Textbook Sections: 15.8, 16.1 and 16.2

Topics: Triple Integrals in Spherical Coordinates, Vector Fields and Line Integrals

Instructions: Try each of the following problems, show the detail of your work.

Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

TRIPLE INTEGRALS IN SPHERICAL COORDINATES:

1. Evaluate $\iiint_E (x^2+y^2)\, dV$ where E is the region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

We first express the surface $x^2+y^2+z^2=4$ in spherical coordinate with $\rho=2$, and the surface $x^2+y^2+z^2=9$ in spherical coordinate with $\rho=3$. In spherical coordinates, the function $f(x,y,z)z=x^2+y^2$ becomes

$$\begin{aligned} f(\rho,\theta,\phi) &= \rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta \\ &= \rho^2\sin^2\phi \end{aligned}$$

We can describe E in spherical coordinates as follows.

$$E=\{(\rho,\theta,\phi): 2\le\rho\le 3, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$$

and the integral becomes

$$\iiint_E (x^2+y^2)dV = \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^2\sin^2\phi\rho^2\sin\phi d\rho d\phi d\theta = \left(\int_0^{2\pi} 1 d\theta\right)\left(\int_0^{\pi}\sin^3\phi d\phi\right)\left(\frac{\rho^5}{5}\bigg|_2^3\right)$$

$$= (2\pi)\frac{3^5-2^5}{5}\left(\int_0^{\pi}(1-\cos^2\phi)\sin\phi d\phi\right) = \frac{1688\pi}{15}$$

2. Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$.

We first express the surface $x^2+y^2+z^2=4$ in spherical coordinate with $\rho=2$. The equation of the cone $z=\sqrt{x^2+y^2}$ in spherical coordinates is $\phi=\pi/4$. We can describe E in spherical coordinates as follows.

$$E=\left\{(\rho,\theta,\phi): 0\le\rho\le 2, 0\le\theta\le 2\pi, \frac{\pi}{4}\le\phi\le\frac{\pi}{2}\right\}$$

and the volume

$$V(E)=\iiint_E dV = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^2 \rho^2\sin\phi d\rho d\phi d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_{\frac{\pi}{4}}^{\pi/2}\sin\phi\ d\phi\right)\left(\frac{\rho^3}{3}\bigg|_0^2\right)$$

$$= (2\pi)\frac{8}{3}\left(-\cos\phi\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right) = \frac{16\pi}{3}\left(0-\frac{\sqrt{2}}{2}\right) = \frac{8\sqrt{2}\pi}{3}$$

3. Find the volume of the part of the ball $\rho\le a$ that lies between the cones $\phi=\frac{\pi}{6}$ and $\phi=\frac{\pi}{3}$.

The solid region is given by

$$E = \left\{ (\rho, \theta, \phi) \mid 0 \leq \rho \leq a, 0 \leq \theta \leq 2\pi, \frac{\pi}{6} \leq \phi \leq \frac{\pi}{3} \right\}$$

and its volume is

$$\begin{aligned} V &= \iiint_E dV = \int_{\pi/6}^{\pi/3} \int_0^{2\pi} \int_0^a \rho^2 \sin\phi d\rho d\theta d\phi = \int_{\pi/6}^{\pi/3} \sin\phi d\phi \int_0^{2\pi} d\theta \int_0^a \rho^2 d\rho \\ &= [-\cos\phi]_{\pi/6}^{\pi/3} [\theta]_0^{2\pi} \left[\frac{1}{3}\rho^3\right]_0^a = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)(2\pi)\left(\frac{1}{3}a^3\right) = \frac{\sqrt{3}-1}{3}\pi a^3 \end{aligned}$$

4. Evaluate the following integrals by switching to spherical coordinates.

(a) $\displaystyle\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2 z + y^2 z + z^3)\, dz\, dx\, dy.$

From bounds, we can see that:

$$\begin{aligned} -\sqrt{a^2-x^2-y^2} &\leq z \leq \sqrt{a^2-x^2-y^2} \\ -\sqrt{a^2-y^2} &\leq x \leq \sqrt{a^2-y^2} \\ -a &\leq y \leq a \end{aligned}$$

describes a region E that is the entire sphere of radius a.

