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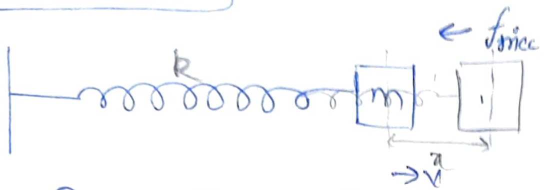
Title of the Paper :- Physics at Work I : Deconstructing Machines.

**Solution 6**




$$F_{net} = F_{spring} + f_{fric.}$$

$$F = -kx - bv$$

Where b is viscosity constent.

the eq$^n$ of motion is

$$m\ddot{x} = -kx - b\dot{x}$$

$$\Rightarrow \ddot{x} + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0$$

Let $\frac{b}{m} = r$ & $\frac{k}{m} = \omega_0^2$

$$\Rightarrow \ddot{x} + r\dot{x} + \omega_0^2 x = 0 \quad \text{——} (1)$$

also, $\ddot{y} + r\dot{y} + \omega_0^2 y = 0 \quad \text{——} (2)$

multiplying (2) by $i$ (iota) and adding to (1).

$$\ddot{z} + r\dot{z} + \omega_0^2 z = 0 \quad \text{——} (3)$$

As $r$ & $\omega_0^2$ are constant.

Let consider a trial sol$^n$.

$$z = z_0 e^{\alpha t} \quad (z_0 \;\&\; \alpha \text{ are const})$$

Substituting in eq (3)

$$\alpha^2 z_0 e^{\alpha t} + \alpha r z_0 e^{\alpha t} + \omega_0^2 z e^{\alpha t} = 0$$

$$z_0 e^{\alpha t}\left(\alpha^2 + \alpha + \omega_0^2\right) = 0.$$

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$\alpha^2 + \alpha\gamma + \omega_0^2 = 0.$

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$$

Let the two roots be $\alpha_1$ & $\alpha_2$

$\therefore$ the sol$^n$ is

$$Z = Z_A e^{\alpha_1 t} + Z_B e^{\alpha_2 t}$$

where, $Z_A$ & $Z_B$ are const.

There are 3 possible case of sol$^n$.

CASE I :- Light damping

$$\omega_0^2 - \left(\frac{\gamma}{2}\right)^2 > 0$$

$\Rightarrow \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$ is imaginary..

$$\Rightarrow \alpha = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

$$\alpha = -\frac{\gamma}{2} \pm i\omega_1 \qquad \left[\omega_1 = \omega_0^2 - \left(\frac{\gamma}{2}\right)^2\right]$$

$\Rightarrow$ the solution is:

$$z = e^{-(\gamma/2)t}\left[Z_1 e^{i\omega_1 t} + Z_2 e^{i\omega_2 t}\right]$$

to find the real part.

$$x + iy = e^{-(\gamma/2)t}\left[(x_1 + iy_1)(\cos\omega_1 t + i\sin\omega_1 t) + (x_2 + iy_2)(\cos\omega_1 t - i\sin\omega_1 t)\right]$$

The real part can be arranged in the form

$$\boxed{x = A e^{-(\gamma/2)t}\cos(\omega_1 t + \phi)}$$

Similarly for imaginary part

$$\boxed{y = A e^{-(\gamma/2)t}\cos(\omega_1 t + \phi)}$$

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CASE 2: heavy damping.

$$\left(\frac{\gamma}{2}\right)^2 > \omega_0^2$$

$\Rightarrow \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$ is real.

$$\therefore \alpha = -\frac{\gamma}{2} \pm \frac{\gamma}{2}\sqrt{1 - \frac{\omega_0^2}{(\gamma/2)^2}}$$

As both roots are -ve.

$$\therefore z = z_1 e^{-|\alpha_1|t} + z_2 e^{-|\alpha_2|t}.$$

As the exponentials are real,

$$\therefore x = Ae^{-|\alpha_1|t} + Be^{-|\alpha_2|t}.$$

↳ This solution has no oscillatory behaviour
$\Rightarrow$ It is called as overdamped

CASE 3: Critical damping

$$\gamma^2/4 = \omega_0^2$$

$\Rightarrow$ we have only single root.

$$\alpha = -\frac{\gamma}{2}.$$

$\Rightarrow$ The corresponding solⁿ is

$$x = Ae^{-(\gamma/2)t}.$$

As this solⁿ is incomplete, because there should be two constants in solⁿ of a 2nd degree D.E.
$\therefore$ we can find the other one.

Let, $x = u(t)\, e^{-(\gamma/2)t}$

Substituting in eq ① & recalling that $\gamma = 2\omega_0$ & for this $u(t)$ must satisfy

$$\ddot{u} = 0.$$

$\Rightarrow u = a + bt$

$\therefore$ General solⁿ for critical damping is

$$\boxed{x = (A+Bt)\, e^{-(\gamma/2)t}.}$$

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Energy of a harmonic oscillator.

