Chapter 6. Introduction to Nonlinear Systems

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Outline of Chapter 6

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6.1 Introduction

6.2 Common nonlinear elements

6.3 Properties of nonlinear systems

6.4 Approaches to analyzing nonlinear systems

6.5 Summary

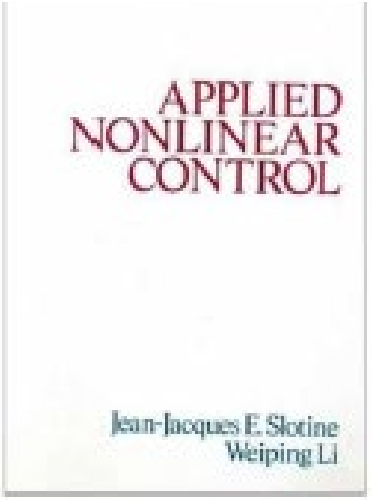
Linear and nonlinear systems

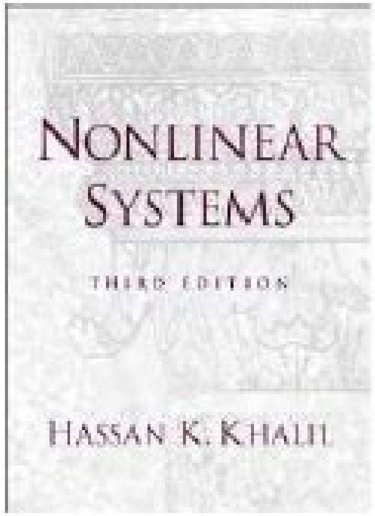
- Main features of linear systems
  - Principle of superposition

$$c_1(t) = f\left[r_1(t)\right],\ c_2(t) = f\left[r_2(t)\right] \quad \Rightarrow c_1(t) + c_2(t) = f\left[r_1(t) + r_2(t)\right]$$

- Typical inputs are used to generate transfer functions.
- Complete set of mathematical tools is available for analysis and design: e.g. ODE, Laplace transformation, etc.

- Main features of nonlinear systems
  - Principle of superposition is not applicable
  - No unified solution





Why to study nonlinearity?

- Most practical systems are nonlinear
    - All linear systems we study are only the approximations to the practical systems to certain extent.
    - For some systems, the nonlinearities are not negligible.

- Special result may be achieved by using nonlinear control.

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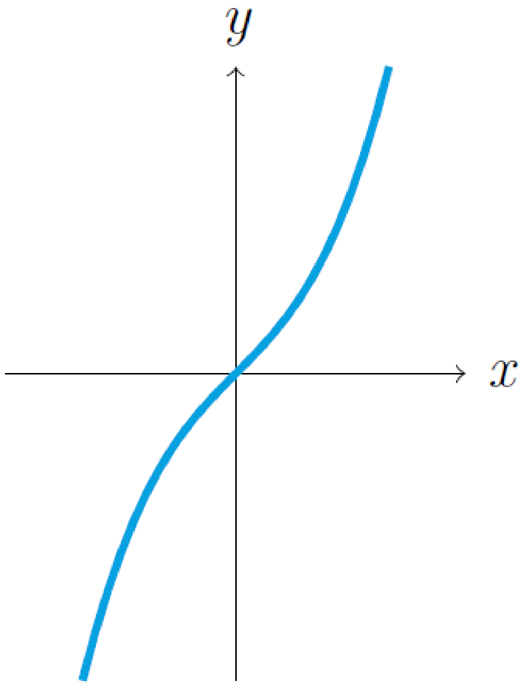
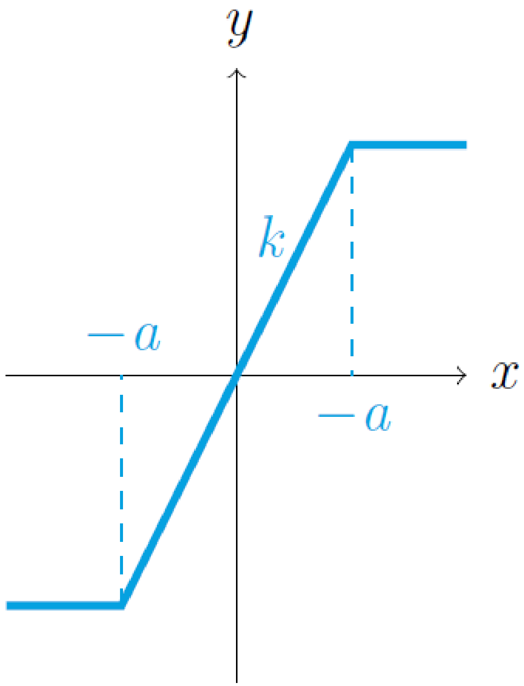
Some typical nonlinearities



