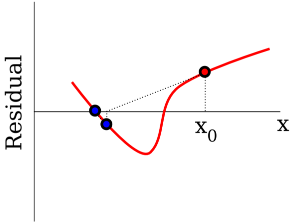
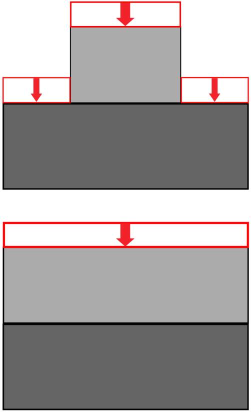


Solution. Correct?

Patch test

Methods passing Taylor's patch test

- mortar;
- Nitsche;
- node-to-node;
- Contact domain method.



Possible schemes for contact patch test

Patch test

Methods passing Taylor's patch test

- mortar;
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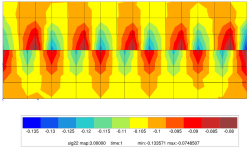
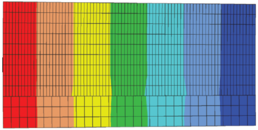
Method not passing Taylor's patch test

- node-to-segment.





NTS not passing patch test - oscilation of contact pressure (top)
Nitsche method passing patch test

Patch test

Methods passing Taylor's patch test

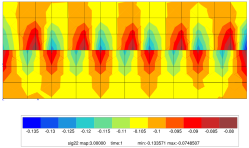
- mortar;
- Nitsche;
- node-to-node;
- Contact domain method.

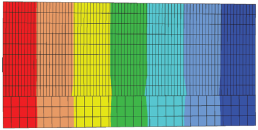
Method not passing Taylor's patch test

- node-to-segment.

But!

- NTS passes the patch test in two pass;
- NTS passing patch test [G. Zavarise, L. De Lorenzis, 2009];
- Revisiting Taylor's patch test [Crisfield].





NTS not passing patch test - oscilation of contact pressure (top)
Nitsche method passing patch test

Master-slave discretization

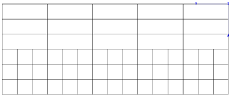
General rules

Rule 1

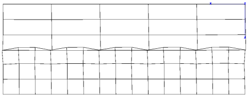
- Contacting surface with higher mesh density is always slave surface.

Rule 1

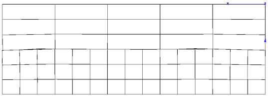
- If the mesh densities are equal on two surfaces, master surface is the surface which deforms less.



