A NEW CONTACT PARADOX

Abstract

There is a well-known variety of contact paradoxes which are significantly linked to topology (Zimmerman 1996). A second, less discussed, class of problems involves contact with bodies composed of a denumerable infinity of parts. Here the emphasis lies not much in topology as in the possibility (or not) of coherently describing the physical interaction with such entities and/or the properties of such interaction (Alper & Bridger 1998, Peijnenburg & Atkinson 2010, Prosser 2006). The aim of this paper is to present a new paradox concerning this second type.

Keywords: Contact, Infinity, Newtonian Forces, Paradox

1. Introduction. The principle of influence

The notion of force is a theoretical construct that corresponds to a certain kind of physical experience. It must therefore be required that this construct has some kind of empirical relevance. Thus, a non-zero force acting on a body must have at least causal effects in the form of movements and/or non-zero internal forces induced within it. This is a requirement for any reasonable theory on forces (even if it is not empirically correct, like Newtonian mechanics). The forces to which we are accustomed causally entail movements and/or the presence of some other forces (typically in the form of internal stresses in the bodies on which they act). This justifies the following intuitively evident principle, which I shall henceforth assume (unless expressly stated otherwise).

PRINCIPLE OF INFLUENCE: any force exerted on a body B induces (causes) movement of B and/or the emergence of internal forces in B.

The role of principle of influence in infinite component systems is clear. Consider for example the following description of an infinite stack of slabs (Benardete 1964). Lying upon the ground there is a slab of stone 1/2 thick, weighing 1/2. Resting squarely on this first slab is a second slab of stone 1/4 thick, weighing 1/4. Resting squarely on this second slab is a third slab of stone 1/8 thick, weighing 1/8, &c. *ad infinitum*. It follows that the i + 1-th slab exerts an upward force of $1/2^{i+1}$ (on the i + 2-th) and a downward force of $1/2^{i}$ (on the i-th). Now, suppose that a man of weight P climbs to the top of the stack of slabs. Obviously, he exerts a (downward) force of magnitude P on it. This force does not cause movement but it does cause new internal forces to emerge in the stack. The i + 1-th slab now exerts an upward force of P + ($1/2^{i+1}$) (on the i + 2-th) and a downward force of P + ($1/2^{i}$) (on the i-th).

Another illustrative example is as follows. Let us consider a set of infinite point particles p_{n+1} of identical unit mass at rest at points $x_n = 1/n$ (n = 1,2,3,4.....). Now let a particle p_1 also of unit mass and velocity v (moving to the left) approach them from the right (Laraudogoitia 1996). Particle p_1 has been set in this state of motion by a given force F exerted on it to the left. Taking into account that in a binary collision between identical particles, the particles simply exchange their velocities, it is clear the final state resulting from the infinite sequence of successive binary collisions that takes place is as follows: an infinite set of point particles p_n of identical unit mass at rest at points x_n = 1/n (n = 1,2,3,4.....). Indeed, velocity v is transferred first to p_2, then to p_3, and so on successively until no particle continues in movement. We shall use the term P for the system of infinite particles p_1, p_2, p_3, ... p_n, ... Since it has been exerted on p_1, force F has also obviously been exerted on P. Even though it does not cause its movement [1], it does cause new internal forces to emerge in P. In effect, p_{i+1} experiences a force exerted on it by p_i and then exerts a force on p_{i+2}.

What makes the principle of influence eminently plausible is its weak character. It speaks of the causal effects of a force without going into much detail. For example, it does not specify exactly where the force that produces such effects acts (beyond generically indicating that it does so "on a body B"). This lack of specification averts numerous problems when dealing with material systems with an infinite number of component parts. For example, in the previous case of the man on the stack of slabs, what this principle says is quite clear and blatantly true. This is precisely because it makes no commitment to any specific statement concerning which part (if any) of the stack of slabs is in contact with the man and therefore feels his "direct and immediate" pressure. Also note that the principle of influence DOES NOT DEFINE the way in which a force acts, what it does is DESCRIBE (very succinctly) the way in which a

[1] F causes the movement of p_1, but not of P, whose center of masses remains fixed at x = 0. By definition, the movement of P is the movement of its center of masses.

force acts. There is therefore no danger of circularity due to the fact that forces appear in it as possible effects of forces.

