Caveat

Everything presented here was derived by me. Some of the formulas, especially the initial ones, were derived thanks to the 'poetry spirit' (poetry means 'creation' in Ancient Greek), and one must perceive the same patterns and order I observed to understand them. However, the proof is presented, and the numerical implementation has been thoroughly tested. Finally, it's worth noting that this is still a draft.

1 Atan,Acot,Atanh,Acoth

Order	Derivative
1	$\frac{1}{1+x^2}$
2	$-\frac{2x}{(1+x^2)^2}$
3	$\frac{8x^2}{(1+x^2)^3} - \frac{2}{(1+x^2)^2}$
4	$-\frac{48x^3}{(1+x^2)^4} + \frac{24x}{(1+x^2)^3}$
5	$\frac{384x^4}{(1+x^2)^5} - \frac{288x^2}{(1+x^2)^4} + \frac{24}{(1+x^2)^3}$
6	$-\frac{3840x^5}{(1+x^2)^6} + \frac{3840x^3}{(1+x^2)^5} - \frac{720x}{(1+x^2)^4}$
7	$\frac{46080x^6}{(1+x^2)^7} - \frac{57600x^4}{(1+x^2)^6} + \frac{17280x^2}{(1+x^2)^5} - \frac{720}{(1+x^2)^4}$
8	$-\frac{645120x^7}{(1+x^2)^8} + \frac{967680x^5}{(1+x^2)^7} - \frac{403200x^3}{(1+x^2)^6} + \frac{40320x}{(1+x^2)^5}$
9	$\frac{10321920x^8}{(1+x^2)^9} - \frac{18063360x^6}{(1+x^2)^8} + \frac{9676800x^4}{(1+x^2)^7} - \frac{1612800x^2}{(1+x^2)^6} + \frac{40320}{(1+x^2)^5}$
10	$-\frac{3628800x(1-12x^2+\frac{126x^4}{5}-12x^6+x^8)}{(1+x^2)^{10}}$
11	$-\frac{3628800(1-55x^2+330x^4-462x^6+165x^8-11x^{10})}{(1+x^2)^{11}}$
12	$\frac{479001600x(1-\frac{55x^2}{3}+66x^4-66x^6+\frac{55x^8}{3}-x^{10})}{(1+x^2)^{12}}$
13	$\frac{479001600(1-78x^2+715x^4-1716x^6+1287x^8-286x^{10}+13x^{12})}{(1+x^2)^{13}}$

Table 1: Derivatives of $\arctan(x)$ up to 13th order

We can rewrite this as

Order	**Derivative**
0	$\arctan(x)$
1	$\frac{1}{x^2+1}$
2	$-\frac{2x}{(x^2+1)^2}$
3	$\frac{2\left(\frac{4x^2}{x^2+1}-1\right)}{(x^2+1)^2}$
4	$\frac{24x\left(-\frac{2x^2}{x^2+1}+1\right)}{(x^2+1)^3}$
5	$\frac{24\left(\frac{16x^4}{(x^2+1)^2}-\frac{12x^2}{x^2+1}+1\right)}{(x^2+1)^3}$
6	$\frac{240x\left(-\frac{16x^4}{(x^2+1)^2}+\frac{16x^2}{x^2+1}-3\right)}{(x^2+1)^4}$
7	$\frac{720\left(\frac{64x^6}{(x^2+1)^3}-\frac{80x^4}{(x^2+1)^2}+\frac{24x^2}{x^2+1}-1\right)}{(x^2+1)^4}$
8	$\frac{40320x\left(-\frac{16x^6}{(x^2+1)^3}+\frac{24x^4}{(x^2+1)^2}-\frac{10x^2}{x^2+1}+1\right)}{(x^2+1)^5}$
9	$\frac{40320\left(\frac{256x^8}{(x^2+1)^4}-\frac{448x^6}{(x^2+1)^3}+\frac{240x^4}{(x^2+1)^2}-\frac{40x^2}{x^2+1}+1\right)}{(x^2+1)^5}$
10	$\frac{725760x\left(-\frac{256x^8}{(x^2+1)^4}+\frac{512x^6}{(x^2+1)^3}-\frac{336x^4}{(x^2+1)^2}+\frac{80x^2}{x^2+1}-5\right)}{(x^2+1)^6}$
11	$\frac{3628800\left(\frac{1024x^{10}}{(x^2+1)^5}-\frac{2304x^8}{(x^2+1)^4}+\frac{1792x^6}{(x^2+1)^3}-\frac{560x^4}{(x^2+1)^2}+\frac{60x^2}{x^2+1}-1\right)}{(x^2+1)^6}$
12	$\frac{159667200x\left(-\frac{512x^{10}}{(x^2+1)^5}+\frac{1280x^8}{(x^2+1)^4}-\frac{1152x^6}{(x^2+1)^3}+\frac{448x^4}{(x^2+1)^2}-\frac{70x^2}{x^2+1}+3\right)}{(x^2+1)^7}$
13	$\frac{479001600\left(\frac{4096x^{12}}{(x^2+1)^6}-\frac{11264x^{10}}{(x^2+1)^5}+\frac{11520x^8}{(x^2+1)^4}-\frac{5376x^6}{(x^2+1)^3}+\frac{1120x^4}{(x^2+1)^2}-\frac{84x^2}{x^2+1}+1\right)}{(x^2+1)^7}$