Initial FE mesh



Incorrect master-slave choice ☹



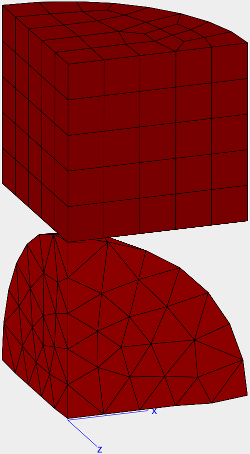
Correct master-slave choice ☺

Plan

Validation

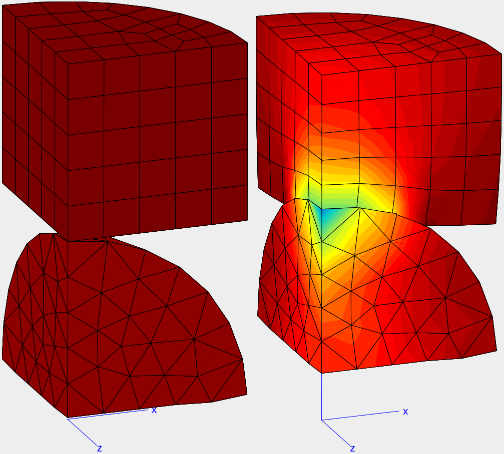
3D contact

Increments



Validation
3D contact

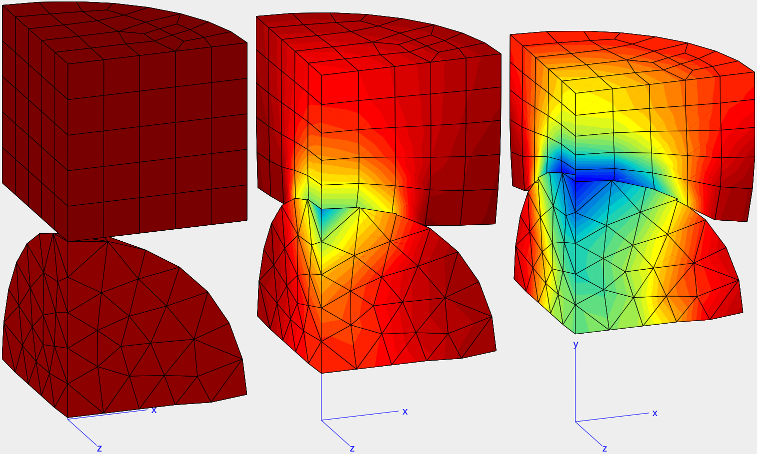
Increments



Validation

3D contact

Increments



Validation

Shallow ironing

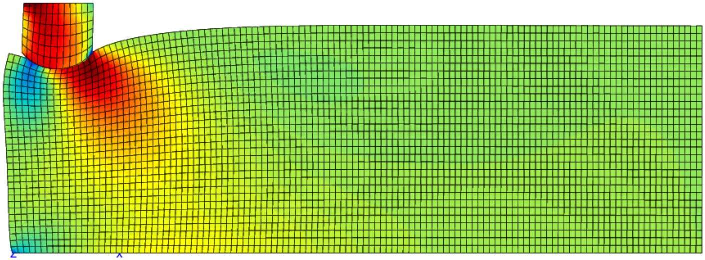
Finite element mesh

Description

- Plane strain
- $E_i = 68.96 \cdot 10^8$ Pa
- $E_s = 68.96 \cdot 10^7$ Pa
- $\nu_i = \nu_s = 0.32$
- $\mu = 0.3$
- $\Delta u_v = 1\text{mm}/10$ incr
- $\Delta u_h = 10\text{mm}/500\text{incr}$
- $NN = 3840$

