A HAND BOOK OF APPLIED MATHEMATICS-III

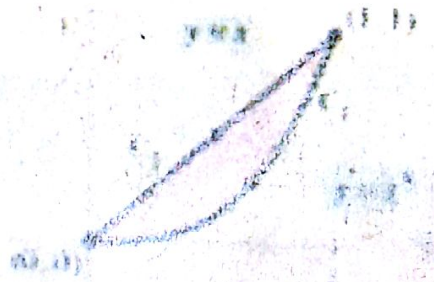
PROBLEM SOLVING APPROACH

For Engineering, Science and Technology Students

Method of Variation of Parameters:

$$y_p(x) = -y_1 \int \frac{y_2 f(x)}{W(y_1, y_2)}\, dx + y_2 \int \frac{y_1 f(x)}{W(y_1, y_2)}\, dx$$

Additivity of Line Integrals



$$C = C_1 \cup C_2$$

$$\int_C r(t)\,dt = \int_{C_1} r(t)\,dt + \int_{C_2} r(t)\,dt$$

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A HAND BOOK OF APPLIED MATHEMATICS-III

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CHAPTER-1

Ordinary Differential Equations (ODE)

1.1 Definition and Classifications of Differential Equations

An equation containing (involving) derivative of the dependent variable with respect to independent variables is known as *differential equation* (DE).

There are two basic types of differential equations.

a) **Ordinary Differential Equation (ODE):** A differential equation involving derivatives of the dependent variable with respect to *one* independent variable.

b) **Partial Differential Equation (PDE):** A differential equation containing the derivatives of the dependent variable with respect to *more than one* independent variables.

Notations: $\frac{dy}{dx}=y', \frac{d^2y}{dx^2}=y'', \frac{d^3y}{dx^3}=y''',..., \frac{d^ny}{dx^n}=y^{(n)}$ in ODE.

Examples:

a) Examples of ODE: i) $\frac{dy}{dx}=2x$ ii) $y''+4y=xe^{3x}$ iii) $\frac{d^3y}{dx^3}-6\frac{dy}{dx}=0$

b) Examples of PDE: i) $\frac{\partial z}{\partial x}=4xy$ ii) $\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial y}=x^2+y^2$ iii) $z_{xx}=2y$

Order and Degree of Differential Equations:

a) **Order of a Differential Equation:** It refers the order of the term with the highest derivative in the Differential Equation.

In a differential equation, y' represents first order, y'' represents second order, y''' represents third order and in general, $y^{(n)}$ represents n^{th} order derivative. So, if we get derivatives of different orders in a single differential equation, we take the highest order as the order of that differential equation.

b) **Degree of a Differential Equation:** It refers the highest power or exponent (positive integer only) of the term with the highest order derivative.

Remarks: Main points about Order and Degree

1) The degree of a DE is identified after the exponents of all the terms with any order derivatives are made to have positive integer exponents. This may need squaring or cubing both sides of the equation.

Besides, if the term with the highest order is part of an operation like multiplication, you have to simplify the operation before reading the degree.

2) Degree of a differential equation is defined only if the DE is a polynomial in all the terms with the derivatives $y',\ y'',\ y''',\ ...,\ y^n$. In general, the degree of a differential equation that contains any of the following exceptions need not be defined. Because of this, every DE has order but may not have degree.

Exceptions: $e^{\frac{dy}{dx}}$ or $e^{y'}$, $\ln\frac{dy}{dx}$ or $\ln y'$, $\sin(y')$ or $\sin(\frac{dy}{dx})$, $\cos(y')$, $\sin^{-1}y'$, $\cos^{-1}y'$.

Examples: Identify the order and degree of the following DEs.

a) $y'''^5+4y''^7-2y^8=x^9$. For this DE, order $n=3$ and degree $d=5$.

Why the degree is 5? Because the term with the highest order derivative is y''' and its exponent is 5.

b) $\left(\frac{dy}{dx}\right)^3-4x\left(\frac{d^3y}{dx^3}\right)^2+\frac{d^4y}{dx^4}=0$. Here, order $n=4$ and degree $d=1$.

Why the degree is 1? This is because the term with the highest order derivative in the given DE is $\frac{d^4y}{dx^4}$ and its power is 1.

c) $y''^4-y'^{\frac{3}{2}}=0$. Here, order $n=2$ and degree $d=8$. How? In this DE, we cannot read the degree directly because the exponent of y' is not integer.

So, first square both sides to make it integer.

That is $y''^4-y'^{\frac{3}{2}}=0 \Rightarrow y''^4=y'^{\frac{3}{2}} \Rightarrow (y''^4)^2=(y'^{\frac{3}{2}})^2 \Rightarrow y''^8-y'^3=0 \Rightarrow d=8$.

d) $y''^3+\sin 4y'=x$. Here, order $n=2$ but its degree is not defined because it is not a polynomial with respect to y'. (This is because of the term $\sin 4y'$).

e) $y''^3+\sin 4y=x$. Here, order $n=2$ and its degree is also defined which is $d=3$ even though it is not polynomial in y.

f*) $y''^4(y''^3-y')^2=x$. Here, order $n=2$ and degree $d=10$. (How?)

1.2 Linear and Non-linear Differential Equations

Definition: A differential equation is said to be linear if and only if the following two conditions are satisfied:

i) The dependent variable and its derivatives in all terms have first degree.

ii) There is no term involving product of the dependent variable and any order of its derivatives.

A differential equation which violates either of these two conditions is said to be *non-linear* differential equation. In general, any differential equation that contains at least one of the following terms is automatically non-linear.

$$\sin(y'), \cos(y'), \sec(y'), \ln y', e^{y'}, \sin^{-1} y', \cos^{-1} y'.$$

Examples:

a) The DE $\frac{d^2y}{dx^2} = 5x\frac{dy}{dx}$ has order $n=2$ and degree $n=1$. Thus, it is linear.

b) The DE $y''^4 + 3y' - y = 0$ has order $n=2$ and degree $n=4$. It violates the first condition of the definition. Thus, it is non-linear.

c) The DE $y'' + yy' = 1$ has order $n=2$ and degree $n=1$ but it contains the term yy' which is the product of the dependent variable and its derivative. It violates the second condition of the definition and thus it is non-linear.

d) The DE $y'' = \sqrt{1+x^2}$ has order $n=2$ and degree $n=1$. Thus, it is linear.

e) The DE $y'' = \sqrt{1+y'^2}$ has order $n=2$ and degree $n=2$. It violates the first condition and thus it is non-linear.

f) Consider the DEs: $\sqrt{y''+x} = y$ and $\sqrt{y''+y} = x$. Which equation is linear and which equation is non-linear. Why?

1.3 Solutions of Differential Equations

Definition: Any function (involving the independent and dependent variables) which satisfies the given DE whenever substituted is called solution of the DE.

Types of solutions: There are two forms of solutions for a given DE (if it has).

a) **General solution:** The solution of a DE which contains arbitrary constants in its expression is called *general solution* (*primitive*).

b) **Particular solution:** The solution of a DE free from arbitrary constants (that does not contain arbitrary constants) is called a *particular solution*. Usually, particular solutions are solutions obtained from the general solution by assigning particular values to the arbitrary constants.

Examples:

a) $y = 3e^{2x} - 4x$ is the solution of the DE $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 8$.

Here, $y = 3e^{2x} - 4x, \frac{dy}{dx} = 6e^{2x} - 4, \frac{d^2y}{dx^2} = 12e^{2x}$.

Then, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 12e^{2x} - 2(6e^{2x} - 4) = 12e^{2x} - 12e^{2x} + 8 = 8$

This means $y = 3e^{2x} - 4x$ satisfies the given DE when substituted.

b) Consider the DE: $y''' = 8y$. Can both $y = 3e^{2x}$ and $y = ce^{2x}$ be solutions?

Here, $y = 3e^{2x}, y' = 6e^{2x}, y'' = 12e^{2x}, y''' = 24e^{2x}$. So, $y''' = 24e^{2x} = 8(3e^{2x}) = 8y$.

Again, $y = ce^{2x}, y' = 2ce^{2x}, y'' = 4ce^{2x}, y''' = 8ce^{2x}$. So, $y''' = 8ce^{2x} = 8(ce^{2x}) = 8y$.

Therefore, both $y = 3e^{2x}$ and $y = ce^{2x}$ are solutions.

Besides, $y = ce^{2x}$ is general solution because it contains arbitrary constant c in its expression. But $y = 3e^{2x}$ is particular solution obtained by assigning $c = 3$.

c) For arbitrary constants c_1 and c_2, $y = c_1 + c_2e^{-x} + x^3$ is the solution of the DE $y'' + y' - 6x = 3x^2$. Therefore, it is a general solution. But if we give $c_1 = 0$ and $c_2 = 2$, we obtain $y = 2e^{-x} + x^3$ which is a particular solution.

d) $y = x^3$ is a solution of $y'' - 2xy' + 6y = 6$ but $y = x^2$ is not.

e) If $y=e^{2x}$ is the solution of the DE $\frac{d^2y}{dx^2}+3y'-ky=0$, then find k.

Solution: Since is $y=e^{2x}$ is given to be a solution, it must satisfy the DE.

That is $y=e^{2x} \Rightarrow y'=2e^{2x}, y''=4e^{2x} \Rightarrow \frac{d^2y}{dx^2}+3y'-ky=0$

$$\Rightarrow 4e^{2x}+6e^{2x}-ke^{2x}=0 \Rightarrow k-10=0 \Rightarrow k=10$$

d) For what value of a does $y=x^2$ is the solution of the differential equation given by $4x\frac{d^2y}{dx^2}+axy'-6y=2x^2+8x$?

Solution: Since $y=x^2$ is a solution, it must satisfy the given DE.

That is $y=x^2 \Rightarrow y'=2x, y''=2 \Rightarrow 4x\frac{d^2y}{dx^2}+axy'-6y=2x^2+8x$

$$\Rightarrow 8x+2ax^2-6x^2=2x^2+8x \Rightarrow 2a-6=2 \Rightarrow a=4$$

1.4 Initial Value Problems (IVP)

The problem of finding a function solution satisfying a differential equation and an initial condition is called *an initial value problem* (IVP).

Consider a differential equation $\frac{dy}{dx}=f(x,y)$. Then the initial value problem for such first order DEs is of the form $\begin{cases}\frac{dy}{dx}=f(x,y)\\ y(x_0)=y_0\end{cases}$. Then, the problem of finding a function which satisfies this differential equation with the given additional condition is said to be *Initial Value Problem*.

The condition $y(x_0)=y_0$ is called initial condition.

Examples: *a)* $xy'+2y=4x^2,\ y(1)=3$ *b)* $y'=2y+6,\ y(0)=2$

Note: To solve IVPs, first find the general solution of the given DE using any method and finally find the constants using the given initial conditions.

1.5 Solving First Order Differential Equations ODE

Solving a differential equation means finding the unknown function which satisfies the given differential equation.

Forms of First Order Differential Equation:

Any DE of the form $M(x,y)dx+N(x,y)dy=0$ or $\frac{dy}{dx}=f(x,y)$ is said to be first order differential equation. Here, under we will discuss how to find the general solution for such form of DE.

Since there is no general method to solve all forms of DEs, we will see different methods for different forms of DEs.

Methods of Solutions for First Order Differential Equations:

1) Separable Differential Equations: Method of Separation
2) Homogeneous Differential Equations: Change of Variable
3) Exact Differential Equations: Method of Exactness
4) Non-Exact Differential Equations: Integrating Factor Method
5) Bernoulli's Differential Equations: Transformation (Reduction)

1.5.1 Separable Differential Equations: Method of Separation

Separable Differential Equation:

Any first order differential equation $M(x,y)dx+N(x,y)dy=0$ which can be expressible in the form $g(y)dy=f(x)dx$ is said to be *separable differential equation*. Here, the form $g(y)dy=f(x)dx$ is said to be the separable form.

Method of solution: The *general solution* of such separable DE is obtained by integrating the separable form $g(y)dy=f(x)dx$ both sides.

That is $\int g(y)dy=\int f(x)dx$.

Examples:

1. Verify that the following DEs are separable and solve them.

$a)\ (xy^4 - y^4)dx - (x^3y^2 - 3x^3)dy = 0$ $\qquad b)\ (x^2+1)\dfrac{dy}{dx} - \dfrac{x}{2y} = 0$

$c)\ 24dy - (x^2y^3 - 4x^2y + y^3 - 4y)dx = 0$ $\qquad d)\ e^{x+2y}dx - e^{2x-y}dy = 0$

$e)\ \tan x\sin^2 y dx + \cos^2 x\cot y dy = 0$ $\qquad f)\ e^y \sin x dx - \cos^2 x dy = 0$

$g)\ (e^y+3)\cos x dx - e^y \sin x dy = 0$ $\qquad h)\ \dfrac{dy}{dx} = e^{x-y} + 2xe^{-y}$

$i)\ (y^2 + xy^2)dx + (x^2 - x^2y)dy = 0$ $\qquad j)\ 3(1+x^2)dy = 2xy(y^3-1)dx$

Solution:

$a)\ (x^2+1)\dfrac{dy}{dx} - \dfrac{x}{2y} = 0 \Rightarrow (x^2+1)dy = \dfrac{x}{2y}dx \Rightarrow 2ydy = \dfrac{x}{x^2+1}dx.$

Hence, the DE is separable.

Thus, $2ydy = \dfrac{x}{x^2+1}dx \Rightarrow \int 2ydy = \int \dfrac{x}{x^2+1}dx \Rightarrow y^2 = \dfrac{1}{2}\ln(x^2+1)+c$

$b)\ (xy^4 - y^4)dx - (x^3y^2 - 3x^3)dy = 0 \Rightarrow y^4(x-1)dx - x^3(y^2-3)dy = 0$

$$\Rightarrow \left(\frac{y^2-3}{y^4}\right)dy = \left(\frac{x-1}{x^3}\right)dx \Rightarrow \left(\frac{1}{y^2} - \frac{3}{y^4}\right)dy = \left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx$$

$c)\ 24dy - (x^2y^3 - 4x^2y + y^3 - 4y)dx = 0 \Rightarrow 24dy = [x^2(y^3-4y) + y^3 - 4y]dx$

$$\Rightarrow 24dy = (y^3-4y)(x^2+1)dx \Rightarrow \frac{24}{y^3-4y}dy = (x^2+1)dx$$

Hence, the DE is separable and by PFD, $\dfrac{24}{y^3-4y} = \dfrac{-6}{y} + \dfrac{3}{y-2} + \dfrac{3}{y+2}.$

$d)\ e^x dx - \dfrac{e^{2x}}{e^{3y}}dy = 0 \Rightarrow e^{-x}dx = e^{-3y}dy \Rightarrow \int e^{-x}dx = \int e^{-3y}dy \Rightarrow -e^{-x} = \dfrac{-1}{3}e^{-3y} + c$

So, $\displaystyle\int \frac{24}{y^3-4y}dy = \int\left(\frac{-6}{y} + \frac{3}{y-2} + \frac{3}{y+2}\right)dy = \int (x^2+1)dx \Rightarrow 3\ln\left|\frac{y^2-4}{y^2}\right| = \frac{x^3}{3} + x.$

$e)\ \dfrac{\cot y}{\sin^2 y}dy = \dfrac{-\tan x}{\cos^2 x}dx \Rightarrow \cos y\sin^{-3} y = -\sin x\cos^{-3} x.$ Hence, it is separable.

So, $\displaystyle\int \cos y\sin^{-3} y dy = -\int \sin x\cos^{-3} x dx \Rightarrow \frac{1}{2\sin^2 y} = \frac{1}{2\cos^2 x} + c$

$f)\ e^{-y}dy = \dfrac{\sin x}{\cos^2 x}dx \Rightarrow \int e^{-y}dy = \int \dfrac{\sin x}{\cos^2 x}dx \Rightarrow -e^{-y} = \sec x + c$

?

$g)\ \dfrac{e^y}{e^y+3}dy = \dfrac{\sin x}{\cos x}dx \Rightarrow \int \dfrac{e^y}{e^y+3}dy = \int \dfrac{\sin x}{\cos x}dx \Rightarrow (e^y+3)\cos x = c$

$h)\ \dfrac{dy}{dx} = e^{x-y} + 2xe^{-y} \Rightarrow \dfrac{dy}{dx} = \dfrac{e^x}{e^y} + \dfrac{2x}{e^y} \Rightarrow e^y dy = (e^x + 2x)dx$

Hence, $e^y dy = (e^x+2x)dx \Rightarrow \int e^y dy = \int (e^x+2x)dx \Rightarrow e^y = e^x + x^2 + c$

$i)\ (y^2 + xy^2)dx + (x^2 - x^2y)dy = 0 \Rightarrow y^2(1+x)dx + x^2(1-y)dy = 0$

So, $\dfrac{1-y}{y^2}dy = -\dfrac{1+x}{x^2}dx \Rightarrow \int \dfrac{1-y}{y^2}dy = -\int \dfrac{1+x}{x^2}dx \Rightarrow -\dfrac{1}{y} - \ln|y| = \dfrac{1}{x} + \ln|x| + cx$

$j)\ \dfrac{3}{y(y^3-1)}dy = \dfrac{2x}{1+x^2}dx \Rightarrow \dfrac{3}{y(y-1)(y^2+y+1)}dy = \dfrac{2x}{1+x^2}dx$

2. Solve the following IVPs using separable method

a) $2xe^{x^2}dx + (y^3-1)dy = 0,\ y(0) = 2$ *b)* $\dfrac{dy}{dx} = 6xe^{y-x^2},\ y(0) = 0$

c) $y' + y^2\cos 2x = 0,\ y(0) = 1$ *d)* $x\dfrac{dy}{dx} + \cot y = 0,\ y(\sqrt{2}) = \dfrac{\pi}{4}$

Solution: Very Important: To solve IVPs with $y(x_0) = y_0$,

First: Find the general solution using appropriate method

Second: Find the constant by putting $x = x_0$ and $y = y_0$ in the general solution.

a) $2xe^{x^2}dx + (y^3-1)dy = 0 \Rightarrow (y^3-1)dy = -2xe^{x^2}dx$

$$\Rightarrow \int (y^3-1)dy = \int -2xe^{x^2}dx \Rightarrow \frac{y^4}{4} - y = -e^{x^2} + c$$

Now, find the arbitrary constant by using $x = 0,\ y = 2$.

That is $\dfrac{y^4}{4} - y = -e^{x^2} + c \Rightarrow \dfrac{(2)^4}{4} - 2 = -e^0 + c \Rightarrow -1 + c = 2 \Rightarrow c = 3$

Therefore, the solution that satisfies the given IVP is $\dfrac{y^4}{4} - y = -e^{x^2} + 3$.

b) $\dfrac{dy}{dx} = 6xe^{y-x^2} \Rightarrow \dfrac{dy}{e^y} = 6xe^{-x^2}dx \Rightarrow \int e^{-y}dy = \int 6xe^{-x^2}dx \Rightarrow -e^{-y} = -3e^{-x^2} + c$

$y(0) = 0 \Rightarrow -1 = -3 + c \Rightarrow c = 2 \Rightarrow -e^{-y} = -3e^{-x^2} + 2 \Rightarrow e^{-y} = 3e^{-x^2} - 2$

Equations Reducible to Separable: Any DE of the form $\frac{dy}{dx}=f(ax+by+c)$ can be reduced to separable form using the substitution $t=ax+by+c$.

Then, $\frac{dt}{dx}=a+bf(t)\Rightarrow dx=\frac{dt}{a+bf(t)}\Rightarrow\int dx=\int\frac{dt}{a+bf(t)}+c$

Examples: By reducing into separable form, solve the following DEs:

a) $\frac{dy}{dx}=(9x+y+5)^2$ *b)* $(x+y+1)^2\frac{dy}{dx}=1$ *c)* $\frac{dy}{dx}=\frac{2x+2y+3}{x+y+1}$

d) $\frac{dy}{dx}=2x+y$ *e)* $\frac{dy}{dx}=\frac{2x+3y+5}{4x+6y-3}$ *f)* $\frac{dy}{dx}=\sec(x+5y)$

Solution: We use the above substitution rule

a) Let $t=9x+y+5\Rightarrow\frac{dt}{dx}=9+\frac{dy}{dx}$. But from the given $\frac{dy}{dx}=t^2$.

So, $\frac{dt}{dx}=9+t^2\Rightarrow\frac{dt}{9+t^2}=dx\Rightarrow\int\frac{dt}{9+t^2}=\int dx\Rightarrow\frac{1}{3}\tan^{-1}\frac{t}{3}=x+c$

$$\Rightarrow\frac{1}{3}\tan^{-1}(\frac{9x+y+5}{3})=x+c\Rightarrow y=3\tan(3x+3c)-9x-5$$

b) Let $t=x+y+1\Rightarrow\frac{dt}{dx}=1+\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{dt}{dx}-1$

Then, $(x+y+1)^2\frac{dy}{dx}=1\Rightarrow t^2(\frac{dt}{dx}-1)=1\Rightarrow\frac{dt}{dx}=1+\frac{1}{t^2}\Rightarrow\frac{t^2dt}{t^2+1}=dx$

$\Rightarrow\int\frac{t^2dt}{t^2+1}=\int dx\Rightarrow\int\left(1-\frac{1}{t^2+1}\right)dt=\int dx=t-\tan^{-1}t=x+c$

$\Rightarrow x+y+1-\tan^{-1}(x+y+1)=x+c\Rightarrow y=\tan^{-1}(x+y+1)-1+c$

c) Let $t=x+y+1\Rightarrow\frac{dy}{dx}=\frac{dt}{dx}-1$.

Then, $\frac{dy}{dx}=\frac{2x+2y+3}{x+y+1}=\frac{2(x+y)+3}{x+y+1}\Rightarrow\frac{dt}{dx}-1=\frac{2(t-1)+3}{t}$

$$\Rightarrow\frac{dt}{dx}=3+\frac{1}{t}\Rightarrow\int(3+\frac{1}{t})dt=\int dx\Rightarrow 3t+\ln|t|=x+c$$

$$\Rightarrow 3(x+y+1)+\ln|x+y+1|=x+c\Rightarrow 2x+3y+\ln|x+y+1|=c$$

1.5.2 Homogeneous Differential Equations

A function $f(x,y)$ is said to be *homogenous* of degree 1 if for any parameter $t \neq 0$, $f(tx,ty) = f(x,y)$. In general, f is *homogenous* function of degree n if and only if for any parameter $t \neq 0$, $f(tx,ty) = t^n f(x,y)$.

Homogenous Differential Equation:

A differential equation of the form $\dfrac{dy}{dx} = f(x,y)$ is said to be *homogenous* if $f(x,y)$ is homogeneous function. Otherwise, it is said to be non-homogeneous.

Test of Homogeneity:

A differential equation $\dfrac{dy}{dx} = f(x,y)$ is *homogenous* of degree 1 if and only if for any parameter $t \neq 0$, $f(tx,ty) = f(x,y)$.

Method of Solutions: By Reducing into Separable DEs

First: Use Test of Homogeneity to check whether it is homogeneous or not.

Second: Change the homogeneous DE into separable DE.

To change into separable DE, use the substitutions: $y = vx, \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

This will change the given DE into separable DE in the variables x and v. Then, integrating the separable form gives the general solution in terms of x and v. Finally, by solving $y = vx$ for v, express the answer in terms of x and y.

Examples:

1. Check whether the differential equations are *homogenous* or not. For these which are homogeneous, solve using homogeneous method.

a) $\dfrac{dy}{dx} = \dfrac{x+y}{x}$

b) $2xy\dfrac{dy}{dx} = x^2 + y^2$

c) $xdy = (y + x\csc\dfrac{y}{x})dx$

d) $xdy = y(1 + \ln\dfrac{y}{x})dx$

e) $x^2dy = (y^2 - xy)dx$

f) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\dfrac{y}{x}$

g) $(x-y)dx = (x+y)dy$

h) $(x^2 - y^2)dx = 2xydy$

i) $(y^2 - 2xy)dx = (x^2 - 2xy)dy$

j) $xy' = y(\ln y - \ln x)$

Solution:

First: Use Test of Homogeneity: To use this test for DE, first express the given DE in the form $\frac{dy}{dx}=f(x,y)$. Then, apply the test on f.

a) Then the DE is the form $\frac{dy}{dx}=f(x,y)$ where $f(x,y)=\frac{x+y}{x}$.

Here, for any $t\neq 0$, $f(tx,ty)=\frac{tx+ty}{tx}=\frac{x+y}{x}=f(x,y)$.

Thus, f is homogeneous which implies the DE itself is *homogenous*.

$$\frac{dy}{dx}=\frac{x+y}{x}=1+\frac{y}{x}\Rightarrow v+x\frac{dv}{dx}=1+v\Rightarrow x\frac{dv}{dx}=1\Rightarrow dv=\frac{dx}{x}$$

$$\Rightarrow\int dv=\int\frac{1}{x}dx\Rightarrow v=\ln|x|+c\Rightarrow\frac{y}{x}=\ln|x|+c\Rightarrow y=x\ln|x|+cx$$

b) $2xy\frac{dy}{dx}=x^2+y^2\Rightarrow\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$. Then the DE is expressible in the form $\frac{dy}{dx}=f(x,y)$ where the function is $f(x,y)=\frac{x^2+y^2}{2xy}$.

Here, for any $t\neq 0$, $f(tx,ty)=\frac{t^2x^2+t^2y^2}{2t^2xy}=\frac{t^2(x^2+y^2)}{2t^2xy}=\frac{x^2+y^2}{2xy}=f(x,y)$

Thus, f is homogeneous which implies the DE itself is *homogenous*.

Second: Use the substitution $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$ to change into separable.

$$\frac{dy}{dx}=\frac{x^2+y^2}{2xy}\Rightarrow v+x\frac{dv}{dx}=\frac{x^2+x^2v^2}{2vx^2}\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{2v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v^2}{2v}\Rightarrow\frac{2v}{v^2-1}dv=-\frac{dx}{x}$$

$$\Rightarrow\int\frac{2v}{v^2-1}dv=-\int\frac{1}{x}dx\Rightarrow\ln|v^2-1|=-\ln|x|+c$$

$$\Rightarrow\ln|v^2-1|+\ln|x|=c\Rightarrow\ln|(v^2-1)x|=c\Rightarrow(v^2-1)x=c'$$

$$\Rightarrow\left(\frac{y^2}{x^2}-1\right)x=c'\Rightarrow\left(\frac{y^2-x^2}{x^2}\right)x=c'\Rightarrow y^2-x^2=c'x$$

c) Here, $xdy = (y + x\csc\frac{y}{x})dx \Rightarrow \frac{dy}{dx} = \frac{y}{x} + \csc\frac{y}{x}$. Here, for any $t \neq 0$,

$f(x,y) = \frac{y}{x} + \csc\frac{y}{x} \Rightarrow f(tx,ty) = \frac{ty}{tx} + \csc\frac{ty}{tx} = \frac{y}{x} + \csc\frac{y}{x} = f(x,y)$.

Thus, f is homogeneous which implies the DE itself is *homogenous.*

Second: Use the substitution $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ to change into separable.

$$\frac{dy}{dx} = \frac{y}{x} + \csc\frac{y}{x} \Rightarrow v + x\frac{dv}{dx} = v + \csc v \Rightarrow x\frac{dv}{dx} = \csc v \Rightarrow \frac{1}{\csc v}dv = \frac{1}{x}dx$$

$$\Rightarrow \int \sin v dv = \int \frac{1}{x}dx \Rightarrow -\cos v = \ln|x| + c \Rightarrow -\cos\frac{y}{x} = \ln|x| + c$$

d) $xdy = y(1 + \ln\frac{y}{x})dx \Rightarrow \frac{dy}{dx} = \frac{y}{x}(1 + \ln\frac{y}{x}) = \frac{y}{x} + \frac{y}{x}\ln\frac{y}{x}$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y}{x}.\ln\frac{y}{x} \Rightarrow v + x\frac{dv}{dx} = v + v\ln v \Rightarrow x\frac{dv}{dx} = v\ln v$$

$$\Rightarrow \frac{1}{v\ln v} = \frac{1}{x}dx \Rightarrow \int \frac{1}{v\ln v}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \int \frac{1}{v\ln v}dv = \int \frac{1}{x}dx \Rightarrow \ln|\ln v| = \ln|x| + c$$

$$\Rightarrow \ln\left|\ln\frac{y}{x}\right| = \ln|x| + c \text{ or } \ln\frac{y}{x} = cx \text{ or } y = xe^{cx}$$

e) $\frac{dy}{dx} = \frac{y^2 - xy}{x^2} \Rightarrow v + x\frac{dv}{dx} = v^2 - v \Rightarrow x\frac{dv}{dx} = v^2 - 2v \Rightarrow \frac{1}{v^2 - 2v}dv = \frac{1}{x}dx$

$$\Rightarrow (\frac{1/2}{v-2} - \frac{1/2}{v})dv = \frac{1}{x}dx \Rightarrow \int(\frac{1/2}{v-2} - \frac{1/2}{v})dv = \int \frac{1}{x}dx$$

$$\Rightarrow \frac{1}{2}\ln|v-2| - \frac{1}{2}\ln|v| = \ln|x| + c \Rightarrow \frac{1}{2}\ln\left|\frac{y}{x} - 2\right| - \frac{1}{2}\ln\left|\frac{y}{x}\right| = \ln|x| + c$$

f) $v + x\frac{dv}{dx} = v + \sin v \Rightarrow x\frac{dv}{dx} = \sin v \Rightarrow \frac{dv}{\sin v} = \frac{dx}{x} \Rightarrow \int \frac{1}{\sin v}dv = \int \frac{1}{x}dx$

$$\Rightarrow \int \csc v dv = \int \frac{1}{x}dx \Rightarrow |\csc v - \cot v| = |x| + c \Rightarrow 1 - \cos\frac{y}{x} = cx\sin\frac{y}{x}$$

g) $\frac{dy}{dx}=\frac{x-y}{x+y}\Rightarrow v+x\frac{dv}{dx}=\frac{1-v}{1+v}\Rightarrow x\frac{dv}{dx}=\frac{1-2v-v^2}{1+v}\Rightarrow \frac{v+1}{v^2+2v-1}dv=-\frac{1}{x}dx$

But by substitution, $u=v^2+2v-1\Rightarrow du=2(v+1)dv$,

$$\int(\frac{v+1}{v^2+2v-1})dv=-\int\frac{1}{x}dx\Rightarrow\int\frac{1}{2u}du=-\ln|x|+c\Rightarrow\frac{1}{2}\ln|u|=-\ln|x|+c$$

$$\Rightarrow\ln\left|v^2+2v-1\right|+2\ln|x|=c\Rightarrow\ln\left|(v^2+2v-1)x^2\right|=c$$

$$\Rightarrow\left(\frac{y^2}{x^2}+\frac{2y}{x}-1\right)x^2=c\Rightarrow y^2+2xy-x^2=c\Rightarrow x^2-y^2=2xy+c$$

h) $\frac{dy}{dx}=\frac{x^2-y^2}{2xy}\Rightarrow v+x\frac{dv}{dx}=\frac{x^2-v^2x^2}{2x^2v}=\frac{1}{2v}-\frac{v}{2}\Rightarrow x\frac{dv}{dx}=\frac{1}{2v}-\frac{v}{2}-v$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2}{2v}\Rightarrow\frac{2vdv}{1-3v^2}=\frac{dx}{x}\Rightarrow\int\frac{2vdv}{1-3v^2}=\int\frac{dx}{x}$$

$$\Rightarrow-\frac{1}{3}\ln\left|1-3v^2\right|=\ln|x|+c\Rightarrow x^3(1-3v^2)=c$$

$$\Rightarrow x^3(\frac{x^2-3y^2}{x^2})=c\Rightarrow x(x^2-3y^2)=c$$

i) $\frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}\Rightarrow v+x\frac{dv}{dx}=\frac{v^2x^2-2vx^2}{x^2-2vx^2}\Rightarrow x\frac{dv}{dx}=\frac{3v^2-3v}{1-2v}$

$$\Rightarrow\frac{1-2v}{3v^2-3v}dv=\frac{1}{x}dx\Rightarrow\int\frac{1-2vdv}{3v^2-3v}=\int\frac{1}{x}dx$$

$$\Rightarrow\int\frac{1dv}{3v^2-3v}-\int\frac{2dv}{3v-3}=\int\frac{1}{x}dx\Rightarrow\frac{1}{3}\int\frac{1}{v(v-1)}dv-\int\frac{2dv}{3v-3}=\int\frac{1}{x}dx$$

$$\Rightarrow\frac{1}{3}\int\left(\frac{1}{v-1}-\frac{1}{v}\right)dv-\int\frac{2dv}{3v-3}=\int\frac{1}{x}dx\Rightarrow\frac{1}{3}\ln|v-1|-\frac{1}{3}\ln|v|-\frac{2}{3}\ln|v-1|=\ln|x|+c$$

$$\Rightarrow-\frac{1}{3}\ln|v-1|-\frac{1}{3}\ln|v|=\ln|x|+c\Rightarrow\ln|v-1|+\ln|v|=-3\ln|x|+c\Rightarrow xy(y-x)=c$$

j) $xy'=y(\ln y-\ln x)\Rightarrow x\frac{dy}{dx}=y(\ln\frac{y}{x})\Rightarrow\frac{dy}{dx}=\frac{y}{x}(\ln\frac{y}{x})\Rightarrow v+x\frac{dv}{dx}=v\ln v$

$$\Rightarrow\int\frac{1}{v(\ln v-1)}dv=\int\frac{1}{x}dx\Rightarrow\ln(\ln v-1)=\ln|x|+c$$

$$\Rightarrow\ln v-1=Cx\Rightarrow\ln\frac{y}{x}=Cx+1\Rightarrow\frac{y}{x}=e^{Cx+1}\Rightarrow y=xe^{Cx+1}$$

2. Solve the following homogenous DEs

a*) $xdy - ydx = \sqrt{x^2+y^2}\,dx$ b) $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$

c) $x\sin\dfrac{y}{x}dy = (x + y\sin\dfrac{y}{x})dx$ d*) $xdy = (y + y(\ln\frac{y}{x})^{-5})dx$

e) $(x^3 - y^3)dx + xy^2dy = 0$ f) $(2y-3x)dx + xdy = 0$

g) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan(\dfrac{y}{x})$ h) $(x^2+y^2)\dfrac{dy}{dx} = xy$

i) $y^2dx + (xy + x^2)dy = 0$ j) $x(x-y)dy + y^2dx = 0$

Solution:

a) $xdy - ydx = \sqrt{x^2+y^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{\sqrt{x^2+y^2}}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}} \Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \Rightarrow \frac{1}{\sqrt{1+v^2}}dv = \frac{1}{x}dx$$

$$\Rightarrow \int\frac{1}{\sqrt{1+v^2}}dv = \int\frac{1}{x}dx \Rightarrow \int\frac{1}{\sqrt{1+v^2}}dv = \ln|x| + c$$

Now, using trig substitution, $v = \tan\theta \Rightarrow dv = \sec^2\theta d\theta$.

$$\int\frac{1}{\sqrt{1+v^2}}dv = \int\frac{\sec^2\theta d\theta}{\sqrt{1+\tan^2\theta}}d\theta = \int\sec\theta d\theta = \ln|\sec\theta + \tan\theta| = \ln\left|\sqrt{1+v^2} + v\right|$$

So, $\displaystyle\int\frac{1}{\sqrt{1+v^2}}dv = \ln|x| + c \Rightarrow \ln\left|\sqrt{1+v^2} + v\right| = \ln|x| + c \Rightarrow \ln\left|\sqrt{1+\frac{y^2}{x^2}} + \frac{y}{x}\right| = \ln|x| + c$

$$\Rightarrow \ln\left|\sqrt{\frac{x^2+y^2}{x^2}} + \frac{y}{x}\right| = \ln|x| + c \Rightarrow \ln\left|\frac{\sqrt{x^2+y^2}}{x} + \frac{y}{x}\right| = \ln|x| + c$$

$$\Rightarrow \ln\left|\frac{\sqrt{x^2+y^2}+y}{x}\right| - \ln|x| = c \Rightarrow \ln\left|\frac{\sqrt{x^2+y^2}+y}{x^2}\right| = c$$

$$\Rightarrow \frac{\sqrt{x^2+y^2}+y}{x^2} = e^c \Rightarrow \sqrt{x^2+y^2} + y = Cx$$

b) $\dfrac{dy}{dx}=\dfrac{y^2}{xy-x^2}\Rightarrow v+x\dfrac{dv}{dx}=\dfrac{v^2}{v-1}\Rightarrow x\dfrac{dv}{dx}=\dfrac{v}{v-1}\Rightarrow \dfrac{v-1}{v}dv=\dfrac{1}{x}dx$

$$\Rightarrow(\frac{v-1}{v})dv=\frac{1}{x}dx\Rightarrow\int(1-\frac{1}{v})dv=\int\frac{1}{x}dx\Rightarrow v-\ln|v|=\ln|x|+c$$

$$\Rightarrow v=\ln|v|+\ln|x|+c\Rightarrow v=\ln|cvx|\Rightarrow\frac{y}{x}=\ln|cy|\Rightarrow y=x\ln|cy|$$

c) $x\sin\dfrac{y}{x}dy=(x+y\sin\dfrac{y}{x})dx\Rightarrow\sin\dfrac{y}{x}.\dfrac{dy}{dx}=1+\dfrac{y}{x}\sin(\dfrac{y}{x}).$

$$\Rightarrow\sin v.(v+x\frac{dv}{dx})=1+v\sin v\Rightarrow v\sin v+x\sin v\frac{dv}{dx}=1+v\sin v$$

$$\Rightarrow x\sin v\frac{dv}{dx}=1\Rightarrow\sin v dv=\frac{1}{x}dx\Rightarrow\int\sin v dv=\int\frac{1}{x}dx$$

$$\Rightarrow-\cos v=\ln|x|+c\Rightarrow-\cos\frac{y}{x}=\ln|x|+c$$

d) $xdy=(y+y(\ln\frac{y}{x})^{-5})dx\Rightarrow\dfrac{dy}{dx}=\dfrac{y}{x}+\dfrac{y}{x}(\ln\frac{y}{x})^{-5}$

$$\Rightarrow v+x\frac{dv}{dx}=v+24v(\ln v)^{-5}\Rightarrow x\frac{dv}{dx}=v(\ln v)^{-5}$$

$$\Rightarrow\frac{1}{v(\ln v)^{-5}}dv=\frac{1}{x}dx\Rightarrow\frac{(\ln v)^5}{v}dv=\frac{1}{x}$$

$$\Rightarrow\int\frac{(\ln v)^5}{v}dv=\int\frac{1}{x}dx\Rightarrow\int t^5dt=\int\frac{1}{x}dx\qquad(\text{Using }\ln v=t\Rightarrow\frac{1}{v}dv=dt)$$

$$\Rightarrow\frac{t^6}{6}=\ln|x|+c\Rightarrow\frac{(\ln v)^6}{6}=\ln|x|+c\Rightarrow\frac{1}{6}(\ln\frac{y}{x})^6=\ln|x|+c$$

e) $\dfrac{dy}{dx}=\dfrac{y^3-x^3}{xy^2}\Rightarrow v+x\dfrac{dv}{dx}=\dfrac{v^3-1}{v^2}\Rightarrow x\dfrac{dv}{dx}=\dfrac{-1}{v^2}\Rightarrow-v^2dv=\dfrac{1}{x}dx$

$$\Rightarrow\int-v^2dv=\int\frac{1}{x}dx\Rightarrow\frac{-v^3}{3}=\ln|x|+c\Rightarrow-\frac{y^3}{3x^3}=\ln|x|+c$$

f) $\frac{dy}{dx}=\frac{3x-2y}{x}\Rightarrow v+x\frac{dv}{dx}=3-2v\Rightarrow x\frac{dv}{dx}=3-3v$

$$\Rightarrow \frac{1}{3-3v}dv=\frac{1}{x}dx\Rightarrow \int\frac{1}{3-3v}dv=\int\frac{1}{x}dx$$

$$\Rightarrow -\frac{1}{3}\ln|1-v|=\ln|x|+c\Rightarrow \ln\left|1-\frac{y}{x}\right|=-3\ln|x|+c$$

g) $\frac{dy}{dx}=\frac{y}{x}+\tan(\frac{y}{x})\Rightarrow v+x\frac{dv}{dx}=v+\tan v\Rightarrow x\frac{dv}{dx}=\tan v\Rightarrow \frac{dv}{\tan v}=\frac{dx}{x}$

$$\Rightarrow \int\frac{\cos v}{\sin v}dv=\int\frac{1}{x}dx\Rightarrow \ln|\sin v|=\ln|x|+c\Rightarrow \sin v=cx\Rightarrow \sin(\frac{y}{x})=cx$$

h) $\frac{dy}{dx}=\frac{xy}{x^2+y^2}\Rightarrow v+x\frac{dv}{dx}=\frac{v}{1+v^2}\Rightarrow x\frac{dv}{dx}=-\frac{v^3}{1+v^2}\Rightarrow \frac{1+v^2}{v^3}dv=-\frac{1}{x}dx$

$$\Rightarrow (\frac{1}{v}+\frac{1}{v^3})dv=-\frac{1}{x}dx\Rightarrow \int(\frac{1}{v}+\frac{1}{v^3})dv=\int-\frac{1}{x}dx$$

$$\Rightarrow \ln|v|-\frac{1}{2v^2}=-\ln|x|+c\Rightarrow \ln\left|\frac{y}{x}\right|+\ln|x|-\frac{x^2}{2y^2}=c\Rightarrow \ln|y|-\frac{x^2}{2y^2}=c$$

i) $\frac{dy}{dx}=\frac{-y^2}{x^2+xy}\Rightarrow v+x\frac{dv}{dx}=\frac{-v^2}{1+v}\Rightarrow x\frac{dv}{dx}=\frac{-2v^2-v}{1+v}\Rightarrow \frac{v+1}{2v^2+v}dv=\frac{-1}{x}dx$

But by Partial Fraction decomposition, $\frac{v+1}{2v^2+v}=\frac{v+1}{v(2v+1)}=\frac{1}{v}-\frac{1}{2v+1}$.

