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Advanced Programming

Finger Trees Persistent Data Structures, Polymorphic Recursion

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SOFTWARE QUALITY RESEARCH

Goals

Finger trees are more of an excuse than a goal...

- To hint how **pure persistent data structures** are designed
- To **train recursion**
- To see **polymorphic recursion** in action
- To see type classes, higher order types, property-based testing, Gen, and monoids in a slightly larger context of delivering an implementation of a data structure
- To compare Haskell and Scala in a case study setting

What is a deque?

- A linear data structure providing efficient access at both ends
- Efficient add-from-left, and efficient add-from-right
- Efficient remove-from-left, and remove-from-right
- Efficient emptiness check
- A double-ended stack

```scala
trait F[A]
  def addL(q: F[A]) (a: A): F[A]
  def addR(q: F[A]) (a: A): F[A]
  def headL(q: F[A]): A
  def headR(q: F[A]): A
  def tailL(q: F[A]): F[A]
  def tailR(q: F[A]): F[A]
  def empty(q: F[A]): Boolean
  def toTree(l: List[A]): F[A] // nice to have
```

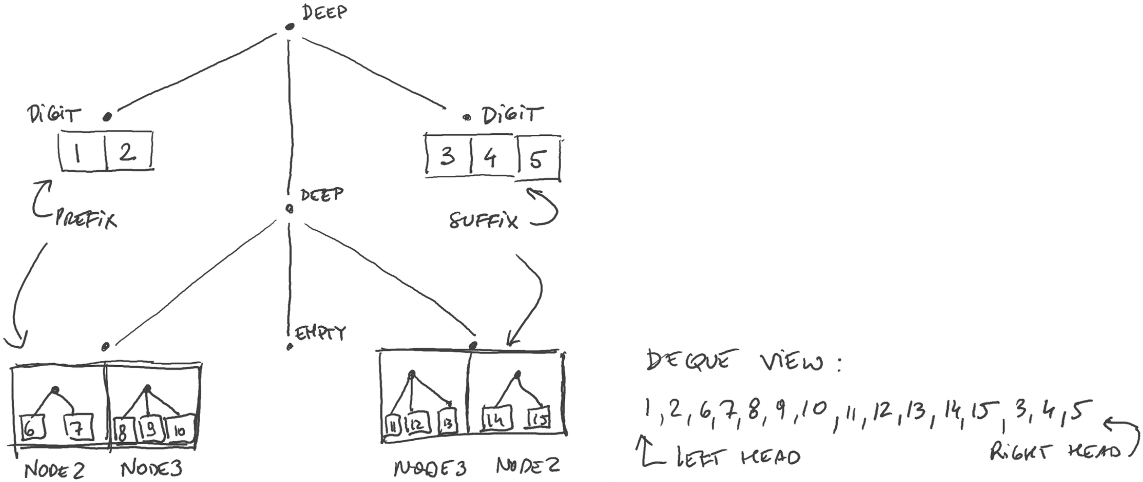
```scala
q match
case ConsL (hd, tl) => ...
case ConsR (tl, hd) => ...
case Nil => ...
```

Finger Trees

- **Balanced trees**, same family as AVL, 2-3-4, red-black trees, etc.
- They maintain the same invariant: logarithmic depth, so depth is in $O(\log n)$ if the tree holds n elements
- Finger trees have four kinds of nodes
  - Leaf nodes (store data elements):
    - **Empty**: stores no elements
    - **Single**: stores exactly one element
    - **Digit**: stores from 1 to 4 data elements
  - Internal nodes (store **finger trees**, with prefix and suffix)
    - **Deep** means somewhere in the middle, not in the prefix, and not in the suffix
    - Deep node contains a **prefix** digit, a **middle** finger tree, and a **suffix** digit
  - Data elements (a twist!):
    - Top level contains just data values directly
    - At depth 1 we use balanced trees of depth 1 as elements
    - Elements at depth n are trees of height n (**Deeper elements are heavier!**)
    - Element trees are always balanced, binary and ternary (Node2, and Node3)

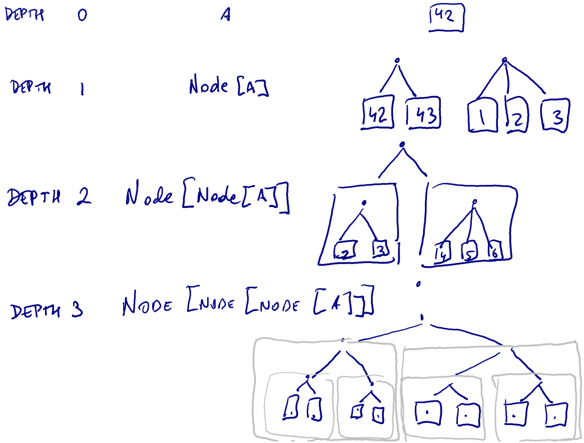
Finger Tree

A Visual Example



Element in the Trees

Representing trees in a type-safe manner



- `Node[ ... Node [A] ... ]`: type of trees of a fixed depth
- The type checker counts the depths of the trees by counting nestings!