Validation
Shallow ironing

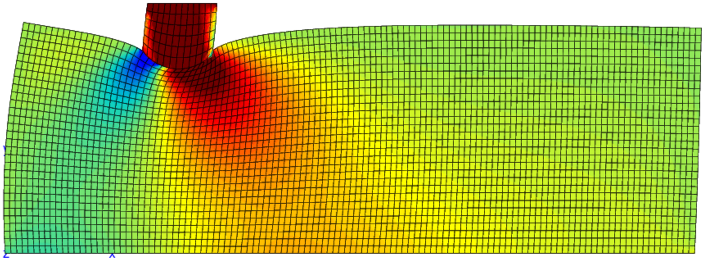
Results <Stress$_{12}$>



Validation

Shallow ironing

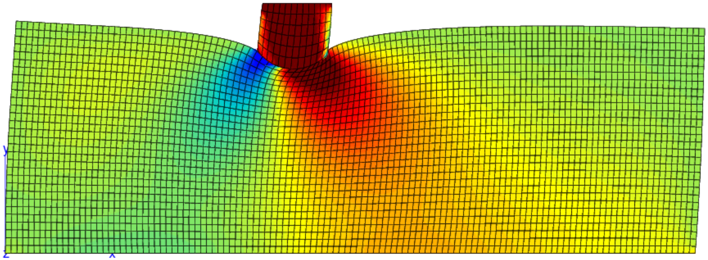
Results <Stress$_{12}$>



Validation

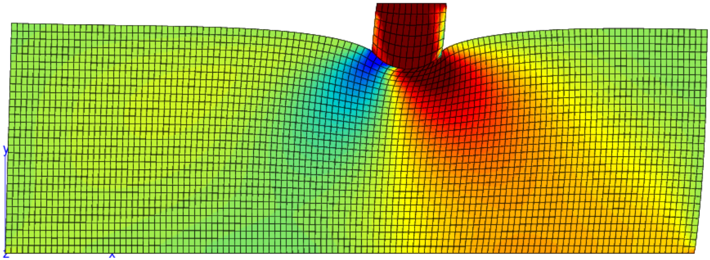
Shallow ironing

Results <Stress$_{12}$>



Validation
Shallow ironing

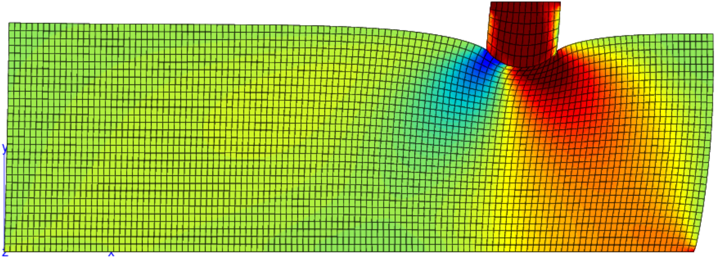
Results <Stress$_{12}$>



Validation

Shallow ironing

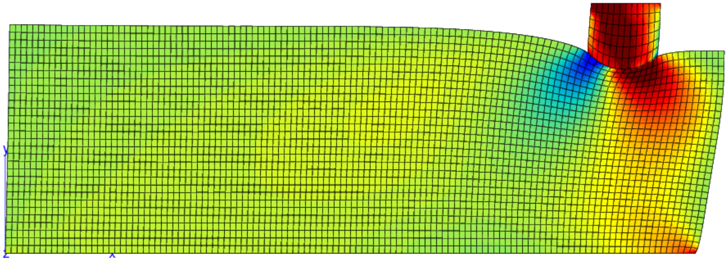
Results <Stress$_{12}$>



Validation

Shallow ironing

Results <Stress$_{12}$>

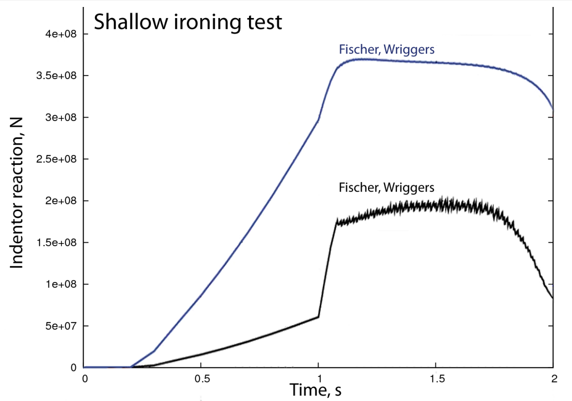


Validation

Shallow ironing

Comparison

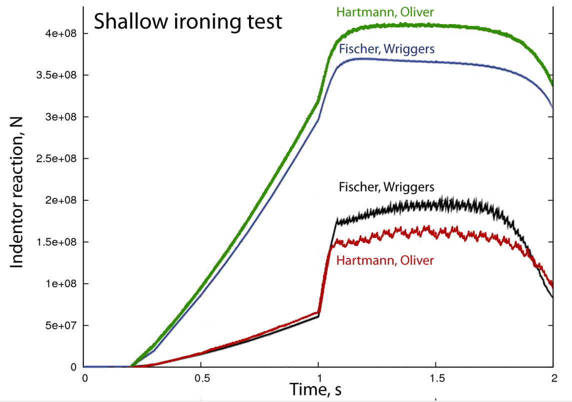
- K.A.Fischer, P. Wriggers [2006]



Validation
Shallow ironing

Comparison

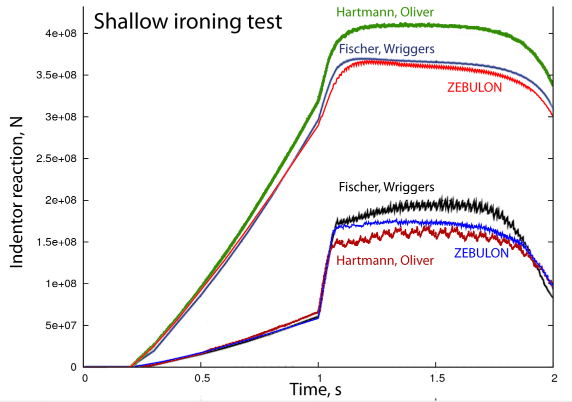
- J. Oliver, S. Hartmann [2009]