Table 2: Derivatives of $\arctan(x)$ up to 13th order

Let us start first with the definition of these numbers.

Definition 1. *We define the n-th hypertriangular number in d dimension as:*

$$\mathcal{T}^d(n) := \begin{cases} 1 & \textit{if } d = 0 \\ \frac{(n)\times(n+1)\times\ldots\times(n+d-1)}{(d)!} & \textit{if } d \geq 1 \end{cases}$$

There are a lot of interesting relations for this hypertriangular numbers. I proved some of them but I could further investigate their properties. For instance, here I provide the simplest property that will be used later.

Lemma 1. *The hypertriangular numbers can be alternatively defined as*

$$\mathcal{T}^d(n) := \binom{n+d-1}{d}$$

Proof. Simple calculation. □

Now we introduce another auxiliary function to have a more compact formula

$$f(n) = (n-1) \times ((n+1) \mod 2) + 1$$

Thus we can finally write the generic derivative of $arctan(x)$.

Theorem 1.

$$arctan^{(n)}(x) = \frac{(n-1)!\left[x^{((n+1) \mod 2)}(-1)^{n+\lfloor\frac{n-3}{2}\rfloor}f(n) + \sum_{i=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^{n+i+1}2^{n-1-2i}\mathcal{T}^i(n-2i)\frac{x^{n-1-2i}}{(x^2+1)^{\lfloor\frac{n-1-2i}{2}\rfloor}}\right]}{(x^2+1)^{\lceil\frac{n+1}{2}\rceil}} \tag{1}$$

Proof. To prove the previous relation we will follow the easy path of induction. At first we observe that this relation obviously holds for the previously showed cases in the table of derivatives. Then we are gonna prove by induction the relation diving the cases in odd and even integers. In this proof we will use 1 as

$$\alpha_i^n := (-1)^{n+i+1}2^{n-1-2i}\mathcal{T}^i(n-2i)$$

We start with an odd n.

Case 1: n odd

In this case we can rewrite 1 as

$$arctan^{(n)}(x) = (n-1)!\left[\frac{(-1)^{n+\frac{n-3}{2}}}{(x^2+1)^{\frac{n+1}{2}}} + \sum_{i=0}^{\frac{n-3}{2}}\alpha_i^n\frac{x^{n-1-2i}}{(x^2+1)^{n-i}}\right]$$

We then focus on the second term to obtain an interesting relation. Indeed deriving we obtain

$$\frac{\partial}{\partial x}\left(\sum_{i=0}^{\frac{n-3}{2}}\alpha_i^n\frac{x^{n-1-2i}}{(x^2+1)^{n-i}}\right) = \sum_{i=0}^{\frac{n-3}{2}}\alpha_i^n\left(\frac{2x^{n-2i}(i-n)}{(x^2+1)^{n-i+1}} + \frac{x^{n-2i-2}(n-2i-1}{(x^2+1)^{n-i}}\right)$$

From this we see that, summing the monomials with the same orders, we get

$$\frac{x^{n-2i-2}(n-2i-1}{(x^2+1)^{n-i}}\left((n-2i-1)\alpha_i^n + 2(i+1-n)\alpha_{i+1}^n\right)$$

Thus we need to check that

$$(n-1)!(n-2i-1)\alpha_i n + 2(i+1-n)\alpha_{i+1}^n = (n!)\alpha_{i+1}^{n+1}$$

To have a better expression for the left-hand side we compute

$$\frac{\alpha_i^n}{\alpha_{i+1}^n} = -4\frac{(i+1)(n-i-1)}{(n-2i-1)(n-2i-2)}$$

Thus the left-hand side reads

$$(n-2i-1)\alpha_i^n + 2(i+1-n)\alpha_{i+1}^n = \alpha_{i+1}^n\left(-4\frac{(i+1)(n-i-1)}{(n-2i-2)} + 2(i+1-n)\right)$$