Neither is what I call the Principle of Influence (POI) a dubious eventual intuition, but rather something deeply rooted in the very nature of Newtonian mechanics. This is traditionally divided into three parts: statics, kinematics and dynamics. Only the first and last parts explicitly consider the role of forces. The first considers them as the cause of internal stresses in bodies in equilibrium and the third as the cause of their movements. It is precisely these two aspects, internal stresses and movement, which are reflected in the POI. Of course, both aspects do not always need to manifest themselves simultaneously. For example, two equal and opposite forces acting on a compressed spring cause only internal stresses within it, but no movement. Thus, it is clear that the POI is a very general principle in Newtonian mechanics, but by no means a vague principle. Its concrete, quantitative manifestations depend on the specific application of the laws of statics and/or dynamics, whose most general features are described by the POI.

According to its logical form, the Principle of Influence establishes that a force acting on B is a sufficient condition to cause (induce) the movement of B and/or the emergence of internal forces in B. Is it also a necessary condition for this or, on the contrary, are there other ways of causing (inducing) such effects? If it were a necessary condition, we could formulate a general principle, which could be termed the Strong Principle of Influence:

STRONG PRINCIPLE OF INFLUENCE: only a force exerted on a body B induces (causes) movement of B and/or the emergence of internal forces in B.

An interesting subtlety regarding the strong principle of influence (which is not an argument against it) is the following. Consider the process described above where the infinite system of particles P ended with all of them at rest. The temporal inversion of this process also describes a possible process (given the temporal symmetry of mechanics) which Laraudogoitia (1996) calls for obvious reasons spontaneous self-excitation. In it we start from an initial state

Initial state (IS): point particles p_n of identical unit mass at rest at points $x_n = 1/n$ (n = 1,2,3,4.....).

From (IS) an evolution occurs in which the binary collisions that characterized the direct process now take place in reverse order. In other words:

a) During an interval of time lasting $([1/n] - [1/(n + 1)])(1/v)$ particle p_{n+1} moves at constant velocity v, beginning its movement on receiving an impact, when at rest, from p_{n+2}, which is at that moment in movement at velocity v, and ending it on colliding in movement with p_n, which is at that moment at rest (n = 1,2,3,4.....).

b) After colliding at rest with p_2, which was at that moment in movement at velocity v, particle p_1 acquires velocity v indefinitely.

a) and b) describe the evolution that leads to the final state

Final state ($FS^{(v)}$): Point particles p_{n+1} of identical unit mass at rest at points $x_n = 1/n$ (n = 1,2,3,4…..). Moving away from them on the right is particle p_1 also of unit mass and at velocity v (to the right).

From IS the system of particles could have evolved with no collisions at all, perpetuating that initial state indefinitely. Therefore, it is reasonable to call the possible evolution, described by a) and b), which begins in initial state IS and ends in final state $FS^{(v)}$, spontaneous self-excitation at velocity v. There is no reason why it should have taken place and, having done so, no external cause brought it about. It corresponds to a sudden spontaneous disturbance that propagates to the right (at velocity $v_{pert} = v$) through the system of particles until it finally reaches particle p_1.[2] Even though P does not move as such (see note (1)), nor is any force exerted on it, internal forces do emerge in P. Indeed, p_{n+1} experiences a force exerted on it by p_{n+2} (in $t = (1/(n + 1))(1/v)$) and then exerts a force on p_n (in $t = (1/n)(1/v)$). However, the strong principle of influence is

[2] As we see, from IS self-excitations are possible with different velocities. All these self-excitations are spontaneous and unpredictable, as there is no particular reason for any of them to occur: the system of particles could remain in the IS permanently. What matters is that the system in the initial state IS can spontaneously self-excite in many ways. It is an unstable system and each of its possible elementary spontaneous self-excitations corresponds to a sudden spontaneous disturbance that propagates to the right at a certain velocity v_{pert} through the system of particles.

not violated because the emergence of this sequence of internal forces (such as the emergence of the sequence of collisions between particles in which these are manifested) is acausal. The strong principle of influence does not prohibit the acausal emergence of internal forces.

