Chapter 8 Phase Plane Analysis

Chengju Liu

Department of Control Science & Engineering

School of Electronic & Information Engineering

Tongji University

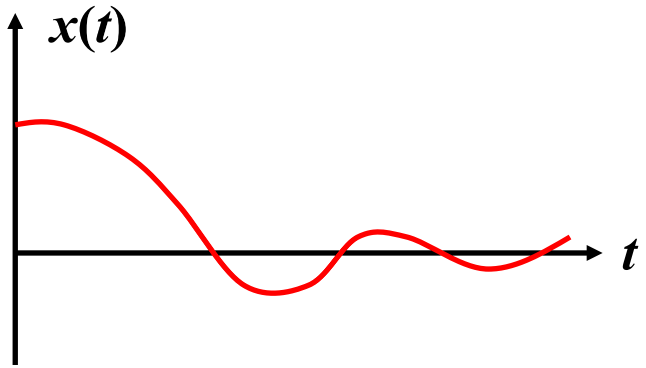
E-mail: liuchengju@tongji.edu.cn

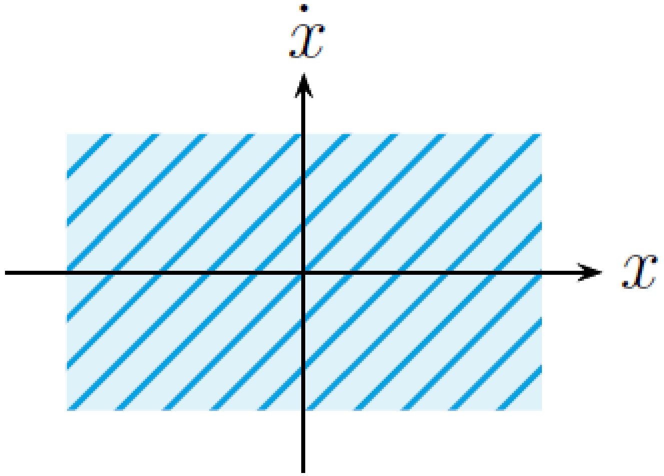
Spring semester, 2023

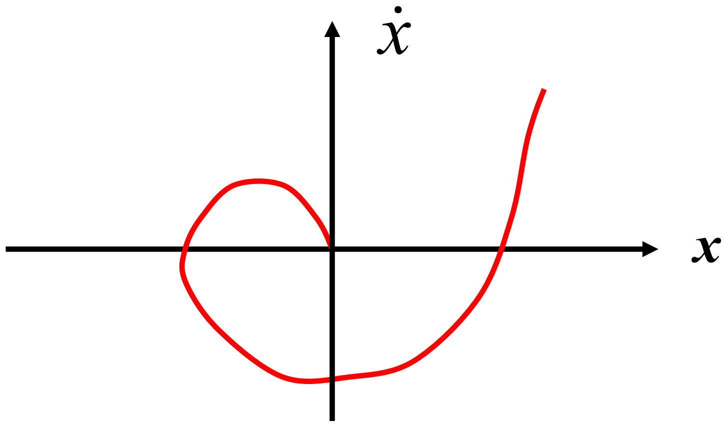
Outline of Chapter 8

Phase plane portraits

- Phase Plane of $\ddot{x} + f(x, \dot{x}) = 0$
  - Phase variables x and $\dot{x}$
  - Phase plane as shown right

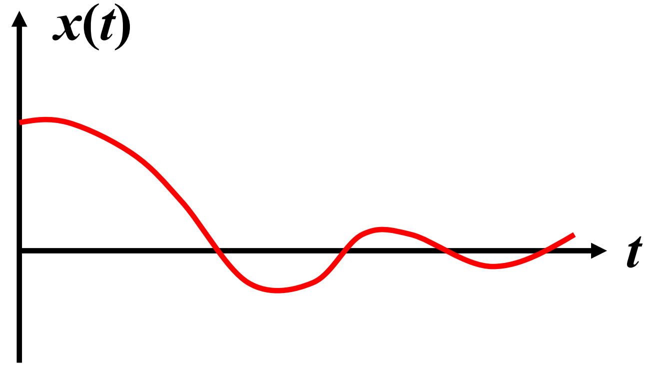






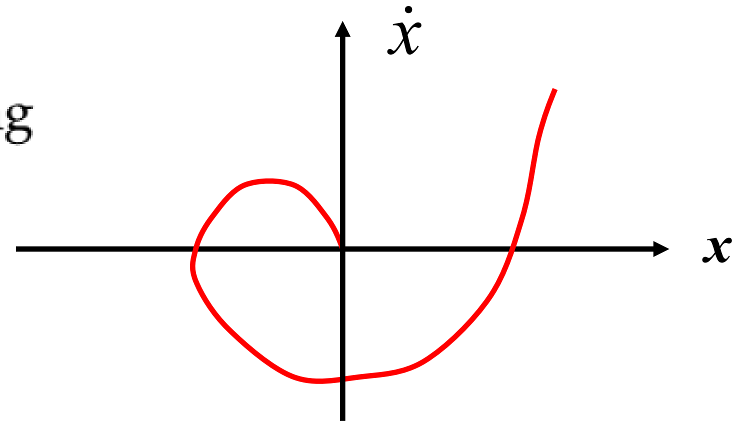
Let $x_1 = x$, $x_2 = \dot{x}$. Equation $\ddot{x} + f(x, \dot{x}) = 0$ becomes

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -f(x, \dot{x}) = -f(x_1, x_2) \end{cases} \xrightarrow{\text{Re-Define}} \begin{cases} \frac{dx_1}{dt} = f_1(x_1, x_2) \\ \frac{dx_2}{dt} = f_2(x_1, x_2) \end{cases}$$



Then we get a trajectory in the phase plane

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \text{ with } (x_{10}, x_{20}) \quad \Rightarrow \quad x_2 = \phi(x_1)$$



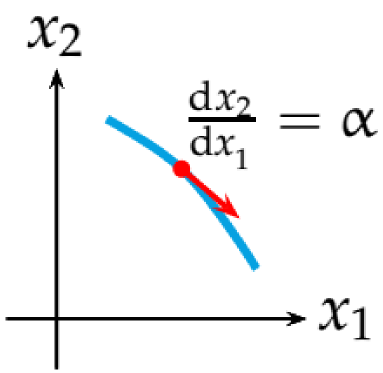
and at a specific point (x_1, x_2), $\frac{dx_2}{dx_1}$ is the direction of the motion along the trajectory.

Hence, we have phase plane portraits

$$\text{A set of } (x_{10}, x_{20}) \Rightarrow \text{A set of trajectories.}$$

Phase plane portraits

- Ordinary point $\frac{dx_2}{dx_1} = \alpha$ for a point (x_1, x_2)
  - Direction at this point is unique.
  - The trajectory started from an ordinary point is unique.



- Singular point $\frac{dx_2}{dx_1} = \frac{0}{0}$
  - The slope of the trajectory (i.e. the direction of the motion) is an uncertain value.
  - Infinite number of trajectories may depart from or arrive at this point.
  - The singular point is an equilibrium point. (Why?)
  - Isolated singular point: a singular point in the neighborhood where no other singular points exist.

Example

Find the singular points of a system $\ddot{x} + x = 0$.

Solutions

Let $x_1 = x$, $x_2 = \dot{x}$, then

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \dot{x} = x_2 \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \ddot{x} = -x = -x_1 \end{cases} \Rightarrow \quad (x_1 = 0, x_2 = 0) \quad \text{is the equilibrium point.}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{x_1}{x_2} \quad \Rightarrow \quad (x_1 = 0, x_2 = 0)$$

is a singular point and the only singular point.

NB: This is an undamped oscillation with $\dot{x}^2 + x^2 = R^2$.

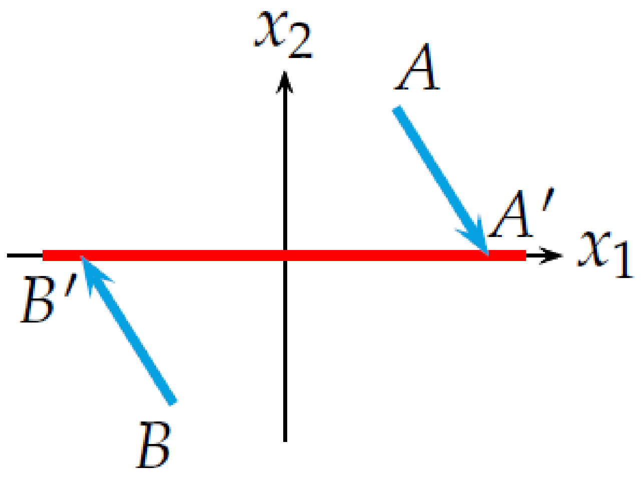
Example

Find the singular points of a system $\ddot{x} + \dot{x} = 0$.

Solutions

Let $x_1 = x$, $x_2 = \dot{x}$, then

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \dot{x} = x_2 \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \ddot{x} = -\dot{x} = -x_2 \end{cases}$$



$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{x_2}{x_2} \quad \Rightarrow$$ All point with $x_2 = 0$ are singular points.

i.e. all points on the x_1 -axis are singular points

Outline of Chapter 8

Motion along a trajectory

- Upper half plane

$$\dot{x} > 0 \quad \Rightarrow \quad x\uparrow$$

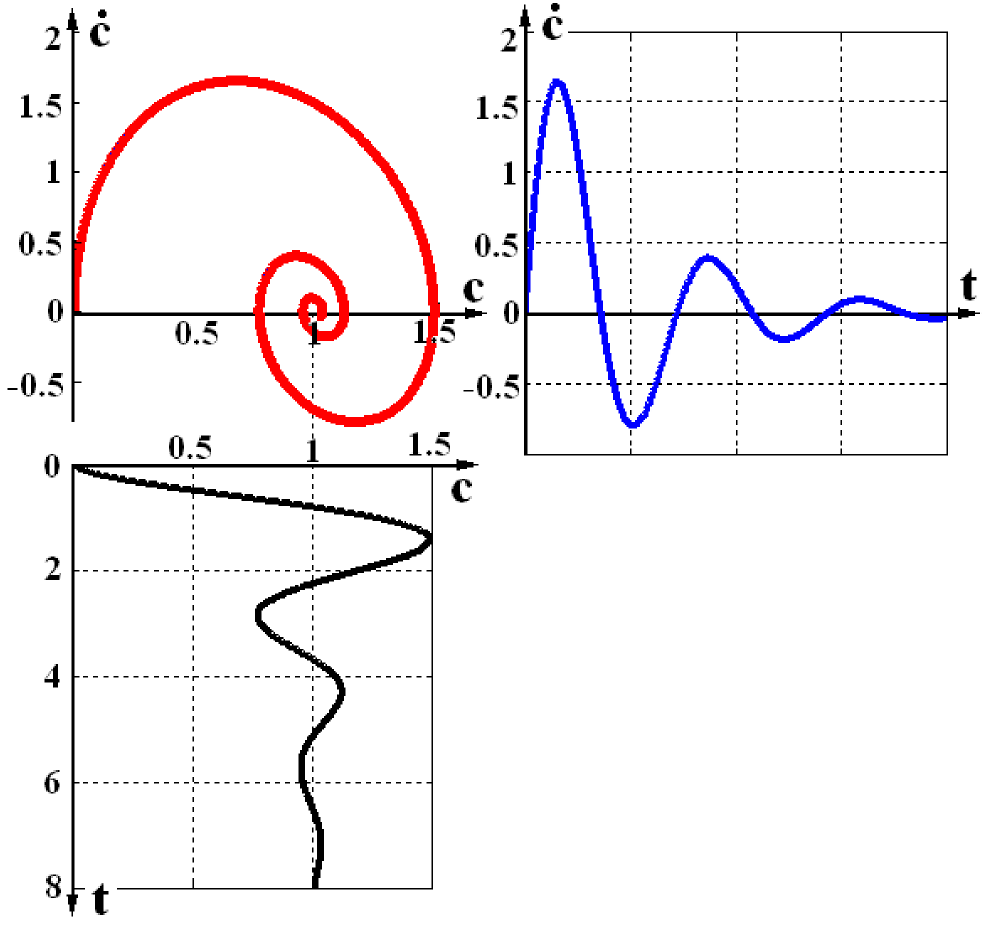
- Lower half plane

$$\dot{x} < 0 \quad \Rightarrow \quad x\downarrow$$

- Clockwise movement

- When the phase locus intersects with x axis, it always passes through with an angle of 90°

$$\frac{d\dot{x}}{dx} = \frac{d\dot{x}/dt}{dx/dt} = \frac{f(x,\dot{x})}{\dot{x}}$$



Symmetry in phase plane portraits

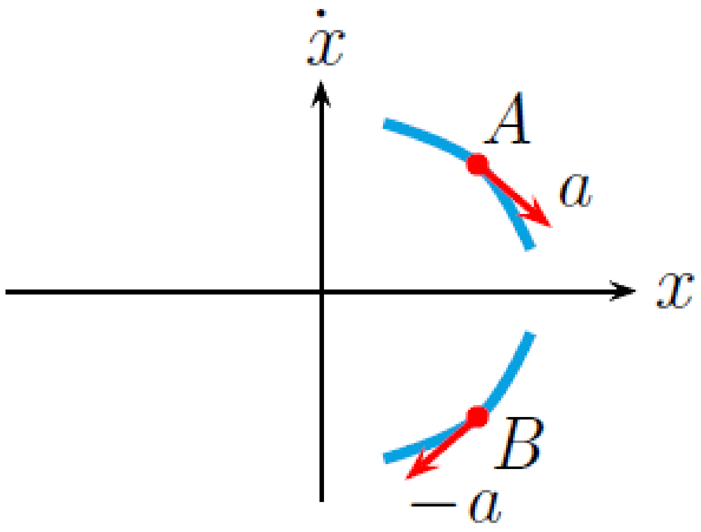
Case 1: Symmetry about the x-axis

$$A:\ \frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(x,\dot{x})}{\dot{x}} = a$$

$$B:\ \frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(x,-\dot{x})}{-\dot{x}} = -a$$

$$\Rightarrow f(x,\dot{x}) = f(x,-\dot{x})$$

i.e. $f(x,\dot{x})$ is even function of $\dot{x}$.



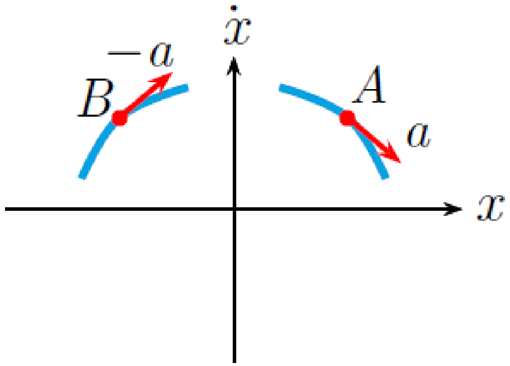
即$f(x,\dot{x})$ 是 $\dot{x}$ 的偶函数，是相轨迹对称于 x 轴的条件。

Symmetry in phase plane portraits

Case 2: Symmetry about the $\dot{x}$-axis

$$A: \ \frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(x,\dot{x})}{\dot{x}} = a$$
$$B: \ \frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(-x,\dot{x})}{-\dot{x}} = -a$$
$$\Rightarrow f(x,\dot{x}) = -f(-x,\dot{x})$$

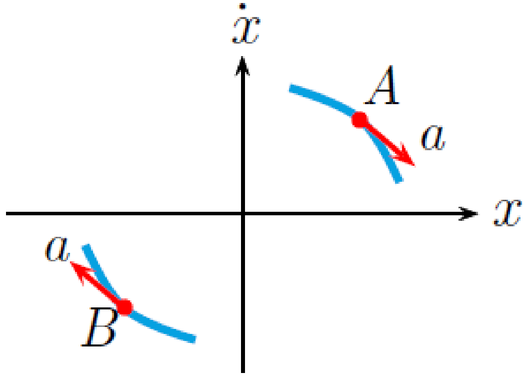
i.e. $f(x,\dot{x})$ is odd function of x.



