Modern Control Theory

Li Yan (1st-9th week) & Liu Chengju (10th-17th week)

Department of Control Science & Engineering

School of Electronic & Information Engineering

Tongji University

https://rail.tongji.edu.cn/

liyan_tongji@tongji.edu.cn, 15201614174

Course Information

Part I: State Space Control of Linear Systems

Chap 1: State Space Description

Chap 2: Solution of State Equations

Chap 3: Controllability and Observability

Chap 4: Design of State Feedback Control

Chap 5: State Space Analysis of Discrete-Time Control System

Part II: Nonlinear Control

Chap 6: Introduction to Nonlinear Control Systems

Chap 7: Describing Functions Analysis

Chap 8: Phase Plane Analysis

Chap 9: Lyapunov Stability Theory

Part I: State Space Control of Linear Systems (Chap 1-5)

Modeling → State space description

- Establishment
- Solution
- Transformation

Analysis

- Controllability
- Observability
- Stability

Design

- Design of State Feedback Control
- Design of State Observer
- Optimal Control

Main content of the State Space Control of Linear Systems

- State space description method → Modeling
- The internal properties of linear systems → Analysis
- The integrated design of linear systems with state space → Design

Chap 1: State space description
Chap 2: Solution of state equations
Chap 3: Controllability and observability
Chap 4: Design of state feedback control
Chap 5: State Space Analysis of Discrete-Time Control System

Outline of Chapter 1

1.1 Introduction

1.2 State space description of dynamic systems

1.3 Relationship between TF & SS equations

1.4 Similarity transformation

1.5 Simulations with MATLAB

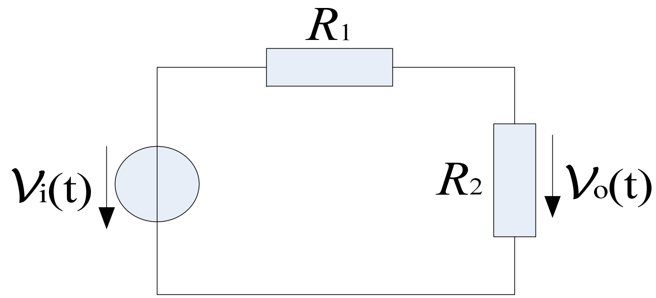
Outline of Chapter 1

Outline of Chapter 1

“Static system” & “dynamic system”

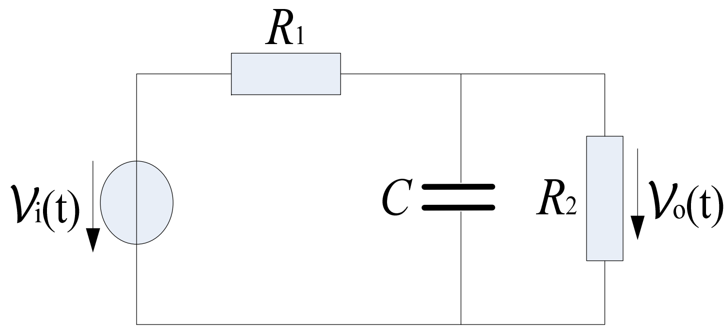
Static system

The current outputs depend solely on the instantaneous values of the current inputs.
(example: a resistance networked system).



Dynamical system

A system’s state summarizes its entire past. It describes the memory-side of dynamic systems.
(example: a RLC networked system).



State & State variable

Definition (State variable)

The state of a system is a set of variables $(x_1(t), x_2(t), \cdots x_n\,(t))$ such that the initial knowledge of theses variables $(x_i(t_0), \mathrm{i} = 1, 2, \cdots n)$ and the input functions $(u_j(t),$ $\mathrm{j} = 1, 2, \cdots p, t \geq t_0)$ will, with the equations describing the dynamics, provide the future state and output of the system.

- The elements of state variable choice:
  - Complete description
  - The choose of state variables is non-unique
  - The only restriction on the choice of state variables is that they are independent
  - The number of the state variable is unique

State vector

Definition (State vector)

The state vector is the column vector $x(t)$ composed of all the state variables $x_1(t), x_2(t), \cdots, x_n(t)$, i.e.

$$x(t) \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \equiv x$$

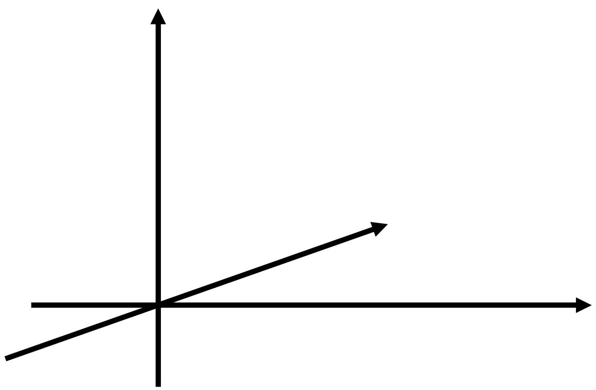
- If the state vector by n components, is called the n-dimensional state vector.
- If given the initial state vector $x(t_0)$ (when $t = t_0$) and the input vector $u(t)$ (when $t \geq t_0$), the state is uniquely determined by the state vector $x(t)$.

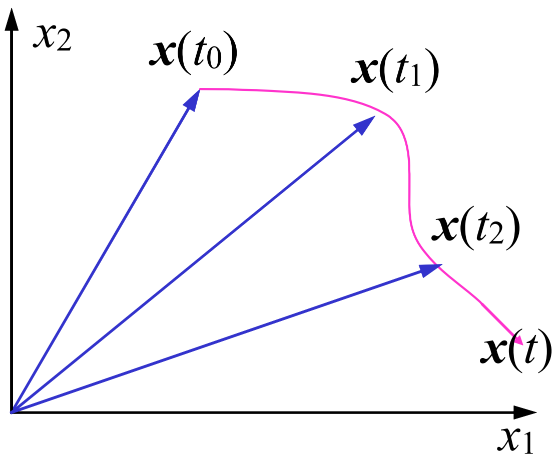
"State space & State trajectory"

Definition (State space & State trajectory)

State space is defined as the *n*-dimensional space ($\mathbf{R}^n$) in which the components of the state vector $(x_1(t), x_2(t), \ldots, xn(t))$represent its coordinate axes.

State trajectory is defined as the path produced in the state space by the state vector $x(t)$ as it changes with the passage of time ($t \geq t_0$).





State equation & Output equation

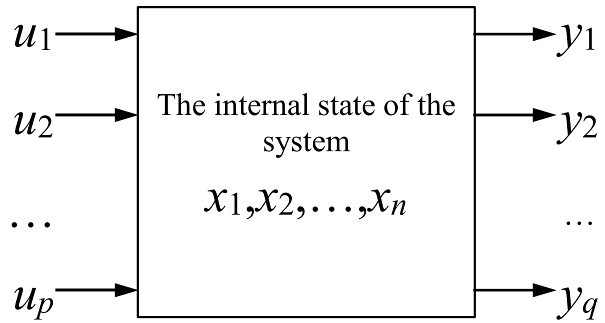
An *n*th-order dynamic system can be described by the following two sets of equations:

- State equations

$$\begin{cases} \dot{x}_1(t) &= f_1[x_1(t), x_2(t), \cdots, x_n(t); u_1(t), u_2(t), \cdots, u_p(t); t] \\ \dot{x}_2(t) &= f_2[x_1(t), x_2(t), \cdots, x_n(t); u_1(t), u_2(t), \cdots, u_p(t); t] \\ \vdots & \vdots \qquad \vdots \\ \dot{x}_n(t) &= f_n[x_1(t), x_2(t), \cdots, x_n(t); u_1(t), u_2(t), \cdots, u_p(t); t] \end{cases}$$

- Output equations

$$\begin{cases} y_1(t) &= g_1[x_1(t), x_2(t), \cdots, x_n(t); u_1(t), u_2(t), \cdots, u_p(t); t] \\ y_2(t) &= g_2[x_1(t), x_2(t), \cdots, x_n(t); u_1(t), u_2(t), \cdots, u_p(t); t] \\ \vdots & \vdots \qquad \vdots \\ y_q(t) &= g_q[x_1(t), x_2(t), \cdots, x_n(t); u_1(t), u_2(t), \cdots, u_p(t); t] \end{cases}$$



State space description of a general dynamic system

For an *n*th-order linear time-invariant (LTI) system with *p* input and *q* output, the dynamic equation are written as

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t), t \geq t_0$$

$$y(t) = \mathrm{C}x(t) + \mathrm{D}u(t)$$

where **A** $\in R^{n\times n}$ is called the system matrix (coefficient matrix, state matrix)
B $\in R^{n\times p}$ is called the input or control matrix
C $\in R^{q\times n}$ is called the output matrix
D $\in R^{q\times p}$ is called the direct connection matrix.

For simplicity, the system can also be denoted as Σ(A, B, C, D).

$$\dot{\mathrm{x}}(t) = \mathrm{Ax(t)} + \mathrm{Bu(t)} \qquad \mathrm{y} = \mathrm{Cx(t)} + \mathrm{Du(t)}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$\mathrm{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1p} \\ d_{21} & d_{22} & \cdots & d_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ d_{q1} & d_{q11} & \cdots & d_{qp} \end{bmatrix} \qquad \mathrm{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{11} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{q1} & c_{q2} & \cdots & c_{qn} \end{bmatrix}$$

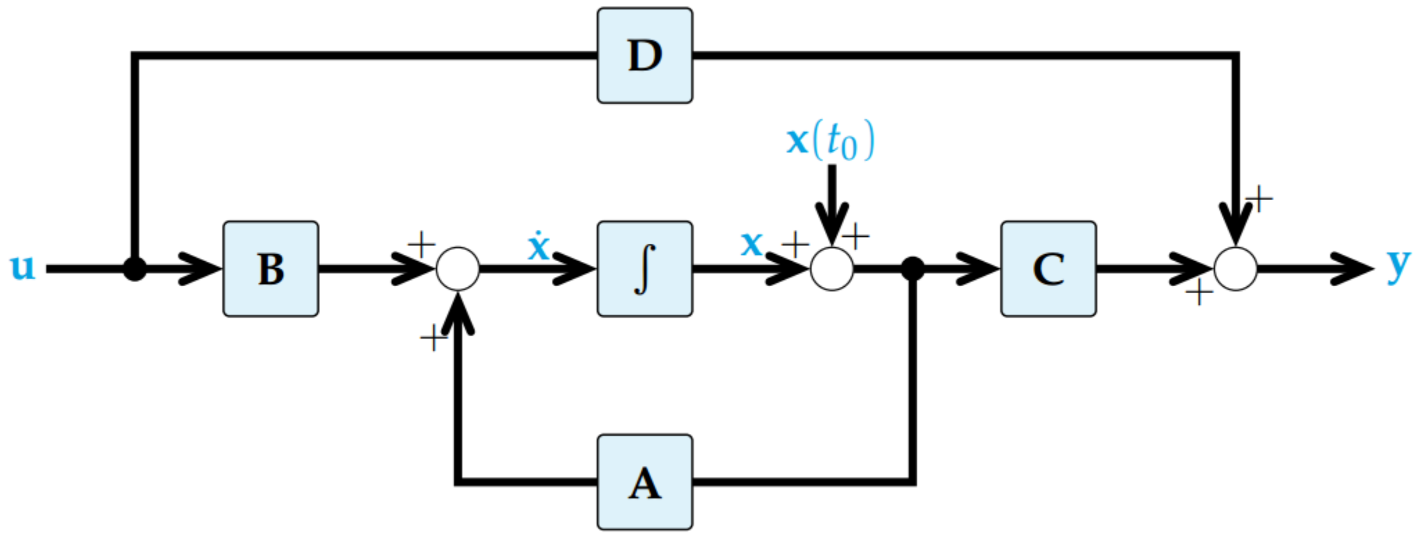
Block diagram for the state space model

Given the state space model of an LTI system

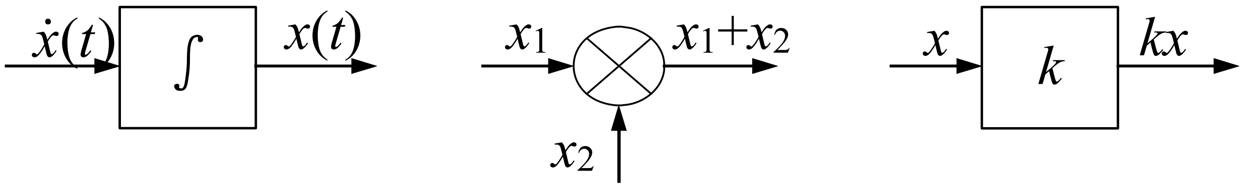
$$\dot{x}(t) = Ax(t) + Bu(t), t \geq t_0$$