Validation
Shallow ironing

Comparison

- Our results [2009]

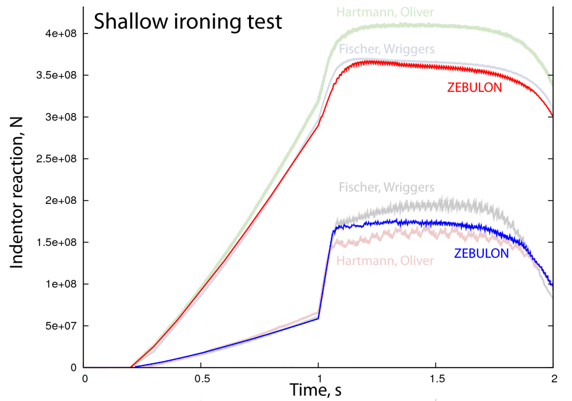


Validation

Shallow ironing

Comparison

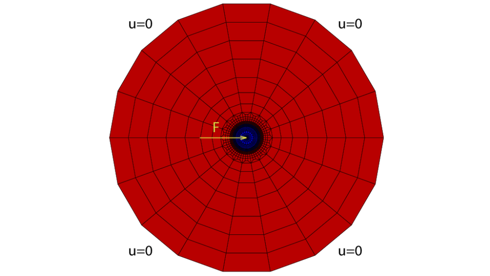
- Our results [2009]



Validation

Klang's problem

Finite element mesh

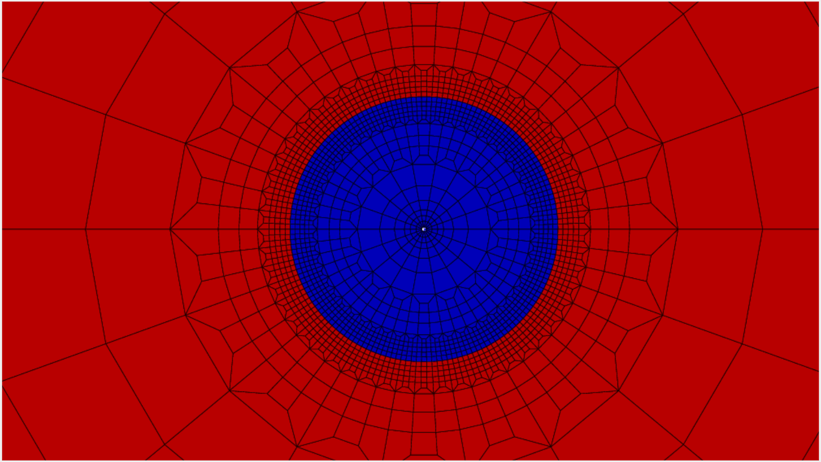


Description

- Plane stress
- $E = 2.1 \cdot 10^{11}$ Pa
- $\nu = 0.3$
- $\mu = 0.4$
- $r = 5.999$ cm
- $R = 6$ cm
- $F = 18750$ N
- $\alpha = 120^\circ$
- $NN = 2500$