Thus, $\int(\frac{v+1}{2v^2+v})dv=-\int\frac{1}{x}dx\Rightarrow \int(\frac{1}{v}-\frac{1}{2v+1})dv=-\ln|x|+c$

$$\Rightarrow \ln|v|-\frac{1}{2}\ln|2v+1|=-\ln|x|+c\Rightarrow \ln|v|+\ln|x|-\frac{1}{2}\ln|2v+1|=c$$

$$\Rightarrow \ln|vx|+\ln\sqrt{\frac{1}{2v+1}}=c\Rightarrow \ln|y|+\ln\sqrt{\frac{x}{2y+x}}=c$$

j) $\frac{dy}{dx}=\frac{y^2}{xy-x^2}\Rightarrow v+x\frac{dv}{dx}=\frac{v^2}{v-1}\Rightarrow x\frac{dv}{dx}=\frac{v}{v-1}\Rightarrow \frac{v-1}{v}dv=\frac{1}{x}dx$

$$\Rightarrow (1-\frac{1}{v})dv=\frac{1}{x}dx\Rightarrow \int(1-\frac{1}{v})dv=\int\frac{1}{x}dx$$

$$\Rightarrow v-\ln|v|=\ln|x|+c\Rightarrow \frac{y}{x}=\ln|y|+c$$

3. Solve the following problems using the Methods of homogenous DEs.

a) $(x\sec\frac{y}{x}+y)dx-xdy=0, y(1)=0$ b) $xdy-(2xe^{\frac{-y}{x}}+y)dx=0, y(1)=0$

c) $6y^2dy-\frac{6y^3}{x}dx=x^2e^{-\frac{y^3}{x^3}}dx$ d) $xdy=(y+x\tan\frac{y}{x})dx$

e) $(x^2+xy)dy=(x^2+y^2)dx$ f) $xdy=(y+x\cos\frac{y}{x})dx$

Solution: First, find the general solution and then use the initial conditions.

a) $\frac{dy}{dx}=\frac{y}{x}+\sec\frac{y}{x}\Rightarrow v+x\frac{dv}{dx}=v+\sec v\Rightarrow x\frac{dv}{dx}=\sec v\Rightarrow \cos v dv=\frac{1}{x}dx$

$$\Rightarrow \int\cos v dv=\int\frac{1}{x}dx\Rightarrow \sin v=\ln|x|+c\Rightarrow \sin\frac{y}{x}=\ln|x|+c$$

Besides, $y(1)=0\Rightarrow \sin 0=\ln 1+c\Rightarrow c=0\Rightarrow \sin\frac{y}{x}=\ln|x|\Rightarrow x=e^{\sin\frac{y}{x}}$

b) $\frac{dy}{dx}=2e^{\frac{-y}{x}}+\frac{y}{x}\Rightarrow x\frac{dv}{dx}=2e^{-v}\Rightarrow e^v dv=\frac{2}{x}dx\Rightarrow \int e^v dv=\int\frac{2}{x}dx\Rightarrow e^{\frac{y}{x}}=\ln x^2+c$

Besides, $y(1)=0\Rightarrow e^0=\ln 1+c\Rightarrow c=1\Rightarrow e^{\frac{y}{x}}=\ln x^2+1$

c) $6y^2dy=(\frac{6y^3}{x}+x^2e^{-\frac{y^3}{x^3}})dx\Rightarrow 6\frac{dy}{dx}=\frac{6y}{x}+\frac{x^2}{y^2}e^{-\frac{y^3}{x^3}}\Rightarrow 6(v+x\frac{dv}{dx})=6v+\frac{1}{v^2}e^{-v^3}$

$$\Rightarrow 6v^2e^{v^3}dv=\frac{1}{x}dx\Rightarrow \int 6v^2e^{v^3}dv=\int\frac{1}{x}dx\Rightarrow 2e^{v^3}=\ln|x|+c\Rightarrow 2e^{\frac{y^3}{x^3}}=\ln|x|+c$$

d) $\frac{dy}{dx}=\frac{y}{x}+\tan(\frac{y}{x})\Rightarrow v+x\frac{dv}{dx}=v+\tan v\Rightarrow x\frac{dv}{dx}=\tan v\Rightarrow \frac{1}{\tan v}dv=\frac{1}{x}dx$

$$\Rightarrow \cot v dv=\frac{1}{x}dx\Rightarrow \int\cot v dv=\int\frac{1}{x}dx\Rightarrow \int\frac{\cos v}{\sin v}dv=\int\frac{1}{x}dx$$

$$\Rightarrow \ln|\sin v|=\ln|x|+c\Rightarrow \ln\left|\sin\frac{y}{x}\right|=\ln|x|+c \text{ or } \sin\frac{y}{x}=cx$$

e) $\frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy}\Rightarrow \int(1+\frac{2}{v-1})dv=-\int\frac{1}{x}dx\Rightarrow \frac{y}{x}+\ln\left|\frac{(y-x)^2}{x}\right|=c$

f) $\int\sec v dv=\int\frac{1}{x}dx\Rightarrow \ln|\sec v+\tan v|=\ln|x|+c\Rightarrow \ln\left|\sec\frac{y}{x}+\tan\frac{y}{x}\right|=\ln|x|+c$

1.5.3 Exact Differential Equations: Method of Exactness

Definition: A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is said to be exact differential equation if there exists a function $u(x,y)$ such that $u_x(x,y)=M(x,y)$ & $u_y(x,y)=N(x,y)$.

The function $u(x,y)$ with such properties is called *potential function.*

But determining exactness by finding the potential function $u(x,y)$ is a difficult task. So, we have the following test for exactness.

Test for exactness: Any DE of the form $M(x,y)dx+N(x,y)dy=0$ is exact if and only if $M_y=N_x$ or $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial X}$.

Solving exact DE: Method of Exactness

Suppose $M(x,y)dx+N(x,y)dy=0$ is exact. If $u(x,y)$ is its potential function, then, the general solution is $u(x,y)=c$ where c is arbitrary constant.

This means that once the DE is exact its general solution is obtained from the knowledge of the potential function $u(x,y)$ that satisfies the definition.

So, the basic task is how to determine the potential function.

Procedures to find the general solution of exact DEs:

Step-1: Integrate $u_x(x,y)=M(x,y)$ w.r.t x to get $u(x,y)=\int M(x,y)dx+g(y)$

Here, assume y as constant and g(y) as constant of integration.

Step-2: To find g(y), equate $u_y(x,y)=N(x,y)$ from step-1.

That is $u_y(x,y)=\dfrac{\partial}{\partial y}\int M(x,y)dx+\dfrac{d}{dy}[g(y)]=N(x,y)$

Step-3: Integrate the result in step-2 with respect to y to obtain $g(y)$.

Therefore, the general solution is $u(x,y)=\int M(x,y)dx+g(y)=c$ where $g(y)$ is to be substituted by the result in step-3.

Short-Cut Formula:

By collecting and rearranging the results from **step-1** to **step-3** in the above procedures, we get the following short-cut formula for the potential function.

Potential function: $u(x,y)=\int \mathrm{M}dx+\int[\mathrm{N}-\int \mathrm{M}_y dx]dy$.

1. Verify that the following DEs are exact and solve them.

a) $(y^3e^x+2\cos x)dx+(3y^2e^x-4y)dy=0$

b) $3x^2ydx+(6y+x^3)dy=0$

c) $(5x+4y)dx+(4x-8y^3)dy=0$

d) $(2y^2x-3)dx+(2yx^2+4)dy=0$

e) $-2xy\sin(x^2)dx+\cos(x^2)dy=0$

f) $ye^{xy}dx+(2y+xe^{xy})dy=0$

g) $(e^x\sin y+3x^2)dx+e^x\cos ydy=0$

h) $2x\sin ydx+x^2\cos ydy=0$

Solution: For your understanding, first let's follow the procedures and then use the short-cut formula. Please compare the results in each problem.

a) $M=y^3e^x+2\cos x, N=3y^2e^x-4y \Rightarrow M_y=3y^2e^x, N_x=3y^2e^x$.

Since $M_y=N_x=3y^2e^x$, the differential equation is exact.

To clarify the above steps, let's solve step by step only this part.

Step-1: $u_x(x,y)=y^3e^x+2\cos x \Rightarrow u(x,y)=y^3e^x+2\sin x+g(y)$

Step-2: Equate $u_y(x,y)=N(x,y)$.

From step-1, $u(x,y)=y^3e^x+2\sin x+g(y) \Rightarrow u_y(x,y)=3y^2e^x+g'(y)$.

Then, $u_y(x,y)=N(x,y) \Rightarrow 3y^2e^x+g'(y)=3y^2e^x-4y \Rightarrow g'(y)=-4y$.

Step-3: Integrate the result in step-2 with respect toy.

That is $g'(y)=-4y \Rightarrow \int g'(y)dy=\int -4ydy \Rightarrow g(y)=-2y^2$.

So, using $g(y)=-2y^2$, the potential function is $u(x,y)=y^3e^x+2\sin x-2y^2$.

Using the short-cut formula: Here, $\mathbf{M}=e^xy^3+2\cos x$, $\mathbf{N}=3y^2e^x-4y$.

$$\begin{aligned}
u(x,y) &= \int \mathbf{M}dx+\int[\mathbf{N}-\int \mathbf{M}_y dx]dy \\
&= \int (e^xy^3+2\cos x)dx+\int[3y^2e^x-4y-\int 3y^2e^x dx]dy \\
&= e^xy^3+2\sin x+\int(3y^2e^x-4y-3y^2e^x)dy \\
&= e^xy^3+2\sin x+\int -4ydy = e^xy^3+2\sin x-2y^2
\end{aligned}$$

Therefore, the general solution is $e^xy^3+2\sin x-2y^2=c$.

b) Here, $M(x,y)=3x^2y$, $N=6y+x^3 \Rightarrow M_y=3x^2=N_x$.

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M}=3x^2y, \mathbf{N}=6y+x^3$. Then

$$u(x,y)=\int \mathbf{M}dx+\int[\mathbf{N}-\int \mathbf{M}_y dx]dy$$

$$=\int 3x^2ydx+\int[6y+x^3-\int 3x^2dx]dy=x^3y+3y^2$$

Therefore, the general solution is $u(x,y)=x^3y+3y^2=c$.

c) $M=5x+4y, M_y=4,\ N=4x-8y^3,\ N_x=4 \Rightarrow M_y=N_x$.

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M}=5x+4y, \mathbf{N}=4x-8y^3$. Then

$$u(x,y)=\int \mathbf{M}dx+\int[\mathbf{N}-\int \mathbf{M}_y dx]dy$$

$$=\int (5x+4y)dx+\int[4x-8y^3-\int 4dx]dy=x^3y+3y^2$$

$$=\frac{5}{2}x^2+4xy+\int -8y^3dy=\frac{5}{2}x^2+4xy-2y^4$$

Therefore, the general solution is $\frac{5x^2}{2}+4xy-2y^4=c$.

d) Here, $M=2y^2x-3, N=2yx^2+4 \Rightarrow M_y=4xy=N_x$.

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M}=2y^2x-3, \mathbf{N}=2yx^2+4$. Then

$$u(x,y)=\int \mathbf{M}dx+\int[\mathbf{N}-\int \mathbf{M}_y dx]dy$$

$$=\int (2y^2x-3)dx+\int[2yx^2+4-\int 4xydx]dy=x^2y^2-3x+4y$$

Hence, $u(x,y)=y^2x^2-3x+4y=c$ is the general solution.

e) $M=-2xy\sin(x^2), M_y=-2x\sin(x^2),\ N=\cos(x^2), N_x=-2x\sin(x^2)$

Using the short-cut formula: Here, $\mathbf{M}=-2xy\sin(x^2), \mathbf{N}=\cos(x^2)$. Then

$$u(x,y)=\int -2xy\sin(x^2)dx+\int[\cos(x^2)-\int -2x\sin(x^2)dx]dy=y\cos(x^2)$$

Hence, $u(x,y)=y\cos(x^2)=c$ is the general solution.

f) $M_y=e^{xy}+xye^{xy},\ N_x=e^{xy}+xye^{xy} \Rightarrow M_y=N_x$

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M} = ye^{xy}, \mathbf{N} = 2y + xe^{xy}$ .Then

$$u(x,y) = \int \mathbf{M}dx + \int [\mathbf{N} - \int \mathbf{M}_y dx]dy$$
$$= \int ye^{xy}dx + \int [2y + xe^{xy} - \int (e^{xy} + xye^{xy})dx]dy$$
$$= x + e^{xy} + \int 2ydy = x + e^{xy} + y^2$$

Hence, the general solution is $u(x,y) = x + y^2 + e^{xy} = c$.

g) $M_y = e^x \cos y$, $N_x = e^x \cos y \Rightarrow M_y = N_x$.

Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M} = e^x \sin y + 3x^2, \mathbf{N} = e^x \cos y$ .Then

$$u(x,y) = \int \mathbf{M}dx + \int [\mathbf{N} - \int \mathbf{M}_y dx]dy$$
$$= \int (e^x \sin y + 3x^2)dx + \int [e^x \cos y - \int e^x \cos ydx]dy = e^x \sin y + x^3$$

Thus, the solution is $e^x \sin y + x^3 = c$.

h) $M = 2x\sin y$, $N = x^2 \cos y \Rightarrow M_y = 2x\cos y$, $N_x = 2x\cos y \Rightarrow M_y = N_x$.

Using the short-cut formula: Here, $\mathbf{M} = 2x\sin y, \mathbf{N} = x^2 \cos y$ .Then

$$u(x,y) = \int \mathbf{M}dx + \int [\mathbf{N} - \int \mathbf{M}_y dx]dy$$
$$= \int 2x\sin ydx + \int [x^2 \cos y - \int 2x\cos ydxdy = x^2 \sin y$$

Therefore, the general solution is $U(x,y) = c \Rightarrow x^2 \sin y = c$.

2. Check the exactness and solve the following DEs

a) $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0, y(0) = 3$

b) $(e^y - ye^x)dx + (xe^y - e^x)dy = 0, y(3) = 0$

c) $(3x^2 + y\cos x)dx + (\sin x - 4y^3)dy = 0, y(2) = 0$

d) $(e^x \sin y - 2y\sin x)dx + (e^x \cos y + 2y\cos x)dy = 0,\ y(0) = \pi$

e) $2xydx + (x^2 + \cos y)dy = 0$

f) $(2xy + 3x^2)dx + x^2dy = 0$

Solution:

a) $M = x^2 - 4xy - 2y^2, N = y^2 - 4xy - 2x^2 \Rightarrow M_y = -4x - 4y = N_x$

The DE is exact.

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Using the short-cut formula:

$$u(x,y)=\int \mathbf{M}dx+\int[\mathbf{N}-\int \mathbf{M}_y dx]dy$$

$$=\int(x^2-4xy-2y^2)dx+\int[y^2-4xy-2x^2-\int(-4x-4y)dx]dy$$

$$=\frac{x^3}{3}-2x^2y-2xy^2+\int y^2dy=\frac{x^3}{3}-2x^2y-2xy^2+\frac{y^3}{3}$$

The general solution is $\frac{x^3}{3}-2x^2y-2xy^2+\frac{y^3}{3}=c\Rightarrow x^3-6x^2y-6xy^2+y^3=c$

But $y(0)=3\Rightarrow c=27\Rightarrow x^3-6x^2y-6xy^2+y^3=27$

$b)$ $M=e^y-ye^x,\ N=xe^y-e^x\Rightarrow M_y=e^y-e^x=N_x$

The DE is exact. So, by themethod of exactness,

$$u(x,y)=\int(e^y-ye^x)dx+\int[xe^y-e^x-\int(e^y-e^x)dx]dy=xe^y-ye^x$$

The general solution is $xe^y-ye^x=c$. But $y(3)=0\Rightarrow c=3\Rightarrow xe^y-ye^x=3$

$c)$ $M_y=\cos x,\ N_x=\cos x\Rightarrow M_y=N_x$. Hence, the equation is exact.

Thus, the general solution is $x^3-y^4+y\sin x=c$.

Besides, $y(2)=0\Rightarrow c=8\Rightarrow x^3-y^4+y\sin x=8$.

d) $M_y=e^x\cos y-2\cos x,\ N_x=e^x\cos y-2\cos x\Rightarrow M_y=N_x$

Thus, the general solution is $u(x,y)=e^x\sin y+2y\cos x=c$

Besides, $y(0)=\pi\Rightarrow c=2\pi\Rightarrow e^x\sin y+2y\cos x=2\pi$.

f) $M_y=2x,\ N_x=2x\Rightarrow M_y=N_x$. Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M}=2xy,\mathbf{N}=x^2+\cos y$. Then

$$u(x,y)=\int 2xydx+\int[x^2+\cos y-\int 2xdx]dy=x^2y+\sin y$$

Hence, the general solution is $x^2y+\sin y=c$.

g) $M_y=2x,\ N_x=2x\Rightarrow M_y=N_x$. Hence, the equation is exact.

Using the short-cut formula: Here, $\mathbf{M}=2xy+3x^2,\mathbf{N}=x^2$.

Then $u(x,y)=\int(2xy+3x^2)dx+\int[x^2-\int 2xdx]dy=x^2y+x^3$

Hence, the general solution is $x^2y+x^3=c$.

1.6 Non-Exact ODEs: Method of Integrating Factors

So far, we discussed how to solve DEs when they are separable, homogeneous or exact. However, there are many situations that do not fit to either of such cases. So, our next discussion focuses on how to solve DEs that are neither of the above forms. Consider the non-exact DE $P(x,y)dx+Q(x,y)dy=0$. Now, multiply this equation by a nonzero function, say μ (it will be a function of x, y, or both) such that the resulting equation $\mu Pdx+\mu Qdy=0$ is exact.

The function μ which is used to change the non-exact differential equation $P(x,y)dx+Q(x,y)dy=0$ into an equivalent exact DE of $\mu Pdx+\mu Qdy=0$ is known as *Integrating Factor*. For example, the equation $2ydx+xdy=0$ is not exact but if we multiply it by $\mu(x)=-x$, it becomes $-2xydx-x^2dy=0$ such that $M(x,y)=-2xy,\ N(x,y)=-x^2 \Rightarrow M_y=-2x=N_x$ which is exact and its solution is obtained easily. But, here the main problem is how to choose or select the function μ which is used as multiplier to change the non-exact DEs into exact DE. Even though there is no hard and fast rule on how to find integrating factor, any way let's see the general procedure to find such function. From the condition of exactness, the DE $\mu Pdx+\mu Qdy=0$ will be exact if and only if $\frac{\partial}{\partial y}(\mu P)=\frac{\partial}{\partial x}(\mu Q)$. Then, by product rule, $\mu_y P+\mu P_y=\mu_x Q+\mu Q_x$.

Now, solving this equation for μ is too complicated. So, to simplify the complication let's consider different cases for μ.

Case-1: Suppose μ is a function of x only. Then, using the relation between partial and ordinary derivatives, we have

$$\mu_y P+\mu P_y=\mu_x Q+\mu Q_x \Rightarrow \mu P_y=\frac{d\mu}{dx}Q+\mu Q_x, \quad (\because \mu_y=0,\ \mu_x=\frac{d\mu}{dx})$$

Again, dividing this result by μQ gives

$$\frac{1}{\mu}\frac{d\mu}{dx}=\frac{P_y-Q_x}{Q} \Rightarrow \frac{\mu'}{\mu}=\frac{P_y-Q_x}{Q} \quad (\because \frac{d\mu}{dx}=\mu')$$

Now, integrating this equation with respect to x gives

$$\int \frac{\mu'}{\mu}dx = \int \frac{P_y - Q_x}{Q}dx \Rightarrow \mu(x) = e^{\int f(x)dx}, \quad where\ f(x) = \frac{P_y - Q_x}{Q}$$

From this explanation, we can state the following theorem

Theorem: [Integrating Factor of the form $\mu(x)$]:

If $P(x,y)dx + Q(x,y)dy = 0$ is non-exact such that $\mu P dx + \mu Q dy = 0$ is exact and $\frac{P_y - Q_x}{Q}$ depends only on x, then the integrating factor is $\mu(x) = e^{\int \frac{P_y - Q_x}{Q}dx}$.

Case-2: Suppose μ is a function of y only.

Theorem: [Integrating Factor of the form $\mu(y)$]:

If $P(x,y)dx + Q(x,y)dy = 0$ is non-exact such that $\mu P dx + \mu Q dy = 0$ is exact and $\frac{Q_x - P_y}{P}$ depends only on y, then the integrating factor is $\mu(y) = e^{\int \frac{Q_x - P_y}{P}dy}$.

Examples:

1. Verify that the following DEs are not exact and solve by finding the appropriate Integrating Factor:

$a)\ (8x^5 + 3y^4)dx + 4xy^3dy = 0$ $\qquad b)\ 6xydx + (4y + 9x^2)dy = 0$

$c)\ (x^4 + xy)dx + (x^2 + xy)dy = 0$ $\qquad d)\ (x^2y^2 + y)dx + (y^2 - x)dy = 0$

$e)\ (x^4 + y^2)dx - xydy = 0$ $\qquad f)\ ydx + (3x - y + 3)dy = 0$

$g)\ (4xy + 2y^2)dx + (2xy + 5y^4)dy = 0$ $\qquad h)(2x^2 + y)dx + (x^2y - x)dy = 0$

Solution:

a) Here, $P = 8x^5 + 3y^4, Q = 4xy^3 \Rightarrow P_y = 12y^3, Q_x = 4y^3$.

Since $P_y \neq Q_x$, the DE is not exact or it non-exact.

So, to solve this DE, first find the integrating factor that changes it into exact.

But, $\frac{P_y - Q_x}{Q} = \frac{12y^3 - 4y^3}{4xy^3} = \frac{8y^3}{4xy^3} = \frac{2}{x} = f(x)$ which depends only on x.

Hence, the integrating factor is $\mu(x) = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$.

Now, multiply by $\mu(x) = x^2$ to get the new exact DE.

Hence, the new exact DE is $(8x^7 + 3x^2y^4)dx + 4x^3y^3dy = 0$.

Next, solve by method of exactness. Let $M = 8x^7 + 3x^2y^4$, $N = 4x^3y^3$.

Then integrating $u_x(x,y) = M(x,y)$ and integrate with respect to x.

That is $\int u_x(x,y)dx = \int (8x^7 + 3x^2y^4)dx \Rightarrow u(x,y) = x^8 + x^3y^4 + g(y)$.

Differentiating $u(x,y) = x^8 + x^3y^4 + g(y)$ with respect to y and equating with N gives $u_y(x,y) = 4x^3y^3 + g'(y) = 4x^3y^3 \Rightarrow g'(y) = 0 \Rightarrow g(y) = k$.

Therefore, the solution is $u(x,y) = c \Rightarrow x^8 + x^3y^4 = c$.

b) $P = 6xy, Q = 4y + 9x^2 \Rightarrow P_y = 6x \neq 18x = Q_x$. This means it is not exact.

Besides, $\frac{Q_x - P_y}{P} = \frac{18x - 6x}{6xy} = \frac{2}{y} = f(y) \Rightarrow \mu(y) = e^{\int \frac{2}{y}dy} = y^2$

Hence, the exact DE is $6xy^3dx + (4y^3 + 9x^2y^2)dy = 0$.

Next, solve by method of exactness. Let $M = 6xy^3$, $N = 4y^3 + 9x^2x^2$.

Then integrating $u_x(x,y) = M(x,y)$ and integrate with respect to x.

That is $\int u_x(x,y)dx = \int 6xy^3dx \Rightarrow u(x,y) = 3x^2y^3 + g(y)$.

Differentiating $u(x,y) = 3x^2y^3 + g(y)$ with respect to y and equating with N gives $u_y(x,y) = 9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2 \Rightarrow g'(y) = 4y^3$.

Integrating with respect to y gives $\int g'(y)dy = \int 4y^3dy \Rightarrow g(y) = y^4$.

Therefore, the general solution is $3x^2y^3 + y^4 = c$.

c) $M_y = x, N_x = 2x + y \Rightarrow M_y \neq N_x$. Hence, the equation is not exact.

Besides, $\frac{M_y - N_x}{N} = \frac{x - 2x - y}{x^2 + xy} = \frac{-(x+y)}{x(x+y)} = \frac{-1}{x}$. Hence, the integrating factor is $\mu(x) = e^{\int \frac{-1}{x}dx} = \frac{1}{x}$ and the new exact DE is $(x^3 + y)dx + (x + y)dy = 0$.

Next, solve by method of exactness. Let $M = x^3 + y$, $N = x + y$.

Then integrating $u_x(x,y) = M(x,y)$ and integrate with respect to x.

That is $\int u_x(x,y)dx = \int (x^3 + y)dx \Rightarrow u(x,y) = \frac{x^4}{4} + xy + g(y)$.

Differentiating $u(x,y)=\frac{x^4}{4}+xy+g(y)$ with respect to y and equating with N

gives $u_y(x,y)=x+g'(y)=x+y \Rightarrow g'(y)=y$.

Integrating $g'(y)=y$ with respect to y gives us $g(y)=\frac{y^2}{2}$.

Therefore, the general solution is $\frac{x^4}{4}+xy+\frac{y^2}{2}=c$.

d) Here, $P=x^2y^2+y, Q=y^2-x \Rightarrow P_y=2xy^2+1, Q_x=-1$.

Since $P_y \neq Q_x$, the DE is not exact or it non-exact.

To solve this DE, first change it into exact using integrating factor.

But, $\frac{Q_x-P_y}{P}=\frac{-1-(2xy^2+1)}{x^2y^2+y}=\frac{-2(1+xy^2)}{y(x^2y+1)}=-\frac{2}{y}=f(y)$ depends on y.

Hence, $\mu(y)=e^{\int f(y)dy}=e^{\int -\frac{2}{y}dy}=e^{-2\ln y}=e^{\ln(\frac{1}{y^2})}=\frac{1}{y^2}$.

Now, multiply by $\mu(y)=\frac{1}{y^2}$ to get the new exact DE.

Hence, the new exact DE is $(x^2+\frac{1}{y})dx+(1-\frac{x}{y^2})dy=0$.

Next, solve by method of exactness. Let $M=x^2+\frac{1}{y}, N=1-\frac{x}{y^2}$.

Then integrating $u_x(x,y)=M(x,y)$ and integrate with respect to x.

That is $\int u_x(x,y)dx=\int(x^2+\frac{1}{y})dx \Rightarrow u(x,y)=\frac{x^3}{3}+\frac{x}{y}+g(y)$.

Differentiating $u(x,y)=\frac{x^3}{3}+\frac{x}{y}+g(y)$ with respect to y and equating with N

gives $u_y(x,y)=-\frac{x}{y^2}+g'(y)=1-\frac{x}{y^2} \Rightarrow g'(y)=1$.

Integrating $g'(y)=1$ with respect to y gives us $g(y)=y$.

Therefore, the general solution is $u(x,y)=c \Rightarrow \frac{x^3}{3}+\frac{x}{y}+y=c$.

e) $P = x^4 + y^2, Q = -xy \Rightarrow P_y = 2y \neq -y = Q_x$. This means it is not exact.

Besides, $\dfrac{P_y - Q_x}{Q} = -\dfrac{3}{x} = f(x) \Rightarrow \mu(x) = e^{\int \frac{-3}{x} dx} = \dfrac{1}{x^3}$

Hence, the exact DE is $(x + \dfrac{y^2}{x^3})dx - \dfrac{y}{x^2} dy = 0$ and thus using the method of exactness, we get the solution to be $x^2 - \dfrac{y^2}{x^2} = c$

f) $P = y, Q = 3x - y + 3 \Rightarrow P_y = 1 \neq 3 = Q_x$. This means it is not exact.

Besides, $\dfrac{Q_x - P_y}{P} = \dfrac{2}{y} = f(y) \Rightarrow \mu(y) = e^{\int \frac{2}{y} dy} = y^2$

Hence, the exact DE is $y^3 dx + (3xy^2 - y^3 + 3y^2)dy = 0$.

Next, solve by method of exactness. Let $M = y^3, N = 3xy^2 - y^3 + 3y^2$.

Then integrating $u_x(x, y) = M(x, y)$ and integrate with respect to x.

That is $\int u_x(x, y)dx = \int y^3 dx \Rightarrow u(x, y) = xy^3 + g(y)$.

Differentiating $u(x, y) = xy^3 + g(y)$ with respect to y and equating with N gives

$u_y(x, y) = 3xy^2 + g'(y) = 3xy^2 - y^3 + 3y^2 \Rightarrow g'(y) = -y^3 + 3y^2$.

Integrate $g'(y) = -2y^3 + 3y^2$ with respect to y

That is $\int g'(y)dy = \int (-y^3 + 3y^2)dy \Rightarrow g(y) = -\dfrac{y^4}{4} + y^3$.

Therefore, the general solution is $xy^3 - \dfrac{y^4}{4} + y^3 = c$.

g) Here, $P = 4xy + 2y^2, Q = 2xy + 5y^4 \Rightarrow P_y = 4x + 4y, Q_x = 2y$.

Since $P_y \neq Q_x$, the DE is not exact or it non-exact.

But, $\dfrac{Q_x - P_y}{P} = \dfrac{2y - (4x + 4y)}{4xy + 2y^2} = \dfrac{-(4x + 2y)}{y(4x + 2y)} = -\dfrac{1}{y} = f(y)$ depends on y.

Hence, the integrating factor is $\mu(y) = e^{\int f(y)dy} = e^{\int -\frac{1}{y} dy} = e^{-\ln y} = e^{\ln(\frac{1}{y})} = \dfrac{1}{y}$.

Hence, the new exact DE is $(4x + 2y)dx + (2x + 5y^3)dy = 0$.

Next, solve by method of exactness. Let $M=4x+2y, N=2x+5y^3$.

Then integrating $u_x(x,y)=M(x,y)$ and integrate with respect to x.

That is $\int u_x(x,y)dx=\int(4x+2y)dx \Rightarrow u(x,y)=2x^2+2xy+g(y)$.

Differentiating $u(x,y)=2x^2+2xy+g(y)$ with respect to y and equating with N gives $u_y(x,y)=2x+g'(y)=2x+5y^3 \Rightarrow g'(y)=5y^3$.

Integrating $g'(y)=5y^3$ with respect to y gives us $g(y)=\frac{5y^4}{4}$.

Therefore, the general solution is $u(x,y)=c \Rightarrow 2x^2+2xy+\frac{5y^4}{4}=c$.

h) $P=2x^2+y, Q=x^2y-x \Rightarrow P_y=1 \neq 2xy-1=Q_x$. This means it is not exact.

Besides, $\frac{P_y-Q_x}{Q}=\frac{1-(2xy-1)}{x^2y-x}=\frac{-2(xy-1)}{x(xy-1)}=-\frac{2}{x}=f(x) \Rightarrow \mu(x)=e^{\int -\frac{2}{x}dy}=\frac{1}{x^2}$

Hence, the new exact DE is $(2+\frac{y}{x^2})dx+(y-\frac{1}{x})dy=0$.

Then integrating $u_x(x,y)=M(x,y)$ and integrate with respect to x.

That is $\int u_x(x,y)dx=\int(2+\frac{y}{x^2})dx \Rightarrow u(x,y)=2x-\frac{y}{x}+g(y)$ and

$u_y(x,y)=-\frac{1}{x}+g'(y)=y-\frac{1}{x} \Rightarrow g'(y)=y \Rightarrow g(y)=\frac{y^2}{2}$.

Therefore, the general solution is $2x-\frac{y}{x}+\frac{y^2}{2}=c$.

2. Verify whether the following DEs are not exact and solve by finding the appropriate Integrating Factor:

$a)$ $(y^2+2xy)dx+(4x^2+5xy+6)dy=0$

$b)$ $ydx+(2xy-e^{-2y})dy=0$

c) $3x^2y^2dx+(2x^3y+x^3y^4)dy=0$

$d)$ $2xydx+y^2dy=0$

$e)$ $(y-e^{x+y})dx-(1+xe^{x+y})dy=0$

$f)$ $(3x^2y-x^2)dx+dy=0$

$g)$ $(y+x^4)dx-xdy=0$

$h)$ $(1+y)dx+(1-x)dy=0$

Solution:

$a)$ $P_y=2y+2x, Q_x=8x+5y \Rightarrow P_y \neq Q_x$. Hence, the equation is not exact.

Besides, $\dfrac{Q_x - P_y}{P} = \dfrac{8x+5y-(2y+2x)}{y^2+2xy} = \dfrac{3(2x+y)}{y(y+2x)} = \dfrac{3}{y} = f(y)$.

Hence, the integrating factor is $\mu(y) = e^{\int \frac{3}{y}dy} = e^{3\ln y} = e^{\ln y^3} = y^3$.

Then, the new exact DE is $(y^5 + 2xy^4)dx + (4x^2y^3 + 5xy^4 + 6y^3)dy = 0$.

Here, $M_y = 5y^4 + 8xy^3$, $N_x = 8xy^3 + 5y^4 \Rightarrow M_y = N_x$. So, it is exact.

Thus, the general solution is obtained as follow:

Using the short-cut formula:

$$
\begin{aligned}
u(x,y) &= \int \mathrm{M}dx + \int [\mathrm{N} - \int \mathrm{M}_y dx]dy \\
&= \int (y^5 + 2xy^4)dx + \int [4x^2y^3 + 5xy^4 + 6y^3 - \int (5y^4 + 8xy^3)dx]dy \\
&= xy^5 + x^2y^4 + \int 6y^3 dy = xy^5 + x^2y^4 + \frac{3}{2}y^4
\end{aligned}
$$

Therefore, the general solution is $u(x,y) = xy^5 + x^2y^4 + \dfrac{3}{2}y^4 = c$.

b) $M_y = 1$, $N_x = 2y \Rightarrow M_y \neq N_x$. Hence, the equation is not exact. Besides,

$\dfrac{N_x - M_y}{M} = \dfrac{2y-1}{y} = 2 - \dfrac{1}{y} = f(y)$. Hence, the integrating factor is

$\mu(y) = e^{\int\left(2-\frac{1}{y}\right)dy} = e^{2y - \ln|y|} = e^{2y}.e^{\ln\left|\frac{1}{y}\right|} = \dfrac{e^{2y}}{y}$ and the new exact DE is

$e^{2y}dx + (2xe^{2y} - \dfrac{1}{y})dy = 0$. Thus, the solution is

$$
\begin{aligned}
&u_x(x,y) = M(x,y) = e^{2y} \Rightarrow u(x,y) = xe^{2y} + g(y) \\
&u_y(x,y) = N(x,y) \Rightarrow 2xe^{2y} + g'(y) = 2xe^{2y} - \frac{1}{y} \Rightarrow g(y) = -\ln|y|
\end{aligned}
$$

Thus, the solution is $xe^{2y} - \ln|y| = c$.

c) $M_y = 6x^2y$, $N_x = 6x^2y + 3x^2y^4 \Rightarrow M_y \neq N_x$. Hence, the equation is not

exact. Besides, $\dfrac{N_x - M_y}{M} = \dfrac{6x^2y + 3x^2y^4 - 6x^2y}{3x^2y^2} = y^2$.

Hence, the integrating factor is $\mu(y)=e^{\int y^2 dy}=e^{\frac{y^3}{3}}$ and the new exact DE is

$(3x^2y^2)e^{\frac{y^3}{3}}dx+(2x^3y+x^3y^4)e^{\frac{y^3}{3}}dy=0$.

$u_x(x,y)=3x^2y^2e^{\frac{y^3}{3}}\Rightarrow u(x,y)=x^3y^2e^{\frac{y^3}{3}}+g(y),$

$u_y(x,y)=(2x^3y+x^3y^4)e^{\frac{y^3}{3}}+g'(y)=(2x^3y+x^3y^4)e^{\frac{y^3}{3}}\Rightarrow g(y)=k$

d) $M_y=2x,\ N_x=0\Rightarrow M_y\neq N_x$. Hence, the equation is not exact. Besides,

$\frac{N_x-M_y}{M}=\frac{0-2x}{2xy}=-\frac{1}{y}=f(y)$. Hence, the integrating factor is

$\mu(y)=e^{\int\left(-\frac{1}{y}\right)dy}=e^{-\ln|y|}=\frac{1}{y}$ and the new exact DE is $2xdx+ydy=0$.

Thus, the solution is $x^2+y^2/2=c$.

e) Since $P_y=1-e^{x+y}\neq-(1+x)e^{x+y}=Q_x$,the DE is not exact. But,

$\frac{P_y-Q_x}{Q}=\frac{1-e^{x+y}+(1+x)e^{x+y}}{-(1+xe^{x+y})}=-1$. Hence, $\mu(x)=e^{\int f(x)dx}=e^{-x}$ and the new exact DE is $(ye^{-x}-e^y)dx-(xe^y+e^{-x})dy=0$.

Here, let $M=ye^{-x}-e^y,\ N=-(xe^y+e^{-x})$. Therefore, integrating $u_x(x,y)=M(x,y)$ with respect to x gives $u(x,y)=-xe^y-ye^{-x}+h(y)$. Then, differentiating $u(x,y)=-xe^y-ye^{-x}+h(y)$ w.r.t y and equating with N gives

$u_y(x,y)=-xe^y-e^{-x}+h'(y)=-(xe^y+e^{-x})\Rightarrow h'(y)=0\Rightarrow h(y)=c$

Therefore, the solution is $u(x,y)=c\Rightarrow xe^y+ye^{-x}=c$

f) Since $P_y=3x^2\neq 0=Q_x$, the DE is not exact.

But, $\frac{P_y-Q_x}{Q}=3x^2=f(x)$. Hence, the integrating factor is $\mu(x)=e^{\int f(x)dx}=e^{x^3}$ and the new exact DE is $(3x^2y-x^2)e^{x^3}dx+e^{x^3}dy=0$.

