Chapter 2 Solution of State Equations

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2.4 Solution of nonhomogeneous state equations

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2.1 Introduction

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2.5 Simulations with MATLAB

Introduction

In Chapter 1, a linear time-invariant (LTI) system is described by

$$\begin{aligned}
&\text{State equation:} && \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0,\ t \geqslant 0 \\
&\text{Output equation:} && \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)
\end{aligned}$$

- In order to analyze the characteristics of the system, we need to solve the equations.
- The key point is to obtain the solution of the state $x(t)$ from the state equation.
- We decompose the solution of $x(t)$ into two parts:

Self-excited dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}_0 \neq \mathbf{0}$$

Forced dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}_0 = \mathbf{0}$$

Outline of Chapter 2

Solution of homogeneous state equations

For a general state-space system $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D})$. If $u(t) = 0$, we can obtain the homogenous state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

$$\mathbf{x}(0) = \mathbf{x}_0, \quad t \geqslant 0$$

- The homogenous state equation can be solved by
  - Linear algebraic approach (to be more specific, Matrix exponential approach)
  - Laplace transform approach

Matrix exponential method(con.)

$$\boxed{\dot{\mathrm{x}}(t) = \mathrm{Ax}(t)}$$

$$x(t) = \left(I + \mathrm{A}t + \frac{1}{2!}\mathrm{A}^2 t^2 + \cdots + \frac{1}{k!}\mathrm{A}^k t^k + \cdots \right) x(0)$$

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots + \frac{1}{k!}\mathbf{A}^k t^k + \cdots$$

Matrix exponential function

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t)$$

State transition matrix

$$\mathrm{x}(t) = e^{\mathrm{A}t}\mathrm{x}(0) = \Phi(t)\mathrm{x}(0)$$

Laplace transform approach

Let’s begin with the following scalar differential equation

$$\dot{x} = ax$$

Taking the Laplace transform of the equation, we obtain

$$sX(s) - x(0) = aX(s), \quad \text{where } X(s) = \mathcal{L}\left[x(t)\right]$$

$$\therefore \; X(s) = \frac{x(0)}{s-a}$$

The inverse Laplace transform of the equation gives the solution

$$x(t) = e^{at}x(0)$$

$$\boxed{\begin{aligned} \frac{1}{s-a} &= \frac{1}{s} + \frac{a}{s^2} + \frac{a^2}{s^3} + \cdots \\ e^{at} &= 1 + at + \frac{a^2t^2}{2!} + \cdots \end{aligned}}$$

$$\Downarrow$$

$$\boxed{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}}$$

Can we extend the solution to the vector differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

Taking the Laplace transform of both sides of the equation, we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s), \text{ where } \mathbf{X}(s) = \mathcal{L}\left[\mathbf{x}(t)\right]$$

$$\therefore \ (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0)$$

Premultiplying both sides by $(s\mathbf{I} - \mathbf{A})^{-1}$, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \quad \xrightarrow{\mathcal{L}^{-1}} \quad \mathbf{x}(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}(0)$$

How to calculate $\Phi(t)$, e^{At}?

- Direct solution method (definition)

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots + \frac{1}{k!}\mathbf{A}^k t^k + \cdots$$

- Laplace transform method

$$\mathbf{x}(t) = L^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{x}(0)$$

- Eigenvalue method

$$e^{At} = \mathbf{\Phi}(t) = P \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

- Cayley-Hamilton method

$$\mathrm{e}^{\mathrm{At}} = a_0(\mathrm{t})\mathrm{I} + a_1(\mathrm{t})\mathrm{A} + \cdots + a_{n-1}(\mathrm{t})\mathrm{A}^{n-1}$$

Laplace transform method

Example

Calculate the state transition matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution

$$\boldsymbol{\Phi}(t) = e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} t + \frac{1}{2!}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^2 t^2 + \frac{1}{3!}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^3 t^3 + \cdots$$

$$= \begin{bmatrix} 1 - t^2 + t^3 + \cdots & t - \frac{3}{2}t^2 - \frac{7}{6}t^3 + \cdots \\ -2t + 3t^2 - \frac{7}{3}t^3 + \cdots & 1 - 3t + \frac{7}{2}t^2 - \frac{5}{2}t^3 + \cdots \end{bmatrix}$$

Laplace transform method

Example

Calculate the state transition matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution

$$[sI - A]^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$e^{At} = L^{-1}\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix} = L^{-1}\begin{bmatrix} \dfrac{2}{s+1} + \dfrac{-1}{s+2} & \dfrac{1}{s+1} + \dfrac{-1}{s+2} \\ \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Eigenvalue method

Suppose J is the Jordan canonical form of A, i.e. $P^{-1}AP = J$

$$
\begin{aligned}
e^{\mathbf{A}t} &= e^{\mathbf{PJP^{-1}}t} \\
&= \mathbf{I} + \mathbf{PJP^{-1}}t + \frac{(\mathbf{PJP^{-1}})^2 t^2}{2!} + \cdots + \frac{(\mathbf{PJP^{-1}})^k t^k}{k!} + \cdots \\
&= \mathbf{P}\left(\mathbf{I} + \mathbf{J}t + \frac{\mathbf{J}^2 t^2}{2!} + \cdots + \frac{\mathbf{J}^k t^k}{k!} + \cdots\right)\mathbf{P}^{-1} \\
&= \mathbf{P}e^{\mathbf{J}t}\mathbf{P}^{-1}
\end{aligned}
$$

How can we use $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{J}t}\mathbf{P}^{-1}$ to simplify the calculation of $e^{\mathbf{A}t}$?

