Chapter 9. Lyapunov Stability Theory

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Outline of Chapter 9

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How to judge the stability of a system?

稳定性：

控制系统本身处于平衡状态。受到扰动，产生偏差，

在扰动消失后，由偏差状态逐渐恢复到原来平衡状态的性能。

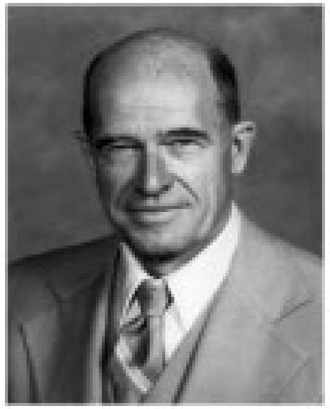
偏差逐渐变大，不能恢复到原来的平衡状态，则不稳定。

稳定性是动态系统的一个重要性能，保证系统的稳定性通常是控制器设计的最基本要求。

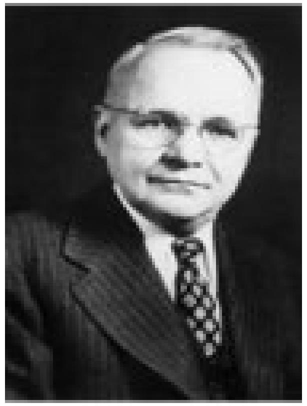
How to judge the stability of a system?

Routh

Hurwitz



Evans



Nyquist

Lyapunov

Routh-Hurwitz Criterion(algebraic criterion, 1895)

Nyquist Criterion (the frequency domain, 1932)

Root-Locus Method (graphic method, 1948)

Lyapunov Stability Theory (1892)

（1）局限于描述线性定常系统

（2）局限于研究系统的外部稳定性（输入输出稳定性）

The development of Lyapunov stability theory

- Lyapunov introduced the most useful and general approach for studying the stability of nonlinear systems.

  - 1892/ *The General Problem of Motion Stability* was first published.
  - 1908/ Translated into French.
  - 1947/ Reprinted by Princeton University Press.
  - 1960's/ Known by the control community.

A. M. Lyapunov
(1857-1918)
Russian Mathematician

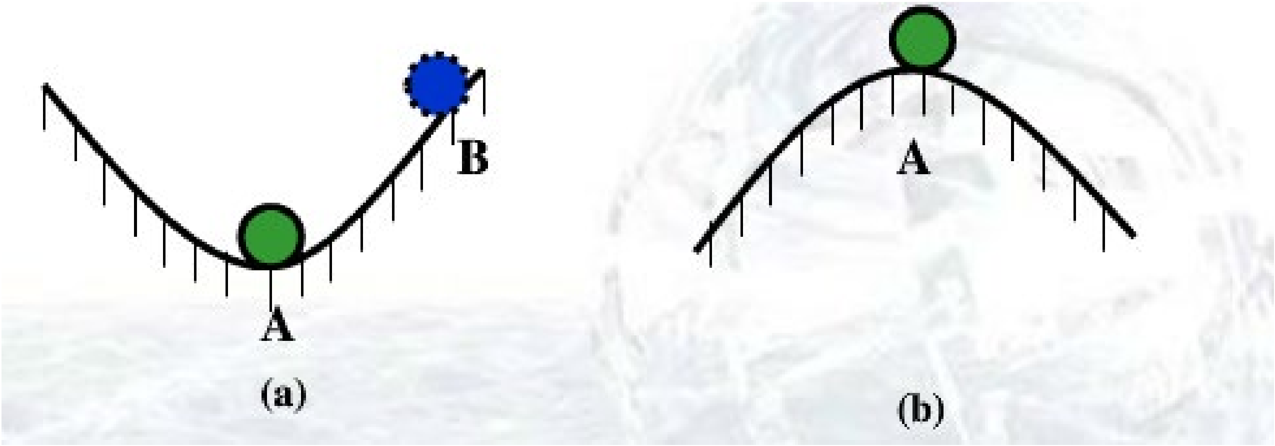
（1）稳定判据可用于线性/非线性，定常/时变系统

（2）研究系统的外部稳定性和内部稳定性（状态稳定性）

The basic idea of Lyapunov stability theory

The basic idea of Lyapunov theory

——using “energy” to judge “ stability”



Outline of Chapter 9

Autonomous and non-autonomous systems

- Linear systems are classified as either time-invariant or time-varying systems.

- For nonlinear systems, these adjectives are traditionally replaced by "autonomous" and "non-autonomous".

Definition

A nonlinear system is autonomous if it does not depend **explicitly** on time, i.e. if the system's state equation can be written

$$\dot{x} = f(x)$$

otherwise, the system is called non-autonomous.

自治系统与非自治系统定义 编辑 讨论 上传视频

本词条缺少**信息栏**、**概述图**，补充相关内容使词条更完整，还能快速升级，赶紧来编辑吧！

简述了线性系统与非线性系统、自治系统与非自治系统的概念。

对于一个微分系统

$$\dot{x} = f(x, t)$$

$\dot{x} = f(x, t)$ f 称为系统的向量场，对于微分系统：

（1）如果系统中 f 与t无关，或者说 f 中不显含时间变量t，则系统 $\dot{x} = f(x)$ 称为自治系统；

（2）否则，如果 f 中显含时间变量t，则系统 $\dot{x} = f(x, t)$ 称为非自治系统。

根据系统矩阵A是否随时间变化，可以把线性系统分为自治的和非自治的，但是对于线性系统一般称为定常的和时变的，也就是说：

（1）自治的线性系统就是定常线性系统

（2）而非自治的线性系统就是时变线性系统。

对于非线性系统，就可以分为自治非线性系统和非自治非线性系统。

Remarks on “autonomous” and “non-autonomous”

- The fundamental difference between autonomous and non-autonomous systems is whether the state trajectory of an autonomous system is independent of the initial time.

- Strictly speaking, all physical systems are non-autonomous as none of their dynamics is strictly time-invariant.

- In practice, system properties often change very slowly and can be regarded as time invariant without causing any practically meaningful error.

Equilibrium points

Definition: A state x_e is an equilibrium state of the system if once $x(t)$ is equal to x_e, it remains equal to x_e for all future time.

- Mathematically, this means that the constant vector x_e satisfies $f(x_e) = 0$.
- Given an LTI system $\dot{x} = Ax$, if A is nonsingular, the system has a single equilibrium point (the origin 0).
- If A is singular, the system has an infinity of equilibrium points. They are not isolated equilibrium states, and form a subspace of the state space.
- For nonlinear systems, usually with one or a few isolated equilibrium states.

Equilibrium points

设系统的齐次状态方程为：

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

*n*维状态向量

*n*维向量函数

展开式为： $\dot{x}_i = f_i(x_1, x_2, \cdots, x_n, t) \quad i = 1,2,\cdots,n$

方程的解（运动或状态轨线）为：$\mathbf{x}(t; \mathbf{x}_0, t_0)$

初始状态向量

初始时刻

$$\Rightarrow \mathbf{x}(t_0; \mathbf{x}_0, t_0) = \mathbf{x}_0$$

Equilibrium points

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

平衡状态：各分量相对于时间不再发生变化

$$\boxed{\dot{\mathbf{x}}_e = \mathbf{f}(\mathbf{x}_e, t) = 0}$$

所有状态的变化速度为零，即是静止状态

线性定常系统： $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

平衡状态： $\dot{\mathbf{x}}_e = \mathbf{A}\mathbf{x}_e = 0$

$|\mathbf{A}| \neq 0 \Rightarrow \mathbf{x}_e = 0$ 一个平衡状态——状态空间原点

$|\mathbf{A}| = 0$ 无穷多个平衡状态

Example

For a nonlinear system

$$\begin{cases} x'_1 = -x_1 \\ x'_2 = x_1 + x_2 - x_2^3 \end{cases}$$

The solutions of the following algebraic equations are the equilibrium state:

$$\begin{cases} -x_1 = 0 \\ x_1 + x_2 - x_2^3 = 0 \end{cases}$$

The three isolated equilibrium states are as follows:

$$x_{\mathrm{e},1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_{\mathrm{e},2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad x_{\mathrm{e},3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Outline of Chapter 9