Validation
Klang's problem

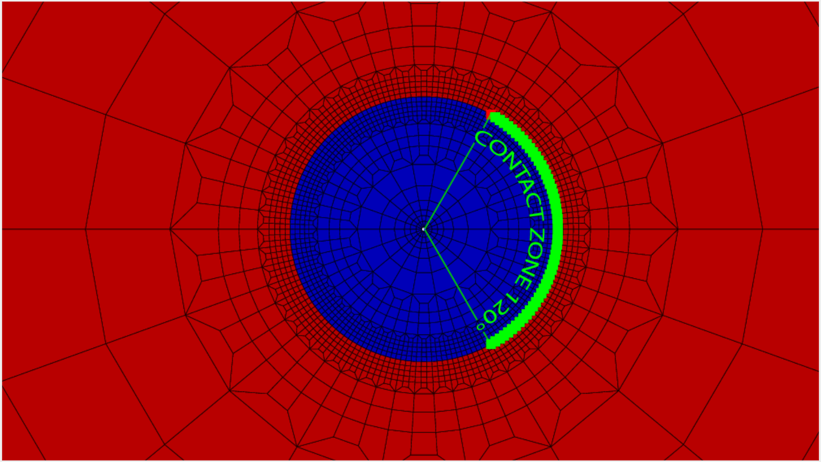
Finite element mesh



Validation

Klang's problem

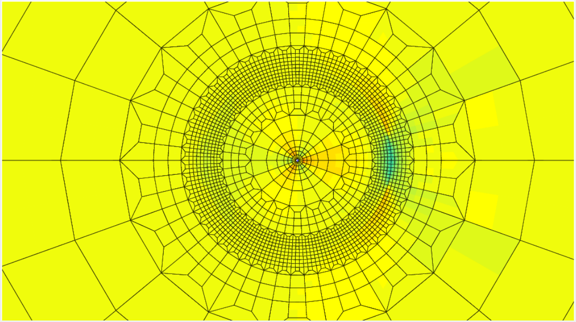
Finite element mesh



Validation

Klang's problem

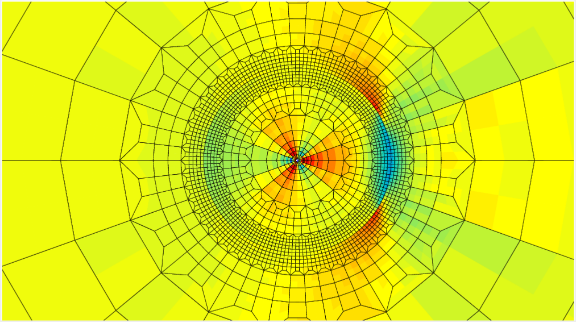
Results <Stress$_{22}$>



Validation

Klang's problem

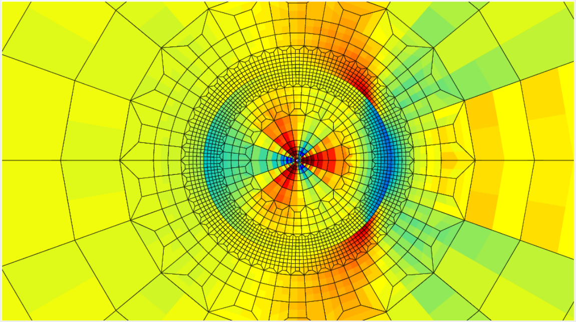
Results <Stress$_{22}$>



Validation

Klang's problem

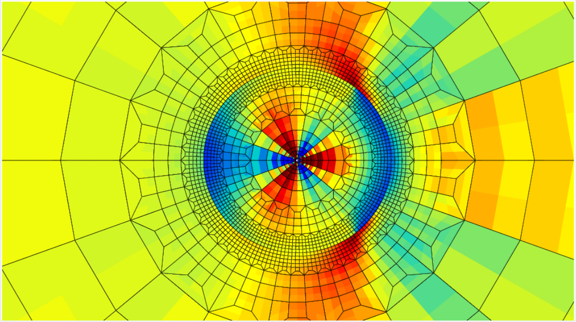
Results <$\mathtt{Stress}_{22}$>



Validation

Klang's problem

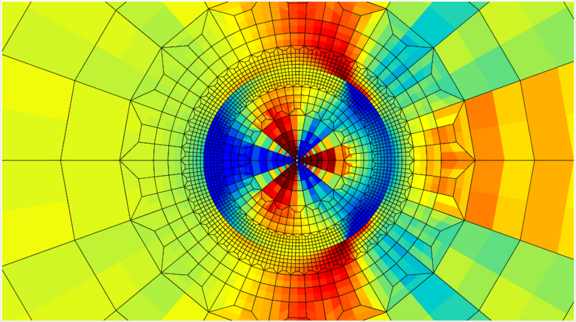
Results <Stress$_{22}$>



Validation

Klang's problem

Results <Stress$_{22}$>

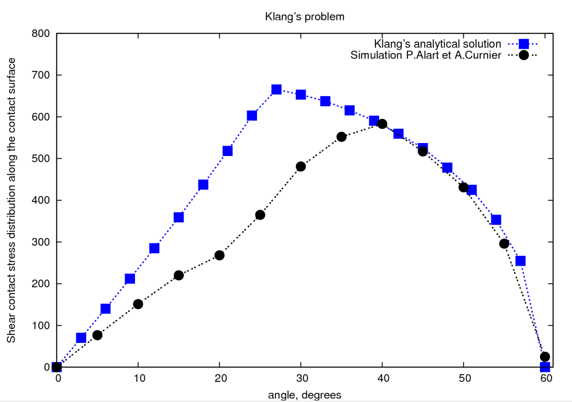


Validation

Klang's problem

Results

Semianalytical, K.M.Klang [1979]. Simulation, P.Alart and A.Curnier [1990]

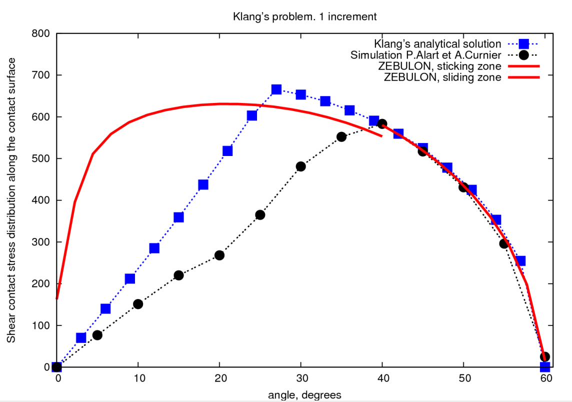


Validation

Klang's problem

Results

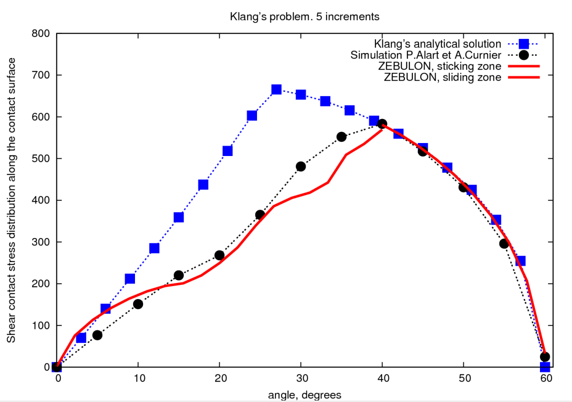
ZEBULON, 1 increment



Validation
Klang's problem

Results

ZEBULON, 5 increment

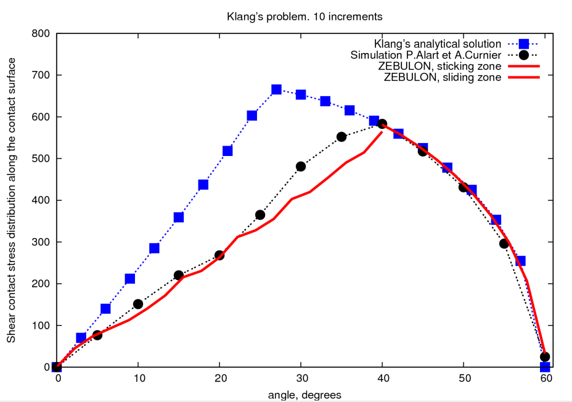


Validation

Klang's problem

Results

ZEBULON, 10 increment

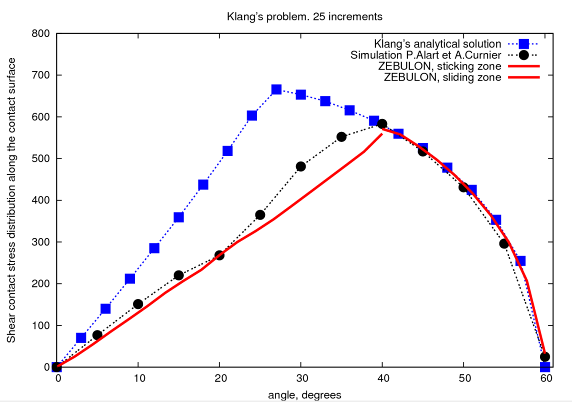


Validation

Klang's problem

Results

