第 7 章习题参考答案

1. Given the linear time-invariant model

$$\dot{x}=\begin{bmatrix}-7 & -2 & 6\\ 2 & -3 & -2\\ -2 & -2 & 1\end{bmatrix}x+\begin{bmatrix}1 & 1\\ 1 & -1\\ 1 & 0\end{bmatrix}u=Ax+Bu$$

$$y=\begin{bmatrix}-1 & -1 & 2\\ 1 & 1 & -1\end{bmatrix}x=Cx$$

Check controllability and observability using

(1) the controllability and observability matrix;

(2) the rows of $\bar{B}=M^{-1}B$ and columns of $\bar{C}=CM$, where M is chosen such that $M^{-1}AM$ is diagonal;

(3) Matlab programs.

解:

(1) $Q_c=\begin{pmatrix}B & AB & A^2B\end{pmatrix}=\begin{pmatrix}1 & 1 & -3 & -5 & 9 & 25\\ 1 & -1 & 3 & 5 & 9 & -25\\ 1 & 0 & -3 & 0 & 9 & 0\end{pmatrix}$, $rank(Q_c)=2$，不可控

$Q_o=(C \quad CA \quad CA^2)^T=\begin{pmatrix}2 & -1 & -2 & 3 & 2 & 9\\ -1 & 1 & 1 & -3 & -1 & 9\\ -1 & 1 & 1 & -3 & -1 & 9\end{pmatrix}^T$, $rank(Q_o)=2$，不可观

(2) 求其特征值与特征向量

$$(\lambda I-A)=\begin{pmatrix}\lambda+7 & 2 & -6\\ -2 & \lambda+3 & 2\\ 2 & 2 & \lambda-1\end{pmatrix}=(\lambda+5)(\lambda+3)(\lambda+1)=0$$

求得特征值为：$\lambda_1=-5$， $\lambda_2=-3$， $\lambda_3=-1$

求得对应的特征向量为： $p_1=(-1 \quad 1 \quad 0)^T$, $p_2=(1 \quad 1 \quad 1)^T$， $p_3=(1 \quad 0 \quad 1)^T$

因此， $P=(p_1,p_2,p_3)=\begin{pmatrix}-1 & 1 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$， $P^{-1}=\begin{pmatrix}-1 & 0 & 1\\ 1 & 1 & -1\\ -1 & -1 & 2\end{pmatrix}$

$\bar{A}=P^{-1}AP=\begin{pmatrix}-5 & & \\ & -3 & \\ & & -1\end{pmatrix}$， $\bar{B}=P^{-1}B=\begin{pmatrix}0 & -1\\ 1 & 0\\ 0 & 0\end{pmatrix}$， $\bar{C}=CP=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$

可判断得，该系统不可控，不可观。

(3) 程序：

```
>>A=[-7 -2 6;2 -3 -2;-2 -2 1];
  B=[1 1;1 -1;1 0];
  C=[-1 -1 2;1 1 -1];
  Tc=ctrb(A,B);
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rank(Tc);
To=obsv(A,C);
rank(To)
```

2. Try to investigate the controllability and observability of the following system.

(1) $A=\begin{bmatrix}-5 & 1\\ 0 & 4\end{bmatrix}$, $b=\begin{bmatrix}1\\1\end{bmatrix}$, $c=[0 \quad -2]$

解：$Q_c=(b \quad Ab)=\begin{pmatrix}1 & -4\\ 1 & 4\end{pmatrix}$, $rank(Q_c)=2$，系统可控

$Q_o=(c \quad cA)^T=\begin{pmatrix}0 & -2\\ 0 & -8\end{pmatrix}$, $rank(Q_o)=1$，系统不可观

(2) $A=\begin{bmatrix}3 & 3 & 6\\ 1 & 1 & 2\\ 2 & 2 & 4\end{bmatrix}$, $b=\begin{bmatrix}0\\0\\1\end{bmatrix}$, $c=[1 \quad 0 \quad -1]$

解：$Q_c=\left(b \quad Ab \quad A^2b\right)=\begin{pmatrix}0 & 6 & 48\\ 0 & 2 & 16\\ 1 & 4 & 32\end{pmatrix}$, $rank(Q_c)=2$，系统不可控