Lyapunov stability and instability

$\forall$: *for all*　对于任意给定

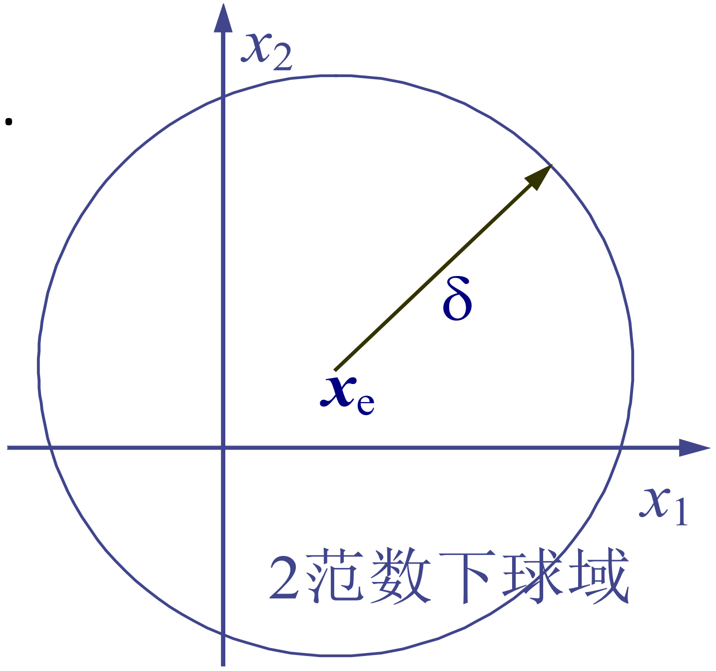
$\exists$: *at least one*　存在一个

$\|\ \|$: *norm* 范数　$\|\mathbf{x}_1 - \mathbf{x}_2\| = \sqrt{\sum_{i=1}^{n}(x_{1,i} - x_{2,i})^2}$

❑范数在数学上定义为度量*n*维空间中的点之间的距离.对n维空间中任意两点$\mathbf{x}_1$和$\mathbf{x}_2$,它们之间距离的范数记为$||\mathbf{x}_1-\mathbf{x}_2||$.

球域

❑以n维空间中的点$\mathbf{x}_e$为中心，在所定义的范数度量意义下的长度δ为半径内的各点所组成空间体称为球域，记为$S(\mathbf{x}_e,\delta)$，即$S(\mathbf{x}_e,\delta)$包含满足$||\mathbf{x}-\mathbf{x}_e||\le\delta$的*n*维空间中的各点$\mathbf{x}$.



Lyapunov stability and instability

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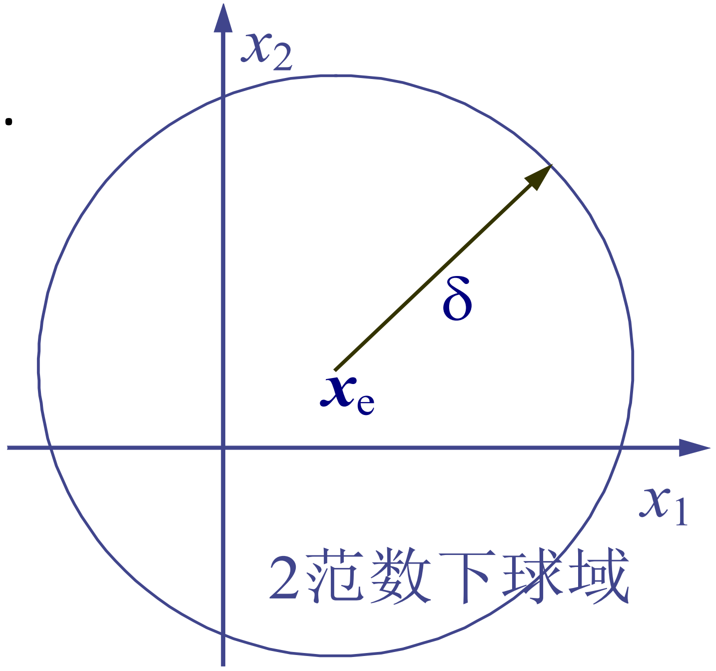


Lyapunov stability and instability

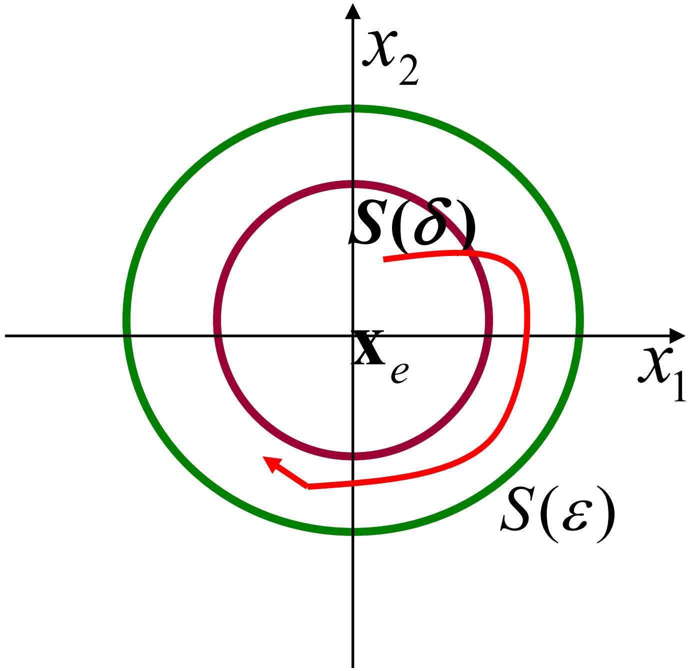
Definition (Lyapunov stability)

If the state equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(x)$$

the described system meets the conditions:

- for any $\varepsilon > 0$ and arbitrary initial time t_0,
- there always have a corresponding real number $\delta > 0$,
- for any initial state $\boldsymbol{x}_0$ which located in the spherical domain $S(\boldsymbol{x}_e,\delta)$ of equilibrium state $\boldsymbol{x}_e$,
- the solutions $x(t)$ of the state equation are all in the spherical domain $S(\boldsymbol{x}_e,\varepsilon)$,



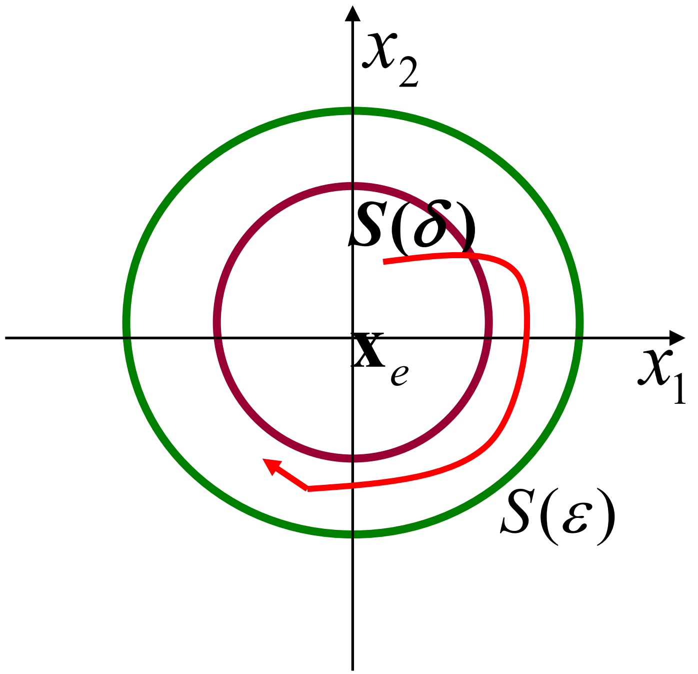
It is defined that the equilibrium state $\boldsymbol{x}_e$ is stable in the sense of Lyapunov.

Lyapunov stability and instability

Definition (Lyapunov stability)

任给一个球域 $S(\varepsilon)$，若存在一个球 域 $S(\delta)$，使得从$S(\delta)$出发的轨迹不离开$S(\varepsilon)$，则称系统的平衡状态是李雅普诺夫意义下稳定的。

初始状态有界，随时间推移，状态向量距平衡点的距离可以维持在一个确定的数值内，而到达不了平衡状态。

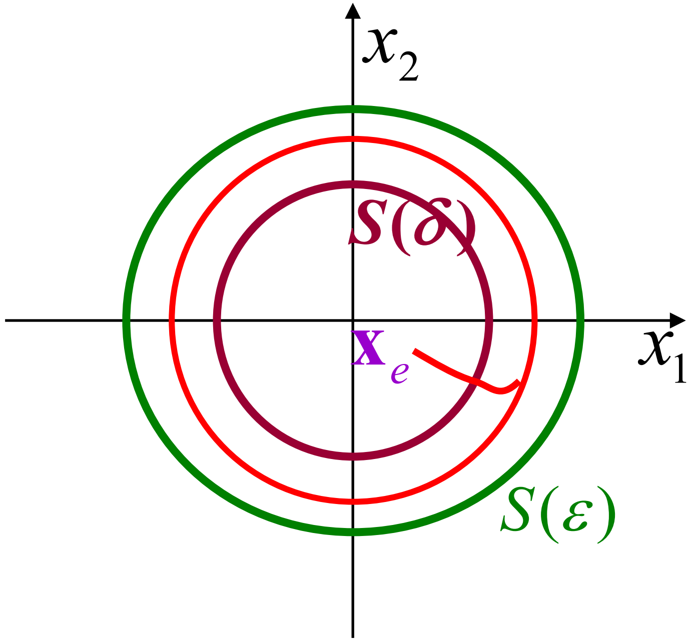


若 δ 与初始时刻 t_0无关，则称系统的平衡状态 $\mathbf{x}_e$ 是一致稳定的。

Lyapunov stability and instability

Definition (Lyapunov stability)