Writing now explicitly α^n_{i+1} we get

$$(n-2i-1)\alpha^n_i + 2(i+1-n)\alpha^n_{i+1} = \frac{-2n(n-i-1)}{n-2i-2}(-1)^{n+i+2}2^{n-2i-3}\mathcal{T}^{i+1}(n-2i-2)$$

Now writing explicitly also α^{n+1}_{i+1}

$$\alpha^{n+1}_{i+1} = (-1)^{n+i+3}2^{n-2i-2}\mathcal{T}^{i+1}(n-2i-1)$$

we see that we only need to check

$$\frac{\mathcal{T}^{i+1}(n-2i-2)\frac{-2n(n-i-1)}{n-2i-2}}{n\mathcal{T}^{i+1}(n-2i-1)} = 1$$

and, using the definition of the hypertriangular number and their expression the through binomial coefficient, this is easily demonstrable.

Now we are left with proving the corner cases, i.e. $i = 0$ and $i = \frac{n-3}{2}$. In the first case we easily see

$$(n-1)!\alpha^n_i \frac{2x^{n-2i}(i-n)}{(x^2+1)^{n-i+1}}\bigg|_{i=0} = -(n!)\frac{2x^n}{(x^2+1)^{n+1}} = -(n!)\alpha^{n+1}_0 \frac{2x^n}{(x^2+1)^{n+1}}$$

that is we have an equality of coefficients for this fraction of monomials. For the case $i = \frac{n-1}{3}$ we will need to take into account the derivative of the piece outside the summation of 1 $\frac{(-1)^{n+\frac{n-3}{2}}}{(x^2+1)^{\frac{n+1}{2}}}$ and sum it with the contribution coming from the summation. In the end we get

$$(n-1)!\frac{x}{x(n+1)}(x^2+1)^{\frac{n+3}{2}}\left(2\alpha^n_{\frac{n-3}{2}} - (n+1)\right)$$

and we need to check that this is equal to the part outside the summation of 1 computed with n+1. Using the previous properties this is easily checked.

Case 2: n even

Now we start to prove the case with n even. In this case the formula 1 reads

$$arctan^{(n)}(x) = (n-1)!\left[\frac{n(-1)^{n+\frac{n-3}{2}-\frac{1}{2}}}{(x^2+1)^{\frac{n+1}{2}+\frac{1}{2}}} + \sum_{i=0}^{\frac{n-3}{2}-\frac{1}{2}} \alpha^n_i \frac{x^{n-1-2i}}{(x^2+1)^{n-i}}\right]$$

. In this case we immediately see the we need to check only the corner cases since the relation between α^n_i, α^n_{i+1} and α^{n+1}_{i+1} is still valid. The case $i = 0$ is as before. We now need to check $\frac{n(-1)^{n+\frac{n-3}{2}-\frac{1}{2}}}{(x^2+1)^{\frac{n+1}{2}+\frac{1}{2}}}$. Derivating this part of the equation will give a monomial with a numerator of 0 degree and and another monomial that will be summed with the contribution coming from the summation. The first term is easily checked since we just need to compute it and compare it with the same monomial of the equation for the derivative of order $n+1$. We thus get

$$\frac{\partial}{\partial x}\left(\frac{n(-1)^{n+\frac{n-3}{2}-\frac{1}{2}}}{(x^2+1)^{\frac{n+1}{2}+\frac{1}{2}}}\right) = \frac{(-1)^{\frac{3n-4}{2}}n}{(x^2+1)^{\frac{n}{2}+1}} + \frac{(-1)^{\frac{3n-4}{2}}2nx^2(-\frac{n}{2}-1)}{(x^2+1)^{\frac{n}{2}+2}}$$

One can easily check that computing the term of the same order of $\frac{(-1)^{\frac{3n-4}{2}}(n!)}{(x^2+1)^{\frac{n}{2}+1}}$ for the formula of degree $n+1$ will have the same coefficient. We thus miss the last piece. To prove that the second piece obtained through the derivation of $\frac{n(-1)^{n+\frac{n-3}{2}-\frac{1}{2}}}{(x^2+1)^{\frac{n+1}{2}+\frac{1}{2}}}$ summed with the coefficient obtained through the derivation of the summation part will yield the correct coefficient α^{n+1}_{i+1}. To prove this we notice that

$$n(-1)^{\frac{3n}{2}} = \alpha^n_{\frac{n-3}{2}+1}$$

. So, since we know that the relation $(n-1)!((n-2i-1)\alpha_i n + 2(i+1-n)\alpha^n_{i+1}) = (n!)\alpha^{n+1}_{i+1}$, we just need to check that in our case

$$(i+1-n) = -(1+\frac{n}{2})$$

and substituting $i = \frac{n-3}{2} + \frac{1}{2}$ we achieve the result we wanted. □

Now we will prove that formula 1 can be rewritten in a simpler way.