$Q_o=\begin{pmatrix}c\\ cA\\ cA^2\end{pmatrix}^T=\begin{pmatrix}1 & 0 & -1\\ 1 & 1 & 2\\ 8 & 8 & 16\end{pmatrix}$, $rank(Q_o)=2$，系统不可观

(3) $A=\begin{bmatrix}-2 & & & & & \\ & -1 & 1 & & & \\ & & -1 & & & \\ & & & -3 & 1 & \\ & & & & -3 & \\ & & & & & -1\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\\ 0 & 1\\ 0 & 0\\ 1 & 0\end{bmatrix}$, $C^T=\begin{bmatrix}1 & 0\\ 2 & 0\\ 0 & 2\\ 0 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}$

解：根据约旦阵的判断规则，可知该系统不可控，不可观。

3. Transform the state space model below into controllable canonical form, and from the resulting equations compute its transfer function.

$$\dot{x}=\begin{bmatrix}0 & 0 & -2\\ 1 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}x+\begin{bmatrix}1\\1\\1\end{bmatrix}u$$
$$y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}x$$

解： $Q_c=\begin{pmatrix}1 & -2 & -6\\ 1 & 2 & 1\\ 1 & 3 & 8\end{pmatrix}$， $rank(Q_c)=3$，该系统可控，能转换成可控规范型。

求得 $Q_c^{-1}=\dfrac{1}{21}\begin{pmatrix}13 & -2 & 10\\ -7 & 10 & -7\\ 1 & -5 & 4\end{pmatrix}$， 因此 $P_{c1}=\dfrac{1}{21}(1 \quad -5 \quad 4)$

$$P_c^{-1}=\begin{pmatrix}P_{c1}\\ P_{c1}A\\ P_{c1}A^2\end{pmatrix}=\frac{1}{21}\begin{pmatrix}1 & -5 & 4\\ -5 & 4 & 1\\ 4 & 1 & 16\end{pmatrix}, \qquad P_c=\begin{pmatrix}-3 & -4 & 1\\ -4 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}$$

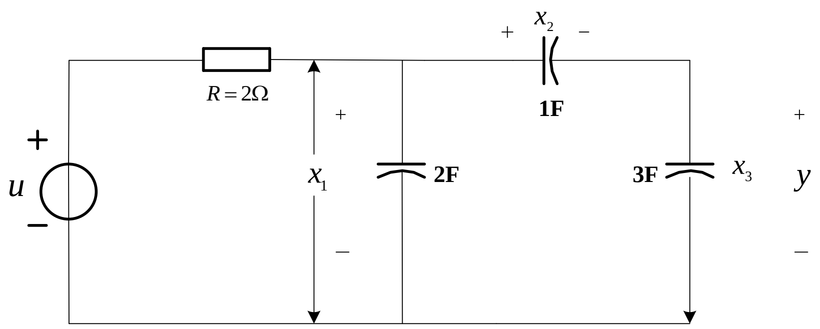
$$\overline{A_c}=P_c^{-1}AP_c=\frac{1}{21}\begin{pmatrix}1 & -5 & 4\\ -5 & 4 & 1\\ 4 & 1 & 16\end{pmatrix}\begin{pmatrix}0 & 0 & -2\\ 1 & 0 & 1\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}-3 & -4 & 1\\ -4 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & 1 & 2\end{pmatrix}$$

$$\overline{B_c}=P_c^{-1}B=\frac{1}{21}\begin{pmatrix}1 & -5 & 4\\ -5 & 4 & 1\\ 4 & 1 & 16\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$\overline{C_c}=CP_c=(0 \quad 0 \quad 1)\begin{pmatrix}-3 & -4 & 1\\ -4 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}=(1 \quad 1 \quad 1)$$

求得传递函数为: $G(s)=\dfrac{s^2+s+1}{s^3-2s^2-s+2}$

4. Find two and three-dimensional state equations to describe the network shown below. Discuss their controllability and observability.



