# Chapter 3 Controllability and Observability

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# Outline of Chapter 3

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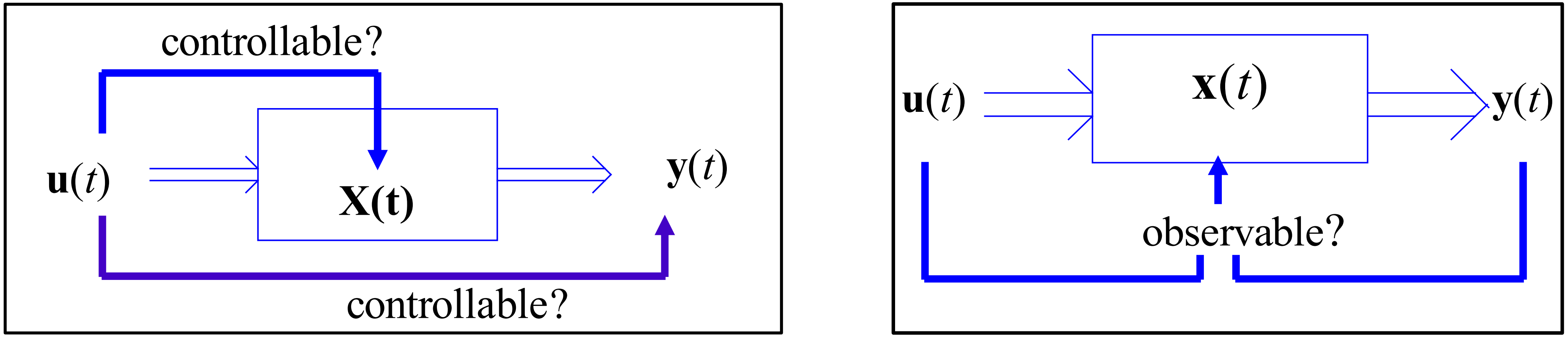
# Who introduced the concepts?

Rudolf Emil Kálmán

| | |
|---|---|
| **Born** | May 19, 1930 (age 84) Budapest, Hungary |
| **Residence** | United States |
| **Nationality** | Hungarian-born American citizen |
| **Fields** | Electrical Engineering; Mathematics; Applied Engineering Systems Theory |
| **Institutions** | Stanford University; University of Florida; Swiss Federal Institute of Technology |
| **Alma mater** | Massachusetts Institute of Technology; Columbia University |
| **Doctoral advisor** | John Ragazzini |
| **Notable awards** | IEEE Medal of Honor (1974) Rufus Oldenburger Medal (1976) Kyoto Prize (1985) Richard E. Bellman Control Heritage Award (1997) National Medal of Science (2008) Charles Stark Draper Prize |

- ✓ “Controllability and Observability” (1960)
- ✓ 《A New Approach to Linear Filtering and Prediction Problems》(1960)

# What are “controllability” and “observability”?




- Controllability determines whether or not the state of a state equation can be controlled by the input.
- Observability determines whether or not the initial state can be observed from the output.
- The conditions of controllability and observability govern the existence of a solution to the control system design problem.

# Outline of Chapter 3

## *Example* (Simple dynamic system)

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \boldsymbol{u}, \qquad \boldsymbol{y} = [c_1, \quad c_2]\, \boldsymbol{x}$$

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 \\ \dot{x}_2 &= \lambda_2 x_2 + b_2 u \end{aligned}$$

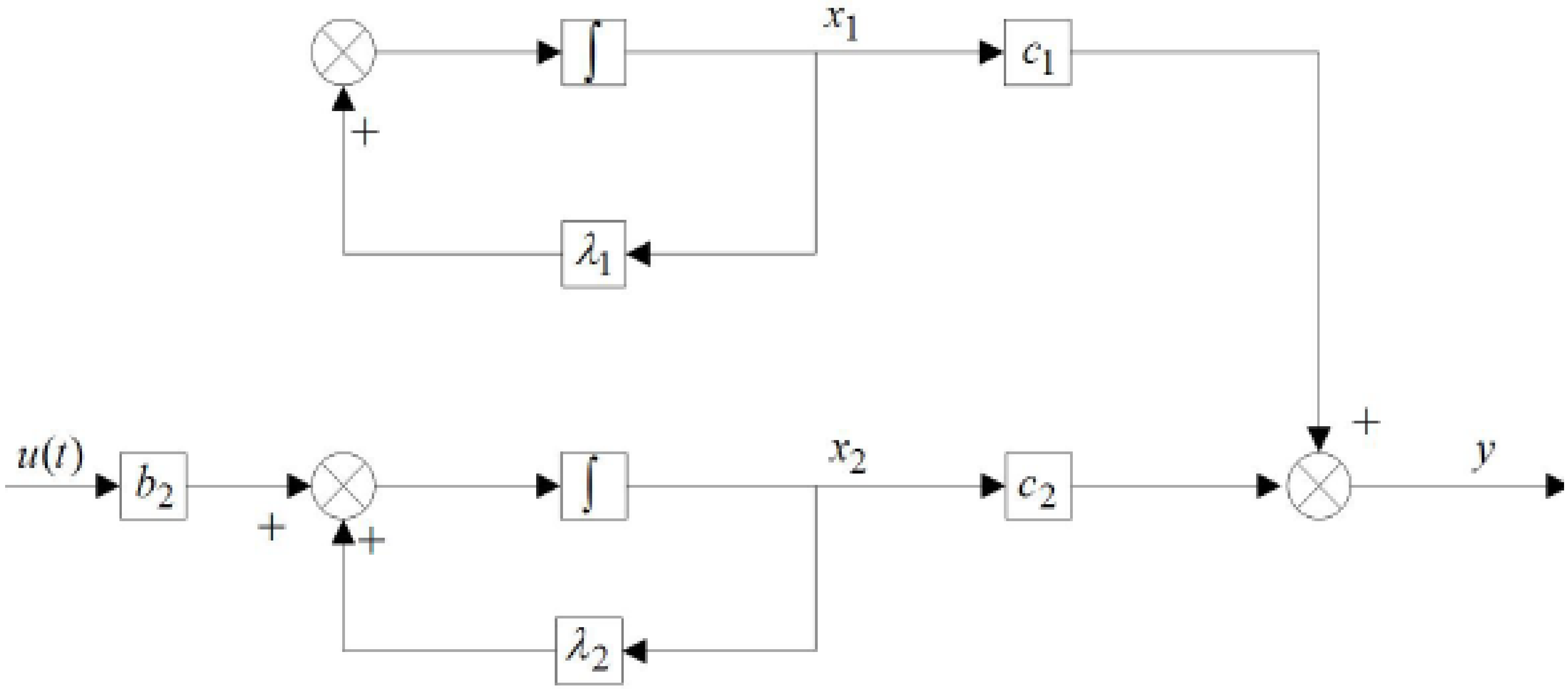
$$\dot{\boldsymbol{x}} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \boldsymbol{u}, \qquad \boldsymbol{y} = [c_1, \quad c_2]\, \boldsymbol{x}$$

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + x_2 \\ \dot{x}_2 &= \lambda_2 x_2 + b_2 u \end{aligned}$$

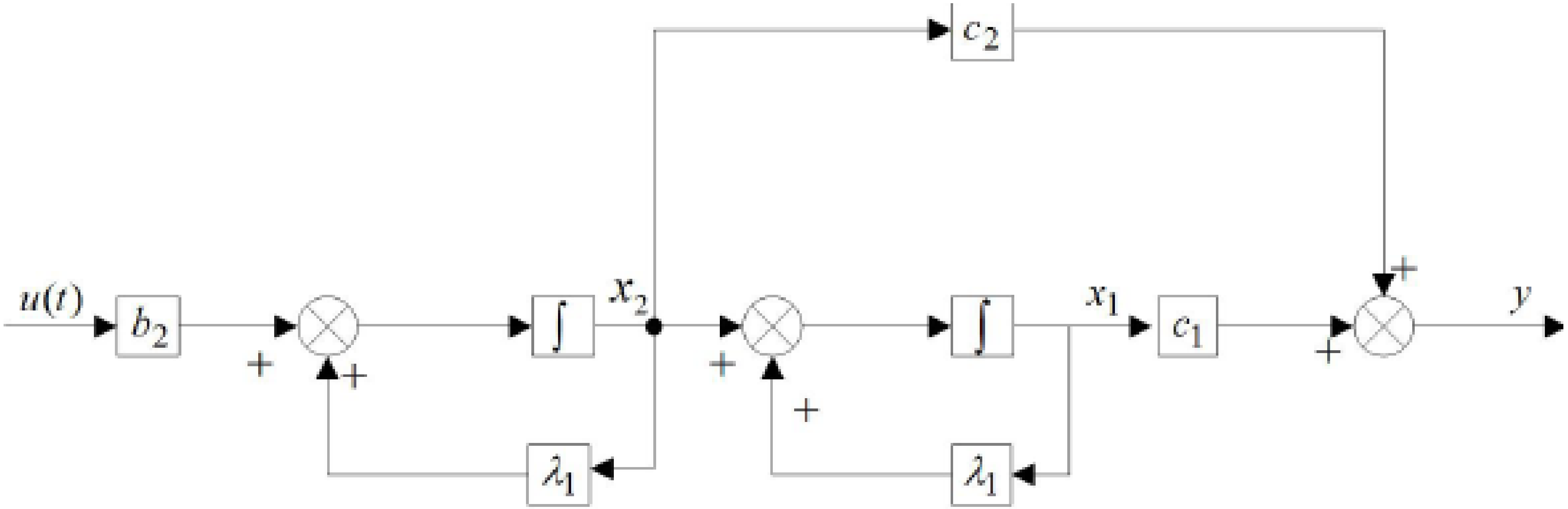
$$\dot{\boldsymbol{x}} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \boldsymbol{u}, \qquad \boldsymbol{y} = [c_1, \quad c_2]\, \boldsymbol{x}$$

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + x_2 + b_1 u \\ \dot{x}_2 &= \lambda_2 x_2 \end{aligned}$$

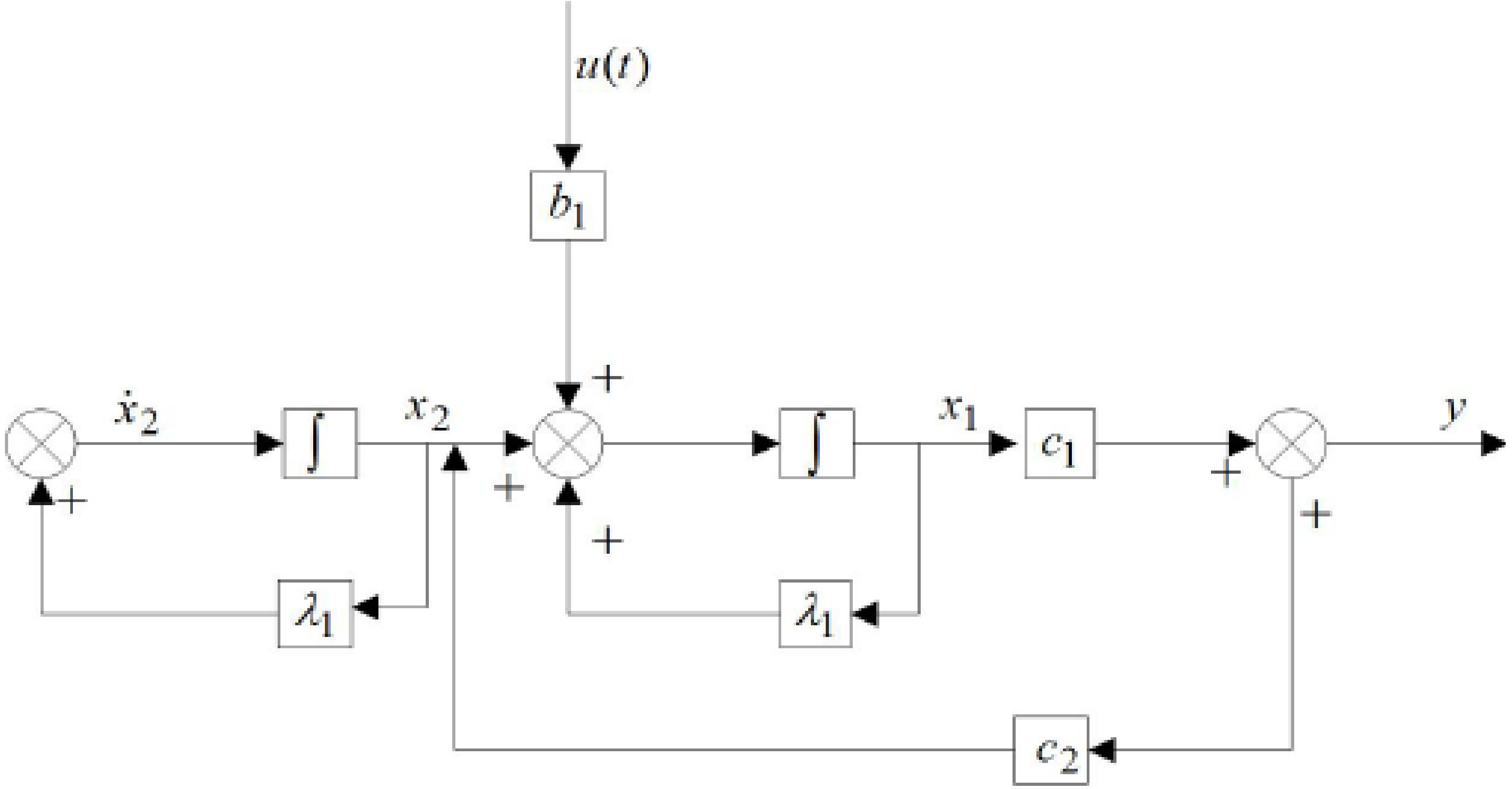
$$\boxed{\begin{aligned}\dot{x}_1 &= \lambda_1 x_1 \\ \dot{x}_2 &= \lambda_2 x_2 + b_2 u\end{aligned}}$$

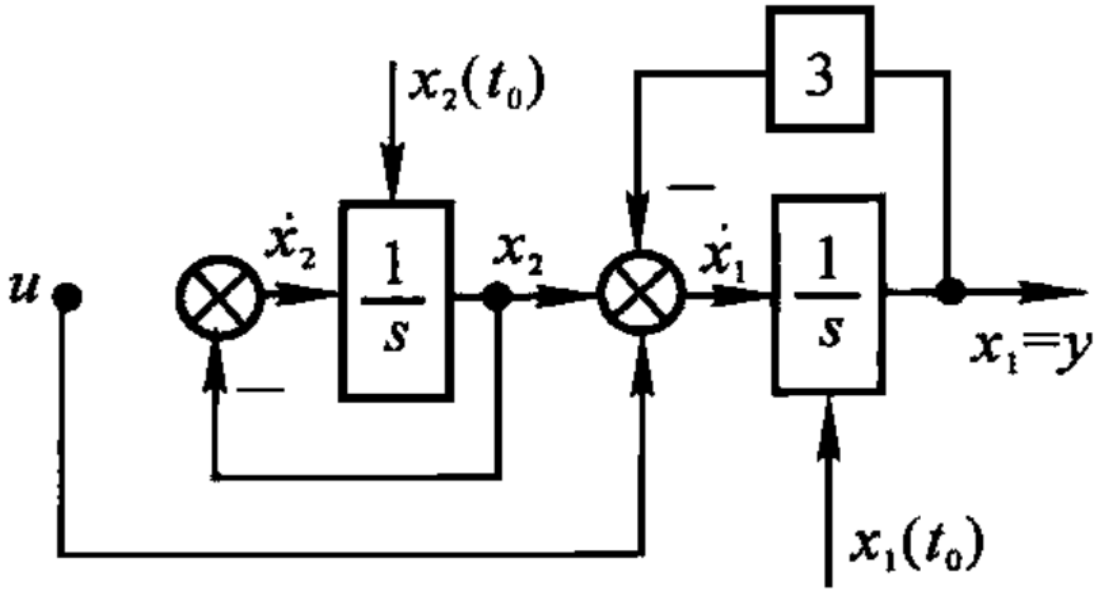


$$\dot{x}_1 = \lambda_1 x_1 + x_2$$
$$\dot{x}_2 = \lambda_2 x_2 + b_2 u$$

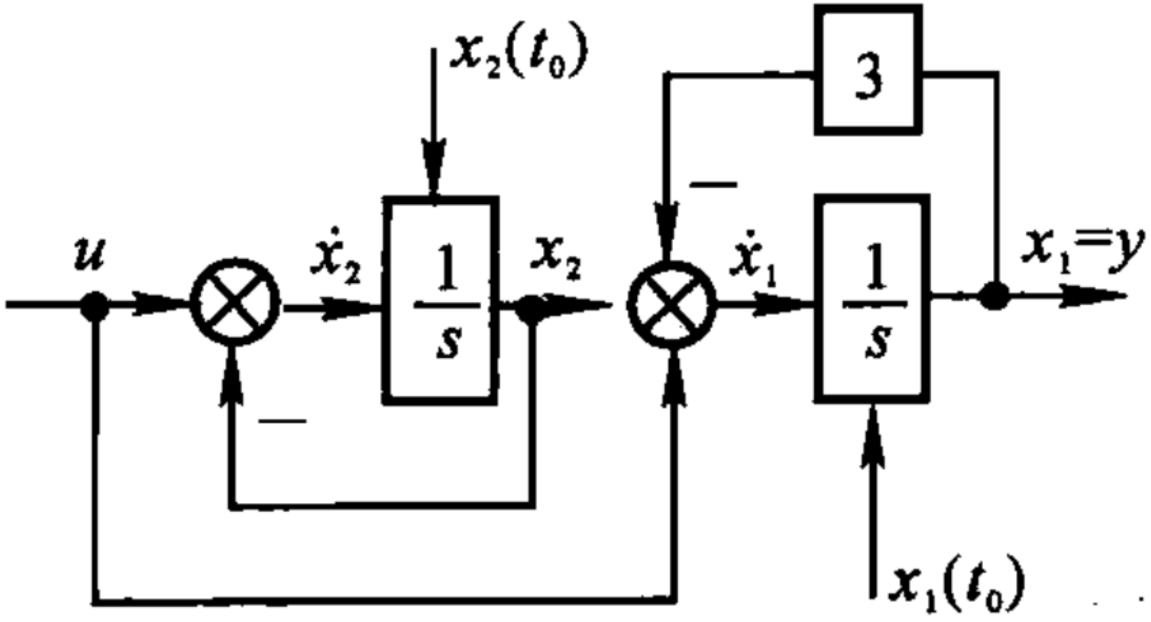


$$\boxed{\begin{aligned}\dot{x}_1 &= \lambda_1 x_1 + x_2 + b_1 u \\ \dot{x}_2 &= \lambda_2 x_2\end{aligned}}$$