当系统做不衰减的震荡运动时，将描绘出一条封闭曲线，只要不超出$S(\varepsilon)$，则认为是稳定的。



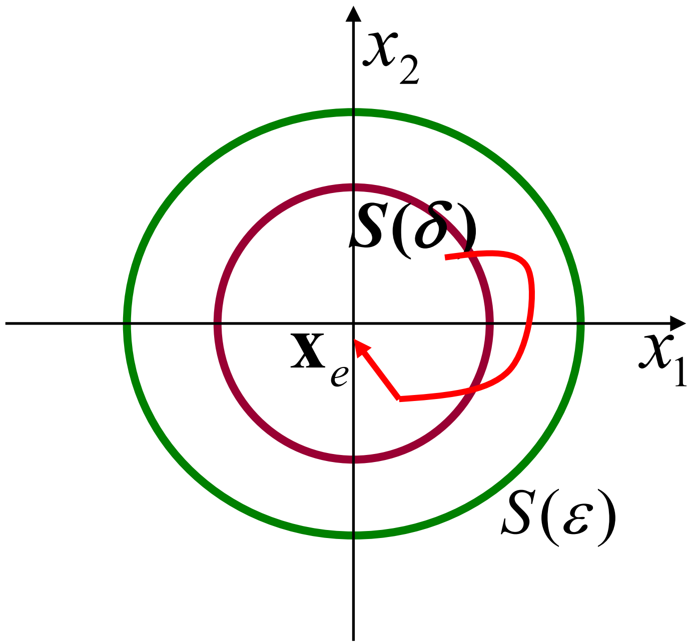
Asymptotic stability

Definition (Asymptotic stability)

An equilibrium point x_e is asymptotically stable if it is Lyapunov stable, and if in addition there exists $\|x(t_0) - x_e\| < \delta(\varepsilon, t_0) \Rightarrow \|x(t) - x_e\| \longrightarrow 0 \; as \; t \to \infty$

From the definition, the asymptotic stability means:

- The equilibrium is Lyapunov stable.
- The states converge to x_e when $t \to \infty$



Asymptotic stability

Definition (Asymptotic stability)

若系统方程的平衡状态 $\mathbf{x}_e$ 不仅具有李雅普诺夫意义下的稳定性，且有

$$\lim_{t\to\infty}\left\|\mathbf{x}(t;\mathbf{x}_0,t_0)-\mathbf{x}_e\right\|=0$$

则称系统的平衡状态 $\mathbf{x}_e$ 是渐近稳定的。

若 δ 与 t_0 无关，则为一致渐近稳定。（定常系统）

几何意义：

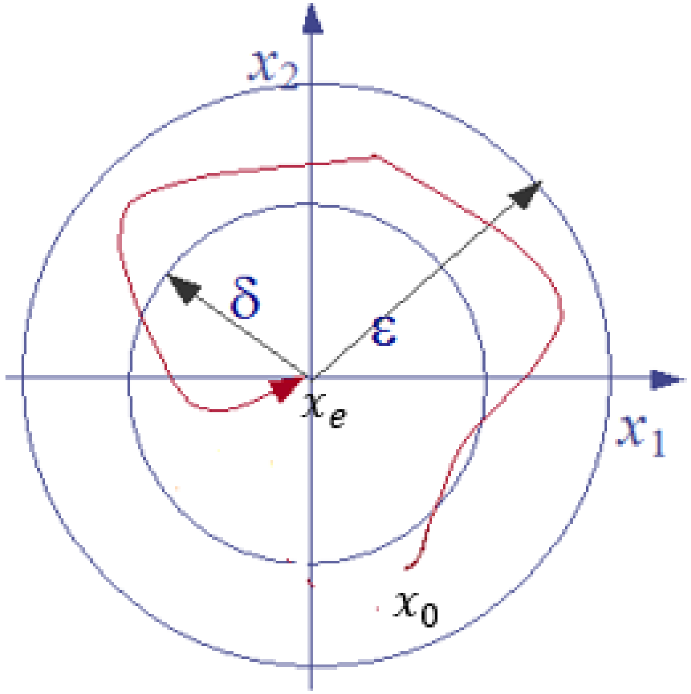
初始状态有界，随时间推移，状态向量距平衡点的距离可以无限接近，直至到达平衡状态后停止运动。

Global asymptotic stability

Definition (Global asymptotic stability)

If asymptotic (or exponential) stability holds for any initial state, the equilibrium point is said to be asymptotically (or exponentially) stable in the large. It is also called *globally* asymptotically (or exponentially) stable.

- For the LTI system, asymptotically stable and globally asymptotically stable are equivalent.
- For nonlinear systems, asymptotic stability usually does not imply globally stable.



Global asymptotic stability

Definition (Global asymptotic stability)

当初始条件扩展到整个状态空间，且平衡状态均具有渐近稳定性时，称此平衡状态是大范围渐近稳定的。

几何意义：系统不管在什么样的初始状态下，经过足够长的时间总能回到平衡状态附近并且向平衡状态靠拢。

- 大范围渐近稳定的必要条件是状态空间中只能有一个平衡状态。
- 线性系统稳定性与初始条件无关，如果渐近稳定，则必然大范围渐近稳定。
- 非线性系统稳定性与初始条件密切相关，如果渐近稳定，不一定大范围渐近稳定。

Exponential stability

Motivations:

- In many engineering applications, it is still not sufficient to know that a system will converge to the equilibrium point.
- There is a need to estimate how fast the system trajectory approaches x_e.

The concept of *exponential stability* can be used for this purpose.

Definition (Exponential stability)

An equilibrium point 0 is exponentially stable if there exist two strictly positive numbers α and λ such that

$$\forall t > 0, \quad \|x(t)\| \leq \alpha \|x(0)\| e^{-\lambda t}$$

Remarks on exponential stability

- The state vector of an exponentially stable system converges to the origin faster than an exponential function.

- The parameter λ is called the *rate* of exponential convergence.

- The exponential stability (e.s.) implies asymptotic stability (a.s.), but a.s. does not guarantee e.s.

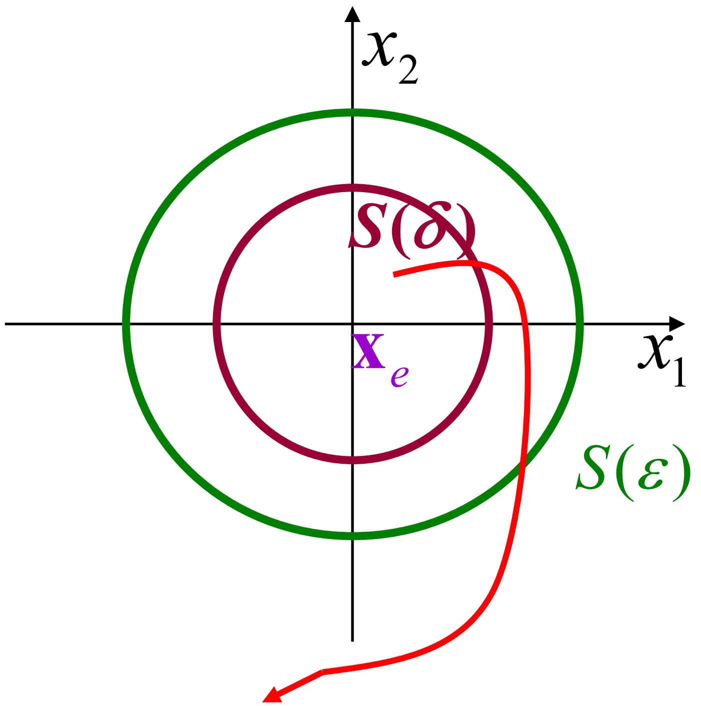
Non-stability

如果对于某个实数 $\varepsilon > 0$ 和任一个实数 $\delta > 0$，不管实数 δ 有多小，在 $S(\delta)$ 内总存在着一个状态 $\mathbf{x}_0$ ，由这一状态出发的轨迹超出 $S(\varepsilon)$，则称次平衡状态是不稳定的。

几何意义：

初始状态有界，随时间推移，

状态向量距平衡状态越来越远。



Outline of Chapter 9

Linearization and local stability (the first method)

- Lyapunov's linearization method is concerned with the *local* stability of a nonlinear system.
- It is a formalization of the intuition that a nonlinear system should behave similarly to its linearized approximation for small range motions.
- Since all physical systems are inherently nonlinear, Lyapunov's linearization method serves as the fundamental justification of using linear control techniques in practice.

Lyapunov’s linearization method

Theorems of Lyapunov’s linearization method

- If the linearized system is strictly stable (all eigenvalues has negative real part), then the equilibrium point is asymptotically stable for the nonlinear system.

