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Matrices

P. N. CHATTERJEE

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MATRICES

[For B. A., B. Sc., & B. E. Students of All Indian Universities]

(WITH OBJECTIVE TYPE QUESTIONS)

By :

Prof. P. N. CHATTERJI

Head of the Department of Mathematics

I. (Post-Graduate) College, Meerut.

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PREFACE TO THE NINETEENTH EDITION

It gives me great pleasure in bringing out the Nineteenth Edition of this book.

The book has been thoroughly revised and a number of new examples and articles selected from recent examination papers, have been added.

Besides giving due credit to the printers and publishers, I express my thanks to the professors and students for the appreciation and patronage of the book.

Suggestions for further improvement of the book will be highly appreciated.

—Author

PREFACE TO THE FIRST EDITION

The present book comprising the subject "Matrices" is meant for the students appearing in the B. A., B. Sc. & B. E. Examinations of All Indian Universities. Efforts have been made to make the treatment logical and simple.

I gratefully acknowledge my indebtedness to various authors and publishers whose books have been freely consulted during the preparation of this book.

I shall be grateful to the readers for pointing out errors and omissions that, inspite of all care, might have crept in.

I look forward to the suggestions from the readers for the improvement of the book.

—Author

Contents

Note – Please note that important questions are marked as* and very important questions as** in this book.

CHAPTER I

Addition and Multiplication

§ 1.01. Introduction.

The matrices (formal definition is given in § 1.02 Page 2) were invented about a century ago in connection with the study of simple changes and movements of geometric figures in coordinate geometry.

J. J. Sylvester was the fitst to use the word "matrix" in 1850 and later on in 1858 Arthur Cayley developed the theory of matrices in a systematic way.

"Matrices" is a powreful tool of modern mathematics and its study is becoming important day by day due to its wide applications in almost every branch of science and especially in physics (atomic) and engineering. These are used by Sociologists in the study of dominance within a group, by Demographers in the study of births and deaths, mobility and class structure etc., by Economist in the study of inter-industry economics, by Statisticians in the study of 'design of experiments' and 'multivariate analysis', by Engineers in the study of 'net work analysis' which is used in electrical and communication engineering.

Rectangular Array.

While defining matrix (see § 1.02 Page 2) we use the word 'rectangular array', which should be understood clearly before we come to the formal definition of 'matrices' and to understand the same we consider the following example :

In an inter-university debate, a student can speak either of the five languages : Hindi, English, Bangla, Marathi and Tamil. A certain university (say A) sent 25 students of which 8 offered to speak in Hindi, 7 in English, 5 in Bangla, 2 in Marathi and rest in Tamil. Another university (say B) sent 20 students of which 10 spoke in Hindi, 7 in English and 3 in Marathi. Out of 25 students from the third university (say C), 5 spoke in Hindi, 10 in English, 6 in Bangla and 4 in Tamil.'

The information given in the above example can be put in a compact way if we give them in a tabular form as follows :

University	Number of speakers in				
	Hindi	English	Bangla	Marathi	Tamil
A	8	7	5	2	3
B	10	7	0	3	0
C	5	10	6	0	4

The numbers in the above arrangement form what is known as a **rectangular array.** In this array the lines down the page are called **columns** whereas those across the page are called **rows.** Any particular number in this arrangement is known as an **entry** or **an element.** Thus in the above arrangement we find that there are 3 rows and 5 columns and also we observe that there are 5 elements in each row and so total number of elements $= 3 \times 5$ *i.e.* 15.

If the data given in the above arrangement is written without lines and enclosed by a pair of square brackets *i.e.* in the form

$$\begin{bmatrix} 8 & 7 & 5 & 2 & 3 \\ 10 & 7 & 0 & 3 & 0 \\ 5 & 10 & 6 & 0 & 4 \end{bmatrix},$$

then this is called a matrix.

§ 1.02. Definition of a Matrix.

A system of any mn numbers arranged in a rectangular array of m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix (which is read as m by n matrix).

Or

A set of mn elements of a set S arranged in a rectangular array of m rows and n columns is called an $m \times n$ matrix over S.

For example : $\begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 7 \end{bmatrix}$ is a 2×3 matrix.

and $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots\dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots\dots & a_{mn} \end{bmatrix}$ is an $m \times n$ matrix.

where the symbols a_{ij} represent any numbers (a_{ij} lies in the ith row and jth column).

Note 1. A matrix may be represented by the symbols $[a_{ij}]$, (a_{ij}), $\| a_{ij} \|$ or by a single capital letter **A,** say.

Generally the first system is adopted.

Note 2. Each of the mn numbers constituting an $m \times n$ matrix is known as an **element of the matrix.**

The elements of matrix may be scalar or vector quantities.

Note 3. The plural of 'matrix' is 'matrices'.

Solved Examples on § 1.02.

Ex. 1. Find a_{23} in $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 5 & 0 & 3 & 6 \end{bmatrix}$

Sol. a_{23} = element in the 2nd row and 3rd column

$= 1$ **Ans.**

Ex. 2. Write down the orders of the matrices :—

(a) $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 3 \end{bmatrix}$; **(b)** $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$; **(c)** $[3 \; 4 \; 5]$; **(d)** $[1]$.

Sol. (a) 2×3 ; (b) 2×1 ; (c) 1×3 ; (d) 1×1. **Ans.**

Ex. 3. How many elements are there in a 5×4 matrix ?

Sol. The required number of elements in 5×4 matrix is 5×4 *i.e.* 20. **Ans.**

Ex. 4. The results of a music competition are given in the following matrix :

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 4 \\ 5 & 0 & 3 & 0 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Here the rows represent the teams A, B, C, D in that order and the columns represent the number of wins, first place, second place, third place and fourth place scored by the teams.

From the above matrix find (a) How many events did the team A win ? (b) How many first places did the team B win ? (c) How many third places did the team C win ? What does 0 represent in second row ?

Sol. (a) $\because$ the first row represents the team A, so the required number

$= $ Sum of the elements of first row

$= 3 + 2 + 1 = 6.$ **Ans.**

(b) As first elements of second row (which represents the team B) is zero, so the team B did not win any first place. **Ans.**

(c) The third row represents the team C and third column represents the third place scored by the teams, so the number of third places won by the team C is 3. **Ans.**

(d) The second row represents the team B and the first column represents the first place scored by teams. So 0 in the second row represents that the team B did not score any first place. **Ans.**

Ex. 5. The order of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is

(i) 2×3, (ii) 3×2, (iii) 2×2, (iv) None of these. **Ans. (i)**

Exercises on § 1.02

Ex. 1. In Example 1 above, find (i) a_{32}, (ii) a_{24}. **Ans.** (i) 0, (ii) 5

Ex. 2. Write down the orders of the matrices :—

(a) $\begin{bmatrix} 2 & 3 & 4 & 2 & 1 \\ 3 & 5 & 5 & 3 & 4 \\ 4 & 7 & 6 & 7 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$; (c) $[5]$

Ans. (a) 3×5 ; (b) 3×1, (c) 1×1.

§ 1.03. Rectangular Matrices.

The number of rows and columns of a matrix need not be equal $\therefore$ when $m \neq n$ *i.e.* the number of rows and columns of the array are not equal, then the matrix is known as a **rectangular matrix.**

Classifications of rectangular matrices are as follows :—

Square Matrix.

If $m = n$ *i.e.* the number of rows and columns of a matrix are equal, then the matrix is of order $n \times n$ and is called a square matrix of order n.

For example $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ 7 & 6 & 9 \end{bmatrix}$ is a square matrix and $\begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 5 & 7 & 9 \end{bmatrix}$ is a rectangular matrix.

Horizontal matrix. If in a matrix the number of columns is more than the number of rows then it is called a horizontal matrix.

For example $\begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 5 & 7 & 9 \end{bmatrix}$ is a horizontal matrix.

Row matrix : If in a matrix, there is only one row it is called a row matrix. For example [1, 2, 3]. This is also called a *row vector.*

Vertical matrix. If in a matrix the number of rows is more than the number of columns it is called a vertical matrix.

For example $\begin{bmatrix} 2 & 3 \\ 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{bmatrix}$ is a vertical matrix.

Column matrix : If there is only one column in a matrix, it is called a column matrix.

For example $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. This is also called *column vector.*

Null (or zero) Matrix : If all the elements of an $m \times n$ matrix are zero, then it is called a null or zero matrix and is denoted by $\mathbf{O}_{m \times n}$ or simply $\mathbf{O}$.

For example $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the 2×3 null matrix.

Unit matrix : A square matrix having unity for its elements in the leading diagonal and all other elements as zero is called an **unit matrix.**

For example $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a four rowed unit matrix and we denote it by $\mathbf{I}_4$.

$\therefore$ an n-rowed square matrix $[a_{ij}]$ is called a unit matrix provided

$$a_{ij} = 1, \text{ whenever } i = j$$
$$= 0, \text{ whenever } i \neq j.$$

Equal matrix : Two matrices are said to be equal if (a) they are of the same type *i.e.* if they have same number of rows and columns and (b) the elements in the corresponding positions of the two matrices are equal.

For example, let two matrices be $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ then the two matrices are said to be equal if $a_{ij} = b_{ij}$, for all values of i and j.

From the definition given above it is evident that

(i) If $\mathbf{A} = \mathbf{B}$, then $\mathbf{B} = \mathbf{A}$ (Symmetry)

(ii) $\mathbf{A} = \mathbf{A}$, where $\mathbf{A}$ is any matrix. (Reflexivity)

(iii) If $\mathbf{A} = \mathbf{B}$ and $\mathbf{B} = \mathbf{C}$, then $\mathbf{A} = \mathbf{C}$ (Transitivity)

i.e. the relation of equality in the set of all matrices is an equivalence relation. **(See Author' Set Theory)**

Matrices over a number field.

A matrix **A** is defined as 'over the field F of numbers' if all the elements of the matrix **A** belong to the field F of the numbers.

Diagonal Element and Principal Diagonal.

Those elements a_{ij} of any matrix $[a_{ij}]$ are called diagonal elements for which $i = j$.

The line along which the above elements lie is called the **Principal diagonal** or the **Diagonal** of the matrix.

Diagonal Matrix : A square matrix in which all elements expect those in the main (or leading) diagonal are zero is known as a diagonal matrix.

For example $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a 3–rowed diagonal matrix.

The sum of the diagonal elements of a square matrix **A** (say) is called the trace of the matrix **A**.

Sub-matrix : A matrix which is obtained from a given matrix by deleting any number of rows and number of columns is called a sub-matrix of the given matrix.

For example $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a sub– matrix of $\begin{bmatrix} 5 & 3 & 2 \\ 1 & 1 & 2 \\ 7 & 3 & 4 \end{bmatrix}$

Exercises on § 1.03

Ex. 1. The unit matrix is

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$;

(iii) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **Ans.** (iv)

Ex. 2. What is the type of the matrix $[a\ b\ c]$

(i) column matrix; (ii) unit matrix;

(iii) square matrix; (iv) row matrix. **Ans.** (iv)

Ex. 3. The unit matrix is

(i) $[\ 1\ ]$; (ii) $[\ 0\ ]$;

(iii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, (iv) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **Ans.** (i)

Ex. 4. The matrix of order $m \times n$ will be a unit matrix if

(i) all its elements are unity;

(ii) $m = n$ and all elements are unity;

(iii) $m = n$, and its diagonal elements are unity;

(iv) $m = n$, diagonal elements are unity and all the remaining elements are zero. **Ans.** (iv)

§ 1.04. Scalar Multiple of a matrix.

If **A** is a matrix and λ is a number then λ**A** is defined as the matrix each element of which is λ times the corresponding element of the matrix **A**.

For example : $2\begin{bmatrix} 3 & 5 & 7 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 10 & 14 \\ 4 & 6 & 8 \end{bmatrix}$

or if $\mathbf{A} = [a_{ij}]$, then $\lambda\mathbf{A} = [\lambda a_{ij}]$, where λ is a number.

§ 1.05. Addition of matrices.

If there be two $m \times n$ matrices given by $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, then the matrix $\mathbf{A} + \mathbf{B}$ is defined as the matrix each element of which is the sum of the corresponding elements of **A** and **B** *i.e.* $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$,

where $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

For example : If $\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$

then $\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_1 + a_3 & b_1 + b_3 & c_1 + c_3 \\ a_2 + a_4 & b_2 + b_4 & c_2 + c_4 \end{bmatrix}$

§ 1.06. Subtraction of matrices.

If there be two $m \times n$ matrices given by $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, then the matrix $\mathbf{A} - \mathbf{B}$ is defined as the matrix each element of which is obtained by subtracting the element of **B** from the corresponding element of **A** *i.e.* $\mathbf{A} - \mathbf{B} = [a_{ij} - b_{ij}]$,

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

For example : If $\mathbf{A} = \begin{Bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Bmatrix}$ and $\mathbf{B} = \begin{Bmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{Bmatrix}$

then $\mathbf{A}-\mathbf{B}=\begin{Bmatrix} a_1-a_3 & b_1-b_3 & c_1-c_3 \\ a_2-a_4 & b_2-b_4 & c_2-c_4 \end{Bmatrix}$.

***Note.** If the two matrices **A** and **B** are of the same order, then only their addition and subtraction is possible and these matrices are said to be **conformable** for addition or subtraction. On the other hand if the matrices **A** and **B** are of different orders, then their addition and subtraction is not possible and these matrices are called **non-conformable** for addition and subtraction.

§ 1.07. Properties of Matrix addition.

Property I. Addition of matrices is commutative.

i.e. $$[a_{ij}]+[b_{ij}]=[b_{ij}]+[a_{ij}],$$

where $[a_{ij}]$ and $[b_{ij}]$ are any two $m\times n$ matrices i.e. matrices of the same order.

(Meerut 95)

Proof : $[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$, by definition of addition

$=[b_{ij}+a_{ij}]$, $\because$ addition of numbers (elements) is commutative

$=[b_{ij}]+[a_{ij}]$

i.e. $[a_{ij}]+[b_{ij}]=[b_{ij}]+[a_{ij}]$ Hence the theorem.

Property II. Addition of matrices is associative.

i.e. $$\{[a_{ij}]+[b_{ij}]\}+[c_{ij}]=[a_{ij}]+\{[b_{ij}]+[c_{ij}]\},$$

where $[a_{ij}]$, $[b_{ij}]$ and $[c_{ij}]$ are any three matrices of the same order $m\times n$, say.

Proof : $\{[a_{ij}]+[b_{ij}]\}+[c_{ij}]$

$=[a_{ij}+b_{ij}]+[c_{ij}]$, by law of addition for matrices

$=[(a_{ij}+b_{ij})+c_{ij}]$, by law of addition for matrices

$=[a_{ij}+(b_{ij}+c_{ij})]$, $\because$ addition of numbers is associative

$=[a_{ij}]+[b_{ij}+c_{ij}]$

$=[a_{ij}]+\{[b_{ij}]+[c_{ij}]\}$. Hence the theorem.

Property III. Addition for matrices obey the distributive law.

i.e. $$k([a_{ij}]+[b_{ij}])=k[a_{ij}]+k[b_{ij}],$$

where $[a_{ij}]$ and $[b_{ij}]$ are any two matrices of the same order $m\times n$, say.

Proof : $k([a_{ij}]+[b_{ij}])=k[a_{ij}+b_{ij}]$, by law of addition

$=[k(a_{ij}+b_{ij})]$, by law of scalar multiplication

$=[ka_{ij}+kb_{ij}]$, by distributive law for numbers.

$=[ka_{ij}]+[kb_{ij}]$

$=k[a_{ij}]+k[b_{ij}]$. Hence the theorem.

Property IV. Existence of additive identity.

If $\mathbf{A}=[a_{ij}]$ be any $m\times n$ matrix and $\mathbf{O}$ be the $m\times n$ null matrix then

$$\mathbf{A}+\mathbf{O}=\mathbf{A}=\mathbf{O}+\mathbf{A}$$

Proof : Here $\mathbf{A}=[a_{ij}]_{m\times n}$ and $\mathbf{O}=[0]_{m\times n}$

Then $\mathbf{A}+\mathbf{O}=[a_{ij}]_{m\times n}+[0]_{m\times n}$.

$$= [a_{ij} + 0]_{m \times n}, \text{ by def. of addition}$$
$$= [a_{ij}]_{m \times n} = \mathbf{A} \quad \text{...(i)}$$

Again $\quad \mathbf{O} + \mathbf{A} = [0]_{m \times n} + [a_{ij}]_{m \times n}$
$$= [0 + a_{ij}]_{m \times n}, \text{ by def. of addition}$$
$$= [a_{ij}]_{m \times n} = \mathbf{A} \quad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we get $\mathbf{A} + \mathbf{O} = \mathbf{A} = \mathbf{O} + \mathbf{A}$

Thus we observe that **O** (the null matrix) is the additive identity.

Property V. Existence of addititive inverse.

If $\mathbf{A} = [a_{ij}]$ *be any* $m \times n$ *matrix, there exists another* $m \times n$ *matrix* **B** *such that*

$$\mathbf{A} + \mathbf{B} = \mathbf{O} = \mathbf{B} + \mathbf{A},$$

where **O** *is the* $m \times n$ *null matrix.*

Here the matrix **B** is called the additive inverse of the matrix **A** or the negative of **A**.

Also the (i, j)th element of **B** is $-a_{ij}$ if $\mathbf{A} = [a_{ij}]$

Property VI. Cancellation Law.

If **A, B, C** *are three matrices of the same order* $m \times n$, *say such that* $\mathbf{A} + \mathbf{B} = \mathbf{A} + \mathbf{C}$, *then* $\mathbf{B} = \mathbf{C}$

Proof : Given $\mathbf{A} + \mathbf{B} = \mathbf{A} + \mathbf{C}$

or $\quad -\mathbf{A} + (\mathbf{A} + \mathbf{B}) = -\mathbf{A} + (\mathbf{A} + \mathbf{C})$, adding $-\mathbf{A}$ from left on both sides

or $\quad (-\mathbf{A} + \mathbf{A}) + \mathbf{B} = (-\mathbf{A} + \mathbf{A}) + \mathbf{C}$, by associative law of adition

or $\quad \mathbf{O} + \mathbf{B} = \mathbf{O} + \mathbf{C}$, by def. of additive inverse

or $\quad \mathbf{B} = \mathbf{C}$, by def. of additive identity.

Solved Examples on § 1.04 to § 1.07.

Ex. 1. If $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ **evaluate 3A – 4B.** *(Avadh 90)*

Sol. $3A - 4B = 3\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -4 \\ 0 & -4 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 9-8 & 3-(-4) \\ 0-0 & -3-(-4) & 15-12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 7 \\ 0 & 1 & 3 \end{bmatrix} \quad \textbf{Ans.}$$

Ex. 2. If $\mathbf{A} = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$ **then show that** $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 0 & 13 \\ 2 & 0 & 7 \end{bmatrix}$, $\mathbf{A} - \mathbf{B} = \begin{bmatrix} 0 & 10 & -1 \\ -14 & 14 & -7 \end{bmatrix}$

Sol. Do yourself as Ex. 1 above.

Ex. 3. Determine the matrix A, where

$$A = 2\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 4 & 5 \end{bmatrix} + 3\begin{bmatrix} 3 & 3 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

Sol. $A = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 8 \\ 2 & 8 & 10 \end{bmatrix} + \begin{bmatrix} 9 & 9 & -3 \\ 6 & 6 & 9 \\ -3 & 9 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2+9 & 4+9 & 6+(-3) \\ 6+6 & 4+6 & 8+9 \\ 2+(-3) & 8+9 & 10+3 \end{bmatrix} = \begin{bmatrix} 11 & 13 & 3 \\ 12 & 10 & 17 \\ -1 & 17 & 13 \end{bmatrix}$$ **Ans.**

Ex. 4. Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, find the matrix C, such that $A + 2C = B$.

Sol. Given that $A + 2C = B$ or $2C = B - A$

or $2C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3-1 & -1-2 & 2-(-3) \\ 4-5 & 2-0 & 5-2 \\ 2-1 & 0-(-1) & 3-1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

or $C = (1/2)\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -(3/2) & (5/2) \\ -(1/2) & 1 & (3/2) \\ (1/2) & (1/2) & 1 \end{bmatrix}$ **Ans.**

Ex. 5. Solve the following equations for A and B;

$$2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}, \quad 2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

Sol. Given $2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$

Multiplying both sides by 2, we get

$$4A - 2B = 2\begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} \quad \text{...(i)}$$

Also given that $2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$ ...(ii)

Adding (i) and (ii) we get

$$5A = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+4 & -6+1 & 0+5 \\ 6-1 & 6+4 & 4-4 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix}$$

or $A = (1/5)\begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ **Ans.**

Again from (ii) we get

$$2\mathbf{B}=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}-\mathbf{A}$$

or $$2\mathbf{B}=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}-\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}4-2 & 1+1 & 5-1\\ -1-1 & 4-2 & -4-0\end{bmatrix}=\begin{bmatrix}2 & 2 & 4\\ -2 & 2 & -4\end{bmatrix}$$

or $$\mathbf{B}=(1/2)\begin{bmatrix}2 & 2 & 4\\ -2 & 2 & -4\end{bmatrix}=\begin{bmatrix}1 & 1 & 2\\ -1 & 1 & -2\end{bmatrix}$$ **Ans.**

Exercises on § 1.04 to § 1.07.

***Ex. 1.** If **X**, **Y** are two matrices given by the equations

$\mathbf{X}+\mathbf{Y}=\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}$ and $\mathbf{X}-\mathbf{Y}=\begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}$, find **X, Y**.

Ans. $\mathbf{X}=\begin{bmatrix}2 & 0\\ 1 & 2\end{bmatrix}$, $\mathbf{Y}=\begin{bmatrix}-1 & -2\\ 2 & 2\end{bmatrix}$

Ex. 2. If $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & 7\\ 6 & 8 & 9\end{bmatrix}$ $\mathbf{B}=\begin{bmatrix}2 & 0 & 3\\ 3 & 0 & 5\\ 5 & 7 & 0\end{bmatrix}$ evaluate $2\mathbf{A}-3\mathbf{B}$.

Ans. $\begin{bmatrix}-4 & 4 & -3\\ -9 & 10 & -1\\ -3 & -5 & 18\end{bmatrix}$

Ex. 3. If $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 3 & 2 & 1\end{bmatrix}$, then $2\mathbf{A}$ equals

(i) $\begin{bmatrix}2 & 4 & 6\\ 2 & 1 & 3\\ 3 & 2 & 1\end{bmatrix}$; (ii) $\begin{bmatrix}2 & 2 & 3\\ 2 & 2 & 3\\ 3 & 2 & 2\end{bmatrix}$;

(iii) $\begin{bmatrix}2 & 4 & 6\\ 4 & 2 & 6\\ 6 & 4 & 2\end{bmatrix}$; (iv) $\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 6 & 4 & 2\end{bmatrix}$ **Ans.** (iii)

Ex. 4. If $\mathbf{A}=\begin{bmatrix}\sec^2\theta & \sin^2\theta\\ 1/3 & \operatorname{cosec}^2\theta\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}-\tan^2\theta & \cos^2\theta\\ 2/3 & -\cot^2\theta\end{bmatrix}$

then $\mathbf{A}+\mathbf{B}$ is

(i) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$; (ii) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$;

(iii) $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$; (iv) $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ **Ans.** (iii)

Ex. 5. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ and $\mathbf{A} + \mathbf{B} = \mathbf{0}$, then $\mathbf{B}$ equals ....

$\frac{1}{\sqrt{3}}$

Ans. $\mathbf{B} = \begin{bmatrix} -1 & -2 & -3 \\ -2 & -3 & -1 \\ -3 & -1 & -2 \end{bmatrix}$

***§ 1.08. Multiplication of matrices.** *(Gorakhpur 95)*

If $\mathbf{A}$ and $\mathbf{B}$ be two matrices such that the number of columns in $\mathbf{A}$ is equal to the number of rows in $\mathbf{B}$ *i.e.* if $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{jk}]$ then the product of $\mathbf{A}$ and $\mathbf{B}$ denoted by $\mathbf{AB}$ is defined as matrix $[c_{ik}]$, where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ or in other words the product $\mathbf{AB}$ is defined as the matrix whose element in the ith row and kth column is $a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \ldots + a_{in} b_{nk}$.

The product matrix will have i rows and k columns.

Thus we conclude that :

'If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times k$ matrix then the product matrix $\mathbf{AB}$ is an $m \times k$ matrix.' **(Remember)**

As an example, consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

Here the number of columns in $\mathbf{A} = 3 =$ the number of rows in $\mathbf{B}$ and thus we can evaluate $\mathbf{AB}$.

Let $\mathbf{AB} = [c_{ij}]$, where $[c_{ij}]$ is 2×2 matrix.

Now to write c_{11}, we take the element of the first row of $\mathbf{A}$ *viz.* 1, 2, 3 in this order and the elements of the first column of $\mathbf{B}$ *viz.* 7, 9, 11 in this order and form the products $1 \cdot 7$, $2 \cdot 9$, $3 \cdot 11$ and finally add them.

i.e. $c_{11} = 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 = 58$

Similarly $c_{12} = 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 = 64;$

$c_{21} = 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 = 139$

and $c_{22} = 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 = 154$

Hence $\mathbf{AB} = [c_{ij}] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$.

Note. The product $\mathbf{AB}$ can be calculated only if the number of columns in $\mathbf{A}$ be equal to the number of rows in $\mathbf{B}$. The two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfying this condition are called *conformable to multiplication.*

Post-multiplication and Pre-multiplication of matrices.

The matrix $\mathbf{AB}$ is the matrix $\mathbf{A}$ post-multiplied by $\mathbf{B}$ whereas the matrix $\mathbf{BA}$ is the matrix $\mathbf{A}$ pre-multiplied by $\mathbf{B}$.

In the product **AB**, the matrix **A** is known as the **pre-factor** and the matrix **B** is known as the **post-factor.**

The product in both the above cases *viz.* **AB** and **BA** may or may not exist and may be equal or different,

i.e. we say $\mathbf{AB} \neq \mathbf{BA}$ *in general.* (*Bundelkhand 93; Gorakhpur 90*)

The same is discussed below :

Case I. If the matrix **A** is $m \times n$ and the matrix **B** is $n \times k$, then the product **AB** exists whereas **BA** does not exist, since we know that **AB** can be calculated only if the numbers of columns in **A** is equal to the number of rows in **B**.

Case II. If the matrix **A** is $m \times n$ and the matrix **B** is $n \times m$, then both **AB** and **BA** exist, but the matrix **AB** is $m \times m$ while the matrix **BA** is $n \times n$. **(Note)**

Hence $\mathbf{AB} \neq \mathbf{BA}$ though **AB** and **BA** exist.

Case III. If both **A** and **B** are square matrices of the same order, then **AB** as well as **BA** exist but are not necessarily equal

i.e. if $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$

then $\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 4 & 1 \cdot 1 + 2 \cdot 7 \\ 3 \cdot 3 + 4 \cdot 4 & 3 \cdot 1 + 4 \cdot 7 \end{bmatrix}$

$= \begin{bmatrix} 11 & 15 \\ 25 & 31 \end{bmatrix}$

and $\mathbf{BA} = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 1 \cdot 3 & 3 \cdot 2 + 1 \cdot 4 \\ 4 \cdot 1 + 7 \cdot 3 & 4 \cdot 2 + 7 \cdot 4 \end{bmatrix}$

$= \begin{bmatrix} 6 & 10 \\ 25 & 36 \end{bmatrix}$

$\therefore$ $\mathbf{AB} \neq \mathbf{BA}$.

But if $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

then $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 4 \\ 0 \cdot 1 - 2 \cdot 0 & 0 \cdot 0 - 2 \cdot 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}$

and $\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0\,(-2) \\ 0 \cdot 1 + 4 \cdot 0 & 0 \cdot 0 + 4\,(-2) \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}$

$\therefore$ $\mathbf{AB} = \mathbf{BA}$.

Hence in general $\mathbf{AB} \neq \mathbf{BA}$. (*Gorakhpur 95, 90*)

Note 1. If $\mathbf{AB} = \mathbf{BA}$, then matrices **A** and **B** are said to commute. If $\mathbf{AB} = -\mathbf{BA}$, the matrices **A** and **B** are said to anticommute.

Note 2. *The product of two non-zero matrices can also be a zero (or null) matrix.* *(Avadh 93; Gorakhpur 91; Meerut 96P)*

Let $\mathbf{A}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$,

then $\mathbf{AB}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}=\begin{bmatrix}1\cdot1+1\cdot(-1) & 1\cdot0+1\cdot0\\ 1\cdot1+1\cdot(-1) & 1\cdot0+1\cdot0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{O}$

i.e. **AB** is zero matrix (or null matrix) where neither **A** nor **B** is a zero matrix.

$\therefore$ $\mathbf{AB}=\mathbf{O}$ does not imply that either $\mathbf{A}=\mathbf{O}$ **or** $\mathbf{B}=\mathbf{O}$.

Here $\mathbf{BA}=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}\times\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

$=\begin{bmatrix}1\cdot1+0\cdot1 & 1\cdot1+0\cdot1\\ -1\cdot1+0\cdot1 & -1\cdot1+0\cdot1\end{bmatrix}=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$

i.e. $\mathbf{BA}\neq\mathbf{O}$

Another Example.

If $\mathbf{A}=\begin{bmatrix}4 & 4\\ 3 & 3\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$, then

$\mathbf{AB}=\begin{bmatrix}4 & 4\\ 3 & 3\end{bmatrix}\times\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$

$=\begin{bmatrix}4(-1)+4(1) & 4(1)+4(-1)\\ 3(-1)+3(1) & 3(1)+3(-1)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{O}$

i.e. the product of two non-zero square matrices can be a zero matrix.

and $\mathbf{BA}=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\times\begin{bmatrix}4 & 4\\ 3 & 3\end{bmatrix}$

$=\begin{bmatrix}(-1)\cdot4+1\cdot3 & (-1)\cdot4+1\cdot3\\ 1\cdot4+(-1)\cdot3 & 1\cdot4+(-1)\cdot3\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\neq\mathbf{O}$

***Note 3.** *The multiplication of matrices generally does not obey the law of cancellation.*

Let $\mathbf{A}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$, $\mathbf{C}=\begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix}$,

where $a\neq b$

Then $\mathbf{AB}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{O}$

and $\mathbf{AC}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{O}$

$\therefore$ It is evident that here $\mathbf{AB}=\mathbf{AC}$ but $\mathbf{B}\neq\mathbf{C}$.

$\therefore$ Law of cancellation is not obeyed in general.

Solved Examples on § 1.08

Ex. 1 (a). A is any $m \times n$ matrix such that AB and BA are both defined. What is the order of B ?

Sol. The required order of **B** is $\mathbf{n \times m}$. (See Case II Page 12)

Ex. 1 (b). Multiply $[3 - 1\ 4]$ and $\begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$

Sol. $[3\ \ -1\ \ 4] \times \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix} = [3(-2) + (-1)\cdot 6 + 4\cdot 3] = [0]$ **Ans.**

Ex. 1 (c). If $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$ **find AB.**

Sol. $AB = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot 0 + 2\cdot 0 + 0\cdot 1 & 1\cdot 0 + 2\cdot 0 + 0\cdot 4 & 1\cdot 0 + 2\cdot 0 + 0\cdot 9 \\ 1\cdot 0 + 1\cdot 0 + 0\cdot 1 & 1\cdot 0 + 1\cdot 0 + 0\cdot 4 & 1\cdot 0 + 1\cdot 0 + 0\cdot 9 \\ -1\cdot 0 + 4\cdot 0 + 0\cdot 1 & -1\cdot 0 + 4\cdot 0 + 0\cdot 4 & -1\cdot 0 + 4\cdot 0 + 0\cdot 9 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$, where **O** is the null matrix of order 3. **Ans.**

Ex. 2. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$

then find AB. Whether BA exists ? Give reason. *(Purvanchal 89)*

Sol. $AB = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3\cdot 1 + 1\cdot 2 + 2\cdot 1 & 3\cdot 4 + 1\cdot 2 + 2\cdot 0 \\ 0\cdot 1 + 1\cdot 2 + 1\cdot 1 & 0\cdot 4 + 1\cdot 2 + 1\cdot 0 \\ 1\cdot 1 + 2\cdot 2 + 0\cdot 1 & 1\cdot 4 + 2\cdot 2 + 0\cdot 0 \end{bmatrix}$$

$= \begin{bmatrix} 7 & 14 \\ 3 & 2 \\ 5 & 8 \end{bmatrix}$ **Ans.**

Here **A** is a matrix of order 3×3 and **B** is a matrix of order 3×2.

Hence **BA** does not exist as number of columns in **B** is not equal to the number of rows in **A**.

***Ex. 3 (a).** If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

find AB and show that AB ≠ BA. *(Rohilkhand 97)*

Sol. $AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot2 + (-2)\cdot4 + 3\cdot2 & 1\cdot3 + (-2)\cdot5 + 3\cdot1 \\ -4\cdot2 + 2\cdot4 + 5\cdot2 & -4\cdot3 + 2\cdot5 + 5\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

and $BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot1 + 3(-4) & 2(-2) + 3(2) & 2(3) + 3(5) \\ 4\cdot1 + 5(-4) & 4(-2) + 5(2) & 4(3) + 5(5) \\ 2\cdot1 + 1(-4) & 2(-2) + 1(2) & 2(3) + 1(5) \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Hence $AB \neq BA$.

Ex. 3 (b). If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

then prove that AB ≠ BA. *(Meerut 97)*

Sol. $AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot1 + 3(-1) + 4\cdot0 & 2\cdot3 + 3\cdot2 + 4\cdot0 & 2\cdot0 + 3\cdot1 + 4\cdot2 \\ 1\cdot1 + 2(-1) + 3\cdot0 & 1\cdot3 + 2\cdot2 + 3\cdot0 & 1\cdot0 + 2\cdot1 + 3\cdot2 \\ (-1)1 + 1(-1) + 2\cdot0 & (-1)3 + 1\cdot2 + 2\cdot0 & (-1)0 + 1\cdot1 + 2\cdot2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3+0 & 6+6+0 & 0+3+8 \\ 1-2+0 & 3+4+0 & 0+2+6 \\ -1-1+0 & -3+2+0 & 0+1+4 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix} \quad \text{...(i)}$$

and $BA = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot2 + 3\cdot1 + 0(-1) & 1\cdot3 + 3\cdot2 + 0\cdot1 & 1\cdot4 + 3\cdot3 + 0\cdot2 \\ (-1)2 + 2\cdot1 + 1(-1) & (-1)3 + 2\cdot2 + 1\cdot1 & (-1)4 + 2\cdot3 + 1\cdot2 \\ 0\cdot2 + 0\cdot1 + 2(-1) & 0\cdot3 + 0\cdot2 + 2\cdot1 & 0\cdot4 + 0\cdot3 + 2\cdot2 \end{bmatrix}$$

$$=\begin{bmatrix} 2+3+0 & 3+6+0 & 4+9+0 \\ -2+2-1 & -3+4+1 & -4+6+2 \\ 0+0-2 & 0+0+2 & 0+0+4 \end{bmatrix}=\begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix} \quad \text{...(ii)}$$

From (i) and (ii) we find that **AB** $\neq$ **BA**. Hence proved.

****Ex. 4. If** $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}\mathbf{A}=\begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$**, find A.**

Sol. From § 1.08 Page 11 we know that if **X** is an $m \times n$ matrix, **Y** is an $n \times k$ matrix, then the product **XY** is an $m \times k$ matrix.

Here $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ is 3×1 matrix and $\begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$ is 3×3 matrix, so **A** must be a 1×3 matrix *i.e.* a row matrix. **(Note)**

$\therefore$ Let $\mathbf{A}=[a \ \ b \ \ c]$

Then $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \times [a \ \ b \ \ c]=\begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$

which gives $\begin{bmatrix} 4a & 4b & 4c \\ a & b & c \\ 3a & 3b & 3c \end{bmatrix}=\begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$

Comparing corresponding elements we have

$4a=-4$, $a=-1$, $3a=-3$, $4b=8$, $b=2$, $3b=6$ and $4c=4$, $c=1$, $3c=3$.

All these are satisfied by $a=-1, b=2, c=1$.

Hence from (i) we have $\mathbf{A}=[a \ \ b \ \ c]=[-1, 2, 1]$. **Ans.**

Ex. 5. If $\mathbf{A}=\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}$**, show that** $\mathbf{A}^2=\mathbf{O}$

Sol. $\mathbf{A}^2=\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot1+1\cdot2+3\cdot(-1) & 1\cdot1+1\cdot2+3(-1) & 1\cdot3+1\cdot6+3(-3) \\ 2\cdot1+2\cdot2+6\cdot(-1) & 2\cdot1+2\cdot2+6(-1) & 2\cdot3+2\cdot6+6(-3) \\ -1\cdot1-1\cdot2-3\cdot(-1) & -1\cdot1-1\cdot2-3(-1) & -1\cdot3-1\cdot6-3(-3) \end{bmatrix}$$

$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\mathbf{O}$, where **O** is 3×3 null matrix.

Hence proved.

Ex. 6. Find the square of the matrix

$$\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Sol. $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}^2$

$= \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} (-1)(-1)+1\cdot1+1\cdot1+1\cdot1 & (-1)1+1(-1)+1\cdot1+1\cdot1 & (-1)1+1\cdot1+1(-1)+1\cdot1 & (-1)1+1\cdot1+1\cdot1+1(-1) \\ 1(-1)+(-1)1+1\cdot1+1\cdot1 & 1\cdot1+(-1)(-1)+1\cdot1+1\cdot1 & 1\cdot1+(-1)1+1(-1)+1\cdot1 & 1\cdot1+(-1)1+1\cdot1+1(-1) \\ 1(-1)+1\cdot1+(-1)1+1\cdot1 & 1\cdot1+1(-1)+(-1)1+1\cdot1 & 1\cdot1+1\cdot1+(-1)(-1)+1\cdot1 & 1\cdot1+1\cdot1+(-1)1+1(-1) \\ 1(-1)+1\cdot1+1\cdot1+(-1)1 & 1\cdot1+1(-1)+1\cdot1+(-1)1 & 1\cdot1+1\cdot1+1(-1)+(-1)1 & 1\cdot1+1\cdot1+1\cdot1+(-1)(-1) \end{bmatrix}$$

$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4\mathbf{I}.$ **Ans.**

***Ex. 7.**

$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix}$

show that AB and CA are null matrices but BA $\neq$ O, AC $\neq$ O.

Sol. $\mathbf{AB} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 1(-1)+1\cdot6+(-1)5 & 1(-2)+1\cdot12+(-1)10 & 1(-1)+1\cdot6+(-1)5 \\ 2(-1)-3\cdot6+4\cdot5 & 2(-2)-3\cdot12+4\cdot10 & 2(-1)-3\cdot6+4\cdot5 \\ 3(-1)-2\cdot6+3\cdot5 & 3(-2)-2\cdot12+3\cdot10 & 3(-1)-2\cdot6+3\cdot5 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which is a null matrix.

(See § 1.03 Page 4)

This is known as 'unusual property' of Matrix Multiplication

$\mathbf{CA} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -1\cdot1-1\cdot2+1\cdot3 & -1\cdot1-1(-3)+1(-2) & -1(-1)-1\cdot4+1\cdot3 \\ 2\cdot1+2\cdot2-2\cdot3 & 2\cdot1+2(-3)-2(-2) & 2(-1)+2\cdot4-2\cdot3 \\ -3\cdot1-3\cdot2+3\cdot3 & -3\cdot1-3(-3)+3(-2) & -3(-1)-3\cdot4+3\cdot3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ which is a null matrix.}$$

Hence proved.

We can prove in a simillar way that $\mathbf{BA} \neq \mathbf{O}$ and $\mathbf{AC} \neq \mathbf{O}$.

Ex. 8. Find the product of the following two matrices

$$\begin{bmatrix} \mathbf{0} & \mathbf{c} & -\mathbf{b} \\ -\mathbf{c} & \mathbf{0} & \mathbf{a} \\ \mathbf{b} & -\mathbf{a} & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} \mathbf{a^2} & \mathbf{ab} & \mathbf{ac} \\ \mathbf{ab} & \mathbf{b^2} & \mathbf{bc} \\ \mathbf{ac} & \mathbf{bc} & \mathbf{c^2} \end{bmatrix}$$

(*Bundelkhand 93; Kanpur 94*)

Sol. The required product

$$= \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \times \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0\cdot a^2 + c\cdot ab - b\cdot ac & 0\cdot ab + c\cdot b^2 - b\cdot bc & 0\cdot ac + c\cdot bc - b\cdot c^2 \\ -c\cdot a^2 + 0\cdot ab + a\cdot ac & -c\cdot ab + 0\cdot b^2 + a\cdot bc & -c\cdot ac + 0\cdot bc + a\cdot c^2 \\ b\cdot a^2 - a\cdot ab + 0\cdot ac & b\cdot cb - a\cdot b^2 + 0\cdot bc & b\cdot ac - a\cdot bc + 0\cdot c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ans.

****Ex. 9. Prove that the product of two matrices**

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is zero when θ and ϕ differ by an odd multiple of $\frac{1}{2}\pi$.

(*Bundelkhand 92; Meerut 91 S*)

Sol. The required product

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \cos\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \end{bmatrix},$$

$$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

If $\theta \sim \phi =$ an odd multiple of $\frac{1}{2}\pi$, then $\cos(\theta \sim \phi) = 0$ and consequently the above product is zero (*i.e.* the null matrix of order 2×2).

***Ex. 10. If** $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

show that AB = BA. (*Gorakhpur 90*)

Sol. $\mathbf{AB} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\phi\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \qquad \text{...(i)}$$

And $\mathbf{BA} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we get **AB = BA.** Hence proved.

****Ex. 11. If A, B, C are three matrices such that**

$\mathbf{A} = [\mathbf{x}, \mathbf{y}, \mathbf{z}]$, $\mathbf{B} = \begin{bmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{h} & \mathbf{b} & \mathbf{f} \\ \mathbf{g} & \mathbf{f} & \mathbf{c} \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}$ **evaluate ABC.**

(*Gorakhpur 94; Kanpur 93; Kumaun 94; Purvanchal 90*)

Sol. $\mathbf{AB} = [x, y, z] \times \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

$$= [x.a + y.h + z.g \quad x.h + y.b + z.f \quad x.g + y.f + z.c]$$

or $\mathbf{ABC} = [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz] \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= [x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz)] \qquad \textbf{(Note)}$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]. \qquad \textbf{Ans.}$$

Ex. 12 If $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{5} \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-6} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$

evaluate (a) $\mathbf{A^2 - B^2}$ **and (b) AB and BA.**

Sol. (a) $\mathbf{A}^2 = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$, which does not exist as number of columns in the first matrix is not equal to number of rows in the second matrix.

Similarly $\mathbf{B}^2$ does not exit.

(b) AB and **BA** both do not exist, the reason being the same as in part (a) above.

***Ex. 13. Evalute $\mathbf{A}^3$ if $\mathbf{A} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$**

Sol. $\mathbf{A}^2 = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \times \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$

$$= \begin{bmatrix} \cosh^2\theta + \sinh^2\theta & \cosh\theta\sinh\theta + \sinh\theta\cosh\theta \\ \sinh\theta\cosh\theta + \cosh\theta\sin\theta & \sinh^2\theta + \cosh^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix} \quad \because \cosh^2\theta + \sinh^2\theta = \cosh^2\theta, \quad 2\sinh\theta\cosh\theta = \sinh 2\theta$$

$$\therefore \quad \mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix} \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh 2\theta\cosh\theta + \sinh 2\theta\sin\theta & \cosh 2\theta\sinh\theta + \sinh 2\theta\cosh\theta \\ \sinh 2\theta\cosh\theta + \cosh 2\theta\sinh\theta & \sinh 2\theta\sinh\theta + \cosh 2\theta\cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(2\theta+\theta) & \sinh(2\theta+\theta) \\ \sinh(2\theta+\theta) & \cosh(2\theta+\theta) \end{bmatrix},$$

$$\because \sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B$$
$$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$$

$$= \begin{bmatrix} \cosh 3\theta & \sinh 3\theta \\ \sinh 3\theta & \cosh 3\theta \end{bmatrix}$$

Ans.

Ex. 14. If $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, evaluate $\mathbf{A}^3$.

Sol. $\mathbf{A}^2 = \mathbf{A}\,\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot2 - 1\cdot0 + 1\cdot1 & 2(-1) - 1\cdot1 + 1\cdot0 & 2\cdot1 - 1\cdot2 + 1\cdot1 \\ 0\cdot2 + 1\cdot0 + 2\cdot1 & 0(-1) + 1\cdot1 + 2\cdot0 & 0\cdot1 + 1\cdot2 + 2\cdot1 \\ 1\cdot2 + 0\cdot1 + 1\cdot1 & 1(-1) + 0\cdot1 + 1\cdot0 & 1\cdot1 + 0\cdot2 + 1\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & -2-1+0 & 2-2+1 \\ 0+0+2 & 0+1+0 & 0+2+2 \\ 2+0+1 & -1+0+0 & 1+0+1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^3 = \mathbf{A}^2\,\mathbf{A} = \begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 5\cdot2-3\cdot0+1\cdot1 & 5(-1)-3\cdot1+1\cdot0 & 5\cdot1-3\cdot2+1\cdot1 \\ 2\cdot2+1\cdot0+4\cdot1 & 2(-1)+1\cdot1+4\cdot0 & 2\cdot1+1\cdot2+4\cdot1 \\ 3\cdot2-1\cdot0+2\cdot1 & 3(-1)-1\cdot1+2\cdot0 & 3\cdot1-1\cdot2+2\cdot1 \end{bmatrix}$$

$$=\begin{bmatrix} 10-0+1 & -5-3+0 & 5-6+1 \\ 4+0+4 & -2+1+0 & 2+2+4 \\ 6-0+2 & -3-1+0 & 3-2+2 \end{bmatrix}=\begin{bmatrix} 11 & -8 & 0 \\ 8 & -1 & 8 \\ 8 & -4 & 3 \end{bmatrix}$$ **Ans.**

Ex. 15. If $\mathbf{A}=\begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & -\mathbf{i} \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$, $\mathbf{C}=\begin{bmatrix} \mathbf{0} & \mathbf{i} \\ \mathbf{i} & \mathbf{0} \end{bmatrix}$,

Prove that

$\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=-\mathbf{I}$ **and** $\mathbf{AB}=-\mathbf{C}=-\mathbf{BA}$**, where** $\mathbf{I}=\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$

(*Kumaun 92*)

Sol. $\mathbf{A}^2=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\times\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$$=\begin{bmatrix} i\cdot i+0\cdot0 & i\cdot0+0(-i) \\ 0\cdot i-i\cdot0 & 0\cdot0+(-i)(-i) \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$=-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ...See § 1.04 Page 6

or $\mathbf{A}^2=-\mathbf{I}.$

and $\mathbf{B}^2=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\times\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0\cdot0-1\cdot1 & 0(-1)+(-1)\cdot0 \\ 1\cdot0+0\cdot1 & 1(-1)+0\cdot0 \end{bmatrix}$

$$=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}=-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=-\mathbf{I}$$

Similarly we can prove that $\mathbf{C}^2=-\mathbf{I}$. Hence $\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=-\mathbf{I}$.

Again $\mathbf{AB}=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\times\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$=\begin{bmatrix} i\cdot0+0\cdot1 & i(-1)+0\cdot0 \\ 0\cdot0-i(1) & 0(-1)-i\cdot0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}=-\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}=-\mathbf{C}$$

and $\mathbf{BA}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\times\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$$=\begin{bmatrix} 0\cdot i-1\cdot0 & 0\cdot0-1(-i) \\ 1\cdot i+0\cdot0 & 1\cdot0+0(-i) \end{bmatrix}=\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}=\mathbf{C}.$$

Hence $\mathbf{AB}=-\mathbf{C}=-\mathbf{BA}$.

***Ex. 16. If** $\mathbf{A}=\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$; $\mathbf{B}=\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}$ **and** $\mathbf{AB}=\begin{bmatrix} \mathbf{6} \\ \mathbf{3} \\ \mathbf{1} \end{bmatrix}$

find the values of x, y, z.

Sol. $AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

or $$\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot x + 2\cdot y + 3\cdot z \\ 0\cdot x + 1\cdot y + 2\cdot z \\ 0\cdot x + 0\cdot y + 1\cdot z \end{bmatrix}$$

or $6 = x + 2y + 3z,\ 3 = y + 2z,\ 1 = z,$ (Note)

comparing the corresponding elements of the matrices on both sides

Solving these we get $x = 1,\ y = 1,\ z = 1.$ **Ans.**

Ex. 17. Find the values of x, y, z in the following equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & -6 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Sol. $$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1\cdot x + 2\cdot y + 3\cdot z \\ 3\cdot x + 1\cdot y + 2\cdot z \\ 2\cdot x + 3\cdot y + 1\cdot z \end{bmatrix} \quad ...(i)$$

And $$\begin{bmatrix} 4 & -2 \\ 0 & -6 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4\cdot 2 + (-2)\, 1 \\ 0\cdot 2 + (-6)\, 1 \\ -1\cdot 2 + \quad 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 0 \end{bmatrix} \quad ...(ii)$$

With the help of (i) and (ii), the given equation reduces to

$$\begin{bmatrix} x + 2y + 3z \\ 3x + y + 2z \\ 2x + 3y + z \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 0 \end{bmatrix}$$

From this on comparing the corresponding elements on both sides we get $x + 2y + 3z = 6$; $3x + y + 2z = -6$ and $2x + 3y + z = 0$.

Solving these we get $x = -4, y = 2, z = 2.$ **Ans.**

Ex. 18. Given $A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$; $A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$

$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$ **and** $A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Show that $A_iA_k + A_kA_i = 2I$ or O according as $i = k$ or $i \neq k$ and I is the unit matrix of order 4 and i and k take the values 1, 2, 3 and 4.

Sol. Let $i = k = 1$ (say). Then $A_iA_k = A_1A_1 = A_kA_i$

$$\therefore \quad A_iA_k = A_1A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0+1 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+1+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+1+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 1+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

$\therefore \quad \mathbf{A_iA_k + A_kA_i = I + I = 2I}$ Hence proved.

If $i \neq k$, let $i = 3$ and $k = 2$

Then $\mathbf{A_iA_k = A_3A_2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0+0+0+0 & 0+0+i+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+i & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & i+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+i+0+0 & 0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} = i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

And $\mathbf{A_kA_i = A_2A_3} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix}$, multiplying in the usual way

$$= -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad A_iA_k + A_kA_i = i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$= \mathbf{O}$ Hence proved.

We can in a similar way prove the above result by giving i and k other values also.

****Ex. 19. If** $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$**, then prove that**

(a) $A_\alpha \cdot A_\beta = A_{\alpha+\beta}$ **and (b)** $A_\alpha \cdot A_{-\alpha}$ **is unit matrix.**

Sol. (a) $A_\alpha \cdot A_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = A_{\alpha+\beta}$$

Hence proved.

(b) Here $A_{-\alpha} = \begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) \\ -\sin(-\alpha) & \cos(-\alpha) \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$\therefore \quad A_\alpha \cdot A_{-\alpha} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is an unit matrix.

Hence proved.

Exercises on § 1.08.

Ex. 1. Multiply $[4 \;\; 5 \;\; 6]$ and $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ **Ans.** [17]

Ex. 2. Multiply $[1 \;\; 2 \;\; 3]$ and $\begin{bmatrix} 4 & -6 & 9 & 6 \\ 0 & -7 & 10 & 7 \\ 5 & 8 & -11 & -8 \end{bmatrix}$

Ans. $[19 \;\; 4 \;\; -4 \;\; -4]$

Ex. 3. If $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, show that **AB** is a null matrix.

Ex. 4. Show that $\begin{bmatrix} -5 & 2 & 3 \\ -5 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Ex. 5. Show that $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Ex. 6. If $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find **AB** and **BA** if they exist.

Ex. 7. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

then prove that $\mathbf{AB} = \mathbf{O}$ but $\mathbf{BA} \neq \mathbf{O}$.

Ex. 8. If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

then prove that $\mathbf{AB} \neq \mathbf{BA}$.

Ex. 9. If $\mathbf{A} = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$

then show that $\mathbf{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *(Meerut 94)*

Ex. 10. Show that $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 0 & (1/2) & (1/2) \\ (1/2) & 0 & (1/2) \\ (1/2) & (1/2) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 11. Form the products **AB** and **BA**, when

$$\mathbf{A} = [1\ \ 2\ \ 3\ \ 4] \text{ and } \mathbf{B} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$$

Ans. $\mathbf{AB} = [30]$

Ex. 12. If $\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$,

then prove that $\mathbf{AB} - \mathbf{AC} = \mathbf{O}$.

Ex. 13. Show that $\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 0 \\ 0 & 4 & 1 \\ -2 & 1 & 0 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}$

(Bundelkhand 94)

Ex. 14. If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, evaluate $\mathbf{A}^2$.

Ans. $\begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$

Ex. 15. If $\mathbf{A}=\begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 2 & 2 & 4\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}1 & -2\\ -1 & 0\\ 1 & -1\end{bmatrix}$, find **AB** or **BA** whichever exists.

Ans. $\mathbf{AB}=\begin{bmatrix}1 & -2\\ 2 & -5\\ 3 & -8\end{bmatrix}$ and **BA** does not exist.

Ex. 16. If $\mathbf{A}=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ 3 & 1 & 2\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 1 & 2 & 0\end{bmatrix}$.

then prove that $\mathbf{AB} \neq \mathbf{BA}$.

****Ex. 17.** If **X, Y** are two matrices given by the equations $\mathbf{X}+\mathbf{Y}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $\mathbf{X}-\mathbf{Y}=\begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}$, find **XY**. **Ans.** $\mathbf{XY}=\begin{bmatrix}-2 & -4\\ 3 & 2\end{bmatrix}$

Ex. 18. In Ex. 11 Page 19 of this chapter, evaluate **A (BC)**.

(*Purvanchal 90*)

Ex. 19. If order of **A** is $m \times n$ and that of **C** is $m \times l$ and $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ then order of **B** will be (i) $l \times n$, (ii) $n \times l$, (iii) 1×3, (iv) 3×1. **Ans.** (ii)

Ex. 20. If **A** is $m \times n$ matrix, **B** is $n \times l$ matrix and **C** is $l \times k$ matrix, then the order of **(AB) C** will be (a) $m \times l$, (b) $n \times p$, (c) $m \times k$, (d) $k \times m$. **Ans.** (c)

§ 1.09. Properties of Multiplication of Matrices.

****Property I Multiplication of matrices is associative.**

(*Agra 96; Avadh 94, 92, 90; Garhwal 91; Gorakhpur 91; Rohilkhand 94*)

Let $\mathbf{A}=[a_{ij}]$; $\mathbf{B}=[b_{jk}]$, $\mathbf{C}=[c_{kr}]$ be three $m \times n$, $n \times p$ and $p \times l$ matrices respectively, then $(\mathbf{AB}) . \mathbf{C} = \mathbf{A} . (\mathbf{BC})$.

Proof. Let $\mathbf{AB}=[d_{ik}]$, where $d_{ik}=\sum_{j=1}^{n} a_{ij}\, b_{jk}$ ...(i)

Then $(\mathbf{AB}) . \mathbf{C} = [d_{ik}] \times [c_{kr}] = [e_{ir}]$,

where
$$e_{ir} = \sum_{k=1}^{p} d_{ik} . c_{kr}$$
$$= \sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij}\, b_{jk} \right) . c_{kr}, \text{ from (i)}$$

i.e. $(i . r)$th element of $(\mathbf{AB}) . \mathbf{C} = \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kr}$ ...(ii)

And let $\mathbf{BC}=[g_{jr}]$, $g_{jr}=\sum_{k=1}^{p} b_{jk}\, c_{kr}$ ...(iii)

Then $\mathbf{A} . \mathbf{BC} = [a_{ij}] \times [g_{jr}] = [h_{ir}]$,

where
$$h_{ir} = \sum_{j=1}^{n} a_{ij}\, g_{jr}$$

$$= \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk} c_{kr} \right), \text{ from (iii)}$$

i.e. (i, r)th element of $\mathbf{A} \cdot (\mathbf{BC}) = \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij} b_{jk} c_{kr}$, ...(iv)

since the summation can be interchanged.

$\therefore$ From (iii) and (iv) we can conclude that the (i, r)th elements of $(\mathbf{AB}) \cdot \mathbf{C}$ and $\mathbf{A} \cdot (\mathbf{BC})$ are the same and their orders are also $m \times l$.

Hence $(\mathbf{AB}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{BC})$.

****Property II. Multiplication of matrices is distributive with respect to matrix addition.** *(Bundelkhand 96, 92)*

(a) Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{jk}]$ and $\mathbf{C} = [c_{jk}]$ be three $m \times n$, $n \times p$ and $n \times p$ matrices respectively, then $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

(Avadh 93; Gorakhpur 93; Rohilkhand 93, 92)

Proof. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = [a_{ij}] \times \{[b_{jk}] + [c_{jk}]\}$

$= [a_{ij}] [b_{jk} + c_{jk}] = [d_{jk}]$, say,

where $$d_{jk} = \sum_{j=1}^{n} a_{ij} (b_{jk} + c_{jk})$$

or (i, k)th element of $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \sum_{j=1}^{n} a_{ij} b_{jk} + \sum_{j=1}^{n} a_{ij} c_{jk}$ ...(i)

Again $\mathbf{AB} = [a_{ij}] [b_{jk}] = [e_{ik}]$, say,

where $e_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ *i.e.* (i, k)th element of $\mathbf{AB} = \sum_{j=1}^{n} a_{ij} b_{jk}$ ...(ii)

Similarly we can prove

(i, k)th element of $\mathbf{AC} = \sum_{j=1}^{n} a_{ij} c_{jk}$ ...(iii)

$\therefore$ From (ii) and (iii) we have

(i, k)th element of $\mathbf{AB} + \mathbf{AC} = \sum_{j=1}^{n} a_{ij} b_{jk} + \sum_{j=1}^{n} a_{ij} c_{jk}$ ...(iv)

Hence from (i) and (iv) we conclude that $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

(b) Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{jk}]$ and $\mathbf{C} = [c_{jk}]$ be three $n \times p$, $m \times n$ and $m \times n$ matrices respectively.

Then $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$.

(**Note.** If $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ and $n \times p$ matrices then $\mathbf{BA}$ can not exist whereas $\mathbf{AB}$ exists).

Proof. Its proof is similar to that of part (a) above.

§ 1.10. Positive integral power of a square matrix.

From § 1.09 we find that if $\mathbf{A}$ is a square matrix, then only the product $\mathbf{AA}$ is defined and we write $\mathbf{A}^2$ for $\mathbf{AA}$.

Also by associative law

$$A^2A = (AA)\ A = A\ (AA) = AA^2$$

So A^2A or AA^2 is written as A^3.

In general AAA ... A is denoted by A^n if there are n factors.

Definition. If A be a square matrix, then AA .... n times $= A^n$

and $A^{m+1} = A^m . A$, where m is a positive integer.

Theorem I. *If* A *be a square matrix* ($n \times n$ *say*), *then*

$A^p . A^q = A^{p+q}$, *for any pair of positive integers p and q.*

Proof. We shall prove this by the method of induction.

From definition we know that $A^p . A = A^{p+1}$, where p is any positive integer.

$\therefore\ \ A^p A^q = A^{p+q}$ holds when $q = 1$, whatever p may be.

We shall now prove that if it holds for a particular value m say of q for all values of p, then it must hold for the value $m + 1$ of q for all values of p.

Now $A^p A^{m+1} = A^p . (A^m . A)$, by definition given above

$= (A^p . A^m) . A$, by associative law

$= (A^{p+m})\ A$, by hypothesis

$= A^{p+m+1}$, by definition given above

$= A^{p+(m+1)}$, by associative law of addition of numbers.

i.e. $A^p . A^q = A^{p+q}$ holds for the value $m + 1$ of q, whatever p may be if it holds for $q = m$.

Hence the proof by mathematical induction.

Theorem II. *If* A *be a square matrix, then*

$(A^p)^q = A^{pq}$, *for every pair of positive integers p and q*

Proof is similar to that of Theorem I above.

Solved Examples on § 1.09 — § 1.10.

***Ex. 1. Evaluate $A^2 - 4A - 5I$, where**

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Garhwal 90)

Sol. $$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot1 + 2\cdot2 + 2\cdot2 & 1\cdot2 + 2\cdot1 + 2\cdot2 & 1\cdot2 + 2\cdot2 + 2\cdot1 \\ 2\cdot1 + 1\cdot2 + 2\cdot2 & 2\cdot2 + 1\cdot1 + 2\cdot2 & 2\cdot2 + 1\cdot2 + 2\cdot1 \\ 2\cdot1 + 2\cdot2 + 1\cdot2 & 2\cdot2 + 2\cdot1 + 1\cdot2 & 2\cdot2 + 2\cdot2 + 1\cdot1 \end{bmatrix}$$

$$=\begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$\therefore\ \mathbf{A^2-4A-5I}$

$$=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}-4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}-5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}+\begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix}+\begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$=\begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\mathbf{O},$$ **Ans.**

where **O** is the null matrix.

Ex. 2. Let $\mathbf{f(x)=x^2-5x+6}$, find $\mathbf{f(A)}$ if $\mathbf{A}=\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Sol. $f(\mathbf{A})=\mathbf{A}^2-5\mathbf{A}+6$

$$=\mathbf{A}^2-5\mathbf{A}+6\mathbf{I}, \text{ where } \mathbf{I}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now proceed is in Ex. 1 above. **Ans.** $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$

***Ex. 3. If** $\mathbf{A}=\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ **and** $\mathbf{B}=\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$, **show that**

$$\mathbf{(A+B)^2=A^2+AB+BA+B^2\neq A^2+2AB+B^2}$$

Sol. $\mathbf{A}^2=\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\times\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 2\cdot2+(-1)\,0 & 2(-1)-1\cdot1 \\ 0\cdot2+1\cdot0 & 0(-1)+1\cdot1 \end{bmatrix}=\begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix};$$

$$\mathbf{AB}=\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 2\cdot1-1(-1) & 2\cdot0-1(-1) \\ 0\cdot1+1(-1) & 0\cdot0+1(-1) \end{bmatrix}=\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix};$$

$$\mathbf{BA}=\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}\times\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1\cdot2+0\cdot0 & 1(-1)+0\cdot1 \\ -1\cdot2-1\cdot0 & -1(-1)-1\cdot1 \end{bmatrix}=\begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix};$$

$$\mathbf{B}^2=\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}\times\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 1\cdot1+0(-1) & 1\cdot0+0(-1) \\ -1\cdot1-1(-1) & -1\cdot0-1(-1) \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 2+1 & -1+0 \\ 0-1 & 1-1 \end{bmatrix}=\begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix};$$

$$(\mathbf{A}+\mathbf{B})^2=\begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}\times\begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 3\cdot3-1(-1) & 3(-1)-1\cdot0 \\ -1\cdot3+0(-1) & -1(-1)+0\cdot0 \end{bmatrix}=\begin{bmatrix} 10 & -3 \\ -3 & 1 \end{bmatrix} \quad \text{...(i)}$$

Now $\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$

$$=\begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}+\begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 4+3+2+1 & -3+1-1+0 \\ 0-1-2+0 & 1-1+0+1 \end{bmatrix}=\begin{bmatrix} 10 & -3 \\ -3 & 1 \end{bmatrix}$$

$= (\mathbf{A}+\mathbf{B})^2$, from (i) Hence proved.

Also $\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$

$$=\begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}+2\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 6 & 2 \\ -2 & -2 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$ See § 1.04 Page 6

$$=\begin{bmatrix} 4+6+1 & -3+2+0 \\ 0-2+0 & 1-2+1 \end{bmatrix}=\begin{bmatrix} 11 & -1 \\ -2 & 0 \end{bmatrix}\neq(\mathbf{A}+\mathbf{B})^2$$

Hence proved.

Ex. 4. If $\mathbf{A}=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, show that

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})\neq\mathbf{A}^2-\mathbf{B}^2$$

Sol. $\mathbf{A}+\mathbf{B}=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}+\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0+0 & 1-1 \\ 1+1 & 1+0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$

$$\mathbf{A}-\mathbf{B}=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}-\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0-0 & 1-(-1) \\ 1-1 & 1-0 \end{bmatrix}=\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})=\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}\times\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0\cdot0+0\cdot0 & 0\cdot2+0\cdot1 \\ 2\cdot0+1\cdot0 & 2\cdot2+1\cdot1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \quad ...(i)$$

$$A^2=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\times\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 0\cdot0+1\cdot1 & 0\cdot1+1\cdot1 \\ 1\cdot0+1\cdot1 & 1\cdot1+1\cdot1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B^2=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\times\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0\cdot0-1\cdot1 & -0\cdot1-1\cdot0 \\ 1\cdot0+0\cdot1 & -1\cdot1+0\cdot0 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore\ A^2-B^2=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}-\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}=\begin{bmatrix} 1+1 & 1-0 \\ 1-0 & 2+1 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad ...(ii)$$

Hence from (i) and (ii), $(A+B)(A-B)\neq A^2-B^2$.

***Ex. 5 (a). If A denotes the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, prove that**

$$A^2-(a+d)A+(ad-bc)I=O.$$

Sol. $A^2=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\times\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$=\begin{bmatrix} a\cdot a+b\cdot c & a\cdot b+b\cdot d \\ c\cdot a+d\cdot c & c\cdot b+d\cdot d \end{bmatrix}=\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & cb+d^2 \end{bmatrix}$$

$\therefore\ A^2-(a+d)A+(ad-bc)I$

$$=\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & cb+d^2 \end{bmatrix}-(a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix}+(ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & cb+d^2 \end{bmatrix}+\begin{bmatrix} -a(a+d) & -b(a+d) \\ -c(a+d) & -d(a+d) \end{bmatrix}+\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$=\begin{bmatrix} a^2+bc-a(a+d)+ad-bc & b(a+d)-b(a+d)+0 \\ c(a+d)-c(a+d)+0 & cb+d^2-d(a+d)+ad-bc \end{bmatrix}$$

$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=O$, where O is the 2×2 null matrix.

Hence proved.

Ex. 5 (b). If $A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$, evaluate $6A^2-25A+42I$. *(Agra 94)*

Sol. Here $A^2=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\times\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot1-2\cdot2-3\cdot3 & -1\cdot2-2\cdot3+3\cdot1 & 1\cdot3+2\cdot1+3\cdot2 \\ 2\cdot1+3\cdot2+1\cdot3 & -2\cdot2+3\cdot3-1\cdot1 & 2\cdot3-3\cdot1-1\cdot2 \\ -3\cdot1+1\cdot2-2\cdot3 & 3\cdot2+1\cdot3+2\cdot1 & -3\cdot3-1\cdot1+2\cdot2 \end{bmatrix}$$

$$=\begin{bmatrix} 1-4-9 & -2-6+3 & 3+2+6 \\ 2+6+3 & -4+9-1 & 6-3-2 \\ -3+2-6 & 6+3+2 & -9-1+4 \end{bmatrix}=\begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$\therefore\ 6\mathbf{A}^2-25\mathbf{A}+42\mathbf{I}$

$$=6\begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}-25\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}+42\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} -72 & -30 & 66 \\ 66 & 24 & 6 \\ -42 & 66 & -36 \end{bmatrix}-\begin{bmatrix} 25 & -50 & 75 \\ 50 & 75 & -25 \\ -75 & 25 & 50 \end{bmatrix}+\begin{bmatrix} 42 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 42 \end{bmatrix}$$

$$=\begin{bmatrix} -72-25+42 & -30+50+0 & 66-75+0 \\ 66-50+0 & 24-75+42 & 6+25+0 \\ -42+75+0 & 66-25+0 & -36-50+42 \end{bmatrix}=\begin{bmatrix} -55 & 20 & -9 \\ 16 & -9 & 31 \\ 33 & 41 & -44 \end{bmatrix}$$

***Ex. 6. If $\mathbf{A}=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then show that $\mathbf{A}^2=2\mathbf{A}$ and $\mathbf{A}^3=4\mathbf{A}$.**

Sol. Given $\mathbf{A}=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ ...(i)

$$\therefore\ \mathbf{A}^2=\mathbf{A}\,.\,\mathbf{A}=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1\cdot1+(-1)(-1) & 1(-1)+(-1)1 \\ (-1)1+1(-1) & (-1)(-1)+1\cdot1 \end{bmatrix}=\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$=2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}=2\mathbf{A}$, from (i) ...(ii)

Hence proved.

Again $\mathbf{A}^3=\mathbf{A}\,.\,\mathbf{A}^2=\mathbf{A}\,.\,(2\mathbf{A})$, from (ii)

$=2\mathbf{A}\,.\,\mathbf{A}=2\mathbf{A}^2=2\,(2\mathbf{A})$, from (ii)

$=4\mathbf{A}$ Hence proved.

Ex. 7 (a). If $\mathbf{I}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{E}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, prove that

100%

$$(a\mathbf{I}+b\mathbf{E})^3=a^3\mathbf{I}+3a^2b\mathbf{E}.$$ *(Avadh 91; Garhwal 96)*

Sol. $a\mathbf{I}+b\mathbf{E}=a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}+\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a+0 & 0+b \\ 0+0 & a+0 \end{bmatrix}=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}=\mathbf{B}\text{ (say)}$$

$$\therefore \quad (a\mathbf{I}+b\mathbf{E})^2=\mathbf{B}^2=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\times\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

$$=\begin{bmatrix} a\cdot a+b\cdot 0 & a\cdot b+b\cdot a \\ 0\cdot a+a\cdot 0 & 0\cdot b+a\cdot a \end{bmatrix}=\begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$\therefore \quad (a\mathbf{I}+b\mathbf{E})^3=\mathbf{B}^3=\mathbf{B}^2\mathbf{B}$$ **(Note)**

$$=\begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\times\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

$$=\begin{bmatrix} a^2\cdot a+2ab\cdot 0 & a^2\cdot b+2ab\cdot a \\ 0\cdot a+a^2\cdot 0 & 0\cdot b+a^2\cdot a \end{bmatrix}=\begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix} \quad ...(i)$$

Now $a^3\mathbf{I}+3a^2b\mathbf{E}=a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+3a^2b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$=\begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix}+\begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} a^3+0 & 0+3a^2b \\ 0+0 & a^3+0 \end{bmatrix}=\begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix} \quad ...(ii)$$

$\therefore$ From (i) and (ii) we get $(a\mathbf{I}+b\mathbf{E})^3=a^3\mathbf{I}+3a^2b\mathbf{E}$.

Ex. 7 (b). If $\mathbf{I}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{E}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Then prove that $(2\mathbf{I}+3\mathbf{E})^3=8\mathbf{I}+36\mathbf{E}$ *(Rohilkhand 95)*

Sol. Do exactly as Ex. 7 (a) above. Here 'a' = 2 and b = 3.

***Ex. 8. If $\mathbf{A}=\begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$, and I is a unit matrix, then prove that $\mathbf{I}+\mathbf{A}=(\mathbf{I}-\mathbf{A})\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$**

Sol. $\mathbf{I}+\mathbf{A}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$

$$=\begin{bmatrix} 1+0 & 0-\tan(\alpha/2) \\ 0+\tan(\alpha/2) & 1+0 \end{bmatrix}=\begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix} \quad ...(i)$$

$$\mathbf{I}-\mathbf{A}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1-0 & 0+\tan(\alpha/2) \\ 0-\tan(\alpha/2) & 1-0 \end{bmatrix}=\begin{bmatrix} 1 & \tan(\alpha/2) \\ -\tan(\alpha/2) & 1 \end{bmatrix}$$

$$\therefore \quad (\mathbf{I}-\mathbf{A})\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$=\begin{bmatrix} 1 & \tan(\alpha/2) \\ -\tan(\alpha/2) & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$=\begin{bmatrix} 1\cdot\cos\alpha+\tan(\alpha/2)\cdot\sin\alpha & 1(-\sin\alpha)+\tan(\alpha/2)\cos\alpha \\ -\tan(\alpha/2)\cdot\cos\alpha+1\cdot\sin\alpha & (\sin\alpha)\cdot\tan(\alpha/2)+1\cdot\cos\alpha \end{bmatrix}$$

$$=\begin{bmatrix} \{1-2\sin^2(\alpha/2)\}+2\sin^2(\alpha/2) & -2\sin(\alpha/2)\cos(\alpha/2) \\ & +\tan(\alpha/2)\cos\alpha \\ -\tan(\alpha/2)\cos\alpha+2\sin(\alpha/2)\cos(\alpha/2) & 2\sin^2(\alpha/2) \\ & +\{1-2\sin^2(\alpha/2)\} \end{bmatrix},$$

writing $\cos\alpha = 1-2\sin^2\frac{1}{2}\alpha$

$$=\begin{bmatrix} 1 & -2\tan(\alpha/2)\cos^2(\alpha/2) \\ & +\tan(\alpha/2)\cos\alpha \\ -\tan(\alpha/2)\cos\alpha & 1 \\ \quad +2\tan(\alpha/2)\cos^2(\alpha/2) & \end{bmatrix},$$

wrting $\sin\frac{1}{2}\alpha$ as $\tan\frac{1}{2}\alpha\cos\frac{1}{2}\alpha$

$$=\begin{bmatrix} 1 & -\tan(\alpha/2)\,[2\cos^2(\alpha/2) \\ & -\{2\cos^2(\alpha/2)-1\}] \\ \tan(\alpha/2)\,[-\{2\cos^2(\alpha/2)-1\} & 1 \\ \quad +2\cos^2(\alpha/2)\} & \end{bmatrix},$$

writing $\cos\alpha = 2\cos^2(\alpha/2)-1$

$$=\begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix} = \mathbf{I}+\mathbf{A}, \text{ from (i)}$$

Hence proved.

****Ex. 9 (a). If $\mathbf{A}=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, show that $\mathbf{A}^n=\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$**

(Agra 96; Avadh 92; Garhwal 91; Kanpur 95, Kumaun 95, 93; Meerut 90)

Sol. $\mathbf{A}^2=\mathbf{A}\,.\,\mathbf{A}=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\times\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$=\begin{bmatrix} 3\cdot3-4(1) & 3(-4)-4(-1) \\ 1\cdot3-1(1) & 1(-4)-1(-1) \end{bmatrix}=\begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} 1+2(2) & -4(2) \\ (2) & 1-2(2) \end{bmatrix}$$

$=\mathbf{A}^n$, when $n=2$ **(Note)**

$\therefore\ \mathbf{A}^n=\begin{bmatrix} 1+2n & 4n \\ n & 1-2n \end{bmatrix}$ holds when $n=2$

Now $\mathbf{A}^{n+1}=\mathbf{A}^n\,.\,\mathbf{A}$ ...See def. § 1.10 Page 28.

$$=\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} (1+2n)\cdot 3-4n(1) & (1+2n)(-4)-4n(-1) \\ n\cdot 3+(1-2n)(1) & n(-4)+(1-2n)(-1) \end{bmatrix}$$

$$=\begin{bmatrix} 3+2n & -4-4n \\ 1+n & -1-2n \end{bmatrix}=\begin{bmatrix} 1+2(n+1) & -4(n+1) \\ (n+1) & 1-2(n+1) \end{bmatrix}$$

i.e. $\quad A^n=\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ holds for 'n' $= n+1$.

Also we have shown above that it holds for $n = 2$.

Hence by mathematical induction it is true for all positive integral values of n.

Hence proved.

Ex. 9 (b). If $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, prove that $A^n=\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, where n is positive integer. *(Kanpur 97, 93)*

Sol. $A^2 = A\bullet A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1.1+1.0 & 1.1+1.1 \\ 0.1+1.0 & 0.1+1.1 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$= A^n$, where $n = 2$.

i.e., $\quad A^n=\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ holds when $n = 2$

Now $A^{n+1} = A^n\bullet A$ ... See def. § 1.10 Page 27

$$=\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1.1+n.0 & 1.1+n.1 \\ 0.1+1.0 & 0.1+1.1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad A^n=\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ holds for 'n' $= n+1$

Also we have showsn above that it holds for $n = 2$. Hence by mathematical induction it is true for all positive integral values of n.

Hence proved.

Ex. 10. Let $A=\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where $a \neq 0$. Show that for $n \geq 0$, $A^n=\begin{bmatrix} a^n & \dfrac{b(a^n-1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$

Sol. $\mathbf{A}^2 = \mathbf{A}\bullet\mathbf{A} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} a.a + b.0 & a.b + b.1 \\ 0.a + 1.0 & 0.b + 1.1 \end{bmatrix} = \begin{bmatrix} a^2 & b\,(a+1) \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & \dfrac{b\,(a^2-1)}{(a-1)} \\ 0 & 1 \end{bmatrix} = \mathbf{A}^n \text{, when } n = 2$$

(Note)

$\therefore \quad \mathbf{A}^n = \begin{bmatrix} a^n & b\,(a^n - 1)/(a-1) \\ 0 & 1 \end{bmatrix}$ holds when $n = 2$.

Now $\mathbf{A}^{n+1} = \mathbf{A}^n\bullet\mathbf{A}$, See def. § 1·10 Page 27

$$= \begin{bmatrix} a^n & b\,(a^n-1)/(a-1) \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^n.a + 0 & a^n b + 1.\{b\,(a^n-1)/(a-1)\} \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{n+1} & b\,\{a^n\,(a-1) + (a^n-1)\}/(a-1) \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{n+1} & b\,(a^{n+1}-1)/(a-1) \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad \mathbf{A}^n = \begin{bmatrix} a^n & b\,(a^n-1)/(a-1) \\ 0 & 1 \end{bmatrix}$ holds for 'n' $= n+1$.

Also we have shown above that it holds for $n = 2$.

Hence by mathematical induction it is true for all positive integral values of $n \geq 0$.

Hence proved.

***Ex. 11. (a) Show that** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{\mathbf{n}} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$,

when n is a positive integer. *(Avadh 95, Gorakhpur 90)*

Sol. Let $\quad \mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ...(i)

Then $(\mathbf{A}^2) = \mathbf{A}\bullet\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -\sin\theta\cos\theta - \sin\cos\theta \\ \sin\theta\cos\theta + \sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix}$$

or $\quad (\mathbf{A})^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ ...(ii)

Similarly $(\mathbf{A})^3 = (\mathbf{A})^2\bullet\mathbf{A}$ ... See def. § 1.10 Page 27

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ from (i) and (ii)}$$

$$= \begin{bmatrix} \cos 2\theta\cos\theta - \sin 2\theta\sin\theta & -\cos 2\theta\sin\theta - \sin 2\theta\cos\theta \\ \sin 2\theta\cos\theta + \cos 2\theta\sin\theta & -\sin 2\theta\sin\theta + \cos 2\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta+\theta) & -\sin(2\theta+\theta) \\ \sin(2\theta+\theta) & \cos(2\theta+\theta) \end{bmatrix}$$

or $$(\mathbf{A})^3 = \begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix} \qquad \text{...(iii)}$$

In the light of (i), (ii) and (iii) let us assume that

$$(\mathbf{A})^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix} \qquad \text{...(iv)}$$

Now $(\mathbf{A})^{n+1} = (\mathbf{A})^n . (\mathbf{A})$

$$= \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos n\theta\cos\theta - \sin n\theta\sin\theta & -\cos n\theta\sin\theta - \sin n\theta\cos\theta \\ \sin n\theta\cos\theta + \cos n\theta\sin\theta & -\sin n\theta\sin\theta + \cos n\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(n\theta+\theta) & -\sin(n\theta+\theta) \\ \sin(n\theta+\theta) & \cos(n\theta+\theta) \end{bmatrix} = \begin{bmatrix} \cos(n+1)\theta & -\sin(n+1)\theta \\ \sin(n+1)\theta & \cos(n+1)\theta \end{bmatrix}$$

i.e. (iv) holds for $n+1$ if is true for n.

We have already proved in (ii) and (iii) that (iv) holds for $n = 2$ and 3. Hence (iv) holds for all positive integral values of n.

i.e. $$(\mathbf{A})^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$ Hence proved.

****Ex. 11. (b) If $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, evaulate $\mathbf{A}^n$.** *(Garhwal 94, 92; Meerut 97)*

Sol. $$\mathbf{A}^2 = \mathbf{A}\bullet\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta - \cos\theta\sin\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix}$$

or $$(\mathbf{A})^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \qquad \text{...(i)}$$

Similarly $(\mathbf{A})^3 = (\mathbf{A})^2 \bullet \mathbf{A}$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta\cos\theta - \sin 2\theta\sin\theta & \cos 2\theta\sin\theta + \sin 2\theta\cos\theta \\ -\sin 2\theta\cos\theta - \cos 2\theta\sin\theta & -\sin 2\theta\sin\theta + \cos 2\theta\cos\theta \end{bmatrix}$$

$$=\begin{bmatrix} \cos(2\theta+\theta) & \sin(2\theta+\theta) \\ -\sin(2\theta+\theta) & \cos(2\theta+\theta) \end{bmatrix}$$

$$=\begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix} \quad \text{...(ii)}$$

In the light of (i), (ii), let us assume that

$$(\mathbf{A})^n=\begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix} \quad \text{...(iii)}$$

Now $(\mathbf{A})^{n+1}=(\mathbf{A})^n.\,\mathbf{A}$

$$=\begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}\times\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$=\begin{bmatrix} \cos n\theta\cos\theta-\sin n\theta\sin\theta & \cos n\theta\sin\theta+\sin n\theta\cos\theta \\ -\sin n\theta\cos\theta-\cos n\theta\sin\theta & -\sin n\theta\sin\theta+\cos n\theta\cos\theta \end{bmatrix}$$

$$=\begin{bmatrix} \cos(n\theta+\theta) & \sin(n\theta+\theta) \\ -\sin(n\theta+\theta) & \cos(n\theta+\theta) \end{bmatrix}=\begin{bmatrix} \cos(n+1)\theta & \sin(n+1)\theta \\ -\sin(n+1)\theta & \cos(n+1)\theta \end{bmatrix}$$

$\therefore$ (iii) holds for $n+1$ if it is true for n.

We have already proved in (i) and (ii) that (iii) holds for $n=2$ and 3. Hence by mathematical induction (iii) holds for all +ve integral values of n and value of $\mathbf{A}^n$ is given by (iii).

***Ex. 12. Show that if $\mathbf{A}=\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$**

then $\mathbf{A}^n=\begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix}$ *(Agra 93)*

Sol. Here $\mathbf{A}^2=\mathbf{A}\bullet\mathbf{A}$

$$=\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\times\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

$$=\begin{bmatrix} \cosh^2\theta+\sinh^2\theta & \cosh\theta\sinh\theta+\sinh\theta\cosh\theta \\ \sinh\theta\cosh\theta+\cosh\theta\sinh\theta & \sinh^2\theta+\cosh^2\theta \end{bmatrix}$$

or $$\mathbf{A}^2=\begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix}$$

Similarly $\mathbf{A}^3=\mathbf{A}^2\bullet\mathbf{A}$ ...(i)

$$=\begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix}\times\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}, \text{ from (i)}$$

$$=\begin{bmatrix} \cosh 2\theta\cosh\theta+\sinh 2\theta\sinh\theta & \cosh 2\theta\sinh\theta+\sinh 2\theta\cosh\theta \\ \sinh 2\theta\cosh\theta+\cosh 2\theta\sinh\theta & \sinh 2\theta\sinh\theta+\cosh 2\theta\cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(2\theta+\theta) & \sinh(2\theta+\theta) \\ \sinh(2\theta+\theta) & \cosh(2\theta+\theta) \end{bmatrix}$$

or $$\mathbf{A}^3 = \begin{bmatrix} \cosh 3\theta & \sinh 3\theta \\ \sinh 3\theta & \cosh 3\theta \end{bmatrix} \qquad \text{...(ii)}$$

In the light of (i), (ii) and the given value of **A**, let us assume that

$$\mathbf{A}^n = \begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix} \qquad \text{...(iii)}$$

Now $\mathbf{A}^{n+1} = \mathbf{A}^n \bullet \mathbf{A}$

$$= \begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix} \times \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh n\theta \cosh\theta + \sinh n\theta \sinh\theta & \cosh n\theta \sinh\theta + \sinh n\theta \cosh\theta \\ \sinh n\theta \cosh\theta + \cosh n\theta \sinh\theta & \sinh n\theta \sinh\theta + \cosh n\theta \cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(n\theta+\theta) & \sinh(n\theta+\theta) \\ \sinh(n\theta+\theta) & \cosh(n\theta+\theta) \end{bmatrix} = \begin{bmatrix} \cosh(n+1)\theta & \sinh(n+1)\theta \\ \sinh(n+1)\theta & \cosh(n+1)\theta \end{bmatrix}$$

i.e. (iii) holds for $n+1$ if it is true for n.

Also from (i) and (ii) we know that (iii) holds for $n=2$ and $n=3$. Hence (iii) holds for all positive integral values of n.

i.e. $$\mathbf{A}^n = \begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix}$$ Hence proved.

[Note. See Ex. 13 Page 20 also].

Exercises on § 1·09 – § 1·10

Ex. 1. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ satisfies the equation $\mathbf{A}^2 - 2\mathbf{A} - 5\mathbf{I} = \mathbf{O}$, where **O** is the 2×2 null matrix.

Ex. 2. Evaluate $\mathbf{A}^2 - 3\mathbf{A} - 13\mathbf{I}$, where **I** is the 2×2 unit matrix and $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$

Ex. 3. Show that matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ satisfies the equation $\mathbf{A}^3 - 3\mathbf{A}^2 + 3\mathbf{A} - \mathbf{I} = \mathbf{O}$, where **I** is the unit matrix and **O** the null matrix of order 3.

Ex. 4. If $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ verify (i) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$; (ii) $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

Ex. 5. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 1 \\ 7 & 4 \end{bmatrix}$, show that

$$A(B + C) = AB + AC.$$

Ex. 6. If $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, show that

$$(A + B)(A + B) = A^2 + 2AB + B^2.$$

Ex. 8. Show that $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n & \frac{1}{2}n(n+1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$

for all natural numbers n.

[Hint. See Ex. 10 Page 35]

SOME TYPICAL SOLVED EXAMPLES

****Ex. 1. A manufacturer produces three products A, B, C which he sells in the market. Annual sale volumes are indicated as follows :**

Markets	**Products**		
	A	**B**	**C**
I	**8,000**	**10,000**	**15,000**
II	**10,000**	**2,000**	**20,000**

(i) If unit sale prices of A, B and C are Rs. 2·25, Rs. 1·50 and Rs. 1·25 respectively, find the total revenue in each market with the help of matrices, (ii) if the units costs of the above three products are Rs. 1·60, Rs. 1·20 and Rs. 0·90 respectively, find the gross profit with the help of matrices.

Sol. (i) The total revenue in each market is given by the products matrix.

$$[2{\cdot}25 \quad 1{\cdot}50 \quad 1{\cdot}25] \times \begin{bmatrix} 8{,}000 & 10{,}000 \\ 10{,}000 & 2{,}000 \\ 15{,}000 & 20{,}000 \end{bmatrix} \qquad \textbf{(Note)}$$

$= [(2{\cdot}25 \times 8{,}000) + (1{\cdot}50 \times 10{,}000) + (1{\cdot}25 \times 15{,}000)$
$\qquad (2{\cdot}25 \times 10{,}000) + (1{\cdot}50 \times 2{,}000) + (1{\cdot}25 \times 20{,}000)]$

$= [18{,}000 + 15{,}000 + 18{,}750 \qquad 22{,}500 + 3{,}000 + 25{,}000]$

$= [51750 \quad 50500]$

$\therefore$ Total revenue from the market I = Rs. 51,750.

and total revenue from the market II = Rs. 50,500. **Ans.**

(ii) Similarly the total cost of products with the manufacturer sells in the markets are :

$$[1{\cdot}60 \quad 1{\cdot}20 \quad 0{\cdot}90] \times \begin{bmatrix} 8{,}000 & 10{,}000 \\ 10{,}000 & 2{,}000 \\ 15{,}000 & 20{,}000 \end{bmatrix}$$

$$= [(1{\cdot}60 \times 8{,}000) + (1{\cdot}20 \times 10{,}000) + (0{\cdot}90 \times 15{,}000)$$
$$(1{\cdot}60 \times 10{,}000) + (1{\cdot}20 + 2{,}000) + (0{\cdot}90 + 20{,}000)]$$
$$= [12{,}800 + 12{,}000 + 13{,}500 \qquad 16{,}000 + 2{,}400 + 18{,}000]$$
$$= [38{,}300 \qquad 36{,}400]$$

$\therefore$ Total cost of products which the manufacturer sells in the market I and II are Rs, 38,300 and Rs. 36,400 respectively.

$\therefore$ Required gross profit = (Total revenue received from both the markets) -- (Total costs of product which the manufactuerr sold in both the markets)

= (Rs. 51,750 + Rs, 50,500) − (Rs. 38,300 + Rs. 36,400).

= Rs. 102,250 − Rs. 74,700 = Rs. 27,550. **Ans.**

Ex. 2. A man buys 8 dozens of mangoes, 10 dozens of apples and 4 dozens of bananas. Mangoes cost Rs. 18 per dozen, apples Rs. 9 per dozen and bananas Rs. 6 per dozen. Represent the quantities bought by a row matrix and the prices by a column matrix and hence obtain the total cost.

(I. C. W. A. Final)

Sol. The quantities bought are represented by 3×1 row matrix $[8 \quad 10 \quad 4]$ and the prices are represented by 3×1 column matrix

$$\begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$$

$\therefore$ The cost of fruits is a single number *i.e.* 1×1 matrix given by the product matrix $[8 \quad 10 \quad 4] \times \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$

i.e. $[(8 \times 18) + (10 \times 9) + (4 \times 6)]$ *i.e.* $[144 + 90 + 24]$ *i.e* $[258]$

$\therefore$ The required total cost = Rs. 258. **Ans.**

****Ex. 3. A store has in stock 30 dozen shirts, 15 dozen trousers and 25 dozen pairs of socks. If the selling prices are Rs. 50 per shirt, Rs. 90 per trouser and Rs. 12 per pair of socks, then find the total amount the store owner will get after selling all the items in the stock.**

Sol. The stock in the store can be written in the form of a row matrix **A** given by $\mathbf{A} = [20 \times 12 \quad 15 \times 12 \quad 25 \times 12]$

or $\mathbf{A} = [240 \quad 180 \quad 300]$, which is a 1×3 matrix.

The prices can be written in the form of a column matrix **B** given by

$$\mathbf{B} = \begin{bmatrix} 50 \\ 90 \\ 12 \end{bmatrix}, \text{ which is a } 3 \times 1 \text{ matrix.}$$

The required amount is a single number *i.e.* a matrix of order 1×1 and so the same can be obtained by multiplying the matrices **A** and **B**, since their product would be a 1×1 matrix. **(Note)**

Now $\mathbf{AB} = [240 \quad 180 \quad 300] \times \begin{bmatrix} 50 \\ 90 \\ 12 \end{bmatrix}$

$= [(240 \times 50) + (180 \times 90) + (300 \times 12)]$

$= [12000 + 16200 + 36000] = [31800]$

$\therefore$ The required amount received by the store owner

$=$ Rs. 31,800. **Ans.**

Ex. 4. A trust fund has Rs. 50,000 that is to be invested into two types of bonds. The first bond pays 5% interest per year and the second bond pays 6% interest per year. Using matrix multipication, determine how to divide Rs. 50,000 among two types of bonds so as to obtain an annual total interest of Rs. 2780.

Sol. Let Rs. 50,000 be divided into two parts Rs. x and Rs. $(50{,}000 - x)$ out of which first part is invested in first type of bonds and the second part is invested in second type of bonds.

The values of these bonds can be written in the form of a row matrix **A** given by $\mathbf{A} = [x \quad 50{,}000 - x]$, which is a 1×2 matrix.

And the amounts received as interest per rupee annually from these two types of bonds can be written in the form of a column matrix **B** given by

$$\mathbf{B} = \begin{bmatrix} 5/100 \\ 6/100 \end{bmatrix}, \text{ which is a } 2 \times 1 \text{ matrix.}$$

Here the interest has been calculated per rupee annually.

Now the interest to be obtained annually is a single number *i.e.* a matrix of order 1×1 and the same can be obtained by the product matrix **AB**, since this product matrix would be a 1×1 matrix. **(Note)**

Here $\mathbf{AB} = [x \quad 50{,}000 - x] \times \begin{bmatrix} 5/100 \\ 6/100 \end{bmatrix}$

$$= \left[x \cdot \frac{5}{100} + (50{,}000 - x) \cdot \frac{6}{100} \right]$$

$$= \left[3000 - \frac{x}{100} \right]$$

Also we are given that the annual interest $= 2{,}780$.

$\therefore$ We must have $\left[3000 - \frac{x}{100}\right] = [2780]$ **(Note)**

or $\quad 3000 - \frac{x}{100} = 2780 \quad$ or $\quad x = (3000 - 2780) \times 100$

or $\quad x = 220 \times 100 = 22{,}000$

Hence the required amounts are

Rs. 22,000 and Rs. (50,000 – 22,000) *i.e.* Rs. 22,000 and Rs. 28,000 **Ans.**

Ex. 5. A finance company has offices located in every division, every district and every taluka in a certain state in India. Assume that there are five divisions, thirty districts and 200 talukas in the state. Each office has one headclerk, one cashier, one clerk and one peon. A divisional office has, in addition, one office superintendent, two clerks, one typist and one peon. A district office, has in addition, one clerk and one peon, The basic monthly salaries are as follows : office superitendent Rs. 500. Head clerk Rs. 200, cashier Rs. 175, clerks and typists Rs. 150 and peon Rs. 100. Using matrix notation find —

(i) The total number of posts of each kind in all the offices taken together, (ii) the total basic monthly salary bill of each kind of office and (iii) the total basic monthly salary bill of all the offices taken together.

(*C. A. Intermediate*)

Sol. Let us use the symbols Div, Dis, Tal for division, district, taluka respectively and O, H, C, Cl, T and P for office superintendent, Head clerk, cashier, clerk, typist and peon respectively.

Then the number of offices can be arranged as elements of a row matrix **A** (say) given by

$$\begin{array}{cccc} & \text{Div.} & \text{Dis.} & \text{Tal.} \\ \mathbf{A} = (& 5 & 30 & 200) \end{array}$$

The composition of staff in various offices can be arranged in a 3×6 matrix **B** (say) given by

$$\mathbf{B} = \begin{array}{c} \begin{array}{cccccc} \text{O} & \text{H} & \text{C} & \text{Cl} & \text{T} & \text{P} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 2+1 & 1 & 1+1 \\ 0 & 1 & 1 & 1+1 & 0 & 1+1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

The basic monthly salaries of various types of employees of these offices correspond to the elements of the column matrix **C** (say) given by

$$\mathbf{C} = \begin{array}{c} \text{O} \\ \text{H} \\ \text{C} \\ \text{Cl} \\ \text{T} \\ \text{P} \end{array} \begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix}$$

(i) Total number of posts of each kind in all the offices are the elements of the product matrix **AB**.

i.e. $$[5 \quad 30 \quad 200] \times \begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$ **(Note)**

i.e. $$[5+0+0, \quad 5+30+200, \quad 5+30+200, \quad 15+60+200, \quad 5+0+0, \quad 10+60+200]$$

i.e. $$\begin{array}{cccccc} O & H & C & Cl & T & P \\ 5 & 235 & 235 & 275 & 5 & 270 \end{array}$$

i.e. Required number of posts in all the offices taken together are 5 offices supdts., 235 Head clerks, 235 cashiers, 275 clerks, 5 typists and 270 peons. **Ans.**

(ii) Total basic monthly salary bill of each kind of office are the elements of the product matrix **BC**

i.e. $$\begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \times 500) + (1 \times 200) + (1 \times 175) + (3 \times 150) + (1 \times 150) + (2 \times 100) \\ (0 \times 500) + (1 \times 200) + (1 \times 175) + (2 \times 150) + (0 \times 150) + (2 \times 100) \\ (0 \times 500) + (1 \times 200) + (1 \times 175) + (1 \times 150) + (0 \times 150) + (1 \times 100) \end{bmatrix}$$

$$= \begin{bmatrix} 500 + 200 + 175 + 450 + 150 + 200 \\ 0 + 200 + 175 + 300 + 0 + 200 \\ 0 + 200 + 175 + 150 + 0 + 100 \end{bmatrix} = \begin{bmatrix} 1675 \\ 875 \\ 625 \end{bmatrix}$$ **Ans.**

i.e. The total basic monthly salary bill of each divisional, district and taluka offices are Rs. 1675, Rs. 875 and Rs. 625 respectively. **Ans.**

(iii) Total basic monthly salary bill of all the officers (*i.e.* of five divisional, 30 district and 200 taluka offices) is the element of the product matrix **ABC**

i.e. $$[5 \quad 30 \quad 200] \times \begin{bmatrix} 1675 \\ 875 \\ 625 \end{bmatrix}$$ **(Note)**

i.e. $$[(5 \times 1675) + (30 \times 875) + (200 \times 625)]$$

i.e. $[8375 + 2650 + 125000]$ *i.e.* $[159625]$

i.e. total basic monthly salary bill of all the offices taken together is Rs. 159,625. **Ans.**

****Ex. 6. In a development plan of a city, a contractor has taken a contract to construct certain houses for which he needs building materials like stones, sand etc. There are three firms A, B, C that can supply him these materials. At one time these firms A B,C, supplied him 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of sand respectively. If the cost of one truck load of stone and sand are Rs. 1,200 and Rs. 500 respectively, then find the total amount paid by the contractor to each of these firms, A, B, C separately.**

Sol. The truck-loads of stone and sand supplied by the firms A, B and C can be written in the from of a matrix **A** (say) given by

$$\mathbf{A} = \begin{array}{r} \\ \text{Stone} \\ \text{Sand} \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix} \end{array}, \text{ which is a } 2 \times 3 \text{ matrix}$$

And the cost per truck of stone and sand can be given in the form of a matrix **B** (say) given by

$$\begin{array}{cc} & \begin{array}{cc} \text{Stone} & \text{Sand} \end{array} \\ \mathbf{B} = & [1200 \quad 500] \end{array}$$

The required total amount paid to each of the firms A, B and C are given by the product matrix **BA**. [Note **AB** can not be calculated).

Now $\mathbf{BA} = [1200 \quad 500] \times \begin{bmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix}$

$= [(1200 \times 40) + (500 \times 10) \quad (1200 \times 35) + (500 \times 5) \quad (1200 \times 25) + (500 \times 8)]$

$= [48000 + 5000 \quad 42000 + 2500 \quad 30000 + 4000]$

$= [53,000 \quad 44,500 \quad 34,000]$

$\therefore$ The amount paid to the firms A, B and C by the contractor are Rs, 53,000, Rs. 44,500 and Rs. 34,000 respectively. **Ans.**

Exercises

Ex. 1. A fruit seller has in stock 20 dozen mangoes, 16 dozen apples and 32 dozen bananas. Suppose the selling prices are Rs. 0.35, Rs. 0.75 and Rs. 0.08 per mango, apple and banana respectively. Find the total amount the fruit seller will get by selling his whole stock. **Ans.** Rs. 258·72

Ex. 2. In Ex. 4 Page 3 write down (i) the row matrix which represents team B's result; (ii) the column matrix which represent the results of first places of various teams. **Ans.** $[0 \quad 3 \quad 2 \quad 4]$ and $\begin{bmatrix} 3 \\ 0 \\ 5 \\ 2 \end{bmatrix}$

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1. If** $\mathbf{A}=\begin{bmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{2} & -\mathbf{1} \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} \mathbf{a} & \mathbf{1} \\ \mathbf{b} & -\mathbf{1} \end{bmatrix}$

and $(\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+\mathbf{B}^2$, find a and b. *(Kanpur 96)*

Sol. Here we have

$$\mathbf{A}^2=\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\times\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 1-2 & -1+1 \\ 2-2 & -2+1 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{B}^2=\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\times\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}=\begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

$$\therefore\quad \mathbf{A}^2+\mathbf{B}^2=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}+\begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

$$=\begin{bmatrix} -1+a^2+b & 0+a-1 \\ 0+ab-b & -1+b+1 \end{bmatrix}=\begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix} \qquad \text{...(i)}$$

Also $\mathbf{A}+\mathbf{B}=\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}+\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

$$=\begin{bmatrix} 1+a & -1+1 \\ 2+b & -1-1 \end{bmatrix}=\begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$

$$\therefore\ (\mathbf{A}+\mathbf{B})^2=\begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}\times\begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$

$$=\begin{bmatrix} (1+a)^2+0 & 0+0 \\ (2+b)(1+a)-2(2+b) & 0+4 \end{bmatrix}$$

$$=\begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix} \qquad \text{...(ii)}$$

Now it is given that $(\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+\mathbf{B}^2$.

or $\begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix}=\begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$, from (i) and (ii)

or $0=a-1$ and $4=b$, comparing the elements of second column on both sides.

or $a=1$ and $b=4$. **Ans.**

Ex. 2. If $\mathbf{A}=\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \\ \mathbf{5} & \mathbf{6} \end{bmatrix}$ **and** $\mathbf{B}=\begin{bmatrix} -\mathbf{3} & -\mathbf{2} \\ \mathbf{1} & -\mathbf{5} \\ \mathbf{4} & \mathbf{3} \end{bmatrix}$,

find $\mathbf{D} = \begin{bmatrix} \mathbf{p} & \mathbf{q} \\ \mathbf{r} & \mathbf{s} \\ \mathbf{t} & \mathbf{u} \end{bmatrix}$**, such that** $\mathbf{A+B-D=O}$**.**

Sol. $\mathbf{A+B-D=O}$ or $\mathbf{D=A+B}$

or $\quad \mathbf{D} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1-3 & 2-2 \\ 3+1 & 4-5 \\ 5+4 & 6+3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix}, \text{ given}$$

$\therefore$ We have $p = -2$, $q = 0$, $r = 4$, $s = -1$, $t = 9$, $u = 9$ which gives **D**.

Ex. 3. If $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{p} & \mathbf{q} & \mathbf{r} \end{bmatrix}$ **and I is the unit matrix of order 3, show that**

$$\mathbf{A^3 = pI + qA + rA^2}.$$

Sol. Here $\mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+1+0 \\ 0+0+p & 0+0+q & 0+0+r \\ 0+0+rp & p+0+rq & 0+q+r^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p+rq & q+r^2 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2 \cdot \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p+rq & q+r^2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+p & 0+0+q & 0+0+r \\ 0+0+rp & p+0+rq & 0+q+r^2 \\ 0+0+pq+pr^2 & rp+0+q^2+r^2q & 0+p+rq+rq+r^3 \end{bmatrix}$$

$$= \begin{bmatrix} p & q & r \\ rp & p+rq & q+r^2 \\ pq+pr^2 & rp+q^2+r^2q & p+2rq+r^3 \end{bmatrix} \qquad \text{...(i)}$$

And $p\mathbf{I} + q\mathbf{A} + r\mathbf{A}^2$

$$= p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + q\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} + r\begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p+rq & q+r^2 \end{bmatrix}$$

$$=\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}+\begin{bmatrix} 0 & q & 0 \\ 0 & 0 & q \\ pq & q^2 & rq \end{bmatrix}+\begin{bmatrix} 0 & 0 & r \\ rp & qr & r^2 \\ r^2p & pr+qr^2 & qr+r^3 \end{bmatrix}$$

$$=\begin{bmatrix} p+0+0 & 0+q+0 & 0+0+r \\ 0+0+rp & p+0+pr & 0+q+r^2 \\ 0+pq+r^2p & 0+q^2+pr+qr^2 & p+2rq+r^3 \end{bmatrix}$$

$= \mathbf{A}^3$, from (i). Hence proved.

Ex. 4. Show that $\mathbf{E}^2\mathbf{F}+\mathbf{F}^2\mathbf{E}=\mathbf{E}$, where

$$\mathbf{E}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{F}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol. $\mathbf{E}^2=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\times\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$=\begin{bmatrix} 0\cdot0+0\cdot0+1\cdot0 & 0\cdot0+0\cdot0+1\cdot0 & 0\cdot1+0\cdot1+1\cdot0 \\ 0\cdot0+0\cdot0+1\cdot0 & 0\cdot0+0\cdot0+1\cdot0 & 0\cdot1+0\cdot1+1\cdot0 \\ 0\cdot0+0\cdot0+0\cdot0 & 0\cdot0+0\cdot0+0\cdot0 & 0\cdot1+0\cdot1+0\cdot0 \end{bmatrix}$$

or $\mathbf{E}^2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ $\mathbf{E}^2\mathbf{F}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\times\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ..(i)

Again $\mathbf{F}^2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot0+0\cdot0+0\cdot0 & 1\cdot0+0\cdot1+0\cdot0 & 1\cdot0+0\cdot0+1\cdot0 \\ 1\cdot0+1\cdot0+0\cdot0 & 0\cdot0+1\cdot1+0\cdot0 & 0\cdot0+1\cdot0+0\cdot1 \\ 0\cdot1+0\cdot0+1\cdot0 & 0\cdot0+0\cdot1+1\cdot0 & 0\cdot0+0\cdot0+1\cdot1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\mathbf{F}^2\mathbf{E}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\times\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot0+0\cdot0+0\cdot0 & 1\cdot0+0\cdot0+0\cdot0 & 1\cdot1+0\cdot1+0\cdot0 \\ 0\cdot0+1\cdot0+0\cdot0 & 0\cdot0+1\cdot0+0\cdot0 & 0\cdot1+1\cdot1+0\cdot0 \\ 0\cdot0+0\cdot0+1\cdot0 & 0\cdot0+0\cdot0+1\cdot0 & 0\cdot1+0\cdot1+1\cdot0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad ...(ii)$$

$\therefore$ From (i) and (ii) we get

$$\mathbf{E^2F+F^2E}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}=\mathbf{E}$$

Hence proved.

Ex. 5. If $A=\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$**, find the matrix X such that $A+X+I=O$, where I and O are unit and zero 3×3 matrices respectively.**

Sol. Given that $\mathbf{A+X+I=O}$ or $\mathbf{X=O-A-I}$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}-\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}-\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

substituting values of **A, I** and **O**

$$=\begin{bmatrix} 0-1-1 & 0-2-0 & 0-3-0 \\ 0-3-0 & 0+2-1 & 0-1-0 \\ 0-4-0 & 0-2-0 & 0-1-1 \end{bmatrix}=\begin{bmatrix} -2 & -2 & -3 \\ -3 & 1 & -1 \\ -4 & -2 & 2 \end{bmatrix}$$

Ans.

****Ex. 6. Show that**

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}=\begin{bmatrix} 1 & -\tan\frac{1}{2}\theta \\ \tan\frac{1}{2}\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\frac{1}{2}\theta \\ -\tan\frac{1}{2}\theta & 1 \end{bmatrix}^{-1}$$

Sol. We have

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\times\begin{bmatrix} 1 & \tan\frac{1}{2}\theta \\ -\tan\frac{1}{2}\theta & 1 \end{bmatrix}$$

$$=\begin{bmatrix} \cos\theta+\sin\theta\tan\frac{1}{2}\theta & \cos\theta\tan\frac{1}{2}\theta-\sin\theta \\ \sin\theta-\cos\theta\tan\frac{1}{2}\theta & \sin\theta\tan\frac{1}{2}\theta+\cos\theta \end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{\cos\theta\cos\frac{1}{2}\theta+\sin\theta\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} & \dfrac{\cos\theta\sin\frac{1}{2}\theta-\sin\theta\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} \\ \dfrac{\sin\theta\cos\frac{1}{2}\theta-\cos\theta\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} & \dfrac{\sin\theta\sin\frac{1}{2}\theta+\cos\theta\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} \end{bmatrix}$$

$$= \frac{1}{\cos\frac{1}{2}\theta}\begin{bmatrix} \cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta & \cos\theta\sin\frac{1}{2}\theta - \sin\theta\cos\frac{1}{2}\theta \\ \sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta & \cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta \end{bmatrix}$$

$$= (\sec\tfrac{1}{2}\theta)\begin{bmatrix} \cos(\theta - \frac{1}{2}\theta) & -\sin(\theta - \frac{1}{2}\theta) \\ \sin(\theta - \frac{1}{2}\theta) & \cos(\theta - \frac{1}{2}\theta) \end{bmatrix}$$

$$= (\sec\tfrac{1}{2}\theta)\begin{bmatrix} \cos\frac{1}{2}\theta & -\sin\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{1}{2}\theta\sec\frac{1}{2}\theta & -\sin\frac{1}{2}\theta\sec\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta\sec\frac{1}{2}\theta & \cos\frac{1}{2}\theta\sec\frac{1}{2}\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{1}{2}\theta \\ \tan\frac{1}{2}\theta & 1 \end{bmatrix}$$

or $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{1}{2}\theta \\ \tan\frac{1}{2}\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\frac{1}{2}\theta \\ -\tan\frac{1}{2}\theta & 1 \end{bmatrix}^{-1}$$ Hence proved.

Ex. 7. If A and B be n-rowed square matrices, then show that

(i) $(A + B)^2 = A^2 + AB + BA + B^2$;

(ii) $(A + B)(A - B) = A^2 - AB + BA - B^2$;

(iii) $(A - B)(A + B) = A^2 + AB - BA - B^2$;

and (iv) $(A - B)^2 = A^2 - AB - BA + B^2$.

Sol. As A and B are n-rowed square matrices therefore $A + B$ and $A - B$ are also n-rowed square matrices and as such distributive law is true.

(i). $(A + B)^2 = (A + B) \times (A + B)$
$= (A + B)A + (A + B)B$, by distributive law
$= AA + BA + AB + BB$, by distributive law
$= A^2 + BA + AB + B^2$.

(ii) $(A + B)(A - B)$
$= (A + B)A + (A + B)(-B)$, by distributive law
$= AA + BA + A(-B) + B(-B)$, by distributive law
$= A^2 + BA - AB - B^2$.

(iii) $(A - B)(A + B) = (A - B)A + (A - B)B$, by distributive law
$= AA - BA + AB - BB$, by distributive law
$= A^2 - BA + AB - B^2$.

(iv) $(A - B)^2 = (A - B) \bullet (A - B)$
$= AA + A(-B) + (-B)A + (-B)(-B)$, by distributive law
$= A^2 - AB - BA + B^2$. Hence proved.

***Ex. 8. If A, B are two n × n matrices and if**

$$C = A + B,\ AB = BA,\ B^2 = O$$

then show that for every integer m, $C^{m+1} = A^m[A + (m + 1)B]$.

Sol. We shall prove that $\mathbf{C}^{m+1} = \mathbf{A}^m [\mathbf{A} + (m+1)\mathbf{B}]$, ...(i)

by mathematical induction.

For $m = 1$, from (i) we get $\mathbf{C}^2 = \mathbf{A}[\mathbf{A} + 2\mathbf{B}]$ ...(ii)

Also $\mathbf{C} = \mathbf{A} + \mathbf{B}$, given

$\therefore \; \mathbf{C}^2 = (\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})$

$= \mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2$, as in Ex. 7 (i) Page 50.

$= \mathbf{A}^2 + 2\mathbf{AB}$, since $\mathbf{AB} = \mathbf{BA}$, $\mathbf{B}^2 = \mathbf{O}$ (given)

or $\quad \mathbf{C}^2 = \mathbf{A}(\mathbf{A} + 2\mathbf{B})$, which in the same as (ii).

Hence (i) is true for $m = 1$.

Let us now assume that (i) holds when $m = k$

i.e. $$\mathbf{C}^{k+1} = \mathbf{A}^k [\mathbf{A} + (k+1)\mathbf{B}] \quad ...(iii)$$

Now $\mathbf{C}^{k+2} = \mathbf{C}^{k+1}\mathbf{C}$, by def. § 1·10 Page 27.

$= \mathbf{A}^k [\mathbf{A} + (k+1)\mathbf{B}].(\mathbf{A}+\mathbf{B})$, from (iii) and $\mathbf{C} = \mathbf{A} + \mathbf{B}$(given)

or $\quad \mathbf{C}^{k+2} = \mathbf{A}^k [\mathbf{A}(\mathbf{A}+\mathbf{B}) + (k+1)\mathbf{B}(\mathbf{A}+\mathbf{B})]$

$= \mathbf{A}^k [\mathbf{A}^2 + \mathbf{AB} + (k+1)\mathbf{BA} + (k+1)\mathbf{B}^2]$

$= \mathbf{A}^k [\mathbf{A}^2 + \mathbf{AB} + (k+1)\mathbf{AB}]$, $\because \; \mathbf{BA} = \mathbf{AB}$, $\mathbf{B}^2 = \mathbf{O}$

$= \mathbf{A}^k [\mathbf{A}^2 + (1 + k + 1)\mathbf{AB}]$

$= \mathbf{A}^k . \mathbf{A}[\mathbf{A} + \{(k+1)+1\}\mathbf{B}]$

or $\quad \mathbf{C}^{k+2} = \mathbf{A}^{k+1}[\mathbf{A} + \{(k+1)+1\}\mathbf{B}]$.

Hence (i) is true for $m = k+1$ provided (iii) is true *i.e.* for $m = k$. Also we have shown that (i) is true for $m = 1$, so it is true for $m = 1 + 1$ *i.e.* $m = 2$ and so on. Hence by induction (i) is true for all positive integral values of m.

Hence proved.

Ex. 9. If $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$

then prove that $\mathbf{AB} = 2\mathbf{B}$.

Sol. $\mathbf{AB} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$

$$= \begin{bmatrix} 2x_1 + 0 + 0 & 2y_1 + 0 + 0 & 2z_1 + 0 + 0 \\ 0 + 2x_2 + 0 & 0 + 2y_2 + 0 & 0 + 2z_2 + 0 \\ 0 + 0 + 2x_3 & 0 + 0 + 2y_3 & 0 + 0 + 2z_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 & 2y_1 & 2z_1 \\ 2x_2 & 2y_2 & 2z_2 \\ 2x_3 & 2y_3 & 2z_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = 2\mathbf{B}$$

Hence proved.

Ex. 10. If A, B are two matrices given below, which of the two statements is true AB = BA or AB ≠ BA.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol. Do yourself. **Ans. AB ≠ BA.**

Ex. 11. Find a if $[a \quad 4 \quad 1] \times \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} a \\ 4 \\ -1 \end{bmatrix} = \mathbf{O}$,

where O is 1 × 1 null matrix.

Sol. $[a \quad 4 \quad 1] \times \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

$= [2a+4+0 \quad a+0+2 \quad 0+8+4]$ **(Note)**

$= [2a+4 \quad a+2 \quad 12]$

$\therefore \quad [a \quad 4 \quad 1] \times \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} a \\ 4 \\ -1 \end{bmatrix}$

$= [2a+4 \quad a+2 \quad 12] \times \begin{bmatrix} a \\ 4 \\ -1 \end{bmatrix}$

$= [\{(2a+4) \times a\} + (a+2)\,4 + 12\,(-1)]$ **Ans.**

$= [2a^2 + 4a + 4a + 8 - 12] = [2a^2 + 8a - 4] = \mathbf{O} = [0]$, given

$\therefore \quad 2a^2 + 8a - 4 = 0 \quad$ or $\quad a^2 + 4a - 2 = 0$

or $\qquad a = \frac{1}{2}[-4 \pm \sqrt{(16+8)}] = -2 \pm \sqrt{6}$. **Ans.**

****Ex. 12. Show that if A, B, C are matrices, such that A (BC) is defined, then (AB) C is also defined and A (BC) = (AB) C.**

Sol. Since **A (BC)** is defined so the matrices **A, B, C** are conformable to multipictions and we can take $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{jk}]$ and $\mathbf{C} = [c_{kl}]$, where **A, B, C** are $m \times n$, $n \times p$, $p \times q$ matrices.

Then $\mathbf{AB} = [a_{ij}]\,[b_{jk}]$ is an $m \times p$ matrix

i.e. (i, k)th element of the product $\mathbf{AB} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$ (Note)

Simmarity (j, l)th element of the product $\mathbf{BC} = \sum_{k=1}^{p} b_{jk}\, c_{kl}$ (Note)

Also $(\mathbf{AB})\,\mathbf{C}$ is the product of an $m \times p$ and a $p \times q$ matrices and so is conformable to multiplication, hence defined.

$\therefore$ (i, l)th element in the product of $(\mathbf{AB})$ and $\mathbf{C}$

= sum of products of corresponding elements in the ith row of $\mathbf{AB}$ and lth column of $\mathbf{C}$ with k common

$$= \sum_{k=1}^{p} \left[\left(\sum_{j=1}^{n} a_{ij}\, b_{jk} \right) c_{kl} \right] \qquad \textbf{(Note)}$$

$$= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kl} \qquad \text{...(i)}$$

Again (i, l)th element in the product of $\mathbf{A}$ and $(\mathbf{BC})$.

= sum of products of corresponding elements in the ith row of $\mathbf{A}$ and lth column of $(\mathbf{BC})$

$$= \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{p} b_{jk}\, c_{kl} \qquad \textbf{(Note)}$$

$$= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kl} \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we conclude that $(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$.

***Ex. 13. If A and B are two matrices such that AB and A + B are both defined, then prove that A and B are square matrices.**

Sol. Let $\mathbf{A}$ be an $m \times n$ matrix.

Since $\mathbf{A} + \mathbf{B}$ is defined *i.e.* $\mathbf{A}$ and $\mathbf{B}$ are conformable to addition, so $\mathbf{B}$ must also be an $m \times n$ matrix.

Again $\mathbf{AB}$ is defined *i.e.* $\mathbf{A}$ and $\mathbf{B}$ are conformable to multiplication and hence the number of columns in $\mathbf{A}$ must be equal to number of rows in $\mathbf{B}$ *i.e.* $n = m$.

Hence $\mathbf{A}$ and $\mathbf{B}$ are $m \times m$ matrices *i.e.* square matrices.

****Ex. 14. If AB = BA then prove that $(\mathbf{AB})^n = \mathbf{A}^n\mathbf{B}^n$.**

Sol. We shall prove this by mathematical induction.

If $n = 1$, then $(\mathbf{AB})^n = \mathbf{A}^n\,\mathbf{B}^n \Rightarrow (\mathbf{AB})^1 = \mathbf{AB}$, which is true.

If $n = 2$, then

$$(\mathbf{AB})^n = (\mathbf{AB})^2 = (\mathbf{AB})\,(\mathbf{AB})$$
$$= (\mathbf{ABA})\,\mathbf{B}, \text{ by associative iaw}$$
$$= (\mathbf{AAB})\,\mathbf{B}, \quad \because \quad \mathbf{BA} = \mathbf{AB}, \text{ given}$$
$$= \mathbf{A}^2\mathbf{B}^2.$$

Hence $(\mathbf{AB})^n = \mathbf{A}^n\mathbf{B}^n$ is true for $n = 2$.

Now suppose that it is true for $n = m$ *i.e.* $(\mathbf{AB})^m = \mathbf{A}^m\mathbf{B}^m$

or $\quad (\mathbf{AB})^m\,(\mathbf{AB}) = (\mathbf{A}^m\,\mathbf{B}^m)\,(\mathbf{AB})$

or $\quad (AB)^{m+1} = A^m (B^m A) B$, by associative law

$$= A^m (B^{m-1} BA) B, \quad \because \quad B^m = B^{m-1} B$$

$$= A^m (B^{m-1} AB) B, \quad \because \quad BA = AB, \text{ given}$$

$$= A^m (B^{m-2} BAB) B, \quad \because \quad B^{m-1} = B^{m-2} B$$

$$= A^m (B^{m-2} ABB) B, \quad \because \quad BA = AB, \text{ given}$$

$$= A^m (B^{m-2} AB^2) B$$

$$= A^m (AB^{m-2} B^2) B = (A^m A)(B^{m-2} B^2 B)$$

or $\quad (AB)^{m+1} = A^{m+1} B^{m+1}$

i.e. if $(AB)^n = A^n B^n$ is true for $n = m$, it is *true* for $n = m + 1$.

Also we have proved that it is true for $n = 1$ and 2.

Hence by mathematical induction it is true for all +ve integral values of n.

***Ex. 15. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, prove that $A^n = I_2,\ A,\ -I_2,\ -A$ according as $n = 4p,\ 4p+1,\ 4p+2$ and $4p+3$ respectively.**

Sol. Given $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ ...(i)

$$\therefore \quad A^2 = A \bullet A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} i.i + 0.0 & i.0 + 0.i \\ 0.i + i.0 & 0.0 + i.i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} \quad \text{...(ii)}$$

$$A^3 = A^2 \bullet A = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} i^2.i + 0.0 & i^2.0 + 0.i \\ 0.i + i^2.0 & 0.0 + i^2.i \end{bmatrix} = \begin{bmatrix} i^3 & 0 \\ 0 & i^3 \end{bmatrix} \quad \text{...(iii)}$$

From (ii) and (iii) we get $A^2 = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}$, $A^3 = \begin{bmatrix} i^3 & 0 \\ 0 & i^3 \end{bmatrix}$

Let us assume that $A^n = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix}$ ...(iv)

and also assume that (iv) is true when $n = k$.

i.e. $\quad A^k = \begin{bmatrix} i^k & 0 \\ 0 & i^k \end{bmatrix}$ ...(v)

$$\therefore \quad A^{k+1} = A^k \bullet A = \begin{bmatrix} i^k & 0 \\ 0 & i^k \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \text{ from (v) and (i)}$$

$$= \begin{bmatrix} i^k.i + 0.0 & i^k.0 + 0.i \\ 0.i + i^k.0 & 0.0 + i^k.i \end{bmatrix} = \begin{bmatrix} i^{k+1} & 0 \\ 0 & i^{k+1} \end{bmatrix}$$

$\because$ (iv) is true for $n = k + 1$ provided (v) is true.