$$y(t) = Cx(t) + Du(t)$$

the block diagram for the system is

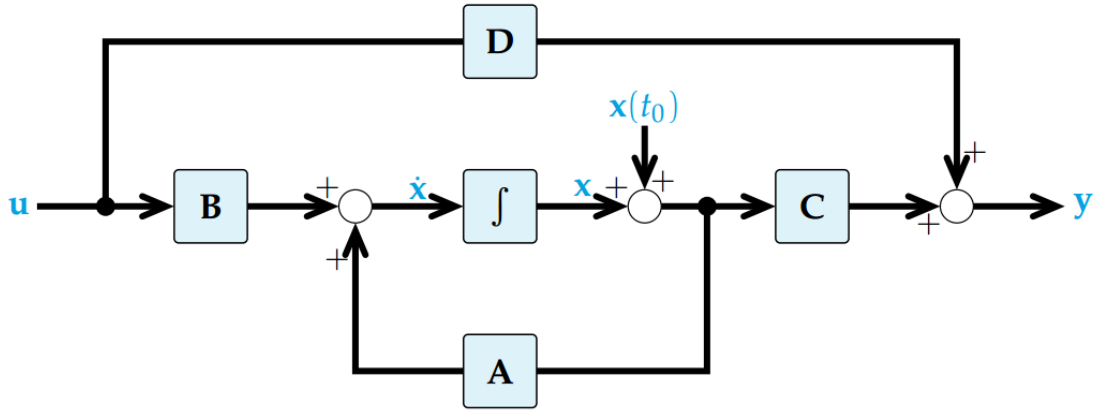


The three basic components of the block diagram



$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t), t \geq t_0$$

$$y(t) = \mathrm{C}x(t) + \mathrm{D}u(t)$$



状态空间描述的模拟结构图绘制步骤：

(1) 画出所有积分器；积分器的个数等于状态变量数，每个积分器的输出表示相应的某个状态变量。

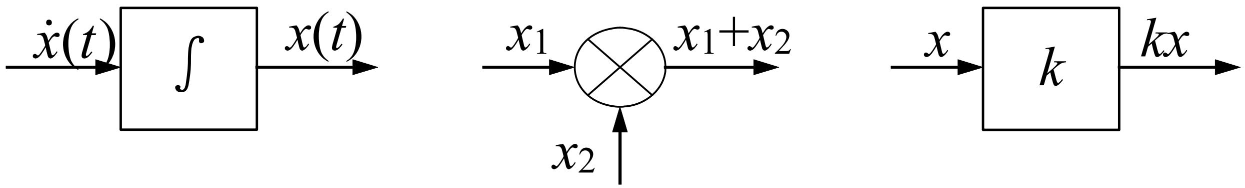
(2) 根据状态方程和输出方程，画出相应的加法器和比例器；

(3) 用箭头将这些元件连接起来。

例1-1　画出一阶微分方程的模拟结构图。

微分方程：　$\dot{x} = ax + bu$

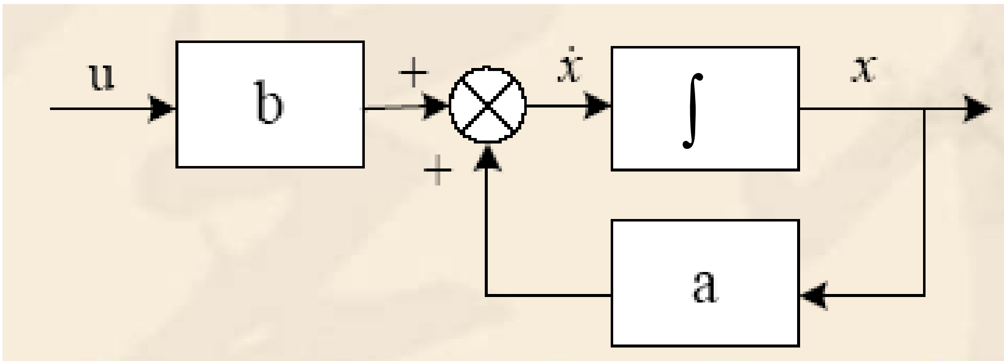
模拟结构图?



例1-1 画出一阶微分方程的模拟结构图。

微分方程：$$\dot{x} = ax + bu$$

模拟结构图



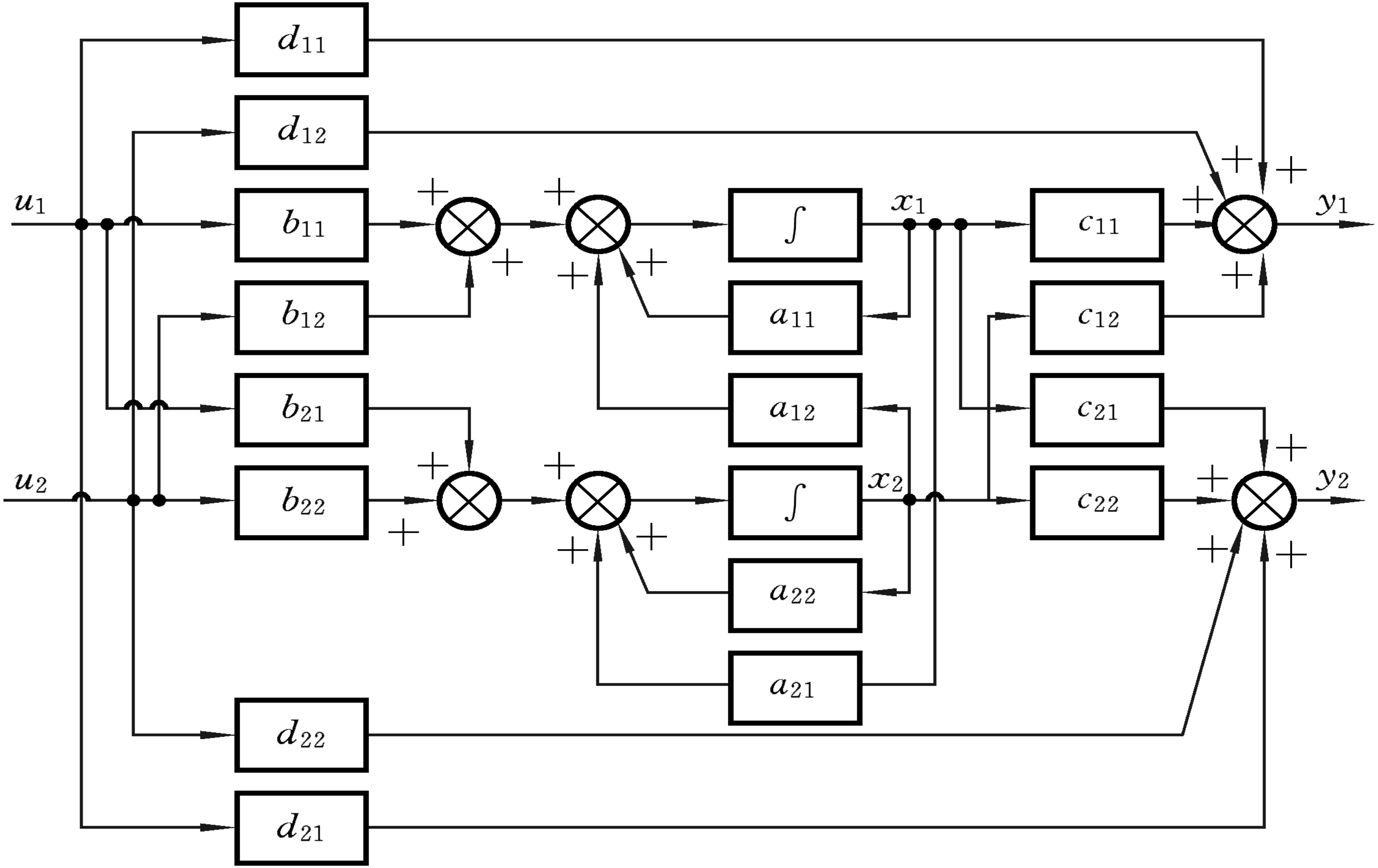
Example Block diagram of double input double output linear constant system

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t), t \geq t_0$$

$$y(t) = \mathrm{C}x(t) + \mathrm{D}u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$


d_{11}
d_{12}
u_1
b_{11}
$+$
$+$
$+$
$+$
$+$
$\int$
x_1
c_{11}
$+$
$+$
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y_1
b_{12}
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c_{12}
b_{21}
a_{12}
c_{21}
u_2
b_{22}
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$+$
$+$
$+$
$\int$
x_2
c_{22}
$+$
$+$
$+$
$+$
y_2
a_{22}
a_{21}
d_{22}
d_{21}

例1-3　某系统的状态空间表达式为

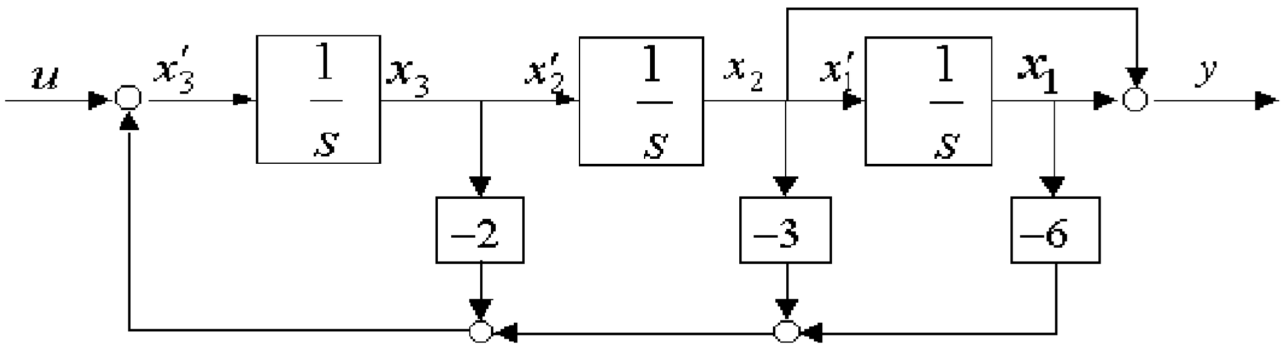
$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -3 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \boldsymbol{x}$$

试绘制其结构图。

分析：本系统状态变量有三个 $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$

一个输入量$\boldsymbol{u}$，一个输出量$\boldsymbol{y}$，（r=1, m=1）

解：系统结构图（或状态变量图）如下：



The establishment methods of the state space description

- **Deduced by the mechanism of the system**
- Set up by the system block diagram
- Evolution by differential equations or the transfer function(TF)

Example 1-1 State space description by system mechanism

Build up the state space model for the following MSD system.