- If the linearized system is unstable (at least one eigenvalue with positive real part), then the equilibrium point is unstable for the nonlinear system.

- If the linearized system is marginally stable (at least one eigenvalue on the imaginary axis, but no eigenvalues with positive real part), then the equilibrium point may be stable, asymptotically stable, or unstable for the nonlinear system.

Example

Given a first order system $\dot{x} = ax + bx^5$, let's decide the stability of the system around the origin.

Solutions

The linearization of the system around the origin is $\dot{x} = ax$. The Lyapunov's linearization method indicates that

$a < 0$ asymptotically stable;

$a > 0$ unstable;

$a = 0$ cannot decide from the linearization

In the 3rd case, the nonlinear system is $\dot{x} = bx$.
The linearization method fails, while the direct method to be described can easily solve this problem.

Lyapunov第一法（间接法）是利用状态方程的解的特性来判断系统稳定性的方法，适用于线性定常、线性时变及可线性化的非线性系统。

线性定常系统的特征值判据：系统 $\dot{x} = Ax$ 渐进稳定的充分必要条件是系统矩阵A的全部特征值均位于复平面的左半部，即 $\mathrm{Re}(\lambda_i) < 0, i = 1, \cdots, n$

证明： 假定*A*有互异特征值 $\lambda_1, \cdots, \lambda_n$ ，根据线性代数理论，存在非奇异线性变换 $x = P\overline{x}$（*P*有特征值 λ_i对应的特征向量构成，为一常数矩阵），可以使 $\overline{A}$ 对角化，有 $\overline{A} = P^{-1}AP = diag(\lambda_1, \cdots, \lambda_n)$

变换后状态方程的解为：$\overline{x}(t) = e^{\overline{A}t}\overline{x}(0) = diag(e^{\lambda_1 t}, \cdots, e^{\lambda_n t})\overline{x}(0)$

由于 $\overline{x} = P^{-1}x, \overline{x}(0) = P^{-1}x(0)$ 故原状态方程的解为 $x(t) = Pe^{\overline{A}t}P^{-1}x(0) = e^{At}x(0)$

有 $e^{At} = Pe^{\overline{A}t}P^{-1} = Pdiag(e^{\lambda_1 t}, \cdots, e^{\lambda_n t})P^{-1}$

将上式展开，e^{At} 的每一个元素都是 $e^{\lambda_1 t}, \cdots, e^{\lambda_n}$ 的线性组合，因而可以写成矩阵多项式

$$e^{At} = \sum_{i=1}^{n} R_i e^{\lambda_i t} = R_1 e^{\lambda_1 t} + \cdots + R_n e^{\lambda_n t}$$

故$x(t)$可以显式表现出与λ_i的关系，即 $x(t) = e^{At}x(0) = [R_1 e^{\lambda_1 t} + \cdots + R_n e^{\lambda_n t}]x(0)$

当 $\mathrm{Re}(\lambda_i) < 0, i = 1, \cdots, n$ 成立时，对于任意的 $x(0)$，均有 $x(t)\big|_{t\to\infty} \to 0$，系统渐进稳定。

例:用间接法判断下列系统的稳定性

1）$\begin{cases}\dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1\end{cases}$，2）$\begin{cases}\dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_1 - x_2\end{cases}$，3）$\begin{cases}\dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = -x_1 + x_2\end{cases}$

解:1）$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \det(sI - A) = s^2 + 1$，$s_{1,2} = \pm i,$

系统所有平衡点稳定。

2）$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \det(sI - A) = (s+1)^2 + 1$，$s_{1,2} = -1 \pm i,$

系统平衡点渐近稳定。

3）$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \det(sI - A) = (s-1)^2 + 1, s_{1,2} = 1 \pm i,$

系统每个平衡点不稳定。

用线性化的方法判断稳定性

(1) $$\begin{cases}\dot{x}_1 = x_1 - x_2 - x_1^{\ 3} \\ \dot{x}_2 = x_1 + x_2 - x_2^{\ 3}\end{cases}$$

(2) $$\begin{cases}\dot{x}_1 = -x_1 + x_2 + x_1(x_1^{\ 2} + x_2^{\ 2}) \\ \dot{x}_2 = -x_1 - x_2 + x_2(x_1^{\ 2} + x_2^{\ 2})\end{cases}$$

【解】:

（1）采用非线性系统线性化的方法，在平衡点原点处线性化得：

$$A=\frac{\partial f}{\partial x^T}\bigg|_{x=0}=\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix}_{x=0}=\begin{bmatrix}1-3x_1^{\ 2} & -1\\ 1 & 1-3x_2^{\ 2}\end{bmatrix}_{x=0}=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

$$|sI-A|=\begin{vmatrix}s-1 & 1\\ -1 & s-1\end{vmatrix}=s^2-2s+2=0$$

系统的两个特征值均在右半平面，则系统在平衡点附近不稳定。

（2）采用非线性系统线性化的方法，在平衡点原点处线性化得：

$$A=\frac{\partial f}{\partial x^T}\bigg|_{x=0}=\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix}_{x=0}=\begin{bmatrix}-1+3x_1^{\ 2}+x_2^{\ 2} & 1+2x_1x_2\\ -1+2x_1x_2 & -1+x_1^{\ 2}+3x_2^{\ 2}\end{bmatrix}_{x=0}=\begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix}$$

$$|sI-A|=\begin{vmatrix}s+1 & -1\\ 1 & s+1\end{vmatrix}=s^2+2s+2=0$$

系统的两个特征值都在左半平面，则系统在平衡点附近渐近稳定。

练习题： 设系统方程为：

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 6 \\ 1 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}u, \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{x}$$

试确定其外部稳定性、内部稳定性。

解

（1）系统的传递函数为：

$$\mathbf{W}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s & -6\\ -1 & s+1\end{bmatrix}^{-1}\begin{bmatrix}-2\\ 1\end{bmatrix}=\frac{(s-2)}{(s-2)(s+3)}=\frac{1}{(s+3)}$$

极点位于s左半平面，s=2的极点被对消掉了。系统是有界输入有界输出稳定的。

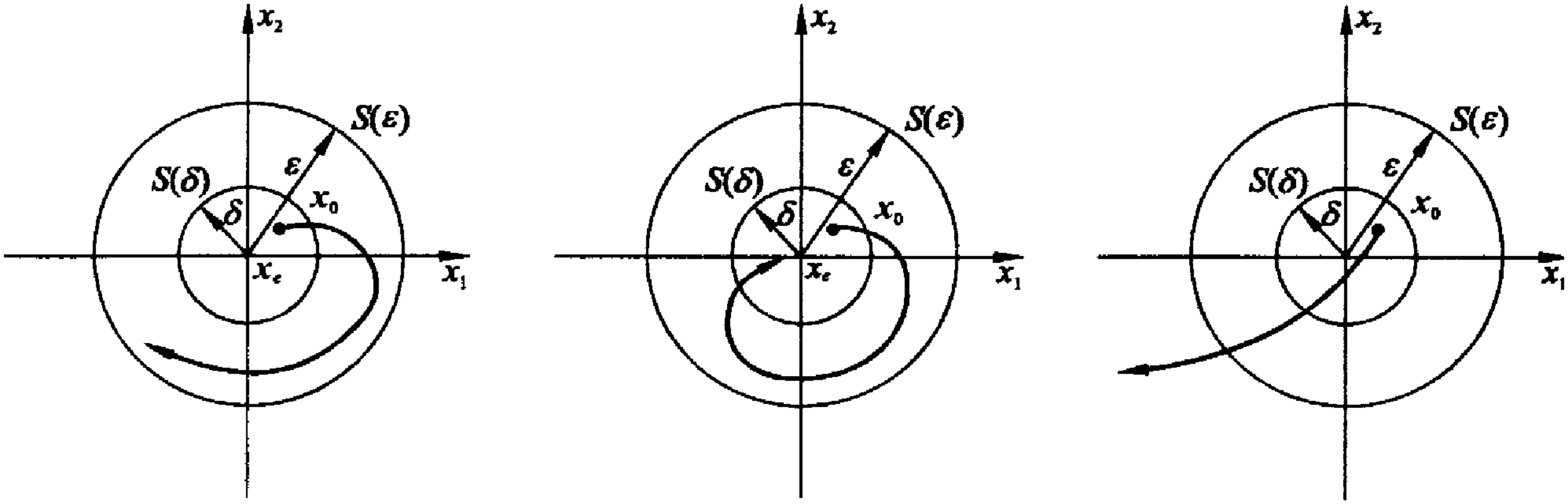
（2）求系统的特征方程：

$$\det(\lambda I - A) = \begin{bmatrix} \lambda & -6 \\ -1 & \lambda + 1 \end{bmatrix} = (\lambda - 2)(\lambda + 3) = 0$$

求得：$\lambda_1 = 2$，$\lambda_2 = -3$

系统不是渐近稳定的。

Review



(a) (b) (c)