Also we have shown in (ii) and (iii) that (iv) is true for $n = 2$ and 3. So it is true for $3 + 1$ *i.e.* 4 and so on.

Hence (iv) is true for all positive integral values of n.

Also if $n = 4p$ then from (iv) we get

$\mathbf{A^n} = \begin{bmatrix} i^{4p} & 0 \\ 0 & i^{4p} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, since $i^{4p} = (i^4)^p = (1)^p = 1$, where $i = \sqrt{(-1)}$

or $\mathbf{A^n = I_2}$. Hence proved

If $n = 4p + 1$, then $i^n = i^{4p+1} = (i)^{4p}.i = 1.i = i$

$\therefore$ From (iv), we get $\mathbf{A^n} = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \mathbf{A}$. Hence proved.

If $n = 4p + 2$, then $i^n = i^{4p+2} = i^{4p} \times i^2$

$= (1)(-1)$, since $i^{4p} = 1$, $i^2 = -1$

$\therefore$ From (iv), we get

$$\mathbf{A^n} = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -\mathbf{I_2}$$ Hence proved.

If $n = 4p + 3$, then $i^n = i^{4p+3} = (i^{4p+2}).i = (-1)\,i$, as above

$= -i$

$\therefore$ From (iv), we get

$$\mathbf{A^n} = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = -\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = -\mathbf{A}$$ Hence proved.

Ex. 16. Evaluate $\begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix}^n$

Sol. Let $\mathbf{A} = \begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix}$ ...(i)

Then $\mathbf{A^2 = A \bullet A}$

$$= \begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta + \sin\theta)^2 - 2\sin^2\theta & (\cos\theta + \sin\theta)\sqrt{2}\sin\theta \\ & \quad + \sqrt{2}\sin\theta(\cos\theta - \sin\theta) \\ -\sqrt{2}\sin\theta(\cos\theta + \sin\theta) & -\sqrt{2}\sin\theta\sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta(\cos\theta - \sin\theta) & \quad + (\cos\theta - \sin\theta)^2 \end{bmatrix}$$

$$= \begin{bmatrix} (\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos\theta & 2\sqrt{2}\sin\theta\cos\theta \\ -2\sqrt{2}\sin\theta\cos\theta & (\cos^2\theta - \sin^2\theta) - 2\cos\theta\sin\theta \end{bmatrix}$$

(Note)

or $$A^2 = \begin{bmatrix} \cos 2\theta + \sin 2\theta & \sqrt{2}\sin 2\theta \\ -\sqrt{2}\sin 2\theta & \cos 2\theta - \sin 2\theta \end{bmatrix} \quad ...(ii)$$

Looking at (i) and (ii) let us assume that

$$A^n = \begin{bmatrix} \cos n\theta + \sin n\theta & \sqrt{2}\sin n\theta \\ -\sqrt{2}\sin n\theta & \cos n\theta - \sin n\theta \end{bmatrix} \quad ...(iii)$$

Let (iii) be true for $n = k$

i.e. $$A^k = \begin{bmatrix} \cos k\theta + \sin k\theta & \sqrt{2}\sin k\theta \\ -\sqrt{2}\sin k\theta & \cos k\theta - \sin k\theta \end{bmatrix} \quad ...(iv)$$

$$\therefore \quad A^{k+1} = A^k \bullet A$$

$$= \begin{bmatrix} \cos k\theta + \sin k\theta & \sqrt{2}\sin k\theta \\ -\sqrt{2}\sin k\theta & \cos k\theta - \sin k\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} (\cos k\theta + \sin k\theta)(\cos\theta + \sin\theta) + (\sqrt{2}\sin k\theta)(-\sqrt{2}\sin\theta) & (\cos k\theta + \sin k\theta)(\sqrt{2}\sin\theta) + (\sqrt{2}\sin k\theta)(\cos\theta - \sin\theta) \\ -\sqrt{2}\sin k\theta(\cos\theta + \sin\theta) + (\cos k\theta - \sin k\theta)(-\sqrt{2}\sin\theta) & (-\sqrt{2}\sin k\theta)(\sqrt{2}\sin\theta) + (\cos k\theta - \sin k\theta)(\cos\theta - \sin\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta\cos\theta + \cos k\theta\sin\theta + \sin k\theta\cos\theta + \sin k\theta\sin\theta - 2\sin k\theta\sin\theta & \sqrt{2}(\sin k\theta\cos\theta + \cos k\theta\sin\theta) \\ -\sqrt{2}(\sin k\theta\cos\theta + \cos k\theta\sin\theta) & -2\sin k\theta\sin\theta + \cos k\theta\cos\theta - \cos k\theta\sin\theta - \sin k\theta\cos\theta + \sin k\theta\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k\theta + \theta) + \sin(k\theta + \theta) & \sqrt{2}\sin(k\theta + \theta) \\ -\sqrt{2}\sin(k\theta + \theta) & \cos(k\theta + \theta) - \sin(k\theta + \theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k+1)\theta + \sin(k+1)\theta & \sqrt{2}\sin(k+1)\theta \\ -\sqrt{2}\sin(k+1)\theta & \cos(k+1)\theta - \sin(k+1)\theta \end{bmatrix}$$

$\therefore$ (iii) is true for $n = k + 1$ provided (iv) is true.

Also we have shown in (ii) that (iii) is true for $n = 2$.

Hence it is true for $n = 2 + 1$ *i.e.* 3 and so on.

Hence (iii) is true for all positive integral values of n.

Hence $$A^n = \begin{bmatrix} \cos n\theta + \sin n\theta & \sqrt{2}\sin n\theta \\ -\sqrt{2}\sin n\theta & \cos n\theta - \sin n\theta \end{bmatrix}$$ **Ans.**

***Ex. 17. If $P(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, then show that**

$$\mathbf{P(x) \bullet P(y) = P(x + y) = P(y) \bullet P(x)}$$

Sol. $\mathbf{P}(x).\mathbf{P}(y)$

$$=\begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}\times\begin{bmatrix}\cos y & \sin y\\ -\sin y & \cos y\end{bmatrix}$$

$$=\begin{bmatrix}\cos x\cos y-\sin x\sin y & \cos x\sin y+\sin x\cos y\\ -\sin x\cos y-\cos x\sin y & -\sin x\sin y+\cos x\cos y\end{bmatrix}$$

$$=\begin{bmatrix}\cos(x+y) & \sin(x+y)\\ -\sin(x+y) & \cos(x+y)\end{bmatrix}=\mathbf{P}(x+y)$$

Similarly we can prove (to be proved in the exam) that

$$\mathbf{P}(y)\bullet\mathbf{P}(x)=\mathbf{P}(x+y)$$

Hence $\mathbf{P}(x).\mathbf{P}(y)=\mathbf{P}(x+y)=\mathbf{P}(y).\mathbf{P}(x)$ Hence proved.

Ex. 18. If $\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, find number a, b so that $(\mathbf{aI}+\mathbf{bA})^2=\mathbf{A}$

Sol. $a\mathbf{I}+b\mathbf{A}=a\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+b\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

$$=\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}+\begin{bmatrix}0 & b\\ -b & 0\end{bmatrix}=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}$$

$$\therefore\quad (a\mathbf{I}+b\mathbf{A})^2=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}\times\begin{bmatrix}a & b\\ -b & a\end{bmatrix}$$

$$=\begin{bmatrix}a^2-b^2 & ab+ba\\ -ab-ab & -b^2+a^2\end{bmatrix}=\begin{bmatrix}a^2-b^2 & 2ab\\ -2ab & a^2-b^2\end{bmatrix}$$

$\therefore$ If $(a\mathbf{I}+b\mathbf{A})^2=\mathbf{A}$, then we have

$$\begin{bmatrix}a^2-b^2 & 2ab\\ -2ab & a^2-b^2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

Equating the corresponding elements, we have

$$a^2-b^2=0,\ 2ab=1\ \Rightarrow\ a=b=1/\sqrt{2}$$ **Ans.**

***Ex. 19. If $\mathbf{e}^{\mathbf{A}}$ is defined as $\mathbf{I}+\mathbf{A}+(\mathbf{A}^2/2\,!)+(\mathbf{A}^3/3\,!)+\ldots$, then show that $\mathbf{e}^{\mathbf{A}}=\mathbf{e}^{\mathbf{x}}\begin{bmatrix}\cosh\mathbf{x} & \sinh\mathbf{x}\\ \sinh\mathbf{x} & \cosh\mathbf{x}\end{bmatrix}$, where $\mathbf{A}=\begin{bmatrix}\mathbf{x} & \mathbf{x}\\ \mathbf{x} & \mathbf{x}\end{bmatrix}$** *(Budelkhand 95)*

Sol. Given that $\mathbf{A}=\begin{bmatrix}x & x\\ x & x\end{bmatrix}$

$$\therefore\quad \mathbf{A}^2=\begin{bmatrix}x & x\\ x & x\end{bmatrix}\times\begin{bmatrix}x & x\\ x & x\end{bmatrix}=\begin{bmatrix}x.x+x.x & x.x+x.x\\ x.x+x.x & x.x+x.x\end{bmatrix}$$

$$= 2\begin{bmatrix} x^2 & x^2 \\ x^2 & x^2 \end{bmatrix},$$

$$\mathbf{A}^3 = \mathbf{A}^2 \bullet \mathbf{A} = 2\begin{bmatrix} x^2 & x^2 \\ x^2 & x^2 \end{bmatrix}\begin{bmatrix} x & x \\ x & x \end{bmatrix} = 2\begin{bmatrix} x^2.x + x^2.x & x^2.x + x^2.x \\ x^2.x + x^2.x & x^2.x + x^2.x \end{bmatrix}$$

$$= 2^2\begin{bmatrix} x^3 & x^3 \\ x^3 & x^3 \end{bmatrix}.$$

In a similar way we can prove that

$$\mathbf{A}^4 = 2^3\begin{bmatrix} x^4 & x^4 \\ x^4 & x^4 \end{bmatrix},\ \mathbf{A}^5 = 2^4\begin{bmatrix} x^5 & x^5 \\ x^5 & x^5 \end{bmatrix}, \text{ etc.}$$

In general $\mathbf{A}^n = 2^{n-1}\begin{bmatrix} x^n & x^n \\ x^n & x^n \end{bmatrix}$ ...(i)

Now we are given that

$$\mathbf{e^A} = \mathbf{I} + \mathbf{A} + (\mathbf{A}^2/2\,!) + (\mathbf{A}^3/3\,!) + \ldots$$

or $$\mathbf{e^A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} x & x \\ x & x \end{bmatrix} + \frac{2}{2\,!}\begin{bmatrix} x^2 & x^2 \\ x^2 & x^2 \end{bmatrix} + \frac{2^2}{3\,!}\begin{bmatrix} x^3 & x^3 \\ x^3 & x^3 \end{bmatrix} + \ldots + \frac{2^{n-1}}{n\,!}\begin{bmatrix} x^n & x^n \\ x^n & x^n \end{bmatrix}$$

$$+ \ldots\ldots = \begin{bmatrix} u & v \\ v & u \end{bmatrix}, \qquad \text{...(ii)}$$

where $u = 1 + x + \frac{2x^2}{2\,!} + \frac{2^2x^3}{3\,!} + \ldots + \frac{2^{n-1}x^n}{n\,!} + \ldots$

$$v = 0 + x + \frac{2x^2}{2\,!} + \frac{2^3x^3}{3\,!} + \ldots + \frac{2^{n-1}x^n}{n\,!} + \ldots$$

or $$u = \frac{1}{2}\left[2 + 2x + \frac{(2x)^2}{2\,!} + \frac{(2x)^3}{3\,!} + \ldots + \frac{(2x)^n}{n\,!} + \ldots\right]$$

$$= \frac{1}{2}\left[1 + \left\{1 + 2x + \frac{(2x)^3}{2\,!} + \frac{(2x)^3}{3\,!} + \ldots + \frac{(2x)^n}{n\,!} + \ldots\right\}\right]$$

$$= \frac{1}{2}\,[1 + e^{2x}]$$

and $$v = \frac{1}{2}\left[\left\{1 + 2x + \frac{(2x)^2}{2\,!} + \ldots + \frac{(2x)^n}{n\,!} + \ldots\right\} - 1\right], \text{ similarly}$$

$$= \frac{1}{2}\,[e^{2x} - 1]$$

$\therefore$ From (ii), we get

$$\mathbf{e^A} = \frac{1}{2}\begin{bmatrix} e^{2x} + 1 & e^{2x} - 1 \\ e^{2x} - 1 & e^{2x} + 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^x(e^x+e^{-x}) & e^x(e^x-e^{-x}) \\ e^x(e^x-e^{-x}) & e^x(e^x+e^{-x}) \end{bmatrix} = e^x\begin{bmatrix} (e^x+e^{-x})/2 & (e^x-e^{-x})/2 \\ (e^x-e^{-x})/2 & (e^x+e^{-x})/2 \end{bmatrix}$$

$$= e^x\begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}$$

Hence proved.

EXERCISES ON CHAPTER I

Ex. 1. Given $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$

find the matrix **C**, such that $\mathbf{A} + \mathbf{C} = \mathbf{B}$.

Ans. $\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

Ex. 2. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & -4 \\ 1 & 5 \\ -2 & 2 \end{bmatrix}$

find **AB** and show that $\mathbf{AB} \neq \mathbf{BA}$.

Ex. 3. Find **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & -2 \\ -2 & -1 & -1 \\ -1 & -3 & -1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Ans. $\mathbf{AB} = \begin{bmatrix} 7 & 7 & 7 \\ -1 & -1 & -1 \\ -6 & -6 & -6 \end{bmatrix}$, $\mathbf{BA} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Ex. 4. If $\mathbf{A} = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

verify that $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$.

Ex. 4. Find **A** and **B**, where

$$\mathbf{A} + 2\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 0 \\ -5 & 3 & 1 \end{bmatrix}, \ 2\mathbf{A} - \mathbf{B} = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

Ans. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

Ex. 6. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$

and $\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, prove that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

Ex. 7. If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -3 & 3 & -1 \end{bmatrix}$, show that $\mathbf{A}^2 = \mathbf{I}_4$, where $\mathbf{I}_4$ is 4×4 identity matrix.

Ex. 8. For two matrices **A** and **B**, state the conditions under which (i) $\mathbf{A} = \mathbf{B}$; (ii) **AB** exists and (iii) $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

Ex. 9. State true or false in the case of the following statement. Justify your answer.

If **A** and **B** are conformable for addition, then

$$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2.$$

Ex. 10. If $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, then find $[\mathbf{AB}]^2$.

Ex. 11. What is the difference between zero matrix and a unit matrix ?

[Hint : See § 1·03 Page 4]

Ex. 12. Find non-zero matrices **A** and **B** of order 3×3 such that $\mathbf{AB} = \mathbf{O}$, where **O** is the zero matrix of order 3×3.

[Hint : See Ex. 1 (c) Page 14 or Ex. 7 Page 17]

Chapter II

Some Types of Matrices

§ 2·01. Triangular Matrices. *(Bundelkhand 94)*

(a) Upper Triangular Matrix. A square matrix **A** whose elements $a_{ij} = 0$ for $i > j$ is called an upper triangular matrix.

For example
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ 0 & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ 0 & 0 & a_{33} & \ldots\ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & a_{nn} \end{bmatrix}$$

(b) Lower Triangular Matrix. A square matrix **A** whose elements $a_{ij} = 0$ for $i < j$ is called a lower triangular matrix.

For example
$$\begin{bmatrix} a_{11} & 0 & 0 \ldots\ldots & 0 \\ a_{21} & a_{22} & 0 \ldots\ldots & 0 \\ a_{31} & a_{32} & a_{33} \ldots\ldots & 0 \\ \ldots & \ldots & \ldots \ldots\ldots & 0 \\ a_{n1} & a_{n2} & a_{n3} \ldots\ldots & a_{nn} \end{bmatrix}$$

§ 2·02. Diagonal Matrix.

Definition. A square matrix which is both upper and lower triangular is called a diagonal matrix. *(Bundelkhand 94)*

For example
$$\begin{bmatrix} a_{11} & 0 & 0 \ldots\ldots & 0 \\ 0 & a_{22} & 0 \ldots\ldots & 0 \\ 0 & 0 & a_{33} \ldots\ldots & 0 \\ \ldots & \ldots & \ldots \ldots\ldots & \ldots \\ 0 & 0 & 0 \ldots\ldots & a_{nn} \end{bmatrix}$$
(See § 1·03 Page 4 also)

Theorem I. *Any two diagonal matrices of the same order commute under multiplication.* *(Bundelkhand 95, 94)*

Proof. Let any two diagonal matrices be

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_1 & 0 & 0 & \ldots & 0 \\ 0 & b_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & b_n \end{bmatrix}$$

Then we have

$$\mathbf{AB} = \begin{bmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{bmatrix} \times \begin{bmatrix} b_1 & 0 & 0 & \ldots & 0 \\ 0 & b_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & b_n \end{bmatrix}$$

or $$\mathbf{AB}=\begin{bmatrix} a_1b_1 & 0 & 0 & \dots & 0 \\ 0 & a_2b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_nb_n \end{bmatrix} \quad \text{...(i)}$$

and $$\mathbf{BA}=\begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & b_n \end{bmatrix} \times \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

$$=\begin{bmatrix} b_1a_1 & 0 & 0 & \dots & 0 \\ 0 & b_2a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_na_n \end{bmatrix} \quad \text{...(ii)}$$

$\therefore$ From (i) and (ii), we find that $\mathbf{AB}=\mathbf{BA}$ and each one of them is a diagonal matrix of order n. **(Note)**

Hence proved.

Theorem II. *Product of any two diagonal matrices of order n is a diagonal matrix of order n.*

Proof. The same as of Theorem I above.

Theorem III. *Sum of any two diagonal matrices of order n is a diagonal matrix of order n and commute under addition.*

Proof. Let any two diagonal matrices be

$$\mathbf{A}=\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix} \text{ and } \mathbf{B}=\begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & b_n \end{bmatrix}$$

$$\therefore \quad \mathbf{A}+\mathbf{B}=\begin{bmatrix} a_1+b_1 & 0 & 0 & \dots & 0 \\ 0 & a_2+b_2 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & a_n+b_n \end{bmatrix} \quad \text{...(i)}$$

and $$\mathbf{B}+\mathbf{A}=\begin{bmatrix} b_1+a_1 & 0 & 0 & \dots & 0 \\ 0 & b_2+a_2 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & b_n+a_n \end{bmatrix} \quad \text{...(ii)}$$

$\therefore$ From (i) and (ii), we get $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ and each one of them is a diagonal matrix of order n.

§ 2·03. Scalar matrix.

Definition. If in a square matrix **A** all the diagonal elements are equal to a (where $a \neq 0$) and all the remaining elements are equal to zero then it is called a scalar matrix.

For example $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$ is a scalar matrix of order 4×4.

Commutative Matrices

Definition. If **A** and **B** are two square matrices such that **AB** = **BA**, then **A** and **B** are called **commutative** matrices or are said to **commute.**

If **AB** = − **BA**, the matrices **A** and **B** are said to **anti-commute.**

Solved Examples on § 2·03.

Ex. 1. If $\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{a} \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & \mathbf{a_{23}} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & \mathbf{a_{33}} \end{bmatrix}$

Then prove that AB = BA = aB.

Sol. $\mathbf{AB} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$= \begin{bmatrix} aa_{11} & aa_{12} & aa_{13} \\ aa_{21} & aa_{22} & aa_{23} \\ aa_{31} & aa_{32} & aa_{33} \end{bmatrix} = a \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$= a\mathbf{B}$.

Similarly $\mathbf{BA} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

$$= \begin{bmatrix} aa_{11} & aa_{12} & aa_{13} \\ aa_{21} & aa_{22} & aa_{23} \\ aa_{31} & aa_{33} & aa_{33} \end{bmatrix} = a \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a\mathbf{B}$$

Hence **AB** = **BA** = a**B**.

Ex. 2. Show that the matrices A and B anti-commute, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{2} & -\mathbf{1} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{4} & -\mathbf{1} \end{bmatrix}$$

Sol. Here $\mathbf{AB} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 \cdot 1 - 1 \cdot 4 & 1 \cdot 1 + (-1).(-1) \\ 2 \cdot 1 - 1 \cdot 4 & 2 \cdot 1 + (-1).(-1) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix} \quad \text{...(i)}$$

And $\mathbf{BA} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 \cdot 1 + 1 \cdot 2 & 1.(-1) + 1.(-1) \\ 4 \cdot 1 + (-1).2 & 4.(-1) + (-1)(-1) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -3 \end{bmatrix}$$

$$= -\begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix} \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we find that **AB** = − **BA**.

Hence **A** and **B** anti-commute.

Exercise on § 2·03

Ex. 1. Show that the matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ anti-commute.

Ex. 2. Show that the matrices $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix}$ commute.

§ 2·04. Unit Matrix or Identity Matrix.

Definition. If in a scalar matrix the diagonal element $a = 1$, then the matrix is called the unit matrix or identity matrix and is denoted by $\mathbf{I_n}$ in the case of $n \times n$ matrix.

For example $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solved Examples on § 2·04.

***Ex. 1. If A be any $n \times n$ matrix and $\mathbf{I_n}$ is the identity matrix of order $n \times n$, then prove that A $\mathbf{I_n}$ = $\mathbf{I_n}$ A = A**

Sol. Let us suppose that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } \mathbf{I_n} = \begin{bmatrix} 1 & 0 & \dots 0 \\ 0 & 1 & \dots 0 \\ \dots & \dots & \dots 0 \\ 0 & 0 & \dots 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{A \bullet I_n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}.1 + a_{12}.0 + \dots + a_{1n}.0 & a_{11}.0 + a_{12}.1 + \dots + a_{1n}.0 & \dots & \dots & a_{11}.0 + a_{12}.0 + \dots + a_{1n}.1 \\ a_{21}.1 + a_{22}.0 + \dots + a_{2n}.0 & a_{21}.0 + a_{22}.1 + \dots + a_{2n}.0 & \dots & \dots & a_{21}.0 + a_{22}.0 + \dots + a_{2n}.1 \\ & \dots & \dots & \dots & \\ a_{n1}.1 + a_{n2}.0 + \dots + a_{nn}.0 & a_{1n}.0 + a_{n2}.1 + \dots + a_{nn}.0 & \dots & \dots & a_{n1}.0 + a_{n2}.0 + \dots + a_{nn}.1 \end{bmatrix}$$

$$=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \mathbf{A}$$

Similarly we can show that $\mathbf{I_n} \bullet \mathbf{A} = \mathbf{A}$.

Hence we have $\mathbf{A} \bullet \mathbf{I_n} = \mathbf{I_n} \bullet \mathbf{A} = \mathbf{A}$.

***Ex. 2. Prove that $\mathbf{I}^m = \mathbf{I}^{m-1} = \dots = \mathbf{I}^2 = \mathbf{I}$, where m is any positive integer and $\mathbf{I_n}$ is the unit matrix of order $n \times n$.**

Sol. Let **A** be any $n \times n$ matrix and **I** be the unit matrix of order $n \times n$ *i.e.* $\mathbf{I} = \mathbf{I_n}$.

Now we know that $\mathbf{AI_n} = \mathbf{I_nA} = \mathbf{A}$ (See Ex. 1 above)

But $\mathbf{I_n} = \mathbf{I}$. ...(i)

$\therefore \mathbf{AI} = \mathbf{IA} = \mathbf{A}$

Taking $\mathbf{A} = \mathbf{I}$, we have $\mathbf{I} \bullet \mathbf{I} = \mathbf{I}$ or $\mathbf{I}^2 = \mathbf{I}$ ...(ii)

Again from (i), taking $\mathbf{A} = \mathbf{I}^2$, where $\mathbf{I}^2 = \mathbf{I}$ (proved), we get

$\mathbf{I}^2 \bullet \mathbf{I} = \mathbf{I}^2$ or $\mathbf{I}^3 = \mathbf{I}^2 = \mathbf{I}$, from (ii).

Proceeding in this way, we can prove that

$\mathbf{I}^m = \mathbf{I}^{m-1} = \dots = \mathbf{I}^2 = \mathbf{I}$, where m is any positive integer.

Exercise on § 2·04

Ex. If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 3 & 1 \end{bmatrix}$, show that $\mathbf{A}^2 = \mathbf{I}$, where **I** is the unit matrix.

§ 2·05. Periodic Matrix.

Definition. A square matrix **A** is called periodic, if $\mathbf{A}^{k+1} = \mathbf{A}$, where k is a positive integer.

If k is the least positive integer for which $\mathbf{A}^{k+1} = \mathbf{A}$, then **A** is said to be of period k.

Idempotent matrix.

Definition. A square matrix **A** is called idempotent provided it satisfies the relation $\mathbf{A}^2 = \mathbf{A}$.

Symmetric Idempotent Matrix.

Definition. A square matrix **A** is called symmetric idempotent if $\mathbf{A} = \mathbf{A}'$ and $\mathbf{A}^2 = \mathbf{A}$, where $\mathbf{A}'$ is the transposed matrix of **A**, (See § 2·08 Page 69).

Solved Examples on § 2·05.

Ex. 1 (a) **Show that the matrix** $\mathbf{A} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ **is idempotent.**

(*Rohilkhand 96*)

Sol. $A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$= \begin{bmatrix} 2\cdot2 - 2(-1) - 4\cdot1 & 2(-2) - 2\cdot3 - 4(-2) & 2(-4) - 2\cdot4 - 4(-3) \\ -1\cdot2 + 3(-1) + 4\cdot1 & -1(-2) + 3\cdot3 + 4(-2) & -1(-4) + 3\cdot4 + 4(-3) \\ 1\cdot2 - 2(-1) - 3\cdot1 & 1(-2) - 2\cdot3 - 3(-2) & 1(-4) - 2\cdot4 - 3(-3) \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$

Hence the matrix **A** is idempotent.

Ex. 1 (b) Show that the matrix $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ **is idempotent.** *(Avadh 91)*

Sol $A^2 = A \cdot A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \times \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$

$= \begin{bmatrix} 4+3-5 & -6-12+15 & -10-15+20 \\ -2-4+5 & 3+16-15 & 5+20-20 \\ 2+3-4 & -3-12+12 & -5-15+16 \end{bmatrix}$

$= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} = A$

Hence the matrix **A** is idempotent.

***Ex. 2. If A and B are idempotent matrices, then show that AB is idempotent if A and B commute.**

Sol. If **A** is the idempotent, then $A^2 = A$ and if **B** is idempotent then

$$B^2 = B. \qquad \text{...(i)}$$

And if **A** and **B** commute, then $AB = BA$ ...(ii)

Now $(AB)^2 = (AB).(AB)$

$= A(BA)B$, by associative law

$= A(AB)B$, from (ii)

$= (AA)(BB)$, by associative law

$= A^2B^2$

$= AB$ by (i)

Hence **AB** is idempotent

Ex. 3. If A is an idempotent matrix, then the matrix $B = I - A$ is idempotent and $AB = O = BA$.

Sol. We know $IA = AI = A$. (See Ex. 1 Page 64) ...(i)

Also $\mathbf{A}$ being an idempotent matrix, we have $\mathbf{A}^2 = \mathbf{A}$. ...(ii)

Since $\mathbf{I}$ and $\mathbf{A}$ are square matrices, so $\mathbf{I} - \mathbf{A}$ is also a square matrix and therefore we have

$$(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A})$$
$$= (\mathbf{I} - \mathbf{A})\,\mathbf{I} - (\mathbf{I} - \mathbf{A})\,\mathbf{A}, \text{ by distributive law}$$
$$= \mathbf{I}^2 - \mathbf{AI} - \mathbf{IA} + \mathbf{A}^2$$
$$= \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A}, \text{ from (i), (ii) and } \mathbf{I}^2 = \mathbf{I}$$

or $(\mathbf{I} - \mathbf{A})^2 = \mathbf{I} - \mathbf{A}$, *i.e.* $\mathbf{I} - \mathbf{A}$ or $\mathbf{B}$ is an idempotent matrix by definition.

Again $\mathbf{AB} = \mathbf{A}(\mathbf{I} - \mathbf{A}) = \mathbf{AI} - \mathbf{A}^2$, by distributive law

$= \mathbf{A} - \mathbf{A}$, from (i) and (ii)

i.e $\mathbf{AB} = \mathbf{O}$.

And $\mathbf{BA} = (\mathbf{I} - \mathbf{A})\,\mathbf{A} = \mathbf{IA} - \mathbf{A}^2$, by distributive law

$= \mathbf{A} - \mathbf{A} = \mathbf{O}$.

Ex. 4. Show that if A and B are matrices of order $n \times n$ and such that $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$, then A and B are idempotent matrices.

Sol. We have $\mathbf{ABA} = (\mathbf{AB})\,\mathbf{A} = (\mathbf{A})\,\mathbf{A}$, $\quad \because \ \mathbf{AB} = \mathbf{A}$ (given)

or $\mathbf{ABA} = \mathbf{A}^2$ ...(i)

Also $\mathbf{ABA} = \mathbf{A}(\mathbf{BA}) = \mathbf{A}(\mathbf{B})$, $\quad \because \ \mathbf{BA} = \mathbf{B}$ (given)

$= \mathbf{AB} = \mathbf{A}$ $\quad \because \ \mathbf{AB} = \mathbf{A}$ (given)

or $\mathbf{ABA} = \mathbf{A}$ ...(ii)

From (i) and (ii), we have $\mathbf{A}^2 = \mathbf{A}$ *i.e.* $\mathbf{A}$ is idempotent.

In a similar manner, we can prove that

$\mathbf{BAB} = \mathbf{B}(\mathbf{AB}) = \mathbf{B}(\mathbf{A})$, $\quad \because \ \mathbf{AB} = \mathbf{A}$ (given)

$= \mathbf{BA} = \mathbf{B}$, $\quad \because \ \mathbf{BA} = \mathbf{B}$ (given)

or $\mathbf{BAB} = \mathbf{B}$ ...(iii)

Also $\mathbf{BAB} = (\mathbf{BA})\,\mathbf{B} = (\mathbf{B})\,\mathbf{B}$, $\quad \because \ \mathbf{BA} = \mathbf{B}$ (given)

or $\mathbf{BAB} = \mathbf{B}^2$ ...(iv)

From (iii) and (iv), we have $\mathbf{B}^2 = \mathbf{B}$ *i.e.* $\mathbf{B}$ is idempotent. Hence proved.

Exercises on § 2·05

Ex. If A and B are idempotent, then $\mathbf{A} + \mathbf{B}$ will be idempotent if $\mathbf{AB} = \mathbf{BA} = \mathbf{O}$, where $\mathbf{O}$ is the null matrix.

[**Hint :** $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 = \mathbf{A} + \mathbf{O} + \mathbf{O} + \mathbf{B}$]

§ 2·06. Involutory Matrix.

Defintion. A square matrix A is called Involutory provided it satisfies the relation $\mathbf{A}^2 = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

For example, the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is involutory matrix, since $A^2=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\times\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

$$=\begin{bmatrix}1\cdot1+0\cdot0 & 1\cdot0+0.(-1)\\ 0\cdot1+(-1).0 & 0\cdot0+(-1).(-1)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I.$$

Solved Examples on § 2·06.

Ex. 1. Show that the matrix $A=\begin{bmatrix}-5 & -8 & 0\\ 3 & 5 & 0\\ 1 & 2 & -1\end{bmatrix}$ is involutory. *(Rohilkhand 91)*

Sol. $A^2=\begin{bmatrix}-5 & -8 & 0\\ 3 & 5 & 0\\ 1 & 2 & -1\end{bmatrix}\times\begin{bmatrix}-5 & -8 & 0\\ 3 & 5 & 0\\ 1 & 2 & -1\end{bmatrix}$

$$=\begin{bmatrix}(-5)(-5)+(-8).3+0\cdot1 & (-5)\cdot(-8)+(-8)\cdot5+0\cdot2 & (-5)\cdot0+(-8)\cdot0+0\cdot(-1)\\ 3\cdot(-5)+5\cdot3+0\cdot1 & 3\cdot(-8)+5\cdot5+0\cdot2 & 3\cdot0+5\cdot0+0\cdot(-1)\\ 1.(-5)+2\cdot3+(-1)\cdot1 & 1\cdot(-8)+2\cdot5+(-1)\cdot2 & 1\cdot0+2\cdot0+(-1)(-1)\end{bmatrix}$$

$$=\begin{bmatrix}25-24+0 & 40-40+0 & 0+0+0\\ -15+15+0 & -24+25+0 & 0+0+0\\ -5+6-1 & -8+10-2 & 0+0+1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

Hence the given matrix **A** is involutory.

Ex. 2. If A is any square matrix of order n and I_n is the identity matrix of order n, such that $(I_n-A)(I_n+A)=O$, then show that A is involutory matrix.

Sol. Given that $(I_n-A)(I_n+A)=O$

or $\quad I_n^2+I_n\bullet A-A\bullet I_n-A^2=O$

or $\quad I_n+A-A-A^2=O, \qquad \because I_n^2=I_n, I_n\bullet A=A=A\bullet I_n.$

(See Ex. 1. Page 64)

or $\quad I_n-A^2=O$ or $A^2=I_n$ *i.e.* **A** is involutory by definition.

§ 2·07. Nilpotent Matrix. *(Avadh 93)*

Definition. A square matrix **A** is called Nilpotent matrix of order m, provided it satisfies the relation $A^m=O$ and $A^{m-1}\neq O$, where m is a positive integer and **O** is the null matrix.

For example, the matrix $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ is a nilpotent matrix,

since $\quad A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\neq O,$

$$A^2=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\times\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0\cdot0+1\cdot0 & 0\cdot1+1\cdot0\\ 0\cdot0+0\cdot0 & 0\cdot1+0\cdot0\end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O},$$

$$\mathbf{A}^3 = \mathbf{A}^2 \bullet \mathbf{A} = \mathbf{O} \bullet \mathbf{A} = \mathbf{O}.$$

i.e. **A** is a matrix which is not itself a zero matrix though its powers are zero matrices and so it is a nilpotent matrix (**Another definition of nilpotent matrix**).

Solved Examples on § 2·07.

Ex. Show that $\mathbf{A}=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ **is a nilpotent matrix of order 2.**

Sol. Given $\mathbf{A}=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} \neq \mathbf{O}$

$$\therefore \quad \mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot1+2\cdot1+3(-1) & 1\cdot2+2\cdot2+3(-2) & 1\cdot3+2\cdot3+3(-3) \\ 1\cdot1+2\cdot1+3(-1) & 1\cdot2+2\cdot2+3(-2) & 1\cdot3+2\cdot3+3(-3) \\ -1\cdot1-2\cdot1-3(-1) & -1\cdot2-2\cdot2-3(-2) & -1\cdot3-2\cdot3-3(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix of order 3.}$$

i.e. $\mathbf{A}^2 = \mathbf{O}$ but $\mathbf{A} \neq \mathbf{O}$. Hence **A** is a nilpotent matrix of order 2.

Exercises on § 2·07

Ex. 1. Show that the matrix $\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is nilpotent.

Ex. 2. Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of order 3. *(Avadh 93, 90)*

[**Hint** : Prove that $\mathbf{A}^3 = \mathbf{O},\ \mathbf{A}^2 \neq \mathbf{O}$].

****§ 2·08. Transposed Matrix.** *(Agra 94)*

Definition. The matrix of order $n \times m$ obtained by interchanging the rows and columns of a matrix **A** of order $m \times n$ is called the *transposed matrix of* **A** or *transpose of the matrix* **A** and is denoted by $\mathbf{A}'$ or $\mathbf{A}^t$ (read as **A transpose**).

Another Definition. If $\mathbf{A} = [a_{ij}]$ be a matrix of order $m \times n$, then the matrix $\mathbf{B} = [b_{ij}]$ of order $n \times m$, such that $b_{ij} = a_{ji}$ is known as *transposed matrix of* **A** *or the transpose of the matrix* **A** and is denoted by $\mathbf{A}'$or $\mathbf{A}^t$.

For example : If $\mathbf{A}=\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$ then $\mathbf{A}'=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$

Note 1. The element a_{ij} in the ith row and jth column of **A** stands in jth row and ith column of **A**′.

Note 2. The transpose of an $m \times n$ matrix is an $n \times m$ matrix.

***§ 2·09. Some Important Theorems on Transposed Matrices.**

Theorem I. *The transpose of the sum of two matrices is the sum of their transpose i.e.* $(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$.

Proof. Let $\mathbf{A}=[a_{ij}]$ and $\mathbf{B}=[b_{ij}]$.

Then $\mathbf{A}+\mathbf{B}=[a_{ij}+b_{ij}],=[c_{ij}]$, say

then $c_{ij}=a_{ij}+b_{ij}$

$\therefore (\mathbf{A}+\mathbf{B})'=[d_{ij}]$, where $d_{ij}=c_{ij}$ for all $1 \le i \le m,\ 1 \le j \le n$

i.e. $d_{ij}=a_{ij}+b_{ij}$, for all $1 \le i \le m,\ 1 \le j \le n$

or $(\mathbf{A}+\mathbf{B})'=[c_{ij}]=[a_{ij}+b_{ij}]$ ...(i)

Also $\mathbf{A}'=[f_{ji}]$, where $f_{ji}=a_{ij}$, for all $1 \le i \le m,\ 1 \le j \le n$

and $\mathbf{B}'=[g_{ji}]$, where $g_{ji}=b_{ij}$ for all $1 \le i \le m,\ 1 \le j \le n$

$\therefore\ \mathbf{A}'+\mathbf{B}'=[f_{ji}]+[g_{ji}]=[f_{ji}+g_{ji}]$

$=[a_{ij}+b_{ij}]$ ...(ii)

$\therefore$ From (i) and (ii) we get $(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$

***Theorem II.** *The transpose of the transpose of a matrix is the matrix itself i.e.* $(\mathbf{A}')'=\mathbf{A}$. *(Meerut 95, 94)*

Proof. Let $\mathbf{A}=[a_{ij}]$ be an $m \times n$ matrix. Then $\mathbf{A}'$ *i.e.* the tranpose of **A** is $n \times m$ matrix and $(\mathbf{A}')'$ *i.e.* the transpose of $\mathbf{A}'$ (or the transpose of **A**) is an $m \times n$ matrix.

Therefore the matrices **A** and $(\mathbf{A}')$ are both $m \times n$ matrices and hence comparable. ...(i)

Also, the element in the ith row and jth column of $(\mathbf{A}')'$.

= the element in the jth row and ith column of $\mathbf{A}'$

= the element in the ith row and jth column of **A**

i.e. the corresponding elements of $(\mathbf{A}')'$ and **A** are equal ...(ii)

$\therefore$ From (i) and (ii), we conclude that $(\mathbf{A}')'=\mathbf{A}$. Hence proved.

Theorem III. *If* **A** *is any* $m \times n$ *matrix, then* $(k\mathbf{A})'=k\mathbf{A}'$, *where* k *is any number.*

Proof. Let $\mathbf{A}=[a_{ij}]$ be any $m \times n$ matrix. Then $k\mathbf{A}$ is also $m \times n$ matrix and therefore $(k\mathbf{A})'$ *i.e.* the transpose of the matrix $k\mathbf{A}$ is an $n \times m$ matrix.

Also $\mathbf{A}'$, the transpose of the matrix **A**, is $n \times m$ matrix and $k\mathbf{A}'$ is also an $n \times m$ matrix.

Thus we find that the matrices $(k\mathbf{A})'$ and $k\mathbf{A}'$ are both $n \times m$ matrices and hence comparable. ...(i)

Again the element in ith row and jth column of $(k\mathbf{A})'$

= the element in jth row and ith column of $k\mathbf{A}$

= k times the element in jth row and ith column of $\mathbf{A}$ (Note)

= k times the element in ith row and jth column of $\mathbf{A}'$ (Note)

= ka_{ij} (Note)

= the element in ith row and jth column of $k\mathbf{A}'$

i.e. the corresponding elements of $(k\mathbf{A})'$ and $k\mathbf{A}'$ are equal ...(ii)

From (i) and (ii), we conclude that $(k\mathbf{A})' = k\mathbf{A}'$. Hence proved.

****Theorem IV.** *The transpose of the product of two matrices is the product in reverse order of their transpose i.e.* $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

(*Garhwal 95, 93; Gorakhpur 96, Rohilkhand 94*)

Proof. Let $\mathbf{A} = [a_{ik}]$ and $\mathbf{B} = [b_{kj}]$ be the two matrices of orders $m \times n$ and $n \times p$ respectively.

Let $\mathbf{C} = \mathbf{AB} = [g_{ik}] \times [b_{kj}] = [c_{ij}]$, say

where $\mathbf{C}$ is a matrix of order $m \times p$.

$\therefore$ The element in the ith row and jth column of $\mathbf{AB}$ is $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$.

This is also the element in the ith row and jth column of $(\mathbf{AB})'$. ...(i)

The elements in the jth row of $\mathbf{B}'$ are $b_{1j}, b_{2j}, b_{3j}, ..., b_{nj}$ and elements in the ith column of $\mathbf{A}'$ are $a_{i1}, a_{i2}, a_{i3}, ..., a_{in}$. Then the element in the jth row and ith column of $\mathbf{B}'\mathbf{A}'$ is

$$\sum_{k=1}^{n} b_{kj}\, a_{ik} = \sum_{k=1}^{n} a_{ik}\, b_{kj} = c_{ij} \quad ...(ii)$$

Hence from (i) and (ii) we conclude that $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

Note. The statement of theorem IV is called the *reversal rule for the transpose of a product.*

Solved Examples on § 2·08 to § 2·09.

Ex. 1. Write down the transpose of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \end{bmatrix}$

Sol. Let $\mathbf{A}'$ be the required transpose of the matrix $\mathbf{A}$. Then $\mathbf{A}'$ = matrix obtained by interchanging the rows and columns of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 2 & 8 \\ 4 & 1 \end{bmatrix}$. **Ans.**

Ex. 2. Verify that $(\mathbf{B})^t (\mathbf{A})^t = (\mathbf{AB})^t$, when

(a) $\mathbf{A} = [2 \quad 1]$, $\mathbf{B} = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 5 & -3 \end{bmatrix}$ (*Budenkhand 91*)

(b) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

(c) $\mathbf{A}=\begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} 3 & 4 & 5 \\ -1 & 2 & 7 \\ 2 & 1 & 0 \end{bmatrix}$ *(Avadh 92)*

Sol. (a) Here $\mathbf{A}^t=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{B}^t=\begin{bmatrix} 1 & 4 \\ -2 & 5 \\ 0 & -3 \end{bmatrix}$

$$\therefore \quad \mathbf{B}^t\mathbf{A}^t=\begin{bmatrix} 1 & 4 \\ -2 & 5 \\ 0 & -3 \end{bmatrix}\times\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 1\cdot2+4\cdot1 \\ -2\cdot2+5\cdot1 \\ 0\cdot2-3\cdot1 \end{bmatrix}=\begin{bmatrix} 6 \\ 1 \\ -3 \end{bmatrix} \quad ...(i)$$

Also $\mathbf{AB}=[2 \quad 1]\times\begin{bmatrix} 1 & -2 & 0 \\ 4 & 5 & -3 \end{bmatrix}$

$$=[2\cdot1+1\cdot4 \quad 2(-2)+1\cdot5 \quad 2\cdot0+1(-3)]$$
$$=[6 \quad 1 \quad -3].$$

$\therefore$ $(\mathbf{AB})^t$ = transposed matrix of **AB**

$$=\begin{bmatrix} 6 \\ 1 \\ -3 \end{bmatrix}=\mathbf{B}^t\mathbf{A}^t, \text{ from (i)}$$

Hence proved.

(b) Here $\mathbf{A}^t=\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$, $\mathbf{B}^t=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

$$\therefore \quad \mathbf{B}^t\mathbf{A}^t=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1\cdot1+2\cdot2-1\cdot3 & 1\cdot3+2(-2)-1\cdot1 \\ 2\cdot1+0\cdot2+1\cdot3 & 2\cdot3+0(-2)+1\cdot1 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix} \quad ...(ii)$$

Also $\mathbf{AB}=\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot1+2\cdot2+3(-1) & 1\cdot2+2\cdot0+3\cdot1 \\ 3\cdot1-2\cdot2+1(-1) & 3\cdot2-2\cdot0+1\cdot1 \end{bmatrix}=\begin{bmatrix} 2 & 5 \\ -2 & 7 \end{bmatrix}$$

$\therefore$ $(\mathbf{AB})^t$ = transposed matrix of **AB**

$$=\begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix}=\mathbf{B}^t\mathbf{A}^t, \text{ from (ii)},$$

Hence proved.

(c) Here $\mathbf{A}^t = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{B}^t = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 1 \\ 5 & 7 & 0 \end{bmatrix}$

$\therefore \quad \mathbf{B}^t\mathbf{A}^t = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 1 \\ 5 & 7 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3 \cdot 2 - 1 \cdot 4 + 2(-1) & 3(-1) - 1 \cdot 0 + 2 \cdot 2 \\ 4 \cdot 2 + 2 \cdot 4 + 1(-1) & 4(-1) + 2 \cdot 0 + 1 \cdot 2 \\ 5 \cdot 2 + 7 \cdot 4 + 0(-1) & 5(-1) + 7 \cdot 0 + 0 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 15 & -2 \\ 38 & -5 \end{bmatrix}$ ...(iii)

Also $\mathbf{AB} = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 4 & 5 \\ -1 & 2 & 7 \\ 2 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 2 \cdot 3 + 4(-1) - 1 \cdot 2 & 2 \cdot 4 + 4 \cdot 2 - 1 \cdot 1 & 2 \cdot 5 + 4 \cdot 7 - 1 \cdot 0 \\ -1 \cdot 3 + 0(-1) + 2 \cdot 2 & -1 \cdot 4 + 0 \cdot 2 + 2 \cdot 1 & -1 \cdot 5 + 0 \cdot 7 + 2 \cdot 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 15 & 38 \\ 1 & -2 & -5 \end{bmatrix}$

$\therefore \quad (\mathbf{AB})^t$ = transposed matrix of $\mathbf{AB}$

$= \begin{bmatrix} 0 & 1 \\ 15 & -2 \\ 38 & -5 \end{bmatrix} = \mathbf{B}^t \mathbf{A}^t$, from (iii).

Hence proved.

***Ex. 3. If $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$**

then verify $(\mathbf{AB})^t = \mathbf{B}^t \mathbf{A}^t$. (*Meerut 93, 91*)

Sol. $\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \times \begin{bmatrix} 4 & 1 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \cdot 4 - 1 \cdot 2 + 0 \cdot 1 & 1 \cdot 1 + 1 \cdot 3 + 0 \cdot 1 & 1 \cdot 0 - 1 \cdot 1 - 0 \cdot 1 \\ 2 \cdot 4 + 1 \cdot 2 + 3 \cdot 1 & 2 \cdot 1 - 1 \cdot 3 + 3 \cdot 1 & 2 \cdot 0 + 1 \cdot 1 - 3 \cdot 1 \\ 4 \cdot 4 + 1 \cdot 2 + 8 \cdot 1 & 4 \cdot 1 - 1 \cdot 3 + 8 \cdot 1 & 4 \cdot 0 + 1 \cdot 1 - 8 \cdot 1 \end{bmatrix}$

or $\quad \mathbf{AB} = \begin{bmatrix} 2 & 4 & -1 \\ 13 & 2 & -2 \\ 26 & 9 & -7 \end{bmatrix}$ and so $(\mathbf{AB})^t = \begin{bmatrix} 2 & 13 & 26 \\ 4 & 2 & 9 \\ -1 & -2 & -7 \end{bmatrix}$ ...(i)

Again $\mathbf{A}^t = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 1 \\ 0 & 3 & 8 \end{bmatrix}$ and $\mathbf{B}^t = \begin{bmatrix} 4 & 2 & 1 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$$\therefore \quad \mathbf{B}^t\mathbf{A}^t = \begin{bmatrix} 4 & 2 & 1 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 1 \\ 0 & 3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4\cdot1 - 2\cdot1 + 1\cdot0 & 4\cdot2 + 2\cdot1 + 1\cdot3 & 4\cdot4 + 2\cdot1 + 1\cdot8 \\ 1\cdot1 + 3\cdot1 + 1\cdot0 & 1\cdot2 - 3\cdot1 + 1\cdot3 & 1\cdot4 - 3\cdot1 + 1\cdot8 \\ 0\cdot1 - 1\cdot1 - 1\cdot0 & 0\cdot2 + 1\cdot1 - 1\cdot3 & 0\cdot4 + 1\cdot1 - 1\cdot8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 13 & 26 \\ 4 & 2 & 9 \\ -1 & -2 & -7 \end{bmatrix} = (\mathbf{AB})^t, \text{ from (i)}$$

Hence proved.

Ex. 4. If $\mathbf{A} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $\mathbf{AA}' = \mathbf{I}_2 = \mathbf{A}'\mathbf{A}$.

Sol. Here $\mathbf{A}' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$\therefore \quad \mathbf{AA}' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

Similarlly we can prove that

$$\mathbf{A}'\mathbf{A} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

Hence $\mathbf{AA}' = \mathbf{I}_2 = \mathbf{A}'\mathbf{A}$.

Exercises on § 2·08 – 2·09

Ex. 1. If $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$, verify that $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$, where $\mathbf{A}', \mathbf{B}'$ are transposes of $\mathbf{A}$ and $\mathbf{B}$.

Ex. 2. If $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

then verify that $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

Ex. 3. If $\mathbf{A}=\begin{bmatrix}2 & 4 & -1\\ -1 & 0 & 2\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}3 & 4 & 5\\ -1 & 2 & 7\\ 2 & 1 & 0\end{bmatrix}$

prove that $(\mathbf{AB})'$ and $\mathbf{B'A'}$ are equal.

Ex. 4. If $\mathbf{A}=\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}3 & 4\\ 2 & 1\end{bmatrix}$, then verify that $[\mathbf{AB}]^t=\mathbf{B}^t\mathbf{A}^t$

Ex. 5. If $\mathbf{A}=\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 3\\ 2 & 4 & 9\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}0 & 1 & -1\\ -3 & 2 & 4\\ 1 & 1 & 0\end{bmatrix}$

then verify that $(\mathbf{AB})^t=\mathbf{B}^t\mathbf{A}^t$.

*§ 2·10. Complex conjugate (or conjugate) of a Matrix.

Definition. The matrix obtained from any given matrix **A** of order $m \times n$ with complex elements a_{ij} by replacing its elements by the corresponding conjugate complex numbers is called the complex conjugate or conjugate of **A** denoted by $\overline{\mathbf{A}}$ and is read as '**A** conjugate.'

or If $\mathbf{A}=[a_{ij}]$ and $\overline{a_{ij}}$ is the complex conjugate of the element a_{ij} then $\overline{\mathbf{A}}=[\overline{a_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$.

For example : If $\mathbf{A}=\begin{bmatrix}1+i & 2+3i\\ 2 & 3i\end{bmatrix}$,

then $$\overline{\mathbf{A}}=\begin{bmatrix}1-i & 2-3i\\ 2 & -3i\end{bmatrix}$$

Real Matrix. *(Avadh 93)*

Definition. A matrix **A** is called real provided it satisfies the relation

$$\mathbf{A}=\overline{\mathbf{A}}$$

Imaginary Matrix. *(Avadh 93)*

Definition. A matrix **A** is called imaginary provided it satisfies the relation $\mathbf{A}=-\overline{\mathbf{A}}$

**§ 2·11. Theorems on complex conjugate of a matrix.

Theorem I. *If* $\mathbf{A}=[a_{ij}]$ *be any* $m \times n$ *matrix with complex elements* a_{ij}, *then the complex conjugate of* $\overline{\mathbf{A}}$ *is the matrix* **A** *itself.*

Proof : By definition (given in § 2·10 above) we know that $\overline{\mathbf{A}}=[\overline{a_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$ and $\overline{a_{ij}}$ is the complex conjugate of a_{ij}.

i.e. the element in the ith row and jth column of complex conjugate of **A** *i.e.* $\overline{\mathbf{A}}$.

= the complex conjugate of element in ith row and jth column of **A**.

$\therefore$ The element in the ith row and jth column of the complex conjugate of $\overline{\mathbf{A}}$ *i.e.* $\overline{\overline{\mathbf{A}}}$.

= the complex conjugate of the element in ith row and jth column of $\overline{\mathbf{A}}$

= the complex conjugate of $\overline{a_{ij}}$ **(Note)**

$= a_{ij}$ *i.e.* the element in the *i*th row and *j*th column of **A**. **(Note)**

i.e. the corresponding elements of **A** and the complex conjugate of $\overline{\mathbf{A}}$ are equal. ...(i)

Also it is evident that **A**, $\overline{\mathbf{A}}$ and its conmplex conjugate are $m \times n$ matrices and hence comparable. ...(ii)

$\therefore$ From (i) and (ii), we conclude that the complex conjugate of $\overline{\mathbf{A}}$ is equal to **A** or $\overline{\overline{\mathbf{A}}} = \mathbf{A}$.

Theorem II. *If* $\mathbf{A} = [a_{ij}]$ *be any* $m \times n$ *matrix with complex elements* a_{ij}, *then* $\overline{\lambda \mathbf{A}} = \bar{\lambda}\, \overline{\mathbf{A}}$.

Proof : By definition, we know

$\overline{\mathbf{A}} = [\bar{a}_{ij}]$, for all $1 \le i \le m$, $1 \le j \le n$ and $\bar{a}_{ij}$ is the complex conjugate of a_{ij}.

Also $\lambda \mathbf{A} = [\lambda a_{ij}]$, for all $1 \le i \le m$, $1 \le j \le n$.

$\therefore\ \overline{\lambda \mathbf{A}} = [\overline{\lambda a_{ij}}] = [\bar{\lambda}\, \bar{a}_{ij}]$, for all $1 \le i \le m$, $1 \le i \le n$ ...(i)

and we know that $\overline{z_1 z_2} = \overline{z_1} . \overline{z_2}$, where z_1, z_2 are any two complex numbers,

Again $\bar{\lambda}\, \overline{\mathbf{A}} = [b_{ij}]$, where $b_{ij} = \overline{\lambda\, a_{ij}}$ for all $1 \le i \le m$, $1 \le j \le n$

$= [\bar{\lambda}\, \overline{a_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$. ...(ii)

$\therefore$ From (i) and (ii) we conclude that the corresponding elements of $\overline{\lambda \mathbf{A}}$ and $\bar{\lambda}\, \overline{\mathbf{A}}$ are equal. Also it is evident that $\overline{\lambda \mathbf{A}}$ and $\bar{\lambda}\, \overline{\mathbf{A}}$ are matrices of the same order. Hence we conclude that $\overline{\lambda \mathbf{A}} = \bar{\lambda}\, \overline{\mathbf{A}}$.

Theorem III. *If* **A** *and* **B** *are two matrices conformable to addition, then*

$$\overline{\mathbf{A} + \mathbf{B}} = \overline{\mathbf{A}} + \overline{\mathbf{B}}.$$

Proof : Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be any two matrices of order $m \times n$. Then as these matrices are given as conformable to addition, so we have $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$, for all $1 \le i \le m$, $1 \le j \le n$. ...(i)

Also $\overline{\mathbf{A}} = [\bar{a}_{ij}]$ and $\overline{\mathbf{B}} = [\bar{b}_{ij}]$, by definition.

$\therefore\ \ \overline{\mathbf{A}} + \overline{\mathbf{B}} = [\bar{a}_{ij} + \bar{b}_{ij}] = [\overline{a_{ij} + b_{ij}}]$, ...(ii)

for all $1 \le i \le m$, $1 \le j \le n$

and also as $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$, where z_1, z_2 are any two complex numbers.

Again from (i), we have

$\overline{\mathbf{A} + \mathbf{B}} =$ complex conjugate of $[a_{ij} + b_{ij}]$

$=$ complex conjugate of $[c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$

$= [\overline{c_{ij}}] = [\overline{a_{ij} + b_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$

i.e. $\overline{\mathbf{A} + \mathbf{B}} = [\overline{a_{ij} + b_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$ ...(iii)

$\therefore$ from (ii) and (iii) we conclude that the corresponding elements of $\overline{A}+\overline{B}$ and $\overline{A+B}$ are equal. Also it is evident that both $\overline{A}+\overline{B}$ and $\overline{A+B}$ are matrices of order $m \times n$ as **A** and **B** are given as conformable to addition. Hence we conclude that $\overline{A+B}=\overline{A}+\overline{B}$

Theorem IV. *If* $\mathbf{A}=[a_{ij}]$ *be any* $m \times n$ *matrix and* $\mathbf{B}=[b_{jk}]$ *be any* $n \times p$ *matrix i.e. if* **A** *and* **B** *are conformable to the product* **AB** *then* $\overline{\mathbf{AB}}=\overline{\mathbf{A}}\,\overline{\mathbf{B}}$.

Proof : Since **A** and **B** are conformable to the product **AB**, so $\mathbf{AB}=[a_{ij}]\times[b_{jk}]=[c_{ik}]$, where $c_{ik}=a_{ij}\,b_{jk}$, for all $1 \le i \le m$, $1 \le k \le p$ and there is summation on j, where $j=1, 2, 3, \ldots\ldots, n$.

Also $\overline{\mathbf{A}}=[\overline{a}_{ij}]$, for all $1 \le i \le m$, $1 \le j \le n$

and $\qquad \overline{\mathbf{B}}=[\overline{b}_{jk}]$, for $1 \le j \le n$, $1 \le k \le p$

$\overline{\mathbf{A}}\,\overline{\mathbf{B}}$ is defined and we have $\overline{\mathbf{A}}\,\overline{\mathbf{B}}=[\overline{a}_{ij}]\times[\overline{b}_{jk}]=[\overline{d}_{ik}]$, ...(i)

where $\overline{d}_{ik}=\overline{a}_{ij}\,\overline{b}_{jk}$ for all $1 \le i \le m$, $1 \le k \le p$ and $j=1, 2, \ldots\ldots, n$.

Again $\overline{\mathbf{AB}}$ = complex conjugate of **AB** *i.e.* $[c_{ik}]$

or $\qquad \overline{\mathbf{AB}}=[\overline{c}_{ik}]$, where $c_{ik}=a_{ij}\,b_{jk}$

$$=[\overline{a_{ij}\,b_{jk}}]=[\overline{a}_{ij}\,\overline{b}_{jk}], \quad \because \quad \overline{z_1 z_2}=\overline{z_1}.\overline{z_2},$$

for any complex numbers z_1 and z_2

$$=[\overline{d}_{ik}], \text{ since } \overline{d}_{ik}=\overline{a}_{ij}\,\overline{b}_{jk} \text{ for all } 1 \le i \le m,$$

$1 \le k \le p$ and $j=1, 2, \ldots, n$ ...(ii)

From (i) and (ii), we conclude the $\overline{\mathbf{AB}}=\overline{\mathbf{A}}\,\overline{\mathbf{B}}$.

§ 2·12. Transposed Conjugate of a Matrix.

Definition. The transpose of conjugage of a matrix **A** *i.e.* $(\overline{\mathbf{A}})'$ is defined as transposed conjugage or tranjugate **A** and is denoted by $\mathbf{A}^{\Theta}$ *i.e.* $\mathbf{A}^{\Theta}=(\overline{\mathbf{A}})'$.

For example : If $\mathbf{A}=\begin{bmatrix} 1+i & 2+3i \\ 2 & 3i \end{bmatrix}$,

then $$\overline{\mathbf{A}}=\begin{bmatrix} 1-i & 2-3i \\ 2 & -3i \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^{\Theta}=\text{transpose of } \overline{\mathbf{A}}=(\overline{\mathbf{A}})'$$
$$=\begin{bmatrix} 1-i & 2 \\ 2-3i & -3i \end{bmatrix}$$

***§ 2·13. Theorems on Transposed conjugate of a matrix.**

Theorem I. *For any matrix* **A**, $(\overline{\mathbf{A}})'=(\overline{\mathbf{A}'})$

i.e. the transposed conjugate of a matrix is equal to conjugate of its transpose.

Proof : Let $\mathbf{A}=[a_{ij}]$ be any $m \times n$ matrix.

Then by definition, $\overline{\mathbf{A}}=[\overline{a}_{ij}]$, for all $1 \le i \le m$ and $1 \le j \le n$.

$\therefore (\overline{\mathbf{A}})'$ = transpose of $\overline{\mathbf{A}}$,

i.e. $\qquad (\overline{\mathbf{A}})'=[b_{ji}]$, where $[b_{ji}]$ is $n \times m$ matrix and $b_{ji}=\overline{a}_{ij}$

for all $1 \le i \le m$, $1 \le j \le n$, ...(i)

Again $\mathbf{A}'$ = transpose of **A** *i.e.* $[a_{ij}]$

$= [c_{ji}]$, where $c_{ji} = a_{ij}$ and $[c_{ji}]$ is $n \times m$ matrix for all

$1 \le i \le m,\ 1 \le j \le n.$

$\therefore\quad (\bar{A})' =$ complex conjugate of $\mathbf{A}'$.

$= \overline{[c_{ji}]}$, by definition.

$= \overline{[a_{ij}]}$, since $c_{ji} = a_{ij}$

or $\quad (\overline{\mathbf{A}})' = [b_{ji}]$, since $b_{ji} = \overline{a_{ij}}$, where $[b_{ji}]$ is $n \times m$ matrix for all $1 \le i \le m,\ 1 \le j \le n$ ...(ii)

$\therefore$ From (i) and (ii), we conclude that $(\overline{\mathbf{A}})' = (\overline{\mathbf{A}'})$.

Theorem II. *For any matrix* $\mathbf{A}$, $(\mathbf{A}^{\Theta})^{\Theta} = \mathbf{A}$.

Proof : Let $\mathbf{A}^{\Theta} = \mathbf{B}$ *i.e.* $\mathbf{B} = (\overline{\mathbf{A}})'$

Then $\mathbf{B}' =$ transpose of $\mathbf{B}$

$=$ transpose of $(\overline{\mathbf{A}})'$

$= \overline{\mathbf{A}}$, since we know $(\mathbf{A}')' = \mathbf{A}$ ...See Th. II Page 70

$\therefore\quad (\overline{\mathbf{B}})' =$ complex conjugate of $\mathbf{B}'$ ...See Th. I above

$=$ complex conjugate of $\overline{\mathbf{A}}$ **(Note)**

$= \mathbf{A}$, since we know $\overline{\overline{\mathbf{A}}} = \mathbf{A}$ ...See Th. I Page 75

i.e. $\quad \mathbf{B}^{\Theta} = \mathbf{A}$, since $\mathbf{B}^{\Theta} = (\overline{\mathbf{B}'}) = (\overline{\mathbf{B}})'$ ...See Th. I above

i.e. $\quad (\mathbf{A}^{\Theta})^{\Theta} = \mathbf{A}$, since $\mathbf{A}^{\Theta} = \mathbf{B}$. Hence proved.

Theorem III. (a). *For any matrix* $\mathbf{A}$, $(k\mathbf{A})^{\Theta} = k\mathbf{A}^{\Theta}$, *where k is a scalar.*

Proof : By definition, we know that

$(k\mathbf{A})^{\Theta} = (\overline{k\mathbf{A}})'$

$= (\bar{k}\,\overline{\mathbf{A}})'$, by Th. II Page 76

$= \bar{k}\,(\overline{\mathbf{A}})'$, by Th. III Page 70

$= k\mathbf{A}^{\Theta}$, since k is a scalar. Hence proved.

Theorem III (b). *For any matrix* $\mathbf{A}$, $(k\,\mathbf{A})^{\Theta} = \bar{k}\,\mathbf{A}^{\Theta}$, *where k is any complex number.*

Proof : By definition, we know that

$(k\mathbf{A})^{\Theta} = (\overline{k\mathbf{A}})'$

$= (\bar{k}\,\overline{\mathbf{A}})'$, by Th. II Page 76

$= (\bar{k}\,\overline{\mathbf{A}})'$, by Th. III Page 70

$= \bar{k}\,\mathbf{A}^{\Theta}$, $\because\ (\overline{\mathbf{A}})' = \mathbf{A}^{\Theta}$, by definition. Hence proved.

Theorem IV. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are two matrices conformable to addition, then*

$$(\mathbf{A} + \mathbf{B})^{\Theta} = \mathbf{A}^{\Theta} + \mathbf{B}^{\Theta}.$$ *(Meerut 90)*

Proof : By definition, we have

$(\mathbf{A} + \mathbf{B})^{\Theta} = (\overline{\mathbf{A} + \mathbf{B}})' = (\overline{\mathbf{A}} + \overline{\mathbf{B}})'$, $\because\ \overline{\mathbf{A} + \mathbf{B}} = \overline{\mathbf{A}} + \overline{\mathbf{B}}$

...See Th. III Page 76

$= (\overline{\mathbf{A}})' + (\overline{\mathbf{B}})'$ by Th. I Page 70

$= \mathbf{A}^{\Theta} + \mathbf{B}^{\Theta}$, by definition. Hence proved.

****Theorem V.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are two matrices conformable to the product* $\mathbf{AB}$, *then* $(\mathbf{AB})^{\Theta} = \mathbf{B}^{\Theta}\mathbf{A}^{\Theta}$

Proof : $(\mathbf{AB})^{\Theta} = (\overline{\mathbf{AB}})'$, by definition

$= (\overline{\mathbf{A}}\,\overline{\mathbf{B}})'$, by Th. IV Page 77

$= (\overline{\mathbf{B}})'(\overline{\mathbf{A}})'$, by Th. IV Page 71 **(Note)**

$= \mathbf{B}^{\Theta}\mathbf{A}^{\Theta}$, by definition. Hence proved

Example : Find $\{\mathbf{A}^{\Theta}\}^{\Theta}$, $\overline{\mathbf{A}}'$ **and** $(\overline{\mathbf{A}})'$ **for the matrix**

$$\mathbf{A} = \begin{bmatrix} 1+i & 3-5i \\ 2i & 5 \end{bmatrix}$$

Solution. $\overline{\mathbf{A}} = \begin{bmatrix} 1-i & 3+5i \\ -2i & 5 \end{bmatrix}$ ... See § 2·10 Page 75

$(\overline{\mathbf{A}})' = \text{Transpose of } \overline{\mathbf{A}} = \begin{bmatrix} 1-i & -2i \\ 3+5i & 5 \end{bmatrix}$ ... See § 2·08 Page 69

$\mathbf{A}' = \text{Transpose of } \mathbf{A} = \begin{bmatrix} 1+i & 2i \\ 3-5i & 5 \end{bmatrix}$ ...See § 2·08 Page 69

$$\overline{\mathbf{A}}' = \text{conjugate of } \mathbf{A}' = \begin{bmatrix} 1-i & -2i \\ 3+5i & 5 \end{bmatrix} = (\overline{\mathbf{A}})'$$

$$\mathbf{A}^{\Theta} = \text{conjugate transpose of } \mathbf{A} = \begin{bmatrix} 1-i & -2i \\ 3+5i & 5 \end{bmatrix} = \overline{\mathbf{A}}'$$

and $\{\mathbf{A}^{\Theta}\}^{\Theta} = \text{conjugate transpose of } \mathbf{A}^{\Theta}$

$$= \begin{bmatrix} 1+i & 3-5i \\ 2i & 5 \end{bmatrix} = \mathbf{A}$$

**§ 2·14. Symmetric and skew-symmetric matrices.

(a) Symmetric Matrix. *(Agra 94; Avadh 92)*

Definition. A square matrix $\mathbf{A} = [a_{ij}]$ is called symmetric provided $a_{ij} = a_{ji}$, for all values of i and j.

For example : $\mathbf{A} = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 2 & 7 \\ 5 & 7 & 3 \end{bmatrix}$

Note. $k\mathbf{A}$ is also symmetric, if k is scalar.

(b) Skew-symmetric Matrix. *(Agra 94; Avadh 92)*

Definition. A square matrix $\mathbf{A} = [a_{ij}]$ is called skew-symmetric provided $a_{ij} = -a_{ji}$, for all values of i and j.

For example : $\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$

Note. $k\mathbf{A}$ is also skew-symmetric, if k is scalar.

§ 2·15. Theorems on Symmetric and Skew-symmetric matrices.

Theorem I. *A square matrix* $\mathbf{A}$ *is symmetric iff* $\mathbf{A} = \mathbf{A}'$. (*Kanpur 90*)

Proof : Let $\mathbf{A}$ be an $n \times n$ square matrix *i.e.* $\mathbf{A} = [a_{ij}]$, for all $1 \le i \le n$ and $1 \le j \le n$.

If $\mathbf{A}$ is symmetric matrix, then by definition, we have

$$[a_{ij}] = [a_{ji}], \text{ for all } 1 \le i \le n \text{ and } 1 \le j \le n \qquad \text{...(i)}$$

Also, by definition,

$\mathbf{A}' = [b_{ij}]$ such that $b_{ij} = a_{ji}$ for all $1 \le i \le n$, $i \le j \le n$ ...See § 2·08 Page 69

or $\mathbf{A}' = [a_{ji}]$, for all $1 \le i \le n$, $1 \le j \le n$

$= [a_{ij}]$, from (i).

Hence $\mathbf{A}' = \mathbf{A}$.

Conversely if $\mathbf{A} = \mathbf{A}'$. Then $\mathbf{A}$ must be a square matrix

Also $\mathbf{A} = \mathbf{A}' \Rightarrow [a_{ij}] = [a_{ji}]$, for all $1 \le i \le n$, $1 \le j \le n$

$\Rightarrow a_{ij} = a_{ji}$, for all $1 \le i \le n$, $1 \le j \le n$

$\Rightarrow$ $\mathbf{A}$ is a symmetric matrix. Hence proved.

Theorem II. *A square matrix* $\mathbf{A}$ *is skew-symmetric iff* $\mathbf{A}' = -\mathbf{A}$.

Proof : Let $\mathbf{A}$ be an $n \times n$ square matrix *i.e* $\mathbf{A} = [a_{ij}]$ for all $1 \le i \le n$ and $1 \le j \le n$.

If $\mathbf{A}$ is a skew-symmetric matrix, then by definition, we have

$$[a_{ij}] = [-a_{ji}], \text{ for all } 1 \le i \le n,\ 1 \le j \le n \qquad \text{...(i)}$$

Also, by definition, $\mathbf{A}' = [b_{ij}]$, such that $b_{ij} = a_{ji}$.

for all $1 \le i \le n$, $1 \le j \le n$. ...See § 2·08 Page 69

or $\mathbf{A}' = [a_{ji}]$ for all $1 \le i \le n$, $1 \le j \le n$

$= -[-a_{ji}] = -[a_{ij}]$, from (i).

Hence $\mathbf{A}' = -\mathbf{A}$.

Conversely if $\mathbf{A}' = -\mathbf{A}$, then $\mathbf{A}$ must be a square matrix.

Also $\mathbf{A}' = -\mathbf{A} \Rightarrow [a_{ji}] = -[a_{ij}]$, for all $1 \le i \le n$, $1 \le j \le n$

$\Rightarrow a_{ji} = -a_{ij}$

$\Rightarrow a_{ij} = -a_{ji}$, for all $1 \le i \le n$, $1 \le j \le n$

$\Rightarrow$ $\mathbf{A}$ is a skew-symmetric matrix. Hence proved.

*****Theorem III.** *Every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrices.*

(*Avadh 94, 92, 90; Bundelkhand 95; Meerut 93*)

Proof : Let $\mathbf{A}$ be a square matrix, then we can write

$$\mathbf{A} = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{A} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}') + \tfrac{1}{2}(\mathbf{A} - \mathbf{A}') \qquad \text{...(i)}$$

since $\frac{1}{2}\mathbf{A}$, $\frac{1}{2}\mathbf{A}'$ are conformable to addition, $\mathbf{A}$ being a square matrix. (**Note**)

Now $\{\frac{1}{2}(\mathbf{A} + \mathbf{A}')\}' =$ transpose of $\frac{1}{2}(\mathbf{A} + \mathbf{A}')$

$= \frac{1}{2}(\mathbf{A} + \mathbf{A}')'$ ...by § 2·09 Th. III Page 70

$= \frac{1}{2}\{\mathbf{A}' + (\mathbf{A}')'\}$ ...by § 2·09 Th. I Page 70

$= \frac{1}{2}(\mathbf{A}' + \mathbf{A})$ ...by § 2·09 Th. II Page 70

or $\{\frac{1}{2}(\mathbf{A} + \mathbf{A}')\}' = \frac{1}{2}(\mathbf{A} + \mathbf{A}')$, as matrix addition is commutative.

Therefore, by definition, $\frac{1}{2}(\mathbf{A} + \mathbf{A}')$ is a symmetric matrix ...(ii)

Again $\{\frac{1}{2}(\mathbf{A} - \mathbf{A}')\}' = \frac{1}{2}(\mathbf{A} - \mathbf{A}')'$ ...by § 2·09 Th. III Page 70

$= \frac{1}{2}\{\mathbf{A} + (-1)\,\mathbf{A}'\}'$ **(Note)**

$= \frac{1}{2}\{\mathbf{A}' + \{(-1)\,(\mathbf{A}')'\}$ ...by § 2·09 Th. I Page 70

$= \frac{1}{2}\{\mathbf{A}' + (-1)\,(\mathbf{A}')'\}$ ...by § 2·09 Th. III Page 70

$= \frac{1}{2}\{\mathbf{A}' + (-1)\,\mathbf{A}\}$ ...by § 2·09 Th. II Page 70

$= \frac{1}{2}(\mathbf{A}' - \mathbf{A}) = \frac{1}{2}\{(-1)^2\,\mathbf{A}' + (-1)\,\mathbf{A}\}$ **(Note)**

$= (-1).\frac{1}{2}(-\mathbf{A}' + \mathbf{A})$

or $\{\frac{1}{2}(\mathbf{A} - \mathbf{A}')\}' = -\frac{1}{2}(\mathbf{A} - \mathbf{A}')$, as matric addition is commutative.

Therefore, by definition, $\frac{1}{2}(\mathbf{A} - \mathbf{A}')$ is a skew-symmetric matric. ...(iii)

Hence from (i), (ii) and (iii), we find that the matrix **A** can be expressed as the sum of a symmetric and a skew symmetric matrices.

To prove that the representation (i) is unique, let

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2. \quad \text{...(iv)}$$

where $\mathbf{A}_1$ is symmetric and $\mathbf{A}_2$ is skew-symmetric.

Then $\mathbf{A}_1 = \mathbf{A}_1'$ ...(v)

and $\mathbf{A}_2 = -\mathbf{A}_2'$ ...(vi)

From (iv), we have $\mathbf{A}' = (\mathbf{A}_1 + \mathbf{A}_2)'$

$= \mathbf{A}_1' + \mathbf{A}_2'$ ... by Th. I § 2·09 Page 70

or $\mathbf{A}' = \mathbf{A}_1 - \mathbf{A}_2$, from (v), (vi) ...(vii)

Adding and subtracting (iv) and (vii), we get

$$\mathbf{A} + \mathbf{A}' = 2\mathbf{A}_1 \text{ and } \mathbf{A} - \mathbf{A}' = 2\mathbf{A}_2$$

or $$\mathbf{A}_1 = \frac{1}{2}(\mathbf{A} + \mathbf{A}') \text{ and } \mathbf{A}_2 = \frac{1}{2}(\mathbf{A} - \mathbf{A}')$$

$\therefore$ From (iv), we get $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}') + \frac{1}{2}(\mathbf{A} - \mathbf{A}')$, which is the same as (i).

Hence the representation (i) is unique. Hence proved.

Solved Examples on § 2·14 and § 2·15

Ex. 1. Show that the matric $\mathbf{A} = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ -7 & -8 & 0 \end{bmatrix}$ **is skew-symmetric.** *(Meerut 94)*

Sol. In the given matrix, we find that

$a_{11} = 0,\ a_{12} = 6 = -a_{21},\ a_{13} = 7 = -a_{31},\ a_{22} = 0,\ a_{23} = 8 = -a_{32},\ a_{33} = 0$

i.e. $a_{ij} = -a_{ji}$ for all $1 \le i \le 3,\ 1 \le j \le 3$.

Hence by definition [See § 2·14 (b) Page 79] the given matrix **A** is skew-symmetric.

Ex. 2. If $\mathbf{A}=\begin{bmatrix}3 & 1 & -1\\ 0 & 1 & 2\end{bmatrix}$**, then show that AA′ and A′A are both symmetric matrices.**

Sol. Here $A'=\begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 2\end{bmatrix}$

$$\therefore \quad AA'=\begin{bmatrix}3 & 1 & -1\\ 0 & 1 & 2\end{bmatrix}\times\begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}3\cdot3+1\cdot1-1(-1) & 3\cdot0+1\cdot1-1\cdot2\\ 0\cdot3+1\cdot1+2(-1) & 0\cdot0+1\cdot1+2\cdot2\end{bmatrix}$$

$$=\begin{bmatrix}11 & -1\\ -1 & 5\end{bmatrix}, \text{ which is a symmetric matrix.}$$

[See § 2·14 (a) Page 79]

Similarly $A'A=\begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 2\end{bmatrix}\times\begin{bmatrix}3 & 1 & -1\\ 0 & 1 & .2\end{bmatrix}$

$$=\begin{bmatrix}3\cdot3+0\cdot0 & 3\cdot1+0\cdot1 & 3(-1)+0\cdot2\\ 1\cdot3+1\cdot0 & 1\cdot1+1\cdot1 & 1(-1)+1\cdot2\\ -1\cdot3+2\cdot0 & -1\cdot1+2\cdot1 & -1(-1)+2\cdot2\end{bmatrix}$$

$$=\begin{bmatrix}9 & 3 & -3\\ 3 & 2 & 1\\ -3 & 1 & 5\end{bmatrix}, \text{ which is a symmetric matrix}$$

[See § 2·14 (a) Page 79]

***Ex. 3. (a). If A and B are both skew-symmetric matrices of same order such that AB = BA, then show that AB is symmetric.**

Sol. If **A** and **B** are both skew-symmetric matrices,

then $$A=-A' \text{ and } B=-B' \qquad \text{...(i)}$$

Also given that $AB = BA$

$$=(-B')(-A'), \text{ from (i)}$$

$$=B'A'=(AB)' \qquad \text{...See Th. IV § 2·09 Page 71}$$

or $AB=(AB)'$ *i.e.* **AB** is a symmetric matrix. Hence proved.

Ex. 3 (b). If A is a symmetric matrix, then show that kA is also symmetric for any scalar k.

Sol. Here $(kA)'=kA'$, See. § 2·09 Th. III Page 70

$$=kA, \quad \because \quad A'=A, \text{ A being symmetric}$$

Hence kA is symmetric, if **A** is so.

****Ex. 4 (a). Find the symmetric and skew-symmetric parts of the matrix**

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 4\\ 6 & 8 & 1\\ 3 & 5 & 7\end{bmatrix}$$

Sol. (Refer Theorem III § 2·15 Pages 80 – 81)

Here $\mathbf{A}'$ = transpose of $\mathbf{A}$

$$= \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}$$

The symmetric part of $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}')$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1+1 & 2+6 & 4+3 \\ 6+2 & 8+8 & 1+5 \\ 3+4 & 5+1 & 7+7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 8 & 7 \\ 8 & 16 & 6 \\ 7 & 6 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & \frac{7}{2} \\ 4 & 8 & 3 \\ \frac{7}{2} & 3 & 7 \end{bmatrix}$$ **Ans.**

And the skew-symmetric part of $\mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}')$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & \text{[illegible]} & \text{[illegible]} \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1-1 & 2-6 & 4-3 \\ 6-2 & 8-8 & 1-5 \\ 3-4 & 5-1 & 7-7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 2 & 0 & -2 \\ -\frac{1}{2} & 2 & 0 \end{bmatrix}$$ **Ans.**

***Ex. 4 (b) Express given matrix A as sum of a symmetric and skew-symmetric matrices.** $\mathbf{A} = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 1 & 7 & 1 \end{bmatrix}$ *(Agra 93)*

Sol. From Theorem III § 2·15 Pages 80 – 81 we find that the symmetric and skew-symmetric parts of a matrix $\mathbf{A}$ are $\frac{1}{2}(\mathbf{A} + \mathbf{A}')$ and $\frac{1}{2}(\mathbf{A} - \mathbf{A}')$ respectively whose sum is eveidently $\mathbf{A}$. **(Note)**

i.e. $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}') + \frac{1}{2}(\mathbf{A} - \mathbf{A}')$ ...(i)

Now $\mathbf{A}'$ = transpose of $\mathbf{A} = \begin{bmatrix} 6 & 4 & 1 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}$

$$\therefore \quad A + A' = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 1 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 4 & 1 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6+6 & 8+4 & 5+1 \\ 4+8 & 2+2 & 3+7 \\ 1+5 & 7+3 & 1+1 \end{bmatrix} = \begin{bmatrix} 12 & 12 & 6 \\ 12 & 4 & 10 \\ 6 & 10 & 2 \end{bmatrix}$$

$$\therefore \quad \frac{1}{2}(A + A)' = \frac{1}{2}\begin{bmatrix} 12 & 12 & 6 \\ 12 & 4 & 10 \\ 6 & 10 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 3 \\ 6 & 2 & 5 \\ 3 & 5 & 1 \end{bmatrix},$$

which is evidently a symmetric matrix as $a_{ij} = a_{ji}$ for all values of i and j

$$\text{And } A - A' = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 1 & 7 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 4 & 1 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-6 & 8-4 & 5-1 \\ 4-8 & 2-2 & 3-7 \\ 1-5 & 7-3 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 \\ -4 & 0 & -4 \\ -4 & 4 & 0 \end{bmatrix}$$

$$\therefore \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 4 & 4 \\ -4 & 0 & -4 \\ -4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix},$$

which is evidently a skew-symmetric matrix as $a_{ij} = -a_{ji}$ for all values of i, j

$\therefore$ From (i), we get

$$A = \begin{bmatrix} 6 & 6 & 3 \\ 6 & 2 & 5 \\ 3 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

= sum of a symmetric and skew-symmetric matrices, as proved above.

****Ex. 5. If A is any square matrix, show that AA′ is a symmetric matrix.**

Sol. $(AA')' =$ transpose of AA'

$= (A')'A'$ ...See Th. IV § 2·09 Page 71

$= AA'$ ... See Th. II § 2·09 Page 70

i.e. $AA' = (AA')'$. Hence AA' is a symmetric matrix by definition.

***Ex. 6. If A be a square matrix, show that A + A′ is symmetric and A − A′ is a skew-symmetric matrix.** *(Meerut 99)*

Sol. If A is a square matrix, then

$(A + A')' = A' + (A')'$ ...See § 2.09 Th. I Page 70

$= A' + A$ ...See § 2.09 Th. II Page 70

$= A + A'$, by commutative law of addition

Hence by definition $A + A'$ is symmetric.

Again $(\mathbf{A}-\mathbf{A}')'=\mathbf{A}'-(\mathbf{A}')'$, ...See § 2.09 Th. I Page 70

$=\mathbf{A}'-\mathbf{A}$ ...See § 2.09 Th. II Page 70

$=-(\mathbf{A}-\mathbf{A}')$

Hence by definition $\mathbf{A}-\mathbf{A}'$ is skew-symmetric.

***Ex. 7. If A is skew-symmetric matrix, then show that $\mathbf{AA}'=\mathbf{A}'\mathbf{A}$ and $\mathbf{A}^2$ is symmetric.**

Sol. If **A** is a skew-symmetric matrix, then we know that

$$\mathbf{A}'=-\mathbf{A} \quad \text{...(i)}$$

Pre-multiplying both sides of (i) by **A**, we get

$$\mathbf{AA}'=-\mathbf{AA}=-\mathbf{A}^2 \quad \text{...(ii)}$$

Post-multiplying both sides of (i) by **A**, we get

$$\mathbf{A}'\mathbf{A}=-\mathbf{AA}=-\mathbf{A}^2 \quad \text{...(iii)}$$

From (ii) and (iii) we conclude that $\mathbf{AA}'=\mathbf{A}'\mathbf{A}$

Further we can prove (as in Ex. 5 Page 84) that $\mathbf{AA}'$ and $\mathbf{A}'\mathbf{A}$ are symmetric matrices. Hence from (ii) and (iii) we find that $-\mathbf{A}^2$ is a symmetric matrix or $\mathbf{A}^2$ is a symmetric matrix, as we know that $k\mathbf{A}$ is also symmetric if k is scalar and **A** is symmetric. **Hence proved.**

Exercises on § 2.14 – § 2.15

***Ex. 1.** If **A** and **B** are symmetric (or skew-symmetric) matrices, then so is $\mathbf{A}+\mathbf{B}$.

Ex. 2. If **A** and **B** are symmetric matrices, then prove that $\mathbf{AB}+\mathbf{BA}$ is symmetric and $\mathbf{AB}-\mathbf{BA}$ is skew-symmetric.

Ex. 3. Show that all positive integral powers of a symmetric matrix are symmetric.

Ex. 4. If **A** is any matrix, then show that $\mathbf{A}'\mathbf{A}$ is a symmetric matrix.

(Hint : See Ex. 5 Page 84)

Ex. 5. If **A** is a symmetric matrix, then show that $\mathbf{AA}'=\mathbf{A}'\mathbf{A}$ and $\mathbf{A}^2$ is symmetric.

(Hint : See Ex. 7 above)

Ex. 6. What is the main diagonal of a skew symmetric matrix ?

(*Kanpur 90*)

[Hint : See § 2.14 (b) Page 79. Each element is zero].

Ex. 7. What is the transpose of a symmetric matrix ? (*Kanpur 90*)

[Hint : See Th. I § 2.15 Page 80]. **Ans.** The matrix itself.

Ex. 8. **A** is a skew symmetric matrix. How will be $\mathbf{A}^n$? n is any positive integer.

***Ex. 9.** Prove that every diagonal element of a skew-symmetric matrix is necessarily zero. (*Garhwal 91; Kanpur 94*)

[**Hint :** In the case of skew-symmetric matrix, we know

$$a_{ij}=-a_{ji} \text{ for all values of } i \text{ and } j$$

$\therefore$ If $i=j$, then $a_{ii}=-a_{ii}$ for all i

i.e. $a_{ii}+a_{ii}=0$ or $2a_{ii}=0$ or $a_{ii}=0$

i.e. all diagonal element of a skew symmetric matrix are necessarily zero.]

**§2.16. Hermitian and Skew-Hermitian Matrices.

(a) Hermitian Matrix. *(Avadh 95, 91, 90)*

Definition. A square matrix **A** such that $\overline{\mathbf{A}}' = \mathbf{A}$ is called Hermitian *i.e.* the matrix $[a_{ij}]$ is Hermitian provided $a_{ij} = \overline{a}_{ji}$, for all values of i and j.

For example : $\mathbf{A} = \begin{bmatrix} l & \alpha + i\beta & \gamma + i\delta \\ \alpha - i\beta & m & x + iy \\ \gamma - i\delta & x - iy & n \end{bmatrix}$

(b) Skew-Hermitian Matrix. *(Avadh 91, 90)*

Definition. A square matrix **A** such that $\overline{\mathbf{A}}' = -\mathbf{A}$ is called skew-Hermitian *i.e.* the matrix $[a_{ij}]$ is skew-Hermitian provided $a_{ij} = -\overline{a}_{ji}$ for all values of i and j.

For example : $\mathbf{A} = \begin{bmatrix} 2i & -\alpha - i\beta & -3 + i \\ \alpha - i\beta & -i & -\gamma + i\delta \\ 3 + i & \gamma + i\delta & 0 \end{bmatrix}$

§ 2.17. Theorems on Hermitian and Skew-Hermitian Matrices.

***Theorem I.** *The diagonal elements of a Hermitian matrix are necessarily real.* *(Avadh 95)*

Proof : Let $[a_{ij}]$ be a $n \times n$ Hermitian matrix, then according to definition [as given in § 2.16 (a) above], we have

$$a_{ij} = \overline{a}_{ji}, \text{ for all } 1 \le i \le n,\ 1 \le j \le n \qquad \text{...(i)}$$

Now the diagonal elements are a_{ii}, where $1 \le i \le n$.

$\therefore$ From (i), we have $a_{ii} = \overline{a}_{ii}$, for all $1 \le i \le n$ ...(ii)

If $a_{ii} = \alpha + i\beta$ where α and β are real,

then $\overline{a}_{ii} = \alpha - i\beta$

$\therefore$ From (ii), we get $\alpha + i\beta = \alpha - i\beta$

or $2i\beta = 0$ or $\beta = 0$

$\therefore$ $a_{ii} = \alpha + i\,(0) = \alpha$, which is purely real.

Hence the diagonal elements of a Hermitian matrix are necessarily real.

Hence proved.

***Theorem II.** *The diagonal elements of a skew-Hermitian matrix are either purely imaginary or zero.* *(Avadh 90)*

Proof : Let $[a_{ij}]$ be an $n \times n$ skew-Hermitian matrix, then according to definition [as given in § 2.16 (b) above] we have

$$a_{ij} = -\overline{a}_{ji}, \text{ for all } 1 \le i \le n,\ 1 \le j \le n. \qquad \text{...(i)}$$

Now the diagonal elements are a_{ii}, where $1 \le i \le n$.

$\therefore$ From (i), we have $a_{ii} = -\overline{a}_{ii}$, for all $1 \le i \le n$. ...(ii)

If $a_{ii} = \alpha + i\beta$, where α and β are real,

then $\bar{a}_{ii} = \alpha - i\beta$.

$\therefore$ From (ii), we get $\alpha + i\beta = -(\alpha - i\beta)$

or $\alpha + i\beta = -\alpha + i\beta$ or $2\alpha = 0$ or $\alpha = 0$

$\therefore$ $a_{ii} = 0 + i\beta = i\beta$, which is purely imaginary and can be zero if $\beta = 0$.

Hence the diagonal elements of a skew-Hermitian matrix **are either** purely imaginary or zero.

****Theorem III.** *Every square matrix (with complex elements)* ***can be*** *uniquely expressed as the sum of a Hermitian and a skew-Hermitian* ***matrices.***

(*Garhwal* 92)

Proof. Let **A** be a square matrix. Then we can write

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^{\Theta}) + \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^{\Theta}) \qquad \text{...(i)}$$

Now $\left(\overline{\mathbf{A} + \mathbf{A}^{\Theta}}\right) = \overline{\mathbf{A}} + \overline{\mathbf{A}^{\Theta}}$ ...See § 2.11 Th. III Page 76

$\therefore \left\{\left(\overline{\mathbf{A} + \mathbf{A}^{\Theta}}\right)\right\}' = \left\{\overline{\mathbf{A}} + \overline{\mathbf{A}^{\Theta}}\right\}' = (\overline{\mathbf{A}})' + (\overline{\mathbf{A}^{\Theta}})'$, ...See § 2.09 Th. I Page 70

$= \mathbf{A}^{\Theta} + (\overline{\mathbf{A}^{\Theta}})'$, by def. $(\overline{\mathbf{A}})' = \mathbf{A}^{\Theta}$, See § 2.12 Page 77

$= \mathbf{A}^{\Theta} + (\overline{\mathbf{A}^{\Theta}})'$. ...(ii)

Now $(\overline{\mathbf{A}^{\Theta}})'$ = transposed conjugate of $\mathbf{A}^{\Theta}$

= transposed conjugate of $(\overline{\mathbf{A}})'$, ...See § 2.12 P. 77

= transposed matrix of $(\mathbf{A})'$,

since conjugate of $\overline{\mathbf{A}}$ is $\mathbf{A}$ ...See Th. I Page 75

$= \mathbf{A}$, $\because$ $(\mathbf{A}')' = \mathbf{A}$ ...See § 2.09 Th. II Page 70

$\therefore$ From (ii) we get, $\left\{\left(\overline{\mathbf{A} + \mathbf{A}^{\Theta}}\right)\right\}' = \mathbf{A}^{\Theta} + \mathbf{A} = \mathbf{A} + \mathbf{A}^{\Theta}$,

as addition of matrices obey commutative law.

$\therefore$ By definition (See § 2.16 (a) Page 86) we find that $\mathbf{A} + \mathbf{A}^{\Theta}$ is a Hermitian matrix.

Again $\left\{\left(\overline{\mathbf{A} - \mathbf{A}^{\Theta}}\right)\right\}' = (\overline{\mathbf{A}} - \overline{\mathbf{A}^{\Theta}})' = (\overline{\mathbf{A}})' - \left(\overline{\mathbf{A}^{\Theta}}\right)'$

$= \mathbf{A}^{\Theta} - \mathbf{A}$, as above

$= -(\mathbf{A} - \mathbf{A}^{\Theta})$.

$\therefore$ By definition (See § 2.16 (b) Page 86) we find that $\mathbf{A} - \mathbf{A}^{\Theta}$ is a skew-Hermitian matrix.

$\therefore$ From (i) we conclude that the square matrix **A** is the sum of a Hermitian and a skew-Hermitian matrices.

Solved Examples on § 2.16 – § 2.17.

Ex. 1 (a). Is $\mathbf{A}=\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3-i \\ -2-5i & 3+i & 5 \end{bmatrix}$

a hermitian matrix ?

Sol. $\mathbf{A}'=\begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3+i \\ -2+5i & 3-i & 5 \end{bmatrix}$

$\therefore \quad \overline{\mathbf{A}}'=\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3-i \\ -2-5i & 3+i & 5 \end{bmatrix}=\mathbf{A}$

Hence by definition [See § 2.16 (a) Page 86], the given matrix **A** is hermitian.

Ex. 1 (b). Prove that the matrix $\mathbf{A}=\begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$

is Hermitian. *(Avadh 91; Rohilkhand 97)*

Sol. $\mathbf{A}'=\begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & -i \\ 2 & i & 0 \end{bmatrix}$

$\therefore \quad \overline{\mathbf{A}}'=\begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}=\mathbf{A}$

$\therefore$ **A** is Hermitian. ...See § 2.16 (a) Page 86

Ex. 2. If $\mathbf{A}=\begin{bmatrix} 3 & 2-3i & 3+5i \\ 2+3i & 5 & i \\ 3-5i & -i & 7 \end{bmatrix}$,

then prove that $\overline{\mathbf{A}}$ **is Hermitian.** *(Meerut 96)*

Sol. $\overline{\mathbf{A}}=\begin{bmatrix} 3 & 2+3i & 3-5i \\ 2-3i & 5 & -i \\ 3+5i & i & 7 \end{bmatrix}=\mathbf{B}$ (say)

Then $\mathbf{B}'=\begin{bmatrix} 3 & 2-3i & 3+5i \\ 2+3i & 5 & i \\ 3-5i & -i & 7 \end{bmatrix}$

$$\therefore \quad \overline{\mathbf{B}}' = \begin{bmatrix} 3 & 2+3i & 3-5i \\ 2-3i & 5 & -i \\ 3+5i & i & 7 \end{bmatrix} = \mathbf{B}$$

$\therefore$ **B** *i.e.* $\overline{\mathbf{A}}$ is Hermitian. ...See § 2.16 (a) Page 86

Ex. 3. Show that $\mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{3+2i} & \mathbf{-2-i} \\ \mathbf{-3+2i} & \mathbf{0} & \mathbf{3-4i} \\ \mathbf{2-i} & \mathbf{-3-4i} & \mathbf{-2i} \end{bmatrix}$,

is skew-Hermitian Matrix. (*Rohilkhand 95*)

Sol. Here $\mathbf{A}' = \begin{bmatrix} i & -3+2i & 2-i \\ 3+2i & 0 & -3-4i \\ -2-i & 3-4i & -2i \end{bmatrix}$

$$\therefore \quad \overline{\mathbf{A}}' = \begin{bmatrix} -i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix} \quad \textbf{(Note)}$$

$$= -\begin{bmatrix} i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix} = -\mathbf{A}$$

Hence by definition [See § 2.16 (b) Page 86], the given matrix **A** is skew-Hermition.

Ex. 4. If A and B are Hermitian, then show that AB is Hermitian if and only if A and B commute.

Sol. If **A** and **B** are Hermitian matrices, then we have

$$\mathbf{A} = (\overline{\mathbf{A}})' = \mathbf{A}^{\Theta} \text{ and } \mathbf{B} = (\overline{\mathbf{B}})' = \mathbf{B}^{\Theta} \qquad \text{...(i)}$$

Then $(\mathbf{AB})^{\Theta} = \mathbf{B}^{\Theta}\mathbf{A}^{\Theta}$, by § 2.13 Th. V Page 79

$= \mathbf{BA}$, by (i) above

$= \mathbf{AB}$, if **A** and **B** commute

i.e. $\quad (\mathbf{AB})^{\Theta} = \mathbf{AB} \quad \text{or} \quad (\overline{\mathbf{AB}})' = \mathbf{AB}, \quad \because \quad \mathbf{A}^{\Theta} = (\overline{\mathbf{A}})'$

Hence by definition **AB** is Hermitian.

Converse of this can be proved to be true by reversing the above calculations.

Ex. 5 (a). If A is a Hermitian matrix, then show that iA is skew-Hermitian. (*Kanpur 90*)

Sol. If **A** is a Hermitian matrix, then

we have $\quad \mathbf{A} = \overline{\mathbf{A}}'$ ...See § 2.16 (a) Page 86

Also $\quad \overline{\mathbf{A}}' = \mathbf{A}^{\Theta}$ ...See § 2.12 Page 77

$\therefore$ Here $\quad \mathbf{A} = \overline{\mathbf{A}}' = \mathbf{A}^{\Theta}$ ...(i)

Now $(iA)^{\Theta} = -iA^{\Theta}, \quad \because \bar{i} = -i$

...See § 2.13 Th. III (a) Page 78

$= -\left(iA^{\Theta}\right)$

or $(iA)^{\Theta} = -(iA)$, from (i) ...(ii)

Also from § 2.16 (b) Page 86 we know that if **A** is a skew-Hermitian matrix, then $\bar{A}' = -A = A^{\Theta}$, from (i)

And from (ii), we find that $-(iA) = (iA)^{\Theta}$, hence (iA) is a skew-Hermitian-matrix.

Ex, 5 (b). If A is a skew-Hermitian matrix, then show that iA is Hermitian.

Sol. If **A** is a skew-Hermitian matrix, then we have

$-A = \bar{A}'$ ...See § 2.16 (b) Page 86

Also $\bar{A}' = A^{\Theta}$ ...See § 2.12 Page 77

$\therefore$ $-A = \bar{A}' = A^{\Theta}$ ...(i)

Now $(iA)^{\Theta} = -iA^{\Theta}, \quad \because \bar{i} = -i$

...See § 2.13 Th. III (a) Page 78

$= -i(-A)$, from (i)

or $(iA)^{\Theta} = iA$ ...(ii)

Also from § 2.16 (a) Page 86 we know that if **A** is a Hermitian matrix, then $\bar{A}' = A = A^{\Theta}$, from (i).

And from (ii) we find that $(iA) = (iA)^{\Theta}$, hence iA is a Hermitian matrix.

Ex. 6. If A is any square matrix, show that AA^{Θ} and $A^{\Theta}A$ are Hermitian.

Sol. $(AA^{\Theta})^{\Theta} = (A^{\Theta})^{\Theta} A^{\Theta}$ ...by § 2.13 Th. V Page 79

$= AA^{\Theta}$ ...by § 2.13 Th. II Page 78

$\therefore$ By definition (See § 2.16 (a) Page 86), AA^{Θ} is Hermitian.

Similarly $(A^{\Theta} A)^{\Theta} = A^{\Theta} (A^{\Theta})^{\Theta}$ ...by § 2.13 Th. V Page 79

$= A^{\Theta} A$ ...by § 2.13 Th. II Page 78

$\therefore$ By definition (See § 2.16 (a) Page 86), $A^{\Theta} A$ is Hermitian.

Ex. 7. Show that A is Hermitian iff $\bar{A}$ is Hermitian.

Sol. Let A be Hermitian, then $A = A^{\Theta}$ ...(i)

Now $(\bar{A})^{\Theta}$ = transposed conjugate of $\bar{A}$

= transposed matrix of **A**, since $(\bar{\bar{A}}) = A$

...See § 2.11 Th. I Page 75

$= \mathbf{A}' = (\mathbf{A}^{\Theta})'$, by (i)

= transpose of transposed conguate of **A**

= conguate of **A**, $\because (\mathbf{B}')' = \mathbf{B}$

i.e., $(\overline{\mathbf{A}})^{\Theta} = \overline{\mathbf{A}}$

Hence by definition, $\overline{\mathbf{A}}$ is a Hermitian matrix

Again if $\overline{\mathbf{A}}$ is Hermitian, then we have

$\overline{\mathbf{A}} = (\overline{\mathbf{A}})^{\Theta}$

= transposed conjuage of $\overline{\mathbf{A}}$

= transponse of **A** ... by § 2·11 Th. I Page 75

or $\overline{\mathbf{A}} = \mathbf{A}'$ ...(ii)

Now $\mathbf{A}^{\Theta} = (\overline{\mathbf{A}})'$, by definition

$= (\mathbf{A}')'$, by (ii)

i.e., $\mathbf{A}^{\Theta} = \mathbf{A}$, ... by § 2·09 Th. II Page 70

Hence by definition **A** is Hermitian. Hence proved

Exercises on § 2.16 – § 2.17

Ex.1. If $\mathbf{A} = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & 1 \\ -2-3i & -1 & 0 \end{bmatrix}$, then slow that $\overline{\mathbf{A}}$ is skew–Hermitian

Ex.2. Show that $\mathbf{A} = \begin{bmatrix} 0 & 2-3i & -2-i \\ -2-3i & 0 & -3+4i \\ 2-i & 3+4i & 0 \end{bmatrix}$ is skew–Hermitian.

Ex. 3. Show that **A** is skew-Hermitian iff $\overline{\mathbf{A}}$ is skew-Hermitian.

[**Hint** : See Ex. 7. Page 90]

Ex. 4. Give an example of matrix which is skew symmetric but not skew-Hermitian.

Ex. 5. If **A** and **B** are Hermitian matrices, show that $\mathbf{AB} + \mathbf{BA}$ is Hermitian and $\mathbf{AB} - \mathbf{BA}$ is skew-Hermitian.

Ex. 6. Show that every square matrix can be uniquely expressed as $P + iQ$. where P, Q are Hermitian. (*Garhwal 95; Rohilkhand 91*)

[**Hint** : See Th. III Page 87, Ex 5(a) Page 89].

***§ 2.18. The inverse of a matrix.**

(*Avadh 91; Bundelkhand 93; Garhwal 91*)

If for a given square matrix **A**, there exists a matrix **B** such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ where **I** is an unit matrix, then **A** is called **non-singular** or **Invertible** and **B** is called **inverse of A** and we write $\mathbf{B} = \mathbf{A}^{-1}$ (read as **B** equals **A** inverse).

Here **A** is the inverse of **B** and we can write $\mathbf{A} = \mathbf{B}^{-1}$

If **B** *i.e.*, $\mathbf{A}^{-1}$ does not exist, then **A** is called **singular.**

Note 1. If **AB** and **BA** are both defined and equal then the matrices **A** and **B** should both be square matrices of the same order.

Note 2. Non-square matrix has no inverse.

For example : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\mathbf{I}$

Each matrix in the product is the inverse of the other.

§ 2.19. Theorems on Inverse of a matrix.

****Theorem I.** *If a given square matrix* **A** *has an inverse, then it is unique or there exists one and only one inverse matrix to a given matrix.*

(*Bundelkhand 93, 91*)

Proof. Let us suppse that **B** and **C** are two possible inverses of **A**. Then we must have (See § 2·18 above).

$$\mathbf{AB}=\mathbf{BA}=\mathbf{I} \quad \text{...(i)}$$

and $$\mathbf{AC}=\mathbf{CA}=\mathbf{I} \quad \text{...(ii)}$$

$\therefore$ From (i) and (ii), we get $\mathbf{AB}=\mathbf{AC}$, each being equal to **I**

or $$\mathbf{B}\,(\mathbf{AB})=\mathbf{B}\,(\mathbf{AC})$$

or $$(\mathbf{BA})\,\mathbf{B}=(\mathbf{BA})\,\mathbf{C} \quad \text{... See § 1·09 Prop. I Page 26}$$

or $$\mathbf{IB}=\mathbf{IC}, \text{ from (i)}$$

or $$\mathbf{B}=\mathbf{C} \quad \text{...See Ex. 1 Page 64}$$

Hence there cannot be two inverses of **A**.

****Theorem II.** *If* **A** *and* **B** *be two non-singluar or invertible matrices of the same order then* **AB** *is also non-singular and*

$$(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\,\mathbf{A}^{-1}$$

(*Avadh 91; Bundelhkand 95; Garhwal 92; Gorakhpur 97; Purvanchal 97, 94*)

Or

The inverse of a product is the product of the inverse taken in the reverse order.

This is also known as the **Reciprocal Law for the inverse of a product.**

Proof. $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist since **A** and **B** are non-singluar.

$\therefore$ $(\mathbf{AB})\,(\mathbf{B}^{-1}\,\mathbf{A}^{-1})=\mathbf{A}\,(\mathbf{BB}^{-1})\,\mathbf{A}^{-1}$, by associative law

$$=\mathbf{AIA}^{-1}=\mathbf{AA}^{-1}, \quad \text{...See Ex. 1. Page 64}$$

$$=\mathbf{I} \quad \text{...See § 2·18 Page 91}$$

And $(\mathbf{B}^{-1}\,\mathbf{A}^{-1})\,(\mathbf{AB})=\mathbf{B}^{-1}\,(\mathbf{A}^{-1}\,\mathbf{A})\,\mathbf{B}$, by associative law

$$=\mathbf{B}^{-1}\,(\mathbf{I})\,\mathbf{B}, \quad \because\ \mathbf{A}^{-1}\,\mathbf{A}=\mathbf{I}$$

$$=\mathbf{B}^{-1}\,(\mathbf{IB})=\mathbf{B}^{-1}\,\mathbf{B}, \quad \text{...See Ex. 1. Page 64}$$

$$=\mathbf{I}. \quad \text{...See § 2·18 Page 91}$$

$\therefore$ $(\mathbf{B}^{-1}\,\mathbf{A}^{-1})\,(\mathbf{AB})=(\mathbf{AB})\,(\mathbf{B}^{-1}\,\mathbf{A}^{-1})=\mathbf{I}$

i.e., $\mathbf{B^{-1} A^{-1}}$ is the inverse of **AB** or $\mathbf{(AB)^{-1} = B^{-1} A^{-1}}$ and as such **AB** is also non-singluar.

Note : For more details on inverse of matrices see chapter V of this book.

****§ 2·20. Orthogonal Matrix.**

Definition. A square matrix **A** is called an orthogonal matrix if $\mathbf{AA' = I}$, where **I** is an identity matrix and **A′** is the transposed matrix of **A**. (*Kanpur 97*)

Theorems on Orthogonal Matrices.

Theorem I. *For any square matrix* **A**, *if* $\mathbf{AA' = I}$, *then* $\mathbf{A'A = I}$.

Proof : Since $\mathbf{AA' = I}$, so **A** is invertible (*i.e.* **A** possesses an inverses) and there exists another matrix **B** such that

$$\mathbf{AB = BA = I} \qquad ...(i)$$

(See § 2·18 Page 91)

Now $\mathbf{B = BI = B\,(AA')}$, $\because \mathbf{AA' = I}$ (given)

$= \mathbf{(BA)\,A' = IA'}$, from (i)

i.e. $\mathbf{B = A'}$

$\therefore$ From (i), we get $\mathbf{AA' = A'A = I}$. Hence proved.

Theorem II. *If* **A** *is an orthogonal matrix, then* **A′** *is also orthogonal.*

Proof : By definition if **A** is an orthogonal matrix, then

$$\mathbf{AA' = A'A = I}$$

or $\mathbf{(AA')' = (A'A)' = I}$, transposing and remembering $\mathbf{I' = I}$

or $\mathbf{(A')'\,A' = A'\,(A')' = I}$. by Th. IV § 2·09 Page 71

or **A′** is orthogonal by definition. Hence proved.

i.e. Transpose of an orthogonal matrix is also orthogonal.

Theorem III. *If* **A** *is an orthogonal matrix, then* $\mathbf{A^{-1}}$ *is also orthogonal.*

Proof : By definition if **A** is orthogonal, then

$$\mathbf{AA' = A'A = I}$$

or $\mathbf{(AA')^{-1} = (A'\,A)^{-1} = I}$,

taking inverse and remembering $\mathbf{I^{-1} = I}$

or $\mathbf{(A')^{-1}\,A^{-1} = A^{-1}\,(A')^{-1} = I}$. by Th. II § 2·19 Page 92

or $\mathbf{(A^{-1})'\,A^{-1} = A^{-1}\,(A^{-1})' = I}$ **(Note)**

or $\mathbf{A^{-1}}$ is orthogonal by definition. Hence proved.

i.e. Inverse of an orthogonal matrix is also orthogonal.

Theorem IV. *For any orthogonal matrices,* **A** *and* **B**, *show that* **AB** *is an orthogonal matrix.*

Proof : If **A** and **B** are orthogonal matrices, then by definition we have

$$\mathbf{AA' = A'A = I} \qquad ...(i)$$

and $$\mathbf{BB' = B'B = I} \qquad ...(ii)$$

$\therefore \mathbf{(AB)\,(AB)' = (AB)\,(B'A')}$ by Th. IV § 2·09 Page 71

$$= \mathbf{AB\,B'A'} = \mathbf{A\,(BB')\,A'} \qquad \textbf{(Note)}$$
$$= \mathbf{AIA'}, \text{ from (ii).}$$
$$= \mathbf{AA'} = \mathbf{I}, \text{ from (i).}$$

Similarly, we can prove that

$$\mathbf{(AB)'\,(AB)} = \mathbf{B'A'\,AB}, \text{ by Th, IV § 2·09 Page 71}$$
$$= \mathbf{B'IB}, \text{ from (i).}$$
$$= \mathbf{B'B} = \mathbf{I}, \text{ from (ii).}$$

Hence **AB** is an orthogonal matrix by definition.

§ 2·21. Unitary Matrix.

Definition. A square matrix **A** is called an unitary matrix if $\mathbf{A}^{\Theta} = \mathbf{I}$, where **I** is an identity matrix and $\mathbf{A}^{\Theta}$ is the transposed conjuage of**A.**

Theorems on Unitary matrices.

Theorem I. *For any square matrix, if* $\mathbf{AA}^{\Theta} = \mathbf{I}$, *then* $\mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$.

Proof : Since $\mathbf{AA}^{\Theta} = \mathbf{I}$, where **I** is the unit matrix, so we find that **A** is invertible and there exists another matrix **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \qquad \text{...(i)}$$

Now $\mathbf{B} = \mathbf{BI} = \mathbf{B}\,(\mathbf{AA}^{\Theta})$, $\because \mathbf{AA}^{\Theta} = \mathbf{I}$ (given)

$$= (\mathbf{BA})\,\mathbf{A}^{\Theta} = \mathbf{IA}^{\Theta}, \text{ from (i).}$$

i.e. $\quad \mathbf{B} = \mathbf{A}^{\Theta}$

$\therefore$ From (i), we get $\mathbf{AA}^{\Theta} = \mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$ Hence proved.

Theorem II. *If* **A** *is an unitary matrix, then* **A′** *is also unitary.*

Proof : By definition if **A** is an unitary matrix, then

$$\mathbf{AA}^{\Theta} = \mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$$

or $\quad (\mathbf{AA}^{\Theta})^{\Theta} = (\mathbf{A}^{\Theta}\,\mathbf{A})^{\Theta} = \mathbf{I}$, taking transposed conguate and remembering that $\mathbf{I}^{\Theta} = \mathbf{I}$ **(Note)**

or $\quad (\mathbf{A}^{\Theta})^{\Theta}\,\mathbf{A}^{\Theta} = \mathbf{A}^{\Theta}\,(\mathbf{A}^{\Theta})^{\Theta} = \mathbf{I}$, using § 2·09 Th. IV Page 71

or $\quad \mathbf{AA}^{\Theta} = \mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$, since $(\mathbf{A}^{\Theta})^{\Theta} = \mathbf{A}$

or $\quad (\mathbf{AA}^{\Theta})' = (\mathbf{A}^{\Theta}\,\mathbf{A})' = \mathbf{I}$, taking transpose of each side

or $\quad (\mathbf{A}^{\Theta})'\,\mathbf{A}' = \mathbf{A}'\,(\mathbf{A}^{\Theta})' = \mathbf{I}$, using § 2·09 Th. IV Page 71

or $\quad (\mathbf{A}')^{\Theta}\,\mathbf{A}' = \mathbf{A}'\,(\mathbf{A}')^{\Theta} = \mathbf{I}$ **(Note)**

or $\quad$ **A′** is an unitary matrix. Hence proved.

Theorem III. *If* **A** *is an unitary matrix then* $\mathbf{A}^{-1}$ *is also unitary.*

Proof : By definition if **A** is an unitary matrix, then

$$\mathbf{AA}^{\Theta} = \mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$$

or $\quad (\mathbf{AA}^{\Theta})^{-1} = (\mathbf{A}^{\Theta}\,\mathbf{A})^{-1} = \mathbf{I}$, taking inverse

or $\quad (\mathbf{A}^{\Theta})^{-1}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,(\mathbf{A}^{\Theta})^{-1} = \mathbf{I}$, by Th. II § 2·19 Page 92 or $$(A^{-1})^{\Theta} A^{-1} = A^{-1} (A^{-1})^{\Theta} = I \qquad \textbf{(Note)}$$

or A^{-1} is an unitary matrix by definition. Hence proved.

Theorem IV. *For any two unitary matrices* **A** *and* **B** *show that* **AB** *is an unitary matrix.* (*Bundelkhand 91*)

Proof : If **A** an **B** are unitary matrices then by definition we have

$$\mathbf{AA}^{\Theta} = \mathbf{A}^{\Theta} \mathbf{A} = \mathbf{I} \qquad ...(i)$$

and $$\mathbf{BB}^{\Theta} = \mathbf{B}^{\Theta} \mathbf{B} = \mathbf{I} \qquad ...(ii)$$

$\therefore$ $(\mathbf{AB})(\mathbf{AB})^{\Theta} = (\mathbf{AB})(\mathbf{B}^{\Theta}\mathbf{A}^{\Theta})$; by Th. V § 2·13 Page 79

$= \mathbf{A}(\mathbf{BB}^{\Theta})\mathbf{A}^{\Theta} = \mathbf{AIA}^{\Theta}$, from (ii)

$= \mathbf{AA}^{\Theta} = \mathbf{I}$, from (i)

Similarly $(\mathbf{AB})^{\Theta}(\mathbf{AB}) = \mathbf{B}^{\Theta}\mathbf{A}^{\Theta}\mathbf{AB}$, by Th. V. § 2·13 Page 79

$= \mathbf{B}^{\Theta}\mathbf{IB}$, from (i)

$= \mathbf{B}^{\Theta}\mathbf{B} = \mathbf{I}$, from (ii)

Hence **AB** is an unitary matrix. Hence proved.

Solved Examples on § 2·20 and § 2·21.

Ex. 1. Show that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ **is orthogonal.** (*Bundelkhand 95*)

Sol. Let $\mathbf{A} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Then $\mathbf{A}' = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$$\therefore \quad \mathbf{A'A} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} (-1).(-1)+2.2+2.2 & (-1).2+2.(-1)+2.2 & (-1).2+2.2+2(-1) \\ 2.(-1)+(-1).2+2.2 & 2.2+(-1).(-1)+2.2 & 2.2+(-1)2+2.(-1) \\ 2.(-1)+2.2+(-1).2 & 2.2+2.(-1)+(-1).2 & 2.2+2.2+(-1)(-1) \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Hence the given matrix **A** is orthogonal.

Ex. 2. Verify that the matrix

$$\mathbf{A}=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \text{ is orthogonal.}$$

Sol. Here $A'=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

$$\therefore \quad A'A=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\times\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{3}+\frac{1}{3}+\frac{1}{3} & \frac{1}{3\sqrt{2}}-\frac{2}{3\sqrt{2}}+\frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{6}}+0+\frac{1}{\sqrt{6}} \\ \frac{1}{3\sqrt{2}}-\frac{2}{3\sqrt{2}}+\frac{1}{3\sqrt{2}} & \frac{1}{6}+\frac{4}{6}+\frac{1}{6} & \frac{-1}{2\sqrt{3}}+0+\frac{1}{2\sqrt{3}} \\ \frac{-1}{\sqrt{6}}+0+\frac{1}{\sqrt{6}} & \frac{-1}{2\sqrt{3}}+0+\frac{1}{2\sqrt{3}} & \frac{1}{2}+0+\frac{1}{2} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\mathbf{I}.$$

Hence **A** is orthogonal.

****Ex. 3. Show that the matrix** $\mathbf{A}=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ **is orthogonal.**

(*Bundelkhand 91; Kanpur 97*)

Sol. $A'=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$\therefore \quad A'A=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$=\begin{bmatrix} \cos^2\alpha+\sin^2\alpha & \cos\alpha\sin\alpha-\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha-\cos\alpha\sin\alpha & \sin^2\alpha+\cos^2\alpha \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\mathbf{I}$$

Hence **A** is orthogonal.

Ex. 4. Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ **is unitary.**

(*Meerut 96*)

Sol. Let $A=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

Then $A^{\Theta}=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$\therefore \quad \mathbf{A}^{\Theta}\mathbf{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \tfrac{1}{3}\begin{bmatrix} 1.1+(1+i).(1-i) & 1.(1+i)+(1+i)(-1) \\ (1-i).1+(-1)(1-i) & (1-i)(1+i)+(-1)(-1) \end{bmatrix}$$

$$= \tfrac{1}{3}\begin{bmatrix} 1+1-i^2 & 0 \\ 0 & 1-i^2+1 \end{bmatrix} = \tfrac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Hence **A** is an unitary matrix.

Exercises on § 2·20 – § 2·21

Ex. 1. Show that the matrix $\mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is unitary.

Ex. 2. Show that the matrix $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ is orthogonal.

Ex. 3. For any two orthogonal matrices **A** and **B**, show that **BA** is an orthogonal matrix.

Ex. 4. For any two unitary matrices A and **B**, show that **BA** is an **unitary** matrix.

Ex. 5. Prove that the following matrix is unitary :—

$$\begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$$

Ex. 6. Prove that a real matrix is unitary if it is orthogonal.

(*Rohilkhand 93*)

§ 2·22. Partitioning of Matrices.

Submatrix.

Definition. A matrix obtained by striking off some of the rows and columns of another matrix **A** is defined as a sub-matrix of **A**.

For example if $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, then

[2], [3], [5] etc.

$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix}$ etc. are all sub– matrices of **A**

It is sometimes found useful to subdivide a matrix into sub-matrices by drawing lines parallel to its rows and columns and to consider these sub-matrices as the elements of the original matrix.

Consider the matrix

$$\mathbf{A}=\begin{bmatrix} x_1\ y_1\ z_1 : \alpha_1\ \beta_1 \\ x_2\ y_2\ z_2 : \alpha_2\ \beta_2 \\ x_3\ y_3\ z_3 : \alpha_3\ \beta_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ p_1\ q_1\ r_1 : a_1\ b_1 \\ p_2\ q_2\ r_2 : a_2\ b_2 \end{bmatrix}$$

Let $\mathbf{A}_{11}=\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$; $\mathbf{A}_{12}=\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$;

$\mathbf{A}_{21}=\begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{bmatrix}$; $\mathbf{A}_{22}=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

Then we may write $\mathbf{A}=\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$

The matrix **A** is then said to have been **partitioned** and the dotted lines indicate the partitions. Here it is obvious that a matrix can be partitioned in several ways. The elements $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ are themselves matrices and are the sub-matrices of A.

Identically partitioned matrices.

Two matrices of the same size are known as identically partitioned matrices if when expressed as matrices of matrices (*i.e.* when partitioned) they are of the same order and the corresponding submatrices (or elements) are also of the same size. Such matrices are said to be **additively coherent.**

For example :

$$\begin{bmatrix} 1\,2\,3 : 7\,5 \\ 4\,5\,6 : 9\,8 \\ \cdots\cdots\cdots \\ 2\,3\,4 : 2\,3 \\ 5\,6\,7 : 4\,5 \\ 4\,5\,8 : 6\,7 \end{bmatrix} \text{ and } \begin{bmatrix} 1\,2\,4 : 3\,0 \\ 2\,0\,5 : 4\,6 \\ \cdots\cdots\cdots \\ 1\,0\,2 : 1\,2 \\ 2\,5\,4 : 3\,4 \\ 2\,6\,2 : 5\,6 \end{bmatrix}$$

Two matrices **A** and **B**, which are conformable to the product **AB**, are called **multiplicative coherent** if **A** and **B** are partioned in such a way that columns of **A** are partitioned in the same way as the rows of **B** are partitioned. Here the rows of **A** and columns of **B** can be partitioned in any way.

For example :

Let $\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 3 & 1 & 1 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 2 & 3 & 5 \\ 3 & 7 & 1 \\ 4 & 0 & 2 \\ 2 & 5 & 1 \end{bmatrix}$

Here **A** is a 3×4 matrix and **B** is a 4×3 matrix, so these are conformable to the product **AB** (*i.e.* the product **AB** exits). Now if write

$$\mathbf{A} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \hline 2 & 3 & 1 & 1 \end{array}\right] \text{ and } \mathbf{B} = \left[\begin{array}{c|cc} 2 & 3 & 5 \\ 3 & 7 & 1 \\ 4 & 0 & 2 \\ \hline 2 & 5 & 1 \end{array}\right]$$

then the partitioning of the columns of **A** is in the same way as the partitioning of the rows of **B.** (Here we note that after third column in **A** the partitioning has been done and in **B** the partitioning has been done after third row). Thus according to definition given above the matrices **A** and **B** are called multiplicative coherent.