即 $f(x,\dot{x})$ 是 x 的奇函数是相轨迹对称于 $\dot{x}$ 轴的条件

Case 3: Symmetry about the origin

$$A: \ \frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(x,\dot{x})}{\dot{x}} = a$$
$$B: \ \frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(-x,-\dot{x})}{-\dot{x}} = a$$
$$\Rightarrow f(x,\dot{x}) = -f(-x,-\dot{x})$$



若相轨迹对称于原点，其条件是：对称点上的斜率应大小相等，符号相同

Outline of Chapter 8

Example: Draw the phase plane portrait of $\ddot{x} + \omega_n^2 x = 0$

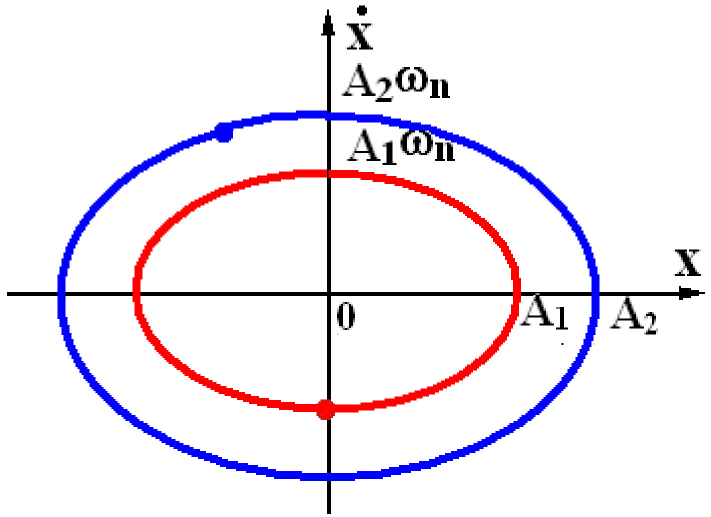
$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \cdot \frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx} = -\omega_n^2 x$$

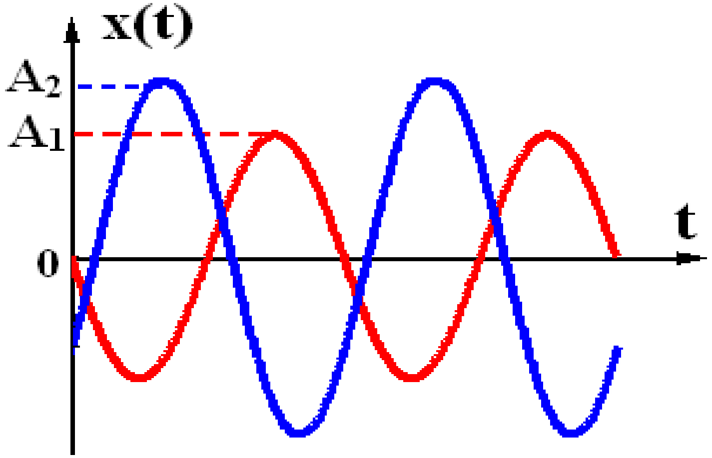
$$\dot{x} \cdot d\dot{x} = -\omega_n^2 x \cdot dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{-\omega_n^2}{2} \cdot x^2 + C$$

$$x^2 + \frac{\dot{x}^2}{\omega_n^2} = \frac{2C}{\omega_n^2} = A^2$$

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2\omega_n^2} = 1$$ — Elliptic equation





Isocline methods for phase plane portraits

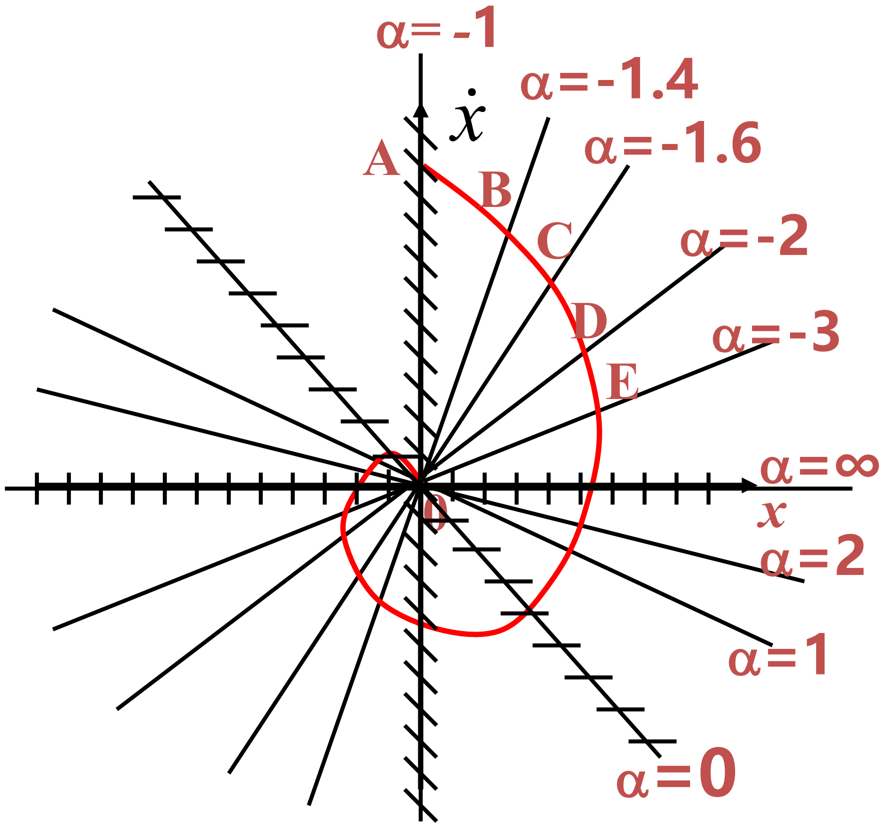
等倾斜线法原理:<u>任一曲线都可以用一系列足够短的折线来近似</u>

$$\ddot{x} + f(x, \dot{x}) = 0$$

$$\ddot{x} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx} = -f(x, \dot{x})$$

$$\frac{d\dot{x}}{dx} = \frac{-f(x, \dot{x})}{\dot{x}} \qquad \alpha = \frac{-f(x, \dot{x})}{\dot{x}}$$

等倾斜线方程



Example: Draw the phase portraits of the dynamic system

$$\ddot{x} + \dot{x} + x = 0$$

Solution

$$\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} = -(x + \dot{x})$$

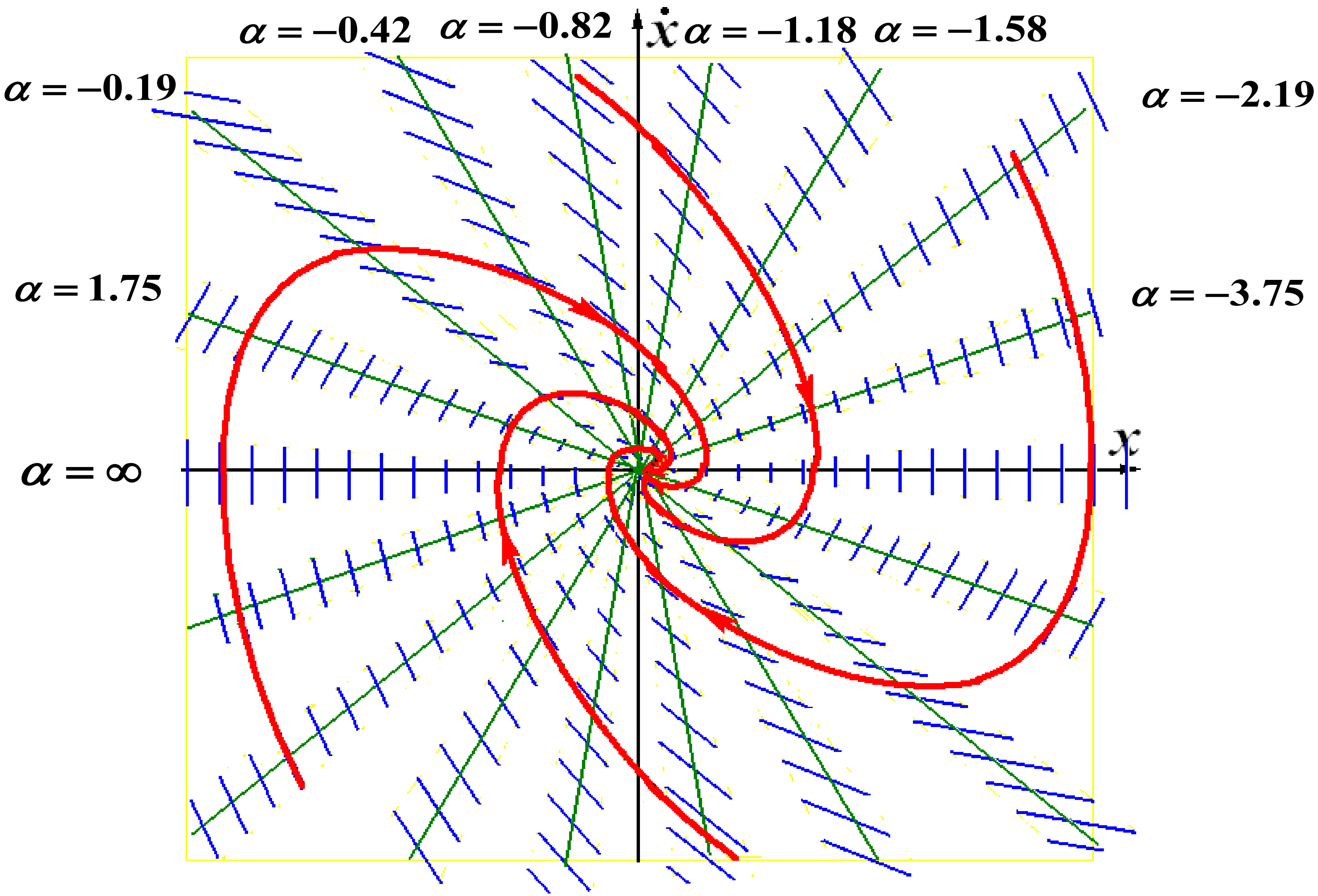
$$\alpha = \frac{-f(x,\,\dot{x})}{\dot{x}} = \frac{-(x + \dot{x})}{\dot{x}}$$

Isocline equation,

$$\dot{x} = \frac{-x}{1 + \alpha} \qquad \left(\theta = \arctan\left(\frac{-1}{1 + \alpha}\right)\right)$$

$$\dot{x} = \frac{-x}{1+\alpha}$$

α	-3.75	-2.19	-1.58	-1.18	-0.82	-0.42	-0.19	1.75	∞
$-1/(1+\alpha)$	0.36	0.84	1.73	5.67	-5.76	-1.73	-0.84	-0.36	0.00
$\arctan\left(\frac{-1}{1+\alpha}\right)$	20°	40°	60°	80°	100°	120°	140°	160°	180°



Isocline methods for phase plane portraits(con.)

NOTES:

- ✓ Choose the same scale for horizontal and vertical axes.
- ✓ The direction of the arrow on the phase trajectories are always in a clockwise direction.
- ✓ Except $\frac{d\dot{x}}{dx} = \frac{0}{0}$, phase plane trajectories intersect perpendicularly the x-axis.
- ✓ The accuracy of the isocline method depends on the distribution density of the isoclinic lines. (Generally the isoclinic line interval set as 5-10 degrees)
- ✓ For linear systems, isocline is a simple straight isocline. For nonlinear systems, the isocline is no longer a simple straight line but curve.

由相轨迹求时间解

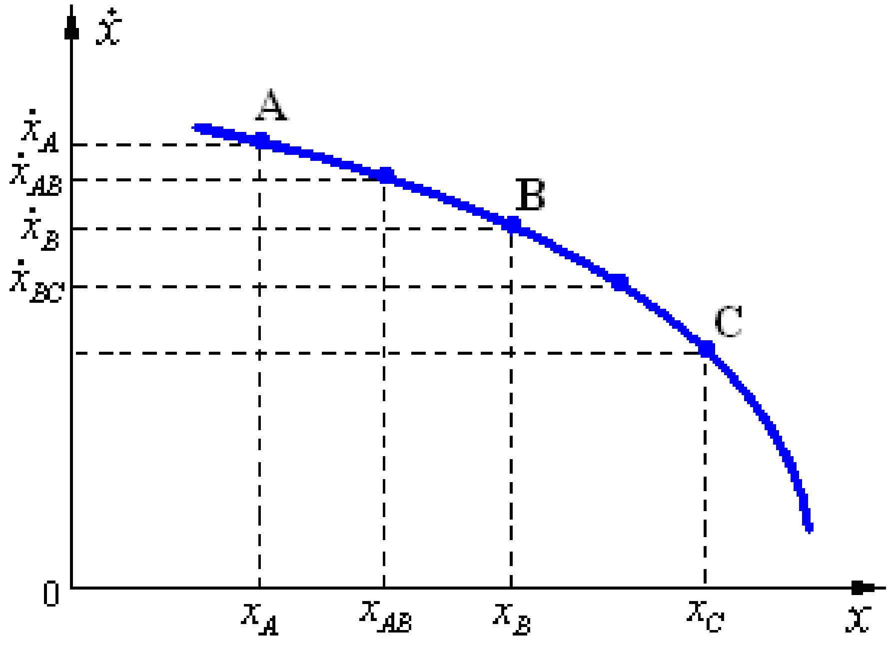
相轨迹A-B段的平均速度：

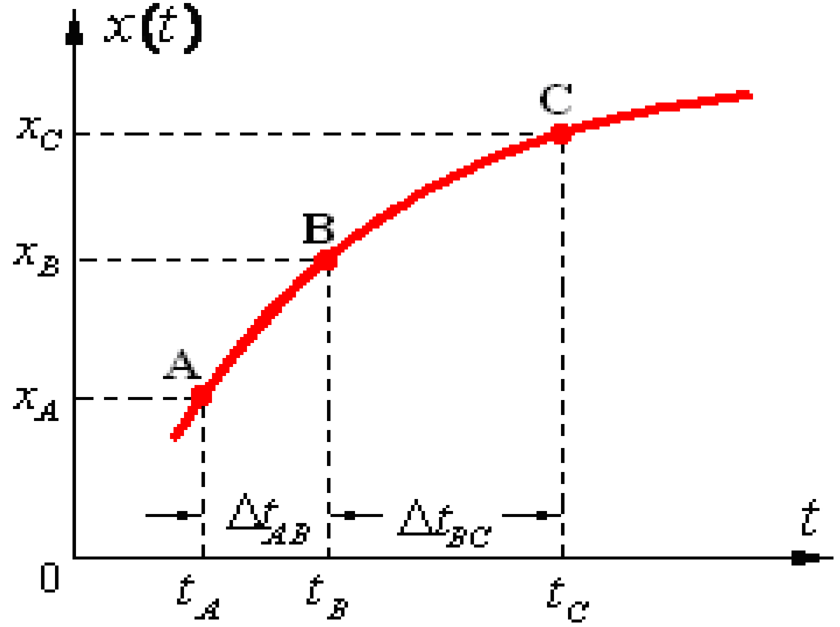
$$\dot{x}_{AB}=\frac{\Delta x}{\Delta t}=\frac{x_B-x_A}{\Delta t_{AB}}$$

$$\dot{x}_{AB}=\frac{\dot{x}_A+\dot{x}_B}{2}$$

相轨迹A-B段所用的时间：

$$\Delta t_{AB}=\frac{2(x_B-x_A)}{\dot{x}_A+\dot{x}_B}$$





Outline of Chapter 8

Singular points

Let $x_1 = x$, $x_2 = \dot{x}$. Equation $\ddot{x} + f(x, \dot{x}) = 0$ becomes

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -f(x, \dot{x}) = -f(x_1, x_2) \end{cases} \xrightarrow{\text{Re-Define}} \begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2) \end{cases}$$

Definition

A singular point is the point, which satisfies

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2) = 0$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2) = 0$$

i.e. $\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{0}{0}$, and motion cannot be determined with $\frac{\mathrm{d}x_2}{\mathrm{d}x_1}$.

Supposing a nonlinear system represented by $\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$

where at least one of $f_1(x), f_2(x)$ is nonlinear.

