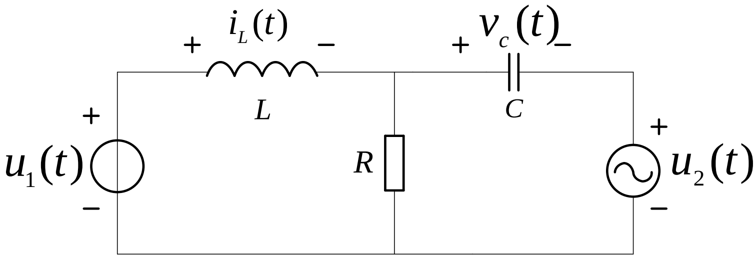
第 5 章习题参考答案

1. Obtain state equations for the following circuit. (Hint: you can use the voltage across the capacitor and the current through the inductor as state variables)



解：微分方程为

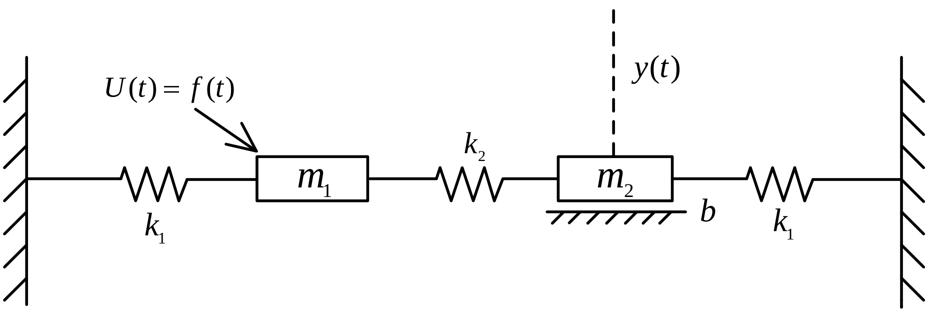
$$L\frac{di_L(t)}{dt}+v_c(t)=u_1(t)-u_2(t)$$

$$(i_L(t)-C\frac{dv_c(t)}{dt})R=v_c(t)+u_2(t)$$

状态方程为：

$$\begin{pmatrix}\dot{i}_L(t)\\ \dot{v}_c(t)\end{pmatrix}=\begin{pmatrix}0 & -\frac{1}{L}\\ \frac{1}{C} & -\frac{1}{RC}\end{pmatrix}\begin{pmatrix}i_L(t)\\ v_c(t)\end{pmatrix}+\begin{pmatrix}\frac{1}{L} & -\frac{1}{L}\\ 0 & -\frac{1}{RC}\end{pmatrix}\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix}$$

2. Consider the mass-spring system shown below. Assume that a force is acting on m_1, and let the horizontal position of m_2 represent the output of this system.



(1) Derive a set of differential equations which describes this input-output system. To solve this problem you will require Newton's law of translational motion, and the following facts: (i) The force exerted by a spring is proportional to its displacement, and (ii) the

force exerted by a frictional source is proportional to the relative speed of the source and mass.

(2) Find the transfer function for the system.

(3) Obtain a state-space description of the system.

解：(1) 微分方程为：

$$m_1\ddot{y}_1 = f(t) - k_1 y_1 - k_2(y_1 - y_2)$$

$$m_2\ddot{y}_2 = -m_2\dot{y}_2 - k_1 y_2 + k_2(y_1 - y_2)$$

其中，y_1表示m_1产生的位移，y_2表示m_2产生的位移。

(2) 对上式两边取拉普拉斯变换得：

$$(m_1 s^2 + k_1 + k_2)Y_1(s) - k_2 Y_2(s) = f(s)$$

$$(m_2 s^2 + k_1 + k_2 + m_2 s)Y_2(s) - k_2 Y_1(s) = 0$$

即 $$\begin{pmatrix} m_1 s^2 + k_1 + k_2 & -k_2 \\ -k_2 & m_2 s^2 + m_2 s + k_1 + k_2 \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} f(s) \\ 0 \end{pmatrix}$$

传递关系为：

$$\begin{pmatrix} m_1 s^2 + k_1 + k_2 & -k_2 \\ -k_2 & m_2 s^2 + m_2 s + k_1 + k_2 \end{pmatrix}^{-1} f(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix}$$

因此，系统的传递函数阵为：

$$G(s) = \frac{1}{\Delta}\begin{pmatrix} m_2 s^2 + m_2 s + k_1 + k_2 & k_2 \\ k_2 & m_1 s^2 + k_1 + k_2 \end{pmatrix}$$

其中，$\Delta = (m_1 s^2 + k_1 + k_2)(m_2 s^2 + m_2 s + k_1 + k_2) - k_2^2$

(3) 令 $x_1 = y_1$， $x_2 = \dot{y}_1$， $x_3 = y_2$， $x_4 = \dot{y}_2$

则有：

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{f(t)}{m_1} - \dfrac{k_1}{m_1}x_1 - \dfrac{k_2}{m_1}(x_1 - x_3) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{k_2}{m_2}(x_1 - x_3) - \dfrac{k_1}{m_2}x_3 - x_4 \end{cases}$$

$y_1 = x_1$

$y_2 = x_3$

所以，状态方程为：

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_1+k_2}{m_1} & 0 & \dfrac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_2}{m_2} & 0 & -\dfrac{k_1+k_2}{m_2} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{m_1} \\ 0 \\ 0 \end{pmatrix} f(t)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

3. Given a system represented by the following state equation, please transform it into the Jordan form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} u$$