Now, let $M(x,y)=(3x^2y-x^2)e^{x^3},\ N(x,y)=e^{x^3}$.

Method of Integrating Factor for First Order Linear DEs:

Suppose $y'+p(x)y=f(x)$ is first order linear DE.

Then, $\frac{dy}{dx}+p(x)y=f(x)\Rightarrow[p(x)y-f(x)]dx+dy=0$.

Here, $P(x,y)=p(x)y-f(x),\quad Q(x,y)=1\Rightarrow P_y=p(x),\ Q_x=0\Rightarrow P_y\neq Q_x$

Thus, the DE is not exact. Besides, $\frac{P_y-Q_x}{Q}=p(x)$ is the function of x.

Hence, the integrating factor is $\mu(x)=e^{\int p(x)dx}$.

Furthermore, multiplying the DE with this integrating factor gives

$e^{\int p(x)dx}[p(x)y-f(x)]dx+e^{\int p(x)dx}dy=0$ which is exact.

Then, the general solution becomes $y(x)=e^{-\int p(x)dx}[\int e^{\int p(x)dx}f(x)dx+C]$.

Examples:

1. Find the general solution of the following first order linear DEs

a) $xy'-2y=x^3e^x$ *b)* $xy'+y=\frac{1}{x}$ *c)* $y'+y=\sin x$

d) $y'+6x^2y=x^2$ *e)* $(2y-3x)dx+xdy=0$ *f)* $xy'+y=x^2+1$

Solution: First change in the standard form $y'+p(x)y=f(x)$

a) $y'-\frac{2}{x}y=x^2e^x\Rightarrow\mu(x)=e^{\int\frac{-2}{x}dx}=\frac{1}{x^2}\Rightarrow y(x)=x^2\left(\int e^xdx+c\right)=x^2e^x+cx^2$

b) $xy'+y=\frac{1}{x}\Rightarrow y'+\frac{1}{x}y=\frac{1}{x^2}\Rightarrow\mu=e^{\int\frac{1}{x}dx}=x\Rightarrow y=\frac{1}{x}\int\frac{1}{x}dx+\frac{c}{x}=\frac{\ln|x|}{x}+\frac{c}{x}$

c) $\mu(x)=e^{\int dx}=e^x\Rightarrow y=e^{-x}\int e^x\sin xdx+Ce^{-x}\Rightarrow y=\frac{1}{2}(\sin x-\cos x)+Ce^{-x}$

d) $\mu(x)=e^{\int 6x^2dx}=e^{2x^3}\Rightarrow y=e^{-2x^3}\int x^2e^{2x^3}dx+ce^{-2x^3}=ce^{-2x^3}+\frac{1}{6}$

e) $(2y-3x)dx+xdy=0\Rightarrow\frac{dy}{dx}+\frac{2}{x}y=3\Rightarrow\mu=e^{\int\frac{2}{x}dx}=x^2\Rightarrow y=x+\frac{c}{x^2}$

2. Solve the following IVPs.

a) $y'+5y=3e^{x}-1,\quad y(0)=1$ *b)* $y'+y\tan x=\sin 2x,\quad y(0)=1$

c) $xe^{x^2}dx+(y^3-1)dy=0,\quad y(0)=0$ *d)* $xy'+2y=4x^2,\quad y(1)=3$

Solution: Use integrating factor method for first order linear DEs

a) The integrating factor is $\mu(x)=e^{\int 5dx}=e^{5x}$. Then, the general solution is

$$y(x)=e^{-5x}[\int e^{5x}(3e^{x}-1)dx+c]=e^{-5x}\left(\frac{1}{2}e^{6x}-\frac{1}{5}e^{5x}+c\right)=\frac{1}{2}e^{x}+ce^{-5x}-\frac{1}{5}.$$

Now, find c using the initial condition. That is $y(0)=1\Rightarrow c=\frac{7}{10}$.

Hence, $y(x)=\frac{1}{2}e^{x}+\frac{7}{10}e^{-5x}-\frac{1}{5}$.

b) The integrating factor is $\mu(x)=e^{\int \tan x dx}=\frac{1}{\cos x}$. Then, the general solution is

$$y(x)=\cos x[\int \frac{1}{\cos x}\sin 2x dx+c]=\cos x(-2\cos x+c)=c\cos x-2\cos^2 x$$

Now, find c using the initial condition. That is $y(0)=1\Rightarrow c=3$.

Hence, $y(x)=3\cos x-2\cos^2 x$.

c) $xe^{x^2}dx+(y^3-1)dy=0\Rightarrow (y^3-1)dy=-xe^{x^2}dx$

$$\Rightarrow \int (y^3-1)dy=-\int xe^{x^2}dx\Rightarrow \frac{y^4}{4}-y=-\frac{e^{x^2}}{2}+c$$

Now, find c using the initial condition. That is $y(0)=0\Rightarrow c=\frac{1}{2}$.

Hence, $\frac{y^4}{4}-y=\frac{1}{2}-\frac{e^{x^2}}{2}$.

d) $xy'+2y=4x^2\Rightarrow y'+\frac{2}{x}y=4x$. Hence, $\mu(x)=e^{\int \frac{2}{x}dx}=x^2$.

Then, $y(x)=e^{-\int p(x)dx}[\int e^{\int p(x)dx}f(x)dx+C]=\frac{1}{x^2}\left(\int 4x^3dx+c\right)=x^2+\frac{c}{x^2}$.

Now, find c. That is $y(1)=3\Rightarrow c=2$. Hence, $y(x)=x^2+\frac{2}{x^2}$.

1.7 Bernoulli's Differential Equations

Definition: Differential equations of the form $y'+p(x)y=f(x)y^n,\ n\in R$ are known as **Bernoulli's Differential Equations.** If $n\neq 0,1$, the DE is nonlinear. Such non linear differential equations are transformed into linear as follow.

First, multiply both sides by $\frac{1}{y^n}$. That is $y^{-n}y'+p(x)y^{1-n}=f(x)$

Using the substitution $z=y^{1-n}$, $z'=\frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}\Rightarrow y'=\frac{z'}{(1-n)y^{-n}}$.

Putting this in the equation $y^{-n}y'+p(x)y^{1-n}=f(x)$, we have

$$y^{-n}y'+p(x)y^{1-n}=f(x)\Rightarrow y^{-n}.\frac{z'}{(1-n)y^{-n}}+p(x)y^{1-n}=f(x)$$

$$\Rightarrow\frac{z'}{1-n}+p(x)z=f(x)\Rightarrow z'+(1-n)p(x)z=(1-n)f(x)$$

Therefore, we get firs order linear DE $z'+(1-n)p(x)z=(1-n)f(x)$.

Simple steps to solve Bernoulli's Equations $y'+p(x)y=f(x)y^n,\ n\in R$

First: Identify $p(x), f(x), n$ from the given problem.

Then, use the following formula to get the general solution.

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}$.

General solution: $y^{1-n}=\frac{1}{\mu(x)}\int(1-n)\mu(x)f(x)dx+\frac{c}{\mu(x)}$ where c is constant.

Examples:

1. Solve the following Bernoulli's Differential Equations.

$a)\ y'+\frac{y}{x}=x^2y^2;x>0$ $\quad b)\ y'-y=xy^2, y(0)=-1$ $\quad c)\ y'-\frac{3y}{x}=x^4y^{1/3}$

$d)\ 2xy\frac{dy}{dx}-y^2=x^2$ $\quad e)\ y'=y-\frac{1}{4}y^{3/2}, y(0)=\frac{1}{4}$ $\quad f)\ y'-2y=6y^3$

$g)\ y'=\frac{y}{x}-y^2$ $\quad h)\ 2xy'=10x^3y^5+y$ $\quad i)\ \frac{dy}{dx}=\frac{x^2+y^2+1}{2xy}$

Solution:

a) Here $p(x)=\frac{1}{x}$, $f(x)=x^2$, $n=2$.

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=\frac{1}{x}$.

General solution:

$$y^{1-n}=\frac{1}{\mu(x)}\int(1-n)\mu(x)f(x)dx+\frac{c}{\mu(x)}$$

$$\Rightarrow y^{1-2}=x\int(1-2).\frac{1}{x}.x^2dx+cx$$

$$\Rightarrow y^{-1}=x\int-xdx+cx\Rightarrow\frac{1}{y}=-\frac{x^3}{2}+cx\Rightarrow y=\frac{2}{2cx-x^3}$$

b) $y'-y=xy^2$. Here $p(x)=-1$, $f(x)=x$, $n=2$.Then,

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}=e^{\int 1dx}=e^x$.

General solution:

$$y^{-1}=\frac{1}{e^x}\int-xe^xdx+\frac{c}{e^x}=-e^{-x}(xe^x-e^x)+ce^{-x}\Rightarrow y=\frac{1}{1-x+ce^{-x}}.$$

Besides, $y(0)=-1\Rightarrow\frac{1}{1+c}=-1\Rightarrow c=-2\Rightarrow y=\frac{1}{1-x-2e^{-x}}$.

c) Here $p(x)=-\frac{3}{x}$, $f(x)=x^4$, $n=\frac{1}{3}$.

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}=e^{\int\frac{-2}{x}dx}=e^{-2\ln x}=\frac{1}{x^2}$.

General solution: $y^{2/3}=x^2\int\frac{2x^2}{3}dx+cx^2=\frac{2x^3}{9}+cx^2\Rightarrow y^2=(\frac{2x^3}{9}+cx^2)^3$.

d) First rearrange in Bernoulli's form to identify $p(x)$, $f(x)$, n.

That is $2xy\frac{dy}{dx}-y^2=x^2\Rightarrow\frac{dy}{dx}-\frac{1}{2x}y=\frac{x}{2}y^{-1}\Rightarrow p(x)=\frac{-1}{2x}, f(x)=\frac{x}{2}, n=-1$.

Integrating Factor: $\mu(x)=e^{\int(1-n)p(x)dx}=e^{\int(2)(-\frac{1}{2x})dx}=e^{\int-\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{x}$.

General solution: $y^2=x\int dx+cx=x^2+cx$.

e) $y' = y - \frac{1}{4}y^{3/2} \Rightarrow y' - y = -\frac{1}{4}y^{3/2}$. Here $p(x) = -1$, $f(x) = \frac{-1}{4}$, $n = \frac{3}{2}$.

Integrating Factor: $\mu(x) = e^{\int (1-n)p(x)dx} = e^{\int \frac{1}{2}dx} = e^{\frac{x}{2}}$.

General solution:

$$y^{1-n} = \frac{1}{\mu(x)}\int (1-n)\mu(x)f(x)dx + \frac{c}{\mu(x)} \Rightarrow y^{-1/2} = e^{\frac{-x}{2}}\left(\int \frac{1}{8}e^{\frac{x}{2}}dx + c\right) = \frac{1}{4} + ce^{\frac{-x}{2}}.$$

f) Here $p(x) = -2$, $f(x) = 6$, $n = 3$.

Integrating Factor: $\mu(x) = e^{\int (1-n)p(x)dx} = e^{\int 4dx} = e^{4x}$.

General solution:

$$y^{-2} = e^{-4x}\int -12e^{4x}dx + ce^{-4x} = ce^{-4x} - 3 \Rightarrow \frac{1}{y^2} = ce^{-4x} - 3 \Rightarrow y = \pm\frac{1}{\sqrt{ce^{-4x} - 3}}$$

g) Here, $y' = \frac{y}{x} - y^2 \Rightarrow y' - \frac{1}{x}y = -y^2 \Rightarrow p(x) = -\frac{1}{x}$, $f(x) = -1$, $n = 2$.

Integrating Factor: $\mu(x) = e^{\int (1-n)p(x)dx} = e^{\int \frac{1}{x}dx} = x$.

General solution: $y^{-1} = \frac{1}{x}\int x + \frac{c}{x} \Rightarrow \frac{1}{y} = \frac{x}{2} + \frac{c}{x} = \frac{x^2 + c}{2x} \Rightarrow y = \frac{2x}{x^2 + c}$

h) Here, $2xy' = 10x^3y^5 + y \Rightarrow y' = 5x^2y^5 + \frac{1}{2x}y \Rightarrow y' - \frac{1}{2x}y = 5x^2y^5$.

Then, we have $p(x) = -\frac{1}{2x}$, $f(x) = 5x^2$, $n = 5$.

Integrating Factor: $\mu(x) = e^{\int (1-n)p(x)dx} = e^{\int \frac{2}{x}dx} = x^2$.

General solution:

$$y^{-4} = \frac{1}{x^2}\int -20x^4dx + \frac{c}{x^2} = -4x^5 + \frac{c}{x^2} \Rightarrow \frac{1}{y^4} = -4x^5 + \frac{c}{x^2} \Rightarrow y^4 = \frac{x^2}{C - 4x^5}$$

1.8 Second Order Linear Differential Equations) with Constant Coefficients (SOLDE

Second–Order-Linear-Differential Equations with constant coefficients are equations of the form $ay''+by'+cy=f(x)$..(*)

Here, if $f(x)=0$, then the differential equation is known as homogeneous and if $f(x)\neq 0$, it is known as non-homogeneous.

Examples:

$$\left.\begin{array}{l}\text{i) } y''-3y'-4y=0\\ \text{ii) } 7y''+6y'+5y=0\end{array}\right\}\text{are homogeneous differential equations.}$$

$$\left.\begin{array}{l}\text{iii) } y''+y'-2y=6x^2\\ \text{iv) } y''-8y'-7y=2e^{3x}\end{array}\right\}\text{are non-homogeneous differential equations.}$$

1.8.1 Solutions and their Properties

Particular and Complementary solutions:

Consider second -order linear non homogeneous differential equation and the corresponding (reduced) homogeneous differential equation of the form

$$\begin{cases} ay''+by'+cy=f(x)\text{..(i)}\\ ay''+by'+cy=0\text{..(ii)}\end{cases}$$

Then, any function free of arbitrary constants that satisfies the DE in **(i)** is said to be particular solution and denoted by y_p.

The general solution of the corresponding homogeneous DE given in **(ii)** is said to be complementary solution and denoted by y_c.

Fundamental Set of Solutions and Superposition Principle

Any set $F=\{y_1,y_2\}$ of *linearly* independent solutions of the homogenous differential equation $ay''+by'+cy=0$ is said to be fundamental set of solutions.

Superposition Principle:

Let y_1 and y_2 be any two solutions of the equation $ay''+by'+cy=0$. Then, the linear combination $y=c_1y_1+c_2y_2y$ is also a solution.

In particular, if y_1 and y_2 are fundamental (linearly independent) solutions, then $y=c_1y_1+c_2y_2$ is its general solution and this is called complementary solution denoted by $y_c=c_1y_1+c_2y_2$.

So, once we have the fundamental solutions to $ay''+by'+cy=0$, we can easily determine its general solution from these fundamental solutions.

Here, to determine the general solution of $ay''+by'+cy=0$, it is sufficient to get any two we fundamental or linearly independent solutions y_1 and y_2.

Question:

How to check whether any two solutions are fundamental or not?

Wronskian Test and Fundamental Solutions:

Wronskian: Let f and g be differentiable functions. Then, the determinant defined by $W(f,g)=\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}=f.g'-f'.g$ is known as Wronskian of f and g.

Linearly Independent Functions: Any two functions are said to be linearly independent if and only if their Wronskian is non-zero.

Example: Verify that $f(x)=e^{-2x}$ and $g(x)=e^{5x}$ are linearly independent.

Solution: The Wronskian of f and g is $W(f,g)(x)=\begin{vmatrix} e^{-2x} & e^{5x} \\ -2e^{-2x} & 5e^{5x} \end{vmatrix}=7e^{3x}$.

Since $W(f,g)(x)=7e^{3x}\neq 0, \forall x\in R$, then f and g are linearly independent.

Wronskian Test (For Fundamental Solutions):

Any two solutions y_1 and y_2 of the equation $ay''+by'+cy=0$ are fundamental or linearly independent solutions if and only if $W(y_1,y_2)(x)\neq 0$ for all x.

1.8.2 Second Order Homogeneous Linear Differential Equations with constant coefficients (SOHLDE)

General Form: $ay''+by'+cy=0$ where a and b are constants.

Form of Solution: The solutions are of the form $y=e^{rx}$. (Oh! How?)

Question: What is the basic task to get the solution? As you see in $y=e^{rx}$, the constant r in the exponent is arbitrary. If we know the value of r, then the solution is known. Therefore, the basic task is to determine r.

Method to determine r: Assume $y=e^{rx}$ is the solution of $ay''+by'+cy=0$.

Since $y=e^{rx}$ is assumed to be the solution of $ay''+by'+cy=0$, it must satisfy this equation whenever substituted. Here, $y=e^{rx}, y'=re^{rx}, y''=r^2e^{rx}$.

Then, substitute these in the equation $ay''+by'+cy=0$.

$$ay''+by'+cy=0 \Rightarrow ar^2e^{rx}+bre^{rx}+ce^{rx}=0$$
$$\Rightarrow e^{rx}(ar^2+br+c)=0$$
$$\Rightarrow e^{rx}=0 \text{ or } ar^2+br+c=0$$

Since $e^{rx}\neq 0$, we must have $ar^2+br+c=0$.

Note:

i) The equation $ar^2+br+c=0$ is called **Auxiliary (Characteristics) equation** of the differential equation $ay''+by'+cy=0$.

ii) The function $y=e^{rx}$ is a solution of $ay''+by'+cy=0$ if and only if the constant r is the solution of the quadratic equation $ar^2+br+c=0$.

That is $ar^2+br+c=0 \Rightarrow r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Here, we may get two distinct real roots, single real root, or complex roots depending on the sign of the expression b^2-4ac under the radical sign. Since the type of the root determines the form of the solution of the DE, let's see the three different cases based on the type of the roots. The forms of the solutions based on the natures of the roots are summarized using table as follow.

Forms of Fundamental and General Solution for $ay''+by'+cy=0$

Cases	Roots of the Auxiliary equation $ar^2+br+c=0$	Fundamental Solutions y_1, y_2	Complementary Solution y_c
I) $b^2-4ac>0$	Two Real Roots $r_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ $r_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$	$\begin{cases} y_1=e^{r_1x} \\ y_2=e^{r_2x} \end{cases}$	$y_c=c_1y_1+c_2y_2$ i.e $y_c=c_1e^{r_1x}+c_2e^{r_2x}$
II) $b^2-4ac=0$	Single Root $r_1=-\dfrac{b}{2a}$	$\begin{cases} y_1=e^{r_1x} \\ y_2=xe^{r_1x} \end{cases}$	$y_c=c_1y_1+c_2y_2$ i.e $y_c=(c_1+c_2x)e^{r_1x}$
III) $b^2-4ac<0$	Complex Roots $r_{1,2}=\alpha\pm\beta i$ $\alpha=\dfrac{-a}{2}$, $\beta=\dfrac{\sqrt{a^2-4b}}{2}$	$\begin{cases} y_1=e^{\alpha x}\cos\beta x \\ y_2=e^{\alpha x}\sin\beta x \end{cases}$	$y_c=c_1y_1+c_2y_2$ i.e $y_c=c_1e^{\alpha x}\cos\beta x$ + $c_2e^{\alpha x}\sin\beta x$

Notice about case-III: For the two complex roots $r_1=\alpha+\beta i, r_2=\alpha-\beta i$.

Here, $y_1=e^{(\alpha+\beta i)x}$ and $y_2=e^{(\alpha-\beta i)x}$ are the fundamental solutions. But these are complex solutions while our problem is real. So, we have to change these solutions in to their real forms. This is possible by using **Euler's formula.**

$$\begin{cases} y_1=e^{(\alpha+\beta i)x}=e^{\alpha x}e^{\beta ix}=e^{\alpha x}(\cos\beta x+i\sin\beta x) \\ y_2=e^{(\alpha-\beta i)x}=e^{\alpha x}e^{-\beta ix}=e^{\alpha x}(\cos\beta x-i\sin\beta x) \end{cases} \Rightarrow \begin{cases} y_1^*=\dfrac{1}{2}(y_1+y_2)=e^{\alpha x}\cos\beta x \\ y_2^*=\dfrac{1}{2i}(y_1-y_2)=e^{\alpha x}\sin\beta x \end{cases}$$

Hence, the corresponding real solutions are $y_1=e^{\alpha x}\cos\beta x,\ y_2=e^{\alpha x}\sin\beta x$

Examples:

1. Solve the differential equations using the above procedures.

a) $y''-5y'+6y=0$ *b)* $y''+8y'+16y=0$ *c)* $y''-4y'+13y=0$

d) $2y''-7y'-4y=0$ *e)* $y''+7y'=0$ *f)* $y''+9y=0$

g) $4y''-4y'+y=0$ *h)* $y''+4y'+7y=0$ *i)* $y''+2\sqrt{2}y'+2y=0$

Solution:

a) Here, the characteristics equation is $r^2-5r+6=0$.

Solving this gives us $r^2-5r+6=0 \Rightarrow (r-2)(r-3)=0 \Rightarrow r_1=2, r_2=3$.

Hence, the fundamental solutions are $y_1=e^{2x}, y_2=e^{3x}$.

Therefore, by *case-I*, the solution is $y=c_1e^{r_1x}+c_2e^{r_2x}=c_1e^{2x}+c_2e^{3x}$.

b) Here, the characteristics equation is $r^2+8r+16=0$.

Solving this gives us $r^2+8r+16=0 \Rightarrow (r+4)^2=0 \Rightarrow r_1=r_2=-4$.

Thus, the fundamental solutions are $y_1=e^{-4x}, y_2=xy_1=xe^{-4x}$.

Hence, by *case-II*, the general solution is $y=c_1e^{-4x}+c_2xe^{-4x}$.

c) Here, the characteristics equation is $r^2-4r+13=0$.

Solve this using quadratic formula give the following complex roots.

$$r^2-4r+13=0 \Rightarrow r=\frac{4\pm\sqrt{-36}}{2}=\frac{4\pm 6i}{2}=2\pm 3i.$$

From these complex roots, we have $\alpha=2$ and $\beta=3$. Thus, the fundamental solutions are $y_1=e^{\alpha x}\cos\beta x=e^{2x}\cos 3x,\ y_2=e^{\alpha x}\sin\beta x=e^{2x}\sin 3x$

Hence, by *case-III*, the general solution is $y=e^{2x}(c_1\cos 3x+c_2\sin 3x)$.

d) Here, the characteristics equation is $2r^2-7r-4=0$.

$$2r^2-7r-4=0 \Rightarrow r=\frac{7\pm\sqrt{81}}{4}=\frac{7\pm 9}{4} \Rightarrow r_1=4,\ r_2=-\frac{1}{2}.$$

Hence, the fundamental solutions are $y_1=e^{4x}, y_2=e^{-\frac{1}{2}x}$.

Therefore, by *case-I*, the general solution is $y=c_1e^{4x}+c_2e^{-\frac{1}{2}x}$.

e) Here, $r^2+7r=0 \Rightarrow r(r+7)=0 \Rightarrow r=0, r=-7$.

Hence, the fundamental solutions are $y_1=1, y_2=e^{-7x}$

Therefore, the general solution is $y=c_1y_1+c_2y_2=c_1+c_2e^{-7x}$.

f) Here, $r^2+9=0 \Rightarrow r^2=-9 \Rightarrow r=\pm\sqrt{-9} \Rightarrow r_1=3i,\ r_2=-3i$.

Hence, by *case-III*, $y=e^{\alpha}(c_1\cos\beta x+c_2\sin\beta x e^{r_1x})=c_1\cos 2x+c_2\sin 2x$.

2. Find the values of the constant k for which the DE $y''+ky'+ky=0$ has a general solution of the form $y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$.

Solution: The characteristics equation is $r^2+kr+k=0$. Then the DE will have a general solution of the form $y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ if and only if $r^2+kr+k=0$ has complex roots. But the roots will be complex if and only if

$$k^2-4k<0 \Leftrightarrow k(k-4)<0 \Leftrightarrow 0<k<4.$$

Obtaining HLDE from its General solutions:

Once we understand the forms of solution of second order homogeneous DEs, we can also determine the reduced DE from its general solution as follow.

First: Identify the fundamental solutions y_1 and y_2 from the given general solution and obtain the roots r_1 and r_2 from these fundamental solutions.

Second: Form the characteristics equation using the roots obtained.

That is $(r-r_1)(r-r_2)=0 \Rightarrow r^2-(r_1+r_2)r+r_1.r_2=0$.

Third: Deduce the DE. It is $y''-(r_1+r_2)y'+r_1.r_2y=0$.

Examples: Find second order LHDE whose general solution is given.

a) $y=c_1e^x+c_2e^{3x}$ *b)* $y=c_1\sin\sqrt{2}x+c_2\cos\sqrt{2}x$ *c)* $y=c_1e^{2x}+c_2xe^{2x}$

Solution: First identify the roots of the characteristics equation.

a) Using *case-I*, $y=c_1e^x+c_2e^{3x} \Rightarrow y_1=e^x, y_2=e^{3x} \Rightarrow r_1=1, r_2=3$.

So, the characteristics equation is determined as follow.

$(r-r_1)(r-r_2)=0 \Rightarrow (r-1)(r-3)=0 \Rightarrow r^2-4r+3=0$.

Therefore, the required homogeneous DE is $y''-4y'+3y=0$.

b) Here, using case-III, $y_1=\sin\sqrt{2}x, y_2=\cos\sqrt{2}x \Rightarrow r_1=\sqrt{2}\mathrm{i},\ r_2=-\sqrt{2}\mathrm{i}$

So, $(r-r_1)(r-r_2)=0 \Rightarrow (r-\sqrt{2}\mathrm{i})(r+\sqrt{2}\mathrm{i})=0 \Rightarrow r^2+4=0$

Therefore, the required Homogeneous DE is $y''+4y=0$.

c) Using *case-II*, $y=c_1e^{2x}+c_2xe^{2x} \Rightarrow y_1=e^{2x}, y_2=xe^{2x} \Rightarrow r_1=r_2=2$.

So, $(r-r_1)(r-r_2)=0 \Rightarrow (r-2)(r-2)=0 \Rightarrow r^2-4r+4=0$.

Therefore, the required Homogeneous DE is $y''-4y'+4y=0$.

tial and Boundary Value Problems (IVP and BVP)

ntial equation together with some specific conditions on the dependent and its derivatives which are given at the same value of the ent variable is known as Initial Value Problems (IVPs). The specific s are said to be initial conditions. On the other hand, if the specific s are given at different values of the independent variable, the is known as Boundary Value Problem (BVPs) and the specific s are said to be boundary conditions.

of Initial Value Problems (IVPs)

$$\left.\begin{cases} ay''+by'+cy=f(x)-\text{DE} \\ y(x_0)=y_0,\ y'(x_0)=y_1-\text{ICs} \\ \qquad\downarrow \\ (\text{Both } y \text{ and } y' \text{at the same value } x=x_0) \end{cases}\right\rfloor \text{IVP}$$

of Boundary Value Problems (BVPs)

$$\left.\begin{cases} ay''+by'+cy=f(x)-\text{DE} \\ y(x_0)=y_0,\ y'(x_1)=y_1-\text{BCs} \\ \qquad\downarrow \\ (\text{Here } y \text{ and } y' \text{t are given at diiferent values } x=x_0, x=x_1) \end{cases}\right\rfloor \text{BVP}$$

s:

ne following IVPs

$-4y=0, y(0)=4, y'(0)=14$ $\quad$ $b)\ y''-3y'=0, y(0)=7, y'(0)=-9$

$0, y(0)=2, y'(0)=0$ $\quad$ $d)\ y''-4y'+4y=0, y(0)=1, y'(0)=4$

$y=0,\ y(1)=3, y'(1)=4$ $\quad$ $f)\ y''+2y=0, y(\pi)=1, y'(\pi)=0$

e characteristics equation is $r^2+3r-4=0 \Rightarrow r=1,-4$.

general solution is $y=c_1e^{x}+c_2e^{-4x}$.

determine the constants $c_1,\ c_2$ using the initial conditions.

$$)=4, y'(0)=14 \Rightarrow \begin{cases} c_1+c_2=4 \\ c_1-4c_2=14 \end{cases} \Rightarrow c_1=6, c_2=-2.$$

the solution is $y=6e^{x}-2e^{-4x}$.

b) Here, the characteristics equation is $r^2 - 3r = 0 \Rightarrow r = 0,3$.

Thus, the general solution is $y = c_1 + c_2 e^{3x}$. Now, determine c_1, c_2.

That is $y(0) = 7, y'(0) = -9 \Rightarrow \begin{cases} c_1 + c_2 = 7 \\ 3c_2 = -9 \end{cases} \Rightarrow c_2 = -3, c_1 = 10$.

Therefore, the solution is $y = 10 - 3e^{3x}$.

c) Here, the characteristics equation is $r^2 + 1 = 0 \Rightarrow r = \pm i$ .Thus, the general solution is $y = c_1 \cos x + c_2 \sin x$. Now, let's determine c_1, c_2.

That is $y(0) = 2 \Rightarrow c_1 = 2$, $y'(0) = 0 \Rightarrow c_2 = 0$ .Therefore, $y = 2\cos x$.

d) Here, $r^2 - 4r + 4 = 0 \Rightarrow r = 2$.Thus, the general solution is $y = c_1 e^{2x} + c_2 x e^{2x}$

and $y(0) = 1, y'(0) = 4 \Rightarrow \begin{cases} c_1 = 1 \\ 2c_1 + c_2 = 4 \end{cases} \Rightarrow c_1 = 1, c_2 = 2$ .So, $y = (1 + 2x)e^{2x}$.

2. Solve the following BVPs

a) $y'' + 4y = 0, y(0) = 2, y'(\pi) = -6$ *b)* $y'' + 4y' + 4y = 0, y(0) = 6, y(3) = 0$

Solution:

a) Here, the characteristics equation is $r^2 + 4 = 0$.

Solving this gives us $r^2 + 4 = 0 \Rightarrow r_1 = 2i,\ r_2 = -2i$.

Hence, the general solution is $y = c_1 \cos 2x + c_2 \sin 2x$.

Now, let's determine the constants c_1, c_2 using the boundary conditions.

$$y(0) = 2, y'(\pi) = -6 \Rightarrow \begin{cases} c_1 \cos 0 + c_2 \sin 0 = 2 \\ -2c_1 \sin(2\pi) + 2c_2 \cos(2\pi) = -6 \end{cases} \Rightarrow c_1 = 2, c_2 = -3.$$

Hence, the solution of the BVP is $y = 2\cos 2x - 3\sin 2x$

b) Here, $r^2 + 4r + 4 = 0 \Rightarrow r = -2$. The solution is $y = c_1 e^{-2x} + c_2 x e^{-2x}$.

Now, let's determine the constants c_1, c_2 using the boundary conditions.

Therefore, the solution of the BVP is $y = 6e^{-2x} - 2xe^{-2x}$.

1.8.4 Solving Non-homogeneous Linear Differential Equations (SONHLDE) with Constant Coefficients

Form of SONHLDE with constant coefficients: $ay''+by'+cy=f(x)$.

Particular solution of $ay''+by'+cy=f(x)$**:**

Any function y_p free of arbitrary constants that satisfies this SOHLDE is said to be a particular solution.

Theorem (The General Solution Theorem, GST):

If y_p is any particular solution of the non-homogeneous DE $ay''+by'+cy=f(x)$ and y_c is the complementary solution of the homogeneous part $ay''+by'+cy=0$, then the general solution of $ay''+by'+cy=f(x)$ is given by $y=y_c+y_p$.

In short, this theorem says that the general solution of $ay''+by'+cy=f(x)$ is the sum of the general solution of the corresponding HLDE $ay''+by'+cy=0$ and any particular solution of $ay''+by'+cy=f(x)$. That is $y=y_c+y_p$.

So far, we have seen how to find y_c of $ay''+by'+cy=0$ but how to get y_p?

Procedures to solve $ay''+by'+cy=f(x)$.

First: Solve $ay''+by'+cy=0$ and obtained the solution $y_c=c_1y_1+c_2y_2$.

Second: Find any particular solution y_p of $ay''+by'+cy=f(x)$.

Third: Form the general solution $y=c_1y_c+c_2y_2+y_p$ using the GST.

There are different methods to find y_p.

1. Method of Undetermined Coefficients (MUCs).
2. Variation of Parameters (VPs)
3. The Operator Method (OM)
4. Diagonalization Method (DM)
5. Laplace Transform Method
6. Power Series Method

1.11.1 Method of Undetermined Coefficients (MUCs)

Suppose we want to solve $ay''+by'+cy=f(x)$ where the coefficients a,b and c are constants using the Method of Undetermined coefficient.

Main principles to notice about MUCs:

i) Assumption: The method of undetermined coefficients assumes that the solution to the DE equation is the same form as $f(x)$.

ii) Starts with trial form: Making educated guess

Once we assume y_p has the same form as $f(x)$, the method proceeds with an educated guess by expressing y_p using undetermined coefficients.

iii) Coefficient Determination: From the trial form obtain, y_p, y'_p and y''_p and substitute in $ay''+by'+cy=f(x)$ to determine the coefficients. With this it is possible to determine the undetermined coefficients to be determined.

For this reason, the method is named as Method of Undetermined Coefficients.

Conditions to use the method: When do we use the method?

The general method is limited to non-homogeneous linear DE of $ay''+by'+cy=f(x)$ with the assumption that the coefficients are constants, and $f(x)$ is only of the form $p_n(x), e^{ax}, \sin\alpha x, \cos\beta x, p_n(x)e^{ax}$, and their combinations like $p_n(x)\sin\alpha x, p_n(x)\cos\beta x, p_n(x)e^{ax}\sin\alpha x + p_n(x)e^{ax}\cos\beta x$.

Exceptions: When does the method fails?

The Method of Undetermined Coefficient (MUCs) is not applicable if the function $f(x)$ is of the form $\frac{1}{x}, \ln x, \sqrt{x}, \tan x, \cos^{-1} x, \sec x, \cot x, \csc x$ or any other transcendental functions.

TABLE-1.2: The basic Trial Forms of Particular Solution

The form of $f(x)$	Trial form of y_p
$f(x)=ae^{kx}$	$y_p=Ae^{kx}$
$f(x)=ax+b$	$y_p=Ax+B$
$f(x)=ax^2+bx+c$	$y_p=Ax^2+Bx+C$
$f(x)=\begin{cases} a\sin kx \text{ or } b\cos kx \\ a\sin kx+b\cos kx \end{cases}$	$y_p=A\sin kx+B\cos kx$
$f(x)=\begin{cases} a\sinh kx \text{ or} \\ b\cosh kx \end{cases}$	$y_p=A\sinh kx+B\cosh kx$
$f(x)=\begin{cases} a\sin kx+b\cos mx \\ a\sin kx+b\sin mx \end{cases}; k\neq m$	$y_p=A\sin kx+B\cos kx+C\sin mx+D\cos mx$
$f(x)=ae^{kx}+be^{mx}, k\neq m$	$y_p=Ae^{kx}+Be^{mx}$
For Product forms	
$f(x)=(ax+b)e^{kx}$	$y_p=(Ax+B)e^{kx}$
$f(x)=\begin{cases} (ax+b)\sin kx \text{ or} \\ (ax+b)\cos kx \end{cases}$	$y_p=(Ax+B)\sin kx+(Cx+D)\cos kx$
$f(x)=ae^{mx}\sin kx \text{ or } be^{mx}\cos kx$	$y_p=Ae^{mx}\sin kx+Be^{mx}\cos kx$
$f(x)=\begin{cases} (ax+b)e^{mx}\sin kx \\ (ax+b)e^{mx}\cos kx \end{cases}$	$y_p=(Ax+B)e^{mx}\sin kx+(Cx+D)e^{mx}\cos kx$
$f(x)=(ax^2+bx+c)e^{kx}$	$y_p=(Ax^2+Bx+C)e^{kx}$

Cautions:

The table will give you only **hints** on how to guess y_p based on $f(x)$.

Always, ASK yourself the following questions about y_p:

Does what we guess for y_p always work? **No!**

How do we know when it does not work? **From analysis of roots!**

How do we correct if it does not work? **Use Modification Rules!**

Examples: Solve the following DEs using MUCs

$a)\ y''-3y'-4y=12e^{2x}$ $b)\ y''-2y'+y=x^2-x+3$ $c)\ y''-3y'-4y=34\sin x$

$d)\ y''+y'-2y=2xe^{-x}$ $e)\ y''+2y=3x\sin x$ $f)\ y''+2y'+5y=e^{x}\cos 2x$

Solution:

a) **Step-1:** Find the complementary solution y_c of $y''-3y'-4y=0$.

Here, the characteristics equation is $r^2-3r-4=0$. Solving this gives us

$r^2-3r-4=0 \Rightarrow (r+1)(r-4)=0 \Rightarrow r_1=-1, r_2=4$.

Hence, the complementary solution is $y_c=c_1e^{r_1x}+c_2e^{r_2x}=c_1e^{-x}+c_2e^{4x}$.

Step-2: Find the particular solution y_p having the same form as $f(x)=12e^{2x}$

From the table, it seems of the form $y_p=ae^{2x}$. Then, $y'_p=2ae^{2x}$, $y''_p=4ae^{2x}$.

Now, determine the constant a by substituting these values in the given DE.

That is $y''-3y'-4y=12e^{2x} \Rightarrow 4ae^{2x}-6ae^{2x}-4ae^{2x}=12e^{2x}$

$$\Rightarrow -6ae^{2x}=12e^{2x} \Rightarrow a=-2$$

Thus, $y_p=ae^{2x}=-2e^{2x}$. Therefore, $y=y_c+y_p=c_1e^{-x}+c_2e^{4x}-2e^{2x}$.

b) **Step-1:** Find the complementary solution y_c of $y''-2y'+y=0$.

Here, $r^2-2r+1=0 \Rightarrow (r-1)^2=0 \Rightarrow r=1$.

Hence, the complementary solution is $y_c=c_1e^{x}+c_2xe^{x}$.

Step-2: Find the particular solution y_p of the form $f(x)=x^2-x+3$.

That is y_p is of the form $y_p=ax^2+bx+c$.

Then, substitute $y_p=ax^2+bx+c$, $y'_p=2ax+b$, $y''_p=2a$ in the DE.

$$y''-2y'+y=x^2-x+3 \Rightarrow 2a-2(2ax+b)+ax^2+bx+c=x^2-x+3$$

$$\Rightarrow ax^2+(b-4a)x+2a-2b+c=x^2-x+3$$

$$\Rightarrow a=1, b-4a=-1, 2a-2b+c=3 \Rightarrow a=1, b=3, c=7$$

So, the particular solution is $y_p=x^2+3x+7$.

Hence, the general solution is $y=y_C+y_p=c_1e^{x}+c_2xe^{x}+x^2+3x+7$

c) **Step-1:** Find the complementary solution y_c of $y''-3y'-4y=0$.

It is the same as in part (a). That is $y_c=c_1e^{-x}+c_2e^{4x}$.

Step-2: Find the particular solution y_p of the form $f(x)=34\sin x$

It seems of the form $y_p=a\sin x+b\cos x$.

Then, $y'_p=a\cos x-b\sin x$, $y''_p=-a\sin x-b\cos x$.

Now, determine a,b by substituting these values in the given DE.

$$y''-3y'-4y=34\sin x$$

$$\Rightarrow -a\sin x-b\cos x-3(a\cos x-b\sin x)-4(a\sin x+b\cos x)=34\sin x$$

$$\Rightarrow (-5a+3b)\sin x+(-3a-5b)\cos x=34\sin x$$

Equation the.coefficients of $\sin x$, $\cos x$ on both sides, we have

$$\begin{cases} -3a-5b=0\Rightarrow b=\dfrac{-3}{5}a, \\ -5a+3b=34\Rightarrow \dfrac{-34}{5}a=34\Rightarrow a=-5, b=\dfrac{-3}{5}a\Rightarrow b=3 \end{cases}$$

Thus, $y_p=a\sin x+b\cos x=-5\sin x+3\cos x$.