- ✓ Determine the singular points according to the definition.
- ✓ For each singular point X_e, using Taylor expansion, the above equation can be rewritten as,

$$\dot{x}_1 = a_1 x_1 + b_1 x_2 + g_1(x_1, x_2)$$
$$\dot{x}_2 = a_2 x_1 + b_2 x_2 + g_2(x_1, x_2)$$

where $g_1 \; and \; g_2$ contain higher order terms,

$$a_1 = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_1}\right|_{X_e} \qquad b_1 = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_2}\right|_{X_e} \qquad a_2 = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_1}\right|_{X_e} \qquad b_2 = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_2}\right|_{X_e}$$

- ✓ The linearized equation:

$$\dot{x}_1 = a_1 x_1 + b_1 x_2$$
$$\dot{x}_2 = a_2 x_1 + b_2 x_2$$

- ✓ For each singular point, the local behavior of the nonlinear system can be approximated by the linearized equation.

$$\dot{x}_1 = a_1 x_1 + b_1 x_2$$
$$\dot{x}_2 = a_2 x_1 + b_2 x_2$$

Let $x = x_1$, then

$$\dot{x} = \dot{x}_1 = a_1 x_1 + b_1 x_2 = a_1 x + b_1 x_2$$
$$\ddot{x} = \ddot{x}_1 = a_1 \dot{x} + b_1 \dot{x}_2 = a_1 \dot{x} + b_1 (a_2 x_1 + b_2 x_2) = a_1 \dot{x} + b_1 a_2 x + b_1 b_2 x_2$$

Since $\dot{x} = a_1 x + b_1 x_2 \longrightarrow b_1 x_2 = \dot{x} - a_1 x$,

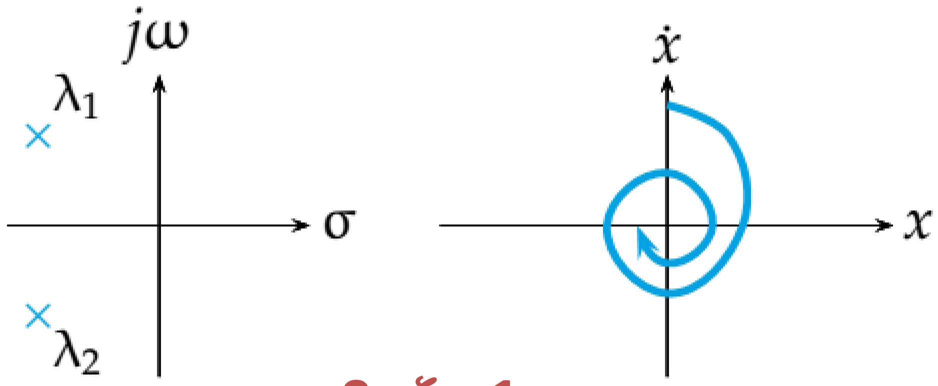
$$\ddot{x} = a_1 \dot{x} + b_1 a_2 x + b_2 (\dot{x} - a_1 x) = (a_1 + b_2)\dot{x} + (b_1 a_2 - b_2 a_1) x$$

That is $\ddot{x} + a\dot{x} + bx = 0$, where $a = -(a_1 + b_2)$ and $b = a_1 b_2 - a_2 b_1$.
The roots of the char. eqn $\lambda^2 + a\lambda + b = 0$ are

$$\boxed{\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}}$$

Classification of singular points

(1) Stable focus



$0<\zeta<1$

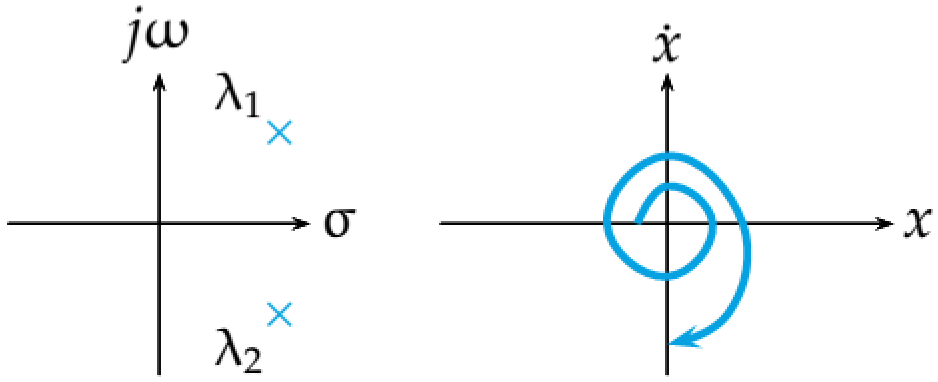
$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n{}^2x=0$$

$$s_{1,2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

$$\ddot{x}+a\dot{x}+bx=0$$

$$\boxed{\lambda_{1,2}=\frac{-a\pm\sqrt{a^2-4b}}{2}}$$

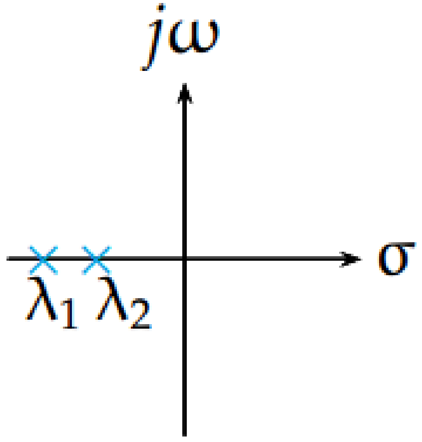
(2) Unstable focus

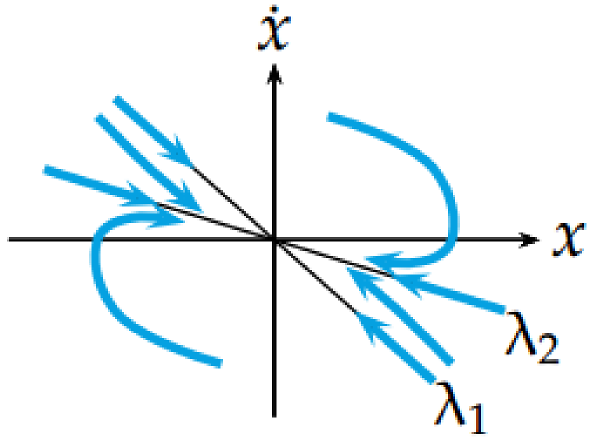


$-1<\zeta<0$

(3) Stable node

ζ>=1





ζ>1

$$\lambda_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$\lambda_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

相轨迹包含两条特殊的等倾线，斜率分别等于两个根，初始值落在这两条直线上，相轨迹沿直线趋于原点。除此之外沿K_1趋于原点。

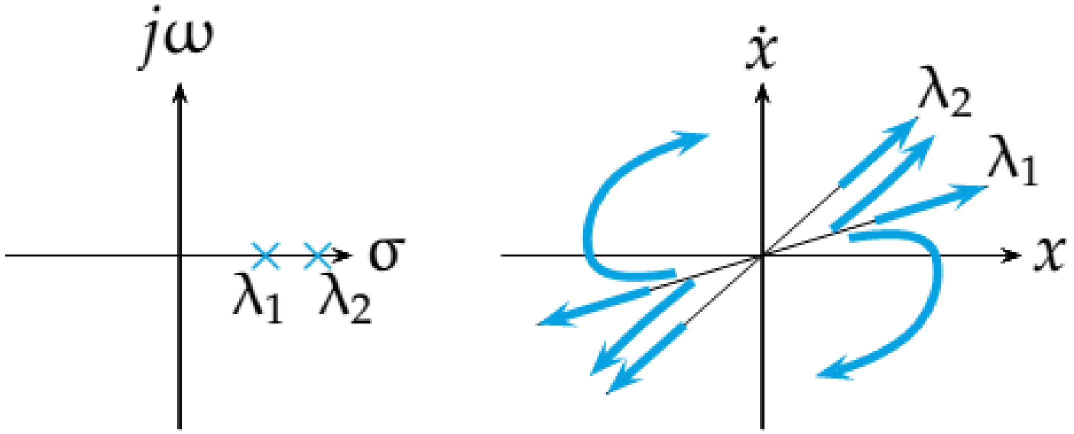
ζ=1

$$\lambda_{1,2} = -\zeta\omega_n$$

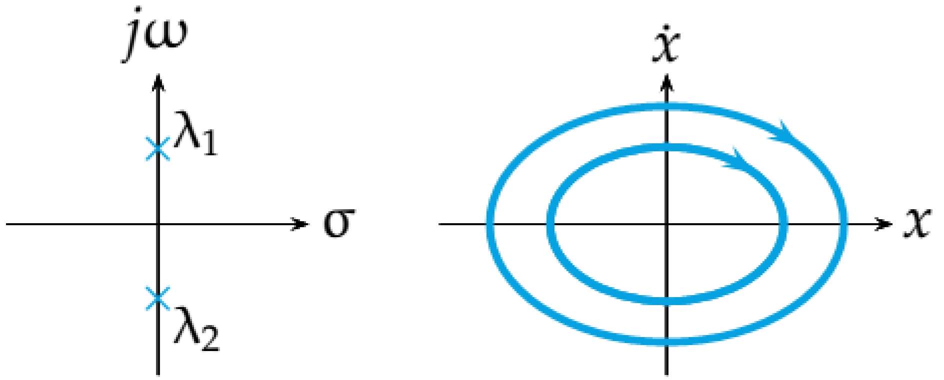
相轨迹包含一根特殊的等倾线，斜率等于根，不同初始条件的相轨迹最终将沿这条特殊的等倾线趋于原点。平衡点为稳定的节点。

(4) Unstable node

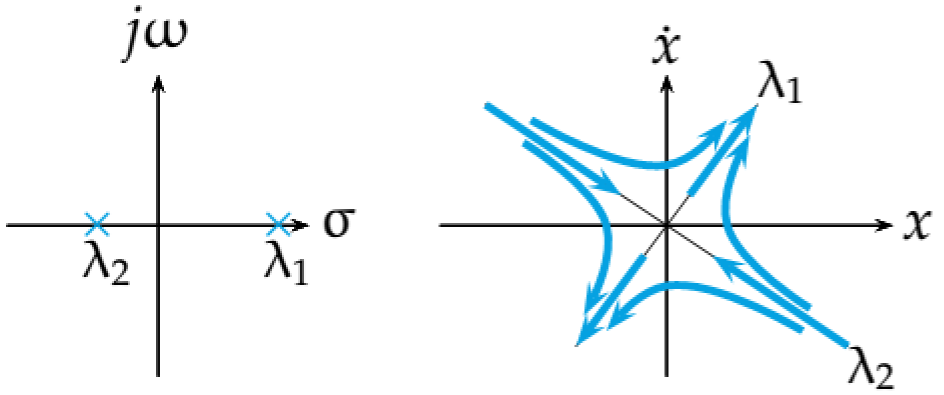
$\zeta \leq$ **-1**



(5) Center



(6) Saddle point

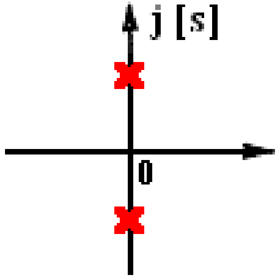
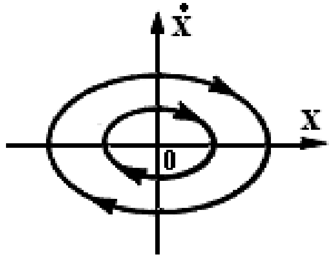
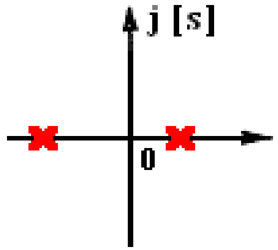
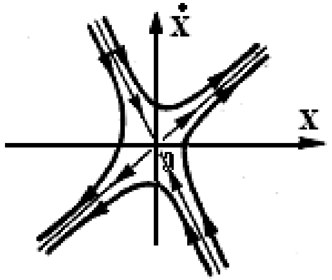
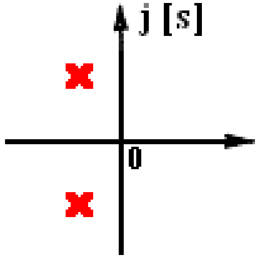
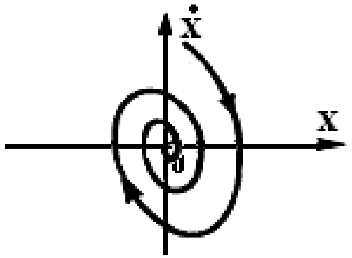
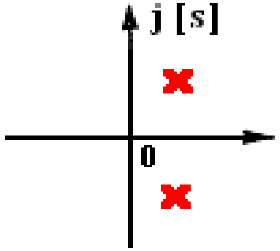
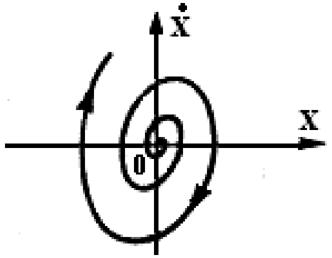
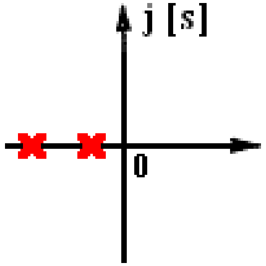
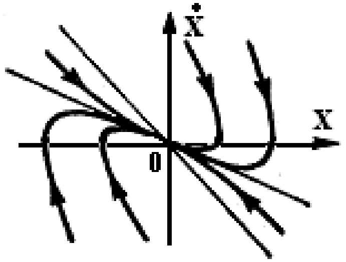
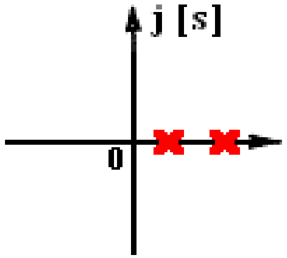
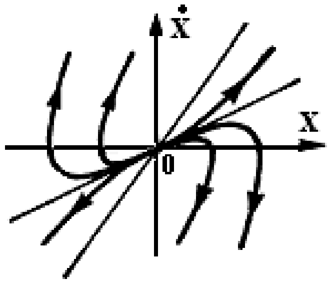


b<0时，两条等倾线既是相轨迹，又将相平面分成四个区域。只有初始值落在负斜率的等倾线上，运动将趋于原点。即使这种情况，如受到微小的扰动，将偏离该轨迹，发散至无穷。

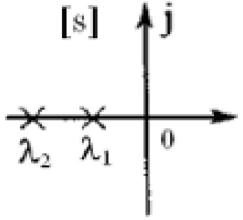
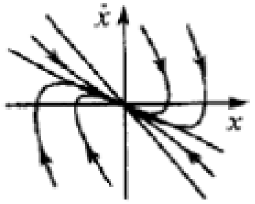
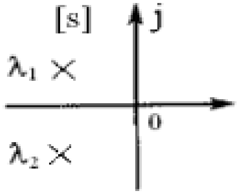
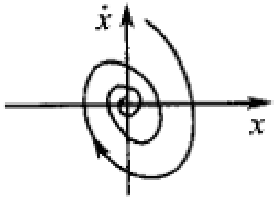
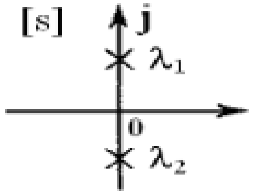
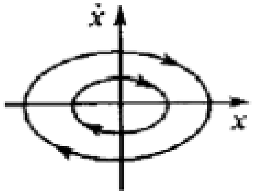
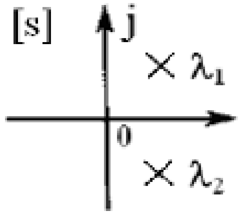
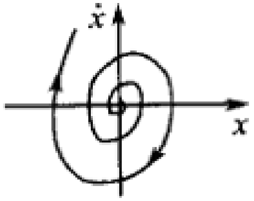
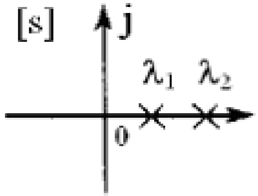
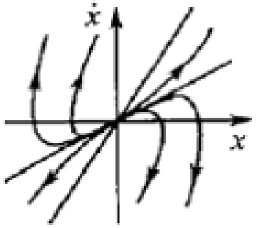
Phase locus of second order linear systems

$$\Phi(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

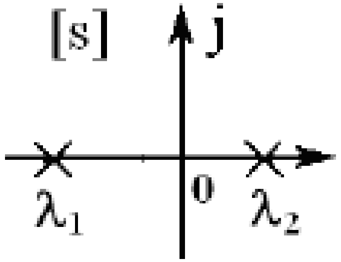
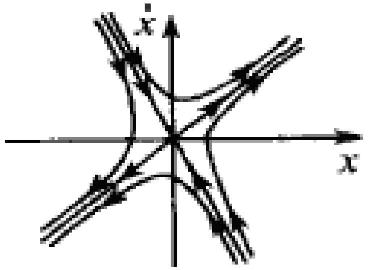
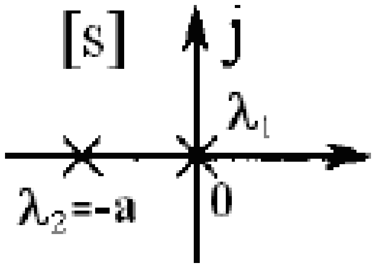
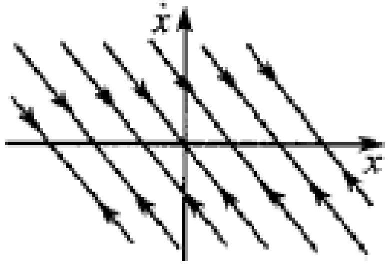
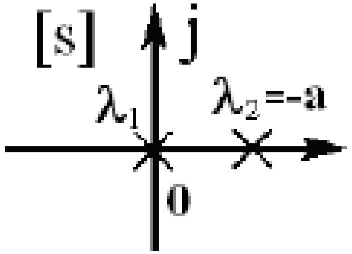
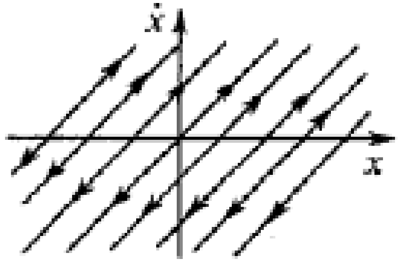
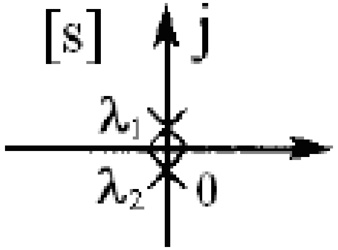
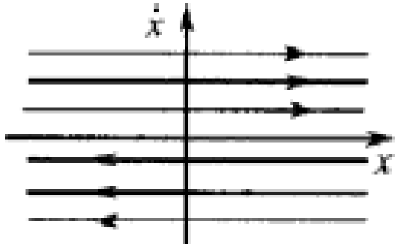
$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n{}^2x=0$$

Location of poles	Singular point	Phase locus	Location of poles	Singular point	Phase locus
	center point			Saddle point	
	stable focus			unstable focus	
	stable node			unstable node	

二阶线性系统的相轨迹

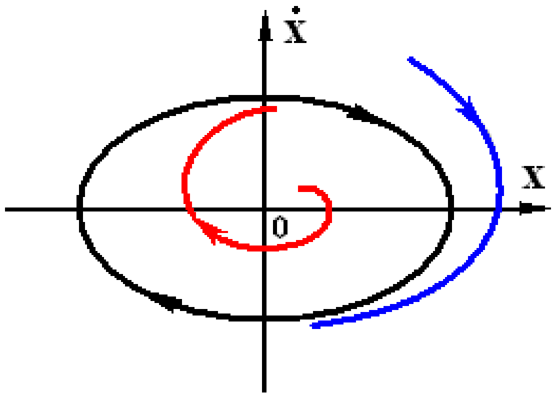
序号	系统方程 方程	系统方程 参数	极点分布	相轨迹	奇点	相轨迹方程
1		$\xi \geq 1$			(0，0) 稳定节点	抛物线（收敛） 特殊相轨迹： $\begin{cases} \dot{x}=\lambda_1 x \\ \dot{x}=\lambda_2 x \end{cases}$
2	$\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2x=0$	$0<\xi<1$			(0，0) 稳定焦点	螺线（收敛）
3		$\xi=0$			(0，0) 中心点	椭圆
4		$-1<\xi<0$			(0，0) 不稳定焦点	螺线（发散）
5		$\xi<-1$			(0，0) 不稳定节点	抛物线（发散） 特殊相轨迹： $\begin{cases} \dot{x}=\lambda_1 x \\ \dot{x}=\lambda_2 x \end{cases}$

Phase locus of second order linear systems (con.)