Intuitively, it seems clear that the strong principle of influence (SPOI) is true, but recent literature on supertasks casts doubt on this. Laraudogoitia (2009) proposes a surprising example of what he calls action without interaction. This example (supposedly) demonstrates that it is possible to act on a material body B by inducing internal forces within it in spite of the fact that no external force is exerted on it. Violation of the strong principle of influence (SPOI) would be obvious. Such cases of (supposed) action without interaction in no way call into question the principle of influence (POI). Whether or not there are actions without interaction that are ultimately a real counterexample to the strong principle of influence (SPOI), the central argument of this paper is unaffected by this. What I will propose in this central argument is a counterexample to the principle of influence (POI) as such. It is therefore a different theoretical task, and my new contact paradox does not require, in any way, that the correctness of the proposed models of action without interaction be assumed.

2. Starting point

As shall be seen, the new contact paradox that I propose violates the principle of influence and is based on a variant of the configuration introduced for other purposes by Laraudogoitia 2019, which I shall describe as follows.

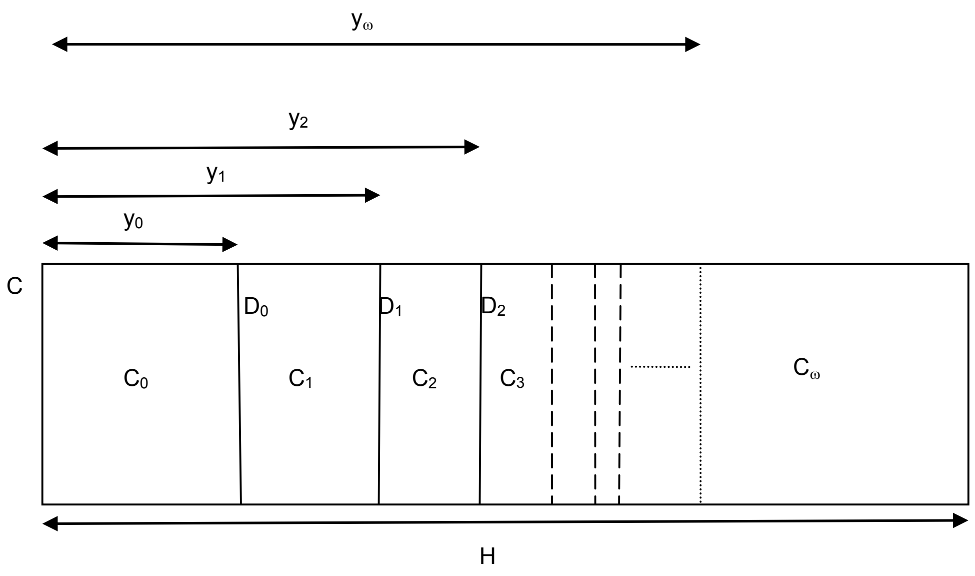

Figure 1

A rigid, hollow cylinder C of unit section and length H contains an infinite number of rigid circular disks D_0, D_1, D_2, D_3, ... , D_n, ... (order type ω) that fit perfectly inside C and divide its interior into an infinity of separate compartments C_0, C_1, C_2, C_3, ... , C_n, ... C_ω (order type ω + 1). The location of each D_i in the interior of C may be identified, for instance, by measuring its horizontal distance y_i from the left end of C, as shown in Figure 1. Note that y_ω is the limit of the D_i locations (namely, $\lim_{i\to\infty} y_i = y_\omega$) but it is not the location of any disks. There is no D_ω. I assume that the masses of the D_i decrease sufficiently for the total mass of the configuration to be finite, and that the thickness of the D_i is decreasing sufficiently. I also assume that the C_i compartments ($0 < i < \omega$) are full of gas at different pressures. As we already know, where there is no friction between the D_i and C's inner surface, the evolution of the whole will direct us to a final location of the former (generally changing their y_i coordinates) that guarantees pressures will be equal in all the C_i compartments. In order to make things more interesting and highlight the new contact paradox presented here, I will use only a marginally different context: suppose there is friction between every D_i and C to the extent that the net horizontal force required to move any D_i inside C must be greater than a certain finite value $k^* > 2$. This will ensure that no D_i slides inside C in any of the cases discussed below. It must also be assumed that C_0 and C_ω are initially empty, so their internal pressure is $p(C_0) = p(C_\omega) = 0$. I will use the following abbreviations to refer to forces exerted on material bodies by material bodies:

$F[A/B]$ = Force that body A exerts on body B

In the following only horizontal forces will be of interest, therefore $F[A/B]$ will be positive or negative depending on whether the force of A on B is directed to the right or to the left respectively. Occasionally, in order to be clear, I shall also add the subscript notation:

$F_x[A/B]$ = X-type force that body A exerts on body B

So the case may arise where $F[A/B] = F_x[A/B] + F_y[A/B] + \ldots$

3. Case I

The following non-problematic case is first considered (which we will call case I): $p(C_i) = 1/i$ $(i > 0)$. Clearly, $p(C_i) < k^*$, hence no D_i slides inside C. Given that $p(C_i) > p(C_{i+1})$ $(i > 0)$, it is evident that every D_i $(i > 0)$ is given a net push to the right E_i by the gases that it is in contact with. However, D_i is at rest, so a force $-E_i$ acts on it to the left, evidently exerted by C. So, each D_i $(i > 0)$ pushes C to the right, in turn, with force E_i. Moreover, $F_{friction}[D_0/C] = -F_{friction}[C/D_0]$. Since the net force on D_0 must also be zero (considering that D_0 does not slide inside C either) $F_{friction}[C/D_0] + F[\text{the gas in } C_1/D_0] = 0$. I shall use the abbreviation GC_α to refer to the material system formed by the gas in compartment C_α. It follows that $F_{friction}[C/D_0] = -F[GC_1/D_0] = 1$. For $i > 0$, $F_{friction}[D_i/C] = E_i = -F_{friction}[C/D_i]$. As the net force on D_i $(i > 0)$ must be null (considering D_i does not slide inside C, which is at rest) and, therefore $F_{friction}[C/D_i] + F[\text{the gases in } C_i \text{ and } C_{i+1}/D_i] = 0$ $(i > 0)$, it follows that $-F_{friction}[C/D_i] = F[\text{the gases in } C_i \text{ and } C_{i+1}/D_i] = (1/i) - [1/(i + 1)] = 1/[i(i + 1)]$ $(i > 0)$. I will also use the abbreviation $S_1 + S_2$ to denote the material system made up of subsystems S_1 and S_2 (S_1 and S_2 being exclusive and exhaustive parts of $S_1 + S_2$). In general, with $S_1 + S_2 + S_3 + \ldots$, I will designate the material system composed of the denumerable infinity of subsystems $S_1, S_2, S_3, \ldots$ ($S_1, S_2, S_3, \ldots$ being exclusive and exhaustive parts of $S_1 + S_2 + S_3 + \ldots$). Therefore, $F_{friction}[D_i/C] = -F_{friction}[C/D_i] = F[GC_i + GC_{i+1}/D_i] = 1/[i(i + 1)]$ $(i > 0)$ and hence $F_{friction}[D_1 + D_2 + D_3 + \ldots /C] = \sum F_{friction}[D_i/C] = \sum E_i = \sum 1/[i(i + 1)] = 1$. This force is annulled by the force D_0 exerts on $C = F_{friction}[D_0/C] = -1$, thus we can deduce the existence of equilibrium.

4. Case II

Case II will differ from Case I in just one respect: compartment $C_\omega$will also contain gas and, furthermore, at pressure $p(C_\omega) = 1$ (therefore identical to $p(C_1)$). However this new pressure affects (obviously, at best, indirectly) the pressures of the gases in the other compartments, in no event would value 2 be exceeded (as $1 + (1/i) \leq 2$). Since $2 < k^*$, it follows that none of the D_i ($i \geq 0$) pistons will alter their position inside cylinder C. Consequently, neither will any of the gas pressures. It will continue to be $p(C_i) = 1/i$ ($i > 0$), $p(C_0) = 0$ y $p(C_\omega) = 1$.

As previously stated, when I mention forces, I will always be referring to horizontal forces (understood as forces parallel to the cylinder axis, the X axis, so to speak). The reason is that the forces in a perpendicular direction to the X axis (which exist, and are exerted by the gases on the side walls of cylinder C) play no part in the problem to be discussed in this paper. Given the known pressures, it is still true that $F_{friction}[D_1 + D_2 + D_3 + ... /C] = 1$ y $F_{friction}[D_0/C] = -1$. But now too $F[GC_\omega/C] = 1$. Since $F_{friction}[D_0/C] + F_{friction}[D_1 + D_2 + D_3 + ... /C] + F[\text{the gas in } C_\omega/C] = 1 \neq 0$, apparently the net force on C is not null, so C would seem to cease to be in equilibrium simply because it has filled compartment C_ω. This semblance will turn out to be deceptive but is directly linked to the contact paradox discussed in this paper.