Therefore, $y=y_c+y_p=c_1e^{-x}+c_2e^{4x}-5\sin x+3\cos x$.

d) First, find the complementary solution y_c of $y''+y'-2y=0$.

Here, $r^2+r-2=0\Rightarrow (r-1)(r+2)=0\Rightarrow r_1=1, r_2=-2$.

Hence, the complementary solution is $y_c=c_1e^{r_1x}+c_2e^{r_2x}=c_1e^{x}+c_2e^{-2x}$.

Next, find the particular solution y_p of the form $f(x)=2xe^{-x}$.

It seems of the form $y_p=(ax+b)e^{-x}$.

Then, $y'_p=ae^{-x}-(ax+b)e^{-x}$, $y''_p=-2ae^{-x}+(ax+b)e^{-x}$.

Thus, substitute these values to determine the coefficients.

$$y''+y'-2y=2xe^{-x}$$

$$\Rightarrow -2ae^{-x}+(ax+b)e^{-x}+ae^{-x}-(ax+b)e^{-x}-2(ax+b)e^{-x}=2xe^{-x}$$

$$\Rightarrow -2axe^{-x}-(a+2b)e^{-x}=2xe^{-x}$$

$$\Rightarrow -2a=2, -a-2b=0\Rightarrow a=-1, b=\frac{1}{2}\Rightarrow y_p=(\frac{1}{2}-x)e^{-x}$$

Therefore, the general solution is $y=y_c+y_p=c_1e^{x}+c_2e^{-2x}+(\frac{1}{2}-x)e^{-x}$.

$\nu=0\Rightarrow r^2+2=0\Rightarrow r=\pm\sqrt{2}i\Rightarrow y_c=c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x$

s' find the particular solution y_p of the form $f(x)=3x\sin x$.

, is of the form $y_p=(ax+b)\cos x+(cx+d)\sin x$. Then, we have

$\cos x-(ax+b)\sin x+c\sin x+(cx+d)\cos x$

$2a\sin x-(ax+b)\cos x+2c\cos x-(cx+d)\sin x$

$2y=3x\sin x$

$2c+b)\cos x+(d-2a)\sin x+ax\cos x+cx\sin x=3x\sin x$

$!c+b=0, d-2a=0, a=0, c=3\Rightarrow a=0, b=-6, d=0$

$y_p=3x\sin x-6\cos x$

the general solution is $y=c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x+3x\sin x-6\cos x$

$-2r+5=0\Rightarrow r=-1+2i,-1-2i$. Hence, the fundamental solutions are

$^{-x}\cos 2x,\ y_2=e^{-x}\sin 2x$. Now, find y_p of the form $f(x)=e^x\cos 2x$.

the form $y_p=ae^x\sin 2x+be^x\cos 2x$.

$$\begin{cases} y'_p=(a-2b)e^x\sin 2x+(2a+b)e^x\cos 2x, \\ y''_p=-(3a+4b)e^x\sin 2x+(4a-3b)e^x\cos 2x \end{cases}$$

$y'+5y=e^x\cos 2x$

$$\Rightarrow (4a-8b)e^x\sin 2x+(8a+4b)e^x\cos 2x=e^x\cos 2x$$

$$\Rightarrow 4a-8b=0,\ 8a+4b=1\Rightarrow a=2b,\ 8a+4b=1$$

$$\Rightarrow 16b+4b=1\Rightarrow b=\frac{1}{20},\ a=\frac{1}{10}$$

$$\Rightarrow y_p=\frac{1}{10}e^x\sin 2x+\frac{1}{20}e^x\cos 2x=\frac{e^x}{10}(2\sin 2x+\cos 2x)$$

Remark (Sum Rule for Trial forms):

In the DE $ay''+by'+cy=f(x)$, the function $f(x)$ may be the sum (difference) of functions like $f(x)=g(x)+h(x)$,in such case, guess the trial forms y_{p1} for $g(x)$ and y_{p2} for $h(x)$ separately as the particular solutions.

Then, their sum $y_p=y_{p1}+y_{p2}$ is a trail form for $f(x)=g(x)+h(x)$.

Examples: Solve the following DEs using MUC.

$a)$ $y''+4y'+5y=30e^x+10x-7$ $\quad$ $b)$ $y''+4y=8x+9\sin x$

$c)$ $y''-y'-2y=2x^2+e^x$ $\quad$ $d)$ $y''+5y'+6y=4e^{-x}+5\sin x$

Solution:

Step-1: Find the complementary solution y_c of $y''+4y'+5y=0$.

Here, $r^2+4r+5=0\Rightarrow r=-2\pm i$. Then, $y_c=c_1e^{-2x}\cos x+c_2e^{-2x}\sin x$

Step-2: Find y_p which is the same form as $f(x)=30e^x+10x-7$.

Here, $f(x)=g(x)+h(x)$ where $g(x)=30e^x$, $h(x)=10x-7$.

So, guess $y_{p1}=Ae^x$ for $g(x)=30e^x$ and $y_{p2}=Bx+C$ for $h(x)=9\sin x$.

Then, y_p for $f(x)=30e^x+10x-7$ becomes $y_p=y_{p1}+y_{p2}=Ae^x+Bx+C$.

That is $y_p=ae^x+bx+c\Rightarrow y'_p=ae^x+b,\ y''_p=ae^x$.

So, $y''+4y'+5y=30e^x+10x-7$

$$\Rightarrow Ae^x+4(Ae^x+B)+5(Ae^x+Bx+C)=30e^x+10x-7$$

$$\Rightarrow Ae^x+4Ae^x+4B+5Ae^x+5Bx+5C=30e^x+10x-7$$

$$\Rightarrow 10Ae^x+5Bx+4B+5C=30e^x+10x-7$$

$$\Rightarrow \begin{cases} 10A=30\Rightarrow A=3 \\ 5B=10\Rightarrow B=2 \\ 4B+5C=-7\Rightarrow C=-3 \end{cases} \Rightarrow y_p=3e^x+2x-3$$

Therefore, $y=y_c+y_p=c_1e^{-2x}\cos x+c_2e^{-2x}\sin x+3e^x+2x-3$.

b) **Step-1:** Find the complementary solution y_c of $y''+4y=0$.

Here, the characteristics equation $r^2+4=0\Rightarrow r=\pm 2i$.

So, the complementary solution is $y_c=c_1\cos 2x+c_2\sin 2x$.

Step-2: Find the particular solution y_p of the form $f(x) = 8x + 9\sin x$.

Guess $y_{p1} = Ax + B$ for $g(x) = 8x$ and $y_{p2} = C\cos x + D\sin x$ for $h(x) = 9\sin x$.

Then, $y_p = y_{p1} + y_{p2} = Ax + B + C\cos x + D\sin x$ for $f(x) = 8x + 9\sin x$.

Hence, $y'_p = A - C\sin x + D\cos x$, $y''_p = -C\cos x - D\sin x$.

Now, determine the constants A, B, C, D.

$$y'' + 4y = 8x + 9\sin x$$

$$\Rightarrow -C\cos x - D\sin x + 4[Ax + B + C\cos x + D\sin x] = 8x + 9\sin x$$

$$\Rightarrow 4A = 8, 4B = 0, 3C = 0, 3D = 9 \Rightarrow A = 2, B = 0, C = 0, D = 3$$

Hence, the particular solution is $y_p = 2x + 3\sin x$.

Therefore, $y = c_1\cos 2x + c_2\sin 2x + 2x + 3\sin x$.

c) **Step-1:** Find the complementary solution y_c of $y'' - y' - 2y = 0$.

Here, the characteristics equation is $r^2 - r - 2 = 0$. Solving this gives us $r^2 - r - 2 = 0 \Rightarrow (r+1)(r-2) = 0 \Rightarrow r_1 = -1, r_2 = 2$. So, $y_c = c_1e^{-x} + c_2e^{2x}$.

Step-2: Find the particular solution y_p having the same form as $f(x) = 4x^2 + 8e^x$. Here, we guess $y_{p1} = ax^2 + bx + c$ for $g(x) = 4x^2$ and $y_{p2} = de^x$ for $h(x) = 8e^x$. Then, $y_p = y_{p1} + y_{p2} = ax^2 + bx + c + de^x$ for $f(x) = 4x^2 + 8e^x$. Hence, $y'_p = 2ax + b + de^x$, $y''_p = 2a + de^x$.

Now, determine the constants a, b, c, d.

$$y'' - y' - 2y = 4x^2 + 8e^x$$

$$\Rightarrow 2a + de^x - [2ax + b + de^x] - 2(ax^2 + bx + c + de^x) = 4x^2 + 8e^x$$

$$\Rightarrow -2ax^2 - (2a + 2b)x + 2a - b - 2c - 2de^x = 4x^2 + 8e^x$$

$$\Rightarrow -2a = 4, -2a - 2b = 0, 2a - b - 2c = 0, -2d = 8$$

$$\Rightarrow a = -2, b = 2, c = -3, d = -4$$

Hence, the particular solution is $y_p = -2x^2 + 2x - 3 - 4e^x$.

Therefore, $y = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3 - 4e^x$.

Modification Rules for Trial Forms: Generalization on MUCs

Now, let's analyze the three questions about y_p that we have posed earlier.

- ✓ Does what we guess for y_p always work? **No!**
- ✓ How do we know when it does not work? **From analysis of roots!**
- ✓ How do we correct if it does not work? **Use Modification Rules!**

As we have discussed for the DE $ay''+by'+cy=f(x)$, we used an educated guess of the particular solution y_p based on the form of $f(x)$. But this is not always true. There are cases where the form of the particular solution y_p that we guessed may not work. In what follow, let's discuss the cases where the form of y_p is determined based on the relation between the roots r_1 and r_2 of the characteristics equation and some part of $f(x)$.

Modification Rule-1: For the form $f(x)=(A_nx^n+...+A_1x+A_0)e^{kx}$.

In such case, the form of the particular solution y_p of $ay''+by'+cy=0$ depends on r_1,r_2 and the exponent k.

i) If $r_1\neq k, r_2\neq k$, then $y_p=(a_nx^n+...+a_1x+a_0)e^{kx}$

ii) If $r_1=k$ or $r_2=k$ but $r_1\neq r_2$, then $y_p=x(a_nx^n+...+a_1x+a_0)e^{kx}$

iii) If $r_1=r_2=k$, then $y_p=x^2(a_nx^n+...+a_1x+a_0)e^{kx}$

Examples: Solve the following DEs using undetermined coefficients.

$a)\ y''-3y'-4y=10e^{-x}$ $\quad b)\ y''-3y'-4y=xe^{-x}$ $\quad c)\ y'-2y'+y=6e^{x}$

$d)\ y''+6y'+9y=xe^{-3x}$ $\quad e)\ y''+y'-2y=e^{x}+e^{-x}$ $\quad f)\ y''-4y'+4y=xe^{2x}$

Solution:

a) **Step-1:** Find the complementary solution y_c of $y''-3y'-4y=0$

Here, the characteristics equation of is $r^2-3r-4=0$. Solving this gives us

$r^2-3r-4=0\Rightarrow(r+1)(r-4)=0\Rightarrow r_1=-1, r_2=4$.

Hence, the complementary solution is $y_c=c_1e^{r_1x}+c_2e^{r_2x}=c_1e^{-x}+c_2e^{4x}$.

Step-2: Find y_p .Here, $f(x)=10e^{kx}$, with $k=-1$, $r_1\neq r_2$ and $r_1=-1, r_2=4$ but here $r_1=k$ .Thus, y_p is of the form $y_p=axe^{-x}$.

Then, $y'_p = ae^{-x} - axe^{-x}$, $y''_p = -2ae^{-x} + axe^{-x}$. So,

$$y''-3y'-4y = 2e^{-x} \Rightarrow -2ae^{-x} + axe^{-x} - 3(ae^{-x} - axe^{-x}) - 4ae^{-x} = 10e^{-x}$$

$$\Rightarrow -5ae^{-x} = 10e^{-x} \Rightarrow -5a = 10 \Rightarrow a = -2$$

Thus, $y_p = -2xe^{-x}$. Therefore, $y = y_c + y_p = c_1e^{-x} + c_2e^{4x} - 2xe^{-x}$.

b) Here, $r^2 - 3r - 4 = 0 \Rightarrow r_1 = -1, r_2 = 4$ and $f(x) = xe^{-x}$, with $k = -1$.

So, by the second part of rule-1, $y_p = x(ax+b)e^{-x} = (ax^2 + bx)e^{-x}$.

Then, $y'_p = (b + 2ax - bx - ax^2)e^{-x}$, $y''_p = (ax^2 - 4ax + bx + 2a - 2b)e^{-x}$.

So, $y''-3y'-4y = xe^{-x}$

$$\Rightarrow (ax^2 - 4ax + bx + 2a - 2b)e^{-x} - 3(b + 2ax - bx - ax^2)e^{-x} - 4(ax^2 + bx)e^{-x} = xe^{-x}$$

$$\Rightarrow (-10ax + 2a - 5b)e^{-x} = xe^{-x} \Rightarrow -10a = 1,\ 2a - 5b = 0 \Rightarrow a = -\frac{1}{10}, b = -\frac{1}{25}$$

Thus, $y_p = (-\frac{x^2}{10} - \frac{x}{25})e^{-x}$.

c) **Step-1:** Find the complementary solution y_c of $y''-2y'+y = 0$

Here, the solution of the corresponding homogenous equation is

$r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0 \Rightarrow r_1 = r_2 = 1$.

Hence, the complementary solution is $y_c = c_1e^x + c_2xe^x$.

Step-2: Find the particular solution y_p.

Here, $f(x) = 6e^{kx}$, with $k = 1$, $r_1 = r_2 = k = 1$. Thus, $y_p = ax^2e^x$.

Then, $y'_p = (2ax + ax^2)e^x$, $y''_p = (2a + 4ax + ax^2)e^x$. So,

$$y''-2y'+y = 6e^x \Rightarrow (2a + 4ax + ax^2)e^x - 2(2ax + ax^2)e^x + ax^2e^x = 6e^x$$

$$\Rightarrow 2ae^x = 6e^x \Rightarrow 2a = 6 \Rightarrow a = 3$$

Thus, $y_p = 3x^2e^x$. Therefore, $y = y_c + y_p = c_1e^x + c_2xe^x + 3x^2e^x$.

d) Here, $r^2 + 6r + 9 = 0 \Rightarrow r_1 = r_2 = -3$ and $f(x) = xe^{-3x}$, with $k = -3$.

So, by third part of rule-1, $y_p = x^2(ax + b)e^{-3x}$. (Complete it !)

e) **Step-1:** Find the complementary solution y_c of $y''+y'-2y = 0$.

Here, the characteristics equation is $r^2+r-2=0$. Solving this gives us $r^2+r-2=0 \Rightarrow (r-1)(r+2)=0 \Rightarrow r_1=1, r_2=-2$. So, $y_c=c_1e^x+c_2e^{-2x}$.

Step-2: Find y_p having the same form as $f(x)=e^x+e^{-x}$. Here, we guess $y_{p1}=axe^x$ for $g(x)=e^x$ and $y_{p2}=be^{-x}$ for $h(x)=e^{-x}$. Then,

$y_p=y_{p1}+y_{p2}=axe^x+be^{-x}$ for $f(x)=e^x+e^{-x}$. Now, determine a and b.

$y''+y'-2y=e^x+e^{-x}$

$$\Rightarrow 2ae^x+axe^x+be^{-x}+ae^x+axe^x-be^{-x}-2[axe^x+be^{-x}]=e^x+e^{-x}$$

$$\Rightarrow 3ae^x-2be^{-x}=e^x+e^{-x} \Rightarrow a=\frac{1}{3}, b=-\frac{1}{2} \Rightarrow y_p=\frac{1}{3}xe^x-\frac{1}{2}e^{-x}$$

Therefore, $y=c_1e^x+c_2e^{-2x}+\frac{1}{3}xe^x-\frac{1}{2}e^{-x}$.

Modification Rule-2: For the form $f(x)=A_nx^n+...+A_1x+A_0$.

In such case, the form of the particular solution y_p of $ay''+by'+cy=f(x)$ depends on the coefficients a, b.

i) If $b \neq 0$, then $y_p=a_nx^n+...+a_1x+a_0$

ii) If $b=0$, but $a \neq 0$, then $y_p=x(a_nx^n+...+a_1x+a_0)$

iii) If $a=b=0$, then $y_p=x^2(a_nx^n+...+a_1x+a_0)$

Examples: Find the particular solution of the following DEs.

a) $y''-3y'=18x^2+2$ *b)* $y''=24x$ *c)* $y''=9x^2+2x-6$

Solution:

a) Here, the characteristics equation is $r^2-3r=0 \Rightarrow a=-3, b=0$ and $f(x)=18x^2+2$. So, y_p is of the form $y_p=x(ax^2+bx+c)=ax^3+bx^2+cx$.

Then, $y'_p=3ax^2+2bx+c$, $y''_p=6ax+2b$. So,

$$y''-3y'-4y=2e^{-x} \Rightarrow 6ax+2b-3(3ax^2+2bx+c)=18x^2+2$$

$$\Rightarrow -9ax^2+(6a-6b)x+2b-3c=18x^2+2$$

$$\Rightarrow -9a=18, 6a-6b=0, 2b-3c=2 \Rightarrow a=-2, b=-2, c=-2$$

Thus, $y_p=x(-2x^2-2x-2)=-2x^3-2x^2-2x$.

b) Here, the characteristics equation is $r^2=0$. So, the coefficients of the characteristics equations are $a=b=0$ and $f(x)=24x$. So, y_p is of the form $y_p=x^2(ax+b)=ax^3+bx^2$. Then, $y'_p=3ax^2+2bx,\ y''_p=6ax+2b$.

But $y''=24x \Rightarrow 6ax+2b=24x \Rightarrow 6a=24, 2b=0 \Rightarrow a=4, b=0$

Thus, $y_p=4x^3$ and the general solution is $y=y_c+y_p=c_1+c_2x+4x^3$.

c) Here, the characteristics equation $r^2=0 \Rightarrow r_1=r_2=0$ which is a single root.

Then, the fundamental solutions are $y_1=e^0=1, y_2=xy_1=x$.

Next, let's determine the particular solution y_p. The direct trial form is

$y_p=ax^2+bx+c$ but it does not work because the coefficients of the characteristics equations $r^2=0$ are $a=b=0$. Besides, $f(x)=9x^2+2x-6$.

Thus, by the above modification rule, the direct trial form of y_p must be modified as $y_p=x^2(ax^2+bx+c)=ax^4+bx^3+cx^2$.

Then, $y'_p=4ax^3+3bx^2+2cx,\ y''_p=12ax^2+6bx+2c$.

So, $y''=9x^2+2x-6 \Rightarrow 12ax^2+6bx+2c=9x^2+2x-6$

$$\Rightarrow 12a=9, 6b=2, 2c=-6 \Rightarrow a=\frac{3}{4}, b=\frac{1}{3}, c=-3$$

Thus, $y_p=\frac{3}{4}x^4+\frac{1}{3}x^3-3x^2$. Therefore, $y=c_1+c_2x+\frac{3}{4}x^4+\frac{1}{3}x^3-3x^2$.

Modification Rule-3: For the form $f(x)=(a\cos\beta x+b\sin\beta x)e^{\alpha x}$.

In such cases, the form of the particular solution y_p of $ay''+by'+cy=f(x)$ depends on the relation between the characteristics roots r_1, r_2 and α, β.

i) If $r_1\neq\alpha+\beta i, r_2\neq\alpha-\beta i$, then $y_p=(A\cos\beta x+B\sin\beta x)e^{\alpha x}$

ii) If $r_1=\alpha+\beta i, r_2=\alpha-\beta i$, then $y_p=x(A\cos\beta x+B\sin\beta x)e^{\alpha x}$

Examples: Find the particular solution of the following DEs.

a) $y''+y=\sin x$ *b)* $y''+2y'+5y=e^{-x}\sin 2x$ *c)* $y''+4y=x\cos 2x$

d) $y''+4y=8\cos 2x$ *e)* $y''-2y'+y=e^x\sin x$ *f)* $y''+16y=4\cos 4x$

Solution:

a) Here, $f(x)=\sin x$, with $\beta=1, \alpha=0$.

But $r^2+1=0 \Rightarrow r=\pm i \Rightarrow r_1=i=\alpha+\beta i, r_2=-i=\alpha-\beta i$

So, by part (ii) of modification rule-3, $y_p=x(a\cos x+b\sin x)$.

$y'_p=a\cos x+b\sin x+x(b\cos x-a\sin x), y''_p=(-2a-bx)\sin x+(2b-ax)\cos x$

So, $y''+y=\sin x \Rightarrow (-2a-bx)\sin x+(2b-ax)\cos x+x(a\cos x+b\sin x)=\sin x$

$$\Rightarrow -2a\sin x+2b\cos x=\sin x \Rightarrow -2a=1, 2b=0 \Rightarrow a=-\frac{1}{2}, b=0$$

Thus, $y_p=x(a\cos x+b\sin x)=-\frac{1}{2}x\cos x$.

b) $y''+2y'+5y=0 \Rightarrow r^2+2r+5=0 \Rightarrow r_1=-1+2i, r_2=-1-2i$

Now, lets' find y_p which is the same form as $f(x)=e^{-x}\sin 2x$.

Here, in $f(x)=e^{-x}\sin 2x$, we have $\alpha=-1$, $\beta=2$. But we have

$r_1=-1+2i, r_2=-1-2i \Rightarrow r_1=\alpha+\beta i, r_2=\alpha-\beta i$. Thus, y_p is of the form

$y_p=x(A\cos 2x+B\sin 2x)e^{-x}$. (Complete the solution!)

c) $y''+4y=0 \Rightarrow r_1=2i, r_2=-2i$. Now, lets' find y_p which is the same form as $f(x)=x\cos 2x$. Here, in $f(x)=x\cos 2x$, we have $\alpha=0$, $\beta=2$. But we have $r_1=2i, r_2=-2i \Rightarrow r_1=\alpha+\beta i, r_2=\alpha-\beta i$. Thus, y_p is of the form

$y_p=x[(Ax+B)\cos 2x+(Cx+D)\sin 2x]$. (Complete the solution!).

Miscellaneous Examples on MUCs:

The following problems will help you to test yourself whether you understood all the main concepts about MUCs. First try by yourself and then see the hints.

1. Solve the following DEs using MUCs.

a) $y''-y'-2y=4x^2$

b) $y''-3y'+2y=x^2+e^x$

c) $y''-5y'+6y=xe^{2x}$

d) $y''-6y'+9y=x+e^x$

e) $y''-2y'-3y=12xe^{2x}+6x-11$

f) $y''-3y'+2y=8x^2+1$

Solution:

a) **Step-1:** Find the complementary solution y_c of $y''-y'-2y=0$.

Here, the characteristics equation is $r^2 - r - 2 = 0$. Solving this gives us $r^2 - r - 2 = 0 \Rightarrow (r+1)(r-2) = 0 \Rightarrow r_1 = -1, r_2 = 2$. So, $y_c = c_1e^{-x} + c_2e^{2x}$.

Step-2: Find the particular solution y_p of the form $f(x) = 4x^2$.

Here, we guess $y_p = ax^2 + bx + c$. Hence, $y'_p = 2ax + b,\ y''_p = 2a$.

Now, determine the constants a, b, c.

$$y''-y'-2y = 4x^2 \Rightarrow 2a - [2ax + b] - 2(ax^2 + bx + c) = 4x^2$$
$$\Rightarrow -2ax^2 - (2a+2b)x + 2a - b - 2c = 4x^2$$
$$\Rightarrow -2a = 4, -2a - 2b = 0, 2a - b - 2c = 0$$
$$\Rightarrow a = -2, b = 2, c = -3$$

Hence, the particular solution is $y_p = -2x^2 + 2x - 3$.

Therefore, $y = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3$.

b) Here, $r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0 \Rightarrow r_1 = 1, r_2 = 2 \Rightarrow y_c = c_1e^x + c_2e^{2x}$.

Here, y_p seems of the form $y_p = y_{p1} + y_{p2}$ where $y_{p1} = ax^2 + bx + c, y_{p2} = de^x$ but not because e^x is already in the solution $y_c = c_1e^x + c_2e^{2x}$.

Thus $y_{p2} = dxe^x$ such that $y_p = ax^2 + bx + c + dxe^x$.

Then, $y'_p = 2ax + b + (d + dx)e^x, y''_p = 2a + (2d + dx)e^x$.

So, $y''-3y'+2y = x^2 + e^x$

$$\Rightarrow 2a + (2d+dx)e^x - 3[2ax + b + (d+dx)e^x] + 2[ax^2 + bx + c + dxe^x] = x^2 + e^x$$
$$\Rightarrow 2ax^2 + (-d-2)xe^x + (2b-6a)x - de^x + 2a - 3b + 2c = x^2 + e^x$$
$$\Rightarrow 2a = 1, , 2b - 6a = 0, -d = 1, 2a - 3b + 2c = 0 \Rightarrow a = \frac{1}{2}, b = \frac{3}{2}, c = \frac{7}{4}, d = -1$$

Hence, $y_p = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4} - xe^x \Rightarrow y = c_1e^x + c_2e^{2x} + \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4} - xe^x$.

c) Here, the characteristics equation is

$r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \Rightarrow r_1 = 2, r_2 = 3$ and $f(x) = xe^{2x}$, with $k = 2$

So y_p is of the form $y_p = x(ax + b)e^{2x} = (ax^2 + bx)e^{2x}$.

Then, $y'_p = (2ax^2 + 2ax + 2bx + b)e^{2x}$, $y''_p = (4ax^2 + 8ax + 4bx + 2a + 4b)e^{2x}$.

So, $y''-5y'+6y = xe^{2x} \Rightarrow (-2ax+2a-b)e^{2x} = xe^{2x}$

$$\Rightarrow -2a = 1,\ 2a-b=0 \Rightarrow a = -\frac{1}{2}, b=-1 \Rightarrow y_p = (-\frac{x^2}{2}-x)e^{2x}.$$

Thus, $y = y_c + y_p = c_1e^{2x} + c_2e^{3x} - (\frac{x^2}{2}+x)e^{2x}$.

d) Here, $r^2-6r+9=0 \Rightarrow (r-3)^2=0 \Rightarrow r=3$. Then, $y_c = c_1e^{3x}+c_2xe^{3x}$.

Now, lets' find y_p which is the same form as $f(x) = x+e^x$.

Here, y_p seems of the form $y_p = y_{p1}+y_{p2}$ where $y_{p1} = ax+b, y_{p2} = ce^x$.

Then, $y'_p = a+ce^x, y''_p = ce^x$.

So, $y''-6y'+9y = x+e^x \Rightarrow 4ce^x+9ax-6a+9b = x+e^x$

$$\Rightarrow 4c=1,\ 9a=1, 9b-6a=0 \Rightarrow c=\frac{1}{4}, a=\frac{1}{9}, b=\frac{2}{27} \Rightarrow y_p = \frac{x}{9}+\frac{e^x}{4}+\frac{2}{27}$$

e) **Step-1:** Find the complementary solution y_c of $y''-2y'-3y=0$.

Here, the characteristics equation is $r^2-2r-3=0$. Solving this gives us $r^2-2r-3=0 \Rightarrow (r+1)(r-3)=0 \Rightarrow r_1=-1, r_2=3$. So, $y_c = c_1e^{-x}+c_2e^{3x}$.

Step-2: Guess y_p having the same form as $f(x) = 12xe^{2x}+6x-11$.

Here, $y_{p1} = (ax+b)e^{2x}$ for $h(x) = 12xe^{2x}$ and $y_{p2} = cx+d$ for $g(x) = 6x-11$.

Then, $y_p = y_{p1}+y_{p2} = (ax+b)e^{2x}+cx+d$ for $f(x) = 12xe^{2x}+6x-11$.

Hence, $y'_p = (2ax+2b+a)e^{2x}+c,\ y''_p = (4ax+4a+4b)e^{2x}$.

$$y''-2y'-3y = 12xe^{2x}+6x-11$$
$$\Rightarrow (-3ax+2a-3b)e^{2x}-3cx-2c-3d = 12xe^{2x}+6x-11$$
$$\Rightarrow -3a=12, 2a-3b=0, -3c=6,\ -2c-3d=-11$$
$$\Rightarrow a=-4, b=-\frac{8}{3}, c=-2=2, d=5 \Rightarrow y_p = (-4x-\frac{8}{3})e^{2x}-2x+5$$

f) **Step-1:** Find the complementary solution y_c of $y''+3y'+4y=0$

Here, $r^2+3r+4=0 \Rightarrow r = \frac{-3\pm\sqrt{7}i}{2} \Rightarrow \alpha = \frac{-3}{2}, \beta = \frac{\sqrt{7}}{2}$.

2. Using ***Method of Undetermined Coefficients***, find the general solution.

$a)\ y''+y'-6y = x+e^{-3x}$ $\quad b)\ y''-y=\cos 2x$ $\quad c)\ y''-y'=x+e^{x}$

$d)\ y''+3y'=4x+6e^{-3x}$ $\quad e)\ y''-2y'=e^{x}\sin x$ $\quad f)\ y''+y'-2y=2xe^{-x}$

$g)\ y''+3y'+2y=4x+e^{3x}$ $\quad h)\ y''+y'=3x+4e^{x}$ $\quad i)\ y''-3y'+2y=2x^2+e^{x}$

Solution:

a) Here, $k^2+k-6=0 \Rightarrow (k+3)(k-2)=0 \Rightarrow k=-3,2$.

Hence, $y_c = c_1e^{-3x}+c_2e^{2x}$. Now, $y_p = ax+b+ce^{-3x}$. But the term ce^{-3x} already exists in y_c. So, it must be multiplied by x. That is $y_p = ax+b+cxe^{-3x}$.

Then, $y'_p = a+ce^{-3x}-3cxe^{-3x}$, $y''_p = -3ce^{-3x}-3ce^{-3x}+9cxe^{-3x}$

$$-3ce^{-3x}-3ce^{-3x}+9cxe^{-3x}+a+ce^{-3x}-3cxe^{-3x}-6(ax+b+cxe^{-3x})=x+e^{-3x}$$

$$\Rightarrow -5ce^{-3x}-6ax+a-6b=x+e^{-3x} \Rightarrow a=\frac{-1}{6}, b=\frac{-1}{36}, c=\frac{-1}{5}$$

Therefore, $y=y_c+y_p=c_1e^{-3x}+c_2e^{2x}-\frac{1}{6}x-\frac{1}{5}xe^{-3x}-\frac{1}{36}$.

b) Here, $r^2-1=0 \Rightarrow r_1=1, r_2=-1$ and $y_p=a\cos 2x+b\sin 2x$.

$y''-y=\cos 2x \Rightarrow -5a\cos 2x-5b\sin 2x=\cos 2x \Rightarrow -5a=1, -5b=0$

$$\Rightarrow a=-1/5, b=0 \Rightarrow y_p=-1/5\cos 2x \Rightarrow y=c_1e^{x}+c_2e^{-x}-1/5\cos 2x$$

c) Here, the characteristics equation $r(r-1)=0 \Rightarrow r_1=0, r_2=1$.

So, $y_c=c_1+c_2e^{x}$. Now, let's find y_p having the same form as $f(x)=x+e^{x}$. Since $b=0$ (as in rule-2) and $r_2=1$ as in rule-1, we guess $y_{p1}=x(ax+b)$ for $g(x)=x$ and $y_{p2}=cxe^{x}$ for $h(x)=e^{x}$. Then,

$y_p=y_{p1}+y_{p2}=ax^2+bx+cxe^{x}$ for $f(x)=x+e^{x}$. Hence,

$y'_p=2ax+b+ce^{x}+cxe^{x}$, $y''_p=2a+2ce^{x}+cxe^{x}$ .Thus,

$y''-y'=x+e^{x} \Rightarrow 2a+2ce^{x}+cxe^{x}-[2ax+b+ce^{x}+cxe^{x}]=x+e^{x}$

$$\Rightarrow -2ax+ce^{x}+2a-b=x+e^{x} \Rightarrow -2a=1,\ c=1,\ 2a-b=0$$

$$\Rightarrow a=-\frac{1}{2}, b=-1,\ c=1 \Rightarrow y_p=-\frac{x^2}{2}-x+xe^{x}$$

d) Here, $r^2+3r=0 \Rightarrow r=0, r=-3$. So, $y_c = c_1 + c_2e^{-3x}$. Now, let's find y_p having the same form as $f(x)=6e^{-3x}+4x$. Since $b=0$(as in form 2) and $r_2=-3$ as in form 1, we guess $y_{p1}=x(ax+b)$ for $g(x)=4x$ and $y_{p2}=cxe^{-3x}$ for $h(x)=6e^{-3x}$.

Then, $y_p = y_{p1}+y_{p2} = ax^2+bx+cxe^{-3x}$ for $f(x)=4x+6e^{-3x}$. Hence,

$y'_p = 2ax+b+ce^{-3x}-3cxe^{-3x}$, $y''_p = 2a-6ce^{-3x}+9cxe^{-3x}$.Thus,

$$y''+3y'=4x+6e^{-3x} \Rightarrow 2a-6ce^{-3x}+9cxe^{-3x}+3(2ax+b+ce^{-3x}-3cxe^{-3x})=4x+6e^{-3x}$$

$$\Rightarrow 2a-3ce^{-3x}+6ax+3b=4x+6e^{-3x}$$

$$\Rightarrow 6a=4, -3c=6, 2a+3b=0 \Rightarrow a=\frac{2}{3}, c=-2, b=-\frac{4}{9}$$

$$\Rightarrow y_p = \frac{2}{3}x^2-\frac{4}{9}x-2xe^{-3x}$$

Therefore, $y=c_1+c_2e^{-3x}+\frac{2}{3}x^2-\frac{4}{9}x-2xe^{-3x}$.

e) $y''-2y'=0 \Rightarrow r^2-2r=0 \Rightarrow r(r-2)=0 \Rightarrow r_1=0, r_2=2 \Rightarrow y_c=c_1+c_2e^{2x}$

Now, lets' find the particular solution y_p which is the same form as $f(x)=e^x\sin x$. Here, y_p is of the form $y_p=(a\cos x+b\sin x)e^x$ and thus

$y'_p=(b\cos x-a\sin x+a\cos x+b\sin x)e^x$, $y''_p=(-2a\sin x+2b\cos x)e^x$

So, $y''-2y'=e^x\sin x$

$$\Rightarrow (-2a\sin x+2b\cos x)e^x-2(b\cos x-a\sin x+a\cos x+b\sin x)e^x=e^x\sin x$$

$$\Rightarrow -2ae^x\cos x-be^x\sin x=e^x\sin x \Rightarrow -2a=0, -2b=1 \Rightarrow a=0, b=-\frac{1}{2}$$

Hence, $y_p=-\frac{1}{2}e^x\sin x \Rightarrow y=c_1+c_2e^{2x}-\frac{1}{2}e^x\sin x$.

g) Here, $f(x)=4x+e^{3x}$ indicates $y_p=ax+b+ce^{3x}$.

Then, using $y'_p=a+3ce^{3x}$, $y''_p=9ce^{3x}$, we have

$$y''+3y'+2y=4x+e^{3x}\Rightarrow 9ce^{3x}+3(a+3ce^{3x})+2(ax+b+ce^{3x})=4x+e^{3x}$$

$$\Rightarrow 2ax+3a+2b+20ce^{3x}=4x+e^{3x}$$

$$\Rightarrow 2a=4, 3a+2b=0, 20c=1\Rightarrow a=2, b=-3, c=\frac{1}{20}$$

$$\Rightarrow y_p=2x+\frac{e^{3x}}{20}-3$$

h) Here, $r^2+r=0\Rightarrow r=0, r=-1$

$f(x)=4x+e^{3x}$ indicates $y_p=ax+b+ce^{3x}$.

Then, using $y'_p=a+3ce^{3x}$, $y''_p=9ce^{3x}$, we have

3. Using Method of Undetermined Coefficients, solve the IVPs and BVPs.

a) $y''-4y'=16x, y(0)=1, y'(0)=3$ *b)* $y''-2y'+y=\sinh x, y(0)=\frac{1}{8}, y'(1)=e$

Solution:

a) Here, $r^2-4r=0\Rightarrow r=0,4$. Thus, $y_c=c_1+c_2e^{4x}$. Besides, as $f(x)=16x$, y_p is of the form $y_p=x(ax+b)$. Then, $y'_p=2ax+b$, $y''_p=2a$.

So, $y''-4y'=2x\Rightarrow 2a-4(2ax+b)=16x\Rightarrow a=-2, b=-1$

Hence, the general solution is $y=c_1+c_2e^{4x}-2x^2-x$

Here, $y(0)=1, y'(0)=3\Rightarrow\begin{cases}c_1+c_2=1\\4c_2-1=3\end{cases}\Rightarrow c_2=1, c_1=0$.

Therefore, the solution is $y=e^{4x}-2x^2-x$

4. Find a DE whose general solution is $y=c_1+c_2e^{-x}+\frac{x^2}{2}-x$

Solution: Here, from $y_c=c_1+c_2e^{-x}$, we can infer the roots $r_1=0,\ r_2=-1$.

So, the characteristics equation is $(r-0)(r+1)=0\Rightarrow r^2+r=0$.

Hence, the corresponding homogeneous DE is $y''+y'=0$.

Now, putting $y_p=\frac{x^2}{2}-x$, in $y''+y'=f(x)$ we get $y''+y'=x$.

1.8.4 Method of Variation of Parameters (VPs)

How the method is developed? Why it is so named?

The particular solution y_p is a *pseudo-linear* combination of the homogeneous equation. By a pseudo-linear combination we mean an expression that has the same form as a linear combination, but the constants in the linear combination are allowed to depend on x: $y_p(x) = u_1(x)y_1 + u_2(x)y_2$.

In the combination $y_p(x) = u_1(x)y_1 + u_2(x)y_2$, the parameters (the constants in the linear combination) u_1 and u_2 are assumed to be variables. That means the particular solution is the combination of the fundamental solutions with variable coefficients. That is why the method is so named.

Suppose y_1 and y_2 are fundamental solutions of $ay''+by'+cy=0$.

Then, we look for a pair of functions u_1 and u_2 that will make the combination given by $y_p(x) = u_1(x)y_1 + u_2(x)y_2$ a solution of $ay''+by'+cy = f(x)$.

Putting the values $y_p(x), y'_p(x), y''_p(x)$ in $ay''+by'+cy = f(x)$ gives the linear system $\begin{cases} u'_1 y_1 + u'_2 y_2 = 0 \\ u'_1 y'_1 + u'_2 y'_2 = f(x) \end{cases}$. In matrix form, $\begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix}\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$. 11

Since y_1 and y_2 are linearly independent, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0$. 14

Hence, solving the above linear system by Cramer's rule, we have

$$u'_1(x) = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}}{W(y_1, y_2)} = \frac{-f(x)y_2}{W(y_1, y_2)}, \quad u'_2(x) = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix}}{W(y_1, y_2)} = \frac{f(x)y_1}{W(y_1, y_2)}.$$

Hence, by integration, $u_1(x) = -\int \frac{f(x)y_2}{W(y_1, y_2)}dx$, $u_2(x) = \int \frac{f(x)y_1}{W(y_1, y_2)}dx$.

Therefore, the particular solution $y_p(x) = u_1(x)y_1 + u_2(x)y_2$ that we are looking for becomes $y_p(x) = -y_1\int \frac{f(x)y_2}{W(y_1, y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1, y_2)}dx$.

Summary of Procedures to find y_p using VPs:

Objective: To solve the DE $ay''+by'+cy=f(x)$.

Step-1: Find y_1 and y_2 of part $ay''+by'+cy=0$ and compute their wronskian.

That is solve $ar^2+br+c=0$ and compute $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$.

Step-2: Find y_p using $y_p=-y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx+y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$.

Finally: The General solution is $y=y_C+y_p$ by superposition principle.