Exercise on § 2·22

Ex. Compute **AB** using partitioning

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}; \ \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 3 & 1 & 2 \end{bmatrix}.$$

MISCELLANEOUS SOLVED EXAMPLES

Ex. 1. Show that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ **is the inverse of** $\begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

Sol. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \times \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1.3 + 2(-4) + 3.2 & 1(-2) + 2.1 + 3.0 & 1(-1) + 2(-1) + 3.1 \\ 2.3 + 5(-4) + 7.2 & 2(-2) + 5.1 + 7.0 & 2(-1) + 5(-1) + 7.1 \\ -2.3 - 4(-4) - 5.2 & -2(-2) - 4.1 - 5.0 & -2(-1) - 4(-1) - 5.1 \end{bmatrix}$$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$, where **I** is an unit matrix.

Hence $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 3 & -2 & 1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

***Ex. 2. If A is a non-singular matrix, then prove that $\mathbf{AB} = \mathbf{AC} \Rightarrow \mathbf{B} = \mathbf{C}$, where B and C are square matrices of the same order.**

(*Kanpur 96*)

Sol. Since **A** is non-singular matrix, so $\mathbf{A}^{-1}$ exists.

Now $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$,

premultiplying both sides by A^{-1}

$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$,

by associative law of multiplication

$\Rightarrow IB = IC, \ \because \ A^{-1}A = I$

$\Rightarrow B = C, \ \because \ IB = B$ etc. Hence proved.

****Ex. 3. If product of two non-zero square matrices is a zero matrix, then prove that both of them are singular matrices.**

Sol. Let A and B be two non-zero $n \times n$ matrices.

Given that $AB = O$, where O is the $n \times n$ null matrix.

Let us suppose that B is non-singluar matrix then B^{-1} exists.

Then $AB = O \Rightarrow (AB)B^{-1} = OB^{-1}$ post multiplying both sides by B^{-1},

$\Rightarrow A(BB^{-1}) = O$, by associative law of multiplication. **(Note)**

$\Rightarrow AI = O, \ \because \ BB^{-1} = I$

$\Rightarrow A = O$,

which is against hypothesis as A is a non-zero matrix.

Hence B is not a non-singular matrix *i.e.* B is a singluar matrix.

Similarly we can prove that A is also a singluar matrix.

****Ex. 4. Express the following matrix as the sum of a hermitian and a skew hermitian matrix :**

$$A = \begin{bmatrix} 2+3i & 1-i & 2+i \\ 3 & 4+3i & 5 \\ 1 & 1+i & 2i \end{bmatrix}$$ (*Kumaun 92*)

Sol. From § 2·17 Theorem III Page 87 we know that

$$A = \tfrac{1}{2}(A + A^{\Theta}) + \tfrac{1}{2}(A - A^{\Theta}) \qquad \text{...(i)}$$

i.e. the hermitian and skew-hermitian parts of the matrix A are $\frac{1}{2}(A + A^{\Theta})$ and $\frac{1}{2}(A - A^{\Theta})$ respectively.

Now we know that $A^{\Theta} = (\overline{A})'$, ...(ii)

where $\overline{A} = \begin{bmatrix} 2-3i & 1+i & 2-i \\ 3 & 4-3i & 5 \\ 1 & 1-i & -2i \end{bmatrix}$ **(Note)**

$\therefore$ From (ii) we have $A^{\Theta} = (\overline{A})' =$ transpose of $\overline{A}$

$$= \begin{bmatrix} 2-3i & 3 & 1 \\ 1+i & 4-3i & 1-i \\ 2-i & 5 & -2i \end{bmatrix} \qquad \text{...(iii)}$$

$$\therefore \quad \mathbf{A}+\mathbf{A}^{\Theta}=\begin{bmatrix} 2+3i & 1-i & 2+i \\ 3 & 4+3i & 5 \\ 1 & 1+i & 2i \end{bmatrix}+\begin{bmatrix} 2-3i & 3 & 1 \\ 1+i & 4-3i & 1-i \\ 2-i & 5 & -2i \end{bmatrix}$$

$$=\begin{bmatrix} 2+3i+2-3i & 1-i+3 & 2+i+1 \\ 3+1+i & 4+3i+4-3i & 5+1-i \\ 1+2-i & 1+i+5 & 2i-2i \end{bmatrix}$$

$$=\begin{bmatrix} 4 & 4-i & 3+i \\ 4+i & 8 & 6-i \\ 3-i & 6+i & 0 \end{bmatrix}.$$

$\therefore$ Hermitian part of the given matrix **A**

$$=\tfrac{1}{2}(\mathbf{A}+\mathbf{A}^{\Theta})=\tfrac{1}{2}\begin{bmatrix} 4 & 4-i & 3+1 \\ 4+i & 8 & 6-i \\ 3-i & 6+i & 0 \end{bmatrix}$$

Again $\mathbf{A}-\mathbf{A}^{\Theta}=\begin{bmatrix} 2+3i & 1-i & 2+i \\ 3 & 4+3i & 5 \\ 1 & 1+i & 2i \end{bmatrix}-\begin{bmatrix} 2-3i & 3 & 1 \\ 1+i & 4-3i & 1-i \\ 2-i & 5 & -2i \end{bmatrix}$

$$=\begin{bmatrix} 2+3i-2+3i & 1-i-3 & 2+i-1 \\ 3-1-i & 4+3i-4+3i & 5-1+i \\ 1-2+i & 1+i-5 & 2i+2i \end{bmatrix}$$

$$=\begin{bmatrix} 6i & -2-i & 1+i \\ 2-i & 6i & 4+i \\ -1+i & -4+i & 4i \end{bmatrix}$$

$\therefore$ Skew-hermitian part of the given matrix **A**

$$=\tfrac{1}{2}(\mathbf{A}-\mathbf{A}^{\Theta})=\tfrac{1}{2}\begin{bmatrix} 6i & -2-i & 1+i \\ 2-i & 6i & 4+i \\ -1+i & -4+i & 4i \end{bmatrix}$$

Hence from (i), we have the given matrix **A**

$$=\tfrac{1}{2}\begin{bmatrix} 4 & 4-i & 3+i \\ 4+i & 8 & 6-i \\ 3-i & 6+i & 0 \end{bmatrix}+\tfrac{1}{2}\begin{bmatrix} 6i & -2-i & 1+i \\ 2-i & 6i & 4+i \\ -1+i & -4+i & 4i \end{bmatrix},$$

which is the sum of a hermitian and a shew-hermitian matrix (as proved above).

EXERCISES ON CHAPTER II

Ex. 1. Show that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ -2 & 3 & 1 & 1 \end{bmatrix} \text{ is the inverse of } \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 8 & -1 & -1 & 1 \end{bmatrix}$$

(Hint. See Ex. 1 Page 99)

Ex. 2. If $\mathbf{A}$ be any square matrix, then show that $\mathbf{A} + \mathbf{A}^{\Theta}$ is Hermitian.

Ex. 3. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric and they commute, then $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{A}^{-1}\mathbf{B}^{-1}$ are symmetric.

Ex. 4. Show that every square matrix can be expressed in one and only one way as $P + iQ$, where P and Q are Hermitian.

Ex. 5. If $\mathbf{B}$ is any square matrix, show that $\mathbf{B'AB}$ is symmetric or skew-symmetric according as $\mathbf{A}$ is symmetric or skew-symmetric provided $\mathbf{B'AB}$ is defined.

Ex. 6. If $\mathbf{A}$ and B are two non-singular square matrices of the same order, which of the following statements is true :—

(i) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$;

(ii) $(\mathbf{AB})' = \mathbf{A}'\mathbf{B}'$;

(iii) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$;

(iv) $\mathbf{A} \,.\, \mathbf{A}' = \mathbf{I} \Rightarrow \mathbf{A}' = \mathbf{A}^{-1}$

(v) $\mathbf{A} + \mathbf{A}'$ is a symmetric matrix,

Ex. 7. If $\mathbf{A}$ is Hermitian, such that $\mathbf{A}^2 = \mathbf{O}$, show that $\mathbf{A} = \mathbf{O}$, where $\mathbf{O}$ is the zero matrix.

Ex. 8. Show that every skew-symmetric matrix of odd order is singular.

Ex. 9. When is a matrix said to be invertible ?

[Hint : See § 2·18 Page 91].

Ex. 10. If $\mathbf{D} = \text{diag}\ [d_1, d_2, \ldots, d_n]$,

$d_1 d_2 \ldots d_n \neq 0$, what will be $\mathbf{D}^{-1}$?

Ex. 11. If non-singular matrices $\mathbf{A}$ and $\mathbf{B}$ commute, then

(i) $\mathbf{A}^{-1}$ and $\mathbf{B}$ and (ii) $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$

also commute.

Chapter III

Equivalence

§ 3·01. Elementary Row operations.

Consider the matrices

$$\mathbf{A}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}4&5&6\\1&2&3\\7&8&9\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}3&6&9\\4&5&6\\7&8&9\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}1&2&3\\6&9&12\\7&8&9\end{bmatrix}.$$

Here we observe that the matrices **B, C, D** are related to the matrix **A** in as much as :

(a) **B** can be obtained from **A** by interchanging first and second rows of **A** ;

(b) **C** can be obtained from **A** by multiplying the first row of **A** by 3 and

(c) **D** can be obtained from **A** by adding two times the first row to the second row of **A**.

Such operations on the rows of a matrix are known as elementary row operations. Formal definition is given below :

Definition. Let $\mathbf{A}_i$ denote the ith row of the matrix $\mathbf{A}=[a_{ij}]$. then the elementary row operations on the marix **A** are defined as :

(i) the interchanging of any two rows $\mathbf{A}_i$ and $\mathbf{A}_j$ (*i.e.* ith and jth rows). The symbols R_{ij} or $R_i \longleftrightarrow R_j$ are generally employed for this operation.

(ii) the multiplication of every element of $\mathbf{A}_i$ by a non-zero scalar c *i.e.* replacing the ith row $\mathbf{A}_i$ by $c\mathbf{A}_i$. The symbols $R_i\,(c)$ or $R_i \to cR_i$ are employed for this operation.

(iii) the addition to the elements of row $\mathbf{A}_i$ of c (a scalar) times the corresponding elements of the row $\mathbf{A}_k$ *i.e.* replacing the row $\mathbf{A}_i$ by $\mathbf{A}_i + c\mathbf{A}_k$.

The symbols $R_{ik}\,(c)$ or $R_i \to R_i + cR_k$ are used for this operation.

Note : The above operation do not change the order of the matrix.

Example : Let $\mathbf{A}=\begin{bmatrix}1&2&3\\3&4&5\\5&6&7\end{bmatrix}$

The effect of the elementary row operation $R_2 - R_1$ or $R_{21}\,(-1)$ is to produce the matrix

$$\mathbf{B}=\begin{bmatrix}1&2&3\\3-1&4-2&5-3\\1&6&7\end{bmatrix}=\begin{bmatrix}1&2&3\\2&2&2\\5&6&7\end{bmatrix}$$

Again the effect of elementary row operation $R_2 + R_1$ or $R_{21}(1)$ is to produce the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 3 \\ 2+1 & 2+2 & 2+3 \\ 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \text{ i.e. the matrix } \mathbf{A}.$$

Thus the above two operations are the inverse elementary row operations.

§ 3·02. Row equivalent Matrices.

Definition. If an $m \times n$ matrix **B** can be obtained from an $m \times n$ matrix **A** by a finite number of elementary row operations, then **B** is called the row equivalent to **A** and is written as

$$\mathbf{B} \overset{row}{\sim} \mathbf{A}.$$

Note : Equivalent matrices have the same order.

Example : $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & -3 & 5 & 6 \\ 1 & 0 & 3 & 2 \end{bmatrix} \overset{row}{\sim} \begin{bmatrix} 2 & -3 & 5 & 6 \\ 1 & 3 & 4 & 7 \\ 1 & 0 & 3 & 2 \end{bmatrix}$

(interchanging first and second rows).

§ 3·03. Elementary Row Matrix.

Definition. The matrix obtained by the application of one elementary row operation to the identity matrix $\mathbf{I}_n$ is called an elementary row matrix.

Example. Examples of elementary matrices obtained from $\mathbf{I}_3$, where

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) $\mathbf{I}_3 \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}_a$ (say),

obtained by interchanging first two rows.

(ii) $\mathbf{I}_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}_b$ (say),

obtained by multiplying the elements of second row by c.

(iii) $\mathbf{I}_3 \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}_c$ (say),

obtained by adding two times the elements of second row to the corresponding elements of first row *i.e.* replacing R_1 by $R_i + 2R_2$ *i.e.* $R_{12}(2)$.

§ 3·04. Types of Elementary Row Matrices and their symbols.

(i) $\mathbf{E}_{ij}$ denotes the elementary matrix obtained by interchanging the ith and jth rows (or columns) of an identity (or unit) matrix.

(ii) $\mathbf{E}_i(c)$ denotes the elementary matrix obtained by multiplying the ith row (or column) of the identity matrix by c.

(iii) $\mathbf{E}_{ik}(c)$ denotes the elementary matrix obtained by adding to the elements of the ith row of the identity matrix c times the corresponding elements of the kth row.

(iv) $\mathbf{E}'_{ik}(c)$ denotes the transpose of $\mathbf{E}_{ik}(c)$ and can be obtained by adding to the elements of the ith column of the identity matrix c times the corresponding elements of the kth column.

§ 3·05. Theorem. *Each elementary row operation on $m \times n$ matrix can be effected by premultiplying it by the corresponding elemntary matrix.*

Example : Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

(i) Interchanging the first and third rows, we have

$$\mathbf{A} \sim \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} = \mathbf{B} \text{ (say)}$$

The corresponding elementary matrix (obtained by interchanging first and third row of $\mathbf{I}_2$) is given by

$$\mathbf{E}_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

[Here students should note that as we are to premultiply **A** therefore the number of columns of $\mathbf{E}_{13}$ should be 3, the number of rows of **A**].

$$\text{Now } \mathbf{E}_{13}.\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} = \mathbf{B}$$

This shows that **B** can be obtained from **A** by pre-multiplying it by $\mathbf{E}_{13}$, the corresponding elementary matrix.

(ii) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Multiplying the elements of second row by 2, we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{bmatrix} = \mathbf{B} \text{ (say)}$$

The corresponding elementary matrix (obtained by multilying the elements of second row of $\mathbf{I}_3$ by 2) is given by

$$\mathbf{E}_2(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $\mathbf{E}_3(2) \times \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot1+0\cdot4+0\cdot7 & 1\cdot2+0\cdot5+0\cdot8 & 1\cdot3+0\cdot6+0\cdot9 \\ 0\cdot1+2\cdot4+0\cdot7 & 0\cdot2+2\cdot5+0\cdot8 & 0\cdot3+2\cdot6+0\cdot9 \\ 0\cdot1+0\cdot4+1\cdot7 & 0\cdot2+0\cdot5+1\cdot8 & 0\cdot3+0\cdot6+1\cdot9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{bmatrix} = \mathbf{B}$$

i.e. **B** can be obtained from**A** by pre multiplying it by $\mathbf{E}_2(2)$.

(iii) Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 4 & 5 \\ 5 & 6 & -7 \end{bmatrix}$

Replacing R_1 by $R_1 + 2R_2$ *i.e.* adding two times the elements of second row to the corresponding elements of first row, we get

$$\mathbf{A} \sim \begin{bmatrix} -5 & 6 & 13 \\ -3 & 4 & 5 \\ 5 & 6 & -7 \end{bmatrix} = \mathbf{B} \text{ (say)}$$

The corresponding elementary matrix (obtained by adding two times the elements of second row of $\mathbf{I}_3$ to the corresponding elements of the first row) is given by

$$\mathbf{E}_{12}(2) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $\mathbf{E}_{12}(2) \times \mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 3 \\ -3 & 4 & 5 \\ 5 & 6 & -7 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot1+2(-3)+0\cdot5 & 1\cdot(-2)+2\cdot4+0\cdot6 & 1\cdot3+2\cdot5+0\cdot(-7) \\ 0\cdot1+1(-3)+0\cdot5 & 0\cdot(-2)+1\cdot4+0\cdot6 & 0\cdot3+1\cdot5+0\cdot(-7) \\ 0\cdot1+0(-3)+1\cdot5 & 0\cdot(-2)+0\cdot4+1\cdot6 & 0\cdot3+0\cdot5+1\cdot(-7) \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 6 & 13 \\ -3 & 4 & 5 \\ 5 & 6 & -7 \end{bmatrix} = \mathbf{B}$$

i.e. **B** can be obtained from **A** by pre-multiplying it by $\mathbf{E}_{12}(2)$.

COROLLARY of Theorem given in 3·05 Page 105.

If the matrix **B** is row equivalent to the matrix **A**, then $\mathbf{B} = \mathbf{S} \bullet \mathbf{A}$, where **S** is a product of the elementary matrices.

§ **3·06. Theorem.** *The elementary matrices* $\mathbf{E}_{ij}$, $\mathbf{E}_i(c)$, $\mathbf{E}_{jk}(1)$ *are non-singular.* **(See § 2·18 Page 91)**

Proof : (i) The elementary matrix $\mathbf{E}_{ij}$ is obtained by interchanging the ith and jth rows of **I**. We shall get back **I** if we now apply the same row operation upon $\mathbf{E}_{ij}$ which can also be effected by pre-multiplying $\mathbf{E}_{ij}$ by $\mathbf{E}_{ij}$

(See § 3·05 Page 105).

$$\therefore \mathbf{E}_{ij} \bullet \mathbf{E}_{ij} = \mathbf{I}.$$

i.e. $\mathbf{E}_{ij}$ its own inverse *i.e.* $\mathbf{E}_{ij}$ is non-singular.

(ii) The elementary matrix $\mathbf{E}_i(c)$ is obtained by multiplying the ith row of the identity matrix by c (where $c \neq 0$). We shall get back **I** if we now multiply the elements of ith row of $\mathbf{E}_i(c)$ by $1/c$ which can also be effected by pre-multiplying $\mathbf{E}_i(c)$ with the corresponding elementary matrix which is obtained from **I** by multiplying its ith row by $1/c$, which is therefore the inverse of $\mathbf{E}_i(c)$.

For example, let $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{E}_3(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$

Then $\{\mathbf{E}_3(c)\}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/c \end{bmatrix}$, where $\{\mathbf{E}_3(c)\}^{-1}$ is the inverse of $\mathbf{E}_3(c)$

(iii) The elementary matrix $\mathbf{E}_{ik}(1)$ obtained from **I** by replacing its jth row by (jth row + kth row).

We shall get back **I** if we not replace the jth row of $\mathbf{E}_{ij}(1)$ by (jth row − kth row). **(Note)**

Hence the inverse of $\mathbf{E}_{jk}(1)$ is the elementary matrix obtained from **I** by replacing its jth row by (jth row − kth row).

For example, let $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{E}_{13}(1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

obtained from **I** by replacing its 1st row by (1st row + 3rd row).

Then $\{E_{13}(1)\}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, obtained from I by replacing its first row by (1st row – 3rd row)

§ 3·07. Theorem : *If the matrix* **B** *is row equivalent to the matrix* **A**, *then* **B** = **SA** *where* **S** *is non-singular.*

From Cor. of § 3·05 Page 107 we know that

if $B \overset{row}{\sim} A$, then $B = SA$, where S is the product of the elementary matrices and in § 3·06 above we have proved that elementary matrices are non-singular and hence their product is also non-singular.

This proves the above theorem.

§ 3·08. Theorem : *If a square matrix* **A** *of order n is row equivalent to the identity matrix* I_n, *then* **A** *is non-singular.*

Proof : From § 3·07 above we know that

$A \overset{row}{\sim} I_n$, then $A = S.I_n$ wherre S is non-singular.

Now $S \bullet I_n$ being the product of two non-singular matrices is non-singular. Therefore **A** is non-singular.

Note. The converse of this theorem is also true.

§ 3·09. Theorem : *If a sequence of row operations applied to a square matrix* **A** *reduces it to the identity matrix* **I**, *then the same sequence of row operations applied to the identity matrix gives the inverse of* **A** *(i.e.* A^{-1}*).*

Proof : From Cor. of § 3·05 Page 107 we know that **SA** = **I**, where **S** is the product of the elementary matrices.

i.e. $(E_k \ldots E_3 E_2 E_1) A = I$, where E_i denotes the elementary matrices

or $(E_k \ldots E_3 E_2 . E_1) AA^{-1} = IA^{-1}$

or $(E_k \ldots E_3 . E_2 . E_1) I = A^{-1}$, since $AA^{-1} = I$ and $IA^{-1} = A^{-1}$

(See § 2·18 Page 91 and Ex. 1 Page 64)

Hence the theorem

Note. With the help of the above theorem we shall find the inverse of the given non-singular matrix A.

In the following examples we shall show the successive matrices row equivalent to **A** and **I** in the left hand and right hand columns respectively. When ultimately **A** is reduced to **I** in the left hand column, **I** is reduced to A^{-1} in the right hand column.

Also R_1, R_2, R_3 ,... etc. stand for first row, second row, third row, etc.

Solved Examples on § 3·09.

***Ex. 1 Find the inverse of the matrix** $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}$

Sol.

$$\begin{array}{c|c} \mathbf{A} & \mathbf{I} \\ \begin{bmatrix}1 & -3 & 2\\ 2 & 0 & 0\\ 1 & 4 & 1\end{bmatrix} & \sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{array}$$

$$\sim\begin{bmatrix}1 & -3 & 2\\ 1 & 0 & 0\\ 1 & 4 & 1\end{bmatrix}\Bigg|\sim\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}$$

(Replacing R_2 by $\frac{1}{2}R_2$)

$$\sim\begin{bmatrix}1 & 0 & 0\\ 1 & -3 & 2\\ 1 & 4 & 1\end{bmatrix}\Bigg|\sim\begin{bmatrix}0 & \frac{1}{2} & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

(Interchanging R_1 and R_2)

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & -3 & 2\\ 0 & 4 & 1\end{bmatrix}\Bigg|\sim\begin{bmatrix}0 & (1/2) & 0\\ 1 & -(1/2) & 0\\ 0 & -(1/2) & 1\end{bmatrix}$$

(Replacing R_2 by $R_2 - R_1$ and R_3 by $R_3 - R_1$)

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & -11 & 0\\ 0 & 4 & 1\end{bmatrix}\Bigg|\sim\begin{bmatrix}0 & \frac{1}{2} & 0\\ 1 & \frac{1}{2} & -2\\ 0 & -\frac{1}{2} & 1\end{bmatrix}$$

(Replacing R_2 by $R_2 - 2R_3$)

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 4 & 1\end{bmatrix}\Bigg|\sim\begin{bmatrix}0 & \frac{1}{2} & 0\\ -\frac{1}{11} & -\frac{1}{22} & \frac{2}{11}\\ 0 & -\frac{1}{2} & 1\end{bmatrix}$$

(Replacing R_2 by $-\frac{1}{11}R_2$)

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\Bigg|\sim\begin{bmatrix}0 & \frac{1}{2} & 0\\ -\frac{1}{11} & -\frac{1}{22} & \frac{2}{11}\\ \frac{4}{11} & -\frac{7}{22} & \frac{3}{11}\end{bmatrix}$$

(Replacing R_3 by $R_3 - 4R_2$)

$$= \mathbf{I} \qquad\qquad | \qquad = \mathbf{A}^{-1}$$

$$\therefore \quad \mathrm{A}^{-1} = \begin{bmatrix}0 & \frac{1}{2} & 0\\ -\frac{1}{11} & -\frac{1}{22} & \frac{2}{11}\\ \frac{4}{11} & -\frac{7}{22} & \frac{3}{11}\end{bmatrix}$$

Ans.

***Ex. 2.** $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$**, evaulate $\mathbf{A}^{-1}$,**

Sol.

$$\begin{array}{c|c} \mathbf{A} & \mathbf{I} \\ \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} & \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(Replacing R_2 by $R_2 - 3R_1$ and R_3 by $R_3 - R_1$)

$$\sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 2 & 0 & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(Replacing R_1 by $R_1 - R_3$ and R_2 by $-\frac{1}{4} R_2$)

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

(Replacing R_1 by $R_1 - 3R_2$ and R_3 by $R_3 + R_2$)

$$= \mathbf{I} \qquad | \qquad = \mathbf{A}^{-1}$$

$$\therefore \quad \mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

Ans.

Ex.3. Find the inverxe of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Sol.

$$\begin{array}{c|c} \mathbf{A} & \mathbf{I} \\ \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} & \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Replacing R_1 by $R_1 + 2R_2$)

$$\sim\begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & -2 & 1\end{bmatrix}\Bigg\|\sim\begin{bmatrix}1 & 0 & 2\\1 & 1 & 2\\0 & 0 & 1\end{bmatrix}$$

(Replacing R_2 by R_2+R_1)

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\Bigg\|\sim\begin{bmatrix}3 & 2 & 6\\1 & 1 & 2\\2 & 2 & 5\end{bmatrix}$$

(Replacing R_1 by R_1+2R_2 and R_3 by R_3+2R_2)

$$=\mathbf{I} \qquad | \qquad =\mathbf{A}^{-1}$$

$$\therefore\quad \mathbf{A}^{-1}=\begin{bmatrix}3 & 2 & 6\\1 & 1 & 2\\2 & 2 & 5\end{bmatrix}$$

Ans.

Exercises on § 3·09

Ex. 1. Find $\mathbf{A}^{-1}$ if $\mathbf{A}=\begin{bmatrix}2 & 4 & 3\\0 & 1 & 1\\2 & 2 & -1\end{bmatrix}$ **Ans.** $\frac{1}{4}\begin{bmatrix}3 & -10 & -1\\-2 & 8 & 2\\2 & -4 & -2\end{bmatrix}$

Ex. 2. Find the reciprocal matrix of $\begin{bmatrix}1 & 1 & 1\\2 & 2 & 3\\2 & 4 & 9\end{bmatrix}$ **Ans.** $\frac{1}{3}\begin{bmatrix}-6 & 5 & -1\\15 & -8 & 1\\-6 & 3 & 0\end{bmatrix}$

Ex. 3. Find $\mathbf{A}^{-1}$, if $\mathbf{A}=\begin{bmatrix}1 & 2 & 1\\3 & 1 & 2\\0 & 0 & 2\end{bmatrix}$ **Ans.** $\frac{1}{9}\begin{bmatrix}0 & 3 & -3\\6 & -2 & -1\\-3 & 1 & 5\end{bmatrix}$

§ 3·10. Elementary Column Operation and Column Equivalent Matrices.

In § 3·01 Page 103, if the word *row* is replaced by the word *column* we get the definition of the elementary column operation.

Similarly in § 3·02 Page 104 repacing the word *row* by the word *column* we get definition of column equivalent matrices.

$\mathbf{B}\overset{col}{\sim}\mathbf{A}$ means the matrix **B** is column equivalent to the matrix **A**.

Symbols for column operations are similar to those given for row operations in § 3·01 Page 103. Here the letter R in the symbols are to be replaced by C *e.g.* C_{ij}, $C_{ij}(c)$, $C_{ik}(c)$, where C_{ij} stands for the interchange of ith and jth columns etc or $C_i \longleftrightarrow C_j$; $C_i \to cC_i$, $C_i \to cC_k$.

§ 3·11. Theorem. *Each elemntary column operation on an $m\times n$ matrix* **A** *can be effected by* **post multiplying A** *by the $n\times n$ matrix obtained from the $n\times n$ identity matrix $\mathbf{I}_n$ by the same elementary column operation.*

Proof : If $\mathbf{B} \overset{col}{\sim} \mathbf{A}$

then $\mathbf{B}' \overset{row}{\sim} \mathbf{A}'$ where $\mathbf{B}'$ and $\mathbf{A}'$ are the transposed matrices of **B** and **A**.

(See § 2·08 Page 69)

Since if **B** is obtained from **A** by elementary column operation, then $\mathbf{B}'$ can be obtained from $\mathbf{A}'$ by an elementary row operation.

Hence $\mathbf{B}' = \mathbf{E}\mathbf{A}'$, where **E** is the elementary matrix obtained from $\mathbf{I}_n$ by an elementary row operation. (See § 3·05 Page 105)

Therefore $\mathbf{B} = \mathbf{A}\mathbf{E}'$, **(Note)**

where $\mathbf{E}'$, the transposed matrix of **E**, can be obtained from $\mathbf{I}_n$ by the same elementary column operation.

Hence the theorem.

§ 3·12. Theorem. If there be two $m \times n$ matrices **A** and **B**, then

$\mathbf{B} \overset{col}{\sim} \mathbf{A}$ if $\mathbf{B} = \mathbf{AT}$, where **T** is an $n \times n$ non-singular matrix

Proof : If $\mathbf{B} \overset{col}{\sim} \mathbf{A}$, then $\mathbf{B}' \overset{col}{\sim} \mathbf{A}'$, where $\mathbf{B}'$ and $\mathbf{A}'$ are the transposed matrices of **B** and **A** respectively.

Therefore $\mathbf{B}' = \mathbf{S}\mathbf{A}'$, where **S** is an $n \times n$ non-singular matrix.

(See § 3·07 Page 108)

Consequently $\mathbf{B} = \mathbf{A}\mathbf{S}'$, where $\mathbf{S}'$ is the transposed matrix of **S**

$= \mathbf{AT}$, where $\mathbf{T} = \mathbf{S}'$, an $n \times n$ singular matrix.

Hence the theorem.

§ 3·13. Equivalent Matrices (General Definition). *(Avadh 95)*

Definition. Two $m \times n$ matrices **A** and **B** are called equivalent if one can be obtained from the other by a finite number of row and column operations (or elementary operations) and written as $\mathbf{B} \sim \mathbf{A}$.

§ 3·14. Triangular Matrix.

Definition. A matrix $[a_{ij}]$ is called a triangular matrix if

$$a_{ij} = 0 \text{ for } i > j.$$

For Example $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 7 \end{bmatrix}$ or $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

Note 1. Triangular matrix need not be square. If it is square, then it is called upper triangular matrix. (See § 2·01 (a) Page 61)

Note 2. The elements a_{ij} for which $i \le j$ are not necessarily zero.

***§ 3·15. Theorem.** *Every matrix can be reduced to triangular form by elementary row operations.*

Proof : We shall prove this theorem by Mathematical intduction.

Assume that this theorem holds for all matrices containing $n-1$ rows and let $\mathbf{A}=[a_{ij}]$ be an $n \times m$ matrix given below —

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & \ldots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \ldots & \ldots & \ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & \ldots & \ldots & a_{nm} \end{bmatrix},$$

Now the following cases arise :—

Case I. If $a_{11} \neq 0$, then replacing R_1 by $(1/a_{11})\, R_1$ (*i.e.* by applying elementary row operation) the matrix **A** reduces to an $n \times m$ matrix

$$\mathbf{B}=[b_{ij}]=\begin{bmatrix} b_{11} & b_{12} & \ldots & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & \ldots & b_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & \ldots & b_{nm} \end{bmatrix},$$

where $b_{11}=1$.

Now applying elementary row-operation $R_k - b_{ki}R_1$ to R_k where $k=1, 2, \ldots, n$ *i.e.* subtract b_{ki} times R_1 from R_k, where k takes values from 1 to n. This reduces the matrix **B** to matrix $\mathbf{C}=[c_{ij}]$ where $c_{k1}=0$ whenever $k>1$ and we have

$$\mathbf{C}=\begin{bmatrix} 1 & c_{12} & c_{13} & \ldots & c_{1m} \\ 0 & c_{22} & c_{23} & \ldots & c_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & c_{n2} & c_{n3} & \ldots & c_{nm} \end{bmatrix}.$$

Now by our assumption that the theorem which we are going to prove holds for matrices containing $(n-1)$ rows we find that $(n-1)$ rowed matrix

$$\begin{bmatrix} 0 & c_{22} & c_{23} & \ldots & c_{2m} \\ 0 & c_{32} & c_{33} & \ldots & c_{3m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & c_{n2} & c_{n3} & \ldots & c_{nm} \end{bmatrix}$$

can always be reduced to triangular form by elementary row operations and hence from (i) the matrix **C** will reduce to triangular form when the same elementary row operations are applied to **C**.

Case II. If $a_{11}=0$ but $a_{k1} \neq 0$ for some value of k then interchanging R_1 and R_k the matrix **A** reduces to the matrix $\mathbf{D}=[d_{ij}]$ where $d_{11} \neq 0$.

Then the matrix **D** can always be reduced to the triangular form as in case I above.

Case III. If $a_{k1}=0$ for all values of k then we have

$$\mathbf{A}=\begin{bmatrix} 0 & a_{12} & a_{13} & \ldots & a_{1m} \\ 0 & a_{22} & a_{23} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & a_{n2} & a_{n3} & \ldots & a_{nm} \end{bmatrix}$$

By hypothesis (inductive) the $(n-1)$ rowed matrix

$$\begin{bmatrix} 0 & a_{22} & a_{23} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix}$$

as in case I above can be reduced to triangular form by elementary row operations and the same elementary operations when applied on **A** will reduce **A** to triangular form.

Hence the matrix **A** can always be reduced to triangular form and the proof is complete by mathematical induction.

Solved Examples on § 3·15.

Ex. 1. Reduce the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$ **to triangular form.**

Sol. Let $\mathbf{A} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 1 & 2 & -5 \\ 0 & 1 & 2 \end{bmatrix}, \text{ replacing } R_1 \text{ by } \tfrac{1}{4} R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & \frac{5}{3} & -\frac{19}{3} \\ 0 & 1 & 2 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & \frac{5}{3} & -\frac{19}{3} \\ 0 & 0 & \frac{29}{5} \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - \tfrac{3}{5} R_2$$

This is the required triangular form.

Aliter $\mathbf{A} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 3 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 19 \\ 0 & 1 & 2 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 19 \\ 0 & 5 & 10 \end{bmatrix}, \text{ replacing } R_3 \text{ by } 5R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 19 \\ 0 & 0 & 29 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 + R_2$$

This is also a triangular matrix.

Note. The above shows that reduction of a matrix to triangular form is not unique.

Ex. 2. Reduce $\mathbf{A} = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix}$ **to triangular form.** *(Agra 95)*

Sol. Let $\mathbf{A} = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}, \text{ replacing } R_1 \text{ by } -R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 8 & 24 & 4 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 8R_2$$

This is a triangular matrix as here $a_{ij} = 0$ for $i > j$. [See definition § 3·14 Page 112]

Exercises on § 3·15

Ex. 1. Reduce the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 3 & -1 & 4 \end{bmatrix}$ to the triangular form.

Ans. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 2. Reduce the matrix $\begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$ to the triangular form.

Ans. $\begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 5 & 1 & 16 \\ 0 & 0 & 12 & 27 \end{bmatrix}$

MISCELLANEOUS SOLVED EXAMPLES

Ex. 1. Apply successively the row transformations (or operation) R_{23}, $R_3(-2)$ and $R_{12}(4)$ to the matrix $\begin{bmatrix} 3 & 1 & 2 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 1 \end{bmatrix}$

Sol. (i). Applying R_{23} opertion to the given matrix we have

$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 3 & 2 \\ 3 & 1 & 4 & 1 \end{bmatrix}$ [Here we have interchanged second and third rows].

(ii) Applying $R_3(-2)$ operation to the given matrix we have

$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 2 & 0 & 3 & 2 \\ -2 & -4 & -6 & -8 \\ 3 & 1 & 4 & 1 \end{bmatrix}$ [Here we have replaced third row R_3 by $-2R_3$]

(iii) Applying $R_{12}(4)$ operation to the given matrix we have

$\begin{bmatrix} 7 & 9 & 14 & 17 \\ 2 & 0 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 1 \end{bmatrix}$ [Here we have replaced the first row R_1 by $R_1 + 4R_3$]

Ex. 2. Apply successively the column operation C_{13}; $C_2(-4)$ and $C_{23}(-2)$ to the matrix $\begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & 1 & -2 & 1 & 3 \\ 3 & 2 & 1 & -2 & 5 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$.

Sol. (i) Applying C_{13} operation to the given matrix, we have

$\begin{bmatrix} 2 & -1 & 1 & 2 & 4 \\ -2 & 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & -2 & 5 \\ 6 & 5 & 4 & 7 & 8 \end{bmatrix}$ [Here we have interchanged C_1 and C_3 *i.e.* first and third columns.]

(ii) Applying $C_2(-4)$ operation to the given matrix, we have

$$\begin{bmatrix} 1 & 4 & 2 & 3 & 4 \\ 2 & -4 & -2 & 1 & 3 \\ 3 & -8 & 1 & -2 & 5 \\ 4 & -20 & 6 & 7 & 8 \end{bmatrix}$$ [Here we have replaced second column C_2 by $-4C_2$].

(iii) Applying $C_{23}(-2)$ operation to the given matrix, we have

$$\begin{bmatrix} 1 & -5 & 2 & 3 & 4 \\ 2 & 5 & -2 & 1 & 3 \\ 3 & 0 & 1 & -2 & 5 \\ 4 & -7 & 6 & 7 & 8 \end{bmatrix}$$ [Here we have replaced second column C_2 by $C_2 - 2C_3$].

Ex. 3. Compute the following elementary matrices of order 4 E_{23}, $E_2(4)$, $E_{34}(-2)$, $E'_{34}(-2)$. **(Refer § 3·04 Page 105)**

Sol. The identity (or unit) matrix of order four is given by

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) $$\mathbf{E}_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [Interchanging R_2 and R_3 or C_2 and C_3] **(Note)**

(ii) $$\mathbf{E}_2(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [Replacing R_2 by $4R_2$ or C_2 by $4C_2$].

(iii) $$\mathbf{E}_{34}(-2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [Replacing R_3 by $R_3 - 2R_4$].

(iv) $$\mathbf{E}'_{34}(-2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$ [Replacing C_3 by $C_3 - 2C_4$].

[Here students should note that $\mathbf{E}'_{34}(-2)$ is nothing but the **transpose** matrix of $E_{34}(-2)$].

Ex. 4. Evaluate the inverse of the following elementary matrices of order four : $E_3(-2)$, $E_{23}(4)$ **(Refer § 3·06 Page 107-108)**

Sol. The identity matrix of order four is given by

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) Then $\mathbf{E_3}(-2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing R_3 by $-2R_3$

$\therefore$ Inverse of $\mathbf{E_3}(-2)$ *i.e.* $\{\mathbf{E_3}(-2)\}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **Ans.**

(obtained by replacing R_3 of $\mathbf{I_4}$ by $-\frac{1}{2}R_3$). **(Note)**

(ii) $\mathbf{E_{23}}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing R_2 of $\mathbf{I_4}$ by $R_2 + 4R_3$.

Then the inverse of $\mathbf{E_{23}}(4)$ *i.e.* $\{\mathbf{E_{23}}(4)\}^{-1}$ is given by

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing R_2 of $\mathbf{I_4}$ by $R_2 - 4R_3$. **(Note)**

****Ex. 5. Find the inverse of the matrix** $\mathbf{A} = \begin{bmatrix} \mathbf{i} & -\mathbf{1} & \mathbf{2i} \\ \mathbf{2} & \mathbf{0} & \mathbf{2} \\ -\mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

Sol.

$$\begin{array}{c} \mathbf{A} \\ \begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \Bigg| \sim \begin{array}{c} \mathbf{I} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\sim \begin{bmatrix} i & -1 & 2i \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Replacing R_2 by $\frac{1}{2}R_2$)

$$\sim \begin{bmatrix} i & -1 & 2i \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

(Replacing R_3 by $R_3 + R_2$)

$$\sim \begin{bmatrix} i & -1 & 2i \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

(Replacing R_3 by $\frac{1}{2}R_3$)

$$\sim\begin{bmatrix} i & -1 & 2i \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & -1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

(Replacing R_2 by $R_2 - R_3$)

$$\sim\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim\begin{bmatrix} 1 & -\frac{3}{4}i & -\frac{1}{2}i \\ 0 & 1/4 & -1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

(Replacing R_1 by $R_1 - iR_2 - 2i\,R_3$) **(Note)**

$$\sim\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim\begin{bmatrix} -1 & \frac{3}{4}i & \frac{1}{2}i \\ 0 & 1/4 & -1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

(Replacing R_1 by $-R_1$)

$$\sim\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I} \Bigg| \sim\begin{bmatrix} 0 & 1/4 & -1/2 \\ -1 & \frac{3}{4}i & \frac{1}{2}i \\ 0 & 1/4 & 1/2 \end{bmatrix} = \mathbf{A}^{-1}$$

(Interchanging R_1 and R_2)

Therefore $\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1/4 & -1/2 \\ -1 & \frac{3}{4}i & \frac{1}{2}i \\ 0 & 1/4 & 1/2 \end{bmatrix}$ **Ans.**

EXERCISES ON CHAPTER III

Ex. 1. Apply the row operation $R_4(-3)$ and $R_{21}(4)$ to the matrix

$$\begin{bmatrix} 4 & -1 & 2 & 3 \\ -1 & 8 & -3 & -4 \\ 2 & 3 & 4 & -1 \\ -3 & -4 & -1 & 8 \end{bmatrix}$$

(Hint ; See Ex. 1 Page 116)

Ans. $\begin{bmatrix} 4 & -1 & 2 & 3 \\ -1 & 8 & -3 & -4 \\ 2 & 3 & 4 & -1 \\ 9 & 12 & 3 & -24 \end{bmatrix}$ and $\begin{bmatrix} 4 & -1 & 2 & 3 \\ 15 & 4 & 5 & 8 \\ 2 & 3 & 4 & -1 \\ -3 & -4 & -1 & 8 \end{bmatrix}$

Ex. 2. Apply the column operation $C_3(4)$ and $C_{12}(-3)$ to the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 0 & 1 & 3 & 4 \end{bmatrix}$$

[Hint : See Ex. 2 Page 116]

$$\text{Ans.}\begin{bmatrix}0 & 1 & 8 & 3 & 4\\ 1 & 2 & 12 & 4 & 0\\ 3 & 4 & 0 & 1 & 2\\ 2 & 0 & 4 & 3 & 4\end{bmatrix} \text{and} \begin{bmatrix}-3 & 1 & 2 & 3 & 4\\ -5 & 2 & 3 & 4 & 0\\ -9 & 4 & 0 & 1 & 2\\ 2 & 0 & 1 & 3 & 4\end{bmatrix}$$

Ex. 3. Compute E_{23}, $E_2(-2)$ and $E_{34}(-1)$ for the identity matrix of order 4.

(Hint. See Ex. 3 Page 117)

$$\text{Ans.}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Ex. 4. Evaluate the inverse of the following elementary matrices of order 4 :

$$E_{14}, E_4(3), E_{22}(2).$$

(Hint : See Ex. 4 Page 117).

$$\text{Ans. } E_{14},\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{1}{3}\end{bmatrix} \text{and} \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Ex. 5. Find the inverse of the matrix

$$A=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 3 & -4\\ 2 & 3 & 5 & -5\\ 3 & -4 & -5 & 8\end{bmatrix} \quad \text{Ans. } \frac{1}{18}\begin{bmatrix}2 & 16 & 6 & 4\\ 22 & 41 & -30 & -1\\ -10 & -44 & 30 & -2\\ 4 & -13 & 6 & -1\end{bmatrix}$$

(Hint : See Ex. 5 Page 118).

***Ex. 6.** Find the inverse of the matrix $\begin{bmatrix}1 & 2 & -1\\ -1 & 1 & 2\\ 2 & -1 & 1\end{bmatrix}$.

(Hint : See Ex. 5 Page 118).

$$\text{Ans. } \frac{1}{14}\begin{bmatrix}3 & -1 & 5\\ 5 & 3 & -1\\ -1 & 5 & 3\end{bmatrix}$$

Ex. 7. Find the inverse of the matrix $\begin{bmatrix}1 & 3 & 3 & 2 & 1\\ 1 & 4 & 3 & 3 & -1\\ 1 & 3 & 4 & 1 & 1\\ 1 & 1 & 1 & 1 & -1\\ 1 & -2 & -1 & 2 & 2\end{bmatrix}$

(Hint : See Ex. 5 Page 118).

Ans. $\frac{1}{15}\begin{bmatrix} 30 & -20 & -15 & 25 & -5 \\ 30 & -11 & -18 & 7 & -8 \\ -30 & 12 & 21 & -9 & 6 \\ -15 & 2 & 6 & -9 & 6 \\ 15 & -7 & -6 & -1 & -1 \end{bmatrix}$

Ex. 8. Has the following matrix an inverse ?

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & -1 & 4 \\ 3 & 3 & 2 & 5 \\ 1 & -1 & 4 & -1 \end{bmatrix}$$

(Hint : It can not be reduced to I_4). **Ans.** No.

CHAPTER IV.

Determinants

§ 4·01. Permutations.

Def. The operation of rearranging n distinct elements of a set among themselves is called permutation.

Let S be a set defined by

$$S = \{i_1, i_2, i_3, \ldots, i_n\}\ ;\ i_m \neq i_k \text{ for } m \neq k.$$

Let P be the transformation on S, such that $P(i_1) = a_1$, $P(i_2) = a_2$, $P(i_3) = a_3, \ldots, P(i_n) = a_n$, where $a_1, a_2, \ldots, a_n$ is some arrangement of the elements $i_1, i_2 \ldots, i_n$ of S.

Then a two line notation for the permutation is

$$P = \begin{pmatrix} i_1 & i_2 & i_3 \ldots i_n \\ a_1 & a_2 & a_3 \ldots a_n \end{pmatrix}$$

(**Note :** The order of columns in this notation is immaterial).

If in a given permutation a larger integer precedes a smaller one then there is an inversion. For example in 452 we see that 5 precedes 2.

If in a given permutation the number of inversions is odd, the permutation is known as odd. For example the permutation 5312 is odd as in this permutation we observe that 5 precedes 3, 5 precedes 1, 5 precedes 2, 3 precedes 1 and 3 precedes 2 *i.e.*, there are five (*i.e.* odd) inversion in 5312

Similarly if in a given permutation the number of inversions is even the permutation is known as even. For example the permutation 5314 is even as in this permutation we observe that 5 precedes 3, 5 precedes 1, 5 precedes 4 and 3 precedes 1.

Note : If there is no inversion the permutation is even, for example 345.

§ 4·02. Determinant of a square matrix.

Let us consider a square matrix **A** of order $n \times n$ given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & \ldots & a_{nn} \end{bmatrix}$$

The product of the elements in the principal diagonal is

$$a_{11}\, a_{22}\, a_{33} \ldots a_{nn}.$$

This is also called the trace of the matrix.

Now obtain $n\,!$ terms of the above type by operating on the row-subscripts of the elements of the above expression by $n\,!$ permutations $p = \begin{pmatrix} 1 & 2 & 3 \ldots n \\ i_1 & i_2 & i_3 \ldots i_n \end{pmatrix}$, where $i_1, i_2 \ldots, i_n$ are one of the $n\,!$ permutations of the integers $1, 2, 3, \ldots, n$.

The sum of n ! signed terms thus obtained is defined as the **determinants of the matrix A** and is denoted by $|\mathbf{A}|$ or $|a_{ij}|$.

Therefore the determinant of the square matrix $\mathbf{A} = [a_{ij}]$ of order $n \times n$ is given by

$$|a_{ij}| = \Sigma \pm a_{\alpha_1}\, a_{\beta_2}\, a_{\gamma_3}\, a_{\delta_4} \cdots a_{k_n}$$

where +or –sign is taken when $\alpha, \beta, \gamma, \delta, \ldots, k$ is an even or odd permutation of $1, 2, 3, \ldots, n$, and the summation extends over n ! permutations of the row subscripts $1, 2, 3, \ldots$

Note : The determinant of a square matrix of order n is known as a determinant of order n.

§ 4·03. Determinant of order two.

Let us consider asquare matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ of order 2×2.

Then 2 ! permutation on two symbols 1 and 2 are

$$I = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},\ p = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

The product of the elements of the principal diagonal are $a_{11}\, a_{22}$.

Operating on the row subscripts of $a_{11}\, a_{22}$ by the permutation I we get $+a_{11}\, a_{22}$, prefixing + sign as the permutation I is even (See § 4·01 Note) and by the permutation p we get $-a_{21}\, a_{12}$, prefixing – sign as the permutation is odd.

Hence $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{21}\, a_{12}$

§ 4·04. Determinant of order three.

Let us consider a sqaure matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ of order 3×3.

Then 3 ! permutations on three symbols 1, 2 and 3 are

$$i = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix};\ p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix};\ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix};$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix};\ p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ and } p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Operating on the row subscripts of $a_{11}\, a_{22}\, a_{33}$ by the permutation $I, p_1, p_2, \ldots p_5$ we have successively

(a) $+\, a_{11}\, a_{22}\, a_{33}$, prefixing + as permutation I is even.

(b) $-\, a_{11}\, a_{32}\, a_{23}$, prefixing – as permutation p_1 is odd

(c) $+\, a_{21}\, a_{32}\, a_{13}$, prefixing + sign as permutation p_2 is even.

(d) $-\, a_{21}\, a_{12}\, a_{33}$, prefixing – sign as permutation p_3 is odd.

(e) $+\, a_{31}\, a_{12}\, a_{23}$, prefixing + sign as permutation p_4 is even.

(f) $-a_{31}\,a_{22}\,a_{13}$, prefixing $-$ sign as permutation p_5 is odd.

Hence we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{32}\,a_{23} + a_{21}\,a_{32}\,a_{13} - a_{21}\,a_{12}\,a_{33} + a_{31}\,a_{12}\,a_{23} - a_{31}\,a_{22}\,a_{13}$$

$$= a_{11}\,(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}\,(a_{21}\,a_{33} - a_{23}\,a_{31}) + a_{13}\,(a_{21}\,a_{32} - a_{22}\,a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Solved Examples on § 4·02 to § 4·04.

Ex. 1. Evaluate $\begin{vmatrix} -5 & 0 \\ 7 & -2 \end{vmatrix}$

Sol. $\begin{vmatrix} -5 & 0 \\ 7 & -2 \end{vmatrix} = (-5)\,(-2) - 0 \times 7 = 10 - 0 = 10.$ **Ans.**

Ex. 2. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Sol. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\,\{5{\cdot}9 - 6{\cdot}8\} - 2\,(4{\cdot}9 - 6{\cdot}7) + 3\,\{4{\cdot}8 - 5{\cdot}7\},$ See § 4·04 Page 123

$$= 1\,\{45 - 48\} - 2\,\{36 - 42\} + 3\,\{32 - 35)$$
$$= -3 + 12 - 9 = 0$$ **Ans.**

Exercise

Ex. Show that $\begin{vmatrix} 2 & 5 \\ -3 & 7 \end{vmatrix} = 29$

§ 4·05. Cofactor of an element.

Definition. If in the expansion of a determinant $|\,a_{ij}\,|$, all the terms containing a_{ij} as a factor are collected and their sum be denoted by $a_{ij}\,C_{ij}$, then the factor C_{ij} is defined as the cofactor of the element a_{ij}.

From the above definition we find that if $[a_{ij}]$ be the $n \times n$ matrix whose determinant is $|\,a_{ij}\,|$ then if from $[a_{ij}]$ the element of its ith row and jth column are removed, the terms of C_{ij} are then composed of elements from the remaining $(n-1) \times (n-1)$sub-matrix M_{ij} of $[a_{ij}]$.

Hence the determinant $|\,a_{ij}\,|$ can be expressed as a function of the elements of ith row, by collecting all the terms containing $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{ij}$ and finding their sum

i.e. $$|\,a_{ij}\,| = a_{i1}\,C_{i1} + a_{i2}\,C_{i2} + \ldots + a_{in}\,C_{in}$$
$$= \sum_{j=1}^{n} a_{ij}\,C_{ij},$$

which is the expansion of the determinant $|a_{ij}|$ by the elements of the ith row and their cofactors.

In a similar manner we can expand the determinant $|a_{ij}|$ by the elements of the kth column and their cofactors and write as

$$|a_{ij}| = \sum_{k=1}^{n} a_{ik} C_{ik}$$

Example : Expand the determinant $\begin{vmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{h} & \mathbf{b} & \mathbf{f} \\ \mathbf{g} & \mathbf{f} & \mathbf{c} \end{vmatrix}$

by the elements of 1st row.

Sol. Elements of first row are a, h, g.

Let A, H and G denote the cofactors of a, h, g.

Then $A = \begin{vmatrix} b & f \\ f & c \end{vmatrix}$; $H = -\begin{vmatrix} h & f \\ g & c \end{vmatrix}$ and $G = \begin{vmatrix} h & b \\ g & f \end{vmatrix}$

Hence $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = aA + hH + gG$ See. § 4·04 above

$$= a \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \begin{vmatrix} h & f \\ g & c \end{vmatrix} + g \begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$$= a(bc - f^2) - h(ch - fg) + g(hf - bg)$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2.$$ **Ans.**

***§ 4·06. Properties of Determinants.**

Prop I. *If the elements of ith row (or ith column) of the determinant* $|a_{ij}|$ *are multiplied by a scalar c then the resulting determinant is* $c\,|a_{ij}|$.

(*Gorakhpur 93*)

Proof : We can write $|a_{ij}| = \sum_{j=1}^{n} a_{ij} C_{ij}$ ... See § 4·05 above.

In this case, the resulting determinant (when the elements of ith row are multiplied by c)

$$= \sum_{j=1}^{n} ca_{ij} C_{ij} = c \sum_{j=1}^{n} a_{ij} C_{ij} = c\,|a_{ij}|.$$

Similarly we can prove that statement when elements of kth column are multiplied by c.

Prop II. *If* $\mathbf{A} = [a_{ij}]$ *is an* $n \times n$ *matrix then* $|\mathbf{A}'| = |\mathbf{A}|$, *where* $\mathbf{A}'$ *is the transpose of the matrix* $\mathbf{A}$. (See § 2·08 Page 69)

Proof : If $\mathbf{A} = [a_{ij}]$, then $\mathbf{A}' = [a'_{ij}]$, where $a'_{ij} = a_{ji}$.

Now the product of elements of the principal diagonal of $\mathbf{A}'$

$$= a'_{11}\, a'_{22}\, a'_{33} \ldots a'_{nn}.$$

Operating on the row subscripts of the elements of this product by the permutation $p = \begin{pmatrix} 1 & 2 & 3 \dots n \\ i_1 & i_2 & i_3 \dots i_n \end{pmatrix}$, where $i_1, i_2, i_3, \dots$ are 1, 2, 3, ... in some order, we have $\pm a'_{i_1 1}\, a'_{i_2 2}\, a'_{i_3 3} \dots a'_{i_n n}$ as a term of $|\mathbf{A}'|$ plus or minus sign to be taken according as p is even or odd.

Now as $a'_{ij} = a_{ji}$, so we have

$a'_{i_1 1}\, a'_{i_2 2}\, a'_{i_3 3} \dots a'_{i_n n} = a_{1 i_1}\, a_{2 i_2}\, a_{3 i_3} \dots a_{n i_n}$

The term $a_{1 i_1}\, a_{2 i_2} \dots a_{n i_n}$ can be obtained from the term $a_{i_1 i_1}\, a_{i_2 i_2}\, a_{i_3 i_3} \dots a_{i_n i_n}$ by opperating on its row subscripts the permutation

$$p' = \begin{pmatrix} i_1 & i_2 & i_3 \dots i_n \\ 1 & 2 & 3 \dots n \end{pmatrix} = p^{-1}.$$

Hence p' is even or odd according as p is even or odd. Therefore the term $\pm a'_{i_1 1}\, a'_{i_2 2}\, a'_{i_3 3} \dots a'_{i_n n}$ of $|\mathbf{A}'|$ is also a term of $|\mathbf{A}|$.

We can thus prove that every one of the $n\,!$ terms of $|\mathbf{A}'|$ is a term of $|\mathbf{A}|$. Hence the property.

For example, if $|\mathbf{A}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1.$$

Then $|\mathbf{A}'| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1$$

$\therefore |\mathbf{A}'| = |\mathbf{A}|$. *(Gorakhpur 95)*

Prop. III. *If* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by interchanging two rows (or columns) then* $|\mathbf{B}| = -|\mathbf{A}|$.

Proof : Let us suppose that sth and tth columns of the determinant $\mathbf{A}$ are interchanged where $s < t$.

$$\text{Let } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} & \dots & a_{1t} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2s} & \dots & a_{2t} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i1} & \dots & a_{is} & \dots & a_{it} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{ns} & \dots & a_{nt} & \dots & a_{nn} \end{bmatrix}$$

Then the trace of $\mathbf{A}$ *i.e.* the product of the elements of the principal diagonal of $\mathbf{A} = a_{11}\, a_{22} \dots a_{ss} \dots a_{tt} .. a_{nn}$.

If the sth and tth columns are interchanged the product of the elements of the principal diagonal of **B**

$$= a_{11}\, a_{22} \ldots a_{ss} \ldots a_{tt} \ldots a_{nn} \qquad \textbf{(Note)}$$

In order to have a term of $|\mathbf{A}|$, let us operate on the row subscripts of the trace of **A** by the permutation

$$p = \begin{pmatrix} 1 & 2 & \ldots & s & \ldots & t & \ldots & n \\ i_1 & i_2 & \ldots & i_s & \ldots & i_t & \ldots & i_n \end{pmatrix}$$

Then we have $a_{i_1 1}\, a_{i_2 2} \ldots a_{i_s s} \ldots a_{i_t t} \ldots a_{i_n n}$.

This terms can also be obtained from the trace of **B** by operating on its row subscripts by the permutation

$$p' = \begin{pmatrix} 1 & 2 & \ldots & s & \ldots & t & \ldots & n \\ i_1 & i_2 & \ldots & i_t & \ldots & i_s & \ldots & i_n \end{pmatrix}$$

Here we observe that $p' = p\,(i_s\, i_t)$, since $(i_s\, i_t)$ is a transposition. Therefore p' is odd or even according as p is even or odd. Hence the term $\pm\, a_{i_1 1}\, a_{i_2 2} \ldots a_{i_s s} \ldots a_{i_t t} \ldots a_{i_n n}$ is also a term of $|\mathbf{B}|$ but with its sign changed. Thus every one of the $n\,!$ terms of $|\mathbf{A}|$ is a term of $|\mathbf{B}|$ but with sign changed.

Hence $\quad |\mathbf{B}| = -\,|\mathbf{A}|$.

Example : Let $|\mathbf{A}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Let the determinant $|\mathbf{B}|$ be formed by interchanging second and third columns.

Then $|\mathbf{B}| = \begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix}$

Exapanding $|\mathbf{B}|$ by the elements of its first row, we get

$$|\mathbf{B}| = a_1 \begin{vmatrix} b_3 & b_2 \\ c_3 & c_2 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$

$$= a_1\,(b_3c_2 - b_2c_3) - a_3\,(b_1c_2 - b_2c_1) + a_2\,(b_1c_3 - b_3c_1)$$

$$= -\,[a_1\,(b_2c_3 - b_3c_2) - a_2\,(b_1c_3 - b_3c_1) + a_3\,(b_1c_2 - b_2c_1)]$$

$$= -\left[a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right]$$

$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\,|\mathbf{A}|\,.$$

Prop. IV. *If two rows or two columns of a determinant* $|\mathbf{A}|$ *are identical then* $|\mathbf{A}| = 0$.

Proof : In Prop. III above we have proved that if any two rows or columns of a determinant are interchanged then the value of the determinant changes in sign only.

Thus if the two *identical* columns (or rows) of a determinant are interchanged, then the determinant does not change but its sign only changes.

Hence $|\mathbf{A}| = -|\mathbf{A}|$ *i.e.* $|\mathbf{A}| + |\mathbf{A}| = 0$ *i.e.* $|\mathbf{A}| = 0$.

Example : Evaluate $\begin{vmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_1} \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_1} \\ \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_1} \end{vmatrix}$

Expanding the determinant with respect to the first row we have the given determinant

$$= a_1 \begin{vmatrix} b_2 & b_1 \\ c_2 & c_1 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_1 \\ c_1 & c_1 \end{vmatrix} + a_1 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 (b_2c_1 - b_1c_2) - a_2 (b_1c_1 - b_1c_1) + a_1 (b_1c_2 - b_2c_1)$$

$$= a_1b_2c_1 - a_1b_1c_2 - a_2 (0) + a_1b_1c_2 - a_1b_2c_1 = 0.$$

***§ 4·07. Minor of an element.**

Definition : If $\mathbf{M}_{ij}$ be the $(n-1) \times (n-1)$ sub-matrix of the matrix $\mathbf{A} = [a_{ij}]$ obtained by removing the ith row and jth column, then the determinant $|\mathbf{M}_{ij}|$ is defined as the minor of the element a_{ij} in the determinant $|a_{ij}|$ of order n.

§ 4·08. Theorem. *The cofactor C_{ij} of the element a_{ij} in the determinant $|a_{ij}|$ is given by $C_{ij} = (-1)^{i+j} |\mathbf{M}_{ij}|$.*

Proof : Let us first of all prove the case $C_{11} = (-1)^2 |\mathbf{M}_{11}|$. *i.e.* $C_{11} = |\mathbf{M}_{11}|$.

The tems in C_{11} are composed of elements taken from the $(n-1) \times (n-1)$ sub-matrix $\mathbf{M}_{11}$ of **A**.

The general term of $a_{11} C_{11} = \pm a_{11} a_{i_2 2} a_{i_3 3} \dots a_{i_n n}$, where $i_2, i_3, \dots, i_n$ are 2, 3, ..., n in some order.

This term can also be obtained from the trace of matrix A *i.e.* the product of the elements of the diagonal of the matrix **A** *i.e.* $a_{11} a_{22} a_{33} \dots a_{nn}$, by operating on its row subscripts by the permutation $p = \begin{pmatrix} 1 & 2 & 3 \dots n \\ i_1 & i_2 & i_3 \dots i_n \end{pmatrix}$, where $i_2, i_3, \dots, i_n$ are defined as above.

Thus the permutation p may be regarded as a permutation on the symbols 2, 3, ..., n. Hence all the terms of $a_{11} C_{11}$ can be obtained by running p through the $(n-1)$! permutations of the symbols 2, 3, ..., n keeping 1 fixed.

Thus the terms of C_{11} can be obtained by operating on the row subscripts of the elements of the product $a_{22} a_{33} \dots a_{nn}$, which is the product of the elements of the diagonal of $\mathbf{M}_{11}$ *i.e.* the trace of $\mathbf{M}_{11}$.

Hence $C_{11} = |\mathbf{M}_{11}|$.

Now let us prove $C_{ij} = (-1)^{i+j} |\mathbf{M}_{ij}|$.

Move the jth column of the matrix **A** to the first column by performing $(j-1)$ successive interchanges of adjacent columns and move the ith row of the matrix **A** to the first row by performing $(i-1)$ successive interchanges of adjacent rows. Then the element a_{ij} is in the first row and first column of the resulting matrix **B**, say. The sub-matrix of **B** obtained by removing the first row and first column is the sub-matrix $\mathbf{M}_{ij}$ of the matrix **A**. Hence $a_{ij} |\mathbf{M}_{ij}|$ is the term of $|\mathbf{B}|$ containing a_{ij}.

Also we know that if two rows or two columns of determinant $|\mathbf{A}|$ are interchanged, the new determinant $= -|\mathbf{A}|$.

$\therefore$ We have $|\mathbf{B}| = (-1)^{i-1+j-1} |\mathbf{A}|$ **(Note)**

$$= (-1)^{i+j} (-1)^{-2} |\mathbf{A}|$$

$$= (-1)^{i+j} |\mathbf{A}|, \quad \because \quad (-1)^{-2} = 1$$

or $|\mathbf{A}| = (-1)^{i+j} |\mathbf{B}|$ **(Note)**

Equating the coefficients of a_{ij} from both sides, we have

$$C_{ij} = (-1)^{i+j} |\mathbf{M}_{ij}|.$$

*§ 4.09. **Theorem.** If C_{ij} is the cofactor of a_{ij} in the determinant $|\mathbf{A}| = |a_{ij}|$ of order n, then (i) the sum of the products of the elements of the ith row with the cofactors of the corresponding elements of the kth row is zero provided $i \neq k$.

(ii) Also the sum of the products of the elements of the jth column with the cofactors of the corresponding elements of the kth column is zero provided $j \neq k$.

i.e. (i) $\sum_{j=1}^{n} a_{ij} C_{kj} = 0$, if $i \neq k$.

and (ii) $\sum_{j=1}^{n} a_{ij} C_{ik} = 0$, if $j \neq k$.

Proof : (i) The given determinant $|\mathbf{A}| = \sum_{j=1}^{n} a_{kj} C_{kj}$.

Now eplace the kth row by ith row, then we have the new determinant

$$= \sum_{j=1}^{n} a_{ij} C_{kj}.$$

But the kth and ith rows of the new determinant are identical, hence its value is zero.

$$\therefore \quad \sum_{i=1}^{n} a_{ij} C_{kj} = 0.$$

(ii) The given determinant $|A| = \sum_{i=1}^{n} a_{ik} C_{ik}$.

Now replace the kth column by jth column, then we have the new determinant $= \sum_{i=1}^{n} a_{ik} C_{ij}$.

But the kth and jth columns of the new determinant are identical, hence its value is zero.

$$\therefore \sum_{i=1}^{n} a_{ij} C_{ik} = 0.$$

****Example. In the determinant** $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ **prove that**

$a_1A_2 + b_1B_2 + c_1C_2 = 0$, $b_1C_1 + b_2C_2 + b_3C_3 = 0$ **and** $c_1B_1 + c_2B_2 + c_3B_3 = 0$, **where capital letters dentoe the cofactors of the corresponding small letters.**

Also prove that

$a_1A_1 + b_1B_1 + c_1C_1 = |A| = a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3$.

Sol. In the det. $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ we have

$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$; $A_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$; $B_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$; $B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$;

$B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$; $C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$; $C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$; $A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

$C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

$\therefore a_1A_2 + b_1B_2 + c_1C_2$

$$= -a_1 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$= -a_1 (b_1c_3 - b_3c_1) + b_1 (a_1c_3 - a_3c_1) - c_1 (a_1b_3 - a_3b_1)$

$= 0$, on simplifying.

$$b_1C_1 + b_2C_2 + b_3C_3 = b_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$= b_1 (a_2b_3 - a_3b_2) - b_2 (a_1b_3 - a_3b_1) + b_3 (a_1b_2 - a_2b_1)$

$= 0$, on simplifying.

In a similar way the remaining part can also be proved.

Also $|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \quad \text{...(i)}$$

expanding with respect to first row.

Again $a_1A_1 + b_1B_1 + c_1C_1$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \left\{ - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \right\} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= |\mathbf{A}|$, from (i)

Similarly $a_2A_2 + b_2B_2 + c_2C_2$

$$= a_2 \left\{ - \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \right\} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + c_2 \left\{ - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \right\}$$

$$= - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \quad \text{...(ii)}$$

Also $|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \quad \text{...(iii)}$$

expanding with respect to second row.

$\therefore$ From (ii) and (iii), we get $a_2A_2 + b_2B_2 + c_2C_2 = |\mathbf{A}|$.

In a similar way we can prove that $a_3A_3 + b_3B_3 + c_3C_3 = |\mathbf{A}|$, by expanding $|\mathbf{A}|$ with respect to third row.

§ 4·10. An Important Property of the Determinant.

(a) *If A_i, the ith row of a determinant $|\mathbf{A}| = |a_{ij}|$ of order n, be replaced by $A_i + cA_k$, where c is a scalar and A_k denotes the kth row of the determinant $|\mathbf{A}|$, then the value of the determinant remains unaltered.* (*Gorakhpur 94*)

Proof : The determinant $|\mathbf{A}| = \sum_{j=1}^{n} a_{ij} C_{ij}$.

Replacing A_i by $A_i + cA_k$ we get the new determinant $|\mathbf{B}|$, say.

Then $|\mathbf{B}| = \sum_{j=1}^{n} (a_{ij} + ca_{kj}) C_{ij}$

$$= \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij}$$

$= |\mathbf{A}| + c.0$ ... See § 4·09 Pages 129-131

i.e. $|\mathbf{B}| = |\mathbf{A}|$.

(b) *If C_i, the ith column of determinant $|\mathbf{A}| = |a_{ij}|$ of order n be replaced by $C_i + \lambda C_k$, where λ is a scalar and C_k dentoes the kth column of $|\mathbf{A}|$, then the value of the determinant remains unaltered.*

Proof is similar to part (a) above.

Solved Examples on the Evaluation of Determinants :

In the following examples $R_1, R_2, R_3, \ldots$ stand for first, second, third, ... rows and $C_1, C_2, C_3, \ldots$ stand for first, second, third... columns.

Ex. 1. Without expanding the determinants, prove that

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{p} & \mathbf{q} & \mathbf{r} \end{vmatrix} = \begin{vmatrix} \mathbf{y} & \mathbf{b} & \mathbf{q} \\ \mathbf{x} & \mathbf{a} & \mathbf{p} \\ \mathbf{z} & \mathbf{c} & \mathbf{r} \end{vmatrix} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \end{vmatrix}$$

Sol. $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = - \begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix}$, interchanging R_1 and R_2

$= \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$, interchanging R_2 and R_3

Hence proved.

Again $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$, interchanging rows and columns

$= - \begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$, interchanging C_1 and C_2

$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$, interchanging R_1 and R_2

Hence proved.

Ex. 2 (a). Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{a} & \mathbf{b+c} \\ \mathbf{1} & \mathbf{b} & \mathbf{c+a} \\ \mathbf{1} & \mathbf{c} & \mathbf{a+b} \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$, replacing C_3 by $C_3 + C_2$

$= (a+b+c) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$, taking out $(a+b+c)$ common from C_3

$= 0$, since two columns are identical. **Ans.**

Ex. 2. (b). Evaluate $\begin{vmatrix} \mathbf{13} & \mathbf{16} & \mathbf{19} \\ \mathbf{14} & \mathbf{17} & \mathbf{20} \\ \mathbf{15} & \mathbf{18} & \mathbf{21} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 13 & 16 & 3 \\ 14 & 17 & 3 \\ 15 & 18 & 3 \end{vmatrix}, \text{ replacing } C_3 \text{ by } C_3 - C_2$$

$$= \begin{vmatrix} 13 & 3 & 3 \\ 14 & 3 & 3 \\ 15 & 3 & 3 \end{vmatrix}, \text{ replacing } C_2 \text{ by } C_2 - C_1$$

$= 0$, since two columns are identical. **Ans.**

Ex. 3. Evaluate $\begin{vmatrix} \mathbf{265} & \mathbf{240} & \mathbf{219} \\ \mathbf{240} & \mathbf{225} & \mathbf{198} \\ \mathbf{219} & \mathbf{198} & \mathbf{181} \end{vmatrix}$ *(Kanpur 90)*

Sol. Given determinant

$$= \begin{vmatrix} 25 & 21 & 219 \\ 15 & 27 & 198 \\ 21 & 17 & 181 \end{vmatrix}, \text{ replacing } C_1, C_2 \text{ by } C_1 - C_2, C_2 - C_3 \text{ respectively}$$

$$= \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}, \text{ replacing } C_1, C_3 \text{ by } C_1 - C_2, C_3 - 10C_2 \text{ respectively}$$

$$= \begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 - R_3, R_2 + 3R_3 \text{ respectively}$$

$$= 4 \begin{vmatrix} 4 & -2 \\ 78 & -39 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= 4 \begin{vmatrix} 0 & -2 \\ 0 & -39 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 + 2C_2.$$

$= 4 \times 0 = 0$ **Ans.**

Ex. 4. Evaluate $\begin{vmatrix} \mathbf{a} & -\mathbf{a} & -\mathbf{a} & -\mathbf{a} \\ \mathbf{b} & -\mathbf{b} & -\mathbf{b} & -\mathbf{b} \\ \mathbf{c} & -\mathbf{c} & -\mathbf{c} & -\mathbf{c} \\ \mathbf{d} & -\mathbf{d} & -\mathbf{d} & -\mathbf{d} \end{vmatrix}$

Sol. Since three columns of the given determinant are identical, so the value of the determinant is zero. **Ans.**

Ex. 5. (a) Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} & \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{vmatrix}$.

Sol. The given determinant

$$= \begin{vmatrix} 10 & 2 & 3 & 4 \\ 10 & 3 & 4 & 1 \\ 10 & 4 & 1 & 2 \\ 10 & 1 & 2 & 3 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 + C_2 + C_3 + C_4$$

$$= 10 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{vmatrix}, \text{ taking out 10 common from } C_1$$

$$= 10 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{vmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2 - R_1, R_3 - R_1 \text{ and } R_4 - R_1 \text{ respectively}$$

$$= 10 \begin{vmatrix} 1 & 1 & -3 \\ 2 & -2 & -2 \\ -1 & -1 & -1 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= -20 \begin{vmatrix} 1 & 1 & -3 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix}, \text{ taking out 2 common from } R_2 \text{ and } -1 \text{ from } R_3$$

$$= -20 \begin{vmatrix} 1 & 1 & -3 \\ 0 & -2 & 2 \\ 0 & 0 & 4 \end{vmatrix}, \text{ replacing } R_2 \text{ and } R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_1 \text{ respectively}$$

$$= -20 \begin{vmatrix} -2 & 2 \\ 0 & 4 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$= -20\,[(-2)\cdot 4 - 0\cdot 2] = -20\,[-8] = 160.$ **Ans.**

Ex. 5 (b) Evaluate $\begin{vmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & 1 & 1 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 1 & 1 & 1 & -4 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 + C_2 + C_3 + C_4 + C_5$$

(Note)

$= 0$, expanding with respect to elements of first column. **Ans.**

***Ex. 6. Evaluate** $\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$ *(Meerut 96)*

Sol. The given determinant

$$= -\begin{vmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{vmatrix}, \text{ interchanging } C_2 \text{ and } C_3$$

$$= \begin{vmatrix} 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \\ 1 & 0 & 0 \end{vmatrix}, \text{ interchanging } C_1 \text{ and } C_2$$

$$= \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}, \text{ expanding with respect to } C_1$$

$= \cos\theta . \cos\theta - (-\sin\theta)\sin\theta = \cos^2\theta + \sin^2\theta =$ **Ans.**

Ex. 7. Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1+x} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1+y} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix}, \text{ replacing } R_2 \text{ by } R_2 - R_1 \text{ and } R_3 \text{ by } R_3 - R_1$$

$$= 1 \times \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix}, \text{ expanding with respect to the first column,}$$

$= xy.$

****Ex. 8. Evaluate** $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ *(Meerut 97; Purvanchal 97)*

Sol. The given determinant

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & -7 & -20 & -39 \\ 9 & -20 & -56 & -108 \\ 16 & -39 & -108 & -207 \end{vmatrix}, \text{ replacing } C_2, C_3, C_4 \text{ by } C_2 - 4C_1, C_3 - 9C_1 \text{ and } C_4 - 16C_1 \text{ respectively.}$$

$$= (-1)^3 \begin{vmatrix} 7 & 20 & 39 \\ 20 & 56 & 108 \\ 39 & 108 & 207 \end{vmatrix}, \text{ expanding with respect to first row}$$

$$= - \begin{vmatrix} 7 & -1 & -1 \\ 20 & -4 & -4 \\ 39 & -9 & -9 \end{vmatrix}, \text{ replacing } C_2 \text{ and } C_3 \text{ by } C_2 - 3C_1 \text{ and } C_3 - 2C_2 \text{ repsectively.}$$

$= 0$, as two columns are identical. **Ans.**

Ex. 9. Evaluate $\begin{vmatrix} \mathbf{5} & \mathbf{7} & \mathbf{10} & \mathbf{14} \\ \mathbf{2} & \mathbf{3} & \mathbf{7} & \mathbf{6} \\ \mathbf{3} & \mathbf{3} & \mathbf{6} & \mathbf{9} \\ \mathbf{5} & \mathbf{6} & \mathbf{11} & \mathbf{20} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 0 & 1 & -1 & -6 \\ 2 & 3 & 7 & 6 \\ 1 & 0 & -1 & 3 \\ 5 & 6 & 11 & 20 \end{vmatrix}, \text{ replacing } R_1 \text{ and } R_3 \text{ by } R_1 - R_4 \text{ and } R_3 - R_2 \text{ repsectively.}$$

$$= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & -10 & 24 \\ 1 & 0 & -1 & 3 \\ 5 & 6 & 17 & 56 \end{vmatrix}, \text{ replacing } C_3 \text{ and } C_4 \text{ by } C_2 + C_3 \text{ and } C_4 + 6C_2 \text{ respectively.}$$

Now expand with respect to 1st row and proceed as in Ex. 8 Page 135

Ans. 96

Ex. 10. Show that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \beta+\gamma & \gamma+\delta & \delta+\alpha & \alpha+\beta \\ \delta & \alpha & \beta & \gamma \end{vmatrix} = 0$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \beta+\gamma & \gamma+\delta & \delta+\alpha & \alpha+\beta \\ \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta \end{vmatrix},$$

replacing R_4 by $R_2 + R_3 + R_4$

$$= (\alpha+\beta+\gamma+\delta) \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \beta+\gamma & \gamma+\delta & \delta+\alpha & \alpha+\beta \\ 1 & 1 & 1 & 1 \end{vmatrix},$$

taking out $(\alpha+\beta+\gamma+\delta)$ common from R_4.

$= 0$, since two rows are identical.

Hence proved.

Ex. 11. Evaluate $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 69 \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 9 & 14 \\ 1 & 3 & 9 & 19 & 34 \\ 1 & 4 & 14 & 34 & 68 \end{vmatrix}, \text{ replacing } C_2, C_3, C_4 \text{ and } C_5 \text{ by } C_2 - C_1, C_3 - C_1, C_4 - C_1 \text{ and } C_5 - C_1 \text{ respectively.}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 9 & 14 \\ 3 & 9 & 19 & 34 \\ 4 & 14 & 34 & 68 \end{vmatrix}, \text{ expanding with respect to first row.}$$

$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 6 \\ 3 & 3 & 10 & 22 \\ 4 & 6 & 22 & 52 \end{vmatrix}$, replacing C_2, C_3 and C_4 by $C_2 - 2C_1$, $C_3 - 3C_1$ and $C_4 - 4C_1$ repsectively.

$= \begin{vmatrix} 1 & 3 & 6 \\ 3 & 10 & 22 \\ 6 & 22 & 52 \end{vmatrix}$, expanding with respect to first row

$= \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 4 \\ 6 & 4 & 16 \end{vmatrix}$, replacing C_2 and C_3 by $C_2 - 3C_1$ and $C_3 - 6C_1$ respectively.

$= \begin{vmatrix} 1 & 4 \\ 4 & 16 \end{vmatrix}$, expanding with respect to first row.

$= 1.(16) - 4.4 = 0.$ **Ans.**

Ex. 12. Show that $\begin{vmatrix} \mathbf{a+b+2c} & \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{b+c+2a} & \mathbf{b} \\ \mathbf{c} & \mathbf{a} & \mathbf{c+a+2b} \end{vmatrix} = \mathbf{2\,(a+b+c)^3}$

(Garhwal 90; Meerut 92)

Sol. The given determinant

$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$, replacing C_1 by $C_1 + C_2 + C_3$

$= (2a+2b+2c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$, taking out $2a+2b+2c$ common.

$= 2\,(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively

$= 2\,(a+b+c) \begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}$, expanding with respect to 1st column.

$= 2\,(a+b+c)\,[(b+c+a)\,(c+a+b)] = 2\,(a+b+c)^3.$

***Ex. 13. Prove that** $\begin{vmatrix} \mathbf{b+c} & \mathbf{c+a} & \mathbf{a+b} \\ \mathbf{q+r} & \mathbf{r+p} & \mathbf{p+q} \\ \mathbf{y+z} & \mathbf{z+x} & \mathbf{x+y} \end{vmatrix} = \mathbf{2} \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{vmatrix}$ *(Meerut 90)*

Sol. L.H.S. $= \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

$= \begin{vmatrix} b & c+a & a+b \\ q & r+p & p+q \\ y & z+x & x+y \end{vmatrix} + \begin{vmatrix} c & c+a & a+b \\ r & r+p & p+q \\ z & z+x & x+y \end{vmatrix}$ **(Note)**

$$= \begin{vmatrix} b & c+a & a \\ q & r+p & p \\ y & z+x & x \end{vmatrix} + \begin{vmatrix} b & c+a & b \\ q & r+p & q \\ y & z+x & y \end{vmatrix} + \begin{vmatrix} c & c & a+b \\ r & r & p+q \\ z & z & x+y \end{vmatrix} + \begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + \begin{vmatrix} b & a & a \\ q & p & p \\ y & x & x \end{vmatrix} + \begin{vmatrix} c & a & a \\ r & p & p \\ z & x & x \end{vmatrix} + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix},$$

second and third determinants vanish as two columns in each are identical.

$$= \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix}$$, second and third determinants vanish as two columns in each are identical.

$$= -\begin{vmatrix} b & a & c \\ q & p & r \\ y & x & z \end{vmatrix} - \begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix}$$, interchanging C_2 and C_3 in first and C_1 and C_2 in second determinant

$$= \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$, interchanging C_1 and C_2 in first and C_2 and C_3 in second determinant

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \text{R.H.S.}$$

Hence proved.

Ex. 14. Evaluate $\begin{vmatrix} \mathbf{y+z} & \mathbf{x} & \mathbf{y} \\ \mathbf{z+x} & \mathbf{z} & \mathbf{x} \\ \mathbf{x+y} & \mathbf{y} & \mathbf{z} \end{vmatrix}$

Hint : Add all the rows to first, take out $(x+y+z)$ common from first row and then subtract sum of second and third columns from first. Then expand.

Ans. $(x+y+z)(x-z)^2$

Ex. 15. Evaluate $\begin{vmatrix} \mathbf{b+c} & \mathbf{a+b} & \mathbf{a} \\ \mathbf{c+a} & \mathbf{b+c} & \mathbf{b} \\ \mathbf{a+b} & \mathbf{c+a} & \mathbf{c} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 2a+2b+2c & 2a+2b+2c & a+b+c \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$$, replacing R_1 by $R_1+R_2+R_3$

$$= (a+b+c)\begin{vmatrix} 2 & 2 & 1 \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$$, taking out $(a+b+c)$ common

$$= (a+b+c)\begin{vmatrix} 0 & 2 & 1 \\ a-b & b+c & b \\ b-c & c+a & c \end{vmatrix}$$, replacing C_1 by C_1-C_2

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a-b & c-b & b \\ b-b & a-c & c \end{vmatrix}$$, replacing C_2 by C_2-2C_3

$= (a+b+c)\,[(a-b)(a-c)-(b-c)(c-b)]$

$= (a+b+c)\,[a^2-ac-ba+bc+b^2+c^2-2bc]$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = a^3+b^3+c^3-3abc.$ **Ans.**

***Ex. 16. Evaluate** $\begin{vmatrix} \mathbf{a-b-c} & \mathbf{2a} & \mathbf{2a} \\ \mathbf{2b} & \mathbf{b-c-a} & \mathbf{2b} \\ \mathbf{2c} & \mathbf{2c} & \mathbf{c-a-b} \end{vmatrix}$

(Kanpur 96; Kumaun 93; Meerut 95)

Sol. The given determinant

$= \begin{vmatrix} a+b+c & b+c+a & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$, replacing R_1 by $R_1+R_2+R_3$

$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$, taking out $(a+b+c)$ common

$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -a-b-c & 0 \\ 2c & 0 & -a-b-c \end{vmatrix}$, replacing C_2 and C_3 by C_2-C_1 and C_3-C_1 respectively.

$= (a+b+c)\begin{vmatrix} -a-b-c & 0 \\ 0 & -a-b-c \end{vmatrix}$, expanding with respect to first row.

$= (a+b+c)(-a-b-c)(-a-b-c) = (a+b+c)^3.$ **Ans.**

Ex. 17. Evaluate $\begin{vmatrix} \mathbf{a} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{a} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{a} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{a} \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} a+3 & 1 & 1 & 1 \\ a+3 & a & 1 & 1 \\ a+3 & 1 & a & 1 \\ a+3 & 1 & 1 & a \end{vmatrix}$, replacing C_1 by $C_1+C_2+C_3+C_4$

Now proceed as in Ex. 5 (a) Page 133. **Ans.** $(a-1)^3(a+3)$

***Ex. 18. Evaluate** $\begin{vmatrix} \mathbf{1+a} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1+a} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1+a} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1+a} \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} 4+a & 1 & 1 & 1 \\ 4+a & 1+a & 1 & 1 \\ 4+a & 1 & 1+a & 1 \\ 4+a & 1 & 1 & 1+a \end{vmatrix}$, replacing C_1 by $C_1+C_2+C_3+C_4$

$= (4+a) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix}$, taking out $(4+a)$ common from first column.

$= (4+a) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & 0 & a & 0 \\ 1 & 0 & 0 & a \end{vmatrix}$, replacing C_2, C_3 and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$

$= (4+a) \begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$, expanding with respect to R_1

$= (4+a)\, a \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix}$, expanding with respect to R_1

$= (4+a)\, a\, (a^2) = (4+a)\, a^3$. **Ans.**

Ex. 19. Prove that

$$\begin{vmatrix} 1+a_1 & a_2 & a_3 & a_4 \\ a_1 & 1+a_2 & a_3 & a_4 \\ a_1 & a_2 & 1+a_3 & a_4 \\ a_1 & a_2 & a_3 & 1+a_4 \end{vmatrix} = 1 + a_1 + a_2 + a_3 + a_4$$

Sol. The given determinant

$= \begin{vmatrix} 1+a_1+a_2+a_3+a_4 & a_2 & a_3 & a_4 \\ 1+a_1+a_2+a_3+a_4 & 1+a_2 & a_3 & a_4 \\ 1+a_1+a_2+a_3+a_4 & a_2 & 1+a_3 & a_4 \\ 1+a_1+a_2+a_3+a_4 & a_2 & a_3 & 1+a_4 \end{vmatrix}$, replacing C_1 by $C_1 + C_2 + C_3 + C_4$

Now proceed further as in Ex. 18 above.

***Ex. 20. Evaluate** $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}$, replacing C_1 by $C_1 + C_2 + C_3 + C_4$

$= (x+3a) \begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}$, taking out $(x+3a)$ common from C_1

$$= (x + 3a) \begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ and } R_4 \text{ by } R_2 - R_1, R_3 - R_1 \text{ and } R_4 - R_1 \text{ respectively.}$$

$$= (x + 3a) \begin{vmatrix} x-a & 0 & 0 \\ 0 & x-a & 0 \\ 0 & 0 & x-a \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (x + 3a)(x - a) \begin{vmatrix} x-a & 0 \\ 0 & x-a \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (x + 3a)(x - a)(x - a)(x - a) = (x - 3a)(x - a)^3.$$ **Ans.**

Ex. 21. Evaluate $\begin{vmatrix} \mathbf{x+a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \mathbf{a} & \mathbf{x+b} & \mathbf{c} & \mathbf{d} \\ \mathbf{a} & \mathbf{b} & \mathbf{x+c} & \mathbf{d} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{x+d} \end{vmatrix}$ *(Meerut 96)*

Sol. The given determinant

$$= \begin{vmatrix} x+a+b+c+d & b & c & d \\ x+a+b+c+d & x+b & c & d \\ x+a+b+c+d & b & x+c & d \\ x+a+b+c+d & b & c & x+d \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 + C_2 + C_3 + C_4$$

$$= (x+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 1 & x+b & c & d \\ 1 & b & x+c & d \\ 1 & b & c & x+d \end{vmatrix}, \text{ taking out } (x+a+b+c+d) \text{ common from } C_1$$

$$= (x+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2 - R_1, R_3 - R_1 \text{ and } R_4 - R_1 \text{ respectively.}$$

$$= (x+a+b+c+d) \begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= x(x+a+b+c+d) \begin{vmatrix} x & 0 \\ 0 & x \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= x(x+a+b+c+d)\,x^2 = x^3(x+a+b+c+d).$$ **Ans.**

Ex. 22. Evaluate $\begin{vmatrix} \mathbf{b+c} & \mathbf{a} & \mathbf{a} \\ \mathbf{b} & \mathbf{c+a} & \mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{a+b} \end{vmatrix}$ *(Garhwal 90)*

Sol. The given determinant

$$= \begin{vmatrix} b+c & 0 & a \\ b & c+a-b & b \\ c & c-a-b & a+b \end{vmatrix}, \text{ replacing } C_2 \text{ by } C_2 - C_3$$

$$= (b+c)\begin{vmatrix} c+a-b & b \\ c-a-b & a+b \end{vmatrix} + a\begin{vmatrix} b & c+a-b \\ c & c-a-b \end{vmatrix},$$

expanding with respect to R_1

$$= (b+c)\begin{vmatrix} c+a-b & b \\ -2a & a \end{vmatrix} + a\begin{vmatrix} b & c+a-b \\ c-b & -2a \end{vmatrix},$$

replacing R_2 by $R_2 - R_1$ in each determinant.

$= (b+c)\,[a\,(c+a-b) - b\,(-2a)] + a\,[b\,(-2a) - (c-b)\,(c+a-b)]$

$= (b+c)\,(ac+a^2+ab) + a\,(-2ab-c^2-ca+bc+bc+ab-b^2)$

$= abc + a^2b + ab^2 + ac^2 + a^2c + abc - 2a^2b - ac^2 - ca^2 + 2abc + a^2b - ab^2$

$= 4abc.$ **Ans.**

Ex. 23. Evaluate $\begin{vmatrix} \mathbf{b+c} & \mathbf{a-c} & \mathbf{a-b} \\ \mathbf{b-c} & \mathbf{c+a} & \mathbf{b-a} \\ \mathbf{c-b} & \mathbf{c-a} & \mathbf{a+b} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 2b & 2a & 0 \\ b-c & c+a & b-a \\ 0 & 2c & 2b \end{vmatrix},$$ replacing R_1 and R_3 by $R_1 + R_2$ and $R_3 + R_2$ respectively.

$$= 4\begin{vmatrix} b & a & 0 \\ b-c & c+a & b-a \\ 0 & c & b \end{vmatrix},$$ taking out 2 common from R_1 and R_3

$$= 4\begin{vmatrix} b & a & 0 \\ -c & c & b-a \\ 0 & c & b \end{vmatrix},$$ replacing R_2 by $R_2 - R_1$

$$= 4\left[b\begin{vmatrix} c & b-a \\ c & b \end{vmatrix} + c\begin{vmatrix} a & 0 \\ c & b \end{vmatrix}\right],$$ expanding with respect to C_1

$$= 4\left[b\begin{vmatrix} c & b-a \\ 0 & a \end{vmatrix} + c\begin{vmatrix} a & 0 \\ c & b \end{vmatrix}\right],$$ replacing R_2 by $R_2 - R_1$ in first determinant.

$= 4\,[b\,(ca) + c\,(ab)] = 4\,(2abc) = 8abc.$ **Ans.**

Ex. 24. Evaluate $\begin{vmatrix} \mathbf{x} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{1} \\ \mathbf{d} & \mathbf{x} & \mathbf{f} & \mathbf{h} & \mathbf{1} \\ \mathbf{d} & \mathbf{e} & \mathbf{x} & \mathbf{k} & \mathbf{1} \\ \mathbf{d} & \mathbf{e} & \mathbf{g} & \mathbf{x} & \mathbf{1} \\ \mathbf{d} & \mathbf{e} & \mathbf{g} & \mathbf{m} & \mathbf{1} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} x-d & a-x & b-f & c-h & 0 \\ 0 & x-e & f-x & h-k & 0 \\ 0 & 0 & x-g & k-x & 0 \\ 0 & 0 & 0 & x-m & 0 \\ d & e & g & m & 1 \end{vmatrix},$$

replacing R_1, R_2, R_3 and R_4 by R_1-R_2, R_2-R_3, R_3-R_4 and R_4-R_5 resplactively.

$$=(x-m)\begin{vmatrix} x-d & a-x & b-f & 0 \\ 0 & x-e & f-x & 0 \\ 0 & 0 & x-g & 0 \\ d & e & g & 1 \end{vmatrix} \text{, exapanding with respect to } R_4. \quad \textbf{(Note)}$$

$$=(x-m)\begin{vmatrix} x-d & a-x & b-f \\ 0 & x-e & f-x \\ 0 & 0 & x-g \end{vmatrix} \text{, expanding with respect to } C_4$$

$$=(x-m)(x-d)\begin{vmatrix} x-e & f-x \\ 0 & x-g \end{vmatrix} \text{, expanding with respect to } C_1$$

$$=(x-m)(x-d)(x-e)(x-g). \quad \textbf{Ans}$$

Ex. 25. Evaluate $\begin{vmatrix} 1 & x & 1 & y \\ x & 1 & y & 1 \\ 1 & y & 1 & x \\ y & 1 & x & 1 \end{vmatrix}$

Sol. The given determinant

$$=\begin{vmatrix} x+y+2 & x+y+2 & x+y+2 & x+y+2 \\ x & 1 & y & 1 \\ 1 & y & 1 & x \\ y & 1 & x & 1 \end{vmatrix} \text{, replacing } R_1 \text{ by } R_1+R_2+R_3+R_4$$

$$=(x+y+2)\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & 1 & y & 1 \\ 1 & y & 1 & x \\ y & 1 & x & 1 \end{vmatrix} \text{, taking out } (x+y+2) \text{ common from } R_1$$

$$=(x+y+2)\begin{vmatrix} 1 & 0 & 0 & 0 \\ x & 1-x & y-x & 1-x \\ 1 & y-1 & 0 & x-1 \\ y & 1-y & x-y & 1-y \end{vmatrix} \text{, replacing } C_2, C_3 \text{ and } C_4 \text{ by } C_2-C_1, C_3-C_1 \text{ and } C_4-C_1 \text{ respectively.}$$

$$=(x+y+2)\begin{vmatrix} 1-x & y-x & 1-x \\ y-1 & 0 & x-1 \\ 1-y & x-y & 1-y \end{vmatrix} \text{, expanding with respect to } R_1$$

$$=(x+y+2)\begin{vmatrix} 0 & y-x & 1-x \\ y-x & 0 & x-1 \\ 0 & x-y & 1-y \end{vmatrix} \text{, replacing } C_1 \text{ by } C_1-C_3$$

$$=-(x+y+2)(y-x)\begin{vmatrix} y-x & 1-x \\ x-y & 1-y \end{vmatrix} \text{, expanding with respect to } C_1$$

$$=-(x+y+2)(y-x)(y-x)\begin{vmatrix} 1 & 1-x \\ -1 & 1-y \end{vmatrix},$$

taking out $(y-x)$ common from C_1

$= -(x+y+2)(y-x)^2[(1-y)-(-1)(1-x)]$

$= (x+y+2)(x-y)^2(x+y-2)$. **Ans.**

Ex. 26. Evaluate $\begin{vmatrix} a & x & y & a \\ x & 0 & 0 & y \\ y & 0 & 0 & x \\ a & y & x & a \end{vmatrix}$

Sol. Then given determinant

$= \begin{vmatrix} 0 & x+y & y & a \\ x-y & 0 & 0 & y \\ y-x & 0 & 0 & x \\ 0 & y+x & x & a \end{vmatrix}$, replacing C_1 and C_2 by $C_1 - C_4$ and $C_2 + C_3$ respectively.

$= (x-y)(x+y)\begin{vmatrix} 0 & 1 & y & a \\ 1 & 0 & 0 & y \\ -1 & 0 & 0 & x \\ 0 & 1 & x & a \end{vmatrix}$, taking out $x-y$ and $x+y$ common from C_1 and C_2 respectively.

$= (x^2-y^2)\begin{vmatrix} 0 & 1 & y & a \\ 1 & 0 & 0 & y \\ 0 & 0 & 0 & x+y \\ 0 & 0 & x-y & 0 \end{vmatrix}$, resplacing R_3 and R_4 by $R_2 + R_3$ and $R_4 - R_1$ rspectively.

$= -(x^2-y^2)\begin{vmatrix} 1 & y & a \\ 0 & 0 & x+y \\ 0 & x-y & 0 \end{vmatrix}$, expanding with respect to C_1

$= -(x^2-y^2)\begin{vmatrix} 0 & x+y \\ x-y & 0 \end{vmatrix}$, expanding with respect to C_1

$= -(x^2-y^2)[-(x+y)(x-y)] = (x^2-y^2)^2$. **Ans**

Ex. 27 (a) Evaluate $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} 1 & bc & a(b+c) \\ 0 & c(a-b) & c(b-a) \\ 0 & b(a-c) & b(c-a) \end{vmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively.

$= \begin{vmatrix} c(a-b) & c(b-a) \\ b(a-c) & b(c-a) \end{vmatrix}$, expanding with respect to C_1

$= (a-b)(a-c)\begin{vmatrix} c & -c \\ b & -b \end{vmatrix}$, taking out $(a-b)$ and $(a-c)$ common from R_1 and R_2 respectively

$= -(a-b)(a-c)\begin{vmatrix} c & c \\ b & b \end{vmatrix} = 0$, two columns being identical.

Ex. 27 (b) Evaluate $\begin{vmatrix} \mathbf{1/a} & \mathbf{a^2} & \mathbf{bc} \\ \mathbf{1/b} & \mathbf{b^2} & \mathbf{ca} \\ \mathbf{1/c} & \mathbf{c^2} & \mathbf{ab} \end{vmatrix}$ (*Kanpur 92*)

Sol. The given determinant

$= \dfrac{1}{abc} \begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & bca \\ 1 & c^3 & cab \end{vmatrix}$, multiplying R_1, R_2 and R_3 by a, b and c respectively. **(Note)**

$= \dfrac{1}{abc} \cdot abc \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix}$, taking out abc common from C_3

$= 0$, since C_1 and C_3 are identical. **Ans.**

***Ex. 28. Evaluate** $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \end{vmatrix}$ (*Meerut 96P*)

Sol. The given determinant

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively.

$= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}$, expanding with respect to R_1

$= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$, taking common factors out

$= (b-a)(c-a)[(c+a)-(b+a)] = (a-b)(c-a)(b-c).$ **Ans.**

***Ex. 29. Show that** $\begin{vmatrix} \mathbf{1} & \mathbf{a} & \mathbf{bc} \\ \mathbf{1} & \mathbf{b} & \mathbf{ca} \\ \mathbf{1} & \mathbf{c} & \mathbf{ab} \end{vmatrix} = \mathbf{(b-c)(c-a)(a-b)}$ (*Meerut 96P; Purvanchal 96*)

Sol. The given determinant

$= \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively.

$= \begin{vmatrix} b-a & -c(b-a) \\ c-a & -b(c-a) \end{vmatrix}$, expanding with respect to C_1

$= (b-a)(c-a) \begin{vmatrix} 1 & -c \\ 1 & -b \end{vmatrix}$, taking common factors out

$= (b-a)(c-a)(-b+c) = (a-b)(b-c)(c-a).$ Hence proved.

Ex. 30. Show that $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \\ \mathbf{a^3} & \mathbf{b^3} & \mathbf{c^3} \end{vmatrix} = \mathbf{abc\,(a-b)(b-c)(c-a)}$

Sol. The given determinant

$= abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$, taking out a, b and c common from C_1, C_2 and C_3 respectively.

Now proceed as in Ex. 28 Page 145.

Ex. 31. Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^3} & \mathbf{b^3} & \mathbf{c^3} \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively.

$= \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$, expanding with respect to R_1

$= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+ac+a^2 \end{vmatrix}$,

taking common factors out from C_1 and C_2

$= (b-a)(c-a)[(c^2+ac+a^2)-(b^2+ab+a^2)]$

$= (b-a)(c-a)[(c^2-b^2)+a(c-b)]$

$= (b-a)(c-a)(c-b)[(c+b)+a]$

$= (a-b)(b-c)(c-a)(a+b+c)$. **Ans.**

***Ex. 32. If a, b, c are all different and**

$\begin{vmatrix} \mathbf{a} & \mathbf{a^2} & \mathbf{1+a^3} \\ \mathbf{b} & \mathbf{b^2} & \mathbf{1+b^3} \\ \mathbf{c} & \mathbf{c^2} & \mathbf{1+c^3} \end{vmatrix} = \mathbf{0}$, **prove that abc = – 1.**

(Meerut 91 S)

Sol. The given determinant

$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$, breaking into two determinants. **(Note)**

$= \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$, interchanging rows and columns in each determinant. **(Note)**

Now proceed as in Ex. 28 Page 145 and Ex. 30 above.

The value of given determinant $= (1+abc)[(a-b)(b-c)(c-a)]$

Ex. 33 (a). Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \\ \mathbf{a^3} & \mathbf{b^3} & \mathbf{c^3} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively.

$$= \begin{vmatrix} b^2-a^2 & c^2-a^2 \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$$, expanding with respect to R_1

$$= (b-a)(c-a) \begin{vmatrix} b+a & c+a \\ b^2+ab+a^2 & c^2+ac+a^2 \end{vmatrix},$$

taking out common factors of C_1, C_2

$$= (b-a)(c-a) \begin{vmatrix} b-c & c+a \\ b^2+ab-c^2-ca & c^2+ca+a^2 \end{vmatrix},$$

replacing C_1 by $C_1 - C_2$

$$= (b-a)(c-a) \begin{vmatrix} b-c & c+a \\ (b^2-c^2)+a(b-c) & c^2+ca+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(b-c) \begin{vmatrix} 1 & c+a \\ b+c+a & c^2+ca+a^2 \end{vmatrix}$$

$= -(a-b)(b-c)(c-$ [illegible] $1(c^2+ca+a^2)-(c+a)(b+c+a)]$

$= -(a-b)(b-c)(c-$ [illegible] $^2+ca+a^2-cb-c^2-ac-ab-ac-a^2]$

$= -(a-b)(b-c)(c$ [illegible] $-ab-bc-ca]$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$ **Ans.**

Ex. 33. (b) If $\begin{vmatrix} \mathbf{a} & \mathbf{a^3} & \mathbf{a^4-1} \\ \mathbf{b} & \mathbf{b^3} & \mathbf{b^4-1} \\ \mathbf{c} & \mathbf{c^3} & \mathbf{c^4-1} \end{vmatrix} = 0$**, then prove that**

$$\mathbf{abc\,(ab+bc+ca) = a+b+c.}$$

Sol. The given equation is

$$\begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} - \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} = 0$$

or $$abc \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0$$

or $abc\,[(a-b)(b-c)(c-a)(ab+bc+ca)]$
$\qquad - [(a-b)(b-c)(c-a)(a+b+c)] = 0$

interchaing rows and columns of the determinant and evaluating determinants as in Ex. 31 Page 146 and Ex. 33 (a) above

or $(a-b)(b-c)(c-a)\,[abc\,(ab+bc+ca)-(a+b+c)] = 0$

or $abc\,(ab+bc+ca)-(a+b+c) = 0, \because \quad a \neq b \neq c$

or $abc\,(ab+bc+ca) = a+b+c.$ Hence proved.

Ex. 34. Prove that

$$\begin{vmatrix} \mathbf{1} & \mathbf{a^2+bc} & \mathbf{a^3} \\ \mathbf{1} & \mathbf{b^2+ca} & \mathbf{b^3} \\ \mathbf{1} & \mathbf{c^2+ab} & \mathbf{c^3} \end{vmatrix} = -(\mathbf{b-c})(\mathbf{c-a})(\mathbf{a-b})(\mathbf{a^2+b^2+c^2})$$

Sol. The given determinant

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} 1 & bc & a^3 \\ 1 & ca & b^3 \\ 1 & ab & c^3 \end{vmatrix}$$, breaking into two determinants **(Note)**

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ a^3 & b^3 & c^3 \end{vmatrix}$$, interchanging rows and columns. **(Note)**

...(i)

The first determinant $= (a-b)(b-c)(c-a)(ab+bc+ca)$. ...(ii)

[See Ex. 33 (a) Page 146]

The second determinant

$$= \begin{vmatrix} 1 & 0 & 0 \\ bc & ca-bc & ab-bc \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$

$$= \begin{vmatrix} c(a-b) & b(a-c) \\ (b-a)(b^2+ab+a^2) & (c-a)(c^2+ac+a^2) \end{vmatrix},$$

expanding with respect to R_1

$$= (a-b)(a-c) \begin{vmatrix} c & b \\ -(b^2+ab+a^2) & -(c^2+ac+a^2) \end{vmatrix},$$

taking out common factors from C_1 and C_2

$$= (a-b)(a-c) \begin{vmatrix} (c-b) & b \\ -(b^2-c^2+ab-ac) & -(c^2+ac+a^2) \end{vmatrix},$$

replacing C_1 by $C_1 - C_2$

$$= (a-b)(a-c) \begin{vmatrix} (c-b) & b \\ (c-b)(c+b+a) & -(c^2+ca+a^2) \end{vmatrix}$$

$$= (a-b)(a-c)(c-b) \begin{vmatrix} 1 & b \\ a+b+c & -(c^2+ca+a^2) \end{vmatrix},$$

taking $(c-b)$ common from C_1.

$$= (a-b)(b-c)(c-a)[-(c^2+ca+a^2) - b(a+b+c)]$$

$$= -(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca) \qquad ...(iii)$$

Substituting values from (ii) and (iii) in (i), we find the given determinant

$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$- (a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$$

$= (a-b)(b-c)(c-a)[(ab+bc+ca)-(a^2+b^2+c^2+ab+bc+ca)]$

$= -(a-b)(b-c)(c-a)(a^2+b^2+c^2).$ Hence proved.

Ex. 35. Show that

$$\begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{x^2} & \mathbf{y^2} & \mathbf{z^2} \\ \mathbf{yz} & \mathbf{zx} & \mathbf{xy} \end{vmatrix} = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{x^2} & \mathbf{y^2} & \mathbf{z^2} \\ \mathbf{x^3} & \mathbf{y^3} & \mathbf{z^3} \end{vmatrix} = \mathbf{(y-z)(z-x)(x-y)(yz+zx+xy)}.$$

(*Gorakhpur 93; Kumaun 96; Meerut 94*)

Sol.

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}$$, multiplying C_1, C_2, C_3 by x, y, z respectively. **(Note)**

$$= \frac{1}{xyz} xyz \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$, taking out xyz common from R_3

$$= \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$, interchanging R_2 and R_3 and then R_1 and R_2

For the 2nd part do as Ex. 33 (a) Pages 146–47. **(Note)**

Ex. 36. Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1+x} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1+y} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1+z} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{vmatrix}$$, replacing R_2, R_3 and R_4 by $R_2 - R_1, R_3 - R_1$ and $R_4 - R_1$ respectively

$$= \begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix}$$, expanding with respect to C_1

$$= x\begin{vmatrix} y & 0 \\ 0 & z \end{vmatrix}$$, expanding with respect to R_1

$= xyz.$ **Ans.**

Ex. 37. Evaluate $\begin{vmatrix} \mathbf{a^3} & \mathbf{3a^2} & \mathbf{3a} & \mathbf{1} \\ \mathbf{a^2} & \mathbf{a^2+2a} & \mathbf{2a+1} & \mathbf{1} \\ \mathbf{a} & \mathbf{2a+1} & \mathbf{a+2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} a^3-3a^2+3a-1 & 3a^2 & 3a & 1 \\ 0 & a^2+2a & 2a+1 & 1 \\ 0 & 2a+1 & a+2 & 1 \\ 0 & 3 & 3 & 1 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1-C_2+C_3-C_4$$

(Note)

$$= (a^3-3a^2+3a-1)\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (a-1)^3\begin{vmatrix} (a^2+2a)-(2a+1)+1 & 2a+1 & 1 \\ (2a+1)-(a+2)+1 & a+2 & 1 \\ 3-3+1 & 3 & 1 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1-C_2+C_3$$

$$= (a-1)^3\begin{vmatrix} a^2 & 2a+1 & 1 \\ a & a+2 & 1 \\ 1 & 3 & 1 \end{vmatrix}, \text{ on simplying}$$

$$= (a-1)^3\begin{vmatrix} a^2-1 & 2a-2 & 0 \\ a-1 & a-1 & 0 \\ 1 & 3 & 1 \end{vmatrix}, \text{ replacing } R_1 \text{ and } R_2 \text{ by } R_1-R_3 \text{ and } R_2-R_3 \text{ respectively.}$$

$$= (a-1)^3\begin{vmatrix} a^2-1 & 2(a-1) \\ a-1 & a-1 \end{vmatrix}, \text{ expanding with respect to } C_3$$

$$= (a-1)^3(a-1)(a-1)\begin{vmatrix} a+1 & 2 \\ 1 & 1 \end{vmatrix}, \text{ taking out common factors.}$$

$$= (a-1)^5[(a+1)-2] = (a-1)^6.$$

Ans.

Ex.38. Evaluate $\begin{vmatrix} \mathbf{1+a} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1+b} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1+c} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1+d} \end{vmatrix}$

(*Gorakhpur 90; Kanpur 97; Meerut 91; Purvanchal 93*)

Sol. The given determinant

$$= abcd\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix}, \text{ taking out } a, b, c \text{ and } d \text{ common from } R_1\, R_2, R_3 \text{ and } R_4 \text{ respectively.}$$

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix}$$

replacing, R_1 by $R_1 + R_2 + R_3 + R_4$ and taking $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)$ common from R_1

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix},$$

replacing C_2, C_3, and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$ respectively.

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$, expanding with respect to R_1

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$, expanding with respect to R_1

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)$$ **Ans.**

Ex. 39. Prove that $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{b} & \mathbf{b} \\ \mathbf{a} & \mathbf{b} & \mathbf{a} & \mathbf{a} \\ \mathbf{a} & \mathbf{a} & \mathbf{b} & \mathbf{a} \\ \mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{a} \end{vmatrix} = -(\mathbf{b}-\mathbf{a})^4.$

Sol. The given determinant

$$= \begin{vmatrix} a & b-a & b-a & b-a \\ a & b-a & 0 & 0 \\ a & 0 & b-a & 0 \\ b & 0 & 0 & a-b \end{vmatrix}$$, replacing C_2, C_3 and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$ respectively.

$$= -(b-a)\begin{vmatrix} a & b-a & 0 \\ a & 0 & b-a \\ b & 0 & 0 \end{vmatrix} + (a-b)\begin{vmatrix} a & b-a & b-a \\ a & b-a & 0 \\ a & 0 & b-a \end{vmatrix},$$

expanding with respect to C_4

$$= -(b-a)\,b\begin{vmatrix} b-a & 0 \\ 0 & b-a \end{vmatrix} + (a-b)\begin{vmatrix} 0 & 0 & b-a \\ a & b-a & 0 \\ a & 0 & b-a \end{vmatrix},$$

expanding first determinant with respect to R_3 and in the second determinant replacing R_1 by $R_1 - R_2$.

$$= -b\,(b-a)^3 + (a-b)\,(b-a)\begin{vmatrix} a & b-a \\ a & 0 \end{vmatrix},$$

expanding the second determinant with respect to R_1

$$= -b\,(b-a)^3 + (a-b)\,(b-a)\,[0 - a\,(b-a)]$$

$$= -b\,(b-a)^3 + a\,(b-a)^3 = -(b-a)^3\,(b-a)$$

$$= -(b-a)^4$$

Hence proved

****Ex. 40. Show that** $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$

(Kumaun 94; Meerut 93; Purvanchal 95)

Sol. The given determinant

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix},$$ replacing C_1 and C_2 by $C_1 - C_3$ & $C_2 - C_3$ respectively.

$$= \begin{vmatrix} (b+c+a)\,(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)\,(c+a-b) & b^2 \\ (c+a+b)\,(c-a-b) & (c+a+b)\,(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix},$$ taking out the common factors from C_1 and C_2.

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix},$$ replacing R_3 by $R_3 - R_1 - R_2$.

$$= (a+b+c)^2\begin{vmatrix} b+c & a^2/b & a^2 \\ b^2/a & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix},$$ replacing C_1 and C_2 by $C_1 + \frac{1}{a}C_3$ and $C_2 + \frac{1}{b}C_3$ respectively.

(Note)

$$= 2ab\,(a+b+c)^2\begin{vmatrix} b+c & a^2/b \\ b^2/a & c+a \end{vmatrix},$$ expanding with respect to R_3

$$= 2ab\,(a+b+c)^2\,[(b+c)\,(c+a) - (b^2/a)\,(a^2/b)]$$

$= 2ab\,(a+b+c)^2\,[bc+ba+c^2+ca-ab]$

$= 2ab\,(a+b+c)^2\,c\,[b+a+c] = 2abc\,(a+b+c)^3.$ Hence proved.

Ex. 41. Evaluate $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{ax+by} \\ \mathbf{b} & \mathbf{c} & \mathbf{bx+cy} \\ \mathbf{ax+by} & \mathbf{bc+cy} & \mathbf{0} \end{vmatrix}$ *(Kanpur 90)*

Sol. The given determinant

$$= \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ ax+by & bx+cy & -x\,(ax+by) \\ & & -y\,(bx+cy) \end{vmatrix}$$, replacing C_3 by $C_3 - xC_1 - yC_2$. **(Note)**

$$= -(ax^2+2bxy+cy^2)\begin{vmatrix} a & b \\ b & c \end{vmatrix}$$, expanding with respect to C_3

$= -(ax^2+2bxy+cy^2)\,(ac-b^2).$ **Ans.**

Ex. 42. Prove that

$$\begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{x} & \mathbf{0} & \mathbf{x} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{x} & \mathbf{0} \\ \mathbf{x} & \mathbf{0} & \mathbf{x+1} & \mathbf{0} & \mathbf{x} \\ \mathbf{0} & \mathbf{x} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{x} & \mathbf{0} & \mathbf{x} & \mathbf{0} & \mathbf{1} \end{vmatrix} = (\mathbf{x}-\mathbf{1})^2\,(\mathbf{x}+\mathbf{1})\,(\mathbf{1}+\mathbf{2x}-\mathbf{x}^2)$$

Sol. The given determinat

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & 0 \\ x & 0 & x+1-x^2 & 0 & -1 \\ 0 & x & 0 & 1 & 0 \\ x & 0 & x-x^2 & 0 & 1-x \end{vmatrix}$$, replacing C_3 and C_5 by $C_3 - xC_1$ and $C_5 - C_3$ respectively.

$$= \begin{vmatrix} 1 & 0 & x & 0 \\ 0 & x+1-x^2 & 0 & -1 \\ x & 0 & 1 & 0 \\ 0 & x-x^2 & 0 & 1-x \end{vmatrix}$$, expanding with respect to R_1

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & x+1-x^2 & 0 & -1 \\ x & 0 & 1-x^2 & 0 \\ 0 & x-x^2 & 0 & 1-x \end{vmatrix}$$, replacing C_3 by $C_3 - xC_1$

$$= \begin{vmatrix} x+1-x^2 & 0 & -1 \\ 0 & 1-x^2 & 0 \\ x\,(1-x) & 0 & 1 \end{vmatrix}$$, expanding with respect to R_1

$$= -(1-x)\begin{vmatrix} 0 & 1-x^2 & 0 \\ x+1-x^2 & 0 & -1 \\ x & 0 & 1 \end{vmatrix}$$, interchanging R_1 and R_2 and taking out$(1-x)$ common from R_3

$$= (1-x)(1-x^2)\begin{vmatrix} x+1-x^2 & -1 \\ x & 1 \end{vmatrix}$$, expanding with respect to R_1

$$= (1-x)^2(1+x)[(x+1-x^2).1-(-1).x]$$
$$= (x-1)^2(1+x)(2x+1-x^2).$$ Hence proved.

Ex. 43. Show that $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{bc\,(b+c)} & \mathbf{ca\,(c+a)} & \mathbf{ab\,(a+b)} \\ \mathbf{b^2c^2} & \mathbf{c^2a^2} & \mathbf{a^2b^2} \end{vmatrix}$

$= \mathbf{abc\,(a-b)\,(b-c)\,(c-a)\,(a+b+c)}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 1 & 1 \\ b^2c+bc^2 & c^2a+ca^2 & a^2b+ab^2 \\ b^2c^2 & c^2a^2 & a^2b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ b^2c+bc^2 & c(a-b)(a+b+c) & b(a-c)(a+b+c) \\ b^2c^2 & c^2(a-b)(a+b) & b^2(a-c)(a+c) \end{vmatrix},$$

replacing C_2, C_3 by C_2-C_1, C_3-C_1

$$= \begin{vmatrix} c(a-b)(a+b+c) & b(a-c)(a+b+c) \\ c^2(a-b)(a+b) & b^2(a-c)(a+c) \end{vmatrix}$$, expanding with respect to R_1

$$= c(a-b)\,b(a-c)\begin{vmatrix} a+b+c & a+b+c \\ c(a+b) & b(a+c) \end{vmatrix}$$, taking out common factors from C_1, C_2

$$= bc(a-b)(a-c)(a+b+c)\begin{vmatrix} 1 & 1 \\ ca+cb & ba+bc \end{vmatrix}$$, taking out $a+b+c$ common from R_1

$$= bc(a-b)(a-c)(a+b+c)\begin{vmatrix} 1 & 0 \\ ca+cb & ba-ca \end{vmatrix}$$, replacing C_2 by C_2-C_1

$$= bc(a-b)(a-c)(a+b+C)(ab-ca)$$
$$= -abc(a-b)(b-c)(c-a)(a+b+c).$$ Hence proved.

***Ex. 44. Evaluate** $\begin{vmatrix} \mathbf{1} & \mathbf{bc+ad} & \mathbf{b^2c^2+a^2d^2} \\ \mathbf{1} & \mathbf{ca+bd} & \mathbf{c^2a^2+b^2d^2} \\ \mathbf{1} & \mathbf{ab+cd} & \mathbf{a^2b^2+c^2d^2} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 0 & ca+bd-bc-ad & c^2a^2+b^2d^2-b^2c^2-a^2d^2 \\ 0 & ab+cd-bc-ad & a^2b^2+c^2d^2-b^2c^2-a^2d^2 \end{vmatrix},$$

replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively

$$= \begin{vmatrix} ca - bc + bd - ad & c^2a^2 - b^2c^2 + b^2d^2 - a^2d^2 \\ ab - bc + cd - ad & a^2b^2 - b^2c^2 + c^2d^2 - a^2d^2 \end{vmatrix},$$

expanding with respect to C_1

$$= \begin{vmatrix} (c-d)(a-b) & (c^2-d^2)(a^2-b^2) \\ (b-d)(a-c) & (b^2-d^2)(a^2-c^2) \end{vmatrix}, \text{ factorising the elements}$$

$$= (c-d)(a-b)(b-d)(a-c) \begin{vmatrix} 1 & (c+d)(a+b) \\ 1 & (b+d)(a+c) \end{vmatrix},$$

taking out the common factors

$$= (c-d)(a-b)(b-d)(a-c) \begin{vmatrix} 1 & ca + bc + da + db \\ 0 & ba + dc - ca - db \end{vmatrix},$$

replacing R_2 by $R_2 - R_1$

$$= (c-d)(a-b)(b-d)(a-c)(ba + dc - ca - db)$$

$$= (c-d)(a-b)(b-d)(a-c)(a-d)(b-c).$$ **Ans.**

***Ex. 45. Prove that**

$$\begin{vmatrix} \mathbf{a}^2 + \lambda & \mathbf{ab} & \mathbf{ac} & \mathbf{ad} \\ \mathbf{ba} & \mathbf{b}^2 + \lambda & \mathbf{bc} & \mathbf{bd} \\ \mathbf{ca} & \mathbf{cb} & \mathbf{c}^2 + \lambda & \mathbf{cd} \\ \mathbf{da} & \mathbf{db} & \mathbf{dc} & \mathbf{d}^2 + \lambda \end{vmatrix} = \lambda^3 (\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2 + \lambda)$$

(*Kanpur* 90)

Sol. The given determinant

$$= abcd \begin{vmatrix} a + \frac{\lambda}{a} & a & a & a \\ b & b + \frac{\lambda}{b} & b & b \\ c & c & c + \frac{\lambda}{c} & c \\ d & d & d & d + \frac{\lambda}{d} \end{vmatrix},$$ taking out a, b, c, d common from C_1, C_2, C_3 and C_4 respectively.

$$= abcd \begin{vmatrix} a + \frac{\lambda}{a} & -\frac{\lambda}{a} & -\frac{\lambda}{a} & -\frac{\lambda}{a} \\ b & \frac{\lambda}{b} & 0 & 0 \\ c & 0 & \frac{\lambda}{c} & 0 \\ d & 0 & 0 & \frac{\lambda}{d} \end{vmatrix},$$ replacing C_2, C_3 and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$ respectively.

$$= abcd \begin{vmatrix} a & 0 & 0 & -\frac{\lambda}{a} \\ b & \frac{\lambda}{b} & 0 & 0 \\ c & 0 & \frac{\lambda}{c} & 0 \\ d+\frac{\lambda}{d} & -\frac{\lambda}{d} & -\frac{\lambda}{d} & \frac{\lambda}{d} \end{vmatrix}$$

, replacing C_1, C_2 and C_3 by $C_1 + C_4$, $C_2 - C_4$ and $C_3 - C_4$ respectively.

$$= a^2bcd \begin{vmatrix} \frac{\lambda}{b} & 0 & 0 \\ 0 & \frac{\lambda}{c} & 0 \\ -\frac{\lambda}{d} & -\frac{\lambda}{d} & \frac{\lambda}{d} \end{vmatrix} + \lambda\, bcd \begin{vmatrix} b & \frac{\lambda}{b} & 0 \\ c & 0 & \frac{\lambda}{c} \\ d+\frac{\lambda}{d} & -\frac{\lambda}{d} & -\frac{\lambda}{d} \end{vmatrix},$$

expanding with respect to R_1

$$= a^2bcd\, \frac{\lambda}{b} \begin{vmatrix} \frac{\lambda}{c} & 0 \\ -\frac{\lambda}{d} & \frac{\lambda}{d} \end{vmatrix} + \lambda bcd.b \begin{vmatrix} 0 & \frac{\lambda}{c} \\ -\frac{\lambda}{d} & \frac{\lambda}{d} \end{vmatrix}$$

$$-\lambda\, bcd \cdot \frac{\lambda}{b} \begin{vmatrix} c & \frac{\lambda}{c} \\ d+\frac{\lambda}{d} & -\frac{\lambda}{d} \end{vmatrix},$$

expanding each determinant with respect to R_1

$$= \lambda\, a^2\, cd \left(\frac{\lambda^2}{cd}\right) + \lambda\, b^2 cd \left(\frac{\lambda^2}{cd}\right) - \lambda^2\, cd \left(-\frac{\lambda c}{d} - \frac{\lambda d}{c} - \frac{\lambda^2}{cd}\right)$$

$$= \lambda^3 a^2 + \lambda^3 b^2 + \lambda^2 cd \left(\frac{\lambda c^2 + \lambda d^2 + \lambda^2}{cd}\right)$$

$$= \lambda^3 a^2 + \lambda^3 b^2 + \lambda^3 (c^2 + d^2 + \lambda) = \lambda^3 (a^2 + b^2 + c^2 + d^2 + \lambda)$$ Hence proved.