A nonlinear spring

$$y = k_1 x + k_2 x^3,$$

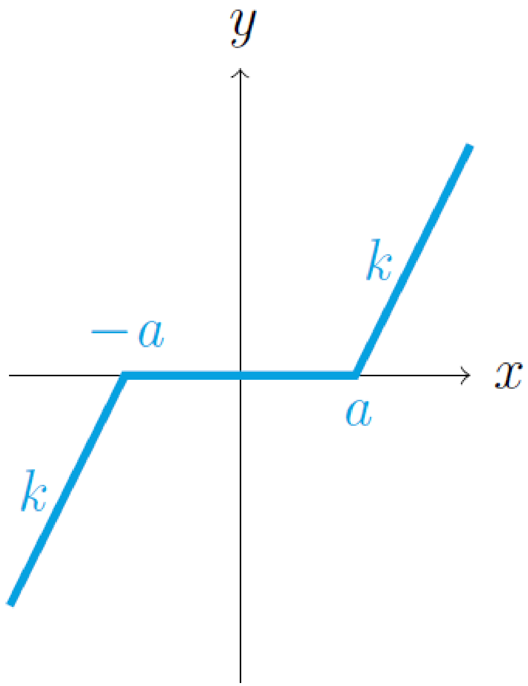
$$k_2 > 0, k_2 > 0$$



Saturation

e.g. Electronic amp.

e.g. Power limitation in servo motors

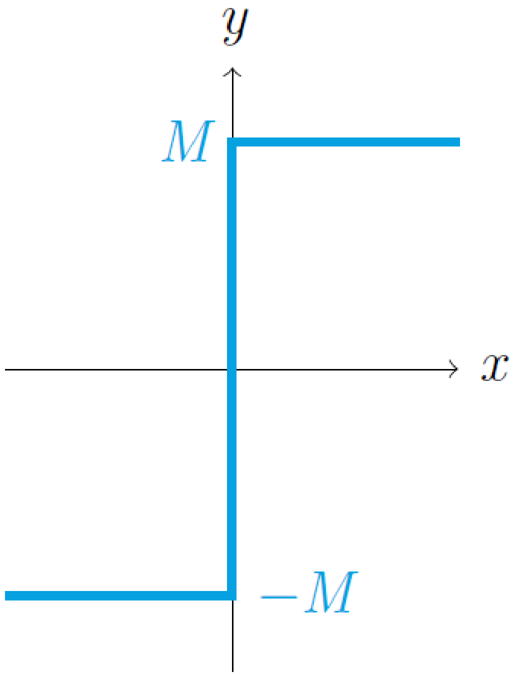


Dead zone

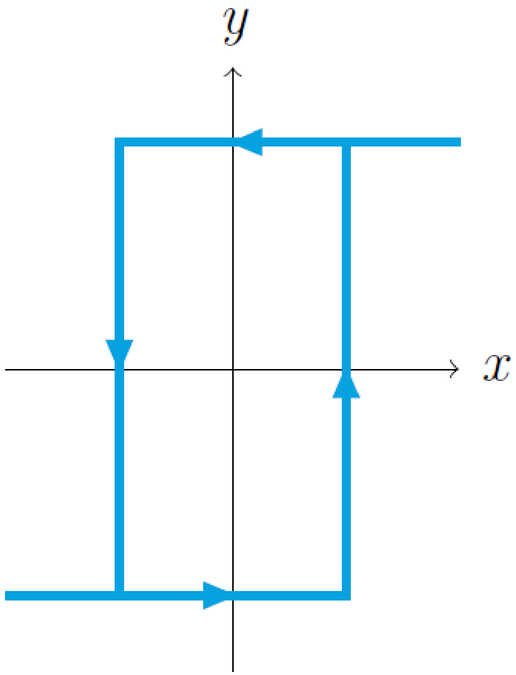
e.g. Relay amp.