Corollary 2. *Defining the coefficients*

$$\alpha_i^n := (-1)^{n+i+1} 2^{n-1-2i} \mathcal{T}^i(n-2i)$$

formula 1 can be rewritten as

$$arctan^{(n)}(x) = (n-1)! \left(\sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor + 1} \alpha_i^n \frac{x^{n-1-2i}}{(x^2+1)^{n-i}} \right) \tag{2}$$

Proof. The proof is easy indeed if we compute

$$\lfloor \frac{n-1-2i}{2} \rfloor + \lceil \frac{n+1}{2} \rceil = n - i$$

. Moreover we now need to prove that $x^{((n+1) \mod 2)}(-1)^{n+\lfloor \frac{n-3}{2} \rfloor} f(n)$ in 1 can be written as $\alpha_{\lfloor \frac{n-3}{2} \rfloor + 1}$. When n is even we already now that this is true since we used it in the proof of 1. We then need just to prove it when n is odd. In this case we see

$$x^{((n+1) \mod 2)}(-1)^{n+\lfloor \frac{n-3}{2} \rfloor} f(n) = (-1)^{\frac{3n-3}{2}}$$

and

$$\begin{aligned} \alpha^n_{\lfloor \frac{n-3}{2} \rfloor + 1} &= && (3) \\ &= \alpha^n_{\frac{n-1}{2}} && (4) \\ &= (-1)^{\frac{3n+1}{2}} 2^0 \mathcal{T}^{\frac{n-1}{2}}(1) && (5) \\ &= (-1)^{\frac{3n-3}{2}} \times 1 && (6) \end{aligned}$$

The last piece to be checked is that the numerator has not any power of x. This is easily checked

$$x^{n-1-2i}\Big|_{i=\frac{n-1}{2}} = x^0$$

For completeness we write the proof also for the case of n even. In this case we have

$$x^{((n+1) \mod 2)}(-1)^{n+\lfloor \frac{n-3}{2} \rfloor} f(n) = x(-1)^{\frac{3n-4}{2}} n$$

and

$$\begin{aligned} \alpha^n_{\lfloor \frac{n-3}{2} \rfloor + 1} &= && (7) \\ &= \alpha^n_{\frac{n-2}{2}} && (8) \\ &= (-1)^{\frac{3n}{2}} 2^1 \mathcal{T}^{\frac{n-2}{2}}(2) && (9) \\ &= (-1)^{\frac{3n-4}{2}} 2 \binom{\frac{n}{2}}{\frac{n-2}{2}} && (10) \\ &= (-1)^{\frac{3n-4}{2}} 2 \frac{n}{2} && (11) \\ &= (-1)^{\frac{3n-4}{2}} n && (12) \end{aligned}$$

And in the end we check that

$$x^{n-2i-1}\Big|_{i=\frac{n-2}{2}} = x^1$$

□

Thus we have some interesting results:

Corollary 3. *The Taylor expansion of atan around x reads:*

$$\mathcal{T}^n_{atan(x)}(y) = atan(x) + \sum_{n=1}^{\infty} \frac{\left[\sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor +1} (-1)^{n+i+1} 2^{n-1-2i} \mathcal{T}^i(n-2i) x^{n-1-2i}\right]}{n(x^2+1)^{n-i}} y^n \tag{13}$$

Proof. Obvious. □

In a trivial way one can get the expansion of *arccot* around x in the same way, obtaining (only a minus sign difference):

Theorem 4. *The Taylor expansion of acot around x reads:*

$$\mathcal{T}^n_{acot(x)}(y) = acot(x) + \sum_{n=1}^{\infty} \frac{\left[\sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor +1} (-1)^{n+i} 2^{n-1-2i} \mathcal{T}^i(n-2i) x^{n-1-2i}\right]}{n(x^2+1)^{n-i}} y^n \tag{14}$$

Proof. Same proof as theorem 1 □

One can extend these formulas also to the hyperbolic counterparts to obtain:

Theorem 5. *The Taylor expansion of atanh around x reads:*

$$\mathcal{T}^n_{atanh(x)}(y) = atanh(x) + \sum_{n=1}^{\infty} \frac{\left[\sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor +1} (-1)^{n+i} 2^{n-1-2i} \binom{n-i-1}{i} x^{n-1-2i}\right]}{n(x^2-1)^{n-i}} y^n \tag{15}$$

Proof. Same proof as theorem 1 □

Theorem 6. *The Taylor expansion of acoth around x reads:*

$$\mathcal{T}^n_{acoth(x)}(y) = acoth(x) + \sum_{n=1}^{\infty} \frac{\left[\sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor +1} (-1)^{n+i} 2^{n-1-2i} \binom{n-i-1}{i} x^{n-1-2i}\right]}{n(x^2-1)^{n-i}} y^n \tag{16}$$

Proof. Same proof as theorem 1 □

Another way to find the formulas is to use the complex formulation of the previous functions. Let take *atan* again as an example. We then have that:

$$\begin{aligned}
\frac{d^n}{dx^n}\arctan(x) &= \frac{d^n}{dx^n}\frac{i}{2}log\left(\frac{x-i}{x+i}\right) \\
&= \frac{1}{2i}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{x-i} - \frac{1}{x+i}\right) \\
&= \frac{(-1)^{n-1}(n-1)!}{2i}\left(\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right) \\
&= \frac{(-1)^{n-1}(n-1)!}{2i} \cdot \frac{(x+i)^n - (x-i)^n}{(x^2+1)^n} \\
&= \frac{(-1)^{n-1}(n-1)!}{2i} \cdot \frac{\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\left(i^k - (-i)^k\right)}{(x^2+1)^n} \\
&= \frac{(n-1)!}{(x^2+1)^n}\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n}{2k+1}x^{n-2k-1}(-1)^{n+k-1} \quad (1,3)
\end{aligned}$$

This formula seems really similar to the previous one but actually we a substantial difference: we notice that here we are summing powers of x without dividing them by powers of (x^2+1). Also the coefficient multiplying the powers of x is different. Thanks to these two different expressions we can write a new formula for the binomial coefficient:

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\left(\binom{n}{2k+1}-2^{n-1-2k}\binom{n-i-k}{i}(x^2+1)^k\right)\frac{x^{n-2k-1}}{(x^2+1)^n}=0$$

Expanding now $(x^2+1)^k$ we find that $\forall k$ and $\forall n$ (with $k \leq \lfloor\frac{n-1}{2}\rfloor$):

$$\binom{n}{2k+1}=\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor-k}(-1)^i 2^{n-1-2(k+i)}\binom{n-(k+i)-1}{k+i}\binom{k+i}{k}$$

We can follow the exact same procedure for $acot$. In this case we get :

$$\begin{aligned}\frac{d^n}{dx^n}acot(x)&=\frac{d^n}{dx^n}\frac{i}{2}log\left(\frac{x-i}{x+i}\right)\\&=-\frac{d^n}{dx^n}\frac{1}{2i}log\left(\frac{x-i}{x+i}\right)\\&\vdots\\&=\frac{(n-1)!}{(x^2+1)^n}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}x^{n-2k-1}(-1)^{n+k}\quad(1,3)\end{aligned}$$

We then proceed to obtain the other 2 alternantive expansions for $atanh$ and $acoth$:

$$\begin{aligned}\frac{d^n}{dx^n}atanh(x)&=\frac{d^n}{dx^n}\frac{1}{2}log\left(\frac{1+x}{1-x}\right)\qquad if\quad |x|<1\\&=\frac{1}{2}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)\\&=\frac{(-1)^{n-1}(n-1)!}{2i}\left(\frac{1}{(1+x)^n}-\frac{}{(x-1)^n}\right)\\&=\frac{(-1)^{n-1}(n-1)!}{2}\cdot\frac{(x-1)^n-(x+1)^n}{(x^2-1)^n}\\&=\frac{(-1)^{n-1}(n-1)!}{2}\cdot\frac{\sum_{k=0}^{n}\binom{n}{k}(x)^{n-k}\left((-1)^k-1\right)}{(x^2-1)^n}\\&=\frac{(n-1)!}{(x^2-1)^n}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}x^{n-2k-1}(-1)^{n+k}\quad(1,3)\end{aligned}$$

and

$$\begin{aligned}\frac{d^n}{dx^n}acoth(x)&=\frac{d^n}{dx^n}\frac{1}{2}log\left(\frac{1+x}{1-x}\right)\qquad if\quad |x|>1\\&\vdots\\&=\frac{(n-1)!}{(x^2-1)^n}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}x^{n-2k-1}(-1)^{n+k}\quad(1,3)\end{aligned}$$

