MATH 243: Worksheet 3 **Answer Key**

Textbook Sections: 12.3, 12.4 and 12.5
Topics: The dot product, scalar and vector projections, and the cross product.
Instructions: Try each of the following problems, show the detail of your work. Clearly mark your choices in multiple choice items. Justify your answers. Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

1. Consider $\mathbf{a} = \langle -1, 4, 8\rangle$ and $\mathbf{b} = \langle 18, 2, 1\rangle$.
   (a) Find the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$.
   (b) Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.

(a) Since $||\mathbf{a}|| = \sqrt{1+16+64} = \sqrt{81} = 9$, then the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$comp_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||} = \frac{(-1)\cdot 18 + 4\cdot(2) + 8\cdot(1)}{9} = -\frac{2}{9}.$$

(b) The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$proj_{\mathbf{a}}\mathbf{b} = comp_{\mathbf{a}}\mathbf{b}\frac{\mathbf{a}}{||\mathbf{a}||} = -\frac{2}{9}\frac{\mathbf{a}}{||\mathbf{a}||} = -\frac{2}{9}\frac{\langle -1,4,8\rangle}{9} = \left\langle \frac{2}{81}, -\frac{8}{81}, -\frac{16}{81}\right\rangle$$

2. Find the work (in J) done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$ that moves an object from the point $(0, 6, 4)$ to the point $(4, 14, 22)$ along a straight line. The distance is measured in meters and the force in newtons. (Hint: The work done by a constant force is the dot product of the force and displacement vectors).

The displacement vector is

$$\mathbf{D} = (4-0)\mathbf{i} + (14-6)\mathbf{j} + (22-4)\mathbf{k} = 4\mathbf{i} + 8\mathbf{j} + 18\mathbf{k}.$$

Since $W = \mathbf{F}\cdot\mathbf{D}$, then the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = 8\cdot 4 + (-6)\cdot 8 + 5\cdot 18 = 32 - 48 + 90 = 74\,\text{joules}.$$

3. Compute the dot product and cross product for the following pairs of vectors:
   (a) $\mathbf{u} = \langle -1, 1, 2\rangle$, $\mathbf{v} = \langle 4, 5, -2\rangle$

(a)
$$\mathbf{u}\cdot\mathbf{v} = \langle -1,1,2\rangle\cdot\langle 4,5,-2\rangle = -4+5-4 = -3.$$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 2 \\ 4 & 5 & -2 \end{vmatrix} = -12\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 9\hat{\mathbf{k}}.$$

(b) $\mathbf{u} = \langle 1, -1, 3\rangle$, $\mathbf{v} = \langle 2, -2, 6\rangle$

$$\mathbf{u}\cdot\mathbf{v} = \langle 1, -1, 3\rangle \cdot \langle 2, -2, 6\rangle = 2 + 2 + 18 = 22.$$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 3 \\ 2 & -2 & 6 \end{vmatrix} = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}.$$

(c) $\mathbf{u} = \langle 1, 2, 3\rangle$, $\mathbf{v} = \langle -3, 0, 1\rangle$

$$\mathbf{u}\cdot\mathbf{v} = \langle 1, 2, 3\rangle \cdot \langle -3, 0, 1\rangle = -3 + 0 + 3 = 0.$$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ -3 & 0 & 1 \end{vmatrix} = 2\hat{\mathbf{i}} - 10\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

(d) Based on your answer for parts (b) and (c), what conclusions can you make about the dot and cross products of two parallel vectors? Perpendicular vectors? Justify your answers.

(d) We claim that: (1) the cross product of two (nonzero) parallel vectors is the zero vector, and (2) the dot product between two (nonzero) perpendicular vectors is zero.

Justifying claim 1: $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin(\theta) = \|\mathbf{u}\|\|\mathbf{v}\|\sin(0) = 0.$
Justifying claim 2: $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta) = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\pi/2) = 0.$

4. Consider $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{j} + \mathbf{k}$.

(a) Find the cross product $\mathbf{a}\times\mathbf{b}$.

We have the following.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 0 & -2 & 1 \end{vmatrix} = (1-(6))\mathbf{i} - (2-(0))\mathbf{j} + (-4-(0))\mathbf{k} = -5\mathbf{i} - 2\mathbf{j} - 4\mathbf{k} = \langle -5, -2, -4\rangle$$

(b) Verify that $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since
$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle -5, -2, -4\rangle \cdot \langle 2, 1, -3\rangle = -10 - 2 + 12 = 0,$$
then $(\mathbf{a}\times\mathbf{b})$ is orthogonal to $\mathbf{a}$.