Examples:

1. Solve the following NHLDEs using Variation of Parameters

a) $y''+y=\sec x$ *b)* $y''-2y+y=\frac{e^x}{x}$ *c)* $y''+2y'+y=e^{-x}\ln x$

d) $y''+3y'+2y=e^{-2x}\cos x$ *e)* $y''-5y'+6y=x$ *f)* $y''-2y'+2y=e^x\tan x$

Solution:

a) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''+y=0$

Here, $r^2+1=0\Rightarrow(r-i)(r+i)=0\Rightarrow r_1=-i, r_2=i$.

Hence, the fundamental solutions are $y_1=\cos x,\ y_2=\sin x$.

Then the Wronskian becomes $W(y_1,y_2)=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}=1$.

Step-2: Find the particular solution y_p using the formula.

$$\begin{aligned} y_p&=-y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx+y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx \\ &=-\cos x\int\sec x\sin x\,dx+\sin x\int\sec x\cos x\,dx \\ &=-\cos x\int\tan x\,dx+\sin x\int dx \\ &=\cos x\ln|\cos x|+x\sin x \end{aligned}$$

Therefore, the solution is $y=c_1\cos x+c_2\sin x+\cos x\ln|\cos x|+x\sin x$

b) **Step-1:** Here, $r^2-2r+1=0\Rightarrow r=1$. Hence, the fundamental solutions are $y_1=e^x$, $y_2=xe^x$ and their Wronskian becomes $W(y_1,y_2)=e^{2x}$.

Step-2: Find y_p.

$$y_p(x) = -y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= -e^x\int\frac{e^x}{xe^{2x}}xe^x dx + xe^x\int\frac{e^x}{xe^{2x}}e^x dx$$

$$= -e^x\int dx + xe^x\int\frac{1}{x}dx = xe^x\ln|x| - xe^x$$

Therefore, the general solution is $y = c_1e^x + c_2xe^x + xe^x\ln|x| - xe^x$

c) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''+2y'+y=0$

That is $r^2+2r+1=0 \Rightarrow (r+1)^2 = 0 \Rightarrow r_1 = r_2 = -1$.

Hence, the fundamental solutions are $y_1 = e^{-x}$, $y_2 = xe^{-x}$.

Then, the wronskian becomes $W(y_1,y_2) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x}-xe^{-x} \end{vmatrix} = e^{-2x}$.

Step-2: Find the particular solution y_p

$$y_p = -y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= -e^{-x}\int\frac{xe^{-x}(e^{-x}\ln x)}{e^{-2x}}dx + xe^{-x}\int\frac{e^{-x}(e^{-x}\ln x)}{e^{-2x}}dx$$

$$= -e^{-x}\int x\ln x dx + xe^{-x}\int\ln x dx$$

Now using by parts, $\int x\ln x dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}$ and $\int\ln x dx = x\ln x - x$

Hence, $y_p = -e^{-x}\int x\ln x dx + xe^{-x}\int\ln x dx = \frac{x^2}{2}e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$

Therefore, the solution is $y = c_1e^{-x} + c_2xe^{-x} + \frac{x^2}{2}e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$.

d) **Step-1:** Here, $r^2+3r+2=0 \Rightarrow r_1 = -2, r_2 = -1$.

Hence, the fundamental solutions are $y_1 = e^{-2x}$, $y_2 = e^{-x}$.

Then, their Wronskian becomes $W(y_1,y_2) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = e^{-3x}$.

Step-2: Find y_p.

$$y_p=-e^{-2x}\int\frac{e^{-2x}\cos x}{e^{-3x}}e^{-x}dx+e^{-x}\int\frac{e^{-2x}\cos x}{e^{-3x}}e^{-2x}dx$$

$$=-e^{-2x}\int\cos x dx+e^{-x}\int e^{-x}\cos x dx$$

$$=-e^{-2x}\sin x+\frac{e^{-x}}{2}(e^{-x}\sin x-e^{-x}\cos x)=\frac{-e^{-2x}}{2}(\sin x+\cos x)$$

Therefore, the general solution is $y=c_1e^{-2x}+c_2e^{-x}-\frac{e^{-2x}}{2}(\sin x+\cos x)$

e) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''-5y'+6y=0$

As we did in part (a), $y_1=e^{3x}$, $y_2=e^{2x}$ and $W(y_1,y_2)=-e^{5x}$.

Step-2: Find the particular solution y_p

$$y_p(x)=-y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx+y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx=-e^{3x}\int\frac{xe^{2x}}{-e^{5x}}dx+e^{2x}\int\frac{xe^{3x}}{-e^{5x}}dx$$

$$=e^{3x}\int xe^{-3x}dx-e^{2x}\int xe^{-2x}dx=e^{3x}(-\frac{xe^{-3x}}{3}-\frac{e^{-3x}}{9})-e^{2x}(-\frac{xe^{-2x}}{2}-\frac{e^{-2x}}{4})$$

$$=\frac{-x}{3}-\frac{1}{9}+\frac{x}{2}+\frac{1}{4}=\frac{x}{6}+\frac{5}{36}$$

Therefore, the general solution is $y=c_1e^{3x}+c_2e^{2x}+\frac{x}{6}+\frac{5}{36}$.

f) Here, $r^2-2r+2=0\Rightarrow r_1=1+i, r_2=1-i$. Hence, the fundamental solutions are $y_1=e^x\cos x$, $y_2=e^x\sin x$ and their Wronskian becomes

$$W(y_1,y_2)=\begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x\cos x-e^x\sin x & e^x\sin x+e^x\cos x\end{vmatrix}=e^{2x}.$$

Now, find the particular solution y_p

$$y_p=-e^x\cos x\int\frac{\sin^2 x}{\cos x}dx+e^x\sin x\int\sin x dx$$

$$=-e^x\cos x\int(\sec x-\cos x)dx+e^x\sin x\int\sin x dx=-e^x\cos x\ln|\sec x+\tan x|$$

Therefore, $y=c_1e^x\cos x+c_2e^x\sin x-e^x\cos x\ln|\sec x+\tan x|$

2. Solve the following DEs using Variation of Parameters

a) $y''+9y=\csc 3x$ *b)* $y''-2y'+2y=8e^{x}\sin x$ *c)* $y''-9y=\dfrac{9x}{e^{3x}}$

d) $y''-y=\sinh 2x$ *e)* $y''+y=\tan x$ *f)* $y''+y=\cot x$

g) $y''+4y'+4y=e^{-x}$ *h)* $y''-4y'+4y=x^{2}e^{x}$ *i)* $y''+y=x\sin x$

Solution:

a) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''+9y=0$

Here, the characteristics equation is $r^2+9=0$. Solving this gives us $r^2+9=0 \Rightarrow r_1=-3i, r_2=3i$.

Hence, the fundamental solutions are $y_1=\cos 3x,\ y_2=\sin 3x$ and their Wronskian becomes $W(y_1,y_2)=\begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}=3$.

Step-2: Find the particular solution y_p

$$y_p(x)=-y_1\int \frac{f(x)y_2}{W(y_1,y_2)}dx+y_2\int \frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$=-\cos 3x\int \frac{1}{3}\csc 3x\sin 3x\,dx+\sin 3x\int \frac{1}{3}\csc 3x\cos 3x\,dx$$

$$=-\frac{1}{3}\cos 3x\int dx+\frac{1}{3}\sin 3x\int \cot 3x\,dx=-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x\ln|\sin 3x|$$

Therefore, $y=c_1\cos 3x+c_2\sin 3x-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x\ln|\sin 3x|$

b) Here, the characteristics equation is $r^2-2r+2=0$. Solving this gives us $r^2-2r+2=0 \Rightarrow r_1=1+i, r_2=1-i$. Hence, the fundamental solutions are $y_1=e^{x}\cos x,\ y_2=e^{x}\sin x$ and their Wronskian becomes

$$W(y_1,y_2)=\begin{vmatrix} e^{x}\cos x & e^{x}\sin x \\ e^{x}\cos x-e^{x}\sin x & e^{x}\sin x+e^{x}\cos x \end{vmatrix}=e^{2x}.$$

Now, find the particular solution y_p

$$y_p = -y_1\int \frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= -e^x \cos x\int 8\sin^2 x dx + e^x \sin x\int 8\sin x\cos x dx$$

$$= -e^x \cos x\int 4(1-\cos 2x)dx + 4e^x \sin x\int \sin 2x dx$$

$$= -e^x \cos x(4x - 2\sin 2x) - 2e^x \sin x\cos 2x$$

Therefore, $y = c_1e^x \cos x + c_2e^x \sin x - e^x \cos x(4x - 2\sin 2x) - 2e^x \sin x\cos 2x$

c) Here, $r^2 - 9 = 0 \Rightarrow r_1 = 3, r_2 = -3$. Hence, the fundamental solutions are $y_1 = e^{3x}$, $y_2 = e^{-3x}$ and $W(y_1,y_2) = -6$. Now, find y_p.

$$y_p = -y_1\int \frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= -e^{3x}\int -\frac{9x}{6e^{3x}}(e^{-3x})dx + e^{-3x}\int -\frac{9x}{6e^{3x}}(e^{3x})dx$$

$$= \frac{3e^{3x}}{2}\int xe^{-6x}dx - \frac{3e^{-3x}}{2}\int x dx = \frac{-e^{-3x}}{24} - \frac{xe^{-3x}}{4} - \frac{3x^2e^{-3x}}{4}$$

d) **Step-1:** Here, $r^2 - 1 = 0 \Rightarrow r_1 = 1, r_2 = -1$.

Hence, the fundamental solutions are $y_1 = e^x$, $y_2 = e^{-x}$ and their Wronskian becomes $W(y_1,y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$.

Step-2: Find y_p. Observe that $f(x) = \sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2}$

$$y_p(x) = -y_1\int \frac{f(x)y_2}{W(y_1,y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1,y_2)}dx$$

$$= e^x\int \frac{1}{2}\left(\frac{e^{2x} - e^{-2x}}{2}\right)e^{-x}dx - e^{-x}\int \frac{1}{2}\left(\frac{e^{2x} - e^{-2x}}{2}\right)e^x dx$$

$$= \frac{e^x}{4}\int (e^x - e^{-3x})dx - \frac{e^{-x}}{4}\int (e^{3x} - e^{-x})dx$$

$$= \frac{e^{2x}}{4} + \frac{e^{-2x}}{12} - \frac{e^{2x}}{12} - \frac{e^{-2x}}{4} = \frac{1}{6}(e^{2x} - e^{-2x}) = \frac{1}{3}\sinh 2x$$

Therefore, the general solution is $y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{3}\sinh 2x$

e) **Step-1:** Find the fundamental solutions y_1, y_2 of $y'' + y = 0$

Here, the characteristics equation is $r^2 + r = 0$. Solving this gives us

$r^2 + r = 0 \Rightarrow (r - i)(r + i) = 0 \Rightarrow r_1 = -i, r_2 = i$.

Hence, $y_1 = \cos x$, $y_2 = \sin x$ and $W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$.

Step-2: Find the particular solution y_p

$$y_p(x) = -y_1 \int \frac{f(x) y_2}{W(y_1, y_2)} dx + y_2 \int \frac{f(x) y_1}{W(y_1, y_2)} dx$$

$$= -\cos x \int \tan x \sin x dx + \sin x \int \tan x \cos x dx$$

$$= -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x dx = -\cos x \int \frac{1 - \cos^2 x}{\cos x} dx - \sin x \cos x$$

$$= \cos x \int (\cos x - \sec x) dx - \sin x \cos x$$

$$= \cos x \left(\int \cos x dx - \int \sec x dx \right) - \sin x \cos x = -\cos x \ln|\sec x + \tan x|$$

Therefore, the general solution is $y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$

f) Here, $r^2 + 1 = 0 \Rightarrow r_1 = -i, r_2 = i$. Hence, the fundamental solutions are

$y_1 = \cos x$, $y_2 = \sin x$ and $W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$. Then,

$$y_p = -\cos x \int \cot x \sin x dx + \sin x \int \cot x \cos x dx$$

$$= -\cos x \int \cos x dx + \sin x \int \frac{\cos^2 x}{\sin x} dx$$

$$= -\cos x \sin x + \sin x \int (\csc x - \sin x) dx, \quad \cos^2 x = 1 - \sin^2 x$$

$$= -\cos x \sin x - \sin x \ln|\csc x + \cot x| + \sin x \cos x = -\sin x \ln|\csc x + \cot x|$$

Therefore, $y = c_1 \cos x + c_2 \sin x - \sin x \ln|\csc x + \cot x|$

g) **Step-1:** Here, $r^2 + 4r + 4 = 0 \Rightarrow (r + 2)^2 = 0 \Rightarrow r_1 = r_2 = -2$.

Hence, the fundamental solutions are $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.

Then, $W(y_1,y_2)=\begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x}-2xe^{-2x}\end{vmatrix}=e^{-4x}$.

Step-2: Find the particular solution y_p

$$y_p=-e^{-2x}\int\frac{e^{-x}(xe^{-2x})}{e^{-4x}}dx+xe^{-2x}\int\frac{e^{-x}(e^{-2x})}{e^{-4x}}dx$$

$$=-e^{-2x}\int xe^{x}dx+xe^{-2x}\int e^{x}dx=-e^{-2x}(xe^{x}-e^{x})+xe^{-2x}(e^{x})=e^{-x}$$

Therefore, the general solution is $y=y_c+y_p=c_1e^{-2x}+c_2xe^{-2x}+e^{-x}$.

h) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''-4y'+4y=0$.

Here, $r^2-4r+4=0\Rightarrow(r-2)^2=0\Rightarrow r=2$ (we have repeated roots).

So. the fundamental solutions are $y_1=e^{2x}$, $y_2=xe^{2x}$ and their Wronskian becomes $W(y_1,y_2)=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x}+e^{2x}\end{vmatrix}=e^{4x}$.

Step-2: Find the particular solution y_p

$$y_p(x)=-y_1\int\frac{f(x)y_2}{W(y_1,y_2)}dx+y_2\int\frac{f(x)y_1}{W(y_1,y_2)}dx=-e^{2x}\int\frac{x^3e^{3x}}{e^{4x}}dx+xe^{2x}\int\frac{x^2e^{3x}}{e^{4x}}dx$$

$$=-e^{2x}\int x^3e^{-x}dx+xe^{2x}\int x^2e^{-x}dx=(x^2+4x+6)e^{2x}$$

Therefore, the general solution is $y=c_1e^{2x}+c_2xe^{2x}+(x^2+4x+6)e^{x}$.

3. Solve the following IVPs using **Variation of parameters.**

a) $y''-5y'+4y=e^{2x}, y(0)=1, y'(0)=0$

b) $y''-2y'+y=e^{x}\sin x, y(0)=3, y'(0)=0$

c) $y''+2y'+5y=e^{-x}\sin x, y(0)=0, y'(0)=1$

Solution:

a) **Step-1:** Find the fundamental solutions y_1, y_2 of $y''-5y'+4y=0$

Here, the characteristics equation is $r^2-5r+4=0$. Solving this gives us $r^2-5r+4=0\Rightarrow(r-4)(r-1)=0\Rightarrow r_1=4, r_2=1$.

Hence, $y_1=e^{4x}$, $y_2=e^{x}$ and $W(y_1,y_2)=\begin{vmatrix} e^{4x} & e^{x} \\ 4e^{4x} & e^{x}\end{vmatrix}=-3e^{5x}$.

Step-2: Find the particular solution y_p

$$y_p(x) = -e^{4x}\int \frac{e^{2x}(e^x)}{-3e^{5x}}dx + e^x\int \frac{e^{2x}(e^{4x})}{-3e^{5x}}dx$$

$$= \frac{e^{4x}}{3}\int e^{-2x}dx - \frac{e^x}{3}\int e^x dx = -\frac{e^{2x}}{6} - \frac{e^{2x}}{3} = -\frac{e^{2x}}{2}$$

Therefore, the general solution is $y = y_c + y_p = c_1e^{4x} + c_2e^x - \frac{e^{2x}}{2}$.

Now, let's determine the constants c_1 and c_2 using the initial conditions.

$$\begin{cases} y(0) = 1 \Rightarrow c_1 + c_2 - \frac{1}{2} = 1 \Rightarrow c_1 + c_2 = \frac{3}{2} \\ y'(0) = 0 \Rightarrow 4c_1 + c_2 - 1 = 0 \Rightarrow 4c_1 + c_2 = 1 \end{cases} \Rightarrow c_1 = -\frac{1}{6}, c_2 = \frac{5}{3}$$

Hence, the solution of the IVP becomes $y = \frac{5}{3}e^x - \frac{1}{6}e^{4x} - \frac{e^{2x}}{2}$.

b) **Step-1:** Here, $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$.

Hence, $y_1 = e^x$, $y_2 = xe^x$ and $W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$.

Step-2: Find the particular solution y_p

$$y_p(x) = -e^x\int \frac{e^x \sin x(xe^x)}{e^{2x}}dx + xe^x\int \frac{e^x \sin x(e^x)}{e^{2x}}dx$$

$$= -e^x\int x\sin x dx + xe^x\int \sin x dx$$

$$= -e^x(\sin x - x\cos x) - xe^x \cos x = -e^x \sin x$$

Therefore, the general solution is $y = y_c + y_p = c_1e^x + c_2xe^x - e^x \sin x$.

Now, let's determine the constants c_1 and c_2 using the initial conditions.,

$$\begin{cases} y(0) = 3 \Rightarrow c_1 = 3 \\ y'(0) = 0 \Rightarrow c_1 + c_2 - 1 = 0 \Rightarrow 3 + c_2 - 1 = 0 \end{cases} \Rightarrow c_1 = 3, c_2 = -2$$

Hence, the solution of the IVP becomes $y = 3e^x - 2xe^x - e^x \sin x$.

c) **Step-1:** Find the fundamental solutions y_1, y_2 of $y'' + 2y' + 5y = 0$

Here, $r^2 + 2r + 5 = 0 \Rightarrow r = -1 + 2i, -1 - 2i$.

Hence, the fundamental solutions are $y_1 = e^{-x}\cos 2x$, $y_2 = e^{-x}\sin 2x$ and

$$W(y_1,y_2)=\begin{vmatrix} e^{-x}\cos 2x & e^{-x}\sin 2x \\ -e^{-x}\cos 2x-2e^{-x}\sin 2x & 2e^{-x}\cos 2x-e^{-x}\sin 2x\end{vmatrix}=2e^{-2x}.$$

Step-2: Find the particular solution y_p

$$y_p=-e^{-x}\cos 2x\int\frac{e^{-x}\sin x(e^{-x}\sin 2x)}{2e^{-2x}}dx+e^{-x}\sin 2x\int\frac{e^{-x}\sin x(e^{-x}\cos 2x)}{2e^{-2x}}dx$$

$$=\frac{-e^{-x}\cos 2x}{2}\int\sin x\sin 2x dx+\frac{e^{-x}\sin 2x}{2}\int\sin x\cos 2x dx \quad \text{(Complete it!)}$$

4. Solve the following NHLDEs using Variation of parameters

a) $y''-4y'+4y=\dfrac{2e^{2x}}{x^2+1}$ b) $y''-y=\dfrac{e^x}{1+e^{2x}}$ c) $y''-3y'+2y=\dfrac{e^x}{1+e^x}$

Solution:

a) **Step-1:** Here, $r^2-4r+4=0\Rightarrow(r-2)^2=0\Rightarrow r=2$. Hence, the fundamental solutions are $y_1=e^{2x}$, $y_2=xe^{2x}$ and $W(y_1,y_2)=\begin{vmatrix} e^{2x} & xe^{2x}\\ 2e^{2x} & e^{2x}+2xe^{2x}\end{vmatrix}=e^{4x}$

Step-2: Find the particular solution y_p

$$y_p=-e^{2x}\int\left(\frac{2e^{2x}}{x^2+1}\right)\frac{xe^{2x}}{e^{4x}}dx+xe^{2x}\int\left(\frac{2e^{2x}}{x^2+1}\right)\frac{e^{2x}}{e^{4x}}dx$$

$$=-e^{2x}\int\frac{2x}{x^2+1}dx+xe^{2x}\int\frac{2}{x^2+1}dx=-e^{2x}\ln(x^2+1)+2xe^{2x}\tan^{-1}x$$

Therefore, the solution is $y=c_1e^{2x}+c_2xe^{2x}-e^{2x}\ln(x^2+1)+2xe^{2x}\tan^{-1}x$.

b) **Step-1:** Here, $r^2-1=0\Rightarrow r_1=1, r_2=-1$. Hence, the fundamental solutions are $y_1=e^x$, $y_2=e^{-x}$ and $W(y_1,y_2)=\begin{vmatrix} e^x & e^{-x}\\ e^x & -e^{-x}\end{vmatrix}=-2$

Step-2: Find the particular solution y_p

$$y_p(x)=-e^x\int\left(\frac{e^x}{1+e^{2x}}\right)\frac{e^{-x}}{-2}dx+e^{-x}\int\left(\frac{e^x}{1+e^{2x}}\right)\frac{e^x}{-2}dx$$

$$=\frac{e^x}{2}\int\frac{1}{1+e^{2x}}dx-\frac{e^{-x}}{2}\int\frac{e^{2x}}{1+e^{2x}}dx$$

Now consider the integrals $\int\frac{1}{1+e^{2x}}dx$ and $\int\frac{e^{2x}}{1+e^{2x}}dx$ separately.

1.9 System of First Order Linear Differential Equations

Revision on Matrix:

System of Linear Equations: Consider a 2×2 system: $\begin{cases} ax+by=k_1 \\ cx+dy=k_2 \end{cases}$

We know that such system of linear equations can be written or expressed in matrix form as $AX=B$ where $X=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, B=\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$.

Eigen-values and Eigen-vectors:
From the matrix representation, we have also seen how to determine the eigen-values and the corresponding eigen-vectors associated with the coefficient matrix A by forming its characteristics equation.
Characteristics Equation: $\det(\mathbf{A}-\lambda\mathbf{I})=0$ where **I** is identity matrix.
Eigen-values: The solution of the characteristics equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$.
Eigen-vectors: The vector with the property $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=0$.
These algebraic concepts are the basis for analysis of systems of DE.

1.12.1 Homogeneous Systems with Constant Coefficients

A system of linear differential equation is a system of equation involving the derivative of two or more variables about the same input parameter, usually denoted by t. Here under, we are going to see a system of two variables x and y about the same parameter t.

Notations: Derivatives wit respect to t: $\frac{dx}{dt}=x', \frac{dy}{dt}=y'$

Form of the system: $\begin{cases} \frac{dx}{dt}=ax+by \\ \frac{dy}{dt}=cx+dy \end{cases}$ **or** $\begin{cases} x'=ax+by \\ y'=cx+dy \end{cases}$

In matrix Form or in vector:

$X'=AX$ or $X'=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ where $X'=\begin{pmatrix} x' \\ y' \end{pmatrix}, A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, X=\begin{pmatrix} x \\ y \end{pmatrix}$.

Fundamental solutions: In general, like that of second order DE, such systems have two fundamental solutions X_1 and X_2. But the form of the fundamental solutions X_1 and X_2 depends on the nature of the roots of the characteristics equation $\det(A-\lambda I)=0$ of the system. So, our next task is how to find such fundamental solutions using eigenvector method.

Since the eigenvalues are the roots of $\det(A-\lambda I)=0$, they could be two distinct real roots, single root or complex roots. That is if the coefficient matrix of the system is $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\det(A-\lambda I)=0 \Rightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}=0 \Rightarrow \lambda^2-(a+d)\lambda+ad-bc=0$$

$$\Rightarrow \lambda^2-\text{trace}(A).\lambda+\det A=0 \quad (\text{Note: trace}(A)=a+d)$$

Now, consider the three cases for this quadratic equation.

Case-I: Two distinct real roots $\lambda_1 \neq \lambda_2$.

Then, the fundamental solutions are $X_1=v_1e^{\lambda_1 t}$, $X_2=v_2e^{\lambda_2 t}$.

Case-II: Single or repeated real root $\lambda_1=\lambda_2=\lambda$.

Then, the fundamental solutions are $X_1=v_1e^{\lambda t}$, $X_2=(v_2t+v_1)e^{\lambda t}$ where v_2 is a vector to be determined from the condition $(A-\lambda I)v_2=v_1$.

Case-III: Complex conjugate roots $\lambda=\alpha\pm\beta i$.

Now, for eigenvalue $\lambda=\alpha+\beta i$, calculate a complex eigenvector $v^*=\begin{pmatrix} a \\ b \end{pmatrix}$.

That is $(A-\lambda I)v^*=0 \Rightarrow (A-\lambda I)\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Now, identify the real and imaginary parts of $v=v^*e^{\alpha t}(\cos\beta t+i\sin\beta t)$.

That is $\begin{cases} X_1=\text{Re}(v)=\text{Re}[v^*e^{\alpha t}(\cos\beta t+i\sin\beta t)] \\ X_2=\text{Im}(v)=\text{Im}[v^*e^{\alpha t}(\cos\beta t+i\sin\beta t)] \end{cases}$

Complementary Solution: Therefore, $X_c=c_1X_1+c_2X_2$.

Eugenvalue Approach to solve system of DE:

The process of solving system of linear differential equations using the characteristics equation is known as eigenvalue or matrix approach.

Procedures:

First: Find the eigenvalues of the coefficient matrix A using $|\mathbf{A}-\lambda\mathbf{I}|=0$.

Second: Find the eigenvectors corresponding to each eigenvalue. But be careful to analyze the three cases of the forms of the eigenvalues.

Third: Find the fundamental solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ and write the complementary solution $\mathbf{X}_c$. That is $\mathbf{X}_c = c_1\mathbf{X}_1 + c_2\mathbf{X}_2$.

Examples:

1. Find the general solution for the systems of differential equations.

a) $\begin{cases} \frac{dx}{dt} = 3x+2y \\ \frac{dy}{dt} = x+4y \end{cases}$ b) $\begin{cases} x' = 3x+y \\ y' = -x+y \end{cases}$ c) $\begin{cases} x' = 2x+3y \\ y' = -3x+2y \end{cases}$ d) $\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{X}$

Solution:

a) This is an example with two distinct real roots.

First, identify the coefficient matrix A and find its exigent-values.

The coefficient matrix is $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(4-\lambda)-2 = 0$$

$$\Rightarrow (12-3\lambda-4\lambda+\lambda^2)-2 = 0$$

$$\Rightarrow \lambda^2-7\lambda+10 = 0 \Rightarrow (\lambda-2)(\lambda-5) = 0$$

$$\Rightarrow \lambda_1 = 2,\ \lambda_2 = 5$$

Now, let's determine the corresponding eigen vectors. Let $V = \begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda = 2$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a+2b=0 \\ a+2b=0 \end{cases} \Rightarrow a = -2b$

Hence, letting $b=1$, we get the basis vector to be $\mathbf{v}_1 = \begin{pmatrix} -2b \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

ii) For $\lambda = 5$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2a+2b=0 \\ a-b=0 \end{cases} \Rightarrow a = b$

Hence, letting $a=1$, we get the basis vector to be $\mathbf{v}_2=\begin{pmatrix}a\\a\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1e^{\lambda_1 t}+\mathbf{v}_2e^{\lambda_2 t}=cC_1\begin{pmatrix}2\\-1\end{pmatrix}e^{2t}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^{5t}=\begin{pmatrix}2C_1e^{2t}+C_2e^{5t}\\-C_1e^{2t}+C_2e^{5t}\end{pmatrix}$$

b) This is an example with single repeated real root.

The coefficient matrix is $A=\begin{pmatrix}3&1\\-1&1\end{pmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}3-\lambda&1\\-1&1-\lambda\end{vmatrix}=0\Rightarrow(3-\lambda)(1-\lambda)+1=0$$
$$\Rightarrow(3-3\lambda-\lambda+\lambda^2)+1=0$$
$$\Rightarrow\lambda^2-4\lambda+4=0\Rightarrow(\lambda-2)^2=0\Rightarrow\lambda=2$$

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix}a\\b\end{pmatrix}$

$$\text{For }\lambda=2,\ (A-\lambda I)v=0\Rightarrow\begin{pmatrix}1&1\\-1&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{cases}a+b=0\\-a-b=0\end{cases}\Rightarrow a=-b$$

Hence, letting $b=1$, we get the basis vector to be $\mathbf{v}_1=\begin{pmatrix}-b\\b\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$.

Question: How do we get the second basis vector $\mathbf{v}_2$?

We need with the condition that $(A-\lambda I)\mathbf{v}_2=\mathbf{v}_1$. Let $\mathbf{v}_2=\begin{pmatrix}c\\d\end{pmatrix}$.

$$(A-\lambda I)\mathbf{v}_2=\mathbf{v}_1\Rightarrow\begin{pmatrix}1&1\\-1&-1\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}\Rightarrow c+d=-1\Rightarrow c=-1-d$$

Hence, letting $d=0$, we get the basis vector $\mathbf{v}_2=\begin{pmatrix}-1-d\\d\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1e^{\lambda t}+c_2(\mathbf{v}_2t+\mathbf{v}_1)e^{\lambda t}=c_1\begin{pmatrix}-1\\1\end{pmatrix}e^{2t}+c_2[\begin{pmatrix}-1\\0\end{pmatrix}t+\begin{pmatrix}-1\\1\end{pmatrix}]e^{2t}$$

c) This is an example with complex conjugate roots.

$$|\mathbf{A}-\lambda\mathbf{I}|=0 \Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix}=0 \Rightarrow (2-\lambda)(2-\lambda)+9=0$$

$$\Rightarrow 4-2\lambda-2\lambda+\lambda^2+9=0$$

$$\Rightarrow \lambda^2-4\lambda+13=0 \Rightarrow \lambda=2\pm 3i$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $\mathbf{v}^*=\begin{pmatrix} a \\ b \end{pmatrix}$

For $\lambda=2+3i$, $(A-\lambda I)v=0$

$$\Rightarrow \begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -3ai+3b=0 \\ -3a-3bi=0 \end{cases} \Rightarrow b=ai$$

Hence, letting $a=-i$, we get $b=ai=-i^2=1$.

So, the complex basis vector becomes $\mathbf{v}^*=\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Now, identify the real and imaginary parts of $\mathbf{v}=\mathbf{v}^* e^{\alpha t}(\cos\beta t+i\sin\beta t)$.

Here, $\lambda=2\pm 3i \Rightarrow \alpha=2,\ \beta=3$. Then, $\mathbf{v}=\mathbf{v}^* e^{2t}(\cos 3t+i\sin 3t)$.

$$\mathbf{v}=\mathbf{v}^* e^{2t}(\cos 3t+i\sin 3t)=\begin{pmatrix} -\mathbf{i} \\ 1 \end{pmatrix}e^{2t}(\cos 3t+\mathbf{i}\sin 3t)$$

$$=\begin{pmatrix} -\mathbf{i}e^{2t}\cos 3t+e^{2t}\sin 3t) \\ e^{2t}\cos 3t+\mathbf{i}e^{2t}\sin 3t) \end{pmatrix}=\begin{pmatrix} e^{2t}\sin 3t) \\ e^{2t}\cos 3t \end{pmatrix}+\begin{pmatrix} -\mathbf{i}e^{2t}\cos 3t \\ \mathbf{i}e^{2t}\sin 3t) \end{pmatrix}$$

$$=\underbrace{\begin{pmatrix} e^{2t}\sin 3t) \\ e^{2t}\cos 3t \end{pmatrix}}_{=\mathbf{Re}(\mathbf{v})}+\mathbf{i}\underbrace{\begin{pmatrix} -e^{2t}\cos 3t \\ e^{2t}\sin 3t) \end{pmatrix}}_{=\mathbf{Im}(\mathbf{v}))}$$

Here, we got $\mathbf{v}_1=\mathbf{Re}(\mathbf{v})=\begin{pmatrix} e^{2t}\sin 3t) \\ e^{2t}\cos 3t \end{pmatrix}, \mathbf{v}_2=\mathbf{Im}(\mathbf{v})=\begin{pmatrix} -e^{2t}\cos 3t \\ e^{2t}\sin 3t) \end{pmatrix}$

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1+c_2\mathbf{v}_2=c_1\begin{pmatrix} e^{2t}\sin 3t) \\ e^{2t}\cos 3t \end{pmatrix}+c_2\begin{pmatrix} -e^{2t}\cos 3t \\ e^{2t}\sin 3t) \end{pmatrix}$$

d) This is an example with complex conjugate roots.

First, identify the coefficient matrix A and find its exigent-values.

$$|\mathbf{A}-\lambda\mathbf{I}|=0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}=0 \Rightarrow (1-\lambda)(1-\lambda)+1=0$$

$$\Rightarrow (1-\lambda-2\lambda+\lambda^2)+1=0$$

$$\Rightarrow \lambda^2-2\lambda+2=0 \Rightarrow \lambda=1\pm i$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda=1+i$, $(A-\lambda I)v=0 \Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -ai-b=0 \\ a-bi=0 \end{cases} \Rightarrow b=-ai$

Hence, letting $a=i$, we get $b=-ai=-i^2=1$.

So, the complex basis vector becomes $\mathbf{v}^*=\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} i \\ 1 \end{pmatrix}$.

Now, identify the real and imaginary parts of $\mathbf{v}=\mathbf{v}^*e^{\alpha t}(\cos\beta t+i\sin\beta t)$.

That is $\lambda=1\pm i \Rightarrow \alpha=1, \beta=1$. Then, $\mathbf{v}=\mathbf{v}^*e^t(\cos t+i\sin t)$.

$$\mathbf{v}=\mathbf{v}^*e^t(\cos t+i\sin t)=\begin{pmatrix} \mathbf{i} \\ 1 \end{pmatrix}e^t(\cos t+\mathbf{i}\sin t)$$

$$=\begin{pmatrix} \mathbf{i}e^t\cos t-e^t\sin t \\ e^t\cos t+\mathbf{i}e^t\sin t \end{pmatrix}=\begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix}+\begin{pmatrix} \mathbf{i}e^t\cos t \\ \mathbf{i}e^t\sin t \end{pmatrix}=\underbrace{\begin{pmatrix} -e^t\sin t) \\ e^t\cos t \end{pmatrix}}_{=\mathbf{Re}(\mathbf{v})}+\mathbf{i}\underbrace{\begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}}_{=\mathbf{Im}(\mathbf{v}))}$$

Here, we got $\mathbf{v}_1=\mathbf{Re}(\mathbf{v})=\begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix}, \mathbf{v}_2=\mathbf{Im}(\mathbf{v})=\begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}$

Therefore, $\mathbf{X}_c=c_1\mathbf{v}_1+c_2\mathbf{v}_2=c_1\begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix}+c_2\begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}$

1.12.1 Non-homogeneous Systems with constant coefficients

A system of differential equation of the form $X'=AX+g(t)$ is said to be non-homogeneous when $g(t)\neq 0$. The function $g(t)\neq 0$ is said to be forcing or input function. In the system $X'=AX+g(t)$, the part $X'=AX$ is said to be the corresponding homogeneous part.

General form of non-homogeneous system: $\begin{cases} \frac{dx}{dt}=x'=ax+by+g_1(t) \\ \frac{dy}{dt}=y'=cx+dy+g_2(t) \end{cases}$

In matrix or in vector Form:

The above non-homogeneous system can be written in matrix or in vector form as follow: $X'=AX+g(t)$ where $X'=\begin{pmatrix} x' \\ y' \end{pmatrix}, A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, X=\begin{pmatrix} x \\ y \end{pmatrix}, g(t)=\begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

It can also be expressed in the form $X'=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

How to solve such non-homogeneous system?

Given: Suppose you a system is given in one of the notations or forms.

$$\begin{cases} x'=ax+by+g_1(t) \\ y'=cx+dy+g_2(t) \end{cases} \text{ or } \begin{cases} \frac{dx}{dt}=ax+by+g_1(t) \\ \frac{dy}{dt}=cx+dy+g_2(t) \end{cases} \text{ or } X'=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$$

Objective: The main objective is to solve this system.

Fundamental solutions: The solutions of the corresponding homogeneous part which is $X'=AX$ or $\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Then, the fundamental solutions are $X_1=\mathbf{v}_1e^{\lambda_1 t}$, $X_2=\mathbf{v}_2e^{\lambda_2 t}$.

Complementary (General) Solution: $\mathbf{X}_c=c_1\mathbf{X}_1+c_2\mathbf{X}_2$

General Solution of the non-homogeneous system:

The general solution of the system $X'=AX+g(t)$ is $\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p$. Here, $\mathbf{X}_c$ is the complementary solution of the homogeneous part and $\mathbf{X}_p$ is any particular solution of the non-homogeneous.

What is the challenge to express the solution using $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$?

Since we have discussed how to determine $\mathbf{X}_c$, the challenge is how to get $\mathbf{X}_p$.

In general, to solve the system $X' = AX + g(t)$:

First: Find $\mathbf{X}_c$ of $X' = AX$. How? Use **Eigenvalue Approach.**

Second: Find $\mathbf{X}_p$ of $X' = AX + g(t)$.

Question: How to determine $\mathbf{X}_p$?

There are different approaches to find particular solution $\mathbf{X}_p$.

1. Method of Undetermined Coefficients
2. Variation of Parameters.
3. Laplace Transform method
4. Fourier Transform method
5. The Operator Method
6. Diagonalization method.
7. Power series Method

1. Method of Undetermined Coefficients

The method works in the same way as we used to solve non-homogeneous second order differential equation.

First: Assumption: Assume the form of the particular solution $\mathbf{X}_p$ based on the form of $\mathbf{g}(t)$. That means make educated guess for $\mathbf{X}_p$.

Second: Substitute $\mathbf{X}_p$ and $\mathbf{X}'_p$ in the system to determine constants.

Hints: To make educated guess, always notice the following points.

i) Since the form of $\mathbf{g}(t)$ is $\mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$, the form of $\mathbf{X}_p$ is also $\mathbf{X}_p = \begin{pmatrix} x_p \\ y_p \end{pmatrix}$

That means guess x_p for $g_1(t)$ and y_p for $g_2(t)$.

Some trial forms: As example consider the following trial forms.

I) $\mathbf{g}(t) = \begin{pmatrix} ae^{kt} \\ be^{kt} \end{pmatrix}$, then the guess will be $\mathbf{X}_p = \begin{pmatrix} Ae^{kt} \\ Be^{kt} \end{pmatrix}$.

II) $\mathbf{g}(t) = \begin{pmatrix} at + b \\ ct + d \end{pmatrix}$, then the guess will be $\mathbf{X}_p = \begin{pmatrix} At + B \\ Ct + D \end{pmatrix}$.

III) $\mathbf{g}(t)=\begin{pmatrix} a\sin kt+b\cos kt \\ c\sin kt+d\cos kt \end{pmatrix}$, then $\mathbf{X}_p=\begin{pmatrix} A\sin kx+B\cos kx \\ C\sin kx+D\cos kx \end{pmatrix}$.

IV) $\mathbf{g}(t)=\begin{pmatrix} (at+b)e^{kt} \\ (ct+d)e^{kt} \end{pmatrix}$, the guess will be $\mathbf{X}_p=\begin{pmatrix} (At+B)e^{kt} \\ (Ct+D)e^{kt} \end{pmatrix}$.

V) $\mathbf{g}(t)=\begin{pmatrix} ae^{k_1t} \\ be^{k_2t} \end{pmatrix}$, then the guess will be $\mathbf{X}_p=\begin{pmatrix} Ae^{k_1t}+Be^{k_2t} \\ Ce^{k_1t}+De^{k_2t} \end{pmatrix}$.

VI*) $\mathbf{g}(t)=\begin{pmatrix} ae^{kt} \\ bt+c \end{pmatrix}$, then the guess will be $\mathbf{X}_p=\begin{pmatrix} Ae^{kt}+Bt+C \\ De^{kt}+Et+F \end{pmatrix}$.

Notice: IN addition to these hints, modification rule also works here in a more generalized way. That means the trial form of $\mathbf{X}_p$ may not work directly. In such case, we use the rule depending on the situation.