The gas in C_ω is at unit pressure. The force exerted on C by this gas is $F[GC_\omega/C] = 1$. [3] The existence of such pressure also implies (again, according to Newton's third law of action and reaction, see note (3)) that GC_ω must exert a force of unit magnitude directed to the left. In a sufficiently generic way, we could say that this force acts on the material system made up of the set of D_i ($i \geq 0$) plus the set of gases enclosed between them. In other words, it acts on system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$ although (clearly) it does not do so on any of the component parts D_0, GC_1, D_1, GC_2, D_2, GC_3, ... [(4)] [4]Hence, $F[GC_\omega/D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...] = -1$.

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[3] The fact that a gas is subjected to pressure p implies that its environment exerts certain forces on it. The fact that it also acts on its environment is a consequence of the law of action and reaction.

[4] This is an example of what has been called "global" interaction in the literature. See, for example, Laraudogoitia (2005). Similarly, in section 1 we saw that the man of

5. The paradox

The force GC_ω exerts on system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$ violates the principle of influence. The presence of gas at unit pressure in C_ω leads to a force acting on itself by system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$. But there is no contact interaction which has been induced by the gas at unit pressure in C_ω between the D_i ($i \geq 0$) and the gases enclosed between them. Moreover, no matter how system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$ is broken down (even purely formally) into exclusive and exhaustive parts, there is no contact interaction between the parts which has been induced by the gas at unit pressure in C_ω. That is, GC_ω at unit pressure does not cause the emergence of internal forces in $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$ that did not previously exist, when C_ω was empty. In fact, this unit pressure, added to the pressure of any of the gases enclosed between two contiguous pistons (which is never higher than 1), in no event exceeds the $k^* > 2$ value required for the displacement of at least some D_i inside C. However, if the D_i do not move, the initial values of the gas pressures in the C_i compartments will not change either. And if these pressures do not change, neither will the friction forces that kept the different D_i pistons in their initial positions (i.e. before compartment C_ω was filled with gas at unit pressure). This means that there is no contact interaction which has been induced by the gas at unit pressure in C_ω between the D_i ($i \geq 0$) and the enclosed gases between them, as stated above. And, as also stated earlier, GC_ω does not cause the emergence of internal forces in $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$ that did not previously exist, when C_ω was empty. We therefore observe an instance of contact interaction that violates the principle of influence. This is a truly unique type of interaction one is tempted to call "phantom". The gas in C_ω exerts a unit force on $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$, the set consisting of the D_i ($i \geq 0$) and the GC_i ($1 \leq i < \omega$) gases, [5] without inducing any internal force between them or,

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weight P exerts a force of magnitude P on the stack of slabs although (clearly) he does not do so on any of the component slabs.

[5] Some philosophers would prefer to speak here of the fusion of material bodies D_i ($i \geq 0$) and gases GC_i ($1 \leq i < \omega$). The difference between the two terms is irrelevant for the purposes of this paper.

indeed, causing any kind of movement (nor any internal stress in the D_i that did not previously exist, when C_ω was empty). Its sole function is purely formal: to ensure compliance with the law of action and reaction. We will describe this phantom-formal character by writing "ph" in subscript to denote this force: $F_{ph}[GC_\omega/ D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...] = -1$. Note that the phantom force is the force that GC_ω exerts on $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$, but not the force that $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$ exerts on GC_ω. This latter force, unsurprisingly, causes the emergence of internal forces in GC_ω in the form of gaseous pressure.

Comparison with Benardete's infinite stack of slabs may be enlightening here. Before the man climbs the stack, the i + 1-th slab exerts an upward force of $1/2^{i+1}$ on the i + 2-th. Once he has climbed up, this force has a value of $P + (1/2^{i+1})$. In other words, the man's weight has caused the emergence of internal forces in the stack, altering those existing previously. In our case, prior to filling C_ω with gas, the internal forces in system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$ had certain defined values. However, these values are not altered in any way when C_ω is filled with gas at unit pressure! The man's weight has a causal influence on the stack of slabs on which it acts (altering the internal stresses within it). However, the pressure of the gas in C_ω has no causal influence on system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...$ on which it acts (it does not alter the internal stresses within it). In short, the paradox of Benardete's slabs does not violate the Principle of Influence, but rather confirms it. As confirmed (as far as I know) by all the examples considered thus far in the relevant literature. The new contact paradox presented above is radically different on this point. It constitutes the first counterexample to the POI, that is to say, to our most fundamental intuitions concerning the causal role of forces in the Newtonian conception of classical mechanics.