The region of integration is the solid sphere $x^2+y^2+z^2 \leq a^2$ so $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$, and $0 \leq \rho \leq a$. Also $f(x,y,z) = x^2z + y^2z + z^3 = \left(x^2+y^2+z^2\right)z = \rho^2 z = \rho^3$ cos ϕ. In spherical coordinates, the integral becomes

$$\int_0^{\pi} \int_0^{2\pi} \int_0^a \left(\rho^3 \cos\phi\right) \rho^2 \sin\phi d\rho d\theta d\phi = \int_0^{\pi} \sin\phi \cos\phi d\phi \int_0^{2\pi} d\theta \int_0^a \rho^5 d\rho = \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi} [\theta]_0^{2\pi} \left[\frac{1}{6}\rho^6\right]_0^a = 0$$

(b) $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\, dz\, dy\, dx.$

The integration bounds of the variables give us the region E:

$$\begin{aligned} 2-\sqrt{4-x^2-y^2} &\leq z \leq 2+\sqrt{4-x^2-y^2} \\ -\sqrt{4^2-y^2} &\leq y \leq \sqrt{4-x^2} \\ -2 &\leq x \leq 2 \end{aligned}$$

By using the bounds for z, we find the equation of a sphere of radius 2 with center $(0,0,2)$.

$$\begin{aligned} z &= 2+\sqrt{4-x^2-y^2} \\ (z-z)^2 &= 4-x^2-y^2 \\ x^2+y^2+(z-z)^2 &= 4 \end{aligned}$$

Because this sphere is above the xy-plane, we have that $0 \leq \phi \leq \pi/2$. Then, we express the equation of the sphere $x^2+y^2+z^2-4z+4=4$ using spherical coordinates:

$$\Rightarrow \begin{aligned} \rho^2 - 4\rho\cos\phi &= 0 \\ \rho &= 4\cos\phi \end{aligned}$$

Thus, $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2}$, and $0 \leq \rho \leq 4\cos\phi$. Also, $f(x,y,z) = \left(x^2+y^2+z^2\right)^{3/2} = \left(\rho^2\right)^{3/2} = \rho^3$, so the integral becomes

$$\begin{aligned}
\int_0^{\pi/2}\int_0^{2\pi}\int_0^{4\cos\phi} \left(\rho^3\right)\rho^2\sin\phi\, d\rho d\theta d\phi &= \int_0^{\pi/2}\int_0^{2\pi} \sin\phi\left[\frac{1}{6}\rho^6\right]_{\rho=0}^{\rho=4\cos\phi} d\theta d\phi \\
&= \frac{1}{6}\int_0^{\pi/2}\int_0^{2\pi} \sin\phi\left(4096\cos^6\phi\right) d\theta d\phi \\
&= \frac{1}{6}(4096)\int_0^{\pi/2}\cos^6\phi\sin\phi d\phi \int_0^{2\pi} d\theta = \frac{2048}{3}\left[-\frac{1}{7}\cos^7\phi\right]_0^{\pi/2}[\theta]_0^{2\pi} \\
&= \frac{2048}{3}\left(\frac{1}{7}\right)(2\pi) = \frac{4096\pi}{21}
\end{aligned}$$

5. A solid lies inside the sphere $x^2+y^2+z^2=4z$ and outside the cone $z=\sqrt{x^2+y^2}$. Write a description of the solid in terms of inequalities involving spherical coordinates.

The sphere $x^2+y^2+z^2=4z \Leftrightarrow x^2+y^2+z^2-4z+4=4 \Leftrightarrow x^2+y^2+(z-2)^2=2^2$ is a sphere with radius 2 centered at $(0,0,2)$.
In spherical coordinates, we have $\rho^2 = 4\rho\cos\phi \Leftrightarrow \rho^2 - 4\rho\cos\phi = 0 \Leftrightarrow \rho = 0$ or $\rho = 4\cos\phi$, so "inside the sphere" is described by $0 \leq \rho \leq 4\cos\phi$.
The cone $z=\sqrt{x^2+y^2}$ is described by $\phi = \frac{\pi}{4}$, so "outside the cone" is described by $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$.
We can describe the solid region E in spherical coordinates as follows.