(c) Find two unit vectors orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The cross product of two vectors is orthogonal to both vectors. Since the cross product is $\mathbf{a} \times \mathbf{b} = \langle -5, -2, -4 \rangle$ from (a) with

$$||\mathbf{a} \times \mathbf{b}|| = \sqrt{(-5)^2 + (-2)^2 + (-4)^2} = \sqrt{45} = 3\sqrt{5}.$$

So two unit vectors orthogonal to both are

$$\pm \frac{\mathbf{a} \times \mathbf{b}}{||\mathbf{a} \times \mathbf{b}||}, \quad \text{that is} \quad \left\langle -\frac{5}{3\sqrt{5}}, -\frac{2}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}} \right\rangle \quad \text{and} \quad \left\langle \frac{5}{3\sqrt{5}}, \frac{2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}} \right\rangle$$

5. Consider points $P(1,2,1), Q(2,5,4), R(6,9,12)$ and $S(5,6,9)$ in $\mathbb{R}^3$.
   (a) Find the area of the parallelogram with vertices $P(1,2,1), Q(2,5,4), R(6,9,12)$ and $S(5,6,9)$.

By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{PQ} = \langle 1, 3, 3 \rangle$ and $\overrightarrow{PS} = \langle 4, 4, 8 \rangle$. Then, the cross product is

$$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 3 \\ 4 & 4 & 8 \end{vmatrix} = (24-12)\mathbf{i} - (8-12)\mathbf{j} + (4-12)\mathbf{k} = 12\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$$

So, the area of parallelogram PQRS is

$$||\overrightarrow{PQ} \times \overrightarrow{PS}|| = \sqrt{12^2 + 4^2 + (-8)^2} = 4\sqrt{14}.$$

   (b) Find the area of the triangle PQS.

The area of the triangle determined by P, Q, and S is equal to half the area of the parallelogram PQRS. Using part (a), since the area of the parallelogram is $4\sqrt{14}$, then the area of triangle PQS is

$$\frac{1}{2}\,\text{(Area of PQRS)} = \frac{1}{2}\,4\sqrt{14} = 2\sqrt{14}.$$

   (c) Show that the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are coplanar.

We have $\overrightarrow{PQ} = \langle 1, 3, 3 \rangle$, $\overrightarrow{PR} = \langle 5, 7, 11 \rangle$, and $\overrightarrow{PS} = \langle 4, 4, 8 \rangle$. Then,

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} 1 & 3 & 3 \\ 5 & 7 & 11 \\ 4 & 4 & 8 \end{vmatrix} = 1(56-44) - 3(40-44) + 3(20-28) = 12 + 12 - 24 = 0.$$

Since $||\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})|| = 0$, which means that the volume of the box determined by the three vectors is 0, i.e. there is no box. Then, the vectors must lie in the same plane; that is, they are coplanar.

6. Find the following equations of the line through the point $P(2,2,4)$ and parallel to the vector $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k}$. Use the parameter t.

(a) Vector equation

(b) Parametric equations

(c) Symmetric equations

We have the position vector of the point P given by $\overrightarrow{OP} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and choose the direction vector $\mathbf{v} = \mathbf{a}$ since the line is parallel to vector $\mathbf{a}$. Any point on the line has the position vector $\mathbf{r}$.
(a) A vector equation of the line through P and direction vector v is

$$\mathbf{r} = \overrightarrow{OP} + t\mathbf{v} = (2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) + t(4\mathbf{i} - \mathbf{j} + 7\mathbf{k})$$

$$\mathbf{r} = (2 + 4t)\mathbf{i} + (2 - t)\mathbf{j} + (4 + 7t)\mathbf{k}.$$

(b) Parametric equations are

$$x(t) = 2 + 4t, \quad y(t) = 2 - t, \quad z(t) = 4 + 7t.$$

(c) Symmetric equations are

$$\frac{x-2}{4} = \frac{y-2}{-1} = \frac{z-4}{7}.$$

7. (a) Find the symmetric equations for the line through $P(3, 4, 0)$ and perpendicular to both $2\mathbf{i} + 2\mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

We have that the direction vector $\mathbf{v}$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

is the direction of the line perpendicular to both $2\mathbf{i} + 2\mathbf{j}$ and $\mathbf{j} + \mathbf{k}$. Symmetric equations are

$$\frac{x-3}{2} = \frac{y-4}{-2} = \frac{z-0}{2}.$$

(b) Find the point of intersection between the line you found in part (a) and the yz-plane.