The consequences do not end here and lead to the need for new forces with the same phantom characteristics. Since system D_i ($i \geq 0$) plus gases GC_i ($1 \leq i < \omega$) was in equilibrium with C_ω empty, it will cease to be so if the only thing that has changed in its respect is the presence of the new force $F_{ph}[GC_\omega/ D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...]$. Therefore, (according to Newton's first law of equilibrium) something must counteract it. The only option is cylinder C, so there must be a new "phantom" force $F_{ph}[C/ D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...] = 1$, which neither induces movement or any internal force in the set made up of the D_i ($i \geq 0$) and gases GC_i ($1 \leq i < \omega$) (i.e. no internal force that did not previously exist, when C_ω was empty). Again, this phantom force has a purely formal role to play, in this case ensuring compliance with the law of equilibrium. Finally, the principle of action and reaction requires a final additional force $F[D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ... /C] = -1$. As $F[GC_\omega/C] = 1$, equilibrium of C is finally reached. [6]

By abstracting the specific details, what characterizes a phantom force becomes clear. The force F exerted on an extended body B is a phantom force if and only if it does not induce (cause) movement of B or the emergence of internal forces in B. Consequently, a phantom force violates the principle of influence. One might believe that such entities are idle. One could populate the world with a multitude of entities that (apparently at least) are of this kind, all of which can be eliminated using Occam's razor. In reality, the gods of Olympus, elves or demons play no role in the causal structure of the world. However, phantom forces are not exactly of the same "empty" nature. Although they violate the principle of influence, they play (at least indirectly) a role in the causal description of the world through the Newtonian laws of equilibrium, and action and reaction. The reason is that, as we have seen, they are required by these laws.

6. Paradoxical non-phantom forces

The role of "phantom" forces in the contact paradox is more subtle than the above comments suggest. We have seen that, by virtue of $F[GC_\omega/C] = 1$, $F_{ph}[GC_\omega/ D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...] = -1$, and that friction is the cause of forces $F_{friction}[D_1 + D_2 + D_3 + ... /C] = 1$ and $F_{friction}[D_0/C] = -1$. As the gases in the compartments do not interact by friction, it follows from the penultimate equality that $F_{friction}[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C] = 1$, a force also caused by friction (evidently, the friction of D_i, $i \geq 1$, with C). Furthermore, as there are no interactions at a distance in our model, $F[GC_\omega/ D_0] = F[GC_\omega/GC_1] = 0$. The principle of superposition of forces therefore leads

[6] Note that we have written $F[D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ... /C]$ and not $F_{ph}[D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ... /C]$; the reason being that this is a force with obvious causal power. It induces internal stresses in cylinder C by opposing $F[GC_\omega/C]$, thus reaching equilibrium.

to $F_{ph}[GC_\omega/\ D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...\ ] = -\ 1$.[7] Since system $D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...$ was already in equilibrium with C_ω empty, it will cease to be so if the only thing that has changed in its respect is the presence of the new force $F_{ph}[GC_\omega/\ D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...\ ]$. Therefore, (according to the law of equilibrium) something must counteract it. It cannot be the gas in compartment C_1 (GC_1) because this gas has already annulled the force that C exerts by friction on system $D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...$ from the beginning. Indeed, it was seen in Case I that $F[GC_1/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...] = -\ F_{friction}[C/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...] = 1$. As before, the only option is cylinder C, so there must be a new phantom force $F_{ph}[C/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...\ ] = 1$, which neither induces movement nor any internal force in the set of D_i ($i \geq 1$) and gases GC_i ($2 \leq i < \omega$). Also, as before, this phantom force has a purely formal role to play, ensuring compliance with the law of equilibrium. Lastly, the principle of action and reaction requires a final, additional force: $F_?[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...\ /C] = -\ 1$. As $F[GC_\omega/C] = 1$, equilibrium of C is finally reached. The interesting point here is that there are two forces of a very different nature acting between the same material bodies. As seen above, $F_{friction}[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...\ /C] = 1$ is a force caused by friction and, as we have just seen, $F_?[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...\ /C] = -\ 1$ is a "non-phantom" force [8] originating in phantom force $F_{ph}[C/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...\ ] = 1$ (which, in turn, originated in the phantom force $F_{ph}[GC_\omega/\ D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...]$).