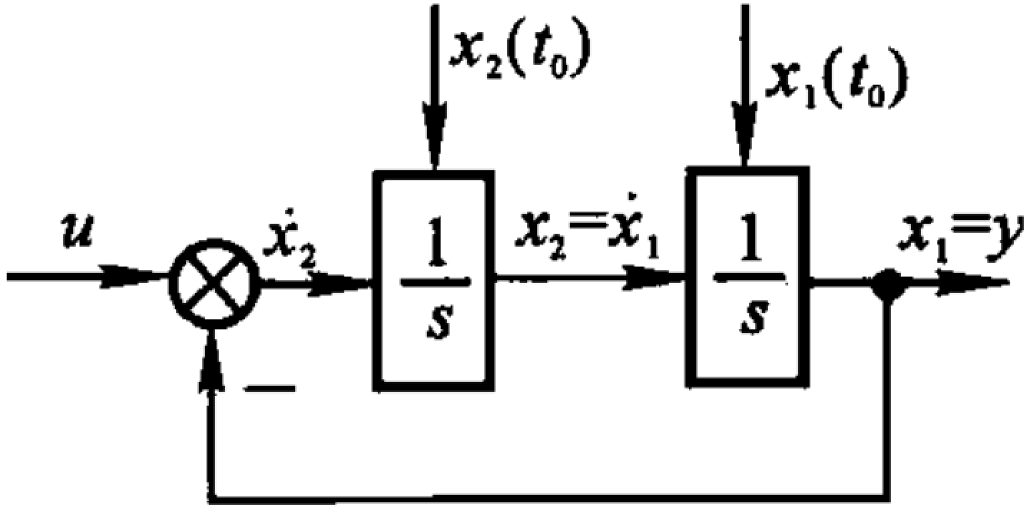





(a)




（b）




（c）

# Definitions of "controllability"

## Definition (Controllability)

A system is completely controllable if there exists an unconstrained control $\boldsymbol{u}(t)$ that can transfer any initial state $\boldsymbol{x}(t_0)$ to any other desired location $\boldsymbol{x}(t)$ in a finite time, $t_0 \le t \le T$.

$$\forall t_0 \in T \ \ \forall \boldsymbol{x}(t_0) \ \ \exists t_1 \in T \cap (t_1 > t_0) \ \exists \boldsymbol{u}(t) \cap (t \in [t_0, t_1]) \ (\boldsymbol{x}(t_1) = 0)$$

- 对初始时刻$t_0$($t_0 \in T$, $T$为时间定义域)和初始状态$\mathbf{x}(t_0)$,
  - ✓ 存在另一有限时刻$t_1$($t_1 > t_0, t_1 \in T$),
  - ✓ 可以找到一个控制量$\mathbf{u}(t)$,
  - ✓ 能在有限时间$[t_0, t_1]$内把系统状态从初始状态$\mathbf{x}(t_0)$控制到原点,即$\mathbf{x}(t_1)=0$,

则称$t_0$时刻的状态$\mathbf{x}(t_0)$能控;

若对$t_0$时刻的状态空间中的所有状态都能控,则称系统在$t_0$时刻状态完全能控。

Why origin ?

# How to judge the controllability of an LTI system?

Consider the continuous-time single-input system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

where the state $\mathbf{x}(t) \in \mathbb{R}^{n\times 1}$, control input $u(t) \in \mathbb{R}$, and the matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^{n\times 1}$.

The solution of the equation is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)\,\mathrm{d}\tau$$

Applying the definition of complete state controllability, we have

$$\mathbf{x}(t_1) = \mathbf{0} = e^{\mathbf{A}t_1}\mathbf{x}(0) + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{b}u(\tau)\,\mathrm{d}\tau$$

or

$$\boxed{\mathbf{x}(0) = -\int_0^{t_1} e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)\,\mathrm{d}\tau}$$

As the matrix exponential function $e^{At}$ can be written as (Cayley-Hamilton theory)

$$\boxed{A^k = \sum_{j=0}^{n-1} \alpha_{jk} A^j \text{ (for any k)}, \quad e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k}$$

we have

$$\begin{aligned}
\mathbf{x}(0) &= -\int_0^{t_1} \sum_{k=0}^{n-1} \alpha_k(\tau) \mathbf{A}^k \mathbf{b} u(\tau)\, \mathrm{d}\tau \\
&= -\sum_{k=0}^{n-1} \left\{ \int_0^{t_1} \alpha_k(\tau) \mathbf{A}^k \mathbf{b} u(\tau)\, \mathrm{d}\tau \right\} \\
&= -\sum_{k=0}^{n-1} \left\{ \mathbf{A}^k \mathbf{b} \int_0^{t_1} \alpha_k(\tau) u(\tau)\, \mathrm{d}\tau \right\} \\
&= -\sum_{k=0}^{n-1} \left\{ \mathbf{A}^k \mathbf{b} \beta_k \right\},
\end{aligned}$$

where $\beta_k = \int_0^{t_1} \alpha_k(-\tau) u(\tau) d\tau$

we are now in a position to get the result by the following steps

$$\begin{aligned}
\mathbf{x}(0) &= -\sum_{k=0}^{n-1} \left\{ \mathbf{A}^k \mathbf{b} \beta_k \right\} \\
&= -\mathbf{b}\beta_1 - \mathbf{A}\mathbf{b}\beta_2 - \cdots - \mathbf{A}^{n-1}\mathbf{b}\beta_{n-1} \\
&= -\begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \end{bmatrix}
\end{aligned}$$

If the system is completely state controllable, the above equation must be satisfied for any initial state $x(0)$, which requires that

$$\operatorname{rank} \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} = n$$

It can be extended to a general case where $u$ is a vector.

# Algebraic controllability criterion

**Theorem**

*The system* $(\mathbf{A}, \mathbf{B})$ *is completely controllable if and only if the rank of the controllability matrix*

$$\mathbf{Q}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

*is* $n$, *i.e.* $\operatorname{rank}(\mathbf{Q}_c) = n$

## Example

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

**Solutions**

For this case,

$$\begin{aligned} \mathbf{Q}_c &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -2 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 & -4 & -7 \\ 1 & 1 & -4 & -7 & 13 & 25 \end{bmatrix} \end{aligned}$$

Since $\text{rank}(\mathbf{Q}_c) = 3$, the system is completely state controllable.

## Example

Prove that an SISO system must be completely controllable if it can be described by a state-space model of the controllable canonical form.

### Solutions

Since the model is in controllable canonical form,

$$
\mathbf{A} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & a_{n-1} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
$$

We have the controllability matrix

$$\mathbf{Q}_c = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \vdots & \iddots & 1 & * \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & * & \cdots & * \\ 1 & * & * & \cdots & * \end{bmatrix}$$

It is clear that

$$\text{rank}\mathbf{Q}_c = n$$

Therefore the system is completely controllable.

# Example

## Consider the system described by

$$\dot{x} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} u$$

**Solutions**

$$[B \quad AB \quad A^2B] = \begin{bmatrix} 2 & 1 & 3 & 2 & 5 & 4 \\ \cancel{1} & \cancel{1} & \cancel{2} & \cancel{2} & \cancel{4} & \cancel{4} \\ \cancel{-1} & \cancel{-1} & \cancel{-2} & \cancel{-2} & \cancel{-4} & \cancel{-4} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 3 & 2 & 5 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $rank(Q_c) = 2$, therefore the system is not completely controllable.

# Jordan canonical controllability criterion

## Corollary (Diagonal canonical criterion)

*If the eigenvalues of* $\mathbf{A}$ *are distinct and the corresponding diagonal canonical form after similarity transformation is*

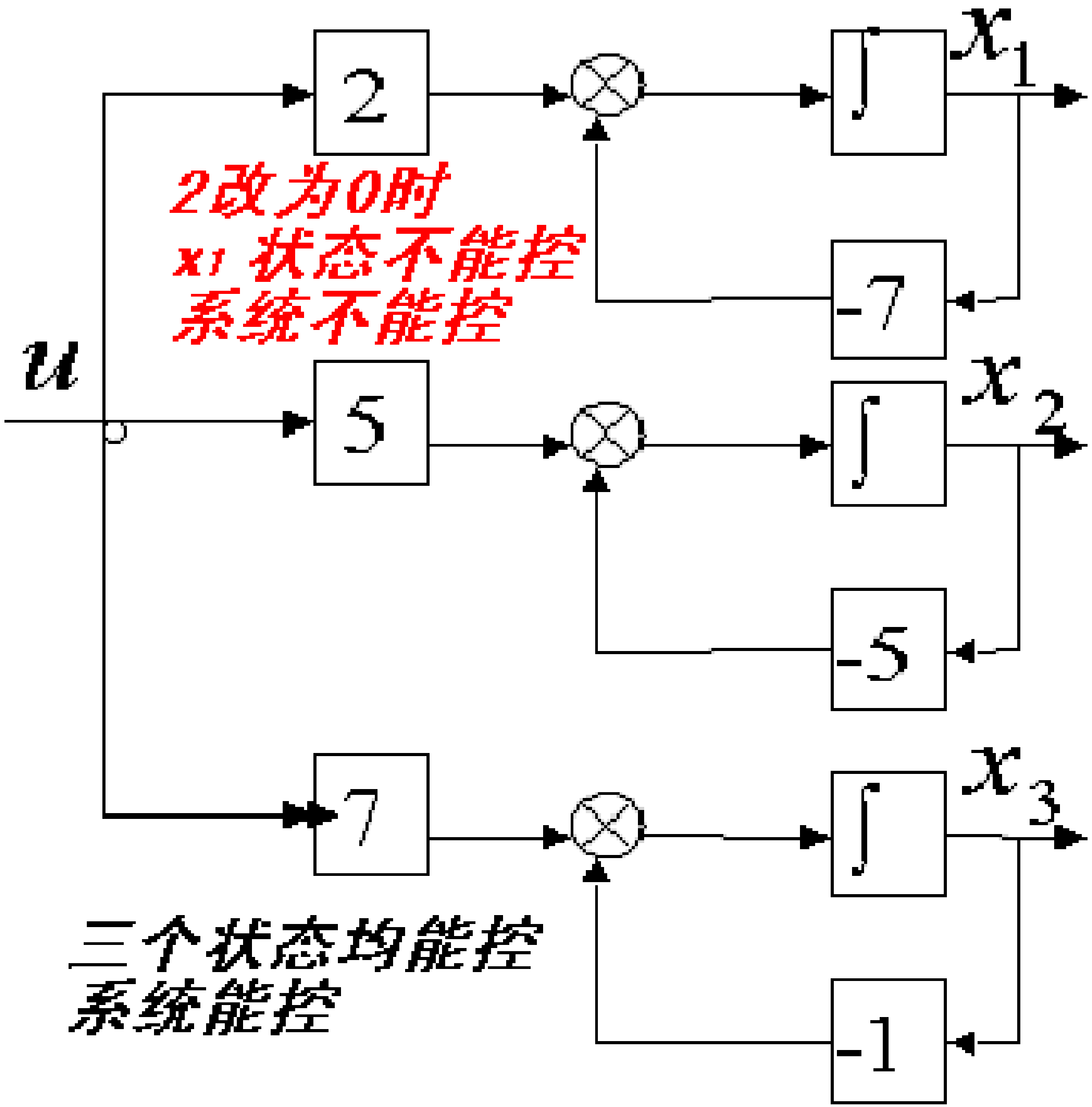
$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\bar{\mathbf{x}}(t) + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}(t)$$

*where*

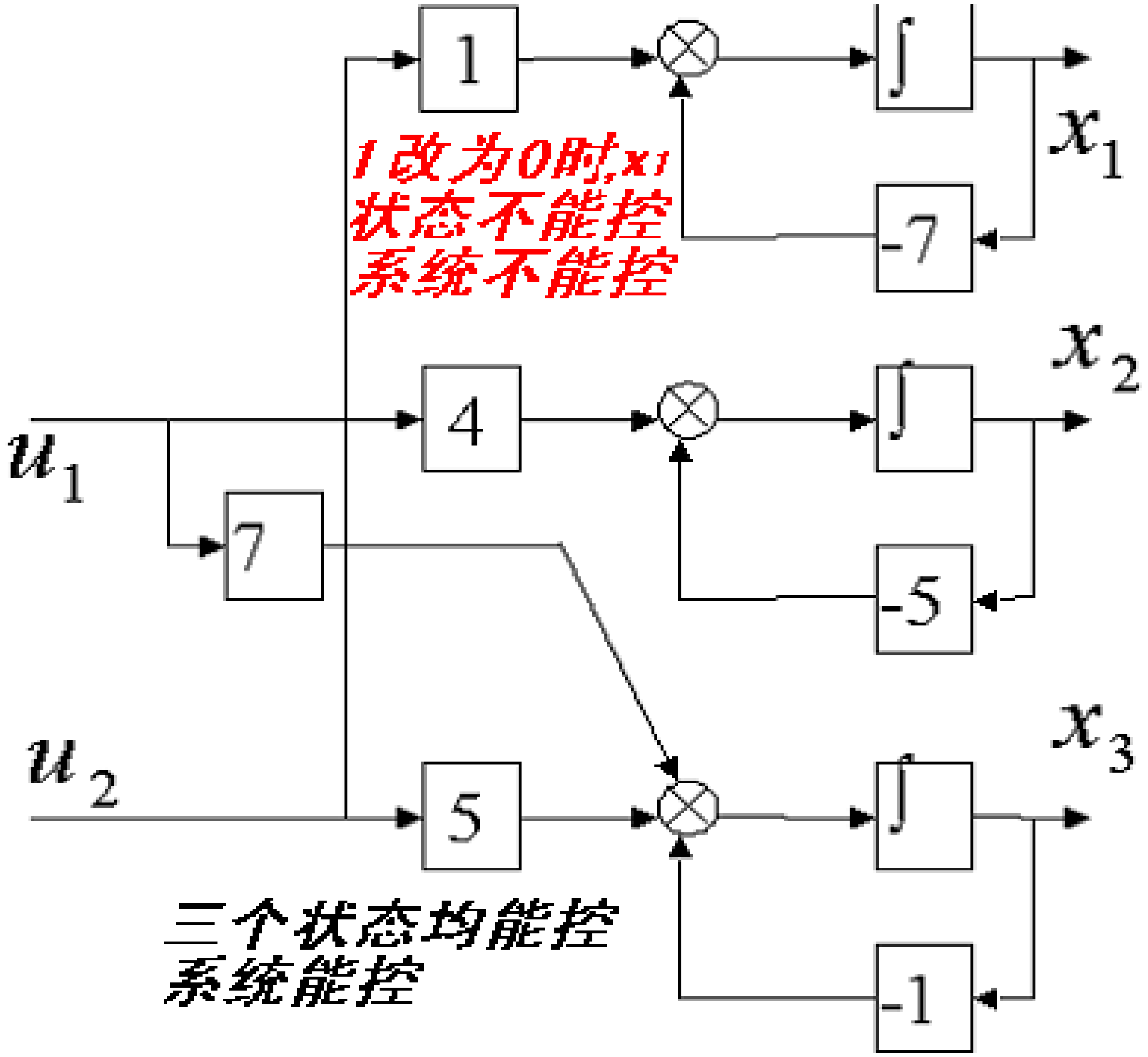
$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \bar{\mathbf{A}} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

*The original system* $(\mathbf{A}, \mathbf{B})$ *is completely state controllable if and only if there is no zero row in* $\bar{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B}$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -7 & & 0 \\ & -5 & \\ 0 & & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} u$$

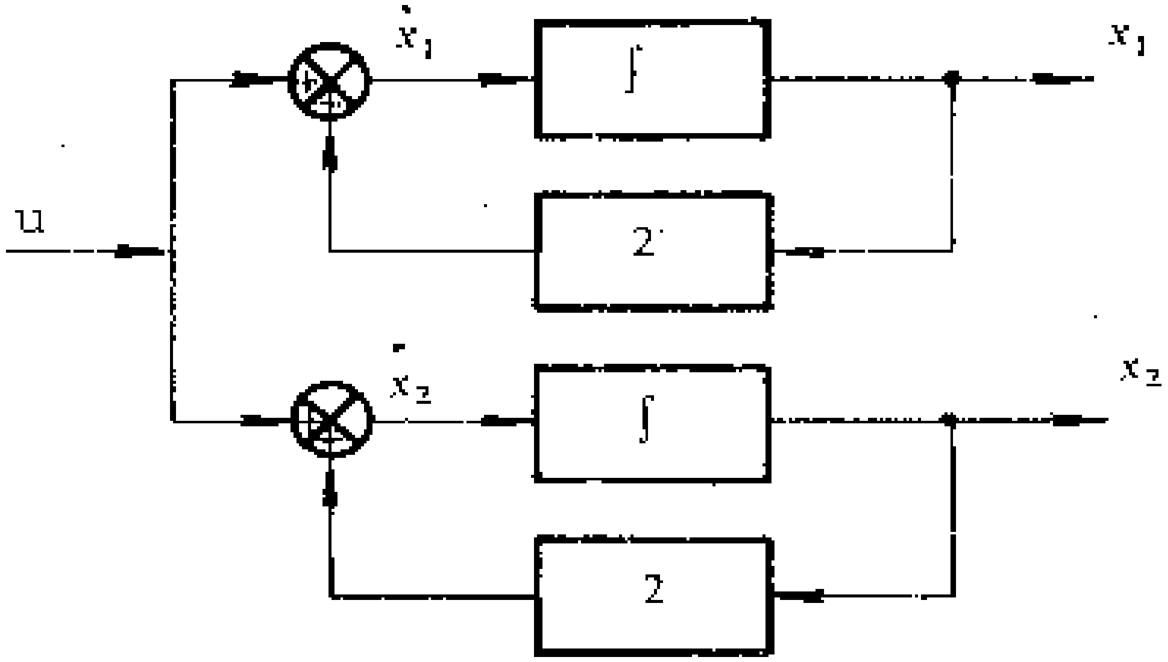


$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -7 & & \mathbf{0} \\ & -5 & \\ \mathbf{0} & & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 4 & 0 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u,$$

**虽然A为对角线阵，但特征值不满足互异条件，故不能由B不包全零行判别系统能控**. 实际上本系统状态不能控，因它由两个完全相同子系统组成，当$x_1(t_0)\neq x_2(t_0)$时，不可能找到一个$u(t)$，将它们转移到零.



# Jordan canonical controllability criterion(con.)

## Corollary (Jordan canonical criterion)

*Suppose the system matrix* $\mathbf{A}$ *has repeated eigenvalues, and the corresponding Jordan canonical form after similarity transformation is*

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\bar{\mathbf{x}}(t) + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}(t)$$

*where*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \bar{\mathbf{A}} = \begin{bmatrix} \mathrm{J}_1 & & & \\ & \mathrm{J}_2 & & \\ & & \ddots & \\ & & & \mathrm{J}_l \end{bmatrix}, \ \mathrm{J}_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}$$

# Jordan canonical controllability criterion (con.)

- If each distinct eigenvalue is associated with only one Jordan block. Then the original system (A, B) is completely state controllable if and only if the row of $\bar{B} = P^{-1}B$ corresponding to the last row of each Jordan block is not zero row.
- If one eigenvalue is associated with more than one Jordan matrix block, then the original system (A, B) is completely state controllable if and only if the row of $\bar{B} = P^{-1}B$ corresponding to the last row of each Jordan block is linear independence.