李雅普诺夫稳定性、BIBS 稳定性、BIBO 稳定性之间的关系

BIBS 稳定性。对任意有界的 $\boldsymbol{x}(0)$，若在任意有界的输入 $\boldsymbol{u}(t)$ 的作用下，$\boldsymbol{x}(t)$ 均有界，则称系统 BIBS 稳定。

BIBO 稳定性。对任意有界的 $\boldsymbol{x}(0)$，若在任意有界的输入 $\boldsymbol{u}(t)$ 的作用下，$\boldsymbol{y}(t)$ 均有界，则称系统 BIBO 稳定。

线性定常系统的 BIBO 稳定性判别主要依据传递函数矩阵进行，如果其极点全部位于左半复平面(不含虚轴)，则系统 BIBO 稳定。

线性定常系统的 BIBS 稳定性判别主要依据系统矩阵 $\boldsymbol{A}$ 进行，如果其特征值全部位于左半复平面(不含虚轴)，则系统 BIBS 稳定。

对线性定常系统，如果系统是渐近稳定的，则系统必然是 BIBS 稳定的和 BIBO 稳定的。如果系统是 BIBS 稳定的，则系统必然是 BIBO 稳定的。即渐近稳定要求的条件严于 BIBS 稳定，而 BIBS 稳定要求的条件又严于 BIBO 稳定的。但是，如果系统是李雅普诺夫意义下稳定的，则系统不一定是 BIBS 稳定的和 BIBO 稳定的。

Review

Theorems of Lyapunov's linearization method

- If the linearized system is strictly stable (all eigenvalues has negative real part), then the equilibrium point is asymptotically stable for the nonlinear system.

- If the linearized system is unstable (at least one eigenvalue with positive real part), then the equilibrium point is unstable for the nonlinear system.

- If the linearized system is marginally stable (at least one eigenvalue on the imaginary axis, but no eigenvalues with positive real part), then the equilibrium point may be stable, asymptotically stable, or unstable for the nonlinear system.

Review

线性定常系统的特征值判据：系统 $\dot{x} = Ax$ 渐进稳定的充分必要条件是系统矩阵A的全部特征值均位于复平面的左半部，即 $\mathrm{Re}(\lambda_i) < 0, i = 1, \cdots, n$

例:用间接法判断下列系统的稳定性

1) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$，2) $\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_1 - x_2 \end{cases}$，3) $\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = -x_1 + x_2 \end{cases}$

解:1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \det(sI - A) = s^2 + 1, \quad s_{1,2} = \pm i,$

系统所有平衡点稳定。

2) $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \det(sI - A) = (s+1)^2 + 1, \quad s_{1,2} = -1 \pm i,$

系统平衡点渐近稳定。

3) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \det(sI - A) = (s-1)^2 + 1, s_{1,2} = 1 \pm i,$

系统每个平衡点不稳定。

Outline of Chapter 9

Lyapunov's direct method

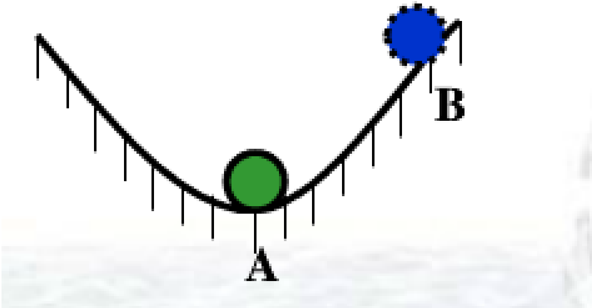
The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation.

- If the total energy of a mechanical (or electrical) system is continuously dissipated, then the system, whether linear or nonlinear, must eventually settle down to an equilibrium point.
- The stability of a system can be decided by examining the variation of a single *scalar* function.

Lyapunov's direct method

The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation.

不必求解微分方程，直接判断系统稳定性。

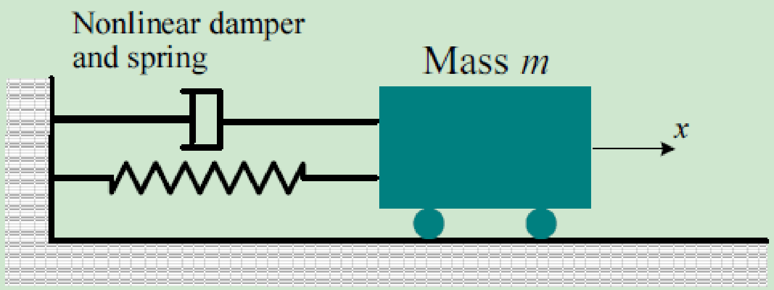


平衡状态能量最小。系统经激励后，其能量若随着时间推移而衰减，最终到达能量最小的平衡状态，则为渐近稳定的。

反之，若系统不断从外界吸收能量，则不稳定。

Example

Let us consider the nonlinear mass-damper-spring system shown below



whose dynamic equation is

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$$

where $b\dot{x}|\dot{x}|$ represents nonlinear dissipation or damping, and $(k_0 x + k_1 x^3)$ represents a nonlinear spring term.

Suppose the mass is pulled away from the natural length of the spring by a large distance. Will the resulting motion be stable?

Solutions

The differential equation of the system is

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$$

It is very difficult to decide the stability of the system because

- The general solution of the equation is not available.
- The linearization method cannot be used because the motion starts outside the linear range (and in any case the systems's linear approximation is only marginally stable).

However, examination of the system's energy can tell us a lot about the motion pattern. The total energy of the system is the sum of its kinetic energy and its potential energy

$$V(x) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_0 x + k_1 x^3)\,\mathrm{d}x = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4$$

Comparing the definitions of stability and mechanical energy, one can easily see some relations between the mechanical energy and the stability concepts described earlier:

- zero energy corresponds to the equilibrium point ($x = 0$, $\dot{x} = 0$)
- asymptotical stability implies the convergence of mechanical energy to zero
- instability is related to the growth of mechanical energy

These relations indicate that the stability properties of the system can be characterized by the variation of the mechanical energy of the system.

$$\because \; V(x) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0x^2 + \frac{1}{4}k_1x^4$$

$$\therefore \; \dot{V}(x) = m\dot{x}\ddot{x} + (k_0x + k_1x^3)\dot{x} = \dot{x}(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3$$

Therefore, the energy of the system, starting from some initial value is continuously dissipated by the damper until the mass settles down.

Lyapunov’s direct method

对于一些纯数学系统，还没有一个定义“能量函数”的简便方法。为了克服这个困难，**Lyapunov**定义了一个虚构的广义能量函数，称为**Lyapunov函数**(能满足稳定性定理的函数)。

$V(x)$ 是标量函数，x 为状态变量，是 t 的函数。

$V(x)$ 是非负数（定号性），反应能量大小；

$\dot{V}(x)=\dfrac{dV(x)}{dt}$ 连续一阶偏导，反应能量变化。

李雅普诺夫直接法：利用$V(x)$ 和$\dot{V}(x)$ 的符号特性来直接判断系统在平衡状态是否稳定。

Positive and negative definite functions

Definition of positive definite:

A scalar continuous function $V(x)$ which has continuous partial derivatives is said to be positive definite in Ω if:

(1) $\mathrm{V(x)} > 0, \forall \mathrm{x} \in \Omega$ and $\mathrm{x} \neq 0$

(2) $\mathrm{V(x)} = 0, \mathrm{x} = 0$

Positive and negative definite functions

Definition of positive semi-definite:

A scalar continuous function $V(x)$ which has continuous partial derivatives is said to be positive definite in Ω if:

(1) $\mathrm{V}(\mathrm{x}) \geq 0, \forall \mathrm{x} \in \Omega$ and $\mathrm{x} \neq 0$

(2) $\mathrm{V}(\mathrm{x}) = 0, \mathrm{x} = 0$

Positive and negative definite functions

Definition of negative definite:

A scalar continuous function $V(x)$ which has continuous partial derivatives is said to be positive definite in Ω if:

(1) $\mathrm{V}(\mathrm{x}) < 0, \forall \mathrm{x} \in \Omega$ and $\mathrm{x} \neq 0$

(2) $\mathrm{V}(\mathrm{x}) = 0, \mathrm{x} = 0$

Positive and negative definite functions

Definition of negative semi-definite:

A scalar continuous function $V(x)$ which has continuous partial derivatives is said to be positive definite in Ω if:

(1) $\mathrm{V(x)} \leq 0, \forall \mathrm{x} \in \Omega$ and $\mathrm{x} \neq 0$

(2) $\mathrm{V(x)} = 0, \mathrm{x} = 0$

Positive and negative definite functions

Definition of non-definite:

A scalar continuous function $V(x)$ which has continuous partial derivatives is said to be Non-definite in Ω if, $V(x)$ can either be positive or negative.

