Linear Nonhomogeneous Recurrence Relations

Connection between Homogeneous and Nonhomogeneous Problems

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2. Main theorem
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Introduction

$$c_m a_{n+m} + c_{m-1} a_{n+m-1} + \cdots + c_1 a_{n+1} + c_0 a_n = g(n), \quad n \geq 0 \qquad (*)$$

$$\sum_{k=0}^{m} c_k a_{n+k} = g(n), \quad c_0 c_m \neq 0$$

$$c_m a_{n+m} + c_{m-1} a_{n+m-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \quad n \geq 0 \qquad (**)$$

$$\sum_{k=0}^{m} c_k a_{n+k} = 0, \quad c_0 c_m \neq 0$$

- The solutions of linear *nonhomogeneous recurrence relations* are closely related to those of the corresponding homogeneous equations.
- First of all, remember Corrolary 3, Section 21:

  If v_n and w_n are two solutions of the nonhomogeneous equation $(*)$,

  then $\varphi_n = w_n - v_n$, $n \geq 0$ is a solution of the homogeneous equation $(**)$.

Main theorem

Theorem

Consider the following linear constant coefficient recurrence relation

$$c_m a_{n+m} + \cdots + c_1 a_{n+1} + c_0 a_n = g(n), \quad c_0 c_m \neq 0, \quad n \geq 0 \qquad (*)$$

and its corresponding homogeneous form

$$c_m a_{n+m} + \cdots + c_1 a_{n+1} + c_0 a_n = 0\ . \qquad (**)$$

If u_n is the general solution of the homogeneous equation (**), and v_n is any `particular solution` *of the nonhomogeneous equation (*), then*

$$a_n = u_n + v_n\ , \quad n \geq 0$$

is the `general solution` *of the nonhomogeneous equation (*).*

Main theorem

Proof.

- For $a_n = u_n + v_n$, we have

$$
\begin{aligned}
c_m a_{n+m} + \cdots + c_1 a_{n+1} + c_0 a_n &= \sum_{i=0}^{m} c_i a_{n+i} = \overbrace{\sum_{i=0}^{m} c_i u_{n+i}}^{\substack{0 \\ \parallel}} + \overbrace{\sum_{i=0}^{m} c_i v_{n+i}}^{\substack{g(n) \\ \parallel}} \\
&= g(n) \ ,
\end{aligned}
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i.e., a_n satisfies the non-homogeneous recurrence relation $(*)$.

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- Since the general solution u_n of the homogeneous problem has m arbitrary constants thus so is $a_n = u_n + v_n$.

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- Hence a_n is the general solution of $(*)$.

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i.e., a_n satisfies the non-homogeneous recurrence relation $(*)$.

- Since the general solution u_n of the homogeneous problem has m arbitrary constants thus so is $a_n = u_n + v_n$.
- Hence a_n is the general solution of $(*)$.
- More precisely, for any solution w_n of $(*)$, since $\varphi_n = w_n - v_n$ satisfies $(**)$, φ_n will just be a special case of the general solution u_n of $(**)$.

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- Since the general solution u_n of the homogeneous problem has m arbitrary constants thus so is $a_n = u_n + v_n$.
- Hence a_n is the general solution of $(*)$.
- More precisely, for any solution w_n of $(*)$, since $\varphi_n = w_n - v_n$ satisfies $(**)$, φ_n will just be a special case of the general solution u_n of $(**)$.
- Hence $w_n = \varphi_n + v_n$ is included in the solution $a_n = u_n + v_n$.
- Therefore, $a_n = u_n + v_n$ is the general solution of the `nonhomogeneous problem` $(*)$.

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- The substitution of $a_n = C3^n$ into the recurrence relation thus gives

$$\underbrace{C \cdot 3^{n+2}}_{\substack{\| \\ a_{n+2}}} - 5 \cdot \underbrace{C \cdot 3^n}_{\substack{\| \\ a_n}} = 2 \times 3^n \ ,$$

i.e., $4C = 2$ or $C = \dfrac{1}{2}$. Hence $\boxed{a_n = \dfrac{1}{2} \times 3^n}$ for $n \geq 0$ is a particular solution.

Example 2

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with constants A and B to be determined.
- Hence f_n be a solution requires
$$\begin{aligned} 6n &= f_{n+1} - 2f_n + 3f_{n-4} \\ &= \big(A(n+1) + B\big) - 2(An + B) + 3\big(A(n-4) + B\big) \\ &= 2An + (2B - 11A) \end{aligned}$$
i.e.
$$\begin{aligned} 2A &= 6 \\ 2B - 11A &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} A &= 3 \\ B &= 33/2 \end{aligned}$$

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- Therefore our particular solution is $\boxed{f_n = 3n + \frac{33}{2}}$.

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Find the particular solution of $\quad a_{n+3} - 7a_{n+2} + 16a_{n+1} - 12a_n = 4^n n$ with

$$a_0 = -2\ , \quad a_1 = 0\ , \quad a_2 = 5\ .$$

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Solution.