Examples:

1. Find the general solution for the following systems of differential equations.

a) $\begin{cases} x'=x+8y+9t \\ y'=x-y+9t \end{cases}$ b) $\begin{cases} x'=2y+4t-5 \\ y'=2x-6t \end{cases}$ c) $\begin{cases} x'=x+y+10\cos t \\ y'=3x-y-10\sin t \end{cases}$

d) $\begin{cases} x'=x+y-2e^{-t} \\ y'=4x+y+3t \end{cases}$

Solution:

a) **First:** Determine the complementary solution using eigenvalue approach.

The coefficient matrix is $A=\begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix}$.

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix} 1-\lambda & 8 \\ 1 & -1-\lambda \end{vmatrix}=0 \Rightarrow (1-\lambda)(-1-\lambda)-8=0$$

$$\Rightarrow (-1-\lambda+\lambda+\lambda^2)-8=0$$

$$\Rightarrow \lambda^2-9=0 \Rightarrow \lambda^2=9 \Rightarrow \lambda=-3,3$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda = 3$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a+8b=0 \\ a-4b=0 \end{cases} \Rightarrow a = 4b$

Hence, letting $b=1$, we get $a = 4b = 4$.

So, the complex basis vector becomes $\mathbf{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

ii) For $\lambda = -3$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4a+8b=0 \\ a+2b=0 \end{cases} \Rightarrow a = -2b$

Hence, letting $b=-1$, we get $a = -2b = -2$.

So, the complex basis vector becomes $\mathbf{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{pmatrix} 4 \\ 1 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-3t}$$

Second: Determine the particular solution X_p using MUCs.

Since $g(t) = \begin{pmatrix} 9t \\ 9t \end{pmatrix}$ is a linear polynomial in both components, X_p is also a linear polynomial. That is $X_p = \begin{pmatrix} at+b \\ ct+d \end{pmatrix}$. Then, $X'_p = \begin{pmatrix} a \\ c \end{pmatrix}$.

$$\begin{cases} x' = x+8y+9t \\ y' = x-y+9t \end{cases} \Rightarrow \begin{cases} x' = at+b+8(ct+d)+9t = a \\ y' = at+b-ct-d+9t = c \end{cases}$$

$$\Rightarrow \begin{cases} (a+8c+9)t+b+8d = a \\ (a-c+9)t+b-d = c \end{cases} \Rightarrow \begin{cases} a+8c+9=0, b+8d=a \\ a-c+9=0, b-d=c \end{cases}$$

$$\Rightarrow \begin{cases} a+8c+9=0 \\ a-c+9=0 \end{cases} \Rightarrow c=0, a=-9$$

$$\Rightarrow \begin{cases} b+8d=a \\ b-d=c \end{cases} \Rightarrow \begin{cases} b+8d=-9 \\ b-d=0 \end{cases} \Rightarrow \begin{cases} b+8d=-9 \\ b=d \end{cases} \Rightarrow b=d=-1$$

Hence, $X_p = \begin{pmatrix} -9t-1 \\ -1 \end{pmatrix}$.

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}=c_1\begin{pmatrix}4\\1\end{pmatrix}e^{3t}+c_2\begin{pmatrix}-2\\1\end{pmatrix}e^{-3t}+\begin{pmatrix}-9t-1\\-1\end{pmatrix}$$

b) **First:** Determine the complementary solution using eigenvalue approach.

The coefficient matrix is $A=\begin{pmatrix}0 & 2\\2 & 0\end{pmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}-\lambda & 2\\2 & -\lambda\end{vmatrix}=0\Rightarrow\lambda^2-4=0\Rightarrow\lambda^2=4\Rightarrow\lambda=2,-2$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix}a\\b\end{pmatrix}$

i) For $\lambda=2$, $(A-\lambda I)v=0\Rightarrow\begin{pmatrix}-2 & 2\\2 & -2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{cases}-2a+2b=0\\2a-2b=0\end{cases}\Rightarrow a=b$

Hence, letting $a=1$, we get $b=a=1$.

C 1
1 0

So, the complex basis vector becomes $\mathbf{v}_1=\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$.

ii) For $\lambda=-2$, $(A-\lambda I)v=0\Rightarrow\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{cases}2a+2b=0\\2a+2b=0\end{cases}\Rightarrow b=-a$

Hence, letting $a=-1$, we get $b=-a=1$.

So, the complex basis vector becomes $\mathbf{v}_2=\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+c_2\begin{pmatrix}-1\\1\end{pmatrix}e^{-2t}$$

Second: Determine the particular solution $\mathbf{X}_p$ using MUCs.

Since $g(t)=\begin{pmatrix}4t-5\\-6t\end{pmatrix}$ is a linear polynomial in both components, $\mathbf{X}_p$ is also a linear polynomial. That is $\mathbf{X}_p=\begin{pmatrix}at+b\\ct+d\end{pmatrix}$. Then, $\mathbf{X}'_p=\begin{pmatrix}a\\c\end{pmatrix}$.

$$\begin{cases} x' = 2y + 4t - 5 \\ y' = 2x - 6t \end{cases} \Rightarrow \begin{cases} x' = 2ct + 2d + 4t - 5 = a \\ y' = 2at + 2b - 6t = c \end{cases} \Rightarrow \begin{cases} (2c+4)t + 2d - 5 = a \\ (2a-6)t + 2b = c \end{cases}$$

$$\Rightarrow \begin{cases} 2c + 4 = 0, 2d - 5 = a \\ 2a - 6 = 0, 2b = c \end{cases} \Rightarrow \begin{cases} c = -2, b = -1 \\ a = 3, d = 4 \end{cases}$$

Hence, the particular solution becomes $\mathbf{X_p} = \begin{pmatrix} 3t - 1 \\ -2t + 4 \end{pmatrix}$.

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X} = \mathbf{X}_c + \mathrm{X_p} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 3t - 1 \\ -2t + 4 \end{pmatrix}$$

c) **First:** Determine the complementary solution using eigenvalue approach.

The coefficient matrix is $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$.

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) - 3 = 0$$

$$\Rightarrow (-1 - \lambda + \lambda + \lambda^2) - 3 = 0$$

$$\Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda^2 = 4 \Rightarrow \lambda = 2, -2$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V = \begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda = 2$, $(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a + b = 0 \\ 3a - 3b = 0 \end{cases} \Rightarrow a = b$

Hence, letting $a = 1$, we get $b = a = 1$.

So, the complex basis vector becomes $\mathbf{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

ii) For $\lambda = -2$, $(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 3a + b = 0 \\ 3a + b = 0 \end{cases} \Rightarrow b = -3a$

Hence, letting $a = -1$, we get $b = -3a = 3$.

So, the complex basis vector becomes $\mathbf{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}-1\\3\end{pmatrix}e^{-2t}$$

Second: Determine the particular solution $\mathrm{X_p}$.

Since $g(t)=\begin{pmatrix}10\cos t\\-10\sin t\end{pmatrix}$ is a linear polynomial in both components, $\mathrm{X_p}$ is also a linear polynomial. That is $\mathrm{X_p}=\begin{pmatrix}a\sin t+b\cos t\\c\sin t+d\cos t\end{pmatrix}$. Then, $\mathrm{X'_p}=\begin{pmatrix}a\\c\end{pmatrix}$.

$$\begin{cases}x'=x+y+10\cos t\\ y'=3x-y-10\sin t\end{cases}$$

$$\Rightarrow\begin{cases}a\cos t-b\sin t=a\sin t+b\cos t+c\sin t+d\cos t+10\cos t\\ c\cos t-d\sin t=3(a\sin t+b\cos t)-c\sin t-d\cos t-10\sin t\end{cases}$$

$$\Rightarrow\begin{cases}a=b+d+10\\-b=a+c\\c=3b-d\\-d=3a-c-10\end{cases}\Rightarrow\begin{cases}-b=b+d+10+3b-d\\-d=3(b+d+10)-(3b-d)-10\end{cases}\Rightarrow\begin{cases}-5b=10\\-d=4d+20\end{cases}$$

$$\Rightarrow\begin{cases}b=-2\\d=-4\end{cases}\Rightarrow\begin{cases}c=3b-d\\a=b+d+10\end{cases}\Rightarrow\begin{cases}c=-2\\a=4\end{cases}$$

Hence, the particular solution becomes $\mathrm{X_p}=\begin{pmatrix}4\sin t-2\cos t\\-2\sin t-4\cos t\end{pmatrix}$.

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+c_2\begin{pmatrix}-1\\3\end{pmatrix}e^{-2t}+\begin{pmatrix}4\sin t-2\cos t\\-2\sin t-4\cos t\end{pmatrix}$$

d) **First:** Determine the complementary solution using eigenvalue approach.

The coefficient matrix is $A=\begin{pmatrix}1&1\\4&1\end{pmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}|=0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(1-\lambda)-4=0$$

$$\Rightarrow (1-2\lambda+\lambda^2)-4=0$$

$$\Rightarrow \lambda^2-2\lambda-3=0 \Rightarrow (\lambda+1)(\lambda-3)=0 \Rightarrow \lambda=-1,3$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda=-1$, $(A-\lambda I)v=0 \Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2a+b=0 \\ 4a+2b=0 \end{cases} \Rightarrow b=-2a$

Hence, letting $a=-1$, we get $a=-2b=2$.

So, the complex basis vector becomes $\mathbf{v}_1=\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

ii) For $\lambda=3$, $(A-\lambda I)v=0 \Rightarrow \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2a+b=0 \\ 4a-2b=0 \end{cases} \Rightarrow b=2a$

Hence, letting $a=1$, we get $b=2a=2$.

So, the complex basis vector becomes $\mathbf{v}_2=\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Therefore, the complementary solution for the homogeneous system is

$$\mathbf{X}_c=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}=c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{-t}+c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t}$$

Second: Determine the particular solution X_p.

2. Variation of Parameters

Consider a non-homogeneous linear system: $\begin{cases} x' = ax + by + g_1(t) \\ y' = cx + dy + g_2(t) \end{cases}$

In matrix Form:

The above system of equations can be written in matrix or in vector form as follow: $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{g}(t)$ where $\mathbf{X}' = \begin{pmatrix} x' \\ y' \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

It can also be expressed in the form $\mathbf{X}' = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$.

What are we going to do?

Given: Suppose you a system is given in one of the notations or forms.

$$\begin{cases} x' = ax + by + g_1(t) \\ y' = cx + dy + g_2(t) \end{cases} \textbf{ or } \begin{cases} \frac{dx}{dt} = ax + by + g_1(t) \\ \frac{dy}{dt} = cx + dy + g_2(t) \end{cases} \textbf{ or } \mathbf{X}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_1(t) \end{pmatrix}$$

Objective: The main objective is to solve this system.

Fundamental solutions: The solutions of the corresponding homogeneous part which is $\mathbf{X}' = \mathbf{A}\mathbf{X}$ **or** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Then, the fundamental solutions are $\mathbf{X}_1 = \mathbf{v}_1 e^{\lambda_1 t}$, $\mathbf{X}_2 = \mathbf{v}_2 e^{\lambda_2 t}$.

Complementary (General) Solution: $\mathbf{X}_c = c_1\mathbf{X}_1 + c_2\mathbf{X}_2$

Particular solution: The general solution of the non-homogeneous is of the form $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$. Here, $\mathbf{X}_c$ is the complementary solution and $\mathbf{X}_p$ is any particular solution to the non-homogeneous system.

Since we have discussed how to get $\mathbf{X}_c$, the challenge is how to get $\mathbf{X}_p$.

Let's see how we can apply variation of Parameters.

Procedures to use Variation of Parameters effectively:

First: Determination of fundamental solutions

Using eigenvalue method, determine $\mathbf{X}_1$ and $\mathbf{X}_2$ of the homogeneous part.

That is solve the homogeneous part: $\mathbf{X}' = \mathbf{A}\mathbf{X}$ **or** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Second: Formation of solution matrix and computing its inverse.

Form the solution matrix **M** by using $\mathbf{X}_1$ as first column and $\mathbf{X}_2$ as second column of **M** and compute its inverse $\mathbf{M}^{-1}$. Use the formula for the inverse of 2×2 matrix. That is $\mathbf{M}=\begin{pmatrix}\mathbf{X}_1 & \mathbf{X}_2\end{pmatrix}=\begin{pmatrix}x_1 & x_2\\ y_1 & y_2\end{pmatrix}\Rightarrow \mathbf{M}^{-1}=\dfrac{1}{\det\mathbf{M}}\begin{pmatrix}y_2 & -x_2\\ -y_1 & x_1\end{pmatrix}$

Third: Determination of the particular solution. .

Apply the formula $\mathbf{X}_p=\mathbf{M}.\int\mathbf{M}^{-1}.\mathrm{g}(t)dt$ where $\mathrm{g}(t)$ is the forcing function in the given system of differential equations.

Examples:

1. Find the general solution for the systems of differential equations.

a) $\begin{cases}x'=2y+e^t\\ y'=-x+3y-e^t\end{cases}$ b) $\mathbf{X}'=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\mathbf{X}+\begin{pmatrix}e^t\cos t\\ e^t\sin t\end{pmatrix}$

c) $\mathbf{X}'=\begin{pmatrix}1 & 8\\ 1 & -1\end{pmatrix}\mathbf{X}+\begin{pmatrix}12e^{-t}\\ 24te^t\end{pmatrix}$ $d)\begin{cases}x'=y+1\\ y'=-x+t\end{cases}$

Solution:

a) **First:** Determination of fundamental solutions using eigenvalue method.

The coefficient matrix is $A=\begin{pmatrix}0 & 2\\ -1 & 3\end{pmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}-\lambda & 2\\ -1 & 3-\lambda\end{vmatrix}=0\Rightarrow-\lambda(3-\lambda)+2=0\Rightarrow\lambda^2-3\lambda+2=0$$

$$\Rightarrow(\lambda-1)(\lambda-2)=0\Rightarrow\lambda_1=1,\lambda_2=2$$

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix}a\\ b\end{pmatrix}$

i) For $\lambda_1=1,\ (A-\lambda I)v=0\Rightarrow\begin{pmatrix}-1 & 2\\ -1 & 2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-a+2b=0\\ -a+2b=0\end{cases}\Rightarrow a=2b$

Hence, letting $b=1$ in $a=2b=2$, we get the basis vector $\mathbf{v}_1=\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}2\\ 1\end{pmatrix}$.

ii) For $\lambda_2=2,\ (A-\lambda I)v=0\Rightarrow\begin{pmatrix}-2 & 2\\ -1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-2a+2b=0\\ -a+b=0\end{cases}\Rightarrow a=b$

Hence, letting $a=b=1$, we get the basis vector $\mathbf{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$. So, the fundamental solutions are $\mathbf{X}_1=\mathbf{v}_1e^{\lambda_1 t}=\begin{pmatrix}2\\1\end{pmatrix}e^t, \mathbf{X}_2=\mathbf{v}_2e^{\lambda_2 t}=\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$ and the complementary solution is $\mathbf{X}_c=c_1\begin{pmatrix}2\\1\end{pmatrix}e^t+c_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$.

Second: Formation of solution matrix and computing its inverse.

Using $\mathbf{X}_1=\begin{pmatrix}2e^t\\e^t\end{pmatrix}$ as first column and $\mathbf{X}_2=\begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix}$ as second column, the solution matrix is $\mathbf{M}=\begin{pmatrix}\mathbf{X}_1 & \mathbf{X}_2\end{pmatrix}=\begin{pmatrix}2e^t & e^{2t}\\ e^t & e^{2t}\end{pmatrix}$ and $\det\mathbf{M}=\det\begin{pmatrix}2e^t & e^{2t}\\ e^t & e^{2t}\end{pmatrix}=e^{3t}$.

Thus, $\mathbf{M}^{-1}=\dfrac{1}{\det\mathbf{M}}\begin{pmatrix}y_2 & -x_2\\ -y_1 & x_1\end{pmatrix}=\dfrac{1}{e^{3t}}\begin{pmatrix}e^{2t} & -e^{2t}\\ -e^t & 2e^t\end{pmatrix}=\begin{pmatrix}e^{-t} & -e^{-t}\\ -e^{-2t} & 2e^{-2t}\end{pmatrix}$.

Third: Determination of $\mathbf{X}_p$ using the formula $\mathbf{X}_p=\mathbf{M}.\int\mathbf{M}^{-1}.\mathbf{g}(t)dt$.

Here, $\mathbf{g}(t)$ is the forcing function in the given system which $\mathbf{g}(t)=\begin{pmatrix}e^t\\-e^t\end{pmatrix}$.

Hence, $\mathbf{X}_p=\mathbf{M}.\int\mathbf{M}^{-1}.\mathbf{g}(t)dt=\begin{pmatrix}2e^t & e^{2t}\\ e^t & e^{2t}\end{pmatrix}.\int\begin{pmatrix}e^{-t} & -e^{-t}\\ -e^{-2t} & 2e^{-2t}\end{pmatrix}.\begin{pmatrix}e^t\\-e^t\end{pmatrix}dt$

$$=\begin{pmatrix}2e^t & e^{2t}\\ e^t & e^{2t}\end{pmatrix}\int\begin{pmatrix}2\\-3e^{-t}\end{pmatrix}dt=\begin{pmatrix}2e^t & e^{2t}\\ e^t & e^{2t}\end{pmatrix}\begin{pmatrix}2t\\3e^{-t}\end{pmatrix}=\begin{pmatrix}4te^t+3e^t\\2te^t+3e^t\end{pmatrix}$$

Thus, $\mathrm{X}_p=\begin{pmatrix}4te^t+3e^t\\2te^t+3e^t\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}te^t+\begin{pmatrix}3\\3\end{pmatrix}e^t$.

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X}=\mathrm{X}_c+\mathrm{X}_p=c_1\begin{pmatrix}2\\1\end{pmatrix}e^t+c_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+\begin{pmatrix}4\\2\end{pmatrix}te^t+\begin{pmatrix}3\\3\end{pmatrix}e^t$$

b) **First:** Determination of fundamental solutions using eigenvalue method.

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{vmatrix}=0\Rightarrow\lambda^2-2\lambda+2=0\Rightarrow\lambda=1\pm i$$

Here, we need how to form fundamental solutions for such complex roots.

i) For $\lambda = 1+i$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -ai-b=0 \\ a-bi=0 \end{cases} \Rightarrow b=-ai$

Hence, letting $a=i$, we get $b=-ai=-i^2=1$.

So, the complex basis vector becomes $\mathbf{v}^* = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

Now, identify the real and imaginary parts of $\mathbf{v} = \mathbf{v}^* e^{\alpha t}(\cos\beta t + i\sin\beta t)$.

That is $\lambda = 1 \pm i \Rightarrow \alpha = 1,\ \beta = 1$. Then, $\mathbf{v} = \mathbf{v}^* e^t(\cos t + i\sin t)$.

$$\mathbf{v} = \mathbf{v}^* e^t(\cos t + i\sin t) = \begin{pmatrix} \mathbf{i} \\ 1 \end{pmatrix} e^t(\cos t + \mathbf{i}\sin t) = \begin{pmatrix} \mathbf{i}e^t\cos t - e^t\sin t \\ e^t\cos t + \mathbf{i}e^t\sin t \end{pmatrix}$$

$$= \begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix} + \begin{pmatrix} \mathbf{i}e^t\cos t \\ \mathbf{i}e^t\sin t \end{pmatrix} = \underbrace{\begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix}}_{=\mathbf{Re}(\mathbf{v})} + \mathbf{i}\underbrace{\begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}}_{=\mathbf{Im}(\mathbf{v})}$$

Hence, we have $\mathbf{X}_1 = \mathbf{Re}(\mathbf{v}) = \begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix}$, $\mathbf{X}_2 = \mathbf{Im}(\mathbf{v}) = \begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}$.

Second: Formation of solution matrix and computing its inverse.

That is $\mathbf{M} = (\mathbf{X}_1 \quad \mathbf{X}_2) = \begin{pmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{pmatrix}$ and $\det\mathbf{M} = -e^{2t}$.

Thus, $\mathbf{M}^{-1} = -\dfrac{1}{e^{2t}}\begin{pmatrix} e^t\sin t & -e^t\cos t \\ -e^t\cos t & -e^t\sin t \end{pmatrix} = \begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}$.

Third: Determination of $\mathbf{X}_p$ using the formula $\mathbf{X}_p = \mathbf{M}.\int \mathbf{M}^{-1}.\mathbf{g}(t)dt$.

Here, $\mathbf{g}(t)$ is the forcing function in the given system which $\mathbf{g}(t) = \begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}$.

$$\mathbf{X}_p = \mathbf{M}.\int \mathbf{M}^{-1}.\mathbf{g}(t)dt = \begin{pmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{pmatrix}.\int \begin{pmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{pmatrix}.\begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}dt$$

$$= \begin{pmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{pmatrix}\int \begin{pmatrix} 0 \\ 1 \end{pmatrix}dt = \begin{pmatrix} -e^t\sin t & e^t\cos t \\ e^t\cos t & e^t\sin t \end{pmatrix}\begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} te^t\cos t \\ te^t\sin t \end{pmatrix}$$

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1\begin{pmatrix} -e^t\sin t \\ e^t\cos t \end{pmatrix} + c_2\begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}te^t$$

c) **First:** Determination of fundamental solutions using eigenvalue method.

$$|\mathbf{A}-\lambda\mathbf{I}|=0 \Rightarrow \begin{vmatrix} 1-\lambda & 8 \\ 1 & -1-\lambda \end{vmatrix}=0 \Rightarrow \lambda^2-9=0 \Rightarrow \lambda_1=3, \lambda_2=-3$$

Now, let's determine the corresponding eigen vectors. Let $V=\begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda_1=3$, $(A-\lambda I)v=0 \Rightarrow \begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2a+8b=0 \\ a-4b=0 \end{cases} \Rightarrow a=4b$

Hence, letting $b=1$ in $a=4b=4$, we get the basis vector $\mathbf{v}_1=\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

ii) For $\lambda_2=-3$, $(A-\lambda I)v=0 \Rightarrow \begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4a+8b=0 \\ a+2b=0 \end{cases} \Rightarrow a=-2b$

Hence, letting $b=1$, we get the basis vector $\mathbf{v}_2=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

So, the fundamental solutions are $\mathbf{X}_1=\mathbf{v}_1e^{\lambda_1 t}=\begin{pmatrix} 4 \\ 1 \end{pmatrix}e^{3t}, \mathbf{X}_2=\mathbf{v}_2e^{\lambda_2 t}=\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-3t}$.

Second: Formation of solution matrix and computing its inverse.

Using $\mathbf{X}_1=\begin{pmatrix} 4e^{3t} \\ e^{3t} \end{pmatrix}$ as first column and $\mathbf{X}_2=\begin{pmatrix} -2e^{-3t} \\ e^{-3t} \end{pmatrix}$ as second column, the solution matrix is $\mathbf{M}=\begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix}$ and $\det\mathbf{M}=\det\begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix}=6$.

Thus, $\mathbf{M}^{-1}=\dfrac{1}{\det\mathbf{M}}\begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix}=\dfrac{1}{6}\begin{pmatrix} e^{-3t} & 2e^{-3t} \\ -e^{3t} & 4e^{3t} \end{pmatrix}=\begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix}$.

Third: Determination of $\mathbf{X}_p$ using the formula $\mathbf{X}_p=\mathbf{M}.\int\mathbf{M}^{-1}.\mathbf{g}(t)dt$.

$$\mathbf{X}_p = \mathbf{M}.\int \mathbf{M}^{-1}.\mathbf{g}(t)dt = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \int \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix} \begin{pmatrix} 12e^{-t} \\ 24te^{t} \end{pmatrix} dt$$

$$= \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \int \begin{pmatrix} 2e^{-4t} + 4te^{-2t} \\ -2e^{2t} + 16te^{4t} \end{pmatrix} dt$$

$$= \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}e^{-4t} - 2te^{-2t} - e^{-2t} \\ -e^{2t} + 4te^{4t} - e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{-t} - 8te^{t} - 4e^{t} + (2e^{-t} - 8te^{t} + 2e^{t}) \\ -\frac{1}{2}e^{-t} + te^{t} - \frac{1}{2}e^{t} + (-e^{-t} + 4te^{t} - e^{t}) \end{pmatrix} = \begin{pmatrix} -16te^{t} - 2e^{t} \\ -\frac{3}{2}e^{-t} + 5te^{t} - \frac{3}{2}e^{t} \end{pmatrix}$$

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} -16te^{t} - 2e^{t} \\ -\frac{3}{2}e^{-t} + 5te^{t} - \frac{3}{2}e^{t} \end{pmatrix}$$

d) **First:** Determination of fundamental solutions using eigenvalue method.

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Here, we need how to form fundamental solutions for such complex roots.

Now, let's determine the corresponding eigen vectors. Let $V = \begin{pmatrix} a \\ b \end{pmatrix}$

i) For $\lambda = i$, $(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -ai + b = 0 \\ -a - bi = 0 \end{cases} \Rightarrow b = ai$

Hence, letting $a = -i$, we get $b = -ai = -i^2 = 1$.

So, the complex basis vector becomes $\mathbf{v}^* = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Now, identify the real and imaginary parts of $\mathbf{v} = \mathbf{v}^* e^{\alpha t}(\cos \beta t + i \sin \beta t)$.
That is $\lambda = i \Rightarrow \alpha = 0, \beta = 1$.

$$\mathbf{v} = \begin{pmatrix} -\mathbf{i} \\ 1 \end{pmatrix} (\cos t + \mathbf{i} \sin t) = \begin{pmatrix} -\mathbf{i}\cos t + \sin t \\ \cos t + \mathbf{i} \sin t \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \mathbf{i} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

Hence, we have $\mathbf{X}_1 = \mathbf{Re}(\mathbf{v}) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$, $\mathbf{X}_2 = \mathbf{Im}(\mathbf{v}) = \begin{pmatrix} -\cos t \\ \sin t) \end{pmatrix}$ as the fundamental solutions.

Second: Formation of solution matrix and computing its inverse.

That is $\mathbf{M}=(\mathbf{X}_1 \quad \mathbf{X}_2)=\begin{pmatrix}\sin t & -\cos t\\ \cos t & \sin t\end{pmatrix}$ and $\det \mathbf{M}=1$.

Thus, $\mathbf{M}^{-1}=\begin{pmatrix}\sin t & \cos t\\ -\cos t & \sin t\end{pmatrix}$.

Third: Determination of $\mathbf{X}_p$ using the formula $\mathbf{X}_p=\mathbf{M}.\int \mathbf{M}^{-1}.\mathbf{g}(t)dt$.

Here, $\mathbf{g}(t)$ is the forcing function in the given system which $\mathbf{g}(t)=\begin{pmatrix}1\\ t\end{pmatrix}$.

$$\mathbf{X}_p=\mathbf{M}.\int \mathbf{M}^{-1}.\mathbf{g}(t)dt=\begin{pmatrix}\sin t & -\cos t\\ \cos t & \sin t\end{pmatrix}\int\begin{pmatrix}\sin t & \cos t\\ -\cos t & \sin t\end{pmatrix}\begin{pmatrix}1\\ t\end{pmatrix}dt$$

$$=\begin{pmatrix}\sin t & -\cos t\\ \cos t & \sin t\end{pmatrix}\int\begin{pmatrix}\sin t+t\cos t\\ -\cos t+t\sin t\end{pmatrix}dt$$

$$=\begin{pmatrix}\sin t & -\cos t\\ \cos t & \sin t\end{pmatrix}\begin{pmatrix}t\sin t\\ -t\cos t\end{pmatrix}=\begin{pmatrix}t\\ 0\end{pmatrix}$$

Therefore, the general solution of the non-homogeneous system is

$$\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p=c_1\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}+c_2\begin{pmatrix}-\cos t\\ \sin t)\end{pmatrix}+\begin{pmatrix}t\\ 0\end{pmatrix}$$

2. Find the general solution for the systems of differential equations.

a) $\begin{cases}x'=-3x-4y+5e^t\\ y'=5x+6y-6e^t\end{cases}$ b) $\begin{cases}x'=x+4y-2\cos t\\ y'=x+y-\cos t+\sin t\end{cases}$

Solution:

a) First, find the complementary solution X_c of the homogeneous system.

That is find the solution of $\begin{cases}x'=-3x-4y\\ y'=5x+6y\end{cases}$

$$\begin{vmatrix}-3-\lambda & -4\\ 5 & 6-\lambda\end{vmatrix}=0\Rightarrow -(3+\lambda)(6-\lambda)+20=0$$

$$\Rightarrow -(18-3\lambda+6\lambda-\lambda^2)+20=0$$

$$\Rightarrow \lambda^2-3\lambda+2=0\Rightarrow(\lambda-1)(\lambda-2)=0$$

$$\Rightarrow \lambda_1=1,\ \lambda_2=2$$

Review Problems on Chapter-1

1. Determine the order n and degree d (if defined) of the following DEs.

$a)\ \dfrac{d^4y}{dx^4}-3x\left(\dfrac{d^2y}{dx^2}\right)^6=2x^7$ $b)\ y'''^2+y'=\ln y'$ $c)\ \left(\dfrac{dy}{dx}\right)^{\frac{7}{3}}=\sqrt{y'^6-x}$

$d)^*\ y''''^4(y'''^7+y')^{\frac{1}{3}}=\sqrt{y''^9}$ $e)\ y''^2=(y''^4-2xy')^{\frac{2}{3}}$ $f)\ \dfrac{d^2y}{dx^2}-e^{y'}=0$

$g)\ \left(\dfrac{d^3y}{dx^3}\right)^2+\cos(3y)=1$ $h)\ \left(\dfrac{d^3y}{dx^3}\right)^2+\cos(3y')=1$ $i)\ \dfrac{d^2y}{dx^2}=0$

Answer : $a)\ n=4, d=6$ $b)\ n=3$, No degree $c)\ n=1, d=18$

$d)\ n=3, d=38$ $e)\ n=2, d=8$ $f)\ n=2$, No degree

$g)\ n=3, d=2$ $h)\ n=2$, No degree $i)\ n=2, d=1$

2*. Show that $y=\tan^{-1}x$ is a solution of the DE $y''+2\sin y\cos^3 y=0$.

3. In each of the following, find the constants k and a.

a) If $y=e^{3x}$ is the solution of the DE $\dfrac{d^2y}{dx^2}-6y'+ky=0$, then find k.

b) If $y=x^2$ is the solution of the DE $8x\dfrac{dy}{dx}-ay=x^2$, then find a.

c) If $y=x^3+kx+1$ is the solution of the DE $y'''+xy''-2y'=0$, then find k.

d) If $y=x^k$ is the solution of the DE $16x^2y''+24xy'+y=0$, then find k.

Answer : $a)\ k=9$ $b)\ a=15$ $c)\ k=3$ $d)\ k=-1/4$

4*. Verify that the following DEs are homogeneous and solve them.

$a)\ xdy=(y+x\tan\dfrac{y}{x})dx,$ $b)\ xdy=(y+x\cot\dfrac{y}{x})dx$

$d)\ x\sin\dfrac{y}{x}dy=(x+y\sin\dfrac{y}{x})dx$ $e)\ ydy=(\dfrac{y^2}{x}+x\tan\dfrac{y^2}{x^2})dx$

5. Verify that the following DEs are homogeneous and solve the IVPs.

a) $\dfrac{dy}{dx}=\dfrac{x+y}{x-y},\ y(1)=0$ b) $\dfrac{dy}{dx}=\dfrac{y^2+2xy}{x^2},\ y(1)=2$

c) $xydy=(x^2+2y^2)dx, y(1)=1$ d) $(x^3+y^3)dx-xy^2dy=0, y(-1)=0$

e) $(3x^2+4xy)dx+(2xy+x^2)dy=0, y(2)=-3$

f) $x\cos\left(\dfrac{y}{x}\right)\dfrac{dy}{dx}=y\cos\left(\dfrac{y}{x}\right)+x, y(1)=\pi$

Answer :a) $2\tan^{-1}(y/x)=\ln(x^2+y^2)$ b) $y=\dfrac{2x^2}{3-2x}$ c) $y^2=2x^4-x^2$

d) $y^3=3x^3\ln|x|$ e) $x^3y^2+x^4y=24$ f) $\sin(y/x)=\ln|x|$

6. Verify that the following DES are exact and solve them.

a) $(6xy+2y^2)dx+(3x^2+4xy)dy=0$ b) $xdx+ydy=(x^2+y^2)dy$

c) $ye^{xy}dx+xe^{xy}dy=0$ d) $2xydx+(1+x^2)dy=0$

e) $(e^x\sin y+3x^2)dx+e^x\cos ydy=0$ f) $(ye^x+2x)dx+e^xdy=0$

g) $(2x+y^2)dx+2xydy=0$

Answer : a) $3x^2y+2xy^2=c$ b) $\ln(x^2+y^2)-2y=c$ c) $e^{xy}=c$

d) $x^2y+y=c$ e) $e^x\sin y+x^3=c.$ f) $ye^x+x^2=c$ g) $x^2+xy^2=c$

7. Verify that the DEs are not exact and solve them.

a) $(x^3-2y^2)dx+2xydy=0$ b) $(x^2y+4xy+2y)dx+(x^2+x)dy=0$

c) $(x^2+xy)y'+3xy+y^2=0$ d) $(x^4y^3+y)dx+(x^5y^2-x)dy=0$

e) $(y^2+2x^2)dx+xydy=0$ f) $(xy^3+y^2)dx+(y-xy)dy=0$

g) $(y^2+2x^2)dx-x(1+xy)dy=0$ h) $2xydx+(y^2-x^2)dy=0$

i) $y(1-xy)dx-xydy=0$ j) $y^3dx+(xy^3+3xy^2+1)dy=0$

Answer :a) $x+\dfrac{y^2}{x^2}=c$ b) $(x+1)e^xx^2y=c$ c) $x^3y+\dfrac{x^2y^2}{2}=c$

d) $x^4y^3-3y=cx$ e) $\dfrac{x^2y^2}{2}+\dfrac{x^4}{2}=c$ f) $\dfrac{x^2}{2}+\dfrac{x}{y}-\dfrac{1}{y}=c$

g) $2\ln x-\dfrac{y^2}{2x^2}=c$ h) $x^2+y^2=cy$ i) $xy^3=c$ j) $e^y(xy^3+1)=c$

8. Find the constants a,m,n so that the following DEs are exact.

$a)\ x^3+3xy+(ax^2+4y)y'=0$ $\qquad b)\ (x^3y^m+x^3)dx+(x^4y^n+y^3)dy=0$

$c)\ (y^3+x^2y)dx+(ax+1)dy=0$ $\qquad d)\ x^m y^3\dfrac{dy}{dx}+7ax^4y^n=0$

Answer: $a)\ a=\frac{3}{2}$ $\quad b)\ m=4, n=3$ $\quad c)\ a=-2$ $\quad d)\ m=5, n=4, a=\frac{5}{28}$

9. Solve the following *Bernoulli's Differential Equations*

$a)\ y'+3y=e^{3x}y^2, y(0)=1/4$ $\qquad b)\ y'+y=xy^4, y(0)=3$

$c)\ xy'+y=xy^3$ $\qquad d)\ xy'+y=x^2y^2\ln x, y(1)=1$

$e)\ y'=xy^2+y, y(0)=1$ $\qquad f)\ y'+2xy=-xy^4$

$g)\ x\dfrac{dy}{dx}+y=x^3y^6$ $\qquad h)\ \dfrac{dy}{dx}+y=xy^2$ $\qquad i)\ y'+y=y^2, y(0)=-1$

Answer: $a)\ y=\dfrac{e^{-3x}}{4-x}$ $\quad b)\ y=\dfrac{3}{\sqrt[3]{27x-8e^{3x}+9}}$ $\quad c)\ y=\dfrac{1}{\sqrt{ce^{2x}-x-1}}$

$d)\ y=\dfrac{1}{x^2-x^2\ln x}$ $\quad e)\ y=\dfrac{1}{1-x}$ $\quad f)\ \dfrac{1}{y^3}=ce^{3x^2}-\dfrac{1}{2}$

$g)\ x^3y^5(cx^2+5/2)=1$ $\quad h)\ y=\dfrac{1}{ce^x+x+1}$ $\quad i)\ y=\dfrac{1}{1-2e^x}$

10. Solve the following DEs using *Methods of Undetermined coefficients*

$a)\ y''-7y'+10y=24e^x$ $\qquad b)\ y''-4y'+4y=2e^{2x}+4x-12$

$c)\ y''+4y'+4y=e^{-2x}\sin 2x$ $\qquad d)\ y''-2y'+y=e^x\sin x$

$e)\ y''-3y'-4y=5e^x$ $\qquad f)\ y''-4y'+4y=e^{2x}\sin x$

$g)\ y''-y=1+2e^x$ $\qquad h)\ y''-2y'+y=2e^x$

$i)\ y''+2y'+y=2x\sin x$ $\qquad j)\ y''-3y'+2y=e^x$

$k)\ y''-2y'+y=e^x+x$ $\qquad m)\ y''+y'-2y=2xe^x$

Answer: $a)\ y=c_1e^{2x}+c_2e^{5x}+6e^x$ $\qquad b)\ y=c_1e^{2x}+c_2xe^{2x}+x^2e^{2x}+x-2$

$c)\ y=(c_1+c_2x)e^{-2x}-\dfrac{1}{4}e^{-2x}\sin 2x$ $\qquad d)\ y=(c_1+c_2x)e^x-e^x\sin x$

$e)\ y=c_1e^{4x}+c_2e^{-x}-xe^{-x}$ $\qquad f)\ y=(c_1+c_2x)e^{2x}-e^{2x}\sin x$

$g)\ y=c_1e^x+c_2e^{-x}+xe^{-x}-1$ $\qquad h)\ y=c_1e^x+c_2xe^x+x^2e^x$

11. Solve the following DEs using *Variation of Parameters*

$a)\ y''-2y'+y=\dfrac{e^x}{x^2}$ $\quad b)\ y''-4y'+4y=\dfrac{12e^{2x}}{x^4}$ $\quad c)\ 4y''+36y=\csc 3x$

$d)\ y''+y=\sin x$ $\quad e)\ y''-4y'+4y=(x+1)e^{2x}$ $\quad f)\ y''+y=\sec x\tan x$

$g)\ y''-y=\cosh x$ $\quad h)\ y''-4y=\dfrac{e^{2x}}{x}$

Answer : $a)\ y(x)=c_1e^x+c_2xe^x-(\ln|x|+1)e^x$ $\quad b)\ y(x)=(c_1+c_2x)e^{2x}+2x^{-2}e^{2x}$

$c)\ y=c_1\cos 3x+c_2\sin 3x-\dfrac{x}{12}\cos 3x+\dfrac{1}{36}\sin 3x\ln|\sin 3x|$

$d)\ y=c_1\cos x+c_2\sin x-\dfrac{x}{2}\cos x$ $\quad e)\ y=c_1e^{2x}+c_2xe^{2x}+\dfrac{x^3e^{2x}}{6}+\dfrac{x^2e^{2x}}{2}$

$f)\ y=c_1\cos x+c_2\sin x+x\cos x-\sin x\ln|\cos x|$

$g)\ y=c_1e^x+c_2e^{-x}-(\dfrac{e^{-x}}{8}+\dfrac{x}{4})e^{-x}+(\dfrac{x}{4}-\dfrac{1}{8})e^x$

12. Find the value(s) of the constant k for which the DE $y''+6y'+ky=0$ has a general solution of the form $y=(c_1+c_2x)e^{-3x}$. **Answer** : $k=9$

13*. Find a and k for which the DE $y''+ay'+ky=0$ has a general solution of the form $y=(c_1\cos 2x+c_2\sin 2x)e^{-3x}$. **Answer** : $a=6,\ k=13$

14. Using the appropriate methods, find the particular solution

$a)\ y''+y'+y=e^x+4$ $\quad b)\ y''-y=\sinh x$ $\quad c)\ y''+2y'+y=e^{-x}\ln x$

$d)\ y''+3y'+2y=\dfrac{1}{1+e^{2x}}$ $\quad e)\ y''-2y'+2y=e^x\tan x$ $\quad f)\ y''-y=\dfrac{1}{1+e^x}$

$g)\ y''-2y+y=4x^2-3$ $\quad h)\ y''+3y+2y=\sin e^x$

Answer : $a)\ y_p=\dfrac{1}{3}e^x$ $\quad b)\ y_p=\dfrac{x}{2}\cosh x$ $\quad c)\ y_p=\dfrac{1}{2}x^2e^{-x}\ln x-\dfrac{3}{4}x^2e^{-x}$

$d)\ y_p=e^{-x}\tan^{-1}e^{-x}-\dfrac{1}{2}[e^{-2x}\ln(1+e^x)-1]$ $\quad e)\ y_p=-e^x\cos x\ln(\sec x+\tan x)$

$f)\ y_p=\dfrac{1}{2}[e^x\ln(1+e^{-x})-e^{-x}\ln(1+e^x)-1]$

$g)\ y_p=4x^2+16x+21$ $\quad h)\ y_p=-e^{-2x}\sin e^x$

CHAPTER-2

Laplace Transform and its Applications

2.1 Definition and Examples of Laplace Transform

Definition: Suppose f is a function defined for all $t \geq 0$. Then, the Laplace Transform of f denoted by $L\{f(t)\}$ is a function $F(s)$ defined by

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \text{ where } s \text{ is a parameter (real or complex).}$$

Note: The interval of integration is infinite which is an improper integral and thus it is evaluated by the rule $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \lim_{n\to\infty} \int_0^n e^{-st} f(t)dt$.