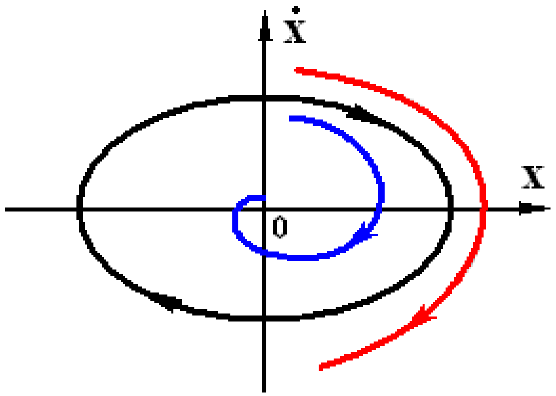
6		$\begin{cases} a \text{ 任意} \\ b > 0 \end{cases}$			(0，0) 鞍点	双曲线 特殊相轨迹： $\begin{cases} \dot{x} = \lambda_1 x \\ \dot{x} = \lambda_2 x \end{cases}$
7	$\ddot{x} + a\dot{x} - bx = 0$	$\begin{cases} a > 0 \\ b = 0 \end{cases}$			x 轴	$\begin{cases} \dot{x} = 0 \\ \dot{x} = -ax + C \end{cases}$
8		$\begin{cases} a < 0 \\ b = 0 \end{cases}$			x 轴	$\begin{cases} \dot{x} = 0 \\ \dot{x} = -ax + C \end{cases}$
9		$\begin{cases} a = 0 \\ b = 0 \end{cases}$			x 轴	$\dot{x} = C$

Limit cycles

The limit cycle is an isolated closed path in the phase plane.



Stable limit cycle



unstable limit cycle

Semi-stable limit cycle

Limit cycles

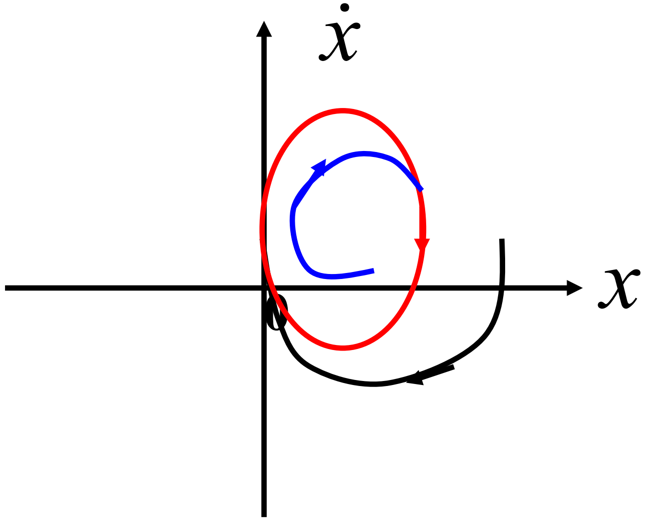
The limit cycle is an isolated closed path in the phase plane.

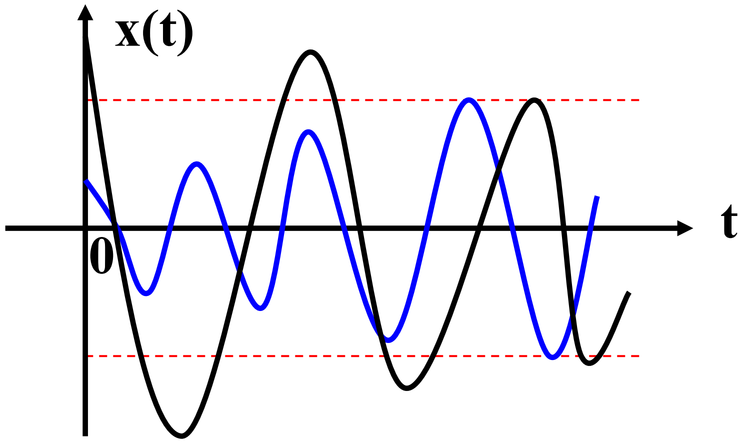
- 将相平面划分为具有不同运动特点的多个区域的特殊相轨迹，称为奇线。
- 极限环对应于非线性系统特有的自振荡现象，它描述了自振荡的振幅和频率。
- 在相平面图中，极限环是孤立的封闭轨迹。
- 非线性系统的极限环情况比较复杂，不同的系统会有不同形式的极限环。

$$\text{极限环可分为：}\begin{cases}\text{稳定极限环}\\\text{不稳定极限环}\\\text{半稳定极限环}\end{cases}$$

Limit cycles

稳定极限环





特点：极限环内外的相轨迹都卷向极限环，自振荡是稳定的.

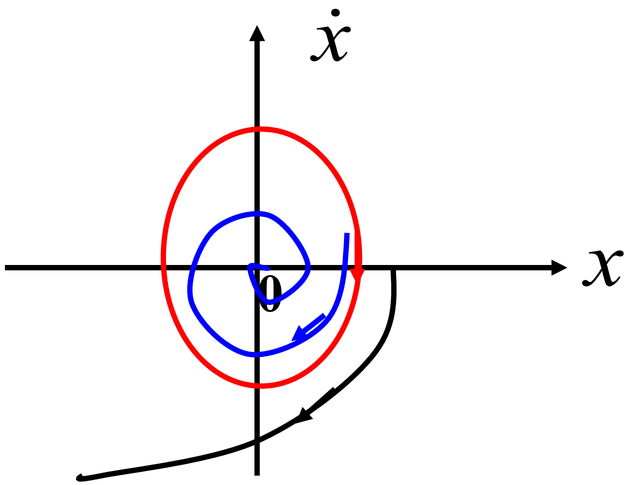
环内：不稳定区域，相轨迹发散

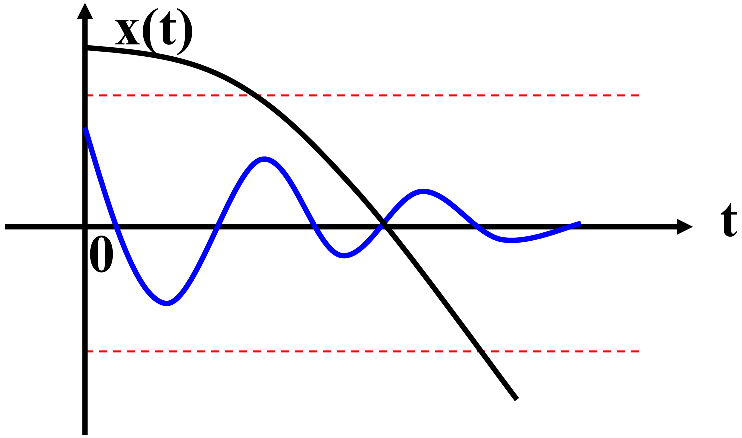
环外：稳定区域，相轨迹收敛

趋向极限环

Limit cycles

不稳定极限环





特点:极限环内外的相轨迹都卷离极限环

环内:稳定区域,相轨迹收敛
环外:不稳定区域,相轨迹发散

卷离极限环

这种系统是小范围稳定,大范围不稳定.设计时应尽量增大稳定区域(即增大极限环).

Limit cycles

The limit cycle is an isolated closed path in the phase plane.

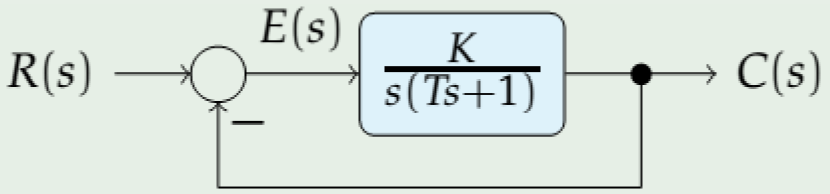
注意:

- 在非线性系统中，可能没有极限环，也可能具有一个或几个极限环。
- 在进行一般系统设计时，应尽量避免产生极限环。如不可能避免时，应尽量缩小稳定的极限环，或加大不稳定的极限环。
- 振荡器是具有稳定极限环的非线性系统的典型例子。

Outline of Chapter 8

Example

Obtain the phase plane trajectories of the step response of the given 2nd-order system.



Solutions

Basic equations:

$$\frac{C(s)}{E(s)} = \frac{K}{Ts^2 + s} \Rightarrow \begin{cases} T\ddot{c} + \dot{c} = Ke & (1) \\ e = r - c & (2) \end{cases}$$

$$C(0) = 0, \dot{C}(0) = 0$$

$$T\ddot{e} + \dot{e} + ke = T\ddot{r} + \dot{r}$$ （在相平面分析法中，一般皆分析误差函数）

由初始条件 $\rightarrow$ 误差的初始条件为：

$$e(0^+) = R, \quad \dot{e}(0^+) = 0 \quad (\because e = r - c)$$

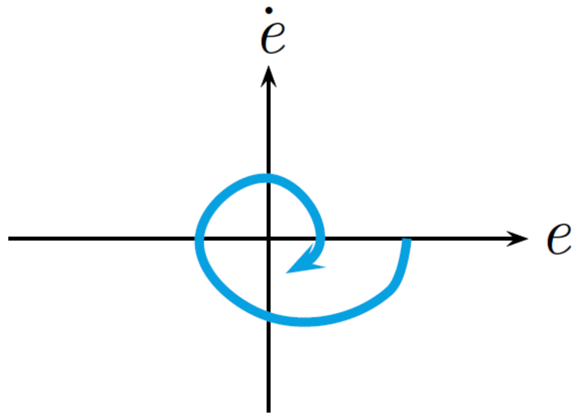
$\because T > 0, K > 0$ 特征根只有以下两种情 况：

- Properties of $(0,0)$

The char. eqn. is $T\lambda^2+\lambda+K=0$.

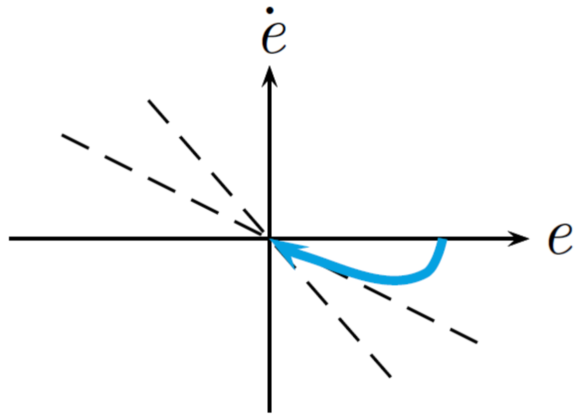
When $1-4KT<0$

$\Rightarrow$ stable focus



When $1-4KT\geqslant 0$

$\Rightarrow$ stable node



线性系统 $\begin{cases}\text{相平面图及奇点的性质取决于特征根的分布.} \\ \text{奇点位置与初始条件取决于输入信号.}\end{cases}$

斜坡响应

输入的斜坡信号为:

$$r(t) = R + vt$$

R, v 皆为常数.

当$t > 0$时， $\dot{r} = v,\ \ddot{r} = 0$

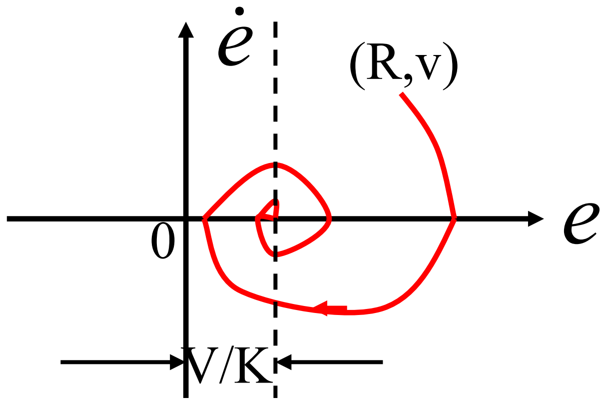
则误差方程变为:

$$T\ddot{e} + \dot{e} + Ke = v, \quad 或\quad T\ddot{e} + \dot{e} + K\left(e - \frac{v}{K}\right) = 0$$

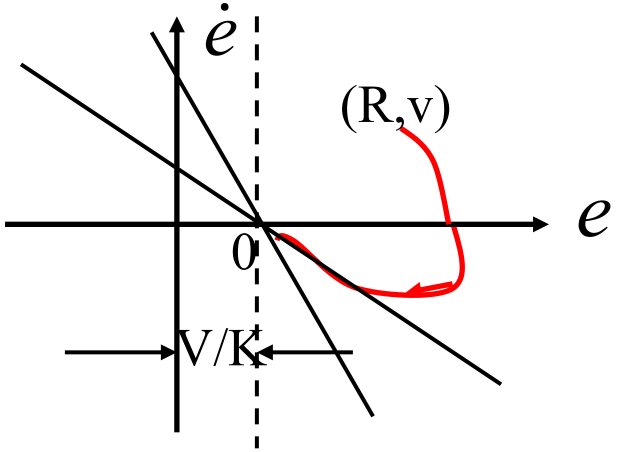
奇点为： $\left({}^{v}\!/\!{}_{K}, 0\right)$

误差的初始条件为： $e(0) = r(0) - c(0) = R - 0 = R$

$\dot{e}(0) = \dot{r}(0) - \dot{c}(0) = v - 0 = v$



a) 欠阻尼



b) 过阻尼

分析可知，相轨迹与阶跃输入时相同，只是向右移动了 v/K 一段距离。

$$\text{线性系统}\begin{cases}\text{相平面图及奇点的性质取决于特征根的分布.}\\ \text{奇点位置与初始条件取决于输入信号.}\end{cases}$$

Outline of Chapter 8

8.1 Phase plane portraits

8.2 Properties of phase plane

8.3 Construction of phase plane portraits

8.4 Singular points and limit cycles

8.5 Phase plane analysis of linear systems

8.6 Phase plane analysis of nonlinear systems

8.7 Simulations with MATLAB

8.8 Summary

Example

Obtain the phase plane portrait of $\ddot{x} + 0.5\dot{x} + 2x + x^2 = 0$

Solutions

(i) Standard form

Let $x_1 = x$ and $x_2 = \dot{x}$, then

$$\dot{x}_1 = \dot{x} = x_2 \triangleq f_1(x_1, x_2)$$

$$\dot{x}_2 = \ddot{x} = -0.5\dot{x} - 2x - x^2 = -2x_1 - 0.5x_2 - x_1^2 \triangleq f_2(x_1, x_2)$$

(ii) Singular points

$$\dot{x}_1 = x_2 = 0, \quad \dot{x}_2 = -2x_1 - 0.5x_2 - x_1^2 = 0$$

Hence, singular points are

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \begin{cases} x_1 = -2 \\ x_2 = 0 \end{cases}, \quad \text{or equivalently} \quad \begin{cases} x = 0 \\ \dot{x} = 0 \end{cases} \begin{cases} x = -2 \\ \dot{x} = 0 \end{cases}$$

(iii) Properties of the singular points

- Singular point (0, 0)

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{(0,0)} = 0, \qquad \left.\frac{\partial f_1}{\partial x_2}\right|_{(0,0)} = 1$$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} = (-2 - 2x_1)\big|_{x_1=0} = -2, \qquad \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} = -0.5$$

The linearized equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - 0.5x_2 \end{cases} \quad \text{i.e.} \quad \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} \\ \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = -2x - 0.5\dot{x} \end{cases}$$

$$a_1 = 0,\ b_1 = 1,\ a_2 = -2,\ b_2 = -0.5 \quad \Rightarrow \quad \ddot{x} + 0.5\dot{x} + 2x = 0$$

Thus the char. eqn. is $\lambda^2 + 0.5\lambda + 2 = 0$, and $\lambda_{1,2} = -0.25 \pm \jmath 1.987$.