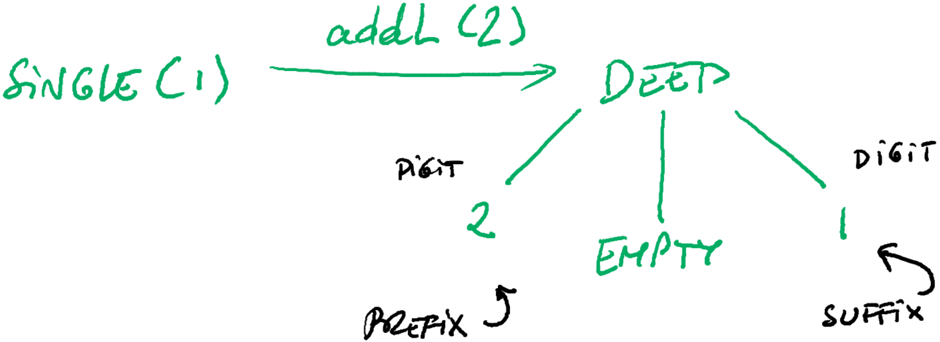
```
enum Node[+A]:
  case Node2(l: A, r: A)
  case Node3(l: A, m: A r: A)


enum FingerTree[+A]
  case Empty
  case Single(data: A)
  case Deep[l: Digit[A], m: FingerTree[Node[A]], r: Digit[A])
```


Adding from the left (addL)

$$\text{EMPTY} \xrightarrow{\text{addL}(1)} \text{SINGLE}(1)$$

Adding from the left (addL)



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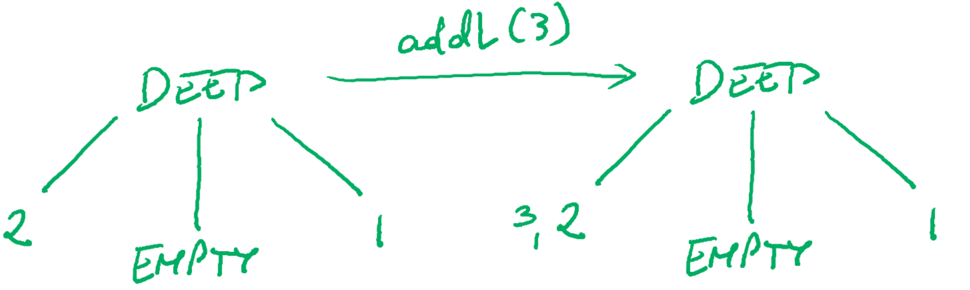

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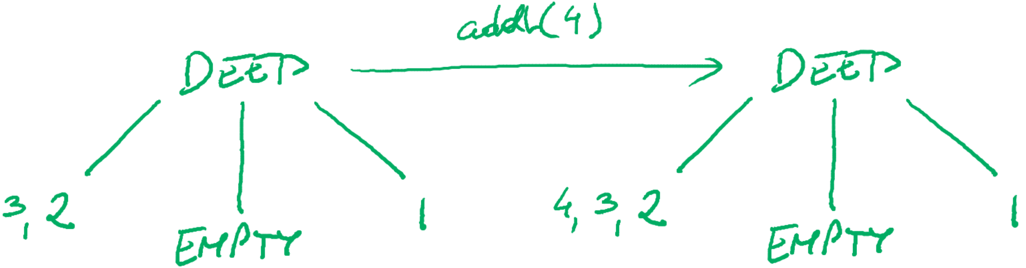

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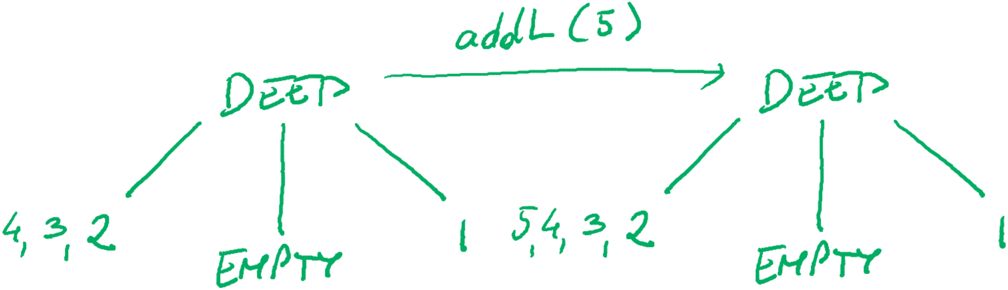