e.g. Actuator

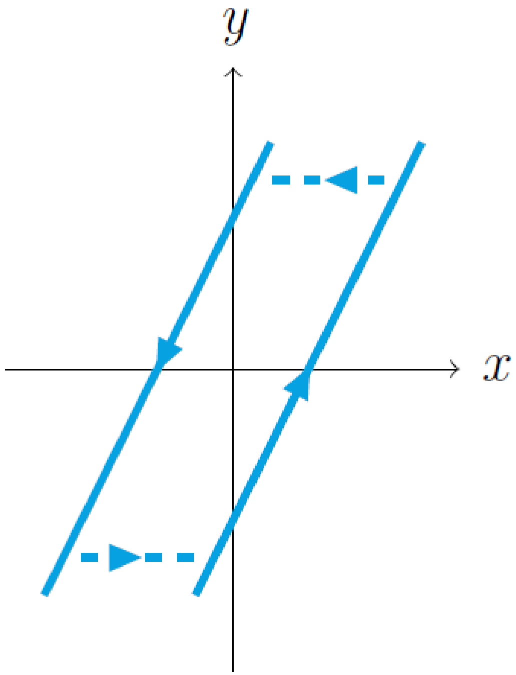
Some typical nonlinearities (con.)



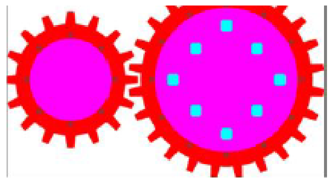
Idea reley



Hysteresis



Backlash



Outline of Chapter 6

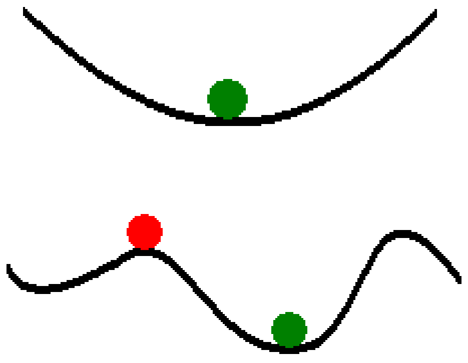
Features of nonlinear systems

Nonlinearity has some distinguished properties different from the linearity.

- Multiple equilibrium points
- Self-excited oscillations (limit cycles)
- Bifurcations
- Chaos
- ...

Multiple equilibrium points

- Nonlinear systems often have more than one equilibrium points.
- The possible equilibrium point depends on the system's parameters, initial conditions and external excitations.



Let us study this feature by an example

Example

Consider the first order system

$$\dot{x}(t) = -x(t) + x^2(t)$$

compare its response with that of the linearized system subject to varied initial conditions x_0.