# Example

The controllability judgment of the following dynamic systems

(1) $$\dot{x} = \begin{bmatrix} -7 & 0 \\ 0 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

(2) $$\boldsymbol{x}' = \left[\begin{array}{cc:c} -4 & 1 & 0 \\ 0 & -4 & 0 \\ \hdashline 0 & 0 & -3 \end{array}\right] \boldsymbol{x} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hdashline 1 & 1 \end{array}\right] \boldsymbol{u}$$

# Example

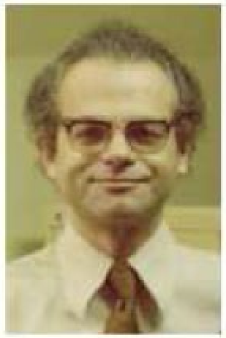
(3) $$\dot{x} = \left[\begin{array}{cc:c:c} -4 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ \hdashline 0 & 0 & -3 & 0 \\ \hdashline 0 & 0 & 0 & -4 \end{array}\right] \boldsymbol{x} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \hdashline 2 & 0 \\ \hdashline 2 & 1 \end{array}\right] \boldsymbol{u}$$

(4) $$\dot{x} = \left[\begin{array}{cc:c:c} -4 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ \hdashline 0 & 0 & -3 & 0 \\ \hdashline 0 & 0 & 0 & -4 \end{array}\right] \boldsymbol{x} + \left[\begin{array}{c} 0 \\ 1 \\ \hdashline 2 \\ \hdashline 3 \end{array}\right] \boldsymbol{u}$$

# Jordan canonical controllability criterion (con.)

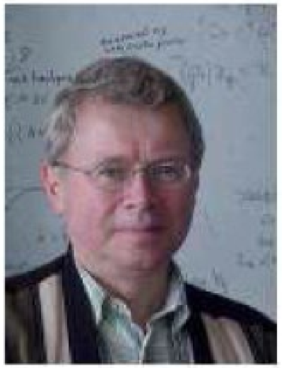
- If each distinct eigenvalue is associated with only one Jordan block. Then the original system (A, B) is completely state controllable if and only if the row of $\bar{B} = P^{-1}B$ corresponding to the last row of each Jordan block is not zero row.
- If one eigenvalue is associated with more than one Jordan matrix block, then the original system (A, B) is completely state controllable if and only if the row of $\bar{B} = P^{-1}B$ corresponding to the last row of each Jordan block is linear independence.

# PBH controllability criterion



V.M. Popov (1928– )

Vitold Belevitch (1921–1999)



Malo Hautus (1940– )

**Theorem**

*The system* $(\mathbf{A}, \mathbf{B})$ *is completely state controllable if and only if all the eigenvalues* $\lambda_i$ *of the state matrix* $\mathbf{A}$ *satisfy*

$$\operatorname{rank}\left[\lambda_i \mathbf{I} - \mathbf{A}, \mathbf{B}\right] = n, \quad i = 1, 2, \cdots, n$$

# Example

The controllability judgment： $\dot{x} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} u$

## Solutions

$|\lambda I - A| = 0$, so $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

$$\mathrm{rank}[\lambda_1 I\text{-}A \quad B] = \mathrm{rank}\begin{bmatrix} 0 & -3 & -2 & 2 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -1 & -2 & -1 & -1 \end{bmatrix} = 3 = n$$

$$\mathrm{rank}[\lambda_2 I\text{-}A \quad B] = \mathrm{rank}\begin{bmatrix} 1 & -3 & -2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 \end{bmatrix} = 3 = n$$

$$\mathrm{rank}[\lambda_3 I\text{-}A \quad B] = \mathrm{rank}\begin{bmatrix} 2 & -3 & -2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & -1 \end{bmatrix} = 2 < n$$

Rank$(\lambda I - A, B) \neq$n therefore the system is incompletely controllable.

# Conclusion-Controllability judgement of an LTI system

| Criterion | Judgement method | Characteristics |
|---|---|---|
| Algebraic criterion | $rankQ_c = rank[B \quad AB \quad \dots \quad A^{n-1}B] = n$ | ➢ The calculation is simple and feasible;<br>➢ Do not know which variables (eigenvalue/pole) controllable |
| Diagonal canonical Criterion<br>Jordan canonical Criterion | ◆ If each distinct eigenvalue is associated with only one Jordan block, the row of $\bar{B} = P^{-1}B$ corresponding to the last row of each Jordan block is not zero row.<br>◆ If one eigenvalue is associated with more than one Jordan matrix block, the row of $\bar{B} = P^{-1}B$ corresponding to the last row of each Jordan block is linear independence. | ➢ Easy to analyze which variables (eigenvalue/pole) controllable;<br>➢ Need to transform into the Jordan canonical form |
| PBH criterion | For all the $\lambda$, $rank[\lambda I - A \quad B] = n$ | ➢ Easy to analyze which variables (eigenvalue / pole) controllable ;<br>➢ Need to calculate the eigenvalue |

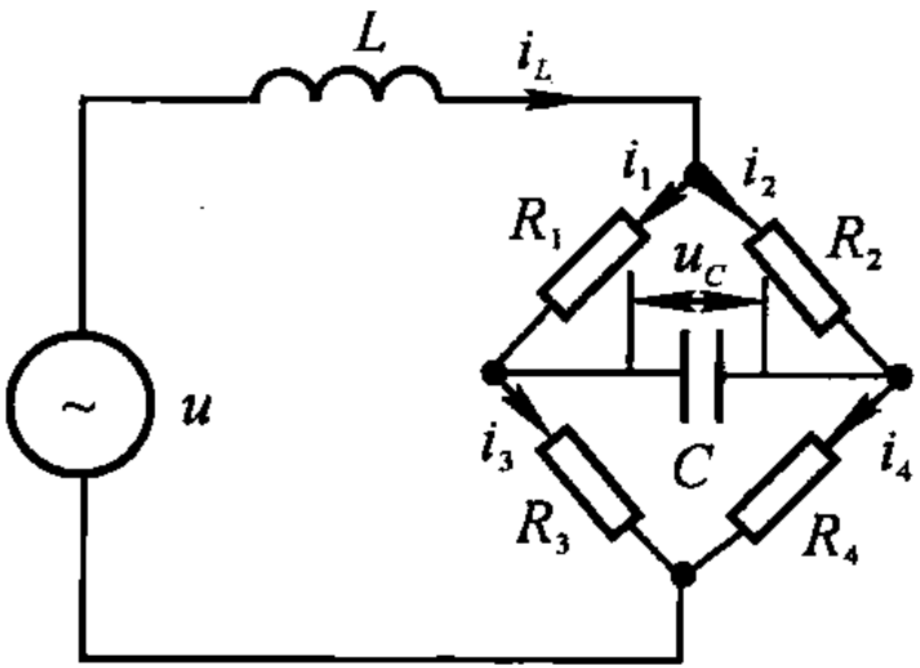



图 8.21　电桥电路

**例 8.22**　试用可控性判据判断图 8.21 所示桥式电路的可控性。

**解**　选取状态变量：$x_1=i_L, x_2=u_C$。电路的状态方程如下：

$$\begin{cases}\dot{x}_1=-\dfrac{1}{L}\left(\dfrac{R_1R_2}{R_1+R_2}+\dfrac{R_3R_4}{R_3+R_4}\right)x_1+\dfrac{1}{L}\left(\dfrac{R_1}{R_1+R_2}-\dfrac{R_3}{R_3+R_4}\right)x_2+\dfrac{1}{L}u\\ \dot{x}_2=\dfrac{1}{C}\left(\dfrac{R_2}{R_1+R_2}-\dfrac{R_4}{R_3+R_4}\right)x_1-\dfrac{1}{C}\left(\dfrac{1}{R_1+R_2}-\dfrac{1}{R_3+R_4}\right)x_2\end{cases}$$

可控性矩阵为

$$\boldsymbol{S}_3=[\boldsymbol{b}\quad \boldsymbol{Ab}]=\begin{bmatrix}\dfrac{1}{L} & -\dfrac{1}{L^2}\left(\dfrac{R_1R_2}{R_1+R_2}+\dfrac{R_3R_4}{R_3+R_4}\right)\\ 0 & \dfrac{1}{LC}\left(\dfrac{R_2}{R_1+R_2}-\dfrac{R_4}{R_3+R_4}\right)\end{bmatrix}$$

当 $R_1R_4\neq R_2R_3$ 时，$\mathrm{rank}\boldsymbol{S}_3=2=n$，系统可控；反之当 $R_1R_4=R_2R_3$，即电桥处于平衡状态时，

$$\mathrm{rank}\boldsymbol{S}_3=\mathrm{rank}[\boldsymbol{b}\quad \boldsymbol{Ab}]=\mathrm{rank}\begin{bmatrix}\dfrac{1}{L} & -\dfrac{1}{L^2}\left(\dfrac{R_1R_2}{R_1+R_2}+\dfrac{R_3R_4}{R_3+R_4}\right)\\ 0 & 0\end{bmatrix}$$

系统不可控，显然，$u$ 不能控制 $x_2$。

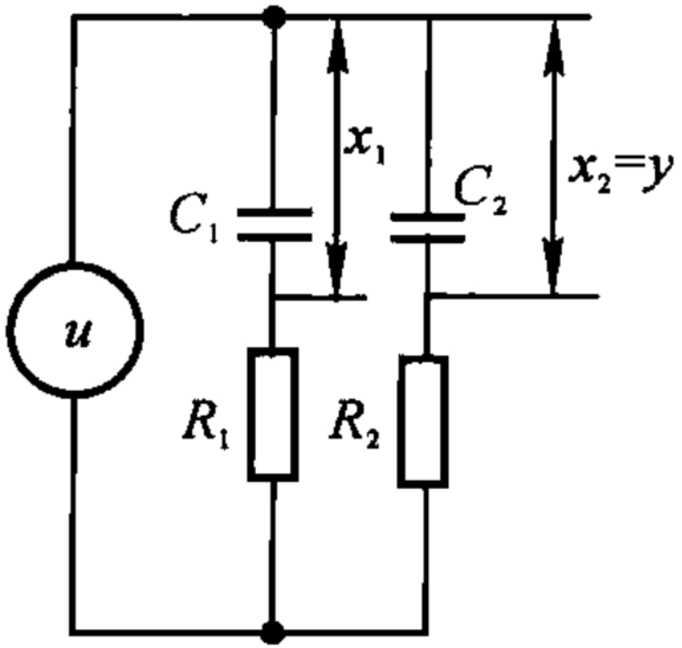



图 8.22　并联电路

**例 8.23** 试判断图 8.22 所示并联网络的可控性。

**解** 网络的微分方程为

$$x_1 + R_1C_1\dot{x}_1 = x_2 + R_2C_2\dot{x}_2 = u$$

式中

$$x_1 = u_{C_1} = \frac{1}{C_1}\int i_1 \mathrm{d}t \qquad x_2 = u_{C_2} = \frac{1}{C_2}\int i_2 \mathrm{d}t$$

状态方程为

$$\begin{cases} \dot{x}_1 = -\dfrac{1}{R_1C_1}x_1 + \dfrac{1}{R_1C_1}u \\ \dot{x}_2 = -\dfrac{1}{R_2C_2}x_2 + \dfrac{1}{R_2C_2}u \end{cases}$$

于是

$$\mathrm{rank}[\boldsymbol{b} \quad \boldsymbol{Ab}] = \mathrm{rank}\begin{bmatrix} \dfrac{1}{R_1C_1} & -\dfrac{1}{R_1^2C_1^2} \\ \dfrac{1}{R_2C_2} & -\dfrac{1}{R_2^2C_2^2} \end{bmatrix}$$

当 $R_1C_1 \neq R_2C_2$ 时，系统可控。当 $R_1 = R_2$，$C_1 = C_2$ 时，有 $R_1C_1 = R_2C_2$，$\mathrm{rank}[\boldsymbol{b} \quad \boldsymbol{Ab}] = 1 < n$，系统不可控；实际上，设初始状态 $x_1(t_0) = x_2(t_0)$，$u$ 只能使 $x_1(t) \equiv x_2(t)$，而不能将 $x_1(t)$ 与 $x_2(t)$ 分别转移到不同的数值，即不能同时控制住两个状态变量。

(1) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 2\end{bmatrix}u$$

(2) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

(3) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\end{bmatrix}=\begin{bmatrix}\lambda_1 & 1 & & & & \\ & \lambda_1 & & & & \\ & & \lambda_2 & & & \\ & & & \lambda_3 & 1 & \\ & & & & \lambda_3 & 1\\ & & & & & \lambda_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{bmatrix}$$

(1) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$$

(2) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

(3) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-3 & 1 & 0\\ 0 & -3 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}2 & -1\\ 0 & 0\\ 3 & 2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

# Review

- Definition (Controllability)
- Controllability judgement of an LTI system

# Definitions of “controllability”

## Definition (Controllability)

A system is completely controllable if there exists an unconstrained control $\boldsymbol{u}(t)$ that can transfer any initial state $\boldsymbol{x}(t_0)$ to any other desired location $\boldsymbol{x}(t)$ in a finite time, $t_0 \le t \le T$.

$$\forall t_0 \in T \;\; \forall \boldsymbol{x}(t_0) \;\; \exists t_1 \in T \cap (t_1 > t_0) \; \exists \boldsymbol{u}(t) \cap (t \in [t_0, t_1])$$

- 对初始时刻$t_0$($t_0 \in T$, $T$为时间定义域)和初始状态**x**($t_0$),
  - ✓ 存在另一有限时刻$t_1$($t_1 > t_0, t_1 \in T$),
  - ✓ 可以找到一个控制量**u**(t),
  - ✓ 能在有限时间[$t_0, t_1$]内把系统状态从初始状态**x**($t_0$)控制到**x**($t_1$)

则称$t_0$时刻的状态**x**($t_0$)能控;
若对$t_0$时刻的状态空间中的所有状态都能控,则称系统在$t_0$时刻状态完全能控。

**Theorem**

*The system* $(\mathbf{A}, \mathbf{B})$ *is completely controllable if and only if the rank of the controllability matrix*

$$\mathbf{Q}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

*is* $n$*, i.e.* $\operatorname{rank}(\mathbf{Q}_c) = n$

## Corollary (Diagonal canonical criterion)

*If the eigenvalues of* $\mathbf{A}$ *are distinct and the corresponding diagonal canonical form after similarity transformation is*

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\bar{\mathbf{x}}(t) + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}(t)$$

*where*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \bar{\mathbf{A}} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

*The original system* $(\mathbf{A}, \mathbf{B})$ *is completely state controllable if and only if there is no zero row in* $\bar{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B}$.