$$E = \left\{(\rho,\theta,\phi) : 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}, 0 \leq \rho \leq 4\cos\phi\right\}$$

VECTOR FIELDS

6. Find the gradient vector ∇f of f.

(a) $f(x,y) = 2y\sin(xy)$

Since $f(x,y) = 2y\sin(xy)$, then

$$\begin{aligned}
\nabla f(x,y) &= f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} \\
&= (2y\cos(xy)\cdot y)\mathbf{i} + (2y\cdot x\cos(xy) + 2\sin(xy)\cdot 1)\,\mathbf{j} \\
&= 2y^2\cos(xy)\mathbf{i} + (2xy\cos(xy) + 2\sin(xy))\,\mathbf{j}
\end{aligned}$$

(b) $f(x,y,z) = x^3ye^{y/z}$

Since $f(x,y,z) = x^3ye^{y/z}$, then

$$\begin{aligned}
\nabla f(x,y,z) &= f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k} \\
&= 3x^2ye^{y/z}\mathbf{i} + x^3\left(ye^{y/z}\frac{1}{z} + e^{y/z}\cdot 1\right)\mathbf{j} + \left(x^3ye^{y/z}\left(-\frac{y}{z^2}\right)\right)\mathbf{k} \\
&= 3x^2ye^{y/z}\mathbf{i} + x^3e^{y/z}\left(\frac{y}{z}+1\right)\mathbf{j} - \frac{x^3y^2}{z^2}e^{y/z}\mathbf{k}
\end{aligned}$$

LINE INTEGRALS:

7. Evaluate $\int_C (x/y)ds$, where $C: x = t^3$, $y = t^4$, $1 \le t \le 2$.

Here $\vec{r}(t) = \langle t^3, t^4 \rangle$, so $\vec{r}'(t) = \langle 3t^2, 4t^3 \rangle$, and since $ds = \|\vec{r}'(t)\|dt$, we have that:

$$\int_C \frac{x}{y} ds = \int_1^2 \frac{x(t)}{y(t)} \|\vec{r}'(t)\| dt = \int_1^2 \frac{1}{t}\sqrt{9t^4 + 16t^6}dt = \int_1^2 t\sqrt{9 + 16t^2}dt.$$

Using the substitution $u = 9 + 16t^2$, the last integral above becomes:

$$\frac{1}{32}\int_{25}^{73} \sqrt{u}du = \frac{1}{48}u^{\frac{3}{2}}\Big|_{25}^{73} = \frac{1}{48}\left(73^{\frac{3}{2}} - 25^{\frac{3}{2}}\right).$$

8. Evaluate $\int_C ydx + zdy + xdz$, where $C: x = \sqrt{t}$, $y = t$, $z = t^2$, $1 \le t \le 4$.

Here we can view:

$$\int_C ydx + zdy + xdz = \int_C \langle y, z, x \rangle \cdot \langle dx, dy, dz \rangle = \int_C \vec{F} \cdot d\vec{r},$$

where $\vec{F}(x, y, z) = \langle y, z, x \rangle$ and $d\vec{r} = \langle dx, dy, dz \rangle$. Our parametrization for C is given by $\vec{r}(t) = \langle \sqrt{t}, t, t^2 \rangle$. Since $\frac{d\vec{r}}{dt} = \vec{r}'(t)$, we have that $d\vec{r} = \vec{r}'(t)dt$. Hence:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_1^4 \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}'(t)dt \\
&= \int_1^4 \langle t, t^2, \sqrt{t} \rangle \cdot \left\langle \frac{1}{2\sqrt{t}}, 1, 2t \right\rangle dt \\
&= \int_1^4 \left(\frac{\sqrt{t}}{2} + t^2 + 2t^{\frac{3}{2}}\right) dt \\
&= \left(\frac{t^{\frac{3}{2}}}{3} + \frac{t^3}{3} + \frac{4t^{\frac{5}{2}}}{5}\right)\Bigg|_1^4 \\
&= \left(\frac{8}{3} + \frac{64}{3} + \frac{128}{5}\right) - \left(\frac{1}{3} + \frac{1}{3} + \frac{4}{5}\right) \\
&= \frac{70}{3} + \frac{124}{5}.
\end{aligned}$$

9. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy^2\mathbf{i} - x^2\mathbf{j}$, $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$, and $0 \le t \le 1$.

$$
\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 \vec{F}(x(t),y(t),z(t))\cdot \vec{r}\,'(t)dt \\
&= \int_0^1 \langle t^3(t^2)^2, -(t^3)^2\rangle \cdot \langle 3t^2, 2t\rangle dt \\
&= \int_0^1 (3t^9 - 2t^7)dt \\
&= \left(\frac{3t^{10}}{10} - \frac{t^8}{4}\right)\Bigg|_0^1 \\
&= \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.
\end{aligned}
$$

10. Find the work done by the force field $\mathbf{F}(x,y) = x\mathbf{i} + (y+2)\mathbf{j}$ in moving an object along an arch of the cycloid $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1-\cos t)\mathbf{j}$, $0 \le t \le 2\pi$

The work done by a force in moving an object along a curve is given by the line integral $W = \int_C \vec{F}\cdot d\vec{r}$, where C is the curve with parametrization $\vec{r}(t)$.Then, we have

$$
\begin{aligned}
W &= \int_C \vec{F}\cdot d\vec{r} \\
&= \int_0^{2\pi} \vec{F}(x(t),y(t),z(t))\cdot \vec{r}\,'(t)dt \\
&= \int_0^{2\pi} \langle t-\sin(t), 1-\cos(t)+2\rangle \cdot \langle 1-\cos(t), \sin(t)\rangle dt \\
&= \int_0^{2\pi} (t - t\cos(t) - \sin(t) + \sin(t)\cos(t) + \sin(t) - \cos(t)\sin(t) + 2\sin(t))dt \\
&= \int_0^{2\pi} (t - t\cos(t) + 2\sin(t))dt \\
&= \int_0^{2\pi} tdt - \int_0^{2\pi} t\cos(t)dt + 2\int_0^{2\pi} \sin(t)dt \\
&= \int_0^{2\pi} tdt - t\sin(t)\Big|_0^{2\pi} + \int_0^{2\pi}\sin(t)dt + 2\int_0^{2\pi}\sin(t)dt \qquad \text{(using parts)} \\
&= \frac{t^2}{2}\Bigg|_0^{2\pi} - t\sin(t)\Big|_0^{2\pi} - 3\cos(t)\Big|_0^{2\pi} = 2\pi^2.
\end{aligned}
$$

Suggested Textbook Problems

Section 15.8	1-13, 15, 17-29, 32, 43-45
Section 16.1	25-28
Section 16.2	1-18, 21-24, 41-45

SOME USEFUL DEFINITIONS, THEOREMS AND NOTATION:

Spherical Coordinates

To change from Cartesian coordinates to spherical coordinates, use the following transformation:

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi,$$

and we note that the distance formula shows that $\rho^2 = x^2 + y^2 + z^2$. In particular, we have that

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \; d\rho \; d\theta \; d\phi,$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) | c \le \phi \le d, \alpha \le \theta \le \beta, a \le \rho \le b\}.$$

Definition of the Vector Field

Let E be a subset of $\mathbb{R}^3$. A **vector field** on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Definition of the Conservative Vector Field

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a potential function for $\mathbf{F}$.

Line Integral with Respect to Arc Length

If f is defined on a smooth curve C given by $x = x(t)$, $y = y(t)$, and $a \le t \le b$, then the *line integral of f along C* is

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Line Integral with Respect to x or y

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of $t : x = x(t)$, $y = y(t)$, $dx = x'(t)dt$, $dy = y'(t)dt$.

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))x'(t)dt \text{ and } \int_C f(x,y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

Line Integral Formula of a Continuous Vector Field

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the *line integral of* $\mathbf{F}$ *along* C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T}ds$$

The work W done by the force field $\mathbf{F}$ is the line integral with respect to arc length of the tangential component of the force.

Line Integral Formula for Vector Fields on $\mathbb{R}^2$

Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$, where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy$