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# Advanced Programming

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## Functional Design: Monoid, Foldable, Functor, Monad

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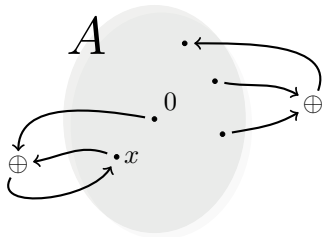
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**S** SOFTWARE  
**Q** QUALITY  
**R** RESEARCH

# Monoid

## Definition

- Type  $A$
- Binary operator  $\oplus : A \times A \mapsto A$  (combine)
- Zero element  $0 \in A$  (empty)
- Laws ...



**Associativity**  $(x \oplus y) \oplus z = x \oplus (y \oplus z)$  for any  $x, y, z \in A$

**Identity**  $x \oplus 0 = x = 0 \oplus x$  for any  $x \in A$

**Associativity:** `forall { (x:A, y:A, z:A) =>  
 combine(combine(x, y), z) == combine(x, combine(y, z)) }`

**Identity:** `forall { (x: A) => combine(x, empty) == x }  
 forall { (x: A) => combine(empty, x) == x }`

**Examples:** (Integers, 0, +), (Integers, 1, \*), (Strings, +, "")

# Monoid

## A type class

```
1 trait Monoid[A]:
2   self =>
3   def combine(a1: A, a2: A): A
4   def empty: A
5
6   object laws:
7     def associative (using Arbitrary[A], Equality[A]) =
8       forAll { (a1: A, a2: A, a3: A) =>
9         self.combine(self.combine(a1, a2), a3) ===
10          self.combine(a1, self.combine(a2, a3)) }
11
12     def unit (using Arbitrary[A], Equality[A]) =
13       forAll { (a: A) =>
14         (self.combine(a, self.empty) === a) &&
15         (self.combine(self.empty, a) === a) }
16
17     def monoid (using Arbitrary[A], using Equality[A]) =
18       associative && unit
```

- Type class: Monoid, type A, op, zero, abstract trait
- Type constructor: Monoid
- Laws: associativity, unit, both (imperative issues)
- Inner object's access to enclosing object
- Type constraints with implicits: Arbitrary, Equality
- Note: Type classes (Equality) are used in Scalatest, despite of imperative style

More on the Scalactic equality test `===` used above: <https://www.scalactic.org/>

# Monoid

## Instances

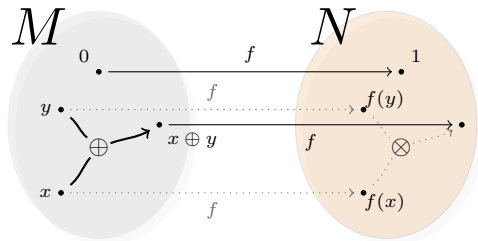
```
1 val stringMonoid: Monoid[String] = new:
2   def combine(a1: String, a2: String): String =
3     a1 + a2
4   val empty: String = ""
5 ...
7 object MonoidSpec extends ...
8   property("stringMonoid is a monoid") =
9     stringMonoid.laws.monoid
```

- A type class instance shows evidence that we can see a type as a Monoid
- Note that a type class instance is a value! How do we create one? We instantiate the trait.
- In place instantiation of abstract traits
- How do we "prove" that an instance is valid? We property test the Monoid Laws!
- Exercise: Take a piece of paper to write an instance for Int, 1, and multiplication
- Mentimeter on more instances

# Monoid

## Morphisms

- Monoid  $(M, \oplus, 0)$ , monoid  $(N, \otimes, 1)$
- **Homomorphism**  $\equiv f : M \rightarrow N$  + laws below
- Homomorphisms both ways  $\equiv$  **isomorphism**



**Distributive**  $f(x \oplus y) = f(x) \otimes f(y)$  for any  $x, y \in M$   
**Preserves identity**  $f(0) = 1$

**Distributive:** `forall { (x:A, y: A) =>f(M.combine(x, y)) ==N.combine(f(x), f(y)) }`

**Preserves identity:** `f(M.empty) ==N.empty`

**Example homomorphism:** `f(s: String): Int =s.size`  
from Strings (with "" and concat) to integers (with 0 and +)  
e.g. `"foo".size + "bar".size ==("foo"+"bar").size`

**Example isomorphisms:** String and List[Char] through `toList` and `mkString`,  
(Boolean, `false`, `||`) and (Boolean, (`true`, `&&`) through negation

# Foldable

## Type Class

```
1 trait Foldable[F[_]]:  
2   extension [A](as: F[A])  
3     // map into a monoid and reduce using 'combine'  
4     def foldMap[B: Monoid](f: A => B): B  
  
6     def foldRight[B](z: B)(f: (A, B) => B): B  
7     def foldLeft[B](z: B)(f: (B, A) => B): B  
  
9     // If A is a monoid then  
10    def concatenate[A](as: F[A])(using m: Monoid[A]): A =  
11      as.foldLeft(m.empty)(m.combine)
```

**Examples:** List, IndexedSeq, Stream, Option (almost all structures implemented in the course so far!) Also trees are foldable, any iterator can be cast as foldable

- Understand the type class and the interface
- Foldable is a **higher kind**, parameterized with a type constructor
- So type classes exist both for types and for higher kinds
- Folds work with Monoids operator, no diff btw left and right. **Why?**
- Reduce = a fold from monoid's empty with combine
- MapReduce (spark) can distribute calculations in monoids
- Behavioral interface, like Enumerable, Iterable, etc. – (Monoid could be called "Addable")
- Mentimeter ( $\times 2$ )

# Functor

## Examples

Consider the following map functions from different parts of the course:

```
def map[A,B] (ga: Gen[A]) (f: A =>B): Gen[B]
def map[A,B] (la: List[A]) (f: A =>B): List[B]
def map[A,B] (oa: Option[A]) (f: A =>B): Option[B]
```

- All very similar, also State, Par, Parser, Tree
- Follow the same interface
- We could call it Mappable
- But for the sake of tradition we settle on Functor

# Functor

## Type Class

```
1 trait Functor[F[_]]:
2
3   extension [A](fa: F[A])
4     def map[B](f: A => B): F[B]
5
6   extension [A, B](fab: F[(A, B)])
7     def distribute: (F[A], F[B]) =
8       (fab.map { _. _1 }, fab.map { _. _2 })
9
10  object functorLaws:
11    def map[A](using Arbitrary[F[A]], Equality[F[A]]): Prop =
12      forAll { (fa: F[A]) => fa.map[A](identity[A]) == fa }
```

**Examples:** List, Gen, Option, Stream are all functors (they have map, mappables)

- Understand the type class and the interface
- Functor is a **higher kind**, parameterized with a type constructor
- The law: structure preservation
- Why the using constraint?
- Even though map is a weak assumption we can derive useful functions (distribute)
- Distribute is a general unzipper for any functor, not just lists
- Distribute is explained in the next slide
- General zippers require map2 (a Monad, or an Applicative Functor—not in curriculum)



# Functor

## Distribute

: F[(Name, Age)]

(John, 35)
(Andrzej, 39)
(Stefan, 18)
...

distribute

: (F[Name], F[Age])

John	35
Andrzej	39
Stefan	18
...	...

```
extension [A, B](fab: F[(A, B)])
  def distribute: (F[A], F[B]) =
    (fab.map { _. _1 }, fab.map { _. _2 })
```

**Note:** This works for List and Stream but also for Gen and Par with the same single implementation! This will continue working for any type constructor in the future for which we add a Functor instance.