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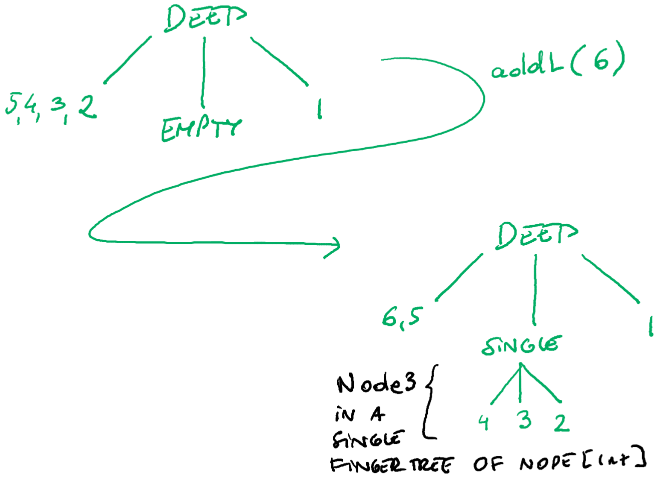

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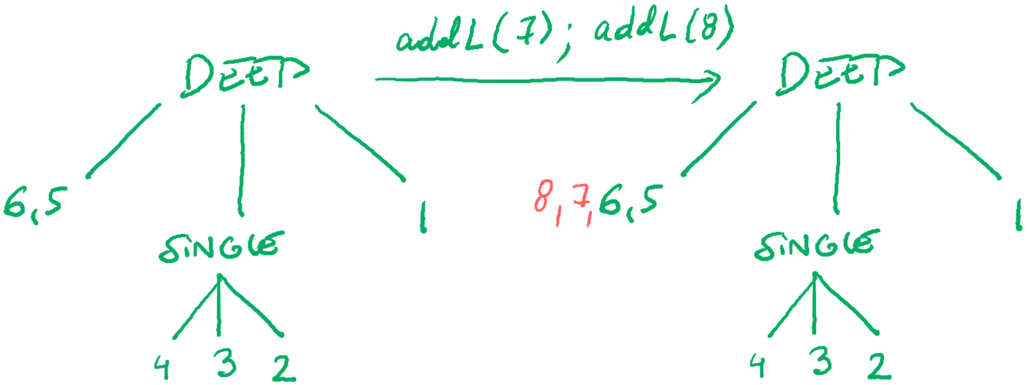

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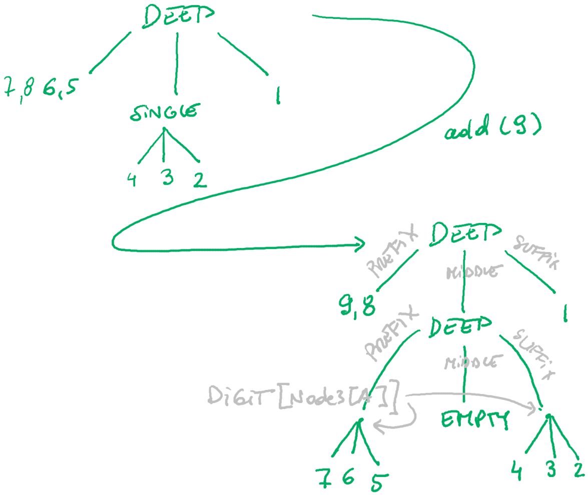


Finger Trees

Sketch of complexity of addition

- Insertion from the right is symmetric (so we don't even implement it)
- Insertion may recurse down the spine and take $O(\log n)$ time worst case
- This gives $O(\log n)$ worst case cost per insertion
- Still, the cost is $O(1)$ amortized time
- Insertion can only propagate to the next level if a Digit is full,
- This makes the digit size 2, so next operation on it will not propagate
- At most half operations descend one level, half of that 2 levels etc.
- For n operations we get: $O(1 + 1/2 + 1/4 + \ldots + 1/2^n) = O(2) = O(1)$
- The amortized cost is constant