Order	Derivative
0	$\operatorname{acosh}(x)$
1	$\frac{1}{\sqrt{(x-1)(x+1)}}$
2	$-\frac{\frac{1}{x+1}+\frac{1}{x-1}}{2\sqrt{(x-1)(x+1)}}$
3	$\frac{\frac{3}{(x+1)^2}+\frac{2}{(x-1)(x+1)}+\frac{3}{(x-1)^2}}{4\sqrt{(x-1)(x+1)}}$
4	$-\frac{3\left(\frac{5}{(x+1)^3}+\frac{3}{(x-1)(x+1)^2}+\frac{3}{(x-1)^2(x+1)}+\frac{5}{(x-1)^3}\right)}{8\sqrt{(x-1)(x+1)}}$
5	$\frac{3\left(\frac{35}{(x+1)^4}+\frac{20}{(x-1)(x+1)^3}+\frac{18}{(x-1)^2(x+1)^2}+\frac{20}{(x-1)^3(x+1)}+\frac{35}{(x-1)^4}\right)}{16\sqrt{(x-1)(x+1)}}$
6	$-\frac{15\left(\frac{63}{(x+1)^5}+\frac{35}{(x-1)(x+1)^4}+\frac{30}{(x-1)^2(x+1)^3}+\frac{30}{(x-1)^3(x+1)^2}+\frac{35}{(x-1)^4(x+1)}+\frac{63}{(x-1)^5}\right)}{32\sqrt{(x-1)(x+1)}}$
7	$\frac{45\left(\frac{231}{(x+1)^6}+\frac{126}{(x-1)(x+1)^5}+\frac{105}{(x-1)^2(x+1)^4}+\frac{100}{(x-1)^3(x+1)^3}+\frac{105}{(x-1)^4(x+1)^2}+\frac{126}{(x-1)^5(x+1)}+\frac{231}{(x-1)^6}\right)}{64\sqrt{(x-1)(x+1)}}$
8	$-\frac{315\left(\frac{429}{(x+1)^7}+\frac{231}{(x-1)(x+1)^6}+\frac{189}{(x-1)^2(x+1)^5}+\frac{175}{(x-1)^3(x+1)^4}+\frac{175}{(x-1)^4(x+1)^3}+\frac{189}{(x-1)^5(x+1)^2}+\frac{231}{(x-1)^6(x+1)}+\frac{429}{(x-1)^7}\right)}{128\sqrt{(x-1)(x+1)}}$
9	$\frac{315\left(\frac{6435}{(x+1)^8}+\frac{3432}{(x-1)(x+1)^7}+\frac{2772}{(x-1)^2(x+1)^6}+\frac{2520}{(x-1)^3(x+1)^5}+\frac{2450}{(x-1)^4(x+1)^4}+\frac{2520}{(x-1)^5(x+1)^3}+\frac{2772}{(x-1)^6(x+1)^2}+\frac{3432}{(x-1)^7(x+1)}+\frac{6435}{(x-1)^8}\right)}{256\sqrt{(x-1)(x+1)}}$

Table 3: Derivatives of arcosh(x) up to 9th order

2 Acosh

We now analyze acosh(x)

In the end the analytical generic formula for derivations of $\operatorname{arcosh}(x)$ reads:

Theorem 7.

$$\operatorname{arcosh}^{(\mathrm{n})}(x) = \frac{(-1)^{(n+1)}}{2^{(n-1)}(x^2-1)^{\frac{1}{2}}} \sum_{i=0}^{\lfloor\frac{(n-1)}{2}\rfloor} \frac{(2n-3-2i)!!\mathcal{T}^i(n-i)(2i-1)!!}{2^{\delta(n-2i-1)}} \left(\frac{1}{(x+1)^{n-i-1}(x-1)^i} + \frac{1}{(x+1)^i(x-1)^{n-i-1}}\right) \tag{17}$$

where $\delta(x)$ is the Kronecker delta.

Proof. As before we define

$$\alpha_i^n := (2n-3-2i)!!\mathcal{T}^i(n-i)(2i-1)!!$$

Then we proceed as before by induction. First we can trivially see that the formula is correct for a starting n_0 case. Then we will prove that if the formula is valid for the n-th order derivative, it is also valid for the $(n+1)$-th order derivative.