The line intersects the yz-plane when $x = 0$ so we need

$$\frac{0-3}{2} = \frac{y-4}{-2} = \frac{z-0}{2} \quad \to \quad -\frac{3}{2} = \frac{y-4}{-2} \text{ and } -\frac{3}{2} = \frac{z-0}{2}$$

So,

$$y = 7, \quad z = -3.$$

Thus, the line intersects the yz-plane at the point $(0, 7, -3)$.

8. Determine whether two lines L_1 and L_2 are parallel, intersecting or skew. If they intersect, find the angle between these two lines.

(a) $L_1 : \mathbf{r}(t) = \langle -1+3t,\ 2+4t,\ 3-2t\rangle$, $\qquad L_2 : \dfrac{x-1}{2} = \dfrac{y}{-3} = \dfrac{z+1}{-3}$

Not parallel; Not Intersecting; Direction vectors for the lines are $\mathbf{v}_1 = \langle 3, 4, -2\rangle$ and $\mathbf{v}_2 = \langle 2, -3, -3\rangle$. They are not parallel. The direction vectors are perpendicular since $\mathbf{v}_1 \cdot \mathbf{v}_2 = 6 - 12 + 6 = 0$. The lines are not intersecting, but the angle between the lines is the same with the angle between their direction vectors, $\theta = \dfrac{\pi}{2}$.

(b) $L_1 : x = 2t,\ y = -3 + t,\ z = 5 - t$

$L_2 : x = 3 - 3s,\ y = 2 - \dfrac{3}{2}s,\ z = \dfrac{3}{2}s$

Parallel; Direction vectors for the lines are $\mathbf{v}_1 = \langle 2, 1, -1\rangle$ and $\mathbf{v}_2 = \left\langle -3, -\dfrac{3}{2}, \dfrac{3}{2}\right\rangle$. The direction vectors are parallel since

$$\mathbf{v}_1 = \langle 2, 1, -1\rangle = -\frac{2}{3}\left\langle -3, -\frac{3}{2}, \frac{3}{2}\right\rangle = -\frac{2}{3}\mathbf{v}_2.$$

Therefore, two lines are parallel.

9. Find an equation for the plane through the points $A(0, 1, 2)$, $B(1, 2, 3)$, and $C(2, 3, 5)$.

First we find the vectors

$$\overrightarrow{AB} = \langle 1, 1, 1\rangle,\ \overrightarrow{AC} = \langle 2, 2, 3\rangle.$$

To find a normal vector to the plane, we compute their cross product

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}\mathbf{k}$$
$$= \mathbf{i} - \mathbf{j} = \langle 1, -1, 0\rangle = \langle a, b, c\rangle.$$

We can choose a fixed point in the plane to be the point A, so the scalar equation of the plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

becomes

$$\begin{aligned} x - (y - 1) + 0(z - 2) &= 0 \\ x - y + 1 &= 0 \\ x - y &= -1 \end{aligned}$$

Suggested Textbook Problems

Section 12.3: 1-13, 17, 19, 23, 25, 27, 28, 33, 40, 41, 43, 45, 47-52

Section 12.4: 1-20, 27-29, 31, 33-39, 41, 43-45, 53

DEFINITIONS AND FORMULAS

The Dot Product for vectors in $\mathbb{R}^3$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The Cross Product (used for vectors in $\mathbb{R}^3$)

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The Scalar and Vector Projections of b onto a are respectively given by

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}, \qquad \text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}$$

Other useful relations

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta), \qquad \|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin(\theta)$$

$$\left\{\text{Area of parallelogram formed by } \mathbf{a} \text{ and } \mathbf{b}\right\} = \|\mathbf{a} \times \mathbf{b}\|$$

$$\left\{\text{Volume of parallelepiped formed by } \mathbf{a}, \mathbf{b}, \text{ and } \mathbf{c}\right\} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$