Therefore, $(0,0)$ is a stable focus.

- Singular point (−2, 0)

Let $y = x + 2$, we obtain $\ddot{y} + 0.5\dot{y} - 2y + y^2 = 0$

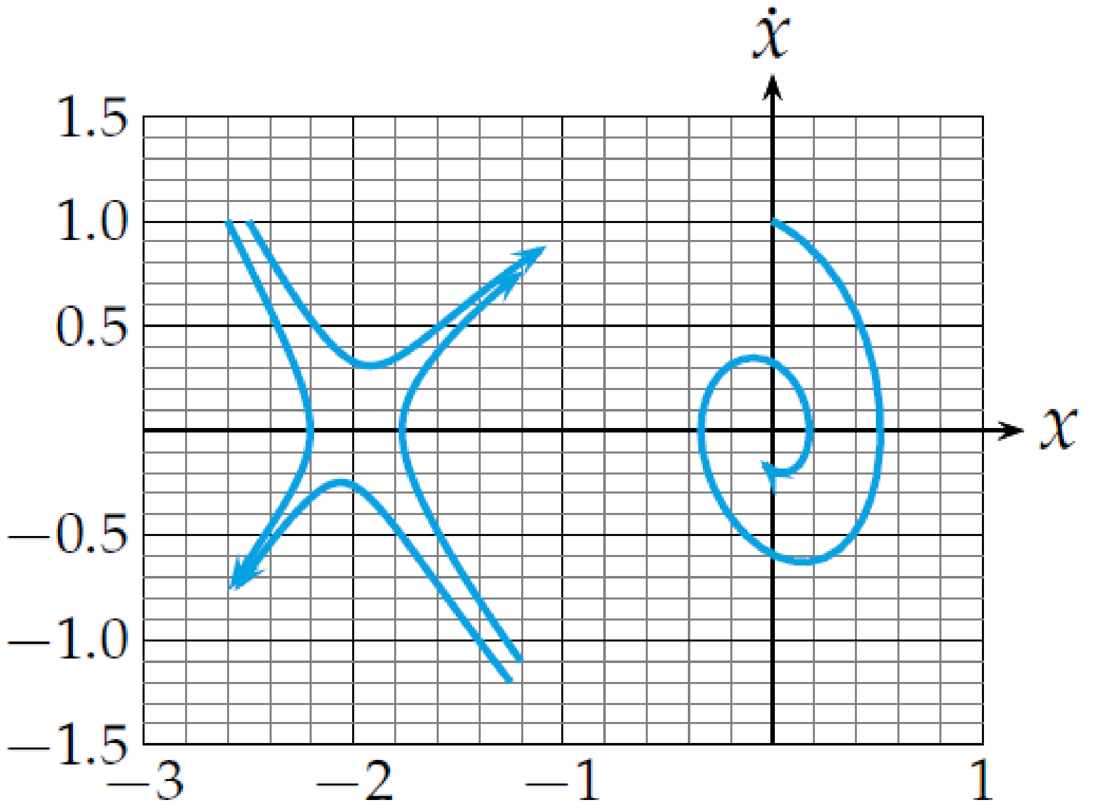
In the similar way, we have

$$\ddot{y} + 0.5\dot{y} - 2y = 0$$

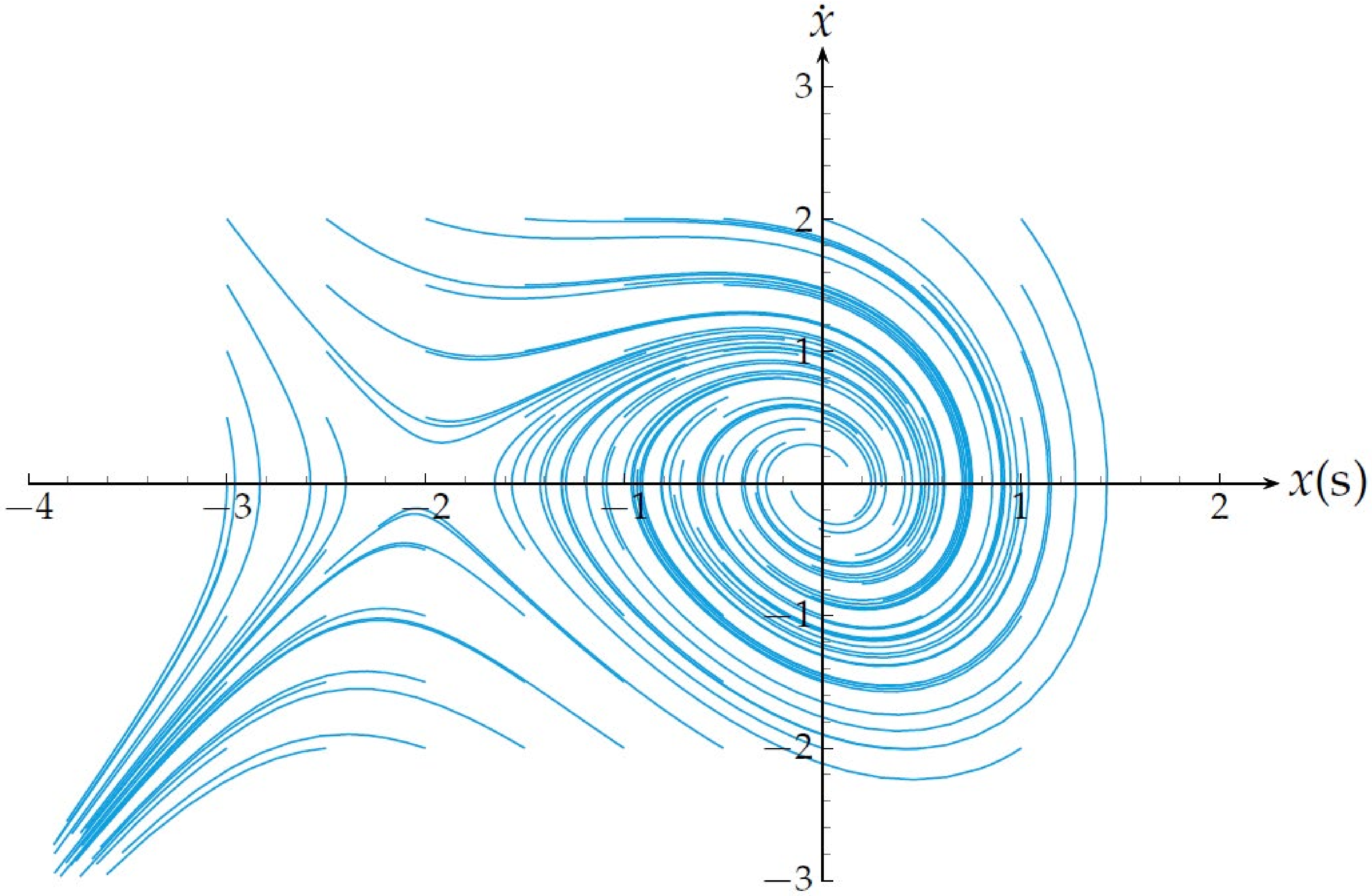
$$\lambda^2 + 0.5\lambda - 2 = 0$$

$$\lambda_{1,2} = 1.186, \ -1.686$$

Therefore, $(-2, 0)$ is a saddle point.



(iv) Actual phase plane portrait



非本质非线性系统的相平面分析

Obtain the phase plane portrait of $\ddot{x}+(3\dot{x}-0.5)\dot{x}+x+x^2=0$

$$\ddot{x}=\dot{x}=0$$
$$x+x^2=x(1+x)=0$$

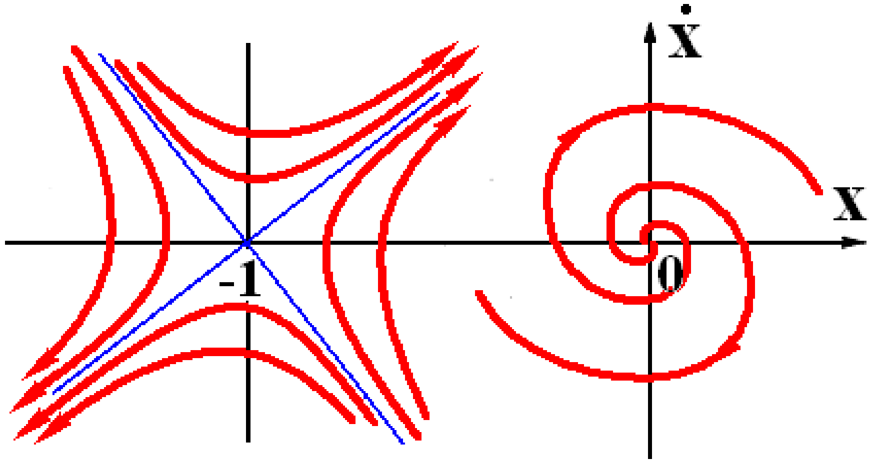
Singular point $\begin{cases} x_{e1}=0 \\ x_{e2}=-1 \end{cases}$ $\begin{cases} x=\Delta x+x_{e1}=\Delta x \\ x=\Delta x+x_{e2}=\Delta x-1 \end{cases}$

$$\begin{cases} \Delta\ddot{x}-0.5\Delta\dot{x}+\Delta x=0 \\ \Delta\ddot{x}-0.5\Delta\dot{x}+(\Delta x-1)+(\Delta x-1)^2=0 \end{cases}$$

$$\begin{cases} \Delta\ddot{x}-0.5\Delta\dot{x}+\Delta x=0 \\ \Delta\ddot{x}-0.5\Delta\dot{x}-\Delta x=0 \end{cases}$$

Characteristic equation $\begin{cases} s^2-0.5s+1=0 \\ s^2-0.5s-1=0 \end{cases}$

Poles $\begin{cases} s=0.25\pm j0.97 \\ s=\begin{cases} 0.78 \\ -1.28 \end{cases} \end{cases}$ unstable focus; Saddle point



非本质非线性系统的相平面分析

Obtain the phase plane portrait of $\ddot{x}+(3\dot{x}-0.5)\dot{x}+x+x^2=0$

$$\ddot{x}=\dot{x}=0$$
$$x+x^2=x(1+x)=0$$

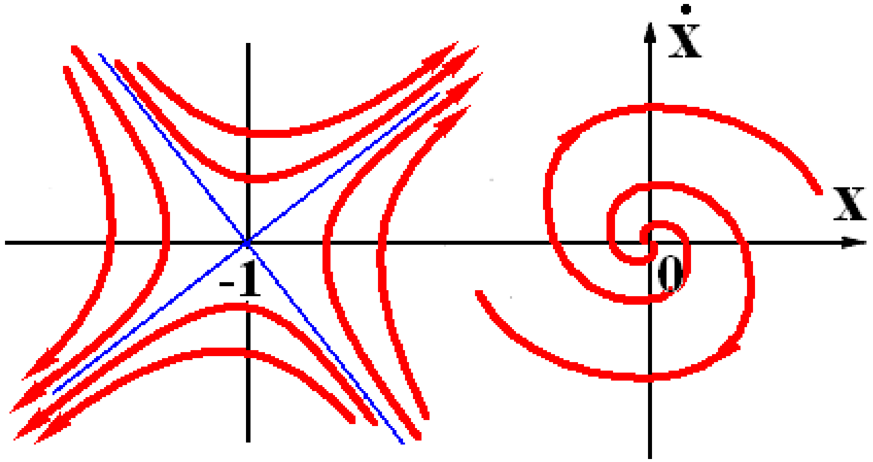
Singular point $\begin{cases} x_{e1}=0 \\ x_{e2}=-1 \end{cases}$ $\begin{cases} x=\Delta x+x_{e1}=\Delta x \\ x=\Delta x+x_{e2}=\Delta x-1 \end{cases}$

$$\begin{cases} \Delta\ddot{x}-0.5\Delta\dot{x}+\Delta x=0 \\ \Delta\ddot{x}-0.5\Delta\dot{x}+(\Delta x-1)+(\Delta x-1)^2=0 \end{cases}$$

$$\begin{cases} \Delta\ddot{x}-0.5\Delta\dot{x}+\Delta x=0 \\ \Delta\ddot{x}-0.5\Delta\dot{x}-\Delta x=0 \end{cases}$$

Characteristic equation $\begin{cases} s^2-0.5s+1=0 \\ s^2-0.5s-1=0 \end{cases}$

Poles $\begin{cases} s=0.25\pm j0.97 & \text{unstable focus} \\ s=\begin{cases} 0.78 \\ -1.28 \end{cases} & \text{Saddle point} \end{cases}$



非本质非线性系统的相平面分析

系统方程：$\ddot{x}+\sin x=0$,求x_e及其附近相轨迹的性质。

非本质非线性系统的相平面分析

系统方程：$\ddot{x}+\sin x=0$,求x_e及其附近相轨迹的性质。

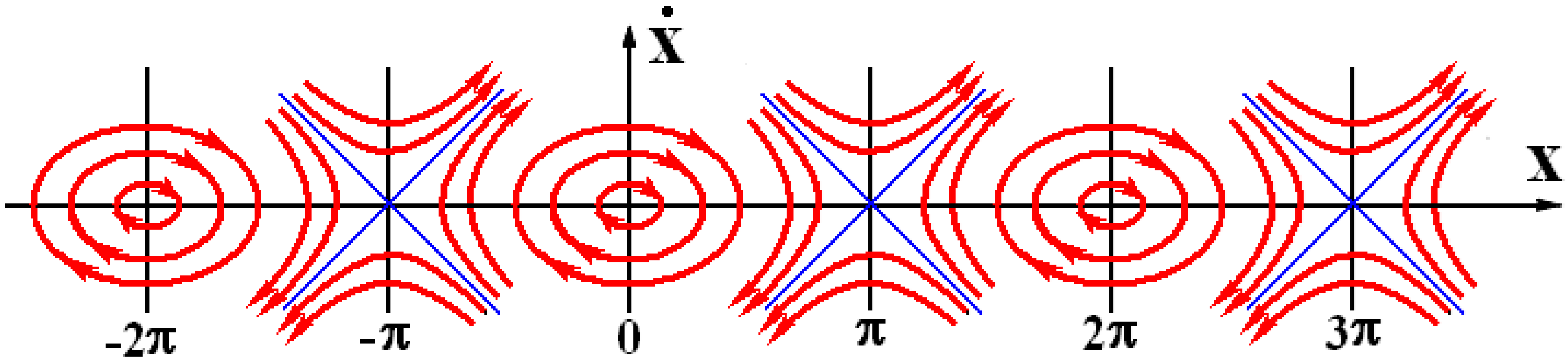
$$\ddot{x}=\dot{x}=0 \;\Rightarrow\; \sin x=0 \;\Rightarrow\; x_e=k\pi$$

$$x_e=\begin{cases} 2k\pi & \sin x=\sin(2k\pi+\Delta x)=\sin\Delta x\approx\Delta x \\ (2k+1)\pi & \sin x=-\sin\Delta x\approx-\Delta x \end{cases}$$

$$\begin{cases}\Delta\ddot{x}+\Delta x=0 \\ \Delta\ddot{x}-\Delta x=0\end{cases}$$

$$\begin{cases}s^2+1=0 \\ s^2-1=0\end{cases}$$

$$\begin{cases}s=\pm j1 \\ s=\pm 1\end{cases}$$



Using Isocline method to draw the phase portraits of the dynamic system $\ddot{x} + \sin x = 0$

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \cdot \frac{dx}{dt} = \dot{x} \cdot \frac{d\dot{x}}{dx} = -\sin x$$