Ex. 46. Evaluate $\begin{vmatrix} \mathbf{a^2} & \mathbf{a^2 - (b-c)^2} & \mathbf{bc} \\ \mathbf{b^2} & \mathbf{b^2 - (c-a)^2} & \mathbf{ca} \\ \mathbf{c^2} & \mathbf{c^2 - (a-b)^2} & \mathbf{ab} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} a^2 & -(b-c)^2 & bc \\ b^2 & -(c-a)^2 & ca \\ c^2 & -(a-b)^2 & ab \end{vmatrix}$$, replacing C_2 by $C_2 - C_1$

$$= -\begin{vmatrix} a^2 & (b^2+c^2)-2bc & bc \\ b^2 & (c^2+a^2)-2ca & ca \\ c^2 & (a^2+b^2)-2ab & ab \end{vmatrix}$$ (Note)

$$= -\begin{vmatrix} a^2 & (b^2+c^2) & bc \\ b^2 & (c^2+a^2) & ca \\ c^2 & (a^2+b^2) & ab \end{vmatrix}$$, replacing C_2 by $C_2 + 2C_3$

$$= -\begin{vmatrix} a^2 & b^2+c^2+a^2 & bc \\ b^2 & c^2+a^2+b^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix}$$, replacing C_2 by $C_2 + C_1$

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$, taking out the common factor from C_2

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2-a^2 & 0 & ca-bc \\ c^2-a^2 & 0 & ab-bc \end{vmatrix}$$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively

$$= (a^2+b^2+c^2)\begin{vmatrix} b^2-a^2 & c\,(a-b) \\ c^2-a^2 & b\,(a-c) \end{vmatrix}$$, expanding with respect to C_2

$$= (a-b)\,(a-c)\,(a^2+L^2+c^2)\begin{vmatrix} -(b+a) & c \\ -(c+a) & b \end{vmatrix}$$, taking out the common factors

$= (a-b)\,(a-c)\,(a^2+b^2+c^2)\,[-b\,(b+a)+c\,(c+a)]$

$= (a-b)\,(a-c)\,(a^2+b^2+c^2)\,[-b^2-ab+c^2+ac]$

$= (a-b)\,(a-c)\,(a^2+b^2+c^2)\,[(c^2-b^2)+a\,(c-b)]$

$= (a-b)\,(a-c)\,(a^2+b^2+c^2)\,(c-b)\,(a+b+c).$

$= (a-b)\,(b-c)\,(c-a)\,(a+b+c)\,(a^2+b^2+c^2).$ **Ans.**

****Ex. 47. Prove that**

$$\begin{vmatrix} \mathbf{0} & -\mathbf{c} & \mathbf{b} & -\mathbf{l} \\ \mathbf{c} & \mathbf{0} & -\mathbf{a} & -\mathbf{m} \\ -\mathbf{b} & \mathbf{a} & \mathbf{0} & -\mathbf{n} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{0} \end{vmatrix} = \mathbf{(al+bm+cn)\,(ax+by+cy)}$$

Sol. The given determinant

$$= \frac{1}{a}\begin{vmatrix} 0 & -ac & ab & -al \\ c & 0 & -a & -m \\ -b & a & 0 & -n \\ x & y & z & 0 \end{vmatrix}$$, taking $1/a$ common from R_1 (Note)

$$=\frac{1}{a}\begin{vmatrix} 0 & 0 & 0 & -al-bm-cn \\ c & 0 & -a & -m \\ -b & a & 0 & -n \\ x & y & z & 0 \end{vmatrix}, \text{ replacing } R_1 \text{ by } R_1+b\,R_2+c\,R_3$$

$$=\frac{(al+bm+cn)}{a}\begin{vmatrix} c & 0 & -a \\ -b & a & 0 \\ x & y & z \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$=\frac{(al+bm+cn)}{a^2}\begin{vmatrix} ac & 0 & -a \\ -ab & a & 0 \\ ax & y & z \end{vmatrix}, \text{ taking } 1/a \text{ common from } C_1$$

(Note)

$$=\frac{(al+bm+cn)}{a^2}\begin{vmatrix} 0 & 0 & -a \\ 0 & a & 0 \\ ax+by+cz & y & z \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1+bC_2+cC_3$$

$$=\frac{(al+bm+cn)\,(ax+by+cz)}{a^2}\begin{vmatrix} 0 & -a \\ a & 0 \end{vmatrix}$$

$= (al+bm+cn)\,(ax+by+cz).$ Hence proved.

Exercises on Evaluation of Determinants

Ex. 1. Evaluate $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ **Ans.** -8

Ex. 2. Show that $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 65 & 54 & 46 \end{vmatrix} = 132$

Ex. 3. Evaluate $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

[Hint : Replace C_3 **by** C_3+C_2**].** **Ans.** 0.

Ex. 4. Show that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0.$

[Hint : Replace C_1 **by** $C_1+C_2+C_3$**].**

Ex. 5. Evaluate $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$, where ω is one of the imaginary cube roots of unity. **Ans.** 0

[Hint : Replace C_1 **by** $C_1+C_2+C_3$ **remembering** $1+\omega+\omega^2=0$**]**

Ex. 6. Prove that $\begin{vmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{vmatrix} = 0$

[Hint : Replace C_1 by $C_1 + C_2 + C_3 + C_4 + C_5$].

Ex. 7. Show that $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix} = 0$

Ex. 8. Calculate the value of the determinant $\begin{vmatrix} 7 & 13 & 10 & 6 \\ 5 & 9 & 7 & 4 \\ 8 & 12 & 11 & 7 \\ 4 & 10 & 6 & 3 \end{vmatrix}$ **Ans.** 0

Ex. 9. Evaluate $\begin{vmatrix} x+a & x+2a & x+3a \\ x+2a & x+3a & x+4a \\ x+4a & x+5a & x+6a \end{vmatrix}$

[Hint : Replace C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_2$]. **Ans.** 0

Ex. 10. Prove that

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 2\,(3\,abc - a^3 - b^3 - c^3).$$

[Hint : See Ex. 13 Page 137].

***Ex. 11.** Prove that $\begin{vmatrix} -2a & a+b & c+a \\ b+a & -2b & c+b \\ c+a & c+b & -2c \end{vmatrix} = 4\,(a+b)\,(b+c)\,(c+a)$

Products of Determinants

§ 4·11. Theorem. *If* **A** *is an* $n \times n$ *matrix and* **E** *is an elementary matrix obtained from the identity matrix* $\mathbf{I_n}$, *then*

$$|\mathbf{EA}| = |\mathbf{AE}| = |\mathbf{E}| \bullet |\mathbf{A}| = |\mathbf{A}| \bullet |\mathbf{E}| \qquad (\textit{Purvanchal 94})$$

Proof : $\because$ $\mathbf{I}_n$ is the $n \times n$ identity matrix.

$\therefore\ |\mathbf{I}_n| = 1.$ ...(i)

Let $\mathbf{E}_a$, $\mathbf{E}_b$, $\mathbf{E}_c$ be three elementary matrices as defined in § 3·03 Page 104.

Then $|\mathbf{E}_a| = -|\mathbf{I}_n|$ ...See § 4·06 Prop. III Page 126.

$\therefore$ From (i) we get $|\mathbf{E}_a| = -1$ ...(ii)

Again $|\mathbf{E}_b| = c\,|\mathbf{I}_n|$ ...See § 4·06 Prop. II, Page 125.

$\therefore$ From (i) we get $|\mathbf{E}_b| = c$ ...(iii)

Similarly $|\mathbf{E}_c| = |\mathbf{I}_n|$ ... See § 4·10 Page 131.

$\therefore$ From (i) we get $|\mathbf{E}_c| = 1$ ...(iv)

Now as the matrix $\mathbf{E}_a\,\mathbf{A}$ can be obtained by interchanging two rows of the matrix **A,** so we have from (ii)

$$|\mathbf{E}_a A| = -|\mathbf{A}| = |\mathbf{E}_a| \bullet |\mathbf{A}| \quad \text{...(v)}$$

Similarly the product $\mathbf{E}_b\,\mathbf{A}$ can be obtained by applying second elementary row operation (as given in § 3·01 Page 103) on the matrix **A**, so we have from (iii)

$$|\mathbf{E}_b\,\mathbf{A}| = c\,|\mathbf{A}| = |\mathbf{E}_b| \bullet |\mathbf{A}| \quad \text{...(vi)}$$

Again the product $\mathbf{E}_c\,\mathbf{A}$ can be obtained by applying third elementary row operation (as given in § 3·01 Page 103) on the matrix **A**, so we have from (iv)

$$|\mathbf{E}_c\,\mathbf{A}| = |\mathbf{A}| = |\mathbf{E}_c|\,|\mathbf{A}| \quad \text{...(vii)}$$

Hence from (v), (vi) and (vii) we conclude that

$$|\mathbf{EA}| = |\mathbf{E}| \bullet |\mathbf{A}|,$$

where **E** is any one of the elementary matrices as defined in § 3·01 Page 103.

In a similar way we can prove that

$$|\mathbf{AE}| = |\mathbf{A}| \bullet |\mathbf{E}|.$$ Hence the theorem.

§ 4·12. Canonical Form (or Normal Form) of a matrix.

Every non-zero $m \times n$ matrix **A** can be reduced by means of elementary transformations (*i.e.* elementary row and column operations) to the form

$$\begin{bmatrix} \mathbf{I_r} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix},$$

where $\mathbf{I}_r$ is the $r \times r$ identity matrix and the remaining sub-matrices are zero matrices.

The above form is called the canonical form or orthogonal form or normal form of the matrix **A.**

Solved Examples on § 4·12.

Ex. 1. Reduce $\mathbf{A} = \begin{vmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{vmatrix}$ **to the canonical form.**

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$, replacing R_2, R_3 and R_4 by $R_2 - 2R_1$, $R_3 - R_1$ and $R_4 + R_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$, replacing C_2, C_3 and C_4 by $C_2 - 2C_1$, $C_3 + C_1$ and $C_4 - 4C_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3 and R_4 by $\frac{1}{4} R_3$ and $R_4 - R_2$ respectively.

$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & -3 & 5 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$, interchanging C_2 and C_4

$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & -3 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$, replacing R_2 by $R_2 - 5R_2$.

$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$, replacing R_2 by $-\frac{1}{3}R_2$.

$\sim\begin{bmatrix}\mathbf{I}_3 & \mathbf{O}\\\mathbf{O} & \mathbf{O}\end{bmatrix}$, which is the required canonical form.

Ex. 2. Reduce A = $\begin{bmatrix}\mathbf{13} & \mathbf{16} & \mathbf{19}\\\mathbf{14} & \mathbf{17} & \mathbf{20}\\\mathbf{15} & \mathbf{18} & \mathbf{21}\end{bmatrix}$ **to the canonical form.**

Sol. A $\sim\begin{bmatrix}13 & 16 & 19\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_2$ respectively.

$\sim\begin{bmatrix}13 & 3 & 3\\1 & 0 & 0\\1 & 0 & 0\end{bmatrix}$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_2$ respectively.

$\sim\begin{bmatrix}13 & 3 & 0\\1 & 0 & 0\\1 & 0 & 0\end{bmatrix}$, replacing C_3 by $C_3 - C_2$

$\sim\begin{bmatrix}13 & 3 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}$, replacing R_3 by $R_3 - R_2$

$\sim\begin{bmatrix}3 & 13 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$, interchanging C_1 and C_2.

$\sim\begin{bmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$, replacing C_2 by $C_2 - (13/3)\,C_1$.

$\sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$, replacing R_1 by $\frac{1}{3}R_1$.

$\sim \begin{bmatrix} \mathbf{I_2} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$, which is the required canonical form.

Exercise on § 4·12

Ex. Reduce $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 2 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & 3 & 3 & 4 \end{bmatrix}$ to the canonical form.

***§ 4·13. Definition.** *If* **A** *and* **B** *be two* $m \times n$ *matrices, then* $\mathbf{B} \sim \mathbf{A}$ *if and only if* $\mathbf{B} = \mathbf{SAT}$, *where* **S** *is an* $m \times m$ *non-singular matrix and* **T** *is an* $n \times n$ *singular matrix.*

With the help of § 3·03 Page 104 and § 3·11 Page 111 it can be proved.

The following two properties of the above relation are fundamental.

1. Symmetry. If $\mathbf{A} \sim \mathbf{B}$, then $\mathbf{B} \sim \mathbf{A}$ for if $\mathbf{A} = \mathbf{PBQ}$.

then $\quad \mathbf{B} = \mathbf{P}^{-1}\,\mathbf{A}\mathbf{Q}^{-1}$, where $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$ are non-singular matrices.

2. Reflexivity. Every matrix **A** is equivalent to itself since we can write $\mathbf{A} = \mathbf{IAI}$, so that $\mathbf{P} = \mathbf{I} = \mathbf{Q}$.

***§ 4·14. Theorem.** *An* $n \times n$ *matrix* **A** *is non-singular (or invertible) if and only if the determinant* $|\mathbf{A}| \neq 0$.

Proof : If **C** be the canonical form of the matrix **A**, then $\mathbf{C} \sim \mathbf{A}$

Therefore $\quad \mathbf{C} = \mathbf{S\,A\,T},$

where **S** and **T** are non singular. (See § 4·13 above)

Hence $\quad \mathbf{A} = \mathbf{S}^{-1}\,\mathbf{C}\,\mathbf{T}^{-1}$

or $\quad \mathbf{A} = \mathbf{E}_r \ldots \mathbf{E}_2\,\mathbf{E}_1\,\mathbf{C}\,\mathbf{D}_1\,\mathbf{D}_2 \ldots \mathbf{D}_s$

where $\mathbf{E}_i$ and $\mathbf{D}_i$ are elementary matrices. ...See § 3·09 Page 108

By the successive application of § 4·11 Page 159, we have

$$|\mathbf{A}| = |\mathbf{E}_r| \ldots |\mathbf{E}_2|\,|\mathbf{E}_1|\,|\mathbf{C}|\,|\mathbf{D}_1|\,|\mathbf{D}_2| \ldots |\mathbf{D}_s|$$

$\therefore$ If $\quad |\mathbf{A}| = 0$, then $|\mathbf{C}| = 0$, as $|\mathbf{E}_i| \neq 0$ and $|\mathbf{D}_i| \neq 0$.

If $|\mathbf{C}| = 0$, then it has at least one row of zero.

$\therefore$ The rank of matrix **A** is less than n (see next chapter) *i.e.* the matrix **A** is singular.

If the matrix **A** is non-singular, then

$\mathbf{C} = \mathbf{I_n}$, where $\mathbf{I_n}$ is the $n \times n$ identity matrix.

i.e. $\quad |\mathbf{C}| = |\mathbf{I_n}| = 1$

$\therefore$ From (i) above, we have $|\mathbf{A}| \neq 0$. Hence the theorem.

***§ 4·15. Theorem.** $|\mathbf{A_1\,A_2}| = |\mathbf{A_1}| \bullet |\mathbf{A_2}|$, *where* $|\mathbf{A_1}|$ *and* $|\mathbf{A_2}|$ *are two determinants.* *(Purvanchal 94)*

Proof : Let $\mathbf{C_1}$ and $\mathbf{C_2}$ be the canonical form of the matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ *i.e.* $\mathbf{A_1} \sim \mathbf{C_1}$ and $\mathbf{A_2} \sim \mathbf{C_2}$.

If $\mathbf{A_1} \sim \mathbf{C_1}$ then from § 4·13 above we have

$A_1 \sim S\,C_1\,T$, where S and T are non-singular matrices.

or $\quad A_1 = E_r \ldots E_2\,E_1\,C_1\,D_1\,D_2 \ldots D_s$,

where E_i and D_i are elementary matrices. (See § 3·09 Page 108)

Similarly $A_2 = F_t \ldots F_2\,F_1\,C_2\,K_1\,K_2 \ldots K_s$,

where F_i and K_i are elementary matrices.

$\therefore\ A_1\,A_2 = E_r \ldots E_2\,E_1\,C_1\,D_1\,D_3 \ldots D_s\,F_t \ldots F_2\,F_1\,C_2\,K_1\,K_2 \ldots K_s$

Hence by § 4·11 Page 159, we have

$$|A_1\,A_2| = |E_r \ldots E_2\,E_1| \bullet |C_1\,D_1\,D_2 \ldots D_s\,F_t \ldots F_2\,F_1\,C_2| \bullet |K_1\,K_2 \ldots K_s| \quad \ldots(i)$$

Now the following cases ariese.

Case I. Let A_1 be a singular matrix. (See § 2·18 Page 91)

Then C_1 has at least one row of zero.

$\therefore\ |C_1\,D_1\,D_2 \ldots D_s\,F_t \ldots F_2\,F_1\,C_2| = 0$,

since the matrix $C_1\,D_1\,D_2 \ldots D_s\,F_t \ldots F_2\,F_1\,C_2$ has a row of zero.

$\therefore$ From (i) above we have $|A_1\,A_2| = 0$.

Case II. If A_2 is a singular matrix. Then C_2 has at least one column of zero, hence as in Case I above

$$|C_1\,D_1\,D_2 \ldots D_s\,F_t \ldots F_1\,C_2| = 0$$

$\therefore$ From (i) above we have $|A_1\,A_2| = 0$.

Case III. If either A_1 or A_2 is singular, then $|A| \bullet |A_2| = 0$

Hence $|A_1\,A_2| = 0 = |A_1| \bullet |A_2|$.

Case IV. If A_1 and A_2 are non-singular matrices . Then C_1 and C_2 are identity matrices. Hence from § 4·11 Page 159, we have

$$|A_1\,A_2| = |E_r \ldots E_2\,E_1\,C_1\,D_1\,D_2 \ldots D_s|\ |F_t \ldots F_2\,F_1\,C_2\,K_1 \ldots K_s|$$
$$= |A_1| \bullet |A_2|$$

$\therefore$ From all the above cases it is clear that

$$|A_1\,A_2| = |A_1| \bullet |A_2|.$$

Cor. $|A_1\,A'_2| = |A_1| \bullet |A_2|$, where A'_2 is the transpose of A_2.

Proof : $|A'_2| = |A_2|$, $\quad \because A'_2$ is the transpose of A_2

$\therefore \quad |A_1\,A'_2| = |A_1|\,|A'_2|$, from § 4·14 above.

$$= |A_1|\,|A_2|, \quad \because\ |A'_2| = |A_2|$$

The corolllary leads to the row by row rule of multiplication of the determinants as given in Examples below :

Solved Examples on Multiplication of Determinants.

***Ex. 1. Evaluate** $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$

Sol. $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0\cdot 0 + c\cdot c + b\cdot b & 0\cdot c + c\cdot 0 + b\cdot a & 0\cdot b + c\cdot a + b\cdot 0 \\ c\cdot 0 + 0\cdot c + a\cdot b & c\cdot c + 0\cdot 0 + a\cdot a & c\cdot b + 0\cdot a + a\cdot 0 \\ b\cdot 0 + a\cdot c + 0\cdot b & b\cdot c + a\cdot 0 + 0\cdot a & b\cdot b + a\cdot a + 0\cdot 0 \end{vmatrix}$$

$$= \begin{vmatrix} c^2 + b^2 & ba & ca \\ ab & c^2 + a^2 & bc \\ ac & bc & b^2 + a^2 \end{vmatrix}$$

Ans.

Ex. 2. Evaluate $\begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2$

Sol. The required product

$$= \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix} \times \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0\cdot 0 + \cos x\cdot\cos x + \sin x\cdot\sin x & 0\cdot\sin x + \cos x\cdot 0 - \sin x\cdot\cos x & 0\cdot\cos x + \cos x\cdot\sin x - \sin x\cdot 0 \\ \sin x\cdot 0 + 0\cdot\cos x - \cos x\cdot\sin x & \sin x\cdot\sin x + 0\cdot 0 + \cos x\cdot\cos x & \sin x\cdot\cos x + 0\cdot\sin x + \cos x\cdot 0 \\ \cos x\cdot 0 + \sin x\cdot\cos x + 0\cdot(-\sin x) & \cos x\cdot\sin x + \sin x\cdot 0 + 0\cdot\cos x & \cos x\cdot\cos x + \sin x\cdot\sin x + 0\cdot 0 \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2 x + \sin^2 x & -\sin x\cos x & \cos x\sin x \\ -\cos x\sin x & \sin^2 x + \cos^2 x & \sin x\cos x \\ \sin x\cos x & \cos x\sin x & \cos^2 x + \sin^2 x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -\lambda & \lambda \\ -\lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{vmatrix}, \text{ where } \lambda = \sin x\cos x$$

Ans.

Ex. 3. Evaluate $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

Sol. The required product

$$= \begin{vmatrix} a_1x_1 + b_1y_1 + c_1z_1 & a_1x_2 + b_1y_2 + c_1z_2 & a_1x_3 + b_1y_3 + c_1z_3 \\ a_2x_1 + b_2y_1 + c_2z_1 & a_2x_2 + b_2y_2 + c_2z_2 & a_2x_3 + b_2y_3 + c_2z_3 \\ a_3x_1 + b_3y_1 + c_3z_1 & a_3x_2 + b_3y_2 + c_3z_2 & a_3x_3 + b_3y_3 + c_3z_3 \end{vmatrix}$$

****Ex. 4. Prove that**

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$$

$$= (a^3 + b^3 + c^3 - 3abc)^2$$ *(Gorakhpur 91; Kanpur 95)*

Sol. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= -\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$, interchanging C_2, C_3 of the second determinant. **(Note)**

$= \begin{vmatrix} -a & b & c \\ -b & c & a \\ -c & a & b \end{vmatrix} \times \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$, multiplying C_1 of first determinant by -1. **(Note)**

$$= \begin{vmatrix} -aa+bc+cb & -ab+ba+cc & -ac+bb+ca \\ -ba+cc+ab & -bb+ca+ac & -bc+cb+aa \\ -ca+ac+bb & -cb+aa+bc & -cc+ab+ba \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

Hence proved.

(See Ex. 9 Page 167 also)

Also $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix}$

$= a(cb - a^2) - b(b^2 - ac) + c(ab - c^2)$

$= abc - a^3 - b^3 + abc + abc - c^3$

$= -(a^3 + b^3 + c^3 - 3abc)$

$\therefore \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (a^3 + b^3 + c^3 - 3abc)^2$

Hence proved.

***Ex. 5. If $\mathbf{u = ax + by + cz}$, $\mathbf{v = ay + bz + cx}$, $\mathbf{w = az + bx + cy}$,**

prove that $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{b} & \mathbf{c} & \mathbf{a} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \end{vmatrix} \times \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{y} & \mathbf{z} & \mathbf{x} \\ \mathbf{z} & \mathbf{x} & \mathbf{y} \end{vmatrix} = \mathbf{u^3 + v^3 + w^3 - 3uvw}$

Sol. By row-by-row multiplication, the product of the given determinants

$$= \begin{vmatrix} ax+by+cz & ay+bz+cx & az+bx+cy \\ bx+cy+az & by+cz+ax & bz+cx+ay \\ cx+ay+bz & cy+az+bx & cz+ax+by \end{vmatrix}$$

$= \begin{vmatrix} u & v & w \\ w & u & v \\ v & w & u \end{vmatrix}$, since $ax + by + cz = u$ etc. (given)

$= u\begin{vmatrix} u & v \\ w & u \end{vmatrix} - v\begin{vmatrix} w & v \\ v & u \end{vmatrix} + w\begin{vmatrix} w & u \\ v & w \end{vmatrix}$

$$= u(u^2 - vw) - v(uw - v^2) + w(w^2 - uv)$$

$= u^3 + v^3 + w^3 - 3uvw.$ Hence proved.

****Ex. 6. Express** $\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

as the product of two determinants. (Gorakhpur 99; *Purvanchal 95*)

Sol. The given determinant.

$$= \begin{vmatrix} a^2 - 2ax + x^2 & b^2 - 2bx + x^2 & c^2 - 2cx + x^2 \\ a^2 - 2ay + y^2 & b^2 - 2by + y^2 & c^2 - 2cy + y^2 \\ a^2 - 2az + z^2 & b^2 - 2bz + z^2 & c^2 - 2cz + z^2 \end{vmatrix}$$

The element in the first row and first column is $a^2 - 2ax + x^2$, which can be written as $1(a^2) + (-2x)(a) + x^2(1)$. **(Note)**

This suggests that the first row of the required determinants are $1 - 2x, x^2$ and $a^2, a, 1$.

Hence proceeding in this way we may write the given determinant

$$= \begin{vmatrix} 1 & -2x & x^2 \\ 1 & -2y & y^2 \\ 1 & -2z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

We can verify by multiplying with the help of row-by-row rule that above two determinants are the required ones.

[**Note :** Such questions are acutally done by trial and error].

Ex. 7. Express $\begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix}$.

as the product of two determinants. (*Gorakhpur 95; Purvanchal 96*)

Sol. The given determinant

$$= \begin{vmatrix} 1 + 2ax + a^2x^2 & 1 + 2ay + a^2y^2 & 1 + 2az + a^2z^2 \\ 1 + 2bx + b^2x^2 & 1 + 2by + b^2y^2 & 1 + 2bz + b^2z^2 \\ 1 + 2cx + c^2x^2 & 1 + 2cy + c^2y^2 & 1 + 2cz + c^2z^2 \end{vmatrix}$$

The element in the first row and first column is $1 + 2ax + a^2x^2$ which can be written as $(1)(1) + (2a).(x) + (a^2).(x^2)$

This suggests that the first rows of the two required determinants are $1, 2a, a^2$ and $1, x, x^2$

Hence the given determinant may be written as

$$\begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \times \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix},$$ **Ans.**

***Ex. 8. Express** $\begin{vmatrix} \mathbf{b^2+c^2} & \mathbf{ab} & \mathbf{ca} \\ \mathbf{ab} & \mathbf{c^2+a^2} & \mathbf{bc} \\ \mathbf{ca} & \mathbf{bc} & \mathbf{b^2+b^2} \end{vmatrix}$

as the square of a determinant.

Hence evaluate. (*Purvanchal 94*)

Sol. The element in first row and first column is $b^2 + c^2$ which can be written as $0 \cdot 0 + c \cdot c + b \cdot b$. **(Note)**

So by trial and error method, we get the given determinant

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

Now $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$

$= - c \begin{vmatrix} c & a \\ b & 0 \end{vmatrix} + b \begin{vmatrix} c & 0 \\ b & a \end{vmatrix}$, expanding with respect to R_1.

$= - a [c \cdot 0 - a \cdot b] + b [c \cdot a - b \cdot 0] = 2abc.$

$\therefore$ The given determinat

$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = (2abc)^2 = 4a^2b^2c^2.$ **Ans.**

****Ex. 9. Express** $\begin{vmatrix} \mathbf{2bc-a^2} & \mathbf{c^2} & \mathbf{b^2} \\ \mathbf{c^2} & \mathbf{2ca-b^2} & \mathbf{a^2} \\ \mathbf{b^2} & \mathbf{a^2} & \mathbf{2ab-c^2} \end{vmatrix}$

as the product of two determiants.

Sol. The elements in the first row and first column is $2bc - a^2$ which can be written as $a(-a) + b(c) + c(b)$ ...(i)

The element in the first row and second column is c^2 which can be written as $a(-b) + b(a) + c(c)$ ...(ii)

The element in the first row and third column is b^2 which can be written as $a(-c) + b(b) + c(a)$ ...(iii) (i), (ii) and (iii) suggest that the given determinant can be written tentatively as

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

But actually multiplying these two determinants we get the given determinant. Hence these determinants are the required ones.

Ex. 10. Show that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ac-b^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$$

Sol. As in Ex. 9. above, we can show that the given determinant

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}, \qquad \text{(Note)}$$

taking – sign common from C_1 and interchanging C_2, C_3 in 2nd determinant

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (a^3+b^3+c^3-3abc)^2,$$

on expanding the determinant. Hence proved.

Ex. 11. Find the product of determinants of different orders

or evaluate
$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Sol. Here the two given determinants are of different orders, so we adopt the following method :

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad \text{(Note)}$$

making the two determinats of the same order

$$= \begin{vmatrix} \alpha_1 a_1 + \beta_1 b_1 + \gamma_1 \cdot 0 & \alpha_1 a_2 + \beta_1 b_2 + \gamma_1 \cdot 0 & \alpha_1 \cdot 0 + \beta_1 \cdot 0 + \gamma_1 \cdot 1 \\ \alpha_2 a_1 + \beta_2 b_1 + \gamma_2 \cdot 0 & \alpha_2 a_2 + \beta_2 b_2 + \gamma_2 \cdot 0 & \alpha_2 \cdot 0 + \beta_2 \cdot 0 + \gamma_2 \cdot 1 \\ \alpha_3 a_1 + \beta_3 b_1 + \gamma_3 \cdot 0 & \alpha_3 a_2 + \beta_3 b_2 + \gamma_3 \cdot 0 & \alpha_3 \cdot 0 + \beta_3 \cdot 0 + \gamma_3 \cdot 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_2\alpha_1 + b_2\beta_1 & \gamma_1 \\ a_1\alpha_2 + b_1\beta_2 & a_2\alpha_2 + b_2\beta_2 & \gamma_2 \\ a_1\alpha_3 + b_1\beta_3 & a_2\alpha_3 + b_2\beta_3 & \gamma_3 \end{vmatrix}$$

Ans.

Exercises on Multiplication of Determinants

Ex. 1. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}^2 = \begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix}.$$

Ex. 2. Show that

$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2-\lambda^2 & bc+a\lambda \\ ac+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$$

$= \lambda^3 (\lambda^2 + a^2 + b^2 + c^2)$

Ex. 3. If ω is one of the imaginary cube roots of unity, show that

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 & 1 \\ \omega^2 & \omega^3 & 1 & \omega \\ \omega^3 & 1 & \omega & \omega^2 \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix} = -27$$

(Gorakhpur 96, 93)

***Ex. 4.** Prove that the determinant

$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$ is a perfect square and find its value.

(Gorakhpur 92)

Ex. 5. Express the product of the following determinats as a single determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ and } \begin{vmatrix} u & r \\ v & s \end{vmatrix}$$

§ 4·16. Theorem. *If C_i be the cofactor of a_{ij} in the $n \times n$ matrix* $\mathbf{A} = [a_{ij}]$ *then* $|C_{ij}| = \{|a_{jk}|\}^{n-1}$

[**Note.** $|C_{ij}|$ is known as reciprocal of the determinant $|a_{ij}|$],

Proof. If $\mathbf{A} = [a_{ij}]$, then $\mathbf{A}' = [a'_{ki}]$, where $\mathbf{A}'$ is the transpose of $\mathbf{A}$ and

$$a'_{ki} = a_{ik}$$

Now $\mathbf{A}'.\ [C_{ij}] = [a'_{ki}]\ [C_{ij}] = [b_{kj}]$, say ...(i)

where $$b_{kj} = \sum_{i=1}^{n} a'_{ki} C_{ij} = \sum_{i=1}^{n} a_{ik} C_{ij}, \because a'_{ki} = a_{ik}$$

Also by § 4·09 Page 129 we know that

$$b_{kj} = \sum_{i=1}^{n} a_{ik}\, C_{ij} = 0, \text{ if } j \neq k$$
$$= |\mathbf{A}|, \text{ if } j = k.$$

$\therefore$ From (i) we conclude that for the product $\mathbf{A}'.\ [C_{ij}]$ *i.e.* $[b_{kj}]$ all the diagonal terms (for which $j = k$) are $|\mathbf{A}|$, whereas the nondiagonal terms (for which $j \neq k$) are zero.

i.e. $$\mathbf{A}'.\ [C_{ij}] = \begin{bmatrix} |\mathbf{A}| & 0 & \ldots & 0 \\ 0 & |\mathbf{A}| & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ .. & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & |\mathbf{A}| \end{bmatrix}$$

Hence $$|\mathbf{A}'.\ [C_{ij}]| = \begin{vmatrix} |\mathbf{A}| & 0 & \ldots & 0 \\ 0 & |\mathbf{A}| & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ .. & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & |\mathbf{A}| \end{vmatrix}$$

or $$|\mathbf{A}'|.\ |C_{ij}| = \{|\mathbf{A}|\}^n, \because |\mathbf{A_1}.\mathbf{A_2}| = |\mathbf{A_1}|.\ |\mathbf{A_2}|$$

or $$|\mathbf{A}|.\ |C_{ij}| = \{|\mathbf{A}|\}^n, \because |\mathbf{A}'| = |\mathbf{A}|$$

r $$|C_{ij}| = \{|\mathbf{A}|\}^{n-1}$$

§ 4·17. Complementary Minor of a Determinant.

Definition. If **B** is $r \times r$ submatrix of an $n \times n$ matrix **A**, then the determinant **E′** of **A** formed by removing the rows and columns of **A** containing the elements of **B** is called the completementary minor of **B**.

For example : In the matrix $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}$

the completemtary minor of the det $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is $\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$; **(Note)**

the completementary minor of $\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$ is a_4

and the complementary minor of $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_1 & d_1 \\ a_4 & d_4 \end{vmatrix}$ **(Note)**

§ 4·18. Laplace's Expansion of a determinant by the minors of first r columns.

If $|\mathbf{B}_i|$ is $r \times r$ minor of an $n \times n$ matrix **A** formed by the elements of the first r columns of **A** and $|\mathbf{B}'_i|$ is the complementary minor of $|\mathbf{B}_i|$, then

$$|\mathbf{A}| = \Sigma \pm |\mathbf{B_i}|.|\mathbf{B'}_i|,$$

where the summation is extended over all the possible $r \times r$ minors of **A** whch can be formed from the elements of the first r columns and + or –sign taken according as an even or odd number of interchanges of adjacent rows of **A** is required to bring the submatrix $\mathbf{B_i}$ into the first r rows of **A**.

The following solved examples explain the above theorem.

Ex. 1. Expand $\begin{vmatrix} a & x & y & a \\ x & 0 & 0 & y \\ y & 0 & 0 & x \\ a & y & x & a \end{vmatrix}$ **Laplace's expansion by the minors of the first two columns. Hence evaluate it.**

Sol. All the possible minors of the first two columns and their complementary minors are given by

$$|B_1| = \begin{vmatrix} a & x \\ x & 0 \end{vmatrix}; |B'_1| = \begin{vmatrix} 0 & x \\ x & a \end{vmatrix};$$

$$|B_2| = \begin{vmatrix} a & x \\ y & 0 \end{vmatrix}; |B'_2| = \begin{vmatrix} 0 & y \\ x & a \end{vmatrix};$$

$$|B_3| = \begin{vmatrix} a & x \\ a & y \end{vmatrix}; |B'_3| = \begin{vmatrix} 0 & y \\ 0 & x \end{vmatrix};$$

$$|B_4| = \begin{vmatrix} x & 0 \\ y & 0 \end{vmatrix}; |B'_4| = \begin{vmatrix} y & a \\ x & a \end{vmatrix};$$

$$|B_5| = \begin{vmatrix} x & 0 \\ a & y \end{vmatrix}; |B'_5| = \begin{vmatrix} y & a \\ 0 & x \end{vmatrix};$$

and $$|B_6| = \begin{vmatrix} y & 0 \\ a & y \end{vmatrix}; |B'_6| = \begin{vmatrix} y & a \\ 0 & y \end{vmatrix};$$

Therefore the given determinant

$$= \begin{vmatrix} a & x \\ x & 0 \end{vmatrix} . \begin{vmatrix} 0 & x \\ x & a \end{vmatrix} - \begin{vmatrix} a & x \\ y & 0 \end{vmatrix} . \begin{vmatrix} 0 & y \\ x & a \end{vmatrix} + \begin{vmatrix} a & x \\ a & y \end{vmatrix} . \begin{vmatrix} 0 & y \\ 0 & x \end{vmatrix}$$

$$+ \begin{vmatrix} x & 0 \\ y & 0 \end{vmatrix} . \begin{vmatrix} y & a \\ x & a \end{vmatrix} - \begin{vmatrix} x & 0 \\ a & y \end{vmatrix} . \begin{vmatrix} y & a \\ 0 & x \end{vmatrix} + \begin{vmatrix} y & 0 \\ a & y \end{vmatrix} . \begin{vmatrix} y & a \\ 0 & y \end{vmatrix} \quad ...(i)$$

The submatrix $\mathbf{B_2}$ requires one interchange of rows viz. of second and third rows to bring it into the first two rows therefore –sign is put before the product $|\mathbf{B_2}||\mathbf{B'}_2|$. Again the submatrix $\mathbf{B_3}$ requires two interchanges of rows to bring fourth row to the position of second row *i.e.* to bring $\mathbf{B_3}$ into first two rows, therefore + sign is put before the product $|\mathbf{B_3}||\mathbf{B'}_3|$.

Similarly $\mathbf{B_4}$ requires two interchanges, $\mathbf{B_5}$ requires three interchanges and $\mathbf{B_6}$ requires four interchanges, hence +, – and + sign are put before $|\mathbf{B_4}|.|\mathbf{B'}_4|$, $|\mathbf{B_5}|.|\mathbf{B'}_5|$ and $|\mathbf{B_6}|.|\mathbf{B'}_6|$ respectively.

Hence from (i) we have (expanding the determinants) the given determinants.

$$= (-x^2)(-x^2) - (-xy)(-xy) + (ay - ax)(0) + (0)(ay - ax) - (xy)(xy) + (y^2)(y^2)$$

$$= x^4 - 2x^2y^2 + y^4 = (x^2 - y^2)^2$$ **Ans.**

Ex. 2. Expand $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \mathbf{e} & \mathbf{f} & \mathbf{g} & \mathbf{h} \\ \mathbf{0} & \mathbf{0} & \mathbf{j} & \mathbf{k} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{m} \end{vmatrix}$ **by Laplace's expansion by the minors of the first two columns.**

Sol. All the possible minors of the first two columns and their complementary minor are given by :

$$|\mathbf{B}| = \begin{vmatrix} a & b \\ e & f \end{vmatrix}, \ |\mathbf{B}'_1| = \begin{vmatrix} j & k \\ l & m \end{vmatrix}$$

$|\mathbf{B_2}| = \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$, hence $|\mathbf{B}'_2|$ need not be calculated

Similarly $|\mathbf{B_3}| = \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$; $|\mathbf{B_4}| = \begin{vmatrix} e & f \\ 0 & 0 \end{vmatrix} = 0$;

$|\mathbf{B_5}| = \begin{vmatrix} e & f \\ 0 & 0 \end{vmatrix} = 0$ and $|\mathbf{B_6}| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$ and therefore their complementary minors need not be calculated.

Then the given determinant by Laplace's Expansion method

$$= \begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} j & k \\ l & m \end{vmatrix}$$ **Ans.**

Ex. 3. Expand $\begin{vmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{4} \\ \mathbf{15} & \mathbf{29} & \mathbf{2} & \mathbf{14} \\ \mathbf{16} & \mathbf{19} & \mathbf{3} & \mathbf{17} \\ \mathbf{33} & \mathbf{39} & \mathbf{8} & \mathbf{38} \end{vmatrix}$ **by Laplace's expansion by the minors of the first two columns.**

Sol. All the possible minors of the first two columns and their complementary minors are given by :

$$|\mathbf{B_1}| = \begin{vmatrix} 3 & 2 \\ 15 & 29 \end{vmatrix}, \ |\mathbf{B}'_1| = \begin{vmatrix} 3 & 17 \\ 8 & 38 \end{vmatrix}$$

$$|\mathbf{B_2}| = \begin{vmatrix} 3 & 2 \\ 16 & 19 \end{vmatrix}, \ |\mathbf{B}'_2| = \begin{vmatrix} 2 & 14 \\ 8 & 38 \end{vmatrix}$$

$$|\mathbf{B_3}| = \begin{vmatrix} 3 & 2 \\ 33 & 39 \end{vmatrix}, \ |\mathbf{B}'_3| = \begin{vmatrix} 2 & 14 \\ 3 & 17 \end{vmatrix}$$

$$|\mathbf{B_4}| = \begin{vmatrix} 15 & 29 \\ 16 & 19 \end{vmatrix}, \ |\mathbf{B}'_4| = \begin{vmatrix} 1 & 4 \\ 8 & 38 \end{vmatrix}$$

$$|\mathbf{B}_5| = \begin{vmatrix} 15 & 29 \\ 33 & 39 \end{vmatrix}, \ |\mathbf{B}'_5| = \begin{vmatrix} 1 & 4 \\ 3 & 17 \end{vmatrix}$$

$$|\mathbf{B}_6| = \begin{vmatrix} 16 & 19 \\ 33 & 39 \end{vmatrix}, \ |\mathbf{B}'_6| = \begin{vmatrix} 1 & 4 \\ 2 & 14 \end{vmatrix}$$

$\therefore$ The given determinat

$$= \begin{vmatrix} 3 & 2 \\ 15 & 29 \end{vmatrix} \bullet \begin{vmatrix} 3 & 17 \\ 8 & 38 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 16 & 19 \end{vmatrix} \bullet \begin{vmatrix} 2 & 14 \\ 8 & 38 \end{vmatrix}$$

$$+ \begin{vmatrix} 3 & 2 \\ 33 & 39 \end{vmatrix} \bullet \begin{vmatrix} 2 & 14 \\ 3 & 17 \end{vmatrix} + \begin{vmatrix} 15 & 29 \\ 16 & 19 \end{vmatrix} \bullet \begin{vmatrix} 1 & 4 \\ 8 & 38 \end{vmatrix}$$

$$- \begin{vmatrix} 15 & 29 \\ 33 & 39 \end{vmatrix} \bullet \begin{vmatrix} 1 & 14 \\ 3 & 17 \end{vmatrix} + \begin{vmatrix} 16 & 19 \\ 33 & 39 \end{vmatrix} \bullet \begin{vmatrix} 1 & 4 \\ 2 & 14 \end{vmatrix}$$

Ex. 4. Expand $\begin{vmatrix} a & 1 & 0 & 0 & 0 \\ b & a & 1 & 0 & 0 \\ 0 & b & a & 1 & 0 \\ 0 & 0 & b & a & 1 \\ 0 & 0 & 0 & b & a \end{vmatrix}$ **by Laplace's expansion by the minors of the first two columns. Hence evaluate it.**

Sol. All the possible minors of the first two columns and their complementary minors are given by :

$$|\mathbf{B}_1| = \begin{vmatrix} a & 1 \\ b & a \end{vmatrix}, \ |\mathbf{B}'_1| = \begin{vmatrix} a & 1 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix}$$

$$|\mathbf{B}_2| = \begin{vmatrix} a & 1 \\ 0 & b \end{vmatrix}, \ |\mathbf{B}'_2| = \begin{vmatrix} 1 & 0 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix}$$

$$|\mathbf{B}_3| = \begin{vmatrix} b & a \\ 0 & b \end{vmatrix}, \ |\mathbf{B}'_3| = \begin{vmatrix} 0 & 0 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix} = 0$$

All other minors of the first two columns are equal to zero as they have at least one row of zero.

Hence the given determinant

$$= \begin{vmatrix} a & 1 \\ b & a \end{vmatrix} \bullet \begin{vmatrix} a & 1 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix} - \begin{vmatrix} a & 1 \\ 1 & b \end{vmatrix} \bullet \begin{vmatrix} 1 & 0 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix} \qquad \text{...(i)}$$

(Note)

Now $\begin{vmatrix} a & 1 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix} = \begin{vmatrix} a & 1 \\ b & a \end{vmatrix} a - \begin{vmatrix} a & 1 \\ 0 & b \end{vmatrix} \bullet 1 + \begin{vmatrix} b & a \\ 0 & b \end{vmatrix} \cdot 0,$

expanding by the minors of first two columns.

$$= (a^2 - b)\, a - (ab) = a^3 - 2ab.$$

$\therefore$ From (i), the given determinant

$$= \begin{vmatrix} a & 1 \\ b & a \end{vmatrix} \cdot (a^3 - 2ab) - \begin{vmatrix} a & 1 \\ 0 & b \end{vmatrix} \cdot \begin{vmatrix} a & 1 \\ b & a \end{vmatrix},$$

expanding the last determinant with respect to R_1.

$$= (a^2 - b)(a^3 - 2ab) - (ab)(a^2 - b) = (a^2 - b)[a^3 - 3ab]$$
$$= a(a^2 - b)(a^2 - 3b).$$ **Ans.**

Exercises on § 4·18

Ex. 1. Expand $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$ by Laplace's method of expansion by the minors of the first two columns.

Ex. 2. Use Laplace's method of expansion of a determinant by means of its second minors to expand

$$\begin{vmatrix} -1 & 0 & 0 & l \\ 0 & -1 & 0 & m \\ 0 & 0 & -1 & n \\ p & q & r & -1 \end{vmatrix}$$

**§ 4·19. Solution of Linear Equations. (Cramer's Rule)

Let the n simultaneous equations in n unknown quantities $x_1, x_2, \ldots, x_n$ be

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n = k_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n = k_2$$
$$a_{31}x_1 + a_{32}x_2 + \ldots + a_{3j}x_j + \ldots + a_{3n}x_n = k_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n = k_i$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nj}x_j + \ldots + a_{nn}x_n = k_n$$

These equations can be written as

$$\sum_{j=1}^{n} a_{ij}\, x_j = k_i, \quad i = 1, 2, \ldots, n \qquad \ldots(i)$$

Let the determinant of the coefficients, $|\mathbf{A}| = |a_{ij}| \neq 0$.

Multiplying (i) by the cofactor of a_{ij} in $|a_{ij}|$ viz. C_{ij}, $i = 1, 2, 3, \ldots, n$ and summing with respect to i, we have

$$\sum_{j=1}^{n} a_{ij}\, C_{ij}\, x_j = \sum_{j=1}^{n} k_i\, C_{ij}$$

or $$|\mathbf{A}| \,.\, x_j = \sum_{j=1}^{n} k_i\, C_{ij}, \because |\mathbf{A}| = \sum_{j=1}^{n} a_{ij}\, C_{ij}$$

= determinant formed by replacing jth column of the det.
$|\mathbf{A}|$ by the constants $k_1, k_2, \ldots, k_n$. **(Note)**

$$= \begin{vmatrix} a_{11} & a_{12} \ldots a_{1j-1} & k_1 & a_{1j+1} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2j-1} & k_2 & a_{2j+1} \ldots a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} \ldots a_{ij-1} & k_i & a_{ij+1} \ldots a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} \ldots a_{nj-1} & k_n & a_{nj+1} \ldots a_{nn} \end{vmatrix}$$

or $$x_j = \frac{1}{|\mathbf{A}|} \cdot \begin{vmatrix} a_{11} & a_{12} \ldots a_{1j-1} & k_1 & a_{1j+1} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2j-1} & k_2 & a_{2j+1} \ldots a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} \ldots a_{nj-1} & k_2 & a_{nj+1} \ldots a_{nn} \end{vmatrix}$$

Solved Examples on § 4·19.

Ex. 1 (a). Solve the following equations by Cramer's Rule.

$$\mathbf{x + y + z = 1,\ ax + by + cz = k,\ a^2x + b^2y + c^2z = k^2.}$$

Sol. The given equations are $x + y + z = 1$

$$ax + by + cz = k$$

$$a^2x + b^2y + c^2z = k^2$$

$\therefore$ By Cramer's Rule we have

$$\frac{x}{\begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}}.$$

or $$\frac{x}{(k-b)(b-c)(c-k)} = \frac{y}{(a-k)(k-c)(c-a)} = \frac{z}{(a-b))(b-k)(k-a)}$$

$$= \frac{1}{(a-b)(b-c)(c-a)}$$ ...See Ex. 28 Page 145

$\therefore$ $$x = \frac{(k-b)(b-c)(c-k)}{(a-b)(b-c)(c-a)} = \frac{(k-b)(c-k)}{(b-c)(c-a)}$$

Similarly $y = \dfrac{(a-k)(k-c)}{(a-b)(b-c)}$ and $z = \dfrac{(b-k)(k-a)}{(b-c)(c-a)}$ **Ans.**

Ex. 1. (b) Solve the following equations by the method of determinants —

$$\mathbf{a_1x + b_1y + c_1z + d_1 = 0}$$

$$\mathbf{a_2x + b_2y + c_2z + d_2 = 0}$$

$$\mathbf{a_3x + b_3y + c_3z + d_3 = 0}$$

Sol. Do exacity as Ex. 1 (a) above

Ex. 2. Solve the equations :

$$\mathbf{x + y + z = 7;\ x + 2y + 3z = 16;\ x + 3y + 4z = 22}$$

Sol. Solving the equations by Cramer's Rule we get

$$\frac{x}{\begin{vmatrix} 7 & 1 & 1 \\ 16 & 2 & 3 \\ 22 & 3 & 4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 7 & 1 \\ 1 & 16 & 3 \\ 1 & 22 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 7 \\ 1 & 2 & 16 \\ 1 & 3 & 22 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}}. \quad \text{...(i)}$$

Now $\begin{vmatrix} 7 & 1 & 1 \\ 16 & 2 & 3 \\ 22 & 3 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 7 & 1 \\ 2 & 16 & 3 \\ 3 & 22 & 4 \end{vmatrix}$, interchanging C_1 and C_2.

$= -\begin{vmatrix} 1 & 7 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - 2R_1$ and $R_3 - 3R_1$ respectively.

$= -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$, expanding w.r to C_1

$= -(2-1) = -1$;

$\begin{vmatrix} 1 & 7 & 1 \\ 1 & 16 & 3 \\ 1 & 22 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 1 \\ 0 & 9 & 2 \\ 0 & 15 & 3 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_1$ respectively.

$= \begin{vmatrix} 9 & 2 \\ 15 & 3 \end{vmatrix}$, expanding w.r. to C_1.

$= 27 - 30 = -3$;

$\begin{vmatrix} 1 & 1 & 7 \\ 1 & 2 & 16 \\ 1 & 3 & 22 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 7 \\ 0 & 1 & 9 \\ 0 & 2 & 15 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_1$ respectively.

$= \begin{vmatrix} 1 & 9 \\ 2 & 15 \end{vmatrix}$

$= 15 - 18 = -3$

and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_2$ respectively.

$= \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$, expanding w.r. to C_1.

$= 1 - 2 = -1$.

$\therefore$ From (i) we have $\dfrac{x}{-1} = \dfrac{y}{-3} = \dfrac{z}{-3} = \dfrac{1}{-1}$

which gives $x = 1, y = 3, z = 3$. **Ans.**

***Ex. 3. Solve the equations (with the help of determinants)**

$$\mathbf{x + y + z = 1;\ x + 2y + 3z = 2;\ x + 4y + 9z = 4.}$$

Sol. The given equations are $x + y + z = 1$.

$$x + 2y + 3z = 2$$

$$x + 4y + 9z = 4$$

$\therefore$ By Cramer's Rule we have

$$\frac{x}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 4 & 4 & 3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 4 & 4 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}}$$

or $\quad \dfrac{x}{0} = \dfrac{y}{D} = \dfrac{z}{0} = \dfrac{1}{D}$, where $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$

This gives $x = 0.\left(\dfrac{1}{D}\right) = 0$; $y = D.\left(\dfrac{1}{D}\right) = 1$, $z = 0.\left(\dfrac{1}{D}\right) = 0$.

i.e. $\quad x = 0,\ y = 1,\ z = 0,$ **Ans.**

***Ex. 4. Solve the equations by determinants $3x + 5y - 7z = 13$, $4x + y - 12z = 6$, $2x + 9y - 3z = 20$** (*Purvanchal 97*)

Sol. The given equation are

$$3x + 5y - 7z = 13$$
$$4x + y - 12z = 6$$
$$2x + 9y - 3z = 20$$

$\therefore$ By Cramer's Rule, we have

$$\frac{x}{\begin{vmatrix} 13 & 5 & -7 \\ 6 & 1 & -12 \\ 20 & 9 & -3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & 13 & -7 \\ 4 & 6 & -12 \\ 2 & 20 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 5 & 13 \\ 4 & 1 & 6 \\ 2 & 9 & 20 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix}} \quad \text{...(i)}$$

Now $\begin{vmatrix} 13 & 5 & -7 \\ 6 & 1 & -12 \\ 20 & 9 & -3 \end{vmatrix} = \begin{vmatrix} -17 & 0 & 53 \\ 6 & 1 & -12 \\ -34 & 0 & 105 \end{vmatrix}$, adding $-5R_2$ to R_1 and $-9R_2$ to R_3

$= \begin{vmatrix} -17 & 53 \\ -34 & 105 \end{vmatrix}$, expanding w.r. to C_2

$= \begin{vmatrix} -17 & 53 \\ 0 & -1 \end{vmatrix}$, adding $-2R_1$ to R_2

$= (-17)(-1) - (0)(53) = 17;$

$\begin{vmatrix} 3 & 13 & -7 \\ 4 & 6 & -12 \\ 2 & 20 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -7 & -4 \\ 0 & -34 & -6 \\ 2 & 20 & -3 \end{vmatrix}$, adding $-2R_3$ to R_2 and $-R_3$ to R_1

$= \begin{vmatrix} 1 & -7 & -4 \\ 0 & -34 & -6 \\ 0 & 34 & 5 \end{vmatrix}$, adding $-2R_1$ to R_3

$= \begin{vmatrix} -34 & -6 \\ 34 & 5 \end{vmatrix}$, expanding w.r. to C_1

$$= \begin{vmatrix} 0 & -1 \\ 34 & 5 \end{vmatrix}, \text{ adding } R_2 \text{ to } R_1$$

$$= 0 \cdot 5 - (-1) \cdot 34 = 34;$$

$$\begin{vmatrix} 3 & 5 & 13 \\ 4 & 1 & 6 \\ 2 & 9 & 20 \end{vmatrix} = \begin{vmatrix} -17 & 0 & -17 \\ 4 & 1 & 6 \\ -34 & 0 & -34 \end{vmatrix}, \text{ adding } -5R_2 \text{ to } R_1 \text{ and } -9R_2 \text{ to } R_3$$

$$= \begin{vmatrix} -17 & -17 \\ -34 & -34 \end{vmatrix}, \text{ expanding w.r. to } C_2$$

$$= 0$$

And
$$\begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix} = \begin{vmatrix} -17 & 0 & 53 \\ 4 & 1 & -12 \\ -34 & 0 & 105 \end{vmatrix}, \text{ adding } -5R_2 \text{ to } R_1 \text{ and } -9R_2 \text{ to } R_3$$

$$= \begin{vmatrix} -17 & 53 \\ -34 & 105 \end{vmatrix}, \text{ expanding w.r. to } C_2$$

$$= \begin{vmatrix} -17 & 53 \\ 0 & -1 \end{vmatrix}, \text{ adding } -2R_1 \text{ to } R_2$$

$$= 17$$

$\therefore$ From (i), we get $\dfrac{x}{17} = \dfrac{y}{34} = \dfrac{z}{0} = \dfrac{1}{17}$

which gives $x = 1, y = 2, z = 0$ **Ans.**

Ex. 5. Solve the eqations $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$.

(*Purvanchal 94*)

Sol. Given equation are
$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

$\therefore$ By Cramer's Rule, we get

$$\frac{x}{\begin{vmatrix} 3 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 4 & 9 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 9 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}}. \qquad \text{...(i)}$$

Now
$$\begin{vmatrix} 3 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ -2 & 2 & 1 \\ -6 & 4 & 5 \end{vmatrix}, \text{ replacing } C_1, C_3, \text{ by } C_1 - 3C_2, C_3 - C_1 \text{ respectively.}$$

$$= -\begin{vmatrix} -2 & 1 \\ -6 & 5 \end{vmatrix} = -[-10 + 6] = 4, \qquad \text{...(ii)}$$

$$\begin{vmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1, R_3 - R_1 \text{ respectively.}$$

$$=\begin{vmatrix}1 & 2\\ 3 & 8\end{vmatrix}=8-6=2; \qquad \text{...(iii)}$$

$$\begin{vmatrix}1 & 1 & 3\\ 1 & 2 & 4\\ 1 & 4 & 6\end{vmatrix}=\begin{vmatrix}1 & 1 & 3\\ 0 & 1 & 1\\ 0 & 3 & 3\end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2-R_1,\ R_3-R_1 \text{ respectively.}$$

$$=\begin{vmatrix}1 & 1\\ 3 & 3\end{vmatrix}=0 \qquad \text{...(iv)}$$

And $$\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{vmatrix}=\begin{vmatrix}1 & 0 & 0\\ 1 & 1 & 2\\ 1 & 3 & 8\end{vmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2-C_1,\ C_3-C_1 \text{ respectively.}$$

$$=\begin{vmatrix}1 & 2\\ 3 & 8\end{vmatrix}=8-6=2 \qquad \text{...(v)}$$

$\therefore$ From (i), (ii), (iii), (iv) and (v) we have

$$\frac{x}{4}=\frac{y}{2}=\frac{z}{0}=\frac{1}{2} \quad \text{or} \quad x=2,\ y=1,\ z=0$$ **Ans.**

Ex. 6 (a). Using determinants, solve the simultaneous equations : $x+2y+3z=6$; $2x+4y+z=7$, $3x+2y+9z=14$. (*Purvanchal 90*)

Sol. By Cramer's Rule, we get

$$\frac{x}{\begin{vmatrix}6 & 2 & 3\\ 7 & 4 & 1\\ 14 & 2 & 9\end{vmatrix}}=\frac{}{\begin{vmatrix}1 & 6 & 3\\ 2 & 7 & 1\\ 3 & 14 & 9\end{vmatrix}}=\frac{z}{\begin{vmatrix}1 & 2 & 6\\ 2 & 4 & 7\\ 3 & 2 & 14\end{vmatrix}}=\frac{1}{\begin{vmatrix}1 & 2 & 3\\ 2 & 4 & 1\\ 3 & 2 & 9\end{vmatrix}} \qquad \text{...(i)}$$

Now $$\begin{vmatrix}6 & 2 & 3\\ 7 & 4 & 1\\ 14 & 2 & 9\end{vmatrix}=-\begin{vmatrix}2 & 6 & 3\\ 4 & 7 & 1\\ 2 & 14 & 9\end{vmatrix}, \text{ interchanging } C_1 \text{ and } C_2$$

$$=-\begin{vmatrix}2 & 6 & 3\\ 0 & -5 & -5\\ 0 & 8 & 6\end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2-2R_1 \text{ and } R_3-R_1 \text{ respectively.}$$

$$=-2\begin{vmatrix}-5 & -5\\ 8 & 6\end{vmatrix}, \text{ expanding with respect to } C_1$$

$$=-2[-30+40]=-20;$$

$$\begin{vmatrix}1 & 6 & 3\\ 2 & 7 & 1\\ 3 & 14 & 9\end{vmatrix}=\begin{vmatrix}1 & 6 & 3\\ 0 & -5 & -5\\ 0 & -4 & 0\end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2-2R_1,\ R_3-3R_1$$

$$=\begin{vmatrix}-5 & -5\\ -4 & 0\end{vmatrix}, \text{ expanding with respect to } C_1$$

$$=-20;$$

$$\begin{vmatrix}1 & 2 & 6\\ 2 & 4 & 7\\ 3 & 2 & 14\end{vmatrix}=\begin{vmatrix}1 & 2 & 6\\ 0 & 0 & -5\\ 0 & -4 & -4\end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2-2R_1,\ R_3-3R_1.$$

$$= \begin{vmatrix} 0 & -5 \\ -4 & -4 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= -20$$

and $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -5 \\ 3 & -4 & 0 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$, $C_3 - 3C_1$ respectively.

$$= \begin{vmatrix} 0 & -5 \\ -4 & 0 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$= -20$$

$\therefore$ From (i) we get $\dfrac{x}{-20} = \dfrac{y}{-20} = \dfrac{z}{-20} = \dfrac{1}{-20}$

or $x = 1, y = 1, z = 1.$ **Ans.**

Ex. 6 (b). Solve the following equation with the help of determinants $2x + y + z = 1$, $x - 2y - 3z = 1$, $3x + 2y - z = 5$. *(Purvanchal 96)*

Sol. By Carmer's Rule, we get

$$\frac{x}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 5 & 2 & -1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & -3 \\ 3 & 5 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 2 & 5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & -1 \end{vmatrix}} \quad ...(i)$$

Now $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 5 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -3 & -4 \\ 5 & -3 & -6 \end{vmatrix}$, replacing C_2, C_3, by $C_2 - C_1$ and $C_3 - C_1$ respectively

$$= \begin{vmatrix} -3 & -4 \\ -3 & -6 \end{vmatrix}, \text{ expanding w.r. to } R_1$$

$$= (-3)(-6) - (-3)(-4) = 18 - 12 = 6;$$

$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & -3 \\ 3 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 7 & 4 & 0 \\ 5 & 6 & 0 \end{vmatrix}$, replacing R_2, R_3 by $R_2 + 3R_1$ and $R_3 + R_1$ respectively

$$= \begin{vmatrix} 7 & 4 \\ 5 & 6 \end{vmatrix}, \text{ expanding w.r. to } C_3$$

$$= (7)(6) - (5)(4) = 42 = 20 = 22;$$

$\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -3 & 0 \\ -7 & -3 & 0 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1$, $R_3 - 5R_1$ respectively.

$$= \begin{vmatrix} -1 & -3 \\ -7 & -3 \end{vmatrix}, \text{ expanding w.r. to } C_3$$

$$= (-1)(-3) - (-7)(-3) = 3 - 21 = -18$$

Also $\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 7 & 1 & 0 \\ 5 & 3 & 0 \end{vmatrix}$, replacing R_2, R_3 by $R_2 + 3R_1$ and $R_3 + R_1$ respectively

$$= \begin{vmatrix} 7 & 1 \\ 5 & 3 \end{vmatrix}, \text{ expanding w.r. to } C_3$$

$$= 7 \cdot 3 - 5 \cdot 1 = 21 - 5 = 16$$

$\therefore$ From (i), we have $\frac{x}{6} = \frac{y}{22} = \frac{z}{-18} = \frac{1}{16}$

which gives $x = \frac{6}{16} = \frac{3}{8}, \; y = \frac{22}{16} = \frac{11}{8}, \; z = -\frac{18}{16} = -\frac{9}{8}$ **Ans.**

Ex. 7. Solve the equations :

$\mathbf{x + y + z + u = 1,}$ $\mathbf{ax + by + cz + du = k,}$

$\mathbf{a^2x + b^2y + c^2z + d^2u = k^2}$ **and** $\mathbf{a^3x + b^3y + c^3z + d^3u = k^3}$

Sol. The given equations are

$$x + y + z + u = 1$$

$$ax + by + cz + du = k,$$

$$a^2x + b^2y + c^2z + d^2u = k^2,$$

and

$$a^3x + b^3y + c^3z + d^3u = k^3$$

Solving these by Cramer's Rule, we get

$$\frac{x}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ k & b & c & d \\ k^2 & b^2 & c^2 & d^2 \\ k^3 & b^3 & c^3 & d^3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & k & c & d \\ a^2 & k^2 & c^2 & d^2 \\ a^3 & k^3 & c^3 & d^3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & k & d \\ a^2 & b^2 & k^2 & d^2 \\ a^3 & b^3 & k^3 & d^3 \end{vmatrix}}$$

$$= \frac{u}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & k \\ a^2 & b^2 & c^2 & k^2 \\ a^3 & b^3 & c^3 & k^3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}} \quad \text{...(i)}$$

Now $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ a^2 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix}$

replacing C_2, C_3 and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$ respectively.

$$= \begin{vmatrix} b-a & c-a & d-a \\ b^2-a^2 & c^2-a^2 & d^2-a^2 \\ b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 \\ b+a & c+a & d+a \\ b^2+ab+a^2 & c^2+ac+a^2 & d^2+ad+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 0 & 0 \\ b+a & c-b & d-b \\ b^2+ab+a^2 & c^2+ac-b^2 & d^2+ad-b^2 \\ & -ab & -ab \end{vmatrix},$$

replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively.

$$= (b-a)(c-a)(d-a)\begin{vmatrix} c-b & d-b \\ (c-b)(a+b+c) & (d-b)(a+b+d) \end{vmatrix},$$

expanding with respect to R_1,

$$= (a-b)(a-c)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & 1 \\ a+b+c & a+b+d \end{vmatrix}$$

$$= (a-b)(a-c)(a-d)(b-c)(d-b)[(a+b+d)-(a+b+c)]$$

$$= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

Similarly we can have $\begin{vmatrix} 1 & 1 & 1 & 1 \\ k & b & c & d \\ k^2 & b^2 & c^2 & d^2 \\ k^3 & b^3 & c^3 & d^3 \end{vmatrix} = (k-b)(k-c)(k-d)(b-c)(b-d)(c-d)$

$\therefore$ From (i), $x = \dfrac{(k-b)(k-c)(k-d)(b-c)(b-d)(c-d)}{(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)}$.

or $$x = \frac{(k-b)(k-c)(k-d)}{(a-b)(a-c)(a-d)}$$ **Ans.**

Similarly from (i) we can find the values of y, z and u.

Exercises on § 4·19

Solve the equations by Cramer's Rule :—

Ex. 1. $x-2y+z=-1$, $3x+y-2z=4$, $y-z=1$. **Ans.** $x=1$, $y=1$, $z=0$,

Ex. 2. $2x+3y-4z=2$, $3x-2y+5z=5$, $x+2y+3z=11$.

Ans. $x=\frac{10}{19}$, $y=\frac{50}{19}$, $z=\frac{33}{19}$.

Ex. 3. $x+2y-z=3$; $3x-y+z=8$, $x+y+z=0$.

Ans. $x=\frac{18}{7}$, $y=-\frac{5}{7}$, $z=-\frac{13}{7}$.

Ex. 4. $x+y+z=3$; $2x+3y+4z=9$, $x+2y-4z=-1$.

Ans. $x=1$, $y=1$, $z=1$

Ex. 5. $2x-y+3z=9$, $x+y+z=6$, $x-y+z=2$.

Ans. $x=1$, $y=2$, $z=3$.

Ex. 6. $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$.

Ans. $x=1$, $y=2$, $z=-1$

Ex. 7. $x_1+2x_2+3x_3+5=0$, $2x_1+x_2+x_3+7=0$. $x_1+x_2+x_3=0$.

Ans. $x_1=-7$, $x_2=19$, $x_3=-12$.

Ex. 8. $6x+y+2z=7$, $3x-y+4z=14$, $5x+2y-3z=-7$.

(*Purvanchal 91*)

Ans. $x=1$, $y=-3$, $z=2$

***Ex. 9.** $x+y+z=9,\ 2x+5y+7z=52,\ 2x+y-z=0$

Ans. $x=1,\ y=3,\ z=5$.

§ 4·20. Derivative of a determinant.

If some elements of the $n\times n$ matrix $\mathbf{A}=[a_{ij}]$ are differentiable functions of a variable x, then the derivative of $|\mathbf{A}|$ with respect to x *i.e.* $\dfrac{d}{dx}|\mathbf{A}|$ is the sum of n determinants formed by replacing in all possible ways the elements of one row (or column) of the det. $|\mathbf{A}|$ by their differential coefficients with respect to x.

The above procedure will be illustrated by the following examples —

Ex. 1. Find the derivative of the det. $\begin{vmatrix} x^3 & 2x+3 \\ 3x^2 & x^4 \end{vmatrix}$.

Sol. $\dfrac{d}{dx}\begin{vmatrix} x^3 & 2x+3 \\ 3x^2 & x^4 \end{vmatrix}=\begin{vmatrix} 3x^2 & 2 \\ 3x^2 & x^4 \end{vmatrix}+\begin{vmatrix} x^3 & 2x+3 \\ 6x & 4x^3 \end{vmatrix}$,

differentiating the elements of R_1 in the first det. whereas differentiating the elements of R_2 in the second det.

$$=[3x^6-6x^2]+[4x^6-6x(2x+3)]$$

$$=3x^6-6x^2+4x^6-12x^2-18x=7x^6-18x^2-18x.$$ **Ans.**

***Ex. 2. Find the derivative of** $\begin{vmatrix} x^2 & x^3 & 2 \\ 2x & 3x+1 & x^3 \\ 0 & 3x-2 & x^2+1 \end{vmatrix}$

Sol. The derivative of the given determinant

$$=\begin{vmatrix} 2x & 3x^2 & 0 \\ 2x & 3x+1 & x^3 \\ 0 & 3x-2 & x^2+1 \end{vmatrix}+\begin{vmatrix} x^2 & x^3 & 2 \\ 2 & 3 & 3x^2 \\ 0 & 3x-2 & x^2+1 \end{vmatrix}+\begin{vmatrix} x^2 & x^3 & 2 \\ 2x & 3x+1 & x^3 \\ 0 & 3 & 2x \end{vmatrix}$$

$$=\begin{vmatrix} 2x & 3x^2 & 0 \\ 0 & 3x+1-3x^2 & x^3 \\ 0 & 3x-2 & x^2+1 \end{vmatrix}+\begin{vmatrix} x^2 & x^3 & 2 \\ 2 & 5-3x & 2x^2-1 \\ 0 & 3x-2 & x^2+1 \end{vmatrix}$$

$$+\begin{vmatrix} x^2 & x^3 & 2 \\ 2x & 3x+1 & x^3 \\ 0 & 3 & 2x \end{vmatrix}$$, replacing R_2 of 1st det. by R_2-R_1 and R_2 of 2nd det. by R_2-R_3

$$=2x\begin{vmatrix} 3x+1-3x^2 & x^3 \\ 3x-2 & x^2+1 \end{vmatrix}+x^2\begin{vmatrix} 5-3x & 2x^2-1 \\ 3x-2 & x^2+1 \end{vmatrix}$$

$$-2\begin{vmatrix} x^3 & 2 \\ 3x-2 & x^2+1 \end{vmatrix}$$

$$+x^2\begin{vmatrix}3x+1 & x^3\\ 3 & 2x\end{vmatrix} - 2x\begin{vmatrix}x^3 & 2\\ 3 & 2x\end{vmatrix}, \text{ expanding det. w.r. to } C_1$$

$$= 2x\,[(x^2+1)(3x+1-3x^2)-x^3(3x-2)]+x^2\,[(5-3x)(x^2+1)$$
$$-(3x-2)(2x^2-1)]-2\,[x^3(x^2+1)-2(3x-2)]$$
$$+x^2\,[2x(3x+1)-3x^3]-2x\,[2x^4-6],$$

$= -30x^5+25x^4-4x^3+9x^2+26x-8$, on similifying. **Ans.**

Exercise on § 4·20

Ex. Find the derivative of $\begin{vmatrix}x^2-1 & x-1 & 1\\ x^4 & x^3 & 2x+5\\ x+1 & x^2 & x\end{vmatrix}$

Ans. $6x^5-5x^4-28x^3+9x^2+20x-2$

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1. Solve** $\begin{vmatrix}1+x & 2 & 3 & 4\\ 1 & 2+x & 3 & 4\\ 1 & 2 & 3+x & 4\\ 1 & 2 & 3 & 4+x\end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix}x+10 & 2 & 3 & 4\\ x+10 & 2+x & 3 & 4\\ x+10 & 2 & 3+x & 4\\ x+10 & 2 & 3 & 4+x\end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1+C_2+C_3+C_4$$

$$= (x+10)\begin{vmatrix}1 & 2 & 3 & 4\\ 1 & 2+x & 3 & 4\\ 1 & 2 & 3+x & 4\\ 1 & 2 & 3 & 4+x\end{vmatrix}, \text{ taking out } (x+10) \text{ common from } C_1$$

$$= (x+10)\begin{vmatrix}1 & 2 & 3 & 4\\ 0 & x & 0 & 0\\ 0 & 0 & x & 0\\ 0 & 0 & 0 & x\end{vmatrix}, \text{ replacing } R_2, R_3 \text{ and } R_4 \text{ by } R_2-R_1, R_3-R_1 \text{ and } R_4-R_1 \text{ respectively.}$$

$$= (x+10)\begin{vmatrix}x & 0 & 0\\ 0 & x & 0\\ 0 & 0 & x\end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (x+10)\,x\begin{vmatrix}x & 0\\ 0 & x\end{vmatrix}, \text{ expanding with respect to } R_1$$

$= (x+10)\,x\,(xx) = x^3\,(x+10).$ **Ans.**

Ex. 2. Show that $-(a+b+c)$ is a root of the equation

$$\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$$

Sol. The given equation can be written as

$$\begin{vmatrix} x+a+b+c & b & c \\ b+x+c+a & x+c & a \\ c+a+x+b & a & x+b \end{vmatrix} = 0, \text{ replacing } C_1 \text{ by } C_1+C_2+C_3 \text{ in the det.}$$

or $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+c & a \\ 1 & a & x+b \end{vmatrix} = 0, \text{ taking out the common factor from } C_1$$

or $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x+c-b & a-c \\ 0 & a-b & x+b-c \end{vmatrix} = 0, \text{ replacing } R_2, R_3 \text{ by } R_2-R_1, R_3-R_1$$

or $$(x+a+b+c)\begin{vmatrix} x+c-b & a-c \\ a-b & x+b-c \end{vmatrix} = 0, \text{ expanding with respect to } C_1$$

or $$(x+a+b+c)\,[(x+c-b)(x+b-c)-(a-c)(a-b)] = 0$$

or $$(x+a+b+c)(x^2+ab+bc+ca-a^2-b^2-c^2) = 0$$

This gives $x = -(a+b+c), \pm\sqrt{(a^2+b^2+c^2-ab-bc-ca)}$

Hence $-(a+b+c)$ is a root of the given equation.

Ex. 3. Show that $$\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma)$$

Sol. The given determinant

$$= \begin{vmatrix} x & l & m & 1 \\ \alpha-x & x-l & n-m & 0 \\ \alpha-x & \beta-l & x-m & 0 \\ \alpha-x & \beta-l & \gamma-m & 0 \end{vmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2-R_1, R_3-R_1, R_4-R_1 \text{ respectively.}$$

$$= -\begin{vmatrix} \alpha-x & x-l & n-m \\ \alpha-x & \beta-l & x-m \\ \alpha-x & \beta-l & \gamma-m \end{vmatrix}, \text{ expanding with respect to } C_4$$

$$= -(\alpha-x)\begin{vmatrix} 1 & x-l & n-m \\ 1 & \beta-l & x-m \\ 1 & \beta-l & \gamma-m \end{vmatrix}, \text{ taking out } (\alpha-x) \text{ common}$$

$$= (x-\alpha)\begin{vmatrix} 1 & x-l & n-m \\ 0 & \beta-x & x-n \\ 0 & \beta-x & \gamma-n \end{vmatrix}, \text{ replacing } R_2 \text{ and } R_3 \text{ by } R_2-R_1 \text{ and } R_3-R_1 \text{ respectively.}$$

$$= (x-\alpha)\begin{vmatrix} \beta-x & x-n \\ \beta-x & \gamma-n \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (x-\alpha)(\beta-x)\begin{vmatrix} 1 & x-n \\ 1 & \gamma-n \end{vmatrix} = (x-\alpha)(\beta-x)\,[(\gamma-n)-(x-n)]$$

$$= (x-\alpha)(\beta-x)(\gamma-x) = (x-\alpha)(x-\beta)(x-\gamma)$$ Hence proved.

****Ex. 4. Evaluate** $\begin{vmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} & \mathbf{z} \\ -\mathbf{x} & \mathbf{0} & \mathbf{c} & \mathbf{b} \\ -\mathbf{y} & -\mathbf{c} & \mathbf{0} & \mathbf{a} \\ -\mathbf{z} & -\mathbf{b} & -\mathbf{a} & \mathbf{0} \end{vmatrix}$

Sol. The given determinant

$$= \frac{1}{a}\begin{vmatrix} 0 & ax-by+cz & y & z \\ -x & 0-bc+cb & c & b \\ -y & -ac+0+ca & 0 & a \\ -z & -ab+ba+0 & -a & 0 \end{vmatrix}$$, replacing C_2 by $aC_2-bC_3+cC_4$. Here $(1/a)$ has been taken common due to aC_2 **(Note)**

$$= \frac{1}{a}\begin{vmatrix} 0 & ax-by+cz & y & z \\ -x & 0 & c & b \\ -y & 0 & 0 & a \\ -z & 0 & -a & 0 \end{vmatrix}$$

$$= -(ax-by+cz)(1/a)\begin{vmatrix} -x & c & b \\ -y & 0 & a \\ -z & -a & 0 \end{vmatrix}$$, expanding with respect to C_2.

$$= (ax+by+cz)(1/a)\begin{vmatrix} x & c & b \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$, taking out -1 common from C_1

$$= \frac{(ax-by+cz)}{a.a}\begin{vmatrix} ax-by+cz & ac-0-ca & ba-ab+0 \\ y & 0 & a \\ z & -a & 0 \end{vmatrix},$$

replacing R_1 by $aR_1-bR_2+cR_3$ and taking out $(1/a)$ common as before

$$= \frac{(ax-by+cz)}{a^2}\begin{vmatrix} ax-by+cz & 0 & 0 \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$

$$= \frac{(ax+by+cz)}{a^2}(ax-by+cz)\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}$$, expanding with respect to R_1

$$= \frac{(ax-by+cz)^2}{a^2}[0\cdot0-a.(-a)] = \frac{(ax-by+cz)^2}{a^2}\cdot[a^2]$$

$$= (ax-by+cz)^2$$ **Ans.**

Ex. 5. Show that $\begin{vmatrix} \mathbf{1} & \mathbf{a} & \mathbf{a^2} & \mathbf{a^3+bcd} \\ \mathbf{1} & \mathbf{b} & \mathbf{b^2} & \mathbf{b^3+cda} \\ \mathbf{1} & \mathbf{c} & \mathbf{c^2} & \mathbf{c^3+dab} \\ \mathbf{1} & \mathbf{d} & \mathbf{d^2} & \mathbf{d^3+abc} \end{vmatrix} = \mathbf{0}$ *(Gorakhpur 92; Kumaun 95)*

given that abcd ≠ 0.

Sol. The given determinant

$$=\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} \qquad \text{...(i)}$$

Now $\begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix}$

$=\dfrac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & bcda \\ c & c^2 & c^3 & cdab \\ d & d^2 & d^3 & dabc \end{vmatrix}$, multiplying R_1, R_2, R_3, R_4 by a, b, c, d respectively and dividing the result by $abcd$, where $abcd \neq 0$.

(Note)

$=\dfrac{1}{abcd}(abcd)\begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$, taking out $abcd$ common from C_4

$=-\begin{vmatrix} a & a^2 & 1 & a^3 \\ b & b^2 & 1 & b^3 \\ c & c^2 & 1 & c^3 \\ d & d^2 & 1 & d^3 \end{vmatrix}$, interchanging C_3 and C_4

$=(-1)^2\begin{vmatrix} a & 1 & a^2 & a^3 \\ b & 1 & b^2 & b^3 \\ c & 1 & c^2 & c^3 \\ d & 1 & d^2 & d^3 \end{vmatrix}$, interchanging C_2 and C_2

$=-\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$, interchanging C_1 and C_2, also $(-1)^3 = -1$.

Substituting this value in (i), the value of the given determinant is zero.

Ex. 6. Show that $\begin{vmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ x & y & z & -1 \end{vmatrix} = 1 - ax - by - cz$

(Gorakhpur 91)

Sol. The given determinant

$$= \begin{vmatrix} -1 & 0 & 0 & -a+0+0+a \\ 0 & -1 & 0 & 0-b+0+b \\ 0 & 0 & -1 & 0+0-c+c \\ x & y & z & ax+by+cz-1 \end{vmatrix}, \text{ replacing } C_4 \text{ by } aC_1+bC_2+cC_3+C_4$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ x & y & z & ax+by+cz-1 \end{vmatrix}$$

$$= (ax+by+cz-1) \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \text{ expanding with respect to } C_4$$

$$= -(ax+by+cz-1) \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (1-ax-by+cz)\,[(-1)(-1)-0\cdot 0]$$

$$= (1-ax-by-cz)\,[1] = (1-ax-by-cz).$$

Hence proved.

***Ex. 7. Prove that**

$$\begin{vmatrix} \mathbf{yz-x^2} & \mathbf{zx-y^3} & \mathbf{xy-z^2} \\ \mathbf{zx-y^2} & \mathbf{xy-z^2} & \mathbf{yz-x^2} \\ \mathbf{xy-z^2} & \mathbf{yz-x^2} & \mathbf{zx-y^2} \end{vmatrix} = \begin{vmatrix} \mathbf{v^2} & \mathbf{u^2} & \mathbf{u^2} \\ \mathbf{u^2} & \mathbf{v^2} & \mathbf{u^2} \\ \mathbf{u^2} & \mathbf{u^2} & \mathbf{v^2} \end{vmatrix} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{y} & \mathbf{z} & \mathbf{x} \\ \mathbf{z} & \mathbf{x} & \mathbf{y} \end{vmatrix}^2,$$

where $\mathbf{v^2 = x^2+y^2+z^2,\ u^2 = yz+zx+xy}$. *(Gorakhpur 90)*

Sol. $$\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2 = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$= \begin{vmatrix} x^2+y^2+z^2 & xy+yz+zx & xz+yx+zy \\ yx+zy+xz & y^2+z^2+x^2 & yz+zx+xy \\ zx+xy+yz & zy+xz+xy & z^2+x^2+y^2 \end{vmatrix}$$

$$= \begin{vmatrix} v^2 & u^2 & u^2 \\ u^2 & v^2 & u^2 \\ u^2 & u^2 & v^2 \end{vmatrix}, \quad \because\ u^2 = yz+zx+xy,\ v^2 = x^2+y^2+z^2, \text{ (given)} \qquad \text{...(i)}$$

Again from § 4·16 Page 169 we knwo that if C_{ij} be the cofactor of a_{ij} in the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, then

$$|C_{ij}| = \{|\mathbf{A}|\}^{n-1}$$

Here $n = 3$, so $|C_{ij}| = \{|\mathbf{A}|\}^2$, ...(ii)

where $$\mathbf{A} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$\therefore\ |C_{ij}| = \begin{vmatrix} zy-x^2 & xz-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$, where C_{ij} is the cofactor of a_{ij} in $|\mathbf{A}|$,

$\therefore$ From (ii), $\begin{vmatrix} zy-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2$

From (i) and this we have the required result.

Ex. 8. Write down as a determinant the product

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} & \mathbf{a} \end{vmatrix} \cdot \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{z} & \mathbf{x} & \mathbf{y} \\ \mathbf{y} & \mathbf{z} & \mathbf{x} \end{vmatrix}$$

Sol. Multiplying by the 'row-by-row' rule we get

$$\begin{vmatrix} ax+by+cz & az+bx+cy & ay+bz+cz \\ cx+ay+bz & cz+ax+by & cy+az+bx \\ bx+cy+az & bz+cx+ay & by+cz+ax \end{vmatrix}$$

$$= (a+b+c)(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ cx+ay+bz & cz+ax+by & cy+az+bx \\ bx+cy+az & bz+cx+ay & by+cz+ax \end{vmatrix}$$

replacing R_1 by $R_1+R_2+R_3$ and taking the common factor out.

Ex. 9. Expand $\begin{vmatrix} \mathbf{0} & \mathbf{1} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & \mathbf{0} & \mathbf{y} & \mathbf{x} \\ \mathbf{z} & \mathbf{w} & \mathbf{0} & \mathbf{0} \\ \mathbf{w} & \mathbf{z} & \mathbf{0} & \mathbf{0} \end{vmatrix}$ **by Laplace's Expansion's by the minors of the fi-st two columns.**

Sol. All the possible minors of the frist two columns and their complementary minors are given below :

$|\mathbf{B}_1| = \begin{vmatrix} 0 & 1 \\ z & w \end{vmatrix}$, $|\mathbf{B}'_1| = \begin{vmatrix} y & x \\ 0 & 0 \end{vmatrix} = 0$

$|\mathbf{B}_2| = \begin{vmatrix} 0 & 1 \\ w & z \end{vmatrix}$, $|\mathbf{B}'_2| = \begin{vmatrix} y & x \\ 0 & 0 \end{vmatrix} = 0$

$|\mathbf{B}_3| = \begin{vmatrix} z & w \\ w & z \end{vmatrix}$, $|\mathbf{B}'_3| = \begin{vmatrix} x & y \\ y & x \end{vmatrix}$

$\therefore$ The given determinat $= \begin{vmatrix} z & w \\ w & z \end{vmatrix} \cdot \begin{vmatrix} x & y \\ y & x \end{vmatrix}$,

the remaing minors or complementary minors are zero.

$$= (z^2 - w^2)(x^2 - y^2).$$ **Ans.**

****Ex. 10 (a). Show that** $\begin{vmatrix} \mathbf{a_1} & \mathbf{b_1} & \mathbf{c_1} \\ \mathbf{a_2} & \mathbf{b_2} & \mathbf{c_2} \\ \mathbf{a_3} & \mathbf{b_3} & \mathbf{c_3} \end{vmatrix}^2 = \begin{vmatrix} \mathbf{A_1} & \mathbf{B_1} & \mathbf{C_1} \\ \mathbf{A_2} & \mathbf{B_2} & \mathbf{C_2} \\ \mathbf{A_3} & \mathbf{B_3} & \mathbf{C_3} \end{vmatrix}$

where the capital letters denote the cofactors of the corresponding small letters. *(Gorakhpur 96, 92; Kanpur 93; Purvanchal 97)*

Sol. Let $|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Then $\Delta \times |\mathbf{A}|$

$$= \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_2A_1 + b_2B_1 + c_2C_1 & a_3A_1 + b_3B_1 + c_3C_1 \\ a_1A_2 + b_1B_2 + c_1C_2 & a_2A_2 + b_2B_2 + c_2C_2 & a_3A_2 + b_3B_2 + c_3C_2 \\ a_1A_3 + b_1B_3 + c_1C_3 & a_2A_3 + b_2B_3 + c_2C_3 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} |\mathbf{A}| & 0 & 0 \\ 0 & |\mathbf{A}| & 0 \\ 0 & 0 & |\mathbf{A}| \end{vmatrix}$$, since $a_1A_1 + b_1B_1 + c_1C_1 = |\mathbf{A}|$ etc. and $a_1A_2 + b_1B_2 + c_1C_2$etc.

(See example on Page 117)

Note. Students are to prove these in the examination.

$$= |\mathbf{A}| \bullet \begin{vmatrix} |\mathbf{A}| & 0 \\ 0 & |\mathbf{A}| \end{vmatrix}$$, expanding with respect to first row.

$$= |\mathbf{A}|\,[|\mathbf{A}| \bullet |\mathbf{A}| - 0 \cdot 0] = [|\mathbf{A}|]^3$$

or $\quad \Delta \times |\mathbf{A}| = \{|\mathbf{A}|\}^3$ or $\Delta = |\mathbf{A}|^2$

or $$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

Hence proved.

***Ex. 10 (b). Prove that**

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} & \mathbf{a} \end{vmatrix}^2 = \begin{vmatrix} \mathbf{a}^2 - \mathbf{bc} & \mathbf{b}^2 - \mathbf{ca} & \mathbf{c}^2 - \mathbf{ab} \\ \mathbf{c}^2 - \mathbf{ab} & \mathbf{a}^2 - \mathbf{bc} & \mathbf{b}^2 - \mathbf{ac} \\ \mathbf{b}^2 - \mathbf{ca} & \mathbf{c}^2 - \mathbf{ab} & \mathbf{a}^2 - \mathbf{bc} \end{vmatrix}$$ *(Gorakhpur 90)*

Sol. We know [See Ex. 10 (a) above] that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix},$$

where capital letters denote the cofactors of the corresponding small letters.

$$\therefore \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2 = \begin{vmatrix} A & B & C \\ C & A & B \\ B & C & A \end{vmatrix}, \qquad \text{...(i)}$$

where A, B, C are the cofactors of a, b, c respectively in the determinant on the left.

$$\therefore A = \begin{vmatrix} a & b \\ c & a \end{vmatrix} = a^2 - bc,\ B = -\begin{vmatrix} c & b \\ b & a \end{vmatrix} = b^2 - ac$$

and $$C = \begin{vmatrix} c & a \\ b & c \end{vmatrix} = c^2 - ab$$ **(Note)**

Hence from (i) we get

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2 = \begin{vmatrix} a^2-bc & b^2-ca & c^2-ab \\ c^2-ab & a^2-bc & b^2-ac \\ b^2-ca & c^2-ab & a^2-bc \end{vmatrix}$$ Hence proved.

Ex. 11. Solve the equation $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$. *(Meerut 97)*

Sol. Given that $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 3x-2 & 3 & 3 \\ 3x-2 & 3x-8 & 3 \\ 3x-2 & 3 & 3x-8 \end{vmatrix} = 0$, replacing C_1 by $C_1 + C_2 + C_3$

$\Rightarrow (3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3x-8 & 3 \\ 1 & 3 & 3x-8 \end{vmatrix} = 0$, taking out $(3x-2)$common.

$\Rightarrow (3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3x-11 & 0 \\ 0 & 0 & 3x-11 \end{vmatrix} = 0$, replacing R_2, R_3 by $R_2 - R_1$, $R_3 - R_1$ respectively.

$\Rightarrow (3x-2)\begin{vmatrix} 3x-11 & 0 \\ 0 & 3x-11 \end{vmatrix} = 0$, expanding with respect to C_1

$\Rightarrow (3x-2)(3x-11)^2 = 0$

$\Rightarrow x = 2/3$ or $11/3$. **Ans.**

Ex. 12. Prove that the value of determinant

$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ **is independent of x.**

Sol. The given determinant

$= \begin{vmatrix} x+1 & 1 & a-1 \\ x+2 & 1 & b-2 \\ x+3 & 1 & c-3 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively

$= \begin{vmatrix} x+1 & 1 & a-1 \\ 1 & 0 & b-a-1 \\ 2 & 0 & c-a-2 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1$, $R_3 - R_1$ respectively.

$$= -\begin{vmatrix} 1 & x+1 & a-1 \\ 0 & 1 & b-a-1 \\ 0 & 2 & c-a-2 \end{vmatrix}, \text{ interchanging } C_1 \text{ and } C_2$$

$$= -\begin{vmatrix} 1 & b-a-1 \\ 2 & c-a-2 \end{vmatrix} = -[(c-a-2)-2(b-a-1)]$$

$$= -[c-a-2-2b+2a+2] = -a+2b-c, \text{ which is independent of } x.$$

Hence proved.

***Ex. 13. Give correct answer to the following :**

The value of the determinant $\begin{vmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{vmatrix}$ **is (A) − 1, (B) 1, (C) 0, (D) 4.**

Sol. The correct answer is (*C*) *i.e.* 0, since replacing C_1 by $C_1 + C_2 + C_3 + C_4$ we find that all the elements of C_1 are zero. Hence the value of the given determinant is zero.

Ex. 14. Show that $(a+b+c)$ and $(a^2+b^2+c^2)$ are factors of determinant $\begin{vmatrix} a^2 & (b+c)^2 & bc \\ b^2 & (c+a)^2 & ca \\ c^2 & (a+b)^2 & ab \end{vmatrix}$ **and find the remaining factors.**

Sol. The given determinant

$$= \begin{vmatrix} a^2 & (b^2+c^2+2bc)+a^2 & bc \\ b^2 & (c^2+a^2+2ca)+b^2 & ca \\ c^2 & (a^2+b^2+2ab)+c^2 & ab \end{vmatrix}, \text{ replacing } C_2 \text{ by } C_2 + C_1$$

$$= \begin{vmatrix} a^2 & b^2+c^2+a^2 & bc \\ b^2 & c^2+a^2+b^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix}, \text{ replacing } C_2 \text{ by } C_2 - 2C_3$$

$$= (a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}, \text{ taking out } (a^2+b^2+c^2) \text{ common from } C_2$$

$$= \frac{a^2+b^2+c^2}{abc}\begin{vmatrix} a^3 & a & abc \\ b^3 & b & bca \\ c^3 & c & cab \end{vmatrix}, \text{ taking } 1/a\ 1/b,\ 1/c \text{ common from } R_1, R_2 \text{ and } R_3 \text{ respectively.}$$

(Note)

$$= \frac{a^2+b^2+c^2}{abc} \times abc \begin{vmatrix} a^3 & a & 1 \\ b^3 & b & 1 \\ c^3 & c & 1 \end{vmatrix}, \text{ taking out } abc \text{ common from } C_3.$$

Now proceed as in Ex. 31 Page 146.

Ex. 15. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$

Sol. The given determinant

$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$, taking out a, b, c common from C_1, C_2, C_3 respectively

$= abc \begin{vmatrix} a+c & c & a+c \\ a+2b & b & a \\ 2b+c & b+c & c \end{vmatrix}$, replacing C_1 by $C_1 + C_2$

$= abc \begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$, replacing C_1 by $C_1 - C_3$

$= abc \begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 0 & c & c-a \end{vmatrix}$, replacing R_3 by $R_3 - R_2$

$= -2ab^2c \begin{vmatrix} c & a+c \\ c & c-a \end{vmatrix}$, expanding with respect to C_1

$= -2ab^2c \begin{vmatrix} 0 & 2a \\ c & c-a \end{vmatrix}$, replacing R_1 by $R_1 - R_2$

$= -2ab^2c\,(-2ac)$, expanding the determinant.

$= 4a^2b^2c^2.$ Hence proved.

***Ex. 16. Solve** $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$

Sol. The given equation is

$\begin{vmatrix} x+2 & -1 & -2 \\ 2x+3 & -x-2 & -2x-4 \\ 3x+5 & -x-2 & x+2 \end{vmatrix} = 0$, replacing C_2, C_3 by $C_2 - 2C_1$, $C_3 - 3C_1$ respectively.

or $\begin{vmatrix} x+2 & -1 & -2 \\ 2x+3 & -x-2 & -2x-4 \\ x+2 & 0 & 3x+6 \end{vmatrix} = 0$, replacing R_3 by $R_3 - R_2$

or $\begin{vmatrix} x+2 & -1 & 0 \\ 2x+3 & -x-2 & 0 \\ x+2 & 0 & 3x+6 \end{vmatrix} = 0$, replacing C_3 by $C_3 - 2C_2$

or $(3x+6) \begin{vmatrix} x+2 & -1 \\ 2x+3 & -x-2 \end{vmatrix} = 0$, expanding with respect to C_3

or $(3x+6)\,[-(x+2)^2 + (2x+3)] = 0$

or $(3x+6)(x^2+2x+1)=0$ or $(3x+6)(x+1)^2=0$

or $x=-1,-2$ **Ans.**

***Ex. 17. Prove that** $\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - x f'(x)$,

where $f(x)=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ and $f'(x)$ is the first derivative of $f(x)$ with respect to x.

Sol. The given determinant

$$= \begin{vmatrix} \alpha & x & 0 & 0 \\ x & \beta & x-\beta & x-\beta \\ x & x & \gamma-x & 0 \\ x & x & 0 & \delta-x \end{vmatrix}, \text{ replacing } C_3 \text{ and } C_4 \text{ by } C_3-C_2 \text{ and } C_4-C_2 \text{ respectively.}$$

$$= \alpha \begin{vmatrix} \beta & x-\beta & x-\beta \\ x & \gamma-x & 0 \\ x & 0 & \delta-x \end{vmatrix} - x \begin{vmatrix} x & x-\beta & x-\beta \\ x & \gamma-x & 0 \\ x & 0 & \delta-x \end{vmatrix}, \text{ expanding w.r. to } R_1$$

$$= \alpha \begin{vmatrix} \beta & x-\beta & 0 \\ x & \gamma-x & x-\gamma \\ x & 0 & \delta-x \end{vmatrix} - x^2 \begin{vmatrix} 1 & x-\beta & 0 \\ 1 & \gamma-x & x-\gamma \\ 1 & 0 & \delta-x \end{vmatrix}, \text{ replacing } C_3 \text{ by } C_3-C_2 \text{ in each determinant}$$

$$= \alpha\beta \begin{vmatrix} \gamma-x & x-\gamma \\ 0 & \delta-x \end{vmatrix} - \alpha(x-\beta) \begin{vmatrix} x & x-\gamma \\ x & \delta-x \end{vmatrix} - x^2 \begin{vmatrix} \gamma-x & x-\gamma \\ 0 & \delta-x \end{vmatrix}$$
$$+ x^2(x-\beta) \begin{vmatrix} 1 & x-\gamma \\ 1 & \delta-x \end{vmatrix}, \text{ expanding each det. w.r. to } R_1$$

$$= (\alpha\beta - x^2)(\gamma-x)(\delta-x) - (x-\beta)\,x\,(\alpha-x)\,[(\delta-x)-(x-\gamma)]$$
$$= (\alpha\beta - x^2)(x-\gamma)(x-\delta) - x(x-\alpha)(x-\beta)(x-\delta)$$
$$- x(x-\alpha)(x-\beta)(x-\gamma)$$
$$= (x^2 - \alpha x - \beta x + \alpha\beta - x^2 + \beta x - x^2 + \alpha x)(x-\gamma)(x-\delta)$$
$$- x(x-\alpha)(x-\beta)(x-\delta) - x(x-\alpha)(x-\beta)(x-\gamma) \quad \text{(Note)}$$
$$= [(x-\alpha)(x-\beta) - x(x-\beta) - x(x-\alpha)](x-\gamma)(x-\delta)$$
$$- x(x-\alpha)(x-\beta)(x-\delta) - x(x-\alpha)(x-\beta)(x-\gamma) \quad \text{(Note)}$$
$$= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) - x[(x-\beta)(x-\gamma)(x-\delta)$$
$$+ (x-\alpha)(x-\gamma)(x-\delta) + (x-\alpha)(x-\beta)(x-\delta)$$
$$+ (x-\alpha)(x-\beta)(x-\gamma)]$$

$= f(x) - x f'(x)$, where $f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ Hence proved.

Ex. 18. Prove that $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$ **= 3, where ω is one of the imaginary cube roots of unity.**

Sol. If ω be one of the imaginary cube roots of unity, then

$$\omega^3 = 1 \quad \text{and} \quad 1 + \omega + \omega^2 = 0 \qquad \text{...(i)}$$

Now the given determinant

$$= \begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}, \text{ from (i) using } \omega^3 = 1$$

$$= \begin{vmatrix} 0 & 1 & \omega^2 \\ 0 & 1 & \omega \\ \omega^2 - \omega & \omega & 1 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 - C_2$$

$$= (\omega^2 - \omega) \begin{vmatrix} 1 & \omega^2 \\ 1 & \omega \end{vmatrix}, \text{ expanding with respect to } C_1$$

$= (\omega^2 - \omega)(\omega - \omega^2) = \omega^3 - \omega^4 - \omega^2 + \omega^3 = 1 - \omega - \omega^2 + 1, \;\because\; \omega^3 = 1$

$= 2 - (\omega + \omega^2) = 2 - (-1), \because 1 + \omega + \omega^2 = 0 \text{ or } \omega + \omega^2 = -1$

$= 2 + 1 = 3.$ Hence proved

****Ex. 19. Prove that the determinant**

$$\begin{vmatrix} 1 & \cos(\beta - \alpha) & \cos(\gamma - \alpha) \\ \cos(\alpha - \beta) & 1 & \cos(\gamma - \beta) \\ \cos(\alpha - \gamma) & \cos(\beta - \gamma) & 1 \end{vmatrix}$$

is a perfect square (of a determinant) and find its value. (*Gorakhpur 94*)

Sol. The given determinant

$$= \begin{vmatrix} \cos\alpha\cos\alpha + \sin\alpha\sin\alpha & \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma \\ \cos\beta\cos\alpha + \sin\beta\sin\alpha & \cos\beta\cos\beta + \sin\beta\sin\beta & \cos\beta\cos\gamma + \sin\beta\sin\gamma \\ \cos\gamma\cos\alpha + \sin\gamma\sin\alpha & \cos\gamma\cos\beta + \sin\gamma\sin\beta & \cos\gamma\cos\gamma + \sin\gamma\sin\gamma \end{vmatrix}$$

(Note)

The element in the first row and first column is $\cos\alpha\cos\alpha + \sin\alpha\sin\alpha$, which can be written as

$$(\cos\alpha)(\cos\alpha) + (\sin\alpha)(\sin\alpha) + 0\cdot 0 \qquad \textbf{(Note)}$$

Similarly the element in the first row and second column is $\cos\alpha\cos\beta + \sin\alpha\sin\beta$, which can be writter as

$$(\cos\alpha)(\cos\beta) + (\sin\alpha)(\sin\beta) + 0\cdot 0, \qquad \textbf{(Note)}$$

Proceeding in this way we can write the given determinant

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}^2, \text{ hence a perfect square of a determinant}$$

$= 0$, since the value of this determinant is zero as all the elements of one of its columns are zero.

***Ex. 20. (a) Show that** $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \times \begin{vmatrix} \alpha+i\beta & \gamma+i\delta \\ -\gamma+i\delta & \alpha-i\beta \end{vmatrix}$

can be expressed as $\begin{vmatrix} A+iB & C+iD \\ -C+iD & A-iB \end{vmatrix}$

Hence prove the Euler's Theorem, 'the product of two sums of four squares each is equal to the sum of the four squares'.

Sol. $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \times \begin{vmatrix} \alpha+i\beta & \gamma+i\delta \\ -\gamma+i\delta & \alpha-i\beta \end{vmatrix}$

$$= \begin{vmatrix} (a+ib)(\alpha+i\beta) + (c+id)(\gamma+i\delta) & (-c+id)(\alpha+i\beta) + (a-ib)(\gamma+i\delta) \\ (a+ib)(-\gamma+i\delta) + (c+id)(\alpha-i\beta) & (-c+id)(-\gamma+i\delta) + (a-ib)(\alpha-i\beta) \end{vmatrix}$$

$$= \begin{vmatrix} (a\alpha-b\beta+c\gamma-d\delta) + i(a\beta+b\alpha+c\delta+d\gamma) & (-c\alpha-d\beta+a\gamma+b\delta) + i(-c\beta+d\alpha+a\delta-b\gamma) \\ (-a\gamma-b\delta+c\alpha+d\beta) + i(a\delta-b\gamma-c\beta+d\alpha) & (a\alpha-b\beta+c\gamma-d\delta) + i(-a\beta-b\alpha-c\delta-d\gamma) \end{vmatrix}$$

$$= \begin{vmatrix} A+iB & C+iD \\ -C+iD & A-iB \end{vmatrix}, \quad \text{...(i)}$$

where $A = a\alpha - b\beta + c\gamma - d\delta$, $B = a\beta + b\alpha + c\delta + d\gamma$,

$C = a\gamma + b\delta - c\alpha - d\beta$, $D = a\delta - b\gamma - c\beta + d\alpha$. ...(ii)

Now $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$= (a+ib)(a-ib) - (c+id)(-c+id)$

$= (a^2 - i^2b^2) - (i^2d^2 - c^2) = a^2 + b^2 + c^2 + d^2$

Similaryly $\begin{vmatrix} \alpha+i\beta & \gamma+i\delta \\ -\gamma+i\delta & \alpha-i\beta \end{vmatrix} = \alpha^2+\beta^2+\gamma^2+\delta^2$

and $\begin{vmatrix} A+iB & c+iD \\ -C+iD & A-iB \end{vmatrix} = A^2+B^2+C^2+D^2$

$\therefore$ From (i) we have

$(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2) = (A^2+B^2+C^2+D^2)$ ...(iii)

i.e. product of two sums of four squares each is equal to the sum of four squares.

Hence proved.

Ex. 20 (b). With the help of determinants express the following as a sum of four squares

$$(1^2+2^2+3^2+4^2)(5^2+6^2+7^2+8^2)$$

Sol. As in Ex. 20 (a) above we can show that,

$(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2) = (A^2+B^2+C^2+D^2)$ ...(i) where A, B, C, D are given by Ex. 20 (a) result (ii).

Now let $a=2,\ b=2,\ c=3,\ d=4,\ \alpha=5,\ \beta=6,\ \gamma=7,\ \delta=8$

Then $A = a\alpha - b\beta + c\gamma - d\delta = 1\,(5) - 2\,(6) + 3\,(7) - 4\,(8)$

$= 5 - 12 + 21 - 32 = -18.$

$B = a\beta + b\alpha + c\delta + d\gamma = 1\,(6) + 2\,(5) + 3\,(8) + 4\,(7)$

$= 6 + 10 + 24 + 28 = 68,$

$C \equiv a\gamma + b\delta - c\alpha - d\beta = 1\,(7) + 2\,(8) - 3\,(5) - 4\,(6)$

$= 7 + 6 - 15 - 24 = -16$

$D = a\delta - b\gamma - c\beta + d\alpha = 1\,(8) - 2\,(7) - 3\,(6) + 4\,(5)$

$= 8 - 14 - 18 + 20 = -4.$

From (I) above we have $(1^2 + 2^2 + 3^2 + 4^2)(5^2 + 6^2 + 7^2 + 8^2)$

$$= (-18)^2 + (68)^2 + (-16)^2 + (-4)^2$$

$$= (18)^2 + (68)^2 + (16)^2 + (4)^2.$$ **Ans.**

Ex. 21. Evaluate $\begin{vmatrix} \mathbf{a} & -\mathbf{a} & -\mathbf{a} & -\mathbf{a} \\ \mathbf{b} & \mathbf{b} & -\mathbf{b} & -\mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{c} & -\mathbf{c} \\ \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{d} \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} a & 0 & 0 & 0 \\ b & 2b & 0 & 0 \\ c & 2c & 2c & 0 \\ d & 2d & 2d & 2d \end{vmatrix}$, replacing C_2, C_3 and C_4 by $C_2 + C_1, C_3 + C_1$ and $C_4 + C_1$ respectively

$= a \begin{vmatrix} 2b & 0 & 0 \\ 2c & 2c & 0 \\ 2d & 2d & 2d \end{vmatrix}$, expanding with respect to R_1

$= 2ab \begin{vmatrix} 2c & 0 \\ 2d & 2d \end{vmatrix}$, expanding with respect to R_1

$= 2ab\,[2c \times 2d - (2d) \times 0] = 8abcd.$ **Ans.**

***Ex. 22. Prove that** $\begin{vmatrix} \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{x} \\ \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{y} \\ \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{z} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{0} \end{vmatrix} = (\mathbf{x} - \mathbf{2y} + \mathbf{z})^2$ *(Gorakhpur 94; Kanpur 94)*

Sol. The given determinant

$= \begin{vmatrix} 10 & 5 & 6 & x \\ 12 & 6 & 7 & y \\ 14 & 7 & 8 & z \\ x+z & y & z & 0 \end{vmatrix}$, replacing C_1 by $C_1 + C_3$

$= \begin{vmatrix} 0 & 5 & 6 & x \\ 0 & 6 & 7 & y \\ 0 & 7 & 8 & z \\ x - 2y + z & y & z & 0 \end{vmatrix}$, replacing C_1 by $C_1 - 2C_2$

$$= -(x-2y+z)\begin{vmatrix} 5 & 6 & x \\ 6 & 7 & y \\ 7 & 8 & z \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= -(x-2y+z)\begin{vmatrix} 12 & 14 & x+z \\ 6 & 7 & y \\ 7 & 8 & z \end{vmatrix}, \text{ replacing } R_1 \text{ by } R_1+R_2$$

$$= -(x-2y+z)\begin{vmatrix} 0 & 0 & x+z-2y \\ 6 & 7 & y \\ 7 & 8 & z \end{vmatrix}, \text{ replacing } R_1 \text{ by } R_1-2R_2$$

$$= -(x-2y+z)^2\begin{vmatrix} 6 & 7 \\ 7 & 8 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$= -(x-2y+z)^2\,[48-49], \text{ expanding the det.}$$

$$= (x-2y+z)^2.$$

Hence proved.

Ex. 23. Evaluate $\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ l & 0 & c & -b \\ m & -c & 0 & a \\ n & b & -a & 0 \end{vmatrix}$ (*Gorakhpur 95; Kanpur 90*)

Sol. The given determinant

$$= \frac{1}{a}\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ al & 0 & ac & -ab \\ m & -c & 0 & a \\ n & b & -a & 0 \end{vmatrix}, \text{ taking } (1/a) \text{ common from } R_2$$

(Note)

$$= \frac{1}{a}\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ al+bm+cm & 0 & 0 & 0 \\ m & -c & 0 & a \\ n & b & -a & 0 \end{vmatrix}, \text{ replacing } R_2 \text{ by } R_2+bR_3+cR_1$$

$$= -\frac{1}{a}(al+bm+cn)\begin{vmatrix} \alpha & \beta & \gamma \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix}, \text{ expanding with respect to } R_2$$

$$= -\frac{1}{a^2}(al+bm+cn)\begin{vmatrix} a\alpha & \beta & \gamma \\ -ac & 0 & a \\ ab & -a & 0 \end{vmatrix}, \text{ taking out } 1/a \text{ common from } C_1$$

$$= -\frac{1}{a^2}(al+bm+cn)\begin{vmatrix} a\alpha+b\beta+c\gamma & \beta & \gamma \\ 0 & 0 & a \\ 0 & -a & 0 \end{vmatrix},$$

replacing C_1 by $C_1+bC_2+cC_3$

$$= -\frac{1}{a^2}(al+bm+cn)(a\alpha+b\beta+c\gamma)\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$= -(1/a^2)(al+bm+cn)(a\alpha+b\beta+c\gamma)a^2$,

$= -(al+bm+cn)(a\alpha+b\beta+c\gamma)$. **Ans.**

Ex. 24. Prove that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}$

$= -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha-\delta)(\beta-\delta)(\gamma-\delta)$

Hence evaluate $\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}$, **where** $s_r = a^r+\beta^r+\gamma^r+\delta^r$

Sol. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ \alpha & \beta-\alpha & \gamma-\alpha & \delta-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 & \delta^2-\alpha^2 \\ \alpha^3 & \beta^3-\alpha^3 & \gamma^3-\alpha^3 & \delta^3-\alpha^3 \end{vmatrix}$, replacing C_2, C_3, C_4 by C_2-C_1, C_3-C_1 and C_4-C_1

$= (\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\begin{vmatrix} 1 & 1 & 1 \\ \beta+\alpha & \gamma+\alpha & \delta+\alpha \\ \beta^2+\alpha^2+\alpha\beta & \gamma^2+\alpha^2+\alpha\gamma & \delta^2+\alpha^2+\alpha\delta \end{vmatrix}$,

expanding w.r. to R_1 and taking out common factors.

$= (\alpha-\beta)(\gamma-\alpha)(\alpha-\delta)\begin{vmatrix} 1 & 0 & 0 \\ \beta+\alpha & \gamma-\beta & \delta-\beta \\ \beta^2+\alpha^2+\alpha\beta & \gamma^2+\alpha\gamma-\beta^2-\alpha\beta & \delta^2+\alpha\delta-\beta^2-\alpha\beta \end{vmatrix}$,

replacing C_2, C_3 by C_2-C_1, C_3-C_1

$= (\alpha-\beta)(\alpha-\delta)(\gamma-\alpha)\begin{vmatrix} 1 & 0 & 0 \\ \beta+\alpha & \gamma-\beta & \delta-\beta \\ \beta^2+\alpha^2+\alpha\beta & (\gamma-\beta)(\alpha+\beta+\gamma) & (\delta-\beta)(\alpha+\beta+\delta) \end{vmatrix}$.

$= (\alpha-\beta)(\alpha-\delta)(\gamma-\alpha)(\beta-\gamma)(\beta-\delta)\begin{vmatrix} 1 & 1 \\ \alpha+\beta+\gamma & \alpha+\beta+\delta \end{vmatrix}$,

expanding w.r.t. to R_1

$= (\alpha-\beta)(\alpha-\delta)(\gamma-\alpha)(\beta-\gamma)(\beta-\delta)[(\alpha+\beta+\delta)-(\alpha+\beta+\gamma)]$

$$= -(\alpha-\beta)(\alpha-\delta)(\gamma-\alpha)(\beta-\gamma)(\beta-\delta)(\gamma-\delta). \quad \text{...(i)}$$

Hence proved.

Squaring both sides of (i), we get

$$(\alpha-\beta)^2(\alpha-\delta)^2(\gamma-\alpha)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}$$

(Note)

$$= \begin{vmatrix} 1+1+1+1 & \alpha+\beta+\gamma+\delta & \alpha^2+\beta^2+\gamma^2+\delta^2 & \alpha^3+\beta^3+\gamma^3+\delta^3 \\ \alpha+\beta+\gamma+\delta & \alpha^2+\beta^2+\gamma^2+\delta^2 & \alpha^3+\beta^3+\gamma^3+\delta^3 & \alpha^4+\beta^4+\gamma^4+\delta^4 \\ \alpha^2+\beta^2+\gamma^2+\delta^2 & \alpha^3+\beta^3+\gamma^3+\delta^3 & \alpha^4+\beta^4+\gamma^4+\delta^4 & \alpha^5+\beta^5+\gamma^5+\delta^5 \\ \alpha^3+\beta^3+\gamma^3+\delta^3 & \alpha^4+\beta^4+\gamma^4+\delta^4 & \alpha^5+\beta^5+\gamma^5+\delta^5 & \alpha^6+\beta^6+\gamma^6+\delta^6 \end{vmatrix}$$

$$= \begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}, \text{ where } s_r = a^r+\beta^r+\gamma^r+\delta^r$$

and $s_0 = \alpha^0+\beta^0+\gamma^0+\delta^0$
$= 1+1+1+1$, etc.

Hence proved.

***Ex. 25. If ω is the cube root of unity, then one root of the equation**

$$\begin{vmatrix} \mathbf{x+1} & \boldsymbol{\omega} & \boldsymbol{\omega^2} \\ \boldsymbol{\omega} & \mathbf{x+\omega^2} & \mathbf{1} \\ \boldsymbol{\omega^2} & \mathbf{1} & \mathbf{x+\omega} \end{vmatrix} = \mathbf{0} \textbf{ is } \mathbf{0}.$$

(MNR 90)

Sol. Adding all the rows of the given determinant to first, the given equation reduces to

$$\begin{vmatrix} x+1+\omega+\omega^2 & x+\omega+\omega^2+1 & \omega^2+1+x+\omega \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$$

or $$\begin{vmatrix} x & x & x \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0, \quad \because \ 1+\omega+\omega^2 = 0$$

or $$x \begin{vmatrix} 1 & 1 & 1 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0 \Rightarrow x = 0$$

Hence proved.

***Ex. 26. The value of θ which lies between $\theta = 0$ and $\theta = \pi/2$ and satisfy the equation**

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0 \text{ is}$$

(a) $7\pi/24$, (b) $5\pi/24$, (c) $11\pi/24$, (d) $\pi/24$. *(I.I.T)*

Sol. Given equation is

$$\begin{vmatrix} 1+\sin^2\theta+\cos^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta+1+\cos^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta+\cos^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0,$$

adding 2nd col. to 1st.

or $\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 2 & 1+\cos^2\theta & 4\sin 4\theta \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0, \quad \because \quad \cos^2\theta+\sin^2\theta = 1$

or $\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$, replacing R_2 by $R_2 - R_1$

or $\begin{vmatrix} 2 & 4\sin 4\theta \\ 1 & 1+4\sin 4\theta \end{vmatrix} = 0$, expanding w.r. to R_2

or $2(1+4\sin 4\theta) - 4\sin 4\theta = 0$ or $4\sin 4\theta + 2 = 0$

or $\sin 4\theta = -1/2 = \sin 210°$ or $\sin 330°$

$= \sin(7\pi/6)$ or $\sin(11\pi/6)$

or $4\theta = 7\pi/6$ or $11\pi/6$ or $\theta = 7\pi/24$ or $11\pi/24$

Hence the required values of θ are given by (a), (c). **Ans.**

Ex. 27. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants then show that $\dfrac{d}{dx}\Delta_1 = 3\Delta_2$ *(M.N.R.)*

Sol. $\dfrac{d}{dx}\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ a & x & b \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ 0 & 1 & 0 \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ 0 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix},$

expanding 1st, 2nd and 3rd determinants w.r. to 1st, 2nd and 3rd row respectively.

$= 3\begin{vmatrix} x & b \\ a & x \end{vmatrix} = 3\Delta_2$

Hence proved.

Ex. 28. The value of determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is zero if

(a) a, b, c are in H.P.

(b) a, b, c, are in G.P.

(c) α is a root of the equation $ax^2+bx+c=0$

(d) $(x-\alpha)$ is a factor of $ax^2+2bx+c$ *(I.I.T.)*

Sol. If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}=0,$

then $\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & -\alpha(a\alpha+b)-(b\alpha+c) \end{vmatrix}=0,$

replacing C_3 by $C_3-\alpha C_1-C_2$

or $\quad -(a\alpha^2+2b\alpha+c)\begin{vmatrix} a & b \\ b & c \end{vmatrix}=0$

or $\quad (a\alpha^2+2b\alpha+c)(ac-b^2)=0$

i.e. either $b^2=ac$ or $a\alpha^2+2b\alpha+c=0$

If $b^2=ac$, then a, b, c are in G.P.

Hence result (b) is true.

If $a\alpha^2+2b\alpha+c=0$, then α is a root of the equation $ax^2+2bx+c=0$ or $(x-\alpha)$ is a factor of $ax^2+2bx+c$.

Hence result (d) is true.

Ans. (b) and (d).

EXERCISES ON CHAPTER IV

***Ex. 1.** If $2s=a+b+c$, prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix}=2s^3(s-a)(s-b)(s-c)$$

Ex. 2. If $a+b+c=0$, solve the equation

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0$$

[Hint : Add all the rows or columns]

Ex. 3. Show that $\begin{vmatrix} a & a^2 & a^3+bc \\ b & b^2 & b^3+ca \\ c & c^2 & c^3+ab \end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca+abc).$

***Ex. 4.** Show that there are three values of t for which the system of equations :

$(a-t)x+by+cz=0,\ bx+(c-t)+az=0,\ cx+ay+(b-t)z=0$

have a common non-zero solution. If the three values of t are t_1, t_2, t_3 show that $t_1 t_2 t_3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Ex. 5. Prove that $\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & -b \\ -y & -c & 0 & a \\ -z & b & -a & 0 \end{vmatrix} = (ax+by+cz)^2$

Ex. 6. Show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1+a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

Ex. 7. Solve $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 3x & x+2 \end{vmatrix} = 0$

(*Meerut 92*)

Ans. $x=-3, 2, 1$

Ex. 8. Show that the roots of the following equations are all real :

$$\begin{vmatrix} a+x & h & g \\ h & b+x & f \\ g & f & c+x \end{vmatrix} = 0,$$

where a, b, c, f, g, h are all real numbers.

Ex. 9. Solve $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+3 \end{vmatrix} = 0$

***Ex. 10.** By the product of determinants estabilish Eulier's theorem that the product of any two sums each of four squares is expressible as the sum of four squares. Does the theorem hold for $(3^2+4^2)(1^2+2^2+3^2)$?

Hence express $(9^2+2^3+3^2+4^2)(5^2+6^2+7^2+8^2)$ as sum of four squares.

(**Hint.** See Ex. 20 (a) and (b) Pages 196–97.

Also $(3^2+4^2)(1^2+2^2+3^2) = (3^2+4^2+0^2+0^2)(1^2+2^2+3^2+0^2)$

Now proceed as in Ex. 20 (b) Page 197.

Ans. $(3^2+4^2)(1^2+2^2+3^2) = 5^2+(10)^2+9^2+(12)^2$

and $(9^2+2^2+3^2+4^2)(5^2+6^2+7^2+8^2)$

$= (22)^2+(116)^2+(40)^2+(60)^2.$

Four possible answers for the following questions are given. Choose the correct answer :—

Ex. 11. The value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 8 & 15 & 20 \end{vmatrix}$ is

(a) 20, (b) 10, (c) 2, (d) 5.

(*M.N.R. 91*)

Ans. (c)

Ex. 12. The value of the determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ is

(a) $a+b+c$, (b) 1, (c) 0, (d) abc

Ans. (c)

Ex. 13. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then Δ is equal to

(a) $-b_1B_1 - b_2B_2 - b_3B_3$; (b) $-b_1B_1 + b_2B_2 - b_3B_3$;

(c) $b_1B_1 - b_2B_2 + b_3B_3$; (d) $b_1B_1 + b_2B_2 + b_3B_3$

Ans. (d)

Ex. 14. The value of the determinant $\begin{vmatrix} 3 & 6 & 12 \\ 5a & 5b & 5c \\ a & b & c \end{vmatrix}$ is

(a) 15, (b) 2, (c) 0, (d) 4.