## Corollary (Jordan canonical criterion)

*Suppose the system matrix* $\mathbf{A}$ *has repeated eigenvalues, and the corresponding Jordan canonical form after similarity transformation is*

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\bar{\mathbf{x}}(t) + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}(t)$$

*where*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \bar{\mathbf{A}} = \begin{bmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_l \end{bmatrix}, \ \mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}$$

**Theorem**

*The system $(\mathbf{A}, \mathbf{B})$ is completely state controllable if and only if all the eigenvalues $\lambda_i$ of the state matrix $\mathbf{A}$ satisfy*

$$\operatorname{rank}\left[\lambda_i \mathbf{I} - \mathbf{A}, \mathbf{B}\right] = n, \quad i = 1, 2, \cdots, n$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 2 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \left[\begin{array}{cc|c} -4 & 1 & 0 \\ 0 & -4 & 0 \\ \hline 0 & 0 & -2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \left[\begin{array}{c} 0 \\ 4 \\ \hline 3 \end{array}\right] u$$

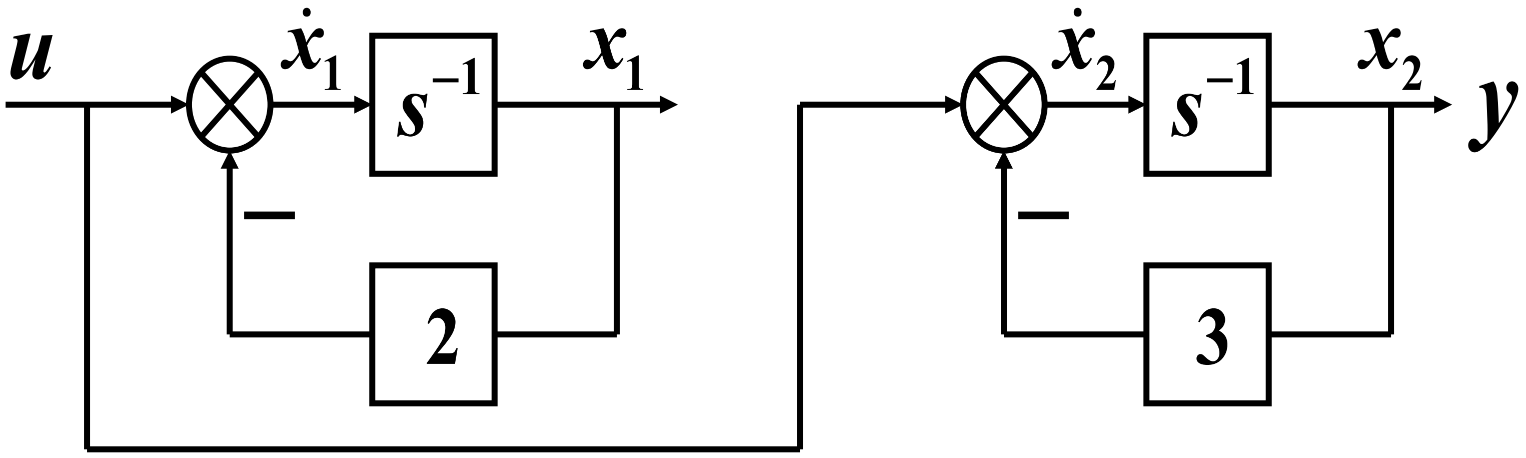
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \left[\begin{array}{cc|cc} -4 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -4 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -1 & 1 \\ & & 0 & -1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 2 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] u$$

# Conclusion-Controllability judgement of an LTI system

| Criterion | Judgement method | Characteristics |
| --- | --- | --- |
| Algebraic criterion | $rankQ_c = rank[B \quad AB \quad \ldots \quad A^{n-1}B] = n$ | ➢ The calculation is simple and feasible;<br>➢ Do not know which variables (eigenvalue/pole) controllable |
| Diagonal canonical Criterion<br>Jordan canonical Criterion | ◆ If each distinct eigenvalue is associated with only one Jordan block, the row of $\bar{B} = P^{-1}B$ corresponding to the last row of each Jordan block is not zero row.<br>◆ If one eigenvalue is associated with more than one Jordan matrix block, the row of $\bar{B} = P^{-1}B$ corresponding to the last row of each Jordan block is linear independence. | ➢ Easy to analyze which variables (eigenvalue/pole) controllable;<br>➢ Need to transform into the Jordan canonical form |
| PBH criterion | For all the $\lambda$, $rank[\lambda I - A \quad B] = n$ | ➢ Easy to analyze which variables (eigenvalue / pole) controllable ;<br>➢ Need to calculate the eigenvalue |

# Outline of Chapter 3

# How to understand the concepts?

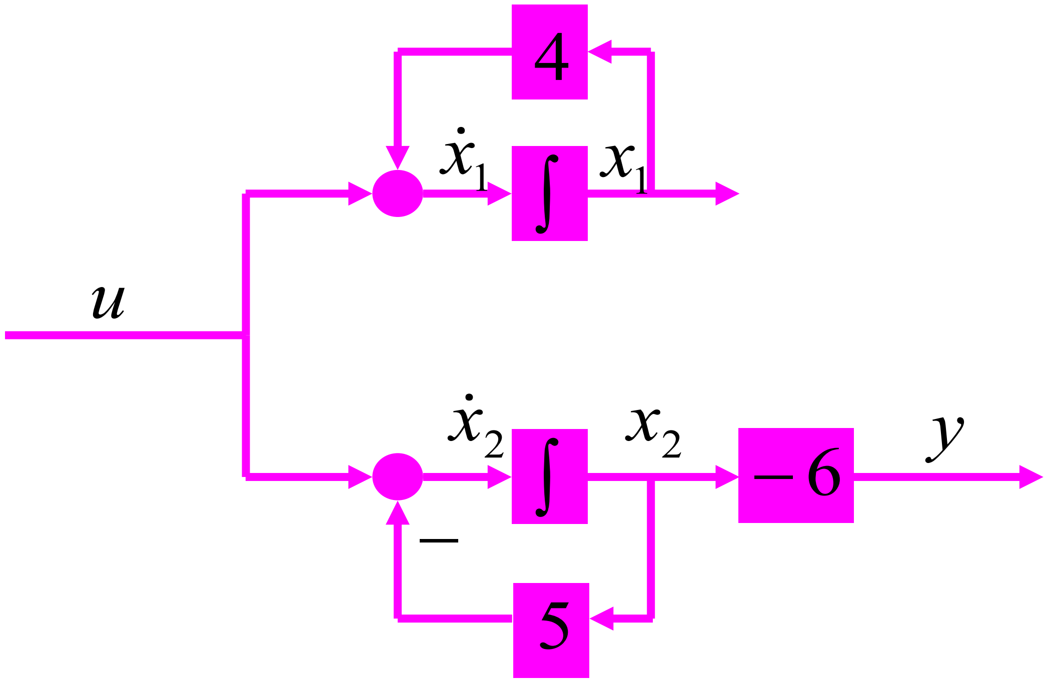



$$\begin{cases} \dot{x}_1 = u - 2x_1 \\ \dot{x}_2 = u - 3x_2 \\ y = x_2 \end{cases}$$

# How to understand the concepts?

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}4 & 0\\ 0 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 2\end{bmatrix}u \quad y=\begin{bmatrix}0 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\dot{x}_1=4x_1+u \qquad \dot{x}_2=-5x_2+2u \qquad y=6x_2$$




$u$ 可以控制$x_1$，$x_2$　　系统完全可控！

$y$ 无法反映 $x_1$　　系统不完全可观！

# Definitions of "observability"

## Definition (Observability)

A system is completely observable if, given the control $\mathbf{u}(t)$, every state $\mathbf{x}(t_0)$ can be determined from the observation of $\mathbf{y}(t)$ over a finite time interval, $t_0 \leqslant t \leqslant T$.

$$\forall t_0 \in T \; \forall \mathbf{x}(t_0) \; \exists t_1 \in T \; (t_1 > t_0) \cap (t \in [t_0, t_1]) \cap (\mathbf{y}(t) \xrightarrow{\text{unique}} \mathbf{x}(t_0))$$

$$\boldsymbol{x}(t) = \Phi(t - t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \Phi(t - \tau) B\boldsymbol{u}(\tau)\mathrm{d}\tau$$

# Algebraic observability criterion Theorem

**Theorem**

*The system* $(\mathbf{A}, \mathbf{C})$ *is completely observable if and only if the observability matrix*

$$\mathbf{Q}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

*or its transpose has rank* $n$*, i.e.* $\mathrm{rank}(\mathbf{Q}_o) = \mathrm{rank}(\mathbf{Q}_o^T) = n$

## Example

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is the system controllable and observable?

## Solutions

Calculating the controllability and observability matrices yields

$$\mathbf{Q}_c = \begin{bmatrix} \mathbf{b} & \mathbf{Ab} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{Q}_o = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Since $\text{rank}(\mathbf{Q}_c) = \text{rank}(\mathbf{Q}_o) = 2$, the system is completely state controllable and observable.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}\mathbf{x}, \quad \mathbf{y} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}\mathbf{x}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 2 & 4 & 4 \\ & \vdots & \end{bmatrix}$$

**条件满足即可，**

**不必写出所有的行！**

前三行已使 $rank\mathbf{N} = n = 3$

# 秩判据

$n \times n$ 阶可观测性矩阵

单输出：

$$rank\mathbf{N} = rank\begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \vdots \\ \mathbf{cA}^{\mathbf{n}-1} \end{bmatrix} = \dim \mathbf{A} = n$$

多输出：

$$rank\mathbf{N} = rank\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{\mathbf{n}-1} \end{bmatrix} = \dim \mathbf{A} = n$$

条件满足即可，

不必写出所有的行！

$nm \times n$ 阶可观测性矩阵

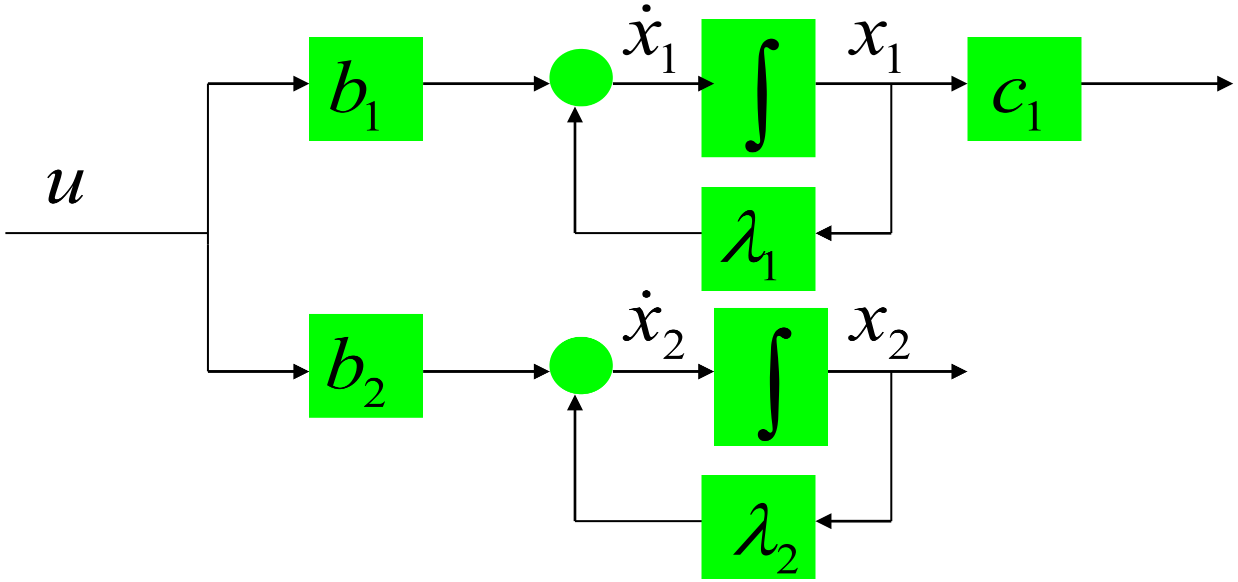
$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & 0 \end{bmatrix} \mathbf{x}$$

$$\dot{x}_1 = \lambda_1 x_1 + b_1 u$$

$$\dot{x}_2 = \lambda_2 x_2 + b_2 u$$

$$y = c_1 x_1$$



1 $$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} -7 & & 0 \\ & -5 & \\ 0 & & -1 \end{bmatrix} \bar{x} \\ y = \begin{bmatrix} \boxed{0} & 4 & 5 \end{bmatrix} \bar{x} \end{cases}$$

2 $$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} -7 & & 0 \\ & -5 & \\ 0 & & -1 \end{bmatrix} \bar{x} \\ y = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \bar{x} \end{cases}$$

## Corollary (Diagonal canonical criterion)

*If the eigenvalues of* $\mathbf{A}$ *are distinct and the corresponding diagonal canonical form after similarity transformation is*

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\bar{\mathbf{x}}(t) + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{P}\bar{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t)$$

*where*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \bar{\mathbf{A}} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

*The original system* $(\mathbf{A}, \mathbf{C})$ *is completely state observable if and only if there is no zero column in* $\bar{\mathbf{C}} = \mathbf{C}\mathbf{P}$.

$$
\mathbf{J}=\begin{bmatrix}
\lambda_1 & 1 & 0 & & & & & \\
0 & \lambda_1 & 1 & & & & 0 & \\
0 & 0 & \lambda_1 & & & & & \\
& & & \lambda_2 & 1 & & & \\
& & & 0 & \lambda_2 & & & \\
& & & & & \lambda_6 & & \\
& 0 & & & & & \ddots & \\
& & & & & & & \lambda_n
\end{bmatrix}
$$

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{J}\mathbf{x}+\mathbf{B}\mathbf{u}\\
\mathbf{y} &= \mathbf{C}\mathbf{x}+\mathbf{D}\mathbf{u}
\end{aligned}
$$

$\Downarrow$

C 矩阵中与约旦块第一列对应的列不全为零

C 矩阵中与互异特征值对应的列不全为零

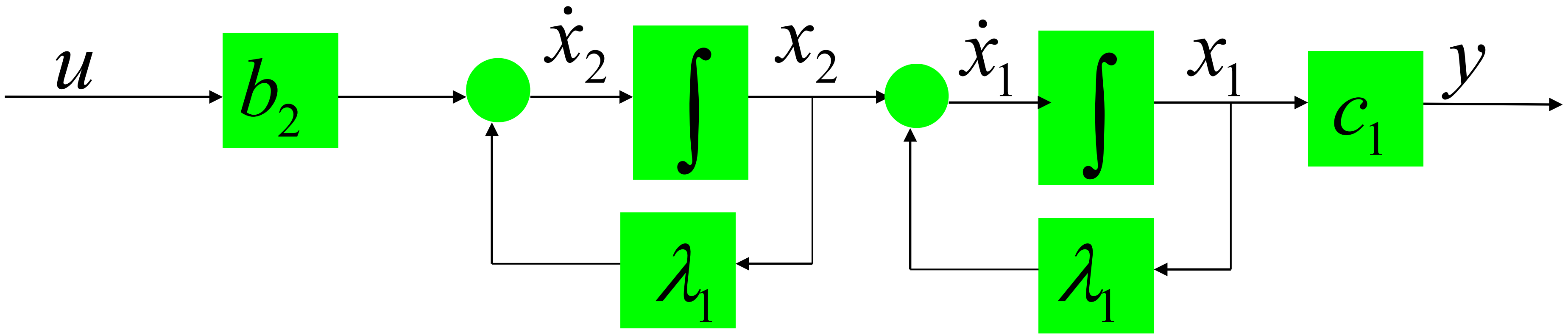
$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda_1 & 1 \\ & \lambda_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & 0 \end{bmatrix} \mathbf{x}$$

$$\dot{x}_1 = \lambda_1 x_1 + x_2$$

$$\dot{x}_2 = \lambda_1 x_2 + b_2 u$$

$$y = c_1 x_1$$




系统可观测！

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda_1 & 1 \\ & \lambda_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u$$

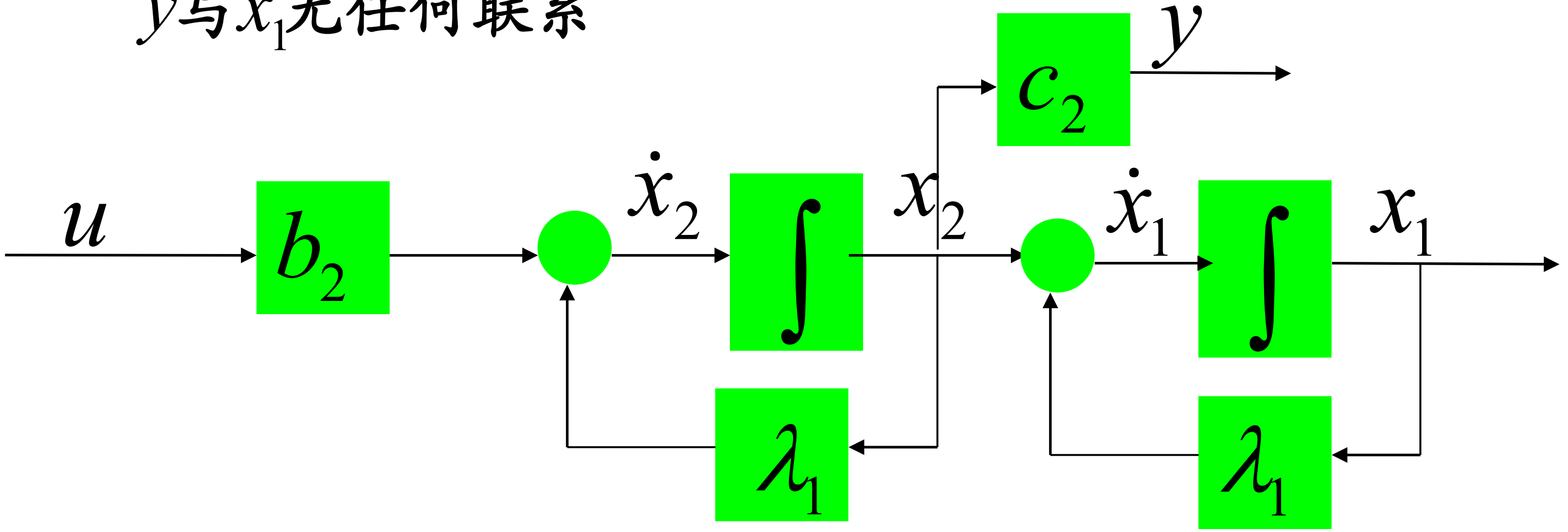
$$\dot{x}_1 = \lambda_1 x_1 + x_2$$

$$\dot{x}_2 = \lambda_1 x_2 + b_2 u$$

$$y = \begin{bmatrix} 0 & c_2 \end{bmatrix} \mathbf{x}$$

$$y = c_2 x_2$$

$y$与$x_1$无任何联系




系统不可观测！

## Corollary (Jordan canonical criterion)

*If the system matrix* $\mathbf{A}$ *has repeated eigenvalues, and the corresponding Jordan canonical form after similarity transformation is*

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\bar{\mathbf{x}}(t) + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{P}\bar{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t)$$

*where*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \bar{\mathbf{A}} = \begin{bmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_l \end{bmatrix}, \ \mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}$$

*and each distinct eigenvalue is associated with only one Jordan block, then the original system* $(\mathbf{A}, \mathbf{C})$ *is completely state observable if and only if the column of* $\bar{\mathbf{C}} = \mathbf{C}\mathbf{P}$ *corresponding to the first column of each Jordan block is not zero column.*

# Example

The observability judgment of the following dynamic systems

(1)
$$\dot{x} = \begin{bmatrix} -7 & 0 \\ 0 & -5 \end{bmatrix} x$$
$$y = \begin{bmatrix} 3 & 0 \end{bmatrix} x$$

(2)
$$\dot{x} = \left[\begin{array}{cc:c} -4 & 1 & 0 \\ 0 & -4 & 0 \\ \hdashline 0 & 0 & -3 \end{array}\right] x$$
$$y = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} x$$

$\Longrightarrow$ Completely state observable

# Example

The observability judgment of the following dynamic systems

(3)
$$\dot{x} = \left[\begin{array}{cc:c:c} -4 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ \hdashline 0 & 0 & -3 & 0 \\ \hdashline 0 & 0 & 0 & -4 \end{array}\right] x$$

$$y = \left[\begin{array}{cc:c:c} 1 & 1 & 0 & -2 \\ 0 & 2 & 1 & 0 \end{array}\right] x$$

- If more than one Jordan matrix block, the row of $\bar{C} = \mathrm{CP}$ corresponding to the first column of each Jordan block is linear correlation, so, the states of the system can not be observed completely .