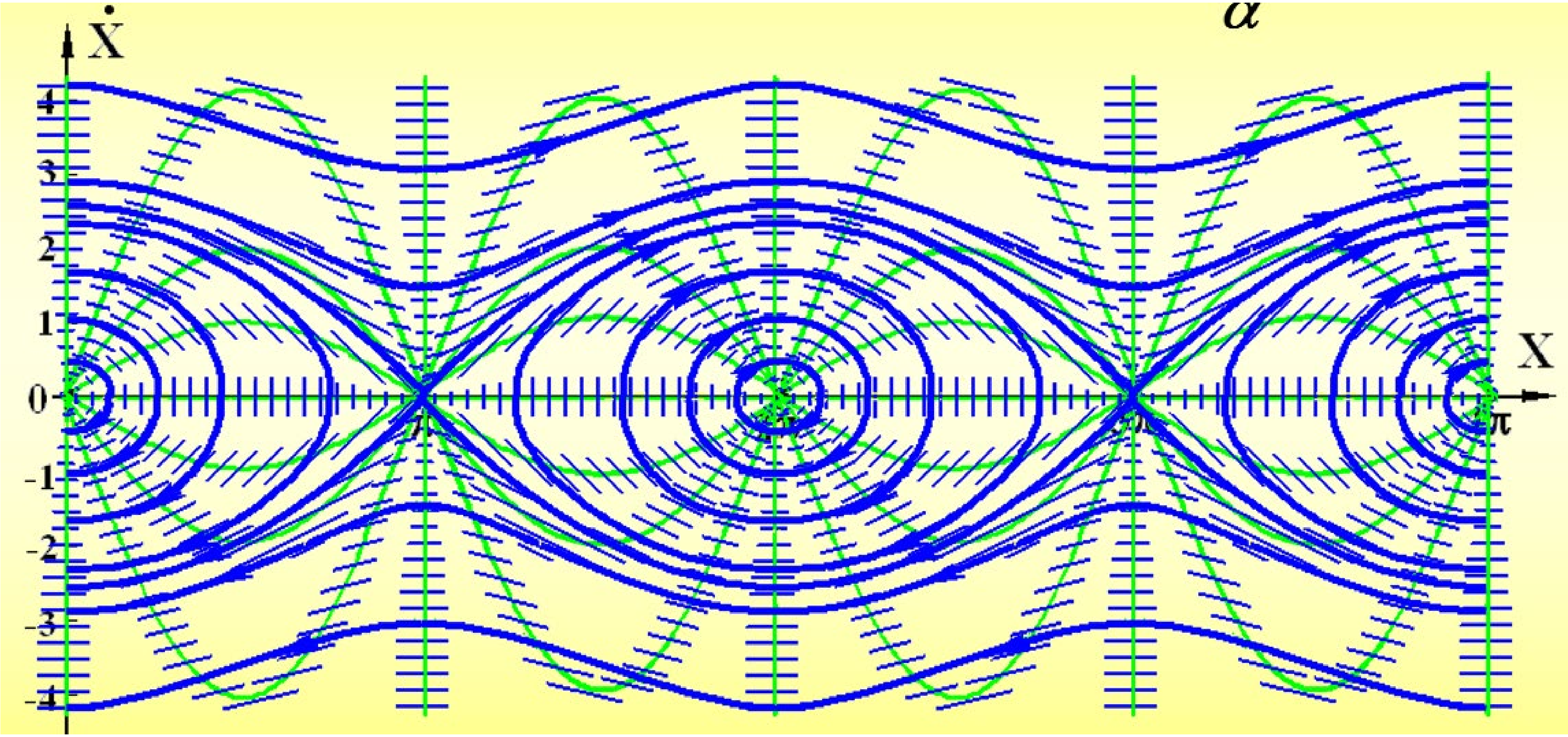
$$\frac{d\dot{x}}{dx} = \alpha \quad \Rightarrow \quad \dot{x} = -\frac{1}{\alpha} \cdot \sin x$$

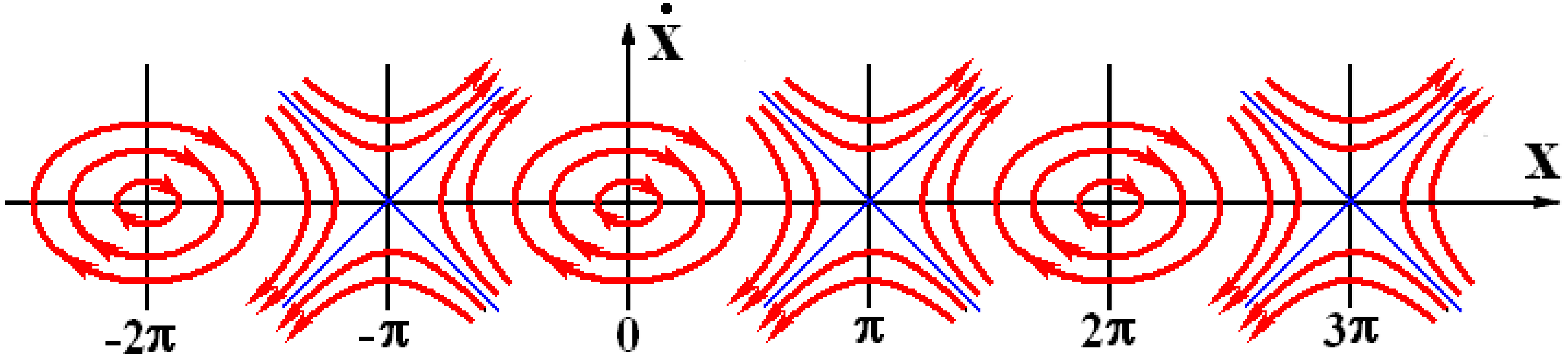
α	值	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	∞
	角	$-45°$	$-26.6°$	$-14°$	$0°$	$14°$	$26.6°$	$45°$	$90°$
$-1/\alpha$		1	2	4	∞	-4	-2	-1	0

α	值	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	∞
	角	$-45°$	$-26.6°$	$-14°$	$0°$	$14°$	$26.6°$	$45°$	$90°$
$-1/\alpha$		1	2	4	∞	-4	-2	-1	0

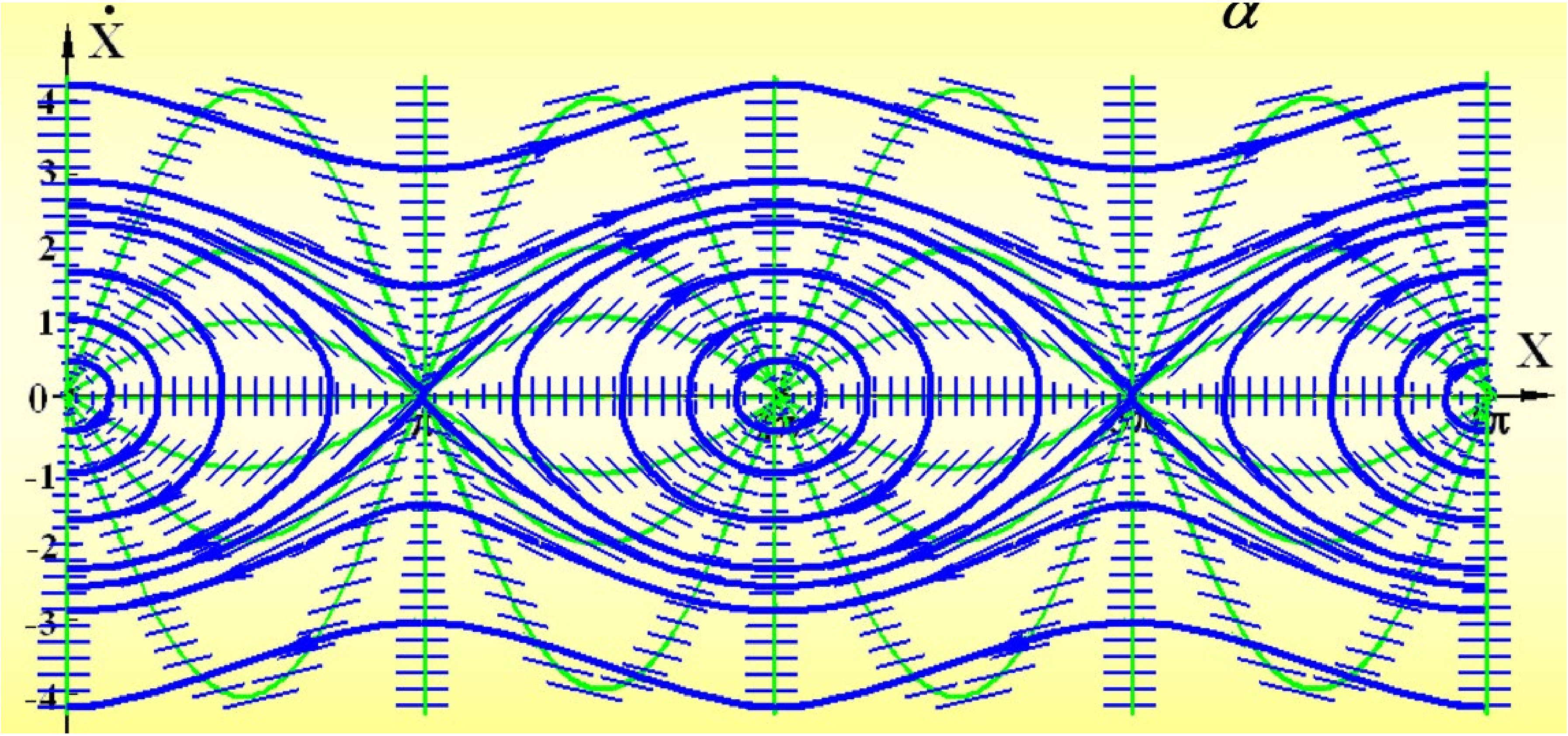
$$\ddot{x} + \sin x = 0$$

$$\dot{x} = -\frac{1}{\alpha} \cdot \sin x$$





$\dot{x}$
x
-2π
-π
0
π
2π
3π

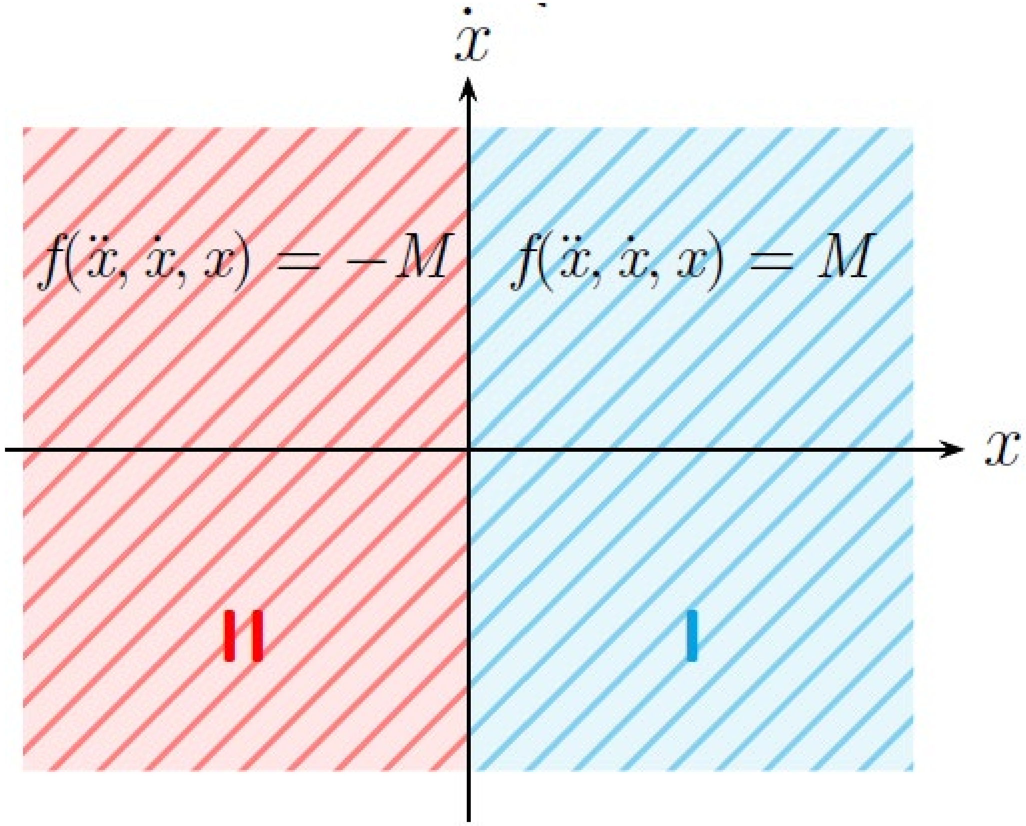


α
$\dot{X}$
X
4
3
2
1
0
-1
-2
-3
-4

Piecewise analysis (本质非线性)

A simple example

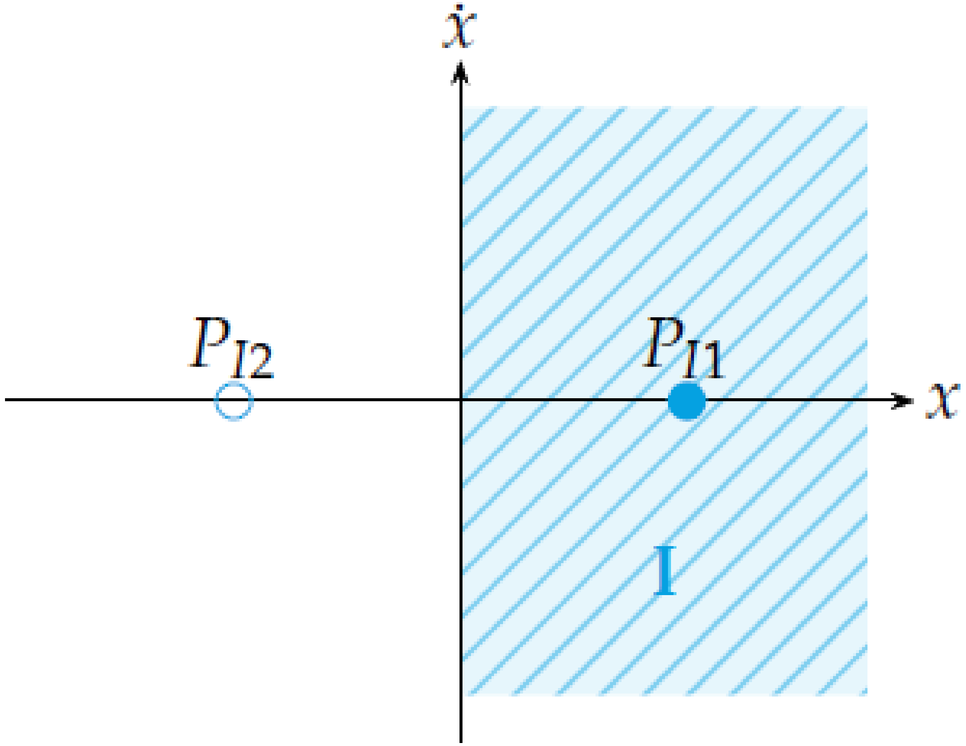
$$f(\ddot{x}, \dot{x}, x) = u = \begin{cases} M & x > 0 \\ -M & x < 0 \end{cases}$$



Actual and virtual singular points

Let $f(\ddot{x}, \dot{x}, x) = M$ for Area I,

and P_{I1} and P_{I2} be singular points of $f(\ddot{x}, \dot{x}, x) = M$.



P_{I1}: Actual singular point

P_{I2}: Virtual singular point

非线性系统的相平面分析

对于分段线性的非线性系统来说，相平面分析法的步骤为：

（1）用n条分界线（开关线，转换线）将相平面分成n个线性区域；

（2）分别写出各个线性区域的微分方程；

（3）求出各线性区的奇点位置并画出相平面图；

（4）将各相邻区的相轨迹联成连续曲线------非线性系统的相轨迹。

Example: Analyze a class of nonlinear systems by the phase locus of 2nd order systems.

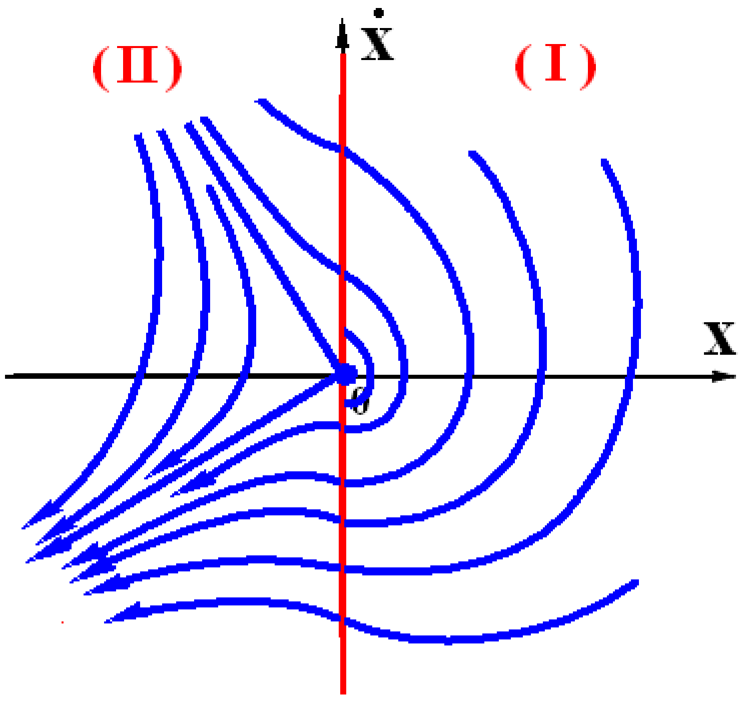
Consider the system $\ddot{x} + \dot{x} + |x| = 0$. Analyze its free response.