Computing the derivative of the formula above we see indeed that

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{1}{(x+1)^{n-i-\frac{1}{2}}(x-1)^{i+\frac{1}{2}}} + \frac{1}{(x+1)^{i+\frac{1}{2}}(x-1)^{n-i-\frac{1}{2}}}\right) \\
&= \left(\frac{1}{2}+i-n\right)\left(\frac{1}{(x-1)^{n-i+\frac{1}{2}}(x+1)^{i+\frac{1}{2}}} + \frac{1}{(x+1)^{n-i+\frac{1}{2}}(x-1)^{i+\frac{1}{2}}}\right) \\
&\quad + \left(-i-\frac{1}{2}\right)\left(\frac{1}{(x-1)^{n-i-\frac{1}{2}}(x+1)^{i+\frac{3}{2}}} + \frac{1}{(x+1)^{n-i-\frac{1}{2}}(x-1)^{i+\frac{3}{2}}}\right)
\end{aligned}$$

From this we notice that the relation we need to check is

$$\alpha_{i+1}^{n+1} = \left(-\frac{1}{2}-i\right)\alpha_i^n + \left(\frac{3}{2}+i-n\right)\alpha_{i+1}^n$$

□

3 Faddeeva function

The Faddeeva function, denoted as $f(z)$, is a complex function widely used in physics and engineering, particularly in problems involving wave propagation and scattering in complex media. It is defined by the expression:

$$f(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}\,dt\right),$$

where z is a complex variable. This function includes an exponential decay term, e^{-z^2}, and an integral that introduces a complex component, leading to its wide applicability in computational and theoretical settings.

To analyze $f(z)$ and related functions, we define two sequences of polynomials, $p_n(z)$ and $q_n(z)$, that follow specific recursive rules. The definitions are structured as follows:

1. The initial values for the sequences are:

$$p_0 = 1, \quad q_1 = 1, \quad p_n = 0 \text{ for } n < 0, \quad \text{and} \quad q_n = 0 \text{ for } n < 1.$$

2. For $n \geq 1$, the polynomials $p_n(z)$ are defined recursively as:

$$p_n = -2z \cdot p_{n-1} + p'_{n-1},$$

where p'_{n-1} denotes the derivative of p_{n-1} with respect to z.

3. The polynomials $q_n(z)$ are defined recursively as a linear combination of the $p_k(z)$ polynomials:

$$q_n = \sum_{k=0}^{n-1} c_k \cdot p_k,$$

where each coefficient c_k depends on the previous recursive steps and the properties of q_n and p_n.

Using these recursive definitions:

$$p_n = \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ -2z \cdot p_{n-1} + p'_{n-1} = -2z \cdot p_{n-1} - 2(n-1)p_{n-2} & \text{for } n \geq 1, \end{cases}$$

$$q_n = \begin{cases} 0 & \text{if } n < 1, \\ 1 & \text{if } n = 1, \\ \sum_{i=0}^{n-1} 2^{n-i-1} \frac{i!}{(2i-n+1)!} \cdot p_{2i-n-1} & \text{for } n \geq 2. \end{cases}$$

Theorem 8.

$$f^{(n)}(z) = p_n(z)f(z) + q_n(z)\frac{2i}{\sqrt{\pi}} \tag{18}$$

Proof. Simple calculation, either by induction or deductively. □

4 Sinc and Sinhc

The numerical instability of these two functions is generated by the division by x when we get near to 0. This problem can be solved by a perturbative approach since the problem is related to only a single point where the functions are differentiable an infinite amount of times. Ind doing so our strategy is as follows: we expand the functions and their derivatives near to 0 and then we try to find a pattern in order to compute as many derivatives as needed. For Sinc and Sinhc this is easy since, when computing the expansion of the derivative we notice that to obtain continuity at $1e-16$ it is more than sufficient to store the first 7 coefficients of the expansion of the first and second derivative (it corresponds respectively to go up to the 13th and 12th order in their expansion) . Indeed, we notice that having the fist 7 coefficients is enough to reach the machine precision up to the limit we impose for the domain of this perturbative approach. Moreover the coefficients are always of the form $\frac{1}{x}$ where x has a formula related to factorials. So storing the coefficients means storing the denominators.
Specifically, if we give an explicit value to everything we get: Let:

- a_0 be the point of evaluation.
- ord_coef[i] the array of coefficients of the tpsa $i \in \{0, \ldots, \texttt{to}\}$.
- The following arrays:

$$\text{odd\_coef}_0 = [-1, 30, -840, 45360, -3991680, 518918400, -93405312000]$$

$$\text{add\_odd} = [2, 12, 240, 10080, 725760, 79833600, 12454041600]$$

z

The update rules of the coefficients for $n > 0$ read:

$$\text{odd\_coef}_{n+1}[i] = (-1)^{i+1} \cdot \text{add\_odd}[i] + \text{odd\_coef}_n[i]$$

