ML II: Exercise 2

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1 Conditional Independence

P(A) = 1st coin flip shows head
P(B) = 2nd coin flip shows head
P(C) = coin is biased (has two heads)

Table 1: C=0

A	B	$P(A\|C)$	$P(B\|C)$	$P(A,B\|C)$
0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Table 2: C=1

A	B	$P(A\|C)$	$P(B\|C)$	$P(A,B\|C)$
0	1	0	1	0
0	0	0	0	0
1	1	1	1	1
1	0	1	0	0

$$
\begin{aligned}
P(A) &= P(A|C)P(C) + P(A|\overline{C})P(\overline{C}) \\
&= \frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \frac{3}{4} \\
P(B) &= P(B|C)P(C) + P(B|\overline{C})P(\overline{C}) \\
&= \frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \frac{3}{4} \\
P(A,B) &= P(A,B|C)P(C) + P(A,B|\overline{C})P(\overline{C}) \\
&= P(A|C)P(B|C)P(C) + P(A|\overline{C})P(B|\overline{C})P(\overline{C}) \\
&= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + 1\cdot 1\cdot\frac{1}{2} = \frac{5}{8} \\
P(A)P(B) &= \frac{9}{16} \neq P(A,B) = \frac{5}{8}
\end{aligned}
$$

2 Boy Problem

2.1 Information Theory

d is the positive Event, either Sunday or Today.

$$
\begin{aligned}
&\frac{p\left(A=b \wedge B=b \wedge \left(\left(A=b \wedge A_d=d\right) \vee \left(B=b \wedge B_d=d\right)\right)\right)}{p\left(\left(A=b \wedge A_d=d\right) \vee \left(B=b \wedge B_d=d\right)\right)} \\
=&\frac{p\left(A=b \wedge B=b \wedge \neg\left(A_d=\neg d \wedge B_d=\neg d\right)\right)}{p\left(\neg\left(\neg\left(A=b \wedge A_d=d\right) \wedge \neg\left(B=b \wedge B_d=d\right)\right)\right)} \\
=&\frac{\frac{1}{2}\frac{1}{2}\left(1-\frac{C-1}{C}\frac{C-1}{C}\right)}{1-\left(1-\frac{1}{2C}\right)^2} \\
=&\frac{\frac{1}{4}\left(1-\frac{(C-1)^2}{C^2}\right)}{1-\left(1-\frac{1}{2C}\right)^2} \\
=&\frac{\frac{1}{4}\left(1-\frac{(C^2-2C+1)}{C^2}\right)}{\frac{1}{C}-\frac{1}{4C^2}} \qquad \leftarrow \text{ expand by } \frac{C^2}{C^2} \\
=&\frac{\frac{1}{4}(2C-1)}{C-\frac{1}{4}} \\
=&\frac{2C-1}{4C-1}
\end{aligned}
$$

2.2 Numerical Evaluation

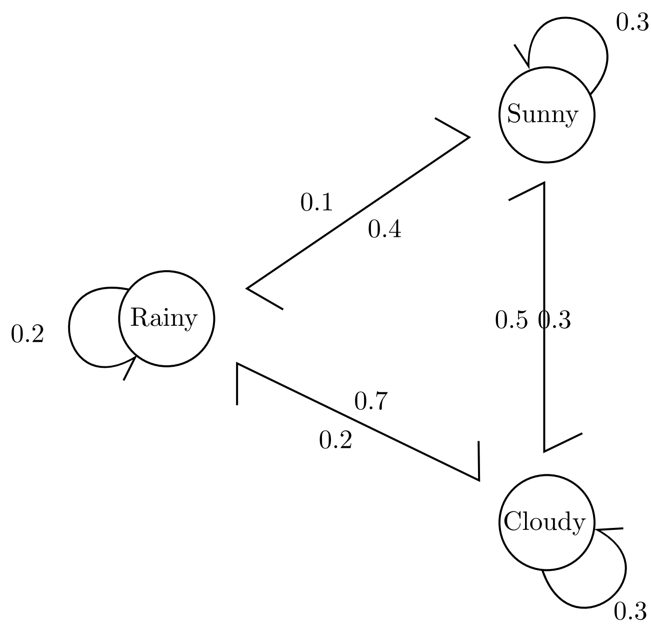
see associated Jupyter notebook.

3 Weather Forecast

a) The missing probabilities are:

$$P(t_n = \text{Cloudy}|t_{n-1} = \text{Rain}) = \frac{7}{10}$$
$$P(t_n = \text{Sunny}|t_{n-1} = \text{Sunny}) = \frac{3}{10}$$
$$P(t_n = \text{Sunny}|t_{n-1} = \text{Cloudy}) = \frac{1}{2}$$

b) Transitions as a diagram



c & d) Rain = R, Sunny = S, Cloudy = C

$$\vec{p}(t_1)$$
$$= \begin{pmatrix} p(t_n = \text{R}|t_{n-1} = \text{R}) \cdot p(t_0 = \text{R}) + p(t_n = \text{R}|t_{n-1} = \text{C}) \cdot p(t_0 = \text{C}) + p(t_n = \text{R}|t_{n-1} = \text{S}) \cdot p(t_0 = \text{S}) \\ p(t_n = \text{C}|t_{n-1} = \text{R}) \cdot p(t_0 = \text{R}) + p(t_n = \text{C}|t_{n-1} = \text{C}) \cdot p(t_0 = \text{C}) + p(t_n = \text{C}|t_{n-1} = \text{S}) \cdot p(t_0 = \text{S}) \\ p(t_n = \text{S}|t_{n-1} = \text{R}) \cdot p(t_0 = \text{R}) + p(t_n = \text{S}|t_{n-1} = \text{C}) \cdot p(t_0 = \text{C}) + p(t_n = \text{S}|t_{n-1} = \text{S}) \cdot p(t_0 = \text{S}) \end{pmatrix}$$
$$= M \times \vec{p}(t_0) = \begin{pmatrix} 0.2 & 0.2 & 0.4 \\ 0.7 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.3 \end{pmatrix} \times \vec{p}(t_0) = \begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix}$$

e) we compute

$$\vec{p}(t_n) = M^n \times \vec{p}(t_0)$$
$$\vec{p}(t_{100}) = M^{100} \times \vec{p}(t_0) = \begin{pmatrix} 0.265625 \\ 0.406250 \\ 0.328125 \end{pmatrix}$$

f) for $\lim n \to \infty$, we solve the equation: $\vec{p}(t_n) = M \times \vec{p}(t_n)$ in a system of linear equations with $p(t_n = \mathrm{R}) = r, p(t_n = \mathrm{C}) = c, p(t_n = \mathrm{S}) = s$:

$$r = 0.2 \cdot r + 0.2 \cdot c + 0.4 \cdot s$$
$$c = 0.7 \cdot r + 0.3 \cdot c + 0.3 \cdot s$$
$$s = 0.1 \cdot r + 0.5 \cdot c + 0.3 \cdot s$$
$$1 = r + c + s$$

Solving these equations results the solution equal to our result for $\vec{p}(t_{100})$ (due to rounding in e)