Tutorial Sheet I

Q1) Let $w = (2 + i)\, z - 2i$. Find the triangle onto which the triangle with vertices $z_1 = -2 + i$, $z_2 = -2 + 2i$, $z_3 = 2 + i$ is mapped.

Q2) Let $w = f(z) = (3 + 4i)\, z - 2 + i$
(a) Find the image of the disk $|z - 1| < 1$
(b) Find the image of the line $x = t$, $y = 1 - 2t$, for $-\infty < t < \infty$
(c) Find the image of the half-plane $\text{Im}(z) > 1$

Q3) Show that under the transformation $w = e^z$ a rectangular region $a \le x \le b$, $c \le y \le d$ is mapped into an angular region $e^a \le r \le e^b$, $c \le \theta \le d$. Draw the domain in the xy plane and the range in the r-θ plane. Hence find the image of the square $-1 \le x, y \le 1$.

Q4) Sketch the region onto which the sector $r \le 1$, $0 \le \theta \le \pi/4$ is mapped by the transformation $w = z^2$

Q5) Does $\lim_{z \to 0} \left(\dfrac{z}{\bar{z}} \right)^2$ exists? If it exists find the limit or else show that the limit does not exist.

Q6) Find the $\lim_{z \to \infty} \dfrac{4z^2}{(1 - z)^2}$.

Q7) Use the definition of derivatives to find the derivative of $f(z) = \dfrac{2z + 1}{z - 1}$.

Q8) Express the function $f(z) = z^5 + z^{-5}$ in polar form $u(r, \theta) + i\, v(r, \theta)$. Hence deduce if $f(z)$ is analytic or not.

Q9) $w = f(z) = u(x, y) + i\, v(x, y)$. If $f(z)$ is analytic everywhere and $u(x, y) = e^x \cos y$, find $v(x, y)$. Is $w(u, v)$ a harmonic function?

Q10) Show that $u(x, y) = 2x - x^3 + 3xy^2$ is harmonic. Find its harmonic conjugate. Find a corresponding analytic function such that $f(0) = 0$.

Q11) Is $\overline{\ln z} = \ln \bar{z}$?

Q12) Under what condition is $\overline{\exp(iz)} = \exp(i\bar{z})$?

Q13) Show that the set of values of $\log(i^{1/2})$ is $\left(n + \dfrac{1}{4}\right)\pi i$ $(n = 0, \pm1, \pm2, \ldots)$ and that the same is true of $(1/2) \log i$. But the set of values of $\log(i^2)$ is not the same as the set of values of $2 \log i$. Give an explanation for the difference.

Q14) Recall that in the class we did that the general power of $z = x + iy$ is given by $z^c = e^{c \ln z}$. Since $\ln z$ is a multiple valued function, z^c is also multivalued. The principle value of z^c is $e^{c\, Ln\, z}$. Use this to find the principle value of $(1 + i)^{1 - i}$. Using the result deduce the value of $(1 - i)^{1 + i}$.

Q15) Find the Maclaurin series expansion of the function $f(z) = \dfrac{z}{z^4 + 16}$.

Q16) Show that when $0 < |z| < 4$, $\dfrac{1}{4z - z^2} = \dfrac{1}{4z} + \sum_{n=0}^{\infty} \dfrac{z^n}{4^{n+2}}$