Ans. (c)

Ex. 15. The cofactor of a in the determinant $\begin{vmatrix} 3 & 4 & 5 \\ 7 & 8 & 9 \\ a & b & c \end{vmatrix}$ is

(a) $\begin{vmatrix} 4 & 5 \\ 8 & 9 \end{vmatrix}$, (b) $\begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix}$, (c) $\begin{vmatrix} 3 & 5 \\ 7 & 9 \end{vmatrix}$, (d) None of These

Ans. (a)

OBJECTIVE TYPE QUESTIONS
CH. I TO IV

(A) SHORT & VERY SHORT ANSWER TYPE QUESTIONS

1. Define matrix. *(Kanpur 2001)* [See § 1·02 Page 2]
2. Define a rectangular and a square matrix. [See § 1·03 P. 4]
3. What are horizontal and vertical matrices ? [See § 1·03 P. 4]
4. What are row and column vectors ? [See § 1·03 P. 4]
5. What is $\mathbf{I}_4$? [See § 1·03 P. 4]
6. Define an unit matrix. [See § 1·03 P. 4]
7. Define a diagonal matrix. [See § 1·03 P. 5]
8. What do you understand by a sub-matrix ? [See § 1·03 P. 5]
9. Write down the properties of matrix addition. [See § 1·07 P. 7]
10. If **A, B, C** be three matrices of the same order and are such that $\mathbf{A}+\mathbf{B}=\mathbf{A}+\mathbf{C}$, then show that $\mathbf{B}=\mathbf{C}$. [See Prop. VI Page 8]
11. When are the two matrices conformable to multiplication ? [See § 1·08 Page 11]
12. Give an example to show that the product of two non-zero matrices can be a zero matrix. [See § 1·08 Note 2 Page 13]
13. If $\mathbf{A}=\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$, then does **BA** exist ?

    Ans. No.
14. If $\mathbf{A}=\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}$, then show that $\mathbf{A}^2=\mathbf{O}$.
15. If $\mathbf{A}=[x, y, z]$, $\mathbf{B}=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $\mathbf{C}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then show that

    $\mathbf{B\,C}\,| = ax^2+by^2+cz^2+2hxy+2gzx+2fyz.$
16. If $\mathbf{A}=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$, calculate integral powers of **A**.

    (Purvanchal 97)
17. When are two matrices said to be equal ? *(Purvanchal 2001)*
18. Define null matrix. *(Purvanchal 2001)* [See § 1·03 Page 4]
19. Define transposed matrix. [See § 2·08 Page 69]
20. Define Nilpotent matrix. [See § 2·07 Page 68]

1o2/VOO

21. State reversal rule for the inverse of product of two matrices.
(Purvanchal 98) [See § 2·19 Th. II Page 92]

22. Show that $\mathbf{AB} = \mathbf{BA}$, where

$$\mathbf{A} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}.$$

(Meerut 2001)

23. State reversal rule for the transpose of a product.
[See § 2·09 Th. IV Page 71]

24. Describe elementary row operations on a matrix.
(Purvanchal 99) [See § 3·01 Page 103]

25. Define cofactor of an element of a determinant.
[See § 4·05 P. 124]

26. Define minor of an element of a determinant.
[See § 4·07 P. 128]

27. Show that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$ [See Ex. 2(a) P. 132]

28. Evaluate $\begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix}$ **Ans.** 0

29. Show that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy.$ [See Ex. 7. Page 135]

30. Evaluate $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$. **Ans.** $(x+2a)(x-a)^2$
[See Ex. 20. P. 140]

31. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$
[See Ex. 28. P. 145]

32. Evaluate $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix}$. **Ans.** abc.
[See Ex. 36 P. 149]

33. Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} c^2+b^2 & ba & ca \\ ab & c^2+a^2 & bc \\ ac & bc & b^2+a^2 \end{vmatrix}$.
[See Ex. 1. P. 163]

34. Express $\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$ as a determinant. *(Purvanchal 97)*

(B) OBJECTIVE TYPE QUESTIONS

(I) MULTIPLE CHOICE TYPE

Select the correct answer of the following :

1. The order of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ is

(i) 3×2 ; (ii) 2×3 ; (iii) 3 ; (iv) none of these.

2. A matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ is called a square matrix, if

(i) $m \leq n$; (ii) $m \geq n$; (iii) $m = n$; (iv) none of these.

3. A matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ is called a vertical matrix, if

(i) $m > n$; (ii) $m = n$; (iii) $m < n$; (iv) none of these.

4. A matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ is called a horizontal matrix, if

(i) $m > n$; (ii) $m = n$; (iii) $m < n$; (iv) none of these.

5. If a is the number of elements in a row and b is the number of elements in a column of a matrix **A**, then the order of **A** is

(i) $a \times b$; (ii) $b \times a$; (iii) $a \times a$; (iv) $b \times b$.

6. A matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ is called a row matrix, if

(i) $m < n$; (ii) $m = n$; (iii) $m = 1$; (iv) $n = 1$.

7. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called a

(i) column matrix; (ii) row matrix;
(iii) null matrix; (iv) unit matrix.

8. The matrix $[a_{ij}]_{m \times n}$ will be an unit matrix, if

(i) all its mn elements are unity;
(ii) $m = n$ and all elements are unity;
(iii) $m = n$, diagonal elements are unity and other elements are zero;
(iv) none of these.

9. The negative of a matrix **A** is

(i) zero ; (ii) $-\mathbf{A}$; (iii) $+\mathbf{A}$; (iv) non-existant.

(Kanpur 2001)

10. If **A** is of order 2×2 and **B** is of order 2×3, then **BA** is of order

(i) 2×3 ; (ii) 3×2 ; (iii) 2×2 ; (iv) none of these.

11. **A** is any $m \times n$ matrix such that **AB** and **BA** are both defined, then the order of **B** is

(i) $m \times m$; (ii) $n \times n$; (iii) $n \times m$; (iv) none of these.

12. If $\mathbf{A} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b & 0 \\ -a & 0 \end{bmatrix}$, then **AB** is

(i) null matrix ; (ii) unit matrix ;
(iii) vertical matrix ; (iv) none of these.

13. If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 5 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 0 & 7 \\ 5 & 4 \end{bmatrix}$, then

(i) $\mathbf{AB} = \mathbf{BA}$; (ii) **AB** does not exist but **BA** exists

(iii) **AB** exists but **BA** does not exist ;

(iv) $\mathbf{AB} \neq \mathbf{BA}$.

14. If $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$, then $\mathbf{A}^2$ is a

(i) diagonal matrix ; (ii) unit matrix ;

(iii) null matrix ; (iv) none of these.

15. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, the value of $\mathbf{A}^2 - 4\mathbf{A}$ is

(i) **I** , (ii) 2 **I**, (iii) 4 **I**, (iv) 5 **I**.

16. Transpose of a column matrix is

(i) square matrix ; (ii) column matrix ;

(iii) row matrix ; (iv) none of these.

17. If $\mathbf{A} = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix}$, then $\mathbf{A}' =$

(i) **A** ; (ii) $-$ **A** ; (iii) 2 **A** ; (iv) none of these.

(Kanpur 2001)

18. If **A** is a square matrix, then $\mathbf{A} + \mathbf{A}'$ will be

(i) diagonal ; (ii) symmetric ;

(iii) skew-symmetric ; (iv) identity matrix. *(Kanpur 2001)*

19. The matrix $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

(i) unitary ; (ii) idempotent ;

(iii) nilpotent ; (iv) identity.

20. In an upper triangular matrix, the elements $a_{ij} = 0$ for

(i) $i > j$; (ii) $i = j$; (iii) $i < j$; (iv) $i \leq j$.

21. A matrix **A** is called involutory if

(i) $\mathbf{A}^2 = \mathbf{A}$; (ii) $\mathbf{A}^2 = \mathbf{O}$; (iii) $\mathbf{A}^2 = \mathbf{I}$; (iv) none of these.

22. Which of the following is a scalar matrix ?

(i) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$; (ii) $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$; (iii) $\begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & a \end{bmatrix}$;

(iv) none of these.

23. If ω be the cube root of unity, then the value of the determinant

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix} \text{ is}$$

(i) 1; (ii) 0; (iii) -1; (iv) none of these.

24. If $\omega^3 = 1$, then the value of the determinant

$$\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix} \text{ is :}$$

(i) 0 ; (ii) 1 ; (iii) 2 ; (iv) 3.

25. The determinant $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ is divisible by

(i) $a - b$; (ii) $a + b$; (iii) ab ; (iv) a/b.

26. If $\begin{vmatrix} 2 & x \\ 3 & 7 \end{vmatrix} = 2$, the value of x is

(i) 1 ; (ii) 2 ; (iii) 3 ; (iv) 4.

27. If two rows of a determinant are proportional, the value of the determinant is

(i) infinite ; (ii) not zero ; (iii) negative ; (iv) zero.

28. One of the roots of $\begin{vmatrix} 2-x & 3 & 3 \\ 3 & 4-x & 5 \\ 3 & 5 & 4-x \end{vmatrix} = 0$ is

(i) -1 ; (ii) 0 ; (iii) 1 ; (iv) none of these.

29. If each element in the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is doubled, the value of the determinant of the matrix is

(i) doubled ; (ii) unchanged ;
(iii) multiplied by 4 ; (iv) none of these.

30. $\Delta = | a_{ij} |$ is a determinant of order three and A^{ij} denotes the cofactors of a_{ij} in Δ, then which of the following is not correct ?

(i) $a_{ij} A^{ij} = \Delta$; (ii) $a_{i1} A^{i2} = 0$;
(iii) $a_{i3} A^{i2} = 0$; (iv) $a_{2j} A^{2j} = 0$. *(Kanpur 2001)*

31. If two rows of a determinant are interchanged, the value of the determinant

(i) remains unchanged ;
(ii) is negative of the value of original determinant ;

(iii) doubles ; (iv) none of these.

32. The system of linear equations can be solved easily by the rule of

(i) Newton ; (ii) Bessel ; (iii) Cramer ; (iv) none of these.

(II) TRUE AND FALSE TYPE

Write "T" or "F" according as the statement is true or false :

1. The elements of matrix may be scalar or vector quantities.

2. Matrix denotes a number.

3. The order of the matrix [2] is 2×1.

4. If in a matrix, the number of columns is more than the number of rows, then it is called a horizontal matrix.

5. In a row matrix there is only one column.

6. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is an unit matrix.

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ is called a diagonal matrix.

8. It is not necessary for the two matrices **A** and **B** to be of the same order so as to be conformable for addition and subtraction.

9. Commutative law holds but associative law does not for addition of matrices.

10. Addition for matrices obeys the distributive law.

11. The product of two non-zero matrices can be a zero matrix.

12. Matrices of different orders can be subtracted.

13. Matrix multiplication in general is commutative.

14. Commutative law does not hold for addition of matrices.

15. The transpose of the transpose of a matrix is the matrix itself.

16. A square matrix $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all values of i and j.

17. If $\mathbf{A} = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, then show that $\mathbf{AA}'$ is a symmetric matrix.

[See Ex. 2 Page 82]

18. The inverse of a matrix is not unique. [See § 2·19 Th. I P. 92]

19. If **A** is an orthogonal matrix, then **A**$'$ is also orthogonal.

20. If **A** is an unitary matrix, then $\mathbf{A}^{-1}$ is not an unitary matrix.

21. A matrix $[a_{ij}]$ is called a triangular matrix if $a_{ij} = 0$ for $i > j$.

22. Non-square matrix has no inverse.

23. If two rows of a determinant $|\mathbf{A}|$ are identical, then $|\mathbf{A}| = 0$.

24. A square matrix **A** is singular if and only if $|\mathbf{A}| \neq 0$.

25. $\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} 1 & -2x & x^2 \\ 1 & -2y & y^2 \\ 1 & -2z & z^2 \end{vmatrix} \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$.

26. A determinant can be expanded using any row or column of the determinant.

27. The value of a determinant changes if the elements of a row are added to or subtracted from the corresponding elements of another row.

28. If each element of a column of a determinant be multiplied by some constant, then the determinant is multiplied by that constant.

29. The value of a determinant changes if its rows and columns are interchanged.

30. The value of a determinant changes in sign if two consecutive rows (or columns) are interchanged.

(III) FILL IN THE BLANKS TYPE

Fill in the blanks in the following :

1. Each of the mn numbers constituting an $m \times n$ matrix is known as an .... of the matrix.

2. The plural of the word 'matrix' is ....

3. The order of the matrix [3] is ....

4. A matrix which is not a square matrix is known as a .... matrix.

5. If $m = 1$ in the matrix $\mathbf{A} = [a_{ij}]_{m \times n}$, then it is called a .... matrix.

6. If $m > n$ in the matrix $\mathbf{A} = [a_{ij}]_{m \times n}$, then it is called a .... matrix.

7. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called .... matrix.

8. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is called a .... matrix.

9. Two matrices are conformable for addition and subtraction if they are of the .... order.

10. Matrices of different orders .... be added.

11. Additive identity .... for addition of matrices.

12. If $\mathbf{B}$ be the additive inverse of the matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ then (i, j)th element of $\mathbf{B}$ is ....

13. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times k$ matrix, then $\mathbf{AB}$ is an .... matrix.

14. If $\mathbf{AB} = -\mathbf{BA}$, the matrices $\mathbf{A}$ and $\mathbf{B}$ are said to ....

15. Multiplication of matrices is .... with respect to matrix addition.

16. Negative matrix is obtained by multiplying it by ....

17. A matrix when added to its negative gives the .... matrix.

18. A matrix when multiplied by .... gives the null matrix.

19. If **A** is a skew-Hermitian matrix, then i**A** is ....

20. If a square matrix **A** is idempotent then ....

21. The square matrix $\mathbf{A} = [a_{ij}]$ is skew-symmetric, if a_{ij} = ..., for all values of i and j.

22. If **A** is a Hermitian matrix, then i**A** is .... *(Meerut 2001)*

23. If **A** is an orthogonal matrix, then $\mathbf{A}^{-1}$ is ....

24. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ = ... $(a-b)(b-c)(c-a)$.

25. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$ = ... $(a-b)(b-c)(c-a)$.

26. If any two columns of a determinant are .... , then value of the determinant is zero.

27. The system of linear equations can easily be solved by the rule of ....

28. A determinant can be expanded using .... row of it.

Answers to Objective Questions

(I) Multiple Choice Type :

1. (ii) ; **2.** (iii) ; **3.** (i) ; **4.** (iii) ; **5.** (ii) ; **6.** (iii) ; **7.** (iv) ; **8.** (iii) ; **9.** (ii) ; **10.** (iv) ; **11.** (iii) ; **12.** (i) ; **13.** (iv) ; **14.** (iii) ; **15.** (iv) ; **16.** (iii) ; **17.** (ii) ; **18.** (ii) ; **19.** (iv) ; **20.** (i) ; **21.** (iii) ; **22.** (ii) ; **23.** (ii) ; **24.** (iv) ; **25.** (i) ; **26.** (iv) ; **27.** (iv) ; **28.** (i) ; **29.** (iii) ; **30.** (iv) ; **31.** (ii) ; **32.** (iii).

(II) True and False Type :

1. T ; **2.** F ; **3.** F ; **4.** T ; **5.** F ; **6.** F ; **7.** T ; **8.** F ; **9.** F ; **10.** T ; **11.** T ; **12.** F ; **13.** F ; **14.** F ; **15.** T ; **16.** T ; **17.** T ; **18.** F ; **19.** T ; **20.** F ; **21.** T ; **22.** T ; **23.** T ; **24.** F ; **25.** T ; **26.** T ; **27.** F ; **28.** T ; **29.** F ; **30.** T.

(III) Fill in the blanks type :

1. element ; **2.** matrices ; **3.** 1×1 ; **4.** rectangular ; **5.** row ; **6.** vertical ; **7.** an unit ; **8.** diagonal ; **9.** same ; **10.** cannot ; **11.** exists ; **12.** $-a_{ij}$; **13.** $m \times k$; **14.** anticommute ; **15.** distributive ; **16.** -1 ; **17.** null ; **18.** zero ; **19.** Hermitian ; **20.** $\mathbf{A}^2 = \mathbf{A}$; **21.** $-a_{ji}$; **22.** skew-Hermitian ; **23.** also orthogonal ; **24.** abc ; **25.** $(a+b+c)$; **26.** identical ; **27.** Cramer ; **28.** any.

Chapter V

Rank and Adjoint of a Matrix

§ 5·01. Order of a minor.

Definition. If any r rows and any r columns from an $m \times n$ matrix **A** are retained and remaining $(m-r)$ rows and $(n-r)$ columns removed, then the determinant of the remaining $r \times r$ submatrix of **A** is called **minor of A of order** r.

For example : In the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix}$$

elements a_{11}, a_{12}, a_{31}, etc. are minors of order unity ;

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{33} & a_{34} \\ a_{53} & a_{54} \end{vmatrix} \text{ etc.}$$

are minors of order 2 ;

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \\ a_{51} & a_{53} & a_{54} \end{vmatrix} \text{ etc.}$$

are minors of order 3 ;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix} \text{ etc.}$$

are minors of order 4.

Note. In the above example there cannot be any minor of order higher than 4.

****§ 5·02. Rank of a matrix.**

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

(Avadh 97; Garhwal 90; Gorakhpur 98; Lucknow 91)

This matrix **A** has only one three-rowed minor *i.e.* minor of order 3, viz. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$ and its value can easily be calculated to be zero, by expanding with respect to first row.

This matrix **A** has 9 minors of order 2 (or two-rowed minors) and one of them is $\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix}$ which has the value

$$(3 \times 7) - (5 \times 4) = 21 - 20 = 1 \neq 0.$$

This fact that **A** is a matrix whose every minor of order 3 is zero and there **is at least** one minor of order 2 which is not equal to zero is also expressed as '**the rank** of the matrix **A** is 2'.

****Definition of Rank of a Matrix :**

(Avadh 92 ; Bundelkhand 96, 95, 94; Purvanchal 98, 96; Rohilkhand 92)

If in an $m \times n$ matrix **A**, *at least one of its $r \times r$ minors is different from zero while all the minors of order $(r+1)$ are zero, then r is defined as the rank of the matrix* **A**.

A number r is defined as the rank of an $m \times n$ matrix **A** provided

(i) **A** has at least one minor of order r which does not vanish and (ii) there is no minor of order $(r+1)$ which is not equal to zero.

Note 1. The rank of a matrix **A** is also denoted by ρ (**A**).

***Note 2.** The rank of a zero matrix by definition is 0 *i.e.* $\rho(\mathbf{O}) = 0$.

Note 3. The rank of a matrix remains unaltered by the application of elementary row or column operations *i.e.* all equivalent matrices have the same rank.

****Note 4.** From the definition of rank of a matrix we conclude that :—

(a) If a matrix **A** does not possess any minor of order $(r+1)$ then $\rho(\mathbf{A}) \leq r$.

(b) If at least one minor of order r of the matrix **A** is not equal to zero, then $\rho(\mathbf{A}) \geq r$.

Note 5. If every minor of order p of a matrix **A** is zero then every minor of order higher than p is definitely zero.

Solved Examples on § 5·02.

***Ex. 1 (a). Find the rank of the matrix** $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{5} & \mathbf{8} \\ \mathbf{4} & \mathbf{10} & \mathbf{18} \end{bmatrix}$ *(Gorakhpur 92)*

Sol. The determinant of order 3 formed by **A**

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 4 & 10 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 4 & 2 & 6 \end{bmatrix}, \text{ replacing } C_2, C_3, \text{ by } C_2 - 2C_1, C_3 - 3C_1 \text{ respectively.}$$

$$= \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

$\therefore$ $$\rho(\mathbf{A}) \geq 3. \qquad \text{...(i)}$$

Also the matrix **A** does not possess any minor of order 4 *i.e.* 3 + 1, so

$$\rho(\mathbf{A}) \leq 3 \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we get $\rho(\mathbf{A}) = 3$. **Ans.**

Ex. 1 (b). Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{4} & \mathbf{10} & \mathbf{18} \end{bmatrix}$

Sol. Do as Ex. 1 (a) above. **Ans. 3**

Ex. 1 (c). Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{4} \end{bmatrix}$

Hint : Do as Ex. 1 (a) above. **Ans. 3**

Ex. 2 (a). Determine the rank of $\mathbf{A} = \begin{bmatrix} \mathbf{6} & \mathbf{1} & \mathbf{8} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{4} & \mathbf{-1} & \mathbf{-8} & \mathbf{-3} \end{bmatrix}$

Sol. The given matrix **A** possesses a minor of order 3 viz.

$$\begin{vmatrix} 6 & 1 & 8 \\ 2 & 1 & 0 \\ 4 & -1 & -8 \end{vmatrix} = \begin{vmatrix} 10 & 0 & 0 \\ 6 & 0 & -8 \\ 4 & -1 & -8 \end{vmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 + R_3, R_2 + R_3$$

$$= 10 \begin{vmatrix} 0 & -8 \\ -1 & -8 \end{vmatrix} = 10\,(0-8) = -80 \neq 0$$

$\therefore$ $\rho\,(\mathbf{A}) \geq 3.$ ...(i)

Also **A** does not possess any minor or order 4 *i.e.* 3 + 1, so

$\rho\,(\mathbf{A}) \leq 3.$ ...(ii)

$\therefore$ From (i) and (ii), we get $\rho\,(\mathbf{A}) = 3.$ **Ans.**

Ex. 2 (b). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} & \mathbf{8} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} & \mathbf{7} & \mathbf{5} \end{bmatrix}$$

Hint : Do as Ex. 2 (a) above. **Ans. 3.**

***Ex. 3 (a). Find the rank of matrix** $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{3} & \mathbf{6} \\ \mathbf{1} & \mathbf{3} & \mathbf{-3} & \mathbf{-4} \\ \mathbf{5} & \mathbf{3} & \mathbf{3} & \mathbf{11} \end{bmatrix}$ *(Kanpur 95)*

Sol. The given matrix **A** possesses a minor of order 3

viz. $$\begin{vmatrix} 1 & 3 & 6 \\ 1 & -3 & -4 \\ 5 & 3 & 11 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 2 \\ 1 & -3 & -4 \\ 6 & 0 & 7 \end{vmatrix}, \text{ replacing } R_1, R_3 \text{ by } R_2 + R_1, R_3 + R_2$$

$$= -3 \begin{vmatrix} 2 & 2 \\ 6 & 7 \end{vmatrix} = -3\,(14-12) = -6 \neq 0$$

$\therefore$ $\rho\,(\mathbf{A}) \geq 3.$ ...(i)

Also **A** does not possess any minor of order 4 *i.e.* 3 + 1, so

$\rho\,(\mathbf{A}) \leq 3$ ...(ii)

$\therefore$ From (i) and (ii) we get $\rho\,(\mathbf{A}) = 3.$ **Ans.**

Ex. 3 (b). Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 6 & 8 \\ 2 & 5 & 3 \\ 7 & 9 & 4 \end{bmatrix}$ *(Purvanchal 96)*

Sol. The determinant of order 3 formed by **A**

$$= \begin{vmatrix} 1 & 6 & 8 \\ 2 & 5 & 3 \\ 7 & 9 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 8 \\ 0 & -7 & -13 \\ 0 & -33 & -52 \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 2R_1, R_3 - 7R_1 \text{ repectively}$$

$$= \begin{vmatrix} -7 & -13 \\ -33 & -52 \end{vmatrix} = \begin{vmatrix} -7 & -13 \\ -5 & 0 \end{vmatrix}, \text{ replacing } R_2 \text{ by } R_2 - 4R_1$$

$$= -65 \neq 0$$

$\therefore \quad \rho(\mathbf{A}) \geq 3.$ ...(i)

Also **A** does not possess any minor of order 4 *i.e.*3 + 1, so

$$\rho(\mathbf{A}) \leq 3 \qquad ...(ii)$$

$\therefore$ From (i) and (ii), we get $\rho(A) = 3$ **Ans.**

Ex. 3 (c). Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 & 8 \\ 5 & 0 & 6 \\ 8 & 9 & 10 \end{bmatrix}$ *(Purvanchal 94)*

Sol. The determinant of order 3 formed by the matrix **A**

$$= \begin{vmatrix} 2 & 3 & 8 \\ 5 & 0 & 6 \\ 8 & 9 & 10 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 8 \\ 5 & 0 & 6 \\ 2 & 0 & -14 \end{vmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 3R_1$$

$$= -3 \begin{vmatrix} 5 & 6 \\ 2 & -14 \end{vmatrix}, \text{ expanding w.r. to } C_2$$

$$= -3(-70 - 12) = 3 \times 82 = 246 \neq 0$$

$\therefore \quad \rho(\mathbf{A}) \geq 3$ ...(i)

Also **A** does not possess any matrix of order 4 *i.e.* 3 + 1 and

$$\rho(\mathbf{A}) \leq 3. \qquad ...(ii)$$

$\therefore$ From (i) and (ii) we get $\rho(\mathbf{A}) = 3$ **Ans.**

Ex. 4 (a). Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{bmatrix}$

Sol. The determinant of order 3 formed by this matrix **A**

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 0 & 0 & 0 \end{vmatrix}, \text{ replacing } R_3 \text{ by } R_3 + R_2$$

$= 0.$

Also there exists a minor of order 2 of **A**.

viz. $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

Hence the rank of the given matrix **A** is 2. **Ans.**

Ex. 4 (b) Find the rank of matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{6} & \mathbf{8} \end{bmatrix}$.

Sol. A minor of order 2 formed by this matrix.

$= \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$. Similarly all minors of order 2 are zero.

Now we are left with minors of order 1 *i.e.* elements of **A** which are not equal to zero.

Hence the rank of the given matrix **A** is 1. **Ans.**

****Ex. 5. Find the rank of the matrix.**

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{2} \end{bmatrix}$$

(Gorakhpur 96)

Sol. In this matrix, a minor of order 3

$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0$, R_1 and R_3 are identical

In a similar way we prove that all the minors of order 3 are zero.

Now a minor of order 2 $= \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$.

But another minor of order 2 $= \begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} \neq 0$,

Hence rank of the given matrix is 2. **Ans.**

Ex. 6. Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{3} & \mathbf{-9} & \mathbf{6} \\ \mathbf{-2} & \mathbf{6} & \mathbf{-4} \end{bmatrix}$

Sol. The determinant of order 3 formed by this matrix **A**.

$= \begin{vmatrix} 1 & -3 & 2 \\ 3 & -9 & 6 \\ -2 & 6 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ -2 & 0 & 0 \end{vmatrix}$ replacing C_2, C_3 by $C_2 + 3C_1$ and $C_3 - 2C_3$ respectively.

$= 0$.

Also there exists no minor of order 2 of **A** which is not equal to zero. (Students can verify for themselves).

Finally all minors of order 1 of the matrix **A** are non-zero, as no element of the matrix **A** is 0.

Hence the rank of **A** is 1. **Ans.**

Ex. 7. Find the rank of the matrix

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$

Sol. In this matrix, a minor of order 3

$$= \begin{vmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ -1 & -3 & -4 \end{vmatrix} = 3 \begin{vmatrix} 1 & 3 & 4 \\ 1 & 3 & 4 \\ -1 & -3 & -4 \end{vmatrix}, \text{ taking 3 common from } R_2$$

$= 0$, as R_1 and R_2 are identical.

In a similar way we can prove that all minors of order 3 are zero.

Now a minor of order 2

$$= \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 3 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}, \text{ taking out 3 common from } R_2$$

$= 0$, as rows are identical.

Similarly all the minors of order 2 are zero.

Hence we are left with minors of order unity, viz. the elements of the given matrix, which are not equal to zero.

Hence rank of the given matrix $= 1$. **Ans.**

Ex. 8 (a). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 16 & 4 & 12 & 15 \\ 5 & 3 & 3 & 4 \\ 4 & 2 & 6 & -1 \end{bmatrix}$$

(Kanpur 96)

Sol. The determinant of order 4 formed by **A**

$$= \begin{bmatrix} 6 & 1 & 3 & 8 \\ 16 & 4 & 12 & 15 \\ 5 & 3 & 3 & 4 \\ 4 & 2 & 6 & -1 \end{bmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -8 & 4 & 0 & -17 \\ -13 & 3 & -6 & -20 \\ -8 & 2 & 0 & -17 \end{vmatrix},$$

, replacing C_1, C_3, C_4 by $C_1 - 6C_2, C_3 - 3C_2$ and $C_4 - 8C_3$ respectively

$$= - \begin{vmatrix} -8 & 0 & -17 \\ -13 & -6 & -20 \\ -8 & 0 & -17 \end{vmatrix} = 0, \text{ as } R_1, R_3 \text{ are indentical.}$$

Also one minor of order 3 *viz.*

$$\begin{vmatrix} 1 & 3 & 8 \\ 3 & 3 & 4 \\ 2 & 6 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & -6 & -20 \\ 2 & 0 & -17 \end{vmatrix} = \begin{vmatrix} -6 & -20 \\ 0 & -17 \end{vmatrix} \neq 0.$$

Hence the rank of given matrix **A** is 3. **Ans.**

Ex. 8 (b). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 9 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

(Garhwal 93)

Sol. The determinant of order 4 formed by **A**

$$= \begin{vmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 9 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{vmatrix}$$, replacing R_2, R_3, R_4 by $R_2 - R_1, R_3 - 2R_1,$ $R_4 - 3R_1$ respectively

$= 0$, R_2, R_4 being identical

Also one minor of order 3 viz.

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0, \text{ as above.}$$

But all minor of order 3 are not zero.

e.g. $$\begin{vmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 9 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ -1 & 0 & -2 \\ 3 & 0 & -2 \end{vmatrix} = - \begin{vmatrix} -1 & -2 \\ 3 & -2 \end{vmatrix}$$

$$= - [2 + 6] = - 8 \neq 0$$

Hence the rank of the given matrix **A** is 3. **Ans.**

Ex. 9 (a). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Sol. The determinant of order 4 formed by **A**

$$= \begin{vmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 2 & -1 & 1 & 1 \\ -2 & 1 & -1 & 1 \end{vmatrix}$$ replacing $C_1, C_2,$ by $C_1 - C_4$ and $C_2 + 2C_4$ respectively

$$= - \begin{vmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{vmatrix}$$, expanding w.r. to R_1

$= 0$, R_1, R_2 being identical.

Also one minor of order 3 viz.

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & 0 \\ 3 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

Similarly all minors of order 3 are zero

Now one minor of order 2 viz $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

$\therefore$ Rank of given matrix **A** is 2. **Ans.**

****Ex. 9 (b). Find the rank of the matrix**

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{2} \end{bmatrix}$$

(Avadh 92; Bundelkhand 92; Gorakhpur 93; Rohilkhand 98)

Hint : Do exactly as Ex. 9 (a) above. **Ans. 2**

Ex. 9 (c). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} & \mathbf{4} \\ \mathbf{-2} & \mathbf{4} & \mathbf{-1} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{2} & \mathbf{7} & \mathbf{6} \end{bmatrix}$$

Hint : Do as Ex. 9 (a) above. **Ans. 2**

Ex. 10. Find the rank of the matrix

$$\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{-2} & \mathbf{-4} \\ \mathbf{3} & \mathbf{1} & \mathbf{3} & \mathbf{-2} \\ \mathbf{6} & \mathbf{3} & \mathbf{0} & \mathbf{-7} \\ \mathbf{2} & \mathbf{3} & \mathbf{-1} & \mathbf{-1} \end{bmatrix}$$

Sol. The determinant of order 4 formed by the given matrix

$$= \begin{vmatrix} 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \\ 2 & 3 & -1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 4 & 9 & 10 \\ 6 & 9 & 12 & 17 \\ 2 & 5 & 3 & 7 \end{vmatrix}, \text{ replacing } C_2, C_3, C_4 \text{ by } C_2 + C_1, \; C_3 + 2C_1, C_4 + 4C_1 \text{ respectively}$$

$$= \begin{vmatrix} 4 & 9 & 10 \\ 9 & 12 & 17 \\ 5 & 3 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 9 & 10 \\ 5 & 3 & 7 \\ 5 & 3 & 7 \end{vmatrix}, \text{ replacing } R_2 \text{ by } R_2 - R_1$$

$= 0$, as its two rows are identical.

A minor of order 3

$$= \begin{vmatrix} 1 & -1 & -2 \\ 3 & 1 & 3 \\ 6 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 4 & 9 \\ 6 & 9 & 12 \end{vmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 + C_1, C_3 + 2C_1 \text{ respectivley}$$

$$= \begin{vmatrix} 4 & 9 \\ 9 & 12 \end{vmatrix} = 48 - 81 = -33 \neq 0$$

Hence the rank of the given matrix is 3. **Ans.**

****Ex. 11. Find the rank of the matrix**

$$\mathbf{A} = \begin{bmatrix} \mathbf{6} & \mathbf{1} & \mathbf{3} & \mathbf{8} \\ \mathbf{4} & \mathbf{2} & \mathbf{6} & \mathbf{-1} \\ \mathbf{10} & \mathbf{3} & \mathbf{9} & \mathbf{7} \\ \mathbf{16} & \mathbf{4} & \mathbf{12} & \mathbf{15} \end{bmatrix}$$

(Avadh 90; Kumaun 90)

Sol. The determinant of order 4 formed by this matrix

$$= \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 6 & 1 & 3 & 8 \\ 6 & 1 & 3 & 8 \end{vmatrix}$$, replacing R_3 and R_4 by $R_3 - R_2$ and $R_4 - R_3$ respectively.

$= 0$, as its three rows are identical

A minor of order 3

$$= \begin{vmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 10 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 1 & 3 \end{vmatrix}$$, replacing R_3 by $R_3 - R_2$

$= 0$, two rows being identical.

In a similar way we can prove that all the minors of order 3 are zero.

Now a minor of order 2 $= \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 12 - 4 = 8 \neq 0$

Hence the rank of the given martix = 2. **Ans.**

Ex. 12. Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{1} & \mathbf{2} & \mathbf{6} & \mathbf{7} \\ \mathbf{1} & \mathbf{5} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

(Gorakhpur 94)

Sol. One minor of order three of **A**

$$= \begin{vmatrix} 1 & 4 & 5 \\ 1 & 6 & 7 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & -4 & -4 \end{vmatrix}$$, replacing C_2, C_3 by $C_2 - 4C_1$ and $C_3 - 5C_1$ respectively.

$$= \begin{vmatrix} 2 & 2 \\ -4 & -4 \end{vmatrix}$$, expending with respect to R_1

$= 2(-4) - 2(-4) = 0$.

In a similar way we can prove that all the minors of order three are zero.

Now a minor of order 2 is $\begin{vmatrix} 2 & 6 \\ 5 & 0 \end{vmatrix} = 2 \cdot 0 - 6 \cdot 5 = -30 \neq 0$

Hence the rank of **A** is 2. **Ans.**

Ex. 13. Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{a} & \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} & \mathbf{d} & \mathbf{1} \\ \mathbf{1} & \mathbf{a} & \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} & \mathbf{d} & \mathbf{1} \end{bmatrix}$$

Sol. $$|\mathbf{A}| = \begin{vmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{vmatrix} = 0, \because \quad R_1, R_3 \text{ are identical}$$

A minor of order 3 of **A**

$$= \begin{vmatrix} a & b & 0 \\ c & d & 1 \\ a & b & 0 \end{vmatrix} = 0, \text{ as } R_1, R_3 \text{ are identical}$$

In a similar way we can show that all the minors of order 3 are zero in value.

A minor of order 2 of $\mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$

Hence the rank of the matrix **A** is 2. **Ans.**

Ex. 14. Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} \\ \mathbf{10} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{14} \\ \mathbf{15} & \mathbf{16} & \mathbf{17} & \mathbf{18} & \mathbf{19} \end{bmatrix}$$

(Kumaun 91)

Sol. One minor of order 3 of **A**

$$= \begin{vmatrix} 5 & 7 & 8 \\ 6 & 8 & 9 \\ 16 & 18 & 19 \end{vmatrix} = \begin{vmatrix} 5 & 7 & 8 \\ 1 & 1 & 1 \\ 11 & 11 & 11 \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_1 \text{ respectively.}$$

$$= \begin{vmatrix} 5 & 7 & 8 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 11R_2$$

$= 0.$

In a similar way we can prove that all the minors of order 3 of **A** are zero. This shows that all minors of order 4 and $|\mathbf{A}|$ of **A** are automatically zero.

(See Note 5 Page 2 of this chapter)

Now one minor of order 2 of **A**

$$= \begin{vmatrix} 7 & 8 \\ 8 & 9 \end{vmatrix} = (7 \times 9) - (8 \times 8) = 63 - 64 = -1 \neq 0$$

Hence the rank of **A** is 2. **Ans.**

Ex. 15. Find the rank of $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{bmatrix}$ *(Kanpur 91)*

Sol. $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix}$

$= -(a-b)(b-c)(c-a)$, on evaluating. ...(i)

Now following cases arise :—

Case I. $a = b = c$.

If $a = b = c$, then $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2a & 2a & 2a \\ a^2 & a^2 & a^2 \end{bmatrix}$.

Therefore all minors of order 2 and 3 of **A** vanish.

Also **A** has non-zero minor of order 1, since no element of **A** is zero.

Hence the rank of **A** in this case is 1. **Ans.**

Case II. Two of numbers a, b, c are equal, but are different from the third.

Let $a = b \neq c$.

Then $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ a+c & c+a & 2a \\ ac & ca & a^2 \end{vmatrix} = 0$, as C_1, C_2 are identical.

Also **A** has a minor of order 2 viz. $\begin{vmatrix} 1 & 1 \\ a+c & 2a \end{vmatrix}$

$= 2a - (a+c) = a - c \neq 0, \quad \because a \neq c.$

Hence the rank of **A** in this case is 2.

Similarly we can discuss the cases $b = c \neq a$, $c = a \neq b$. **Ans.**

Case III. a, b, c are all different.

In this case $|\mathbf{A}| \neq 0$, as is evident from (i) above.

i.e. **A** has a non-zero minor of order 3 and there exists no minor of order greater than 3.

Hence the rank of **A** in this case in 3. **Ans.**

****Ex. 16. Find the rank of the matrix**

$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$, **where a, b, c are all real.** *(Rohilkhand 97)*

Sol.

$|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - C_1, C_3 - C_1$

$$= \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+ca+a^2 \end{vmatrix},$$

taking $(b-a)$, $(c-a)$ common from C_1 and C_2

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 \\ b^2+ab+a^2 & c^2+ca-b^2-ab \end{vmatrix},$$

replacing C_2 by $C_2 - C_1$

$$= (b-a)(c-a)\,[(c^2+ca-b^2-ab)-0]$$

$$= (b-a)(c-a)\,[(c^2-b^2)+a(c-b)] \qquad \textbf{(Note)}$$

$$= (b-a)(c-a)\,[(c-b)(c+b+a)]$$

or $\quad |\mathbf{A}| = (a-b)(b-c)(c-a)(a+b+c)$ ...(i)

Now following cases arise :

Case I. $a=b=c$, **then** $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ a & a & a \\ a^3 & a^3 & a^3 \end{bmatrix}$

Therefore all minors of order 3 and 2 of **A** are zero.

Also as no element of **A** is zero, so **A** has non-zero minors of order 1.

Hence in this case the rank of **A** is 1. **Ans.**

Case II. Two of the numbers a, b, c are equal but are different from the third.

Let $\quad a = b \neq c$.

Then $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ a & a & a \\ a^3 & a^3 & a^3 \end{vmatrix} = 0$, as C_1 and C_2 are identical.

Also **A** has a minor of order 2, viz. $\begin{vmatrix} 1 & 1 \\ a & c \end{vmatrix} = c - a \neq 0 \qquad \because\ a \neq c$

Hence in this case the rank of **A** is 2. **Ans.**

Similarly we can discuss the cases $b = c \neq a$, $c = a \neq b$.

Case III. a, b, c are all different but $a+b+c=0$.

In this case from (i), it is evident that $|\mathbf{A}| = 0$. **(Note)**

Also **A** has a minor of order 2, viz $\begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} = b - a \neq 0,\ \because\ a \neq b$

Hence in this case the rank of **A** is 2. **Ans.**

Case IV. a, b, c are all different but $a+b+c \neq 0$.

In this case from (i), it is evident that $|\mathbf{A}| \neq 0$. **(Note)** *i.e.* **A** has a non-zero minor of order 3.

Also **A** has no minor of order greater than 3.

Hence in this case the rank of **A** is 3. **Ans.**

****Ex. 17. Prove that the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear if the rank of the matrix is $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3.**

(Agra 91; Kanpur 95, 93)

Sol. If the rank of the given matrix is less than 3, then the minor of order 3 of this matrix must be zero. (See § 5·02 Page 1 of this chapter)

i.e.
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \qquad \text{...(i)}$$

Now the area of triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3)

$$= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ **(See Authors Co-ordinate Geometry)**

$= 0$, from (i).

Since the area of this triangle is zero, so its vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear. Hence proved.

Ex. 18. Under what conditions the rank of the following matrix A is 3? Is it possible for the rank to be 1 ? Why ?

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$$ *(Kanpur 94)*

Sol. If the rank of the matrix **A** is 3, then the minor of order 3 of **A** should be non-zero.

i.e.
$$\begin{vmatrix} 2 & 4 & 3 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{vmatrix} \neq 0,$$ which is the required condition.

Also the rank of **A** can not be 1 as at least one minor of order 1 of **A** *i.e.*, one element of **A** is zero.

[If we are to find the condition under which the rank of **A** is 2, then the same is $|\mathbf{A}| = 0$ *i.e.* minor of order 3 of **A** must be zero.

i.e.
$$\begin{vmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{vmatrix} = 0, \text{ i.e. } \begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ 1 & 0 & x \end{vmatrix} = 0, \text{ replacing } R_2 \text{ by } R_2 - R_1$$

i.e.
$$\begin{vmatrix} 0 & 10 & 2 \\ 1 & -3 & 0 \\ 1 & 0 & x \end{vmatrix} = 0, \text{ replacing } R_1 \text{ by } R_1 - 2R_2$$

i.e. $\begin{vmatrix} 10 & 2 \\ -3 & 0 \end{vmatrix} - 0 + x \begin{vmatrix} 0 & 10 \\ 1 & -3 \end{vmatrix} = 0$, expanding with respect to R

i.e. $6 - 10x = 0$ *i.e.* $x = 6/10 = 3/5$. **Ans.]**

Ex. 19. Are the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 9 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix} \text{ equivalent ?}$$

Sol. Since **A** is a 3×3 matrix and **B** is a 4×4 matrix *i.e.* their dimensions are different, so these can not be equivalent.

Exercises on § 5·02

Find the rank of the following matrices :

Ex. 1. (a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 4 & 10 & 18 \end{bmatrix}$, **(b)** $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix}$

Ans. (a) 3; (b) 3; (c) 2.

Ex. 2. $\begin{bmatrix} 3 & 11 & 1 & 5 \\ 5 & 13 & -1 & 11 \\ -2 & 2 & 4 & -8 \end{bmatrix}$

Ans. 2

Ex. 3. $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

Ans. 2

Ex. 4. $\begin{bmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{bmatrix}$

Ans. 2

Ex. 5. $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$

(Lucknow 90) **Ans. 2**

Ex. 6. $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$, where a, b, c are all real.

(Kanpur 90)

[Hint : See Ex. 16 Page 11 of this chapter]

Ex. 7. $\begin{bmatrix} 0 & c & -b & \alpha \\ -c & 0 & a & \beta \\ b & -a & 0 & \gamma \\ -\alpha & -\beta & -\gamma & 0 \end{bmatrix}$, where a, b, c are, all positive numbers and $a\alpha + b\beta + c\gamma = 0$.

Ans. 2

Ex. 8. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$ **Ans. 3**

(Agra 96)

§ 5·03. Normal Form of a Matrix.

Every non-zero matrix **A** of order $m \times n$ can be reduced by application of elementary row and column operations into equivalent matrix of one of the following forms :

(i) $\begin{bmatrix} \mathbf{I_r} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$, (ii) $\begin{bmatrix} \mathbf{I_r} \\ \mathbf{O} \end{bmatrix}$, (iii) $[\mathbf{I_r}, \mathbf{O}]$, (iv) $[\mathbf{I_r}]$,

where $\mathbf{I_r}$ is $r \times r$ identity matrix and **O** is null matrix of any order.

These four forms are called **Normal** or **canonical form** of **A**.

Important Theorem (without Proof). *(Avadh 94)*

Th. I. If $m \times n$ matrix **A** is reduced to the canoncial form or normal form $\begin{bmatrix} \mathbf{I_r} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ by the application of elementary row or column operations, then r, the order of the identity sub-matrix $\mathbf{I}_r$ is the rank of the matrix **A**

Th. II. If a non-singular matrix of order $n \times n$ is reduced to the identity matrix $\mathbf{I_n}$ (which is its canonical or normal form), then the rank of the matrix is n.

Solved Examples on § 5·03.

Ex. 1 (a). Find rank of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{bmatrix}$ *(Gorakhpur 95)*

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 1 \end{bmatrix}$, replacing C_3 by $C_3 - C_2$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$, replacing C_2 by $C_2 - C_1$

$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$, replacing C_3 by $C_3 - C_2$ and C_1 by $C_1 - C_2$

$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - R_2$

$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, replacing R_2 by $R_2 - R_1$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_2$$

$$\sim \begin{bmatrix} \mathbf{I_2} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Hence the rank of **A** is 2. **Ans.**

Ex. 1 (b). Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{4} & \mathbf{6} \\ \mathbf{0} & \mathbf{6} & \mathbf{9} \end{bmatrix}$

Sol. $\mathbf{A} \sim \begin{bmatrix} 0 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 6 & 3 \end{bmatrix}$, replacing C_3 by $C_3 - C_2$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \text{ replacing } C_2 \text{ by } C_2 - 2C_3$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 2R_1, R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_1 \text{ and } C_3$$

$$\sim \begin{bmatrix} \mathbf{I_1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Hence the rank of **A** is 1. Ans.

***Ex. 1 (c). Find the rank of the matrix** $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \mathbf{3} & \mathbf{3} & \mathbf{3} \end{bmatrix}$

Hint : Do as Ex. 1 (b) above. Replace C_2, C_3 by $C_2 - C_1$, $C_3 - C_1$ respectively and then from the result so obtained replace R_2, R_3 by $R_2 - 2R_1$, $R_3 - 3R_1$ respectively. Ans. 1.

Ex. 1 (d). Find the rank of the matrix.

$$\mathbf{A} = \begin{bmatrix} \mathbf{-1} & \mathbf{-2} & \mathbf{-1} \\ \mathbf{6} & \mathbf{12} & \mathbf{6} \\ \mathbf{5} & \mathbf{10} & \mathbf{5} \end{bmatrix}$$

Hint : Do Ex. 1 (b) above. **Ans. 1.**

Ex. 2. Find the rank of tthe matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$$

Sol. $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 2 & -3 & -4 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$, $C_3 - 3C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -6 \\ 0 & -3 & -4 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 4R_1$, $R_3 - 2R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -3 & -4 \end{bmatrix}$, replacing R_2 by $R_2 - R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, replacing C_2, C_3 by $-\frac{1}{3} C_2$ $-\frac{1}{2} C_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, interchanging C_2 and C_3

$\sim [\mathbf{I}_3]$.

Hence the rank of **A** is 3. **Ans.**

Ex. 3 (a) Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ *(Bundelkhand 94)*

Sol. $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$ and $C_3 - 3C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2 \text{ by } C_2 + 2C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_3 \text{ by } -C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_2 \text{ and } C_3$$

$$\sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Hence the rank of A is 2.

Ans.

***Ex. 3 (b). Find the rank of the matrix A $= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ after reducing it to the normal form.** *(Avadh 97; Garhwal 90; Meerut 92)*

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \text{ replacing } R_2 \text{ and } R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_3 \text{ by } C_3 - C_2$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_1 \text{ by } C_1 - C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_1 \text{ and } C_2$$

$$\sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Hence the rank of **A** is 2.

Ans.

Ex. 3 (c). Reduce matrix A to its normal form and then find its rank, where $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{2} \end{bmatrix}$ *(Agra 93)*

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 5 \\ 3 & 1 & 2 & 5 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 - C_1$, $C_3 - C_1, C_4 + C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 5 \\ 2 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing $C_1, C_3\ C_4$ by $C_1 - C_2$, $C_3 - 2C_2, C_4 - 5C_2$ respectively

$\sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$, which is in the normal form.

Hence the rank of **A** is 2. **Ans.**

Ex. 4. (a). Reduce the matrix A to the normal form.

where $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{4} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{3} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$, **hence find the rank of A.** *(Meerut 92 P)*

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 4 & 5 & 0 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing C_2, C_4 by $C_2 + C_1, C_4 + 3C_1$ respectively,

$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 4 & 5 & 0 & 7 \\ 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, replacing C_3, C_4 by $\frac{1}{2} C_3, \frac{1}{2} C_4$ respectively

$\sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 4 & 5 & 0 & 2 \\ 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_1, C_4 by $C_1 - C_3, C_4 - C_2$ respectively

$\sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 5 & 0 & 2 \\ 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_1 by $\frac{1}{4} C_1$

$$\sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2, C_4 \text{ by } C_2 - 5C_1, C_4 - 2C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2, C_4 \text{ by } C_2 + 3C_4, -C_4 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ rearranging columns}$$

$\sim [\mathbf{I_4}]$.

$\therefore$ **Rank of A** is 4. **Ans.**

Ex. 4 (b). Express the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & 3 \\ -1 & -4 & -2 & -7 \\ 2 & 1 & 3 & 0 \\ -1 & -2 & 3 & 0 \end{bmatrix}$$

in the normal form and find its rank. *(Purvanchal 93)*

Sol. $\mathbf{A} \sim \begin{bmatrix} 0 & -1 & 0 & 3 \\ 6 & -4 & 2 & -7 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & 5 & 0 \end{bmatrix}$, replacing C_1, C_3 by $C_1 - C_4$, $C_3 - C_2$ respectively

$$\sim \begin{bmatrix} 0 & -1 & 0 & 3 \\ 4 & -4 & 2 & -7 \\ 0 & 1 & 2 & 0 \\ -6 & -2 & 5 & 0 \end{bmatrix}, \text{ replacing } C_1 \text{ by } C_1 - C_3$$

$$\sim \begin{bmatrix} 0 & -1 & 0 & 3 \\ 4 & -5 & 0 & -7 \\ 0 & 1 & 2 & 0 \\ -6 & 0 & 9 & 0 \end{bmatrix}, \text{ replacing } R_2, R_4 \text{ by } R_2 - R_3\ R_4 + 2R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 2 & 3 \\ 4 & -5 & 0 & -7 \\ 0 & 1 & 2 & 0 \\ -2 & -5 & 9 & -7 \end{bmatrix}, \text{ replacing } R_1, R_4 \text{ by } R_1 + R_3\ R_4 + R_2 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & -5 & 10 & -7 \\ 0 & 1 & 0 & 0 \\ -2 & -5 & 19 & -7 \end{bmatrix}, \text{ replacing } C_3 \text{ by } C_3 - 2C_2$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & 0 & 10 & -7 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 19 & -7 \end{bmatrix}, \text{ replacing } R_2, R_4 \text{ by } R_2 + 5R_3,\ R_4 + 5R_3 \text{ repectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & 0 & 10 & -7 \\ 0 & 1 & 0 & 0 \\ -6 & 0 & 9 & 0 \end{bmatrix}, \text{ replacing } R_4 \text{ by } R_4 - R_2$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 4 & 0 & 10 & -7 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 \end{bmatrix}, \text{ replacing } R_1, R_4 \text{ by } \tfrac{1}{3} R_1, \tfrac{1}{3} R_4 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 16 & -7 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 + 2R_4$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 16 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 + 7R_1$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } (1/16)\, R_2$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_4 \text{ by } R_4 - 3R_2$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_4 \text{ by } -(1/2)\, R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_4 \text{ and interchanging } R_2 \text{ and } R_3$$

or $\mathbf{A} \sim [\mathbf{I}_4]$ $\quad \therefore$ The rank of **A** is 4. **Ans.**

Ex. 5. Find the rank of the matrix

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

(Garhwal 94)

Sol.

$$A \sim \begin{bmatrix} 0 & -1 & -1 & 1 \\ 0 & 2 & 2 & -2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 + 2R_3, R_2 - R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 + C_4, C_3 + C_4 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2 + 2R_1, R_3 - R_1, R_4 + R_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 - C_1, C_3 - 2C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_1, R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_2, C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_2, R_3$$

$$\sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Hence the rank of A is 2. **Ans.**

Ex. 6 (a). Reduce $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ **to normal form.**

(Garhwal 91)

Sol. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 6 & -9 & 6 \\ 2 & 4 & -6 & 2 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 + C_1, C_3 - 2C_2, C_4 + C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 6 \\ 2 & 2 & 3 & 2 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - C_4, C_3 - (3/2)\, C_2$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 6 \\ 2 & 2 & 0 & 0 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - (3/2)\, C_2, C_4 - C_2$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, replacing R_2, R_3 by $\frac{1}{2} R_2, \frac{1}{2} R_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_3 - 2R_1, R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_3 by $\frac{1}{3} C_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, interchanging C_2, C_4

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, interchaning C_3, C_4

$\sim [\mathbf{I}_3 \ \mathbf{O}]$ which is the required normal form **Ans.**

Ex. 6 (b). **Reduce the matrix A to the normal form and hence find the rank of the matrix A, where**

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 2 \end{bmatrix}.$$

Hint : Do as Ex. 6 (a) above. **Ans. 3**

Ex. 6 (c). **Find the rank of the matrix**

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 5 & 8 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

Hint : Do as Ex. 6 (a) above. **Ans. 3**

****Ex. 7. By elementary operations, find the rank of the matrix.**

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(Avadh 95, 93, 91; Garhwal 95, Gorakhpur 90; Kanpur 97; Meerut 98)

Sol. $A \sim \begin{bmatrix} 6 & 3 & 0 & -7 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$, replacing R_1 by $R_1 + R_2 + R_3$

or $\sim \begin{bmatrix} 6 & 3 & 0 & -7 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_4 by $R_4 - R_1$

$\sim \begin{bmatrix} 0 & 9 & 12 & 17 \\ 1 & -1 & -2 & -4 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_1 and R_3 by $R_1 - 6R_2$, $R_3 - 3R_2$ respectively.

$\sim \begin{bmatrix} 0 & 9 & 12 & 17 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3 and C_4 by $C_2 + C_1$, $C_3 + 3C_1$ and $C_4 + 4C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 12 & 17 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging R_1 and R_2

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_2 by $R_2 - 2R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_3 and C_4 by $C_3 + 6C_2$ and $C_4 + 3C_2$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_3 and C_4 by $\frac{1}{33} C_3$ and $\frac{1}{22} C_2$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2 and C_4 by $C_2 - 4C_3$ and $C_4 - C_3$ respectively.

$$\sim \begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$$

Hence the rank of the given matrix = 3. **Ans.**

Ex. 8. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ 2 & 2 & 8 & 0 \end{bmatrix}$$

Hint : Do as Ex. 7. on Pages 23-24 **Ans. 3**

****Ex. 9 (a). Find the rank of** $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.

(Agra 91; Bundelkhand 91; Garhwal 92; Kumaun 96; Lucknow 92; Meerut 90; Purvanchal 98; Rohilkhand 91)

Sol. $A \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - C_1$ and $C_4 - C_1$ respectively

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3, R_4 by $R_3 - R_1$ and $R_4 - R_1$ respectively

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - 3C_2$ $C_4 + C_2$ respectively

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3, R_4 by $R_3 - 3R_2$ and $R_4 - R_2$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging C_1, C_2

$\sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$

Hence the rank of A is 2. **Ans.**

Ex. 9 (b). Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

(Purvanchal 95)

Sol. $A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$, replacing R_2, R_3, R_4 by $R_2 - R_1, R_3 - R_1, R_4 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 - C_1, C_3 - C_1, C_4 - C_1$ respectively

$\sim \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_1, R_3, R_4 by $R_1 - R_2, R_3 - 2R_2, R_4 - 3R_2$ respectively

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - 2C_2, C_4 - 3C_2$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging C_1 and C_2

$\sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$

Hence the rank of **A** is 2. **Ans.**

***Ex. 10. Find the rank of matrix** $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$.

by reducing to canonical form. Also show that it is not equivalent to the matrix $\mathbf{B} = \begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 0 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$

Sol. $A \sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 2 & 4 & 1 \end{bmatrix}$, replacing R_3 and R_4 by $R_3 - R_1$ and $R_4 - R_3$ respectively

$$\sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}, \text{ replacing } R_2 \text{ and } R_3 \text{ by } R_2 - R_4 \text{ and } R_3 - R_4 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}, \text{ replacing } C_1 \text{ and } R_3 \text{ by } \tfrac{1}{2} C_1 \text{ and } \tfrac{1}{2} R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchang } R_3 \text{ and } R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ and } C_4 \text{ by } C_2 + C_1, C_3 - 3C_1 \text{ and } C_4 - 4C_1 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2 \text{ and } C_3 \text{ by } C_2 - 2C_4 \text{ and } C_3 - 4C_4 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchaining } C_3 \text{ and } C_4$$

$$\sim \begin{bmatrix} \mathbf{I}_3 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

∴ The required rank of the matrix **A = 3.**

Also we can prove as in Ex. 5 Page 5 Chapter V that the rank of the matrix **B** *i.e.* $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 0 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ is 2.

Since the ranks of the two matrices **A** and **B** are different so these are not equivalent. (See Note 3 of § 5·02 Page 2 Chapter V)

Ex. 11 (a). Find the rank of matrix $\mathbf{A}=\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

(Gorakhpur 97; Lucknow 92)

Sol. $\mathbf{A} \sim \begin{bmatrix} 0 & 4 & 9 & -7 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$, replacing R_1, R_2 by $R_1 - 3R_3$, $R_2 - R_4$ respectively

$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, interchanging R_1, R_3 and replacing R_4 by $R_4 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 + 2C_1$, $C_3 + 3C_1$ and $C_4 - 2C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & -7 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, replacing R_3 by $R_3 - 4R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 23 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, replacing R_3 by $R_3 + 7R_4$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, replacing R_3 by $\frac{1}{23} R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing C_3 by $C_3 - 2C_4$

$\sim [\mathbf{I}_4]$

Hence the rank of $\mathbf{A}$ is 4. Ans.

Ex. 11 (b). Reduce the following matrix A to the form

$$\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

and hence determine its rank.

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(Kumaun 92)

Sol. $A \sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 4 & 5 & 0 & 2 \\ 0 & 3 & 1 & 5 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing C_2, C_4 by $C_2 + C_1$, $C_4 + C_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 4 & 5 & 0 & 6 \\ 0 & 3 & 1 & 5 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing C_4 by $C_4 + C_1$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 4 & 0 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 5R_4$ $R_3 - 3R_4$ respectively

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 4 & 0 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_4 by $C_4 - 2C_2$

$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, replacing C_4 by $C_4 + C_1 + C_2 + C_3$

$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, replacing R_2 by $(1/4)\, R_2$

$\sim \begin{bmatrix} 0 & 0 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing R_1, C_4 by $R_1 - R_2$ and $C_4 - C_2$ respectively

$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing R_1, C_4 by $R_1 - 2R_3$ $(1/3)\, C_4$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, interchanging R_1 and R_2, R_3 and R_4

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_3 \text{ and } R_4$$

$$\sim [\mathbf{I}_4]$$

Hence the rank of **A** is 4. **Ans.**

***Ex. 12. Is the matrix** $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 0 \\ 2 & -3 & 1 \end{bmatrix}$ **equivalent to $\mathbf{I}_3$?**

Sol. Here we find that the minor of order 3 of **A**

$$= \begin{vmatrix} 1 & 1 & 2 \\ -1 & 2 & 0 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 3 & 2 \\ 2 & -5 & -3 \end{vmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 - C_1, C_3 - 2C_1 \text{ respectively}$$

$$= \begin{vmatrix} 3 & 2 \\ -5 & -3 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$= 3(-3) - 2(-5) = -9 + 10 = 1 \neq 0.$$

Also from § 5·03 Th. II Paper 15 Chapter V we know that this matrix **A** can be recduced to $\mathbf{I}_3$ by elementary row or column operations.

Hence **A** is equivalent to $\mathbf{I}_3$.

Ex. 13. Determine by reducing to normal form the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 3 & 2 & 1 \\ -1 & -1 & -3 & 2 \end{bmatrix}$, replacing C_1 by $\frac{1}{8} C_1$ and C_4 by $\frac{1}{2} C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ -1 & 0 & 0 & 5 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ and } C_4 \text{ by } C_2 - C_1, C_3 - 3C_2, C_4 - 3C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \text{ replaing } C_2 \text{ by } \tfrac{1}{3} C_2 \text{ and } C_3 \text{ by } \tfrac{1}{2} C_3$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 5\end{bmatrix}$$, replacing C_3 and C_4 by $C_3 - C_2$ and $C_4 - C_2$ respectively

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 5 & 0\end{bmatrix}$$, interchanging C_3 and C_4

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$, replacing C_3 by $\frac{1}{3} C_5$

$\sim [\mathbf{I}_3 \ \ \mathbf{O}]$, **(Note)**

Hence the rank of **A** is 3. **Ans.**

Ex. 14. Find the rank of the matrix $\mathbf{A} = \begin{bmatrix}1 & 2 & -1 & 3\\ 2 & 4 & -4 & 7\\ -1 & -2 & -1 & 1\end{bmatrix}$

(Bundellkhand 95; Garhwal 96; Purvanchal 97; Rohilkhand 9

Sol. $\mathbf{A} \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 0 & -2 & 1\\ -1 & 0 & -2 & 4\end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 - 2C_1, C_3 + C_1$ and $C_4 - 3C_1$ respectively

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 0 & -2 & 1\\ -3 & 0 & 0 & 3\end{bmatrix}$$, replacing R_3 by $R_3 - R_2$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & -2 & 1\\ 0 & 0 & 0 & 3\end{bmatrix}$$, replacing R_2, R_3 by $R_2 - 2R_1$ and $R_3 + 3R_1$ respectively

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}$$, replacing C_3 by $-\frac{1}{2} C_3$ and R_3 by $\frac{1}{3} R_3$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$, replacing C_4 by $C_4 - C_3$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$, interchanging C_2 and C_3

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$, interchanging C_3 and C_4

$\sim [\mathbf{I}_3 \ \ \mathbf{O}]$. **(Note)**

Hence the rank of A is 3. **Ans.**

Ex. 15. Use elementary transformations to reduce the following matrix to triangular form and hence find the rank of A.

$$A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

Sol. $A \sim \begin{bmatrix} 5 & 3 & 8 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 4 & 1 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - 2C_2, C_4 - C_2$

$\sim \begin{bmatrix} 5 & 8 & -12 & -4 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 + C_1, C_3 - 4C_1, C_4 - C_1$ respectively.

$\sim \begin{bmatrix} 5 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3 by $C_2 + 2C_4, C_3 - 3C_4$ respectively.

$\sim \begin{bmatrix} 5 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_1 by $-\frac{1}{4} C_4$

$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_1 by $C_1 - 5C_4$

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, interchanging R_1, R_2

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, interchanging R_2, R_3

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, interchanging R_1, R_2

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, interchanging C_3, C_4

$\sim [I_3 \;\; O]$

Hence the rank A is 3. **Ans.**

****Ex. 16. Reduce the matrix A to the normal (or canonical) form and hence obtain its rank.**

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

Sol. $$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ -2 & 7 & 2 & 3 \end{bmatrix}, \text{ replacing } C_2 \text{ and } C_4 \text{ by } C_2 - 2C_1 \text{ and } C_4 + C_1 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 11 & 2 & -7 \end{bmatrix}, \text{ replacing } C_2 \text{ and } C_4 \text{ by } C_2 + 2C_3 \text{ and } C_4 - 5C_3 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}, \text{ replacing } C_1 \text{ by } C_1 + C_3 \text{ and } C_4 \text{ by } C_4 + \tfrac{7}{11} C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 11 & 0 \end{bmatrix}, \text{ interchanging } C_2 \text{ and } C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ replacing } C_3 \text{ by } \tfrac{1}{11} C_3$$

$$\sim [\mathbf{I}_3 \quad \mathbf{O}]$$ **(Note)**

Hence the rank of **A** is 3. **Ans.**

***Ex. 17. Is rank of** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **equal to 2 ?** *(Agra 90)*

Sol. Here $\mathbf{A} = \mathbf{I}_3$, so rank of **A** is 3 and not 2. **Ans.**

Ex. 18. If $\mathbf{A} = \begin{bmatrix} -1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & -2 & 6 & -7 \end{bmatrix}$**, find its rank.** *(Rohilkhand 92)*

Sol. $$A \sim \begin{bmatrix} -1 & 2 & -1 & 4 \\ 0 & 8 & 1 & 12 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2 + 2R_1, R_3 + R_1, R_4 + R_1 \text{ respectively.}$$

$$\sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 8 & 1 & 12 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \text{ replacing } C_2, C_3, C_4 \text{ by } C_2 + 2C_1, C_3 - C_1, C_4 + 4C_1 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 12 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \text{ replacing } C_1, C_2 \text{ by } -C_1 \text{ and } (1/4)\, C_2 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \text{ replacing } C_3, C_4 \text{ by } C_3 - 2C_2, C_4 - 8C_2 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5/3 & 1 \end{bmatrix}, \text{ replacing } R_2, R_4 \text{ by } R_2 - 2R_3, (-1/3)\, R_4 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -29/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5/3 & 1 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 + 4R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5/3 & 1 \end{bmatrix}, \text{ replacing } R_2 \text{ by } -(3/29)\, R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3 \text{ and replacing } R_4 \text{ by } R_4 + (5/3)\, R_3$$

$$\sim [I_4]$$

Hence the rank of A is 4. **Ans.**

Exercises on § 5·03

Find the rank of the following matrices by reducing these to the normal (or canonical) form :—

Ex. 1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ **Ans.** 0, 1, 1, 1

Ex. 2. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{bmatrix}$ **Ans.** 2

Ex. 3. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$ **Ans.** 3

Ex. 4. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ *(Kumaun 93)* **Ans.** 3

***Ex. 5.** $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ **Ans.** 2

Ex. 6. $\begin{bmatrix} 1 & -3 & 2 \\ 3 & -9 & 6 \\ -2 & 6 & -4 \end{bmatrix}$ **Ans.** 1

***Ex. 7.** $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ **Ans.** 2

Ex. 8. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$ **Ans.** 3

Ex. 9. $\begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 5 \end{bmatrix}$ *(Garakhpur 99)* **Ans.** 3

Ex. 10. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ *(Meerut 91S)* **Ans.** 3

Ex. 11. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$ **Ans.** 3

***Ex. 12.** $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ *(Garhwal 91)* **Ans.** 2

Ex. 13. $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 5 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ *(Avadh 98)* **Ans.** 3

Ex. 14. $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$ *(Agra 92)* **Ans.** 2

Ex. 15. $\begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 9 & 2 & -1 & 6 & 4 \\ 7 & -4 & -5 & 6 & 5 \\ 17 & 1 & -4 & 12 & 7 \end{bmatrix}$ **Ans.** 2

Ex. 16. $\begin{bmatrix} 9 & 7 & 3 & 6 \\ 5 & -1 & 4 & 1 \\ 6 & 8 & 2 & 4 \end{bmatrix}$ *(Rohilkhand 96)* **Ans.** 3

Ex. 17. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 2 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & 3 & -3 & 4 \end{bmatrix}$ *(Lucknow 90)* **Ans.** 4

§ 5·04. Echelon Form of a Matrix.

Definition. If in a matrix.

(i) all the non-zero rows, if any, precede the zero rows,

(ii) the number of zero preceeding the first non-zero element in a row is less than the number of such zero in the succeeding row.

(iii) the first non-zero element in a row is unity, then it is in the Echelon form.

Note. The number of non-zero rows of a matrix given in the Echelon form is its rank. **(Remember)**

Example of a matrix in the Echelon Form :—

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In this matrix we observe that

(i) the first three non-zero rows precede the fourth row which is a zero row.

(ii) the number of zero in R_4, R_3 and R_2 are 5, 2 and 1 respectively which are in descending order.

(iii) the first non-zero term in each row is unity.

Hence all the three conditions of the Echelon form are satisfied.

Also there being three non-zero rows in this matrix, its rank is 3. This fact can be proved by actually finding the rank of this matrix.

In this matrix, a minor of order 4

$$= \begin{vmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0, \text{ one row being of zero.}$$

In a similar way we can show that all minors of order 4 are zero.

Now a minor of order 3 $= \begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}, \text{ expanding w.r. to } C_1$$

$$= 1 \neq 0.$$

Hence the rank of this matrix = 3.

Ex. 1. Find the rank of the matrix.

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Sol. In the given matrix we observe that

(i) the first two non-zero rows precede the third row which is a **zero row**,

(ii) the number of zero in R_3, R_2 and R_1 are 4, 2 and 1 respectively which are in descending order, and

(iii) the first non-zero term in each row is unity.

Hence all the three conditions of the Echelon form are satisfied.

Also there being two non-zero rows in this matrix, its rank is 2. **Ans.**

Ex. 2. Reduce the following matrix to its Echelon form and find its rank :

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{3} & \mathbf{9} & \mathbf{12} & \mathbf{9} \\ \mathbf{-1} & \mathbf{-3} & \mathbf{-4} & \mathbf{-3} \end{bmatrix}$$

(Meerut 93)

Sol. $A \sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 3R_1, R_3 + R_1$ respectively

$\sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing R_2, R_3 by $R_2 + 3R_3, (1/2)\, R_3$ respectively.

$\sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging R_2 and R_3

In the above matrix we observe that.

(i) the first two non-zero rows precede the third row which is a zero row,

(ii) the number of zero in R_3 and R_2 are 4 and 3 respectively which are in descending order, and

(iii) the first non-zero term in each row is unity.

Hence all the three conditions of the Echelon form are satisfied and then the given matrix is reduced to its Echelon form.

Also there being two non-zero rows in this matrix, its rank is 2. **Ans.**

****§ 5·05. Invariance of rank under elementary operations.**

Theorem. *All equivalent matrices have the same ranks i.e. the rank of a matrix remains unaltered by the application of elementary row and column operations.* *(Avadh 99; Bundelkhamd 93)*

Proof. Let r be the rank of $m \times n$ matrix $\mathbf{A} = [a_{ij}]$

Case I. *If ith and jth rows are interchanged (which may be written symbolically as R_{ij} or $(R_i \longleftrightarrow R_j)$ then it does not effect the rank.*

Let **B** denote the matrix obtained from the matrix **A** by the elementary operation $R_i \longleftrightarrow R_j$ and let p be the rank of **B**.

Also if **D** be any $(r+1)$ rowed square sub-matrix of **B**, then $|\mathbf{D}| = \pm |\mathbf{C}|$, where **C** is a particular $(r+1)$ rowed sub matrix of **A**.

As r is the rank of the matrix **A** so every $(r+1)$ rowed minor of **A** vanishes and therefore p, the rank of **B**, cannot exceed r, the rank of **A** *i.e.* $p \le r$.

Also we can obtain **A** from **B** by the elementary operation $R_i \longleftrightarrow R_j$, therefore in that case interchanging the roles of **A** and **B** we shall get $r \le p$.

Hence $r = p$.

Case II. *If the elements of a row are multiplied by a non-zero number λ (which may be written symbolically, as $R_i \to \lambda R_i$, $\lambda \ne 0$) then it does not effect the rank.*

Let **B** denote the matrix obtained from the matrix **A** by the elementary operation $R_i \to \lambda R_i$ and let p be the rank of **B**.

Let **D** be any $(r+1)$ rowed square sub-matrix of **B** and let **C** be the sub-matrix of **A** having the same position as **D**. Then either $|\mathbf{D}| = |\mathbf{C}|$ or $|\mathbf{D}| = \lambda |\mathbf{C}|$.

[Here $|\mathbf{D}| = |\mathbf{C}|$ happens if the ith row of **A** is one of those rows which are removed to obtain **D** from **B** and $|\mathbf{D}| = \lambda |\mathbf{C}|$ happens when the ith row is not removed while obtaining **C** from **A**].

Also as r is the rank of the matrix **A** so every $(r+1)$–rowed minor of **A** vanishes and therefore in particular $|\mathbf{C}| = 0$ and consequently in both the above cases $|\mathbf{D}| = 0$.

$\therefore$ p, the rank of **B**, cannot exceed r, the rank of **A**.

i.e. $p \le r$.

Also we can obtain **A** from **B** by the elementary operation $R_i \to \lambda^{-1} R_i$, therefore in that case interchanging the roles of **A** and **B** we shall get $r \le p$.

Hence $r = p$.

Case III. *If to the elements of the ith row are added the products by any non-zero number λ of the corresponding elements of jth row (which may be written symbolically as $R_i \to R_i + \lambda R_j$; $\lambda \neq 0$) then it does not effect the rank.*

Let **B** denote the matrix obtained from the matrix **A** by the elementary operation $R_i \to R_i + \lambda R_j$ and let p be the rank of **B**.

Let **D** be any $(r + 1)$ rowed square submatrix of **B** and let **C** be the submatrix of **A** having the same position as **D**.

Now three sub-cases arise :—

(i) If **A** and **B** differ only in the ith row *i.e.* if ith row of **B** is one of those rows which have been removed while obtaining **C**.

In this case $\mathbf{D} = \mathbf{C}$ and therefore $|\mathbf{D}| = |\mathbf{C}|$.

$\therefore$ The rank of **A** is r, so $|\mathbf{C}| = 0$ and consequently $|\mathbf{D}| = 0$

(ii) If ith row of **B** has not been removed but jth row has been removed while obtaining **D**.

In this case $|\mathbf{D}| = |\mathbf{C}| + \lambda |\mathbf{C_0}|$, where $\mathbf{C_0}$ is an $(r + 1)$ rowed matrix which is obtained from **C** by replacing a_{ik} by a_{jk} *i.e.* $\mathbf{C_0}$ is obtained from **C** by performing the elementary operation R_{ij} or $R_i \longleftrightarrow R_j$ and then removing those rows and columns of the new matrix which are removed to obtain **D** from **B**.

$\therefore$ $|\mathbf{C_0}|$ is negative of some $(r + 1)$-rowed minor of **A** and as the rank of **A** is r so every $(r + 1)$-rowed minor of **A** is zero *i.e.* $|\mathbf{C}| = 0, |\mathbf{C_0}| = 0$ and consequently $|\mathbf{D}| = 0$.

(iii) If neither the ith row not the jth row of **B** has been removed while obtaining **D**.

Here $|\mathbf{D}| = |\mathbf{C}|$ and so as before $|\mathbf{D}| = 0$.

$\therefore$ Every $(r + 1)$-rowed minor of **B** vanishes so p, the rank of **B**, cannot exceed r, the rank of **A** *i.e.* $p \leq r$.

Also we can obtain **A** from **B** by the elementary operation $R_i \to R_i - \lambda R_j$, therefore in that case interchanging the roles of **A** and **B** we shall get $r \leq p$.

Hence $r = p$.

Thus we have observed that the rank of a matrix remains invariant under elementary row operations. Similarly it can be shown that the rank of a matrix remains invariant under elementary column operations too.

Note. By the applications of the above theorem we can easily obtain the rank of a matrix for if we can obtain a matrix **B** by elementary operations on a matrix **A** and of the rank of **B** can be easily determined by inspection or simple calculations as given in previous articles in this chapter, then we can determine the rank of **A**.

§ 5·06. Some Important Theorems.

Theorem I. *The rank of a matrix is equal to the rank of the transposed matrix.*

or $\rho(\mathbf{A}) = \rho(\mathbf{A}')$, *where* $\rho(\mathbf{A})$ *denotes rank of* **A**. *(Kanpur 94; Rohilkhand 92)*

Proof. Let $\mathbf{A} = [a_{ij}]$ be any $m \times n$ matrix.

Then the transposed matrix $\mathbf{A}' = [a_{ji}]$ is an $n \times m$ matrix.

Let the rank of **A** be r and let **B** be the $r \times r$ sub-matrix of **A** such that $|\mathbf{B}| \neq 0$.

Also we know that the value of a determinant remains unaltered if its rows **and columns** are interchanged. (See Prop. II of Determinants)

i.e. $|\mathbf{B}'| = |\mathbf{B}| \neq 0$, where **B** is evidently a $r \times r$ sub-matrix of $\mathbf{A}'$.

$\therefore$ **The** rank of $\mathbf{A}' \geq r$, (See Note 4 (b) Page 2 Ch. V)

Again if **C** be a $(r+1) \times (r+1)$ sub-matrix of **A**, then by definition of rank (**See** § 5·02 Page 1 Ch. V) we must have all $|\mathbf{C}| = 0$.

Also $\mathbf{C}'$ is a $(r+1) \times (r+1)$ submatrix of $\mathbf{A}'$ so we have

$$|\mathbf{C}'| = |\mathbf{C}| = 0, \text{ as explained above.}$$

$\therefore$ **We** conclude that there cannot be any $(r+1) \times (r+1)$ sub-matrix of $\mathbf{A}'$ **with non-zero** determinant.

$\therefore$ **The** rank of $\mathbf{A}' \geq r$ and it cannot be greater then r as above.

$\therefore$ **The** rank of $\mathbf{A}'$ is r which is also the rank of **A**. Hence proved.

Theorem II. *The rank of the product matrix* **AB** *of two matrices* **A** *and* **B** *is less than the rank of either of the matrices* **A** *and* **B**.

Proof. Let r_1 and r_2 be the ranks of the matrices **A** and **B**.

$\because$ r_1 is the rank of **A** therefore $\mathbf{A} \sim \begin{bmatrix} \mathbf{M} \\ \mathbf{O} \end{bmatrix}$, where **M** is a submatrix of rank r_1 and contains r_1 rows.

Post multiplying it by **B**, we get

$$\mathbf{AB} \sim \begin{bmatrix} \mathbf{M} \\ \mathbf{O} \end{bmatrix} \mathbf{B}.$$

But $\begin{bmatrix} \mathbf{M} \\ \mathbf{O} \end{bmatrix} \mathbf{B}$ can have r_1 non-zero rows at the most which are obtained on multiplying r_1 non-zero rows of **M** with columns of **B**.

$\therefore$ **Rank of AB** = Rank of $\begin{bmatrix} \mathbf{M} \\ \mathbf{O} \end{bmatrix} \mathbf{B} \leq \mathbf{r_1}$

i.e. Rank of **AB** $\leq$ Rank of **A** ...(i)

In a similar way we get $\mathbf{B} \sim [\mathbf{N} \;\; \mathbf{O}]$, where **N** is a submatrix of rank r_2 and **contains** r_2 columns.

Pre-multiplying it by **A**, we get

$$\mathbf{AB} \sim \mathbf{A}\, [\mathbf{N} \;\; \mathbf{O}]$$

But $[\mathbf{N} \;\; \mathbf{O}]$ can have r_2 non-zero columns at the most which are obtained on multiplying the rows of **A** with r_2 non-zero columns of $[\mathbf{N} \;\; \mathbf{O}]$

$\therefore$ Rank of $\mathbf{AB} =$ Rank of $\mathbf{A}$ [N O] $\leq r_2$

i.e. Rank of $\mathbf{AB} \leq$ Rank of $\mathbf{B}$. ...(ii)

Hence the theorem from (i) and (ii).

Solved Examples on § 5·05 and § 5·06.

Ex. 1. Show that the rank of a matrix A does not alter by pre or post multiplying it with any non-singular matrix R.

Sol. Let $\mathbf{B} = \mathbf{RA}$.

Then rank of $\mathbf{B} =$ rank of $\mathbf{RA} \leq$ rank $\mathbf{A}$. ...(i)

...See § 5.06 Th II above

Also $\mathbf{A} = \mathbf{R}^{-1}\mathbf{B}$, where $\mathbf{R}^{-1}$ is the inverse matrix of $\mathbf{R}$.

$\therefore$ rank of $\mathbf{A} =$ rank of $(\mathbf{R}^{-1}\mathbf{B}) \leq$ rank of $\mathbf{B}$. ...(ii)

$\therefore$ From (i) and (ii) we conclude that

rank of $\mathbf{A} =$ rank of $\mathbf{B}$. Hence proved.

Ex. 2. Show that AA′ has the same rank as A, where A′ is the transpose of A.

Sol. Let $\mathbf{B} = \mathbf{AA}'$, then the rank of $\mathbf{B} \leq$ rank of $\mathbf{A}$. ...(i)

Also $\mathbf{A}^{-1} = \mathbf{A}'$, and so we have

rank of $\mathbf{A} =$ rank of $\mathbf{A}' \leq$ rank of $\mathbf{B}$. ...(ii)

$\therefore$ From (i) and (ii), rank of $\mathbf{A} =$ rank of $\mathbf{B}$.

Ex. 3. Show that AA^{Θ} has the same rank as A, where A^{Θ} is the transpose conjugate of A.

[**Hint :** Do as Ex. 2 above]

Ex. 4. Prove that if A is a matrix of order n × n and if B non-singular matrix of order n, then the product P = AB has the same rank as A.

Sol. Here $\mathbf{A} \sim (m \times n)$, $\mathbf{B} \sim (n \times n)$

$$\mathbf{P} = \mathbf{AB} \sim (m \times n)$$

If $m < n$, rank of $\mathbf{A} \leq m$ but rank of $\mathbf{B} = n$

$\therefore$ rank of $\mathbf{A} <$ rank of $\mathbf{B}$,

Now rank $(\mathbf{P}) =$ rank $(\mathbf{AB}) \leq$ rank $\mathbf{A}$ ...(i)

But we can write $\mathbf{A} = \mathbf{PB}^{-1}$

$\therefore$ rank of $\mathbf{A} =$ rank of $(\mathbf{PB}^{-1}) \leq$ rank of $\mathbf{P}$. ...(ii)

$\therefore$ From (i) and (ii) we get rank of $\mathbf{P} =$ rank of $\mathbf{A}$.

§ 5·07. Sweep out method of finding the rank of a matrix.

In the process of evaluation of the rank of a matrix by means of elementary row and column transformations, if certain rows or columns are zero-rows or zero-columns *i.e.* each element of these rows or columns are zero, then we can remove these rows or columns without any effect on the rank of the matrix. (See § 5·05 Page 38 Ch. V). This method is generally called the *Sweep out* method.

Solved Examples on § 5·07.

Ex. 1. Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{6} & \mathbf{1} & \mathbf{3} & \mathbf{8} \\ \mathbf{4} & \mathbf{2} & \mathbf{6} & -\mathbf{1} \\ \mathbf{10} & \mathbf{3} & \mathbf{9} & \mathbf{7} \\ \mathbf{16} & \mathbf{4} & \mathbf{12} & \mathbf{15} \end{bmatrix}$$

Sol. $\mathbf{A} \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 4 & 2 & 6 & -1 \\ 4 & 2 & 6 & -1 \end{bmatrix}$, replacing R_3 and R_4 by $R_3 - R_1$ and $R_4 - 2R_1$ respectively.

$\therefore \quad \mathbf{A} \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \end{bmatrix}$ ...See § 5·07 above

Now a minor of order 2 is $\begin{vmatrix} 1 & 8 \\ 2 & -1 \end{vmatrix} = -1 - 16 = -17 \neq 0.$

Hence its rank is 2. **Ans.**

Ex. 2. Find the rank of the matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & -\mathbf{3} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & -\mathbf{2} & \mathbf{0} \end{bmatrix}$$

Sol. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.

Now $\mathbf{A} \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 1 & -2 & 0 \\ 3 & 3 & -6 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 + R_1$ and $R_3 + 2R_1$ respectively.

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3 and R_4 by $\frac{1}{3} R_3$ and $R_4 - R_2$ respectively.

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - R_2$ and then C_2 and C_3 by $C_2 - C_1$ and $C_3 + 2C_1$ respectively.

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. ... See § 5·07 Page 41 Ch. V.

Now a minor of order 2 is

$\sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$. Hence its rank is 2.

Ans.

Ex. 3. Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{4} & \mathbf{7} \\ \mathbf{9} & \mathbf{1} & \mathbf{2} & \mathbf{0} \end{bmatrix}$ *(Meerut 95, 94)*

Sol. Here $\mathbf{A} \sim \begin{bmatrix} 28 & -3 & 10 & 7 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_1, C_3 by $C_1 - 9C_2$, $C_3 - 2C_2$ respectively.

$\sim \begin{bmatrix} 0 & -3 & 3 & 7 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_1, C_3 by $C_1 - 4C_4$, $C_3 - C_4$ respectively.

$\sim \begin{bmatrix} 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_2, C_4 by $C_2 + C_3$, $C_4 - 2C_3$ respectively.

$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_3 by $C_3 - 3C_4$

Now a minar of order 2 is $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

Hence its rank is 2. **Ans.**

Exercises on § 5·07

Find the rank of the following matrices :—

Ex. 1. $\begin{bmatrix} 4 & 3 & 0 & 2 \\ 3 & 4 & -1 & -3 \\ -7 & -7 & 1 & 5 \end{bmatrix}$ **Ans. 3**

Ex. 2. $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & -3 \\ -2 & 4 & 2 \\ 5 & -2 & 4 \end{bmatrix}$ **Ans. 3**

Ex. 3. $\begin{bmatrix} 3 & -2 & 0 & -7 \\ 0 & 2 & 1 & -5 \\ 1 & -2 & -2 & 1 \\ 0 & 1 & 1 & -6 \end{bmatrix}$ **Ans. 4**

§ 5·08. Adjoint of a Matrix. *(Agra 94, 92; Rohilkhand 91, 90)*

Definition. If C_{ij} be the cofactor of the element a_{ij} in $|a_{ij}|$ of the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, then

adjoint of $\mathbf{A} = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$

This is also rewritten as Adj. **A**

or adjoint of **A** = transposed of **C**, where $\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$.

While solving problems we generally use this definition.

Here students should note carefully that the cofactors of the elements of the first row of $|a_{ij}|$ are the elements of the first column of Adj **A**.

Similarly the cofactors of the elements of the first column of $|a_{ij}|$ are the elements of first row of Adj. **A**.

Solved Examples on § 5·08.

Ex. 1 (a). If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 8 \\ 9 & 10 & 12 \end{bmatrix}$, find Adj. A. *(Avadh 95)*

Sol. For the given matrix **A**, we have

$$C_{11} = \begin{vmatrix} 7 & 8 \\ 10 & 12 \end{vmatrix} = 4;\ C_{12} = -\begin{vmatrix} 5 & 8 \\ 9 & 12 \end{vmatrix} = 12;\ C_{13} = \begin{vmatrix} 5 & 7 \\ 9 & 10 \end{vmatrix} = -13;$$

$$C_{21} = -\begin{vmatrix} 2 & 4 \\ 10 & 12 \end{vmatrix} = 16;\ C_{22} = \begin{vmatrix} 1 & 4 \\ 9 & 12 \end{vmatrix} = -24;\ C_{23} = -\begin{vmatrix} 1 & 2 \\ 9 & 10 \end{vmatrix} = 8;$$

$$C_{31} = \begin{vmatrix} 2 & 4 \\ 7 & 8 \end{vmatrix} = -12;\ C_{32} = -\begin{vmatrix} 1 & 4 \\ 5 & 8 \end{vmatrix} = 12;\ C_{33} = \begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix} = -3$$

$$\therefore \quad C = \begin{bmatrix} 4 & 12 & -13 \\ 16 & -24 & 8 \\ -12 & 12 & -3 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = C' = \begin{bmatrix} 4 & 16 & -12 \\ 12 & -24 & 12 \\ -13 & 8 & -3 \end{bmatrix}$$ Ans.

***Ex. 1 (b). Find the adjoint of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$**

Sol. Do as Ex. 1 (a) above. $\begin{bmatrix} 2 & -6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$ **Ans.**

Ex. 2. Find the adjoint of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ *(Kanpur 96)*

Sol. For the given matrix **A**, we have

$$C_{11} = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 3;\ C_{12} = -\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = -9;\ C_{13} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5;$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4;\ C_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1;\ C_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3;$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5;\ C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = 4;\ C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$\therefore \quad C = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = C' = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

Ans.

Ex. 3. Find the adjoint of the matrix A, if

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{2} & \mathbf{0} & \mathbf{4} \end{bmatrix}$$

Sol. For the matrix **A**, we have

$$C_{11} = \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} = 4;\ C_{12} = -\begin{vmatrix} 0 & -1 \\ 2 & 4 \end{vmatrix} = -2;\ C_{13} = \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2;$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4;\ C_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2;\ C_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2;$$

$$C_{31} = \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -4;\ C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = 1;\ C_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \quad C = \begin{bmatrix} 4 & -2 & -2 \\ -4 & -2 & 2 \\ -4 & 1 & 1 \end{bmatrix}$$

$$\therefore \text{Adj. } \mathbf{A} = C' = \begin{bmatrix} 4 & -4 & -4 \\ 2 & -2 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

Ans.

Ex. 4. Find the adjoint of $\mathbf{A} = \begin{bmatrix} \mathbf{5} & \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

Sol. For the matrix **A**, we have

$$C_{11} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2;\ C_{12} = -\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2;$$

$$C_{13} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1; \quad C_{14} = - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2;$$

$$C_{21} = - \begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0; \quad C_{22} = \begin{vmatrix} 5 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = 6$$

$$C_{23} = - \begin{vmatrix} 5 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0; \quad C_{24} = - \begin{vmatrix} 5 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0;$$

$$C_{31} = \begin{vmatrix} 0 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 0; \quad C_{32} = - \begin{vmatrix} 5 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 0;$$

$$C_{33} = \begin{vmatrix} 5 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = 3; \quad C_{34} = - \begin{vmatrix} 5 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0;$$

$$C_{41} = - \begin{vmatrix} 0 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -4;$$

$$C_{42} = \begin{vmatrix} 5 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -2 \begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} = -16;$$

$$C_{43} = - \begin{vmatrix} 5 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 5 & 2 \\ 0 & 1 \end{vmatrix} = -5;$$

$$C_{44} = \begin{vmatrix} 5 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 5 & 0 \\ 1 & 1 \end{vmatrix} = 10$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 2 & 2 & 1 & -2 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ -4 & -16 & -5 & 10 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 2 & 0 & 0 & -4 \\ 2 & 6 & 0 & -16 \\ 1 & 0 & 3 & -5 \\ -2 & 0 & 0 & 10 \end{bmatrix}$$

Ans.

***Ex. 5. Verify that the adjoint of a diagonal matrix of order 3 is a diagonal matrix.**

Sol. Let A be a diagonal matrix of order 3 given by

$$\mathbf{A} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Then for the matrix **A** we have

$$C_{11} = \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} = bc;\; C_{12} = -\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} = 0;\; C_{13} = \begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} = 0;$$

$$C_{21} = -\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} = 0;\; C_{22} = \begin{vmatrix} a & 0 \\ 0 & c \end{vmatrix} = ac;\; C_{23} = -\begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} = 0;$$

$$C_{31} = \begin{vmatrix} 0 & 0 \\ b & 0 \end{vmatrix} = 0;\; C_{32} = -\begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} = 0;\; C_{33} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$$

$\therefore$ Adj. $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$, which is evidently a diagonal matrix

Hence proved.

Ex. 6. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$, find $\mathbf{A}^2 - 2\mathbf{A} + \text{Adj. } \mathbf{A}$

(Agra 95)

Sol. $$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+6 & 2+10+12 & 3+0+9 \\ 0+0+0 & 0+25+0 & 0+0+0 \\ 2+0+6 & 4+20+12 & 6+0+9 \end{bmatrix} = \begin{bmatrix} 7 & 24 & 12 \\ 0 & 25 & 0 \\ 8 & 36 & 15 \end{bmatrix} \quad \text{...(i)}$$

Also $$C_{11} = \begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} = 15,\; C_{12} = -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = 0;\; C_{13} = \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} = -10$$

$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6,\; C_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3;\; C_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0;$$

$$C_{33} = \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = -15,\; C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0;\; C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix} \quad \text{...(ii)}$$

$\therefore\ A^2 - 2A + \text{Adj. } A$

$$= \begin{bmatrix} 7 & 24 & 12 \\ 0 & 25 & 0 \\ 8 & 36 & 15 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}, \quad \text{from (i) and (ii)}$$

$$= \begin{bmatrix} 7 & 24 & 12 \\ 0 & 25 & 0 \\ 8 & 36 & 15 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 0 & 10 & 0 \\ 4 & 8 & 6 \end{bmatrix} + \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 7-2+15 & 24-4+6 & 12-6-15 \\ 0-0+0 & 25-10-3 & 0-0+0 \\ 8-4-10 & 36-8+0 & 15-6+5 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 26 & -9 \\ 0 & 12 & 0 \\ -6 & 28 & 14 \end{bmatrix}$$

Ans.

Exercises on § 5·08

Find the adjoint of the following matrices

Ex. 1. $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 2 & 1 \\ 4 & -5 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 9 & -11 & -8 \\ 8 & -14 & -5 \\ 2 & -13 & -6 \end{bmatrix}$

Ex. 2. $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$

Ex. 3. $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$. **Ans.** $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$

Ex. 4. $\begin{bmatrix} 1 & 5 & 7 \\ 2 & 3 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 3 & 11 & -16 \\ 0 & -26 & 13 \\ -6 & 17 & -7 \end{bmatrix}$

Ex. 5. $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$

Ex. 6. $\begin{bmatrix} 3 & 3 & 4 \\ 2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} -7 & 1 & 24 \\ -2 & 3 & -4 \\ 2 & -3 & -15 \end{bmatrix}$

Ex. 7. $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & 2 & 2 \end{bmatrix}$

Ex. 8. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 8 & -5 & 2 \\ -4 & -3 & 1 \\ -7 & 3 & -1 \end{bmatrix}$

Ex. 9. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ **Ans.** $\begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

Ex. 10. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Ex. 11. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 3 & 3 & -3 \\ 0 & -9 & 6 \\ -2 & 5 & -3 \end{bmatrix}$

Ex. 12. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ *(Agra 95; Bundelkhand 92; Garhwal 92)*

Ans. $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

Ex. 13. $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 7 & -11 & -5 \\ 0 & 10 & -5 \\ 14 & 3 & -5 \end{bmatrix}$

***Ex. 14.** $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$

§ 5·09. Theorems on Adjoint of a Matrix.

****Theorem I.** *If* $\mathbf{A} = [a_{ij}]$ *be an* $n \times n$ *matrix, then*

$\mathbf{A} \bullet (Adj\ \mathbf{A}) = (Adj\ \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}_n$; *where* $\mathbf{I}_n$ *is an* $n \times n$ *identity matrix.*

(Agra 94, 91; Avadh 94, 92, 90; Bundelkhand 94, 93; Garhwal 90; Gorakhpur 97, 92; Kanpur 96; Meerut 91; Purvanchal 95; Rohilkhand 90)

Proof. We know Adj $\mathbf{A} = [C'_{jk}]$,

where C_{kj} is the cofactor of a_{kj} in $|\mathbf{A}|$ and $C'_{jk} = C_{kj}$.

Therefore $\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = [a_{ij}]\,[C'_{jk}]$

$$= [B_{ik}], \text{ say.} \qquad \ldots(i)$$

where $B_{ik} = \sum_{j=1}^{n} a_{ij} C'_{jk} = \sum_{j=1}^{n} a_{ij} C_{kj}, \qquad \because C'_{jk} = C_{kj}$

$$\left.\begin{aligned} &= |\mathbf{A}|, \text{ if } i = k \\ &= 0, \text{ if } i \neq k \end{aligned}\right\}$$ ... See § 4·05 and § 4·09 in Ch. IV

$\therefore$ From (i), (i, k)th element of $\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = |\mathbf{A}|$, or 0 according as $i = k$ or $i \neq k$.

i.e. All diagonal terms of $\mathbf{A} \bullet (\text{Adj } \mathbf{A})$ are $|\mathbf{A}|$ and non-diagonal terms are zero.

$$\mathbf{A} \bullet (\text{Adj. } \mathbf{A}) = \begin{bmatrix} |\mathbf{A}| & 0 & 0 & \dots & 0 \\ 0 & |\mathbf{A}| & 0 & \dots & 0 \\ 0 & 0 & |\mathbf{A}| & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |\mathbf{A}| \end{bmatrix}$$

$$= |\mathbf{A}| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \text{ See Chapter I}$$

$$= |\mathbf{A}| \bullet \mathbf{I} \qquad \text{...(ii)}$$

Similarly we can prove that $(\text{Adj. } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}$ ...(iii)

Hence from (ii) and (iii), we get

$$\mathbf{A} \bullet (\text{Adj. } \mathbf{A}) = (\text{Adj. } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}$$

or $$\mathbf{A} \bullet \frac{(\text{Adj. } \mathbf{A})}{|\mathbf{A}|} = \frac{(\text{Adj. } \mathbf{A})}{|\mathbf{A}|} \bullet \mathbf{A} = \mathbf{I}$$

or $$\mathbf{A}^{-1} = \frac{(\text{Adj. } \mathbf{A})}{|\mathbf{A}|}, \quad \because \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

i.e. The inverse of $\mathbf{A} = \dfrac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|}$ ...(iv)

Note : The result (iv) gives us another method of finding the inverse of a given matrix.

****Theorem II.** *If* $\mathbf{A} = [a_{ij}]$ *be an* $n \times n$ *matrix, then*

$|Adj\ \mathbf{A}| = |\mathbf{A}|^{n-1}$, *if* $|\mathbf{A}| \neq 0$. *(Agra 96; Gorakhpur 92; Rohilkhand 99, 91)*

Proof. We know that $|\mathbf{A}| \bullet |\mathbf{B}| = |\mathbf{AB}|$ ...See Ch. on Determinants

$\therefore$ $|\mathbf{A}| \bullet |Adj\ \mathbf{A}| = |\mathbf{A} \bullet Adj.\ \mathbf{A}|$

$$= \begin{bmatrix} |\mathbf{A}| & 0 & 0 & \dots & 0 \\ 0 & |\mathbf{A}| & 0 & \dots & 0 \\ 0 & 0 & |\mathbf{A}| & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |\mathbf{A}| \end{bmatrix}, \text{ as proved in Theorem I above}$$

or $|\mathbf{A}| \bullet |Adj\ \mathbf{A}| = (|\mathbf{A}|)^n$ **(Note)**

Dividing both sides by $|\mathbf{A}|$, since $|\mathbf{A}| \neq 0$, we get

$$|Adj\ \mathbf{A}| = |\mathbf{A}|^{n-1}.$$ Hence proved.

****Theorem III.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are two* $n \times n$ *matrices, then*

$Adj\ (\mathbf{AB}) = (Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$. *(Agra 93; Rohilkhand 98; Gorakhpur 98)*

Proof. We know $\mathbf{A} \bullet (Adj\ \mathbf{A}) = |\mathbf{A}| \bullet \mathbf{I}$ ...See Th. I Page 49 Ch. V

So we have $(\mathbf{AB}) \bullet (Adj\ \mathbf{AB}) = |\mathbf{AB}| \bullet \mathbf{I}$ ...(i)

Now $(\mathbf{AB}) \bullet (Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$

$= \mathbf{A} \bullet \mathbf{B} \bullet Adj\ \mathbf{B} \bullet Adj\ \mathbf{A}$

$= \mathbf{A} \bullet (\mathbf{B} \bullet Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$ **(Note)**

$= \mathbf{A} \bullet |\mathbf{B}| \bullet \mathbf{I} \bullet Adj\ \mathbf{A}, \quad \because \mathbf{B} \bullet Adj\ \mathbf{B} = |\mathbf{B}| \bullet \mathbf{I}$

$= \mathbf{A} \bullet |\mathbf{B}| \bullet Adj\ \mathbf{A}, \quad \because \mathbf{I} \bullet Adj\ \mathbf{A} = Adj\ \mathbf{A}$ as $\mathbf{I} \bullet \mathbf{A} = \mathbf{A}$ always

$= |\mathbf{B}| \bullet \mathbf{A} \bullet Adj\ \mathbf{A}$ **(Note)**

$= |\mathbf{B}| \bullet |\mathbf{A}| \bullet \mathbf{I} \quad \because \mathbf{A} \bullet Adj\ \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}$.

$= |\mathbf{A}| \bullet |\mathbf{B}| \bullet \mathbf{I},$

$= |\mathbf{AB}| \bullet \mathbf{I}. \quad \because |\mathbf{A}| \bullet |\mathbf{B}| = |\mathbf{AB}|$ ...(ii)

$\therefore$ From (i) and (ii) we get

$(\mathbf{AB}) \bullet (Adj\ \mathbf{AB}) = (\mathbf{AB}) \bullet (Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$

or $Adj\ (\mathbf{AB}) = (Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$. Hence proved.

Solved Examples on § 5·09.

****Ex. 1 (a). For the matrix A given in Ex. 2 Page 44 Ch. V verify the theorem A • (Adj A) = (Adj A) • A = | A | • I.**

Sol. In Ex. 2 Page 45 Ch. V. we have proved that if

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}, \text{ then Adj. } \mathbf{A} = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{A} \bullet (Adj\ \mathbf{A}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-9-5 & -4+1+3 & -5+4+1 \\ 3-18+15 & -4+2-9 & -5+8-3 \\ 6+9-15 & -8-1+9 & -10-4+3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} \quad \text{...(i)}$$

$$\text{Also } (Adj\ \mathbf{A}) \bullet \mathbf{A} = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4-10 & 3-8+5 & 3+12-15 \\ -9+1+8 & -9+2-4 & -9-3+12 \\ -5+3+2 & -5+6-1 & -5-9+3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} \quad \text{...(ii)}$$

Also $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -4 \\ 2 & -3 & 1 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - C_1, C_3 - C_1$ ...(iii)

$$= \begin{vmatrix} 1 & -4 \\ -3 & 1 \end{vmatrix} = 1 - 12 = -11$$

$\therefore$ From (i), (ii) and (iii) we get

$$\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = (\text{Adj. } \mathbf{A}) \bullet \mathbf{A} = -11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Note)

$$= (-11)\, \mathbf{I}_3 = |\mathbf{A}| \bullet \mathbf{I}.$$

Hence proved.

Ex. 1 (b). Find the adjoint of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and verify the theorem $\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = (\text{Adj } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}|\, \mathbf{I}$. *(Bundelkhand 93)*

Sol. For the given matrix **A**, we have

$C_{11} = -5, C_{12} = -3, C_{21} = -2, C_{22} = 1$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$$

And so $\text{Adj } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$

$$\therefore \quad \mathbf{A} \bullet (\text{Adj } \mathbf{A}) = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \times \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5-6 & -2+2 \\ -15+15 & -6-5 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} \quad \text{...(i)}$$

Also $\quad (\text{Adj } \mathbf{A}) \bullet \mathbf{A} = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$

$$= \begin{bmatrix} -5-6 & -10+10 \\ -3+3 & -6-5 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} \quad \text{...(ii)}$$

Also $|\mathbf{A}| = \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -5 - 6 = -11$

$\therefore$ From (i) and (ii), we get

$$\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = (\text{Adj } \mathbf{A}) \bullet \mathbf{A} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$$

$$= -11 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |\mathbf{A}|\, \mathbf{I}_2 = |\mathbf{A}|\, \mathbf{I}$$

Hence proved.

Ex. 1 (c). Verify the theorem $\mathbf{A} \bullet (\mathbf{Adj.\ A}) = (\mathbf{Adj.\ A}) \bullet \mathbf{A}$
$= |\mathbf{A}| \bullet \mathbf{I}$ **when** $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

Sol. Do as Ex. 1 (a) above.

Ex. 2 (a). Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ *(Agra 91)*

Sol. For the given matrix **A**, we have

$C_{11} = \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = -1; C_{12} = -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = 3; C_{13} = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = -2;$

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 3; C_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} = -3; C_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 1;$

$C_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2; C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 1; C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$\therefore \quad \mathbf{C} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

Also $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & -3 \end{vmatrix}$, replacing C_2, C_3, by $C_2 - 2C_1, C_3 - 3C_1$

$= \begin{vmatrix} 0 & -1 \\ -1 & -3 \end{vmatrix} = -1$

$\therefore \quad \mathbf{A}^{-1} = \dfrac{\text{Adj } \mathbf{A}}{|\mathbf{A}|}$

$= -\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ **Ans.**

Ex. 2 (b). Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ *(Agra 96)*

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ -7 & 3 & -1 \\ 5 & -2 & 1 \end{vmatrix}$,

replacing C_1, C_2 by $C_1 + 3C_3, C_2 - 2C_1$ respectively.

$$= -\begin{vmatrix} -7 & 3 \\ 5 & -2 \end{vmatrix}, \text{ expanding w. r. to } R_1$$

$$= -[14 - 15] = 1$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1;\ C_{12} = -\begin{vmatrix} -4 & -1 \\ 2 & 1 \end{vmatrix} = 2;\ C_{13} = \begin{vmatrix} -4 & 1 \\ 2 & 0 \end{vmatrix} = -2;$

$C_{21} = -\begin{vmatrix} -2 & -1 \\ 0 & 1 \end{vmatrix} = 2;\ C_{22} = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5;\ C_{23} = -\begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix} = -4;$

$C_{31} = \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} = 3;\ C_{32} = -\begin{vmatrix} 3 & -1 \\ -4 & -1 \end{vmatrix} = 7;\ C_{13} = \begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} = -5$

$\therefore \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$

$\therefore \quad \mathbf{A}^{-1} = \dfrac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ **Ans.**

Ex. 3 (a). Find the inverse of $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} \end{bmatrix}$

(Avadh 98, 91; Purvanchal 96)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix},$

replacing R_2, R_3 by $R_2 - R_1, R_3 - R_1$

or $\quad |\mathbf{A}| = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \quad$ ...(i)

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7;\ G_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1;\ C_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1;$

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6;\ C_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0;\ C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2;$

$C_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1;\ C_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1;\ C_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

$\therefore \quad C = \begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$

$\therefore \quad \mathbf{A}^{-1} = \dfrac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|}$

$$= -\frac{1}{2}\begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & -3 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

Ans

Ex. 3 (b). Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ *(Avadh 92,*

Hint : Do as Ex. 3 (a) above. **Ans.** $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Ex. 3 (c). Find the adjoint of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$ **and hence evaluate** $\mathbf{A}^{-1}$**.** *(Kumaun 94,*

Hint. Do as Ex. 3 (a). above. **Ans.** $\begin{bmatrix} \frac{11}{3} & -3 & \frac{1}{3} \\ -\frac{7}{3} & 3 & -\frac{2}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{bmatrix}$

Ex. 3 (d). Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

Hint. Do as Ex. 3 (a) above. **Ans.** $-\frac{1}{15}\begin{bmatrix} 15 & -6 & -15 \\ 0 & -3 & 0 \\ -10 & 4 & 5 \end{bmatrix}$

Ex. 4 (a). Find the adjoint of the matrix A and evaluate $\mathbf{A}^{-1}$**, where**

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix}$$

Sol. Here for the matrix **A**, we have

$$C_{11} = \begin{vmatrix} 5 & 5 \\ 5 & 11 \end{vmatrix} = 30;\ C_{12} = -\begin{vmatrix} 2 & 5 \\ 2 & 11 \end{vmatrix} = -12;\ C_{13} = \begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} = 0;$$

$$C_{21} = -\begin{vmatrix} 2 & 2 \\ 5 & 11 \end{vmatrix} = -12;\ C_{22} = \begin{vmatrix} 2 & 2 \\ 2 & 11 \end{vmatrix} = 18;\ C_{23} = -\begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = -6;$$

$$C_{31} = \begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} = 0;\ C_{32} = -\begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = -6;\ C_{33} = \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = 6$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 30 & -12 & 0 \\ -12 & 18 & -6 \\ 0 & -6 & 6 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 30 & -12 & 0 \\ -12 & 18 & -6 \\ 0 & -6 & 6 \end{bmatrix}$$

Ans.

Also $|\mathbf{A}| = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 3 & 3 \\ 2 & 3 & 9 \end{vmatrix}$, applying $C_2 - C_1$, $C_3 - C_1$

$$= 2\begin{vmatrix} 3 & 3 \\ 3 & 9 \end{vmatrix} = 2\,[27 - 9] = 36$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{36}\begin{bmatrix} 30 & -12 & 0 \\ -12 & 18 & -6 \\ 0 & -6 & 6 \end{bmatrix}$$

$$= \frac{1}{36} \times 6\begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 5/6 & -1/3 & 0 \\ -1/3 & 1/2 & -1/6 \\ 0 & -1/6 & 1/6 \end{bmatrix}$$

Ans.

Ex. 4 (b). Find the inverse of matrix A, where

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$$

(Agra 94)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 \\ 0 & 0 & -1 \\ -1 & -4 & -3 \end{vmatrix}$, replacing C_1 by $C_1 - C_3$

$$= \begin{vmatrix} 1 & 3 \\ -1 & -4 \end{vmatrix}, \text{ expanding w.r. to } R_2$$

$$= -4 + 3 = -1 \qquad \text{...(i)}$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 0 & -1 \\ -4 & -3 \end{vmatrix} = -4; C_{12} = -\begin{vmatrix} -1 & -1 \\ -4 & -3 \end{vmatrix} = 1; C_{13} = \begin{vmatrix} -1 & 0 \\ -4 & -4 \end{vmatrix} = 4;$

$C_{21} = -\begin{vmatrix} 3 & 3 \\ -4 & -3 \end{vmatrix} = -3; C_{22} = \begin{vmatrix} 4 & 3 \\ -4 & -3 \end{vmatrix} = 0; C_{23} = -\begin{vmatrix} 4 & 3 \\ -4 & -4 \end{vmatrix} = 4;$

$C_{31} = \begin{vmatrix} 3 & 3 \\ 0 & -1 \end{vmatrix} = -3; C_{32} = -\begin{vmatrix} 4 & 3 \\ -1 & -1 \end{vmatrix} = 1: C_{33} = \begin{vmatrix} 4 & 3 \\ -1 & 0 \end{vmatrix} = 3$

$\therefore \quad \mathbf{C} = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ ...(ii)

$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = -\frac{1}{1}\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, from (i) and (ii)

$= \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ **Ans.**

Ex. 5 (a). Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$

Sol. Here $|\mathbf{A}|$

$= \begin{vmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 5 & 1 & 1 \\ 9 & 1 & 2 \end{vmatrix}$, replacing C_1, C_3 by $C_1 + C_2, C_3 + C_2$

or $\quad |\mathbf{A}| = \begin{vmatrix} 5 & 1 \\ 9 & 2 \end{vmatrix} = 10 - 9 = 1 \neq 0$ ...(i)

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1; C_{22} = -\begin{vmatrix} 4 & 0 \\ 8 & 1 \end{vmatrix} = -4; C_{13} = \begin{vmatrix} 4 & 1 \\ 8 & 1 \end{vmatrix} = -4;$

$C_{21} = -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2; C_{22} = \begin{vmatrix} 1 & 1 \\ 8 & 1 \end{vmatrix} = -7; C_{23} = -\begin{vmatrix} 1 & -1 \\ 8 & 1 \end{vmatrix} = -9:$

$C_{31} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1, C_{32} = -\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = 4; C_{33} = \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} = 5$

$\mathbf{C} = \begin{bmatrix} 1 & -4 & -4 \\ 2 & -7 & -9 \\ -1 & 4 & 5 \end{bmatrix}$

$\therefore$ Adj. $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$

$\therefore$ $\mathbf{A}^{-1} = \dfrac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \mathbf{C}' = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$, from (i).

Ans.

E. 5 (b). Find the inverse $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{5} & \mathbf{7} & \mathbf{8} \\ \mathbf{9} & \mathbf{10} & \mathbf{12} \end{bmatrix}$

Hint : Do as Ex. 5 (a) above. Ans. $\begin{bmatrix} -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{13}{24} & -\frac{1}{3} & \frac{1}{8} \end{bmatrix}$

Ex. 5 (c). Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \end{bmatrix}$

Hint : Do as Ex, 5 (a) Page 57 **Ans.** $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Ex. 6 (a). Find adj A and $\mathbf{A}^{-1}$ **when** $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \end{bmatrix}$

(Bundelkhand 94; Kanpur 93)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_1$ respectively.

$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ ...(i)

Also for the matrix $\mathbf{A}$, we have

$C_{11} = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$; $C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$; $C_{13} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$;

$C_{21} = -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3$; $C_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$; $C_{23} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$;

$C_{31} = \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3$: $C_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$, $C_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$.

$\therefore \quad \mathbf{C}=\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$\therefore \quad \text{Adj } \mathbf{A}=\mathbf{C}'=\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ ...(ii)

$\therefore \quad \mathbf{A}^{-1}=\dfrac{\text{Adj } \mathbf{A}}{|\mathbf{A}|}=\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, from (i) and (ii) **Ans.**

Ex. 6 (b). Find the inverse of the matrix $\mathbf{A}=\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ *(Lucknow 91)*

Sol. Here $|\mathbf{A}|=\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}=\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$, expanding with respect to C_1

or $\qquad |\mathbf{A}|=1-0=1.$ ...(i)

Also for the matrix **A**, we have

$C_{11}=\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}=1;\ C_{12}=-\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}=0;\ C_{13}=\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}=0;$

$C_{21}=-\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}=-1;\ C_{22}=\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}=1;\ C_{23}=-\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}=0;$

$C_{31}=\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=0;\ C_{32}=-\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}=-1;\ C_{33}=\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}=1$

$\therefore \quad \mathbf{C}=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A}=\mathbf{C}'=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ ...(ii)

$\therefore \quad \mathbf{A}^{-1}=\dfrac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|}=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. from (i) and (ii) **Ans.**

***Ex. 7 (a). If** $\mathbf{A}=\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-15} & \mathbf{6} & \mathbf{-5} \\ \mathbf{5} & \mathbf{-2} & \mathbf{2} \end{bmatrix}$, **find adj. A and** $\mathbf{A}^{-1}$. *(Garhwal 95, 91; Meerut 95)*

Sol. Here $|\mathbf{A}|$

$$= \begin{vmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 3 & 6 & 1 \\ -1 & -2 & 0 \end{vmatrix}, \text{ replacing } C_1, C_3 \text{ by } C_1 + 3C_2, C_2 + C_3$$

$$= \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} = 1 \qquad \text{...(i)}$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 6 & -5 \\ -2 & 2 \end{vmatrix} = 2$; $C_{12} = -\begin{vmatrix} -15 & -5 \\ 5 & 2 \end{vmatrix} = 5$; $C_{13} = \begin{vmatrix} -15 & 6 \\ 5 & -2 \end{vmatrix} = 0$

$C_{21} = -\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = 0$; $C_{22} = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 1$; $C_{23} = -\begin{vmatrix} 3 & -1 \\ 5 & -2 \end{vmatrix} = 1$;

$C_{31} = \begin{vmatrix} -1 & 1 \\ 6 & -5 \end{vmatrix} = -1$; $C_{32} = -\begin{vmatrix} 3 & 1 \\ -15 & -5 \end{vmatrix} = 0$; $C_{33} = \begin{vmatrix} 3 & -1 \\ -15 & 6 \end{vmatrix} = 3$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \text{ from (i)}$$

Ans.

Ex. 7 (b). Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$$

(Gorakhpur 97)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{vmatrix}$, replacing R_2 by $R_2 + R_1$

$$= \begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix} = 5 - 4 = 1. \qquad \text{...(i)}$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3$; $C_{12} = -\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $C_{13} = \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2$;

$C_{21} = -\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = 2$; $C_{22} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$; $C_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = 2$;

$C_{31} = \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 6; C_{32} = -\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = 2; C_{33} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5$

$\therefore \quad \mathbf{C} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ ...(ii)

$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$, from (i) and (ii) **Ans.**

Ex. 7 (c). If $\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$, find $\mathbf{A}^{-1}$.

Hint : Do as Ex. 7 (a) Page 59. **Ans.** $(1/6)\begin{bmatrix} 2 & -4 & -4 \\ 1 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

***Ex. 7 (d). If $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$, find adj. A and $\mathbf{A}^{-1}$**

(Avadh 94)

Hint : Do as Ex. 7 (a). Page 60. **Ans.** $(1/20)\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & 8 \\ -18 & 6 & 4 \end{bmatrix}$

Ex. 8 (a). Find the reciprocal or inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

(Kumaun 92)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -3 & -2 \\ 0 & -2 & -3 \\ 1 & 2 & 2 \end{vmatrix}$, applying $R_1 - 2R_3, R_2 - 2R_3$

$= \begin{vmatrix} -3 & -2 \\ -2 & -3 \end{vmatrix} = 9 - 4 = 5$

Also we have

$C_{11} = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2; C_{12} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3; C_{13} = \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2;$

$C_{21} = -\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2; C_{22} = \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2; C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3;$

$C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3;\ C_{32} = -\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 2;\ C_{33} = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2$

$\therefore\ \mathbf{C} = \begin{bmatrix} 2 & -3 & 2 \\ 2 & 2 & -3 \\ -3 & 2 & 2 \end{bmatrix}$

$\therefore$ Adj. $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$

$\therefore$ Reciprocal of $\mathbf{A} = \mathbf{A}^{-1}$

$$= \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{5}\begin{bmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$$

Ans.

Ex. 8 (b). Find the adjoint and inverse of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$.

Hint : Do as Ex. 8 (a) above.

Ans. $\begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$

Ex. 9 (a). Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

(Avadh 90; Bundelkhand 96, 95; Garhwal 96, 94; Gorakhpur 96; Purvanchal 97)

Sol. For the given matrix **A**, we have

$C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1;\ C_{12} = -\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} = -3;\ C_{13} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 1;$

$C_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3;\ C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1;\ C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1;$

$C_{31} = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4;\ C_{32} = -\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0;\ C_{33} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4$

$\therefore\ \mathbf{C} = \begin{bmatrix} 1 & -3 & 1 \\ -3 & 1 & 1 \\ 4 & 0 & -4 \end{bmatrix}$

$\therefore$ Adj. $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$

and $\quad |\mathbf{A}| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix}$, replacing C_3 by $C_3 - C_1$

$$= \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4$$

The inverse of $\mathbf{A} = \dfrac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|}$

$$= -\frac{1}{4}\begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

Ans.

Ex. 9 (b). Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ *(Meerut 98*

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -5 & 3 \end{vmatrix}$, replacing R_2, R_3 by $R_2 + R_1$, $R_3 - 2R_1$ respectively.

$$= \begin{vmatrix} 3 & 1 \\ -5 & 3 \end{vmatrix} = 9 + 5 = 14 \qquad \text{...(i)}$$

Also for the matrix $\mathbf{A}$, we have

$C_{11} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$; $C_{12} = -\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = 5$; $C_{13} = \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = -1$;

$C_{21} = -\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = -1$; $C_{22} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$; $C_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 5$;

$C_{31} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$; $C_{32} = -\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = -1$; $C_{33} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$

$\therefore \qquad \mathbf{C} = \begin{bmatrix} 3 & 5 & -1 \\ -1 & 3 & 5 \\ 5 & -1 & 3 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix} \qquad \text{...(ii)}$

$\therefore \quad \mathbf{A}^{-1} = \dfrac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = \dfrac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$, from (i), (ii)

Ans.

Ex. 9 (c). Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ *(Purvanchal 95)*

Hint : Do as Ex. 9 (a) above.

Ans. $\frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$.

Ex. 10. If $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, **find** $\mathbf{A}^2$, **and show that** $\mathbf{A}^2 = \mathbf{A}^{-1}$.

Sol.
$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1-2+1 & -1+1+0 & 1+0+0 \\ 2-2+0 & -2+1+0 & 2+0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad ...(i)$$

Also $|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{vmatrix}$, replacing C_2, C_3 by $C_2 + C_1, C_3 - C_1$ respectively.

$$= \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = -1 + 2 = 1. \quad ...(ii)$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} = 0$; $C_{12} = -\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = 0$; $C_{13} = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 1$;

$C_{21} = -\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = 0$; $C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$; $C_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1$;

$C_{31} = \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 1$; $C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2$; $C_{33} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1$

$\therefore$ $\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$\therefore$ $$\text{Adj } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad ...(iii)$$

$\therefore$ $$\mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}, \text{ from (ii) and (iii)}$$

$= A^2$, from (i) **Hence proved.**

Ex. 11. Find the adjoint of matrix A and hence find A^{-1}.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Meerut 96)

Sol. Here $|A|$

$$= \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}, \text{ expanding w.r. to } C_3$$

$$= \cos^2\theta - (-\sin^2\theta) = 1 \qquad \text{...(i)}$$

Also we have

$$C_{11} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta;\ C_{12} = -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin\theta;$$

$$C_{13} = \begin{vmatrix} \sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} = 0;\ C_{21} = -\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta;$$

$$C_{22} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta;\ C_{23} = -\begin{vmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{vmatrix} = 0;$$

$$C_{31} = \begin{vmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} = 0;\ C_{32} = -\begin{vmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{vmatrix} = 0;$$

$$C_{33} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1$$

$$\therefore \quad C = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans.

And $$A^{-1} = \frac{\text{Adj. } A}{|A|} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ from (i)}$$

Ans.

