## 7.4 Permutations and Combinations

The multiplication principle discussed in the preceding section can be used to develop two additional counting devices that are extremely useful in more complicated counting problems. Both of these devices use **factorials**.

### Factorials

When using the multiplication principle, we encountered expressions such as

$$26 \cdot 25 \cdot 24 \qquad \text{or} \qquad 8 \cdot 7 \cdot 6$$

where each natural number factor is decreased by 1 as we move from left to right. The factors in the following product continue to decrease by 1 until a factor of 1 is reached:

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

Products like this are encountered so frequently in counting problems that it is useful to express them in a concise notation. The product of the first n natural numbers is called ***n* factorial** and is denoted by ***n*!**. Also, we define **zero factorial**, 0!, to be 1.

**Definition (Factorial)**

For a natural number $n$,

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

or, equivalently,

$$n! = n \cdot (n-1)!,$$

provided

$$0! = 1.$$

### Example 1

Compute:

(a) $1!$

(b) $6!$

(c) $\frac{10!}{9!}$

(d) $\frac{10!}{7!}$

(e) $\frac{5!}{0!3!}$

(f) $\frac{20!}{3!17!}$

## Permutations

### Definition (Permutation of a Set)

Given a set $S$, a **permutation** of $S$, is an arrangement of the elements of $S$ in a specific order without repetition.

**Note:** Recall that set $S$ itself cannot have repeated elements. A set in which some elements are repeated is called a **multiset**.

**Note:** Two permutations of the same set are distinct if order of elements in these permutations is different.

**Note:** The textbook uses the term "permutations of $n$ distinct objects without repetition" which is essentially the same as "permutations of a set $S$ of size $n(S) = n$".

For example, given set $S = \{a, b, c, d, e\}$,

**Question:** How would we count all possible permutations of a given set?

Consider $S = \{a, b, c, d, e\}$.

**Theorem 1 (Number of Permutations of a Set)**

The number of permutations of set $S$ of size $n(S) = n$, denoted by ${}_n\mathbf{P}_n$, is

$$ {}_n\mathbf{P}_n = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = n! $$

Consider $S = \{a, b, c, d, e\}$. Suppose that we want to construct ordered arrangements (sequences) of length 3 using elements from $S$, i.e.,

**Definition ($r$-permutation of a Set)**

Given a set $S$ of size $n(S) = n$, an ***r*-permutation** ($r \leq n$) of $S$ is an arrangement of $r$ elements of $S$ in a specific order without repetition.

**Note:** The textbook uses the term "permutation of $n$ objects taken $r$ at a time" which is essentially the same thing.

**Question:** How would we count all possible $r$-permutations of a given set?

Consider $S = \{a, b, c, d, e\}$. Suppose that we want to count all ordered arrangements (sequences) of length 3 using elements from $S$.

**Theorem 2 (Number of $r$-permutations of a Set)**

The number of $r$-permutations of set $S$ of size $n(S) = n$, denoted by ${}_n\mathbf{P}_r$, is

$${}_n\mathbf{P}_r = \frac{n!}{(n-r)!} = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1).$$

**Note:** ${}_n\mathbf{P}_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$ is the number of $n$-permutations of set $S$.

**Example 2**

Given a set $\{A, B, C, D\}$ how many permutations are possible for this set of 4 objects taken 2 at a time? Answer the question

(a) Using a tree diagram

(b) Using the multiplication principle

(c) Using the two formulas for ${}_n\mathbf{P}_r$

## Example 3

In how many ways can 3 people sit in a row of 7 chairs?

## Example 4

Find the number of permutations of 30 objects taken 4 at a time. Compute the answer using a calculator.

# Combinations

Consider $S = \{a, b, c\}$. Suppose that we want to construct sets (subsets of $S$) using 2 letters from $S$, where **we do not care about the order** of letters in the resulting sets, i.e.,

### Definition ($r$-combination of a Set)

Given a set $S$ of size $n(S) = n$, an ***r*-combination** ($r \leq n$) of $S$ is a subset of $k$ distinct elements of S (no repetition of elements allowed).

**Note:** The arrangement (order) of the elements in the subset does not matter. That is, two $r$-combinations are assumed to be the same if they composed of the same elements.

**Note:** The textbook uses the term "combination of $n$ objects taken $r$ at a time" which is essentially the same thing.

**Question:** How would we count all possible $r$-combinations of a given set?

Consider $S = \{a, b, c\}$ again. Let's construct all possible 2-combinations of $S$.

### Theorem 3 (Number of $r$-combinations of a Set)

The number of $r$-combinations of set $S$ of size $n(S) = n$, denoted by ${}_n\mathbf{C}_r$, is

$$ {}_n\mathbf{C}_r = \binom{n}{r} = \frac{{}_n\mathbf{P}_r}{r!} = \frac{n!}{r!(n-r)!}. $$

### Example 5

From a committee of 12 people,

(a) In how many ways can we choose a chairperson, a vice-chairperson, a secretary, and a treasurer, assuming that one person cannot hold more than one position?

(b) In how many ways can we choose a subcommittee of 4 people?