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[7] The principle of superposition of forces applies here to a finite number of forces, namely: $F[GC_\omega/\ D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...\ ] = F[GC_\omega/\ D_0] + F[GC_\omega/GC_1] + F[GC_\omega/\ D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...\ ]$. When the number of forces considered is infinite, it is not always satisfied. As seen at the end of section 4, $F[GC_\omega/D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...] = -\ 1$ even though $\forall i \geq 0\ \ F[GC_\omega/D_i] = 0$ and $\forall i\ (1 \leq i < \omega)\ \ F[GC_\omega/GC_i] = 0$. This is characteristic of many standard examples in the literature of systems with infinite components (such as the slab stack seen in section 1).

[8] This is exactly the same as $F[D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3 + ...\ /C]$, whose non-phantom character was substantiated in note 6.

This highlights an additional paradoxical dimension of the philosophy of contact in the presence of phantom forces. The reaction force to a phantom force is not in itself phantom but may also be of an enigmatic nature. This is exactly what happens in the case of force $F_?[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C]$. As we have seen, $F_?[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C]$ is neither a frictional force nor a usual contact force originating in the impenetrability of matter (such as that which typically arises when two material bodies press against each other). However, it can coexist with them, as exemplified by the case of the friction force that $D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...$ exerts on C. Even though the reaction force to a phantom force does not violate the principle of influence, it may in itself be a mysterious entity. We lack an adequate causal mechanism to explain its genesis in the philosophy of contact. This is the reason for subscript "?" in $F_?[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C]$.

Note that $F[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C]$, the total force that $D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...$ exerts on C, has a value of

$F[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C] = F_{friction}[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C] + F_?[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C] = 1 - 1 = 0$. Also, (as should be the case, given C's state of equilibrium) $F[D_0 + GC_1/C] = F[D_0/C] = F_{friction}[D_0/C] = F_{friction}[D_0 + GC_1/C] = - F[GC_\omega/C] = -1$, so that $F[D_0 + GC_1/C] + F[GC_\omega/C] = - 1 + 1 = 0$. We have seen that friction is the physical cause of term $F_{friction}[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C]$, but that the physical cause of $F_?[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C]$ remains an enigma. In particular, it is an enigma how $D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...$ can exert equal and opposite forces on C. Likewise, when GC_ω was empty, $F_{friction}[D_0 + GC_1/C]$ and $F_{friction}[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C]$ were already creating internal forces of stress in C. However, on filling GC_ω with gas, new internal stresses in C are generated by $F[GC_\omega/C]$ and enigmatic force $F_?[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C]$, confirming the latter's non-phantom character (in accordance with notes (8) and (6)).

7. Some simple variations

There is nothing special regarding value $p(C_\omega) = 1$ in the contact paradox. Many other values will serve equally well to illustrate it. Most certainly, any that do not entail the possibility of pressures equal to or greater than k*. Let us take for example $p(C_\omega) = 1/3$.

The gas in C_ω now exerts a 1/3 force on system $D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...$ ($F_{ph}[GC_\omega/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...] = -1/3$) but without exerting it directly on any of the D_i, GC_{i+1} ($i \geq 1$) or inducing any force between them (or inside them). The forces involving only C, D_i and GC_{i+1} ($i \geq 1$) remain unaltered (taking $p(C_\omega) = 1/3$ changes nothing to this effect, like taking for example $p(C_\omega) = 1$ or $p(C_\omega) = 0$). The single role of $F_{ph}[GC_\omega/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...]$ is purely formal: to ensure compliance with the law of action and reaction. Moreover, as $F[GC_1/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...] = -F_{friction}[C/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...] = 1$, it follows that $F_{ph}[C/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...] = 1/3$ (this is analogous to the result $F_{ph}[C/D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ...] = 1$ seen earlier for the case of $p(C_\omega) = 1$). Thus, $F[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C]$ (the sum of a force caused by friction and another that is not) $= F_{friction}[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C] + F_?[D_1 + GC_2 + D_2 + GC_3 + D_3 + GC_4 + ... /C] = 1 - 1/3$, which again ensures equilibrium of C, because $F[D_0 + GC_1/C] = -1$ and $F[GC_\omega/C] = 1/3$. The phantom forces that violate the principle of influence have a purely formal role: to ensure compliance with the law of action and reaction and law of equilibrium.