***Ex. 12. How will you use the notion of determinant to compute the inverse of a non-singular square matrix ? Compute the inverse of the matrix**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

Sol. For the first part See Theorem I, result (iv) Page 50 of this chapter

For the second part we have for the matrix **A**

$C_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} = 2$; $C_{12} = -\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix} = 2$; $C_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3$;

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 8 & 10 \end{vmatrix} = 4$; $C_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 10 \end{vmatrix} = -11$; $C_{23} = -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 6$;

$C_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3$; $C_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6$; $C_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 2 & 2 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$\therefore \quad \textbf{Adj. A} = \mathbf{C}' = \begin{bmatrix} 2 & 4 & -3 \\ 2 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix} \qquad \text{...(i)}$$

Also $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 7 & -6 & -11 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$, $C_3 - 3C_1$ respectively.

$$= \begin{vmatrix} -3 & -6 \\ -6 & -11 \end{vmatrix} = 33 - 36 = -3$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\textbf{Adj. A}}{|\mathbf{A}|} = -\frac{1}{3}\begin{bmatrix} 2 & 4 & -3 \\ 2 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Ans.

***Ex. 13. If $\mathbf{A}^t$ denotes the transpose of a matrix A and**

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}, \textbf{ find } (A^t)^{-1}$$

Sol. $\mathbf{A}' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$, by definition of transpose of a matrix

$= \mathbf{B}$ (say).

Now $|\mathbf{B}| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & -1 & -2 \\ 3 & 4 & 7 \end{vmatrix}$, replacing C_3 by $C_3 + 2C_1$

$$= \begin{vmatrix} -1 & -2 \\ 4 & 7 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$= (-1)(7) - (-2)(4) = -7 + 8 = 1 \neq 0.$$

Also we have

$$C_{11} = \begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = -9; C_{12} = -\begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} = 8; C_{13} = \begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} = -5;$$

$$C_{21} = -\begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix} = -8; C_{22} = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 7; C_{23} = -\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -4;$$

$$C_{31} = \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} = -2; C_{32} = -\begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = 2; C_{33} = \begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} = -1$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{B} = \mathbf{C}' = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$\therefore \quad \mathbf{B}^{-1} = \frac{\text{Adj. } \mathbf{B}}{|\mathbf{B}|} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

or $$(\mathbf{A}^t)^{-1} = \mathbf{B}^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

Ans.

***Ex. 14. Find the inverse of the matrix A, where**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{bmatrix}$$

Hint : Do as Ex. 12 Page 65.

Ans. $(1/25)\begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}$

Ex. 15. Find the adjoint and inverse of the matrix

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

(Bundelkhand 92)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$

$$= \cos^2\alpha + \sin^2\alpha = 1 \neq 0. \quad \text{...(i)}$$

$C_{11} = \cos\alpha$, $C_{12} = -\sin\alpha$, $C_{21} = -(-\sin\alpha) = \sin\alpha$ and $C_{22} = \cos\alpha$ **(Note)**

$\therefore \quad C = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ ...(ii)

$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{\text{Adj. } A}{|A|} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$,

substituting values from (i) and (ii). **Ans.**

****Ex. 16. Find the inverse of** $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ *(Meerut 91S)*

Sol. For the given matrix A, we have

$C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1;\ C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8;\ C_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5;$

$C_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1;\ C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6;\ C_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3;$

$C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1;\ C_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2;\ C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$

$\therefore \quad C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$

$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

and $\quad |A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix}$, replacing C_3 by $C_3 - 2C_2$

or $\quad |A| = -\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} = -2 \neq 0$

$\therefore$ Inverse of $A = \dfrac{\text{Adj. } A}{|A|} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$. **Ans.**

Ex. 17 (a). Find the inverse of the matrix $A = \begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ **over the field of the complex numbers.**

Sol. Here $|\mathbf{A}| = \begin{vmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & i & 2i \\ 0 & 2 & 2 \\ 0 & -1 & 1 \end{vmatrix}$, interchanging C_1 and C_2

or $\quad |\mathbf{A}| = \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}$, expanding with respect to C_1

$$= 2 - (-2) = 4 \neq 0. \qquad \text{...(i)}$$

Also for this matrix **A**, we have

$C_{11} = \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0$; $C_{12} = -\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} = -4$; $C_{13} = \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} = 0$;

$C_{21} = -\begin{vmatrix} -1 & 2i \\ 0 & 1 \end{vmatrix} = 1$; $C_{22} = \begin{vmatrix} i & 2i \\ -1 & 1 \end{vmatrix} = 3i$; $C_{23} = -\begin{vmatrix} i & -1 \\ -1 & 0 \end{vmatrix} = 1$;

$C_{31} = \begin{vmatrix} -1 & 2i \\ 0 & 2 \end{vmatrix} = -2$; $C_{32} = -\begin{vmatrix} i & 2i \\ 2 & 2 \end{vmatrix} = 2i$, $C_{33} = \begin{vmatrix} i & -1 \\ 2 & 0 \end{vmatrix} = 2$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 0 & -4 & 0 \\ 1 & 3i & 1 \\ -2 & 2i & 2 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 0 & 1 & -2 \\ -4 & 3i & 2i \\ 0 & 1 & 2 \end{bmatrix}.$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{4}\begin{bmatrix} 0 & 1 & -2 \\ -4 & 3i & 2i \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ -1 & \frac{3}{4}i & \frac{1}{2}i \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Ans.

***Ex. 17 (b). If $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then show that $\mathbf{A}^{-1} = \mathbf{A}$** *(Bundelkhand 91)*

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad$ **...(i)**

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$; $C_{12} = -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$; $C_{13} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$;

$C_{21} = -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$; $C_{22} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$; $C_{23} = -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$;

$C_{31} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$; $C_{32} = -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$; $C_{33} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$\therefore \quad C = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{\text{Adj } A}{|A|} = \dfrac{1}{(-1)} \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = A$

Hence proved.

Ex. 18. Find the inverse of the matrix

$$A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}, \text{ if } a^2+b^2+c^2+d^2 = 1$$

Sol. For this matrix, we have

$C_{11} = a - ib;\ C_{12} = -(-c+id) = c - id;$

$C_{21} = -(c+id);\ C_{22} = a + ib$

$\therefore \quad C = \begin{bmatrix} a-ib & c-id \\ -c-id & a+ib \end{bmatrix}$

$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

Also $|A| = \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$$= (a+ib)(a-ib) - (c+id)(-c+id)$$

$$= a^2 - i^2b^2 + c^2 - i^2d^2 = a^2 + b^2 + c^2 + d^2 = 1 \neq 0.$$

$\therefore \quad \text{Inverse of } A = \dfrac{\text{Adj. } A}{|A|} = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

Ans.

****Ex. 19. If** $\alpha + i\beta = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, **verify that**

$$(\alpha + i\beta)^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}^{-1}$$

Sol. $(\alpha + i\beta)^{-1} = \dfrac{1}{\alpha + i\beta} = \dfrac{(\alpha - i\beta)}{(\alpha + i\beta)(\alpha - i\beta)}$,

multiplying num. and denom. by $\alpha - i\beta$

$$= (\alpha - i\beta)/(\alpha^2 + \beta^2). \qquad \text{...(i)}$$

Again let $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

Then $|\mathbf{A}| = \begin{vmatrix} \alpha & \beta \\ -\beta & \alpha \end{vmatrix} = \alpha(\alpha) - \beta(-\beta) = \alpha^2 + \beta^2 \neq 0.$ ...(ii)

Also for the matrix **A**, we have

$C_{11} = \alpha;\ C_{12} = \beta;\ C_{21} = -\beta;\ C_{22} = \alpha$

$\therefore\quad \mathbf{C} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ and Adj $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

$$\therefore\quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}}{\alpha^2 + \beta^2}, \text{ from (ii)}$$

i.e. $$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}^{-1} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \div (\alpha^2 + \beta^2) \quad \text{...(iii)}$$

Also we are given $\alpha + i\beta = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

Replacing β by $-\beta$ we get $(\alpha - i\beta) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$.

$\therefore\quad$ From (i) we have $(\alpha + i\beta)^{-1} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \div (\alpha^2 + \beta^2)$ ...(iv)

Hence from (iii) and (iv), we have

$$(\alpha + i\beta)^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}^{-1}$$

Hence proved

Ex. 20. If $\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ -9 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \end{bmatrix}$, find $\mathbf{A}^{-1}$.

Sol. Here $|\mathbf{A}| = \begin{vmatrix} -1 & 0 & 0 & 2 \\ -9 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ -9 & 1 & 0 & -17 \\ 1 & 0 & 2 & 1 \\ -4 & 1 & -3 & -7 \end{vmatrix}$,

replacing C_4 by $C_4 + 2C_1$

$$= -\begin{vmatrix} 1 & 0 & -17 \\ 0 & 2 & 1 \\ 1 & -3 & -7 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$= -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & -3 & 10 \end{vmatrix}, \text{ replacing } C_3 \text{ by } C_3 + 17C_1$$

$$= - \begin{vmatrix} 2 & 1 \\ -3 & 10 \end{vmatrix} = -[20+3] = -23 \neq 0. \qquad \text{...(i)}$$

Also for the matrix, A, we have

$$C_{11} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} = -3;$$

$$C_{12} = - \begin{vmatrix} -9 & 0 & 1 \\ 1 & 2 & -1 \\ -4 & -3 & 1 \end{vmatrix} = - \begin{vmatrix} 0 & 0 & 1 \\ -8 & 2 & -1 \\ 5 & -3 & 1 \end{vmatrix} = - \begin{vmatrix} -8 & 2 \\ 5 & -3 \end{vmatrix} = -14;$$

$$C_{13} = \begin{vmatrix} -9 & 1 & 1 \\ 1 & 0 & -1 \\ -4 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 5 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & -1 \\ 5 & 0 \end{vmatrix} = -5;$$

$$C_{14} = - \begin{vmatrix} -9 & 1 & 0 \\ 1 & 0 & 2 \\ -4 & 1 & -3 \end{vmatrix} = - \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 5 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 5 & -3 \end{vmatrix} = -13;$$

$$C_{21} = - \begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 1 & -3 & 1 \end{vmatrix} = - \begin{vmatrix} 0 & 2 \\ 2 & -1 \end{vmatrix} = 4;$$

$$C_{22} = \begin{vmatrix} -1 & 0 & 2 \\ 1 & 2 & -1 \\ -4 & -3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ -4 & -3 & -7 \end{vmatrix} = - \begin{vmatrix} 2 & 1 \\ -3 & -7 \end{vmatrix} = 11;$$

$$C_{23} = - \begin{vmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ -4 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ -4 & 1 & -7 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -7 \end{vmatrix} = -1;$$

$$C_{24} = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ -4 & 1 & -3 \end{vmatrix} = - \begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} = 2;$$

$$C_{31} = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & -3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix} = -6;$$

$$C_{32} = - \begin{vmatrix} -1 & 0 & 2 \\ -9 & 0 & 1 \\ -4 & -3 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 0 & 0 \\ -9 & 0 & -17 \\ -4 & -3 & -7 \end{vmatrix} = \begin{vmatrix} 0 & -17 \\ -3 & -7 \end{vmatrix} = -51;$$

$$C_{33} = \begin{vmatrix} -1 & 0 & 2 \\ -9 & 1 & 1 \\ -4 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -9 & 1 & -17 \\ -4 & 1 & -7 \end{vmatrix} = - \begin{vmatrix} 1 & -17 \\ 1 & -7 \end{vmatrix} = -10;$$

$$C_{34} = - \begin{vmatrix} -1 & 0 & 0 \\ -9 & 1 & 0 \\ -4 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix} = -3;$$

$$C_{41} = -\begin{vmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -4;$$

$$C_{42} = \begin{vmatrix} -1 & 0 & 2 \\ -9 & 0 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -2\begin{vmatrix} -1 & 2 \\ -9 & 1 \end{vmatrix} = -2\,(17) = -34;$$

$$C_{43} = -\begin{vmatrix} -1 & 0 & 2 \\ -9 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 0 \\ -9 & 1 & -17 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -17 \\ 0 & 1 \end{vmatrix} = 1;$$

$$C_{44} = \begin{vmatrix} -1 & 0 & 0 \\ -9 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -2$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} -3 & -14 & -5 & -13 \\ 4 & 11 & -1 & 2 \\ -6 & -51 & -10 & -3 \\ -4 & -34 & 1 & -2 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} -3 & 4 & -6 & -4 \\ -14 & 11 & -51 & -34 \\ -5 & -1 & -10 & 1 \\ -13 & 2 & -3 & -2 \end{bmatrix}$$

$$= -\begin{bmatrix} 3 & -4 & 6 & 4 \\ 14 & -11 & 51 & 34 \\ 5 & 1 & 10 & -1 \\ 13 & -2 & 3 & 2 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{23}\begin{bmatrix} 3 & -4 & 6 & 4 \\ 14 & -11 & 51 & 34 \\ 5 & 1 & 10 & -1 \\ 13 & -2 & 3 & 2 \end{bmatrix}$$

Ans.

***Ex. 21. Porve that $|Adj\,(Adj\,\mathbf{A})| = |\mathbf{A}|^{(n-1)^2}$, if $|\mathbf{A}| \neq 0$ and is any $n \times n$ matrix.** *(Agra 90)*

Sol. We know that

$$|Adj\,\mathbf{A}| = |\mathbf{A}|^{n-1}, \text{ if } |\mathbf{A}| \neq 0. \qquad \text{...(i)}$$

(See Th. II. Page 50 Ch. V)

Replacing **A** by Adj **A** in (i), we get

$$|Adj\,(Adj\,\mathbf{A})| = |Adj\,\mathbf{A}|^{n-1}$$

$$= \{|Adj\,\mathbf{A}|\}^{n-1} \qquad \textbf{(Note)}$$

$$= \{|\mathbf{A}|^{n-1}\}^{n-1}, \text{ from (i)}$$

$$= \{|\mathbf{A}|\}^{(n-1)^2} = |\mathbf{A}|^{(n-1)^2}.$$

Hence proved.

***Ex. 22. Prove that** $Adj\,(Adj\,\mathbf{A}) = |\mathbf{A}|^{n-2} \bullet \mathbf{A}$, **where A is any n × n matrix.** *(Agra 92, 90; Kanpur 90)*

Sol. We know that

$$\mathbf{A}\,.\,(Adj\,\mathbf{A}) = |\mathbf{A}| \bullet \mathbf{I} \qquad \text{(See Th. I Page 49 Ch. V)}$$

or $$Adj\,\{\mathbf{A} \bullet (Adj\,\mathbf{A})\} = Adj\,\{|\mathbf{A}| \bullet \mathbf{I}\}$$

or $$Adj\,(Adj\,\mathbf{A}) \bullet (Adj\,\mathbf{A}) = |\mathbf{A}|^{n-1} \bullet \mathbf{I} \qquad \text{(See Th. III. Page 50 Ch. V)}$$

or $$Adj\,(Adj\,\mathbf{A}) \bullet (Adj\,\mathbf{A})\,\mathbf{A} = |\mathbf{A}|^{n-1} \bullet \mathbf{I} \bullet \mathbf{A}$$

or $$Adj\,(Adj\,\mathbf{A}) \bullet |\mathbf{A}| \bullet \mathbf{I} = |\mathbf{A}|^{n-1}\,\mathbf{AI}, \qquad \text{See Th. I P. 49 Ch. V}$$

or $$Adj\,(Adj\,\mathbf{A}) \bullet |\mathbf{A}| = |\mathbf{A}|^{n-1} \bullet \mathbf{A} \qquad \textbf{(Note)}$$

or $$Adj\,(Adj\,\mathbf{A}) = |\mathbf{A}|^{n-2} \bullet \mathbf{A}.$$ Hence proved.

Exercises on § 5·09

Find the inverse of the following matrices

Ex. 1. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \\ 0 & -1 & 3 \end{bmatrix}$ **Ans.** $\frac{1}{10}\begin{bmatrix} 9 & 1 & 2 \\ -3 & 3 & -4 \\ -1 & 1 & 2 \end{bmatrix}$

Ex. 2. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Ex. 3. $\begin{bmatrix} 3 & 2 & -1 \\ -1 & 2 & 3 \\ -3 & 1 & 3 \end{bmatrix}$ **Ans.** $\frac{1}{8}\begin{bmatrix} -3 & 7 & -8 \\ 6 & -6 & 8 \\ -5 & 9 & -8 \end{bmatrix}$

Ex. 4. $\begin{bmatrix} 2 & -4 & -2 \\ 4 & 6 & 2 \\ 0 & 10 & -4 \end{bmatrix}$ **Ans.** $-\frac{1}{58}\begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & -5 & -7 \end{bmatrix}$

Ex. 5. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ *(Gorakhpur 91; Kanpur 94)* **Ans.** $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Ex. 6. $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ **Ans.** $\frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & 5 \\ -5 & 7 & -1 \end{bmatrix}$

Ex. 7. $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **Ans.** $\frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$

***Ex. 8.** $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ **Ans.** $\frac{1}{3}\begin{bmatrix} -6 & 5 & -1 \\ 15 & -8 & 1 \\ -6 & 3 & 0 \end{bmatrix}$

Ex. 9. $\begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$ **Ans.** $\frac{1}{29}\begin{bmatrix} 11 & 9 & -1 \\ -4 & 2 & 3 \\ -10 & 5 & -7 \end{bmatrix}$

Ex. 10. $\begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

Ex. 11. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ **Ans.** Not possible as $|\mathbf{A}| = 0$.

Ex. 12. $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$

Ex. 13. $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix}$

Ex. 14. $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$

(Kumaun 90)

Ex. 15. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ *(Kumaun 93)* **Ans.** $\begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Ex. 16. Verify that $\mathbf{A} \bullet (\text{Adj. } \mathbf{A}) = (\text{Adjj. A}) \bullet \mathbf{A} = |\mathbf{A}|\, \mathbf{I}_3$, where $\mathbf{I}_3$ is the identity matrix of order 3, and $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 17. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$, verify that

$\mathbf{A} \bullet (\text{Adj. A}) = (\text{Adj. } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}$ *(Meerut 96P)*

***Ex. 18.** Verify that $\mathbf{A} \bullet (\text{Adj. } \mathbf{A}) = (\text{Adj. A}) \bullet \mathbf{A} = |\mathbf{A}|\, \mathbf{I}_2$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \text{ and } \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**§ 5·10. Existence of Inverse.

An Important Theorem.

The necessary and sufficient condition that a square matrix may possess an inverse is that it be non-singular. *(Bundelkhand 96, 92; Kumaun 96; Gorakhpur 99; Meerut 92; Purvanchal 98)*

Proof. The condition is necessary.

If **A** is an $n \times n$ matrix and **B** is its inverse then by definition of the inverse we have $\quad \mathbf{AB} = \mathbf{I}_n \quad$ (See Chapter II)

Taking the determinants of both sides we get

$$|\mathbf{AB}| = |\mathbf{I}_n|. \quad \text{...(i)}$$

But $\quad |\mathbf{AB}| = |\mathbf{A}| \bullet |\mathbf{B}| \quad$ .... See Chapter IV

and $\quad |\mathbf{I}_n| = 1$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix

$\therefore$ From (i) we get $|\mathbf{A}| \bullet |\mathbf{B}| = 1$,

which implies that $|\mathbf{A}| \neq 0$.

$\therefore$ The matrix **A** is non-singular. .... See Chapter IV

The condition is sufficient.

If **A** is an $n \times n$ non-singular matrix and there be another matrix **B** defined

by
$$\mathbf{B} = \frac{1}{|\mathbf{A}|} (\text{Adj. } \mathbf{A})$$

Then
$$\mathbf{AB} = \mathbf{A} \frac{1}{|\mathbf{A}|} (\text{Adj. } \mathbf{A}) = \frac{1}{|\mathbf{A}|} (\mathbf{A} \bullet \text{Adj. } \mathbf{A})$$

$$= \frac{1}{|\mathbf{A}|} \bullet |\mathbf{A}| \mathbf{I}_n \quad \text{... See § 5·09 Th. I Page 49 Ch. V}$$

$$= \mathbf{I}_n$$

Similarly
$$\mathbf{BA} = \frac{1}{|\mathbf{A}|} (\text{Adj } \mathbf{A}) \bullet \mathbf{A} = \frac{1}{|\mathbf{A}|} [(\text{Adj. } \mathbf{A}) \bullet \mathbf{A}]$$

$$= \frac{1}{|\mathbf{A}|} \bullet |\mathbf{A}| \mathbf{I}_n \quad \text{...See § 5.09 Th. I Page 49 Ch. V}$$

$$\therefore \quad \mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

$\therefore$ **B** is the inverse of **A** and it exists.

§ 5·11. Some Important Theorems.

***Theorem I.** *If* **A** *is a non-singular matrix of order n such that* $\mathbf{AX} = \mathbf{AY}$, *then* $\mathbf{X} = \mathbf{Y}$.

Proof. If A is non-singular matrix, then $\mathbf{A}^{-1}$ exists. ...See § 5·10 above

Given $\quad \mathbf{AX} = \mathbf{AY}$

or $\quad \mathbf{A}^{-1} (\mathbf{AX}) = \mathbf{A}^{-1} (\mathbf{AY})$

or $\quad (\mathbf{A}^{-1} \mathbf{A}) \mathbf{X} = (\mathbf{A}^{-1} \mathbf{A}) \mathbf{Y}$

or $\quad \mathbf{IX} = \mathbf{IY} \quad \because \mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$

or $\quad \mathbf{X} = \mathbf{Y}$, by left cancellation law. Hence proved.

Theorem II. *The inverse of transpose of a matrix is the transpose of the inverse.*

Proof. Let **A** be the given matrix. Then its inverse is $\mathbf{A}^{-1}$.

Also we have $\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$, by definition.

$\therefore \quad (\mathbf{AA}^{-1})' = \mathbf{I}' = (\mathbf{A}^{-1}\mathbf{A})'$, taking transpose.

or $\quad (\mathbf{A}^{-1})'\mathbf{A}' = \mathbf{I} = \mathbf{A}'(\mathbf{A}^{-1})', \qquad \because (\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$ and $\mathbf{I}' = \mathbf{I}$.

Hence $\mathbf{A}'$ is invertible *i.e.* $\mathbf{A}'$ possesses inverse

and $\quad (\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$

i.e. the inverse of a transpose of a matrix is the transpose of the inverse.

Hence proved.

Theorem III. *If* **A**, **B** *are any two* $n \times n$ *matrices such that* $\mathbf{BA} = \mathbf{O}$, *where* **O** *is the null matrix, then at least one of them is singular.*

Proof. Since **A**, **B** are two $n \times n$ matrices

so $\quad \mathbf{AB} = \mathbf{O}$, where **O** is the null matrix

$\Rightarrow \quad |\mathbf{A}| \bullet |\mathbf{B}| = 0$ **(Note)**

$$\Rightarrow \quad \begin{cases} \text{either } |\mathbf{A}| = 0, \text{ which means } \mathbf{A} \text{ is singular} \\ \text{or } \quad |\mathbf{B}| = 0, \text{ which means } \mathbf{B} \text{ is singular} \\ \text{or both } |\mathbf{A}| \text{ and } |\mathbf{B}| \text{ are zero which means both } \mathbf{A} \text{ and } \mathbf{B} \text{ are singular.} \end{cases}$$

Hence at least one of **A** and **B** is singular.

Theorem IV. *The inverse of the inverse of a matrix is the matrix itself i.e.* $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$, *where* $\mathbf{A}^{-1}$ *is the inverse of* **A**.

Proof. Let **A** be the given matrix. Then its inverse is $\mathbf{A}^{-1}$.

Also by definition $\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$.

$\therefore \quad \mathbf{A}^{-1}$ is invertible and we have $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

i.e. the inverse of the inverse of **A** is **A** itself. Hence proved.

Theorem V. *If a non singular matrix* **A** *is symmetric, then* $\mathbf{A}^{-1}$ *is also symmetric.*

Proof. If **A** is symmetric, then $\mathbf{A} = \mathbf{A}'$ ...(i)

Also by definition if **A** is non-singular, then

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A} &= \mathbf{I} \\ &= \mathbf{I}', \text{ since } \mathbf{I}' = \mathbf{I} \\ &= (\mathbf{AA}^{-1})', \text{ since } \mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} \\ &= (\mathbf{A}^{-1})'\mathbf{A}', \text{ since } (\mathbf{AB}') = \mathbf{B}'\mathbf{A}' \end{aligned}$$

i.e $\quad \mathbf{A}^{-1}\mathbf{A} = (\mathbf{A}^{-1})'\mathbf{A}$, since $\mathbf{A} = \mathbf{A}'$, from (i).

or $\quad \mathbf{A}^{-1} = (\mathbf{A}^{-1})'$, by right cancellation law.

Hence $\mathbf{A}^{-1}$ is symmetric by definition. Hence proved.

Theorem VI. *The inverse of the transposed conjugate of a non-singular matrix* **A** *is the transposed conjugate of the inverse of* **A**

i.e. $$(\mathbf{A}^{\Theta})^{-1} = (\mathbf{A}^{-1})^{\Theta}$$

Proof. If A is a non-singular matrix, then **A** is invertible and we have

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

or $$(\mathbf{AA}^{-1})^{\Theta} = \mathbf{I}^{\Theta} = (\mathbf{A}^{-1}\mathbf{A})^{\Theta}$$

or $$(\mathbf{A}^{-1})^{\Theta}\,\mathbf{A}^{\Theta} = \mathbf{I} = \mathbf{A}^{\Theta}\,(\mathbf{A}^{-1})^{\Theta}, \text{ since } (\mathbf{AB})^{\Theta} = \mathbf{B}^{\theta}\mathbf{A}^{\Theta}, \mathbf{I}^{\Theta} = \mathbf{I}.$$

$\therefore$ $\mathbf{A}^{\Theta}$ is invertible and we have $(\mathbf{A}^{\Theta})^{-1} = (\mathbf{A}^{-1})^{\Theta}$. Hence proved.

§5·12. Theorem. *If r be the rank of a matrix* **A** *of order* $m \times n$; $\mathbf{A}_r$ *be the normal form of* **A**, **R** *be the product of elementary matrices of order m and* **S** *be the product of elementary matrices of order n, then* $\mathbf{A}_r = \mathbf{RAS}$.

Proof. Since **R** and **S** are non-singular (*i.e.* their inverses exist), therefore

$\mathbf{R}^{-1}\mathbf{A}_r\mathbf{S}^{-1} = \mathbf{A}$, where $\mathbf{R}^{-1}$ and $\mathbf{S}^{-1}$ are the inverses of **R** and **S** respectively.

or $$\mathbf{A} = \mathbf{B}\,\mathbf{A}_r\,\mathbf{C}, \text{ where } \mathbf{B} = \mathbf{R}^{-1}, \mathbf{C} = \mathbf{S}^{-1}$$

or $$\mathbf{A}_r = \mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{C}^{-1}.$$ **(Note)**

Now if **A** is a non-singular matrix of order *n*, then $r = n$ and

$$\mathbf{A}_r = \mathbf{I}_n$$

Hence $$\mathbf{A} = \mathbf{B}\,\mathbf{I}_n\,\mathbf{C},$$

which is of the form $\mathbf{A} = \mathbf{B}$, since **B** and **C** are the product of elementary matrices.

Cor. *If two matrices* **A** *and* **B** *are of the same order* $m \times n$ *and same rank, then there exists non-singular square matrices* **P, Q** *such that* $\mathbf{B} = \mathbf{PAQ}$.

Proof. From above theorem we find that

$$\mathbf{A} = \mathbf{CA}_r\,\mathbf{D}, \mathbf{B} = \mathbf{C}_1\,\mathbf{A}_r\,\mathbf{D}_1$$

where $\mathbf{C}, C_1$ are product of elementary matrices of order m and $\mathbf{D}, \mathbf{D}_1$ of order n.

From $\mathbf{A} = \mathbf{CA}_r\,\mathbf{D}$, we get $\mathbf{A}_r = \mathbf{C}^{-1}\,\mathbf{AD}^{-1}$

Substituting this in $\mathbf{B} = \mathbf{C}_1\,\mathbf{A}_r\,\mathbf{D}_1$, we get

$$\mathbf{B} = \mathbf{C}_1\,(\mathbf{C}^{-1}\,\mathbf{A}\,\mathbf{D}^{-1})\,\mathbf{D}_1 = (\mathbf{C}_1\,\mathbf{C}^{-1})\,\mathbf{A}\,(\mathbf{D}^{-1}\,\mathbf{D}_1)$$

which is of the form $\mathbf{B} = \mathbf{PAQ}$.

Solved Examples on § 5·12

Ex. 1 (a). Find the non-singular matrices R and S, such that RAS is the normal form, where $\mathbf{A} = \begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix}$

Sol. Here we find that A is a 2×3 matrix

$\therefore \qquad [A]_{2\times 3} = I_2 \, A \, I_3$

or $\qquad \begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now we are to bring L.H.S. to the normal form by applying elementary row and column operations.

$\therefore \qquad \begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, by $R_1\left(\frac{1}{2}\right)$

or $\qquad \begin{bmatrix} 1 & 1 & -3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, by $R_2 + R_1$

or $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, by $C_2 - C_1$ and $C_3 + 3C_1$

or $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & -\frac{1}{3} & 3 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, replacing C_2 by $\frac{1}{3} C_2$

or $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & -\frac{1}{3} & \frac{8}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$, by $C_3 + C_2$

Since L. H. S. is in the normal form, so

$R = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & -\frac{1}{3} & \frac{8}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$ **Ans.**

Ex. 1 (b). Determine two non-singular matrices P and Q such that PAQ is in the normal form, where

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$$ *(Garhwal 93)*

Sol. Here we find that A is a 3×4 matrix

$\therefore \qquad [A]_{3\times 4} = I_3 \, A \, I_4$

or $\qquad \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

or
$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 17 & 9 & 4 & 18 \\ 34 & 18 & 11 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $C_1 + 3C_3$, $C_2 + 2C_3$, $C_4 + 5C_3$

or
$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 17 & 9 & 4 & 18 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ applying $R_3 - 2R_2$

or
$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 17 & 9 & 0 & 18 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $R_2 + 4R_1$, $R_3 + 3R_1$

or
$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & -2 \\ -1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $C_1 - 2C_2$, $C_4 - 2C_2$

or
$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & -17 & 0 & -2 \\ -1 & -7 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $C_2 + 9C_1$

or
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & -17 & 0 & -2 \\ -1 & -7 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $-R_1$ and $-R_2$

or
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & -17 & -2 \\ 1 & -1 & -7 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

interchanging columns

$\because$ L.H.S. is in the normal form, so we have

$$\mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \text{ and } \mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & -17 & -2 \\ 1 & -1 & -7 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ans.

***Ex. 2. Find two non-singular matrices P and Q such that PAQ is in the normal form, where**

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$

(Garhwal 96; Meerut 91)

Sol. Here we find that **A** is a 3×3 metrix

$\therefore \quad [\mathbf{A}]_{3\times 3} = \mathbf{I}_3 \, \mathbf{A} \, \mathbf{I}_3$

or $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ applying } R_2 + R_1, R_3 - R_1$$

or $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{applying } R_3 - R_2$$

or $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ applying } R_2 \left(\tfrac{1}{2}\right)$$

or $$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ applying } R_1 - R_2$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_2$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \text{ applying } C_3 - C_2$$

(Note)

Since L. H. S. is in the normal form, so we have

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Ans.

Ex. 3 (a). Using the matrix $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix}$, **find two non-singular matrices P and Q such that PAQ is in the normal form.** *(Agra 95)*

Sol. Here we find that A is a 3×4 matrix

$$\therefore \ [A]_{3 \times 4} = I_3 \bullet A \bullet I_4$$

or
$$\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

or
$$\begin{bmatrix} 0 & 8 & 24 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ applying } R_1 + 5R_3$$

or
$$\begin{bmatrix} 0 & 8 & 24 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ applying } C_2 + C_1, C_3 + 2C_1$$

(Note)

or
$$\begin{bmatrix} 0 & 0 & 0 & -4 \\ 0 & 1 & 3 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ applying } R_1 - 8R_2$$

$$\sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -8 & 5 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1/4 & 2 & -5/4 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ applying } -R_1 \text{ and } -(1/4)\, R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1/4 & 2 & -5/4 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $C_3 - 3C_2, C_4 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1/4 & 2 & -5/4 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

interchanging C_3 and C_4

$\therefore$ L.H.S. is in the normal form, so we have

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1/4 & 2 & -5/4 \end{bmatrix} \text{ and } \mathbf{Q} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Ans.

Ex. 3. (b). Find non-singular matrices R and S such that RAS is in normal form, where $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \end{bmatrix}$

Sol. Here we find that **A** is a 4×3 matrix

$$\therefore \ [\mathbf{A}]_{4 \times 3} = \mathbf{I}_4 \bullet \mathbf{A} \bullet \mathbf{I}_3$$

or
$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 1 & -1 \\ 0 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

applying $R_2 - 3R_1, R_3 - R_1, R_4 - 2R_1$

or
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -5 & 0 & 3 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

applying $R_2 (-\frac{1}{4})$ and $R_4 + 3R_3$

or
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{7}{4} & -\frac{1}{4} & -1 & 0 \\ -1 & 0 & 1 & 0 \\ \frac{5}{6} & 0 & -\frac{1}{2} & -\frac{1}{6} \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

applying $R_2 - R_3, R_4 (-\frac{1}{6})$

or
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 & \frac{3}{2} & \frac{1}{2} \\ \frac{7}{12} & -\frac{1}{12} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{6} \\ \frac{5}{4} & 0 & -\frac{1}{2} & -\frac{1}{6} \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

applying $R_1 - 3R_4, R_2 (\frac{1}{3}), R_3 + R_4$

or
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}-\frac{7}{6}&0&\frac{1}{2}&\frac{5}{6}\\\frac{7}{12}&-\frac{1}{12}&-\frac{1}{3}&0\\-\frac{1}{6}&0&\frac{1}{2}&-\frac{1}{6}\\\frac{3}{12}&\frac{1}{12}&-\frac{1}{6}&-\frac{1}{6}\end{bmatrix}\bullet\mathbf{A}\bullet\begin{bmatrix}1&0&0\\0&1&0\\0&1&0\end{bmatrix},$$

applying R_1-2R_3, R_4-R_2

or
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}-\frac{7}{6}&0&\frac{1}{2}&\frac{5}{6}\\\frac{7}{12}&-\frac{1}{12}&-\frac{1}{2}&0\\-\frac{1}{6}&0&\frac{1}{2}&-\frac{1}{6}\\\frac{3}{12}&\frac{1}{12}&-\frac{1}{6}&-\frac{1}{6}\end{bmatrix}\bullet\mathbf{A}\bullet\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

interchanging C_2 and C_3

$\because$ L.H.S. is in the normal form, so we have

$$\mathbf{R}=\begin{bmatrix}-\frac{7}{6}&0&\frac{1}{2}&\frac{5}{6}\\\frac{7}{12}&-\frac{1}{12}&-\frac{1}{2}&0\\-\frac{1}{6}&0&\frac{1}{2}&-\frac{1}{6}\\\frac{1}{4}&\frac{1}{12}&-\frac{1}{6}&-\frac{1}{6}\end{bmatrix}, \mathbf{S}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

Ans.

Exercise on § 5·12

***Ex. 1. Reduce** $\mathbf{A}=\begin{bmatrix}1&1&2\\1&2&3\\0&-1&-1\end{bmatrix}$ **to normal form N and compute the matrices P and Q, such that PAQ = N.**

Ex. 2. Determine two non-singular matrics P and Q such that PAQ is in the normal form, where

$$\mathbf{A}=\begin{bmatrix}1&1&2\\1&2&3\\0&-1&-1\end{bmatrix}$$

(Garhwal 94)

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1. Find the reciprocal (or inverse) of the matrix**
$\mathbf{S}=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$ **and show that the transform of the matrix**

$\mathbf{A}=\frac{1}{2}\begin{bmatrix}b+c&c-a&b-a\\c-b&c+a&a-b\\b-c&a-c&a+b\end{bmatrix}$ **by S *i.e.* SAS^{-1} is a diagonal matrix.**

Sol. In the usual way we can show that

S^{-1} = inverse of $S = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (To be proved in the examination)

$$\therefore \quad SA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{bmatrix}, \text{ multiplying the matrices in the usual way.}$$

or
$$SA = \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix}$$

$$\therefore \quad SAS^{-1} = \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix}, \text{ multiplying the two matrices in the usual way.}$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \text{ which is a diagonal matrix,}$$

[See Chapter II]

Ex. 2. If A is invertible show that $\overline{A}$ is invertible.

Sol. If **A** is inverible, then we know that

$$AA^{-1} = I = A^{-1}A$$

or
$$(\overline{AA^{-1}}) = I = \overline{(A^{-1}A)} \qquad \textbf{(Note)}$$

or
$$\overline{A}\,(\overline{A^{-1}}) = I = (\overline{A^{-1}})\,\overline{A}, \qquad \because \overline{AB} = \overline{A} \bullet \overline{B}$$

Hence $\overline{A}$ is invertible and we have $(\overline{A})^{-1} = (\overline{A^{-1}})$ Hence proved.

Ex. 3. (a). If $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$, where none of a's is zero, then show that A is invertible. Also evaluate A^{-1}

Sol. $|A| = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{vmatrix} = a_1a_2a_3$, on evaluating

i.e. $|A| \neq 0$. Hence **A** is invertible. ....Sec Ch. IV

Also

$C_{11} = \begin{vmatrix} a_2 & 0 \\ 0 & a_3 \end{vmatrix} = a_2\, a_3$; $C_{12} = -\begin{vmatrix} 0 & 0 \\ 0 & a_3 \end{vmatrix} = 0$; $C_{13} = \begin{vmatrix} 0 & a_2 \\ 0 & 0 \end{vmatrix} = 0$;

$C_{21} = -\begin{vmatrix} 0 & 0 \\ 0 & a_3 \end{vmatrix} = 0$; $C_{22} = \begin{vmatrix} a_1 & 0 \\ 0 & a_3 \end{vmatrix} = a_1\, a_3$; $C_{23} = -\begin{vmatrix} a_1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = \begin{vmatrix} 0 & 0 \\ a_2 & 0 \end{vmatrix} = 0$; $C_{32} = -\begin{vmatrix} a_1 & 0 \\ 0 & 0 \end{vmatrix} = 0$; $C_{33} = \begin{vmatrix} a_1 & 0 \\ 0 & a_2 \end{vmatrix} = a_1\, a_2$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} a_2a_3 & 0 & 0 \\ 0 & a_3a_1 & 0 \\ 0 & 0 & a_1a_2 \end{bmatrix}$$

$$\therefore \quad Adj\ \mathbf{A} = \mathbf{C}' = \begin{bmatrix} a_2a_3 & 0 & 0 \\ 0 & a_3a_1 & 0 \\ 0 & 0 & a_1a_2 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{Adj\ \mathbf{A}}{|\mathbf{A}|} = \frac{1}{a_1a_2a_3}\begin{bmatrix} a_2a_3 & 0 & 0 \\ 0 & a_3a_1 & 0 \\ 0 & 0 & a_1a_2 \end{bmatrix}$$

or
$$\mathbf{A}^{-1} = \begin{bmatrix} 1/a_1 & 0 & 0 \\ 0 & 1/a_2 & 0 \\ 0 & 0 & 1/a_3 \end{bmatrix}$$

Ans.

***Ex. 3 (b). Show that the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is its own inverse.**

Sol. $|\mathbf{A}| = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, expanding w.r. to C_1

$= -1 \neq 0$ ...(i)

Also $C_{11} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$; $C_{12} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$; $C_{13} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$;

$C_{21} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$; $C_{22} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$; $C_{23} = -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$;

$C_{31} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$; $C_{32} = -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$; $C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$\therefore \quad \mathbf{C} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and adj $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{(-1)}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$$

Hence proved.

Ex. 3 (c). Compute the inverse of the matrix A, if

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Sol. $|A| = \begin{vmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 4 & 9 & 5 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$, replacing R_1 by $R_1 - 3R_2$

$= \begin{vmatrix} 4 & 9 & 5 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix}$, expanding w.r. to C_1

$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix}$, applying $R_1 - 4R_3$ and $R_2 - R_3$

$= \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}$, applying $C_2 - C_3$

$= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$, expanding w.r. to R_1

$= 1 \neq 0$ ...(i)

Also $C_{11} = \begin{vmatrix} 2 & 2 & 1 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ 2 & 1 \end{vmatrix}$

$= -3 + 4 = 1$

$C_{12} = -\begin{vmatrix} 0 & 2 & 1 \\ 1 & -3 & -2 \\ 0 & 2 & 1 \end{vmatrix} = 0$; $C_{13} = \begin{vmatrix} 0 & 2 & 1 \\ 1 & -2 & -2 \\ 0 & 1 & 1 \end{vmatrix} = -1$;

$$C_{14} = -\begin{vmatrix} 0 & 2 & 2 \\ 1 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2.$$

$$C_{21} = -\begin{vmatrix} -2 & 0 & -1 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 4 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 1;$$

$$C_{22} = \begin{vmatrix} 3 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ -5 & -3 & -2 \\ 3 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} -5 & -3 \\ 3 & 2 \end{vmatrix} = 1;$$

$$C_{23} = -\begin{vmatrix} 3 & -2 & -1 \\ 1 & -2 & -2 \\ 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & -1 \\ -5 & 2 & -2 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -5 & 2 \\ 3 & -1 \end{vmatrix} = -1;$$

$$C_{24} = \begin{vmatrix} 3 & -2 & 0 \\ 1 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 9 \\ 1 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 4 & 9 \\ 1 & 2 \end{vmatrix} = 1;$$

$$C_{31} = \begin{vmatrix} -2 & 0 & -1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -2;$$

$$C_{32} = -\begin{vmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 0$$

$$C_{33} = \begin{vmatrix} 3 & -2 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 3;$$

$$C_{34} = -\begin{vmatrix} 3 & -2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{vmatrix} = -3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -6;$$

$$C_{41} = -\begin{vmatrix} -2 & 0 & -1 \\ 2 & 2 & 1 \\ -2 & -3 & -2 \end{vmatrix} = -\begin{vmatrix} 0 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = -4;$$

$$C_{42} = \begin{vmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 9 & 5 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{vmatrix} = \begin{vmatrix} 9 & 5 \\ 2 & 1 \end{vmatrix} = -1;$$

$$C_{43} = -\begin{vmatrix} 3 & -2 & -1 \\ 0 & 2 & 1 \\ 1 & -2 & -2 \end{vmatrix} = -\begin{vmatrix} 0 & 4 & 5 \\ 0 & 2 & 1 \\ 1 & -2 & -2 \end{vmatrix} = -\begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix} = 6;$$

$$C_{44} = \begin{vmatrix} 3 & -2 & 0 \\ 0 & 2 & 2 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 9 \\ 0 & 2 & 2 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 4 & 9 \\ 2 & 2 \end{vmatrix} = -10$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & -1 & 1 \\ -2 & 0 & 3 & -6 \\ -4 & -1 & 6 & -10 \end{bmatrix}$$

$$\therefore \quad \text{Adj } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & 1 & -2 & -4 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & 3 & 6 \\ 2 & 1 & -6 & -10 \end{bmatrix} \qquad \text{...(ii)}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj.} \bullet \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 1 & 1 & -2 & 4 \\ 0 & 1 & 0 & 1 \\ -1 & -1 & 3 & -6 \\ -2 & 1 & -6 & -10 \end{bmatrix}, \text{ from (i) and (ii).}$$

Ans.

****Ex. 4. Find $\mathbf{A}^{-1}$, if $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$, where ω is the cube root of unity.** *(Agra 93)*

Sol. $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & \omega-1 & \omega^2-1 \\ 1 & \omega^2-1 & \omega-1 \end{vmatrix}$; **replacing** C_2, C_3 by $C_2 - C_1, C_3 - C_1$ respectively.

$= \begin{vmatrix} \omega-1 & \omega^2-1 \\ \omega^2-1 & \omega-1 \end{vmatrix}$; expanding w.r. to R_1

$= (\omega-1)^2 \begin{vmatrix} 1 & \omega+1 \\ \omega+1 & 1 \end{vmatrix}$, taking out common factors

$= (\omega-1)^2 [1-(\omega+1)^2]$

$= -(\omega-1)^2(\omega^2+2\omega) = -(\omega-1)^2(\omega-1),$

$\because \omega^2+\omega+1=0$ or $\omega^2+2\omega = \omega-1$

$= -(\omega-1)^3 \neq 0.$ ...(i)

Also $C_{11} = \begin{vmatrix} \omega & \omega^2 \\ \omega^2 & \omega \end{vmatrix} = \omega^2 - \omega^4 = \omega^2 - \omega, \because \omega^3 = 1$

$$C_{12} = -\begin{vmatrix} 1 & \omega^2 \\ 1 & \omega \end{vmatrix} = -(\omega - \omega^2) = \omega^2 - \omega;$$

$$C_{13} = \begin{vmatrix} 1 & \omega \\ 1 & \omega^2 \end{vmatrix} = \omega^2 - \omega;\ C_{21} = -\begin{vmatrix} 1 & 1 \\ \omega^2 & \omega \end{vmatrix} = \omega^2 - \omega;$$

$$C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} = \omega - 1;\ C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} = -(\omega^2 - 1);$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ \omega & \omega^2 \end{vmatrix} = \omega^2 - \omega;\ C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} = -(\omega^2 - 1);$$

$$C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} = \omega - 1$$

$$\therefore \qquad \mathbf{C} = \begin{bmatrix} \omega^2 - \omega & \omega^2 - \omega & \omega^2 - \omega \\ \omega^2 - \omega & \omega - 1 & -(\omega^2 - 1) \\ \omega^2 - \omega & -(\omega^2 - 1) & \omega - 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} \omega^2 - \omega & \omega^2 - \omega & \omega^2 - \omega \\ \omega^2 - \omega & \omega - 1 & -(\omega^2 - 1) \\ \omega^2 - \omega & -(\omega^2 - 1) & \omega - 1 \end{bmatrix}$$

$$= (\omega - 1)\begin{bmatrix} \omega & \omega & \omega \\ \omega & 1 & -(\omega + 1) \\ \omega & -(\omega + 1) & 1 \end{bmatrix}$$

$$= (\omega - 1)\begin{bmatrix} \omega & \omega & \omega \\ \omega & 1 & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix}, \qquad \because\ 1 + \omega + \omega^2 = 0$$

or $\quad \text{Adj. } \mathbf{A} = (\omega - 1)\,\omega \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/\omega & \omega \\ 1 & \omega & 1/\omega \end{bmatrix}$, where $\dfrac{1}{\omega} = \dfrac{\omega^2}{\omega^3} = \dfrac{\omega^2}{1}$

$$= \omega\,(\omega - 1)\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}, \qquad \text{...(ii)}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\,\mathbf{A}\,|} = \frac{\omega\,(\omega - 1)}{-(\omega - 1)^3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}, \text{ from (i) and (ii)}$$

$$= \frac{\omega}{-(\omega - 1)^2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \qquad \text{...(iii)}$$

Now $-(\omega-1)^2 = -(\omega^2+1-2\omega)$

$= -[(-\omega)-(2\omega)], \qquad \because \omega^2+\omega+1=0 \text{ or } \omega^2+1=-\omega$

$= 3\omega$

$\therefore$ From (iii), we get

$$\mathbf{A}^{-1} = \frac{\omega}{3\omega}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$ **Ans.**

Ex. 5 (a). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$$

(Agra 96; Bundelkhand 96)

Sol. Given $\mathbf{A} = \begin{bmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -7 & -20 & -39 \\ 9 & -20 & -56 & -108 \\ 16 & -39 & -108 & -207 \end{bmatrix}$, replacing C_2, C_3, C_4 by C_2-4C_3, C_3-9C_1 and C_4-16C_1 respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -20 & -39 \\ 1 & -6 & -16 & -30 \\ 0 & -11 & -28 & -51 \end{bmatrix}$, replacing R_2, R_3, R_4 by R_2-4R_1, R_3-2R_2 and R_4-4R_2 respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -20 & -39 \\ 0 & -6 & -16 & -30 \\ 0 & -4 & -8 & -12 \end{bmatrix}$, replacing R_3, R_4 by R_3-R_1 and R_4-R_2 respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 9 \\ 0 & 2 & 8 & 18 \\ 0 & 4 & 8 & 12 \end{bmatrix}$, replacing R_2, R_3, R_4 by $-(R_2-R_3)$, $-(R_3-R_4)$ and $-R_4$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & -24 \end{bmatrix}$, replacing R_3, R_4 by R_3-2R_2, R_4-4R_2 respectively

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & -8 & -24\end{bmatrix}, \text{ replacing } C_3, C_4 \text{ by } C_3-4C_2, C_4-9C_2 \text{ respectively}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 3\end{bmatrix}, \text{ replacing } R_4 \text{ by } -\tfrac{1}{8}R_4$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 3\\0 & 0 & 0 & 0\end{bmatrix}, \text{ interchanging } R_3 \text{ and } R_4$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}, \text{ replacing } C_4 \text{ by } C_4-3C_3$$

$$\sim\begin{bmatrix}\mathbf{I}_3 & \mathbf{O}\\\mathbf{O} & \mathbf{O}\end{bmatrix}$$

$\therefore$ The rank of martix **A** is 3. **Ans.**

Ex. 5 (b). Find the rank of the matrix

$$\mathbf{A}=\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0}\\\mathbf{2} & \mathbf{4} & \mathbf{3} & \mathbf{2}\\\mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{3}\\\mathbf{6} & \mathbf{8} & \mathbf{7} & \mathbf{5}\end{bmatrix}$$

(Agra 94; Bundelkhand 93)

Sol. Given
$$\mathbf{A}\sim\begin{bmatrix}1 & 2 & 3 & 0\\1 & 2 & 0 & 2\\1 & -2 & -2 & 1\\3 & 6 & 6 & 2\end{bmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2-R_1, R_3-R_2, R_4-R_3 \text{ respectively}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\1 & 0 & -3 & 2\\1 & -4 & -5 & 1\\3 & 0 & -3 & 2\end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2-2C_1, C_3-3C_1 \text{ respectively}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & -3 & 2\\0 & -4 & -5 & 1\\0 & 0 & -3 & 2\end{bmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2-R_1, R_3-R_1, R_4-3R_1 \text{ respectively}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & -3 & 2\\0 & -4 & -2 & -1\\0 & 0 & 0 & 0\end{bmatrix}, \text{ replacing } R_3, R_4 \text{ by } R_3 - R_2, R_4 - R_2 \text{ respectively}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 3 & 2\\0 & 1 & 2 & -1\\0 & 0 & 0 & 0\end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } -(1/4)\, C_2 \text{ and } -C_3 \text{ respectively}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 3 & 2\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}, \text{ replacing } C_3, C_4 \text{ by } C_3 - 2C_2 \text{ and } C_4 + C_1 \text{ respectiely}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 2\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}, \text{ replacing } C_3 \text{ by } C_3 - C_4$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}, \text{ replacing } C_4 \text{ by } C_4 - 2C_3$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$\sim\begin{bmatrix}\mathbf{I}_3 & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{bmatrix}$$

$\therefore$ The rank of the given martix **A** is 3. **Ans.**

***Ex. 6. If** $\mathbf{A} = \begin{bmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix}$, **show that** $\mathbf{A}^3 = \mathbf{A}^{-1}$.

Sol. $|\mathbf{A}| = \begin{vmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{vmatrix}$

or $|\mathbf{A}| = \begin{vmatrix}1 & 0 & 0\\2 & -3 & 4\\0 & -1 & 1\end{vmatrix}$, replacing R_1 by $R_1 - R_2$

$= \begin{vmatrix}-3 & 4\\-1 & 1\end{vmatrix} = -3 + 4 = 1 \neq 0.$ ...(i)

Also for the matrix A, we have

$C_{11} = \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = 1; C_{12} = -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -2; C_{13} = \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2;$

$C_{21} = -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -1; C_{22} = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3; C_{23} = -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = 3;$

$C_{31} = \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = 0; C_{32} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4; C_{33} = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -3;$

$\therefore \quad \mathbf{C} = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$ ...(ii)

$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$, from (i), (ii) ...(iii)

Also $\mathbf{A}^2 = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9-6+0 & -9+9-4 & 12-12+4 \\ 6-6+0 & -6+9-4 & 8-12+4 \\ 0-2+0 & 0+3-1 & 0-4+1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$

$\therefore \quad \mathbf{A}^3 = \mathbf{A}^2 \bullet \mathbf{A} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} \times \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9-8+0 & -9+12-4 & 12-16+4 \\ 0-2+0 & 0+3+0 & 0-4+0 \\ -6+4+0 & 6-6+3 & -8+8-3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

i.e. $\quad \mathbf{A}^3 = \mathbf{A}^{-1}$, from (iii). Hence proved.

****Ex. 7. If two non-singular symmetric matrices A and B be such that AB = BA (*i.e.* commute under multiplication), then prove that $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{A}^{-1}\mathbf{B}^{-1}$ are symmetric.**

Sol. Here we are given that **AB = BA**.

$\therefore$ We have $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{BA}$, premultiplying by $\mathbf{A}^{-1}$

or $\quad \mathbf{IB} = \mathbf{A}^{-1}\mathbf{BA}, \qquad \because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

or $\quad \mathbf{B} = \mathbf{A}^{-1}\mathbf{BA}, \qquad \because \mathbf{IB} = \mathbf{B}$

or $\quad \mathbf{BA}^{-1} = \mathbf{A}^{-1}\mathbf{BAA}^{-1}, \qquad$ post multiplying by $\mathbf{A}^{-1}$

$$= \mathbf{A}^{-1}\mathbf{BI} = \mathbf{A}^{-1}\mathbf{B}, \qquad \text{...(i)}$$

since $\quad \mathbf{AA}^{-1} = \mathbf{I}$ and $\mathbf{BI} = \mathbf{B}$.

Again $(\mathbf{A}^{-1}\mathbf{B})' = \mathbf{B}'(\mathbf{A}^{-1})', \qquad \because (\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

$= \mathbf{B}'(\mathbf{A}')^{-1} \qquad \because (\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$

...See Th. II Page 77 Chapter V

$= \mathbf{BA}^{-1}, \qquad \because \mathbf{A}' = \mathbf{A}, \mathbf{B}' = \mathbf{B}$ as $\mathbf{A}$ and $\mathbf{B}$ are symmetric

i.e. $\quad (\mathbf{A}^{-1}\mathbf{B})' = \mathbf{A}^{-1}\mathbf{B}$, from (i)

Hence $\mathbf{A}^{-1}\mathbf{B}$ is symmetric.

Similarly $(\mathbf{A}^{-1}\mathbf{B}^{-1})' = (\mathbf{B}^{-1})'(\mathbf{A}^{-1})'$, as $(\mathbf{CD})' = \mathbf{D}'\mathbf{C}'$

or $\quad (\mathbf{A}^{-1}\mathbf{B}^{-1})' = (\mathbf{B}')^{-1}(\mathbf{A}')^{-1}, \qquad$ See Th. II Page 77 Ch. V

$= \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \because \mathbf{A}' = \mathbf{A}, \mathbf{B}' = \mathbf{B}$

$= (\mathbf{AB})^{-1}, \qquad \because (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$= (\mathbf{BA})^{-1}, \qquad \because \mathbf{AB} = \mathbf{BA}$ (given)

or $\quad (\mathbf{A}^{-1}\mathbf{B}^{-1})' = \mathbf{A}^{-1}\mathbf{B}^{-1}$.

Hence $\mathbf{A}^{-1}\mathbf{B}^{-1}$ is symmetric.

Ex. 8. Find the rank of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$

Sol. $\quad \mathbf{A} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 2R_1$ and $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing R_1 by $R_1 - R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$ and $C_3 - 3C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, interchanging C_2 and C_4

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$\sim\begin{bmatrix}\mathbf{I}_2 & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{bmatrix}$$

$\therefore$ The rank of matrix **A** is 2. **Ans.**

****Ex. 9. Find the rank of an $m \times n$ matrix, every element of which is unity.**

Sol. Let an $m \times n$ matrix be $\mathbf{A} = \begin{bmatrix}1 & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & 1\\ \dots & \dots & \dots & \dots & \dots\\ 1 & 1 & 1 & \dots & 1\end{bmatrix}$

Then we find that every square submatrix of **A** higher than 1×1 will be a matrix each element of which is unity and therefore the value of the determinant will be always zero, since its rows and columns are identical. But the square sub-matrices of order 1×1 are $[\,1\,]$ and the determinants of these are $|\,\mathbf{A}\,| = 1 \neq 0$.

Hence the rank of **A** is 1. **Ans.**

Ex. 10. Show that the matrix $\mathbf{A} = \begin{bmatrix}1 & a & \alpha & a\alpha\\ 1 & b & \beta & b\beta\\ 1 & c & \gamma & c\gamma\end{bmatrix}$ is of rank 3 provided no two of a, b, c are equal and no two of α, β, γ are equal.

Sol. $$\mathbf{A} \sim \begin{bmatrix}1 & a & \alpha & a\alpha\\ 0 & b-a & \beta-\alpha & b\beta-a\alpha\\ 0 & c-a & \gamma-\alpha & c\gamma-a\alpha\end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1, R_3 - R_1 \text{ respectively}$$

$$\sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & b-a & \beta-\alpha & b\beta-a\alpha\\ 0 & c-a & \gamma-\alpha & c\gamma-a\alpha\end{bmatrix}, \text{ replacing } C_2, C_3, C_4 \text{ by } C_2 - aC_1, C_3 - \alpha C_1 \text{ and } C_4 - a\alpha C_1 \text{ respectively}$$

$$\sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & b-a & \beta-\alpha & b\beta-b\alpha\\ 0 & c-a & \gamma-\alpha & c\gamma-c\alpha\end{bmatrix}, \text{ replacing } C_4 \text{ by } C_4 - \alpha C_2$$

$$\sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & b-a & \beta-\alpha & 0\\ 0 & c-a & \gamma-\alpha & c\gamma-c\alpha-b\gamma+b\alpha\end{bmatrix}, \text{ replacing } C_4 \text{ by } C_4 - bC_3$$

$$\sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & b-a & \beta-\alpha & 0\\ 0 & c-a & \gamma-\alpha & (c-b)(\gamma-\alpha)\end{bmatrix} = \mathbf{B} \text{ (say)};$$

Now a minor of order 3 of **B**

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & b-a & 0 \\ 0 & c-a & (c-b)(\gamma-\alpha) \end{vmatrix} = \begin{vmatrix} b-a & 0 \\ c-a & (c-b)(\gamma-\alpha) \end{vmatrix},$$

expanding with respect to R_1

$= (b-a)(c-b)(\gamma-\alpha) \neq 0,$ as no two of a, b, c and no two of α, β, γ are equal (given)

$\therefore \quad \rho(\mathbf{B}) \geq 3$ ...(i)

Also the matrix **B** does not possess any minor of order 4 *i.e.* of order 3 + 1,

so $\quad \rho(\mathbf{B}) \leq 3.$ ...(ii)

$\therefore$ From (i) and (ii) we get $\rho(\mathbf{B}) = 3$

and therefore $\rho(\mathbf{A}) = 3$, as $\mathbf{A} \sim \mathbf{B}$. Hence proved.

Ex. 11 (a). Find A^{-1} if $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix}$

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{vmatrix}$, replacing R_3 by $R_3 - R_1$

$= 0$, since two rows are identical.

Hence the matrix **A** is not non-singular (*i.e.* is singular) and so $\mathbf{A}^{-1}$ does not exist. (See § 5·10 Page 76 Ch. V)

Ex. 11 (b). Find adjoint and inverse of the matrix

$$\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Sol. Do yourself.

Ans. Adj. $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$;

***Ex. 12. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$**

then show that $\rho(\mathbf{AB}) \neq \rho(\mathbf{BA})$, where ρ denotes its rank.

(Rohilkhand 93)

Sol. $\mathbf{A}\,\mathbf{B} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -1+6-5 & -2+12-10 & -1+6-5 \\ -2-18+20 & -4-36+40 & -2-18+20 \\ -3-12+15 & -6-24+30 & -3-12+15 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= \mathbf{O}$

$\therefore$ By definition ρ (**AB**) *i.e.* rank of **AB** is 0. ...(i)

... Sec § 5.02 Note 2 Page 2.

Again $\mathbf{BA} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} -1-4-3 & -1+6+2 & 1-8-3 \\ 6+24+18 & 6-36-12, & -6+48+18 \\ 5+20+15 & 5-30-10 & -5+40+15 \end{bmatrix}$

$= \begin{bmatrix} -8 & 7 & -10 \\ 48 & -42 & 60 \\ 40 & -35 & 50 \end{bmatrix} = \mathbf{C}$, say ....(ii)

Now $\mathbf{C} \sim \begin{bmatrix} -8 & 7 & -10 \\ 8 & -7 & 10 \\ 40 & -35 & 50 \end{bmatrix}$, replacing R_2 by $R_2 - R_3$

$\sim \begin{bmatrix} -8 & 7 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_2 + R_1$ and $R_3 + 5R_1$ respectively

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, replacing C_1, C_2, C_3 by $(-1/8)\,C_1$ $(1/7)\,C_2$, $(-1/10)\,C_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively

$\sim \begin{bmatrix} \mathbf{I}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$.

$\therefore$ $\rho(\mathbf{C}) = 1$ or $\rho(\mathbf{BA}) = 1$, from (ii) ...(iii)

$\therefore$ From (i) and (iii), $\rho(\mathbf{AB}) \neq \rho(\mathbf{BA})$ Hence proved.

Ex. 13. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, **then show that**

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$$

Sol. Here $\mathbf{A} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 6; C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -6; C_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2;$

$C_{21} = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -5; C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 8; C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -3;$

$C_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1; C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2; C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1;$

$$\therefore \qquad \mathbf{C} = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} \qquad \text{...(i)}$$

Again $|\mathbf{B}| = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -5 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -5 & -1 \end{vmatrix} = 4$

Also for the matrix **B**, we have

$C_{11} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3; C_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -1; C_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5;$

$C_{21} = -\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = 1; C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1; C_{23} = -\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 1;$

$C_{31} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7; C_{32} = -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5; C_{33} = \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13.$

$$\therefore \qquad \mathbf{C} = \begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{B} = \mathbf{C}' = \begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$$

$$\therefore \quad \mathbf{B}^{-1} = \frac{\text{Adj. } \mathbf{B}}{|\mathbf{B}|} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix} \qquad \text{...(ii)}$$

Again $\mathbf{AB} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \times \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2+3+1 & 5+1+2 & 3+2+1 \\ 2+6+3 & 5+2+6 & 3+4+3 \\ 2+12+9 & 5+4+18 & 3+8+9 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 6 \\ 11 & 13 & 10 \\ 23 & 27 & 20 \end{bmatrix} = \mathbf{D} \text{ (say)}$$

Now $|\mathbf{D}| = \begin{vmatrix} 6 & 8 & 6 \\ 11 & 13 & 10 \\ 23 & 27 & 20 \end{vmatrix} = \begin{vmatrix} 6 & 8 & 6 \\ 11 & 13 & 10 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 8 & 6 \\ 1 & 13 & 10 \\ 1 & 1 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 8 & 6 \\ 0 & 12 & 10 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 8 & 6 \\ 12 & 10 \end{vmatrix} = 80 - 72 = 8 \neq 0.$$

For this matrix **D**, we have

$C_{11} = -\begin{vmatrix} 13 & 10 \\ 27 & 20 \end{vmatrix} = -10$; $C_{12} = -\begin{vmatrix} 11 & 10 \\ 23 & 20 \end{vmatrix} = 10$; $C_{13} = \begin{vmatrix} 11 & 13 \\ 23 & 27 \end{vmatrix} = -2$;

$C_{21} = -\begin{vmatrix} 8 & 6 \\ 27 & 20 \end{vmatrix} = 2$; $C_{22} = \begin{vmatrix} 6 & 6 \\ 23 & 20 \end{vmatrix} = -18$; $C_{23} = -\begin{vmatrix} 6 & 8 \\ 23 & 27 \end{vmatrix} = 22$;

$C_{31} = \begin{vmatrix} 8 & 6 \\ 13 & 10 \end{vmatrix} = 2$; $C_{32} = -\begin{vmatrix} 6 & 6 \\ 11 & 10 \end{vmatrix} = 6$; $C_{33} = \begin{vmatrix} 6 & 8 \\ 11 & 13 \end{vmatrix} = -10$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} -10 & 10 & -2 \\ 2 & -18 & 22 \\ 2 & 6 & -10 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{D} = \mathbf{C}' = \begin{bmatrix} -10 & 2 & 2 \\ 10 & -18 & 6 \\ -2 & 22 & -10 \end{bmatrix}$$

$$\therefore \quad (\mathbf{AB})^{-1} = \mathbf{D}^{-1} = \frac{\text{Adj. } \mathbf{D}}{|\mathbf{D}|} = \frac{1}{8}\begin{bmatrix} -10 & 2 & 2 \\ 10 & -18 & 6 \\ -2 & 22 & -10 \end{bmatrix} \quad \text{...(iii)}$$

From (i) and (ii) we get

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -18-6+14 & 15+8-21 & -3-2+7 \\ -6+6+10 & 5-8-15 & -1+2+5 \\ 30-6-26 & -25+8+39 & 5-2-13 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -10 & 2 & 2 \\ 10 & -18 & 6 \\ -2 & 22 & -10 \end{bmatrix}$$

$= (\mathbf{AB})^{-1}$, from (iii). Hence proved.

***Ex. 14. Prove that $(\text{Adj. } \mathbf{A})^{-1} = (\text{Adj. } \mathbf{A}^{-1})$, where A is any $n \times n$ matrix.**

(Agra 95; Avadh 99; Bundelkhand 92; Purvanchal 96)

Sol. We know from Ex. 22 Page 74 Ch. V (To be proved in the examination) that

$$Adj.\ (Adj.\ \mathbf{A}) = |\mathbf{A}|^{n-2} \bullet \mathbf{A}$$

Also from Theorems I and II Pages 49–50, Ch. V we know that

$$|Adj.\ \mathbf{A}| = |\mathbf{A}|^{n-1} \qquad \text{...(ii)}$$

and $$\mathbf{A}^{-1} = \frac{Adj.\ \mathbf{A}}{|\mathbf{A}|} \qquad \text{...(iii)}$$

Now $Adj.\ \mathbf{A}^{-1} = Adj. \left\{\dfrac{Adj\ \mathbf{A}}{|\mathbf{A}|}\right\}$, from (iii)

$= Adj. \left\{\dfrac{1}{|\mathbf{A}|}\right\} \bullet Adj.\ (Adj.\ \mathbf{A})$ ... See Th. III **Page 50 Ch. V**

$= \dfrac{1}{|\mathbf{A}|^{n-1}} \bullet |\mathbf{A}|^{n-2} \bullet \mathbf{A}$, from (i) and (ii)

or $$Adj.\ \mathbf{A}^{-1} = \frac{\mathbf{A}}{|\mathbf{A}|} \qquad \text{...(iv)}$$

Also $(Adj.\ \mathbf{A})^{-1} = \dfrac{Adj.\ (Adj.\ \mathbf{A})}{|Adj.\ \mathbf{A}|}$, from (iii). **(Note)**

or $(Adj.\ \mathbf{A})^{-1} = \dfrac{|\mathbf{A}|^{n-2} \bullet \mathbf{A}}{|\mathbf{A}|^{n-1}}$, from (i) and (ii)

or $$(Adj.\ \mathbf{A})^{-1} = \frac{\mathbf{A}}{|\mathbf{A}|}. \qquad \text{...(v)}$$

Hence from (iv) and (v), we get $(Adj.\ \mathbf{A})^{-1} = (Adj.\ \mathbf{A}^{-1})$. **Hence proved.**

Ex. 15. If A is of order $m \times n$, R is a non-singular matrix of order m, show that Rank of RA = Rank of A.

Sol. Let $\mathbf{A} = \mathbf{E}\ \mathbf{A}_r\ \mathbf{F}$ and $\mathbf{R} = \mathbf{E}_1$ See § 5.12 **Page** 78

Then $\mathbf{RA} = \mathbf{E}_1\ (\mathbf{E}\ \mathbf{A}_r\ \mathbf{F}) = \mathbf{E}_1\ \mathbf{E}\ \mathbf{A}_r\ \mathbf{F}$

i.e. **RA** has been expressed as the result of elementary operations **on** $\mathbf{A}_r$.

Thus Rank of (**RA**) = Rank $\mathbf{A}_r$ = Rank **A**.

****Ex. 16. Prove that the rank of a matrix remains unaltered by the application of elementary row and column operations.**

or Prove that two equivalent matrices have the same rank. *(Avadh 99)*

Sol. Let an $m \times n$ matrix **A** be given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Let **M** be any minor of order r belonging in the first r rows of $|\mathbf{A}|$.

Now firstly if we inerchange any two rows or columns of **A**, then the minor **M** either remains unaltered or changes sign.

Secondly if we multiply one row or column of **A** by a number λ, then either the minor **M** remains unaltered or changes into λ**M**.

Thirdly if we replace any row R_i (for column C_i) by $R_i + \lambda R_i$ (or $C_i + \lambda C_i$), then either the minor **M** remains unaltered or changes into a sum or difference of two of the original minors.

Let **B** be the matrix obtained from **A** by the application of any one of the above three elementary row or column operations.

Thus if all the minor of order r in $|\mathbf{B}|$ are zero, then all the minors of order r in $|\mathbf{B}|$ are also zero.

$\therefore$ rank of $\mathbf{B} \le$ rank of $\mathbf{A}$. ...(i)

Similarly if all the minors of order r in $|\mathbf{B}|$ are zero, then all the minors of order r in $|\mathbf{A}|$ are also zero.

$\therefore$ rank of $\mathbf{A} \le$ rank of $\mathbf{B}$ ...(ii)

$\therefore$ From (i) and (ii) we get

rank of $\mathbf{A}$ = rank of $\mathbf{B}$. Hence proved.

Exercises on Chapter V

Ex. 1. Are the following pairs of matrices equivalent ?

$$\begin{bmatrix} 4 & -1 & 2 \\ 3 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 & 4 & 7 \\ 3 & 6 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Ans. No.

Ex. 2. Show that the rank of a matrix is not altered if a column of it is multiplied by a non-zero scalar.

Ex. 3. Show that the inverse of

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & -1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$$

Ex. 4. Compute the adjoint and inverse of

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Ans. $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Ex. 5. Show that the adjoint and inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 9 \end{bmatrix} \textit{ are } \begin{bmatrix} 1 & 3 & -2 \\ 3 & -9 & 4 \\ -3 & 5 & -2 \end{bmatrix} \text{ and } -\frac{1}{2}\begin{bmatrix} 1 & 3 & -2 \\ 3 & -9 & 4 \\ -3 & 5 & -2 \end{bmatrix}$$

Ex. 6. If $\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$, where $a_i \neq 0$ for all $1 \le i \le n$, then show that **A** is invertible. Also evaluate $\mathbf{A}^{-1}$

Ans. $\mathbf{A}^{-1} = \begin{bmatrix} 1/a_1 & 0 & 0 & \dots & 0 \\ 0 & 1/a_2 & 0 & \dots & 0 \\ 0 & 0 & 1/a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1/a_n \end{bmatrix}$

Ex. 7. Show that the reciprocal (or inverse) of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \text{ is } \begin{bmatrix} -2 & \frac{5}{3} & -\frac{1}{3} \\ 5 & -\frac{8}{3} & -\frac{1}{3} \\ -2 & 1 & 0 \end{bmatrix}$$

Ex. 8. Show that the inverse of

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} -1 & 1 & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Ex. 9. If $\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, prove that $\mathbf{AA}^{-1} = \mathbf{I}_3$

Ex. 10. Prove that $Adj.\ (\mathbf{A}') = (Adj.\ \mathbf{A})'$.

Ex. 11. Let **A** be a non-singular matrix. Will the adjoint of **A** also be non-singular ?

Ex. 12. Show that $\mathbf{A} = \begin{bmatrix} 3 & 7 & 1 \\ 5 & 9 & -1 \\ 7 & 13 & -5 \end{bmatrix}$ is non-singular.

[**Hint** : Prove that $|\mathbf{A}| \neq 0$].

Ex. 13. Show that if **A** is a square matrix of order n then

$$Adj.\ \mathbf{A}\ \{Adj.\ (Adj.\ \mathbf{A})\} = (\det.\ \mathbf{A})^{n-1}\mathbf{I}.$$

Ex. 14. What is the rank of a non-singular matrix of order n ?

Ex. 15 (a). Show that the inverse of

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix} \text{ is } \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$$

Ex. 15 (b). Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$$ *(Agra 95)*

Ans. $\frac{1}{2}\begin{bmatrix} 24 & 10 & -2 & -6 \\ -5 & -3 & 1 & 1 \\ -16 & -6 & 2 & 4 \\ -11 & -5 & 1 & 3 \end{bmatrix}$

Ex. 16. Compute rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **Ans.** 2

Ex. 17. Prove that each non-singular matrix has a unique inverse matrix.

***Ex. 18.** Define Rank of a matrix. Determine rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

[**Hint** : See § **5.02 Pages 1-2 Ex. 13. Page 10 Ch V**]

***Ex. 19.** Find the rank of the matrix **A**, given by

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 7 & 1 \\ 4 & 1 & 3 & 2 \\ 1 & -1 & -4 & 1 \end{bmatrix}$$

Ans. 2

Ex. 20. Find an invertible matrix **P** such that $\mathbf{PAP}^{-1}$ is a diagonal matrix, where $\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$.

Ex. 21. Prove that every matrix of rank r can be reduced by means of elementary transformation to the form $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$.

Ex. 22. Show that for any matrix **A**, rank $(\mathbf{A}'\mathbf{A})$ = rank $(\mathbf{A})$.

Hence or otherwise show that if n be the rank of an $m \times n$ matrix **A**, then $\mathbf{A}'\mathbf{A}$ is a non-singular matrix.

Ex. 23. Find inverse of

(i) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & 4 \\ 0 & -4 & -7 \end{bmatrix}$

Ex. 24. If adj. $\mathbf{B} = \mathbf{A}$ and $|\mathbf{P}| = 1 = |\mathbf{Q}|$, then prove that adj. $(\mathbf{Q}^{-1}\mathbf{B}\mathbf{P}^{-1}) = \mathbf{PAQ}$. *(Kanpur 95, 93)*

Ex. 25. Prove that the inverse of a matrix is unique.

Ex. 26. Prove that for every matrix **A** there exist two non-singular matrices **P** and **Q** such that **PAQ** is in normal form. *(Rohilkhand 90)*

Ex. 27. Show that every elementary matrix is non-singular *i.e.* it is invertible and its inverse is also an elementary matrix of the same type. *(Purvanchal 93)*

Ex. 28. Find the rank of the matrix **A**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

(Rohilkhand 99) **Ans.** 1

Chapter VI

Solution of Linear Equations

§ 6.01. Matrix of coefficients of a system of equations.

Definiton. Let the system of m simultaneous equations in n unknowns $x_1, x_2, ..., x_n$ be

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= k_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= k_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= k_3, \\ \ldots \ldots \ldots \ldots \ldots \ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= k_m \end{aligned}$$

or written in a compact form

$$\sum_{j=1}^{n} a_{ij}\, x_j = k_i, \text{ where } i = 1, 2, \ldots m \qquad \ldots \text{(i)}$$

Then the matrix $\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

of order $m \times n$ is known as the **matrix of coefficients of the system of equations** given by (i).

The determinant of the matrix $\mathbf{A}$, [if there be n equations in (i)] viz.

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix}$$

is called the **determinant of coefficients of the** system of equations given by (i).

Note. If all the k's are zero, then the system of equations given by (i) is said to be **homogeneous** and if at least one of k's is not zero, then the above system of equations is said to be **non-homogeneous.**

§ 6.02. System of equations in the Matrix Form.

The system of equations given by (i) in § 6.01 above can be written in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \ldots \\ \ldots \\ \ldots \\ k_m \end{bmatrix} \qquad \begin{matrix} \textbf{(Note)} \\ \ldots\text{(i)} \end{matrix}$$

or in more compact form it may be written as

$$\mathbf{AX} = \mathbf{K}.$$

where $\mathbf{A} = [a_{ij}]$ *i.e.* the matrix of coefficients of the system of equations given by (i) in § 6.01 on Page 105;

$\mathbf{X}$ = the transposed matrix of $[x_1, x_2, x_3, \ldots x_n]$ and

$\mathbf{K}$ = the transposed matrix of $[k_1, k_2, k_3, \ldots, k_m]$.

Here students should note that the **product AX** is a matrix of order $m \times 1$, as **A** is a matrix of order $m \times n$ and **X** is a **matrix** of order $n \times 1$. And **X** is also a matrix of order $m \times 1$.

§ 6.03. Consistent and Inconsistent Equations.

Consider the system of equations **given by** (i) of § 6.01 Page 105;

If the above system has a solution (*i.e.* a set of values of $x_1, x_2, x_3, \ldots, x_n$, satisfy simultaneously these m equations), **then** the equations are said to be **consistent** otherwise the equations are **said to be inconsistent.**

A consistent system of equations **has either** one solution or infinitely many solutions.

§ 6.04. Solution of non-homogeneous Simultaneous equations.

Solution of equation given by (i) *of* § 6.01 *Page* 105 *Ch. VI when* $m = n$ *and the matrix* **A** *is non-singular.*

We know from § 6.02 Page 105 Ch. VI that the matrix form of the given equations is $\qquad \mathbf{AX} = \mathbf{K} \qquad$ ...(i)

Also we know that if **A** is non-singular, its inverse matrix *i.e.*, $\mathbf{A}^{-1}$ exists such that $\qquad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}, \qquad$ ...(ii)

where **I** is the identity matrix.

Hence by multiplying both sides of (i) by $\mathbf{A}^{-1}$, we have

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K}$$

or $\qquad \mathbf{IX} = \mathbf{A}^{-1}\mathbf{K}$, from (ii)

or $\qquad \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$, which is the required solution of the given equations and is unique.

Solved Examples on § 6.02—§ 6.04.

Ex. 1. Express in matrix form the system of equations

$\mathbf{9x + 7y + 3z = 6;\ 5x + y + 4z = 1;\ 6x + 8y + 2z = 4}$. *(Gorakhpur 97, 94)*

Sol. The given equations are

$$9x + 7y + 3z = 6$$
$$5x + y + 4z = 1$$
$$6x + 8y + 2z = 4$$

$\therefore$ The required matrix form of these equations is

$$\mathbf{AX} = \mathbf{K},$$

where $A = \begin{bmatrix} 9 & 7 & 3 \\ 5 & 1 & 4 \\ 6 & 8 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $K = \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$

Ex. 2 (a). Find the matrix X from the equations AX = B,

where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$.

Sol. Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then from **AX = B**, we have

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$$

$\Rightarrow$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$, by the elementary row operation $R_3 \to R_3 + R_2$

$\Rightarrow$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 12 \end{bmatrix}$, by the elementary row operations $R_3 \to R_3 + 2R_1$

$\Rightarrow$ $x - y = 2, y - z = 1, 3x = 12$ **(Note)**

$\Rightarrow$ $y = x - 2, z = y - 1, x = 4$

$\Rightarrow$ $y = 4 - 2 = 2, z = 2 - 1 = 1, x = 4$

$\Rightarrow$ $x = 4, y = 2, z = 1.$

$\therefore$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ **Ans.**

Ex. 2 (b). Solve by matrix method

$$x - 2y + 3z = 2, 2x - 3z = 0, x + y + z = 0.$$

Sol. Do as Ex. 2 (a) above. **Ans.** $x = (6/19), y = -(10/19), z = (4/19)$

****Ex. 3 (a). Solve by matrix method :**

$$x + y + z = 6, x - y + z = 2, 2x + y - z = 1$$

(Gorakhpur 96; Kanpur 95; Rohilkhand 95)

Sol. Given equations are

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $K = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$ and assume that there exists a matrix

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $AX = K$

Then $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -11 \end{bmatrix}$, by the elementary row operations $R_2 \to R_2 - R_1$ $R_3 \to R_3 - 2R_1$

$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ -11 \end{bmatrix}$, by the elementary row operations $R_1 \to R_1 + R_3$

$\Rightarrow \quad x - 2z = -5, -2y = -4, -y - 3z = -11$

$\Rightarrow \quad x - 2z = -5, y = 2, 2 + 3z = 11$

$\Rightarrow \quad x = 2z - 5, y = 2, z = 3$

$\Rightarrow \quad x = 6 - 5, y = 2, z = 3 \Rightarrow x = 1, y = 2, z = 3.$ **Ans.**

***Ex. 3 (b). Solve by matrix method**

$$x + 2y + 3z = 14, 3x + y + 2z = 11, 2x + 3y + z = 11.$$

(Bundelkhand 92; Purvanchal 93; Rohilkhand 98)

Sol. The given equations are

$$x + 2y + 3z = 14$$
$$3x + y + 2z = 11$$
$$2x + 3y + z = 11$$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ and $K = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$ and assume that there exists a matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $AX = K$.

Then $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 5 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 25 \\ 11 \end{bmatrix}$, by the elementary row operation $R_2 \to R_1 + R_2$

$\Rightarrow \begin{bmatrix} 2 & 4 & 6 \\ 2 & 0 & 4 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 14 \\ 11 \end{bmatrix}$, by the elementary row operation $R_1 \to 2R_1$ $R_2 \to R_2 - R_3$

$$\Rightarrow \begin{bmatrix} 0 & 4 & 2 \\ 2 & 0 & 4 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 14 \\ 14 \\ -3 \end{bmatrix}, \text{ by the elementary row operation } R_1 \to R_1 - R_2, R_3 \to R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 7 \\ 7 \\ -1 \end{bmatrix}, \text{ by the elementary row operation } R_1 \to \tfrac{1}{2} R_1, R_2 \to \tfrac{1}{2} R_2, R_3 \to \tfrac{1}{3} R_3$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix}, \text{ by the elementary row operations } R_1 \to R_1 + R_3$$

$\Rightarrow$ $3y = 6, x + 2z = 7, y - z = -1$ **(Note)**

$\Rightarrow$ $y = 2, z = y + 1 = 2 + 1 = 3, x = 7 - 2z = 7 - 6 = 1$

$\Rightarrow$ $x = 1, y = 2, z = 3.$ **Ans.**

Ex. 3 (c). Solve the following equations by matrix method :

$x + 2y + 3z = 4, 2x + 3y + 8z = 7, x - y + 9z = 1$. *(Agra 96, 93)*

Sol. The given equations are

$$x + 2y + 3z = 4$$
$$2x + 3y + 8z = 7$$
$$x - y + 9z = 1$$

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & -1 & 9 \end{bmatrix}; \mathbf{K} = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$ and assume that there exists a matrix $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

such that $\mathbf{AX} = \mathbf{K}$.

Then $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & -1 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}, \text{ by elementary row operations } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \text{ by elementary row operations } R_3 \to R_3 - 3R_2$$

$\Rightarrow$ $x + 2y + 3z = 4, -y + 2z = -1$

$\Rightarrow$ $x + 7z = 2, -y + 2z = -1$

$\Rightarrow$ $x = 2 - 7z, y = 2z + 1$ and z can take any finite value. **Ans.**

****Ex. 4 (a). Using matrix method, solve the following equations—**

$2x - y + 3z = 9, x + y + z = 6$ and $x - y + z = 2$.

(Avadh 98; Agra 95; Garhwal 95; Gorakhpur 99; Kanpur 90; Meerut 92P, 91; Rohilkhand 97)

Sol. The given equations are

$$2x - y + 3z = 9$$
$$x + y + z = 6$$
$$x - y + z = 2$$

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$ and assume that there exists a matrix $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{K}$.

Then $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$ *(Kanpur 90)*

$\Rightarrow$ $\begin{bmatrix} 3 & 0 & 4 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \\ 8 \end{bmatrix}$, by the elementary row operations $R_1 \to R_1 + R_2$, $R_3 \to R_3 + R_2$

$\Rightarrow$ $3x + 4z = 15,\ x + y + z = 6,\ 2x + 2z = 8$

$\Rightarrow$ $3x + 4z = 15,\ x + y + z = 6,\ x + z = 4$

$\Rightarrow$ $y = 6 - (x + z) = 6 - 4,$ $\quad \because x + z = 4$

or $y = 2$

Also $3x + 4z = 15$ gives $3x + 4(4 - x) = 15,$ $\quad \because x + z = 4$

or $3x + 16 - 4x = 15$ or $x = 1$

$\therefore$ $z = 4 - x = 4 - 1 = 3$

$\Rightarrow$ $x = 1, y = 2, z = 3.$

***Ex. 4 (b). Solve by matrix method only the equations :**

$$x + y + z = 6;\ x + 2y + 3z = 14,\ x + 4y + 9z = 36.$$

(Gorakhpur 98, 91; Kanpur 94; Rohilkhand 99)

Hint : Do as Ex. 4 (a) above. **Ans.** $x = 1, y = 2, z = 3.$

****Ex. 5 (a). Solve the following equations by matrix method :**

$$x + 2y + z = 2,\ 3x + 5y + 5z = 4,\ 2x + 4y + 3z = 3.$$ *(Bundelkhand 91)*

Sol. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 5 \\ 2 & 4 & 3 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ and assume that there exists a matrix $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, such that $\mathbf{AX} = \mathbf{K}$.

Then $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 5 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \text{ by the elementary row operations } R_2 \to R_2 - 3R_1 \text{ and } R_3 \to R_3 - 2R_1$$

$\Rightarrow \quad x + 2y + z = 2, -y + 2z = -2, z = -1$

$\Rightarrow \quad x + 2y - 1 = 2, -y - 2 = -2, z = -1$

$\Rightarrow \quad x + 2y = 3, y = 0, z = -1$

$\Rightarrow \quad x + 0 = 3, y = 0, z = -1$

$\Rightarrow \quad x = 3, y = 0, z = -1.$ **Ans**

****Ex. 5 (b). Solve the equations :**

$\mathbf{x + y + z = 9, 2x + 5y + 7z = 52, 2x + y - z = 0.}$ *(Gorakhpur 92)*

Sol. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$ and assume that there exists a matrix

$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{K}$.

Then $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 34 \\ -18 \end{bmatrix}, \text{ by the elementary row operations } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 2R_1$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ -18 \end{bmatrix}, \text{ by the elementary row operations } R_2 \to R_2 + 3R_3$$

$\Rightarrow \quad x + y + z = 9; -4z = -20; -y - 3z = -18$

$\Rightarrow \quad z = 5; y = 18 - 3z = 18 - 3(5) = 3$

and $\quad x = 9 - y - z = 9 - 3 - 5 = 1$

$\Rightarrow \quad x = 1, y = 3, z = 5.$ **Ans.**

Ex. 6. Solve the following equations by matrix method—

$\mathbf{2x - y + 3z = 8, -x + 2y + z = 4, 3x + y - 4z = 0.}$ *(Rohilkhand 91)*

Sol. The given equations are

$2x - y + 3z = 8$

$-x + 2y + z = 4$

$3x + y - 4z = 0$

Sol. Let $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$ and assume that there exists a matrix

$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{K}$.

Then
$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$\Rightarrow$
$$\begin{bmatrix} 0 & 3 & 5 \\ -1 & 2 & 1 \\ 0 & 7 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 12 \end{bmatrix}$$, by the elementary row operations $R_1 \to R_1 + 2R_2$ $R_3 \to R_3 + 3R_2$

$\Rightarrow$
$$\begin{bmatrix} 0 & 3 & 5 \\ -1 & 2 & 1 \\ 0 & 1 & -11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ -20 \end{bmatrix}$$, by the elementary row operation $R_3 \to R_3 - 2R_1$

$\Rightarrow$
$$\begin{bmatrix} 0 & 0 & 38 \\ -1 & 2 & 1 \\ 0 & 1 & -11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 76 \\ 4 \\ -20 \end{bmatrix}$$, by the elementary row operation $R_1 \to R_1 - 3R_3$

$\Rightarrow$ $38z = 76, -x + 2y + z = 4, y - 11z = -20$

$\Rightarrow$ $z = 2, -x + 2y = 4 - z = 2, y = -20 + 11z = -20 + 22 = 2$

$\Rightarrow$ $-x = 2 - 2y = 2 - 4 = -2, y = 2, z = 2$

$\Rightarrow$ $x = 2, y = 2, z = 2$

Ans.

Ex. 7. Solve the equations—

$$x_1 + 2x_2 + x_3 = 4, x_1 - x_2 + x_3 = 5, 2x_1 + 3x_2 - x_3 = 1$$

Sol. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$ and assume that there exists a matrix $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{K}$

Then
$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

$\Rightarrow$
$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & -1 & 1 \\ 0 & 5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -9 \end{bmatrix}$$, by the elementary row operations $R_1 \to R_1 - R_2$; $R_3 \to R_3 - 2R_2$

$\Rightarrow$ $3x_2 = -1, x_1 - x_2 + x_3 = 5, 5x_2 - 3x_3 = -9$

$\Rightarrow$ $x_2 = -1/3, x_1 + (1/3) + x_3 = 5, 5(-1/3) - 3x_3 = -9$

$\Rightarrow$ $x_2 = -1/3, x_1 + x_3 = 14/3, x_3 = 22/9.$

$\Rightarrow \quad x_2 = -1/3, x_1 + (22/9) = 14/3, x_3 = 22/9.$

$\Rightarrow \quad x_1 = 20/9, x_2 = -1/3, x_3 = 22/9.$ **Ans.**

Ex. 8. Solve the following equations by matrix method :

$2x + 3y + z = 9; x + 2y + 3z = 6, 3x + y + 2z = 8.$ *(Gorakhpur 95)*

Sol. Let $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}; \mathbf{K} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$ and assume that there exists a matrix

$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, such that $\mathbf{AX} = \mathbf{K}$

Then $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & -1 & -5 \\ 1 & 2 & 3 \\ 0 & -5 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -10 \end{bmatrix}$, by the elementaty row operations $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 - 3R_2$

$\Rightarrow \begin{bmatrix} 0 & -1 & -5 \\ 1 & 2 & 3 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 5 \end{bmatrix}$, by the elementaty row operation $R_3 \to R_3 - 5R_1$

$\Rightarrow \quad -y - 5z = -3, x + 2y + 3z = 6, 18z = 5$

$\Rightarrow \quad -y - \frac{25}{18} = -3, x + 2y + \frac{5}{6} = 6, z = \frac{5}{18}$

$\Rightarrow \quad y = \frac{29}{18}, x + 2\left(\frac{29}{18}\right) = \frac{31}{6}, z = \frac{5}{18}$

$\Rightarrow \quad x = \frac{35}{18}, y = \frac{29}{18}, z = \frac{5}{18}$ **Ans.**

Exercises on § 6.02—§ 6.04

Solve the following equations by the matrix method —

Ex. 1. $3x + y - z = 2, x + 2y + z = 3, -x + y + 4z = 9$ *(Purvanchal 98)*

Ans. $x = 2, y = -1, z = 3$

Ex. 2. $x + 2y - z = 1, x + y + 2z = 9, 2x + y - z = 2$ **Ans.** $x = 2, y = 1, z = 3$

Ex. 3. $x + 2y + 3z = 14, 2x - y + 5z = 15, 3x - 2y + 4z = -13$

Ans. $x = -17\frac{6}{17}, y = \frac{10}{17}, z = 10\frac{1}{17}$

***Ex. 4.** $x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6.$ **Ans.** $x = 2, y = 1, z = 0$

Ex. 5. $4x + 3y + 6z = 25, x + 5y + 7z = 13, 2x + 9y + z = 1$

Ans. $x = 4, y = -1, z = 2$

Ex. 6. $x + y = 5, 2x - y = 1.$ **Ans.** $x = 2, y = 3$

Ex. 7. $x - y + 2z = 3, 2x + z = 1, 3x + 2y + z = 4.$ **Ans.** $x = -1, y = 2, z = 3$

Ex. 8. $x - 2y + 3z = 11,\ 3x + y - z = 2,\ 5x + 3y + 2z = 3.$

Ans. $x = 2,\ y = -3,\ z = 1$

Ex. 9. $x + y + z = 9,\ 2x + 5y + 7z = 50,\ 2x + y - z = 2.$ **Ans.** $x = 1,\ y = 4,\ z = 4$

Ex. 10. $x + y + z = 4,\ 2x - y + 2z = 5,\ x - 2y - z = -3.$ *(Agra 92)*

Ans. $x = 1,\ y = 1,\ z = 2$

Ex. 11. $x - y + 2z = 4,\ 3x + y + 4z = 6,\ x + y + z = 1.$ *(Meerut 94)*

Ans. $x = 0,\ y = -2/3,\ z = 5/3$

Ex. 12. $x + 2y - z = 3,\ 3x - y + 2z = 1,\ 2x - 2y + 3z = 2.$ *(Gorakhpur 93)*

Ans. $x = -1,\ y = 4,\ z = 4$

Ex. 13. $x + 2y + 3z = 6,\ 2x + 4y + z = 7,\ 3x + 2y + 9z = 14.$ *(Meerut 97)*

Ans. $x = 1,\ y = 1,\ z = 1$

Ex. 14. $x - 2y + 3z = 6,\ 3x + y - 4z = -7,\ 5x - 3y + 2z = 5.$ *(Meerut 95)*

Ans. $x = -8/7,\ y = -25/7,\ z = 0$

Ex. 15. $5x - 6y + 4z = 15,\ 7x + 4y - 3z = 19;\ 2x + y + 6z = 46.$ *(Meerut 93)*

Ans. $x = 3;\ y = 4,\ z = 6.$

****§ 6.05. To compute inverse of a square matrix with the help of the linear equations.**

Let a system of n linear equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ be

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= k_2 \\ a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n &= k_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= k_n. \end{aligned}$$

Then this system can be written in the matrix from as

$$\mathbf{AX} = \mathbf{K}, \qquad \ldots(\text{i})$$

where $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$; $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \\ \ldots \\ k_n \end{bmatrix}$

...See § 6.02 Page 105 Ch. VI

If $|\mathbf{A}| \neq 0$, the matrix $\mathbf{A}$ is non-singular and the inverse of $\mathbf{A}$ exists,

...(See Ch. V and Ch. IV)

Hence premultiplying (i) by $\mathbf{A}^{-1}$, we have

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K} \qquad \text{or} \quad (\mathbf{A}^{-1}\mathbf{A})\,\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$

or $$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{K}, \qquad \because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

or $$\mathbf{X} = \mathbf{A}^{-1}\mathbf{K},$$ which gives the value of $\mathbf{A}^{-1}$

Solved Examples on § 6.05.

Ex.1. If $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$**, then find** $\mathbf{A}^{-1}$ **and hence solve the equations**

$\mathbf{2x - y + 3z = 9, x + y + z + 6}$ **and** $\mathbf{x - y + z = 2.}$ *(Kanpur 97)*

Sol. The matrix equation $\mathbf{AX} = \mathbf{K}$ is here equivàlent to the equations

$$2x - y + 3z = k_1 \quad \text{...(i)}$$

$$x + y + z = k_2 \quad \text{...(ii)}$$

$$x - y + z = k_3 \quad \text{...(iii)}$$

Adding (i) and (ii) we get $3x + 4z = k_1 + k_2$ ...(iv)

Adding (ii) and (iii) we get $2x + 2z = k_2 + k_3$ ...(v)

Multiplying (v) by 2, we get $4x + 4z = 2k_2 + 2k_3$ ...(vi)

Subtracting (iv) from (vi), we get $x = -k_1 + k_2 + 2k_3$ ...(vii)

$\therefore$ From (vi), (vii), we get $4z = 2k_2 + 2k_3 - 4(-k_1 + k_2 + 2k_3)$

or $$4z = 4k_1 - 2k_2 - 6k_3$$

or $$z = k_1 - \frac{1}{2}k_2 - \frac{3}{2}k_3 \quad \text{...(viii)}$$

From (iii), $$y = x + z - k_3$$

$$= (-k_1 + k_2 + 2k_3) + (k_1 - \frac{1}{2}k_2 - \frac{3}{2}k_3) - k_3$$

or $$y = 0\,k_1 + \frac{1}{2}k_2 - \frac{1}{2}k_3 \quad \text{...(ix)}$$

$\therefore$ From (vii), (viii) and (ix) we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \text{ i.e. } \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$

$\therefore$ $$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \quad \text{...(x) } \textbf{Ans.}$$

Also given equations can be written in the matrix from as $\mathbf{AX} = \mathbf{K}$, ...(xi)

where $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

Now from (xi), we also have $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$.

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$\Rightarrow \quad x = (-1).9 + 1.6 + 2.2 = 1$

$y = (0).9 + (1/2).6 + (-1/2).2 = 2$

$z = (1).9 + (-1/2).6 + (-3/2).2 = 3$

$\Rightarrow \quad x = 1, y = 2, z = 3.$ **Ans.**

***Ex. 2. Find the invesre of the matrix** $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

Sol. The matrix equation $\mathbf{AX} = \mathbf{K}$ is here eqaivalent to the equations

$$x_1 + 2x_2 + 3x_3 = k_1 \quad \text{...(i)}$$

$$0.x_1 + 5x_2 + 0.x_3 = k_2 \quad \text{...(ii)}$$

$$2x_1 + 4x_2 + 3x_3 = k_3 \quad \text{...(iii)}$$

From (ii) we get $x_2 = \frac{1}{5} k_2 = 0.k_1 + \frac{1}{5} k_2 + 0 . k_3$ ...(iv)

Subtracting (i) from (iii), we get

$$x_1 + 2x_2 = k_3 - k_1$$

or $$x_1 = k_3 - k_1 - 2x_2 = k_3 - k_1 - 2 . \tfrac{1}{5} k_2, \text{ from (iv)}$$

or $$x_1 = -k_1 - (2/5) k_2 + k_3 \quad \text{...(v)}$$

Also from (i), $3x_3 = k_1 - x_1 - 2x_2$

$$= k_1 + k_1 + \tfrac{2}{5} k_2 - k_3 - \tfrac{2}{5} k_2, \text{ from (iv) and (v)}$$

or $$x_3 = \tfrac{2}{3} k_1 + 0 . k_2 - \tfrac{1}{3} k_2 \quad \text{...(vi)}$$

From (iv), (v) and (vi) we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{2}{5} & 1 \\ 0 & \frac{1}{5} & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \qquad i.e.\ \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$

$\therefore$ $$\mathbf{A}^{-1} = \begin{bmatrix} -1. & -\frac{2}{5} & 1 \\ 0 & \frac{1}{5} & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}$$ **Ans.**

Ex. 3. Solve the equations by finding the inverse of the coefficient matrix :

$$5x - 6y + 4z = 15; 7x + 4y - 3z = 19, 2x + y + 6z = 46.$$ *(Gorakhpur 90)*

Sol. The coefficient matrix $= \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix} = \mathbf{A}$ (say).

The matrix equation $\mathbf{AX} = \mathbf{K}$ is here equivalent to the equations

$$5x - 6y + 4z = k_1 \quad \text{...(i)}$$

$$7x + 4y - 3z = k_2 \quad \text{...(ii)}$$

$$2x + y + 6z = k_3 \quad \text{...(iii)}$$

Multiplying (iii) by 6 and adding to (i) we get

$$17x + 40z = k_1 + 6k_3 \quad \text{...(iv)}$$

Multiplying (iii) by -4 and adding to (ii) we get

$$x + 27z = 4k_3 - k_2 \quad \text{...(v)}$$

Multiplying (v) by 17 and subtracting from (iv) we get

$$z = -\frac{1}{419}(k_1 + 17k_2 - 62k_3) \quad \text{...(vi)}$$

Substituting this values of z in (v) and simplifying we get

$$x = \frac{1}{419}(27k_1 + 40k_2 + 2k_3) \quad \text{...(vii)}$$

Substituting values of x and z in (iii) we get on simplifying,

$$y = \frac{1}{419}(-48k_1 + 22k_2 + 43k_3) \quad \text{...(viii)}$$

$\therefore$ From (vi), (vii) and (viii) we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

i.e. $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

$\therefore$ $$\mathbf{A}^{-1} = (1/419)\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$ **Ans.**

Also the given equations can be written in the matrix form as $\mathbf{AX} = \mathbf{K}$ or $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$, where

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix};\ \mathbf{A}^{-1} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix};\ \mathbf{K} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$\Rightarrow$ $x = (1/419)[27(15) + 40(19) + 2(46)] = 3;$

$y = (1/419)[-48(15) + 22(19) + 43(46)] = 4;$

$z = (1/419)[-1(15) - 17(19) + 62(46)] = 6;$

$\Rightarrow$ $x = 3, y = 4, z = 6.$ **Ans.**

Exercises on § 6.05

Find the inverses of the following matrices :

Ex. 1. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$ **Ans.** $\frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

Ex. 2. $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$. **Ans.** $\frac{1}{37}\begin{bmatrix} 1 & -4 & 6 \\ -4 & 16 & 13 \\ 6 & 13 & -1 \end{bmatrix}$

Ex. 3. $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$ **Ans.** $\frac{1}{5}\begin{bmatrix} -17 & -3 & 15 & 7 \\ 9 & 1 & -5 & -4 \\ -10 & -5 & 10 & 5 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Ex. 4. Find the inverse of the coefficient matrix of the following system of equations—

$$x + y + z = 1, x + 2y + 2z = 1, x + 2y + 3z = 0$$

and hence solve them. **Ans.** $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$; $x = 1, y = 1, z = -1$

Ex. 5. Solve the following equations by finding the inverse of coefficient matrix :

$x + y + z = 9, 2x + 5y + 7z = 52, 2x + y - z = 0$ **Ans.** $x = 1, y = 3, z = 5$

Ex. 6. Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ and apply the results to solve the equations.

$2x + 5y + 3z = 9, 3x + y + 2z = 3; x + 2y - z = 6.$ **Ans.** $x = 1, y = 2, z = -1$

§ 6.06. Augmented Matrix.

Definition : The matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

augmented by the matrix $\mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_m \end{bmatrix}$ is called augmented matrix of **A** and is written as $\mathbf{A}^*$ or $[\mathbf{A}, \mathbf{K}]$.

$$\therefore \quad \mathbf{A}^* = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & k_1 \\ a_{21} & a_{22} & \dots & a_{2n} & k_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & k_m \end{bmatrix}$$

Also it is evident that the order of the matrix $\mathbf{A}^*$ or $[\mathbf{A}, \mathbf{K}]$ is $m \times (n + 1)$.

****§ 6.07. Fundamental Theorem.**

A system of m linear equations in n unknowns given by **AX = K** *is consistent (i.e. has a solution) if and only if the matrix of coefficients* **A** *and the augmented matrix* **A*** *of the system have the same rank.* *(Agra 94, 92)*

[If the above common rank is r then r of the unknowns can be expressed as linear combinations of the remaining $n-r$ unknowns. When these $n-r$ unknowns are assigned arbitrary values, the system has an infinite number of solutions out of which $(n-r+1)$ are linearly independent whereas the rest are linear combinations of them.]

Proof. Consider m non-homogeneous linear equations in n unknowns given by $\mathbf{AX}=\mathbf{K}$, where

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \mathbf{X}=\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ and } \mathbf{K}=\begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_n \end{bmatrix}$$

Let r be the rank of the matrix $\mathbf{A}$ and $C_1, C_2, C_2 \dots, C_n$ be the column vectors of the matrix $\mathbf{A}$, then $\mathbf{A}=[C_1, C_2, \dots C_n]$ and so $\mathbf{AX}=\mathbf{K}$ reduces to

$$[\mathbf{C_1}, \mathbf{C_2}, \dots, \mathbf{C_n}]\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}=\mathbf{K}$$

or $$\mathbf{C_1}x_1+\mathbf{C_2}x_2+\dots+\mathbf{C_n}x_n=\mathbf{K}. \qquad \text{...(i)}$$

Necessary Condition. Let the given system of equations possess a solution (*i.e.* be consistent), then there must exist n scalars $b_1, b_2, \dots, b_n$ which satisfy (i)

i.e. $$\mathbf{C}_1b_1+\mathbf{C}_2b_2+\dots+\mathbf{C}_nb_n=\mathbf{K}. \qquad \text{...(ii)}$$

Since rank of $\mathbf{A}$ is r, so each $n-r$ columns viz. $\mathbf{C_{r+1}}, \mathbf{C_{r+2}}, \dots, \mathbf{C_n}$ is a linear combination of $\mathbf{C_1}, \mathbf{C_2}, \dots, \mathbf{C_n}$.

$\therefore$ From (ii) we find that $\mathbf{K}$ is a linear combination of $\mathbf{C_1}, \mathbf{C_2}, \dots, \mathbf{C}_r$, since $\mathbf{C}_{r+1}, \mathbf{C}_{r+2}, \dots \mathbf{C}_n$. in (ii) can be expressed in terms of $\mathbf{C_1}, \mathbf{C_2}, \dots\dots, \mathbf{C}_n$

$\therefore$ The maximum number of linearly independent columns of the augmented matrix $[\mathbf{A}, \mathbf{K}]$ or $\mathbf{A}^*$ is also r. Hence the rank of $\mathbf{A}^*$ is r.

Thus $\mathbf{A}$ and $\mathbf{A}^*$ are of the same rank r.

Sufficient Condition. Let the matrices $\mathbf{A}$ and $\mathbf{A}^*$ be of the same rank r. Then the number of linearly independent columns of the matrix $\mathbf{A}^*$ is r. But the column vectors $\mathbf{C_1}, \mathbf{C_2}, \dots \mathbf{C_r}$ of the matrix $\mathbf{A}^*$ already form a linearly independent set and thus the matrix $\mathbf{K}$ can be expressed as a linear combination of the columns $\mathbf{C_1}, \mathbf{C_2}, \dots, \mathbf{C}_r$.

$\therefore$ There exist r scalars $b_1, b_2, \ldots, b_r$ such that

$$b_1C_1 + b_2C_2 + \ldots + b_rC_r + 0C_{r+1} + \ldots + 0C_n = K \qquad \text{....(iii)}$$

From (i) and (ii) on comparing, we get

$x_1 = b_1, x_2 = b_2, \ldots, x_r = b_r, x_{r+1} = 0, \ldots, x_n = 0$ and these are the solutions of the system of equations given by $AX = K$.

***§ 6.08. Theorem.**

If A *be an* $n \times n$ *matrix,* X *and* K *be* $n \times 1$ *matrices, then the system of equations* $AX = K$ *possess a unique solution if matrix* A *is non-singular.*

Proof. Let $A = [a_{ij}]$ and $|A| \neq 0$.

Then rank of A and augmented matrix $[A, K]$ or A^* are both n. Thus from §6.07 Page 120 Ch. VI we conclude that the system of the equations $AX = K$ is consistent.

From $AX = K$, we have

$$A^{-1}(AX) = A^{-1}K, \text{ premultiplying both sides by } A^{-1}$$

or $\quad (A^{-1}A)X = A^{-1}K \quad$ or $\quad IX = A^{-1}K, \quad \because A^{-1}A = I$

or $\quad X = A^{-1}K$ is the solution of the given system of equations.

Now let X_1 and X_2 be two sets of solutions of $AX = K$.

then $\qquad AX_1 = K;\ AX_2 = K$

$\Rightarrow \quad AX_1 = AX_2$, as each is equal to K

$\Rightarrow \quad A^{-1}(AX_1) = A^{-1}(AX_2)$, premultiplying both sides by A^{-1}

$\Rightarrow \quad (A^{-1}A)X_1 = (A^{-1}A)X_2$

$\Rightarrow \quad IX_1 = IX_2, \qquad \because A^{-1}A = I$

$\Rightarrow \quad X_1 = X_2$

$\Rightarrow$ the solution is unique.

§ 6.09. Reduced Echelon Form of a Matrix.

Definition. If in an Echelon Form matrix (See § 5.04 Page 36 Ch. V) the first non-zero element in the ith row lies in jth column and all other elements in the jth column are zero, then the matrix is said to be in reduced Echelon form.

For example : In $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ the first non-zero element in the second row lies in the second column and all other elements in the second column are zero.

Solved Examples on § 6.07—§ 6.09.

Ex. 1. Solve the system of equations :

$$x + 2y - 3z - 4w = 6$$
$$x + 3y + z - 2w = 4$$

$$2x + 5y - 2z - 5w = 10$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ is

$$\begin{bmatrix} 1 & 2 & -3 & -4 \\ 1 & 3 & 1 & -2 \\ 2 & 5 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}$$

The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 4 & 3 & -2 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - 2R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & -11 & -8 & 10 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, replacing R_1, R_2 by $R_1 - 2R_2$, and $R_3 - R_2$ respectively

This is a matrix in the reduced Echelon form having three non-zero row and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get the reduced Echelon from of $\mathbf{A}$ viz.

$\begin{bmatrix} 1 & 0 & -11 & -8 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ having three non–zero rows and hence the rank of $\mathbf{A}$ is also 3

Thus we observe that $\mathbf{A}$ and $\mathbf{A}^*$ have the same rank and as such the giv equations have solutions which can be obtained as follows :

The matrix equation is

$$\begin{bmatrix} 1 & 0 & -11 & -8 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 0 \end{bmatrix}$$

or $\quad x + 0y - 11z - 8w = 10;$

$\quad y + 4z + 2w = -2$ and $w = 0$

or $\quad y = -4z - 2;\ w = 0z;\ x = 11z + 10.$ **(Note)**

Thus we find (See § 6.07 Page 119 Ch. VI) that as the rank of $\mathbf{A}$ and $\mathbf{A}^*$ is 3, so three of the unknowns viz, x, y and w are expressed as a linear function of the remaining $4 - 3$ *i.e.* one unknown viz. z.

By assigning arbitrary values to z, an infinite number of corresponding values of x, y and w can be obtained. Hence the system of equations has infinite number of solutions.

Now we can show that the system has only $n - r + 1$ *i.e.* $4 - 3 + 1$ *i.e.* 2 linearly independent solutions. (See § 6.07 Page 119 Ch. VI)

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the given equations as

x	10	21
y	-2	-6
z	0	1
w	0	0

Let any other solution of the given equations be

$$x = -1, y = 2, z = -1, w = 0,$$

corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions then a, b can be found as follows :

$$\left.\begin{array}{l} 10a + 21b = -1 \\ -2a - 6b = 2 \\ 0.a + 1.b = -1 \\ 0.a + 0.b = 0 \end{array}\right\} \textbf{(Note)}$$

or $$10a + 21b = -1 \qquad \text{...(i)}$$

$$-2a - 6b = 2 \qquad \text{...(ii)}$$

and $$b = -1. \qquad \text{...(iii)}$$

Solving (i) and (iii) we get $a = 2, b = -1$.

These values of a and b satisfy (ii) also. Hence the third solution is a linear combination of the first two solutions.

Ex. 2 (a). Examine if the following equations are consistent ? If yes, solve it :

$$\boldsymbol{x + y + 4z = 6, 3x + 2y - 2z = 9, 5x + y + 2z = 13.}$$ *(Meerut 96)*

Sol. The given equations in the matrix from $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 1 & 4 \\ 3 & 2 & -2 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 13 \end{bmatrix}$$

The augmented matrix

$$\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 4 & 6 \\ 3 & 2 & -2 & 9 \\ 5 & 1 & 2 & 13 \end{bmatrix}$$

or $$\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 4 & 0 & -2 & 7 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 3R_1, R_3 - R_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & -4 & -18 & -17 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & 0 & 38 & 19 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_3 \text{ by } (1/38)\, R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 - 4R_3,\ R_2 + 14R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_1 \text{ by } R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_2 \text{ by } - R_2$$

This is a matrix in the reduced Echelon form having three non-zero rows and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get the reduced Echelon form of $\mathbf{A}$ viz $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is equal to $\mathbf{I}_3$ and so the rank of $\mathbf{A}$ is also 3.

Thus we observe that $\mathbf{A}$ and $\mathbf{A}^*$ have the same rank and as such the given equations are consistent *i.e.* have solutions which can be obtained as follows :

The matrix equation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1/2 \end{bmatrix}$$

$\Rightarrow \quad x = 2, y = 2, z = 1/2.$ **Ans.**

Ex. 2 (b). Examine, if the system of following equations is consistent. If consistent find the solution :

$$x + y + z = 6,\ 2x + 3y - 2z = 2,\ 5x + y + 2z = 13.$$

Sol. Do as Ex. 2 (a) above.

Ans. Given equations are consistent. $x = 1, y = 2, z = 3$

Ex. 2 (c). Apply rank test to examine if the following system of equations is consistent and if consistent then find the complete solution :

$$2x - y + 3z = 8,\ -x + 2y + z = 4,\ 3x + y - 4z = 0. \qquad \textit{(Garhwal 92)}$$

Sol. The given equations in the matrix form can be written as

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 0 & 3 & 5 & 16 \\ -1 & 2 & 1 & 4 \\ 0 & 7 & -1 & 12 \end{bmatrix}$, replacing R_1, R_3 by $R_1 + 2R_2$ and $R_3 + 3R_2$ respectively

$\sim \begin{bmatrix} 0 & 3 & 5 & 16 \\ -1 & 2 & 1 & 4 \\ 0 & 1 & -11 & -20 \end{bmatrix}$, replacing R_3 by $R_3 - 2R_1$

$\sim \begin{bmatrix} 0 & 0 & 38 & 76 \\ -1 & 2 & 1 & 4 \\ 0 & 1 & -11 & -20 \end{bmatrix}$, replacing R_1 by $R_1 - 3R_2$

$\sim \begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 2 & 1 & 4 \\ 0 & 1 & -11 & -20 \end{bmatrix}$, replacing R_1 by $(1/38)R_1$

$\sim \begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1$ $R_3 + 11R_1$ respectively

$\sim \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing R_2 by $-(R_2 - R_3)$

$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, rearranging rows

$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, replacing R_1 by $R_1 + R_2$

This is a matrix in the reduced Echelon form having three non-zero rows and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get the reduced Echelon from of $\mathbf{A}$ viz.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} i.e\ \mathbf{I}_3$$

and so the rank of $\mathbf{A}$ is also 3.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are the same and so the given equations are consistent *i.e.* have solutions which can be obtained as follows—

The matrix equation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

or $\quad x=2, y=2, z=2.$ **Ans.**

***Ex. 3 (a). Apply rank test to examine if the following system of equations is consistent and if consistent, find the complete solution :**

$\boldsymbol{x+2y-z=6, 3x-y-2z=3, 4x+3y+z=9.}$ *(Meerut 98)*

Sol. Given equations can be written in the matrix form as

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^*=\begin{bmatrix} 1 & 2 & -1 & 6 \\ 3 & -1 & -2 & 3 \\ 4 & 3 & 1 & 9 \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 1 & -5 & 0 & -9 \\ 5 & 5 & 0 & 15 \end{bmatrix}$, replacing R_2, R_3 by R_2-2R_1 and R_3+R_1 respectively

$\sim \begin{bmatrix} 0 & 7 & -1 & 15 \\ 1 & -5 & 0 & -9 \\ 1 & 1 & 0 & 3 \end{bmatrix}$, replacing R_1, R_3 by R_1-R_2 $(1/5)\, R_3$ respectively

$\sim \begin{bmatrix} 0 & 7 & -1 & 15 \\ 0 & -6 & 0 & -12 \\ 1 & 1 & 0 & 3 \end{bmatrix}$, replacing R_2 by R_2-R_3

$\sim \begin{bmatrix} 0 & 1 & -1/7 & 15/7 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix}$, replacing R_1, R_2 by $(1/7)\, R_1$ $(-1/6)\, R_2$ respectively

$\sim \begin{bmatrix} 0 & 1 & -1/7 & 15/7 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1/7 & 6/7 \end{bmatrix}$, replacing R_3 by R_3-R_1

$\sim \begin{bmatrix} 0 & 0 & -1/7 & 1/7 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1/7 & 6/7 \end{bmatrix}$, replacing R_1 by R_1-R_2

$\sim \begin{bmatrix} 1 & 0 & 1/7 & 6/7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1/7 & 1/7 \end{bmatrix}$, rearranging rows

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \text{ replacing } R_1 \text{ by } R_1 + R_3 \text{ and } R_3 \text{ by } -7R_3$$

This is a matrix in the reduced Echelon form having three non-zero rows and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneoausly we get the reduced Echelon form of **A** viz. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i.e. $\mathbf{I}_3$ and so the rank of **A** is also 3.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are the same and so the given system of equations is consistent *i.e.* have solutions which can be obtained as follows—

The matrix equation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\Rightarrow$ $x = 1, y = 2, z = -1.$ **Ans.**

***Ex. 3 (b). Apply rank test to examine if the following system of equations is consistent, solve them :**

$$\mathbf{2x + 4y - z = 9,\ 3x - y + 5z = 5,\ 8x + 2y + 9z = 12.}$$

Sol. The given equations in the matrix form $\mathbf{AX = K}$ can be written as

$$\begin{bmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 8 & 2 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 19 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 2 & 4 & -1 & 9 \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 19 \end{bmatrix}$

or $$\mathbf{A}^* = \begin{bmatrix} 2 & 4 & -1 & 9 \\ 3 & -1 & 5 & 5 \\ 2 & 4 & -1 & 9 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 2R_2$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 9 \\ 1 & -5 & 6 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 9 \\ 0 & -7 & 13/2 & -17/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 - \tfrac{1}{2} R_1$$

$$\sim \begin{bmatrix} 2 & 0 & 19/7 & 29/7 \\ 0 & 1 & -13/14 & 17/14 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } -(1/7)\, R_2 \text{ and then } R_1 \text{ by } R_1 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{19}{14} & \frac{29}{14} \\ 0 & 1 & -\frac{13}{14} & \frac{17}{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_1 \text{ by } \tfrac{1}{2} R_1$$

This is a matrix in the reduced Echelon form having two non-zero rows and hence the rank of $\mathbf{A}^*$ is 2.

Simultaneously we get the reduced Echelon form of **A** viz. $\begin{bmatrix} 1 & 0 & (19/14) \\ 0 & 1 & -(13/14) \\ 0 & 0 & 0 \end{bmatrix}$ which also has two non-zero rows and as such the rank of **A** is also 2.

Thus we observe that the rank of **A** and $\mathbf{A}^*$ are the same and as such the given equations are consistent *i.e.* have solutions which can be obtained as follows—

The matrix equation is

$$\begin{bmatrix} 1 & 0 & (19/14) \\ 0 & 1 & -(13/14) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (29/14) \\ (17/14) \\ 0 \end{bmatrix}$$

or $\quad x + (19/14)z = (29/14);\ y - (13/14)z = (17/14)$

or $\quad y = \dfrac{13}{14}z + \dfrac{17}{14};\ x = \dfrac{29}{14} - \dfrac{19}{14}z$

or $\quad x = -\dfrac{19}{14}z + \dfrac{29}{14};\ y = \dfrac{13}{14}z + \dfrac{17}{14}.$

Thus we find that as the rank of $\mathbf{A}^*$ and **A** is 2, so two of the unknowns viz x and y are expressed as a linear function of the remaining $3 - 2$ *i.e.* one unknown viz. z.

By assining arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence the given system of equations has an infinite number of solutions.

Now we can show that the system has only $n - r + 1$ *i.e.* $3 - 2 + 1$ *i.e.* 2 linearly independent solution (See § 6.07 Page 120 Ch VI).

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the given equations as

x	$\frac{29}{14}$	$\frac{10}{14}$
y	$\frac{17}{14}$	$\frac{30}{14}$
z	0	1

Let any other solution of the given equation be $x=(24/7)$, $y=(2/7)$, $z=-1$, corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions then a and b can be found as follows —

$$\left.\begin{aligned}(29/14)\,a+(10/14)\,b&=(24/7)\\(17/14)\,a+(30/14)\,b&=(2/7)\\0.a+1.b&=-1\end{aligned}\right\}\qquad\textbf{(Note)}$$

or
$$29a+10b=48 \quad ...(i)$$
$$17a+30b=4 \quad ...(ii)$$
$$b=-1. \quad ...(iii)$$

Solving (i) and (iii) we get $a=2$, $b=-1$, which satisfy (ii) also. Hence the third solution is a linear combination of the first two solutions.

***Ex. 4. Apply rank test to examine if the following system of equations is consistent and if consistent, find the complete solution.**

$$\boldsymbol{x+y+z=6,\ x+2y+3z=0,\ x+2y+4z=1.}$$

Sol. The given equations in the matrix form $\mathbf{AX}=\mathbf{K}$ can be written as

$$\begin{bmatrix}1&1&1\\1&2&3\\1&2&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\1\end{bmatrix}$$

The augmented matrix $\mathbf{A}^*=\begin{bmatrix}1&1&1&6\\1&2&3&10\\1&2&4&1\end{bmatrix}$

or
$$\mathbf{A}^*\sim\begin{bmatrix}1&1&1&6\\1&2&3&10\\0&0&1&-9\end{bmatrix},\ \text{replacing } R_3 \text{ by } R_3-R_2$$

$$\sim\begin{bmatrix}1&1&1&6\\0&1&2&4\\0&0&1&-9\end{bmatrix},\ \text{replacing } R_2 \text{ by } R_2-R_1$$

$$\sim\begin{bmatrix}1&0&-1&2\\0&1&2&4\\0&0&1&-9\end{bmatrix},\ \text{replacing } R_1 \text{ by } R_1-R_2.$$

or
$$\mathbf{A}^*\sim\begin{bmatrix}1&0&0&-7\\0&1&0&22\\0&0&1&-9\end{bmatrix},\ \text{replacing } R_1, R_2 \text{ by } R_1+R_3 \text{ and } R_2-2R_3 \text{ respectively}$$

This is a matrix in the reduced Echelon form having three non-zero rows and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get the reduced Echelon form of the matrix $\mathbf{A}$ viz.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *i.e.* $\mathbf{I}_3$ and hence the rank of **A** is 3.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are the same and so the given equations are consistent.