**Exercise:** Doodle an instance for Option (go back to the previous slide)

# Map2

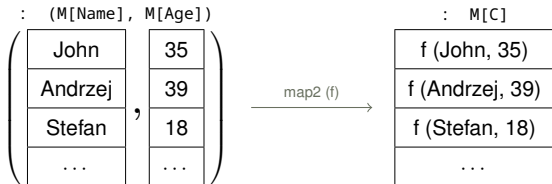
Some of the map2 functions from different parts of the course

```
def map2[A,B,C] (fa: Option[A], fb: Option[B]) (f: (A,B) => C): Option[C] =  
  flatMap (fa) (a => map (fb) (b => f (a,b)))
```

```
def map2[A,B,C] (fa: List[A], fb: List[B]) (f: (A,B) => C): List[C] =  
  flatMap (fa) (a => map (fb) (b => f (a,b)))
```

```
def map2[A,B,C] (fa: Gen[A], fb: Gen[B]) (f: (A,B) => C): Gen[C] =  
  flatMap (fa) (a => map (fb) (b => f (a,b)))
```

- All very similar, also State, Par, Parser, the Map2-appable trait/interface (ApplicativeFunctor)
- map2 is a 'kind-of' generalized binary zipWith (Beware! A different intuition for each instance!)



- Pic makes "sense" for Gen
- What happens for List ?
- What happens for Option ?
- Parser ?

# Monad

## Type Class

```
1 trait Monad[F[_]]
2   extends Functor[F]:

4   def unit[A](a: => A): F[A]

6   extension [A](fa: F[A])
7     def flatMap[B](f: A => F[B]): F[B]

9     def map[B](f: A => B): F[B] =
10      fa.flatMap[B] { a => unit(f(a)) }
```

**Examples:** All types in ADPRO!

Monads in the Scala standard library [21]: FilterMonadic, Stream, StreamWithFilter, TraversableMethods, Iterator, ParIterableLike, ParIterableLike, ParIterableViewLike, TraversableLike, WithFilter, MonadOps, TraversableProxyLike, TraversableViewLike, LeftProjection, RightProjection, Option, WithFilter, Responder, Zipped, ControlContext, Parser

- Understand the type class and the interface
- Monad is a higher kind, parameterized with a type constructor
- Monad is both a functor and an applicative functor
- It could be called "FlatMappable"
- It captures computations that can be sequenced, and transformed

# Monad Laws

- Laws for flatMap and unit: associative, identity
- Discussion of type constraints, where is arbAFA used?

```
1  object monadLaws:

3    def associative[A, B, C](using Arbitrary[F[A]],
4      Arbitrary[A => F[B]], Arbitrary[B => F[C]], Equality[F[C]]): Prop =
5      forAll { (x: F[A], f: A => F[B], g: B => F[C]) =>
6        ( x.flatMap[B](f).flatMap[C](g) ) === ( x.flatMap[C] { a => f(a).flatMap[C](g) } )
7      }

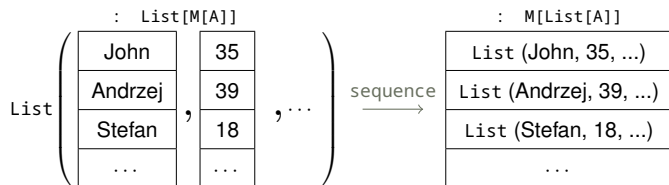
9    def identityRight[A](using Arbitrary[F[A]], Arbitrary[A => F[A]], Equality[F[A]]) =
10      forAll { (x: F[A], f: A => F[A]) => x.flatMap[A](unit[A]) === x }

12    def identityLeft[A: Arbitrary](using Arbitrary[A => F[A]], Equality[F[A]]) =
13      forAll { (y: A, f: A => F[A]) => unit[A](y).flatMap[A](f) === f(y) }
```

# Sequence is generalized n-ary zip

```
def sequence[A] (lfa: List[F[A]]): F[List[A]]
```

- We can derive sequence for any Monad instance.
- The function is implemented in terms of the above interface
- The function produces a computation that produces lists. The computed list is build by polling each of the computations in the parameter for their head, and simply recomposing to the list.



- Makes “sense” for Gen (values incorrect)
- What happens for Option ?
- What happens for List ?
- Parser is a bit similar to Option and Gen