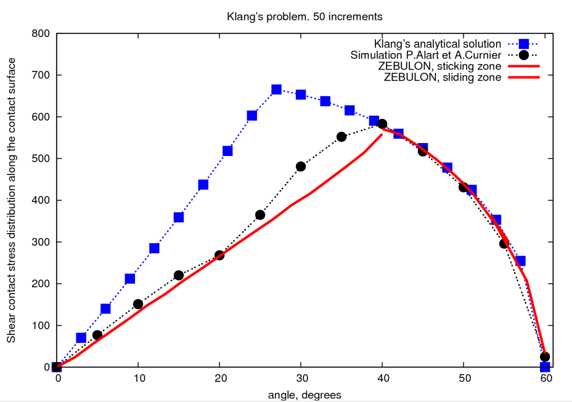
ZEBULON, 25 increment



Validation
Klang's problem

Results

ZEBULON, 50 increment

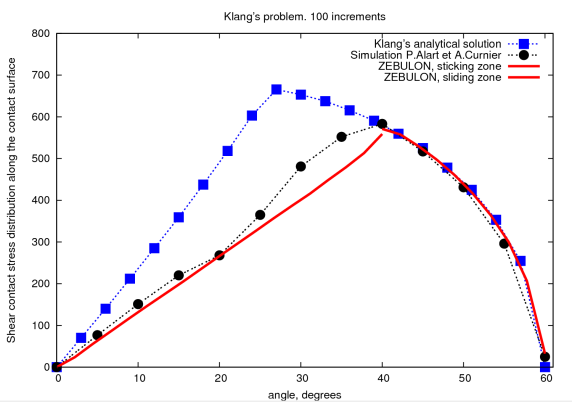


Validation

Klang's problem

Results

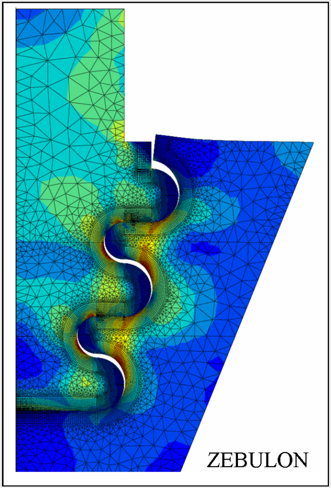
ZEBULON, 100 increment



Performance

Disk-blade contact

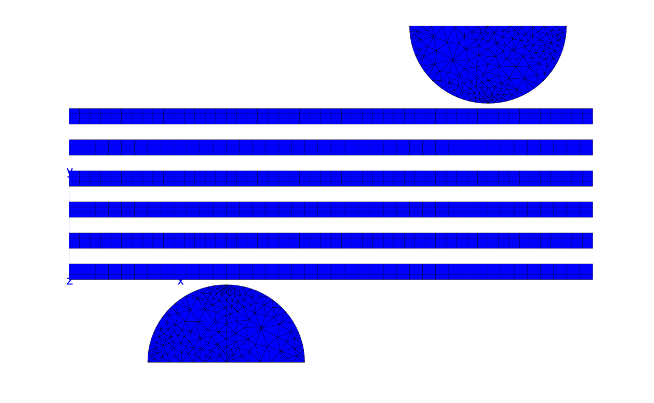
Disk-blade frictional contact, elasto-plastic material