解：

(1) 三维：

$$\begin{cases} 2(2\dfrac{dx_1}{dt}+\dfrac{dx_2}{dt})=u-x_1 \\ x_1=x_2+x_3 \\ \dfrac{dx_2}{dt}=3\dfrac{dx_3}{ty} \end{cases}$$

得：
$$\dot{x}=\begin{bmatrix} -\dfrac{2}{11} & 0 & 0 \\ 0 & -\dfrac{3}{22} & -\dfrac{3}{22} \\ 0 & -\dfrac{1}{22} & -\dfrac{1}{22} \end{bmatrix}x+\begin{bmatrix} \dfrac{2}{11} \\ \dfrac{3}{22} \\ \dfrac{1}{22} \end{bmatrix}u$$
$$y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}x$$

因 $rankQ_c=1$， $rankQ_o=2$，该系统不可控，不可观。

(2) 二维：

$$\begin{cases} 2(2\dfrac{dx_1}{dt}+\dfrac{dx_2}{dt})=u-x_1 \\ x_1=x_2+y \\ \dfrac{dx_2}{dt}=3\dfrac{dx_3}{ty} \end{cases}$$

得：
$$\dot{x}=\begin{bmatrix} -\dfrac{2}{11} & 0 \\ -\dfrac{3}{22} & 0 \end{bmatrix}x+\begin{bmatrix} \dfrac{2}{11} \\ \dfrac{3}{22} \end{bmatrix}u$$
$$y=\begin{bmatrix}1 & -1\end{bmatrix}x$$

因 $rankQ_c=1$， $rankQ_o=2$，该系统不可控，可观。

5. Subdivide the following system into

(1) controllable and uncontrollable subsystems;

(2) observable and unobservable subsystems.

$$\dot{x}=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}x+\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}u$$
$$y=\begin{bmatrix}1 & 0 & 1\end{bmatrix}x$$

解：

(1) $rank(Q_c)=rank\begin{pmatrix}b & Ab & A^2b\end{pmatrix}=rank\begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}=2<3$ 不可控

取 $P=(b \quad Ab \quad q_1)=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$， $rankP=3$，求得 $P^{-1}=\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$$\overline{A}=P^{-1}AP=\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}=\begin{pmatrix} -0 & -2 \\ 1 & 3 \\ 0 & 0 \end{pmatrix}$$

$$\overline{B}=P^{-1}B=\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\overline{C}=CP=\begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\1 & 1 & 0\\0 & 1 & 1\end{pmatrix}=\begin{pmatrix}0 & 2 & 1\end{pmatrix}$$

因此，$A_c=\begin{pmatrix}0 & -2\\1 & 3\end{pmatrix}$，$B_c=\begin{pmatrix}1\\0\end{pmatrix}$，$C_c=\begin{pmatrix}0 & 2\end{pmatrix}$

(2) $Q_o=\begin{pmatrix}1 & 0 & 1\\2 & 2 & 2\\4 & 6 & 4\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 & 1\\0 & 2 & 0\\0 & 0 & 0\end{pmatrix}$ $\qquad rankQ_o=2<3$

取 $P^{-1}=\begin{pmatrix}1 & 0 & 1\\2 & 2 & 2\\0 & 0 & 1\end{pmatrix}$ $\qquad$ 求得 $P=\begin{pmatrix}1 & 0 & -1\\-1 & 1/2 & 0\\0 & 0 & 1\end{pmatrix}$

$$\overline{A}=P^{-1}AP=\begin{pmatrix}1 & 0 & 1\\2 & 2 & 2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\0 & 1 & 0\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\-1 & 1/2 & 0\\0 & 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\-2 & 3 & 0\\0 & 1/2 & 0\end{pmatrix}$$

$$\overline{B}=P^{-1}B=\begin{pmatrix}1 & 0 & 1\\2 & 2 & 2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$\overline{C}=CP=\begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\-1 & 1/2 & 0\\0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\end{pmatrix}$$

因此，$A_o=\begin{pmatrix}0 & 1\\-2 & 3\end{pmatrix}$，$B_o=\begin{pmatrix}0\\2\end{pmatrix}$，$C_o=\begin{pmatrix}1 & 0\end{pmatrix}$