$\therefore$ The matrix equation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 22 \\ -9 \end{bmatrix}$

or $\quad x = -7, y = 22, z = -9.$ **Ans.**

Ex. 5. Are the following equations consistent ?

$$\begin{aligned} x + y + 2z + w &= 5 \\ 2x + 3y - z - 2w &= 2 \\ 4x + 5y + 3z &= 7. \end{aligned}$$

(Agra 91)

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

(Note)

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -5 & -4 & -13 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - 2R_1$ and $R_3 - 4R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 7 & 5 & 13 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$, replacing R_1, R_3 by $R_1 - R_2$, $R_3 - R_2$ respectively

This is a matrix in the reduced Echelon form having three non-zero rows, hence its rank is 3.

Simultaneously we get the reduced Echelon form of **A** viz. $\begin{bmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has two non-zero rows hence its rank is 2.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are not the same, hence the given equations are not consistent *i.e.* they cannot have any solutions.

Ex. 6. Discuss the consistency and find the solution set of the following equations :—

$$\mathbf{x + 2y + 2z = 1, 2x + y + z = 2, 3x + 2y + 2z = 3, y + z = 0.}$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_4$ and $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, replacing R_1, R_2, R_3 by $R_1 - 2R_4$ $\frac{1}{2}R_2, \frac{1}{2}R_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1$ $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging R_2 and R_4

This is a matrix in the reduced Echelon form having two non-zero rows, hence its rank is 2.

Simultaneously we get the reduced Echelon form of $\mathbf{A}$ viz.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which has two non-zero rows and hence its rank is also 2.

Thus we find that the ranks of $\mathbf{A}$ and $\mathbf{A}^*$ are the same and so the given equations are consistent.

$\therefore$ The matrix equation reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$x = 1, y + z = 0. \qquad ...(i)$$

Also here we find that the rank of **A** and **A*** is each 2 *i.e.* less than the number of unknowns viz. x, y and z. So the number of solutions of given equations will be **infinite** given by (i) above, which gives $x = 1$ and $y + z = 0$ can be satisfied by an infinite number of values *e.g.* 0, 0; 1, – 1; 2, – 2; etc.

Ex. 7. Show that the equations $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$ are consistent and solve them. *(Bundelkhand 96)*

Sol. The given equations in the matrix form **AX** = **K** can be written as

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} -4 & -75 & 1 & -23 \\ 3 & 26 & 2 & 9 \\ -8 & -128 & 0 & -40 \end{bmatrix}$, replacing R_1, R_3 by $R_1 - 3R_2$ and $R_3 - 5R_2$ respectively.

$\sim \begin{bmatrix} 4 & 75 & -1 & 23 \\ 3 & 26 & 2 & 9 \\ 0 & 22 & -2 & 6 \end{bmatrix}$, replacing R_1, R_3 by $-R_1$ and $R_3 - 2R_1$ respectively

$\sim \begin{bmatrix} 4 & 75 & -1 & 23 \\ 3 & 26 & 2 & 9 \\ 0 & 11 & -1 & 3 \end{bmatrix}$, replacing R_3 by $\frac{1}{2} R_2$

$\sim \begin{bmatrix} 4 & 64 & 0 & 20 \\ 3 & 48 & 0 & 15 \\ 0 & 11 & -1 & 3 \end{bmatrix}$, replacing R_1, R_2 by $R_1 - R_3$ $R_2 + 2R_3$ respectively.

$\sim \begin{bmatrix} 1 & 16 & 0 & 5 \\ 1 & 16 & 0 & 3 \\ 0 & 1 & -(1/11) & (3/11) \end{bmatrix}$, replacing R_1, R_2 and R_3 by $\frac{1}{4} R_1, \frac{1}{3} R_2$ and $(1/11) R_3$ respectively.

$\sim \begin{bmatrix} 1 & 0 & (16/11) & (7/11) \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -(1/11) & (3/11) \end{bmatrix}$, replacing R_2 by $R_2 - R_1$ and then R_1 by $R_1 - 16R_3$

$\sim \begin{bmatrix} 1 & 0 & (16/11) & (7/11) \\ 0 & 1 & -(1/11) & (3/11) \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging R_2 and R_3

This is a matrix in the reduced Echelon form having two non-zero rows, hence its rank is 2.

Simultaneously we get the reduced Echelon form of A

viz. $\begin{bmatrix} 1 & 0 & (16/11) \\ 0 & 1 & -(1/11) \\ 0 & 0 & 0 \end{bmatrix}$ which also has two non zero rows and so its rank is also 2.

Thus we find that the rank of A and A^* are the same and as such the given equations are consistent. And so the matrix equation reduces to

$$\begin{bmatrix} 1 & 0 & (16/11) \\ 0 & 1 & -(1/11) \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} (7/11) \\ (3/11) \\ 0 \end{bmatrix}$$

or $$x+(16/11)z=(7/11);\ y-(1/11)z=(3/11)$$

or $$11x=7-16z,\ 11y=z+3.$$

Thus we find that as the ranks of A and A^* is 2, so two of the unknowns viz. x and y are expressed as a linear function of remaining $3-2$ *i.e.*, one known viz. z.

By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Thus given system of equations has an infinite number of solutions. Now we can show that the system has only $(n-r+1)$ *i.e.* $(3-2+1)$ *i.e.* 2 linearly independent solutions (See § 6.07 Page 119 Ch. VI).

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the given equations as

x	$\frac{7}{11}$	$-\frac{9}{11}$
y	$\frac{3}{11}$	$\frac{4}{11}$
z	0	1

Let any other solution of the given equation be $x=(23/11)$, $y=(2/11)$, $z=-1$ corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions, then a and b can be found as follows :—

$$(7/11)a-(9/11)b=(23/11) \qquad ...(i)$$

$$(3/11)a+(4/11)b=(2/11) \qquad ...(ii)$$

$$0.a+1.b=-1. \qquad ...(iii)$$

From (i) and (iii) we get $a=2, b=-1$, which satisfy (ii) also. Hence the third solution is a linear combination of the first two solutions.

***Ex. 8. Apply test of rank to examine if the equations $x+y+z=6$, $x+2y+3z=14$, $x+4y+7z=30$ are consistent and if consistent find the complete solution.** *(Kumaun 91; Meerut 96P)*

Sol. In the matrix form $AX=K$, the given equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_1 and R_3 by $R_1 - R_2$ and $R_3 - 3R_2$ respectively.

This is a matrix in the reduced Echelon form having two non-zero rows, hence its rank is 2.

Simultaneously we get the reduced Echelon form **A** viz.

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, which has two non-zero rows and hence its rank is 2.

Thus we find that the ranks of **A** and **A** are the same and as such the given equations are consistent.

Now the matrix form of the given equations reduce to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}$$

which is equivalent to

$$1 \,.\, x + 0.\, y - 1 \,.\, z = -2;\ 0 \,.\, x + 1 \,.\, y + 2 \,.\, z = 8;\ 0.x + 0.y + 0.z = 0$$

or $$x - z = -2,\ y + 2z = 8$$

$\therefore$ As the rank of **A** and $\mathbf{A}^*$ is 2, so two of the unknowns viz. x and y are expressed as a linear function of the remaining unknown z viz. $x = -2 + z,\ y = 8 - 2z$, where z is arbitrary.

By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence the system of equations has infinite number of solutions.

And $x = -2 + k,\ y = 8 - 2k,\ z = k$ forms the general solution of the given equations.

In the matrix form the solution can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix} + k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

***Ex. 9. Find the values of λ so that the equations**

$$ax + hy + g = 0, hx + by + f = 0, gx + fy + c = \lambda$$

are consistent.

Sol. The given equations in the matrix form **AX = K** can be written as

$$\begin{bmatrix} a & h \\ h & b \\ g & f \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -g \\ -f \\ \lambda - c \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} a & h & -g \\ h & b & -f \\ g & f & \lambda - c \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 1 & h/a & -g/a \\ 1 & b/h & -f/h \\ 1 & f/g & \{(\lambda - c)/g\} \end{bmatrix}$, replacing R_1, R_2 and R_3 by R_1/a, R_2/h and R_3/g respectively. (Note)

$$\sim \begin{bmatrix} 1 & h/a & -g/u \\ 0 & (b/h) - (h/a) & -(f/h) + (g/a) \\ 0 & (f/g) - (h/a) & \{(\lambda - c)/g\} + (g/a) \end{bmatrix},$$ replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$

$$\sim \begin{bmatrix} 1 & h/a & -g/a \\ 0 & (ba - h^2)/ha & (gh - af)/ah \\ 0 & (af - gh)/ga & \{(\lambda a + g^2 - ac)/ag\} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & h/a & -g/a \\ 0 & (ba - h^2)/ha & (gh - af)/ah \\ 0 & \dfrac{(ba - h^2)}{ha} & \dfrac{(\lambda a + g^2 - ac)(ba - h^2)}{ah(af - gh)} \end{bmatrix},$$

replacing R_2 by $\dfrac{g}{h}\dfrac{(ba - h^2) R_2}{(af - gh)}$

$$\sim \begin{bmatrix} 1 & h/a & -g/a \\ 0 & (ba - h^2)/ha & (gh - af)/ah \\ 0 & 0 & \{\lambda(ab - h^2) - (abc + 2fgh \\ & & - af^2 - bg^2 - ch^2)\}/\{h(af - gh)\} \end{bmatrix},$$ replacing R_3 by $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & h/a & -g/a \\ 0 & 1 & (gh - af)/(ba - h^2) \\ 0 & 0 & \mu \end{bmatrix},$$ replacing R_2 by $ah\, R_2/(ba - h^2)$ ...(i)

Here $\mu = \dfrac{\lambda\,(ab - h^2) - (abc + 2fgh - af^2 - bg^2 - ch^2)}{h\,(af - gh)}$ ...(ii)

Simultaneously we get the reduced Echelon form of **A** viz.

$\begin{bmatrix} 1 & h/a \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, which has two non-zero rows hence its rank is 2 ...(iii)

From (i) and (iii) we conclude that if the given equations have solution then the ranks of **A** and $\mathbf{A}^*$ must be the same viz. 2 and from (i) if the rank of $\mathbf{A}^*$ is 2, then it must have two non-zero rows *i.e.* $\mu = 0$.

i.e. $\lambda = (abc + 2fgh - af^2 - bg^2 - ch^2)/(ab - h^2)$, from (i). **Ans.**

****Ex. 10. For what values of λ, the equations $x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$ have a solution and solve completely in each case.** *(Garhwal 90; Kanpur 97, 93, 91; Rohilkhand 92)*

Sol. The given equation in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & -2 & 2 - \lambda \\ 0 & 3 & 9 & 3\lambda - 3 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{bmatrix}$, replacing R_1 by $R_1 - R_2$ and then R_2 by $3R_2$

$\sim \begin{bmatrix} 1 & 0 & -2 & 2 - \lambda \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{bmatrix}$, replacing R_3 by $R_3 - R_2$ and R_2 by $\frac{1}{3} R_2$ ...(i)

Simultaneously we get the reduced Echelon form of **A** viz.

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ which has two non-zero rows hence its rank is 2. ...(ii)

From (i), (ii) we conclude that if the given equations have solution then the ranks of **A** and $\mathbf{A}^*$ must be the same viz. 2 and from (i) if the rank $\mathbf{A}^*$ is 2, then it must have two non-zero rows *i.e.* $\lambda^2 - 3\lambda + 2 = 0$ or $\lambda = 1, 2.$ **Ans.**

The matrix form of the given equations reduces to

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$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2-\lambda \\ \lambda - 1 \\ \lambda^2 - 3\lambda + 2 \end{bmatrix}$$

which is equivalent to

$$\left.\begin{array}{l} 1 \,.\, x + 0 \,.\, y - 2 \,.\, z = 2 - \lambda,\ 0 \,.\, x + 1 \,.\, y + 3 \,.\, z = \lambda - 1 \\ 0 \,.\, x + 0 \,.\, y + 0 \,.\, z = \lambda^2 - 3\lambda + 2 \end{array}\right\}$$

and

If $\lambda = 1$, then these are $x - 2z = 1, y + 3z = 0$.

As the rank of **A** and **A*** is 2, so two of the unknowns viz, x and y are expressed as a linear function of the remaining unknown z viz $x = 2z + 1, y = -3z$.

By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence the system of equations has infinite number of solutions.

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the given equations as

x	1	3
y	0	-3
z	0	1

Let any other solution of the given equations be $x = -1, y = 3, z = -1$ corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions then a, b can be found as follows :

$$\left.\begin{array}{l} 1.a + 3.b = -1 \\ 0.a - 3.b = 3 \\ 0.a + 1.b = -1 \end{array}\right\} \qquad \text{...(iv)}$$

or $a + 3b = -1, 3b = -3$ or $b = -1$

i.e. $b = -1, a = 2$. These values of a and b satisfy all the three equations given by (iv). Hence the third solution is a linear combination of the first two solutions.

We can similarly solve for $\lambda = 2$ also.

***Ex. 11. Express the following system of equations into the matrix equations** $\mathbf{AX = K}$

$$\mathbf{4x - y + 6z = 16,\ x - 4y - 3z = -16.}$$
$$\mathbf{2x + 7y + 12z = 48,\ 5x - 5y + 3z = 0}$$

Determine if the system of equations is consistent and if so find its solution.

Sol. In the given system of equations, we observe that the number of unknowns are not equal to the number of equations.

The single matrix equation of these is

$$\begin{bmatrix} 4 & -1 & 6 \\ 1 & -4 & -3 \\ 2 & 7 & 12 \\ 5 & -5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -16 \\ 48 \\ 0 \end{bmatrix} \quad i.e.\ \mathbf{AX} = \mathbf{K}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 4 & -1 & 6 & 16 \\ 1 & -4 & -3 & -16 \\ 2 & 7 & 12 & 48 \\ 5 & -5 & 3 & 0 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 0 & 15 & 18 & 80 \\ 1 & -4 & -3 & -16 \\ 0 & 15 & 18 & 80 \\ 0 & 15 & 18 & 80 \end{bmatrix}$ replacing R_1, R_3 and R_4 by $R_1 - 4R_2$, $R_3 - 2R_2$ and $R_4 - 5R_2$ respectively.

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & -4 & -3 & -16 \\ 0 & 15 & 18 & 80 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ interchanging R_1 and R_2 and then replacing R_3, R_4 by $R_3 - R_2$, $R_4 - R_2$

$\sim \begin{bmatrix} 1 & 0 & (9/5) & (16/3) \\ 0 & 1 & (6/5) & (16/3) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ replacing R_2 by $(1/15)\, R_2$ and then R_1 by $R_1 + 4R_2$

This is a matrix in the reduced Echelon form having two non-zero rows hence its rank is 2.

Simultaneously we get the reduced Echelon form of **A** viz.

$\begin{bmatrix} 1 & 0 & (9/5) \\ 0 & 1 & (6/5) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which has two non-zero rows and hence its rank is also 2.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are the same and as such the given equations are consistent.

Now the matrix form of the given equations reduce to :

$$\begin{bmatrix} 1 & 0 & (9/5) \\ 0 & 1 & (6/5) \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (16/3) \\ (16/3) \end{bmatrix}$$

which is equivalent to

$$1.x + 0.y + (9/5)\, z = (16/3);\ 0.x + 1.y + (6/5)z = (16/3)$$

or $\quad 15x + 27z = 80;\ 15y + 18z = 80.$

As the rank of **A** and $\mathbf{A}^*$ is 2, so two of the unknowns viz. x and y are expressed as a linear function of the remaining unknown z viz.

$$15x = 80 - 27z \text{ and } 15y = 80 - 18z$$

or $$x = -(9/5)z + (16/3) \text{ and } y = -(6/5)z + (16/3),$$

where z is arbitrary.

By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence the given equations have infinite number of solutions.

Also $x = (16/3) - (9/5)k$, $y = (16/3) - (6/5)k$, $z = k$ forms the general solutions of the given equations.

In the matrix form the solutions can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (16/3) \\ (16/3) \end{bmatrix} + k \begin{bmatrix} -(9/5) \\ -(6/5) \\ 1 \end{bmatrix}$$

Ans.

***Ex. 12. Examine if the system of equations $x + y + 4z = 6$, $3x + 2y - 2z = 9$, $5x + y + 2z = 13$ is consistent ? Find also the solution if it is consistent.**

Sol. The given equations are

$$x + y + 4z = 6$$
$$3x + 2y - 2z = 9$$
$$5x + y + 2z = 13$$

In the matrix form $\mathbf{AX} = \mathbf{K}$, these can be written as

$$\begin{bmatrix} 1 & 1 & 4 \\ 3 & 2 & -2 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 13 \end{bmatrix}$$

$\therefore$ The arugmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 4 & 6 \\ 3 & 2 & -2 & 9 \\ 5 & 1 & 2 & 13 \end{bmatrix}$

or $$\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & -4 & -18 & -17 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 3R_1 \text{ and } R_3 - 5R_1 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & -10 & -3 \\ 0 & -1 & -14 & -9 \\ 0 & 0 & 38 & 19 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 + R_2, R_3 - 4R_2 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & -10 & -3 \\ 0 & 1 & 14 & 9 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } -R_2, (1/38)R_3 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 14 & 9 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_1 \text{ by } R_1 + 10R_3$$

This is a matrix having three non-zero rows and in the reduced Echelon form, hence its rank is 3.

Simultaneously we get the reduced Echelon form of **A** viz.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 14 \\ 0 & 0 & 1 \end{bmatrix}$, which also has three non-zero rows and hence its rank is also 3.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are the same and as such the given equations are consistent.

Now the matrix form of the given equations reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 14 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ \frac{1}{2} \end{bmatrix}$$

which is equivalent to

$$1.x + 0.y + 0.z = 2;\ 0.x + 1.y + 14.z = 9;\ 0.x + 0.y + 1.z = \frac{1}{2}$$

or $\quad x = 2;\ y + 14z = 9;\ z = \frac{1}{2} \quad$ or $\quad x = 2;\ y = 9 - 14z;\ z = \frac{1}{2}$

or $\quad x = 2,\ y = 9 - 7 = 2,\ z = \frac{1}{2}$.

Ans.

****Ex. 13. Show the equations $-2x + y + z = a$, $x - 2y + z = b$, $x + y - 2z = c$ have no solution unless $a + b + c = 0$ in which case they have infinitely many solutions. Find the solution when $a = 1, b = 1, c = -2$.** *(Lucknow 90)*

Sol. The given equations are

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$

In the matrix form $\mathbf{AX} = \mathbf{K}$, these can be written as

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 0 & 0 & 0 & a+b+c \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{bmatrix}$, replacing R_1 by $R_1 + R_2 + R_3$

$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & 0 & 0 & a+b+c \\ 0 & 1 & -1 & \frac{1}{3}(c-b) \end{bmatrix}$, interchanging R_1 and R_2 and then replacing R_3 by $\frac{1}{3}(R_3 - R_1)$

$$\sim\begin{bmatrix} 1 & 0 & -1 & \frac{1}{3}(2c+b) \\ 0 & 1 & -1 & \frac{1}{3}(c-b) \\ 0 & 0 & 0 & a+b+c \end{bmatrix},$$ interchanging R_2 and R_3 and then replacing R_1 by $R_1 + 2R_2$ ...(i)

This is a matrix in the reduced Echelon form and has three non-zero rows, hence its rank is 3.

Simultaneously we get the reduced form of **A** viz.

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which has two non-zero rows and hence its rank is 2.

Thus we find that the rank of $\mathbf{A}^*$ and $\mathbf{A}$ are not the same and as such the given equations are inconsistent *i.e.* have no solution.

But in case $a+b+c=0$, the augmented matrix $\mathbf{A}^*$ has two non-zero rows *i.e.* the rank of $\mathbf{A}^*$ is also 2 and thus $\mathbf{A}$ and $\mathbf{A}^*$ have the same rank 2. Consequently the given equations have solutions if $a+b+c=0$ and in this case from (i) we have

$$\mathbf{A}^* = \begin{bmatrix} 1 & 0 & -1 & \frac{1}{3}(2c+b) \\ 0 & 1 & -1 & \frac{1}{3}(c-b) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\therefore$ The matrix form of given equations reduce to,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(2c+b) \\ \frac{1}{3}(c-b) \\ 0 \end{bmatrix},$$ which is equivalent to

$$1.x + 0.y - 1.z = \frac{1}{3}(2c+b);\ 0.x + 1.y - 1.z = \frac{1}{3}(c-b)$$

or $$x - z = \frac{1}{3}(2c+b),\ y - z = \frac{1}{3}(c-b). \qquad \text{...(ii)}$$

As a, b, c can take different values we shall get different solutions rather infinitely many solutions (as will be evident below also) of the given equations.

If $a=1, b=1, c=-2$, then from (ii) we get

$$x - z = -1,\ y - z = -1$$

or $$x = z - 1,\ y = z - 1$$

$\therefore$ By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence in this case the given system of equations has infinite number of solutions.

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the equations as

x	-1	0
y	-1	0
z	0	1

Let any other solution of the given equations be $x = -2,\ y = -2,\ z = -1$, corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions, then two constants λ, μ can be found as follows —

$$\left.\begin{array}{r} -1\,.\,\lambda + 0.\mu = -2 \\ -1.\lambda + 0.\mu = -2 \\ 0.\lambda + 1.\mu = -1 \end{array}\right\} \quad \text{...(iii)}$$

$$\Rightarrow \ -\lambda = -2,\ \mu = -1, \Rightarrow \lambda = 2,\ \mu = -1.$$

These values of λ, μ satisfy all the equations of (iii) and as such third solution corresponding to $z = -1$ is a linear solution of the first two solutions.

***Ex. 14. Solve the equations with the help of matrices considering specially the case when $\lambda = 2$:—**

$$\lambda x + 2y - 2z = 1,\ 4x + 2\lambda y - z = 2,\ 6x + 6y + \lambda z = 3. \qquad \textit{(Kumaun 90)}$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} \lambda & 2 & -2 & 1 \\ 4 & 2\lambda & -1 & 2 \\ 6 & 6 & \lambda & 3 \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 6\lambda & 12 & -12 & 6 \\ 12\lambda & 6\lambda^2 & -3\lambda & 6\lambda \\ 6\lambda & 6\lambda & \lambda^2 & 3\lambda \end{bmatrix}$, replacing R_1, R_3 by $6R_1, 3\lambda R_2, \lambda R_3$ respectively.

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 6\lambda & 12 & -12 & 6 \\ 0 & 6\lambda^2 - 24 & 24 - 3\lambda & 6\lambda - 12 \\ 0 & 6\lambda - 12 & \lambda^2 + 12 & 3\lambda - 6 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 2R_1, R_3 - R_1$ respectively.

This is a matrix having three non-zero rows and in the reduced Echelon form. Hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get reduced Echelon form of $\mathbf{A}$ viz.

$\begin{bmatrix} 6\lambda & 12 & -12 \\ 0 & 6\lambda^2 - 24 & 24 - 3\lambda \\ 0 & 6\lambda - 12 & \lambda^2 + 12 \end{bmatrix}$ which also has three non-zero rows and so its rank is also 3.

Thus the ranks of $\mathbf{A}^*$ and $\mathbf{A}$ are the same and as such the given equations have solutions.

The matrix form of the given equations then reduce to

$$\begin{bmatrix} 6\lambda & 12 & -12 \\ 0 & 6\lambda^2-24 & 24-3\lambda \\ 0 & 6\lambda-12 & \lambda^2+12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 6\lambda-12 \\ 3\lambda-6 \end{bmatrix}$$

This gives $\quad 6\lambda x + 12y - 12z = 6;$

$$(6\lambda^2 - 24)\,y + (24 - 3\lambda)\,z = 6\lambda - 12$$

and $\quad (6\lambda - 12)\,y + (\lambda^2 + 12)\,z = 3\lambda - 6$

If $\lambda = 2$, then these equations reduce to

$$12x + 12y - 12z = 6;\ 18z = 0$$

which gives $\quad z = 0$ and $2x + 2y = 1$

i.e. $\quad x = -y + \frac{1}{2},\ z = 0.y$ **(Note)**

By assigning arbitrary values to y, an infinite number of corresponding values of x and z can be obtained. Hence the given system of equations has an infinite number of solutions. (This can be ascertained from the ranks of $\mathbf{A}^*$ and $\mathbf{A}$ also in the case when $\lambda = 2$).

Assigning two arbitrary values 0, 1 to y we have two sets of solutions of the given equations as

x	$\frac{1}{2}$	$-\frac{1}{2}$
y	0	1
z	0	0

Let any other solution of the given equation be $x = 3/2$, $y = -1$, $z = 0$ corresponding to the value -1 of y.

If this third solution is a linear combination of the first two solutions, then a and b can be found as follows :

$$\tfrac{1}{2}a - \tfrac{1}{2}b = 3/2;\ 0.a + 1.b = -1;\ 0.a + 0.b = 0.$$

Solving the first two of these we get $a = 1$, $b = -1$ which satisfy the third also.

Hence this solution is a linear combination of the first two solutions. In this way we can get two linearly independent solutions of the given set of equations for $\lambda = 2$.

****Ex. 15. Solve the system of linear equations :**

$$\mathbf{2x - 3y + 4z = 3,\ x - 3z = -2.}$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 2 & -3 & 4 & 3 \\ 1 & 0 & -3 & -2 \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 0 & -3 & 10 & 7 \\ 1 & 0 & -3 & -2 \end{bmatrix}$, replacing R_1 by $R_1 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -(10/3) & (-7/3) \end{bmatrix}$, interchanging R_1 and R_2 and replacing R_2 by $\frac{1}{3} R_3$

This is a matrix having two non-zero rows, hence its rank is 2.

Simultaneously we get $\mathbf{A} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -(10/3) \end{bmatrix}$, which is also in the reduced Echelon form having two non-zero rows. Hence its rank is 2.

Thus we find that $\mathbf{A}$ and $\mathbf{A}^*$ have the same rank 2 and then two of th unknowns can be expressed as a linear function of the remaining unknown.

Now the matrix form of the given equations reduce to

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -(10/3) \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -2 \\ (-7/3) \end{bmatrix}$$

which is equivalent to $1.x + 0.y - 3.z = -2;\ 0.x + 1.y - \frac{10}{3} z = -\frac{7}{3}$

i.e. $x - 3z = -2,\ -3y + 10z = 7$ *i.e.* $x = 3z - 2,\ y = \frac{10}{3} z - \frac{7}{3}$, for all z

i.e. $x = 3k - 2,\ y = (10/3)k - (7/3),\ z = k,$ for all k. **Ans.**

Hence the complete solution of the given system of equations is $x = 3k - 2$, $y = (10/3)\,k - (7/3),\ z = k$, for all k. **Ans.**

Exercises on § 6.07 – § 6.09

Ex. 1. Show that the following equations are consistant :

$3x - 4y = 2,\ 5x - 2y = 12,\ -x + 3y = 1.$

Ex. 2. Show that the equations

$2x + 6y + 11 = 0,\ 6x + 20y - 6z + 3 = 0,\ 6y - 13z + 1 = 0$

are not consistent.

Show that if in the following problems the given equations are consistent, then solve them.

Ex. 3. $5x + 3y + 7z = 4,\ 3x + 20y + 2z = 9,\ 7x + 2y + 10z = 5.$

(Kanpur 84; Meerut 86) **Ans.** $x = 5, y = 0, z = -3$

Ex. 4. $x_1 + 2x_2 + x_3 = 2,\ 2x_1 + 4x_2 + 3x_3 = 3,\ 3x_1 + 6x_2 + 5x_3 = 4.$

Ex. 5. $x_1 - x_2 + x_3 = 2,\ 3x_1 - x_2 + 2x_3 = -6,\ 3x_1 + x_2 + x_3 = -18.$

Ex. 6. $x_1 - 3x_2 + x_3 = 2,\ 2x_1 + x_2 + 3x_3 = 3,\ x_1 + 5x_2 + 5x_3 = 2.$

Ans. $x_1 = 1,\ x_2 = -(1/5),\ x_3 = 2/5$

Ex. 7. Can the following equations have solutions ?

(i) $x - y + 3z + 2w = 3$, $3x + 2y + z + w = 1$, $4x + y + 2z + 2w = 3$.

(ii) $x - 4y + 7z = 8$, $3x + 8y - 2z = 6$, $7x - 8y + 16z = 31$.

Ex. 8. Prove, without actually solving that the following system of equations have a unique solution—

$$5x + 3y + 14z = 4,\ y + 2z = 1,\ x - y + 2z = 0.$$

Ex. 9. Can the following equations have solutions ?

(i) $x + 2y + 3z = 4$, $2x + 3y + 8z = 7$, $x - y + 9z = 1$. **Ans.** Yes

(ii) $x + 2y + 3z = 2$, $2x + 3y + 4z = 5$, $3x + 4y + 5z = 9$. *(Agra 92)* **Ans.** No.

Ex. 10. Show that the following equations

$$x + 2y - z = 3,\ 3x - y + 2z = 1,\ 2x - 2y + 3z = 2,\ x - y + z = -1$$

are consistent and solve them by the use of matrices.

(Rohilkhand 94; Garhwal 93)

Ex. 11. Show that the following equations are consistent and find their solutions by matrix method.

$$x_1 + x_2 + x_3 = 2,\ 4x_1 - x_2 + 2x_3 = -6,\ 3x_1 + x_2 + x_3 = -18.$$

Ans. $x_1 = -10$, $x_2 = -10/3$, $x_3 = 46/3$

Ex. 12. Solve $3x - 4y = 2$, $5x + 2y = 12$, $-x + 3y = 1$. **Ans.** $x = 2$, $y = 1$

Solution of Homogeneous Linear Equations

§ 6.10. Definition : A linear equation of the type

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots + a_nx_n = 0$$

is called a homogeneous linear equation.

§ 6.11. Definition : A system of homogeneous linear equations is given in the matrix form by $\mathbf{AX} = \mathbf{O}$, where $\mathbf{A}$ and $\mathbf{X}$ are the same notations as used in §6.02 Page 105 Chapter VI.

Note 1 : Here $\mathbf{K}$ is zero matrix.

Note 2 : The matrix of coefficients $\mathbf{A}$ and the augmented matrix $\mathbf{A}^*$ being the same have equal ranks and thus the system is always consistent.

Note 3 : $x_1 = 0 = x_2 = \ldots = x_n$ is always a solution and is called the **trivial solution.**

An Important Theorem (Without Proof).

A homogeneous system of n linear equations in n unknowns, whose determinants of coefficients does not vanish, has only the trivial solution.

§ 6.12. Theorem (Without Proof)

A system m of m homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$ has a solution other than the trivial solution viz. $x_1 = 0 = x_2 = \ldots = x_n$ if and only if the rank r of the matrix of coefficients $\mathbf{A}$ is less than n, the number of unknowns.

If $r = n$, then n of the equations can be solved by Cramer's Rule for the unique solution $x_1 = 0 = x_2 = \ldots = x_n$ and the given system has non-trival solutions.

If r is the rank of **A**, then r of the n unknowns $x_1, x_2, \ldots, x_n$ can be expressed as linear combination of the remaining $n-r$ unknowns to which arbitrary values may be assigned. Hence the system will have an infinite number of solutions out of which $n-r$ are linearly independent and the remaining can be expressed as linear combination of these $n-r$.

The above theorem is illustrated in the following examples :—

Ex. 1. Solve

$$\begin{aligned} \mathbf{x+y+z+w} &= \mathbf{0} \\ \mathbf{x+3y+2z+4w} &= \mathbf{0} \\ \mathbf{2x+z-w} &= \mathbf{0} \end{aligned}$$

Sol. The given equations in the matrix form $\mathbf{AX = O}$ is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix **A** of coefficients $= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & -1 \end{bmatrix}$

or $\quad \mathbf{A} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & -2 & -1 & -3 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - 2R_1$ respectively

$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 + R_2$

$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1/2 & (3/2) \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_2 by $\frac{1}{2}R_2$

or $\quad \mathbf{A} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1/2 & (3/2) \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2 by $C_2 - C_1$

This is a matrix in the reduced Echelon form having two non-zero rows, hence the rank of **A** is 2 and is less than the number of unknowns x, y, z and w *i.e.* 4.

$\therefore$ The matrix form of the given equations reduces to

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & (3/2) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} 1.x + 0.y + 1.z + 1.w &= 0; \\ 0.x + 1.y + \tfrac{1}{2}.z + \tfrac{3}{2}.w &= 0; \\ 0.x + 0.y + 0.z + 0.w &= 0. \end{aligned}$$

From the first two equations we have

$$\left.\begin{array}{l} x = -z - w \\ y = -\frac{1}{2}z - (3/2)\,w \end{array}\right\} \quad \text{...(i)}$$

i.e. two of the unknowns viz. x and y have been expressed as linear combinations of the remaining two unknowns viz. z and w.

An infinite number of solutions of the given equations can be obtained by assigning arbitrary values to z and w.

Also according to § 6.11 Page 144 Ch. VI we know that the system has $n - r$ *i. e.* $4 - 2$ *i.e.* 2 linearly independent solutions.

Take any two solutions of the system by assigning the following arbitrary values to z and w

$$z = 2, 4$$
$$w = 0, 2$$

Then the solutions are given in the tabular form as

x	-2	-6
y	-1	-5
z	2	4
w	0	2

(**Note :** The corresponding values of x and y are calculated from the equations (i) above)

Let any other solution be

$$x = -2,\ y = -3,\ z = 0,\ w = 2, \quad \text{...(ii)}$$

obtained by assigning z and w the value 0 and 2 in equations (i).

If this solution is a linear combination of the first two solutions (given in the above table) then we can always find two constants λ and μ such that

$$\left.\begin{array}{lr} -2\lambda - 6\mu = -2 & \text{...(iii)} \\ -\lambda - 5\mu = -3 & \text{...(iv)} \\ 2\lambda + 4\mu = 0 & \text{...(v)} \\ 0.\lambda + 2\mu = 2 & \text{(vi)} \end{array}\right\} \quad \textbf{(Note)}$$

From (vi) we get $\mu = 1$.

From (v) we get $\lambda = -2$.

These values of λ and μ satisfy (iii) and (iv) also.

Hence the third solution [given by (ii) above] is a linear combination of the first two solutions (given in the tabular form above).

Ex. 2. Find the solution of the following equations by the matrix method :

$$2x_1 - x_2 + x_3 = 0,\ 3x_1 + 2x_2 + x_3 = 0,\ x_1 - 3x_2 + 5x_3 = 0.$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{O}$ is given by

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix A of coefficients

$$= \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ -9 & 2 & 5 \end{bmatrix}, \text{ replacing } C_1, C_2 \text{ by } C_1 - 2C_3 \text{ and } C_2 + C_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 9 & 29 & 14 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 - 3C_1 \text{ and } C_3 - C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 29 & 14 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 9R_2$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 14R_1 \text{ and } C_2 \text{ by } (1/29)\, C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$= \mathbf{I}_3$$

The rank of A is 3 and is equal to the number of unknowns viz. x_1, x_2, x_3.

Hence $x_1 = 0 = x_2 = x_3$. (See § 6.12 Page 144 Chapter V

Ex. 3. Show by considering rank of an appropriate matrix, that the following system of equations, possesses no solution other than the trivial solutions $x = 0 = y = z$:—

$$3x - y + z = 0, \; -15x + 6y - 5z = 0, \; 5x - 2y + 2z = 0$$

Sol. The give equations in the matrix form $\mathbf{AX} = \mathbf{O}$ is given by

$$\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix A of coefficients

$$= \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ -1 & 0 & 2 \end{bmatrix}, \text{ replacing } C_1, C_2 \text{ by } C_1 - 3C_3, C_2 + C_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 + 5R_1, -R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_3,$$

$$= \mathbf{I}_3$$

$\therefore$ The rank of A is 3 and is equal to the number of unknowns viz. x, y, z. Hence $x = 0 = y = z$ and the given system has no non-trivial solutions. (See § 6.12 Page 144 Chapter VI).

Ex. 4. Solve :

$$\begin{aligned} 2x - 2y + 5z + 3w &= 0 \\ 4x - y + z + w &= 0 \\ 3x - 2y + 3z + 4w &= 0 \\ x - 3y + 7z + 6w &= 0 \end{aligned}$$

(Kumaun 94)

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{O}$ is given by

$$\begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix A of coefficients

$$= \begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 4 & -9 & -9 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 1 & -3 & 7 & 6 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ and } R_4 \text{ by } R_1 - 2R_4, R_2 - 4R_4 \text{ and } R_3 - 3R_4 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_4$$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 7 & -18 & -14 \\ 0 & 0 & 0 & -0 \end{bmatrix}, \text{ replacing } R_4 \text{ by } R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 7 & -18 & -14 \\ 0 & 0 & 0 & -0 \end{bmatrix}, \text{ replacing } C_2 \text{ by } C_2 + 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & -9/4 & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - \tfrac{7}{4} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 6 \\ 0 & 1 & -9/4 & -9/4 \\ 0 & 0 & 1 & -7/9 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } \tfrac{1}{4}R_2 \text{ and } R_3 \text{ by } -\tfrac{4}{9} R_3$$

This is a matrix in the reduced Echelon form having three non-zero rows, hence the rank of **A** is 3 and is less than the number of unknowns viz. 4.

$\therefore$ The matrix form of the given equations reduces to

$$\begin{bmatrix} 1 & 0 & 7 & 6 \\ 0 & 1 & -9/4 & -9/4 \\ 0 & 0 & 1 & -7/9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$x + 7z + 6w = 0; \qquad \text{...(i)}$$

$$y - \frac{9}{4} z - \frac{9}{4} w = 0 \qquad \text{...(ii)}$$

and $$z - \frac{7}{9} w = 0 \qquad \text{...(iii)}$$

From (iii) we get $z = (7/9)\, w$

$\therefore$ From (i) $x = -7z - 6w = [-(49/9) - 6]\, w = -(103/9) w$

and $\qquad y = (9/4)\, z + (9/4\, w) = [(7/4) + (9/4)]\, w = 4w.$

Thus we find that the three of the unknowns viz. x, y and z are expressed in terms of the 4th unknown viz. w.

An infinite number of solutions of the given equations can be obtained by assigning arbitrary values to w.

Also we know that the system has $n-r$ *i. e.* $4-3$ *i. e.* 1 linearly independent solution.

Assigning w one arbitrary value 9, we have a set of solution as $x=-103, y=36, z=7, w=9$.

Let another solution (by assigning 18 to w) be

$$x=-206, y=72, z=14 \text{ and } w=18.$$

It is evident that this second set of values are nothing but double of the first set of values. Hence the theorem of § 6.12 Page 144 Chapter VI is fully verified.

Ex. 5. Solve completely the system of equations :

$$\begin{aligned} x-2y+z-w&=0\\ x+y-2z+3w&=0\\ 4x+y-5z+8w&=0\\ 5x-7y+2z-w&=0 \end{aligned}$$

(Kumaun 93)

Sol. The given equation in the matrix form $AX=O$ is given by

$$\begin{bmatrix} 1 & -2 & 1 & -1\\ 1 & 1 & -2 & 3\\ 4 & 1 & -5 & 8\\ 5 & -7 & 2 & -1 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0\\ 0 \end{bmatrix}$$

The matrix A of coefficients

$$=\begin{bmatrix} 1 & -2 & 1 & -1\\ 1 & 1 & -2 & 3\\ 4 & 1 & -5 & 8\\ 5 & -7 & 2 & -1 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & -2 & 1 & -1\\ 0 & 3 & -3 & 4\\ 0 & 9 & -9 & 12\\ 0 & 3 & -3 & 4 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ and } R_4 \text{ by } R_2-R_1, R_3-4R_1 \text{ and } R_4-5R_1 \text{ respectively}$$

$$\sim\begin{bmatrix} 1 & 0 & -1 & 5/3\\ 0 & 1 & -1 & 4/3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ and } R_4 \text{ by } R_3-3R_2 \text{ and } R_4-R_2 \text{ respectively, then } R_2 \text{ by } \tfrac{1}{3}R_2 \text{ and finally } R_1 \text{ by } R_1+2R_2.$$

This is a matrix in the reduced Echelon form having two non-zerorows, hence the rank of A is 2 and is less than the number of unknowns viz. 4.

$\therefore$ The matrix form of the given equations reduces to

$$\begin{bmatrix} 1 & 0 & -1 & (5/3)\\ 0 & 1 & -1 & (4/3)\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0\\ 0 \end{bmatrix}$$

which is equivalent to

$$x - z + (5/3)\,w = 0,\; y - z + (4/3)w = 0$$

i.e.
$$y = z - (4/3)\,w,\; x = z - (5/3)\,w.$$

$$\therefore\; \mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} z - (5/3)\,w \\ z - (4/3)\,w \\ z \\ w \end{bmatrix}$$

By multiplication we find that

$$\mathbf{AX} = \begin{bmatrix} 1 & 0 & -1 & (5/3) \\ 0 & 1 & -1 & (4/3) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z - (5/3)\,w \\ z - (4/3)\,w \\ z \\ w \end{bmatrix}$$

$$= \begin{bmatrix} z - (5/3)\,w - z + (5/3)\,w \\ 0 + z - (4/3)\,w - z + (4/3)\,w \\ 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{O},$$

whatever the values of z and w may be.

$\therefore$ We have $x = \lambda + (5/3)\,\mu,\; y = \lambda - (4/3)\,\mu,\; z = \lambda,\; w = \mu$, where λ and μ can take any values, as the complete solution of the given system of equations.

****Ex. 6. Find the general solution of the matrix :**

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 9 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \\ t \end{bmatrix} = \mathbf{O}$$

Sol. The given equation in the matrix form $\mathbf{AX} = \mathbf{O}$ is given by

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 9 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix $\mathbf{A}$ of coefficients

$$= \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 9 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 5 & 3 & 7 \\ 1 & -1 & -2 & -4 \\ 0 & 4 & 9 & 10 \\ 0 & 8 & 21 & 17 \end{bmatrix}$$ replacing R_1, R_2 and R_4 by $R_1 - 2\,R_2,\; R_3 - 3\,R_2$ and $R_4 - 6\,R_2$ respectively.

$$\sim \begin{bmatrix} 0 & 1 & -6 & -3 \\ 1 & -1 & -2 & -4 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$ replacing R_1, R_4 by $R_1 - R_2$ and $R_4 - 2\,R_2$ respectively.

$$\sim \begin{bmatrix} 0 & 1 & -6 & -3 \\ 1 & 0 & -8 & -7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 3 & -3 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 4R_1 \text{ and } R_2 \text{ by } R_2 + R_1.$$

$$\sim \begin{bmatrix} 0 & 1 & -9 & -0 \\ 1 & 0 & -8 & -7 \\ 0 & 0 & 0 & 55 \\ 0 & 0 & 3 & -3 \end{bmatrix}, \text{ replacing } R_1 \text{ and } R_3 \text{ by } R_1 - R_4 \text{ and } R_3 - 11\, R_4 \text{ respectively.}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & -9 \\ 1 & 0 & 0 & -15 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \text{ replacing } R_1, R_2, R_3 \text{ and } R_4 \text{ by } R_1 + 3\, R_4, R_2 + (8/3)\, R_4, (1/55)\, R_3 \text{ and } (1/3)\, R_4 \text{ respectively.}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ replacing } R_1, R_2, \text{ and } R_4 \text{ by } R_1 + 9\, R_3, R_2 + 15\, R_3 \text{ and } R_4 + R_3 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ rearranging the rows}$$

$$\sim [\mathbf{I}_4].$$

$\therefore$ The rank of the matrix **A** is 4 and is equal to the number or unknowns viz. x, y, v, t.

Hence $x = 0, y = 0, v = 0, t = 0$ [See § 6.12 Page 144 Ch. VI].

Exercises on § 6.10 – § 6.12

Ex. 1. Solve the following equations :—

$x_1 - x_2 + x_3 = 0, x_1 + 2x_2 - x_3 = 0, 2x_1 + x_2 + 3x_3 = 0.$ *(Lucknow 92)*

Ex. 2. Find the rank of the coefficient matrix for the following system of homogeneous equations over the field of real numbers and compute all the solutions :—

$$x_1 + 2x_2 - 3x_3 + 4x_4 = 0, x_1 + 3x_2 - x_3 = 0, 6x_1 + x_2 + 2x_3 = 0.$$

Ex. 3. Solve completely the following equations using matrices :—

$x + 3y - 2z = 0, 2x - y + 4z = 0, x - 11y + 14z = 0.$ *(Lucknow 90)*

Ex. 4. Solve completely the following equations with the help of matrices :

(i) $x - y - z + t = 0, x - y + 2z - t = 0, 3x + y + t = 0.$

(ii) $2w + 3x - y - z = 0, 4w - 6x - 2y + 2z = 0, -6w + 12\,x + 3y - 4z = 0.$

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1. Show that the only real value of λ for which the following equations have non-zero solution is 6 :**

$$\mathbf{x + 2y + 3z = \lambda x, 3x + y + 2z = \lambda y, 2x + 2y + z = \lambda z.}$$ *(Kanpur 95)*

Sol. The given equations can be rewritten as

$$(1-\lambda)x + 2y + 3z = 0;$$
$$3x + (1-\lambda)y + 2z = 0;$$
and $$2x + 3y + (1-\lambda)z = 0.$$

The equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be rewritten as

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the given system of equations has a non-zero solution then the matrix **A** must have a rank < the number of unknown quantities x, y, z *i.e.* 3 and $|\mathbf{A}| = 0$

$$i.e. \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0, \text{ replacing } R_1 \text{ by } R_1 + R_2 + R_3$$

or $$(6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0, \text{ taking out } (6-\lambda) \text{ common from } R_1$$

or $$(6-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0, \text{ applying } C_2 - C_1 \text{ and } C_3 - C_1$$

or $$(6-\lambda) \begin{vmatrix} -2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

or $$(6-\lambda)[(2+\lambda)(1+\lambda) + 1] = 0$$

or $$(6-\lambda)[\lambda^2 + 3\lambda + 3] = 0 \text{ or } \lambda = 6, \tfrac{1}{2}[-3 \pm \sqrt{(9-12)}]$$

or $\lambda = 6$ (the other roots being imaginary) is the only real value of λ for which the given system of equations has a non-zero solution.

Ex. 2. Prove that if the system of equations

$$\mathbf{x = ay + z, y = z + ax, z = x + y}$$

is consistent (having non-zero solutions) then a + 1 = 0.

Sol. The given equations can be rewritten as

$$x - ay - z = 0$$
$$ax - y + z = 0$$
$$x + y - z = 0$$

These equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be rewritten as

$$\begin{bmatrix} 1 & -a & -1 \\ a & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the given system of equations has a non-zero solution then the matrix **A** have a rank < the number of unknown quantities x, y, z *i.e.* 3 and $|\mathbf{A}| = 0$. Here we have

$$\mathbf{A} = \begin{bmatrix} 1 & -a & -1 \\ a & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

or

$$\mathbf{A} \sim \begin{bmatrix} 0 & -a-1 & -1 \\ a+1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \text{ replacing } C_1, C_2 \text{ by } C_1 + C_3, C_2 + C_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & -(a+1) & 0 \\ a+1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 - R_3, R_2 + R_3 \text{ respectively} \qquad ...(i)$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ replacing } C_1, C_2, C_3 \text{ by } C_1/(a+1), -C_2/(a+1), -C_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_2$$

$$= \mathbf{I}_3$$

i.e. the rank of **A** is 3 *i.e.* equal to the number of unknowns viz. x, y, z.

But if $a + 1 = 0$, then from (i) we get

$$\mathbf{A} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ which has one non-zero row and so the rank of } \mathbf{A} \text{ is } 1$$

i.e. < 3, the number of unknowns viz. x, y, z.

Also
$$|\mathbf{A}| = \begin{vmatrix} 1 & -a & -1 \\ a & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} a & -a-1 & -1 \\ a+1 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix}, \text{adding } C_3 \text{ to } C_1 \text{ and } C_2$$

$$= \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix}, \text{ if } a + 1 = 0$$

$= 0$, two rows being identical

Hence the given equations are consistent if $a + 1 = 0$.

***Ex. 3. Investigate for what values of λ, μ the simultaneous equations :**

$$x + 2y + z = 8,\ 2x + y + 3z = 13,\ 3x + 4y - \lambda z = \mu$$

have (i) no solution (ii) a unique solution and (iii) infinitely many solutions.

Sol. The given equations in the matrix form $AX = K$ can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 4 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ \mu \end{bmatrix}$$

$\therefore$ The augmented matrix $A^* = \begin{bmatrix} 1 & 2 & 1 & 8 \\ 2 & 1 & 3 & 13 \\ 3 & 4 & -\lambda & \mu \end{bmatrix}$

or $A^* \sim \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 1 & -3 \\ 0 & -2 & -\lambda-3 & \mu-24 \end{bmatrix}$, replacing R_2, R_3, by $R_2 - 2R_1, R_3 - 3R_1$ respectively;

$\sim \begin{bmatrix} 1 & 5 & 0 & 11 \\ 0 & -6 & 2 & -6 \\ 0 & -6 & -3\lambda-9 & 3\mu-72 \end{bmatrix}$, replacing R_1, R_2, R_3 by $R_1 - R_2$, $2R_2$ and $3R_3$ respectively;

$\sim \begin{bmatrix} 1 & 5 & 0 & 11 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & -3\lambda-11 & 3\mu-66 \end{bmatrix}$, replacing R_3 and R_2 by $R_3 - R_2$, and $-\dfrac{1}{6}R_2$ respectively ...(i)

Case I If $3\lambda + 11 \neq 0$, $3\mu - 66 \neq 0$ *i.e.* $\lambda \neq -(11/3)$, $\mu \neq 22$, then from (i) matrix A^* has three non-zero rows and is in the reduced Echelon form. Thus the matrix A^* is of rank 3. Also then the matrix $A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & -3\lambda-11 \end{bmatrix}$ is also in reduced Echelon form having three non-zero rows and thus its rank is also 3. Also there are three unknown quantities x, y, z so, in the case $\lambda \neq -(11/3)$, $\mu \neq 22$, the given equations have a **unique solution.**

Case II. If $3\lambda + 11 = 0$, $3\mu - 66 \neq 0$ *i.e.* $\lambda = -(11/3)$, $\mu \neq 22$, then the rank of the matrix A^* is 3 but that of A is 2, since in this case both A^* and A are in the reduced Echelon form but A^* has three non-zero rows, wheares A has two non-zero rows. Thus the ranks of A^* and A are not the same and so there is **no solution** of the given equations.

Case III. If $3\lambda + 11 = 0$, $3\mu - 66 = 0$ *i.e.* $\lambda = -(11/3)$, $\mu = 22$, then the ranks of A as well as A^* are the same and each is 2 *i.e.* less than the number of unknowns viz. x, y and z. Hence in this case two unknowns will be expressed in terms of the third and thus we shall have an **infinite number of solutions.**

***Ex. 4. Investigate for what values of λ and μ, the simultaneous equations : $x + y + z = 6, x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$ have (i) no solution, (ii) unique solution (iii) infinite solutions.**

(Agra 96, 93, 91; Garhwal 91, 90; Kanpur 96, 94; Meerut 91 S, 90; Rohilkhand 96, 90)

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

or $\mathbf{A}^* = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_2$ respectively and then R_1 by $R_1 - R_2$ ...(i)

Now following cases arise :—

Case I. If $\lambda - 3 = 0, \mu - 10 \neq 0$ *i.e.* $\lambda = 3, \mu \neq 10$, then from (i), $\mathbf{A}^*$ is in the reduced Echelon form having three non-zero rows and $\mathbf{A}$ is in the reduced Echelon form having two non-zero rows.

$\therefore$ The ranks of $\mathbf{A}^*$ and $\mathbf{A}$ are 3 and 2 respectively which being different, the given equations have **no solution**.

Case II. If $\lambda - 3 \neq 0, \mu - 10 \neq 0$ *i.e.* $\lambda \neq 3, \mu \neq 10$, then from (i) we find that both $\mathbf{A}^*$ and $\mathbf{A}$ are in the reduced Echelon form having three non-zero rows and hence the ranks of $\mathbf{A}^*$ and $\mathbf{A}$ are each 3 and these being the same the given equations are consistent. Also there are three unknowns viz. x, y, z so the solution is **unique** in this case.

Case III. If $\lambda - 3 = 0, \mu - 10 = 0$ *i.e.* $\lambda = 3, \mu = 10$, then from (i) we find that the matrices $\mathbf{A}^*$ and $\mathbf{A}$ are in the reduced Echelon form having two non-zero rows each. Hence the ranks of $\mathbf{A}^*$ and $\mathbf{A}$ are the same each being 2, which is less than the number of unknowns x, y, z. Therefore in this case two unknowns will be expressed in terms of the third and thus we shall have an **infinite** number of solutions.

Ex. 5. Show that the following equations are consistent $x + y + z = -3, x + y - 2z = -2, 2x + 4y + 7z = 7$. *(Kumaun 96)*

Sol. Given equations can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 1 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 0 & -3 & 1 \\ 1 & 3 & 6 & 10 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1$, $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 0 & -3 & 1 \\ 1 & 3 & 0 & 12 \end{bmatrix}$, replacing R_3 by $R_3 + 2R_2$

$\sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 0 & -1 & 1/3 \\ 0 & 2 & -1 & 15 \end{bmatrix}$, replacing R_2, R_3 by $(1/3)\, R_2$, $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 1 & 0 & -8/3 \\ 0 & 0 & -1 & 1/3 \\ 0 & 2 & 0 & 44/3 \end{bmatrix}$, replacing R_1, R_3 by $R_1 + R_2$ $R_3 - R_2$ respectively

$\sim \begin{bmatrix} 1 & 1 & 0 & -8/3 \\ 0 & 0 & 1 & -1/3 \\ 0 & 1 & 0 & 22/3 \end{bmatrix}$, replacing R_2, R_3 by $-R_2$, $(1/2)\, R_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 0 & 1 & 1/3 \\ 0 & 1 & 0 & 22/3 \end{bmatrix}$, replacing R_1 by $R_1 - R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 22/3 \\ 0 & 0 & 1 & -1/3 \end{bmatrix}$, interchanging R_2 and R_3

This is a matrix in the reduced Echelon form having three non-zero rows, so its rank is 3.

Simultaneously we get the reduced form of **A** viz. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *i.e.* $\mathbf{I}_3$ and so the rank of **A** is also 3.

Thus we find that the ranks of $\mathbf{A}^*$ and **A** are the same and so the given equations are consistent i.e. have solutions given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 22/3 \\ -1/3 \end{bmatrix}$,

which gives

$$x = -10,\ y = 22/3,\ z = -1/3.$$

Ans.

EXERCISES ON CHAPTER VI

Ex. 1. Examine whether the following linear equations are consistent, and if consistent solve them :—

$$x_1 + x_2 + x_3 + x_4 = 0,\ 2x_1 - x_2 + 3x_3 + 4x_4 = 4$$

and $$3x_1 + 4x_3 + 5x_4 = 1.$$

Ex. 2. Solve the equations by matrix method :—

$$x + 2y + 3z = 14;\ x + y + z = 6,\ x + 3y + 6z = 25.$$

Ex. 3. Solve by matrix method :—

$$x - 2y + 3z = 2,\ 2x - 3z = 3;\ x + y + z = 0.$$

Ex. 4. Solve by matrix method :—

$x + 2y - z = 3,\ 3x - y + 2z = 1,\ 2x - 2y + 3z = 2.$ **Ans.** $x = -1, y = 4, z = 4$

Ex. 5. Solve $x + 2y + 3z = 14,\ 2x + y + 2z = 11,\ 2x + 3y + z = 11$ with the help of matrices. **Ans.** $x = 18/11, y = 17/11, z = 34/11$

Ex. 6. Investigate k such that the following system of linear equations is consistent and obtain its solution :—

$$2x + y - z = 12,\ x - y - 2z = -3;\ 3y + 3z = k.$$

Ex. 7. Investigate for which values of λ and μ the following system of equations will have

(i) No solution and (ii) a unique solution :—

$$x + 2y + 3z = 5,\ 3x - y + 2z = 12,\ 3x - y + \lambda z = \mu.$$

Ex. 8. Investigate for what values of λ and μ, the simultaneous equations $x + y + z = 16,\ x + 2y + 5z = \mu,\ x + 2y + \lambda z = 10$ have unique solution. *(Agra 90)*

Ex. 9. Does the following system of linear equations possess a unique solution ? If so, then solve them. If not, why ?

$$x + 2y + 3z = 6,\ 2x + 4y + z = 7,\ 3x + 2y + 9z = 14.$$

Ex. 10. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ and use it to solve the equations $x + 2y + 3z = 3,\ 2x + 3y + 2z = 0,\ 3x + 3y + 4z = 5.$

Ex. 11. Show that the following system of equations have unique solution :

$$x + 2z = 0,\ y + 2z = 1;\ 5x + 3y + 14z = 4.$$

Ex. 12. Solve : $2x_1 - 3x_2 + 4x_3 + x_4 = 0;\ x_1 + x_2 - x_4 = 0;$

$$3x_1 - 3x_2 + 5x_3 = 0;\ 4x_1 - 3x_2 + 6x_3 - x_4 = 0.$$

(Hint. See Ex. 1 Page 145 Chapter VI).

Ex. 13. State the conditions under which a system of non-homogeneous equations will have (i) no solution, (ii) a unique solution, (iii) infinity of solutions.

Ex. 14. For what values of a, b do the system of equations $x + 2y + 3z = 6,\ x + 3y + 5z = 9;\ 2x + 5y + az = b$ have (i) no solution; (ii) a unique solution; (iii) more than one solutions ?

*Ex. 15. Solve $x+2y-z=3; 3x-y+2z=1; 2x-2y+3z=2; x-y+z=-1$.

Ex. 16. Find the inverse of the coefficient matrix and hence solve the following equations :—

$$x+2y-3z=1,\ 3x+y+z=8,\ x-2y=0.$$

Ex. 17. Apply rank-test to examine if the following system of equations is consistent and if consistent, then only find the complete solution :—

$$x+y+z=6,\ 4x+3y+z=9,\ 2x+2y+3z=8.$$

Ex. 18. Apply rank test to find whether

$$x+2y+3z+4w=0,\ 8x+5y+z+4w=0,\ 5x+6y+8z+w=0,$$
$$8x+3y+7z+2w=0$$

have any solution other than $x=y=z=w=0$.

Ex. 19. Solve the following equations by the use of matrices

(a) $x-y-2z-4t=0,\ 2x+3y-z-t=0,\ 6x+3y-7t=0,$
$3x+y+3z-2t=0.$

(b) $x+y+z=0,\ 2x-y-3z=0,\ 3x-5y+4z=0,\ x+17y+4z=0$

(c) $x+2y+3z=0, 3x+4y+4z=0,\ 7x+10y+12z=0.$

Ex. 20. Solve the equations by matrix inversion :

$$x+y+z=4,\ 2x-y+3z=1,\ 3x+2y-z=1.$$

Ex. 21. Show by matrix method that the following system of equations is consistent and solve it

$$4x_1+3x_2+3x_3+x_4=9,\ x_1+2x_2+x_3+x_4=2;$$
$$3x_1+4x_2+2x_3-x_4=8;\ 2x_1+3x_2+4x_3+5x_4=5;$$
$$x_1-x_2+x_3-x_4=4.$$

Ex. 22. Show that the equations $x-3y-8z+10=0,\ 3x+y-4z=0,$ $2x+5y+6z-13=0$ are consistent and solve them. *(Meerut 92)*

Ex. 23. Solve by matrix method :—

$$x+y+z=4,\ x-y+z=5,\ 2x+3y-z=1.$$

Ex. 24. Show that the system of equations is consistent

$$2x+6y=-11,\ 6x+20y-6z=-3,\ 6y-18z=-1.$$

Ex. 25. Show by matrix method, the following equations are consistent and have infinite number of solutions

$$x_1+x_2+x_3=0,\ 2x_1+5x_2+6x_3=0.$$

Ex. 26. Solve by matrix method ;

$$x+y+z+w=1,\ 2x-y+z-2w=2,\ 3x+2y-z-w=3.$$

Ex. 27. Slove the following equations :—

$$2x_1-2x_2+3x_3=3,\ x_1+2x_2-x_3-5x_4=4,\ x_1+2x_2-2x_3+7x_4=5$$

(Rohilkhand 93)

Ex. 28. If x, y, z are not all zero and if $ax+by+cz=0,$
$bx+cy+az=0,\ cx+ay+bz=0$, prove that,

$$x:y:z=1:1:1 \quad \text{or} \quad 1:\omega:\omega^2 \quad \text{or}\ 1:\omega^2:\omega$$

where ω is a complex cube root of unity. *(Purvanchal 98)*

Chapter VII

Characteristic Equation of a Matrix

§ 7.01. Zero Divisors.

We have previously read in § 5.10 Page 76 of Chapter V that the necessary and sufficient condition for a square matrix to possess an inverse is that it must be non-singular. From this result, we get a very important result which does not hold for ordinary multiplication of numbers viz. 'If **A** and **B** are two singular matrices, it is possible to obtain the result $\mathbf{AB} = \mathbf{O}$, where neither $\mathbf{A} = \mathbf{O}$ nor $\mathbf{B} = \mathbf{O}$, $\mathbf{O}$ being the null matrix'. In such a case **A** and **B** are called **proper divisors of zero.**

If however, **A** and **B** be two square matrices of order n such that

$$\mathbf{AB} = \mathbf{O}, \qquad \text{...(i)}$$

then if **A** is non-singular, $\mathbf{A} \neq \mathbf{O}$, $\mathbf{A}^{-1}$ exists and $\mathbf{A}^{-1} \neq \mathbf{O}$.

Pre-multiplying (i) with $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{O} \text{ or } \mathbf{IB} = \mathbf{O}, \because \ \mathbf{A}^{-1}.\mathbf{O} = \mathbf{O} \text{ and } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

or $\qquad \mathbf{B} = \mathbf{O}.$

Hence we conclude that

If $\qquad \mathbf{A} \neq \mathbf{O}$, then $\mathbf{AB} = \mathbf{O} \Rightarrow \mathbf{B} = \mathbf{O}$

Similarly if $\mathbf{B} \neq \mathbf{O}$, then $\mathbf{AB} = \mathbf{O} \Rightarrow \mathbf{A} = \mathbf{O}$

Hence non-singular matrices are not proper divisiors of zero.

Note. If $\mathbf{AB} = \mathbf{O}$ and $\mathbf{B} \neq \mathbf{O}$ then **A** is called a **left zero divisor** and if $\mathbf{AB} = \mathbf{O}$ and $\mathbf{A} \neq \mathbf{O}$, then **B** is called a **right zero divisor**.

§ 7.02. Characteristic equation and roots of a matrix.

(Agra 94; Rohilkhand 92)

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix.

(i) Characteristic Matrix of A :— The matrix $\mathbf{A} - \lambda\mathbf{I}$ is called the **characteristic matrix of** A, where **I** is the identity matrix.

(ii) Characteristic polynomial of A :— The determinant $|\mathbf{A} - \lambda\mathbf{I}|$ is called the **characteristic polynomial of A**.

(iii) Characteristic equation of A :— The equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is known as the characterisctic equation of **A** and its roots are called the **characteristic roots** or **latent roots** or **eigenvalues** or **characteristic values** or **latent values** or **proper values of A.** *(Avadh 99)*

(iv) Spectrum of A :— The set of all eigen values of the matrix **A** is called the **spectrum of A.**

(v) Eigen value problems :— The problem of finding the eigen values of a matrix is known as an **eigen-value problem.**

Solved Examples on § 7.02.

***Ex. 1. Find the characteristic roots of the matrix**

$$\mathbf{A}=\begin{bmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{bmatrix}$$

Sol. Here $\mathbf{A}=\begin{bmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{bmatrix}$ and $\mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$\therefore\quad |\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}\cos\theta-\lambda & -\sin\theta-0\\ -\sin\theta-0 & -\cos\theta-\lambda\end{vmatrix}$$

$$=(\cos\theta-\lambda)(-\cos\theta-\lambda)-(\sin^2\theta)$$

$$=-(\cos^2\theta-\lambda^2)-\sin^2\theta=\lambda^2-1.$$

$\therefore$ The characteristic equaiton of the matrix **A** is $\lambda^2-1=0$ and its roots *i.e.* characterisctic roots are ± 1. **Ans.**

Ex. 2. (a) Find the characteristic roots of the matrix

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 3\\ 0 & 0 & 2\end{bmatrix}$$ *(Kanpur 96)*

Sol. Here $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 3\\ 0 & 0 & 2\end{bmatrix}$ and $\mathbf{I}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$$\therefore\quad |\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}1-\lambda & 2-0 & 3-0\\ 0-0 & 2-\lambda & 3-0\\ 0-0 & 0-0 & 2-\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda & 2 & 3\\ 0 & 2-\lambda & 3\\ 0 & 0 & 2-\lambda\end{vmatrix}$$

$$=(1-\lambda)\begin{vmatrix}2-\lambda & 3\\ 0 & 2-\lambda\end{vmatrix}, \text{ expanding with respect to } C_1$$

$$=(1-\lambda)(2-\lambda)^2.$$

$\therefore$ The characteristic equation of the matrix **A** is

$(1-\lambda)(2-\lambda)^2=0$ .... See § 7.02 (iii) Page 160 Ch. VII and its roots *i.e.* required characteristic roots are 1, 2, 2. **Ans.**

Ex. 2 (b). Find the characteristic roots of the matrix

$$\mathbf{A}=\begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix}$$

Hint : Do as Ex. 2 (a) above. **Ans.** $-2, 4, -2$

Ex. 3. Find the characteristic roots of the matrix

$$\mathbf{A}=\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$$

Sol. Here $\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \quad |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & 2-0 & 1-0 \\ 1-0 & 3-\lambda & 1-0 \\ 1-0 & 2-0 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -3+2\lambda & -2+2\lambda & 2-\lambda \end{vmatrix}, \text{ applying } C_1 - 2C_3 \text{ and } C_2 - 2C_3$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \lambda-1 & 1-\lambda & 1 \\ -3+4\lambda-\lambda^2 & -2+2\lambda & 2-\lambda \end{vmatrix}, \text{ applying } C_1 - \lambda C_3$$

$$= \begin{vmatrix} \lambda-1 & 1-\lambda \\ (\lambda-1)(3-\lambda) & -2(1-\lambda) \end{vmatrix} = (\lambda-1)^2 \begin{vmatrix} 1 & -1 \\ 3-\lambda & 2 \end{vmatrix}$$

$$= (\lambda-1)^2 (2+3-\lambda) = (\lambda-1)^2 (5-\lambda).$$

$\therefore$ The characteristic equation of the matrix **A** is $(\lambda-1)^2(5-\lambda) = 0$ and its roots (or characteristic roots of **A**) are 1, 5. **Ans.**

Ex. 4. Obtain the characteristic roots of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{b} & \mathbf{c} & \mathbf{a} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix}$$

Sol. Here

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} b-\lambda & c-0 & a-0 \\ c-0 & a-\lambda & b-0 \\ a-0 & b-0 & c-\lambda \end{vmatrix} = \begin{vmatrix} b-\lambda & c & a \\ c & a-\lambda & b \\ a & b & c-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} b-\lambda+c+a & c & a \\ c+a-\lambda+b & a-\lambda & b \\ a+b+c-\lambda & b & c-\lambda \end{vmatrix}, \text{ applying } C_1 + C_2 + C_3$$

$$= (a+b+c-\lambda) \begin{vmatrix} 1 & c & a \\ 1 & a-\lambda & b \\ 1 & b & c-\lambda \end{vmatrix}$$

$$= (a+b+c-\lambda) \begin{vmatrix} 1 & c & a \\ 0 & a-\lambda-c & b-a \\ 0 & b-c & c-\lambda-a \end{vmatrix}, \text{applying } R_2 - R_1, R_3 - R_1$$

$$= (a+b+c-\lambda)\,[(a-c-\lambda)(c-a-\lambda) - (b-a)(b-c)]$$

$$= (a + b + c - \lambda)\,[\,(a - c)(c - a) - \lambda\,\{(c - a) + (a - c)\} + \lambda^2 - (b - a)(b - c)]$$

$$= (a + b + c - \lambda)\,(\lambda^2 + 2ac - a^2 - c^2 - b^2 + bc + ab - ac]$$

$$= (a + b + c - \lambda)\,(\lambda^2 - a^2 - b^2 - c^2 + ab + bc + ca).$$

$\therefore$ The characteristic equation of **A** is $|\mathbf{A} - \lambda \mathbf{I}| = 0$

i.e. $(a + b + c - \lambda)\,(\lambda^2 - a^2 - b^2 - c^2 + ab + bc + ca) = 0$

and the characteristic roots of A are

$$a + b + c,\ \pm \sqrt{(a^2 + b^2 + c^2 - ab - bc - ca)}$$ **Ans.**

Ex. 5. Find the eigenvalues (or latent roots) of

$$\mathbf{A} = \begin{bmatrix} 8 & 6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(Avadh 99; Kumaun 91, Lucknow 90)

Sol. Here $|\mathbf{A} - \lambda\,\mathbf{I}| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}$

$$= (8 - \lambda)\,\{(7 - \lambda)(3 - \lambda) - 16\} + 6\,\{-6\,(3 - \lambda) + 8\} + 2\,\{24 - 2\,(7 - \lambda)\}$$

$$= (8 - \lambda)\,(5 - 10\,\lambda + \lambda^2) + 6\,(6\,\lambda - 10) + 2\,(10 - 2\lambda)$$

$$= -45\,\lambda + 18\lambda^2 - \lambda^3.$$

$\therefore$ The characteristic equation of **A** is $-45\,\lambda + 18\,\lambda^2 - \lambda^3 = 0$

which gives

$$\lambda\,(\lambda^2 - 18\lambda + 45) = 0 \text{ or } \lambda\,(\lambda - 3)(\lambda - 15) = 0 \text{ or } \lambda = 0, 3, 15.$$

$\therefore$ The required latent roots or eigenvalues of **A** are 0, 3 and 15.

Ex. 6. If $a_1, a_2, a_3, \ldots, a_n$ are the characteristic roots of the n-square matrix A and μ is a scalar, then show that the characteristic roots of $A - \mu I$ are $a_1 - \mu, \ldots, a_n - \mu$.

Sol. Since $a_1, a_2, \ldots, a_n$ are the characteristic roots of the matrix **A**, so form § 7.02 Page 162 we have

$$|\mathbf{A} - \lambda \mathbf{I}| = (\lambda - a_1)(\lambda - a_2) \ldots (\lambda - a_n). \quad \ldots(\text{i})$$

Now the characteristic function of $\mathbf{A} - \mu \mathbf{I}$

$$= |(\mathbf{A} - \mu \mathbf{I}) - \lambda \mathbf{I}| = |\mathbf{A} - (\mu + \lambda)\,\mathbf{I}|$$

$$= \{(\mu + \lambda) - a_1\}\,\{(\mu + \lambda) - a_2\}\,\{(\mu + \lambda) - a_3\} \ldots \{(\mu + \lambda) - a_n\}, \text{ from (i)}$$

$$= \{\lambda - (a_1 - \mu)\}\,\{\lambda - (a_2 - \mu)\} \ldots \{(\lambda - (a_n - \mu)\},$$

rearranging the terms in each bracket.

$\therefore$ The characteristic roots of $\mathbf{A} - \mu \mathbf{I}$ are $(a_1 - \mu), (a_2 - \mu)$, etc.

Exercises on § 7.02

Ex. 1. Find the characteristic roots of $\mathbf{A}=\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$ **Ans.** a_1, b_2, c_3

Ex. 2. Find the eigenvalues of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ **Ans.** $2, -1 \pm \sqrt{3}$.

Ex. 3. Find the eigen values of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **Ans.** $5, -3, -3$.

***Ex. 4.** Find the eigenvalues of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ **Ans.** $1, -1 \pm \sqrt{2}$

Ex. 5. Find the characteristic roots of the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Ans. $1, 3, -4$

Ex. 6. Determine the characteristic equation and roots of the matrix

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 3 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

Ans. $(\lambda-1)(\lambda-3)(\lambda-5)=0$; $1, 3, 5$.

Ex. 7. Find the eigenvalues of the matrix

$$\begin{bmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{bmatrix}$$

Ans. $1, 2, 3$.

Ex. 8. Find the eigenvalues of the matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

(Agra 92; Lucknow 92) **Ans.** $2, 3, 5$

Ex. 9. Find the latent roots of the matrix

$$\mathbf{A}=\begin{bmatrix} 0 & \sin\alpha & \cos\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\alpha\cos\beta & 0 \end{bmatrix}$$

Ans. $0, i, -i$.

Ex. 10. Show that the matrices

$$\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & c \\ a & 0 & b \\ c & b & 0 \end{bmatrix}, \begin{bmatrix} 0 & b & a \\ b & 0 & c \\ a & c & 0 \end{bmatrix}$$

have the same characteristic equation. *(Kumaun 90)*

§ 7.03. †Cayely Hamilton Theorem *(Agra 95, 93, 91; Avadh 99; Garhwal 96; Kanpur 97, 90; Kumaun 92; Lucknow 92; Meerut 98, 92, 91; Rohilkhand 94, 93, 92, 90)*

Statement : *Every square matrix satisfies its characteristic equation or if* $|\mathbf{A} - \lambda\mathbf{I}| = (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n]$ *be the characteristic polynomial of* $n \times n$ *matrix* $\mathbf{A} = [a_{ij}]$, *then the matrix equation* $\mathbf{X}_n + a_1 \mathbf{X}^{n-1} + \ldots + a_n \mathbf{I} = \mathbf{O}$ *is satisfied by* $\mathbf{X} = \mathbf{A}$

i.e.
$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + \ldots + a_n \mathbf{I} = \mathbf{O}.$$

Proof. $\because$ the elements $(\mathbf{A} - \lambda\mathbf{I})$ are at the most of first degree in λ.

$\therefore$ The elements of Adj $(\mathbf{A} - \lambda\mathbf{I})$ are at the most of degree $(n-1)$ in λ and the coefficients of various powers of λ being polymomials in the a_{ij}.

$\therefore$ Adj $(\mathbf{A} - \lambda\mathbf{I})$ can be written as
$$\mathbf{B} = \mathbf{B}_0 \lambda^{n-2} + \mathbf{B}_1 \lambda^{n-1} + \ldots + \mathbf{B}_{n-1},$$
where $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{n-1}$ are $n \times n$ matrices, their elements being polynomials in a_{ij}.

Also from § 5.09 Page 49 Ch. V we know that if $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix, then $\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = (\text{Adj } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}_0$, where $\mathbf{I}$ is an $n \times n$ identity matrix.

Therefore $(\mathbf{A} - \lambda\mathbf{I}) \bullet \text{Adj } (\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}| \bullet \mathbf{I}$

or
$$(\mathbf{A} - \lambda\mathbf{I}) \bullet \mathbf{B} = |\mathbf{A} - \lambda\mathbf{I}| \bullet \mathbf{I}, \quad \because \mathbf{B} = \text{Adj } (\mathbf{A} - \lambda\mathbf{I})$$

or
$$(\mathbf{A} - \lambda\mathbf{I}) (\mathbf{B}_0 \lambda^{n-1} + \mathbf{B}_1 \lambda^{n-2} + \ldots + \mathbf{B}_{n-1})$$
$$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2 \lambda^{n-2} + \ldots a_n] \mathbf{I}.$$

Comparing coefficients of like powers of λ on both sides, we get
$$-\mathbf{I}\mathbf{B}_0 = (-1)^n \mathbf{I};$$
$$\mathbf{A}\mathbf{B}_0 - \mathbf{I}\mathbf{B}_1 = (-1)^n a_1 \mathbf{I};$$
$$\mathbf{A}\mathbf{B}_1 - \mathbf{I}\mathbf{B}_2 = (-1)^n a_2 \mathbf{I};$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\mathbf{A}\mathbf{B}_{n-1} = (-1)^n a_n \mathbf{I}.$$

Now pre-multiplying these equations by $\mathbf{A}^n, \mathbf{A}^{n-1}, \ldots, \mathbf{A}, \mathbf{I}$ respectively and adding the results so obtained we get
$$\mathbf{A}^n (-\mathbf{I}\mathbf{B}_0) + \mathbf{A}^{n-1} (\mathbf{A}\mathbf{B}_0 - \mathbf{I}\mathbf{B}_1) + \mathbf{A}^{n-2} (\mathbf{A}\mathbf{B}_1 - \mathbf{I}\mathbf{B}_2) + \ldots + \mathbf{I} (\mathbf{A}\mathbf{B}_{n-1})$$
$$= (-1)^n [\mathbf{I}\mathbf{A}^n + a_1 \mathbf{I}\mathbf{A}^{n-1} + a_2 \mathbf{I}\mathbf{A}^{n-2} + a_n \mathbf{I} \bullet \mathbf{I}]$$

†This theorem was first established by Hamilton in 1883 for a particular type of matrices and was later on stated by Cayley in 1885.

or $\quad \mathbf{O} = (-1)^n [\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + a_2 \mathbf{A}^{n-2} + \ldots + a_n \mathbf{I}]$,

where $\mathbf{O}$ is the null matrix.

Hence $\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + a_2 \mathbf{A}^{n-2} + \ldots + a_n \mathbf{I} = \mathbf{O}$. Hence the theorem.

Cor. I. Multiplying the result of § 7.03 above by $\mathbf{A}^{m-n}$, where $m \geq n$ and m is a positive integer, we get

$$\mathbf{A}^m + a_1 \mathbf{A}^{m-1} + a_2 \mathbf{A}^{m-2} + \ldots + a_n \mathbf{A}^{m-n} = \mathbf{O}$$

i.e. any positive integral power $\mathbf{A}^m$ of $\mathbf{A}$ can be linearly expressed in terms of $\mathbf{I}, \mathbf{A}, \ldots, \mathbf{A}^{n-1}$.

Cor. II. In § 7.03 above we have proved that

$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + a_2 \mathbf{A}^{n-2} + \ldots + a_n \mathbf{I} = \mathbf{O} \quad \text{...(i)}$$

or $$-a_n \mathbf{I} = \mathbf{A} [\mathbf{A}^{n-1} + a_1 \mathbf{A}^{n-2} + a_2 \mathbf{A}^{n-3} + \ldots + a^{n-1} \mathbf{I}] \quad \textbf{(Note)}$$

or $$-a_n \mathbf{A}^{-1} \mathbf{I} = \mathbf{A}^{n-1} + a_1 \mathbf{A}^{n-2} + \ldots + a_{n-1} \mathbf{I} \quad \text{...(ii)}$$

or $$(-1)^n \mathbf{AB}_{n-1} \mathbf{A}^{n-1} = -\mathbf{A}^{n-1} - a_1 \mathbf{A}^{n-2} + \ldots - a_{n-1} \mathbf{I},$$

$\because \mathbf{AB}_{n-1} = (-1)^n a_n \mathbf{I}$ (§ 7.03 above)

or $$\mathbf{B}_{n-1} = (-1)^n [-\mathbf{A}^{n-1} - a_1 \mathbf{A}^{n-2} - \ldots - a_{n-1} \mathbf{I}] = \text{Adj } \mathbf{A} \quad \text{...(iii)}$$

Cor. III. From result (ii) of cor. II above we have

$$\mathbf{A}^{-1} = -\frac{1}{a_n} [\mathbf{A}^{n-1} + a_1 \mathbf{A}^{n-2} + \ldots + a_{n-1} \mathbf{I}] \quad \text{...(iv)}$$

which show that $\mathbf{A}^{-1}$ can be expressed linearly in terms of $\mathbf{A}^{n-1}, \mathbf{A}^{n-2}, \ldots, \mathbf{I}$.

§ 7.04. Characteristic vectors (or Eigenvectors). *(Agra 94)*

Let us consider the linear transformation

$$\mathbf{K} = \mathbf{AX} \quad \text{...(i)}$$

which transforms a column vector $\mathbf{X}$ by means of a square matrix A into another column vector $\mathbf{K}$.

If $\mathbf{X}$ be a vector which transforms to its multiple $\mu \mathbf{X}$ by the above transformation (i), then we have $\mu \mathbf{X} = \mathbf{AX}$ ...(ii)

or $$\mathbf{AX} - \mu \mathbf{IX} = \mathbf{O} \quad \text{or} \quad (\mathbf{A} - \mu \mathbf{I}) \mathbf{X} = \mathbf{O}. \quad \text{...(iii)}$$

This equation (iii) when written in full gives n homogeneous equations in $x_1, x_2, \ldots, x_n$ which are n unknowns. The n equations will have a non-zero solution only if $|\mathbf{A} - \mu \mathbf{I}| = 0$ *i.e.* the coefficients matrix is singular. **(Note)**

This equation is called the characteristic equation of transformation and is the same as the characteristic equation of the matrix $\mathbf{A}$. (See § 7.02 Page 160 Ch. VII). This equation has n roots and corresponding to each root, the equation (iii) has a non-zero solution

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}$$

which is defined as **characteristic vector** or **Eigenvector** or **latent vector** or **invariant vector.**

****§ 7.05. Theorems on latent roots (or characteristics roots).**

Theorem I. *If square matrix* **A** *of order n has latent roots* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *then* **A′** *has also the same latent roots.* *(Kumaun 95)*

Proof. Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots & a_{nn} \end{bmatrix}$

$\therefore$ The characteristic equation of **A** is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

Also $\mathbf{A}' = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots\cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots\cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots\cdots & a_{nn} \end{bmatrix}$

$\therefore$ The characteristic equation of **A′** is

$$|\mathbf{A}' - \lambda\mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{21} & \cdots\cdots & a_{n1} \\ a_{12} & a_{22} - \lambda & \cdots\cdots & a_{n2} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{1n} & a_{2n} & \cdots\cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad \text{...(ii)}$$

Also we know that the value of a determinant remains unaltered if rows are changed into columns and thus we find that the determinants given by (i) and (ii) are the same, the diagonal elements being the same.

Hence from (i) and (ii) we conclude that the characteristic equations of **A** and **A′** are the same. Consequently the latent roots of **A** and **A′** are the same.

Theorem II. *If* **A** *is an* $n \times n$ *triangular matrix, then the elements of the principal diagonals are the characteristic roots of* **A**.

Proof. Let $A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots\dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots\dots\dots & a_{2n} \\ 0 & 0 & a_{33} & \dots\dots\dots & a_{3n} \\ \dots & \dots & \dots & \dots\dots\dots & \dots \\ 0 & 0 & 0 & \dots\dots\dots & a_{nn} \end{bmatrix}$

(Here we have taken upper triangular matrix).

$\therefore$ **The characteristic equation of A is**

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} - \lambda & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

or $$(a_{11} - \lambda) \begin{vmatrix} a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ 0 & a_{33} - \lambda & \dots & a_{3n} \\ \dots\dots & \dots\dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0,$$ expanding with respect to C_1

or $$(a_{11} - \lambda)(a_{22} - \lambda) \begin{vmatrix} a_{33} - \lambda & a_{34} & \dots & a_{3n} \\ 0 & a_{44} - \lambda & \dots & a_{4n} \\ \dots\dots & \dots\dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0,$$ expanding with respect to C_2

or $(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \dots (a_{nn} - \lambda) = 0$, proceeding in this way

or $$\lambda = a_{11}, a_{22}, a_{33}, \dots, a_{nn}$$

i.e. the element roots (or characteristic roots) of **A** are the elements of the principal diagonal of **A**. Hence proved.

Theorem III. *The characteristic roots of a hermitian matrix are all real.*

(Agra 95, Kanpur 95, 94)

Proof. Let **A** be the hermitian matrix. Then from § 7.04 (ii) Page 166 Ch. VII] we know that $\mathbf{AX} = \lambda\mathbf{X}$, ...(i)

where λ is a characteristic root of **A** and **X** the corresponding characteristic vector.

From (i) we get $\mathbf{X}^{\Theta}\mathbf{AX} = \mathbf{X}^{\Theta}\lambda\mathbf{X}$, premultiplying both sides by $\mathbf{X}^{\Theta}$

or $$\mathbf{X}^{\Theta}\mathbf{AX} = \lambda\mathbf{X}^{\Theta}\mathbf{X}, \quad \text{...(ii)}$$

where $\mathbf{X}^{\Theta}$ is the transposed conjugate of **X** (See chapter II)

Also if **A** is a hermitian matrix, then by definition we have

$$\mathbf{A}^{\Theta} = \mathbf{A} \text{ (See Chapter II)}. \quad \text{...(iii)}$$

Now taking transposed conjugate of both sides of (ii) we get

$$\mathbf{X}^{\Theta}\mathbf{A}\mathbf{X} = \bar{\lambda}\,\mathbf{X}^{\Theta}\,\mathbf{X}, \text{ using (iii) also} \qquad \text{...(iv)}$$

$\therefore$ From (ii) and (iv) we get $\lambda\mathbf{X}^{\Theta}\,\mathbf{X} = \bar{\lambda}\,\mathbf{X}^{\Theta}\,\mathbf{X}$

or $\qquad (\lambda - \bar{\lambda})\,\mathbf{X}^{\Theta}\,\mathbf{X} = \mathbf{O}$ or $\lambda - \bar{\lambda} = 0$, $\qquad \because \mathbf{X}^{\Theta}\,\mathbf{X} \neq \mathbf{O}$

or $\qquad \lambda = \bar{\lambda}$ or λ is real. Hence proved.

****Theorem IV.** *The characteristic roots of a real symmetric matrix are all real.*

Proof. Do as Theorem III above. Here all the elements of **A** are real and as such it is particular case of Theorem III above.

****Theorem V.** *The characteristic roots of a skew-hermitian matrix are either purely imaginary or zero.*

Proof. Let **A** be a skew-hermitian matrix, then (see Chapter II) we know that $i\mathbf{A}$ is hermitian.

If λ be a charcteristic root of **A**, then $|\mathbf{A} - \lambda\mathbf{I}| = 0$

or $\qquad i\,|\mathbf{A} - \lambda\,\mathbf{I}| = 0$ or $|\,i\mathbf{A} - (i\lambda)\,\mathbf{I}| = 0$, where $i\mathbf{A}$ is hermitian.

or . $(i\lambda)$ is real, since the characteristic roots of a hermitian matrix are all real.

(See Theorem III Page 168 Ch. VII)

or λ is either purely imaginary or zero.

i.e. the characteristic roots of a skew hermitian matrix **A** are either purely imaginary or zero. Hence proved.

Theorem VI. *The characteristic roots of real skew-symmetric matrix are purely imaginary or zero.*

Proof. Do as Theorem **V** above. Here all the elements of **A** are real and as such it is a particular case of Theorem **V** above.

****Theorem VII.** *The characteristic roots of a unitary matrix are of unit modulus.*

Proof. Let **A** be a unitary matrix (See Chapter II). Let λ be a characteristic root of **A** and **X** the corresponding characteristic vector.

Then $\qquad \mathbf{A}\mathbf{X} = \lambda\mathbf{X}$ $\qquad$ ...(i) $\qquad$ (See § 7.04 (ii) Page 166 Ch. VII)

Taking transposed conjugate of both sides of (i), we get

$$(\mathbf{A}\mathbf{X})^{\theta} = (\lambda\mathbf{X})^{\theta} \text{ or } \quad \mathbf{X}^{\theta}\,\mathbf{A}^{\theta} = \bar{\lambda}\,\mathbf{X}^{\theta} \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we get $\mathbf{X}^{\theta}\,\mathbf{A}^{\theta}\,\mathbf{A}\mathbf{X} = \bar{\lambda}\,\mathbf{X}^{\theta}\,\lambda\mathbf{X}$ **(Note)**

or $\qquad \mathbf{X}^{\theta}\,(\mathbf{A}^{\theta}\mathbf{A})\,\mathbf{X} = \bar{\lambda}\,\lambda\,\mathbf{X}^{\theta}\,\mathbf{X}$

or $\qquad \mathbf{X}^{\theta}\,(\mathbf{I})\,\mathbf{X} = \bar{\lambda}\,\lambda\,\mathbf{X}^{\theta}\,\mathbf{X}$, $\qquad \because$ **A** is unitary (See Chapter II)

or $\qquad \mathbf{X}^{\theta}\,\mathbf{X}\,(1 - \bar{\lambda}\,\lambda) = \mathbf{O}$

or $\qquad 1 - \lambda\,\bar{\lambda} = 0$, $\qquad \because \mathbf{X}^{\theta}\,\mathbf{X} \neq \mathbf{O}$

or $\qquad \lambda\,\bar{\lambda} = 1$ $\quad$ or $|\,\lambda\,|^2 = \lambda\,\bar{\lambda} = \mathbf{I}$.

i.e the characteristic roots of **A** are of unit modulus. Hence proved.

****Theorem VIII.** *The characterise roots of an orthogonal matrix are of unit modulus.*

Proof. Do as Theorem VII above remembering that if all the elements of the unitary matrix **A** are real then **A** is an orthogonal matrix.

***§ 7.06. An Important Theorem.**

The scalar λ is a characteristic root of the matrix **A** *if and only if the matrix* $\mathbf{A}-\lambda\mathbf{I}$ *is singular.*

Proof. Let $\mathbf{A}=[a_{ij}]_{n\times n}$ $\mathbf{X}=[x_1, x_2, \ldots, x_n]$.

then $$\mathbf{AX}=\lambda\mathbf{X}$$ ...See § 7.04 Page 166

reduces to
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

or
$$\left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \mathbf{O}$$

or
$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \mathbf{O}$$

or $$(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{X}=\mathbf{O}$$

i.e. $$\mathbf{AX}=\lambda\mathbf{X} \Rightarrow (\mathbf{A}-\lambda\,\mathbf{I})\,\mathbf{X}=\mathbf{O},$$

which is a homogeneous system of linear equations whose coefficient matrix is $\mathbf{A}-\lambda\mathbf{I}$.

Now as we require a vector $\mathbf{X}\neq\mathbf{O}$, so we must have

$$|\mathbf{A}-\lambda\mathbf{I}|=0$$

i.e. the matrix $\mathbf{A} - \lambda\mathbf{I}$ must be singular.

Solved Examples on § 7.03 to § 7.06

Ex. 1. (a). Verify Cayley Hamilton's Theorem for the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

(Agra 93)

Sol. Here $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$
$$|\mathbf{A} - \lambda\,\mathbf{I}| = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}, \text{ expanding w.r. to } C_1$$

$$= (1-\lambda)^3$$

$\therefore$ Characteristic equation of $\mathbf{A}$ is $(1-\lambda)^3 = 0$

or
$$\lambda^3 - 3\,\lambda^2 + 3\lambda - 1 = 0 \qquad \text{...(i)}$$

Now
$$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2 . \mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\mathbf{A}^3 - 3\mathbf{A}^2 + 3\mathbf{A} - \mathbf{I}$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+3-1 & 0-0+0-0 & 3-6+3-0 \\ 0-0+0-0 & 1-3+3-1 & 0-0+0-0 \\ 0-0+0-0 & 0-0+0-0 & 1-3+3-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O},$$

where $\mathbf{O}$ is the null matrix.

Hence the given matrix $\mathbf{A}$ satisfies its characteristics equation given by (i).

Hence proved.

Ex. 1 (b) Show that the matrix $\mathbf{A}=\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$ **satisfies Cayley Hamilton Theorem.** *(Meerut 92 P)*

Sol. Here $\mathbf{I}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{A}=\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$

$\therefore$ $$|\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & -\lambda & 3 \\ 2 & -1 & 1-\lambda \end{vmatrix}$$ **(Note)**

$$=\begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & -\lambda & 3 \\ 0 & -1-2\lambda & 7-\lambda \end{vmatrix}, \text{ adding } 2R_2 \text{ to } R_3$$

$$=(1-\lambda)\begin{vmatrix} -\lambda & 3 \\ -1-2\lambda & 7-\lambda \end{vmatrix}+\begin{vmatrix} 2 & 1 \\ -1-2\lambda & 7-\lambda \end{vmatrix}$$

$$=(1-\lambda)[-7\lambda+\lambda^2+3+6\lambda]+[14-2\lambda+1+2\lambda]$$

$$=(1-\lambda)(\lambda^2-\lambda+3)+15=-\lambda^3+2\lambda^2-4\lambda+18$$

$\therefore$ Characteristic equation of $\mathbf{A}$ is

$$\lambda^3-2\lambda^2+4\lambda-18=0$$

Now $$\mathbf{A}^2=\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 1 & 8 \\ 5 & -5 & 2 \\ 5 & 3 & 0 \end{bmatrix}$$

Now $$\mathbf{A}^3=\mathbf{A}^2\mathbf{A}=\begin{bmatrix} 1 & 1 & 8 \\ 5 & -5 & 2 \\ 5 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}=\begin{bmatrix} 16 & -6 & 12 \\ 14 & 8 & -8 \\ 2 & 10 & 14 \end{bmatrix}$$

$\therefore$ $$\mathbf{A}^3-2\mathbf{A}^2+4\mathbf{A}-18\mathbf{I}$$

$$=\begin{bmatrix} 16 & -6 & 12 \\ 14 & 8 & -8 \\ 2 & 10 & 14 \end{bmatrix}-2\begin{bmatrix} 1 & 1 & 8 \\ 5 & -5 & 2 \\ 5 & 3 & 0 \end{bmatrix}+4\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}-18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 16 & -6 & 12 \\ 14 & 8 & -8 \\ 2 & 10 & 14 \end{bmatrix}+\begin{bmatrix} -2 & -2 & -16 \\ -10 & 10 & -4 \\ -10 & -6 & 0 \end{bmatrix}+\begin{bmatrix} 4 & 8 & 4 \\ -4 & 0 & 12 \\ 8 & -4 & 4 \end{bmatrix}+\begin{bmatrix} -18 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -18 \end{bmatrix}$$

$$=\begin{bmatrix} 16-2+4-18 & -6-2+8+0 & 12-16+4+0 \\ 14-10-4+0 & 8+10+0-18 & -8-4+12+0 \\ 2-10+8+0 & 10-6-4+0 & 14+0+4-18 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O} \text{ where } \mathbf{O} \text{ is the null matrix}$$

Hence the given matrix **A** satisfies its characteristic equaiton given by (i).

Ex. 2 (a). Use Cayley Hamilton Theorem to find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} \\ \mathbf{1} & \mathbf{5} & \mathbf{12} \end{bmatrix}$ *(Rohilkhand 95)*

Sol. Here $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

$$\therefore \quad |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 1 & 3-\lambda & 5 \\ 1 & 5 & 12-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 1 & 3-\lambda & 5 \\ 0 & 2+\lambda & 7-\lambda \end{vmatrix}, \text{ applying } R_3 - R_2$$

$$= (1-\lambda)[(3-\lambda)(7-\lambda) - 5(2+\lambda)] - [2(7-\lambda) - 3(2+\lambda)],$$

expanding w.r. to C_1

$$= (1-\lambda)[21 - 10\lambda + \lambda^2 - 10 - 5\lambda] - [14 - 2\lambda - 6 - 3\lambda]$$

$$= (1-\lambda)(11 - 15\lambda + \lambda^2) - (8 - 5\lambda)$$

$$= 3 - 21\lambda + 16\lambda^2 - \lambda^3, \text{ on simplifying.}$$

$\therefore$ The characteristic equation of **A** is

$$\lambda^3 - 16\lambda^2 + 21\lambda - 3 = 0. \qquad \text{...(i)}$$

Now as **A** must satisfy its characteristic equation (i), so we have

$$\mathbf{A}^3 - 16\mathbf{A}^2 + 21\mathbf{A} - 3\mathbf{I} = \mathbf{0}$$

or

$$3\mathbf{I} = \mathbf{A}^3 - 16\mathbf{A}^2 + 21\mathbf{A}$$

Multiplying both sides by $\mathbf{A}^{-1}$, we get

$$3\mathbf{A}^{-1} = \mathbf{A}^2 - 16\mathbf{A} + 21\mathbf{I}, \qquad \because \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

or

$$\mathbf{A}^{-1} = (1/3)\mathbf{A}^2 - (16/3)\mathbf{A} + 7\mathbf{I}. \qquad \text{...(ii)}$$

Now $\mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

$$= \begin{bmatrix} 1+2+3 & 2+6+15 & 3+10+36 \\ 1+3+5 & 2+9+25 & 3+15+60 \\ 1+5+12 & 2+15+60 & 3+25+144 \end{bmatrix} = \begin{bmatrix} 6 & 23 & 49 \\ 9 & 36 & 78 \\ 18 & 77 & 172 \end{bmatrix}.$$

$\therefore$ From (ii), we get

$$A^{-1}=\frac{1}{3}\begin{bmatrix}6 & 23 & 49\\ 9 & 36 & 78\\ 18 & 77 & 172\end{bmatrix}-\frac{16}{3}\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 5\\ 1 & 5 & 12\end{bmatrix}+7\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2-(16/3)+7 & (23/6)-(32/3)+0 & (49/3)-(48/3)+0\\ 3-(16/3)+0 & 12-16+7 & 26-(80/3)+0\\ 6-(16/3)+0 & (77/3)-(80/3)+0 & (172/3)-(192/3)+7\end{bmatrix}$$

$$=\begin{bmatrix}11/3 & -3 & 1/3\\ -7/3 & 3 & -2/3\\ 2/3 & -1 & 1/3\end{bmatrix}=(1/3)\begin{bmatrix}11 & -9 & 1\\ -7 & 9 & -2\\ 2 & -3 & 1\end{bmatrix}$$ **Ans.**

Ex. 2 (b). Verify Cayley Hamilton's Theorem for the matrix

$$\mathbf{A}=\begin{bmatrix}0 & 0 & 1\\ 3 & 1 & 0\\ -2 & 1 & 4\end{bmatrix}.$$ **Hence compute $\mathbf{A}^{-1}$**

(Agra 94; Kanpur 96; Meerut 96)

Sol. Here $\mathbf{I}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $\mathbf{A}=\begin{bmatrix}0 & 0 & 1\\ 3 & 1 & 0\\ -2 & 1 & 4\end{bmatrix}$

$$\therefore \quad |\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}0-\lambda & 0 & 1\\ 3 & 1-\lambda & 0\\ -2 & 1 & 4-\lambda\end{vmatrix}$$

$$=-\lambda\{(1-\lambda)(4-\lambda)\}+1\{3+2(1-\lambda)\}$$

$$=-\lambda(4-5\lambda+\lambda^2)+5-2\lambda$$

$$=5-6\lambda+5\lambda^2-\lambda^3$$

$\therefore$ The characteristic equation of $\mathbf{A}$ is

$$\lambda^3-5\lambda^2+6\lambda-5=0 \quad \text{...(i)}$$

Also we have

$$\mathbf{A}^2=\begin{bmatrix}0 & 0 & 1\\ 3 & 1 & 0\\ -2 & 1 & 4\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 3 & 1 & 0\\ -2 & 1 & 4\end{bmatrix}=\begin{bmatrix}-2 & 1 & 4\\ 3 & 1 & 3\\ -5 & 5 & 14\end{bmatrix} \quad \text{...(ii)}$$

and $$\mathbf{A}^3=\mathbf{A}^2.\mathbf{A}=\begin{bmatrix}-2 & 1 & 4\\ 3 & 1 & 3\\ -5 & 5 & 14\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 3 & 1 & 0\\ -2 & 1 & 4\end{bmatrix}=\begin{bmatrix}-5 & 5 & 14\\ -3 & 4 & 15\\ -13 & 19 & 51\end{bmatrix}$$

$\therefore$ $\mathbf{A}^3-5\mathbf{A}^2+6\mathbf{A}-5\mathbf{I}$

$$=\begin{bmatrix}-5 & 5 & 14\\ -3 & 4 & 15\\ -13 & 19 & 51\end{bmatrix}-5\begin{bmatrix}-2 & 1 & 4\\ 3 & 1 & 3\\ -5 & 5 & 14\end{bmatrix}+6\begin{bmatrix}0 & 0 & 1\\ 3 & 1 & 0\\ -2 & 1 & 4\end{bmatrix}-5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-5 & 5 & 14\\ -3 & 4 & 15\\ -13 & 19 & 51\end{bmatrix}-\begin{bmatrix}-10 & 5 & 20\\ 15 & 5 & 15\\ -25 & 25 & 70\end{bmatrix}+\begin{bmatrix}0 & 0 & 6\\ 18 & 6 & 0\\ -12 & 6 & 24\end{bmatrix}-\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$$= \begin{bmatrix} -5+10+0-5 & 5-5+0-0 & 14-20+6-0 \\ -3-15+18-0 & 4-5+6-5 & 15-15+0-0 \\ -13+25-12-0 & 19-25+6-0 & 51-70+24-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix.}$$

Hence the matrix **A** satisfies its characterstic equation given by (i). Hence Cayley-Hamilton's Theorem is satisfied by the matrix **A**.

Again $$\mathbf{A}^3 - 5\mathbf{A}^2 + 6\mathbf{A} - 5\mathbf{I} = \mathbf{O}$$

or $$5\mathbf{I} = \mathbf{A}^3 - 5\mathbf{A}^2 + 6\mathbf{A}$$

Multiplying both sides by $\mathbf{A}^{-1}$, we get

$$5\mathbf{A}^{-1} = \mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}, \qquad \because \mathbf{AA}^{-1} = \mathbf{I}$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} - 5\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ from (ii)}$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -5 \\ -15 & -5 & 0 \\ 10 & -5 & -20 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+6 & 1+0+0 & 4-5+0 \\ 3-15+0 & 1-5+6 & 3+0+0 \\ -5+10+0 & 5-5+0 & 14-20+6 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$$

or $$\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$$

Ans.

Ex. 3 (a). Verify Cayley-Hamilton Theorem for the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and hence find $\mathbf{A}^{-1}$.

(Kanpur 97)

Sol. Here we have

$$[\mathbf{A} - \lambda\mathbf{I}] = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

$$\therefore \quad |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\,[(1-\lambda)(-1-\lambda) - 4], \text{ expanding w.r. to } C_3$$

$$= (-1-\lambda)[-1+\lambda^2-4] = (1+\lambda)(5-\lambda^2)$$
$$= 5+5\lambda-\lambda^2-\lambda^3.$$

$\therefore$ The characteristic equation of **A** is $\lambda^3+\lambda^2-5\lambda-5=0$ ...(i)

Also we have

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ $A^3 + A^2 - 5A - 5I$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+5-5-5 & 10+0-10-0 & 0+0-0-0 \\ 10+0-10-0 & -5+5+5-5 & 0+0-0-0 \\ 0+0-0-0 & 0+0-0-0 & -1+1+5-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix.}$$

Hence the matrix **A** satisfies its characteristic equation given by (i). Hence Cayley Hamilton theorem is verified by the matrix **A**.

Again $A^3 + A^2 - 5A - 5I = \mathbf{O}$ gives

$$5I = A^3 + A^2 - 5A$$

or $5A^{-1} = A^2 + A - 5I$, multiplying both sides by A^{-1} and $AA^{-1} = I$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$\therefore$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Ans.

Ex. 3 (b). Show that the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ **satisfies Cayley-Hamilton Theorem and hence compute** A^{-1}**.**

Hint : Do as Ex. 3(a) above. **Ans.** $(1/9)\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{bmatrix}$

Ex. 4 (a). Determine the characteristic equation of the matrix $= \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ **and verify that A satisfies its characteristic equation.**

(Garhwal 96)

Sol. Here we have $|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix}$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 1 & 1-\lambda \end{vmatrix} + 7\begin{vmatrix} 4 & 2-\lambda \\ 1 & 2 \end{vmatrix},$$

expanding w.r. to R_1

$$= (1-\lambda)[(2-\lambda)(1-\lambda) - 6] - 3[4 - 4\lambda - 3] + 7[8 - 2 + \lambda]$$

$$= (1-\lambda)[-3\lambda + \lambda^2 - 4] - 3[1 - 4\lambda] + 7[6 + \lambda]$$

$= 35 + 20\lambda + 4\lambda^2 - \lambda^3$, on simplifying.

Characteristic equation of matrix **A** is

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0. \qquad \text{...(i)}$$

Also we have

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2 . \mathbf{A} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$\mathbf{A}^3 - 4\mathbf{A}^2 - 20\mathbf{A} - 35\,\mathbf{I}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix}$$

Hence the matrix **A** satisfies its characteristic equation (i). Hence proved

Ex. 4 (b). Find the characteristic equaiton of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ and show that A satisfies this equation.

Hint : Do as Ex. 4 (a) above.

Ans. $\lambda^3 = 0$

Ex. 4 (c). Verify that matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ satisfies its characteristic equation. *(Rohilkhand 99)*

Hint : Do as Ex. 4 (a) above.

Ex. 4 (d). Determine the characteristic equation of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ and verify that A satisfies its characteristic equation. *(Garhwal 94)*

Hint : Do as Ex. 4(a) above.

Ans. $\lambda^3 - 6\lambda + 4 = 0$

****Ex. 5. Find the characteristic vectors of the matrix A given in Ex. 4 (c) above.**

Sol. As in Ex. 4 (a) above we can find that the characteristic equation of A is $(\lambda - 1)^2 (5 - \lambda) = 0$ and so the characteristic roots of **A** are 1, 1, 5.

The equation $(\mathbf{A} - \lambda \mathbf{I})\, \mathbf{X} = \mathbf{O}$ of the matrix **A** is

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(i)}$$

...See § 7.04 Page 166 Chapter VII

Putting $\lambda = 1$ in the above equation, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding characteristic vector is given by the equation

$$x_1 + 2x_2 + x_3 = 0.$$

$\therefore$ The characteristic vector corresponding to $\lambda = 1$ may be taken as (1, − 1, 1). **Ans.**

Putting $\lambda = 5$ in (i), we get

$$\begin{bmatrix} 2-5 & 2 & 2 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding characteristic vector is given by the equations $-3x_1+2x_2+x_3=0$, $x_1-2x_2+x_3=0$ and $x_1+2x_2-3x_3=0$.

Solving these we get $\dfrac{x_1}{1}=\dfrac{x_2}{1}=\dfrac{x_3}{1}$.

$\therefore$ The characteristic vector corresponding to $\lambda=5$ may be taken as (1, 1, 1).

Ex. 6 (a). Verify Cayley Hamilton's Theorem for the matrix $A=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1}.

(Agra 96; Garhwal 93; Kanpur 95, 93; Kumaun 95; Lucknow 91; Meerut 98, 97; Rohilkhand 97)

or find the characteristic equation of the matrix $A=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A. *(Kanpur 91)*

Sol. Here $|A-\lambda I|=\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$

$$=\begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & 1-\lambda & 2-\lambda \end{vmatrix}, \text{ replacing } C_2 \text{ by } C_2+C_3$$

$$=(1-\lambda)\begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$=(1-\lambda)\left\{(2-\lambda)\begin{vmatrix} 1 & -1 \\ 1 & 2-\lambda \end{vmatrix}+\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}\right\}$$

$$=(1-\lambda)\{(2-\lambda)(3-\lambda)+(-1-1)\}$$

$$=(1-\lambda)\{4-5\lambda+\lambda^2\}=4-5\lambda+\lambda^2-4\lambda+5\lambda^2-\lambda^3$$

$$=-\lambda^3+6\lambda^2-9\lambda+4.$$

$\therefore$ The characteristic equation of the matrix **A** is

$$\lambda^3-6\lambda^2+9\lambda-4=0 \qquad \text{...(i)}$$

Now $\quad A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

and $\quad A^3 = A^2 . A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore \quad A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} + \begin{bmatrix} -36 & 30 & -30 \\ 30 & -36 & 30 \\ -30 & 30 & -36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9+0 & 21-30+9+0 \\ -21+30-9+0 & 22-36+18-4 & -21+30-9+0 \\ 21-30+9+0 & -21+30-9+0 & 22-36+18-4 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$, where $\mathbf{O}$ is the null matrix

Hence A satisfies its characteristic equation given by (i).

Hence Cayley Hamilton theorem is satisfied by the matrix A.

Again $A^3 - 6A^2 + 9A - 4I = O$ gives $4I = A^3 - 6A^2 + 9A$

Multiplying both sides by A^{-1}, we get

$$4A^{-1} = A^2 - 6A + 9I, \quad \because AA^{-1} = I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

or $$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ **Ans.**

Ex. 6 (b). Find the characteristic equation of the matrix $A = \begin{bmatrix} 0 & 0 & 2 \\ 2 & -3 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ and verify that it is satisfied by A and hence obtain A^{-1} *(Garhwal 95)*

Hint : Do as Ex. 6 (a) above.

Ans. $\lambda^3 + 4\lambda^2 + \lambda - 10 = 0,\ \frac{1}{10}\begin{bmatrix} 3 & 2 & 6 \\ 2 & -2 & 4 \\ 5 & 0 & 0 \end{bmatrix}$

Ex. 6 (c). Verify Cayley Hamilton Theorem for the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & -2 \end{bmatrix}$. Find out the inverse if possible. *(Rohilkhand 94)*

Hint : Do as Ex. 6 (a) above.

Ans. $\frac{1}{2}\begin{bmatrix} -6 & 4 & -6 \\ 4 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$

Ex. 7. Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley Hamilton theorem for the matrix.

Sol. Do as Ex. 6 (a) above. **Ans. 5, − 1**

Ex. 8. Find the characteristic root and inverse of the matrix $A = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$

Sol. Here $$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 6 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (5-\lambda)(2-\lambda) - 6 = 4 - 7\lambda + \lambda^2$$

$\therefore$ The characteristic equation of **A** is $\lambda^2 - 7\lambda + 4 = 0$ ...(i)

Now as **A** must satisfy Cayley Hamilton's Theorem, so we get

$\mathbf{A}^2 - 7\mathbf{A} + 4\mathbf{I} = \mathbf{O}$, where **O** is the null matrix

or $$\mathbf{I} = -\tfrac{1}{4}\mathbf{A}^2 + \tfrac{7}{4}\mathbf{A}.$$

Multiplying both sides by $\mathbf{A}^{-1}$ we get

$$A^{-1} = -\tfrac{1}{4}A + \tfrac{7}{4}I \qquad \because AA^{-1} = I$$

$$= -\frac{1}{4}\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} + \frac{7}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\dfrac{5}{4} & -\dfrac{3}{2} \\ -\dfrac{1}{4} & -\dfrac{1}{2} \end{bmatrix} + \begin{bmatrix} \dfrac{7}{4} & 0 \\ 0 & \dfrac{7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{5}{4} + \dfrac{7}{4} & -\dfrac{3}{2} + 0 \\ -\dfrac{1}{4} + 0 & -\dfrac{1}{2} + \dfrac{7}{4} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} & -\dfrac{3}{2} \\ -\dfrac{1}{4} & \dfrac{5}{4} \end{bmatrix} = (1/4)\begin{bmatrix} 2 & -6 \\ -1 & 5 \end{bmatrix}$$ **Ans.**

Also the characteristic roots of **A** are the roots of (i).

i.e. $$\lambda = \frac{1}{2}[7 \pm \sqrt{(49 - 16)}] = \frac{1}{2}\sqrt{[7 \pm \sqrt{(33)}]}.$$ **Ans.**

Ex. 9. Using Cayley Hamilton's Theorem find A^{-2} where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ *(Agra 95)*

Sol. Here $$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= (1 - \lambda\begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 0 & -1-\lambda \end{vmatrix}$$

$$= (1 - \lambda)(1 + \lambda)^2 + 4(1 + \lambda) = (1 + \lambda)[(1 - \lambda^2) + 4]$$

$$= (1 + \lambda)(5 - \lambda^2) = 5 - \lambda^2 + 5\lambda - \lambda^3.$$

$\therefore$ The characteristic equation of **A** is

$$\lambda^3 + \lambda^2 - 5\lambda - 5 = 0 \qquad \text{...(i)}$$

Now $$A^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And $$A^3 = A^2 . A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

By Cayley Hamilton's Theorem from (i) we have

$$A^3 + A^2 - 5A - 5I = O$$

or $$5I = A^3 + A^2 - 5A. \qquad \text{...(ii)}$$

Multiplying both sides by A^{-1}, we get

$$5A^{-1} = A^2 + A - 5I, \qquad \because AA^{-1} = I$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+1-5 & 0+2+0 & 0+0+0 \\ 0+2+0 & 5-1-5 & 0+0+0 \\ 0+0+0 & 0+0+0 & 1-1-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

or $$\mathbf{A}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad ...(iii)$$

Again multiplying both sides of (ii) by $\mathbf{A}^{-2}$ we get

$$5\mathbf{A}^{-2} = \mathbf{A} + \mathbf{I} - 5\mathbf{A}^{-1}. \qquad \textbf{(Note)}$$

or $$5\mathbf{A}^{-2} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \text{ from (iii)}$$

$$= \begin{bmatrix} 1+1-1 & 2+0-2 & 0+0+0 \\ 2+0-2 & -1+1+1 & 0+0+0 \\ 0+0+0 & 0+0+0 & -1+1+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

or $$\mathbf{A}^{-2} = \frac{1}{5} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$ **Ans.**

Ex. 10. Verify Cayley Hamilton's Theorem for the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

Hence of otherwise compute $\mathbf{A}^{-1}$. *(Garhwal 92; Kumaun 94, 92; Lucknow 92; Meerut 96 P; Rohilkhand 98)*

Sol. Here $$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\{(2-\lambda)(3-\lambda) - 0\} + 2\{0 - 2(2-\lambda)\}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda) - 4(2-\lambda)$$

$$= (2-\lambda)\{(1-\lambda)(3-\lambda) - 4\} = (2-\lambda)\{3 - 4\lambda + \lambda^2 - 4\}$$

$$= (2-\lambda)(\lambda^2 - 4\lambda - 1) = -\lambda^3 + 6\lambda^2 - 7\lambda - 2$$

∴ The characteristic equation of $\mathbf{A}$ is

$$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0. \qquad ...(i)$$

Now $$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

and $$A^3 = A^2 . A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$\therefore$ $$A^3 - 6A^2 + 7A + 2I$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} + \begin{bmatrix} -30 & 0 & -48 \\ -12 & -24 & -30 \\ -48 & 0 & -78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0+0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0+0+0+0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O},$$ where $\mathbf{O}$ is the null matrix.

Hence **A** satisfies its characteristic equation given by (i)

Hence Cayley Hamilton's Theorem is satisfied by **A** *i.e.*

$$A^3 - 6A^2 + 7A + 2I = O \text{ or } 2I = -A^3 + 6A^2 - 7A.$$

Multiplying both sides by A^{-1}, we get

$$2A^{-1} = -A^2 + 6A - 7I, \qquad \because A A^{-1} = I, IA^{-1} = A^{-1}$$

$$= -\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & -8 \\ -2 & -4 & -5 \\ -8 & 0 & -13 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 12 \\ 0 & 12 & 6 \\ 12 & 0 & 18 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} -5+6-7 & 0+0+0 & -8+12-0 \\ -2+0+0 & -4+12-7 & -5+6-0 \\ -8+12+0 & 0+0+0 & -13+18-7 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 4 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix}$$

or $$A^{-1} = (1/2)\begin{bmatrix} -6 & 0. & 4 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & 1/2 & 1/2 \\ 2 & 0 & -1 \end{bmatrix}$$ Ans

Ex. 11 (a). Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

and hence find $\mathbf{A}^{-1}$. Also verify Cayley Hamilton's Theorem for A.

(Kanpur 90)

Sol. Here $|\mathbf{A}-\lambda\mathbf{I}|$

$$=\begin{vmatrix}1-\lambda & 3 & 7\\ 4 & 2-\lambda & 3\\ 1 & 2 & 1-\lambda\end{vmatrix}=\begin{vmatrix}-\lambda & 1 & 6+\lambda\\ 0 & -6-\lambda & -1+4\lambda\\ 1 & 2 & 1-\lambda\end{vmatrix},\text{ applying } R_1-R_3,\ R_2-4R_3$$

$$=-\lambda\begin{vmatrix}-6-\lambda & -1+4\lambda\\ 2 & 1-\lambda\end{vmatrix}+\begin{vmatrix}1 & 6+\lambda\\ -6-\lambda & -1+4\lambda\end{vmatrix}$$

$$=-\lambda[-(6+\lambda)(1-\lambda)-2(4\lambda-1)]+[(4\lambda-1)+(6+\lambda)^2]$$

$$=-\lambda(-6+6\lambda-\lambda+\lambda^2-8\lambda+2)+(\lambda^2+16\lambda+35)$$

$$=-\lambda^3+4\lambda^2+20\lambda+35.$$

$\therefore$ The characteristic equation of the matrix **A** is

$$\lambda^3-4\lambda^2-20\lambda-35=0 \qquad \text{...(i)}$$

$\therefore$ By Cayley Hamilton's Theorem, we have

$\mathbf{A}^3-4\mathbf{A}^2-20\mathbf{A}-35\mathbf{I}=\mathbf{O}$, where **O** is the null matrix.

or $$35\mathbf{I}=\mathbf{A}^3-4\mathbf{A}^2-20\mathbf{A}.$$

Multiplying both sides by $\mathbf{A}^{-1}$, we get

$$35\mathbf{A}^{-1}=\mathbf{A}^2-4\mathbf{A}-20\mathbf{I}, \qquad \because\ \mathbf{A}\mathbf{A}^{-1}=\mathbf{I} \qquad \text{...(ii)}$$

Now $$\mathbf{A}^2=\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix}\times\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1+12+7 & 3+6+14 & 7+9+7\\ 4+8+3 & 12+4+6 & 28+6+3\\ 1+8+1 & 3+4+2 & 7+6+1\end{bmatrix}=\begin{bmatrix}20 & 23 & 23\\ 15 & 22 & 37\\ 10 & 9 & 14\end{bmatrix}$$

$\therefore$ From (ii) we get

$$35\mathbf{A}^{-1}=\begin{bmatrix}20 & 23 & 23\\ 15 & 22 & 37\\ 10 & 9 & 14\end{bmatrix}-4\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix}-20\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}20-4-20 & 23-12-0 & 23-28-0\\ 15-16-0 & 22-8-20 & 37-12-0\\ 10-4-0 & 9-8-0 & 14-4-20\end{bmatrix}=\begin{bmatrix}-4 & 11 & -5\\ -1 & -6 & 25\\ 6 & 1 & -10\end{bmatrix}\ \textbf{Ans.}$$

Verify Cayley Hamilton's Theorem for yourself.

Ex. 11 (b). Find the characteristic equation of the matrix $\mathbf{A}=\begin{bmatrix}1 & 1 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{bmatrix}$ **and hence find $\mathbf{A}^{-1}$.**

(Meerut 91 S)

Sol. Do as Ex. 11 (a) above.

Ex. 11 (c). Using the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$**.**

find A^{-1}**.**

Sol. Do as Ex. 11 (a) above. **Ans.** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 12. If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix}$**, find the characteristic roots of A.**

Verify Cayley Hamilton's Theorem and hence find A^{-1}**.**

Sol. Here $|A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -8 \\ 2 & -4 & 3-\lambda \end{vmatrix}$

$$= (8-\lambda)\{(7-\lambda)(3-\lambda) - 32\} + 6\{-6(3-\lambda) + 16\} + 2\{24 - 2(7-\lambda)\}$$

$$= (8-\lambda)\{-11 - 10\lambda + \lambda^2\} + 6\{6\lambda - 2\} + 2\{10 + 2\lambda\}$$

$$= -88 - 69\lambda + 18\lambda^2 - \lambda^3 + 36\lambda - 12 + 20 + 4\lambda = -\lambda^3 + 18\lambda^2 - 29\lambda - 80$$

$\therefore$ The characteristic equation of the matrix **A** is

$$\lambda^3 - 18\lambda^2 + 29\lambda + 80 = 0. \qquad \text{...(i)}$$

Its roots are the required characteristic roots of **A** (students can calculate it if they have read solution of cubic equations, so left as an exercise for the students).

Now $A^2 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 104 & -98 & 70 \\ -106 & 117 & -92 \\ 46 & -52 & 45 \end{bmatrix}$

And $A^3 = A^2 \cdot A = \begin{bmatrix} 104 & -98 & 70 \\ -106 & 117 & -92 \\ 46 & -52 & 45 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1560 & -1590 & 1202 \\ -1734 & 1823 & -1424 \\ 770 & -820 & 643 \end{bmatrix}$, on evaluating

$\therefore$ $A^3 - 18A^2 + 29A + 80I$

$= \begin{bmatrix} 1560 & -1590 & 1202 \\ -1734 & 1823 & -1424 \\ 770 & -820 & 643 \end{bmatrix} - 18\begin{bmatrix} 104 & 98 & 70 \\ -106 & 117 & -92 \\ 46 & -52 & 45 \end{bmatrix}$

$+ 29\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix} + 80\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1560 & -1590 & 1202 \\ -1734 & 1823 & -1424 \\ 760 & -820 & 643 \end{bmatrix} + \begin{bmatrix} -1872 & 1764 & -1260 \\ 1908 & -2160 & 1656 \\ -828 & 936 & -810 \end{bmatrix}$$

$$+ \begin{bmatrix} 232 & -174 & 58 \\ -174 & 203 & -232 \\ 58 & 116 & 87 \end{bmatrix} + \begin{bmatrix} 80 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 80 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$, where $\mathbf{O}$ is the null matrix.

Hence $\mathbf{A}$ satisfies the characteristic equation given by (i).

Hence Cayley Hamilton's Theorem is satisfied by the given matrix $\mathbf{A}$ *i.e.* from (i) we get $\mathbf{A}^3 - 18\mathbf{A}^2 + 29\mathbf{A} + 80\mathbf{I} = \mathbf{O}$

or $$80\mathbf{I} = -\mathbf{A}^3 + 18\mathbf{A}^2 - 29\mathbf{A}$$

or $$80\,\mathbf{A}^{-1} = -\mathbf{A}^2 + 18\mathbf{A} - 29\mathbf{I}, \text{ multiplying both sides by } \mathbf{A}^{-1}$$

$$= -\begin{bmatrix} 104 & -98 & 70 \\ -106 & 117 & -92 \\ 46 & -52 & 45 \end{bmatrix} + 18\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix} - 29\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -104+144-29 & 98-108+0 & -70+36+0 \\ 106-108+0 & -117+126-29 & 92-144+0 \\ -46+36+0 & 52-72+0 & -45+54-29 \end{bmatrix}$$

or $$80\mathbf{A}^{-1} = \begin{bmatrix} 11 & -10 & -34 \\ -2 & -20 & -52 \\ -10 & -20 & -20 \end{bmatrix}$$

or $$\mathbf{A}^{-1} = (1/80)\begin{bmatrix} 11 & -10 & -34 \\ -2 & -20 & -52 \\ -10 & -20 & -20 \end{bmatrix}$$

Ans.