Solution
$$\begin{cases} \ddot{x} + \dot{x} + x = 0 & x \geq 0 \quad \mathrm{I} \\ \ddot{x} + \dot{x} - x = 0 & x < 0 \quad \mathrm{II} \end{cases}$$

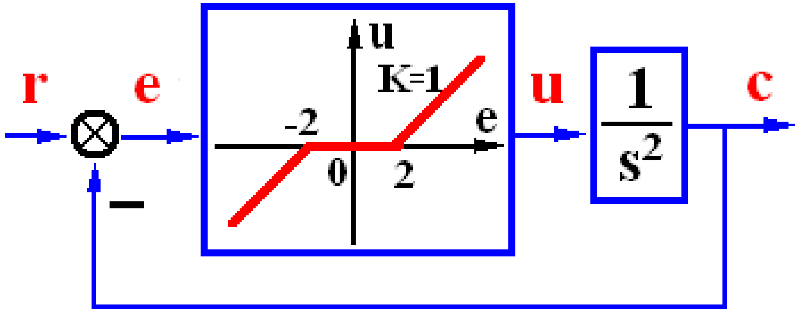
Singular point
$$\begin{cases} \mathrm{I} & x_{e1} = 0 \\ \mathrm{II} & x_{e2} = 0 \end{cases}$$

Characteristic equation
$$\begin{cases} \mathrm{I} & s^2 + s + 1 = 0 \\ \mathrm{II} & s^2 + s - 1 = 0 \end{cases}$$

Poles
$$\begin{cases} s_{1,2} = -0.5 \pm j0.866 & \text{Stable focus} \\ s_{1,2} = \begin{cases} 0.62 \\ -1.62 \end{cases} & \text{Saddle point} \end{cases}$$



Example Consider the system shown in the figure, if $c(0)=0,\ r(t)=4\times 1(t)$ obtain the switch line equation and the equilibrium. Sketch the phase locus.



Solution

Linear part $$\frac{C(s)}{U(s)}=\frac{1}{s^2} \quad\Rightarrow\quad \ddot{c}(t)=u(t)$$

Nonlinear part $$u=\begin{cases}0 & |e|\le 2 \quad (\mathrm{I})\\ e-2 & e>2 \quad (\mathrm{II})\\ e+2 & e<-2 \quad (\mathrm{III})\end{cases}$$

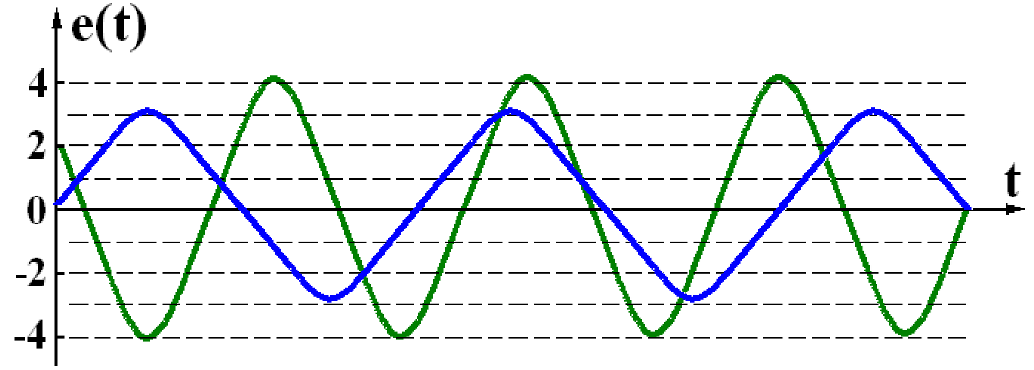
Summing point $e=r-c=4-c$

Switch line equation $\begin{cases}e=\ 2\\ e=-2\end{cases}$

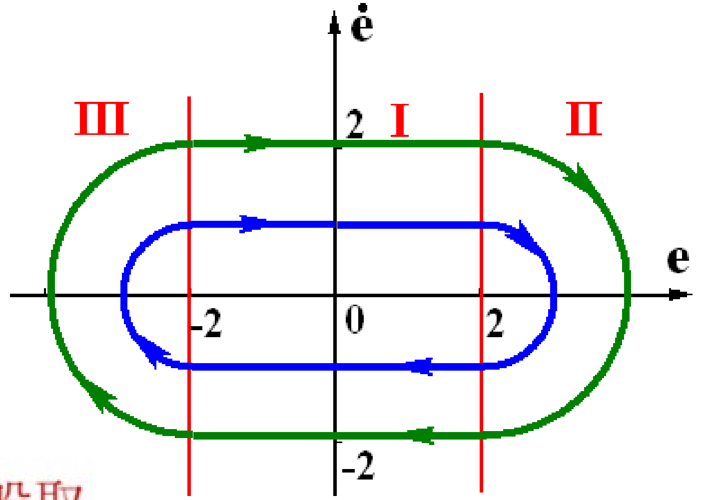
$$\begin{cases}c=4-e\\ \dot{c}=-\dot{e}\\ \ddot{c}=-\ddot{e}\end{cases} \qquad \ddot{e}=-\ddot{c}=-u=\begin{cases}0 & |e|\le 2 \quad (\mathrm{I})\\ 2-e & e>2 \quad (\mathrm{II})\\ -2-e & e<-2 \quad (\mathrm{III})\end{cases}$$

$$\ddot{e} = -\ddot{c} = -u = \begin{cases} 0 & |e| \le 2 \quad (\mathrm{I}) \\ 2-e & e > 2 \quad (\mathrm{II}) \\ -2-e & e < -2 \quad (\mathrm{III}) \end{cases}$$

Area	Differential equation	Singular point	Characteristic equation	Poles	Characteristic of Poles
I	$\ddot{e} = 0$	e_1	$s^2 = 0$	$s = 0$	
II	$\ddot{e} + e - 2 = 0$	$e_2 = 2$	$s^2 + 1 = 0$	$s = \pm j$	Center point
III	$\ddot{e} + e + 2 = 0$	$e_3 = -2$	$s^2 + 1 = 0$	$s = \pm j$	Center point



$$c(t) \begin{cases} e = r - c \\ c = r - e = 4 - e \end{cases}$$



NOTE: 输入给定的情况下，一般取相变量为$(e, \dot{e})$

Outline of Chapter 8

Simulations with MATLAB

- Problems to solve
  - How to solve differential equations?
  - How to plot phase plane portraits?
  - How to simulate nonlinear systems?
- Tools to solve the problems
  - “ode45” command
  - Simulink nonlinear blocks

ODE45 solves ordinary differential equations with medium-order method. It integrates the system of differential equations $\dot{y} = f(t, y)$ from time t_0 to t_f with initial conditions y_0.

```
[tout,yout] = ode45(function,[t0,tf],y0)
```

The solution array “yout” corresponding to a time vector “tout”

- `function` Function handle of differential equations programmed in a separate “.m” file
- `t0, tf` Initial and final time of integral
- `y0` Initial conditions of the DEs

Example

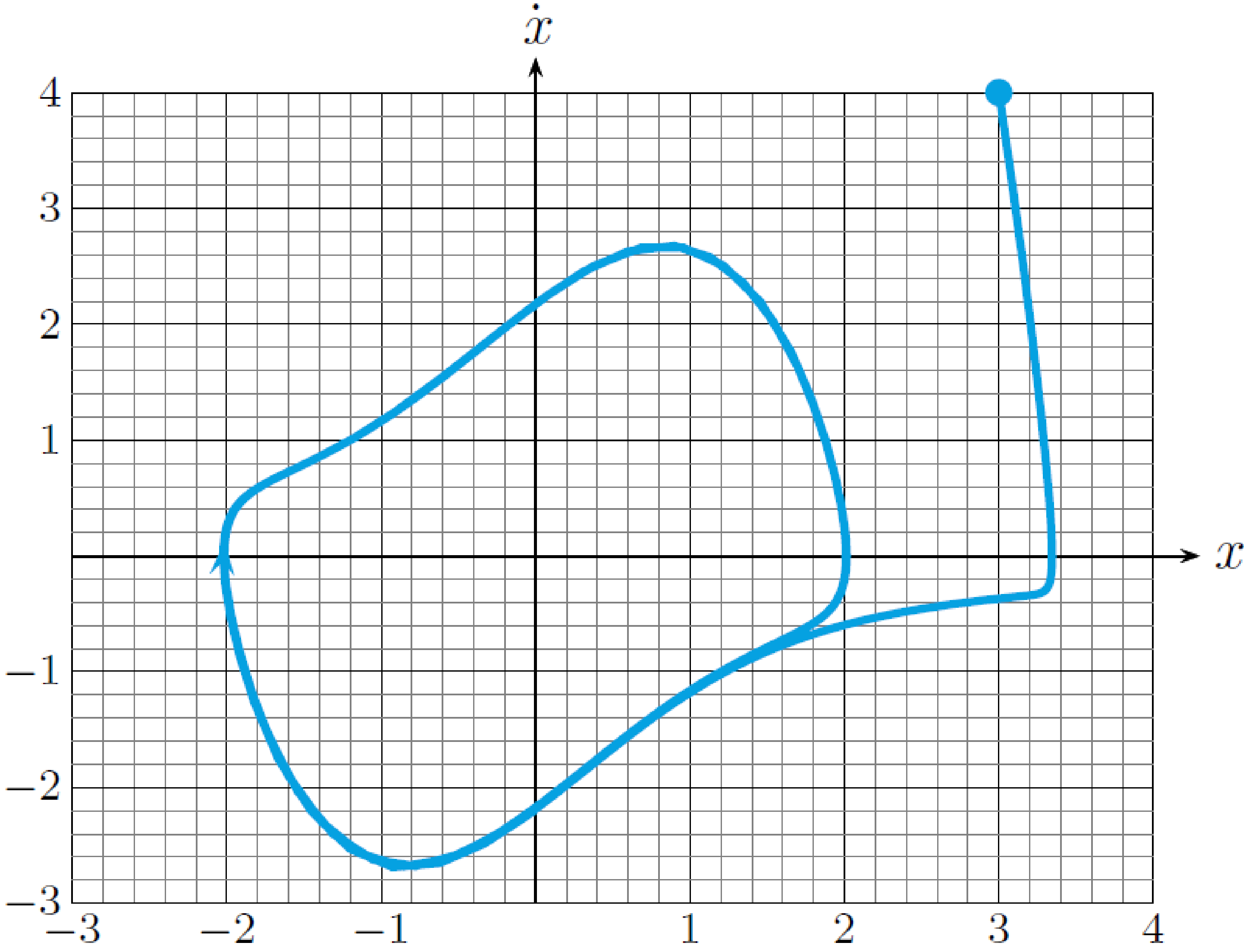
An example of Van de Pol equation $\ddot{x} - (1 - x^2)\dot{x} + x = 0$.

Let $x_1 = x$, $x_2 = \dot{x}$, then $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (1 - x_1^2)x_2 - x_1 \end{cases}$.

```
% This is script, saved as ¡°VandePol.m¡± in a separate file,
% defines the ODEs for the Van de Pol equations.
function [sys,x0]= VandePol(t,x)
sys=[x(2);(1-x(1)*x(1))*x(2)-x(1)];

% This script solves the DEs defined in VandePol.m
%  with the ODE45 method.
[t, x] = ode45(¡®VandePol', [0, 20], [3 ;4] );
plot(x(:,1),x(:,2),'-b');
```

The simulation result is shown below

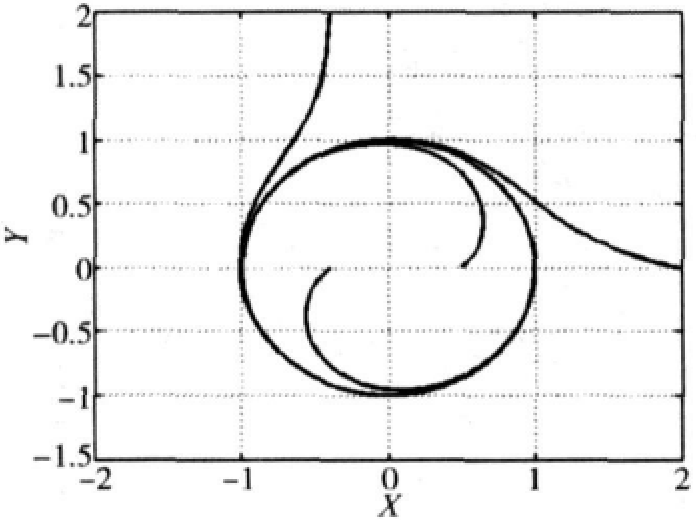


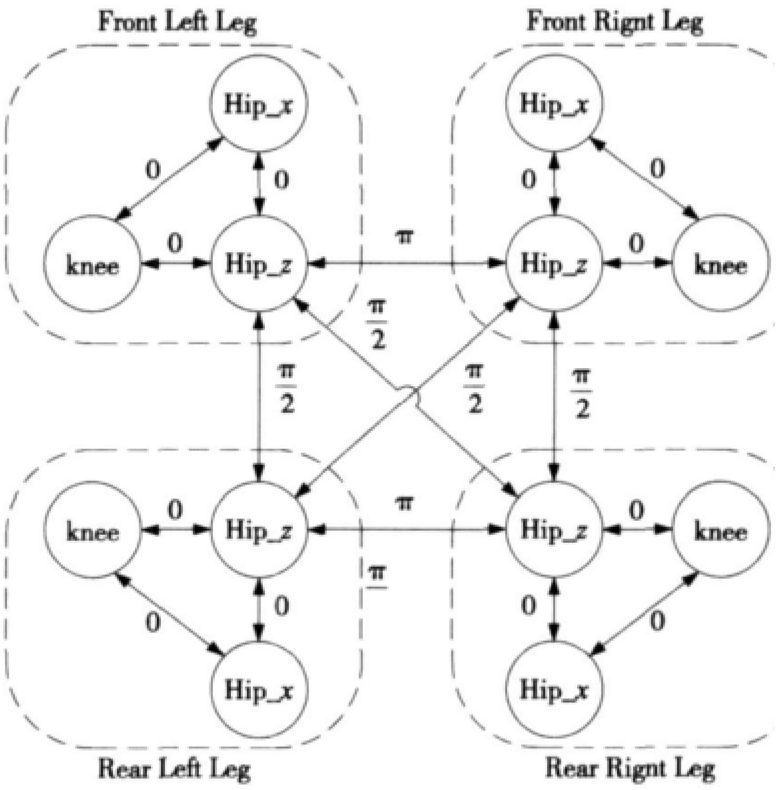
Example: Hopf oscillation model

$$\dot{x} = \gamma(\mu - r^2)x - \omega y$$

$$\dot{y} = \gamma(\mu - r^2)y + \omega x$$

$$r = \sqrt{x^2 + y^2}$$





$$\dot{x}_i = \gamma(\mu - r_i^2)x_i - \omega_i y_i + \epsilon f(t$$

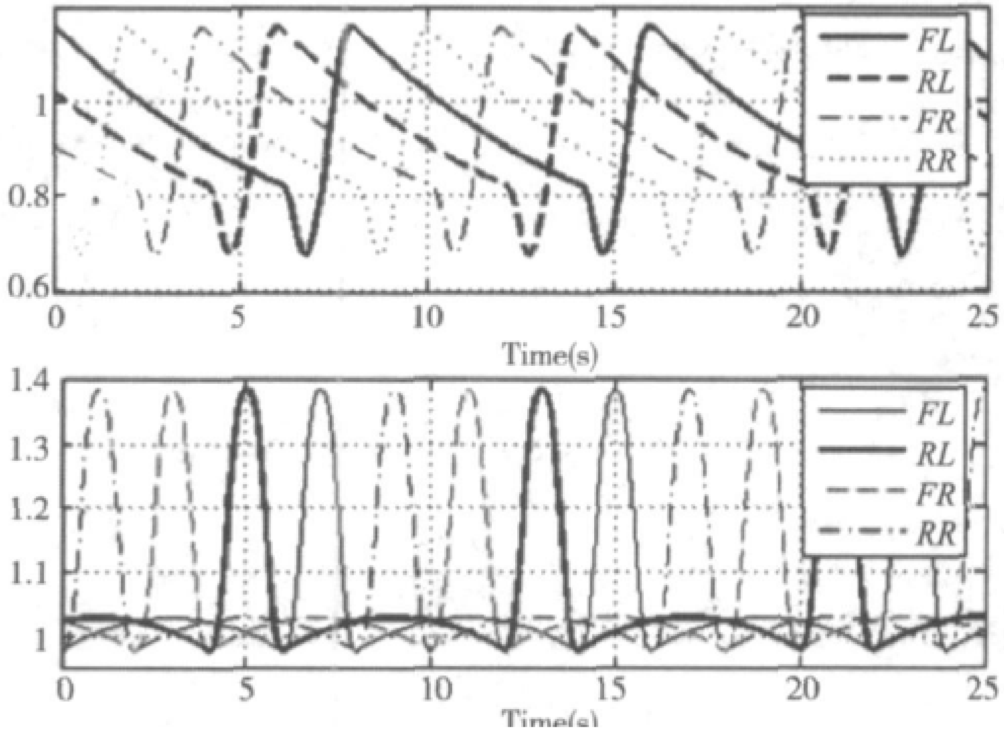
$$\dot{y}_i = \gamma(\mu - r_i^2)y_i + \omega_i x_i$$

$$r_i = \sqrt{x_i^2 + y_i^2}$$

$$\dot{\omega}_i = -\epsilon f(t)\frac{y_i}{r_i}$$

$$\dot{\alpha}_i = \eta x_i f(t)$$

$$f(t) = F_{tech}(t) - \sum_{i=0}^{N} \alpha_i x_i$$



Outline of Chapter 8

Summary of DF and phase-plane methods

	DF method	Phase-plane analysis
Plant complexity	✓	1st & 2nd order system
NL complexity	×	Piecewise linearity
Time response	×	✓
Stability analysis	✓	✓
Limit cycle analysis	✓	✓
Accuracy	×	✓
Method used	Equi. linearization	Graphical solution