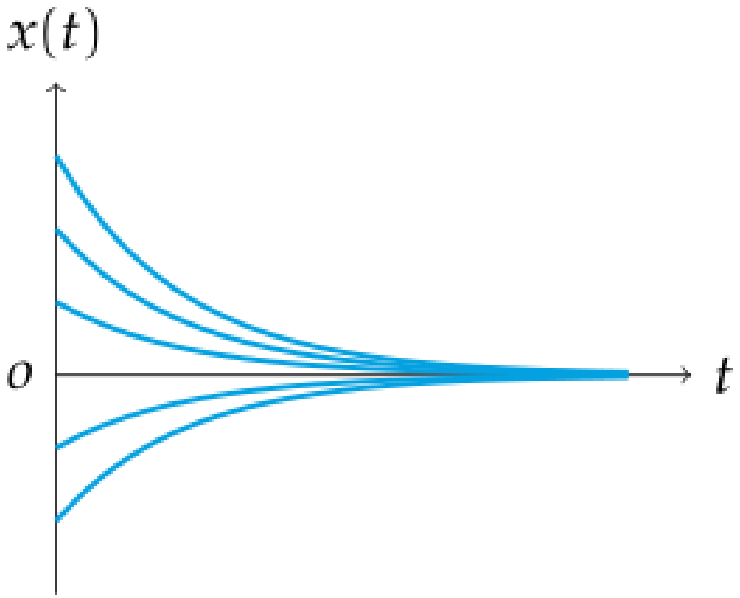
Solutions

The linearized model of the given system around the state $x(t) = 0$ is

$$\dot{x}(t) = -x(t)$$

and its response to any initial condition x_0 is

$$x(t) = x_0 e^{-t}$$



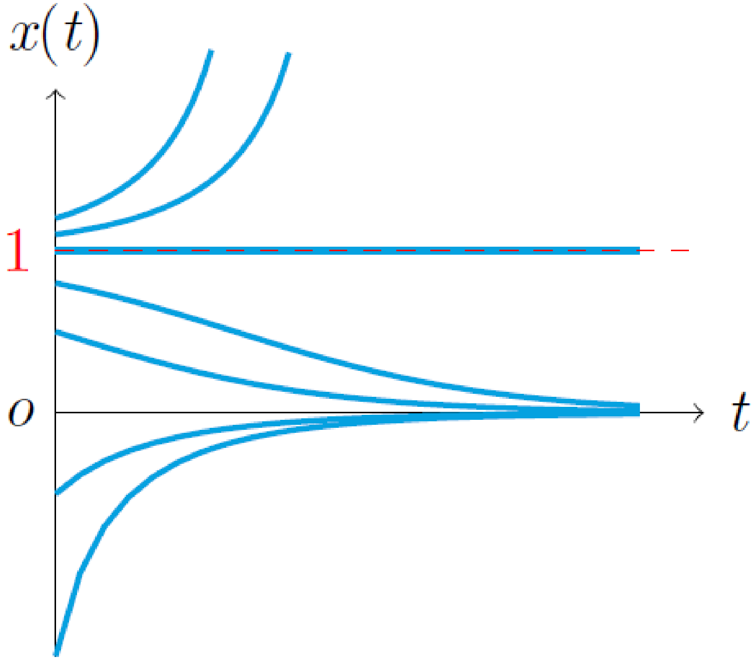
What can be observe?

- The linearized system has a unique equilibrium point.
- The equilibrium point is $x = 0$

The response of the nonlinear system is

$$x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$$

Depending on initial conditions, the trajectory will end in one of the two possible equilibrium points $x = 0$ and $x = 1$.



For $x_0 > 1$ $\lim_{t\to\infty} x(t) = \infty$

For $x_0 = 1$ $x(t) \equiv 1$

For $x_0 < 1$ $\lim_{t\to\infty} x(t) = 0$

Hence, the system has two equilibrium points $x_e = 0$ and $x_e = 1$, which depend on the initial condition x_0.

Self-excited oscillations (limit cycles)

What are self-excited oscillations or limit cycles?

- Nonlinear systems possibly have oscillations with amplitude and frequency independent of the initial conditions of the system. (Question: How about linear system?)
- The occurrence of the oscillations relies on the initial conditions.

Example (Van der Pol Equation)

Consider the second-order nonlinear differential equation

$$m\ddot{x}(t) + 2c\left(x^2(t) - 1\right)\dot{x}(t) + kx(t) = 0$$

where m, c and k are positive constants.

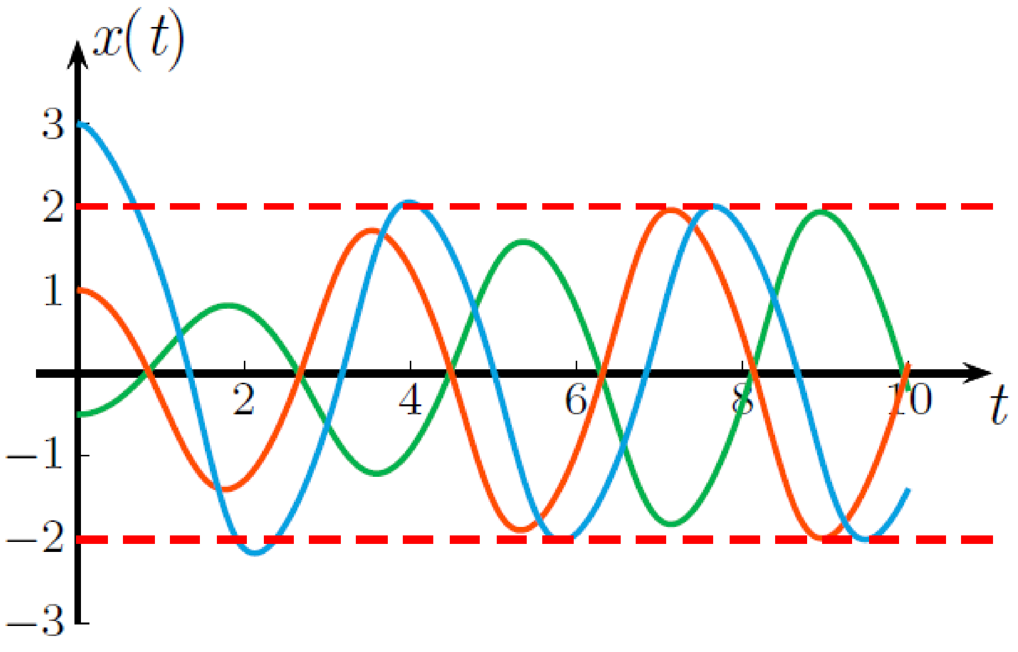
Solutions

What does the equation $m\ddot{x}(t) + 2c\left(x^2(t) - 1\right)\dot{x}(t) + kx(t)$ look like?