Ex. 13 (a) Verify that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies its own characteristic equation. (b) Is it true of every square matrix ? (c) State the theorem that applies here. (d) find $\mathbf{A}^{-1}$. *(Meerut 93, 90; Rohilkhand 91)*

Sol. (a). Here as in Ex. 9 Page 182 Ch. VII we can prove that the characteristic equation of the matrix $\mathbf{A}$ is

$$\lambda^3 + \lambda^2 - 5\lambda - 5 = 0 \qquad \text{...(i)}$$

Now $$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And $\quad A^3 = A^2 . A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\therefore$
$$A^3 + A^2 - 5A - 5I$$
$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 5+5-5-5 & 10+0-10+0 & 0+0+0+0 \\ 10+0-10+0 & -5+5+5-5 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & -1+1+5-5 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$= \mathbf{O}$, where $\mathbf{O}$ is the null matrix.

Hence the matrix **A** satisfies its characteristic equation given by (i).

(b) Every square matrix satisfies its characteristic equation.

(c) Cayley Hamilton's Theorem. (d) Do yourself

Ex. 14. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ and hence compute its cube.

Sol. Here we have
$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{vmatrix}$$
$$= (1-\lambda)\{-\lambda(1-\lambda) - 4\} + 1\{-2(1-\lambda)\}$$
$$= -\lambda(1-\lambda)^2 - 6(1-\lambda) = -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

$\therefore$ The characteristic equation of the matrix **A** is
$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0. \qquad \text{...(i)}$$

$\therefore$ By Cayley-Hamilton theorem we have
$$A^3 - 2A^2 - 5A + 6I = O \qquad \text{...(ii)}$$

Now $\quad A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix}$

$\therefore$ From (ii) we have $A^3 = 2A^2 + 5A - 6I$
$$= 2\begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 & 4 \\ 4 & 10 & 4 \\ 2 & 4 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 10 \\ 0 & 5 & 10 \\ 5 & 10 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 & 14 \\ 4 & 9 & 14 \\ 7 & 14 & 6 \end{bmatrix}$$

Ans.

Ex. 15 (a). Using Cayley Hamilton Theorem calculate $2A^5 + 3A^4 + A^2 - 11I$, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. *(Rohilkhand 93)*

Sol. Here $\quad |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$

$$= (3-\lambda)(2-\lambda) - (-1)(1) = \lambda^2 - 5\lambda + 7.$$

$\therefore$ The characteristic equation of the matrix **A** is $\lambda^2 - 5\lambda + 7 = 0$. ...(i)

Now $A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$$\therefore \quad A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O},$$

where **O** is the null matrix.

Hence **A** satisfies the characteristic equation of **A** given by (i). Thus we have

$$A^2 - 5A + 7I = O. \qquad \text{...(ii)}$$

Now $\qquad 2A^5 + 3A^4 - A^2 - 11I$

$$= 2A^3(A^2 - 5A + 7I) + 13A^4 - 14A^3 - A^2 - 11I$$

$$= 2A^3(O) + 13A^2(A^2 - 5A + 7I) + 51A^3 - 92A^2 - 11I, \quad \text{from (i)}$$

$$= 13A^2(O) + 51A(A^2 - 5A + 7I) + 163A^2 - 355A - 11I$$

$$= 51A(O) + 163(A^2 - 5A + 7I) + 460A - 1152I$$

$$= 163(O) + 460A - 1152I = 460A - 1152I$$

$$= 460\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 1152\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1380-1152 & 460-0 \\ -460-0 & 920-1152 \end{bmatrix}$$

$$= \begin{bmatrix} 228 & 460 \\ -460 & -232 \end{bmatrix} = 4\begin{bmatrix} 57 & 115 \\ -115 & -58 \end{bmatrix}.$$

Ans.

Ex. 15 (b). Verify Cayley Hamilton's Theorem for the following and compute $2A^8 - 3A^5 + A^4 + A^2 - 4I$.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ *(Kumaun 90)*

Sol. Here $|A - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)[\lambda + \lambda^2 - 1] = -\lambda^3 + 2\lambda - 1$$

$\therefore$ The characteristic equation of the matrix **A** is

$$\lambda^3 - 2\lambda + 1 = 0. \qquad ...(i)$$

Now $A^2 = A \,.\, A$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$\therefore \quad A^3 = A^2 \,.\, A$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$\therefore \quad A^3 - 2A + I$

$$= \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2+1 & 0+0+0 & 4-4+0 \\ 0+0+0 & -3+2+1 & 2-2+0 \\ 0+0+0 & 2-2+0 & -1+0+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= \mathbf{O}$, where **O** is the null matrix.

Hence **A** satisfies the characteristic equation of **A** given by (i) and so Cayley Hamilton's Theorem for the matrix **A** is verified.

Thus we have $\qquad A^3 - 2A + I = O \qquad ...(ii)$

Now $\qquad 2A^8 - 3A^5 + A^4 + A^2 - 4I$

$$= 2A^5(A^3 - 2A + I) + 4A^6 - 5A^5 + A^4 + A^2 - 4I$$

$$= 2A^5(O) + 4A^3(A^3 - 2A + I) - 5A^5 + 9A^4 - 4A^3 + A^2 - 4I,$$

from (ii)

$$= 4A^3(O) - 5A^2(A^3 - 2A + I) + 9A^4 - 14A^3 + 6A^2 - 4I$$

$$= -5A^2(O) + 9A(A^3 - 2A + I) - 14A^3 + 24A^2 - 9A - 4I$$

$$= 9A(O) - 14(A^3 - 2A + I) + 24A^2 - 37A + 10I$$

$$= -14\,(\mathbf{O}) + 24\begin{bmatrix}1 & 2 & 2\\0 & 2 & -1\\0 & -1 & 1\end{bmatrix} - 37\begin{bmatrix}1 & 0 & 2\\0 & -1 & 1\\0 & 1 & 0\end{bmatrix} + 10\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}24-37+10 & 48+0+0 & 48-74+0\\0+0+0 & 48+37+10 & -24-37+0\\0+0+0 & -24-37+0 & 24+0+10\end{bmatrix}$$

$$= \begin{bmatrix}-3 & 48 & -26\\0 & 85 & -61\\0 & -61 & 34\end{bmatrix}$$ **Ans.**

Ex. 16 Evaluate the matrix $2A^4 - 7A^3 - 4A^2$, where $A = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 2\\1 & 2 & 0\end{bmatrix}$.

Sol. As in Ex. 10 Page 185 Ch. VII it can be calculated that the matrix A satisfies

$$A^3 - 2A^2 - 5A + 6I = O \quad \ldots$$

Now $\quad 2A^4 - 7A^3 - 4A^2 = (A^3 - 2A^2 - 5A + 6I)(2A - 3I) - 27A + 1$

$= -27A + 18I$, from (i), Now calculate.

Ex. 17. If $A = \begin{bmatrix}1 & 0 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$, show that for every integer $n \geq 4$, $A^n = A^{n-2} + A^3 - A$. Hence evaluate A^{20}.

Sol. Here we have

$$|A - \lambda I| = \begin{vmatrix}1-\lambda & 0 & 0\\1 & 0-\lambda & 1\\0 & 1 & 0-\lambda\end{vmatrix}$$

$$= (1-\lambda)\{(-\lambda)(-\lambda) - 1\} = (1-\lambda)(\lambda^2 - 1)$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1$$

$\therefore$ The characteristic equation of A is $\lambda^3 - \lambda^2 - \lambda + 1 = 0$.

And so by Cayley-Hamilton's theorem we have

$$A^3 - A^2 - A + I = O$$

or $\quad A(A^2 - I) = A^2 - I \quad \ldots(i) \quad$ **(Note)**

Premultiplying both sides of (i) by A^{r-3}, we have

$$A^{r-2}(A^2 - I) = A^{r-3}(A^2 - I). \quad \ldots(ii)$$

Putting $r = n, n-1, n-2, \ldots, 4$ in (ii) we get

$$A^{n-2}(A^2 - I) = A^{n-3}(A^2 - I)$$

$$A^{n-3}(A^2 - I) = A^{n-4}(A^2 - I)$$

....

....

$$\mathbf{A}^2(\mathbf{A}^2-\mathbf{I})=\mathbf{A}(\mathbf{A}^2-\mathbf{I})$$

Multiplying these $n-3$ identities we have

$$\mathbf{A}^{n-2}(\mathbf{A}^2-\mathbf{I})=\mathbf{A}(\mathbf{A}^2-\mathbf{I}). \quad \textbf{(Note)}$$

or $\quad \mathbf{A}^n-\mathbf{A}^{n-2}=\mathbf{A}^3-\mathbf{A}$ or $\mathbf{A}^n=\mathbf{A}^{n-2}+\mathbf{A}^3-\mathbf{A}$, for all $n \geq 4$.

Now we have $\quad \mathbf{A}^n-\mathbf{A}^{n-2}=\mathbf{A}(\mathbf{A}^2-\mathbf{I})$ ...(iii)

Putting $n=20, 18, 16, \ldots, 4$ in (iii) we get **(Note)**

$$\mathbf{A}^{20}-\mathbf{A}^{18}=\mathbf{A}(\mathbf{A}^2-\mathbf{I})$$

$$\mathbf{A}^{18}-\mathbf{A}^{16}=\mathbf{A}(\mathbf{A}^2-\mathbf{I})$$

....

....

$$\mathbf{A}^{4}-\mathbf{A}^{2}=\mathbf{A}(\mathbf{A}^2-\mathbf{I})$$

Adding these nine identities, we get

$$\mathbf{A}^{20}-\mathbf{A}^{2}=9\mathbf{A}(\mathbf{A}^2-\mathbf{I}) \quad \textbf{(Note)}$$

$$=9\mathbf{A}^3-9\mathbf{A}=9(\mathbf{A}^2+\mathbf{A}-\mathbf{I})-9\mathbf{A}, \text{ from (i)}$$

or $\quad \mathbf{A}^{20}=10\mathbf{A}^2-9\mathbf{I}.$ ...(iv)

Now $\quad \mathbf{A}^2=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}\times\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}$

$\therefore$ From (iv) we have

$$\mathbf{A}^{20}=10\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}-9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}10&0&0\\10&10&0\\10&0&10\end{bmatrix}+\begin{bmatrix}-9&0&0\\0&-9&0\\0&0&-9\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\10&1&0\\10&0&1\end{bmatrix} \quad \textbf{Ans.}$$

Ex. 18. Compute $\mathbf{A}^{-2}$ in Example 17 above.

Sol. In Ex. 17 above we have already proved that

$$\mathbf{A}^3-\mathbf{A}^2-\mathbf{A}+\mathbf{I}=\mathbf{O}. \quad \text{...(i)}$$

Also in Cor. III Page 168 Ch. VII we have proved that

$$\mathbf{A}^{-1}=-\frac{1}{a_n}\left\{\mathbf{A}^{n-1}+a_1\mathbf{A}^{n-2}+\ldots+a_{n-1}\mathbf{I}\right\}, \quad \text{...(ii)}$$

provided $\mathbf{A}^n+a_1\mathbf{A}^{n-1}+a_2\mathbf{A}^{n-2}+\ldots+a_{n-1}\mathbf{A}+a_n\mathbf{I}=\mathbf{O}.$ ...(iii)

Comparing (i) and (iii) we have

$$a_n = -1; a_{n-1} = 1, a_{n-2} = -1 \text{ etc. or } a_1 = -1, a_2 = 1 \text{ etc.}$$

$\therefore$ From (ii) we have

$$\mathbf{A}^{-1} = -\frac{1}{(-1)}\left\{\mathbf{A}^{n-1} + (-1)\,\mathbf{A}^{n-2} + \ldots + (1)\,\mathbf{I}\right\}, \text{ where } n = 3$$

$$= \{\mathbf{A}^2 - \mathbf{A} + \mathbf{I}\}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From Ex. 17 Pages 191-192

or $$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \quad \text{...(iv)}$$

Also from (i) pre-multiplying each term by $\mathbf{A}^{-2}$ we have

$$\mathbf{A} - \mathbf{I} - \mathbf{A}^{-1} + \mathbf{A}^{-2} = \mathbf{O}. \quad \textbf{(Note)}$$

or $$\mathbf{A}^{-2} = -\mathbf{A} + \mathbf{I} + \mathbf{A}^{-1}$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

Ans.

Ex. 19 (a). Determine the eigen values and the corresponding eigen vectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ *(Garhwal 95)*

Sol. Here $|\mathbf{A} - \lambda\mathbf{I}|$

$$= \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & -2+2\lambda & 2-\lambda \end{vmatrix}, \text{ applying } C_2 - 2C_3$$

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 2\lambda-2 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 1 & 2\lambda-2 \end{vmatrix}$$

$$= (2-\lambda)\,[(2-3\lambda+\lambda^2) - (2\lambda-2)] + [(2\lambda-2) - (1-\lambda)]$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = -(\lambda-1)^2(\lambda-5)$$

$\therefore$ The characteristic equation of the matrix $\mathbf{A}$ is $(\lambda-1)^2(\lambda-5) = 0$

Its roots *i.e.* required eigen values of $\mathbf{A}$ are 1, 5. **Ans.**

Now the equation $(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{X}=\mathbf{O}$, for the matrix $\mathbf{A}$ is

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{...(i)}$$

...See §7.04 P. 166 Ch. VII

Puting $\lambda = 1$ in (i), we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding eigen-vector are given by the equations

$x_1 + 2x_2 + x_3 = 0$, which does not give any non-zero solution.

$\therefore$ The eigen vector corresponding to $\lambda = 1$ cannot be evaluated.

Putting $\lambda = 5$ in (i), we get

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ applying } R_1 \to R_1 + R_2,\ R_3 \to R_3 - R_2$$

$$\Rightarrow -2x_1 + 2x_3 = 0,\ x_1 - 2x_2 + x_3 = 0,\ 4x_2 - 4x_3 = 0$$

$$\Rightarrow x_1 = x_3,\ x_1 - 2x_2 + x_3 = 0,\ x_2 = x_3$$

$$\Rightarrow x_1 = x_2 = x_3,\ x_1 - 2x_1 + x_1 = 0$$

$$\Rightarrow x_1 = x_2 = x_3 \text{ and } x_1 \text{ can take any value.}$$

$\therefore$ Corresponding eigen vector is (x_1, x_1, x_1), where x_1 can take any non-zero value.

Ex. 19 (b). Find the eigen-values and eigen vectors of the matrix

$$\mathbf{A}=\begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{3} \end{bmatrix}$$

Sol. Here $|\mathbf{A}-\lambda\mathbf{I}|$

$$= \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & \lambda-2 & 3-\lambda \end{vmatrix}, \text{ applying } C_2 - C_3.$$

$$= (3-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ \lambda-2 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 4-\lambda \\ 1 & \lambda-2 \end{vmatrix}$$

$$= (3-\lambda)\,[(4-\lambda)(3-\lambda)-(\lambda-2)] + [(\lambda-2)-(4-\lambda)]$$

$$= (3-\lambda)\,[\lambda^2 - 8\lambda + 14] + (2\lambda - 6)$$

$$= (3-\lambda)[\lambda^2 - 8\lambda + 14 - 2] = (3-\lambda)(\lambda^2 - 8\lambda + 12)$$
$$= (3-\lambda)(\lambda-6)(\lambda-2) = -(\lambda-3)(\lambda-2)(\lambda-6)$$

$\therefore$ The characteristic equation of **A** is $(\lambda-2)(\lambda-3)(\lambda-6)=0$.

Its roots *i.e.* required eigen values of **A** are 2, 3, 6. **Ans.**

Now the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{O}$, for the matrix A is

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} \quad \text{...(i)}$$

(See § 7.04 Page 166 Ch. VII)

Putting $\lambda = 2$ in (i), we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding eigen-vector is given by the equations

$$x_1 + x_2 + x_3 = 0, x_1 + 3x_2 + x_3 = 0$$

These give $\dfrac{x_1}{2} = \dfrac{x_2}{0} = \dfrac{x_3}{2}$ or $\dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{1}$

$\therefore$ The characteristic vector corresponding to $\lambda = 2$ may be taken as (1, 0, 1). **Ans.**

Similarly calculate for $\lambda = 3$ and 6 also.

Ex. 20. Find the eigen-values and eigen vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-5} & \mathbf{-4} \\ \mathbf{-5} & \mathbf{-6} & \mathbf{-5} \\ \mathbf{-4} & \mathbf{-5} & \mathbf{3} \end{bmatrix}$$

Sol. Here $|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & -5 & -4 \\ -5 & -6-\lambda & -5 \\ -4 & -5 & 3-\lambda \end{vmatrix}$

$$= (3-\lambda)\begin{vmatrix} -6-\lambda & -5 \\ -5 & 3-\lambda \end{vmatrix} + 5\begin{vmatrix} -5 & -5 \\ -4 & 3-\lambda \end{vmatrix} - 4\begin{vmatrix} -5 & -6-\lambda \\ -4 & -5 \end{vmatrix}$$

$$= (3-\lambda)[-(6+\lambda)(3-\lambda) - 25] + 5[-15 + 5\lambda - 20] - 4[25 + 4(-6-\lambda)]$$

$$= -\lambda^3 + 93\lambda - 308 = -(\lambda-4)(\lambda^2 + 4\lambda - 77)$$
$$= -(\lambda-4)(\lambda-7)(\lambda+11).$$

$\therefore$ Characteristic equation of **A** is $(\lambda-4)(\lambda-7)(\lambda+11) = 0$

Its roots *i.e.* required eigen values of **A** are 4, 7, − 11.

Now the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{O}$, for the matrix **A** is

$$\begin{bmatrix} 3-\lambda & -5 & -4 \\ -5 & -6-\lambda & -5 \\ -4 & -5 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} \quad \text{...(ii)}$$

...See § 7.04 Page 166 Ch. VII

Putting $\lambda = 4$ in (i), we get

$$\begin{bmatrix} -1 & -5 & -4 \\ -5 & -10 & -5 \\ -4 & -5 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding eigen-vectors is given by the equations

$$x_1 + 5x_2 + 4x_3 = 0,\ 5x_1 + 10x_2 + 5x_3 = 0$$

and $4x_1 + 5x_2 + x_3 = 0$, which give $x_1 = 0 = x_2 = x_3$.

These being all zero, the eigen vector corresponding to $\lambda = 4$ cannot be evalutated.

Putting $\lambda = 7$ in (i), we get

$$\begin{bmatrix} -4 & -5 & -4 \\ -5 & -13 & -5 \\ -4 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding eigen vectors is given by the equations

$$4x_1 + 5x_2 + 4x_3 = 0 \text{ and } 5x_1 + 13x_2 + 5x_3 = 0$$

which give $x_2 = 0$ and $x_3 = -x_1$

These give $\dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{-1}$.

$\therefore$ The characteristic vector corresponding to $\lambda = 7$ may be taken as $(1, 0, -1)$. **Ans.**

Similarly calculate for $\lambda = -11$ also.

Ex. 21. Find the eigen-vectors of $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Sol. Here we can calculate that

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$$

and the eigen values are given by $\lambda = 5, -3, -3$.

(Students are to find these in exam.)

Now the equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{O}$, for the matrix $\mathbf{A}$ is

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} \qquad \text{...(i)}$$

(See § 7.04 Page 166 Ch. VII)

Putting $\lambda = 5$, in (i) we get

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding eigen-vector is given by the equations

$$-7x_1+2x_2-3x_3=0,\ 2x_1-4x_2-6x_3=0,\ -x_1-2x_2-5x_3=0$$

These give $\dfrac{x_1}{1}=\dfrac{x_2}{2}=\dfrac{x_3}{-1}$

$\therefore$ The eigen-vector corresponding to $\lambda=5$ may be taken as $(1, 2, -1)$.

Ans.

Putting $\lambda=-3$, in (i) we get $\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\mathbf{O}$

The corresponding eigen-vector is given by the equations

$$x_1+2x_2-3x_3=0,\ 2x_1+4x_2-6x_3=0,\ -x_1-2x_2+3x_3=0.$$

These reduce to $x_1+2x_2-3x_3=0$ only and so non-zero solution of this cannot be found, hence no eigen-vector can be derived for $\lambda=-3$.

Ex. 22 (a). Determine the characteristic roots and characteristic vector of the matrix $\mathbf{A}=\begin{bmatrix}\mathbf{5} & \mathbf{4}\\ \mathbf{1} & \mathbf{2}\end{bmatrix}$

(Garhwal 93, 92)

Sol. Here $|\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}5-\lambda & 4\\ 1 & 2-\lambda\end{vmatrix}$

$$=(5-\lambda)(2-\lambda)-4=6-7\lambda+\lambda^2$$

$\therefore$ The characteristic equation of the matrix $\mathbf{A}$ is

$$\lambda^2-7\lambda+6=0 \quad \text{or} \quad (\lambda-1)(\lambda-6)=0 \quad \text{or} \quad \lambda=1, 6.$$

$\therefore$ The characteristic roots of the matrix $\mathbf{A}$ are 1, 6. **Ans.**

Now the equation $(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{X}=\mathbf{O}$, for the matrix $\mathbf{A}$ is

$$\begin{bmatrix}5-\lambda & 4\\ 1 & 2-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\mathbf{O}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{...(i)}$$

Putting $\lambda=1$ in (i), we get

$$\begin{bmatrix}4 & 4\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\Rightarrow x_1+x_2=0\Rightarrow x_1=-x_2$$

$$\Rightarrow \quad \frac{x_1}{1}=\frac{x_2}{-1}$$

$\therefore$ The eigen-vector corresponding to $\lambda=1$ may be taken as $(1, -1)$. **Ans.**

Putting $\lambda=6$ in (i), we get

$$\begin{bmatrix}-1 & 4\\ 1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\Rightarrow x_1-4x_2=0$$

$$\Rightarrow \quad \frac{x_1}{-4}=\frac{x_2}{1}$$

$\therefore$ The eigen-vector corresponding to $\lambda=6$ may be taken as $(-4, 1)$. **Ans.**

Ex. 22 (b). Find the characteristic vector of $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \end{bmatrix}$ *(Agra 90)*

Sol. Here we get $|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix}$

and the eigen-values are given by $\lambda = 1, 2, 2$.

(Students are to find these in the exam.)

Now the equaiton $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{O}$, for the matrix $\mathbf{A}$ is

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$, see § 7.04 Page 166 Ch. VII.

Putting $\lambda = 1$ in (i) we get ...(i)

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding characteristic vector is given by the equations $2x_2 + x_3 = 0, x_2 + x_3 = 0, x_3 = 0$ which do not give non-zero solution of these equations and hence no characteristic vector can be derived for $\lambda = 1$.

Putting $\lambda = 2$ in (i) we get

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding characteristic vector is given by the equations

$$-x_1 + 2x_2 - 3x_3 = 0,\ 3x_3 = 0.$$

These gives $\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{0}$.

$\therefore$ The characteristic vector corresponding to $\lambda = 2$ may be taken as (2, 1, 0).

Ans.

Ex. 23. Find the eigen-vector of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{bmatrix}$ *(Agra 92)*

Sol. Here we have $|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix}$

and the characteristic roots (or eigen values) are given by $\lambda = 2, 3, 5$.

(Students are to find these in the exam.)

Now the equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{O}$, for the matrix $\mathbf{A}$ is

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} \qquad \text{...(i)}$$

(See § 7.04 Page 166 Ch. VII)

Putting $\lambda = 5$ in (i) we get

$$\begin{vmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding eigen-vector is given by the equations

$$-2x_1 + x_2 + 4x_3 = 0, -3x_2 + 6x_3 = 0,$$

which give $\quad 3x_2 = 6x_3$ or $\dfrac{x_2}{2} = \dfrac{x_3}{1}$. ...(ii)

Now $\dfrac{x_2}{2} = \dfrac{x_3}{1} = k$ (say), then from $-2x_1 + x_2 + 4x_3 = 0$ we get

$$2x_1 = x_2 + 4x_3 = 2k + 4k = 6k \quad \text{or} \quad x_1 = 3k$$

or $\quad \dfrac{x_1}{3} = k$. So we get $\dfrac{x_1}{3} = \dfrac{x_2}{2} = \dfrac{x_3}{1}$.

$\therefore$ The characteristic vector corresponding to $\lambda = 5$ may be taken as (3, 2, 1).

For $\lambda = 2, 3$ we find that $|\mathbf{A} - \lambda \mathbf{I}| = 0$ and so non-zero solutions of (i) cannot be evaluated in these cases *i.e.* eigen-vectors cannot be calculated.

***Ex. 24. If a + b + c = 0, find the characteristic roots of the matrix**

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{b} \\ \mathbf{c} & \mathbf{b} & \mathbf{a} \\ \mathbf{b} & \mathbf{a} & \mathbf{c} \end{bmatrix}$$

(Garhwal 96)

Sol. Here we have

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a-\lambda & c-0 & b-0 \\ c-0 & b-\lambda & a-0 \\ b-0 & a-0 & c-\lambda \end{vmatrix} = \begin{vmatrix} a-\lambda & c & b \\ c & b-\lambda & a \\ b & a & c-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c-\lambda & c & b \\ c+b+a-\lambda & b-\lambda & a \\ b+a+c-\lambda & a & c-\lambda \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 + C_2 + C_3$$

$$= \begin{vmatrix} -\lambda & c & b \\ -\lambda & b-\lambda & a \\ -\lambda & a & c-\lambda \end{vmatrix}, \text{ since } a+b+c = 0 \text{ (given)}$$

$$= \begin{vmatrix} -\lambda & c & b \\ 0 & b-\lambda-c & a-b \\ 0 & a-c & c-\lambda-b \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_1$$

$$= -\lambda \begin{vmatrix} b-c-\lambda & a-b \\ a-c & c-b-\lambda \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= -\lambda [(b-c-\lambda)(c-b-\lambda)-(a-b)(a-c)]$$

$$= -\lambda [bc - b^2 - b\lambda - c^2 + cb + c\lambda - \lambda c + b\lambda + \lambda^2 - a^2 + ac + ba - bc]$$

or $$|\mathbf{A} - \lambda \mathbf{I}| = \lambda [(a^2 + b^2 + c^2 - ab - bc - ca) - \lambda^2]. \qquad \text{...(i)}$$

Also $a + b + c = 0 \Rightarrow (a + b + c)^2 = 0$

$$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 0$$

$$\Rightarrow 2(ab + bc + ca) = -(a^2 + b^2 + c^2) \qquad \textbf{(Note)}$$

$$\Rightarrow -(ab + bc + ca) = \frac{1}{2}(a^2 + b^2 + c^2)$$

$\therefore$ From (i) we get

$$|\mathbf{A} - \lambda \mathbf{I}| = \lambda [\{(a^2 + b^2 + c^2) + \frac{1}{2}(a^2 + b^2 + c^2)\} - \lambda^2] \qquad \textbf{(Note)}$$

$$= \lambda [(3/2)(a^2 + b^2 + c^2) - \lambda^2]$$

$\therefore$ The characteristic equation of **A** is $\lambda [(3/2)(a^2 + b^2 + c^2) - \lambda^2] = 0$ which gives $\lambda = 0$ or $\lambda^2 = (3/2)(a^2 + b^2 + c^2)$.

The required roots are $0, \pm [(3/2)(a^2 + b^2 + c^2)]^{1/2}$. **Ans.**

Ex. ?⁻ (a). Find latent roots and latent vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{0} & \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{c} \end{bmatrix}$$ *(Kanpur 94)*

Sol. Here $|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a-\lambda & h & g \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{vmatrix}$

$$= (a-\lambda)(b-\lambda)(c-\lambda)$$

$\therefore$ The characteristic equation of the matrix A is

$$(\lambda - a)(\lambda - b)(\lambda - c) = 0$$

and the characteristic or latent roots of **A** are a, b, c. **Ans.**

Again the equation $(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = \mathbf{O}$ for the matrix **A** is

$$\begin{bmatrix} a-\lambda & h & g \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{...(i)}$$

Putting $\lambda = a$ in the above equation we get

$$\begin{bmatrix} 0 & h & g \\ 0 & b-a & 0 \\ 0 & 0 & c-a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding characteristic vector is given by the equations

$$0x_1 + hx_2 + gx_3 = 0, (b-a)x_2 = 0, (c-a)x_3 = 0.$$

$\therefore$ The characteristic vector corresponding $\lambda = a$ may be taken as $(\alpha, 0, 0)$.

Ans.

Similarly we can find the characteristic vector corresponding to $\lambda = b$ and $\lambda = c$ as $(-h, a-b, 0)$ and $(-h, 0, a-c)$. **Ans.**

Ex. 25 (b). Find the latent roots and latent vector of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{0} & \mathbf{b} & \mathbf{f} \\ \mathbf{0} & \mathbf{0} & \mathbf{c} \end{bmatrix}$$

Sol. Do as Ex. 25 (a), above.

Ex. 25 (c). Find the latent roots and latent vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{5} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{4} \end{bmatrix}$$ *(Agra 95)*

Hint : Do as Ex. 25(a) above.

Ex. 26. If $\mathbf{B} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ then find the characteristic equation of B and verify that the matrix B satisfies the equation. Also find the characteristic roots and the corresponding characteristic vectors of B. *(Garhwal 94)*

Sol. Here $\mathbf{B} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad |\mathbf{B} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{vmatrix}$$

$$= [(2-\lambda)(1-\lambda) - \sqrt{2}\sqrt{2}]$$

$$= 2 - 2\lambda - \lambda + \lambda^2 - 2$$

$$= \lambda^2 - 3\lambda = \lambda(\lambda - 3).$$

$\therefore$ The characteristic equation of the matrix **B** is $\lambda(\lambda - 3) = 0$ **Ans.**

and the characteristic roots of **B** are 0 and 3. **Ans.**

Also $$\mathbf{B}^2 = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \times \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2 + \sqrt{2}.\sqrt{2} & 2.\sqrt{2} + \sqrt{2}.1 \\ \sqrt{2}.2 + 1.\sqrt{2} & \sqrt{2}.\sqrt{2} + 1.1 \end{bmatrix} = \begin{bmatrix} 6 & 3\sqrt{2} \\ 3\sqrt{2} & 3 \end{bmatrix}$$

$$\therefore \quad \mathbf{B}^2 - 3\mathbf{B} = \begin{bmatrix} 6 & 3\sqrt{2} \\ 3\sqrt{2} & 3 \end{bmatrix} - 3\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3\sqrt{2} \\ 3\sqrt{2} & 3 \end{bmatrix} + \begin{bmatrix} -6 & -3\sqrt{2} \\ -3\sqrt{2} & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 6-6 & 3\sqrt{2}-3\sqrt{2} \\ 3\sqrt{2}-3\sqrt{2} & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$= \mathbf{O}$, where $\mathbf{O}$ is the null matrix.

Hence the matrix $\mathbf{B}$ satisfies its characteristic equation given by

$$\lambda(\lambda-3)=0 \text{ or } \lambda^2-3\lambda=0.$$

The equation $(\mathbf{B}-\lambda\mathbf{I})=\mathbf{O}$ for the matrix $\mathbf{B}$ is

$$\begin{bmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\mathbf{O} \qquad \text{...(i)}$$

(see § 7.04 Page 166 chapter VII)

Putting $\lambda=0$ in (i), we get $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\mathbf{O}$

The corresponding characteristic vector is given by the equations

$2x_1+\sqrt{2}x_2=0$ and $\sqrt{2}x_1+x_2=0$. **(Note)**

Taking any one of them we get

$$\frac{x_1}{\sqrt{2}}=\frac{x_2}{-2} \text{ or } \frac{x_1}{1}=\frac{x_2}{-\sqrt{2}}.$$

$\therefore$ The characteristic vector corresponding to $\lambda=0$ may be taken as $(1,-\sqrt{2})$.

Putting $\lambda=3$ in (i), we get

$$\begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\mathbf{O}$$

The corresponding characteristic vector is given by the equations $-x_1+\sqrt{2}x_2=0$ and $\sqrt{2}x_1-2x_2=0$.

Taking any one of them we get $\dfrac{x_1}{2}=\dfrac{x_2}{\sqrt{2}}$ or $\dfrac{x_1}{\sqrt{2}}=\dfrac{x_2}{1}$

$\therefore$ The characteristic vector corresponding to $\lambda=3$ can be taken as $(\sqrt{2},1)$. **Ans.**

****Ex. 27. Show that the matrix** $\mathbf{A}=\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ **satisfies its characteristic equation. Also find** $\mathbf{A}^{-1}$. *(Agra 91; Kumaun 91; Rohilkhand 92)*

Sol. Here $|\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix} 0-\lambda & c-0 & -b-0 \\ -c-0 & 0-\lambda & a-0 \\ b-0 & -a-0 & 0-\lambda \end{vmatrix}$

$$=-\lambda\begin{bmatrix} -\lambda & a \\ -a & -\lambda \end{bmatrix}-c\begin{bmatrix} -c & a \\ b & -\lambda \end{bmatrix}-b\begin{bmatrix} -c & -\lambda \\ b & -a \end{bmatrix}$$

$$=-\lambda(\lambda^2+a^2)-c(c\lambda-ab)-b(ca+b\lambda)$$

$$= -\lambda^3 - \lambda (a^2 + b^2 + c^2)$$

$\therefore$ The characteristic equation of $\mathbf{A}$ is

$$\lambda^3 + \lambda (a^2 + b^2 + c^2) = 0. \qquad ...(i)$$

Also we have $\mathbf{A}^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

$$= \begin{bmatrix} -(c^2 + b^2) & ab & ca \\ ab & -(c^2 + a^2) & bc \\ ac & bc & -(a^2 + b^2) \end{bmatrix}$$

$\therefore$ $\mathbf{A}^3 = \mathbf{A}^2 . \mathbf{A}$

$$= \begin{bmatrix} -(b^2 + c^2) & ab & ca \\ ab & -(c^2 + a^2) & bc \\ ac & bc & -(a^2 + b^2) \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c (b^2 + c^2 + a^2) & b (b^2 + c^2 + a^2) \\ c (c^2 + a^2 + b^2) & 0 & -a (b^2 + c^2 + a^2) \\ -b (c^2 + a^2 + b^2) & a (c^2 + a^2 + b^2) & 0 \end{bmatrix}$$

$$= -(a^2 + b^2 + c^2) \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix},$$

taking out $-(a^2 + b^2 + c^2)$ common. **(Note)**

or $$\mathbf{A}^3 = -(a^2 + b^2 + c^2)\, \mathbf{A} \qquad ...(ii)$$

or $\mathbf{A}^3 + (a^2 + b^2 + c^2)\, \mathbf{A} = \mathbf{O}$, which shows that the matrix $\mathbf{A}$ satisfies its characteristic equation given by (i).

Again multiplying both sides of (ii) by $\mathbf{A}^{-2}$, we get

$$\mathbf{A} = -(a^2 + b^2 + c^2)\, \mathbf{A}^{-1} \qquad \textbf{(Note)}$$

which gives $$\mathbf{A}^{-1} = -\frac{1}{(a^2 + b^2 + c^2)} \mathbf{A}$$

$$= -\frac{1}{(a^2 + b^2 + c^2)} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \qquad \textbf{Ans.}$$

Exercises on § 7.03–7.06

Ex. 1. Show that the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \text{ is } (\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$$

Ex. 2. Show that the matrix $\mathbf{A}=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$ satisfies Cayley-Hamilton Theorem.

Ex. 3. Let $\mathbf{A}=\begin{bmatrix}2 & 2 & 0\\ 2 & 1 & 1\\ -7 & 2 & -3\end{bmatrix}$. Find the characteristic equation of **A** and verify that matrix **A** satisfies this equation. Also find the characteristic roots and the corresponding vectors of **A**.

Ex. 4. Using Cayley-Hamilton Theorem or otherwise determine the inverse of the matrix $\mathbf{A}=\begin{bmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{bmatrix}$

Ex. 5. Verify Cayley-Hamilton Theorem in the case of the matrix $\mathbf{A}=\begin{bmatrix}0 & 1 & 2\\ 2 & -3 & 0\\ 1 & 1 & 1\end{bmatrix}$. Hence find $\mathbf{A}^{-1}$.

***Ex. 6.** If matrix $\mathbf{A}=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$, find the characteristic roots of **A**. Verify Cayley-Hamilton's Theorem and hence compute $\mathbf{A}^{-1}$.

Ans. $1, 1, 4, \frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$

Ex. 7. Verify the Cayley-Hamilton Theorem and find the characteristic roots, where $\mathbf{A}=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 1 & 1\end{bmatrix}$

(Garhwal 91)

Ex. 8. Show that the matrix $\mathbf{A}=\begin{bmatrix}3 & 0 & -1\\ 2 & 0 & 1\\ 0 & 0 & 4\end{bmatrix}$ satisfies Cayley Hamilton Theorem.

Ex. 9. Using Cayley-Hamilton Theorem find the inverse of

$$\mathbf{A}=\begin{bmatrix}2 & 4 & 3\\ 0 & -1 & 1\\ 2 & 2 & -1\end{bmatrix}$$

(Rohilkhand 96, 90)

Ex. 10. Verify Cayley-Hamilton Theorem and verify it for the matrix **A** and hence find $\mathbf{A}^{-1}$, where

$$\mathbf{A}=\begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Lucknow 92, 90)

****Ex. 11.** Using Cayley-Hamilton's Theorem, compute the inverse of the matrix $\mathbf{A}=\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

Ans. $\frac{1}{4}\begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -2 \\ -3 & 2 & 3 \end{bmatrix}$

Ex. 12. Show that $\mathbf{A}=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^2-4\mathbf{A}-5\mathbf{I}=\mathbf{O}$, where $\mathbf{I}$ is the unit matirx, Deduce $\mathbf{A}^{-1}$.

Ans. $\frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

Ex. 13. Obtain the characteristic equation of the matrix $\mathbf{A}=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ and hence evaluate $\mathbf{A}^{-1}$.

(Kumaun 93)

Ans. $\lambda^3-2\lambda^2-5\lambda+6=0$; $\mathbf{A}^{-1}=\begin{bmatrix} 4 & -4 & 2 \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

Ex. 14 If $\mathbf{A}=\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, calculate $\mathbf{A}^{-1}$ with the help of Cayley-Hamilton's Theorem.

Ans. $\frac{1}{6}\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$

Ex. 15. If $\mathbf{A}=\begin{bmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$, calculate $\mathbf{A}^{-1}$ with the help of Cayley-Hamilton's Theorem.

Ans. $\frac{1}{4}\begin{bmatrix} 5 & 0 & 0 \\ -3 & 4 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

Ex. 16. Find the eigen-values and eigen-vectors of the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$$

Ans. $-1, \frac{1}{2}[7 \pm \sqrt{(-39)}]$, no eigen vectors.

Ex. 17. Find the eigen-vectors of the following matrices :—

(a) $\begin{bmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ *(Agra 92)*

Ex. 18. Find the characteristic vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ *(Kumaun 91)*

Ex. 19. Find the eigen values and eigen-vectors for the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -5 & -4 \\ -5 & -6 & -5 \\ -4 & -5 & 3 \end{bmatrix}$$

Ex. 20. Find the characteristic roots of

$$\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$ *(Kumaun 96)*

Ex. 21. Find the invariant vector of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Ex. 22. Find the characteristic roots and vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix}$$ **Ans.** $-1, 3$; $(1, -1)$, $(1 - 5)$

Ex. 23. Determine the characteristic roots and associated invariant vectors, given

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$ *(Lucknow 91)*

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1 Prove that matrices A and $\mathbf{B}^{-1}\mathbf{AB}$ have the same latent roots.**

Sol. We know that two matrices have the same latent roots (or characteristic roots) if their characteristic equations are the same. (See definition of latent roots in § 7.02 Page 162).

Let $\mathbf{B}^{-1}\mathbf{AB} = \mathbf{C}$, then $\mathbf{C} - \lambda\mathbf{I} = \mathbf{B}^{-1}\mathbf{AB} - \lambda\mathbf{I}$. ...(i)

Also $\qquad \mathbf{B}^{-1}\lambda\mathbf{IB} = \mathbf{B}^{-1}\lambda\mathbf{B} = \lambda\mathbf{B}^{-1}\mathbf{B} = \lambda\mathbf{I}$

$\therefore$ From (i) we get $\mathbf{C} - \lambda\mathbf{I} = \mathbf{B}^{-1}\mathbf{AB} - \mathbf{B}^{-1}\lambda\mathbf{IB}$

$$= \mathbf{B}^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{B}$$

or $$|\mathbf{C} - \lambda\mathbf{I}| = |\mathbf{B}^{-1}|\,|\mathbf{A} - \lambda\mathbf{I}|\,|\mathbf{B}|$$

$$= |\mathbf{A} - \lambda\mathbf{I}|\,|\mathbf{B}^{-1}|\,|\mathbf{B}|$$

$$= |\mathbf{A} - \lambda\mathbf{I}|\,|\mathbf{B}^{-1}\mathbf{B}|$$
$$= |\mathbf{A} - \lambda\mathbf{I}|\,|\mathbf{I}| = |\mathbf{A} - \lambda\mathbf{I}|$$

$\therefore$ $$|\mathbf{C} - \lambda\mathbf{I}| = 0 \Rightarrow |\mathbf{A} - \lambda\mathbf{I}| = 0$$

Hence the characteristic equation of **C** and **A** are the same *i.e.* **C** and **A** or $\mathbf{B}^{-1}$ **AB** and **A** have the same latent roots. Hence proved.

Ex. 2. Prove that the eigen values of a diagonal matrix are given by its diagonal elements.

Sol. Let $$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

be a diagonal matrix with $a_{11}, a_{22}, \dots, a_{nn}$ as diagonal elements.

Then the characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \text{ or } \begin{vmatrix} a_{11} - \lambda & 0 & \dots & 0 \\ 0 & a_{22} - \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

or $$(a_{11} - \lambda)(a_{22} - \lambda)\dots(a_{nn} - \lambda) = 0$$

or $\lambda = a_{11}, a_{22}, \dots, a_{nn}$ are the eigen values of the matrix **A** and are given by the diagonal elements of the diagonal matrix **A**. Hence proved

Ex. 3. Find the spectrum of the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol. We can find that the eigen values of the matrix **A** are $5, -3, -3$.

Also we know spectrum of **A** is the set of eigen values of **A**.

[See § 7.02 (iv) Page 160]

$\therefore$ Required spectrum of $\mathbf{A} = \{5, -3, -3\} = \{5, -3\}$ **Ans.**

Ex. 4. The equation $\mathbf{AK} = \lambda\,\mathbf{X}$ has non-trivial solution X iff λ is a characteristic value of A.

Sol. Let λ_1 be a characteristic value of **A** and $\mathbf{X}_1$ be the corresponding characteristic vector of **A**, then

$$\mathbf{AX} = \lambda_1\mathbf{X}_1 = \lambda_1\,\mathbf{I}X_1 = (\lambda_1\mathbf{I})\,\mathbf{X}_1$$

or $$\mathbf{AX}_1 - (\lambda_1\mathbf{I})\,\mathbf{X}_1 = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix}$$

or $$(\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{X}_1 = \mathbf{O} \quad \text{or} \quad (\mathbf{A} - \lambda_1\mathbf{I}) = \mathbf{O}, \ \because \ \mathbf{X}_1 \neq \mathbf{O}$$

or $$|\mathbf{A} - \lambda_1| = 0$$

Hence every characteristic value λ of $\mathbf{A}$ is a root of its characteristic equation.

Conversely if λ_1 be any root of the characteristic equation $|\mathbf{A} - \lambda_1| = 0$, then the equation $(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{X} = \mathbf{O}$ must possess a non-zero vector $\mathbf{X}_1$,

such that $$\mathbf{A}\mathbf{X}_1 = \lambda_1\,\mathbf{I}\mathbf{X}_1 = \lambda_1\,\mathbf{X}_1.$$

Hence every root λ of the characteristic equation of $\mathbf{A}$ is a characteristic value of $\mathbf{A}$.

EXERCISES ON CHAPTER VII

Ex. 1. Evaluate the matrix $\mathbf{A}^5 - 27\mathbf{A}^3 + 65\mathbf{A}^2$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} \qquad \textbf{Ans.} \begin{bmatrix} 40 & 2 & 48 \\ 128 & -3 & 0 \\ 86 & -43 & -132 \end{bmatrix}$$

Ex. 2. Evaluate $\mathbf{A}^{50}$ in Ex. 17 Page 193 Ch. VII.

Ex. 3. If the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find the characteristic roots of $\mathbf{A}$.

Verify Cayley-Hamilton Theorem and hence compute $\mathbf{A}^{-1}$.

****Ex. 4.** When do you say that two matrices $\mathbf{A}$ and $\mathbf{B}$ are smilar ? Prove that the similar matrices have the same characteristic roots.

Ex. 5. Find the characteristic polynomials of the matrix $\mathbf{A}$ and hence compute $2\mathbf{A}^6 - 3\mathbf{A}^5 + \mathbf{A}^4 + \mathbf{A}^2 - 4\mathbf{I}$, where $\mathbf{A}$ is the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Also determine one of the characteristic roots corresponding the characteristic vectors.

Ex. 6. Verify the Cayley Hamilton's Theorem and find the latent roots where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Ex. 7. Determine the characteristic roots of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Ex. 8. Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices over the field of real numbers, and let $\mathbf{B}$ be non singular, obtain the characteristic roots of $\mathbf{A}$, if

$$\mathbf{B}^{-1}\mathbf{A}\mathbf{B} = -\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Ex. 9. Use Cayley-Hamilton Theorem to find $\mathbf{A}^{-1}$ if

$$\mathbf{A}=\begin{bmatrix}2 & 1 & 2\\ 0 & 2 & 3\\ 0 & 0 & 5\end{bmatrix}$$

Ex. 10. Find the characteristic roots and characteristic vectors of the following matrix :— $\mathbf{A}=\begin{bmatrix}1 & 0 & -2\\ 0 & 0 & 2\\ -2 & 0 & 4\end{bmatrix}$.

Ex. 11. Find eigen-vectors of the matrix

$$\mathbf{A}=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -8\\ 2 & -4 & 1\end{bmatrix}$$

****Ex. 12.** If $\lambda_1, \lambda_2, ..., \lambda_n$ are the characteristic roots of a square matrix $\mathbf{A}$ of order n, then show that $1/\lambda_1, 1/\lambda_2, ..., 1/\lambda_n$ are the characteristic roots of the matrix $\mathbf{A}^{-1}$. *(Agra 92; Kumaun 94)*

Ex. 13. Does the matrix $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 3\\ 0 & 0 & 3\end{bmatrix}$ satisfy Cayley Hamilton Theorem ?

Find eigen-values and eigen vectors of $\mathbf{A}$. *(Agra 90)*

Ex. 14. Prove that one characteristic root of $\mathbf{A}$ is 2, and find the corresponding characteristic vectors where $\mathbf{A}=\begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$

Ex. 15. Find the characteristic roots and characteristic vectors of the matrix

$$\begin{bmatrix}2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$$

Ex. 16. Show that if $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the latent roots of a matrix $\mathbf{A}$, then $\mathbf{A}'_3$ has the latent roots

$$\lambda_1^3, \lambda_2^3, \lambda_3^3, \ldots \lambda_n^3.$$ *(Agra 96)*

Ex. 17. Find the characteristic roots and characteristic vectors of the matrix

$$\mathbf{A}=\begin{bmatrix}1 & -1 & -1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix}$$ *(Agra 96)*

Ex. 18. Show that the characteristic roots of an idempotent matix are either zero or unity. *(Bundelkhand 91)*

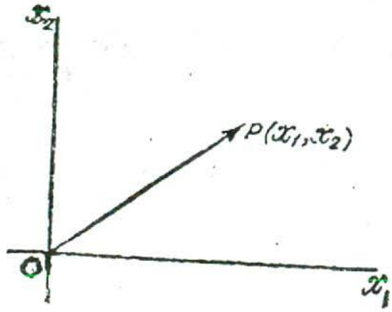
Chapter VIII

Linear Dependence of Vectors

§ 8.01. Two dimensional vector.

We know that the ordered pair of real numbers (x_1, x_2) is used to denote a point P in a plane where Ox_1 and Ox_2 are the coordinate-axes.

A two dimensional vector or **2-vector** OP is denoted by the same pair of numbers written as $[x_1, x_2]$.

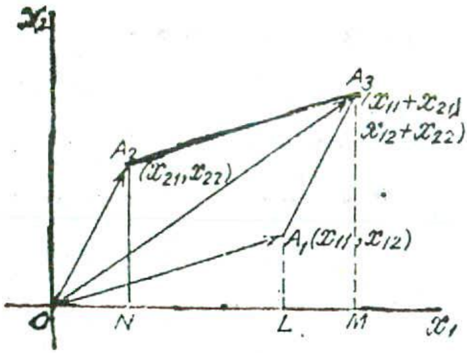


(Fig. 1)

If $\mathbf{A}_1 = [x_{11}, x_{12}]$ and $\mathbf{A}_2 = [x_{21}, x_{22}]$ are two distinct two-dimensional vectors, then their sum by parallelogram law of addition is given by

$$\mathbf{A}_3 = \mathbf{A}_1 + \mathbf{A}_2$$
$$= [x_{11} + x_{21}, x_{12} + x_{22}]$$

[Here $OM = OL + LM$
$= OL + ON = x_{11} + x_{21}$ etc.]



(Fig. 2)

If we treat $\mathbf{A}_1$ and $\mathbf{A}_2$ as 1×2 matrices, we find that the above is the rule for adding matrices as given in chapter I.

Also we observe that $k \cdot \mathbf{A}_1 = [k\,x_{11}, k\,x_{12}]$, where k is any scalar.

§ 8.02. n-dimensional vector or n-vector.

Definition. An ordered set of n elements x_i of a field F, written as

$$\mathbf{A} = [x_1, x_2, x_3, \dots, x_n] \quad \text{...(i)}$$

is called an **n-dimensional vector** or **n-vector A** over F and the elements $x_1, x_2, \dots, x_n$ are called the first, second, ..., nth **components** of **A**.

We find it more convenient to write the components of a vector in a column as

$$\mathbf{A}' = [x_1, x_2, \dots, x_n]' = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix} \quad \text{...(ii)}$$

(i) is called a **row-vector** and (ii) is called a **column-vector.**

Thus we consider the $p \times q$ matrix as defining q column vectors or p row vectors. **(Note)**

Note 1. The sum or difference of two rwo (or column) vectors is formed by the rule governing matrices as given in chapter I.

Note 2. The product of a scalar and a vector is formed by the rule governing matrices as given in chapter I.

Note 3. The vector whose all the components are zero is known as **the null vector** or **zero vector** and is written as **O.**

Solved Examples on § 8.01 – § 8.02

Ex. 1. Given the 3-vectors

$A_1 = [1, 2, 1]$, $A_2 = [2, 1, 4]$, $A_3 = [2, 3, 6]$, evaluate $2A_1 + A_2$, $5A_1 - 2A_3$

Solution. $2A_1 + A_2 = 2\,[1, 2, 1] + [2, 1, 4]$

$= [2, 4, 2] + [2, 1, 4]$

$= [2 + 2, 4 + 1, 2 + 4] = [4, 5, 6]$. **Ans.**

$5A_1 - 2A_3 = 5\,[1, 2, 1] - 2\,[2, 3, 6]$

$= [5, 10, 5] - [4, 6, 12]$

$= [5 - 4, 10 - 6, 5 - 12] = [1, 4, -7]$. **Ans.**

Ex. 2. Given the four-dimensional column-vectors

$A_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ **and** $A_2 = \begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}$**, evaluate $3A_1 + 2A_2$.**

Solution. $3A_1 + 2A_2 = 3\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ 10 \\ 14 \\ 18 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 0+10 \\ 6+14 \\ 9+18 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 20 \\ 27 \end{bmatrix}$ **Ans.**

Ex. 3. Given the three-dimensional row vectors

$A_1 = [3, 1, -4]$, $A_2 = [0, -4, 1]$; $A_3 = [2, 2, -3]$, evaluate $2A_1 - A_2 - 3A_3$.

Solution. $2A_1 - A_2 - 3A_3 = 2\,[3, 1, -4] - [0, -4, 1] - 3\,[2, 2, -3]$

$= [6, 2, -8] - [0, -4, 1] - [6, 6, -9]$

$= [0, 0, 0] = \mathbf{O}$. **Ans.**

***§ 8.03. Linear dependence and independence of vectors.**

(Agra 94; Purvanchal 97)

The nm-vectors over the field F,

$A_1 = [x_{11}, x_{12}, \ldots, x_{1m}]$, $A_2 = [x_{21}, x_{22}, \ldots, x_{2m}]$, ...,

$A_n = [x_{n1}, x_{n2}, \ldots, x_{nm}]$

are called **linearly dependent** over F if there exists a set of n elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ of F, λ's being not all zero, such that

$$\lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2 + \ldots + \lambda_n \mathbf{A}_n = \mathbf{O}.$$

Otherwise the n-vectors are called **linearly independent** over F.

For example the 3-vectors given in Ex. 3 above are linearly dependent.

Note : A vector $\mathbf{A}_{n+1}$ can be expressed as a **linear combination** of the vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ if there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ of F such that

$$\mathbf{A}_{n+1} = \lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2 + \ldots + \lambda_n \mathbf{A}_n$$

Solved Examples on § 8.03.

Ex. 1. Examine whether the set of vector $\mathbf{V}_1 = \{1, 2, 3\}$, $\mathbf{V}_2 = \{1, 0, 1\}$ and $\mathbf{V}_3 = \{0, 1, 0\}$ are linearly dependent or not. *(Purvanchal 94)*

Solution. Let the given set of vectors be linearly dependent, so that

$$\lambda_1 \mathbf{V}_1 + \lambda_2 \mathbf{V}_2 + \lambda_3 \mathbf{V}_3 = \mathbf{O} \qquad \ldots(i)$$

or $\lambda_1 \{1, 2, 3\} + \lambda_2 \{1, 0, 1\} + \lambda_3 \{0, 1, 0\} = \mathbf{O} = \{0, 0, 0\}$ **(Note)**

or $\{\lambda_1 + \lambda_2, 2\lambda_1 + \lambda_3, 3\lambda_1 + \lambda_2\} = \{0, 0, 0\}$

$\therefore$ $\lambda_1 + \lambda_2 = 0, 2\lambda_1 + \lambda_3 = 0, 3\lambda_1 + \lambda_2 = 0$

$\Rightarrow$ $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ *i.e.* λ's are all zero.

Hence the given set of vectors are not linearly dependent *i.e.* these are linearly independent.

Ex. 2. Examine the following set of vectors of the real field for linear dependence or independence :—

$\mathbf{A}_1 = [2, -1, 3, 2]$; $\mathbf{A}_2 = [1, 3, 4, 2]$; $\mathbf{A}_3 = [3, -5, 2, 2]$

Also express $\mathbf{A}_3$ as a linear combination of $\mathbf{A}_1, \mathbf{A}_2$.

Solution. Suppose the given set of vectors is linearly dependent, so that

$$\lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2 + \lambda_3 \mathbf{A}_3 = \mathbf{O}. \qquad \ldots(i)$$

or $\lambda_1 [2, -1, 3, 2] + \lambda_2 [1, 3, 4, 2] + \lambda_3 [3, -5, 2, 2] = \mathbf{O} = [0, 0, 0, 0]$ **(Note)**

or $[2\lambda_1 + \lambda_2 + 3\lambda_3, -\lambda_1 + 3\lambda_2 - 5\lambda_3, 3\lambda_1 + 4\lambda_2 + 2\lambda_3, 2\lambda_1 + 2\lambda_2 + 2\lambda_3] = [0, 0, 0, 0]$

$\therefore$ $2\lambda_1 + \lambda_2 + 3\lambda_3 = 0$ ...(i); $-\lambda_1 + 3\lambda_2 - 5\lambda_3 = 0$ ...(ii)

$3\lambda_1 + 4\lambda_2 + 2\lambda_3 = 0$ ...(iii); and $2(\lambda_1 + \lambda_2 + \lambda_3) = 0$ ...(iv)

From (i) and (iv) we get, $\lambda_1 + 2\lambda_3 = 0$ or $\lambda_1 = -2\lambda_3$ ...(v)

$\therefore$ From (iv) we get $\lambda_2 = -\lambda_1 - \lambda_3 = 2\lambda_3 - \lambda_3 = \lambda_3$ ...(vi)

$\therefore$ From (ii), (v) and (vi) we get

$-\lambda_1 + 3\lambda_2 - 5\lambda_3 = -(-2\lambda_3) + 3\lambda_3 - 5\lambda_3 = 0$. Hence (ii) is satisfied.

Again from (iii) we get $3\lambda_1 + 4\lambda_2 + 2\lambda_3$

$= 3(-2\lambda_3) + 4(\lambda_3) + 2\lambda_3$, from (v), (vi)

$= 0$. Hence (iii) is also satisfied.

Thus for $\lambda_1 = -2\lambda_3$ and $\lambda_2 = \lambda_3$ all the equations (i), (ii), (iii) and (iv) are satisfied and therefore the given set of vectors are linearly dependent.

$\therefore$ From (i) we get $-2\lambda_3\mathbf{A}_1 + \lambda_3\mathbf{A}_2 + \lambda_3\mathbf{A}_3 = \mathbf{O}$ or $\mathbf{A}_3 = 2\mathbf{A}_1 - \mathbf{A}_2$, which expresses $\mathbf{A}_3$ as a linear combination of $\mathbf{A}_1$ and $\mathbf{A}_2$.

Ex. 3. Show, using a matrix, that the set of vectors $\mathbf{X}_1 = [2, 3, 1, -1], \mathbf{X}_2 = [2, 3, 1, -2], \mathbf{X}_3 = [4, 6, 2, -3]$ is linearly dependent. *(Agra 96)*

Solution. Let the given set of vectors be linearly dependent, so that

$$\lambda_1\mathbf{X}_1 + \lambda_2\mathbf{X}_2 + \lambda_3\mathbf{X}_3 = \mathbf{O} \quad \text{...(i)}$$

or $\lambda_1 [2, 3, 1, -1] + \lambda_2 [2, 3, 1, -2] + \lambda_3 [4, 6, 2, -3] = \mathbf{O} = [0, 0, 0, 0]$

or $[2\lambda_1 + 2\lambda_2 + 4\lambda_3, 3\lambda_1 + 3\lambda_2 + 6\lambda_3, \lambda_1 + \lambda_2 + 2\lambda_3, -\lambda_1 - 2\lambda_2 - 3\lambda_3] = [0, 0, 0, 0]$

$\therefore$ $2\lambda_1 + 2\lambda_2 + 4\lambda_3 = 0$; $\quad 3\lambda_1 + 3\lambda_2 + 6\lambda_3 = 0$

$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$; $\quad -\lambda_1 - 2\lambda_2 - 3\lambda_3 = 0$

which reduce to $\lambda_1 + \lambda_2 + 2\lambda_3 = 0$, $\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$

whence solving we get $\lambda_1 + \lambda_3 = 0$, $\lambda_2 + \lambda_3 = 0$

which give $\dfrac{\lambda_1}{1} = \dfrac{\lambda_2}{1} = \dfrac{\lambda_3}{-1}$

$\therefore$ From (i) we get $\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{X}_3 = \mathbf{O}$, where $\lambda_1, \lambda_2, \lambda_3$ are not all zero. Hence given set of vectors is linearly dependent.

Ex. 4. Examine the following set of vectors over the real field for linear dependence or independence :—

$\mathbf{A}_1 = [1, 2, 1]$; $\mathbf{A}_2 = [2, 1, 4]$; $\mathbf{A}_3 = [4, 5, 6]$; $\mathbf{A}_4 = [1, 8, -3]$

Solution. Suppose that the given set of vectors is linearly dependent, so that

$$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 + \lambda_4\mathbf{A}_4 = \mathbf{O} \quad \text{...(i)}$$

or $\lambda_1 [1, 2, 1] + \lambda_2 [2, 1, 4] + \lambda_3 [4, 5, 6] + \lambda_4 [1, 8, -3] = \mathbf{O} = [0, 0, 0]$

or $[\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_4, 2\lambda_1 + \lambda_2 + 5\lambda_3 + 8\lambda_4, \lambda_1 + 4\lambda_2 + 6\lambda_3 - 3\lambda_4] = [0, 0, 0]$

$\therefore$ $\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_4 = 0$ ...(ii); $\quad 2\lambda_1 + \lambda_2 + 5\lambda_3 + 8\lambda_4 = 0$ ...(iii)

and $\lambda_1 + 4\lambda_2 + 6\lambda_3 - 3\lambda_4 = 0$ ...(iv)

From (ii) and (iv) we have $2\lambda_2 + 2\lambda_3 - 4\lambda_4 = 0$

or $\lambda_2 + \lambda_3 - 2\lambda_4 = 0$ ...(v)

From (iii) and (v) we have $2\lambda_1 + \lambda_2 + 5\lambda_3 + 4(\lambda_2 + \lambda_3) = 0$

or $2\lambda_1 + 5\lambda_2 + 9\lambda_3 = 0$ ...(vi)

From (v) and (vi) we have $2\lambda_1 + 5\lambda_2 + 9(2\lambda_4 - \lambda_2) = 0$

or $18\lambda_4 - 4\lambda_2 + 2\lambda_1 = 0$ or $9\lambda_4 = 2\lambda_2 - \lambda_1$ ...(vii)

(v), (vi) and (vii) are satisfied by $\lambda_1 = 0 = \lambda_2 = \lambda_3 = \lambda_4$

Hence the given set of vectors are linearly independent.

§ 8.04. Basic Theorems on Linear dependence of vectors.

Theorem I. *If there be n linearly dependent vectors, then some one of them can always be expressed as a linear combination of the remaining ones.*

Proof. Let $\mathbf{A}_1=[x_{11}, x_{12}, ..., x_{1m}]$, $\mathbf{A}_2=[x_{21}, x_{22}, ..., x_{2m}]$, ... $\mathbf{A}_n=[x_{n1}, x_{n2}, ..., x_{nm}]$ be n m-vectors over the field F, such that

$$\lambda_1\mathbf{A}_1+\lambda_2\mathbf{A}_2+\lambda_3\mathbf{A}_3+\ldots+\lambda_n\mathbf{A}_n=\mathbf{O}, \qquad ...(i)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are elements of F and not all zero.

Let $\lambda_r \neq 0$ then solving (i) we get

$$\mathbf{A}_r=-\frac{1}{\lambda_r}\left[\lambda_1\mathbf{A}_1+\lambda_2\mathbf{A}_2+..+\lambda_{r-1}\mathbf{A}_{r-1}+\lambda_{r+1}\mathbf{A}_{r+1}+\ldots+\lambda_n\mathbf{A}_n\right]$$

or $\mathbf{A}_r=\mu_1\mathbf{A}_1+\mu_2\mathbf{A}_2+..+\mu_{r-1}\mathbf{A}_{r-1}+\mu_{r+1}\mathbf{A}_{r+1}+\ldots+\mu_n\mathbf{A}_n$ ...(ii)

Hence proved.

Theorem II. *If there be n linearly independent vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, whereas the set obtained by adding another vector $\mathbf{A}_{n+1}$ is linearly dependent, then $\mathbf{A}_{n+1}$ can be expressed as linear combination of $\mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_n$.*

Proof. Given $\lambda_1\mathbf{A}_1+\lambda_2\mathbf{A}_2+\ldots+\lambda_n\mathbf{A}_n\neq\mathbf{O}$,. ...(i)

where $\lambda_1, \lambda_2, ..., \lambda_n$ are elements of the field F.

And $(\lambda_1\mathbf{A}_1+\lambda_2\mathbf{A}_2+\ldots+\lambda_n\mathbf{A}_n)+\mathbf{A}_{n+1}=\mathbf{O}$ (Note)

or $\mathbf{A}_{n+1}=-[\lambda_1\mathbf{A}_1+\lambda_2\mathbf{A}_2+\ldots+\lambda_n\mathbf{A}_n]$ Hence proved.

Example. Consider three 3-vectors

$$\mathbf{A}_1=[4, 5, 6], \mathbf{A}_2=[2, 1, 4], \mathbf{A}_3=[1, 2, 1]$$

Let $\mathbf{A}_1$ and $\mathbf{A}_3$ be linearly dependent then we must have

$\lambda_1\mathbf{A}_1+\lambda_2\mathbf{A}_3=\mathbf{O}$, where λ_1 and λ_2 are to be determined.

or $\lambda_1[4, 5, 6]+\lambda_2[1, 2, 1]=\mathbf{O}=[0, 0, 0]$

$\therefore$ $4\lambda_1+\lambda_2=0$; $5\lambda_1+2\lambda_2=0$ and $6\lambda_1+\lambda_2=0$

Solving first and third of these we get $\lambda_1=0=\lambda_2$ which satisfies the second also. But as all λ's are zero, so $\mathbf{A}_2$ and $\mathbf{A}_3$ are not linearly dependent [See § 8.03 Page 211 of this chapter]

But we find that if we take $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ to be linearly dependent

then $\lambda_1\mathbf{A}_1+\lambda_2\mathbf{A}_2+\lambda_3\mathbf{A}_3=\mathbf{O}$. ...(i)

or $\lambda_1[4, 5, 6]+\lambda_2[2, 1, 4]+\lambda_3[1, 2, 1]=\mathbf{O}=[0, 0, 0]$

$\therefore$ $4\lambda_1+2\lambda_2+\lambda_3=0$ ...(ii), $5\lambda_1+\lambda_2+2\lambda_3=0$ ...(iii)

and $6\lambda_1+4\lambda_2+\lambda_3=0$. ...(iv)

From (ii) and (iv) on subtracting we get $\lambda_1+\lambda_2=0$

or $\lambda_2=-\lambda_1$. ...(v)

From (ii) and (v) we get

$$4\lambda_1 + 2(-\lambda_1) + \lambda_3 = 0 \text{ or } \lambda_3 = -2\lambda_1 \quad ...(vi)$$

Substituting values from (v) and (vi) in (i) we get

$$\lambda_1\mathbf{A}_1 - \lambda_1\mathbf{A}_2 - 2\lambda_1\mathbf{A}_3 = \mathbf{O} \text{ or } \mathbf{A}_1 - \mathbf{A}_2 - 2\mathbf{A}_3 = \mathbf{O} \quad ...(vii)$$

or

$$\mathbf{A}_2 = \mathbf{A}_1 - 2\mathbf{A}_3. \quad ..(viii)$$

Thus we find that though $\mathbf{A}_1$ and $\mathbf{A}_3$ are not linearly dependent yet $\mathbf{A}_1$ $\mathbf{A}_2$, $\mathbf{A}_3$ satisfy the relation (vii) and from (viii) we find that $\mathbf{A}_2$ can be expressed as a linear combination of $\mathbf{A}_1$ and $\mathbf{A}_3$.

***Theorem III.** *If there be a subset of r linearly dependent vectors among the n vectors $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ such that $r < n$, then the vectors of the whole set are linearly dependent.*

Proof. Let the subsets $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ of the given n vectors $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ be linearly dependent, then we have

$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \dots + \lambda_r\mathbf{A}_r = \mathbf{O}$, where all λ's are not zero

We can rewrite this as

$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \dots + \lambda_r\mathbf{A}_r + 0.\mathbf{A}_{r+1} + 0 . \mathbf{A}_{r+2} + \dots + 0.\mathbf{A}_n = \mathbf{O}$, **(Note)**

where all λ's are not zero.

Hence the set of vectors $\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_r, \mathbf{A}_{r+1}, \dots, \mathbf{A}_n$ by definition are linearly dependent. Hence proved.

****Theorem IV.** *If the rank of the matrix associated with a set of n m-vector, is r where $r < n$, then there are exactly r vectors which are linearly independent while each of remaining $n - r$ vectos can be expressed as a linear combination of these r vectors.*

****Theorem V.** *A necessary and sufficient condition that the vectors $\mathbf{A}_1 = [x_{11}, x_{12}, \dots, x_{1m}]$, $\mathbf{A}_2 = [x_{21}, x_{22}, \dots, x_{2m}] \dots$, $\mathbf{A}_n = [x_{n1}, x_{n2}, \dots, x_{nm}]$ be linearly dependent is that the matrix* $\mathbf{A} = \begin{bmatrix} x_{11} & x_{12} & \dots & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & \dots & x_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & \dots & x_{nm} \end{bmatrix}$ *of the vectors is of rank $r < n$. If the rank is n, the vectors are linearly independent.*

Proofs of Theorem IV and V above are beyond the scope of this book.

§ 8.05. Linear Form.

Definition. A linear form over F in m variables $x_1, x_2, \dots, x_m$ is a polynomial of the type

$$\sum_{i=1}^{m} a_i x_i = a_1 x_1 + a_2 x_2 + \dots + a_m x_m,$$

where the coefficients are in F.

Consider a system of n linear forms in m variables

$$\begin{bmatrix} f_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m \\ f_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ f_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m \end{bmatrix} \qquad \ldots(\text{i})$$

and the associated matrix formed by their coefficients is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

If there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ in F, λ's being not all zero such that $\lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n = 0$,

then the forms (i) are said to be **linearly dependent**, otherwise they are said to be **linearly independent**.

Thus, the linear dependence or independence of the forms (i) is equivalent to the linear dependence or independence of the row vectors of the matrix **A**.

More Solved Examples :

Ex. 1. Show that the set of vectors $\mathbf{A}_1 = (1, 1, 1), \mathbf{A}_2 = (1, 2, 3)$, $\mathbf{A}_3 = (2, 3, 8)$ is linearly independent.

Solution. Suppose that the given set of vectors is linearly dependent, so that $\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 = \mathbf{O}$, ...(i)

where λ's are to be determined.

or $\lambda_1 (1, 1, 1) + \lambda_2 (1, 2, 3) + \lambda_3 (2, 3, 8) = (0, 0, 0)$

or $(\lambda_1 + \lambda_2 + 2\lambda_3, \lambda_1 + 2\lambda_2 + 3\lambda_3, \lambda_1 + 3\lambda_2 + 8\lambda_3) = (0, 0, 0)$

$\therefore$ $\lambda_1 + \lambda_2 + 2\lambda_3 = 0$ ...(ii); $\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$ ...(iii)

and $\lambda_1 + 3\lambda_2 + 8\lambda_3 = 0$. ...(iv)

From (ii) and (iii) we get $\lambda_2 + \lambda_3 = 0$. ...(v)

From (ii) and (iv) we get $2\lambda_2 + 6\lambda_3 = 0$ or $\lambda_2 + 3\lambda_3 = 0$ ...(vi)

From (v) and (vi) we get $\lambda_3 = 0$.

$\therefore$ From (v) we get $\lambda_2 = 0$ and from (ii) we get $\lambda_1 = 0$, when $\lambda_2 = 0 = \lambda_3$.

Thus all the λ's are zero and hence the given set of vectors is linearly independent.

Ex. 2 (a). Show that set of vectors $\mathbf{A}_1 = [1, 2, 3], \mathbf{A}_2 = [3, 2, 1]$, $\mathbf{A}_3 = [1, 1, 1]$ is linearly dependent. *(Agra 93)*

Solution : Suppose that the given set of vectors is linearly dependent, so that $\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 = \mathbf{O}$

or $\lambda_1 [1, 2, 3] + \lambda_2 [3, 2, 1] + \lambda_3 [1, 1, 1] = \mathbf{O} = [0, 0, 0]$

or $[\lambda_1 + 3\lambda_2 + \lambda_3, 2\lambda_1 + 2\lambda_2 + \lambda_3, 3\lambda_1 + \lambda_2 + \lambda_3] = [0, 0, 0]$

$\therefore$ $\lambda_1 + 3\lambda_2 + \lambda_3 = 0$ ...(i); $2\lambda_1 + 2\lambda_2 + \lambda_3 = 0$ ...(ii)

and $3\lambda_1 + \lambda_2 + \lambda_3 = 0$ ...(iii)

From (i) and (ii) we get $\lambda_1 - \lambda_2 = 0$ or $\lambda_1 = \lambda_2$ ...(iv)

From (iv) and (iii) we get $4\lambda_1 + \lambda_3 = 0$ or $\lambda_3 = -4\lambda_1$ ....(v)

Then we have $\dfrac{\lambda_1}{1} = \dfrac{\lambda_2}{1} = \dfrac{\lambda_3}{-4}$ and from here we do not get all the values of λ as zero.

Hence the given set of vectors is linearly dependent. Hence proved.

Ex. 2 (b). Show that the set of vectors [1, 2, 3], [3, − 2, 1], [1, − 6, − 5] is linearly dependent. *(Agra 92)*

Solution : Let $\mathbf{A}_1 = [1, 2, 3]$, $\mathbf{A}_2 = [3, -2, 1]$ and $\mathbf{A}_3 = [1, -6, -5]$ be a set of linearly dependent vectors.

Then $\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 = \mathbf{O}$, where all λ's are not zero.

or $\lambda_1 [1, 2, 3] + \lambda_2 [3, -2, 1] + \lambda_3 [1, -6, -5] = \mathbf{O} = [0, 0, 0]$

or $[\lambda_1 + 3\lambda_2 + \lambda_3, 2\lambda_1 - 2\lambda_2 - 6\lambda_3, 3\lambda_1 + \lambda_2 - 5\lambda_3] = [0, 0, 0]$.

$\therefore$ $\lambda_1 + 3\lambda_2 + \lambda_3 = 0$ ...(i), $2\lambda_1 - 2\lambda_2 - 6\lambda_3 = 0$ ...(ii

and $3\lambda_1 + \lambda_2 - 5\lambda_3 = 0$ ...(iii)

From (i) and (ii), we get $8\lambda_2 + 8\lambda_3 = 0$ or $\lambda_2 + \lambda_3 = 0$ ...(iv

From (ii) and (iii), we get $8\lambda_1 - 16\lambda_3 = 0$ or $\lambda_1 = 2\lambda_3$ ...(v

$\therefore$ From (iv) and (v) we get $\dfrac{\lambda_1}{2} = \dfrac{\lambda_2}{-1} = \dfrac{\lambda_3}{1}$ and this does not give all the valu of λ as zero.

Hence the given set of vectors is linearly dependent. Hence prove

Ex. 3. Find a linear relation, if any, between the linear forms of th following system $f_1 = x + y + z, f_2 = y - 2z, f_3 = 2x + 3y$;

Sollution : Let $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$.

Then $\lambda_1 (x + y + z) + \lambda_2 (y - 2z) + \lambda_3 (2x + 3y) = 0$

or $(\lambda_1 + 2\lambda_3)\, x + (\lambda_1 + \lambda_2 + 3\lambda_3)\, y + (\lambda_1 - 2\lambda_2)\, z = 0$

$\Rightarrow \lambda_1 + 2\lambda_3 = 0, \lambda_1 + \lambda_2 + 3\lambda_3 = 0, \lambda_1 - 2\lambda_2 = 0$

whence we get $\lambda_3 = -\frac{1}{2}\lambda_1, \lambda_2 = \frac{1}{2}\lambda_1$, which satisfy $\lambda_1 + \lambda_2 + 3\lambda_3 = 0$

Hence from (i) we get $\lambda_1 f_1 + \frac{1}{2}\lambda_1 f_2 - \frac{1}{2}\lambda_1 f_3 = 0$

or $2f_1 + f_2 - f_3 = 0$, which is the required relation.

Ex. 4. Find a linear relation, if any, between the polynomials $f_1 = 2x^3 - 3x^2 + 4x - 2; f_2 = 3x^3 + 2x^2 - 2x + 5; f_3 = 5x^3 - x^2 + 2x + 1$.

Solution. Let $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$ ...(i)

Then $\lambda_1 (2x^3 - 3x^2 + 4x - 2) + \lambda_2 (3x^3 + 2x^2 - 2x + 5)$
$+ \lambda_3 (5x^3 - x^2 + 2x + 1) = 0$

or $(2\lambda_3 + 3\lambda_2 + 5\lambda_3) x^3 + (-3\lambda_1 + 2\lambda_2 - \lambda_3) x^2 + (4\lambda_1 - 2\lambda_2 + 2\lambda_3) x$
$+ (-2\lambda_1 + 5\lambda_2 + \lambda_3) = 0$

$\Rightarrow$ $2\lambda_1 + 3\lambda_2 + 5\lambda_3 = 0$ ...(ii); $3\lambda_1 - 2\lambda_2 + \lambda_3 = 0$ ...(iii)

$2\lambda_1 - \lambda_2 + \lambda_3 = 0$ ...(iv); and $2\lambda_1 - 5\lambda_2 - \lambda_3 = 0$ ...(v)

Solving (ii) and (iv) we get $\lambda_2 - \lambda_3 = 0$ ...(vi)

From (iii) and (iv) we get $3\lambda_1 + 3\lambda_3 = 0$ or $\lambda_1 + \lambda_3 = 0$ (vii)

From (v), (vi) and (vii) we get

$2(-\lambda_3) - 5(-\lambda_3) - \lambda_3 = 0$ or $2\lambda_3 = 0$ or $\lambda_3 = 0$

which gives $\lambda_1 = 0 = \lambda_2$.

Hence from (i) no linear relation exists between f_1, f_2 and f_3.

Ex. 5. If the vectors $(0, 1, a), (1, a, 1), (a, 1, 0)$ are linearly dependent, then find the value of a. *(Agra 94)*

Solution : Let $\mathbf{A}_1 = (0, 1, a), \mathbf{A}_2 = (1, a, 1), \mathbf{A}_3 = (a, 1, 0)$ be a set of linearly dependent vectors.

Then $\lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2 + \lambda_3 \mathbf{A}_3 = \mathbf{O}$, where λ's are not all zero.

or $\lambda_1 (0, 1, a) + \lambda_2 (1, a, 1) + \lambda_3 (a, 1, 0) = \mathbf{O} = (0, 0, 0)$

or $(\lambda_2 + a\lambda_3, \lambda_1 + a\lambda_2 + \lambda_3, a\lambda_1 + \lambda_2) = (0, 0, 0)$

$\therefore$ $\lambda_2 + a\lambda_3 = 0$ ...(i), $\lambda_1 + a\lambda_2 + \lambda_3 = 0$ ...(ii)

and $a\lambda_1 + \lambda_2 = 0$ ...(iii)

From (i), $\lambda_3 = -(1/a)\lambda_2$

From (iii), $\lambda_1 = -(1/a)\lambda_2$

$\therefore$ From (ii), $-(1/a)\lambda_2 + a\lambda_2 - (1/a)\lambda_2 = 0$

or $[a - (2/a)]\lambda_2 = 0$ or $(a^2 - 2)\lambda_2 = 0$

$\therefore$ Either $a^2 - 2 = 0$ or $\lambda_2 = 0$.

But $\lambda_2 = 0$ gives $\lambda_1 = 0, \lambda_3 = 0$, from (i), (iii)

Then $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are not linearly dependent.

Hence $a^2 - 2 = 0$ or $a = \pm\sqrt{2}$. Ans.

Exercises on Chapter VIII

Ex. 1. Show that the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are linearly independent.

Ex. 2. Show that the vectors $[1, 2, 0]$, $[8, 13, 0]$ and $[2, 3, 0]$ are linearly dependent.

Ex. 3. Prove that the set of three vectors

$$[1, 2, -1, 3], [0, -2, 1, -1] \text{ and } [2, 2, -1, 5]$$

is linearly dependent and obtain a relation connecting these vectors.

Ex. 4. Find a linear relation, if any, between the linear forms of the system :—

$$f_1 = 2x_1 - 2x_2 - x_3 + x_4; f_2 = x_1 - x_2 + x_3 + x_4; f_3 = 5x_2 + 3x_3 + x_4.$$

Ex. 5. Prove that any non-empty subset of a linearly independent set is linearly independent. *(Agra 94)*

Chapter IX

Quadratic Forms

§ 9.01. Quadratic Form.

Definiton. A homogeneous polynomial of the type

$$q = \mathbf{X}' \mathbf{A} \mathbf{X} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \qquad \text{...(i)}$$

whose coefficients a_{ij} are elements of the field F is known as a quadratic form over F in the variables $x_1, x_2, \ldots, x_n$.

For Example. $x_1^2 + 2x_2^2 + 5x_3^2 + 8x_1x_3 - 6x_2x_3$ is a quadratic form in the variables x_1, x_2, x_3. Here the matrix of the form can be written in many ways according as the cross product terms $8x_1 x_3$ and $-6x_2x_3$ are seperated to form the terms $a_{13}x_1x_3$, $a_{31}x_3x_1$ and $a_{23} x_2 x_3$, $a_{32} x_3 x_2$.

Here we shall agree that the matrix **A** of quadratic form be **symmetric** and shall always separate the cross-product terms so that $a_{ij} = a_{ji}$

$$\therefore \quad q = x_1^2 + 2x_2^2 + 5x_3^2 + 8x_1x_3 - 6x_2x_3$$

$$= [x_1\ x_2\ x_3] \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & -3 \\ 4 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ ...see Ex. 1 (b) Page 223 of this chapter

$$= \mathbf{X}' \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & -3 \\ 4 & -3 & 5 \end{bmatrix} \mathbf{X} = \mathbf{X}'\mathbf{A}\mathbf{X}$$

The symmetric matrix $\mathbf{A} = [a_{ij}]$ is known as the matrix of the quadratic form and the rank of **A** is called the **rank of the quadratic form.**

If the rank of the form is $r < n$, then the quadratic form is called **singular** otherwise **non-singular.**

Transformations.

The linear transformation $\mathbf{X} = \mathbf{BY}$ over F carries the quadratic form (i) above with symmetric matrix **A** over F into the quadratic form

$$(\mathbf{BY})'\mathbf{A}\,(\mathbf{BY}) = (\mathbf{Y}'\mathbf{B}')\,\mathbf{A}\,(\mathbf{BY}) = \mathbf{Y}'\,(\mathbf{B}'\,\mathbf{AB})\,\mathbf{Y} \qquad \text{...(ii)}$$

with symmetric matrix $\mathbf{B}'\,\mathbf{AB}$.

Equivalent Quadratic Forms.

Definition. Two quadratic forms in the same variables $x_1, x_2, \ldots, x_n$ are called **equivalent** if and only if there exists a non-singular linear transformation $\mathbf{X} = \mathbf{BY}$ which together with $\mathbf{Y} = \mathbf{IX}$, where **I** is the identity matrix, carries one of the forms into the other.

As $\mathbf{B'AB}$ is congruent to $\mathbf{A}$, we have

1. The rank of a quadratic form is invariant under a non-singular transformation of the variables.

2. Two quadratic forms over F are equivalent over F iff their matrices are congruent over F.

A quadratic form of rank r can be reduced to the form

$$b_1y_1^2 + b_2y_2^2 + b_3y_3^2 + \dots + b_ry_r^2,\ b_i \neq 0 \qquad \text{...(iii)}$$

in which only terms in the squares of the variables occur by a non-singular linear transformation $\mathbf{X} = \mathbf{BY}$.

§ 9.02. Lagrange's Reduction.

The reduction of a quadratic form to the form (iii) of § 9.01 above can be carried out by a method or procedure called Lagrange's Reduction, which consists of repeated completing of the square.

From example $q = x_1^2 + 2x_2^2 + 5x_3^2 + 8x_1x_3 - 6x_2x_3$ can be reduced to form (iii) of § 9.01 above as follows :—

$$\begin{aligned} q &= x_1^2 + 2x_2^2 + 5x_3^2 + 8x_1x_3 - 6x_2x_3 \\ &= (x_1^2 + 8x_1x_3 + 16x_3^2) + 2x_2^2 - 11x_3^2 - 6x_2x_3 \\ &= (x_1 + 4x_3)^2 + \frac{1}{2}(4x_2^2 - 12x_2x_3) - 11x_3^2 \\ &= (x_1 + 4x_3)^2 + \frac{1}{2}(4x_2^2 - 12x_2x_3 + 9x_3^2) - \frac{9}{2}x_3^2 - 11x_3^2 \\ &= (x_1 + 4x_3)^2 + \frac{1}{2}(2x_2 - 3x_3)^2 - \frac{31}{2}x_3^2 \\ &= y_1^2 + (1/2)\,y_2^2 - (31/2)\,y_3^2, \end{aligned}$$

where $y_1 = x_1 + 4x_3,\ y_2 = 2x_2 - 3x_3,\ y_3 = x_3$.

§ 9.03. Real Quadratic Forms.

Let a real quadratic form $q = \mathbf{X'AX}$ be reduced by a real non-singular transformation to the form $b_1y_1^2 + b_2y_2^2 + \dots + b_ry_r^2,\ b_i \neq 0$. If one or more of the b_i are negative, then there exists a non-singular transformation $\mathbf{X} = \mathbf{CZ}$, where $\mathbf{C}$ is obtained from $\mathbf{B}$ (see § 9.01 Page 220 of this chapter) by a sequence of row and column transformations which carries q into

$$h_1z_1^2 + h_2z_2^2 + \dots + h_pz_p^2 - h_{p+1}z_{p+1}^2 - \dots - h_rz_r^2 \qquad \text{...(i)}$$

in which the terms with positive coefficients precede those with negative coefficients.

Now the non-singular transformation

$$s_i = \sqrt{(h_i)}\, z_i,\ i = 1, 2, \dots, r$$

$$s_j = z_j,\ j = r + 1, r + 2, \dots, n$$

carries (i) into the canonical form

$$s_2^2 + s_2^2 + s_3^2 + \ldots + s_p^2 - s_{p+1}^2 - \ldots - s_r^2. \quad \ldots(ii)$$

Thus, as the product of non-singular transformations is a non-singular transformation, we have every real quadratic form can be reduced by a non-singular transformation to the canonical form (ii) above, where p, the number of positive terms is called the index and r is the rank of the given quadratic form.

Example. In Ex. 3 (c) Page 225 the quadratic form $q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$ was reduced to $q_1 = y_1^2 - 2y_2^2 + 9y_3^2$. The non-singular transformation $y_1 = z_1, y_2 = z_3, y_3 = z_2$ carries q_1 into $q_2 = z_1^2 + 9z_2^2 - 2z_3^2$ and the non-singular transformation $z_1 = s_1$, $z_2 = s_2/3, z_3 = s_3/\sqrt{2}$ reduces q_2 to $q_3 = s_1^2 + s_2^3 - s_3^2$.

Also in Ex. 3 (c) Page 225 we have

$$y_1 = x_1 - 2x_2 + 4x_3, y_2 = x_2 - 4x_3, y_3 = x_3$$

or $\quad x_1 = y_1 + 2y_2 + 4y_3, x_2 = y_2 + 4y_3, x_3 = y_3$

or $\quad x_1 = z_1 + 2z_3 + 4z_2, x_2 = z_3 + 4z_2, x_3 = z_2$

or $\quad x_1 = s_1 + 2(s_3/\sqrt{2}) + 4(s_2/3), x_2 = (s_3/\sqrt{2}) + 4(s_2/3), x_3 = s_2/3$

or $\quad x_1 = s_1 + (4/3)s_2 + \sqrt{2}s_3, x_2 = (4/3)s_2 + (1/2)\sqrt{2}s_3, x_3 = (1/3)s_2$

or $\quad \mathbf{X} = \begin{bmatrix} 1 & 4/3 & \sqrt{2} \\ 0 & 4/3 & (1/2)\sqrt{2} \\ 0 & 1/3 & 0 \end{bmatrix} \mathbf{S}$

is the non-singular linear transformation that reduces q to $q_3 = s_1^2 + s_2^2 - s_3^2$.

$\therefore$ The quadratic form is of rank 3 and index 2.

§ 9.04. Complex Quadratic Forms

Let the complex quadratic form $\mathbf{X'AX}$ be reduced by a non-singular transformation to the form $b_1y_1^2 + b_2y_2^2 + \ldots + b_ry_r^2, b_i \neq 0$.

Evidently the non-singular transformation

$$z_i = \sqrt{(b_i)}\, y_i,\ i = 1, 2, \ldots, r$$

$$z_j = y_j, \qquad j = r+1, r+2, \ldots, n$$

carries $b_1y_1^2 + b_2y_2^2 + \ldots + b_ry_r^2$ into $z_1^2 + z_2^2 + \ldots + z_r^2$. ...(i)

Solved Examples on § 9.01 to § 9.04

***Ex. 1. Write the following quadratic forms in matrix notation :**

(a) $x_1^2 + 4x_1x_2 + 3x_2^2$;

(b) $x_1^2 - 2x_2^2 - 3x_3^3 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$. *(Garhwal 95)*

Sol. Let $x_1^2 + 4x_1x_2 + 3x_2^2 = \mathbf{X'AX} = [x_1\ x_2]\begin{bmatrix} a_1 & a_2 \\ a_2 & b_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ...(i) since $\mathbf{A}$ is a symmetric matrix

i.e. $$x_1^2 + 4x_1x_2 + 3x_2^2 = [a_1x_1 + a_2x_2 \quad a_2x_1 + b_1x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [(a_1x_1 + a_2x_2)\, x_1 + (a_2x_1 + b_1x_2)\, x_2]$$

$$\Rightarrow \quad x_1^2 + 4x_1x_2 + 3x_2^2 = a_1x_1^2 + 2a_2x_1x_2 + b_1x_2^2$$

Comparing coefficients of x_1^2, x_1x_2 and x_2^2 on both sides, we get

$$a_1 = 1, a_2 = 2, b_1 = 3.$$

$\therefore$ From (i), $\quad x_1^2 + 4x_1x_2 + 3x_2^2 = \mathbf{X}'\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\mathbf{X}$ **Ans**

(b) Let $x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$

$$= \mathbf{X}'\mathbf{A}\mathbf{X} = [x_1 \;\; x_2 \;\; x_3]\begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ ...

$\because$ $\mathbf{A}$ is a symmetric matrix

$$= [a_1x_1 + b_1x_2 + c_1x_3 \quad b_1x_1 + b_2x_2 + c_2x_3 \quad c_1x_1 + c_2x_2 + c_3x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$$= [(a_1x_1 + b_1x_2 + c_1x_3)\, x_1 + (b_1x_1 + b_2x_2 + c_2x_3)\, x_2 + (c_1x_1 + c_2x_2 + c_3x_3)\, x_3]$$

$$\Rightarrow \quad x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$$

$$= a_1x_1^2 + b_2x_2^2 + c_3x_3^2 + 2b_1x_1x_2 + 2c_1x_1x_3 + 2c_2x_2x_3$$

$$\Rightarrow \quad a_1 = 1, b_2 = -2, c_3 = -3, b_1 = 2, c_1 = 3, c_2 = -4.$$

$\therefore$ From (ii), we have $x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$

$$= \mathbf{X}'\begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{bmatrix}\mathbf{X}$$ **Ans.**

Ex. 2 (a). Find the matrix of the quadratic form $\mathbf{x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1}$ and verify that it can be written as a matrix product $\mathbf{X'AX}$.

(Garhwal 94, 93)

Sol. Let $x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1$

$$= \mathbf{X}'\mathbf{A}\mathbf{X} = [x_1 \;\; x_2 \;\; x_3]\begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{...(i)}$$

since $\mathbf{A}$ is a symmetric matrix

$$= [a_1x_1 + b_1x_2 + c_1x_3 \quad b_1x_1 + b_2x_2 + c_2x_3 \quad c_1x_1 + c_2x_2 + c_3x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [(a_1x_1 + b_1x_2 + c_1x_3)\, x_1 + (b_1x_1 + b_2x_2 + c_2x_3)\, x_2 + (c_1x_1 + c_2x_2 + c_3x_3)\, x_3]$$

$$\Rightarrow \quad x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1$$

$$= a_1x_1^2 + b_2x_2^2 + c_3x_3^2 + 2b_1x_1x_2 + 2c_2x_2x_3 + 2c_1x_3x_1$$

Equating coefficients of like terms on both sides, we get

$a_1 = 1, b_2 = 2, c_3 = -5, 2b_1 = -1, 2c_2 = 4, 2c_1 = -3$

$\therefore$ From (i), we have the given quadratic form

$$= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & -1/2 & -3/2 \\ -1/2 & 2 & 2 \\ -3/2 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\therefore$ The required matrix of the given quadratic form

$$= \begin{bmatrix} 1 & -1/2 & -3/2 \\ -1/2 & 2 & 2 \\ -3/2 & 2 & -5 \end{bmatrix}$$ **Ans.**

Also $[x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & -1/2 & -3/2 \\ -1/2 & 2 & 2 \\ -3/2 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= [x_1 - \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 \quad -\tfrac{1}{2}x_1 + 2x_2 + 2x_3 \quad -\tfrac{3}{2}x_1 + 2x_2 - 5x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= (x_1 - \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3)\, x_1 + (-\tfrac{1}{2}x_1 + 2x_2 + 2x_3)\, x_2 + (-\tfrac{3}{2}x_1 + 2x_2 - 5x_3)\, x_3$$

$= x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1 =$ Given quadratic from.

Hence proved.

Ex. 2 (b). Find the matrix of the quadratic form $G = x^2 + y^2 + 3z^2 + 4xy + 5yz + 6zx$ and express it is the form $G = X'AX$, where $X' = (x, y, z)$

(Garhwal 96)

Sol. Let $G = x^2 + y^2 + 3z^2 + 4xy + 5yz + 6zx$

$$= \mathbf{X'AX} = [x \quad y \quad z] \begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{...(i)}$$

since A is a symmetric matrix

$$= [a_1x + b_1y + c_1z \quad b_1x + b_2y + c_2z \quad c_1x + c_2y + c_3y] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [(a_1x + b_1y + c_1z)x + (b_1x + b_2y + c_2z)\,y + (c_1x + c_2y + c_3z)\,z]$$

$$\Rightarrow\ x^2 + y^2 + 3z^2 + 4xy + 5yz + 6zx$$
$$= a_1x^2 + b_2y^2 + c_3z^2 + 2b_1xy + 2c_2yz + 2c_1zx$$

$\Rightarrow\ a_1 = 1, b_2 = 1, c_3 = 3, 2b_1 = 4, 2c_2 = 5, 2c_1 = 6$

$\Rightarrow\ a_1 = 1, b_2 = 1, c_3 = 3, b_1 = 2, c_2 = 5/2, c_1 = 3.$

$\therefore$ From (i), we get $G = \mathbf{X'AX} = [x\ \ y\ \ z]\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ...(ii)

which gives the required matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$ **Ans.**

and G can be expressed in the form $\mathbf{X'\,A\,X}$ by (ii).

Ex. 2 (c). Write out in full the quadratic form in x_1, x_2, x_3 whose matrix is $\begin{bmatrix} 2 & -2 & 5 \\ -2 & 3 & 0 \\ 5 & 0 & 4 \end{bmatrix}$.

Sol. Here $\mathbf{X'AX} = [x_1\ \ x_2\ \ x_3]\begin{bmatrix} 2 & -2 & 5 \\ -2 & 3 & 0 \\ 5 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= [2x_1 - 2x_2 + 5x_3 \quad -2x_1 + 3x_2 + 0.x_3 \quad 5x_1 + 0.x_2 + 4x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [(2x_1 - 2x_2 + 5x_3)\,x_1 + (-2x_1 + 3x_2)\,x_2 + (5x_1 + 4x_3)\,x_3]$$

$$= [2x_1^2 - 2x_2x_1 + 5x_3x_1 - 2x_1x_2 + 3x_2^2 + 5x_1x_3 + 4x_3^2]$$

$$= [2x_1^2 + 3x_2^2 + 4x_3^2 - 4x_1x_2 + 10x_1x_3]$$

$\therefore$ Required quadratic form is

$$2x_1^2 + 3x_2^2 + 4x_3^2 - 4x_1x_2 + 10x_1x_3.$$ **Ans.**

Ex. 3. Reduce the following by Lagrange's Reduction :

(a) $\mathbf{X}'\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}\mathbf{X}$; **(b)** $\mathbf{X}'\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}\mathbf{X}$

(c) $q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

Sol. (a) $\mathbf{X}'\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}\mathbf{X} = [x_1\ x_2\ x_3]\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= [x_1 + 2x_2 + 4x_3 \quad 2x_1 + 6x_2 - 2x_3 \quad 4x_1 - 2x_2 + 18x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [(x_1 + 2x_2 + 4x_3)\, x_1 + (2x_1 + 6x_2 - 2x_3)\, x_2 + (4x_1 - 2x_2 + 18x_3)\, x_3]$$

$$= [x_1^2 + 6x_2^2 + 18x_3^2 + 4x_1x_2 - 4x_2x_3 + 8x_1x_3]$$

$$\therefore \quad q = x_1^2 + 6x_2^2 + 18x_3^2 + 4x_1x_2 - 4x_2x_3 + 8x_1x_3$$

$$= \{x_1^2 + 4x_1(x_2 + 2x_3)\} + 6x_2^2 + 18x_3^2 - 4x_2x_3$$

$$= \{x_1^2 + 4x_1(x_2 + 2x_3) + 4(x_2 + 2x_3)^2\} + 6x_2^2 + 18x_3^2 - 4x_2x_3 - 4(x_2 + 2x_3)^2$$

$$= \{x_1 + 2(x_2 + 2x_3)\}^2 + 6x_2^2 + 18x_3^2 - 4x_2x_3 - 4x_2^2 - 16x_3^2 - 16x_3x_2$$

$$= (x_1 + 2x_2 + 4x_3)^2 + 2x_2^2 + 2x_3^2 - 20x_2x_3$$

$$= (x_1 + 2x_2 + 4x_3)^2 + 2(x_2^2 - 10x_2x_3) + 2x_3^2$$

$$= (x_1 + 2x_2 + 4x_3)^2 + 2(x_2^2 - 10x_2x_3 + 25x_3^2) - 48x_3^2$$

$$= (x_1 + 2x_2 + 4x_3)^2 + 2(x_2 - 5x_3)^2 - 48x_3^2$$

$$= y_1^2 + 2y_2^2 - 48y_3^2.$$

Ans.

where $y_1 = x_1 + 2x_2 + 4x_3$, $y_2 = x_2 - 5x_3$, $y_3 = x_3$.

(b) $$\mathbf{X}'\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}\mathbf{X} = [x_1 \ x_2 \ x_3]\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1 0 + x_2 0 + x_3 1 \quad x_1 0 + x_2 1 - 2x_3 \quad x_1 - 2x_2 + 3x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [(x_3)\, x_1 + (x_2 - 2x_3)\, x_2 + (x_1 - 2x_2 + 3x_3)\, x_3]$$

$$= [x_2^2 + 3x_3^2 + 2x_1x_3 - 4x_2x_3]$$

$$= [z_1^2 + 3z_2^2 + 2z_3z_1 - 4z_1z_2], \text{ using } x_1 = z_3,\ x_2 = z_1,\ x_3 = z_2$$

$$= [\{z_1^2 - 4z_1z_2 + 4z_2^2\} - (z_2^2 - 2z_2z_3)]$$

$$= [(z_1 - 2z_2)^2 - (z_2^2 - 2z_2z_3 + z_3^2) + z_3^2]$$

$$= [(z_1 - 2z_2)^2 - (z_2 - z_3)^2 + z_3^2]$$

$$\therefore \quad q = (z_1 - 2z_2)^2 - (z_2 - z_3)^2 + z_3^2$$

$$= y_1^2 - y_2^2 + y_3^2,$$

where $y_1 = z_1 - 2z_2 = x_2 - 2x_3$, $y_2 = z_2 - z_3 = x_3 - x_1$, $y_3 = z_3 = x_1$

(c) $q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

$$= \{x_1^2 - 4x_1(x_2 - 2x_3)\} + 2x_2^2 - 7x_3^2$$
$$= \{x_1^2 - 4x_1(x_2 - 2x_3) + 4(x_2 - 2x_3)^2\} + 2x_2^2 - 7x_3^2 - 4(x_2 - 2x_3)^2$$
$$= \{x_1 - 2(x_2 - 2x_3)\}^2 + 2x_2^2 - 7x_3^2 - 4(x_2^2 - 4x_2x_3 + 4x_3^2)$$
$$= (x_1 - 2x_2 + 4x_3)^2 - 2x_2^2 + 16x_2x_3 - 23x_3^2$$
$$= (x_1 - 2x_2 + 4x_3)^2 - 2(x_2^2 - 8x_2x_3 + 16x_3^2) + 9x_3^2 \qquad \textbf{(Note)}$$
$$= (x_1 - 2x_2 + 4x_3)^2 - 2(x_2 - 4x_3)^2 + 9x_3^2$$
$$= y_1^2 - 2y_2^2 + 9y_3^2. \qquad \textbf{Ans.}$$

where $y_1 = x_1 - 2x_2 + 4x_3$. $y_2 = x_2 - 4x_3$, $y_3 = x_3$.

Exercise on § 9.01 – § 9.02

Ex. 1. Write $2x_1^2 - 6x_1x_2 + x_3^2$ in matrix notation

Ans. $\mathbf{X}'\begin{bmatrix} 2 & -3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{X}$

Ex. 2. Write out in full the quadratic form in x_1, x_2, x_3 whose matrix is $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 2 & 4 \\ 1 & 4 & -5 \end{bmatrix}$

Ans. $2x_1^2 - 6x_1x_2 + 2x_1x_3 + 2x_2^2 + 8x_2x_3 - 5x_3^2$

Ex. 3. Reduce the following by Lagrange's reduction :

$$\mathbf{X}'\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}\mathbf{X}$$

Ans. $y_1^2 - y_2^2 + 8y_3^2$

[Hint : Use $x_1 = z_3, x_2 = z_1, x_3 = z_2$]

§ 9.05. Definite and Semi-definite Forms.

Definition (i) A real non-singular quadratic form $q = \mathbf{X'AX}$, $|\mathbf{A}| \neq 0$, in n variables is known as **positive definite** if its rank and index are equal. Thus, in the real field a positive definite quadratic form can be reduced to the form $y_1^2 + y_2^2 + \ldots + y_n^2$ and for any non-trivial set of values of the x's, $q > 0$.

(ii) A real singluar quadratic form $q = \mathbf{X'AX}, |\mathbf{A}| = 0$ is as **positive semi-definite** if its rank and index are equal *i.e.* $r = p < n$.

Thus in the real field a positive semi-definite qudratic form can be reduced to the form $y_1^2 + y_2^2 + \ldots + y_r^2$, $r < n$ and for any non-trivial set of values of the x's, $q \geq 0$.

(iii) A real non-singular quadratic form $q = \mathbf{X'AX}$ is known as **negative defnite** if its index $p = 0$ *i.e.* $r = n, p = 0$.

Thus in the real field a negative definite form can be reduced to the form $-y_1^2 - y_2^2 - \ldots - y_n^2$ and for any non-trivial set of values of the x's, $q < 0$.

(iv) A real singular quadratic form $q = \mathbf{X'AX}$ is known as **negative semi-definite** if its index $p = 0$ *i.e.* $r < n, p = 0$.

Thus in the real field a negative semi-defnite form can be reduced to the form $-y_1^2 - y_2^2 - \dots - y_n^2$ and for any non-trivial set of values of the x's, $q \le 0$.

Note 1. If q is negative defnite (semi-definite), then $-q$ is positive definine (semi-defnite).

Note 2. For positive definite quadratic form, if $q = \mathbf{X'AX}$ is positive definite then $|\mathbf{A}| > 0$.

§ 9.06. Definite and Semi-definite Matrices.

Definition. The matrix A of a real quadratic form $q = \mathbf{X'AX}$ is known as definite or semi-definite according as the quadratic form is definite or semi-definite. Thus

(i) A real symmetric matrix **A** is positive definite iff there exists a non-singular matrix **C,** such that $\mathbf{A} = \mathbf{C'C}$.

(ii) A real symmetric matrix of rank r is positive semi-definite iff there exists a matrix **C** of rank r, such that $\mathbf{A} = \mathbf{C'C}$.

§ 9.07. Principal Minors.

Definition. A minor of matrix **A** is known as **principal** if it is obtained by deleting certain rows and the same numbered columns of the matrix **A.**

Note 1. The diagonal elements of a principal minor of the matrix **A** are diagonal elements of the matrix **A.**

Note 2. Every symmetric matrix of rank r has at least one principal minor of order r different from zero.

Note 3. If the matrix **A** is positive definite, then every principal minor of the matrix **A** is positive.

Note 4. If the matrix **A** is positive semi-definite, then every principal minor of the matrix **A** is non-negative.

Exercises on Chapter IX

Ex. 1. Reduce the square matrix **A** into diagonal form and interpret the result in terms of quadratic form :

$$\mathbf{A} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ *(Garhwal 94).*

Ex. 2. Reduce the quardative form $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ to normal form. *(Garhwal 93)*

(A) VERY SHORT AND SHORT ANSWER TYPE QUESTIONS

Ch. V Rank and Adjoint of a Matrix

1. Define rank of a matrix. *(Purvanchal 2000)* [See § 5·02 Pages 1–2]

2. Find the rank of the matrix **A**, where

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(Meerut 2001)

[**Hint :** Do as Ex. 11 (b) P. 28]

Ans. 4

3. Reduce the matrix $\begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix}$ to normal form. *(Kanpur 2001)*

Ans. $\begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$

4. When is a matrix said to be in Echelon form ? *(Purvanchal 98)*

[See § 5·04 Page 36]

5. Write down the four normal forms of a matrix. [See §5·03 Page 15]

6. Define adjoint of a matrix. [See § 5·08 Page 43]

7. Show that adjoint of the matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is $\begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$.

8. How will you use the notion of determinant to complete the inverse of a non-singular square matrix ? [See Th. I result (iv) Page 50]

9. Find the inverse of the matrix $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ [See Ex. 15 Page 67]

Ans. $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

10. If $a^2 + b^2 + c^2 + d^2 = 1$, then show that the inverse of the matrix

$\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ is $\begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$ [See Ex. 18 Page 70]

11. Find the rank of an $m \times n$ matrix, every element of which is unity.

Ans. 1.

[**Hint :** See Ex. 9 Page 96]

12. **A, B, P** and **Q** are matrices such that adj. $\mathbf{B} = \mathbf{A}$, $|\mathbf{P}| = |\mathbf{Q}| = 1$, then adj $(\mathbf{Q}^{-1}\,\mathbf{B}\,\mathbf{P}^{-1}) = \mathbf{PAQ}$. *(Kanpur 2001)*

Ch.VI Solution of Linear Equations

13. Express in matrix form the system of equations :

$9x + 7y + 3z = 6,\ 5x + y + 4z = 1;\ 6x + 8y + 2z = 4$ [See Ex. 1 Page 106]

14. Define a homogeneous linear equation. [See § 6·10 Page 144]

15. Solve the simultaneous equations given below :

$x + y + 2z = 3;\ 2x + 2y + 3z = 7;\ 3x - y + 2z = 1,\ 2x - y - z = 2$

(Kanpur 2001) **Ans.** $x = 2, y = 3, z = 1$

Ch. VII Characteristics Equations of a Matrix

16. What do you understand by the characteristic equation of the matrix **A**?

[See § 7·02 (iii) Page 160]

17. What is eigen value problem ? [See § 7·02 (v) Page 160]

18. Obtain the characteristic equation of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

(Meerut 2001) **Ans.** $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

19. State Cayely-Hamilton's Theorem.

20. Find latent vectors of the matrix $\begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ [See Ex. 25(a) Page 200]

Ch. VIII Linear Dependence of Vectors

21. Define linearly dependent and linearly independent set of vectors.

(Kanpur 2001) [See § 8·03 Page 211]

22. Show that the set of vectors $\mathbf{V}_1 = \{1, 2, 3\}, \mathbf{V}_2 = \{1, 0, 1\}$ and $\mathbf{V}_3 = \{0, 1, 0\}$ are linearly independent. [See Ex. 1 Page 212]

23. Find a linear relation, if any, between the linear forms of the following system $f_1 = x + y + z, f_2 = y - 2z, f_3 = 2x + 3y.$

Ans. $2f_1 + f_2 = f_3$ [See Ex. 3 Page 217]

Ch. IX Quadratic Forms

24. Define a quadratic form. [See § 9·01 Page 220]

25. What do you understand by the rank of a quadratic form.

[See § 9·01 Page 220]

26. Write the quadratic form corresponding to the matrix $\begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 2 \\ -4 & -2 & 0 \end{bmatrix}$.

(Kanpur 2001) **Ans.** $-4x_1x_3$

(B) OBJECTIVE TYPE QUESTIONS

(I) MULTIPLE CHOICE TYPE :

Select (i), (ii), (iii) or (iv) whichever is correct :

Ch. V. Rank and Adjoint of a Martix

1. The rank of the matrix $\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 2 & 3 & 4 \end{bmatrix}$ is

(i) 0 (ii) 1 (iii) 2 (iv) none of these

(Kanpur 2001)

2. The rank of the matrix $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$ is

(i) 0 (ii) 1 (iii) 2 (iv) 3

3. If **A** be a matrix, which one of the following is a number :

(i) A^{-1} (ii) adj **A** (iii) rank **A** (iv) none of these

4. If **A**′ be the transpose of the matrix **A**, then

(i) rank **A**′ > rank **A** (ii) rank **A**′ = rank **A**

(iii) rank **A**′ < rank **A** (iv) none of these

5. If by a series of elementary transformations an n-rowed square matrix **A** is reduced to the form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$, the rank of **A** is

(i) $n + r$ (ii) r (iii) $n - r$ (iv) n

6. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{bmatrix}$ is

(i) 0 (ii) 1 (iii) 2 (iv) 3

7. The rank of the matrix $\begin{bmatrix} 2 & -1 & 3 & 15 \\ 3 & 2 & 0 & 21 \end{bmatrix}$ is

(i) 1 (ii) 2 (iii) 3 (iv) none of these

8. The rank of the matrix $\begin{bmatrix} 1 & 6 & 5 & 5 \\ 3 & 18 & 15 & 3 \\ 1 & 6 & 5 & 1 \end{bmatrix}$ is.

(i) 4 (ii) 3 (iii) 2 (iv) 1

9. The value of a for which the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & -1 \\ a & 1 & 3 \end{bmatrix}$ is singular, is

(i) − 2 (ii) − 1 (iii) 1 (iv) 2

10. The necessary and sufficient condition that a square matrix may possess an inverse is that it be

(i) singular (ii) triangular

(iii) non-singular (iv) none of these

11. If **A** is non-singular matrix, then $(A^{-1})^{-1}$ is

(i) I (ii) A^{-1} (iii) A (iv) AA^{-1}

(Kanpur 2001)

12. If a non-singular matrix **A** is symmetric, then A^{-1} is

(i) skew-symmetric (ii) Hermitian

(iii) diagonal (iv) symmetric

[**Hint :** See § 5·11 Th. V Page 77]

Ch. VI Solution of Linear Equations

13. The system of equations

$x + 2y + z = 2,\ 3x + 5y + 5z = 4,\ 2x + 4y + 3z = 3$ has a

(i) unique solution **(ii)** infinite solution

(iii) trivial solution **(iv)** none of these

[See Ex. 5(a) P.110]

14. The system of equations

$3x - y + z = 0,\ -15x + 6y - 5z = 0,\ 5x - 2y + 2z = 0$ has a

(i) unique solution **(ii)** trivial solution

(iii) infinite solution **(iv)** none of these

15. The theorem 'every square matrix satisfies its characteristic equation' is named after

(i) Cramer **(ii)** Hamilton **(iii)** Newton **(iv)** none of them

Ch. VII Characteristic Equations of a Matrix

16. If **A** be any matrix and **I** the identity matrix, then $\mathbf{A} - \lambda\mathbf{I}$ is known as

(i) characteristic polynomial of **A** **(ii)** spectrum of **A**

(iii) characteristic matrix of **A** **(iv)** none of these

(II) TRUE OR FALSE TYPE :

Write "T" or "F" according as the following statements are true or false :

Ch. V Rank and Adjoint of a Matrix

1. The rank of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is 2. *(Agra 90)*

2. All equivalent matrices have the same rank.

3. If every minor of order p of a matrix **A** is zero, then every minor of order higher than p is not necessarily zero.

4. If at least one minor of order r of the matrix **A** is not equal to zero, then rank of $\mathbf{A} \geq r$.

5. The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 4 & 10 & 18 \end{bmatrix}$ is 2.

[See Ex. 1(a) P. 2]

6. The matrices $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 9 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ are equivalent.

7. If the elements of a row of a matrix are multiplied by a non-zero number, then the rank of the matrix remains unaffected.

8. The rank of a matrix is equal to the rank of the transposed matrix.

9. The rank of the product matrix **AB** of two marices **A** and **B** is less than the rank of either of the matrices **A** and **B**.

10. If **A** and **B** are two $n \times n$ matrices, then

$$\text{Adj}(\mathbf{AB}) \neq (\text{Adj } B) \bullet (\text{Adj } \mathbf{A})$$

11. If $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then $\mathbf{A}^{-1} = \mathbf{A}$

12. The necessary and sufficient condition that a square matrix may possess an inverse is that it be singular.

13. The inverse of transpose of a matrix is not the transpose of the inverse.

14. The inverse of the inverse of a matrix is the matrix itself.

15. If **A** is any square matrix, then $(\text{adj. } \mathbf{A})^{-1} = \text{adj}(\mathbf{A}^{-1})$.

16. Inverse of a matrix **A** exists if **A** is singular.

17. If **A** is a matrix of order $n \times n$, then $\mathbf{A}^{-1}$ is also of the same order.

Ch. VI Solution of Linear Equations

18. A consistent system of equations has no solution.

19. A consistent system of equations has either one solution or infinitely many solutions.

20. A system of m linear equations in n unknowns given by $\mathbf{AX} = \mathbf{K}$ is consistent if the matrix **A** and the augmented matrix $\mathbf{A}^*$ of the system have the same rank.

Ch. VII Characteristic Equation of a Matrix

21. The matrix $\mathbf{A} - \lambda\mathbf{I}$ is known as the characteristics matrix of **A**, when **I** is the identity matrix.

22. Every square matrix satisfies its characteristic equation.

23. The characteristic roots of a Hermitian matrix are either purely imaginary or zero.

24. The characteristic roots of real skew-symmetric matrix are purely imaginary or zero.

25. The characteristic roots of a unitary matrix are of unit modulus.

Ch. VIII Linear Dependence of Vectors

26. The set of vectors $\mathbf{V}_1 = \{1, 2, 3\}$, $\mathbf{V}_2 = \{1, 0, 1\}$ and $\mathbf{V}_3 = \{0, 1, 0\}$ are linearly dependent.

27. If there be n linearly dependent vectors, then none of these can be expressed as a linear combination of the remaining ones.

[See Th. I § 8·04 Page 214]

Ch. IX Quadratic Forms

28. $x_1^2 + 2x_2^2 + 5x_3^2 - 8x_1x_2 + 6x_1x_3$ is a quadratic form in the variables x_1, x_2, x_3.

(III) FILL IN THE BLANKS TYPE :

Fill in the blanks in the following : —

Ch. V Rank and Adjoint of Matrix

1. Rank of the null martix is .... *(Kanpur 2001)*

2. Rank of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is .... *(Kanpur 2001)*

3. If a matrix A of order $m \times n$ can be expressed as $\begin{bmatrix} \mathbf{I_r} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$, then rank of A is .... *(Meerut 2001)*

4. All .... matrices have the same rank.

5. If every minor of order p of a matrix A is zero, then every minor of order .... p is definitely zero.

6. If a matrix A does not possess any minor of order $(r + 1)$ then rank of A .... r.

7. The rank of the matrix $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}$ is ....

8. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ is ....

9. The rank of a matrix is .... to the rank of the transposed matrix.

10. The adjoint of the matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is ....

11. If A be a square matrix of order n, then

$$\mathbf{A}\ (\text{adj } \mathbf{A}) = (\text{adj } \mathbf{A})\ \mathbf{A} = \ldots\ldots\ldots\ldots$$ *(Meerut 2001)*

12. If **A** and **B** are two $n \times n$ matrices, then

$$(\text{Adj } \mathbf{B}) \bullet (\text{Adj } \mathbf{A}) = \ldots\ldots\ldots\ldots$$ [See § 5·09 Th. III P. 50]

13. If **A** be an $n \times n$ matrix and $|\mathbf{A}| \neq 0$, then

$$|\text{Adj } \mathbf{A}| = \ldots.$$ [See § 5·09 Th. II P. 50]

14. **A** and **B** are two matrices such that $\mathbf{AB} = \mathbf{I}$, then

$$\text{adj } \mathbf{B} = \ldots\ldots\ldots\ldots$$ *(Kanpur 2001)*

[Hint : We know $\mathbf{B}^{-1} = \dfrac{\text{adj } \mathbf{B}}{|\mathbf{B}|}$ and here $\mathbf{B}^{-1} = \mathbf{A}$, $\because\ \mathbf{AB} = \mathbf{I}$

$$\therefore \qquad \text{adj } \mathbf{B} = \mathbf{A} \cdot |\mathbf{B}|]$$

15. If A is a non-singular matrix, then $(\mathbf{A}^{-1})^{-1} = \ldots\ldots$, where $\mathbf{A}^{-1}$ is the inverse of A.

16. The inverse of a matrix is ....

17. A singular matrix has no ....

18. A matrix when multiplied by its inverse given the .... matrix.

Ch. VI Solution of Linear Equations

19. Inconsistent equations have .... solution.

20. A consistant system of equations has either one solution or .... solutions.

21. A homogeneous system of n linear equations in n unknowns, whose determinants of coefficients does not vanish, has only the .... solution.

22. A set of simultaneous homogeneous equations expressed in matrix form as $\mathbf{AX} = \mathbf{O}$ has non-trivial solution if .... *(Kanpur 2001)*

23. A system of m non-homogeneous linear equation $\mathbf{AX} = \mathbf{B}$ in n unknowns is called .... iff ranks of $\mathbf{A}$ and $[\mathbf{A}, \mathbf{B}]$ are equal. *(Meerut 2001)*

[See § 6·07 P. 119, § 6·06 P. 118]

Ch. VII Characteristic Equation of a Matrix

24. The set of all eigen values of the matrix $\mathbf{A}$ is called the .... of $\mathbf{A}$.

[See § 7·02 (iv) Page 160]

25. The determinant $|\mathbf{A} - \lambda\mathbf{I}|$ is called the characteristic .... of the matrix $\mathbf{A}$, when $\mathbf{I}$ is the identity matrix.

26. Every square matrix .... its characteristic equation. *(Meerut 2001)*

27. Characteristic roots of skew-Hermitian matrix are either zero or .... *(Kanpur 2001)*

28. All the characteristic roots of a real symmetric matrix are ....

29. The characteristic roots of a Hermitian matrix are all ....

30. The characteristic roots of an orthogonal matrix are of .... modulus.

31. Two .... matrices have the same characteristic roots. *(Kanpur 2001)*

[See § 7·05 Th. I Page 167]

Ch. VIII Linear Dependence of Vectors

32. The set of vectors $\mathbf{V}_1 = \{1, 2, 3\}$, $\mathbf{V}_2 = \{1, 0, 1\}$ and $\mathbf{V}_3 = \{0, 1, 0\}$ are linearly .... [See Ex. 1 P. 212]

33. The set of vectors $\mathbf{X}_1 = [2, 3, 1, -1]$, $\mathbf{X}_2 = [2, 3, 1, -2]$ and $\mathbf{X}_3 = [4, 6, 2, -3]$ is linearly .... [See Ex. 1 Page 213]

Ch. IX Quadratic Forms

34. The quadratic form in x_1, x_2, x_3 of the matrix $\begin{bmatrix} 2 & -2 & 5 \\ -2 & 3 & 0 \\ 5 & 0 & 4 \end{bmatrix}$ is ....

ANSWERS TO OBJECTIVE TYPE QUESTIONS

(I) Multiple Choice Type :

1. (iii); **2.** (ii) **3.** (iii); **4.** (ii); **5.** (ii); **6.** (iv

7. (ii) [illegible] (iii) 12. (iv);
13. (i); 1[illegible] [illegible] 16. (iii).

(II) True & False Type :

1.	F;	**2.**	T;	**3.**	F;	**4.**	T;	**5.**	F;	**6.**	F;	**7.**	T;
8.	T;	**9.**	T;	**10.**	F;	**11.**	T;	**12.**	F;	**13.**	[illegible]	**14.**	T;
15.	T;	**16.**	F;	**17.**	T;	**18.**	F;	**19.**	T;	**20.**	T;	**21.**	T;
22.	T;	**23.**	F;	**24.**	T;	**25.**	T;	**26.**	F;	**27.**	F;	**28.**	T.

(III) Fill in the blanks Type :

1. 0; **2.** 2; **3.** r; **4.** equivalent; **5.** higher than; **6.** $\leq$; **7.** 1; **8.** 1; **9.** equal;
10. $\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$;
11. $|A| \bullet I_n$; **12.** Adj (AB); **13.** $|A|^{n-1}$; **14.** $A \bullet |B|$; **15.** A; **16.** unique;
17. inverse; **18.** unit; **19.** no; **20.** infinitely many; **21.** trivial;
22. the rank of A < number of unknowns; **23.** consistent; **24.** spectrum;
25. polynominal; **26.** satisfies; **27.** purely imaginary; **28.** real; **29.** real;
30. unit; **31.** mutually reciprocal; **32.** independent; **33.** dependent;
34. $2x_1^2 + 3x_2^2 + 4x_3^2 - 4x_1 x_2 + 10x_1 x_3$.