(1) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 0\\ 0 & 0 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

(3) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\mathbf{y}=\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

(2) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\end{bmatrix}=\begin{bmatrix}-1 & 1 & & & \\ & -1 & & & \\ & & -2 & 1 & \\ & & & -2 & 1\\ & & & & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}$$

$$y=\begin{bmatrix}-5 & 0 & 2 & 0 & 0\end{bmatrix}\overline{\mathbf{x}}$$

(4) $$\begin{cases}\dot{\overline{x}}=\begin{bmatrix}2 & 1 & & 0\\ 0 & 2 & & \\ & & 2 & 1\\ 0 & & 0 & 2\end{bmatrix}\overline{x}\\ y=\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}\overline{x}\end{cases}$$

# PBH observability criterion

## Theorem

*The system* $(\mathbf{A}, \mathbf{C})$ *is completely state observable if and only if all the eigenvalues* $\lambda_i$ *of the state matrix* $\mathbf{A}$ *satisfy*

$$\operatorname{rank} \begin{bmatrix} \lambda_i \mathbf{I} - \mathbf{A} \\ \mathbf{C} \end{bmatrix} = n, \quad i = 1, 2, \cdots, n$$

## Example

The observability judgment of the following dynamic systems

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x$$

$$y = [\,4 \quad 5 \quad 1\,]x$$

**Solutions**

$|\lambda I - A| = 0$, so $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$.

$$rank\begin{bmatrix} \lambda_1 I - A \\ C \end{bmatrix} = rank\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 6 & 11 & 5 \\ 4 & 5 & 1 \end{bmatrix} = 2 < n$$

column 3=column 2-column 1

Can not completely observable

# Conclusion-Observability judgement of an LTI system

| Criterion | Judgement method | Characteristics |
|---|---|---|
| Algebraic criterion | $rankQ_o = rank[C \quad CA \quad \dots \quad C^{n-1}A] = n$ | ➢ The calculation is simple and feasible;<br>➢ Do not know which variables (eigenvalue/pole) observable |
| Diagonal canonical Criterion<br>Jordan canonical Criterion | ◆ If each distinct eigenvalue is associated with only one Jordan block, the row of $\bar{C} = CP$ corresponding to the first column of each Jordan block is not zero row.<br>◆ If one eigenvalue is associated with more than one Jordan matrix block, the row of $\bar{C} = CP$ corresponding to the first column of each Jordan block is linear independence. | ➢ Easy to analyze which variables (eigenvalue/pole) observable;<br>➢ Need to transform into the Jordan canonical form |
| PBH criterion | For all the $\lambda$, $rank\begin{bmatrix} \lambda_1 I - A \\ C \end{bmatrix}$ | ➢ Easy to analyze which variables (eigenvalue / pole) observable ;<br>➢ Need to calculate the eigenvalue |

## 非奇异线性变换不改变系统可观测性！

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{y} = \mathbf{C}\mathbf{x}$$

非奇异变换后： $\overline{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}, \overline{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B}, \overline{\mathbf{C}} = \mathbf{C}\mathbf{P}$

$$rank\overline{\mathbf{N}} = rank\begin{bmatrix} \overline{\mathbf{C}} \\ \overline{\mathbf{C}}\overline{\mathbf{A}} \\ \vdots \\ \overline{\mathbf{C}}\overline{\mathbf{A}}^{n-1} \end{bmatrix} = rank\begin{bmatrix} \mathbf{CP} \\ \mathbf{CPP}^{-1}\mathbf{AP} \\ \vdots \\ \mathbf{CP}(\mathbf{P}^{-1}\mathbf{AP})^{\mathbf{n}-1} \end{bmatrix} = rank\begin{bmatrix} \mathbf{CP} \\ \mathbf{CAP} \\ \vdots \\ \mathbf{CA}^{\mathbf{n}-1}\mathbf{P} \end{bmatrix}$$

$$= rank\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{\mathbf{n}-1} \end{bmatrix}\mathbf{P} = rank\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} = rank\mathbf{N}$$

试证明：非奇异线性变换不改变系统可控性

# Outline of Chapter 3

3.1 Introduction

3.2 Analysis of controllability

3.3 Analysis of observability

3.4 Principle of duality

3.5 Obtaining controllable and observable canonical forms

3.6 Canonical decomposition

3.7 Simulations with MATLAB

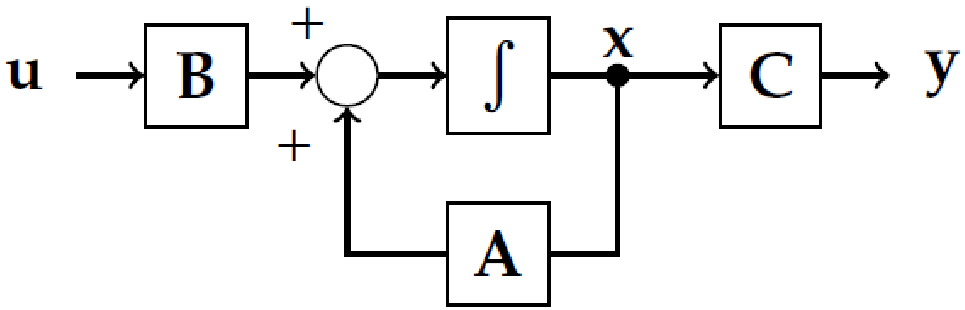
# Principle of duality

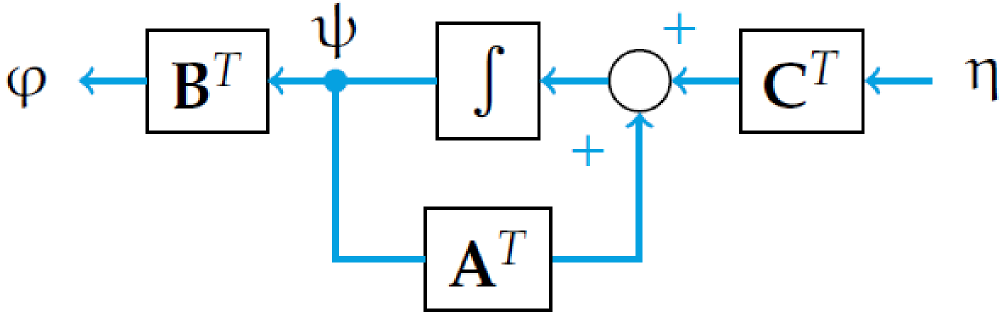
## Definition (Dual systems)

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{\psi}(t) = \mathbf{A}^T\psi(t) + \mathbf{C}^T\eta(t) \\ \varphi(t) = \mathbf{B}^T\psi(t) \end{cases}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^l, \mathbf{y} \in \mathbb{R}^m. \qquad \psi \in \mathbb{R}^n, \eta \in \mathbb{R}^m, \varphi \in \mathbb{R}^l.$$

The LTI systems $\Sigma_1(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and $\Sigma_2(\mathbf{A}^{\mathrm{T}}, \mathbf{C}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}})$ are dual of each other and the dimensions of the input and the output are exchanged between dual systems.



# Principle of duality

1）互为对偶的系统，其传递函数阵是互为转置的。

$$\boxed{\mathbf{G}_2(s) =} \mathbf{C}^*(s\mathbf{I}-\mathbf{A}^*)^{-1}\mathbf{B}^* \quad = \mathbf{B}^T(s\mathbf{I}-\mathbf{A}^T)^{-1}\mathbf{C}^T$$

$$= \mathbf{B}^T\left((s\mathbf{I}-\mathbf{A})^{-1}\right)^T\mathbf{C}^T = \left(\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\right)^T \boxed{= \mathbf{G}_1{}^T(s)}$$

2）互为对偶的系统，其特征方程是相同的。

$$\left|s\mathbf{I}-\mathbf{A}^*\right| = \left|s\mathbf{I}-\mathbf{A}^T\right| = \left|(s\mathbf{I}-\mathbf{A})^T\right| = \left|s\mathbf{I}-\mathbf{A}\right| = 0$$

## Theorem (Principle of duality)

*The system $\Sigma_1(\mathbf{A},\mathbf{B},\mathbf{C})$ is completely controllable (observable) if and only if its dual system $\Sigma_2(\mathbf{A}^\mathrm{T},\mathbf{C}^\mathrm{T},\mathbf{B}^\mathrm{T})$ are completely observable (controllable).*

For system $\Sigma_1(\mathbf{A},\mathbf{B},\mathbf{C})$

Controllability criterion:

$$\operatorname{rank}\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} = n$$

Observability criterion:

$$\operatorname{rank}\begin{bmatrix}\mathbf{C}\\ \mathbf{CA}\\ \vdots\\ \mathbf{CA}^{n-1}\end{bmatrix} = n$$

For system $\Sigma_2(\mathbf{A}^\mathrm{T},\mathbf{C}^\mathrm{T},\mathbf{B}^\mathrm{T})$

Controllability criterion:

$$\operatorname{rank}\begin{bmatrix}\mathbf{C}^T & \mathbf{A}^T\mathbf{C}^T & \cdots & \left(\mathbf{A}^T\right)^{n-1}\mathbf{C}^T\end{bmatrix} = n$$

Observability criterion:

$$\operatorname{rank}\begin{bmatrix}\mathbf{B}^T\\ \mathbf{B}^T\mathbf{A}^T\\ \vdots\\ \mathbf{B}^T\left(\mathbf{A}^T\right)^{n-1}\end{bmatrix} = n$$

# Outline of Chapter 3

Consider the following state space model of a system

$$\dot{x} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} x$$

Find the transformation matrix **P** so that it can be transformed into the diagonal form.

## Solutions

From the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

we obtain the distinct eigenvalues $\lambda_1 = -2, \lambda_2 = -3, \lambda_1 = -4$.

The corresponding eigenvectors are chosen as

$$\boxed{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 7 \\ 12 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}}$$

The transform matrix is

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 7 & 6 & 5 \\ 12 & 8 & 6 \end{bmatrix} \qquad \Rightarrow \mathbf{P}^{-1} = -\frac{1}{2}\begin{bmatrix} -4 & 2 & -1 \\ 18 & -6 & 2 \\ -16 & 4 & -1 \end{bmatrix}$$

Then

$$\bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \quad \bar{\mathbf{b}} = \mathbf{P}^{-1}\mathbf{b} = \begin{bmatrix} -1 \\ -3 \\ 6 \end{bmatrix}, \quad \bar{\mathbf{c}} = \mathbf{c}\mathbf{P} = [\,3\ 5\ 5\,]$$

The corresponding eigenvectors are chosen as

$$\boxed{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 7 \\ 12 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}}$$

The transform matrix is

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Then

$$\bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \quad \bar{\mathbf{b}} = \mathbf{P}^{-1}\mathbf{b} = \begin{bmatrix} -1 \\ -3 \\ 6 \end{bmatrix}, \quad \bar{\mathbf{c}} = \mathbf{c}\mathbf{P} = \begin{bmatrix} 3 & 5 & 5 \end{bmatrix}$$

# Obtain diagonal form by similarity transformation

Case 1: When system matrix **A** has distinct eigenvalues, it can always be transformed into the diagonal form by appropriate similarity transformation matrix **P**. That is,

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

From $\mathrm{P}^{-1}\mathrm{AP} = \Lambda$, it is easy to obtain

$$\mathrm{AP} = \mathrm{P}\Lambda$$

Assuming $$\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

where $v_i$ is the eigenvector of A corresponding to the eigenvalue $\lambda_i$.

# Obtain diagonal form by simil. trans. (cont.)

Substituting $v_i$ into $\mathrm{AP} = \mathrm{P}\Lambda$ yields

$$\mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix}$$

$$= \begin{bmatrix}\lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n\end{bmatrix}$$

Equating columns on the both sides of the above equation yields

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

This equation can be put in the form

$$\left(\lambda_i\mathbf{I} - \mathbf{A}\right)\mathbf{v}_i = \mathbf{0}, \quad i = 1, 2, \cdots, n$$

When all the eigenvalues of $\mathbf{A}$ are distinct, we can search $n$ independent eigenvectors $v_i$. Therefore $\mathbf{P}$ must be nonsingular.

Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix} x$$

Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & 0 & \ddots & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} x$$

# Controllable canonical form for SISO systems

Given a completely controllable SISO system $(\mathbf{A}, \mathbf{b}, \mathbf{c})$, find a similarity transformation matrix $\mathbf{P}_c$, which can transform it into the controllable canonical form $(\mathbf{A}_c, \mathbf{b}_c, \mathbf{c}_c)$, i.e.

$$\mathbf{A}_c = \mathbf{P}_c^{-1}\mathbf{A}\mathbf{P}_c, \quad \mathbf{b}_c = \mathbf{P}_c^{-1}\mathbf{b}, \quad \mathbf{c}_c = \mathbf{c}\mathbf{P}_\mathbf{c}$$

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \quad \mathbf{b}_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

From the definition, we have

$$\mathbf{A}_c = \mathbf{P}_c^{-1}\mathbf{A}\mathbf{P}_c \qquad \longrightarrow \qquad \mathbf{P}_c^{-1}\mathbf{A} = \mathbf{A}_c\mathbf{P}_c^{-1}$$

Let $\mathbf{p}_{ci}$ denotes the $i$th row vector of $\mathbf{P}_c^{-1}$. Hence

$$\begin{bmatrix} \mathbf{p}_{c1} \\ \mathbf{p}_{c2} \\ \vdots \\ \mathbf{p}_{cn} \end{bmatrix} \mathbf{A} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} \mathbf{p}_{c1} \\ \mathbf{p}_{c2} \\ \vdots \\ \mathbf{p}_{cn} \end{bmatrix}$$

Expanding the above equation and comparing the two sizes, we have

$$\mathbf{p}_{c1}\mathbf{A} = \mathbf{p}_{c2}$$

$$\mathbf{p}_{c2}\mathbf{A} = \mathbf{p}_{c3}$$

$$\vdots$$

$$\mathbf{p}_{c(n-1)}\mathbf{A} = \mathbf{p}_{cn}$$

$$\mathbf{p}_{cn}\mathbf{A} = -a_0\mathbf{p}_{c1} - a_1\mathbf{p}_{c2} \cdots - a_{n-1}\mathbf{p}_{cn}$$

If $\mathbf{p}_{c1}$ is known, then we can work out $\mathbf{p}_{c2}, \mathbf{p}_{c3}, \cdots, \mathbf{p}_{cn}$ successively.

Next, we will use equation $\mathbf{b}_c = \mathbf{P}_c^{-1}\mathbf{b}$ to obtain $\mathbf{p}_{c1}$.

$$\mathbf{P}_c^{-1}\mathbf{b} = \mathbf{b}_c \quad \longrightarrow \quad \begin{bmatrix} \mathbf{p}_{c1} \\ \mathbf{p}_{c2} \\ \vdots \\ \mathbf{p}_{cn} \end{bmatrix} \cdot \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Then we have

$$\begin{aligned} \mathbf{p}_{c1}\mathbf{b} &= 0 \\ \mathbf{p}_{c2}\mathbf{b} = \mathbf{p}_{c1}\mathbf{A}\mathbf{b} &= 0 \\ &\vdots \\ \mathbf{p}_{c(n-1)}\mathbf{b} = \mathbf{p}_{c1}\mathbf{A}^{n-2}\mathbf{b} &= 0 \\ \mathbf{p}_{cn}\mathbf{b} = \mathbf{p}_{c1}\mathbf{A}^{n-1}\mathbf{b} &= 1 \end{aligned}$$

$$\therefore \quad \mathbf{p}_{c1}\begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Therefore, we have

$$\begin{aligned}\mathbf{p}_{c1} &= \begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b}\end{bmatrix}^{-1} \\ &= \begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}\mathbf{Q}_c^{-1}\end{aligned}$$

$$\begin{aligned}\mathbf{p}_{c1}\mathbf{A} &= \mathbf{p}_{c2} \\ \mathbf{p}_{c2}\mathbf{A} &= \mathbf{p}_{c3} \\ &\vdots \\ \mathbf{p}_{c(n-1)}\mathbf{A} &= \mathbf{p}_{cn}\end{aligned} \qquad \Longrightarrow \qquad \mathbf{P}_c^{-1} = \begin{bmatrix}\mathbf{p}_{c1} \\ \mathbf{p}_{c1}\mathbf{A} \\ \mathbf{p}_{c1}\mathbf{A}^2 \\ \vdots \\ \mathbf{p}_{c1}\mathbf{A}^{n-1}\end{bmatrix}$$

## Similarity transformation matrix for controllable canonical form

Given a completely controllable SISO system $\Sigma : (\mathbf{A}, \mathbf{b}, \mathbf{c})$, the following similarity transformation matrix $\mathbf{P}_c$ can transform $\Sigma$ into the controllable canonical form $\Sigma_c : (\mathbf{A}_c, \mathbf{b}_c, \mathbf{c}_c)$.

$$\mathbf{P}_c^{-1} = \begin{bmatrix} \mathbf{p}_{c1} \\ \mathbf{p}_{c1}\mathbf{A} \\ \mathbf{p}_{c1}\mathbf{A}^2 \\ \vdots \\ \mathbf{p}_{c1}\mathbf{A}^{n-1} \end{bmatrix}$$

where

$$\mathbf{p}_{c1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{Q}_c^{-1}$$

## Example

Consider a system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Find the similar transformation matrix $\mathbf{P}_c$ which can transform the system into the controllable canonical form.

### Solutions

(1) Check the controllability of the system

$$\mathbf{Q}_c = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

$\because \operatorname{rank}(\mathbf{Q}_c) = 3$

$\therefore$ The system is controllable and can be transformed into the controllable canonical form.

(2) Compute $\mathbf{p}_{c1}$.

$$\mathbf{Q}_c^{-1} = \frac{1}{5}\begin{bmatrix} 5 & 5 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{2}{5} \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$\therefore\ \mathbf{p}_{c1} = [\,0\ 0\ 1\,]\,\mathbf{Q}_c^{-1} = [\,0\ 0\ \tfrac{1}{5}\,]$$

(3) Compute $\mathbf{P}_c^{-1}$ and $\mathbf{P}_c$.

$$\mathbf{P}_c^{-1} = \begin{bmatrix} \mathbf{p}_{c1} \\ \mathbf{p}_{c1}\mathbf{A} \\ \mathbf{p}_{c1}\mathbf{A}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}, \quad \mathbf{P}_c = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{bmatrix}$$

(4) Compute the controllable canonical form.

$$\bar{\mathbf{A}}_c = \mathbf{P}_c^{-1}\mathbf{A}\mathbf{P}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad \mathbf{b}_c = \mathbf{P}_c^{-1}\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

# Observable canonical form for SISO systems

Given a completely controllable SISO system (A, b, c), find a similarity transformation matrix $P_o$, which can transform it into the controllable canonical form $(\mathrm{A}_o, \mathrm{b}_o, \mathrm{c}_o)$, i.e.

$$\mathrm{A}_o = P_o^{-1}\mathrm{A}P_o,\ b_o = P_o^{-1}b,\ c_o = cP_o$$

$$\mathrm{A}_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix}, \quad c_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

# Observable canonical form for SISO systems

## Similarity transformation matrix for observable canonical form

Given a completely observable SISO system $\Sigma : (\mathbf{A}, \mathbf{b}, \mathbf{c})$, the following similarity transformation matrix $\mathbf{P}_o$ can transform $\Sigma$ into the observable canonical form $\Sigma_o : (\mathbf{A}_o, \mathbf{b}_o, \mathbf{c}_o)$.