Solutions

By applying the **Newton's second law**, we obtain the following equations:

$$m\ddot{y} + b\dot{y} + ky = u$$

If we choose

$$\boxed{x_1(t) = y(t),\ x_2(t) = \dot{y}(t)}$$

State variables:
Choose the output of the system (y) and its derivative (v)

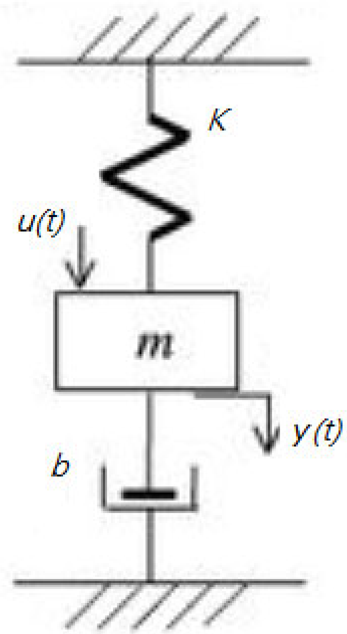
we will have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}\left(-ky - b\dot{y}\right) + \frac{1}{m}u$$

The output equation

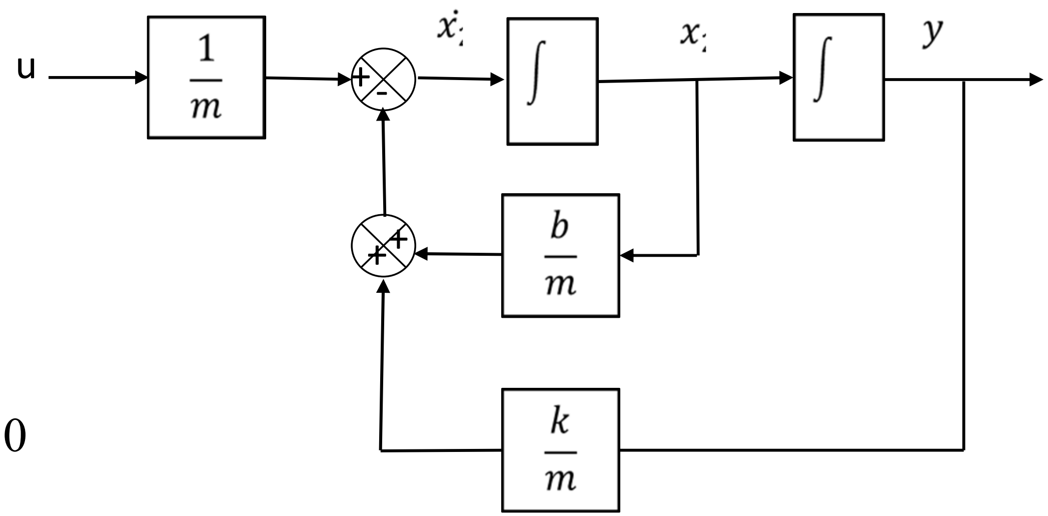
$$y = x_1$$



Represented by the vector-matrix

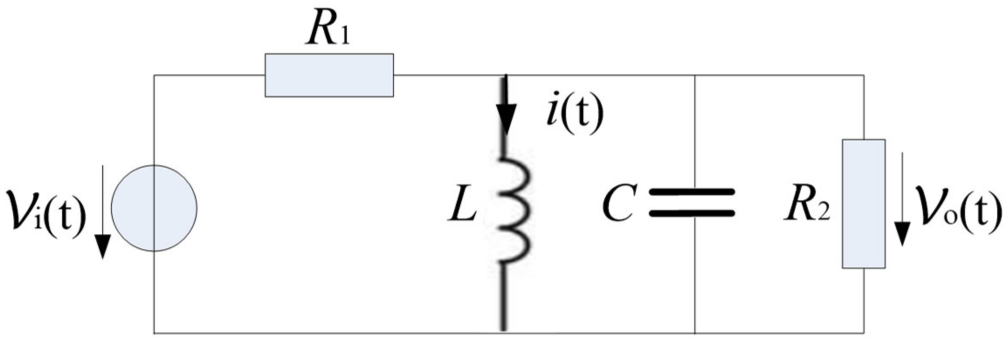
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{\mathrm{x}} = \mathrm{A}x + \mathrm{B}u \\ \mathrm{y} = \mathrm{C}x + \mathrm{D}u \end{cases} \quad \begin{cases} A = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \\ B = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} \\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0 \end{cases}$$



Example 1-2 State space description by system mechanism

Let's build up the state space model for the following circuit.



Solutions

By applying the Kirchhoff's laws, we obtain the following equations:

$$
\begin{aligned}
v_o(t) &= L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \\
\frac{v_i(t)-v_o(t)}{R_1} &= i(t)+C\frac{\mathrm{d}v_o(t)}{\mathrm{d}t}+\frac{v_o(t)}{R_2}
\end{aligned}
$$

The above equations can be rearranged as follows

$$\begin{aligned}\frac{\mathrm{d}i(t)}{\mathrm{d}t} &= \frac{1}{L}v_o(t)\\ \frac{\mathrm{d}v_o(t)}{\mathrm{d}t} &= -\frac{1}{C}i(t) - \left(\frac{1}{R_1C}+\frac{1}{R_2C}\right)v_o(t) + \frac{1}{R_1C}v_i(t)\end{aligned}$$

State variables: Choose the physical quantity output of the energy storage element

If we choose

$$\boxed{\mathbf{x}(t) = \begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix} i(t)\\ v_o(t)\end{bmatrix}, \quad u(t) = v_i(t), \quad y(t) = v_o(t)}$$

we will have

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & \frac{1}{L}\\ -\frac{1}{C} & -\left(\frac{1}{R_1C}+\frac{1}{R_2C}\right)\end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0\\ \frac{1}{R_1C}\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix} 0 & 1\end{bmatrix}\mathbf{x}(t)\end{aligned}$$

which yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), t \geqslant t_0$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

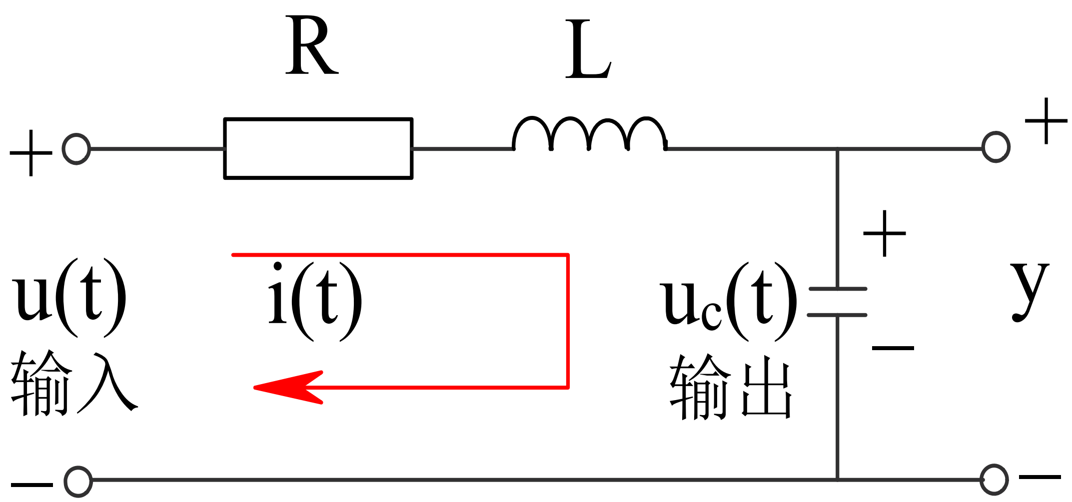
where

$$\mathrm{A} = \begin{bmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\left(\dfrac{1}{R_1 C} + \dfrac{1}{R_2 C} +\right) \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} 0 \\ \dfrac{1}{R_1 C} \end{bmatrix}$$

$$\mathrm{C} = [0 \quad 1] \qquad \mathrm{D} = [0]$$

Exercise 1-1

Build up the state space model for the following RLC system.



例1-1：建立如图所示的RCL电路的状态方程和输出方程。

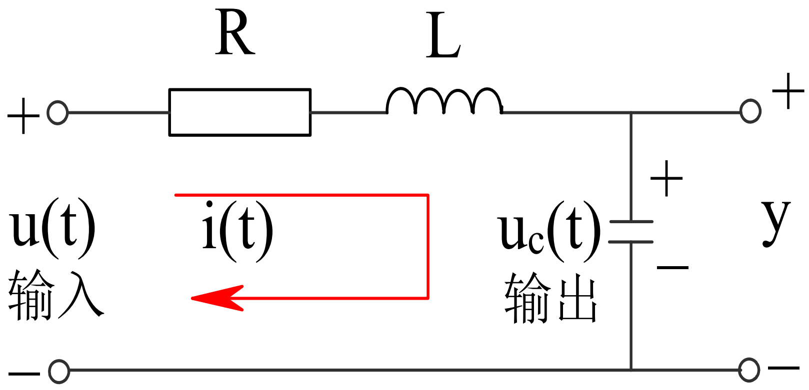


图1

解：

$$LC\ddot{u}_c(t) + RC\dot{u}_c(t) + u_c(t) = u(t) \qquad \text{微分方程}$$

$$\frac{U_C(s)}{U(s)} = \frac{1}{LCs^2 + RCs + 1} \qquad \text{传递函数}$$

定义状态变量 $x_1(t)=u_c(t)$ $x_2(t)=i(t)$

二阶微分方程，选择两个状态变量

状态向量 $\mathbf{x}(t)=[x_1(t),x_2(t)]^T$

定义输出变量 $y(t)=x_1(t)$

整理得一阶微分方程组为

$$Ri(t)+L\frac{di(t)}{dt}+u_c=u(t)$$

$$i(t)=C\frac{du_c}{dt}$$

$$\Rightarrow$$

$$\frac{\mathrm{d}u_c(t)}{dt}=\frac{1}{C}i(t)$$

$$\frac{\mathrm{d}i(t)}{dt}=-\frac{1}{L}u_c(t)-\frac{R}{L}i(t)+\frac{1}{L}u(t)$$

状态方程

$$\dot{x}_1(t)=\frac{1}{C}x_2(t)$$

$$\dot{x}_2(t)=-\frac{1}{L}x_1(t)-\frac{R}{L}x_2(t)+\frac{1}{L}u(t)$$

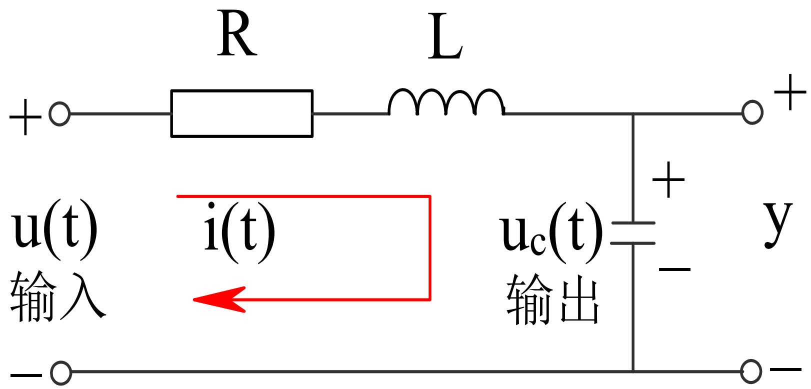
输出方程

$$y=x_1(t)$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & \frac{1}{C}\\ -\frac{1}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{L}\end{bmatrix}u(t)$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