If $\mathbf{A}$ has distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, i.e.

$$\mathbf{J} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{bmatrix}$$

we can easily calculate

$$\boxed{e^{\mathbf{J}t} = \begin{bmatrix} e^{\lambda_1 t} & & & \mathbf{0} \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\lambda_n t} \end{bmatrix} \Longrightarrow e^{\mathbf{A}t} = \mathbf{P} \begin{bmatrix} e^{\lambda_1 t} & & & \mathbf{0} \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\lambda_n t} \end{bmatrix} \mathbf{P}^{-1}}$$

Example:

Calculate the state transition matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution

1) $\lambda_1 = -1, \lambda_2 = -2$

2) $P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, \qquad P^{-1} = -\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

3) $$e^{At} = Pe^{\bar{A}t}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Example:

Calculate the state transition matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$

Solution

$$|\boldsymbol{\lambda I} - \boldsymbol{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -2 & 5 & \lambda - 4 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2) = 0$$

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 2$$

$$\mathbf{P} = \begin{bmatrix} P_1 & \dfrac{dP_1}{d\lambda_1} & P_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_2 \\ \lambda_1^2 & 2\lambda_1 & \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\boldsymbol{\Phi}(t) = e^{At} = \boldsymbol{P} \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \boldsymbol{P}^{-1}$$

$$= \begin{bmatrix} -2te^t + e^{2t} & 3te^t + 2e^t - e^{2t} & -te^t - e^t + e^{2t} \\ 2(-e^t - te^t + e^{2t}) & 3te^t + 5e^t - 4e^{2t} & -te^t - 2e^t + 2e^{2t} \\ -2te^t - 4e^t + 4e^{2t} & 3te^t + 8e^t - 8e^{2t} & -te^t - 3e^t + 4e^{2t} \end{bmatrix}$$

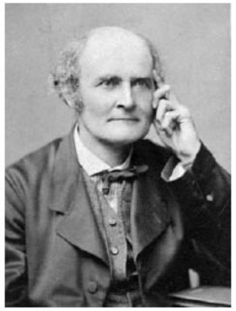
Eigenvalue method (con.)

If A has multiple eigenvalues and J has normal Jordan form, the form of e^{Jt} is much more complex. We will take an example to demonstrate such situations.

Assume that A has distinct eigenvalues λ_1 (2 multiple) and λ_2(3 multiple) and the corresponding Jordan form is

$$J = \left[\begin{array}{cc|ccc} \lambda_1 & 1 & & & \\ & \lambda_1 & & & \\ \hline & & \lambda_2 & 1 & \\ & & & \lambda_2 & 1 \\ & & & & \lambda_2 \end{array}\right] \Rightarrow e^{At} = P\left[\begin{array}{cc|ccc} e^{\lambda_1 t} & te^{\lambda_1 t} & & & \\ & e^{\lambda_1 t} & & & \\ \hline & & e^{\lambda_2 t} & te^{\lambda_2 t} & \dfrac{e^{\lambda_2 t}t^2}{2!} \\ & & & e^{\lambda_2 t} & te^{\lambda_2 t} \\ & & & & e^{\lambda_2 t} \end{array}\right]P^{-1}$$

Cayley-Hamilton method



Arthur Cayley
(British, 1821-1895)

Field Mathematian

Inst University of Cambridge

Known for Projective geometry, Group theory, Cayley - Hamilton theorem

Notable awards Copley Medal (1882)

William R. Hamilton
(Irish, 1805-1865)

Field Physicist, astronomer, and mathematician

Inst Trinity College, Dublin

Known for Hamilton's principle, Hamiltonian mechanics, Hamilton-Jacobi equation, Cayley - Hamilton theorem ...