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{p}_{o1} & \mathbf{A}\mathbf{p}_{o1} & \mathbf{A}^2\mathbf{p}_{o1} & \cdots & \mathbf{A}^{n-1}\mathbf{p}_{o1} \end{bmatrix}$$

where

$$\mathbf{p}_{o1} = \mathbf{Q}_o^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \vdots \\ \mathbf{c}^{n-1}\mathbf{A} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

## Example

Consider a system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find the similar transformation matrix $p_o$ which can transform the original state equation into observable canonical form.

### Solutions

1) Check the observability of the system.

$$Q_o = \begin{bmatrix} c \\ c\mathrm{A} \\ c\mathrm{A}^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & -1 \\ 4 & 1 & 9 \end{bmatrix}, \quad rank(Q_o) = 3.$$

2) The inverse of $Q_o$ should be calculated in order to determine its last column.

$$\because P_{o1} = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & -1 \\ 4 & 1 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{1}{32} \begin{bmatrix} * & * & -2 \\ * & * & 3 \\ * & * & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{16} \\ -\frac{3}{32} \\ \frac{3}{32} \end{bmatrix}$$

$$P_o = \begin{bmatrix} P_{o1} & AP_{o1} & A^2 P_{o1} \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 2 & 8 & 0 \\ -3 & 4 & 16 \\ 3 & -4 & 16 \end{bmatrix}$$

$$\therefore P_o^{-1} = \begin{bmatrix} 4 & -4 & 4 \\ 3 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\therefore A_o = P_o^{-1} A P_o = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & 5 \\ 0 & 1 & 0 \end{bmatrix}, c_o = cP_o = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

系统经过非奇异线性变换，系统的特征值、传递矩阵、可控性、可观测性等重要性质均保持不变。下面进行证明。

（1）变换后系统传递矩阵不变。

（2）变换后系统特征值不变。

（3）变换后系统可控性不变。

（4）变换后系统可观测性不变。

系统经过非奇异线性变换，系统的特征值、传递矩阵、可控性、可观测性等重要性质均保持不变。下面进行证明。

(1) 变换后系统传递矩阵不变。

**证明**　列出变换后系统传递矩阵 $\bar{G}$ 为

$$
\begin{aligned}
\bar{G} = & CP(sI - P^{-1}AP)^{-1}P^{-1}B + D = \\
& CP(P^{-1}sIP - P^{-1}AP)^{-1}P^{-1}B + D = \\
& CP[P^{-1}(sI - A)P]^{-1}P^{-1}B + D = \\
& CPP^{-1}(sI - A)^{-1}PP^{-1}B + D = C(sI - A)^{-1}B + D = G
\end{aligned}
$$

(2) 变换后系统特征值不变。

**证明**　列出变换后系统的特征多项式，即

$$\begin{aligned}|\lambda \boldsymbol{I}-\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}| &= |\lambda \boldsymbol{P}^{-1}\boldsymbol{P}-\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}| = |\boldsymbol{P}^{-1}\lambda \boldsymbol{I}\boldsymbol{P}-\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}| = \\ &|\boldsymbol{P}^{-1}(\lambda \boldsymbol{I}-\boldsymbol{A})\boldsymbol{P}| = |\boldsymbol{P}^{-1}||\lambda \boldsymbol{I}-\boldsymbol{A}||\boldsymbol{P}| = \\ &|\boldsymbol{P}^{-1}||\boldsymbol{P}||\lambda \boldsymbol{I}-\boldsymbol{A}| = |\boldsymbol{I}|\cdot|\lambda \boldsymbol{I}-\boldsymbol{A}| = |\lambda \boldsymbol{I}-\boldsymbol{A}|\end{aligned}$$

表明变换前、后的特征多项式相同，故特征值不变。由此可以推出，非奇异变换后，系统的稳定性不变。

(3) 变换后系统可控性不变。

**证明**　列出变换后系统可控性矩阵的秩为

$$\begin{aligned}\operatorname{rank}\boldsymbol{S}_4=&\operatorname{rank}[\boldsymbol{P}^{-1}\boldsymbol{B}\quad(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})\boldsymbol{P}^{-1}\boldsymbol{B}\quad(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})^2\boldsymbol{P}^{-1}\boldsymbol{B}\quad\cdots\quad(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})^{n-1}\boldsymbol{P}^{-1}\boldsymbol{B}]=\\&\operatorname{rank}[\boldsymbol{P}^{-1}\boldsymbol{B}\quad\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{B}\quad\boldsymbol{P}^{-1}\boldsymbol{A}^2\boldsymbol{B}\quad\cdots\quad\boldsymbol{P}^{-1}\boldsymbol{A}^{n-1}\boldsymbol{B}]=\\&\operatorname{rank}\boldsymbol{P}^{-1}[\boldsymbol{B}\quad\boldsymbol{A}\boldsymbol{B}\quad\boldsymbol{A}^2\boldsymbol{B}\quad\cdots\quad\boldsymbol{A}^{n-1}\boldsymbol{B}]=\operatorname{rank}[\boldsymbol{B}\quad\boldsymbol{A}\boldsymbol{B}\quad\boldsymbol{A}^2\boldsymbol{B}\quad\cdots\quad\boldsymbol{A}^{n-1}\boldsymbol{B}]\end{aligned}$$

表明变换前、后的可控性矩阵的秩相同，故可控性不变。

(4) 变换后系统可观测性不变。

**证明** 列出变换后可观测性矩阵的秩为

$$\begin{aligned}\operatorname{rank}\boldsymbol{V}_2 &= \operatorname{rank}\left[(\boldsymbol{CP})^{\mathrm{T}} \quad (\boldsymbol{P}^{-1}\boldsymbol{AP})^{\mathrm{T}}(\boldsymbol{CP})^{\mathrm{T}} \quad \cdots \quad ((\boldsymbol{P}^{-1}\boldsymbol{AP})^{n-1})^{\mathrm{T}}(\boldsymbol{CP})^{\mathrm{T}}\right] = \\ &\operatorname{rank}\left[\boldsymbol{P}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}} \quad \boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}} \quad \cdots \quad \boldsymbol{P}^{\mathrm{T}}(\boldsymbol{A}^{n-1})^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\right] = \\ &\operatorname{rank}\boldsymbol{P}^{\mathrm{T}}\left[\boldsymbol{C}^{\mathrm{T}} \quad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}} \quad \cdots \quad (\boldsymbol{A}^{n-1})^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\right] = \\ &\operatorname{rank}\left[\boldsymbol{C}^{\mathrm{T}} \quad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}} \quad \cdots \quad (\boldsymbol{A}^{n-1})^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\right]\end{aligned}$$

表明变换前、后可观测性矩阵的秩相同,故可观测性不变。

## G(s)与系统可控性和可观测性的关系

设 $\dot{x} = Ax + bu \qquad y = \mathbf{c}x$ (单输入单输出)

$$G(s) = \mathbf{c}(sI - A)^{-1}\mathbf{b}$$

$$= \mathbf{c}\frac{adj(sI - A)}{\left|sI - A\right|}\mathbf{b} = \frac{N(s)}{D(s)}$$

**定理: 系统能控能观的充要条件是G(s)中没有零极点对消**

例子：写出以下传递函数的能控标准型。

$$G(s)=\frac{s^2+4s+5}{s^3+6s^2+11s+6}$$

[解]： $$G(s)=\frac{s^2+4s+5}{s^3+6s^2+11s+6}=\frac{(s+2)^2+1}{(s+3)(s+2)(s+1)}$$

无零极点相约，故能控且能观测。可以化为能控标准型。

所以： $a_0=6, a_1=11, a_2=6$

$\beta_0=5, \beta_1=4, \beta_2=1$

能控标准型为：

$$\mathbf{A}_o=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}, \qquad \mathbf{B}_o=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\mathbf{C}_o=[\beta_0 \quad \beta_1 \quad \beta_2]=[5 \quad 4 \quad 1]$$

例子：写出以下传递函数的能观测标准型。

$$G(s)=\frac{s^2+4s+5}{s^3+6s^2+11s+6}$$

[解]：

$$G(s)=\frac{s^2+4s+5}{s^3+6s^2+11s+6}=\frac{(s+2)^2+1}{(s+3)(s+2)(s+1)}$$

无零极点相约，故能控且能观测,可以化为能观测标准型。

所以：
$$\alpha_0=6,\alpha_1=11,\alpha_2=6$$
$$\beta_0=5,\beta_1=4,\beta_2=1$$

能观测标准型为：

$$A_o=\begin{bmatrix}0 & 0 & -\alpha_0\\ 1 & 0 & -\alpha_1\\ 0 & 1 & -\alpha_2\end{bmatrix}=\begin{bmatrix}0 & 0 & -6\\ 1 & 0 & -11\\ 0 & 1 & -6\end{bmatrix},\qquad B_o=\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix}=\begin{bmatrix}5\\ 4\\ 1\end{bmatrix}$$

$$C_o=[0\quad 0\quad 1]$$

单输入-单输出系统可控、可观测的充分必要条件是由动态方程导出的传递函数不存在零、极点对消(即传递函数不可约),系统可控的充分必要条件是$(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}$不存在零、极点对消,系统可观测的充分必要条件是$\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^{-1}$不存在零、极点对消。

以上判据仅适用于单输入-单输出系统,对多输入-多输出系统一般不适用。

由不可约传递函数列写的动态方程一定是可控、可观测的,不能反映系统中可能存在的不可控和不可观测的特性。由动态方程导出可约传递函数时,表明系统或是可控、不可观测的,或是可观测、不可控的,或是不可控、不可观测的,三者必居其一;反之亦然。

传递函数可约时,传递函数分母阶次将低于系统特征方程阶次。若对消掉的是系统的一个不稳定特征值,便可能掩盖了系统固有的不稳定性而误认为系统稳定。通常说用传递函数描述系统特性不完全,就是指它可能掩盖系统的不可控性、不可观测性及不稳定性。只有当系统是可控又可观测时,传递函数描述与状态空间描述才是等价的。

(1) $\dot{\boldsymbol{x}}=\begin{bmatrix}0 & 1\\ 2.5 & -1.5\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\ 1\end{bmatrix}u,\quad y=\begin{bmatrix}2.5 & 1\end{bmatrix}\boldsymbol{x}$

(2) $\dot{\boldsymbol{x}}=\begin{bmatrix}0 & 2.5\\ 1 & -1.5\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}2.5\\ 1\end{bmatrix}u,\quad y=\begin{bmatrix}0 & 1\end{bmatrix}\boldsymbol{x}$

(3) $\dot{\boldsymbol{x}}=\begin{bmatrix}1 & 0\\ 0 & -2.5\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}1\\ 0\end{bmatrix}u,\quad y=\begin{bmatrix}1 & 0\end{bmatrix}\boldsymbol{x}$

**解**　三个系统的传递函数均为 $G(s)=\dfrac{Y(s)}{U(s)}=\dfrac{s+2.5}{(s+2.5)(s-1)}$，存在零、极点对消。

(1) 系统 $\boldsymbol{A},\boldsymbol{b}$ 矩阵为可控标准型，故可控、不可观测。

(2) 系统 $\boldsymbol{A},\boldsymbol{c}$ 矩阵为可观测标准型，故可观测、不可控。

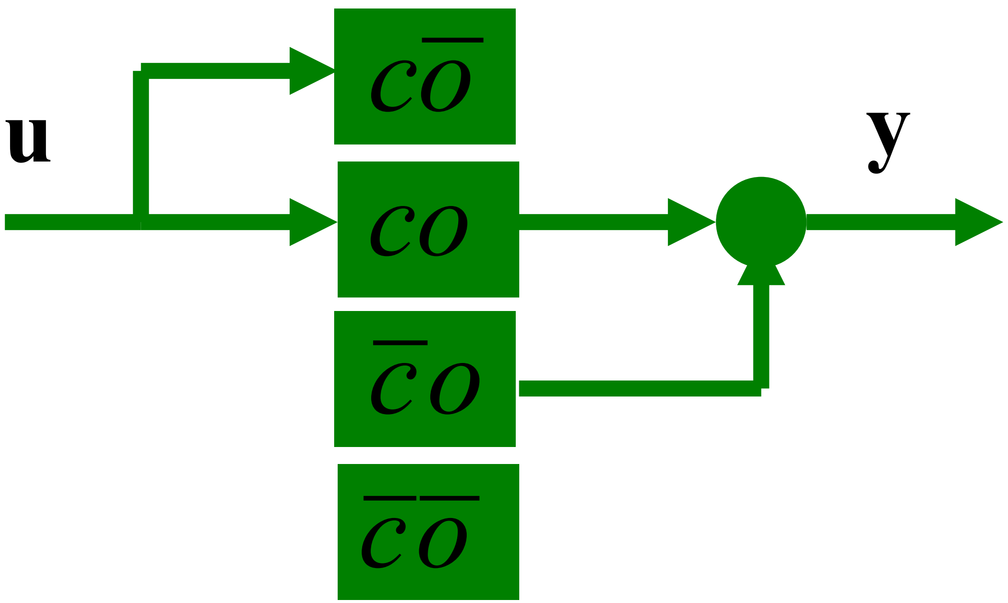
(3) 由系统 $\boldsymbol{A}$ 矩阵对角化时的可控、可观测判据可知，系统不可控、不可观测，$x_2$ 为不可控、不可观测的状态变量。

# Outline of Chapter 3

# 系统的结构分解

| | | | | |
|---|---|---|---|---|
| 系统状态变量 | 可控 | 可观 | 可控可观 | $\mathbf{x}_{co}$ |
| | | 不可观 | 可控不可观 | $\mathbf{x}_{c\bar{o}}$ |
| | 不可控 | 可观 | 不可控可观 | $\mathbf{x}_{\bar{c}o}$ |
| | | 不可观 | 不可控不可观 | $\mathbf{x}_{\bar{c}\bar{o}}$ |

## 系统的结构分解




依据可控可观性，将系统分解为四个子系统

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

特殊的线性变换

$$\mathbf{x} \Rightarrow \begin{bmatrix} \mathbf{x}_{co}^T & \mathbf{x}_{c\bar{o}}^T & \mathbf{x}_{\bar{c}o}^T & \mathbf{x}_{\bar{c}\bar{o}}^T \end{bmatrix}^T = \begin{bmatrix} \mathbf{x}_{co} \\ \mathbf{x}_{c\bar{o}} \\ \mathbf{x}_{\bar{c}o} \\ \mathbf{x}_{\bar{c}\bar{o}} \end{bmatrix}$$

分解步骤：

1. 将系统分解成可控与不可控子系统；

2. 分别将两个子系统分解成可观与不可观子系统.

## 按能控性分解

目的：将系统显性分解为能控和不能控两部分.

如果线性定常系统：$\begin{cases}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x}\end{cases}$

是状态不完全能控的, 它的能控性判别矩阵的秩

$$\boxed{rank\mathrm{Q}_c = n_1 < n}$$

则存在非奇异变换： $\mathbf{x} = P_c \hat{\mathbf{x}}$

将状态空间描述变换为： $\begin{cases} \dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\ \mathbf{y} = \hat{\mathbf{C}}\hat{\mathbf{x}} \end{cases}$

其中：

$$\hat{\mathbf{x}} = \begin{array}{c} \left[\begin{array}{c} \hat{\mathbf{x}}_1 \\ \hdashline \hat{\mathbf{x}}_2 \end{array}\right] \end{array} \begin{array}{l} \}\ n_1 \\ \}\ n-n_1 \end{array}$$

$$\hat{\mathbf{A}} = P_c^{-1}\mathbf{A}P_c = \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{n_1} & \overbrace{\quad}^{n-n_1} \end{array} \\ \left[\begin{array}{c:c} \hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{12} \\ \hdashline \mathbf{0} & \hat{\mathbf{A}}_{22} \end{array}\right] \end{array} \begin{array}{l} \\ \}\ n_1 \\ \}\ n-n_1 \end{array}$$

$$\hat{\mathbf{B}} = P_c^{-1}\mathbf{B} = \left[\begin{array}{c} \hat{\mathbf{B}}_1 \\ \hdashline \mathbf{0} \end{array}\right] \begin{array}{l} \}\ n_1 \\ \}\ n-n_1 \end{array}$$

$$\hat{\mathbf{C}} = \mathbf{C}P_c = \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{n_1} & \overbrace{\quad}^{n-n_1} \end{array} \\ \left[\begin{array}{c:c} \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 \end{array}\right] \end{array}$$

非奇异变换阵： $P_c = \begin{bmatrix} P_1 & \cdots & P_{n1} & \cdots & P_n \end{bmatrix}$