$|x(t)| > 1$ system stable $\longrightarrow$ system energy $\downarrow \longrightarrow |x(t)| \downarrow$

$|x(t)| < 1$ system unstable $\longrightarrow$ system energy $\uparrow \longrightarrow |x(t)| \uparrow$

Therefore, the system motion can neither grow unboundedly or decay to zero.



The responses of the system with different initial conditions display sustained oscillation with the same amplitude and frequency.

Concept of “Chaos”

Some observations

- For stable linear systems, small variations in initial conditions will lead to small variations in the output.
- For nonlinear systems, small variations in initial conditions can lead to huge variations in the output.

Definition

A commonly used definition says that, for a dynamical system to be classified as chaotic, it must have the following properties:

1. It must be sensitive to initial conditions;
2. It must be topologically mixing; and
3. Its periodic orbits must be dense.

Example

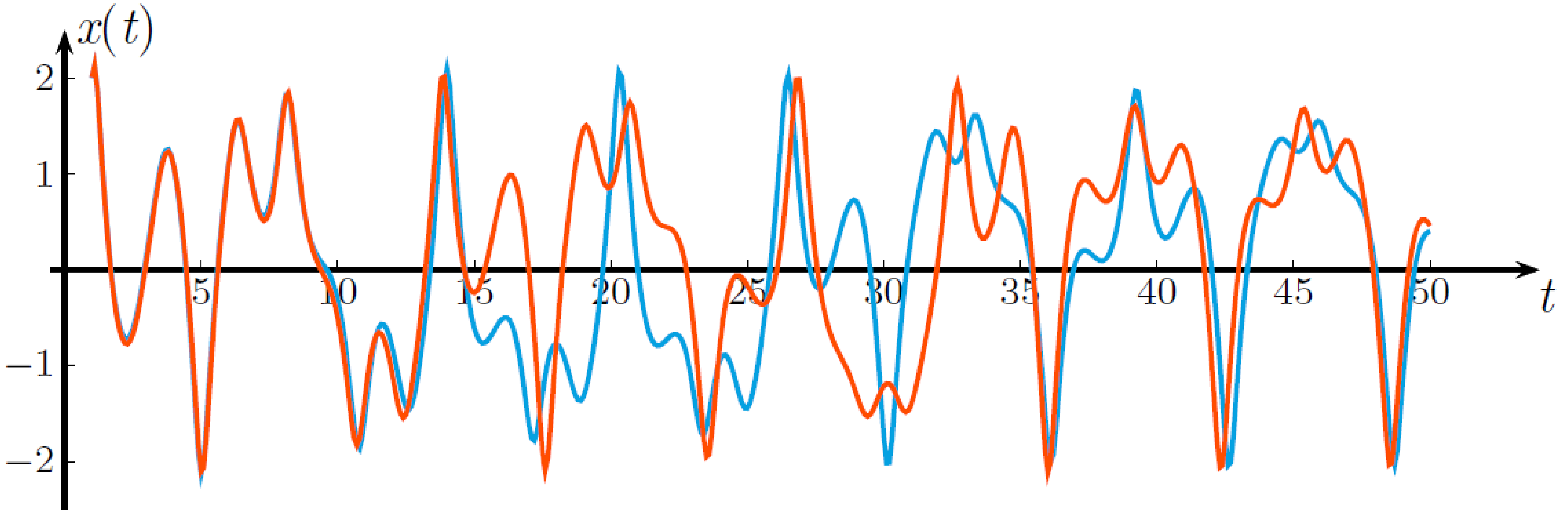
Let us consider the following simple nonlinear system

$$\ddot{x} + 0.1\dot{x} + x^5 = 6\sin t$$

with the two sets of initial conditions $x(0) = 2$, $\dot{x}(0) = 3$ and $x(0) = 2.01$, $\dot{x}(0) = 3.01$.

Solutions

The response of the system subject to initial conditions is shown below



Outline of Chapter 6

How to study nonlinear control systems?

- Linearization by Taylor's Expansion
- The research method for nonlinear system

  Phase Plane
  Describing function
  Popov method
  Feedback linearization
  Differential geometry method

- Simulation method: Digital simulation, Hardware-in-loop simulation

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How to study nonlinear control systems?

- Almost everything is nonlinear!
- There is NO unified methods for nonlinear system analysis and synthesis!
- Nonlinear control is still an active and developing researching field!
- In this course, we will study three fundamental methods for nonlinear systems
    - Describing functions (equivalent frequency response)
    - Phase plane method (graphical representation of state space)
    - Lyapunov methods

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- Any further interest? Please read the famous book "Applied Nonlinear Control" by Slotine and Li.

End of Chapter 6