Cayley-Hamilton theorem

Every matrix satisfies its own characteristic equation, that is, if the characteristic equation of a given matrix A is

$$f(\lambda) = |\lambda \mathrm{I} - \mathrm{A}| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0 = 0$$

then

$$f(\mathrm{A}) = \mathrm{A}^n + a_{n-1}\mathrm{A}^{n-1} + \cdots a_1\mathrm{A} + a_0\mathrm{I} = 0$$

方阵A的特征多项式是A的零化多项式。 方阵A满足自身的特征方程,无穷级数化为矩阵A的有限项之和进行计算

Compute e^{At} by the Cayley-Hamilton method

Compute e^{At} by the Cayley-Hamilton method

$$\boxed{e^{At} = a_0(t)I + a_1(t)A + \cdots + a_{n-1}(t)A^{n-1}}$$

- If A has n distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots \lambda_n$,we have n equations

$$\boxed{e^{\lambda_i t} = a_0(t) + a_1(t)\lambda_i + \cdots + a_{n-1}(t)\lambda_i^{n-1},\ i=1,2,\cdots,n}$$

If we have λ_i, Hence we can solve all the coefficient a_i

$$\boxed{e^{At} = a_0(t)I + a_1(t)A + \cdots + a_{n-1}(t)A^{n-1}}$$

$$\begin{cases} e^{\lambda_1 t} = \alpha_0(t) + \alpha_1(t)\lambda_1 + \cdots + \alpha_{n-1}(t)\lambda_1^{\ n-1} \\ e^{\lambda_2 t} = \alpha_0(t) + \alpha_1(t)\lambda_2 + \cdots + \alpha_{n-1}(t)\lambda_2^{\ n-1} \\ \qquad\qquad\qquad \vdots \\ e^{\lambda_n t} = \alpha_0(t) + \alpha_1(t)\lambda_n + \cdots + \alpha_{n-1}(t)\lambda_n^{\ n-1} \end{cases}$$

$$\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ & & \vdots & \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_0(t) \\ \alpha_1(t) \\ \vdots \\ \alpha_{n-1}(t) \end{bmatrix}$$

$$\begin{bmatrix} \alpha_0(t) \\ \alpha_1(t) \\ \vdots \\ \alpha_{n-1}(t) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ & & \vdots & \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

Examples of the Cayley-Hamilton method

Example

Calculate the state transition matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

Solutions

It is obvious that $\mathbf{A}$ has two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

From the Cayley-Hamilton theorem, we have

$$\begin{aligned} e^{\mathbf{A}t} &= a_0(t)\mathbf{I} + a_1(t)\mathbf{A} \\ e^{\lambda_1 t} &= a_0(t) + a_1(t)\lambda_1 \\ e^{\lambda_2 t} &= a_0(t) + a_1(t)\lambda_2 \end{aligned}$$

By takeing the values of λ_1 and λ_2 to the above equations, we have

$$\begin{cases} e^{-t} &= a_0(t) - a_1(t) \\ e^{-2t} &= a_0(t) - 2a_1(t) \end{cases} \longrightarrow \begin{cases} a_0(t) &= 2e^{-t} - e^{-2t} \\ a_1(t) &= e^{-t} - e^{-2t} \end{cases}$$

Hence

$$\begin{aligned} e^{\mathbf{A}t} &= a_0(t)\mathbf{I} + a_1(t)\mathbf{A} \\ &= \left(2e^{-t} - e^{-2t}\right) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(e^{-t} - e^{-2t}\right) \cdot \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

Compute e^{At} by the Cayley-Hamilton method

$$e^{At} = a_0(t)I + a_1(t)A + \cdots + a_{n-1}(t)A^{n-1}$$

- When λ_i is a repeated eigenvalue with algebraic multiplicity n, we can form the following $n-1$ linearly independent equations:

$$e^{\lambda_1 t} = \alpha_0(t) + \alpha_1(t)\lambda_1 + \cdots + \alpha_{n-1}(t)\lambda_1^{\ n-1}$$

$$te^{\lambda_1 t} = \alpha_1 + 2\alpha_2\lambda_1 + 3\alpha_3\lambda_1^2 + 4\alpha_4\lambda_1^3 + \cdots + (n-2)\alpha_{n-2}\lambda_1^{n-3} + (n-1)\alpha_1^{n-1}\lambda_1^{n-2}$$

$$t^2e^{\lambda_1 t} = 2\alpha_2 + 3\times 2\alpha_3\lambda_1 + 4\times 3\alpha_4\lambda_1^2 + \cdots + (n-2)(n-3)\alpha_{n-2}\lambda_1^{n-4} + (n-1)(n-2)\alpha_1^{n-1}\lambda_1^{n-3}$$

$$\Downarrow$$

$$\frac{1}{2!}t^2e^{\lambda_1 t} = \alpha_2 + 3\alpha_3\lambda_1 + \frac{4\times 3}{2!}\alpha_4\lambda_1^2 + \cdots + \frac{(n-2)(n-3)}{2!}\alpha_{n-2}\lambda_1^{n-4} + \frac{(n-1)(n-2)}{2!}\alpha_1^{n-1}\lambda_1^{n-3}$$

Example

Calculate the state transition matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

Solutions

It is obvious that $\mathbf{A}$ has three eigenvalues $\lambda_1 = -1$ and $\lambda_{2,3} = -2$.