Examples: Find the Laplace Transform $L\{f(t)\}$ of the following functions.

$a)\ f(t) = 1, t \geq 0$ $\quad b)\ f(t) = e^{at}$ $\quad c)\ f(t) = t, t \geq 0$

$d)\ f(t) = \begin{cases} 1, 0 \leq t < 2 \\ t-2, 2 \leq t \end{cases}$ $\quad e)\ f(t) = \begin{cases} -2, 0 < t < 4 \\ 2, t > 4 \end{cases}$ $\quad f)\ f(t) = \begin{cases} 0, 0 < t < 1 \\ 2t-2, t > 1 \end{cases}$

Solution: Let's apply the definition.

$$a)\ F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{n\to\infty} \int_0^n e^{-st} dt = \lim_{n\to\infty} \left[\frac{-e^{-st}}{s}\right]_{t=0}^{t=n} = \lim_{n\to\infty} \left[\frac{-e^{-sn}+1}{s}\right] = \frac{1}{s}$$

$$b)\ F(s) = L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{t(a-s)} dt = \lim_{u\to\infty} \int_0^u e^{t(a-s)} dt$$

$$= \lim_{u\to\infty} \left(\frac{-e^{t(a-s)}}{s-a}\right)\Bigg|_0^u = \lim_{u\to\infty} \left(\frac{-e^{u(a-s)}}{s-a}\right) + \frac{1}{s-a} = \begin{cases} \infty, \text{if } s \leq a \\ \dfrac{1}{s-a}, \text{if } s > a \end{cases}$$

Therefore, $L\{e^{at}\} = \frac{1}{s-a}; s > a$. For instance, $L\{e^{2t}\} = \frac{1}{s-2}, L\{e^{-t}\} = \frac{1}{s+1}$.

c) $F(s)=L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^\infty e^{-st}tdt=\lim\limits_{n\to\infty}\int_0^n e^{-st}tdtx$

$$=\lim_{n\to\infty}\left[\frac{-te^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_{t=0}^{t=n}=\lim_{n\to\infty}\left[\frac{-ne^{-sn}}{s}-\frac{e^{-sn}}{s^2}+\frac{1}{s^2}\right]=\frac{1}{s^2}$$

d) $L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^2 e^{-st}f(t)dt+\int_2^\infty e^{-st}f(t)dt=\int_2^\infty e^{-st}dt+\int_2^\infty (t-2)e^{-st}dt$

$$=\left.\frac{e^{-st}}{-s}\right|_0^2+\left.\left(\frac{-(t-2)e^{-st}}{s}-\frac{e^{-st}}{s^2}\right)\right|_{t=2}^\infty=\frac{1}{s}(1-e^{-2s})+\frac{e^{-2s}}{s^2}$$

e) $L\{f(t)\}=\int_0^4 -2e^{-st}dt+\lim\limits_{u\to\infty}\int_4^u 2e^{-st}dt=\left.\left(\frac{2e^{-st}}{s}\right)\right|_0^4+\lim\limits_{u\to\infty}\left.\left(-\frac{2e^{-st}}{s}\right)\right|_4^u$

$$=\left(\frac{2e^{-4s}}{s}-\frac{2}{s}\right)+\lim_{u\to\infty}\left(-\frac{2e^{-us}}{s}+\frac{2e^{-4s}}{s}\right)=\frac{4e^{-4s}}{s}-\frac{2}{s}$$

f) $L\{f(t)\}=\int_0^1 e^{-st}f(t)dt+\int_1^\infty e^{-st}f(t)dt=\int_1^\infty (2t-2)e^{-st}dt=\frac{2e^{-s}}{s^2},\ s>0$

2.2 Properties of Laplace Transform

Property-I: Linearity Property:

If f and g are functions whose Laplace Transforms exist, then for $a\in R$

i) $L\{af(t)\}=aL\{f(t)\}$

ii) $L\{f(t)+g(t)\}=L\{f(t)\}+L\{g(t)\}$

Example: Find the Laplace transform of $f(t)=2e^{3t}-3t+4$

Solution: In the above example, we got $L\{1\}=\frac{1}{s}, L\{e^{3t}\}=\frac{1}{s-3}, L\{t\}=\frac{1}{s^2}$.

Then, using linearity properties, we have

$L\{f(t)\}=L\{2e^{3t}-3t+4\}=L\{2e^{3t}\}+L\{-3t\}+L\{4\}$

$$=2L\{e^{3t}\}-3L\{t\}+4L\{1\}=\frac{2}{s-3}-\frac{3}{s^2}+\frac{4}{s}$$

The Laplace Transform of Trigonometric Functions:

The Laplace Transform of $f(t) = \sin at$ and $f(t) = \cos at$ can be derived directly by the definition using Integration by Parts which is too demanding. But it can be done easily using the **Euler's formula** with linearity properties.

Recall: Euler's formula: $e^{iat} = \cos at + i\sin at$.

Using linearity properties, we have

$$e^{iat} = \cos at + i\sin at \Rightarrow L\{e^{iat}\} = L\{\cos at + i\sin at\}$$

$$\Rightarrow L\{e^{iat}\} = L\{\cos at\} + iL\{\sin at\} \text{.......................(i)}$$

But as we discussed above, $L\{e^{at}\} = \dfrac{1}{s-a} \Rightarrow L\{e^{iat}\} = \dfrac{1}{s-ia}$.

Now rationalizing, $L\{e^{iat}\} = \dfrac{1}{s-ia}$ using the conjugate of $s-ia$, we have

$$L\{e^{iat}\} = \frac{1}{(s-ia)(s+ia)}(s+ia) = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + \frac{a}{s^2+a^2}.i \text{..........(ii)}$$

Equating (i) and (ii), we have $L\{\cos at\} + iL\{\sin at\} = \dfrac{s}{s^2+a^2} + i\dfrac{a}{s^2+a^2}$.

But we know that two complex numbers are equal if their corresponding real parts at the same time imaginary parts are equal. Using this property,

$$\underbrace{L\{\cos at\}}_{\text{Real part}} + \underbrace{L\{\sin at\}}_{\text{Imaginary part}}.i = \underbrace{\frac{s}{s^2+a^2}}_{\text{Real part}} + \underbrace{\frac{a}{s^2+a^2}}_{\text{Imaginary part}}.i$$

$$\Rightarrow L\{\cos at\} = \frac{s}{s^2+a^2},\ L\{\sin at\} = \frac{a}{s^2+a^2}$$

Therefore, $L\{\cos at\} = \dfrac{s}{s^2+a^2}, L\{\sin at\} = \dfrac{a}{s^2+a^2}; s > 0$

Examples:

$$L\{\cos t\} = \frac{s}{s^2+1}, L\{\sin t\} = \frac{1}{s^2+1}, L\{\cos 2t\} = \frac{s}{s^2+4}, L\{\sin 3t\} = \frac{3}{s^2+9}; s > 0$$

The Laplace Transform of Hyperbolic Functions:

Consider the functions $f(t)=\sinh at$, $f(t)=\cosh at$.

Recall: By definition, we know that $\sinh at=\dfrac{e^{at}-e^{-at}}{2}, \cosh at=\dfrac{e^{at}+e^{-at}}{2}$.

Then, by linearity property, we have

i) $L\{\sinh at\}=L\left\{\dfrac{e^{at}-e^{-at}}{2}\right\}=\dfrac{1}{2}L\{e^{at}\}-\dfrac{1}{2}L\{e^{-at}\}=\dfrac{1}{2(s-a)}-\dfrac{1}{2(s+a)}=\dfrac{a}{s^2-a^2}$

ii) $L\{\cosh at\}=L\left\{\dfrac{e^{at}+e^{-at}}{2}\right\}=\dfrac{1}{2}L\{e^{at}\}+\dfrac{1}{2}L\{e^{-at}\}=\dfrac{1}{2(s-a)}+\dfrac{1}{2(s+a)}=\dfrac{s}{s^2-a^2}$

Examples:

i) $L\{\sinh 2t\}=\dfrac{2}{s^2-4}$, $L\{\sinh 3t\}=\dfrac{3}{s^2-9}$, $L\{\sinh\sqrt{3}t\}=\dfrac{\sqrt{3}}{s^2-3}$

ii) $L\{\cosh 2t\}=\dfrac{s}{s^2-4}$, $L\{\cosh 3t\}=\dfrac{s}{s^2-9}$, $L\{\cosh\sqrt{2}t\}=\dfrac{s}{s^2-2}$

Property-II: Power property: $L\{t^n\}=\dfrac{n!}{s^{n+1}}, n=0,1,2,...$

Examples: $L\{t^2\}=\dfrac{2!}{s^{2+1}}=\dfrac{2}{s^3}, L\{t^3\}=\dfrac{3!}{s^{3+1}}=\dfrac{6}{s^4}, L\{t^4\}=\dfrac{4!}{s^{4+1}}=\dfrac{12}{s^5}; s>0$

Property-III: S-Shifting Properties (First Shifting Rule):

Suppose $L\{f(t)\}=F(s)$. Then,

i) $L\{e^{at}f(t)\}=F(s-a)$ ii) $L\{e^{-at}f(t)\}=F(s+a)$

Examples:

1. Find the following Laplace transforms

a) $L\{e^{3t}\cos 2t\}$ b) $L\{e^{-2t}\sin 5t\}$ c) $L\{e^{-t}\cosh 6t\}$ d) $L\{e^{4t}t^5\}$

Solution: Identify f and compute $L\{f(t)\}=F(s)$. Then apply property-III.

a) Here, let $f(t)=\cos 2t$. Then $F(s)=L\{f(t)\}=L\{\cos 2t\}=\dfrac{s}{s^2+4}$.

Therefore, by S-shifting property, $L\{e^{3t}\cos 2t\}=F(s-3)=\dfrac{s-3}{(s-3)^2+4}$.

b) Here, let $f(t)=\sin 5t$. Then, $F(s)=L\{f(t)\}=L\{\sin 5t\}=\frac{5}{s^2+25}$.

Therefore, by S-shifting property, $L\{e^{-2t}\sin 5t\}=F(s+2)=\frac{5}{(s+2)^2+25}$.

c) Let $f(t)=\cosh 6t, F(s)=L\{\cosh 6t\}=\frac{s}{s^2-36}$.

Therefore, $L\{e^{-t}\cosh 6t\}=F(s+1)=\frac{s+1}{(s+1)^2-36}$.

d) Let $f(t)=t^5, F(s)=L\{t^5\}=\frac{5!}{s^{5+1}}=\frac{120}{s^6}\Rightarrow L\{e^{4t}t^5\}=F(s-4)=\frac{120}{(s-4)^6}$

2. Find the Laplace Transform of

a) $f(t)=\sin 3t\cos 2t$ b) $f(t)=\cos 4t\cos 2t$ c) $f(t)=\cos^2 t$

d) $f(t)=\sin 2t\cosh 3t$ e) $f(t)=\sin 2t\cos 2t$ f) $f(t)=4e^{-6t}\sin^2 t$

Solution: Recall: P*roduct to sum* formula of trigonometric functions.

$$\sin x\cos y=\frac{1}{2}[\sin(x-y)+\sin(x+y)], \cos x\cos y=\frac{1}{2}[\cos(x-y)+\cos(x+y)$$

a) $L\{\sin 3t\cos 2t\}=\frac{1}{2}L\{\sin t\}+\frac{1}{2}L\{\sin 5t\}=\frac{1}{2}(\frac{1}{s^2+1}+\frac{5}{s^2+25})$

b) $L\{\cos 4t\cos 2t\}=\frac{1}{2}L\{\cos 2t\}+\frac{1}{2}L\{\cos 6t\}=\frac{1}{2}(\frac{s}{s^2+4}+\frac{s}{s^2+36})$

c) $\cos^2 t=\frac{1}{2}+\frac{\cos 2t}{2}\Rightarrow L\{\cos^2 t\}=\frac{1}{2}L\{1\}+\frac{1}{2}L\{\cos 2t\}=\frac{1}{2s}+\frac{s}{2(s^2+4)}$

d) $f(t)=\sin 2t\cosh 3t=\sin 2t\left(\frac{e^{3t}+e^{-3t}}{2}\right)=\frac{1}{2}e^{3t}\sin 2t+\frac{1}{2}e^{-3t}\sin 2t.$

Then, $L\{f(t)\}=L\left\{\frac{1}{2}e^{3t}\sin 2t\right\}+L\left\{\frac{1}{2}e^{-3t}\sin 2t\right\}=\frac{1}{(s-3)^2+4}+\frac{1}{(s+3)^2+4}$.

e) $\sin 2t\cos 2t=\frac{\sin 4t}{2}\Rightarrow L\{\sin 2t\cos 2t\}=L\left\{\frac{\sin 4t}{2}\right\}=\frac{1}{2}L\{\sin 4t\}=\frac{2}{s^2+16}$

Property-IV: t-Shifting Property:

If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}[F(s)]$

Examples: Find the following Laplace Transforms using t-shifting property.

a) $L\{t\cos 2t\}$ b) $L\{te^{2t}\sin 3t\}$ c) $L\{t^4 e^{-t}\}$ d) $L\{t\cosh 3t\}$

Solution:

a) Let $f(t) = \cos 2t$. Then, $L\{f(t)\} = L\{\cos 2t\} = \dfrac{s}{s^2+4}$.

Therefore, by t-shifting property, $L\{t\cos 2t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+4}\right) = \dfrac{s^2-4}{(s^2+4)^2}$

b) Using $L\{\sin 3t\} = \dfrac{3}{s^2+9}$ and s-shifting, $F(s) = L\{e^{2t}\sin 3t\} = \dfrac{3}{(s-2)^2+9}$

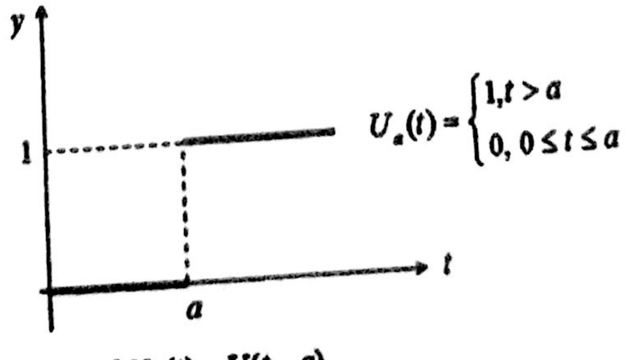
Therefore, by t-shifting, $L\{te^{2t}\sin 3t\} = -\dfrac{d}{ds}\left(\dfrac{3}{(s-2)^2+9}\right) = \dfrac{6(s-2)}{[(s-2)^2+9]^2}$

c) Here, $f(t) = e^{-t}$, $F(s) = L\{e^{-t}\} = \dfrac{1}{s+1} \Rightarrow L\{t^4 e^{-t}\} = (-1)^4 \dfrac{d^4}{ds^4}\left(\dfrac{1}{s+1}\right) = \dfrac{24}{(s+1)^5}$

d) Here, $L\{\cosh 3t\} = \dfrac{s}{s^2-9} \Rightarrow L\{t\cosh 3t\} = (-1)^1 \dfrac{d}{ds}\left(\dfrac{s}{s^2-9}\right) = \dfrac{s^2+9}{(s^2-9)^2}$

2.3 Unit-Step (Heaviside) Function and its Laplace Transform

Definition: The unit-step function (also known as Heaviside function), denoted by $U_a(t) = U(t-a)$ or $H_a(t) = H(t-a)$ is defined as $U_a(t) = \begin{cases} 1, t > a \\ 0, 0 \le t \le a \end{cases}$



Graph of $U_a(t) = U(t-a)$

Laplace Transform of unit-step function $U_a(t)$

For any unit step function $U_a(t) = H(t-a)$, $L\{U_a(t)\} = L\{H(t-a)\} = \dfrac{e^{-as}}{s}$

This is obtained using the definition of transform as follow.

$$L\{U_a(t)\} = \int_0^\infty e^{-st}U_a(t)dt = \int_0^a e^{-st}U_a(t)dt + \int_a^\infty e^{-st}U_a(t)dt = \int_a^\infty e^{-st}dt$$

$$= \lim_{n\to\infty}\int_a^n e^{-st}dt = \lim_{n\to\infty}\left[\frac{-e^{-ts}}{s}\right]_{t=a}^{t=n} = \lim_{n\to\infty}\left[\frac{-e^{-ns}}{s} + \frac{e^{-as}}{s}\right] = \frac{e^{-as}}{s}$$

Examples: $L\{U_2(t)\} = \dfrac{e^{-2s}}{s}, L\{H(t-3)\} = \dfrac{e^{-3s}}{s}, L\{H(t-\pi)\} = \dfrac{e^{-\pi s}}{s}$

Theorem (Second-Shifting Rule): Let $L\{f(t)\} = F(s)$. Then,

i) $L\{f(t-a)U_a(t)\} = L\{f(t-a)H(t-a)\} = e^{-as}L\{f(t)\} = e^{-as}F(s)$

ii) $L\{f(t)U_a(t)\} = L\{f(t)H(t-a)\} = e^{-as}L\{f(t+a)\}$

Examples: Find the following Laplace transforms

a) $L\{4(t-2)H(t-2)\}$ *b*) $L\{2(t-3)^2H(t-3)\}$ *c*) $L\{t^2H(t-4)\}$

d) $L\{(5t+3)H(t-1)\}$ *e*) $L\{\cos 2(t-\pi)H(t-\pi)\}$ *f*) $L\{e^{2t}H(t-4)\}$

Solution: To use the above shifting rule, first identify $f(t)$ from the problem.

a) Here, $f(t-2) = 4(t-2) \Rightarrow f(t) = 4t \Rightarrow L\{f(t)\} = L\{4t\} = \dfrac{4}{s^2}$

Therefore, $L\{4(t-2)H(t-2)\} = e^{-2s}L\{f(t)\} = \dfrac{4e^{-2s}}{s^2}$

b) Here, $f(t-3) = 2(t-3)^2 \Rightarrow f(t) = 2t^2 \Rightarrow L\{f(t)\} = L\{2t^2\} = \dfrac{4}{s^3}$

Therefore, $L\{2(t-3)^2H(t-3)\} = e^{-3s}L\{f(t)\} = e^{-3s}L\{2t^2\} = \dfrac{4e^{-3s}}{s^3}$

c) In this case, let' use the second property with $a = 4,\ f(t) = t^2$.

Here, $f(t) = t^2 \Rightarrow f(t+4) = (t+4)^2 = t^2 + 8t + 16 \Rightarrow L\{f(t+4)\} = \dfrac{2}{s^3} + \dfrac{8}{s^2} + \dfrac{16}{s}$

Therefore, $L\{t^2H(t-4)\} = e^{-4s}L\{f(t+4)\} = e^{-4s}\left(\dfrac{2}{s^3} + \dfrac{8}{s^2} + \dfrac{16}{s}\right)$

d) Here, $a = 1, f(t) = 5t+3 \Rightarrow f(t+1) = 5t+8 \Rightarrow L\{f(t+1)\} = \frac{5}{s^2} + \frac{8}{s}$

Therefore, $L\{(5t+3)H(t-1)\} = e^{-s}L\{f(t+1)\} = e^{-s}L\{5t+8\} = e^{-s}\left(\frac{5}{s^2}+\frac{8}{s}\right)$

e) Here, $f(t-\pi) = \cos 2(t-\pi) \Rightarrow f(t) = \cos 2t \Rightarrow L\{f(t)\} = L\{\cos 2t\} = \frac{s}{s^2+4}$

Therefore, $L\{\cos 2(t-\pi)H(t-\pi)\} = e^{-\pi s}L\{f(t)\} = \frac{se^{-\pi s}}{s^2+4}$

f) Here, $a = 4, f(t) = e^{2t} \Rightarrow f(t+4) = e^{2(t+4)} = e^{2t}e^4 \Rightarrow L\{f(t+4)\} = \frac{e^4}{s-2}$

Therefore, $L\{e^{2t}H(t-4)\} = e^{-4s}L\{f(t+4)\} = e^{-4s}L\{e^{2t}e^4\} = \frac{e^4e^{-4s}}{s-2} = \frac{e^{4-4s}}{s-2}$

Laplace Transform of Piecewise Functions

First, let's see how any piecewise function is expressible in terms of step functions. Suppose $f(t) = \begin{cases} f_1(t), 0 \le t \le a \\ f_2(t), \quad t > a \end{cases}$ is any piecewise continuous function.

Then, $f(t)$ can be expressed in terms of step function as

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]U_a(t) = f_1(t)[1-U_a(t)] + f_2(t)U_a(t)$$

Laplace Transform:

$$L\{f(t)\} = L\{f_1(t) + [f_2(t) - f_1(t)]U_a(t)\} = L\{f_1(t)\} + L\{[f_2(t) - f_1(t)]U_a(t)\}$$

Examples: Express using step functions and find their Laplace Transforms.

a) $f(t) = \begin{cases} 3, 0 \le t < 1 \\ -1, t \ge 1 \end{cases}$ b) $f(t) = \begin{cases} t, 0 \le t < 2 \\ t+3, t \ge 2 \end{cases}$ c) $f(t) = \begin{cases} 0, 0 \le t < 1 \\ (t-1)^2, t \ge 1 \end{cases}$

d) $f(t) = \begin{cases} \sin t, 0 \le t < \pi/2 \\ \sin t + \cos(t-\pi/2), t \ge \pi/2 \end{cases}$ e) $f(t) = \begin{cases} 0, 0 \le t < 1 \\ t-1, 1 \le t < 8 \\ 7, t \ge 8 \end{cases}$

Solution:

$a)$ $f(t)=f_1(t)+[f_2(t)-f_1(t)]U_1(t)=3+[-1-3]u_1(t)=3-4u_1(t)$

Therefore, $L\{f(t)\}=L\{3-4u_1(t)\}=L\{3\}-4L\{u_1(t)\}=\dfrac{3}{s}-\dfrac{4e^{-s}}{s}$

$b)$ $f(t)=f_1(t)+[f_2(t)-f_1(t)]U_2(t)=t+[t+3-t]u_2(t)=t+3u_2(t)$

Therefore, $L\{f(t)\}=L\{t+3u_2(t)\}=L\{t\}+3L\{u_2(t)\}=\dfrac{1}{s^2}+\dfrac{3e^{-2s}}{s}$

$c)$ $f(t)=(t-1)^2u_1(t)\Rightarrow L\{f(t)\}=L\{(t-1)^2u_1(t)\}=e^{-s}L\{t^2\}=\dfrac{2e^{-s}}{s^3}$

$d)$ $f(t)=f_1(t)+[f_2(t)-f_1(t)]U_a(t)=\sin t+\cos(t-\pi/2)u_{\pi/2}(t)$

Therefore, $L\{f(t)\}=L\{\sin t\}+L\{\cos(t-\pi/2)u_{\pi/2}(t)\}=\dfrac{1+se^{-\pi s/2}}{s^2+1}$

$e)$ $f(t)=f_1(t)[U_0(t)-U_1(t)]+f_2(t)[U_1(t)-U_8(t)]+f_3(t)U_8(t)$

$=(t-1)[U_1(t)-U_8(t)]+7U_8(t)=(t-1)U_1(t)-(t-8)U_8(t)+7U_8(t)$

Therefore, $L\{f(t)\}=L\{(t-1)U_1(t)\}-L\{(t-8)U_8(t)\}=(e^{-s}-e^{-8s})/s^2$

Laplace Transform of functions of the form $\dfrac{f(t)}{t}$

If $L\{f(t)\}=F(s)$, then, $L\left\{\dfrac{f(t)}{t}\right\}=\int_s^{\infty}F(r)dr$.

Examples: Find the Laplace Transform of

$a)$ $\dfrac{\sin 2t}{t}$ $\quad b)$ $\dfrac{e^t-1}{t}$ $\quad c)$ $\dfrac{2\sin^2 2t}{t}$ $\quad d)$ $\dfrac{e^{-2t}}{t}$

Solution:

$a)$ Here, $f(t)=\sin 2t\Rightarrow F(s)=L\{f(t)\}=L\{\sin 2t\}=\dfrac{2}{s^2+4}\Rightarrow F(r)=\dfrac{2}{r^2+4}$

Then, $L\left\{\dfrac{\sin 2t}{t}\right\}=\int_s^{\infty}F(r)dr=\int_s^{\infty}\dfrac{2}{r^2+4}dr=\lim_{n\to\infty}\int_s^{n}\dfrac{2}{r^2+4}dr=\dfrac{\pi}{2}-\tan^{-1}\dfrac{s}{2}$

b) $f(t)=e^t-1 \Rightarrow F(s)=L\{f(t)\}=L\{e^t-1\}=\dfrac{1}{s-1}-\dfrac{1}{s} \Rightarrow F(r)=\dfrac{1}{r-1}-\dfrac{1}{r}$

Then, $L\left\{\dfrac{e^t-1}{t}\right\}=\displaystyle\int_s^\infty F(r)dr=\int_s^\infty\left(\frac{1}{r-1}-\frac{1}{r}\right)dr=\lim_{n\to\infty}\ln\left(\frac{r-1}{r}\right)\Bigg|_{r=s}^{r=n}$.

$$=\lim_{n\to\infty}\ln\left(\frac{r-1}{r}\right)-\ln\left(\frac{s-1}{s}\right)-\ln\left(\frac{s-1}{s}\right)=\ln\left(\frac{s}{s-1}\right)$$

c) $f(t)=2\sin^2 2t=2\left(\dfrac{1-\cos 4t}{2}\right)=2(1-\cos 4t) \Rightarrow F(s)=\dfrac{1}{s}-\dfrac{s}{s^2+16}$

Then, $L\left\{\dfrac{2\sin^2 2t}{t}\right\}=\displaystyle\int_s^\infty\left(\frac{1}{r}-\frac{r}{r^2+16}\right)dr=\lim_{n\to\infty}[\ln r-\frac{1}{2}\ln(r^2+16)]\Bigg|_{r=s}^{r=n}$

$$=\lim_{n\to\infty}[\ln\frac{n}{\sqrt{n^2+16}}-\ln\frac{s}{\sqrt{s^2+16}}]=\ln\frac{\sqrt{s^2+16}}{s}$$

d) $f(t)=e^{-2t} \Rightarrow F(s)=L\{e^{-2t}\}=\dfrac{1}{s+2} \Rightarrow F(r)=\dfrac{1}{r+2}$

$L\left\{\dfrac{e^{-2t}}{t}\right\}=\displaystyle\int_s^\infty\frac{1}{r+2}dr=\infty$. This means that $L\left\{\dfrac{e^{-2t}}{t}\right\}$ does not exist.

2.4 Inverse Laplace Transform and Their Properties

If $L\{f(t)\} = F(s)$, then the function f is said to be the inverse Laplace transform of $F(s)$ and is given by $f(t) = L^{-1}\{F(s)\}$.

Examples: $a)\ L\{1\} = \frac{1}{s} \Rightarrow L^{-1}\{\frac{1}{s}\} = 1 \qquad b)\ L\{e^{2t}\} = \frac{1}{s-2} \Rightarrow L^{-1}\{\frac{1}{s-2}\} = e^{2t}$

Properties of Inverse Laplace Transforms

Let $L\{f(t)\} = F(s),\ L\{g(t)\} = G(s)$ and let a be any constant. Then,

i) $L^{-1}\{aF(s)\} = aL^{-1}\{F(s)\} = af(t)$

ii) $L^{-1}\{F(s)+G(s)\} = L^{-1}\{F(s)\} + L^{-1}\{G(s)\} = f(t) + g(t)$

iii) $L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{F(s)\} = e^{at}f(t),\ L^{-1}\{F(s+a)\} = e^{-at}f(t)$

iv) $L^{-1}\{\frac{e^{-as}}{s}\} = U(t-a) = H(t-a)$

v) $L^{-1}\{e^{-as}F(s)\} = U(t-a)f(t-a)$

How to find Inverse Laplace Transforms?

Simple Cases: Using manipulations or rearrangements and then properties of Laplace Transforms and basic tables of transforms.

Examples-1: Observe the following inverse computations.

$$a)\ L^{-1}\{\frac{4+s}{s^2+1}\} = L^{-1}\{\frac{4}{s^2+1} + \frac{s}{s^2+1}\} = L^{-1}\{\frac{4}{s^2+1}\} + L^{-1}\{\frac{s}{s^2+1}\} = 4\sin t + \cos t$$

$$b)\ L^{-1}\{\frac{6}{s^2-2s+5}\} = L^{-1}\{\frac{6}{(s-1)^2+4}\} = 3L^{-1}\{\frac{2}{(s-1)^2+4}\} = 3e^t\sin 2t$$

$$c)\ L^{-1}\{\frac{s+3}{s^2+6s+13}\} = L^{-1}\{\frac{s+3}{(s+3)^2+4}\} = e^{-3t}\cos 2t$$

$$d)\ L^{-1}\{\frac{1+e^{-2s}}{s^4}\} = L^{-1}\{\frac{1}{s^4} + \frac{e^{-2s}}{s^4}\} = L^{-1}\{\frac{1}{s^4}\} + L^{-1}\{\frac{e^{-2s}}{s^4}\} = \frac{t^3}{6} + \frac{(t-2)^3}{6}H(t-2)$$

$$e)\ L^{-1}\left\{\frac{4s}{4s^2+1}\right\} = L^{-1}\left\{\frac{s}{s^2+(1/2)^2}\right\} = \cos\frac{1}{2}t$$

$$f)\ L^{-1}\left(\frac{s+7}{s^2+2s+5}\right) = L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) + L^{-1}\left(\frac{6}{(s+1)^2+4}\right) = e^{-t}\cos 2t + 3e^{-t}\sin 2t$$

General Cases: Inverse Laplace Transform by Partial Fractions

The tricky procedures used in the above examples to find inverse transforms are not useful for many cases. So, let's see the procedures to find inverse transform.

The procedure to find the inverse Laplace Transform of $F(s)=\dfrac{P(s)}{Q(s)}$

Step-1: Factorize the denominator $Q(s)$ completely.

Step-2: Decompose $F(s)=\dfrac{P(s)}{Q(s)}$ as sum of rational functions.

Step-3: Determine the constants in the decomposition.

Step-4: Take the Laplace Transform of each term separately.

In doing so, we may get different cases based on the factors of $Q(s)$

i) For each ***non-repeated linear factors*** like $s-a$ of $Q(s)$ assign a rational function of the form $\dfrac{A}{s-a}$ where A is a constant to be determined.

ii) For each ***repeated linear factors*** like $(s-a)^n$ assign a rational function of the form $\dfrac{A_0}{(s-a)^n}+\dfrac{A_1}{(s-a)^{n-1}}+\dfrac{A_2}{(s-a)^{n-2}}+\ldots+\dfrac{A_n}{s-a}$.

iii) For each ***non-repeated quadratic factors*** like as^2+bs+c assign a rational function of the form $\dfrac{As+B}{as^2+bs+c}$

Examples: Find the inverse Laplace Transforms.

$a)\ \dfrac{2s-7}{s^2-5s+6}$ $\quad b)\ \dfrac{18}{s^3-6s^2+9s}$ $\quad c)\ \dfrac{5s^2-6s+2}{(s-2)(s^2+1)}$ $\quad d)\ \dfrac{8s+10}{(s+1)(s+2)^3}$

Solution:

a) Here, $Q(s)=s^2-5s+6=(s-2)(s-3)$. Then,

$$\frac{2s-7}{s^2-5s+6}=\frac{2s-7}{(s-2)(s-3)}=\frac{A_1}{s-2}+\frac{A_2}{s-3}=\frac{A_1(s-3)+A_2(s-2)}{(s-2)(s-3)}$$

$$\Rightarrow A_1(s-3)+A_2(s-2)=2s-6\Rightarrow (A_1+A_2)s-3A_1-2A_2=2s-7$$

$$\Rightarrow A_1+A_2=2,-3A_1-2A_2=-7\Rightarrow A_1=3,\ A_2=-1$$

Hence, $L^{-1}\left\{\frac{2s-7}{s^2-5s+6}\right\}=L^{-1}\left\{\frac{3}{s-2}\right\}-L^{-1}\left\{\frac{1}{s-3}\right\}=3e^{2t}-e^{3t}$

b) Here, $Q(s)=s^3-6s^2+9s=s(s-3)^2$. Then,

$$\frac{18}{s(s-3)(s-3)}=\frac{A_1}{s}+\frac{B_1}{s-3}+\frac{B_2}{(s-3)^2}=\frac{A_1(s-3)(s-3)+B_1s(s-3)+B_2s}{s(s-3)(s-3)}$$

$$\Rightarrow A_1(s-3)(s-3)+B_1s(s-3)+B_2s=18$$

$$\Rightarrow A_1=2,\ B_1=-2,\ B_2=6$$

$$\therefore L^{-1}\left\{\frac{18}{s^3-6s^2+9s}\right\}=L^{-1}\left\{\frac{2}{s}\right\}-L^{-1}\left\{\frac{2}{s-3}\right\}+L^{-1}\left\{\frac{6}{(s-3)^2}\right\}=2-2e^{3t}+6te^{3t}$$

c) Here, we have mixed factors. So, the partial fraction decomposition is of the form

$$\frac{5s^2-6s+2}{(s-2)(s^2+1)}=\frac{A}{s-2}+\frac{Bs+C}{s^2+1}\Rightarrow A=2, B=3, C=0$$

Hence, $L^{-1}\left\{\frac{5s^2-6s+2}{(s-2)(s^2+1)}\right\}=L^{-1}\left\{\frac{2}{s-2}\right\}+L^{-1}\left\{\frac{3s}{s^2+1}\right\}=2e^{2t}+3\cos t$

d) Here, $\frac{8s+10}{(s+1)(s+2)^3}=\frac{A_1}{s+1}+\frac{B_1}{s+2}+\frac{B_2}{(s+2)^2}+\frac{B_3}{(s+2)^3}$

$$\Rightarrow A_1=2, B_1=-2, B_2=-2, B_3=6$$

$$\therefore L^{-1}\left\{\frac{8s+10}{(s+1)(s+2)^3}\right\}=L^{-1}\left\{\frac{2}{s+1}\right\}-L^{-1}\left\{\frac{2}{s+2}\right\}-L^{-1}\left\{\frac{2}{(s+2)^2}\right\}+L^{-1}\left\{\frac{6}{(s+2)^3}\right\}$$

$$=2e^{-t}-2e^{-2t}-2te^{-2t}+3t^2e^{-2t}$$

Inverse using Second -Shfting Rule: Suppose $L\{f(t)\}=F(s)$.

Then, $L\{f(t-a)u_a(t)\}=e^{-as}F(s)\Leftrightarrow L^{-1}\{e^{-as}F(s)\}=f(t-a)u(t-a)$. We use this property to compute inerse Laplace Transforms of the form $L^{-1}\{e^{-as}F(s)\}$. To use this rule, first identify $F(s)$ and e^{-as} from the given problem, second find $f(t)$ using $f(t)=L^{-1}\{F(s)\}$, finally use the rule.

Examples: Find the inverse Laplace Transform of

a) $\frac{e^{-2s}}{s(s^2+3s+2)}$ b) $\frac{e^{-3s}-e^{-7s}}{s^2-4s+3}$ c) $\frac{e^{-3s}}{(s+1)^2+1}$ d) $\frac{4s+3e^{-20s}}{s^2-9}$

Solution:

a) Here, $\frac{e^{-2s}}{s(s^2+3s+2)} = e^{-2s}.\frac{1}{s(s^2+3s+2)} = e^{-2s}F(s)$ where

$$F(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \Rightarrow A = \frac{1}{2}, B = -1, C = \frac{1}{2}$$

Thus, $f(t) = L^{-1}\left\{\frac{1/2}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1/2}{s+2}\right\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

Thus, $L^{-1}\{e^{-2s}F(s)\} = f(t-2)u(t-2) = (\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)})u(t-2)$

b) Here, $\frac{e^{-3s} - e^{-7s}}{s^2-4s+3} = e^{-3s}.\frac{1}{s^2-4s+3} - e^{-7s}.\frac{1}{s^2-4s+3} = e^{-3s}F(s) - e^{-7s}F(s)$

$$F(s) = \frac{1}{s^2-4s+3} = \frac{1}{(s-3)(s-1)} = \frac{A}{s-3} + \frac{B}{s-1} \Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}$$

$$\Rightarrow f(t) = L^{-1}\left\{\frac{1}{s^2-4s+3}\right\} = L^{-1}\left\{\frac{1/2}{s-3}\right\} + L^{-1}\left\{\frac{-1/2}{s-1}\right\} = \frac{1}{2}e^{3t} - \frac{1}{2}e^{t}$$

Thus, $L^{-1}\{e^{-3s}F(s) - e^{-7s}F(s)\} = L^{-1}\{e^{-3s}F(s)\} - L^{-1}\{e^{-7s}F(s)\}$

$$= f(t-3)U(t-3) - f(t-7)u(t-7)$$

c) Here, $\frac{e^{-3s}}{(s+1)^2+1} = e^{-3s}.\frac{1}{(s+1)^2+1} = e^{-3s}F(s)$ where

$$\mathrm{F(s) = L\{f(t)\}} = \frac{1}{(s+1)^2+1} \Rightarrow f(t) = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}\sin t$$

Thus, $L^{-1}\{e^{-3s}F(s)\} = f(t-3)u(t-3) = e^{3-t}\sin(t-3)u(t-3)$

d) Here, $\frac{4s+3e^{-20s}}{s^2-9} = \frac{4s}{s^2-9} + e^{-20s}.\frac{3}{s^2-9} = 4G(s) + e^{-20s}F(s)$ where

$$g(t) = L^{-1}\left\{\frac{s}{s^2-9}\right\} = \cosh 3t,\ f(t) = L^{-1}\left\{\frac{3}{s^2-9}\right\} = \sinh 3t$$

Thus, $L^{-1}\{4G(s) + e^{-20s}F(s)\} = 4L^{-1}\{G(s)\} + L^{-1}\{e^{-20s}F(s)\}$

$$= 4\cosh 3t + \sinh 3(t-20)u(t-20)$$

2.5 Convolution Product and Laplace Transforms

Definition: Suppose f and g are continuous functions on $[a,b]$. Then for $t \in [a,b]$, the convolution product of f and g, denoted by $f*g$, is defined as $(f*g)(t)=\int_0^t f(t-x)g(x)dx$.