一、标量函数 $V(\mathrm{x})$ 的定号性

在零平衡状态的邻域内 $x=0,\quad V(x)=0;$

① 正定： $x \neq 0,\quad V(x)>0$

② 负定： $x \neq 0,\quad V(x)<0$

③ 半正定： $x \neq 0$ 时，$V(x) \geq 0$

④ 半负定： $x \neq 0$ 时，$V(x) \leq 0$

⑤ 不定： $x \neq 0$ 时，$V(x)$ 可正可负

Example

$$V(x) = (x_1 + x_2)^2 + x_2^2 + x_3^2$$

Positive definite (p.d.)

$$V(x) = (x_1 + x_2 + x_3)^2$$

Positive semi-definite (p.s.d)

$$V(x) = -\left(x_1^2 + x_2^2 + x_3^2\right)$$

negative definite (n.d.)

$$V(x) = -\left(x_1 + x_2 + x_3\right)^2$$

negative semi-definite (n.s.d.)

例：已知 $\mathbf{x}=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}^T$，确定标量函数的定号性。

(1) $V(\mathbf{x})=x_1^4+2x_2^2+x_3^2$

解： $\mathbf{x}\neq 0,\ V(\mathbf{x})>0$ $\quad\Rightarrow V(\mathbf{x})$ 正定

$\mathbf{x}=0,\ V(\mathbf{x})=0$

(2) $V(\mathbf{x})=x_1^2+x_3^2$

解： $\mathbf{x}=0,\ V(\mathbf{x})=0$

$x_1=0, x_2\neq 0, x_3=0,\ V(\mathbf{x})=0$

其余 $V(\mathbf{x})>0$

$\therefore \mathbf{x}=0,\ V(\mathbf{x})=0$

$\mathbf{x}\neq 0,\ V(\mathbf{x})\geq 0$

$\Rightarrow V(\mathbf{x})$ 半正定

(3) $V(\mathbf{x}) = -x_1^2 - (x_1 + 2x_2 + x_3)^2$

解： $\mathbf{x} = 0, \ V(\mathbf{x}) = 0$

$x_1 = 0, x_3 = -2x_2 \neq 0, \ V(\mathbf{x}) = 0$

其余 $V(\mathbf{x}) < 0$

$\therefore \mathbf{x} = 0, \ V(\mathbf{x}) = 0$

$\mathbf{x} \neq 0, \ V(\mathbf{x}) \leq 0$

$\Rightarrow V(\mathbf{x})$ 半负定

(4) $V(\mathbf{x}) = x_1^2 + 2x_2^2 - x_3^2$

解： $x_1^2 + 2x_2^2 > x_3^2 \qquad V(\mathbf{x}) > 0$

$x_1^2 + 2x_2^2 < x_3^2 \qquad V(\mathbf{x}) < 0$

$\Rightarrow V(\mathbf{x})$ 不定

Lyapunov's direct method

Quadratic function (二次型V函数)

In the stability analysis of Lyapunov's direct method, quadratic function plays an important role, i.e.

$$\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x}$$

where $\boldsymbol{P}$ is a real symmetrical matrix.

The properties of $V(x)$ is determined by **Sylvester criterion** （赛尔维斯特判据）, i.e. when all the principle of P is positive, the $V(x) = x^T P x$ is positive definite. If all the principle determinant of P is non-negative, then $V(x)$ is positive smi-definite.

Lyapunov's direct method

二次型V函数

二次型：各项均为自变量的二次单项式的标量函数

$$V(x)=V(x_1,\cdots,x_n)=p_{11}x_1^2+2p_{12}x_1x_2+\cdots+2p_{1n}x_1x_n$$
$$+p_{22}x_2^2+2p_{23}x_2x_3+2p_{24}x_2x_4+\cdots+2p_{2n}x_2x_n+\cdots+p_{nn}x_n^2$$
$$=\begin{bmatrix}x_1 & x_2 & \cdots & x_n\end{bmatrix}\begin{bmatrix}p_{11} & p_{12} & \cdots & p_{1n}\\ p_{21} & p_{22} & \cdots & p_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ p_{n1} & p_{n2} & \cdots & p_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}=x^TPx$$

P为实对称矩阵 $p_{ij}=p_{ji}$ $\mathbf{P}=\mathbf{P}^T$

Lyapunov's direct method

Sylvester criterion （赛尔维斯特判据）

二次型函数的定号性与矩阵P的定号性是一致的。

P的定号性判断准则一，**Sylvester准则**：

① P为正定的充要条件为：P的各阶主子式大于0，即：

$$
\begin{aligned}
&\Delta_1 = P_{11} > 0 \\
&\Delta_2 = \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} > 0 \\
&\qquad \vdots \\
&\Delta_n = |\boldsymbol{P}| > 0
\end{aligned}
$$

② P为负定的充要条件为：P的各阶主子式负、正相间，

$$
\Delta_{\mathrm{i}} \begin{cases} < 0 & i = 1, 3, \cdots \text{奇数} \\ > 0 & \mathrm{i} = 2, 4, \cdots \text{偶数} \end{cases}
$$

Lyapunov's direct method

Sylvester criterion （赛尔维斯特判据）

二次型函数的定号性与矩阵P的定号性是一致的。P的定号性判断准则一，**Sylvester准则：**

③ P为半正定的充要条件为：P的各阶主子式为正或零，即 $\Delta_i \geq 0.$

④ P为半负定的充要条件为：P的各阶主子式满足负定的条件，但其中可以有等于0的。

Lyapunov's direct method

P的定号性判断准则二，特征值判据：

1) A 正定 $\Leftrightarrow A$ 的特征值均为正数
2) A 负定 $\Leftrightarrow A$ 的特征值均为负数
3) A 半正定 $\Leftrightarrow A$ 的特征值均为非负数
4) A 半负定 $\Leftrightarrow A$ 的特征值均为非正数

试确定下列二次型是否正定

$$(1)\ v(x)=x_1^{\ 2}+4x_2^{\ 2}+x_3^{\ 2}+2x_1x_2-6x_3x_2-2x_1x_3$$

$$(2)\ v(x)=-x_1^{\ 2}-10x_2^{\ 2}-4x_3^{\ 2}+6x_1x_2+2x_3x_2$$

$$(3)\ v(x)=10x_1^{\ 2}+4x_2^{\ 2}+x_3^{\ 2}+2x_1x_2-2x_3x_2-4x_1x_3$$

【解】:

（1）

$$P=\begin{bmatrix}1 & 1 & -1\\ 1 & 4 & -3\\ -1 & -3 & 1\end{bmatrix},\Rightarrow |1|>0,\begin{vmatrix}1 & 1\\ 1 & 4\end{vmatrix}=3>0,\begin{vmatrix}1 & 1 & -1\\ 1 & 4 & -3\\ -1 & -3 & 1\end{vmatrix}=-4<0$$

二次型函数不定。

(2)

$$P=\begin{bmatrix}-1 & 3 & 0\\ 3 & -10 & 1\\ 0 & 1 & -4\end{bmatrix},\Rightarrow |-1|<0,\begin{vmatrix}-1 & 3\\ 3 & -10\end{vmatrix}=1>0,\begin{vmatrix}-1 & 3 & 0\\ 3 & -10 & 1\\ 0 & 1 & -4\end{vmatrix}=-3<0$$

二次型函数为负定。

(3)

$$P=\begin{bmatrix}10 & 1 & -2\\ 1 & 4 & -1\\ -2 & -1 & 1\end{bmatrix},\Rightarrow |10|>0,\begin{vmatrix}10 & 1\\ 1 & 4\end{vmatrix}=39>0\begin{vmatrix}10 & 1 & -2\\ 1 & 4 & -1\\ -2 & -1 & 1\end{vmatrix}=17>0$$

二次型函数正定。

例： 确定下列二次型的定号性

$$V(x) = x_1^2 + 2x_2^2 - x_3^2$$

解：

$$V(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

判别方法二

$$|\lambda \mathrm{I} - \mathrm{P}| = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda - 2)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$$

矩阵P的特征值的符号有正有负，

即符号不定 $\Rightarrow V(\mathbf{x})$ 不定

例：确定下列二次型为正定时，待定常数的取值范围

$$V(\mathbf{x})=a_1x_1^2+b_1x_2^2+c_1x_3^2+2x_1x_2-4x_1x_3-2x_2x_3$$

解：

$$V(x)=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}a_1 & 1 & -2\\ 1 & b_1 & -1\\ -2 & -1 & c_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\Delta_1=|a_1|>0 \quad \Delta_2=\begin{vmatrix}a_1 & 1\\ 1 & b_1\end{vmatrix}>0 \quad \Delta_3=\begin{vmatrix}a_1 & 1 & -2\\ 1 & b_1 & -1\\ -2 & -1 & c_1\end{vmatrix}>0$$

$$\begin{cases}a_1>0\\ a_1b_1>1\\ a_1b_1c_1+4>b_1+4a_1+c_1\end{cases}$$

Basic theorem of Lyapunov

Globally asymptotically stability

⇓

Locally asymptotically stability

⇓

Lyapunov stability

⇓

Unstable

Lyapunov theorem for local stability

Theorem

If there exists a scalar function $V(x)$ in the domain Ω around the equilibrium state such that:

- $V(x)$ is positive
- $\dot{V}(x)$ is negative semi-definite

Then the nonlinear system is Lyapunov sable at the equilibrium.