Polymorphic Recursion

- Go back to finger tree types slide: what happens in line 9?
- A type constructor `T` over `A` nests a value of type `T[Node[A]]`
- How different from `List`! `Cons[A](hd:A, tl:List[A]) extends List[A]`
- **Mentimeter:** What will be the type nested by `FingerTree[Node[A]]` ?
- Imagine a recursive function `f[A]` traversing `FingerTree[A]`
- When it hits `Deep`, it will call itself recursively on the middle value `m`
- That value `m` will be of type `Node[A]`
- We call `f[Node[A]]` from within `f[A]` **changing the type of the current function at the recursive call**
- **Polymorphic recursion** (AKA Milner–Mycroft type-ability or the Milner–Mycroft calculus) *refers to a recursive parametrically polymorphic function in which the type parameter changes with each recursive invocation made instead of staying constant*

Polymorphic Recursion (2)

- Languages supporting polymorphic recursion known to AW: Haskell, Scala, Ocaml (and maybe F# as per Internet rumours)
- Hinz/Paterson use a common pattern to use polymorphic recursion to ensure that the trees are balanced.
- This limits the difference between the right and left side of the tree
- You get a typing error, if your code can produce an unbalanced tree
- Ingenious!

Removing from the Left

(headL, tailL, ConsL)

$$\text{EMPTY} \xrightarrow{\text{removeL}} \text{Fail!}$$

Removing from the Left

(headL, tailL, ConsL)

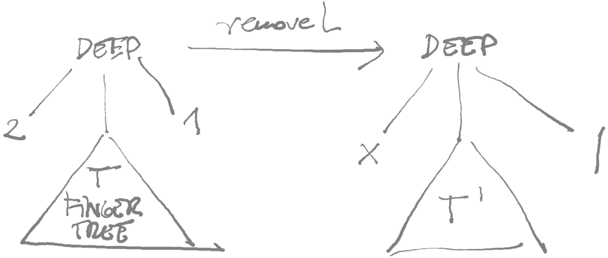
$$\text{Single}(a) \xrightarrow{\text{removeL}} \text{Empty},\ a$$

Removing from the Left

(headL, tailL, ConsL)

Removing from the Left

(headL, tailL, ConsL)



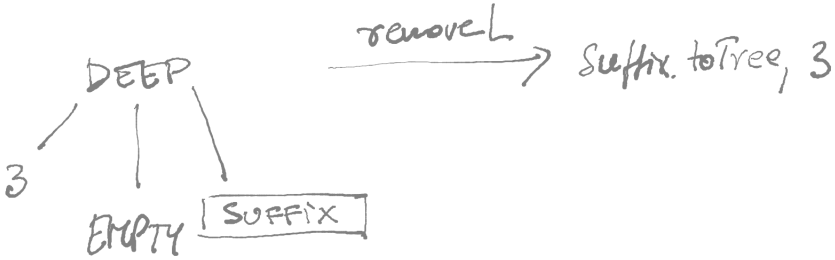
where removeL(T) = T', x

where x has at most 3 elements

(why?)

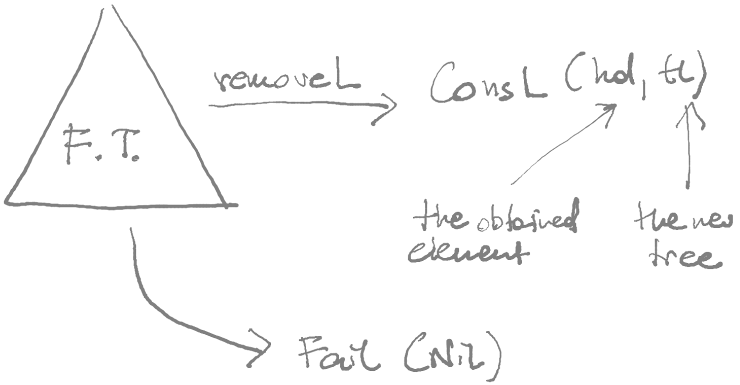
Removing from the Left

(headL, tailL, ConsL)



Removing from the Left

(headL, tailL, ConsL)



Removing Elements

Extractors

Hinze and Paterson implement the previous as a **view function**:

```
data ViewL a = Nil | ConsL a (FingerTree a)
viewL :: FingerTree a -> ViewL a
```

Not entirely convenient. In Scala using this would look like this:

```
viewL(t) match { case Nil =>...; case ConsL(a, t) =>... }
```

Note: For lists we do not need to explicitly match on the view.
In Scala we can make views automatic using **extractors**, so they behave like for lists:

```
t match { case Nil =>...; case ConsL(a,t) =>... }
```

To do this we need two **objects** `Nil` and `ConsL` implementing the `unapply` method.
`unapply` **takes** value to be matched as parameter.
`unapply` **returns** `Option[T]` where `T` the type of parameters of “matching constructor”

For `ConsL(a:A,t:T)` the return type should be `Option[(A,T)]`
In the method, perform matching and return `Some` if successful, `None` if failed.