Performance

Multi contact

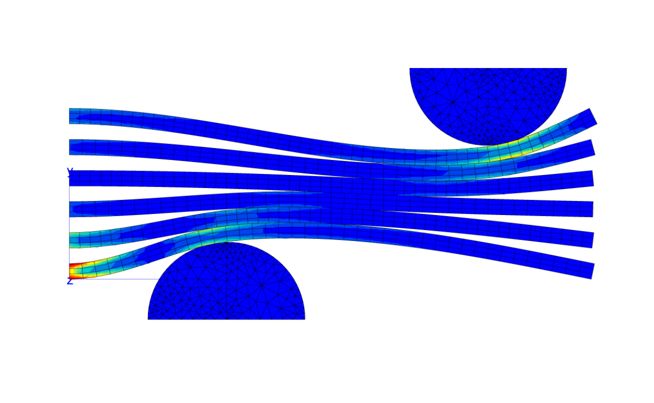
Multi plate frictionless contact



Performance

Multi contact

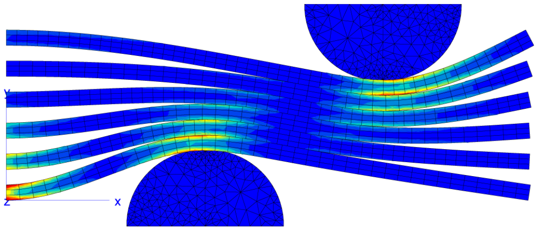
Multi plate frictionless contact



Performance

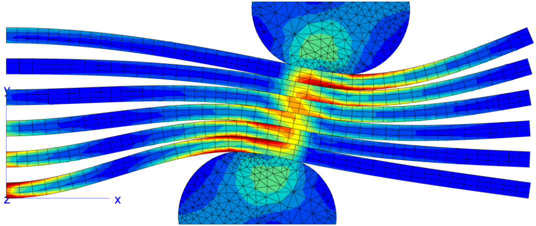
Multi contact

Multi plate frictionless contact



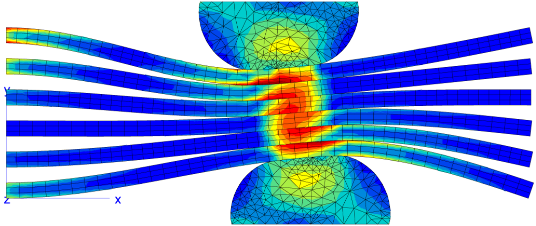
Performance
Multi contact

Multi plate frictionless contact



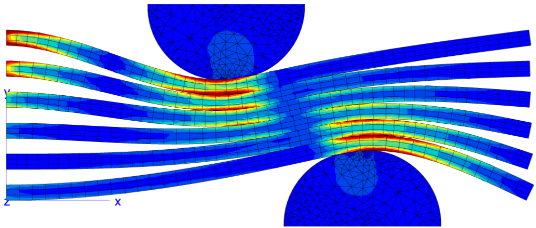
Performance
Multi contact

Multi plate frictionless contact



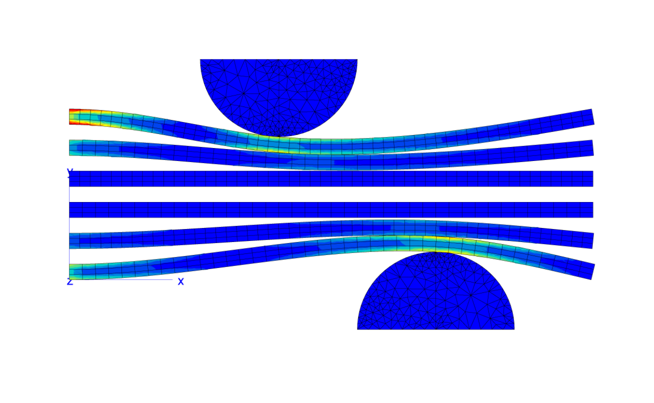
Performance
Multi contact

Multi plate frictionless contact



Performance
Multi contact

Multi plate frictionless contact



Performance

Multi contact

Multi plate frictionless contact