We have,

$$v = -x_0 e^{-(\gamma/2)t}\left[\omega_1 \sin(\omega_1 t+\phi) + \frac{\gamma}{2}\cos(\omega_1 t+\phi)\right]$$

For the case of light damping,

$\left(\omega_1 \gg \frac{\gamma}{2}\right)$

$$\omega_1^2 = \omega_0^2 - \left(\frac{\gamma}{2}\right)^2 \Rightarrow \omega_1 \approx \omega_0.$$

also as $\omega_1 \gg \frac{\gamma}{2}$.

$$\therefore v = v_0 e^{-(\gamma/2)t}\sin(\omega_0 t+\phi)$$

where $(v_0 = \omega_0 x_0)$.

Here the Potential Energy is.

$$U(t) = \frac{1}{2}k x_0^2 e^{-\gamma t}\cos^2(\omega_0 t+\phi) \quad \text{—— (A)}$$

& Kinetic Energy.

$$K(t) = \frac{1}{2}m v_0^2$$

$$= \frac{1}{2}m\omega_0^2 x_0^2 e^{-\gamma t}(\sin^2(\omega_0 t+\phi))$$

$$= \frac{1}{2}k x_0^2 e^{\gamma t}(\sin^2(\omega_0 t+\phi)) \quad (\because k = m\omega_0^2). \quad \text{—— (B)}$$

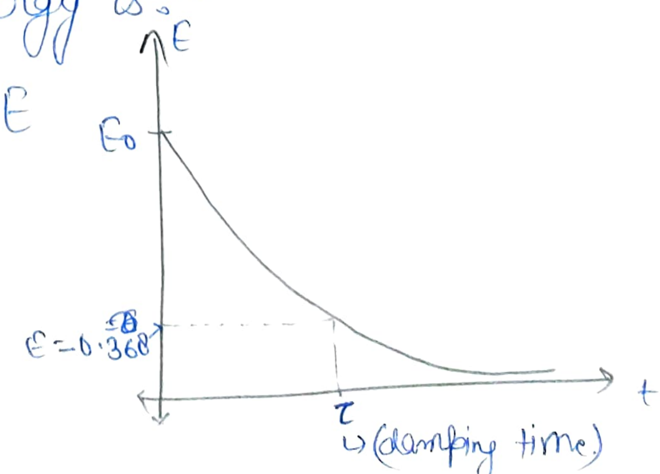
Total energy, (Adding (A) & (B))

$$E(t) = \frac{1}{2}k x_0^2 e^{-\gamma t}$$

The decay of energy is:-

$$\frac{dE}{dt} = -\gamma E$$

$$\Rightarrow E = E_0 e^{-\gamma t}$$



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Q factor of an Oscillator.

Q factor is the degree of damping of an oscillator

$$\text{Q-factor} = \frac{\text{average energy stored in the oscillator}}{\text{avg. energy dissipited during 1 rad of motion}}$$

~~for~~ to mean the time avg. over on cycle,

$\langle \sin^2\theta \rangle = \langle \cos^2\theta \rangle = \frac{1}{2}$.

Energy dissipated per ~~rad~~ rad is the energy lost during that time,

During the period $T = \frac{2\pi}{\omega_0}$, the system oscillates through $2\pi$ rad.

$\therefore$ for 1 radian time $= \frac{1}{\omega_0}$

the energy decay at the rate ~~Q~~ $\dot{E} = -\gamma E$

in time $\Delta t = 1/\omega_0$.

$$\Delta E \approx \frac{dE}{dt}\Delta t = \gamma E \frac{1}{\omega_0}.$$

$\therefore$ the Q-factor is given by

$$Q = \frac{E}{\Delta E} = \frac{\cancel{E}}{\gamma \cancel{E}/\omega_0} = \frac{\omega_0}{\gamma}$$