Then we have:

$$\frac{Sinc^{(2n+1)}(a0)}{(2n+1)!} = \text{ord\_coef}[2n+1] = \frac{(-1)^{\lfloor \frac{2n+1}{2} \rfloor}}{(2n+1)!} \left(\sum_{k=0}^{6} \frac{a_0^{2k+1}}{\text{odd\_coef}_n[k]} \right)$$

$$\frac{Sinc^{(2n+2)}(a0)}{(2n+2)!} = \text{ord\_coef}[2n+2] = \frac{(-1)^{\lfloor \frac{2n+1}{2} \rfloor}}{(2n+2)!} \left(\sum_{k=0}^{6} \frac{(2k+1) \cdot a_0^{2k}}{\text{odd\_coef}_n[k]} \right)$$

For Sinhc the formulas slightly differ. The coefficients become all positive

- The following arrays:

$$\text{odd\_coef}_0 = [1, 30, 840, 45360, 3991680, 518918400, 93405312000]$$

$$\text{add\_odd} = [2, 12, 240, 10080, 725760, 79833600, 12454041600]$$

the update becomes

$$\text{odd\_coef}_{n+1}[i] = \text{add\_odd}[i] + \text{odd\_coef}_n[i]$$

and the formulas read

$$\frac{Sinhc^{(2n+1)}(a0)}{(2n+1)!} = \text{ord\_coef}[2n+1] = \frac{1}{(2n+1)!} \left(\sum_{k=0}^{6} \frac{a_0^{2k+1}}{\text{odd\_coef}_n[k]} \right)$$

$$\frac{Sinhc^{(2n+2)}(a0)}{(2n+2)!} = \text{ord\_coef}[2n+2] = \frac{1}{(2n+2)!} \left(\sum_{k=0}^{6} \frac{(2k+1) \cdot a_0^{2k}}{\text{odd\_coef}_n[k]} \right)$$

5 Asinc and Asinhc

For Asinc and Asinhc we will proceed in a similar manner to Sinc and Sinhc since the problem is of the same nature (actually Asinhc has one more source of instability due to the singularity of the derivatives in 1). In this, as done before, we first compute the derivatives andtheir expansion around 0. Then one can easily prove that the following pattern holds: first we define the coefficientts. First we examine the case of Asinc. Let:

- a_0 be the point where we evaluate the function.
- ord_coef[i] the array of coefficients of the tpsa $i \in \{0, \ldots, \texttt{to}\}$.
- ord is a predefined order used to control the accuracy of the series expansion (in the code set to 80).

- temp_coef[i] represents temporary coefficients calculated iteratively for different orders i, with the following recurrence relation:

$$\text{temp\_coef}[0] = \frac{1}{3}, \quad \text{temp\_coef}[i] = \text{temp\_coef}[i-1] \cdot \frac{(2i+1)^2}{i(4i+6)} \quad \text{for } i \geq 1$$

$$\text{fact} = o(o+1)$$

- mult is a multiplicative factor, dependent on i and o, given by:

$$\text{mult} = \begin{cases} 1, & \text{if } o = 1 \\ \frac{(2i+o)^3}{(2i+o+2)}, & \text{otherwise} \end{cases}$$

- The contribution to the coefficients ord_coef[o] is computed iteratively as a summation over the terms i:

$$\frac{Asinc^{2o+1}(a0)}{(2o+1)!} = \text{ord\_coef}[2o+1] = \sum_{i=0}^{\text{ord}} \frac{a_0^{2i+1} \cdot \text{temp\_coef}[i] \cdot \text{mult}}{(2o+1)!}$$

and

$$\frac{Asinc^{2o+2}(a0)}{(2o+1)!} = \text{ord\_coef}[2o] = \sum_{i=0}^{\text{ord}} \frac{a_0^{2i} \cdot \text{temp\_coef}[i] \cdot (2i+1) \cdot \text{mult}}{(2o)!}$$

For Asinhc we only need to change temp_coef as follows:

$$\text{temp\_coef}[0] = -\frac{1}{3}, \quad \text{temp\_coef}[i] = -\text{temp\_coef}[i-1] \cdot \frac{(2i+1)^2}{i(4i+6)} \quad \text{for } i \geq 1$$

With this trick Asinc is passing al the tests with very few ϵ ($< 10\epsilon$). For Asinhc we have some more difficulties because it is unstable also near to 1 and this causes the general performance to worsen in the interval (0,1) with a performance that is described in the table.