From the Cayley-Hamilton theorem, we have

$$\begin{aligned} e^{\mathbf{A}t} &= a_0(t)\mathbf{I} + a_1(t)\mathbf{A} + a_2(t)\mathbf{A}^2 \\ e^{\lambda_1 t} &= a_0(t) + a_1(t)\lambda_1 + a_2(t)\lambda_1^2 \\ e^{\lambda_2 t} &= a_0(t) + a_1(t)\lambda_2 + a_2(t)\lambda_2^2 \\ te^{\lambda_2 t} &= a_1(t) + 2a_2(t)\lambda_2 \end{aligned}$$

By takeing the values of λ_1 and λ_2 to the above equations, we have

$$\begin{cases} e^{-t} &= a_0 - a_1 + a_2 \\ e^{-2t} &= a_0 - 2a_1 + 4a_2 \\ te^{-2t} &= a_1 - 4a_2 \end{cases} \longrightarrow \begin{cases} a_0 &= 4e^{-t} - 3e^{-2t} - 2te^{-2t} \\ a_1 &= 4e^{-t} - 4e^{-2t} - 3te^{-2t} \\ a_2 &= e^{-t} - e^{-2t} - te^{-2t} \end{cases}$$

For simplicity, let denote $x = e^{-t}$ and $y = e^{-2t}$.

Hence

$$\begin{aligned} e^{\mathbf{A}t} &= a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 \\ &= a_0 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + a_1 \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

Outline of Chapter 2

State-transition matrix

We can write the solution of the homogenous state equation $\dot{\mathrm{x}} = \mathrm{A}\mathrm{x}$ as

$$\mathrm{x}(t) = \mathbf{\Phi}(t)\mathrm{x}(0)$$

where $\Phi(t)$ is an $n \times n$ matrix and is the unique solution of

$$\dot{\mathbf{\Phi}}(t) = \mathrm{A}\mathbf{\Phi}(t),\ \mathbf{\Phi}(0) = \mathrm{I}$$

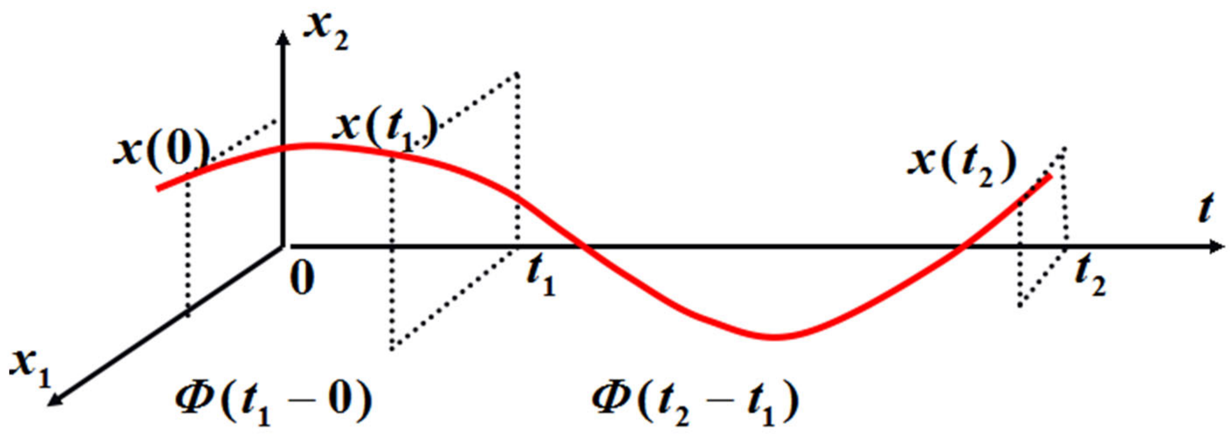
For a linear time-invariant system, we have

$$\begin{cases} e^{\mathbf{A}t} = \mathbf{\Phi}(t) \\ e^{\mathbf{A}(t-t_0)} = \mathbf{\Phi}(t-t_0) \end{cases} \qquad \text{so,} \quad \begin{cases} \mathrm{x}(t) = e^{\mathrm{A}t}\mathrm{x}(0) = \Phi(t)\mathrm{x}(0) \\ \mathrm{x}(t) = e^{\mathrm{A}(t-t_0)}\mathrm{x}(0) = \Phi(t-t_0)\mathrm{x}(t_0) \end{cases}$$

State-transition matrix

$$\begin{cases} x(t) = e^{At}x(0) = \Phi(t)x(0) \\ x(t) = e^{A(t-t_0)}x(0) = \Phi(t-t_0)x(t_0) \end{cases}$$

- After given the initial state, the state transition characteristics of the system is completely determined by $\Phi(t)$.
- The calculation of $\Phi(t)$ is the key point to solving the homogeneous state equation.