解：系统的特征多项式为 $\det(\lambda I - A) = (\lambda - 1)^2 (\lambda - 2) = 0$

解的特征值为：$\lambda_1 = \lambda_2 = 1$，$\lambda_3 = 2$

对应于特征值 $\lambda_1 = 1$ 的特征向量 p_1 有 $Ap_1 = \lambda_1 p_1$，

$(\lambda_1 I - A) p_1 = 0$， 即 $\begin{bmatrix} -1 & -4 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} = 0$

解得： $p_1 = (-5, 0, 1)^T \qquad p_2 = (-4, 1, 0)^T$

对应于特征值 $\lambda_3 = 2$ 的特征向量 p_3 有 $Ap_3 = \lambda_3 p_3$，

解得： $p_3 = (1, 0, 0)^T$

$$P=(p_1,\ \ p_2,p_3)=\begin{pmatrix}-5 & -4 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},\quad P^{-1}=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 4 & 5\end{pmatrix}$$

$$\bar{b}=P^{-1}b=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 4 & 5\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}=\begin{pmatrix}3\\ 2\\ 24\end{pmatrix},\qquad \bar{A}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$$

4. An SISO LTI system is described by the transfer function

$$G(s)=\frac{s+4}{(s+1)(s+2)(s+3)}$$

(1) Obtain a state-space representation in the controllable canonical form;

(2) Now obtain one in the observable canonical form;

(3) Use partial fractions to obtain a representation of this model in the diagonal canonical form.

解： $G(s)=\dfrac{s+4}{(s+1)(s+2)(s+3)}=\dfrac{s+4}{s^3+6s^2+11s+6}$

(1)能控标准型：

$$\dot{x}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}x+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}4 & 1 & 0\end{bmatrix}x$$

(2)能观标准型：

$$\dot{x}=\begin{bmatrix}0 & 0 & -6\\ 1 & 0 & -11\\ 0 & 1 & -6\end{bmatrix}x+\begin{bmatrix}4\\ 1\\ 0\end{bmatrix}u$$

$$y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}x$$

(3) 传递函数可化成 $G(s)=\dfrac{c_1}{(s+1)}+\dfrac{c_2}{(s+2)}+\dfrac{c_3}{(s+3)}$

$$c_1=\lim_{s\to -1}G(s)(s+1)=\frac{3}{2}$$

$$c_2=\lim_{s\to -2}G(s)(s+2)=-2$$

$$c_3 = \lim_{s \to -3} G(s)(s+3) = \frac{1}{2}$$

状态方程为：

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

5. Given a system presented by the following state-space mode

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x$$

(1) Find the controllable-form and diagonal-form equivalent equations of the system by hand;

(2) Find the controllable-form and diagonal-form equivalent equations of the system by using Matlab and give the main command lines.

解：

(1) 能控标准型

$$Q = \begin{pmatrix} b & Ab & Ab^2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 2 & -4 \\ 1 & -2 & 0 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1/2 & 1/2 & -1/2 \\ 1/4 & 0 & -1/4 \end{pmatrix}$$

$$p_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} Q^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} p_1^{-1} \\ p_1^{-1} A \\ p_1^{-1} A^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -1 & -\frac{1}{2} \end{pmatrix}, \quad P = \begin{pmatrix} 2 & 2 & 1 \\ 4 & 2 & 0 \\ -2 & 2 & 1 \end{pmatrix}$$

可得 $\overline{A}=P^{-1}AP=\begin{pmatrix}0&1&0\\0&0&1\\-4&-6&-4\end{pmatrix}$

$\overline{b}=P^{-1}b=\begin{pmatrix}0\\0\\1\end{pmatrix}$， $\bar{c}=cP=\begin{pmatrix}-2&0&1\end{pmatrix}$

(2) 系统的特征多项式为 $\det(\lambda I-A)=(\lambda+2)(\lambda^2+2\lambda+2)=0$

解的特征值为：$\lambda_1=-1+i$，$\lambda_1=-1-i$，$\lambda_3=-2$

对应于特征值 $\lambda_1=-1+i$ 的特征向量 p_1 有 $Ap_1=\lambda_1 p_1$，

$(\lambda_1 I-A)p_1=0$， 即 $\begin{bmatrix}1+i&0&0\\0&-1+i&-1\\0&2&1+i\end{bmatrix}\begin{bmatrix}p_{11}\\p_{12}\\p_{13}\end{bmatrix}=0$

解得： $p_1=(0,1,-1+i)^T$ $p_2=(0,1,-1-i)^T$

对应于特征值 $\lambda_3=-2$ 的特征向量 p_3 有 $Ap_3=\lambda_3 p_3$，

解得：$p_3=(1,0,-1)^T$

$P=(p_1,\ p_2,p_3)=\begin{pmatrix}0&0&1\\1&1&0\\-1+i&-1-i&-1\end{pmatrix}$， $P^{-1}=\begin{pmatrix}-0.5i&0.5-0.5i&-0.5i\\0.5i&0.5+0.5i&0.5i\\1&0&0\end{pmatrix}$

$\overline{b}=P^{-1}b=\begin{pmatrix}-i\\i\\1\end{pmatrix}$， $\bar{c}=cP=(-1\quad -1\quad 1)$， $\Lambda=\begin{pmatrix}-1+i&0&0\\0&-1-i&0\\0&0&-2\end{pmatrix}$

(3) Matlab 程序为：

①

```
>> A=[-2 0 0;1 0 1;0 -2 -2];
   B=[1;0;1];
   C=[1 -1 0];
   D=[0];
   sys=ss(A,B,C,D);
```

```
T =ctrb(A,B);
sysT = ss2ss(sys,inv(T))
```

②

```
>> A=[-2 0 0;1 0 1;0 -2 -2];
   B=[1;0;1];
   C=[1 -1 0];
   [T, A1]=eig(A)
   B1=inv(T)*B
   C1=C*T
```