考虑系统 $\dot{x}=f(x)$，设$x_e=0$为一平衡状态,如果存在连续可微的标量函数$V(x)$满足

1) $V(x)$是正定的；

2) $\dot{V}(x)=\dfrac{\partial V(x)}{\partial x}f(x)$是半负定的；

则系统的平衡状态$x_e=0$是李雅普诺夫意义下稳定的。

Lyapunov theorem for local stability

Theorem

If there exists a scalar function $V(x)$ in the domain Ω around the equilibrium state such that:

- $V(x)$is positive
- $\dot{V}(x)$ is negative definite

Then the nonlinear system is asymptotic sable at the equilibrium.

考虑系统 $\dot{x} = f(x)$，设$x_e = 0$为一平衡状态。如果存在连续可微的标量函数$V(x)$满足

1) $V(x)$是正定的；

2) $\dot{V}(x) = \dfrac{\partial V(x)}{\partial x} f(x)$是负定的；

则系统的平衡状态$x_e = 0$是渐近稳定的。

Lyapunov theorem for local stability (Cont.)

Theorem

If, in a range, there exists a scalar function V(x) with continuous fist partial derivatives such that

- $V(x)$is positive
- $\dot{V}(x)$ is negative semi-definite
- $\dot{V}(x(\mathrm{t}; x_0, 0)) \not\equiv 0$

then the equilibrium point $x_e$is asymptotically stable, if

$$\dot{V}\left[\mathbf{x}\left(t;\ \mathbf{x}_0,\ t_0\right),\ t\right] \equiv 0, (when\ \mathbf{x} \neq 0)$$

then the equilibrium point $x_e$is local stable in the sense of Lyapunov

Lyapunov theorem for local stability (Cont.)

Theorem

- $V(x)$ is positive
- $\dot{V}(x)$ is negative semi-definite
- $\dot{V}(x(t; x_0, 0)) \not\equiv 0$

用“$\dot{V}(x)$半负定，且$\dot{V}(x)$不恒等于0”代替了“$\dot{V}(x)$负定”
即允许运动过程中在某些状态点上能量速率为0，
而由$\dot{V}(x)$不恒等于0保证运动能脱离这类状态点，继续收敛到原点。

Lyapunov theorem for local stability

Theorem

If there exists a scalar function $V(x)$ in the domain Ω around the equilibrium state such that:

- $V(x)$is positive
- $\dot{V}(x)$ is negative definite
- $\|x\| \to \infty, V(x) \to \infty$

Then the nonlinear system is global asymptotic sable at the equilibrium.

考虑系统 $\dot{x} = f(x)$，设$x_e = 0$为一平衡状态。如果存在连续可微的标量函数$V(x)$满足

1) $V(x)$是正定的；

2) $\dot{V}(x) = \dfrac{\partial V(x)}{\partial x} f(x)$是负定的；

3) 当$\|x\| \to \infty$时，$V(x) \to \infty$；

则平衡状态$x_e = 0$是大范围渐近稳定的。

Remarks on Lyapunov's direct method

- Many Lyapunov functions may exist for the same system
- For a given system, specific choices of Lyapunov functions may yield more precise results than others
- The theorems in Lyapunov analysis are all sufficient theorems.

NOTE: if for a particular choice of Lyapunov function candidate $V(x)$, the conditions on $\dot{V}(x)$ are not met, one cannot draw any conclusions on the stability or instability of the system

Lyapunov稳定性定理总结:

设系统状态方程为 $\dot{x}=f(x,t)$,其平衡状态满足 $f(0,t)=0$,不失一般性,把状态空间原点作为平衡状态,并设系统在原点邻域存在 $V(x,t)$ 对 x 的连续的一阶偏导数。

定理 1 若①$V(x,t)$ 正定,②$\dot{V}(x,t)$ 负定,则原点是渐近稳定的。

$\dot{V}(x,t)$ 负定表示能量随时间连续单调地衰减,故与渐近稳定性定义叙述一致。

定理 2 若 ①$V(x,t)$ 正定,②$\dot{V}(x,t)$ 负半定,且在非零状态不恒为零,则原点是渐近稳定的。

$\dot{V}(x,t)$ 负半定表示在非零状态存在 $\dot{V}(x,t)\equiv 0$,但在从初态出发的轨迹 $x(t;x_0,t_0)$ 上,不存在 $\dot{V}(x,t)\equiv 0$ 的情况,于是系统将继续运行至原点。状态轨迹仅是经历能量不变的状态,而不会维持在该状态。

定理 3 若 ①$V(x,t)$ 正定,(2)$\dot{V}(x,t)$ 负半定,且在非零状态恒为零,则原点是李雅普诺夫意义下稳定的。

沿状态轨迹能维持 $\dot{V}(x,t)\equiv 0$,表示系统能维持等能量水平运行,使系统维持在非零状态而不运行至原点。

定理 4 若 ①$V(x,t)$ 正定,②$\dot{V}(x,t)$ 正定,则原点是不稳定的。

$\dot{V}(x,t)$ 正定表示能量函数随时间增大,故状态轨迹在原点邻域发散。

注意：

- Lyapunov函数的选取不是唯一的，但只要找到一个$V(x,t)$满足定理所述条件，便可对原点的稳定性做出判断，并不会因为选取的$V(x,t)$不同而影响结果；
- 至今尚无构造Lyapunov函数的通用方法，这是Lyapunov稳定性理论的主要障碍；
- 如果$V(x,t)$选取不当，会导致$\dot{V}(x,t)$ 不定的结果，这时便做不出确定的判断，需要重新选择$V(x,t)$；
- Lyapunov稳定性理论按照$\dot{V}(x,t)$连续单调衰减的要求来确定系统稳定性，并未考虑实际稳定系统可能存在衰减的情况，因此其条件是过于保守的， Lyapunov第二法稳定性定理所述条件都是充分条件；
- 具体分析时，先构造一个Lyapunov函数，通常选择二次型函数，求其导数，再将状态方程代入，最后根据$\dot{V}(x,t)$的定号性来判断稳定性。
- 对于如何判断在非零状态下$\dot{V}(x,t)$是否有恒为零的情况，可以按照如下方法进行：令$\dot{V}(x,t)\equiv 0$，将状态方程代入，若能接触非零解，表示对$x\neq 0$, $\dot{V}(x,t)\equiv 0$的条件是成立的；若导出的解全是零解，表示只有原点处满足$\dot{V}(x,t)\equiv 0$的条件。

Outline of Chapter 9

Lyapunov functions for LTI systems

Given an LTI system $\dot{x} = Ax$, let us consider a quadratic Lyapunov function candidate

$$V = \mathbf{x}^T \mathbf{P} \mathbf{x}$$

where ***P*** is a given symmetric positive definite matrix.

Differentiating the function ***V*** yields another quadratic form

$$\dot{V} = \dot{\mathbf{x}}^T \mathbf{P} \mathbf{x} + \mathbf{x}^T \mathbf{P} \dot{\mathbf{x}} = \mathbf{x}^T \mathbf{A}^T \mathbf{P} \mathbf{x} + \mathbf{x}^T \mathbf{P} \mathbf{A} \mathbf{x} \triangleq -\mathbf{x}^T \mathbf{Q} \mathbf{x}$$

where $-\mathrm{Q} = \mathrm{A}^{\mathrm{T}}\mathrm{P} + \mathrm{PA}$ is called the Lyapunov equation.

The question is to determine whether the symmetric matrix Q is positive definite.

Theorem

A necessary and sufficient condition for an LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ to be strictly stable is that, for any symmetric positive-definite matrix $\mathbf{Q}$, the unique matrix $\mathbf{P}$ solution to the Lyapunov equation

$$-\mathbf{Q} = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}$$

be symmetric positive definite.

Example

Given an LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$. Decide its stability by the Lyapunov direct method.

Solutions

Let us take $\mathbf{Q} = \mathbf{I}$ and denote $\mathbf{P} \triangleq \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$.

The the Lyapunov equation is

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

whose solution is

$$p_{11} = \frac{5}{16},\ p_{12} = p_{22} = \frac{1}{16}$$

The corresponding matrix

$$\mathbf{P} = \frac{1}{16}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

is positive definite, and the system is globally asymptotically stable.

3. Given a linear system described below

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 \end{cases}$$

try to determine the stability of the system by the Lyapunov's direct method.