$$\boxed{\text{Q-factor} = \frac{\omega_0}{\gamma}}$$

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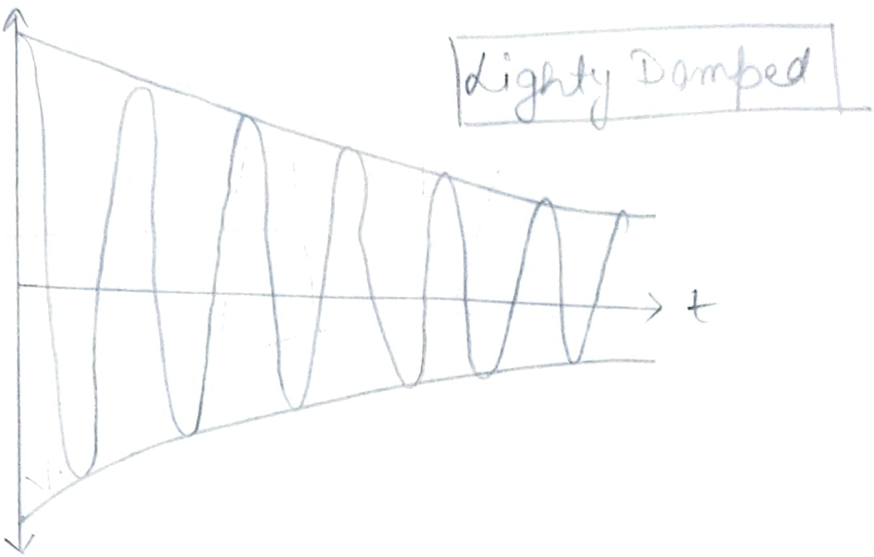
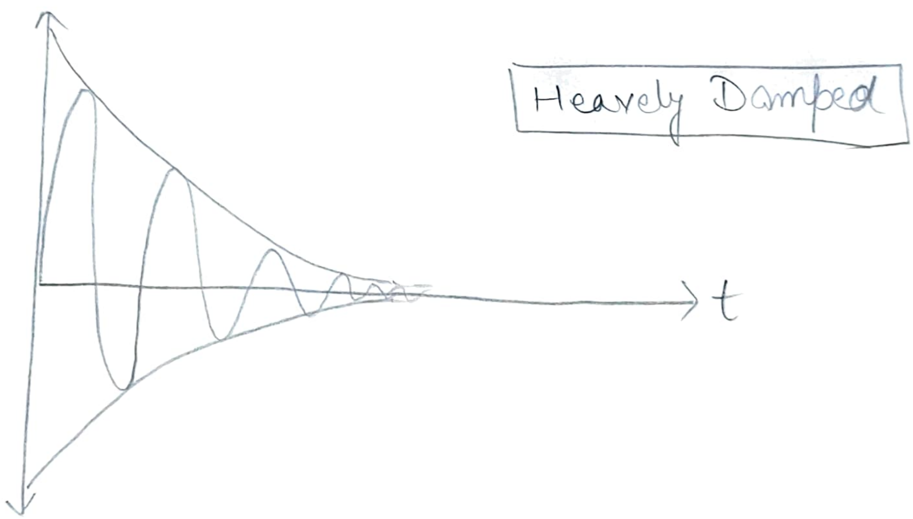
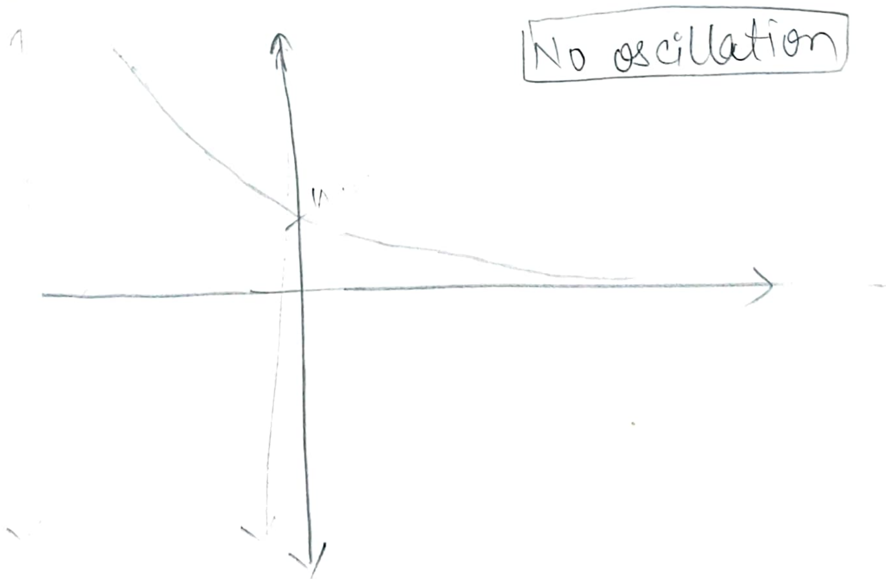
Graphs:-



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→ To find the frequency of oscillation of a damped harmonic oscillator, we can oscillate a system of harmonic oscillator in the lab and can note down several time(t) for a fixed no. of oscillation let's say (20). then, we can repeat the process several times and find the time period of the harmonic oscillator, T, which will be average of all values of (t/20) i.e time taked divided by No. of oscillation.
To find the frequency we will find (1/T).

i.e, $T_1 = \frac{t_1}{20}$ ; $T_2 = \frac{t_2}{20}$ ...... $T_n = \frac{t_n}{20}$

$$T = \frac{T_1 + T_2 + \cdots + T_n}{n}$$

$$f = \frac{1}{T}.$$