图1所示电路，若 $u_c(t)$为输出，
取 $x_1(t)=u_c(t), x_2(t)=i(t)$ 作为状态变量，
则其状态空间表达式为



$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

状态变量选择不同，状态方程也不同。

若按照如下所示的微分方程：

$$\frac{\mathrm{d}u_c(t)}{dt} = \frac{1}{C} i(t)$$

$$\frac{\mathrm{d}i(t)}{dt} = -\frac{1}{L} u_c(t) - \frac{R}{L} i(t) + \frac{1}{L} u(t)$$

选 $\overline{x}_1 = u_c, \overline{x}_2 = \dot{u}_c$ ，则得到一阶微分方程组：

$$\dot{\overline{x}}_1 = \overline{x}_2$$

$$\dot{\overline{x}}_2 = -\frac{1}{LC}\overline{x}_1 - \frac{R}{L}\overline{x}_2 + \frac{1}{LC}u$$

即

$$\dot{\overline{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & -\dfrac{R}{L} \end{bmatrix} \overline{\mathbf{x}} + \begin{bmatrix} 0 \\ \dfrac{1}{LC} \end{bmatrix} u$$

两组状态变量之间的关系：

$$x_1 = u_c \qquad \bar{x}_1 = u_c$$

$$x_2 = i \qquad \bar{x}_2 = \dot{u}_c = \frac{1}{C} i$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \mathbf{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} u_c \\ \frac{1}{C} i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{C} \end{bmatrix} \begin{bmatrix} u_c \\ i \end{bmatrix}$$

$$\bar{\mathbf{x}} = \mathbf{P}\mathbf{x} \qquad \mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{C} \end{bmatrix}$$

P：非奇异矩阵

状态变量的选取原则:

- 选择系统储能元件的输出物理量;
- 选择系统输出及其各阶导数;
- 使系统状态方程成为某种标准形式的变量

（对角线标准型和约当标准型）

状态变量不唯一

状态变量的选取不同，状态空间表达式也不同！

建立状态空间表达式的步骤

1) 选取 n个状态变量；确定输入、输出变量；

2) 根据系统微分方程列出n个一阶微分方程；

$\boxed{\text{状态变量、输入变量、参数}}$

3) 根据系统微分方程，列出状态空间描述。

$\boxed{\text{输出变量、状态变量、输入变量、参数}}$

结论：

(1) 状态变量选取具有非唯一性。状态变量个数➔系统的阶次；

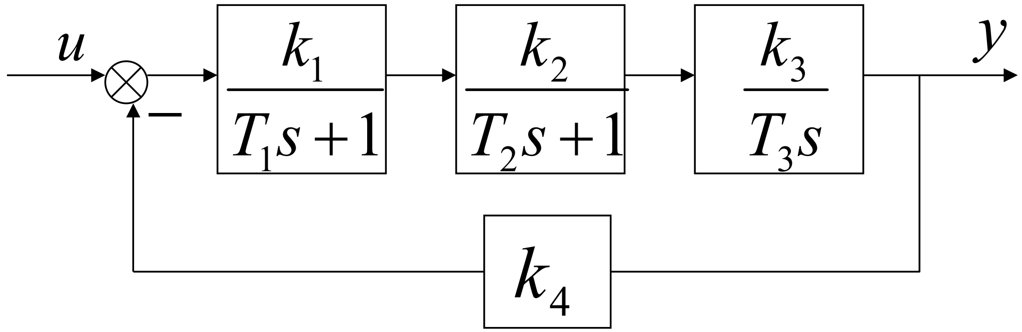
(2) 状态变量具有独立性；

(3) 不同组状态变量之间可做等价变换➔线性变换。

The establishment methods of the state space description

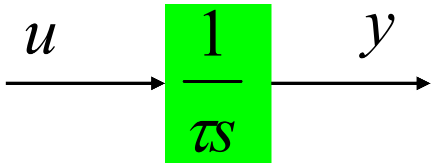
- Deduced by the mechanism of the system
- **Set up by the system block diagram**
- Evolution by differential equations or the transfer function(TF)

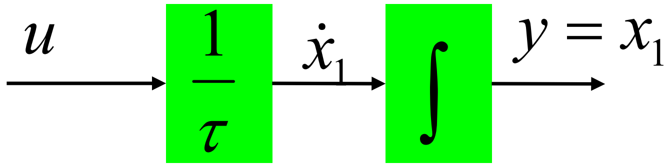
［例1-4］：系统框图如下：



1. 积分环节

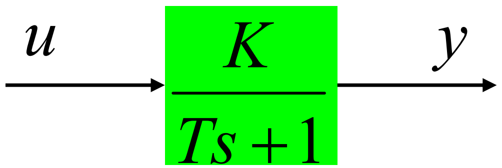
$$\begin{cases} \dot{x}_1 = \dfrac{1}{\tau} u \\ y = x_1 \end{cases}$$

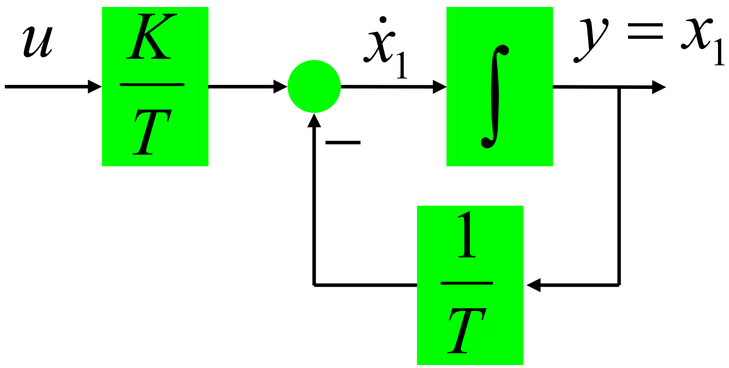




2. 惯性环节

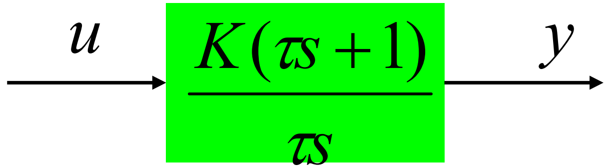
$$\begin{cases} \dot{x}_1 = -\dfrac{1}{T} x_1 + \dfrac{K}{T} u \\ y = x_1 \end{cases}$$