解法一： 选择 Lyapunov 函数为：

$$v(x)=\frac{1}{2}\left[(x_1+x_2)^2+2x_1^2+x_2^2\right]$$

它是正定的。而 $\dot{v}(x)=(x_1+x_2)(\dot{x}_1+\dot{x}_2)+2x_1\dot{x}_1+2x_2\dot{x}_2;=-(x_1^2+x_2^2)$

$\dot{v}(x)$ 是负定的。又因为当 $\|X\|\to\infty$，有 $v(x)\to\infty$，所以是大范围渐近稳定的。

解法二： 该系统为线性系统。由微分方程组，系统矩阵为 $A=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}$

令 $Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$，且设 Lyapunov 矩阵 $P=\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}$

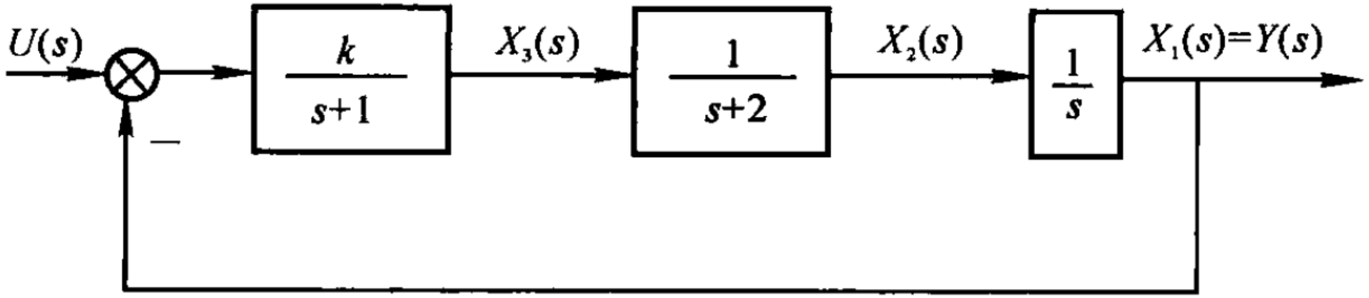
由 Lyapunov 定理可知

$$-Q=A^T P+PA$$

即 $$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}+\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}$$

求解可得 $P=\begin{bmatrix}\frac{3}{2} & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}$，该矩阵为正定矩阵，因此系统是全局渐近稳定的。

例子：试用Lyapunov方程确定，使得下图所示的系统渐进稳定的k的取值范围。



解 由图示状态变量列写状态方程为

$$\dot{\boldsymbol{x}}=\begin{bmatrix}0 & 1 & 0\\ 0 & -2 & 1\\ -k & 0 & -1\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\0\\k\end{bmatrix}\boldsymbol{u}$$

因系统的稳定性与输入无关，可令$\boldsymbol{u}=0$。由于$\det\boldsymbol{A}=-k\neq0$，故$\boldsymbol{A}$非奇异，原点为唯一的平衡状态。

取$\boldsymbol{Q}$矩阵为正半定矩阵，即 $\boldsymbol{Q}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$

则$\dot{V}(\boldsymbol{x})=-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}=-x_3^2$，$\dot{V}(\boldsymbol{x})$负半定。令$\dot{V}(\boldsymbol{x})\equiv0$，有$x_3\equiv0$，考虑状态方程中$\dot{x}_3=-kx_1-x_3$，解得$x_1\equiv0$；考虑到$\dot{x}_1=x_2$，解得$x_2\equiv0$，表明唯有原点存在$\dot{V}(\boldsymbol{x})\equiv0$。令

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}=-\boldsymbol{Q}$$

$$\begin{bmatrix}0 & 0 & -k\\ 1 & -2 & 0\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}+\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & -2 & 1\\ -k & 0 & -1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$$

展开的代数方程为6个，即

$$-2kp_{13}=0 \qquad -kp_{23}+p_{11}-2p_{12}=0 \qquad -kp_{33}+p_{12}-p_{13}=0$$

$$2p_{12}-4p_{22}=0 \qquad p_{13}-3p_{23}+p_{22}=0 \qquad 2p_{23}-2p_{33}=0$$

解得

$$
\boldsymbol{P}=\begin{bmatrix} \dfrac{k^2+12k}{12-2k} & \dfrac{6k}{12-2k} & 0 \\ \dfrac{6k}{12-2k} & \dfrac{3k}{12-2k} & \dfrac{k}{12-2k} \\ 0 & \dfrac{k}{12-2k} & \dfrac{6k}{12-2k} \end{bmatrix}
$$

使 $\boldsymbol{P}$ 矩阵正定的条件为：$12-2k>0$ 及 $k>0$。故当 $0<k<6$ 时，系统渐近稳定。由于是线性定常系统，系统大范围一致渐近稳定。

Lyapunov functions for nonlinear systems

- There is no general solutions to this problem!!!
- Several methods can be tried:
  - The physically motivated method
  - The variable gradient method (Schultz-Gilbson's method)
  - Krasovskii's method

The variable gradient method

- The variable gradient method assumes a certain form for the gradient of an unknown Lyapunov function, i.e. $\nabla V(\mathbf{x})$.
- $\nabla V(\mathbf{x})$ is usually assumed to be a linear function.
- The Lyapunov function can be found by integrating the assumed gradient, and so does $\dot{V}(\mathbf{x})$.
- For low order systems, this approach sometimes leads to the successful discovery of a Laypunov function.
- Of course, this method can be applied to higher-order systems.

Let's consider a system with the form of $\dot{x} = f(x)$.

Suppose the gradient of the Lyapunov function is represented by

$$\frac{\partial V(\mathbf{x})}{\partial x_i} = \sum_{j=1}^{n} a_{ij} x_j$$

In order to ensure that the right terms of the above equation is the part differential of $\mathrm{V}(x)$, the following curl condition must be satisfied

$$\frac{\partial}{\partial x_i}\left(\frac{\partial V(\mathbf{x})}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial V(\mathbf{x})}{\partial x_i}\right), \quad i \neq j$$

Let $a_{ij} = a_{ji}$ anda_{ii} is only the function of x_i, then the curl condition will be satisfied, and

$$\dot{V}(\mathbf{x}) = [\nabla V(\mathbf{x})]^T \dot{\mathbf{x}} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij} x_j\right) f_i(\mathbf{x})$$

Now, we can choose appropriate coefficient a_{ij} such that $\dot{V}\ (\boldsymbol{x})$.is either negative or negative semi-definite in a domain as large as possible.

The full differential of $V(x)$ is

$$\mathrm{d}V(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\,\mathrm{d}x_i$$

Due to the satisfaction of the curl condition, the above equation can be integrated as follows

$$V(\mathbf{x}) = \int_0^x \nabla V\,\mathrm{d}\mathbf{x} = \sum_{i=1}^{n} \left(\int_0^{x_i} \frac{\partial V}{\partial x_i}\,\mathrm{d}x_i \right)$$

Finally, let’s check whether $V(x)$ is positive definite.

Example

Consider a nonlinear system

$$\begin{cases} \dot{x}_1 &= -2x_1 \\ \dot{x}_2 &= -2x_2 + 2x_1x_2^2 \end{cases}$$

Try to use the variable gradient method to find a Lyapunov function.

Solutions

Assume the gradient of the unknown Lyapunov function is

$$\begin{cases} \frac{\partial V(\mathbf{x})}{\partial x_1} &= a_{11}x_1 + a_{12}x_2 \\ \frac{\partial V(\mathbf{x})}{\partial x_2} &= a_{12}x_1 + a_{22}x_2 \end{cases}$$

It is clear $\frac{\partial}{\partial x_2}\left(\frac{\partial V(\mathbf{x})}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial V(\mathbf{x})}{\partial x_2}\right)$

Let's choose $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = 0$, which leads to

$$\frac{\partial V(\mathbf{x})}{\partial x_1} = x_1, \quad \frac{\partial V(\mathbf{x})}{\partial x_2} = x_2$$

$$\therefore \dot{V}(\mathbf{x}) = \nabla V(\mathbf{x})\dot{\mathbf{x}} = -2x_1^2 - 2x_2^2(1 - x_1 x_2)$$

Clearly, $\dot{V}(\mathbf{x})$ is negative definite in the region $(1 - x_1 x_2) > 0$.

The function $V(\mathbf{x})$ can be determined by integrating, as follows

$$V(\mathbf{x}) = \int_0^{x_1} x_1 \, \mathrm{d}x_1 + \int_0^{x_2} x_2 \, \mathrm{d}x_2 = \frac{x_1^2 + x_2^2}{2}$$

Therefore, the function $V(\mathbf{x})$ is positive definite and the system is asymptotic stability at the origin.

Outline of Chapter 9

Summary

- Since analytical solutions of nonlinear differential equations usually cannot be obtained, the two methods of *Lyapunov* are of major importance in the determination of nonlinear system stability.
- The linearization method is concerned with the small motion of nonlinear systems around equilibrium points.
- The direct method is applicable to essentially all dynamic systems, whether linear or nonlinear, continuous or discrete-time, and small or large motion.
- The direct method suffers from the common difficulty of finding a *Lyapunov* function for a given system.