8. Tentative conclusion. Comments on infinitism and the new contact paradox.

Finally, I would like to mention a positive interpretation of the contact paradox. Perhaps this is a paradox that we should accept in our theoretical treatment of mechanical forces. By unambiguously allowing what I have called phantom forces (which violate the principle of influence), we are allowing the Newtonian laws involved (the first and third) to be formulated with total generality (as they usually are). Otherwise, by rejecting such forces we would be forced to complicate the highly intuitive and simple axiomatic format of Newton's laws (their extreme generality) in view of configurations (such as in Case II), where some of these laws would fail. The situation is suggestively similar to what can be found in other theoretical domains. The closest probably being that of projective geometry, where "ideal elements" are accepted without any visual or "empirical" correlate (ideal points and ideal lines as opposed to ordinary points and ordinary lines) in order to enable axioms of the projective plane to be formulated simply and generally.

The new contact paradox arises in some physical systems with an infinite denumerable number of component parts. Faced with this, there is always the drastic option of avoiding difficulties by rejecting the actual infinitude in physical models (like the Newtonian models considered here). There are at least three overriding objections to this course of action. First, the fact that it ignores (and condemns) from the outset the existence of a multitude of infinite physical systems studied in the philosophical, physical and mathematical literature (not even always in the form of supertasks). In many of these cases interesting results are obtained that help frame the scope of validity and interpretation of the physical theories involved. Second, the fact that it even rejects cases of uninteresting infinite physical systems that pose no problem of inconsistency in the physical theories admitting them. And third, the fact that (as a result) it fails to discriminate between both types of systems. In particular, it puts on an equal footing (unchecked, which is scarcely compatible with philosophical practice):

a) cases of interesting infinite systems from which something can be learned;

b) other cases that are quite simply incompatible with the physical theory in which they are studied;

c) cases that are perfectly consistent, albeit trivial.

My personal intuition is that the new contact paradox falls within the framework of type a) cases. Moreover, it is conceptually different from all other cases of this type seen in the literature. In addition, the differentiation criterion is clear: none of the other infinite Newtonian systems studied to date (and, of course, neither non-infinite systems) contradicts the principle of influence (POI). Only the new contact paradox does so. Given this contradiction, one might consider including the new contact paradox within type b) cases. In such a circumstance, it would simply be argued that the infinite material configuration of the new contact paradox goes against Newtonian physics. There are, however, two immediate objections to this:

1) The paradox does not arise if (everything else remaining the same) compartment C_ω is simply left empty (with no gas) in such a configuration. This already seems to imply that it is not the presence of the actual infinite as such that is the problem; rather that the root of the paradox lies elsewhere. Furthermore, it would be necessary to explain how it is possible that the mere absence or not of a single material body (in this case the gas in

C_ω) is what makes the difference between an infinite material system that does not go against Newtonian physics and one that does. [9]

2) This incompatibility does not occur, strictly speaking, with Newtonian physics (summarily condensed into Newton's three laws) but with the principle of influence (POI). I have in fact proposed something along these lines here. Continuing to uphold Newton's laws (Newtonian mechanics) and rejecting the universal validity of POI.

Note also that, in the new contact paradox, the set of gas-filled compartments' ordinal type is $\omega + 1$. In the absence of gas in compartment $C_{\omega,}$ the ordinal type is ω. Yet the contrast between ω and $\omega + 1$ cannot be what makes the difference between an infinite material system that does not go against Newtonian physics and one that does. To see this, one only needs to go back to Benardete's example (1964) considered in the introduction, where a man of weight P climbs to the top of the infinite stack of slabs. Here, the set of relevant material bodies' ordinal type (man and slabs) is $\omega + 1$. And, as made clear from my analysis, this material system not only does it not go against Newtonian physics, it also does not go against the POI. We conclude that the contrast between ω and $\omega + 1$ can in noway be what makes the difference between an infinite material system that does not go against the POI and one that does.

[9] The analogy (yet to be explored) is interesting between this and what Sainsbury (2009) calls the "principle of tolerance" in the analysis of paradoxes of vagueness ("a gram cannot make the difference between not enough wood to make a table and enough wood", p.48).

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