Properties of state-transition matrices

For the time-invariant system $\dot{\mathrm{x}} = \mathrm{Ax}$, the state-transition matrix $e^{\mathrm{A}t} = \Phi(t)$ has the following properties: $\Phi(t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots + \frac{1}{k}\mathbf{A}^k t^k + \cdots$

(1) $\mathbf{\Phi}(0) = \mathbf{I}$

(2) $\dot{\mathbf{\Phi}}(t) = \mathbf{A\Phi}(t) = \mathbf{\Phi}(t)\mathbf{A}$ $\Rightarrow \dot{\mathbf{\Phi}}(0) = \mathbf{A}$

$$
\begin{aligned}
\dot{\mathbf{\Phi}}(t) &= \mathbf{A} + \mathbf{A}^2 t + \frac{1}{2!}\mathbf{A}^3 t^2 + \cdots + \frac{1}{(i-1)!}\mathbf{A}^i t^{i-1} + \cdots \\
&= A\left[I + At + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots + \frac{1}{(i-1)!}\mathbf{A}^{i-1} t^{i-1} + \cdots \right] \\
&= \mathbf{A\Phi}(t) = \mathbf{\Phi}(t)\mathbf{A}
\end{aligned}
$$

Properties of state-transition matrices

Example: Judging the following matrixes meet the conditions of state transition matrix or not. Then find the corresponding matrix A .

$$\Phi(t)=\begin{bmatrix}1 & 0 & 0\\ 0 & \sin t & \cos t\\ 0 & -\cos t & \sin t\end{bmatrix}$$

$$\Phi(t)=\begin{bmatrix}1 & 0.5(1-e^{-2t})\\ 0 & e^{-2t}\end{bmatrix}$$

(1) $$\because \Phi(0)=\begin{bmatrix}1 & 0 & 0\\ 0 & \sin t & \cos t\\ 0 & -\cos t & \sin t\end{bmatrix}_{t=0}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}\neq I$$

$\therefore$ Does not meet the condition of the state transition matrix

(2) $$\because \Phi(0)=\begin{bmatrix}1 & 0.5(1-e^{-2t})\\ 0 & e^{-2t}\end{bmatrix}_{t=0}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

$\therefore$ Meet the condition of the state transition matrix

$$\dot{\Phi}(0)=A\Phi(0)=A$$

$$\dot{\Phi}(t)=\begin{bmatrix}0 & e^{-2t}\\ 0 & -2e^{-2t}\end{bmatrix},\quad \Rightarrow \quad A=\dot{\Phi}(0)=\begin{bmatrix}0 & e^{-2t}\\ 0 & -2e^{-2t}\end{bmatrix}_{t=0}=\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}$$

Properties of state-transition matrices(con.)

(3) $\mathbf{\Phi}^{-1}(t)=\mathbf{\Phi}(-t),\quad \mathbf{\Phi}^{-1}(-t)=\mathbf{\Phi}(t)$

Proof: $\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(-t)=e^{At}e^{A(-t)}=\boldsymbol{\Phi}(t-t)=\boldsymbol{\Phi}(0)=\boldsymbol{I}$

Example:

Using the state transition matrix to calculate $\mathbf{\Phi}^{-1}(t)$

$$\mathbf{\Phi}(t)=\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

$$\mathbf{\Phi}^{-1}(t)=\mathbf{\Phi}(-t)=\begin{bmatrix} 2e^{t}-e^{2t} & e^{t}-e^{2t} \\ -2e^{t}+2e^{2t} & -e^{t}+2e^{2t} \end{bmatrix}$$

$\mathbf{\Phi}(t)$ is reversible

Properties of state-transition matrices(con.)

(4) $\mathbf{\Phi}(t_1 + t_2) = \mathbf{\Phi}(t_1)\mathbf{\Phi}(t_2) = \mathbf{\Phi}(t_2)\mathbf{\Phi}(t_1)$

$$\mathbf{\Phi}(t_1)\cdot\Phi\left(t_2\right) = e^{At_1}.e^{At_2}$$

$$= \left(\mathbf{I} + \mathbf{A}t_1 + \frac{1}{2!}\mathbf{A}^2 t_1{}^2 + \cdots\right)\left(\mathbf{I} + \mathbf{A}t_2 + \frac{1}{2!}\mathbf{A}^2 t_2{}^2 + \cdots\right)$$

$$= \mathbf{I} + \mathbf{A}(t_1 + t_2) + \mathbf{A}^2\left(\frac{t_1}{2!} + t_1 t_2 + \frac{t_2}{2!}\right) + A^3\left(\frac{t_1^3}{3!} + \frac{1}{2!}t_1^2 t_2 + \frac{1}{2!}t_1 t_2^2 + \frac{t_2^3}{3!}\right) + \cdots$$

$$= I + A(t_1 + t_2) + A^2\frac{\left(t_1 + t_2\right)^2}{2!} + A^3\frac{\left(t_1 + t_2\right)^3}{3!} + \cdots$$

$$= e^{A(t_1 + t_2)} = \Phi\left(t_1 + t_2\right)$$

State transition can be segmented!