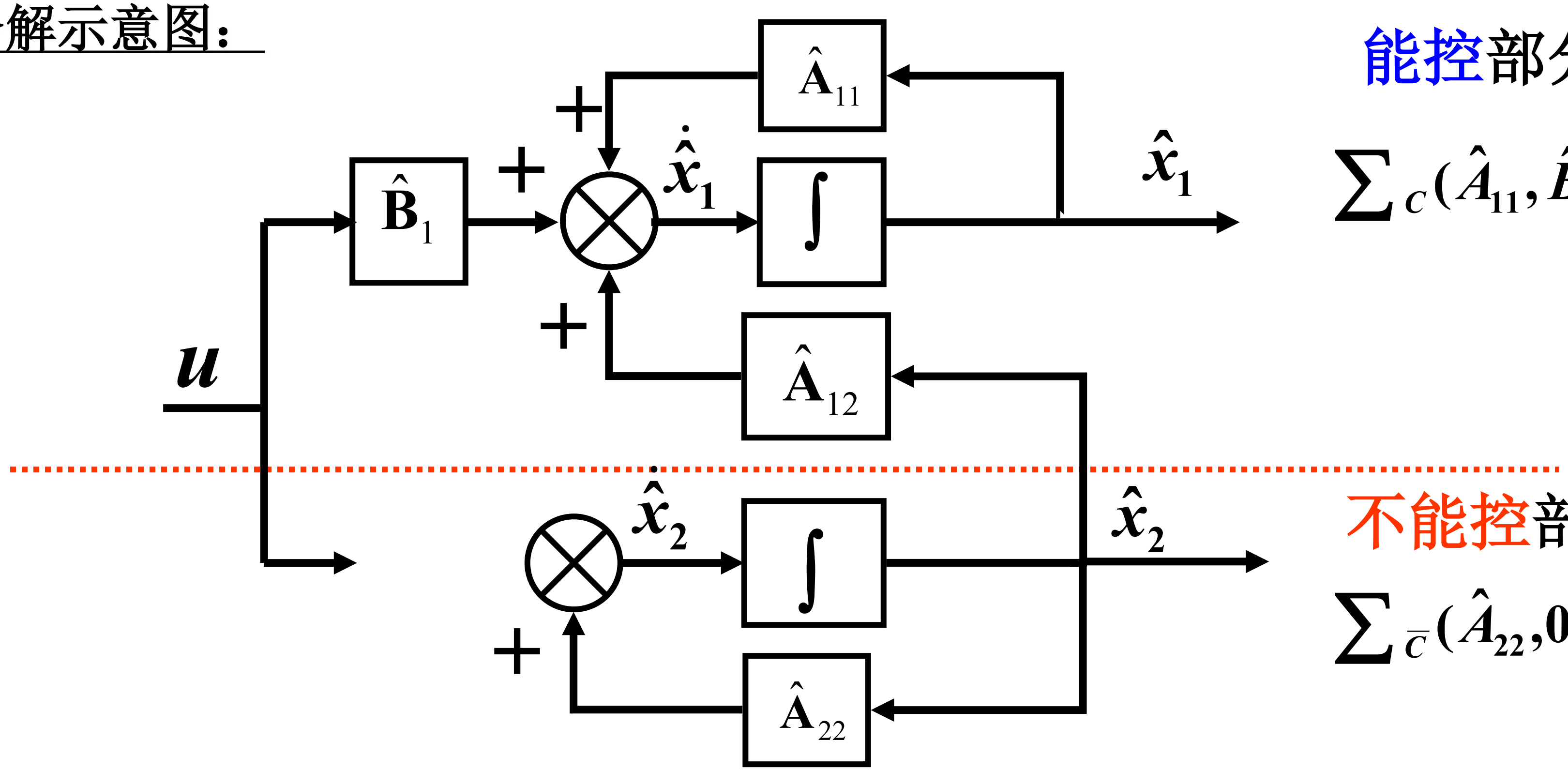
前$n_1$列为Q中$n_1$个线性无关的列，其余列在保证$P_c$非奇异下任选。

其中 $(\hat{\mathbf{A}}_{11},\hat{\mathbf{B}}_1,\hat{\mathbf{C}}_1)$ 是$n_1$维能控部分：$\dot{\hat{\mathbf{x}}}_1=\hat{\mathbf{A}}_{11}\hat{\mathbf{x}}_1+\hat{\mathbf{A}}_{12}\hat{\mathbf{x}}_2+\hat{\mathbf{B}}_1\mathbf{u}$

其中 $(\hat{\mathbf{A}}_{22},\mathbf{0},\hat{\mathbf{C}}_2)$ 是$n-n_1$维不能控部分：$\dot{\hat{\mathbf{x}}}_2=\hat{\mathbf{A}}_{22}\hat{\mathbf{x}}_2$

u不能直接控制 $\hat{\mathbf{x}}_2$ ，而 $\hat{\mathbf{x}}_2$ 未来信息中又不含 $\hat{\mathbf{x}}_1$ 的信息。

能控性分解示意图：




能控部分

$\sum_C(\hat{A}_{11},\hat{B}_1,\hat{C}_1)$

不能控部分

$\sum_{\bar{C}}(\hat{A}_{22},0,\hat{C}_2)$

**例1：对以下系统进行可控性分解。**

$$\mathbf{A}=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & -4 & 3\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad \mathbf{c}=\begin{bmatrix}1 & 1 & -1\end{bmatrix}$$

**解： 可控性矩阵**

$$Q=\begin{bmatrix}\mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b}\end{bmatrix}=\begin{bmatrix}0 & -1 & -4\\ 0 & 0 & 0\\ 1 & 3 & 8\end{bmatrix}$$

$$rank\mathbf{M}=2=n_1<\dim\mathbf{A}=n=3$$ **不可控**

**构造变换矩阵**

$$\mathbf{R}_c=\begin{bmatrix}0 & -1 & 0\\ 0 & 0 & 1\\ 1 & 3 & 0\end{bmatrix}$$

✓与前2个列向量线性无关；
✓尽可能简单

$$P_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix} \qquad P_c^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\hat{\mathbf{A}} = P_c^{-1}\mathbf{A}P_c = \begin{bmatrix} \hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{12} \\ \mathbf{0} & \hat{\mathbf{A}}_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & -4 & 2 \\ 1 & 4 & -2 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$\hat{\mathbf{b}} = P_c^{-1}\mathbf{b} = \begin{bmatrix} \hat{\mathbf{b}}_1 \\ 0 \end{bmatrix} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] \qquad \hat{\mathbf{c}} = \mathbf{c}P_c = \begin{bmatrix} \hat{\mathbf{c}}_1 & \hat{\mathbf{c}}_2 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 2 & -1 \end{array}\right]$$

**可控子系统**

$$\begin{cases} \dot{\hat{\mathbf{x}}}_1 = \begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix}\hat{\mathbf{x}}_1 + \begin{bmatrix} 2 \\ -2 \end{bmatrix}\hat{\mathbf{x}}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u \\ y_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}\hat{\mathbf{x}}_1 \end{cases}$$

**不可控子系统**

$$\begin{cases} \dot{\hat{\mathbf{x}}}_2 = \hat{\mathbf{x}}_2 \\ y_2 = -\hat{\mathbf{x}}_2 \end{cases}$$

# 按能观测性分解

**目的**：将系统显性地分解为能观测和不能观测两部分。

观测器设计基础。

如果线性定常系统：$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$ 是状态不完全能观测的，

它的能观测性判别矩阵的秩：$rankQ_o = n_1 < n$

则存在非奇异变换：$\mathbf{x} = P_o \tilde{\mathbf{x}}$

将状态空间描述变换为：$\begin{cases} \dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u} \\ \mathbf{y} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} \end{cases}$

其中：$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{\mathbf{x}}_1 \\ \tilde{\mathbf{x}}_2 \end{bmatrix} \begin{matrix} \}\, n_1 \\ \}\, n - n_1 \end{matrix}$

$$\tilde{A} = P_o^{-1} A P_o = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad}^{n_1} & \overbrace{\qquad}^{n-n_1} \end{array} \\ \left[\begin{array}{c:c} \tilde{A}_{11} & 0 \\ \hdashline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] \end{array} \begin{array}{l} \}\, n_1 \\ \}\, n-n_1 \end{array}$$

$$\tilde{B} = P_o^{-1} B = \left[\begin{array}{c} \tilde{B}_1 \\ \hdashline \tilde{B}_2 \end{array}\right] \begin{array}{l} \}\, n_1 \\ \}\, n-n_1 \end{array}$$

$$\tilde{C} = C P_o = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad}^{n_1} & \overbrace{\qquad}^{n-n_1} \end{array} \\ \left[\begin{array}{c:c} \tilde{C}_1 & 0 \end{array}\right] \end{array}$$

**非奇异变换阵：** $$P_o^{-1} = \begin{bmatrix} R_1 \\ \vdots \\ R_{n1} \\ \vdots \\ R_n \end{bmatrix}$$

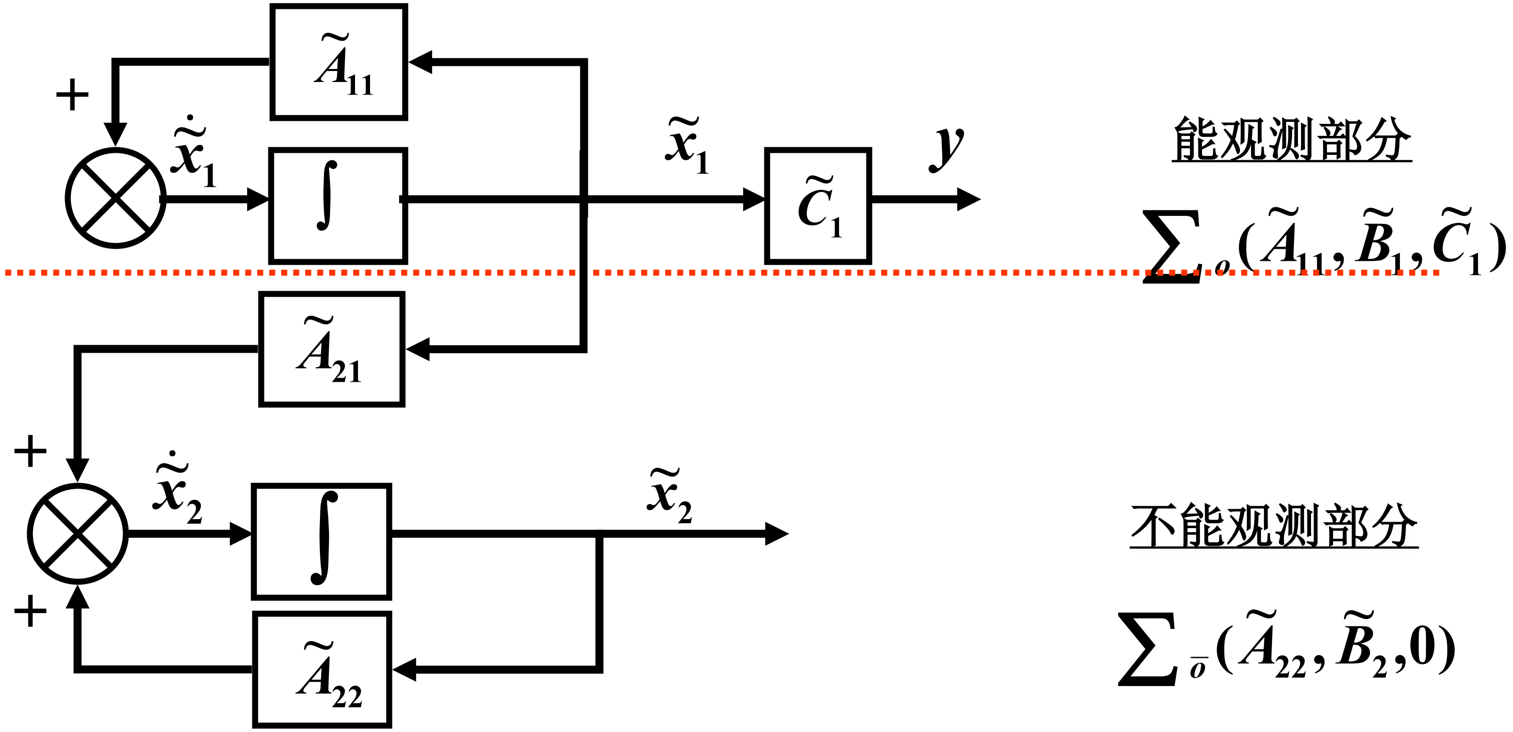
**前 $n_1$ 列为 N 中 $n_1$ 个线性无关的行，其余行在保证 $P_o$ 非奇异下任选。**

其中 $(\tilde{A}_{11},\tilde{B}_1,\tilde{C}_1)$ 是$n_1$维能观测部分：$\dot{\tilde{x}}_1=\tilde{A}_{11}\tilde{x}_1+\tilde{B}_1u,\ y=\tilde{C}_1\tilde{x}_1$

其中 $(\tilde{A}_{22},\tilde{B}_2,0)$ 是$n$-$n_1$维不能观测部分：$\dot{\tilde{x}}_2=\tilde{A}_{21}\tilde{x}_1+\tilde{A}_{22}\tilde{x}_2+\tilde{B}_2u,\ y\equiv 0$

$\tilde{x}_2$ 对y没有直接影响，而 $\tilde{x}_1$ 中又不含 $\tilde{x}_2$ 的信息。

能观测性分解示意图：



## 三、按能控性和能观性进行分解

（系统的标准分解）

假设系统 $\dot{x} = Ax + Bu \quad y = Cx$

不完全能控也不完全能观

---

$$(1)x = P_c \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix}$$

能控性分解

$$(2)x_c = P_{o1}^{-1} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix}$$

能控子系统能观性分解

$$(3)x_{\bar{c}} = P_{o2}^{-1} \begin{bmatrix} x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}$$

不能控子系统能观性分解

$$x = P_c \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} = \begin{bmatrix} P_c x_c \\ P_c x_{\bar{c}} \end{bmatrix} = \begin{bmatrix} P_c P_{o1}^{-1} x_{co} \\ P_c P_{o1}^{-1} x_{c\bar{o}} \\ P_c P_{o2}^{-1} x_{\bar{c}o} \\ P_c P_{o2}^{-1} x_{\bar{c}\bar{o}} \end{bmatrix}$$

$$x = \begin{bmatrix} P_C P_{o1}^{-1} & & & \\ & P_C P_{o1}^{-1} & & \\ & & P_C P_{o2}^{-1} & \\ & & & P_C P_{o2}^{-1} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}$$

$$P \qquad x = P\bar{x} \qquad \bar{x}$$

经过$x = P\overline{x}$的线性变换后，系统化为：

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & 0 & \overline{A}_{13} & 0 \\ \overline{A}_{21} & \overline{A}_{22} & \overline{A}_{23} & \overline{A}_{24} \\ 0 & 0 & \overline{A}_{33} & 0 \\ 0 & 0 & \overline{A}_{43} & \overline{A}_{44} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \overline{B}_{1} \\ \overline{B}_{2} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = y_1 + y_2 + y_3 + y_4 = \begin{bmatrix} \overline{C}_1 & 0 & \overline{C}_3 & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}$$

$$\begin{cases}\text{能控}\\ \text{能观:}\end{cases} \quad \sum_{co}: \dot{x}_{co} = \bar{A}_{11}x_{co} + \bar{A}_{13}x_{\bar{c}\bar{o}} + \bar{B}_1 u \qquad y_1 = \bar{C}_1 x_{\bar{c}o}$$

$$\begin{cases}\text{能控}\\ \text{不能观:}\end{cases} \quad \sum_{c\bar{o}}: \begin{array}{l} \dot{x}_{c\bar{o}} = \bar{A}_{21}x_{co} + \bar{A}_{22}x_{c\bar{o}} + \bar{A}_{23}x_{\bar{c}o} + \bar{A}_{24}x_{\bar{c}\bar{o}} + \bar{B}_2 u \\ y_2 = 0 \end{array}$$

$$\begin{cases}\text{不能控}\\ \text{能观}\end{cases} \quad \sum_{\bar{c}o}: \dot{x}_{\bar{c}o} = \bar{A}_{33}x_{\bar{c}o}\ ,\ y_3 = \bar{C}_3 x_{\bar{c}o}$$

$$\begin{cases}\text{不能控}\\ \text{不能观}\end{cases} \quad \sum_{\bar{c}\bar{o}}: \dot{x}_{\bar{c}\bar{o}} = \bar{A}_{43}x_{\bar{c}o} + \bar{A}_{44}x_{\bar{c}\bar{o}}, \quad y_4 = 0$$

# Decomposition according to controllability

Suppose an $nth$-order SISO system : $\sum$(A, b, c) is not completely state controllable, say

$$\operatorname{rank} \mathbf{Q}_c = \operatorname{rank} \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} = n_1 < n$$

Define a nonsingular matrix

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \cdots & \mathbf{A}^{n_1-1}\mathbf{b} & \mathbf{q}_1 & \cdots & \mathbf{q}_{n-n_1} \end{bmatrix} \\ &\triangleq \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} \end{aligned}$$

where $(n - n_1)$ columns $q_1, \cdots, q_{n-n_1}$ are entirely arbitrary as long as the matrix P is nonsingular.

By using the similar matrix **P** defined above, the system can be transformed into $\bar{\Sigma} : (\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}})$.

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{nc} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{nc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{nc} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{\mathbf{c}}_c & \bar{\mathbf{c}}_{nc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{nc} \end{bmatrix} \quad \text{where } \bar{\mathbf{A}}_c \in \mathbb{R}^{n_1 \times n_1}, \bar{\mathbf{b}}_c \in \mathbb{R}^{n_1 \times 1}, \bar{\mathbf{c}}_c \in \mathbb{R}^{1 \times n_1}.$$

Note that, the $n_1$ dimensional subsystem $\bar{\Sigma}_c$

$$\dot{\bar{\mathbf{x}}}_c = \bar{\mathbf{A}}_c \bar{\mathbf{x}}_c + \bar{\mathbf{b}}_c u$$

$$y = \bar{\mathbf{c}}_c \bar{\mathbf{x}}_c$$

is completely state controllable.