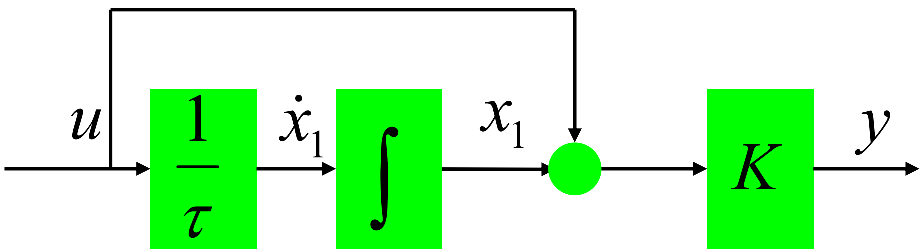




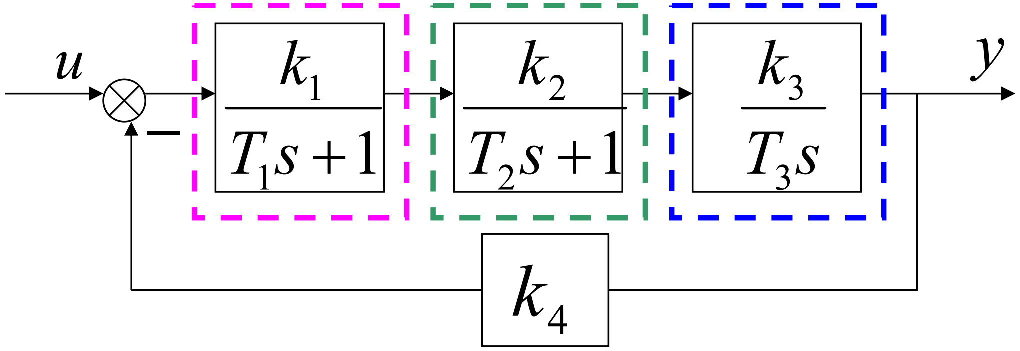
3. 比例积分环节

$$
\begin{cases}
\dot{x}_1 = \dfrac{1}{\tau} u \\
y = K x_1 + K u
\end{cases}
$$

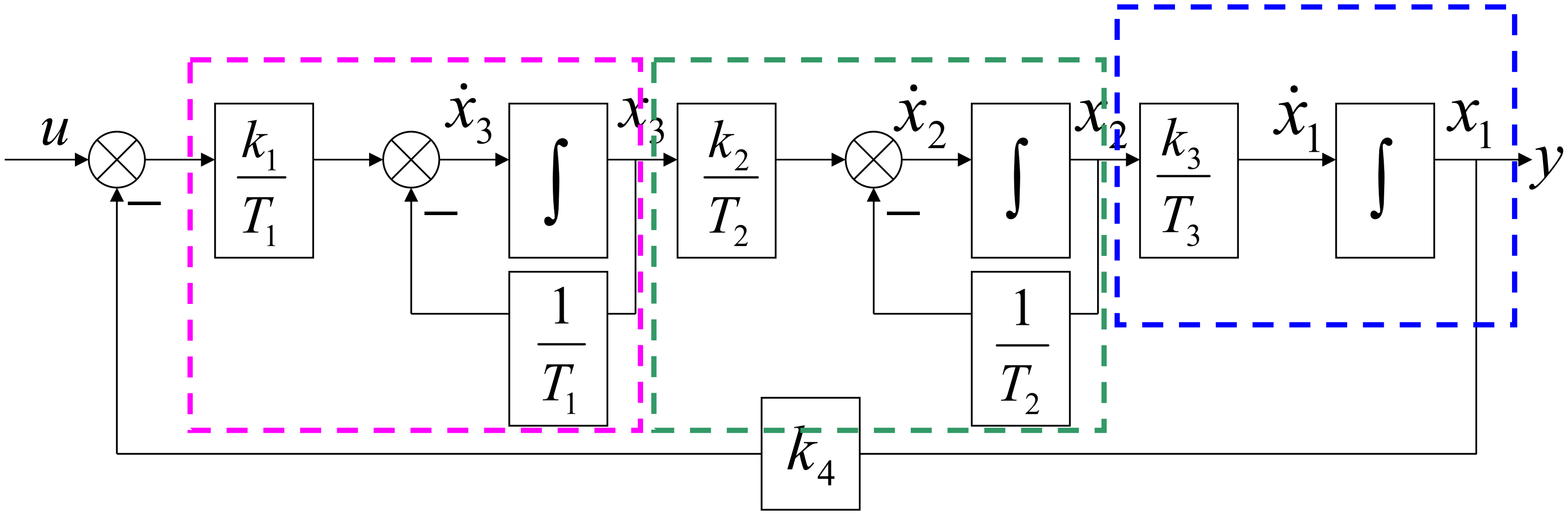


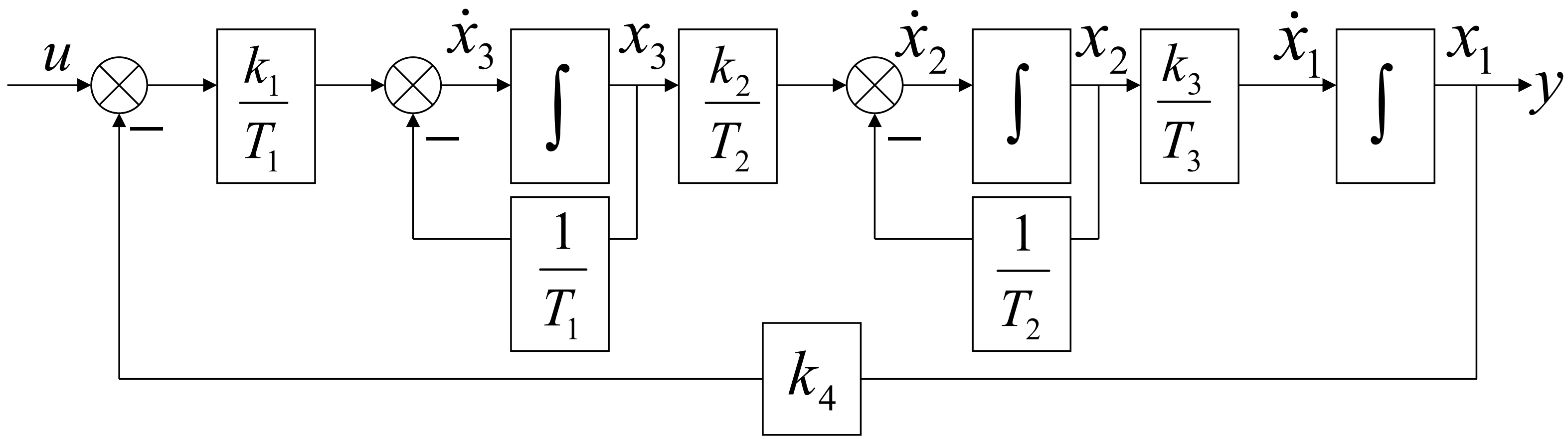


［例1-4］：系统框图如下：



等效变换如下：





图中有三个积分环节，三阶系统，取三个状态变量如上图（选择积分环节后的变量为状态变量），则有：

$$\dot{x}_1 = \frac{k_3}{T_3} x_2$$

$$\dot{x}_2 = -\frac{1}{T_2} x_2 + \frac{k_2}{T_2} x_3 \qquad\qquad y = x_1$$

$$\dot{x}_3 = -k_4 \frac{k_1}{T_1} x_1 - \frac{1}{T_1} x_3 + \frac{k_1}{T_1} u$$

$$\dot{x}_1 = \frac{k_3}{T_3} x_2$$

$$\boldsymbol{\dot{x}_2 = -\frac{1}{T_2} x_2 + \frac{k_2}{T_2} x_3} \qquad \boldsymbol{y = x_1}$$

$$\dot{x}_3 = -k_4 \frac{k_1}{T_1} x_1 - \frac{1}{T_1} x_3 + \frac{k_1}{T_1} u$$

写成矩阵形式： $$\dot{x} = \begin{bmatrix} 0 & \frac{k_3}{T_3} & 0 \\ 0 & -\frac{1}{T_2} & \frac{k_2}{T_2} \\ \frac{-k_1 k_4}{T_1} & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{k_1}{T_1} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

The establishment methods of the state space description

- Deduced by the mechanism of the system
- Set up by the system block diagram
- **Evolution by differential equations or the transfer function(TF)**

From TF(or DE) to SS model-Realizations

The differential equation and TF of the n-order SISO system

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + \cdots + b_{m-1}s^{m-1} + b_m s^m}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

The condition for the realization: $m \leq n$

Canonical form of SISO state space representation

In this section, we study an nth-order SISO system $\sum(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$, i.e.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), t \geqslant t_0$$

$$y(t) = \mathbf{c}\mathbf{x}(t) + du(t)$$

The system $\sum(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ may be represented by the following canonical form:

- Controllable canonical form
- Observable canonical form
- Jordan (or Diagonal) canonical form

1. Transforming into the controllable canonical form

Special case: G(s) has no zeros

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 = b_0 u$$

1) Choose the state variable

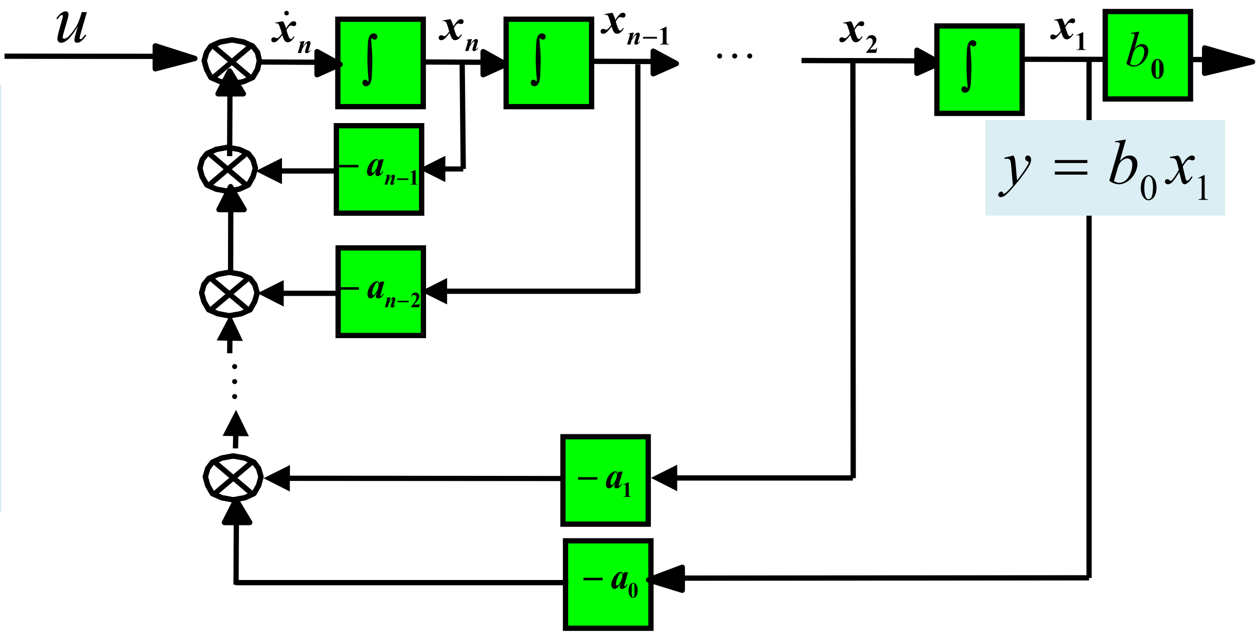
$$x_1 = y/b_0\,,$$
$$x_2 = \dot{y}\,/b_0,$$
$$\cdots,$$
$$x_n = y^{(n-1)}/b_0$$

The state variables are the output y and the ith order derivative of the y;

2) State space representation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \quad \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u \end{cases}$$

$$y = b_0 x_1$$



3) Represented into matrix form

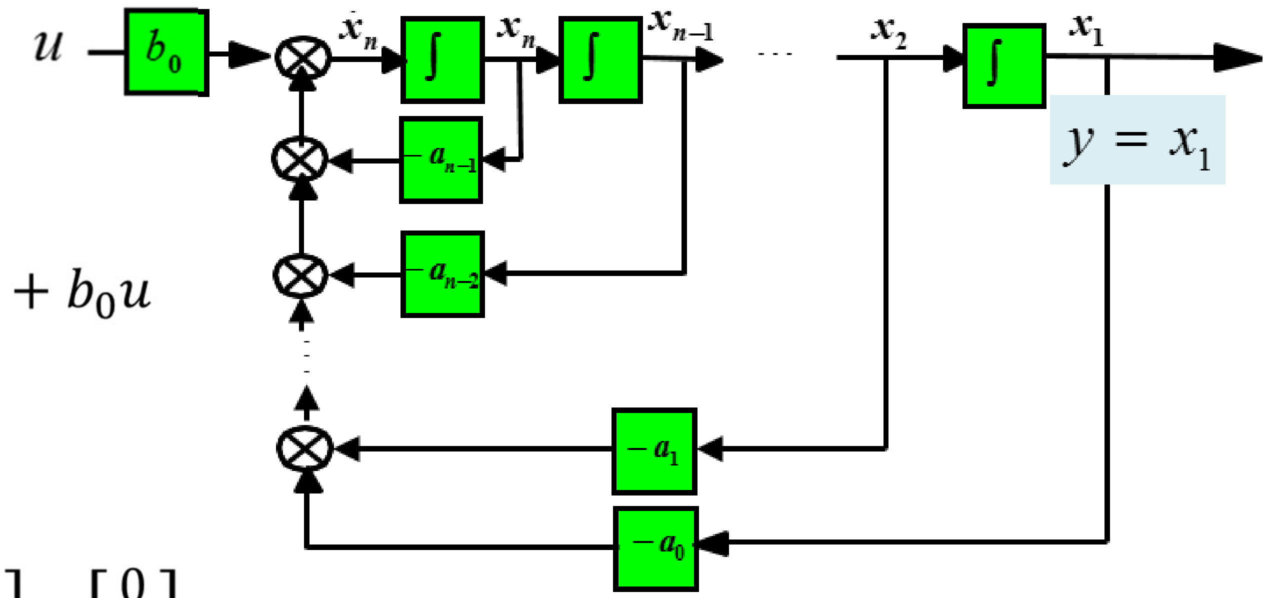
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_0 \quad 0 \quad \cdots \quad 0] x$$

NOTE

- The state variables are the output y and the ith order derivative of the y;
- Companion matrix A: the elements that above the main diagonal elements are all 1, the elements of the last row are the negative coefficients of the differential equations,

$$
\begin{aligned}
&y = x_1, \\
&\dot{x}_1 = x_2 \\
&\vdots \\
&\dot{x}_{n-1} = x_n \\
&\dot{x}_n = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + b_0 u
\end{aligned}
$$