$$(5)\quad \mathbf{x}(t)=\mathbf{\Phi}(t-t_0)\mathbf{x}(t_0)$$

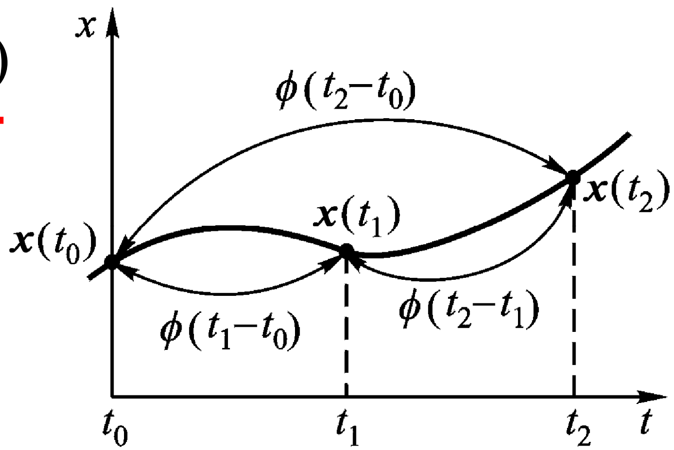
Proof: $\mathbf{x}(t_0)=\mathbf{\Phi}(t_0)\mathbf{x}(0)$

$$\mathbf{x}(t)=\mathbf{\Phi}(t)\mathbf{x}(0)=\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}(t_0)=\mathbf{\Phi}(t)\mathbf{\Phi}(-t_0)\mathbf{x}(t_0)=\mathbf{\Phi}(t-t_0)\mathbf{x}(t_0)$$

$$(6)\quad \mathbf{\Phi}(t_2-t_0)=\mathbf{\Phi}(t_2-t_1)\mathbf{\Phi}(t_1-t_0)$$

Proof: $\mathbf{x}(t_2)=\underline{\mathbf{\Phi}(t_2-t_0)}\mathbf{x}(t_0)\qquad \mathbf{x}(t_1)=\mathbf{\Phi}(t_1-t_0)\mathbf{x}(t_0)$

$$\mathbf{x}(t_2)=\mathbf{\Phi}(t_2-t_1)\mathbf{x}(t_1)=\underline{\mathbf{\Phi}(t_2-t_1)\mathbf{\Phi}(t_1-t_0)\mathbf{x}(t_0)}$$



State transition can be segmented!

$$(7)\ \left[\mathbf{\Phi}(t)\right]^k = \mathbf{\Phi}(kt)$$

Proof: $\because \mathbf{\Phi}(t) = e^{\mathbf{A}t} \quad \therefore \left[\mathbf{\Phi}(t)\right]^k = (e^{\mathbf{A}t})^k = e^{k\mathbf{A}t} = e^{\mathbf{A}(kt)} = \mathbf{\Phi}(kt)$

$$(8)\ \begin{array}{ll} e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t}e^{\mathbf{B}t} = e^{\mathbf{B}t}e^{\mathbf{A}t} & (\mathbf{AB} = \mathbf{BA}) \\ e^{(\mathbf{A}+\mathbf{B})t} \neq e^{\mathbf{A}t}e^{\mathbf{B}t} \neq e^{\mathbf{B}t}e^{\mathbf{A}t} & (\mathbf{AB} \neq \mathbf{BA}) \end{array}$$

Proof: $\boxed{e^{(\mathbf{A}+\mathbf{B})t}} = \mathbf{I} + (\mathbf{A}+\mathbf{B})t + \frac{1}{2!}\underline{(\mathbf{A}+\mathbf{B})^2}t^2 + \frac{1}{3!}\underline{(\mathbf{A}+\mathbf{B})^3}t^3 + \cdots$

$$e^{\mathbf{A}t}e^{\mathbf{B}t} = \left[\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots\right]\left[\mathbf{I} + \mathbf{B}t + \frac{1}{2!}\mathbf{B}^2t^2 + \frac{1}{3!}\mathbf{B}^3t^3 + \cdots\right]$$

$$= \mathbf{I} + (\mathbf{A}+\mathbf{B})t + \frac{1}{2!}\underline{(\mathbf{A}^2 + \mathbf{B}^2 + 2\mathbf{AB})}t^2 + \frac{1}{3!}\underline{(\mathbf{A}^3 + 3\mathbf{A}^2\mathbf{B} + 3\mathbf{AB}^2 + \mathbf{B}^3)}t^3 + \cdots$$

$$\mathbf{AB} = \mathbf{BA} \Rightarrow (\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + \mathbf{B}^2 + \mathbf{AB} + \mathbf{BA} = \mathbf{A}^2 + \mathbf{B}^2 + 2\mathbf{AB}$$

$$(\mathbf{A}+\mathbf{B})^3 = \mathbf{A}^3 + \mathbf{B}^3 + 3\mathbf{A}^2\mathbf{B} + 3\mathbf{AB}^2$$

(9) Two common state transition matrix

If A has distinct eigenvalues and with the diagonal form, that is $\mathbf{A} = diag[\lambda_1 \cdots \lambda_n]$

$$\mathbf{\Phi}(t) = P \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

If A has multiple eigenvalues and with the $(n \times n)$ Jordan form, that is

$$\mathbf{\Phi}(t) = P \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} & \cdots & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & te^{\lambda t} \\ 0 & & & & e^{\lambda t} \end{bmatrix} P^{-1}$$