# Features of controllability decomposition

$$\text{System } \bar{\Sigma}: \begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{nc} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{nc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{nc} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} u$$

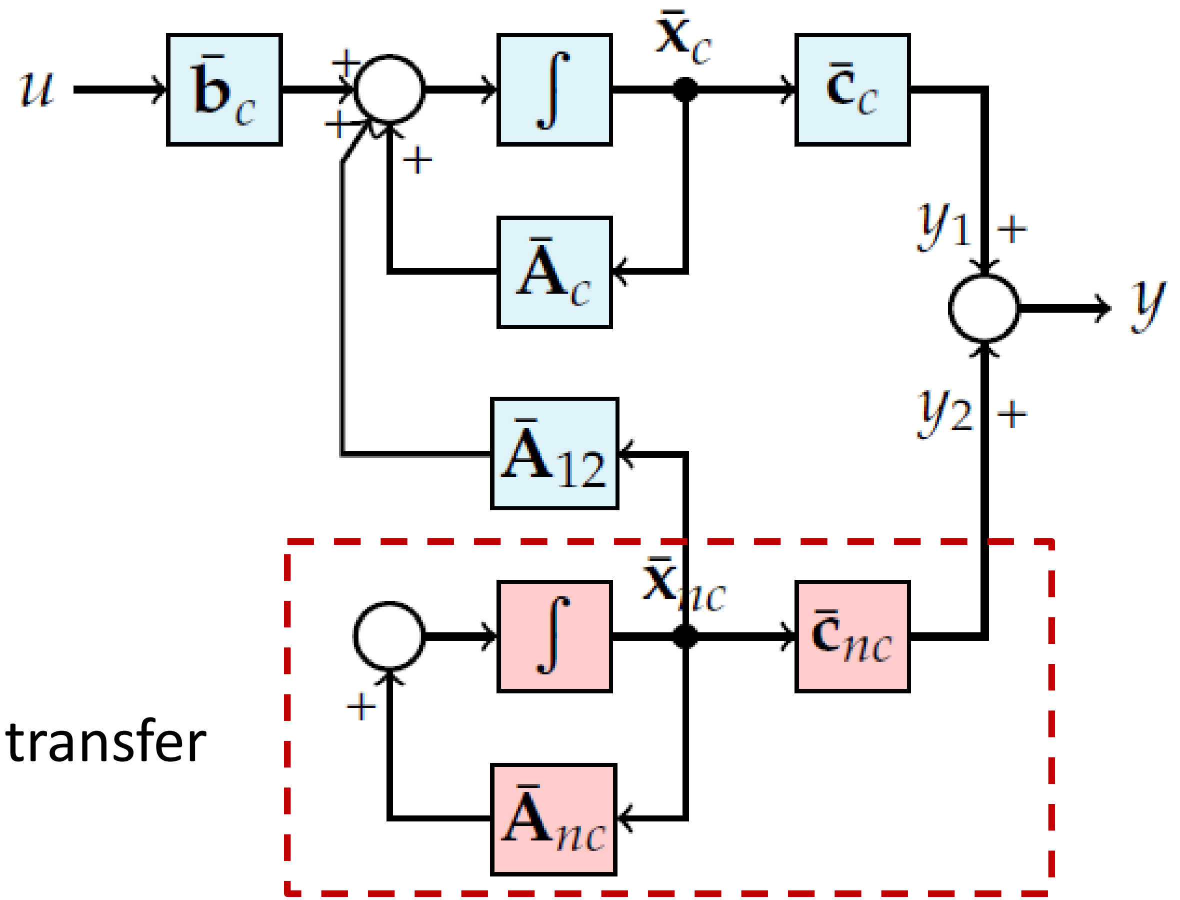
$$y = \begin{bmatrix} \bar{\mathbf{c}}_c & \bar{\mathbf{c}}_{nc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{nc} \end{bmatrix}$$

$$\text{System } \bar{\Sigma}_c: \quad \dot{\bar{\mathbf{x}}}_c = \bar{\mathbf{A}}_c \bar{\mathbf{x}}_c + \bar{\mathbf{b}}_c u$$

$$y = \bar{\mathbf{c}}_c \bar{\mathbf{x}}_c$$

$$G_{\Sigma}(s) = G_{\bar{\Sigma}}(s) = G_{\bar{\Sigma}_c}(s)$$

$$= \bar{\mathbf{c}}_c (s\mathbf{I} - \bar{\mathbf{A}}_c)^{-1} \bar{\mathbf{b}}_c$$




Note: Controllable subsystem has the same transfer function with the original system.

## Example

Consider the following state-space model

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [\,1\ 1\ 1\,]\,\mathbf{x}$$

Determine whether the system is controllable. If not, make a decomposition according to controllability.

## Solutions

Since $\mathrm{rank}(\mathbf{Q}_c) = 2 < 3$, the system is not completely controllable.

The similar matrix can be constructed as follows

$$\mathbf{P} = [\,\mathbf{b}\ \ \mathbf{Ab}\ \ \mathbf{q}_1\,] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Letting $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ yields

$$\bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{\mathbf{b}} = \mathbf{P}^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{c}} = \mathbf{c}\mathbf{P} = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$$

The controllable subsystem is

$$\dot{\bar{\mathbf{x}}}_c = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \bar{\mathbf{x}}_c + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \bar{\mathbf{x}}_c$$

The transfer function of the original system

$$G_\Sigma(s) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s-1 & -1 & 0 \\ 0 & s-1 & 0 \\ 0 & -1 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{(s-1)(s+1)}{(s-1)^3} = \frac{s+1}{(s-1)^2}$$

The transfer function of the controllable subsystem is

$$G_{\bar{\Sigma}_c}(s) = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} s & 1 \\ -1 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s+1}{(s-1)^2}$$

It is apparent that

$$G_\Sigma(s) = G_{\bar{\Sigma}_c}(s)$$

# Decomposition according to observability

Suppose an *n*th-order LTI system $\sum$ (A, b, c) is not completely state observable, i.e.

$$\operatorname{rank} \mathbf{Q}_o = \operatorname{rank} \left[ \mathbf{c}^T \ \mathbf{A}^T\mathbf{c}^T \ \left(\mathbf{A}^T\right)^2\mathbf{c}^T \ \cdots \ \left(\mathbf{A}^T\right)^{n-1}\mathbf{c}^T \right]^T = n_1 < n$$

Define a nonsingular matrix:

$$P_o^{-1} = \begin{bmatrix} C \\ \vdots \\ CA^{n_1-1} \\ \hline p_{o1} \\ \vdots \\ p_{o(n-n_1)} \end{bmatrix}$$

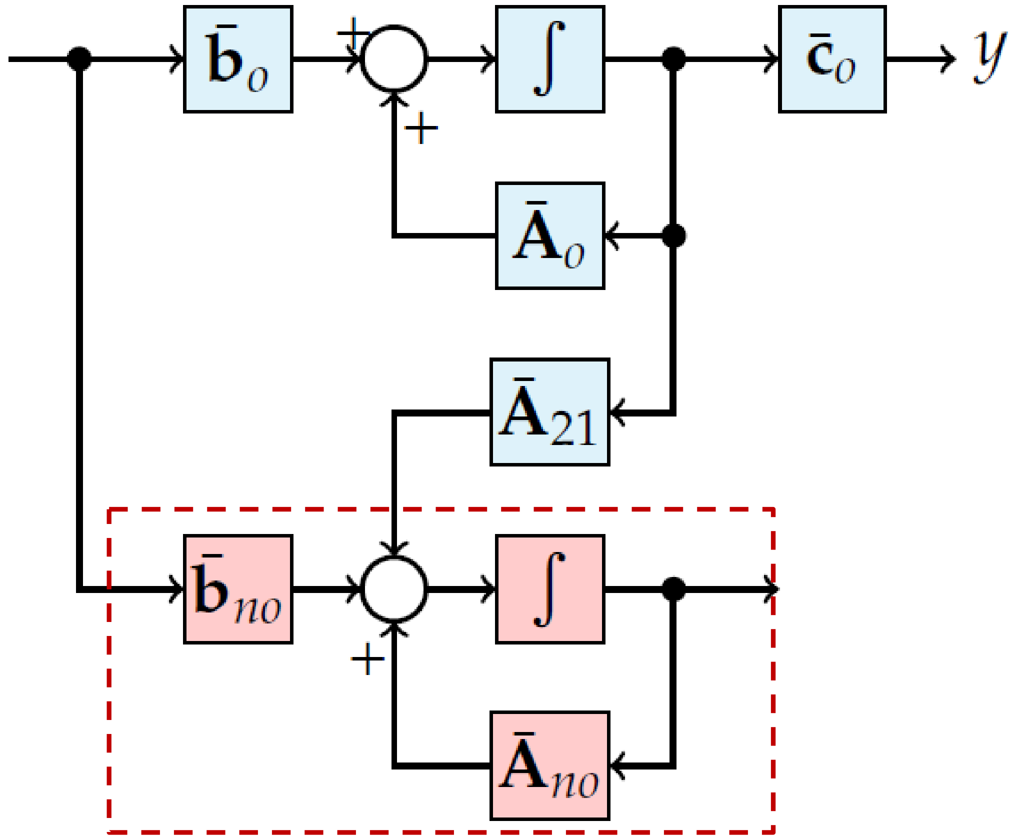
where $(n - n_1)$ row $t_1, \cdots, t_{n-n_1}$ are entirely arbitrary as long as the matrix T is nonsingular.

# Decomposition according to observability

The system$\sum$ (A, b, c) can be transformed into

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{no} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{no} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{no} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_o \\ \bar{\mathbf{b}}_{no} \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{\mathbf{c}}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{no} \end{bmatrix}$$



## Example

Consider the following state-space model

$$\dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u, \quad y = [0 \quad 1 \quad -2]x$$

Determine whether the system is observable. If not, make a decomposition according to observability.

### Solutions

$$rankQ_O = rank \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = rank \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix} = 2 < 3$$

$$\text{let } P_o^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore P_o = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

# Example

Consider the following state-space model

$$\tilde{A} = P_o^{-1} A P_o = \left[\begin{array}{cc:c} 0 & 1 & \boxed{0} \\ -1 & -2 & \boxed{0} \\ \hdashline 1 & 0 & -1 \end{array}\right] \qquad \tilde{B} = P_o^{-1} B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\tilde{C} = C P_o = [1 \quad 0 \;\vdots\; \boxed{0}]$$

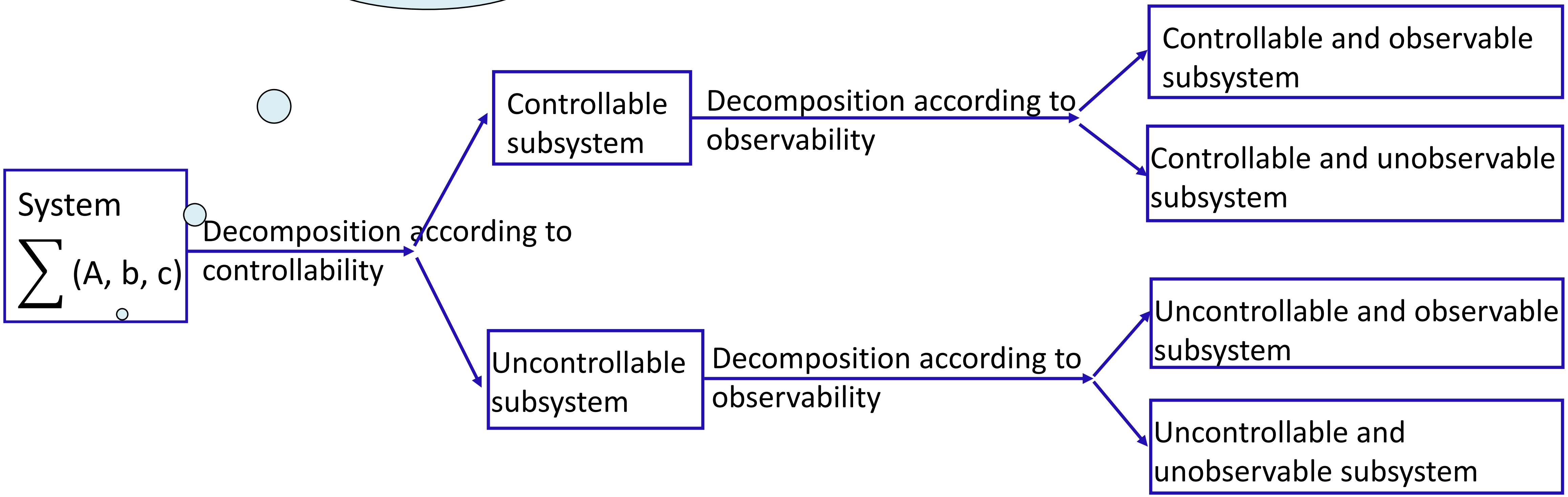
The observable subsystem is

$$\begin{bmatrix} \tilde{x}_1' \\ \tilde{x}_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = [\, 1 \quad 0 \,] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

# Canonical structure of system

Also can first make the Decomposition according to observability



# Canonical structure of system

If a system is neither completely controllable nor completely observable, the system can be decomposed as follows

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \left[\begin{array}{cc:cc} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \hdashline \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{array}\right] \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \left[\begin{array}{c} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \hdashline \mathbf{0} \\ \mathbf{0} \end{array}\right] \mathbf{u}$$

$$\mathbf{y} = \left[\begin{array}{cc:cc} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{array}\right] \bar{\mathbf{x}}$$

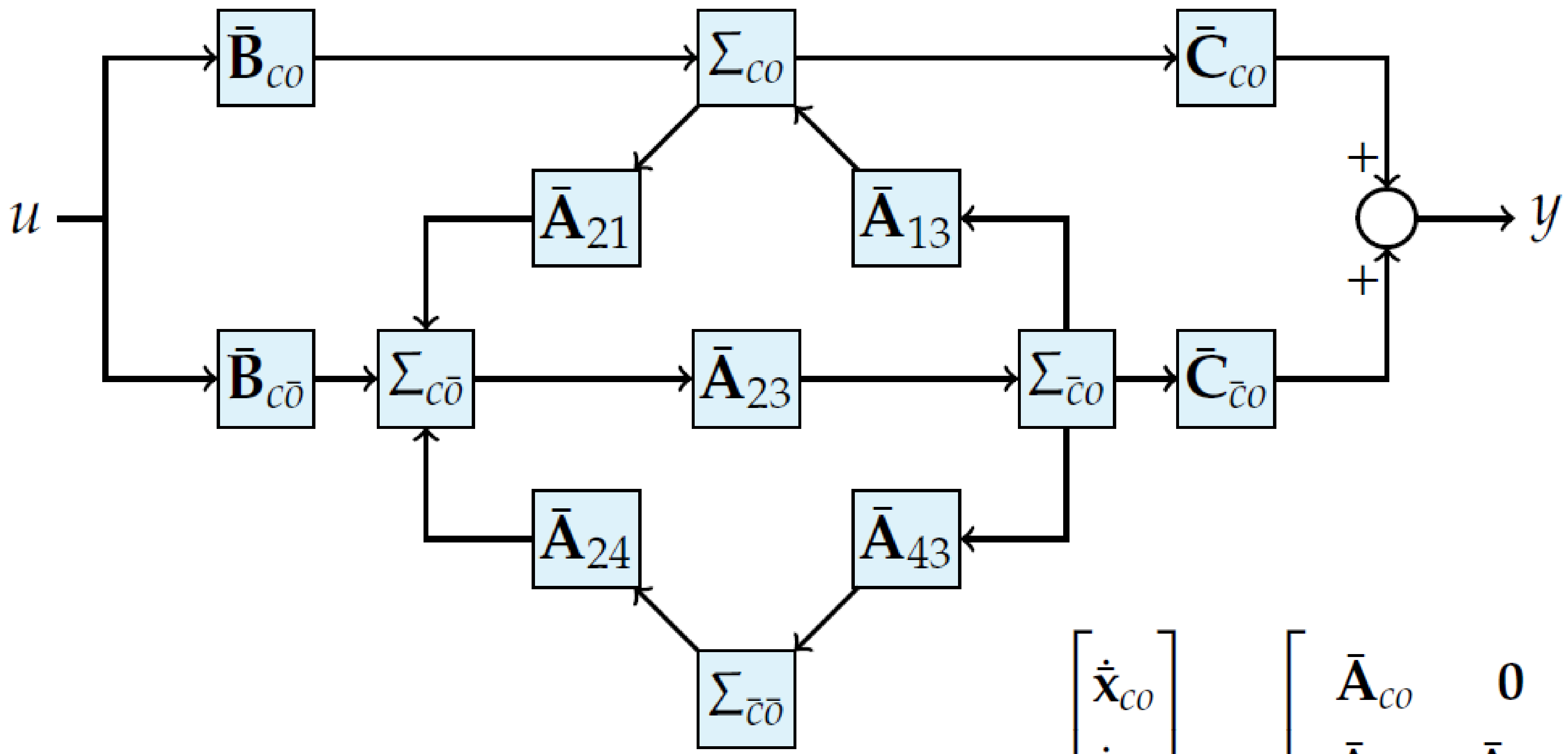
where $\bar{\mathbf{x}}_{co}$ is controllable and observable,

$\bar{\mathbf{x}}_{c\bar{o}}$ is controllable but not observable,

$\bar{\mathbf{x}}_{\bar{c}o}$ is not controllable, but observable,

$\bar{\mathbf{x}}_{\bar{c}\bar{o}}$ is neither controllable nor observable.

# Block diagram of the canonical decomposition




$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \left[\begin{array}{cc|cc} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \hline \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{array}\right] \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \left[\begin{array}{c} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \hline \mathbf{0} \\ \mathbf{0} \end{array}\right] \mathbf{u}$$

$$\mathbf{y} = \left[\begin{array}{cc|cc} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{array}\right] \bar{\mathbf{x}}$$

$$G(s) = \bar{\mathbf{c}}_{co}(s\mathbf{I} - \bar{\mathbf{A}}_{co})^{-1}\bar{\mathbf{b}}_{co}$$

# Example

Consider the following state-space model

$$\dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u, \quad y = [0 \quad 1 \quad -2] x$$

Make a decomposition according to controllability and observability.

## Solutions

1) Check the observability of the system.

$$Q_c = \left(b, Ab, A^2 b\right) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 0 & 1 & -2 \end{pmatrix}, \quad rank(Q_c) = 2 < n$$

So the system incompletely controllable.

2) $P_c = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

3) Decomposition according to controllability

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \quad y = [0 \quad -1 \quad -2] \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix}$$

4) Controllable subsystem, decomposition according to observability

$$\dot{x}_c = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x_c + \begin{bmatrix} -1 \\ -2 \end{bmatrix} x_{\bar{c}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = [1 \quad -1] x_c$$

$$\Rightarrow P_o^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} x_{\bar{c}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y_1 = [1 \quad 0] \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix}$$

5) Uncontrollable subsystem, decomposition according to observability

$$\dot{x}_{\bar{c}o} = -x_{\bar{c}o}, y = -2x_{\bar{c}o}$$

6) The system according to decomposition according to controllability and observability is

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \quad y = [1 \quad 0 \quad -2] \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \end{bmatrix}$$

# Outline of Chapter 3

# MATLAB commands used in this chapter

| Command | Description |
| --- | --- |
| canon | Canonical state-space realizations |
| ctrb | Compute the controllability matrix |
| obsv | Compute the observability matrix |
| rank | Matrix rank |

# A numerical example

## Example

Examine the controllability and observability of the system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{C} = [5\ 6\ 1]$.

## Solutions

The following MATLAB Program computes the controllability and observability of this system, which turns out to be controllable, but not observable.

```
>> A = [0 1 0;0 0 1;-6 -11 -6];
>> B = [0;0;1];
>> C = [5 6 1];
>> D = [0];
>> CONT = ctrb(A,B)
CONT =
    0 0 1
    0 1 -6
    1 -6 25
>> rank(CONT)
ans =
    3
>> OBSER = obsv(A,C)
OBSER =
    5 6 1
    -6 -6 0
    0 -6 -6
>> rank(OBSER)
ans =
    2
```

The End of Chap. 3