$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x
$$

特点：

- 状态变量是输出y及y的各阶导数
- 系统矩阵A特点：主对角线上方的元素为1，最后一行为微分方程系数的负值，其它元素全为0，称为友矩阵或相伴矩阵。

Example: Find the state space description.

$$\dddot{y} + 5\ddot{y} + 8\dot{y} + 6y = u$$

Solution

1) Choose the state variables

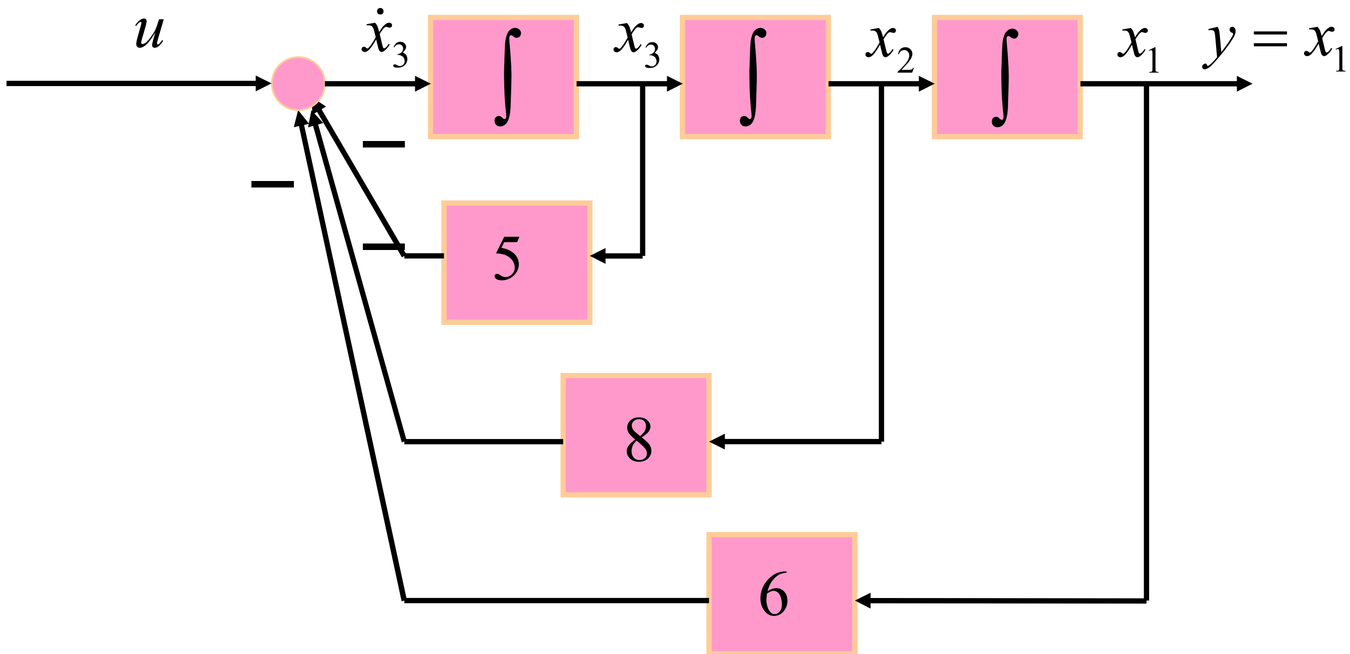
$$x_1 = y,\ x_2 = \dot{y},\ x_3 = \ddot{y}$$

2) The EQ can be represented as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -6x_1 - 8x_2 - 5x_3 + u$$
$$y = x_1$$

3) Represented as matrix form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -8 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = [1 \quad 0 \quad 0]x$$



1. Transforming into the controllable canonical form

Special case: G(s) has zeros

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1\dot{y}+a_0 y=\underline{b_m u^{(m)}+b_{m-1}u^{(m-1)}+\cdots+b_1\dot{u}+b_0 u}$$

- State variable selection principle

Do not appear the derivative term of u on right of the derived first order differential equations

使导出的一阶微分方程组右边不出现u的导数项。

If $\boldsymbol{m = n}$

严格有理真分式

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$\Rightarrow$ Strictly rational fraction

Using long division

综合除法

$$= b_n + \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = b_n + \frac{N(s)}{D(s)}$$

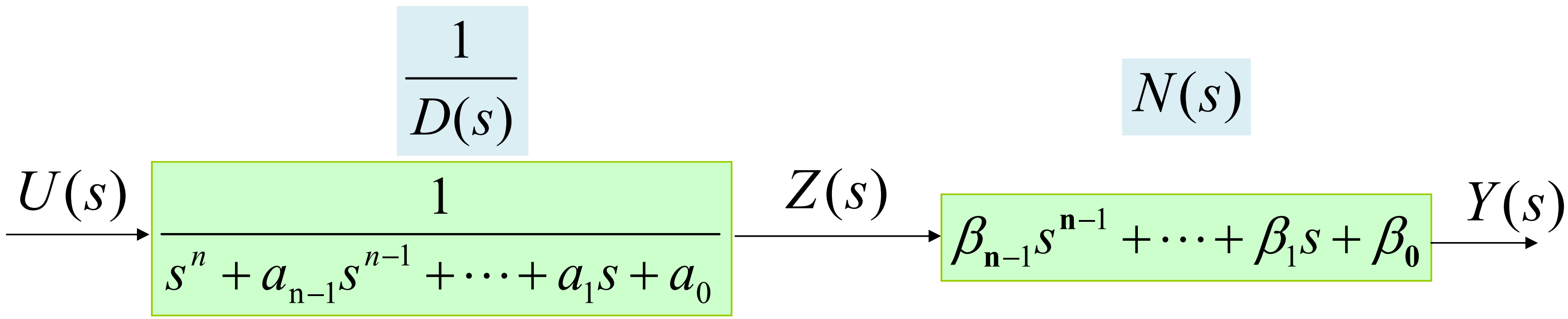
$$\begin{gathered} \beta_0 = b_0 - a_0 b_n \\ \beta_1 = b_1 - a_1 b_n \\ \vdots \\ \beta_{n-1} = b_{n-1} - a_{n-1} b_n \end{gathered}$$

If $b_n = 0$ $\Rightarrow$ 则 $\dfrac{Y(s)}{U(s)} = \dfrac{N(s)}{D(s)}$ $\quad \beta_i = b_i$

$$U(s) \longrightarrow \boxed{\frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}} \longrightarrow Y(s)$$

Introduce a interim variable $Z(s)$

串联分解：



$$\frac{Z(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$\Rightarrow z^{(n)} + a_{n-1}z^{(n-1)} + \cdots + a_1 \dot{z} + a_0 z = u$$

$$\frac{Y(s)}{Z(s)} = \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0$$

$$\Rightarrow y = \beta_{n-1}z^{(n-1)} + \cdots + \beta_1 \dot{z} + \beta_0 z$$

Choose the state variables

$$\begin{cases} x_1 = z \\ x_2 = \dot{z} \\ x_3 = \ddot{z} \\ \vdots \\ x_n = z^{(n-1)} \end{cases}$$

状态方程：

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_n &= -a_0 z - a_1\dot{z} - \cdots - a_{n-1}z^{(n-1)} + u\\ &= -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + u\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix} \quad \mathbf{b} = \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

输出方程：

$$\mathbf{c} = \begin{bmatrix}\beta_0 & \beta_1 & \cdots & \beta_{n-1}\end{bmatrix}$$

$$y = \beta_0x_1 + \beta_1x_2 + \cdots + \beta_{n-1}x_n$$

可控标准型

If $b_n \neq 0$

$$G(s) = \frac{Y(s)}{U(s)} = b_n + \frac{N(s)}{D(s)}$$

so $y = \mathbf{cx} + du = \mathbf{cx} + b_nu$

Example: Get the state space equation of the system

$$\dddot{y} + 9\ddot{y} + 8\dot{y} = \ddot{u} + 4\dot{u} + u$$

Solution $\quad n = 3,\ a_2 = 9, a_1 = 8, a_0 = 0$

$$b_2 = 1,\ b_1 = 4,\ b_0 = 1 \qquad \Rightarrow \beta_i$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix} \qquad d = b_n$$

Example: Get the state space equation of the system

$$\dddot{y} + 9\ddot{y} + 8\dot{y} = \ddot{u} + 4\dot{u} + u$$

Solution

$$n = 3,\ a_2 = 9, a_1 = 8, a_0 = 0$$

$$b_2 = 1,\ b_1 = 4,\ b_0 = 1 \qquad \Rightarrow \beta_i (b_3 = 0)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Controllable canonical form

The transfer function

能控标准型

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}\quad (b_n=0)$$

can be transformed into the following state-space model

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}b_0 & b_1 & b_2 & \cdots & b_{n-1}\end{bmatrix}x$$

Controllable canonical form (con.)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \quad (b_n \neq 0)$$

$$\begin{aligned}
\frac{Y(s)}{U(s)} &= \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \\
&= b_n + \frac{(b_{n-1} - b_n a_{n-1}) s^{n-1} + \cdots + (b_1 - b_n a_1) s + (b_0 - b_n a_0)}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} b_0 - b_n a_0 & b_1 - b_n a_1 & \cdots & b_{n-1} - b_n a_{n-1} \end{bmatrix} \mathbf{x} + b_n u
\end{aligned}$$

From TF to observable canonical form

Now, let's derive the observable canonical form for the (DE) transfer function

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1\dot{y} + a_0 y = b_{n-1}u^{(n-1)} + b_{n-2}u^{(n-2)} + \cdots + b_1\dot{u} + b_0 u$$

$$G(s) \triangleq \frac{N(s)}{D(s)} = \frac{y(s)}{u(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}$$

We set

$$x_n = y$$

$$x_i = \dot{x}_{i+1} + a_i y - b_i u \qquad i = 1, \cdots, n-1$$

$$x_{n-1} = \dot{x}_n + a_{n-1}y - b_{n-1}u = \dot{y} + a_{n-1}y - b_{n-1}u$$

$$x_{n-2} = \dot{x}_{n-1} + a_{n-2}y - b_{n-2}u = \ddot{y} + a_{n-1}\dot{y} - b_{n-1}\dot{u} + a_{n-2}y - b_{n-2}u$$

$$\vdots$$

$$x_2 = \dot{x}_3 + a_2 y - b_2 u = y^{(n-2)} + a_{n-1}y^{(n-3)} - b_{n-2}u^{(n-3)} + a_{n-2}y^{(n-4)} - b_{n-2}u^{(n-4)} + \cdots + a_2 y - b_2 u$$

$$x_1 = \dot{x}_2 + a_1 y - b_1 u = y^{(n-1)} + a_{n-1}y^{(n-2)} - b_{n-1}u^{(n-2)} - a_{n-2}y^{(n-3)} - b_{n-2}u^{(n-3)} + \cdots + a_1 y - b_1 u$$

$$\dot{x}_1 = -a_0 y + b_0 u = -a_0 x_n + b_0 u$$

$$\left(y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1\dot{y} + a_0 y = b_{n-1}u^{(n-1)} + b_{n-2}u^{(n-2)} + \cdots + b_1\dot{u} + b_0 u \right)$$

The state equation:

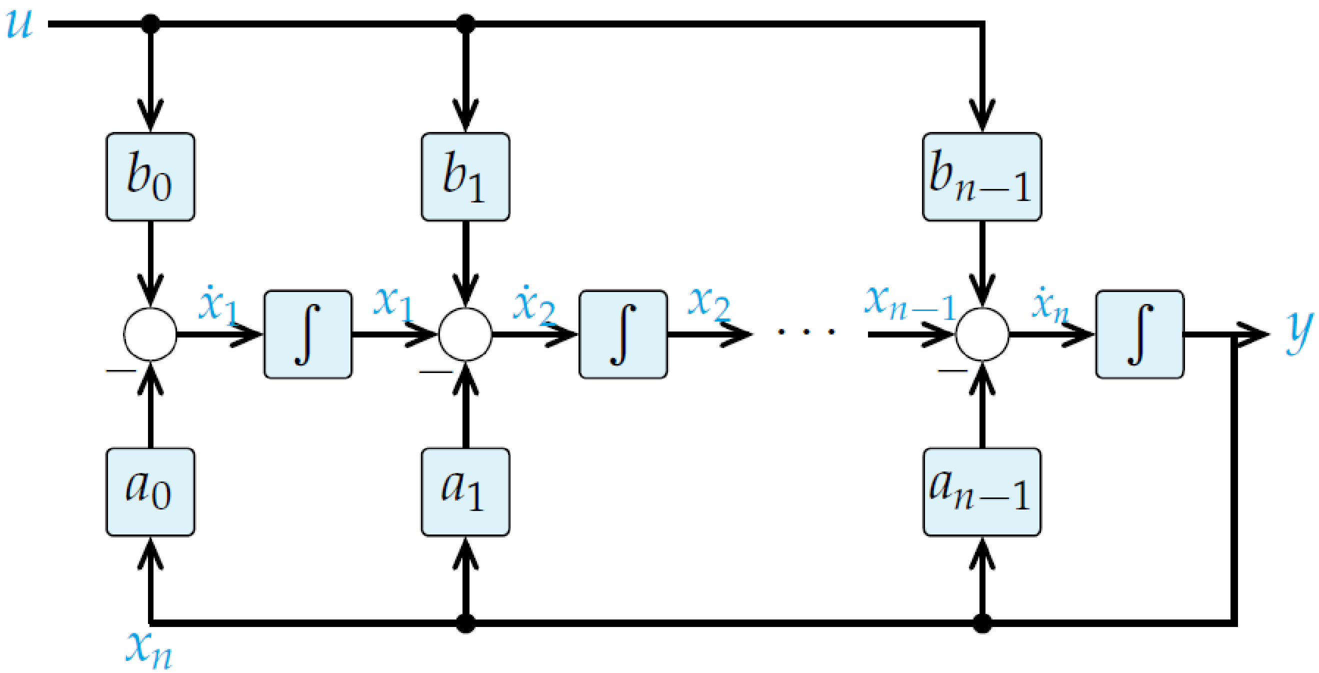
$$\dot{x}_1 = -a_0 x_n + b_0 u$$
$$\dot{x}_2 = x_1 - a_1 x_n + b_1 u$$
$$\vdots$$
$$\dot{x}_{n-1} = x_{n-2} - a_{n-2} x_n + b_{n-2} u$$
$$\dot{x}_n = x_{n-1} - a_{n-1} x_n + b_{n-1} u$$

The output equation: $y = x_n$

The vector-matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}\mathbf{x}$$

$$\left\{ \begin{array}{l} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} \\ \\ \mathbf{c} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \end{array} \right.$$



Now, we have

$$\begin{cases} \dot{x}_1 &= -a_0 x_n + b_0 u \\ \dot{x}_2 &= x_1 - a_1 x_n + b_1 u \\ &\vdots \\ \dot{x}_n &= x_{n-1} - a_{n-1} x_n + b_{n-1} u \\ y &= x_n \end{cases}$$

Observable canonical form

The transfer function

能观标准型

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$$

can be transformed into the following state-space model

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & \cdots & -a_0\\ 1 & 0 & 0 & \cdots & -a_1\\ 0 & 1 & 0 & \ddots & -a_2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 1 & -a_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_n\end{bmatrix}+\begin{bmatrix}b_0\\ b_1\\ b_2\\ \vdots\\ b_{n-1}\end{bmatrix}u$$

$$y=\begin{bmatrix}0 & 0 & 0 & \cdots & 1\end{bmatrix}x$$

Dual systems

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix} x$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & 0 & \ddots & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} x$$

$$\boxed{\mathbf{A}_c = \mathbf{A}_0^T, \quad \mathbf{b}_c = \mathbf{c}_0^T, \quad \mathbf{c}_c = \mathbf{b}_0^T}$$

From TF to Diagonal or Jordan canonical form

Let's begin with the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Suppose the poles of G(s), i.e. $p_1, p_2, \cdots, p_n$, are real and distinct, then the transfer function can be factorized as follows

$$\mathrm{G(s)} = \frac{Y(s)}{U(s)} = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2} + \cdots + \frac{\alpha_n}{s-\lambda_n} + \delta$$

where

$$\boldsymbol{\alpha_i = \lim_{s\to\lambda_i}(s-\lambda_i)\cdot g(s) \qquad i = 1,2,\cdots,n}$$

Diagonal canonical form

The transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2} + \cdots + \frac{\alpha_n}{s-\lambda_n} + \delta$$

can be transformed into the following state-space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} x + \delta u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} x + \delta u$$

Example

Given a system with the following transfer function

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

find its state space description using the partial fraction expansion.

Solutions

It is obvious that G(s) can be expanded into

$$G(s) = \frac{c_1}{s+1} + \frac{c_2}{s+2} + \frac{c_3}{s+3}$$

The coefficients can be calculated as

$$\begin{cases} c_1 = \lim_{s\to -1} G(s)(s+1) = \frac{1}{2} \\ c_2 = \lim_{s\to -2} G(s)(s+2) = -1 \\ c_3 = \lim_{s\to -3} G(s)(s+3) = \frac{1}{2} \end{cases}$$

Therefore, we have

$$G(s) = \frac{\frac{1}{2}}{s+1} + \frac{-1}{s+2} + \frac{\frac{1}{2}}{s+3}$$

The diagonal form can be obtained as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Jordan canonical form

The transfer function

$$\frac{Y(s)}{U(s)} = \frac{c_1}{(s-\lambda_1)^n} + \frac{c_2}{(s-\lambda_1)^{n-1}} + \cdots + \frac{c_n}{s-\lambda_1}$$

can be transformed into the following state-space model

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$\boxed{c_i = \frac{1}{(i-1)!} \frac{d^{i-1}}{ds^{i-1}} \left[\frac{Y(s)}{U(s)} (s-\lambda_i)^n \right] \Bigg|_{s=\lambda_i}}$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} x$$

Jordan canonical form(con.)

$$\frac{Y(s)}{U(s)} = \frac{c_1}{(s-\lambda_1)^r} + \frac{c_2}{(s-\lambda_1)^{r-1}} + \cdots + \frac{c_r}{s-\lambda_1} + \frac{c_{r+1}}{s-\lambda_{r+1}} + \cdots + \frac{c_n}{s-\lambda_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_r \\ \hline \dot{x}_{r+1} \\ \dot{x}_{r+2} \\ \vdots \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{cccc|cccc} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & \lambda_1 & & & & \\ \hline & & & & \lambda_{r+1} & & & \\ & & & & & \lambda_{r+2} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \hline x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \hline 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

Jordan block

Diagonal block

$$y = \left[\begin{array}{cccc|cccc} c_1 & c_2 & \cdots & c_r & c_{r+1} & c_{r+2} & \cdots & c_n \end{array}\right] x$$

$$c_i = \frac{Y(s)}{U(s)}(s-\lambda_i)\bigg|_{s=\lambda_i}$$

$$c_i = \frac{1}{(i-1)!}\frac{d^{i-1}}{ds^{i-1}}\left[\frac{Y(s)}{U(s)}(s-\lambda_i)^n\right]\bigg|_{s=\lambda_i}$$

Outline of Chapter 1

1.1 Introduction

1.2 State space description of dynamic systems

1.3 Relationship between TF & SS equations

1.4 Similarity transformation

1.5 Simulations with MATLAB

From SS model to TF

Let's begin with an LTI system

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \overset{\mathcal{L}}{\Longrightarrow} \begin{cases} s\mathbf{X}(s) - \mathbf{x}(0) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

By assuming the initial condition of the state to be zero, the state equation can be rearranged as follows

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

Submitting it into the output equation produces

$$\mathbf{Y}(s) = \left[\mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

Hence we have the transfer function matrix for the above system

$$\boxed{\mathbf{G}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

例　考虑这样一个系统，它的状态方程、输出方程分别为：

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix},\quad \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$$

求系统的传递函数矩阵。

例 考虑这样一个系统，它的状态方程、输出方程分别为：

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求系统的传递函数矩阵。

解：因为 $A=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$，$B=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$，$C=\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 2\end{bmatrix}$，$D=\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}$

且 $$(SI-A)^{-1}=\begin{bmatrix}S & -1\\ 2 & S+3\end{bmatrix}^{-1}=\begin{bmatrix}\dfrac{S+3}{(S+1)(S+2)} & \dfrac{1}{(S+1)(S+2)}\\ \dfrac{-2}{(S+1)(S+2)} & \dfrac{S}{(S+1)(S+2)}\end{bmatrix}$$

因此，系统的传递矩阵为：

$$G(s)=C(SI-A)^{-1}B+D$$

$$=\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}\frac{S+3}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)}\\ \frac{-2}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)}\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{S+4}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)}\\ \frac{S+4}{S+2} & \frac{1}{S+2}\\ \frac{2(S-2)}{(S+1)(S+2)} & \frac{S^2+5S+2}{(S+1)(s+2)}\end{bmatrix}$$

此传递函数矩阵有六个元素，每个都是一个传递函数。

Outline of Chapter 1

What's similarity transformation

Definition (Similarity transformation)

Given an LTI system with the following state-space model

$$\begin{cases} \dot{\mathrm{x}} = \mathrm{A}x + \mathrm{B}u \\ \mathrm{y} = \mathrm{C}x + \mathrm{D}u \end{cases}$$

we can choose a nonsingular matrix P such that $x = \mathrm{P}\bar{x}$ and the model with the new state $\bar{x}$ is as follows

$$\begin{cases} \dot{\bar{x}} = \bar{\mathrm{A}}\bar{x} + \bar{\mathrm{B}}u \\ \mathrm{y} = \bar{\mathrm{C}}\bar{x} + \bar{\mathrm{D}}u \end{cases}$$

where $\bar{\mathrm{A}}=\mathrm{P}^{-1}\mathrm{AP},\ \bar{\mathrm{B}}=\mathrm{P}^{-1}\mathrm{B},\ \bar{\mathrm{C}}=\mathrm{CP},\ \bar{\mathrm{D}}=\mathrm{D}.$

What's similarity transformation

Definition (Similarity transformation)

Given an LTI system with the following state-space model

$$\begin{cases} \dot{\mathrm{x}} = \mathrm{A}x + \mathrm{B}u \\ \mathrm{y} = \mathrm{C}x + \mathrm{D}u \end{cases}$$

we can choose a nonsingular matrix P such that $x = \mathrm{P}\bar{x}$ ($\bar{x} = P^{-1}x$)and the model with the new state $\bar{x}$ is as follows

$$\begin{cases} \dot{\bar{x}} = \bar{\mathrm{A}}\bar{x} + \bar{\mathrm{B}}u \\ \mathrm{y} = \bar{\mathrm{C}}\bar{x} + \bar{\mathrm{D}}u \end{cases}$$

where $\bar{\mathrm{A}}=\mathrm{P}^{-1}\mathrm{AP},\ \bar{\mathrm{B}}=\mathrm{P}^{-1}\mathrm{B},\ \bar{\mathrm{C}}=\mathrm{CP},\ \bar{\mathrm{D}}=\mathrm{D}.$

What's changed after the similarity transformation

- The state vector is changed, but its dimension remains the same.
- The eigenvalues of the system are NOT changed, i.e. A and $\bar{\mathrm{A}}$ have the same eigenvalues.

The characteristic equation of A is

$$\left|\lambda \mathrm{I} - \mathrm{A}\right| = 0$$

The characteristic equation of $\bar{\mathrm{A}}$ is

$$|\lambda \mathbf{I} - \bar{\mathbf{A}}| = |\lambda \mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}| = 0$$

We have

$$\lambda \mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \lambda \mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^{-1}(\lambda \mathbf{I} - \mathbf{A})\,\mathbf{P}$$

$$= \left|P^{-1}P\right|\left|\lambda \mathrm{I} - \mathrm{A}\right| = \left|\lambda \mathrm{I} - \mathrm{A}\right|$$

What's changed after the similarity transformation

- The value of the coefficients $(a_0, a_1, \cdots a_{n-1}, a_n)$ of the characteristic polynomial are NOT changed.

$$\left|\lambda \mathrm{I} - \mathrm{A}\right| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

- The transfer function G(s) is NOT changed.

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ \bar{G}(s) &= \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} \\ &= CP(s\mathrm{I} - \mathrm{P}^{-1}\mathrm{AP})^{-1}P^{-1}B + D \\ &= C(s\mathrm{I} - \mathrm{A})^{-1}B + D \end{aligned}$$

Obtain diagonal form by similarity transformation

Case 1: When system matrix **A** has distinct eigenvalues, it can always be transformed into the diagonal form by appropriate similarity transformation matrix **P**. That is,

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

From $\mathrm{P}^{-1}\mathrm{AP} = \Lambda$, it is easy to obtain

$$\mathrm{AP} = \mathrm{P}\Lambda$$

Assuming $$\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

where v_i is the eigenvector of A corresponding to the eigenvalue λ_i.

Obtain diagonal form by simil. trans. (cont.)

Substituting v_i into $\mathrm{AP} = \mathrm{P\Lambda}$ yields

$$\mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix}$$

$$= \begin{bmatrix}\lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n\end{bmatrix}$$

Equating columns on the both sides of the above equation yields

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

This equation can be put in the form

$$(\lambda_i\mathbf{I} - \mathbf{A})\,\mathbf{v}_i = \mathbf{0}, \quad i = 1, 2, \cdots, n$$

When all the eigenvalues of $\mathbf{A}$ are distinct, we can search n independent eigenvectors v_i. Therefore $\mathbf{P}$ must be nonsingular.

Obtain diagonal form by simil. trans. —An Example

Example: Consider the following state space model of a system

$$\dot{x} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} x$$

Find the transformation matrix **P** so that it can be transformed into the diagonal form.

Solutions

From the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

we obtain the distinct eigenvalues $\lambda_1 = -2, \lambda_2 = -3, \lambda_1 = -4.$

The corresponding eigenvectors are chosen as

$$\boxed{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 7 \\ 12 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}}$$

The transform matrix is

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 7 & 6 & 5 \\ 12 & 8 & 6 \end{bmatrix} \qquad \Rightarrow \mathbf{P}^{-1} = -\frac{1}{2} \begin{bmatrix} -4 & 2 & -1 \\ 18 & -6 & 2 \\ -16 & 4 & -1 \end{bmatrix}$$

Then

$$\bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \quad \bar{\mathbf{b}} = \mathbf{P}^{-1}\mathbf{b} = \begin{bmatrix} -1 \\ -3 \\ 6 \end{bmatrix}, \quad \bar{\mathbf{c}} = \mathbf{c}\mathbf{P} = \begin{bmatrix} 3 & 5 & 5 \end{bmatrix}$$

Obtain Jordan form by similarity transformation(con.)

Case 2: When **A** has one or more multiple eigenvalues

If λ_1 (q-multiple) , the other (n-q) are distinct eigenvalues.The transformation matrix :

T=$(P_1, P_2, \cdots, P_q, P_{q+1} \cdots, P_n)$

where the eigenvectors $P_{q+1} \cdots, P_n$ are calculated using the same method of the distinct eigenvalues. The eigenvectors $P_1 \cdots, P_q$ are calculated according to the following equations:

$$
\begin{aligned}
&\lambda_1 P_1 - AP_1 = 0 \\
&\lambda_1 P_2 - AP_2 = -P_1 \\
&\vdots \\
&\lambda_1 P_q - AP_q = -P_{q-1}
\end{aligned}
$$

- there is only one eigenvector P_1 associated with λ_1.
- the other $q-1$ $(P_2, \cdots P_q)$ vectors (called generalized eigenvector) can be obtained by the left equations.

Obtain Jordan form by simil. trans. —An Example

Example: Consider a system with the system matrix **A** and control matrix **b.**

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the transformation matrix P so that it can be transformed into the Jordan or diagonal form.

Solutions

The matrix **A** has the following characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = (\lambda - 2)^3 = 0$$

we obtain the multiple eigenvalues $\lambda_1 = 2$ with the algebraic multiplicity $m_1 = 3$.

$$\lambda_1 \mathbf{I} - \mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -8 & 12 & -4 \end{bmatrix}$$

As $rank(\lambda_1 \mathrm{I} - \mathrm{A})$ = 2, we have the geometric multiplicity $q_1 = n - rank(\lambda_1 \mathrm{I} - \mathrm{A}) = 1.$

The only eigenvector is obtained by

$$(\lambda_1 \mathbf{I} - \mathbf{A})\mathbf{v}_{11} = \mathbf{0} \quad \Rightarrow \mathbf{v}_{11} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

The generalized eigenvectors can be solved as follows

$$(\lambda_1 \mathbf{I} - \mathbf{A})\mathbf{v}_{12} = \mathbf{v}_{11} \quad \Rightarrow \mathbf{v}_{12} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$(\lambda_1 \mathbf{I} - \mathbf{A})\mathbf{v}_{13} = \mathbf{v}_{12} \quad \Rightarrow \mathbf{v}_{13} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

The transform matrix is

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & \frac{3}{4} \\ 2 & -1 & \frac{1}{2} \\ 4 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \mathbf{P}^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{4} \\ 2 & -3 & 1 \\ 4 & -4 & 1 \end{bmatrix}$$

Then

$$\bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \bar{\mathbf{b}} = \mathbf{P}^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{4} \\ 1 \\ 1 \end{bmatrix}$$

Obtain Jordan form by similarity transformation(cont.)

Case 3: When A is the special form of companion matrix (controllable canonical form)

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

1) With n distinct eigenvalue $\lambda_1, \lambda_2, \cdots, \lambda_n$, the similarity transformation matrix is called Vandermonde matrix

$$\mathbf{P}=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \vdots & \lambda_n^{n-1} \end{bmatrix}$$

Obtain Jordan form by similarity transformation(cont.)

2) A With q multi-roots (λ_1) and with n-q distinct eigenvectors ($\lambda_{q+1}, \cdots, \lambda_n$),

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad \mathbf{p}_1 = \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{n-1} \end{bmatrix} \qquad \mathbf{p}_i = \begin{bmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i^{n-1} \end{bmatrix} (i = q+1, \cdots, n)$$

$$\mathbf{P} = \left[\mathbf{p}_1 \quad \frac{\partial \mathbf{p}_1}{\partial \lambda_1} \quad \frac{1}{2!}\frac{\partial^2 \mathbf{p}_1}{\partial \lambda_1^2} \quad \cdots \quad \frac{1}{(q-1)!}\frac{\partial^{q-1} \mathbf{p}_1}{\partial \lambda_1^{q-1}} \quad \middle| \quad \mathbf{p}_{q+1} \quad \cdots \quad \mathbf{p}_n \right]$$

Obtain Jordan form by similarity transformation(cont.)

Take 3-multi roots as example to derive the similarity transformation matrix is

$$\mathbf{P}=\begin{bmatrix} 1 & 0 & 0 & 1 & \cdots & 1 \\ \lambda_1 & 1 & 0 & \lambda_4 & \cdots & \lambda_n \\ \lambda_1^2 & 2\lambda_1 & 1 & \lambda_4^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{n-1} & \dfrac{d}{d\lambda_1}\lambda_1^{n-1} & \dfrac{1}{2}\dfrac{d^2}{d\lambda_1^2}\left(\lambda_1^{n-1}\right) & \lambda_4^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

$$\boxed{\mathbf{P}=\left[\mathbf{p}_1 \quad \frac{\partial \mathbf{p}_1}{\partial \lambda_1} \quad \frac{1}{2!}\frac{\partial^2 \mathbf{p}_1}{\partial \lambda_1^2} \quad \cdots \quad \frac{1}{(q-1)!}\frac{\partial^{q-1}\mathbf{p}_1}{\partial \lambda_1^{q-1}} \quad \Big| \quad \mathbf{p}_{q+1} \quad \cdots \quad \mathbf{p}_n\right]}$$

Example:

Consider a system with the system matrix $\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 3 & 0\end{bmatrix}$, find the transformation matrix P so that it can be transformed into the diagonal form.

Solution

$$|\lambda\mathbf{I}-\mathbf{A}|=\begin{vmatrix}\lambda & -1 & 0\\ 0 & \lambda & -1\\ -2 & -3 & \lambda\end{vmatrix}=(\lambda+1)^2(\lambda-2)=0$$

$$\lambda_1=\lambda_2=-1,\lambda_3=2 \quad \Rightarrow \mathbf{P}=\begin{bmatrix}\mathbf{p}_1 & \dfrac{d\mathbf{p}_1}{d\lambda} & \mathbf{p}_2\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\\ \lambda_1 & 1 & \lambda_3\\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 2\\ 1 & -2 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\frac{1}{9}\begin{bmatrix}8 & -2 & -1\\ 6 & 3 & -3\\ 1 & 2 & 1\end{bmatrix}$$

$$\bar{\mathrm{A}}=\mathbf{P}^{-1}\mathbf{AP}=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Outline of Chapter 1

MATLAB commands used in this chapter

Command	Description
canon	Canonical state-space realizations
ss2tf	State space to transfer function
ss2zp	State space to pole-zero conversion
ss	Conversion to state space
ss2ss	Change of state coordinates for state-space models
tf2ss	Transfer function to state space
tf2zp	Transfer function to pole-zero conversion
tf	Conversion to transfer function
zp2tf	Zero-pole to transfer function