Outline of Chapter 2

Solution of nonhomogeneous state equations

$$\dot{\mathrm{x}}(t) = \mathrm{A}\mathrm{x}(t) + \mathbf{Bu}(t) \qquad \mathrm{x}(t) = ?$$

- The nonhomogeneous state equation can be solved by

  - Integral method
  - Laplace transform approach

Solution of nonhomogeneous state equations

Let's consider the nonhomogeneous state equation described by

$$\dot{\mathrm{x}}(t) = \mathrm{A}\mathrm{x}(t) + \mathbf{Bu}(t) \qquad \mathrm{x}(t) = ?$$

Premultiply both sides of this equation by $e^{-\mathbf{A}t}$

$$e^{-\mathrm{A}t}\left[\dot{\mathrm{x}}(t) - \mathrm{A}\mathrm{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{Bu}(t)$$

$$\because \frac{d}{dt} e^{-\mathbf{A}t}\mathrm{x}(t) = e^{-\mathbf{A}t}\dot{\mathrm{x}}(t) - \mathrm{A}e^{-\mathrm{A}t}\mathrm{x}(t)$$

$t_0 = 0 \quad \int_0^t$

$t_0 \neq 0 \quad \int_{t_0}^t$

$$\boxed{\Rightarrow \int_0^t \frac{d}{d\tau}\left(e^{-\mathbf{A}\tau}\mathrm{x}(\tau)\right)d\tau = e^{-\mathbf{A}\tau}\mathrm{x}(\tau)\Big|_0^t} = e^{-\mathbf{A}t}\mathrm{x}(t) - \mathrm{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{Bu}(\tau)d\tau$$

$$\boxed{\mathrm{x}(t) = e^{\mathbf{A}t}\mathrm{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{Bu}(\tau)d\tau = e^{\mathbf{A}t}\mathrm{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau}$$

If $t_0 = 0$

$$\mathrm{X}(t) = e^{\mathbf{A}t}\mathrm{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)\, d\tau$$

$$= \mathbf{\Phi}(t)\mathrm{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{Bu}(\tau)\, d\tau$$

$$= \mathbf{\Phi}(t)\mathrm{x}(0) + \int_0^t \mathbf{\Phi}(\tau)\mathbf{Bu}(t-\tau)\, d\tau$$

If $t_0 \neq 0$

$$\mathrm{X}(t) = e^{\mathbf{A}(t-t_0)}\mathrm{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)\, d\tau$$

$$= \mathbf{\Phi}(t-t_0)\mathrm{x}(t_0) + \int_{t_0}^t \mathbf{\Phi}(t-\tau)\mathbf{Bu}(\tau)\, d\tau$$

Laplace transform approach

Let's consider the nonhomogeneous state equation described by

$$\dot{\mathrm{x}}(t) = \mathrm{A}\mathrm{x}(t) + \mathbf{Bu}(t)$$

The Laplace transform of the equation yields

$$s\mathrm{x}(s) - \mathrm{x}(0) = \mathbf{A}\mathrm{x}(s) + \mathbf{B}\mathrm{U}(s) \quad \Rightarrow (s\mathbf{I} - \mathbf{A})\mathrm{x}(s) = \mathrm{x}(0) + \mathbf{B}\mathrm{U}(s)$$

$$\Rightarrow \mathrm{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathrm{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathrm{U}(s)$$

The inverse Laplace transform of the equation

$$\mathbf{x}(t) = \boxed{L^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]}\mathrm{x}(0) + L^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(s)]$$

Convolution property of the Laplace transform(multiplication in the Laplace domain equals convolution in the transform domain)

$$L\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right] = L\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right] = W_1(s)W_2(s)$$

$$\therefore \quad L^{-1}[F_1(s)F_2(s)] = \int_{t_0}^{t} f_1(t-\tau)f_2(\tau) = \int_{t_0}^{t} f_1(\tau)f_2(t-\tau)$$

$$\therefore \quad L^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)\right] = \int_0^t e^{\mathrm{A}(t-\tau)}\mathbf{B}u(\tau)d\tau = \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{B}u(\tau)d\tau = \int_{t_0}^t \Phi(\tau)\mathbf{B}\mathbf{u}(t-\tau)d\tau$$

$$\boxed{\mathbf{x}(t) = \Phi(t-t_0)\mathbf{x}(t_0) + \int_{t_0}^t \Phi(\tau)\mathbf{B}\mathbf{u}(t-\tau)d\tau \qquad (*a)}$$

$$\mathbf{x}(t) = \Phi(t-t_0)\mathbf{x}(t_0) + \int_{t_0}^t \Phi(t-\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau \qquad (*b)$$

Solution of nonhomogeneous state equations(con.)