课程小结

相平面的基本概念　（相平面、相轨迹和相平面图）

相轨迹的性质　（斜率，奇点，运动方向，垂直过横轴）

相轨迹的绘制　（解析法 ，图解法(等倾斜线法)）

由相轨迹求时间解　（增量法）

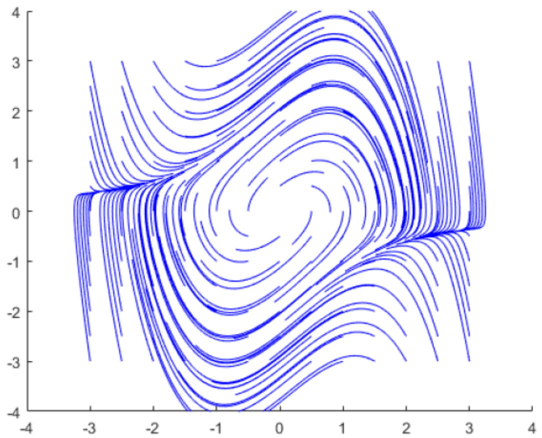
二阶系统的相轨迹　（极点分布，奇点性质，相轨迹）

非线性系统相平面分析

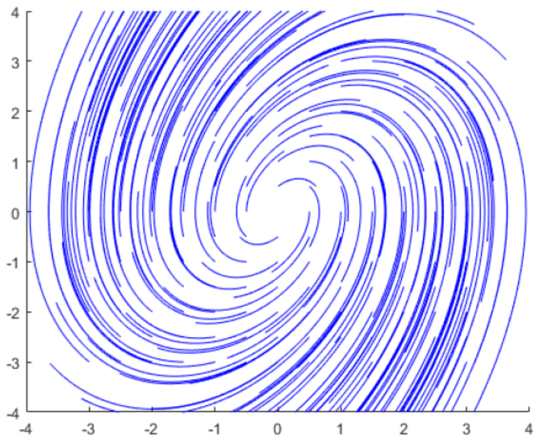
- 非本质非线性　平衡点分析，绘相平面图分析
- 本质非线性　开关线，分区域绘相平面图
- 控制系统　转换成相变量方程，绘相平面图

The phase plane portrait of the system $\ddot{x} - \dot{x} - 2x = 0$ is

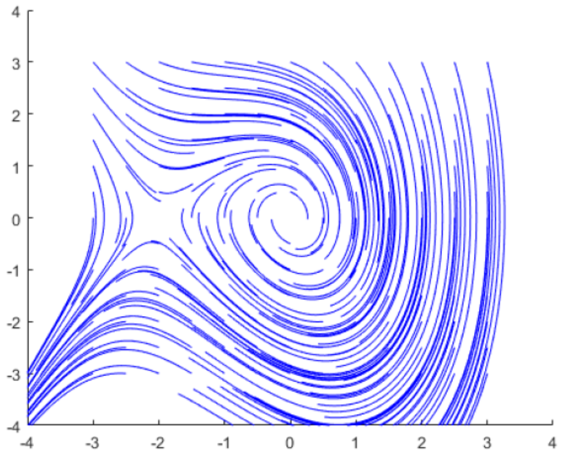
(A)



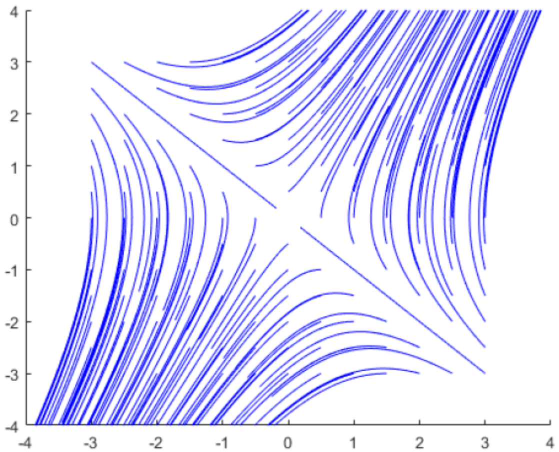
(B)



(C)



(D)



(4). Which of the following non-linear system will produce a saddle point on its phase plane portrait?

(A) $\ddot{x} - \dot{x} - 2x = 0;$

(B) $\ddot{x} + 2\dot{x} + 2x = 0;$

(C) $\ddot{x} - 7\dot{x} + 12x = 0;$

(D) $\ddot{x} - \dot{2}x + 2x = 0.$

6. 【10Marks】

Given the following differential equation of a nonlinear system

$$\ddot{x} + 2\dot{x} + x + x^2 = 0$$

(1) Calculate the equilibrium points of the system; (4 marks)

(2) Sketch the phase-plane portraits of the system around the original point and discuss the stability of system around this point; (Hint: Determine the type of singular points first.) (6 marks)

(1) 平衡点

求平衡点： $\ddot{x} = \dot{x} = 0$， $x + x^2 = x(1+x) = 0$

Singular point $\begin{cases} x_{e1} = 0 \\ x_{e2} = -1 \end{cases}$

(2) 非本质非线性系统，可以先利用在平衡点附近小偏差线性化的方法进行线性化处理。[6]

Singular point $\begin{cases} x_{e1} = 0 \\ x_{e2} = -1 \end{cases}$

线性化： $\begin{cases} x = \Delta x + x_{e1} = \Delta x \\ x = \Delta x + x_{e2} = \Delta x - 1 \end{cases}$ $\begin{cases} \Delta\ddot{x} + 2\Delta\dot{x} + \Delta x = 0 \\ \Delta\ddot{x} + \Delta\dot{x} - \Delta x = 0 \end{cases}$

Characteristic equation

$\begin{cases} s^2 + 2s + 1 = 0 \\ s^2 + 2s - 1 = 0 \end{cases}$， 求解极点并判断奇点类型： $\begin{cases} \mathbf{s} = -1, -1 \quad \textbf{\textit{stable node}} \\ \mathbf{s} = \begin{cases} -1 + \sqrt{2} \\ -1 - \sqrt{2} \end{cases} \quad \textbf{\textit{saddle point}} \end{cases}$

画出完整两个平衡点附近的相轨迹

Example

Obtain the phase plane portrait of the given system $T\ddot{e} + \dot{e} = P$.

NB: No item e !

Solutions

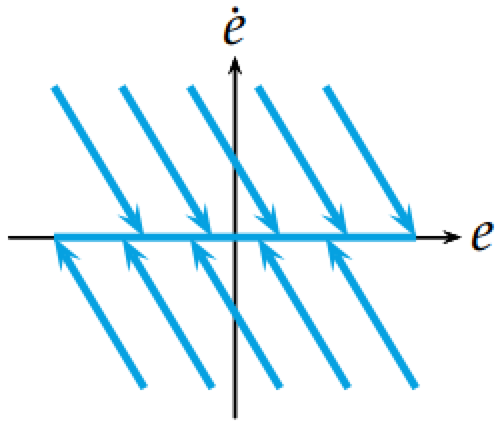
(i) When $P = 0$

$$T\ddot{e} + \dot{e} = 0 \quad \Rightarrow \quad T\dot{e}\frac{\mathrm{d}\dot{e}}{\mathrm{d}e} + \dot{e} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\dot{e}}{\mathrm{d}e} = -\frac{\dot{e}}{T\dot{e}}$$

For $\dot{e} = 0$ Continuous singular points:

$\dot{e} = 0.$

For $\dot{e} \neq 0$ Isoclines: $\frac{\mathrm{d}\dot{e}}{\mathrm{d}e} = -\frac{1}{T}$



(ii) When $P \neq 0$, $T\ddot{e} + \dot{e} = P$

No singular points

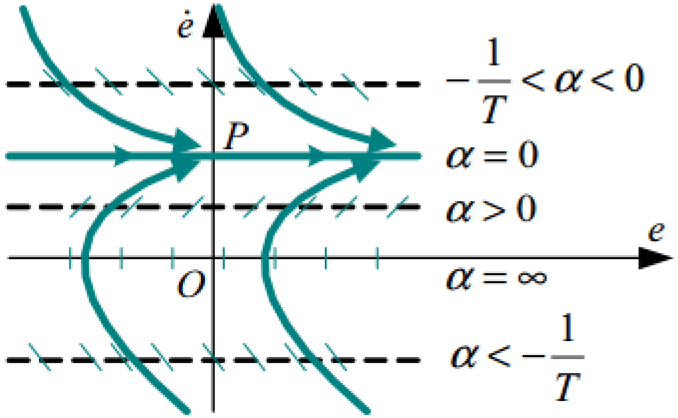
Isoclines:

$$\frac{\mathrm{d}\dot{e}}{\mathrm{d}e} = \frac{P - \dot{e}}{T\dot{e}} = \alpha \quad \Rightarrow \quad \dot{e} = \frac{P}{1 + \alpha T}$$

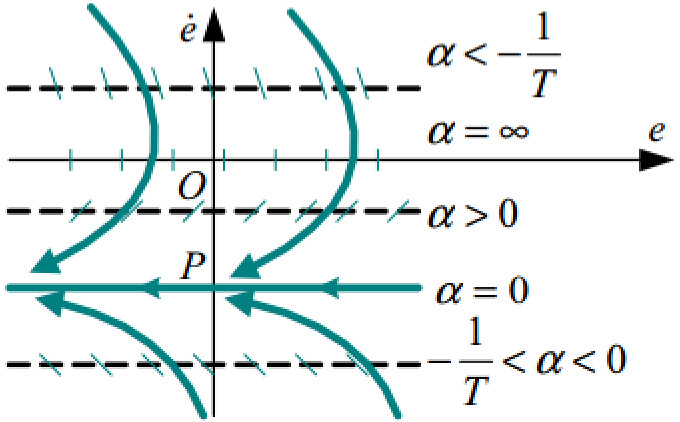
When $\alpha = 0$, $\dot{e} = P$.

Therefore, $\dot{e} = P$ is a phase plane trajectory.

- $P > 0$

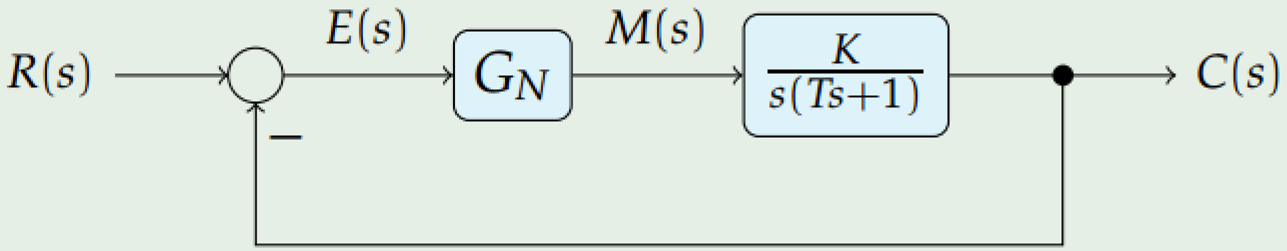


- $P < 0$



Example

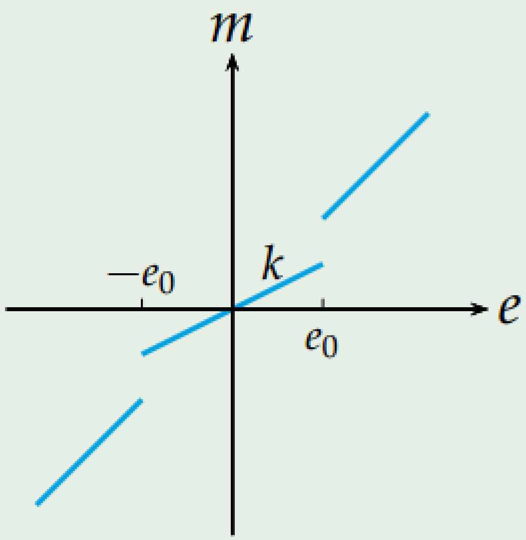
Obtain the phase plane portrait for the following system



where G_N is defined as follows

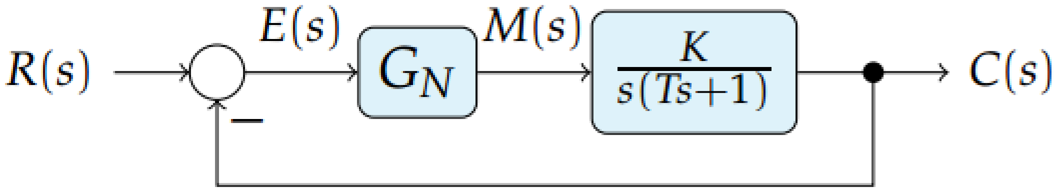
$$m = \begin{cases} e & |e| > e_0 \\ ke & |e| < e_0 \end{cases}$$

$$k < 1$$



Solutions

System equations:

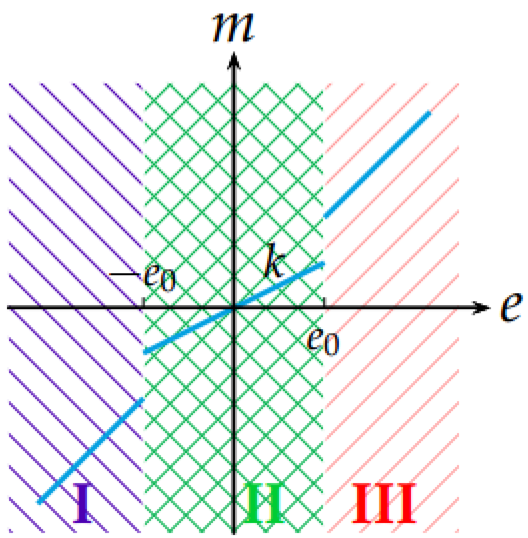


$$E(s) = R(s) - C(s) = R(s) - \frac{K}{s(Ts+1)}M(s)$$

$$T\ddot{e} + \dot{e} = T\ddot{r} + \dot{r} - Km$$

In $e - \dot{e}$ plane

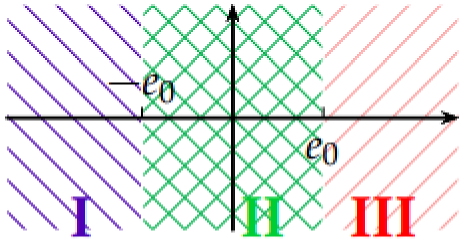
- 3 areas
- 2 different equations



- Step response

$$
\begin{aligned}
r(t) &= 1(t),\ \dot{r} = 0,\ \ddot{r} = 0 \\
e(0) &= E_0,\ \dot{e}(0) = 0
\end{aligned}
\Rightarrow
\begin{cases}
T\ddot{e} + \dot{e} + Ke = 0 & \text{Areas I and III} \\
T\ddot{e} + \dot{e} + kKe = 0 & \text{Area II}
\end{cases}
$$

- Singular point: $e = 0$ and $\dot{e} = 0$
  - Actual singular point for system in II
  - Virtual singular point for system in I, III



- Properties of the singular point:

  Suppose $1 - 4kKT = 0$. Since $k < 1$, so $1 - 4KT < 0$
  - Small input: $|e| < e_0$

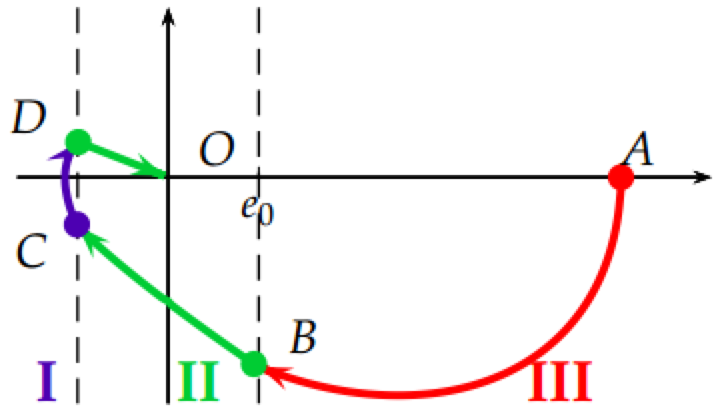
    $T\ddot{e} + \dot{e} + kKe = 0 \quad \Rightarrow$ (0,0) is the stable node
  - Large input: $|e| > e_0$

    $T\ddot{e} + \dot{e} + Ke = 0 \quad \Rightarrow$ (0,0) is the stable focus

- Phase trajectory

  Let A be the initial point

  - For A, $(0,0)$ is the stable focus
  - For B, $(0,0)$ is the stable node
  - For C, $(0,0)$ is the stable focus
  - For D, $(0,0)$ is the stable node



- Features: Speeds up the regulation
  - When large signal in the loop
    - The origin is the stable focus (underdamped)
    - Error decreases fast
  - When small signal in the loop
    - The origin is the stable node (critically damped)
    - Oscillation avoided