- We first find the general solution u_n for the corresponding homogeneous problem.
- Then we look for a particular solution v_n for the nonhomogeneous problem without concerning ourselves with the *initial conditions*.
- Once these two are done, we obtain the general solution $a_n = u_n + v_n$ for the nonhomogeneous recurrence relation, and we just need to use the initial conditions to determine the arbitrary constants in the general solution a_n so as to derive the final particular solution.

Example 3

(a) The associated characteristic equation $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$ can be shown to admit the following roots $\lambda_1 = 3$, $m_1 = 1$, (simple root), $\lambda_2 = 2$, $m_2 = 2$, (double root):

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = \lambda^3 - 3\lambda^2 - 4\lambda^2 + 16\lambda - 12 =$$

$$\lambda^2(\lambda - 3) - 4(\lambda^2 - 4\lambda + 3) = \lambda^2(\lambda - 3) - 4(\lambda - 3)(\lambda - 1) =$$

$$(\lambda - 3)(\lambda^2 - 4\lambda + 4) = (\lambda - 3)(\lambda - 2)^2$$

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$$u_n = A3^n + (B + Cn)2^n, \quad n \geq 0\ .$$

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$$u_n = A3^n + (B + Cn)2^n, \quad n \geq 0 \; .$$
- That is, u_n solves $a_{n+3} - 7a_{n+2} + 16a_{n+1} - 12a_n = 0$.

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(b) Since the r.h.s. of the nonhomogeneous recurrence relation is $4^n \cdot n$, which fits into the description of $4^n \times$ (`first order polynomial in` n), we'll try a particular solution in a similar form, i.e.,

$$v_n = 4^n(Dn + E).$$

The substitution of v_n into the original recurrence relation then gives

$$\begin{aligned} 4^n \cdot n &= v_{n+3} - 7v_{n+2} + 16v_{n+1} - 12v_n \\ &= 4^{n+3}(D(n+3) + E) - 7 \times 4^{n+2}(D(n+2) + E) \\ &\quad + 16 \times 4^{n+1}(D(n+1) + E) - 12 \times 4^n(Dn + E), \text{ i.e.,} \end{aligned}$$

$$\begin{aligned} n &= 64(Dn + 3D + E) - 112(Dn + 2D + E) + 64(Dn + D + E) - 12(Dn + E) \\ &= 4Dn + 4E + 32D\,. \end{aligned}$$

Hence we have

$$4D = 1\,, \quad 4E + 32D = 0 \qquad \Leftrightarrow \qquad D = \frac{1}{4}\,, \quad E = -2$$

and consequently $v_n = 4^n(\frac{n}{4} - 2)\,.$

Example 3

(c) The general solution for the nonhomogeneous problem is then given by $a_n = u_n + v_n$, i.e.

$$a_n = 4^n \left(\frac{n}{4} - 2 \right) + A3^n + (B + Cn)2^n \ , \quad n \geq 0 \ .$$

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$$a_n = 4^n \left(\frac{n}{4} - 2\right) + A3^n + (B + Cn)2^n \ , \quad n \geq 0 \ .$$

(d) We now determine A, B, C by the `initial conditions` and the use of the solution expression in (c)

Initial Conditions	Induced Equations	Solutions
$a_0 = -2$	$A + B - 2 = -2$	$A = 1$
$a_1 = 0$	$3A + 2B + 2C - 7 = 0$	$B = -1$
$a_2 = 5$	$9A + 4B + 8C = 29$	$C = 3$

Finally the particular solution satisfying both the nonhomogeneous recurrence relations and the initial conditions is given by

$$a_n = 4^n \left(\tfrac{n}{4} - 2\right) + 3^n + (3n - 1)2^n \ , \quad n \geq 0 \ .$$

Notes

1. In all the examples in this lecture, it is easy to verify that the $g(n)$ function in $(*)$ is in the form of

$$g(n) = \mu^n(\alpha_k n^k + \cdots + \alpha_1 n + \alpha_0) \ ,$$

where μ is *not* a root of the associated characteristic equation.

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2. If $g(n) = \mu_1^n n + \mu_2^n(3n^2 + 1)$, for instance, with μ_1 and μ_2 neither being a root of the characteristic equation, then the `particular solution` should be tried in the form
$$v_n = \mu_1^n(A_1 n + A_0) + \mu_2^n(B_2 n^2 + B_1 n + B_0) \ .$$

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3. If $g(n) = \cos(\alpha n) \cdot n$, for another instance, then we can treat it as

$$g(n) = \frac{(e^{i\alpha n} + e^{-i\alpha n})}{2} n = \frac{n}{2} \times \mu_1^n + \frac{n}{2} \times \mu_2^n$$

in which $\mu_1 = e^{i\alpha}$ and $\mu_2 = e^{-i\alpha}$.
Alternatively, we could try the particular solution in the form

$$v_n = \sin(\alpha n)(A_1 n + A_0) + \cos(\alpha n)(B_1 n + B_0) .$$