Example:

Let $f(t)=3, g(t)=e^t$. Then, $(f*g)(t)=\int_0^t f(t-x)g(x)dx=\int_0^t 3e^x dx=3e^t-3$

Properties of convolution product

i) $f*g=g*f$ *ii)* $f*(g+h)=f*g+f*h$ *iii)* $f*(g*h)=(f*g)*h$

Convolution Theorem: If $L\{f(t)\}=F(s)$ and $L\{g(t)\}=G(s)$, then

i) $L\{(f*g)(t)\}=L\{f(t)\}L\{g(t)\}=F(s)G(s)$ ii) $L^{-1}\{F(s)G(s)\}=(f*g)(t)$

Examples:

Find $L\{(f*g)(t)\}$ using Convolution Theorem where

a) $f(t)=1, g(t)=t^4$ *b)* $f(t)=t^2, g(t)=te^t$ *c)* $f(t)=e^{3t}, g(t)=\sin 2t$

Solution: First find $L\{f(t)\}$ and $L\{g(t)\}$ so as to apply Convolution Theorem.

a) $L\{f(t)\}=L\{1\}=\frac{1}{s}, L\{g(t)\}=L\{t^4\}=\frac{24}{s^5}$

$$\Rightarrow L\{(f*g)(t)\}=L\{f(t)\}.L\{g(t)\}=L\{1*t^4\}=L\{1\}.L\{t^4\}=\frac{24}{s^6}$$

b) $L\{f(t)\}=L\{t^2\}=\frac{2}{s^3}, L\{g(t)\}=L\{te^t\}=\frac{1}{(s-1)^2}$

$$\Rightarrow L\{(f*g)(t)\}=L\{t^2*te^t\}=L\{t^2\}.L\{te^t\}=\frac{2}{s^3}.\frac{1}{(s-1)^2}=\frac{2}{s^3(s-1)^2}$$

c) $L\{f(t)\}=L\{e^{3t}\}=\frac{1}{s-3}, L\{g(t)\}=L\{\sin 2t\}=\frac{2}{s^2+4}$

$$\Rightarrow L\{(f*g)(t)\}=L\{e^{3t}*\sin 2t\}=L\{e^{3t}\}L\{\sin 2t\}=\frac{2}{(s-3)(s^2+4)}$$

2.6 Laplace Transform of Derivatives and Integrals

2.6.1 Laplace Transform of Derivatives

If $F(s)$ is the Laplace transform of f and if f has a value $f(0)$, when t=0, then $L\{f'(t)\}=sF(s)-f(0)$, $L\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$.

In general, $L\left[f^{(n)}(t)\right]=s^nL[f(t)]-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0)$ where $f^{(n)}(0)$ is the value of the n^{th} derivative of $f(t)$ at $t=0$.

Example: Given f with $f''(t)-6f'(t)+5f(t)=0, f(0)=3, f'(0)=7$. Find the Laplace Transform of f and deduce $f(t)$ itself.

Solution: Take transform both sides.

$$L\{f''(t)-6f'(t)+5f(t)\}=0$$

$$\Rightarrow L\{f''(t)\}-6L\{f'(t)\}+5L\{f(t)\}=0$$

$$\Rightarrow s^2L\{f(t)\}-sf(0)-f'(0)-6[sL\{f(t)\}-f(0)]+5L\{f(t)\}=0$$

$$\Rightarrow s^2L\{f(t)\}-3s-7-6sL\{f(t)\}+18+5L\{f(t)\}=0$$

$$\Rightarrow (s^2-6s+5)L\{f(t)\}=3s-11$$

$$\Rightarrow L\{f(t)\}=\frac{3s-11}{s^2-6s+5}$$

Now let's deduce $f(t)$ itself by using inverse transform. That is

$$L\{f(t)\}=\frac{3s-11}{s^2-6s+5}$$

$$\Rightarrow f(t)=L^{-1}\left\{\frac{3s-11}{s^2-6s+5}\right\}=L^{-1}\left\{\frac{1}{s-5}+\frac{2}{s-1}\right\}$$

$$=L^{-1}\left\{\frac{1}{s-5}\right\}+L^{-1}\left\{\frac{2}{s-1}\right\}=e^{5t}+2e^{t}$$

2.6.2 Laplace Transform of Integrals

Suppose $f(t)$ is piece wise continuous for all $t \ge 0$ and $L\{f(t)\} = F(s)$.

Then, $L\left\{\int_0^t f(x)dx\right\} = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$ and thus $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(x)dx$.

Examples:

1. Find the following Laplace Transforms

a) $L\left\{\int_0^t \cos 3x dx\right\}$ b) $L\left\{\int_0^t x^2e^{2x}dx\right\}$ c) $L\left\{\int_0^t x\sin x dx\right\}$ d) $L\left\{\int_0^t e^x \cos x dx\right\}$

Solution:

a) $L\left\{\int_0^t \cos 3x dx\right\} = \frac{1}{s}L\{\cos 3t\} = \frac{1}{s}\cdot\frac{s}{s^2+9} = \frac{1}{s^2+9}$

b) $L\left\{\int_0^t x^2e^{2x}dx\right\} = \frac{1}{s}L\{t^2e^{2t}\} = \frac{1}{s}\frac{d^2}{ds^2}(L\{e^{2t}\}) = \frac{1}{s}\frac{d^2}{ds^2}(\frac{1}{s-2}) = \frac{2}{s(s-2)^3}$

c) $L\left\{\int_0^t x\sin x dx\right\} = \frac{1}{s}L\{t\sin t\} = -\frac{1}{s}\cdot\frac{-2s}{(s^2+1)^2} = \frac{2}{(s^2+1)^2}$

d) $L\left\{\int_0^t e^x \cos x dx\right\} = \frac{1}{s}L\{e^t\cos t\} = \frac{1}{s}\cdot\frac{s-1}{(s-1)^2+1} = \frac{s-1}{s[(s-1)^2+1]}$

2*. Find the Laplace Transform of the following functions.

a) $f(t) = e^{-6t}\int_0^t x^2e^{2x}dx$ b) $f(t) = 3t\int_0^t x\sin 2x dx$ c) $f(t) = \int_0^t \frac{\sin ax}{x}dx$

Solution: Transform step by step.

a) First using *t-shifting*, $L\{t^2e^{2t}\} = \frac{d^2}{ds^2}(L\{e^{2t}\}) = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$

Second, using Laplace Transform of integrals, find $L\left\{\int_0^t x^2e^{2x}dx\right\}$.

That is $F(s) = L\left\{\int_0^t x^2e^{2x}dx\right\} = \frac{1}{s}L\{t^2e^{2t}\} = \frac{1}{s}\cdot\frac{2}{(s-2)^3} = \frac{2}{s(s-2)^3}$

Then, by *S-shifting*, $L\left\{e^{-6t}\int_0^t x^2e^{2x}\,dx\right\} = F(s+6) = \dfrac{2}{(s+6)(s+4)^3}$

b) Using *t-shifting*, $L\{t\sin 2t\} = -\dfrac{d}{ds}(L\{\sin 2t\}) = -\dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right) = \dfrac{4s}{(s^2+4)^2}$

Second, using Laplace Transform of integrals, find $L\{\int_0^t x\sin 2x\,dx\}$.

That is $F(s) = L\{\int_0^t x\sin 2x\,dx\} = \dfrac{1}{s}L\{t\sin 2t\} = \dfrac{1}{s}\cdot\dfrac{4s}{(s^2+4)^2} = \dfrac{4}{(s^2+4)^2}$

Finally, by *t-shifting*,

$$L\{f(t)\} = L\{3t\int_0^t x\sin 2x\,dx\} = -3\frac{d}{ds}\left(\frac{4}{(s^2+4)^2}\right) = \frac{48s}{(s^2+4)^3}$$

c) First find the transform of $\dfrac{\sin ax}{x}$. That is $L\{\dfrac{\sin ax}{x}\} = \int_s^\infty F(r)\,dr$ where

$$F(r) = L\{\sin ax\} = \frac{a}{r^2+a^2}.$$

So, $L\{\dfrac{\sin ax}{x}\} = \displaystyle\int_s^\infty \frac{a}{r^2+a^2}\,dr = \lim_{n\to\infty}\tan^{-1}\frac{r}{a}\Big|_{r=s}^{r=n} = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$

Therefore, $L\{f(t)\} = L\{\displaystyle\int_0^t \frac{\sin ax}{x}\,dx\} = \frac{1}{s}L\{\frac{\sin ax}{x}\} = \frac{\pi}{2s} - \frac{1}{s}\tan^{-1}\frac{s}{a}$

3. Given $L\{f(t)\} = \dfrac{se^{-6s}}{s^4+2s^2+1}$, $\displaystyle\int_0^2 f(x)\,dx = 8$, $g(t) = \displaystyle\int_2^t f(x)\,dx$. Find $L\{g(t)\}$.

Solution:

$$L\{g(t)\} = L\{\int_2^t f(x)\,dx\} = L\{\int_0^t f(x)\,dx - \int_0^2 f(x)\,dx\}$$

$$= L\{\int_0^t f(x)\,dx\} - L\{\int_0^2 f(x)\,dx\}$$

$$= \frac{1}{s}L\{f(t)\} - L\{8\} = \frac{1}{s}L\{f(t)\} - \frac{8}{s} = \frac{e^{-6s}}{s^4+2s^2+1} - \frac{8}{s}$$

2.6 Applications of Laplace Transform

Laplace Transform is widely applicable to solve many real life problems. Especially it is useful to solve Initial Value Problems, Integral-Equations, System of Differential Equations. Here, let's see some applications.

I) Initial value problems: Here, let's see how Laplace transform can be used to solve initial value problems for linear differential equations.

Procedures to solve IVPs using Laplace Transform:

Given the IVP: $ay''+by'+cy=f(x), y(0)=m, y'(0)=n$.

Then, to find $y(t)$ satisfying the IVP, we use the following procedures.

Step-1: Take Laplace Transform of the DE both sides.

That is $L\{ay''+by'+cy\}=L\{f(x)\} \Rightarrow aL\{y''\}+bL\{y'\}+cL\{y\}=y(s)$

At this step use Laplace transform of derivatives and solve for $y(s)$ by substituting the given initial conditions $y(0)=m,\ y'(0)=n$.

Step-2: Solve for $y(t)$ by taking inverse Laplace Transform of $Y(s)$

Take inverse Laplace Transform of $Y(s)$. That is $y(t)=L^{-1}\{Y(s)\}$

At this step, apply inverse finding techniques to solve $L^{-1}\{Y(s)\}$ for $y(t)$

Examples:

1. Solve the following initial value problem using Laplace Transforms.

a) $y'+3y=e^{t}$, $y(0)=1$

b) $y''-2y'-3y=3e^{2t}$, $y(0)=0, y'(0)=5$

c) $y''+4y=24e^{2t}$, $y(0)=0, y'(0)=0$

d) $y''+y'-2y=9te^{2t}$, $y(0)=y'(0)=0$

e) $y''-3y'+2y=6e^{-t}$, $y(0)=0, y'(0)=0$

f) $y''+2y'+5y=e^{-t}\sin t$, $y(0)=0, y'(0)=1$

g) $y''+4y'+5y=25t^{2}$, $y(0)=0, y'(0)=1$

h) $y''-2y'+y=t^{3}e^{t}$, $y(0)=0,\ y'(0)=0$

i) $y''+2y'+y=te^{-2t}$, $y(0)=1,\ y'(0)=1$

Solution:

a) **Step-1:** Determine the Laplace Transform $Y(s)$ of $y(t)$. That is take Laplace Transform both sides and solve for $Y(s)$ by substituting initial conditions.

$$L\{y'+3y\} = L\{e^t\} \Rightarrow L\{y'\} + L\{3y\} = sy(s) - y(0) + 3y(s) = \frac{1}{s-1}$$

$$\Rightarrow sy(s) - 1 + 3y(s) = \frac{1}{s-1} \Rightarrow sy(s) + 3y(s) = \frac{1}{s-1} + 1$$

$$\Rightarrow y(s)(s+3) = \frac{s}{s-1} \Rightarrow y(s) = \frac{s}{(s+3)(s-1)}$$

Step-2: Solve for $y(t)$ from inverse Laplace Transform of $Y(s)$.

That is $y(t) = L^{-1}\{y(s)\} = L^{-1}\left(\frac{s}{(s+3)(s-1)}\right)$

Apply inverse finding techniques to solve $L^{-1}\{Y(s)\}$ for $y(t)$

That is find the inverse transform $L^{-1}\left(\frac{s}{(s+3)(s-1)}\right)$.

To determine the inverse, let's use Partial Fraction Decomposition.

Here, $y(s) = \frac{s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$ and determine A, B and C.

Now, determine the constants A, B and C.

By using Cover-Up Method, at $s = -3$, we have $A = \frac{3}{4}$ and at $s = 1$, $B = \frac{1}{4}$.

So, the decomposition is $y(s) = \frac{s}{(s+3)(s-1)} = \frac{\frac{3}{4}}{s+3} + \frac{\frac{1}{4}}{s-1}$.

Therefore, the function $y(t)$ is determined as

$$y(t) = L^{-1}\left(\frac{s}{(s+3)(s-1)}\right) = L^{-1}\left(\frac{\frac{3}{4}}{s+3} + \frac{\frac{1}{4}}{s-1}\right)$$

$$= \frac{3}{4}L^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{4}L^{-1}\left(\frac{1}{s-1}\right) = \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

b) **Step-1:** Determine the Laplace Transform $Y(s)$ of $y(t)$. That is take Laplace Transform both sides and solve for $Y(s)$ by substituting initial conditions.

$$L\{y''-2y'-3y\} = L\{3e^{2t}\} \Rightarrow L\{y''\} - L\{2y'\} + L\{-3y\} = \frac{3}{s-2}$$

$$\Rightarrow s^2y(s) - sy(0) - y'(0) - 2[sy(s) - y(0)] - 3y(s) = \frac{3}{s-2}$$

$$\Rightarrow s^2y(s) - 5 - 2sy(s) - 3y(s) = \frac{3}{s-2} \Rightarrow (s^2-2s-3)y(s) = \frac{3}{s-2} + 5$$

$$\Rightarrow y(s) = \frac{5s-7}{(s^2-2s-3)(s-2)} = \frac{5s-7}{(s+1)(s-3)(s-2)}.$$

Hence, the transform of $y(t)$ is found to be $y(s) = \dfrac{5s-7}{(s+1)(s-3)(s-2)}$.

Step-2: Solve for $y(t)$ by inverse Laplace Transform of $Y(s)$.

That is $y(t) = L^{-1}\{y(s)\} = L^{-1}\left(\dfrac{5s-7}{(s-3)(s+1)(s-2)}\right)$

To evaluate the inverse let's use Partial Fraction Decomposition.

Here, $y(s) = \dfrac{5s-7}{(s-3)(s+1)(s-2)} = \dfrac{A}{s-3} + \dfrac{B}{s+1} + \dfrac{C}{s-2}$.

By using cover-up method, at $s = 3$, we have $A = \dfrac{15-7}{(4)(1)} = \dfrac{8}{4} = 2$.

At $s = -1$, $B = \dfrac{-5-7}{-4(-3)} = -\dfrac{12}{12} = -1$ and at $s = 2$, we have $C = \dfrac{10-7}{-3} = -1$.

So, the decomposition is $y(s) = \dfrac{5s-7}{(s-3)(s+1)(s-2)} = \dfrac{2}{s-3} - \dfrac{1}{s+1} - \dfrac{1}{s-2}$

Therefore, $y(t)=L^{-1}\left(\dfrac{5s-7}{(s-3)(s+1)(s-2)}\right)=L^{-1}\left(\dfrac{2}{s-3}-\dfrac{1}{s+1}-\dfrac{1}{s-2}\right)$

$$=2L^{-1}\left(\frac{1}{s-3}\right)-L^{-1}\left(\frac{1}{s+1}\right)-L^{-1}\left(\frac{1}{s-2}\right)$$

$$=2e^{3t}-e^{-t}-e^{2t}$$

c) $L\{y''+4y\}=L\{24e^{2t}\}\Rightarrow L\{y''\}+L\{4y\}=24L\{e^{2t}\}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)+4y(s)=\frac{24}{s-2}\Rightarrow s^2y(s)+4y(s)=\frac{24}{s-2}$$

$$\Rightarrow (s^2+4)y(s)=\frac{24}{s-2}\Rightarrow y(s)=\frac{24}{(s^2+4)(s-2)}$$

But $\dfrac{24}{(s^2+4)(s-2)}=\dfrac{As+B}{s^2+4}+\dfrac{C}{s-2}\Rightarrow A=-3,B=-6,C=3$

So, we get $y(s)=\dfrac{-3s-6}{s^2+4}+\dfrac{3}{s-2}=-\dfrac{3s}{s^2+4}-\dfrac{6}{s^2+4}+\dfrac{3}{s-2}$

Therefore, $y(t)=L^{-1}\left\{-\dfrac{3s}{s^2+4}\right\}+L^{-1}\left\{-\dfrac{6}{s^2+4}\right\}+L^{-1}\left\{\dfrac{3}{s-2}\right\}$

$$=-3L^{-1}\left\{\frac{s}{s^2+4}\right\}-3L^{-1}\left\{\frac{2}{s^2+4}\right\}+3L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$=-3\cos 2t-3\sin 2t+3e^{2t}$$

d) $L\{y''+y'-2y\}=L\{8te^{2t}\}$

$$\Rightarrow L\{y''\}+L\{y'\}-2L\{y\}=8L\{te^{2t}\}$$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)+sy(s)-y(0)-2y(s)=\frac{8}{(s-2)^2}$$

$$\Rightarrow (s^2+s-2)y(s)=\frac{8}{(s-2)^2}\Rightarrow y(s)=\frac{8}{(s-2)^2(s^2+s-2)}$$

Now, using partial fraction decomposition,

$$y(s)=\frac{8}{(s-2)^2(s-1)(s+2)}=\frac{a}{s-2}+\frac{b}{(s-2)^2}+\frac{c}{s-1}+\frac{d}{s+2}$$

$$\Rightarrow a=-\frac{5}{2}, b=2, c=\frac{8}{3}, d=-\frac{1}{6}$$

Therefore, $y(t)=-\frac{5}{2}L^{-1}\left(\frac{1}{s-2}\right)+2L^{-1}\left(\frac{1}{(s-2)^2}\right)+\frac{8}{3}L^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{6}L^{-1}\left(\frac{s}{s+2}\right)$

$$=-\frac{5}{2}e^{2t}+2te^{2t}+\frac{8}{3}e^{t}-\frac{1}{6}e^{-2t}$$

e) **Step-1:** Determine the Laplace Transform $y(s)$.

$L\{y''-3y'+2y\}=L\{6e^{-t}\}\Rightarrow L\{y''\}-3L\{y'\}+2L\{y\}=6L\{e^{-t}\}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)-3[sy(s)-y(0)]+2y(s)=\frac{6}{s+1}$$

$$\Rightarrow s^2y(s)-3sy(s)+2y(s)=\frac{6}{s+1}\Rightarrow (s^2-3s+2)y(s)=\frac{6}{s+1}$$

$$\Rightarrow y(s)=\frac{6}{(s^2-3s+2)(s+1)}=\frac{6}{(s-2)(s-1)(s+1)}$$

Step-2: Solve for $y(t)$ from $y(t)=L^{-1}\{y(s)\}$

To evaluate the inverse let's use Partial Fraction Decomposition.

Here, $\frac{6}{(s-2)(s-1)(s+1)}=\frac{A}{s-2}+\frac{B}{s-1}+\frac{C}{s+1}$.

Then, by using cover-up method, find the constants A, B and C.

At $s=2$, we have $A=2$, at $s=1, B=-3$ and at $s=-1, C=1$.

So, the decomposition is $y(s)=\frac{6}{(s-2)(s-1)(s+1)}=\frac{2}{s-2}-\frac{3}{s-1}+\frac{1}{s+1}$.

Therefore, $y(t)=L^{-1}\left\{\frac{6}{(s-2)(s-1)(s+1)}\right\}=L^{-1}\left\{\frac{2}{s-2}-\frac{3}{s-1}+\frac{1}{s+1}\right\}$

$$=2L^{-1}\left\{\frac{1}{s-2}\right\}-3L^{-1}\left\{\frac{1}{s-1}\right\}+L^{-1}\left\{\frac{1}{s+1}\right\}=2e^{2t}-3e^{t}+e^{-t}$$

f) $L\{y''+2y'+5y\}=L\{e^{-t}\sin t\}\Rightarrow L\{y''\}+L\{2y'\}+L\{5y\}=L\{e^{-t}\sin t\}$

$$\Rightarrow (s^2+2s+5)y(s)=\frac{s^2+2s+3}{(s+1)^2+1}\Rightarrow y(s)=\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

By partial fraction,

$$\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}=\frac{As+B}{s^2+2s+5}+\frac{Cs+D}{s^2+2s+2}\Rightarrow A=C=0,B=\frac{2}{3},D=\frac{1}{3}$$

So, $y(t)=y(t)=\frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=\frac{1}{3}e^{-t}(\sin 2t+\sin t)$

g) $L\{y''+4y'+5y\}=L\{25t^2\}\Rightarrow L\{y''\}+4L\{y'\}+5L\{y\}=25L\{t^2\}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)+4[sy(s)-y(0)]+5y(s)=\frac{50}{s^3}$$

$$\Rightarrow (s^2+4s+5)y(s)=\frac{50}{s^3}+1\Rightarrow y(s)=\frac{50}{s^3(s^2+4s+5)}+\frac{1}{s^2+4s+5}$$

But by partial fraction decomposition, we have

$$\frac{50}{s^3(s^2+4s+5)}=\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}+\frac{Ds+E}{s^2+4s+5}$$

$$\Rightarrow A=10,B=-8,C=\frac{22}{5},D=-\frac{22}{5},E=-\frac{48}{5}$$

So, $y(t)=L^{-1}\left(\frac{50}{s^3(s^2+4s+5)}\right)+L^{-1}\left(\frac{1}{s^2+4s+5}\right)$

$$=10L^{-1}\left(\frac{1}{s^3}\right)-8L^{-1}\left(\frac{1}{s^2}\right)+\frac{22}{5}L^{-1}\left(\frac{1}{s}\right)-\frac{22}{5}L^{-1}\left(\frac{s}{s^2+4s+5}\right)-\frac{43}{5}L^{-1}\left(\frac{1}{s^2+4s+5}\right)$$

$$=10L^{-1}\left(\frac{1}{s^3}\right)-8L^{-1}\left(\frac{1}{s^2}\right)+\frac{22}{5}L^{-1}\left(\frac{1}{s}\right)-\frac{22}{5}L^{-1}\left(\frac{s+2-2}{(s+2)^2+1}\right)-\frac{43}{5}L^{-1}\left(\frac{1}{(s+2)^2+1}\right)$$

$$=5t^2-8t+\frac{22}{5}-\frac{22}{5}(e^{-2t}\cos t-2e^{-2t}\sin t)-\frac{43}{5}e^{-2t}\sin t \quad \text{(How?)}$$

h) $L\{y''-2y'+y\}=L\{t^3e^t\}\Rightarrow L\{y''\}-2L\{y'\}+L\{y\}=L\{t^3e^t\}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)-2[sy(s)-y(0)]+y(s)=\frac{6}{(s-1)^4}$$

$$\Rightarrow (s^2-2s+1)y(s)=\frac{6}{(s-1)^4}\Rightarrow y(s)=\frac{6}{(s^2-2s+1)(s-1)^4}$$

$$\Rightarrow y(s)=\frac{6}{(s-1)^2(s-1)^4}=\frac{6}{(s-1)^6}$$

Therefore, using s-shifting and power properties of Laplace transform, we have $y(t)=L^{-1}\left(\frac{6}{(s-1)^6}\right)=6L^{-1}\left(\frac{1}{(s-1)^6}\right)=\frac{6}{120}t^5e^t=\frac{1}{20}t^5e^t$

i) $L\{y''+2y'+y\}=L\{te^{-2t}\}\Rightarrow L\{y''\}+2L\{y'\}+L\{y\}=L\{te^{-2t}\}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)+2[sy(s)-y(0)]+y(s)=\frac{1}{(s+2)^2}$$

$$\Rightarrow (s^2+2s+1)y(s)=\frac{1}{(s+2)^2}+s+3\Rightarrow y(s)=\frac{s^3+7s^2+16s+12}{(s+1)^2(s+2)^2}$$

Continue with partial fraction and complete it.

2. Using ***Method of Laplace Transform***, solve

a) $y''-y=te^t,\ y(0)=y'(0)=1$ *b*) $y''-3y'-4y=t^2,\ y(0)=2,\ y'(0)=1$

c) $y''+4y'+3y=e^{-t},\ y(0)=y'(0)=1$ *d*) $y''+9y=\sin 2t,\ y(0)=7,\ y'(0)=-3$

e) $y''+y'-2y=te^{2t},\ y(0)=0,\ y'(0)=0$ *f*) $y''-2y'-3y=e^{-3t},\ y(0)=0,\ y'(0)=0$

Solution;

a) Taking Laplace transform on both sides, we get

$$L\{y''-y\}=L\{te^t\}\Rightarrow s^2y(s)-sy(0)-y'(0)-y(s)=\frac{1}{(s-1)^2}$$

$$\Rightarrow y(s)=\frac{s+1}{s^2-1}+\frac{1}{(s-1)^2(s^2-1)}=\frac{1}{s-1}+\frac{1}{(s+1)(s-1)^3}$$

But by Partial Fraction decomposition,

$$\frac{1}{(s+1)(s-1)^3}=\frac{A}{s+1}+\frac{B}{(s-1)^3}+\frac{C}{(s-1)^2}+\frac{D}{s-1}\Rightarrow A=-\frac{1}{8},B=\frac{1}{2},C=-\frac{1}{4},D=\frac{1}{8}$$

So, $y(t)=\frac{9}{8}e^{t}-\frac{1}{8}e^{-t}+\frac{1}{4}t^{2}e^{t}-\frac{1}{4}te^{t}$

b) $L\{y''-3y'-4y\}=L\{t^{2}\}\Rightarrow s^{2}y(s)-sy(0)-y'(0)-3[sy(s)-y(0)]-4y(s)=\frac{2}{s^{3}}$

$$\Rightarrow (s^{2}-3s-4)y(s)=2s-5+\frac{2}{s^{3}}\Rightarrow y(s)=\frac{2s-5}{(s-4)(s+1)}+\frac{2}{s^{3}(s-4)(s+1)}$$

But by Partial Fraction decomposition,

$$\frac{2s-5}{(s-4)(s+1)}=\frac{A}{s-4}+\frac{B}{s+1}\Rightarrow A=\frac{3}{5},B=\frac{7}{5}$$

$$\frac{2}{s^{3}(s-4)(s+1)}=\frac{C}{s-4}+\frac{D}{s+1}+\frac{E}{s^{3}}+\frac{F}{s^{2}}+\frac{G}{s}$$

$$\Rightarrow C=\frac{1}{160},D=\frac{2}{5},E=-\frac{1}{2},F=\frac{3}{8},G=-\frac{13}{32}$$

Thus, $y(t)=\frac{97}{160}e^{4t}+\frac{9}{5}e^{-t}-\frac{1}{4}t^{2}+\frac{3}{8}t-\frac{13}{32}$

c) Taking Laplace transform both sides, we get

$L\{y''+4y'+3y\}=L\{e^{-t}\}\Rightarrow L\{y''\}+L\{4y'\}+L\{3y\}=L\{e^{-t}\}$

$$\Rightarrow (s^{2}+4s+3)y(s)=\frac{s^{2}+6s+6}{s+1}\Rightarrow y(s)=\frac{s^{2}+6s+6}{(s+1)^{2}(s+3)}$$

But $\frac{s^{2}+6s+6}{(s+1)^{2}(s+3)}=\frac{A}{s+1}+\frac{B}{(s+1)^{2}}+\frac{C}{s+3},\ A=\frac{7}{4},\ B=\frac{1}{2},\ C=-\frac{3}{4}$

So, $y=\frac{7}{4}L^{-1}\left(\frac{1}{s+1}\right)+\frac{1}{2}L^{-1}\left(\frac{1}{(s+1)^{2}}\right)-\frac{3}{4}L^{-1}\left(\frac{1}{s+3}\right)=\frac{7}{4}e^{-t}-\frac{3}{4}e^{-3t}+\frac{1}{2}te^{-t}$

d) $L\{y''+9y\}=L\{\sin 2t\}\Rightarrow y(s)=\frac{2}{(s^{2}+4)(s^{2}+9)}+\frac{7s-3}{s^{2}+9}$

$$\Rightarrow y(s)=\frac{2/5}{s^{2}+4}-\frac{2/5}{s^{2}+9}+\frac{7s-3}{s^{2}+9}=\frac{2/5}{s^{2}+4}-\frac{17/5}{s^{2}+9}+\frac{7s}{s^{2}+9}$$

$$\Rightarrow y(t)=\frac{1}{5}\sin 2t-\frac{17}{15}\sin 3t+7\cos 3t$$

e) $L\{y''\}+L\{y'\}-2L\{y\}=L\{te^{2t}\}\Rightarrow L\{y\}=\dfrac{1}{(s-2)^2(s^2+s-2)}$

$$\frac{1}{(s-2)^2(s^2+s-2)}=\frac{a}{s-2}+\frac{b}{(s-2)^2}+\frac{c}{s-1}+\frac{d}{s+2}$$

$$\Rightarrow d=-\frac{1}{48},c=\frac{1}{3},b=\frac{1}{4},a=-\frac{5}{16}$$

Thus, $y=L^{-1}\left[\dfrac{-5/16}{s-2}+\dfrac{1/4}{(s-2)^2}+\dfrac{1/3}{s-1}-\dfrac{1/48}{s+2}\right]=-\dfrac{5}{16}e^{2t}+\dfrac{1}{4}te^{2t}+\dfrac{1}{3}e^{t}-\dfrac{1}{48}e^{-2t}$

3. Using ***Method of Laplace Transform***, solve the following IVPs.

a) $y''-y=f(t), y(0)=1, y'(0)=0$ where $f(t)=\begin{cases}1, & 0\le t<1\\ 0, & t>1\end{cases}$

b) $y''+9y=f(t), y(0)=0, y'(0)=0$ where $f(t)=\begin{cases}1, & 0\le t<2\\ 0, & t\ge 2\end{cases}$

c) $y''+y=f(t), y(0)=0, y'(0)=0$ where $f(t)=\begin{cases}t, & 0\le t<\pi\\ 0, & t>\pi\end{cases}$

d) $y'-2y=\dfrac{\sin t}{t}, y(0)=1$

Solution:

a) First observe that using unti step function $f(t)=1-u(t-1)$.

$$L\{f(t)\}=L\{1-u(t-1)\}=L\{1\}\}-L\{u(t-1)\}=\frac{1}{s}-\frac{e^{-s}}{s}$$

Then, $L\{y''-y\}=L\{f(t)\}\Rightarrow L\{y''\}-L\{y\}=L\{f(t)\}=\dfrac{1}{s}-\dfrac{e^{-s}}{s}$

$$\Rightarrow s^2y(s)-sy(0)-y'(0)-L(y)=\frac{1}{s}-\frac{e^{-s}}{s}$$

$$\Rightarrow (s^2-1)y(s)-s=\frac{1}{s}-\frac{e^{-s}}{s}\Rightarrow y(s)=\frac{s}{s^2-1}+\frac{1}{s(s^2-1)}-\frac{e^{-s}}{s(s^2-1)}$$

$$\Rightarrow y(t)=L^{-1}\left(\frac{s}{s^2-1}\right)+L^{-1}\left(\frac{1}{s(s^2-1)}\right)-L^{-1}\left(\frac{e^{-s}}{s(s^2-1)}\right)$$

Hence, $y(t) = L^{-1}\left(\frac{s}{s^2-1}\right) + L^{-1}\left(\frac{1}{s(s^2-1)}\right) - L^{-1}\left(\frac{e^{-s}}{s(s^2-1)}\right)$

$$= e^t + e^{-t} - 1 - \left(\frac{1}{2}e^{t-1} + \frac{1}{2}e^{1-t} - 1\right)H(t-1) = \begin{cases} e^t + e^{-t} - 1, 0 \le t < 1 \\ e^t + e^{-t} - \frac{1}{2}(e^{t-1} + e^{1-t}), t > 1 \end{cases}$$

b) First observe that

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = \int_0^{2} e^{-st} f(t)dt + \int_2^{\infty} e^{-st} f(t)dt = \int_0^{2} e^{-st} dt = \frac{1}{s} - \frac{e^{-2s}}{s}$$

Then, $L\{y''+9y\} = L\{f(t)\} \Rightarrow L\{y''\} + L\{9y\} = L\{f(t)\} = \frac{1}{s} - \frac{e^{-2s}}{s} = \frac{1-e^{-2s}}{s}$

$$\Rightarrow s^2y(s) - sy(0) - y'(0) + 9L(y) = \frac{1}{s} - \frac{e^{-2s}}{s} \Rightarrow s^2y(s) + 9y(s) = \frac{1-e^{-2s}}{s}$$

$$\Rightarrow (s^2+9)y(s) = \frac{1}{s} - \frac{e^{-2s}}{s} \Rightarrow y(s) = \frac{1-e^{-2s}}{s(s^2+9)} = \frac{1}{s(s^2+9)} - \frac{e^{-2s}}{s(s^2+9)}$$

$$\Rightarrow y(t) = L^{-1}\left(\frac{1}{s(s^2+9)} - \frac{e^{-2s}}{s(s^2+9)}\right) = L^{-1}\left(\frac{1}{s(s^2+9)}\right) - L^{-1}\left(\frac{e^{-2s}}{s(s^2+9)}\right)$$

Now, by partial fraction, $\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9} \Rightarrow A = \frac{1}{9}, B = -\frac{1}{9}, C = 0$

Thus, $L^{-1}\left(\frac{1}{s(s^2+9)}\right) = L^{-1}\left(\frac{1/9}{s}\right) - \left(\frac{s/9}{s^2+9}\right) = \frac{1}{9} - \frac{1}{9}\cos 3t$

Therefore, $y(t) = \frac{1}{9} - \frac{1}{9}\cos 3t - \left(\frac{1}{9} - \frac{1}{9}\cos 3(t-2)\right)u(t-2)$

c) First observe that using unti step function $f(t) = t - tu(t-\pi)$ and using the t-shifting propety that $L\{tf(t)\} = -\frac{d}{ds}[L\{f(t)\}]$ we have

$$L\{f(t)\} = L\{t - tu(t-\pi)\} = L\{t\} - L\{tu(t-\pi)\}$$

$$= \frac{1}{s^2} + \frac{d}{ds}[L\{u(t-\pi)\}] = \frac{1}{s^2} + \frac{d}{ds}\left(\frac{e^{-\pi s}}{s}\right) = \frac{1}{s^2} - e^{-\pi s}\left(\frac{\pi}{s} + \frac{1}{s^2}\right)$$

Then, $L\{y''+y\}=L\{f(t)\}\Rightarrow L\{y''\}+L\{y\}=L\{f(t)\}=\dfrac{1}{s^2}-e^{-\pi s}\left(\dfrac{\pi}{s}+\dfrac{1}{s^2}\right)$

$$\Rightarrow (s^2+1)y(s)=\frac{1}{s^2}-e^{-\pi s}\left(\frac{\pi}{s}+\frac{1}{s^2}\right)$$

$$\Rightarrow y(s)=\frac{1}{s^2(s^2+1)}-e^{-\pi s}\left(\frac{\pi}{s(s^2+1)}+\frac{1}{s^2(s^2+1)}\right)$$

$$\Rightarrow y(t)=L^{-1}\left(\frac{1}{s^2(s^2+1)}\right)-L^{-1}\left[e^{-\pi s}\left(\frac{\pi}{s(s^2+1)}+\frac{1}{s^2(s^2+1)}\right)\right]$$

Now, by partial fraction,

$$\frac{1}{s^2(s^2+1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+1}\Rightarrow A=0,B=1,C=0,D=-1$$

$$\frac{\pi}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}\Rightarrow A=\pi,B=-\pi,C=0$$

$$y(t)=L^{-1}\left(\frac{1}{s^2}\right)-L^{-1}\left(\frac{1}{s^2+1}\right)-L^{-1}\left[e^{-\pi s}\left(\frac{\pi}{s}-\frac{\pi s}{s^2+1}+\frac{1}{s^2}-\frac{1}{s^2+1}\right)\right]$$

$$=t-\sin t-\left(t-\pi\cos(t-\pi)u(t-\pi)-\sin(t-\pi)\right)$$

$$=t-\sin t-\left(t+\pi\cos t+\sin t\right)u(t-\pi)=\begin{cases}t-\sin t, 0<t<\pi\\ -2\sin t-\pi\cos t, t>\pi\end{cases}$$

d) $L\{y'-2y\}=L\left\{\dfrac{\sin t}{t}\right\}\Rightarrow sy(s)-y(0)-2y(s)=\dfrac{\pi}{2}-\tan^{-1}s$

$$\Rightarrow y(s)=\left(\frac{1}{s-2}+\frac{\pi}{2(s-2)}-\frac{\tan^{-1}s}{s-2}\right)$$

$$\Rightarrow y(t)=L^{-1}\left(\frac{1}{s-2}\right)+\frac{\pi}{2}L^{-1}\left(\frac{1}{s-2}\right)-L^{-1}\left(\frac{\tan^{-1}s}{s-2}\right)$$

11. Solving Integral or Integro-DEs

An equation that involves an integral and derivative of a function is known as integral (integro) DE. Such equations are also solved easily by using Laplace Transforms of derivatives and integrals at the same time.

Examples: Solve the integral-DEs of IVPs using Laplace Transforms.

a) $y' - \int_0^t y(x)dx = t^2, y(0) = 1$ *b)* $y(t) = t + \int_0^t \sin(t-x)y(x)dx$

c) $y(t) = \cos t + \int_0^t (t-x)y(x)dx$ *d)* $\int_0^t e^{t-x}y(x)dx = \sin t$

Solution: a) Using Laplace Transform of derivatives and integrals,

$$L\{y' - \int_0^t y(x)dx\} = L\{t^2\} \Rightarrow sY(s) - y(0) - \frac{Y(s)}{s} = \frac{2}{s^3}$$

$$\Rightarrow Y(s)[s - \frac{1}{s}] = \frac{2}{s^3} + 1 \Rightarrow Y(s) = \frac{s^3+2}{s^2(s^2-1)}$$

Then, $y(t) = L^{-1}\{y(s)\} = L^{-1}\left(\frac{s^3+2}{s^2(s^2-1)}\right)$.

But by Partial Fraction $\frac{s^3+2}{s^2(s^2-1)} = \frac{3/2}{s-1} - \frac{1/2}{s+1} - \frac{2}{s^2}$. Therefore,

$$y(t) = L^{-1}\left(\frac{s^3+2}{s^2(s^2-1)}\right) = L^{-1}\left(\frac{3/2}{s-1}\right) - L^{-1}\left(\frac{1/2}{s+1}\right) - 2L^{-1}\left(\frac{2}{s^2}\right)$$

$$= \frac{3}{2}e^t - \frac{1}{2}e^{-t} - 2t$$

b) Using the definition of convolution the given equation is transformed into $y(t) = t + \sin t * y(t)$. Now, by taking the Laplace transform both sides, we get

$$y(s) = \frac{1}{s^2} + \frac{y(s)}{1+s^2} \Rightarrow y(s)(1 - \frac{1}{1+s^2}) = \frac{1}{s^2} \Rightarrow y(s) = \frac{1+s^2}{s^4} = \frac{1}{s^4} + \frac{1}{s^2}.$$

Finally, taking inverse Laplace transform gives us

$$y(t) = L^{-1}\{y(s)\} = L^{-1}\left\{\frac{1}{s^4} + \frac{1}{s^2}\right\} = L^{-1}\left\{\frac{1}{s^4}\right\} + L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{t^3}{6} + t$$