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,\mathrm{d}\tau$$

$$\mathbf{x}(t) = \mathrm{e}^{At}\mathbf{x}(0) + \int_0^t \mathrm{e}^{A(t-\tau)}B\mathbf{u}(\tau)\mathrm{d}\tau$$

The solution of LTI continuous system state equations is composed of two parts:

- The first part is the free movement caused by the $x(0)$,
  - ✓ it is the effect of the $x(0)$ to the transfer of the system state.
  - ✓ regardless of the $u(t)$ after t_0,
  - ✓ called as the zero input response of the state.
- The second part is the forced movement caused by $u(t)$. The value of the second part is the convolution of $u(t)$ and $\Phi(t)$.
  - ✓ Thus, it is related to $u(t)$, independent of $x(0)$,
  - ✓ called as the zero state response.

Solution of nonhomogeneous state equations(con.)

The output equation

From the non-homogeneous state equation $x(t)$, we can get the output equation:

If $t_0 = 0$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

$$y(t) = C\Phi(t)x_0 + \int_0^t C\Phi(t-\tau)Bu(\tau)d\tau + Du(t)$$

If $t_0 \neq 0$

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

$$y(t) = C\Phi(t-t_0)x(t_0) + \int_{t_0}^t C\Phi(t-\tau)Bu(\tau)d\tau + Du(t)$$

- The output solution is composed of three parts :
  - The first part is the freedom of movement caused by the initial state
  - The second part is the forced motion caused by the input of the system
  - The third part is the feedforward response caused by the direct connection.

Example

Solving the unit step response of the system with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solutions

(1) Using Integral method ($t_0 = 0$)

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}u(\tau)d\tau \qquad u(\tau) = 1(\tau)$$

$$set \quad \tau' = t - \tau$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_t^0 -\mathbf{\Phi}(\tau')Bd\tau' = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(\tau)\mathbf{B}d\tau$$

$$\Phi(t) = L^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$s\mathbf{I}-\mathbf{A}=\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}-\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}=\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\left(s\mathbf{I}-\mathbf{A}\right)^{-1}=\frac{adj\left(s\mathbf{I}-\mathbf{A}\right)}{\left|s\mathbf{I}-\mathbf{A}\right|}=\frac{1}{\left(s+1\right)\left(s+2\right)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}=\begin{bmatrix} \frac{2}{s+1}-\frac{1}{s+2} & \frac{1}{s+1}-\frac{1}{s+2} \\ \frac{-2}{s+1}+\frac{2}{s+2} & \frac{-1}{s+1}+\frac{2}{s+2} \end{bmatrix}$$

$$\mathbf{\Phi}\left(t\right)=\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

$$\int_0^t \mathbf{\Phi}\left(t\right)\mathbf{B}d\tau=\int_0^t \begin{bmatrix} e^{-\tau}-e^{-2\tau} \\ -e^{-\tau}+2e^{-2\tau} \end{bmatrix}d\tau=\begin{bmatrix} -e^{-\tau}+\frac{1}{2}e^{-2\tau} \\ e^{-\tau}-e^{-2\tau} \end{bmatrix}\Bigg|_0^t=\begin{bmatrix} -e^{-t}+\frac{1}{2}e^{-2t}+\frac{1}{2} \\ e^{-t}-e^{-2t} \end{bmatrix}$$

$$\mathbf{x}\left(t\right)=\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}\begin{bmatrix} x_1\left(0\right) \\ x_2\left(0\right) \end{bmatrix}+\begin{bmatrix} -e^{-t}+\frac{1}{2}e^{-2t}+\frac{1}{2} \\ e^{-t}-e^{-2t} \end{bmatrix}$$

(2) Using Laplace transformation method

$$\mathbf{x}(t) = L^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{x}(0) + L^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)]$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\mathbf{\Phi}(t) = L^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{b}U(s)=\begin{bmatrix}\dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)}\\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)}\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix}\cdot\frac{1}{s}\quad=\begin{bmatrix}\dfrac{1}{s(s+1)(s+2)}\\ \dfrac{1}{(s+1)(s+2)}\end{bmatrix}$$

$$L^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{b}U(s)\right]=\begin{bmatrix}-e^{-t}+\dfrac{1}{2}e^{-2t}+\dfrac{1}{2}\\ e^{-t}-e^{-2t}\end{bmatrix}$$

$$\mathbf{x}(t)=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}+\begin{bmatrix}-e^{-t}+\dfrac{1}{2}e^{-2t}+\dfrac{1}{2}\\ e^{-t}-e^{-2t}\end{bmatrix}$$

2. Use the following methods to find the unit-step response of

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 2 & 3 \end{pmatrix} x$$

(1) The Laplace transform method;

(2) The eigenvalue method;

(3) The Cayley-Hamilton method.

Outline of Chapter 2

MATLAB commands used in this chapter

Command	Description
inv	Matrix inverse
det	Determinant of a square matrix
eig	Eigenvalues and eigenvectors
exp	Exponential function
ode45	Solve differential equations, medium order method

The End of Chap. 2