Textbook Sections: 13.3 and 13.4
Topics: Arc length and curvature; velocity and acceleration
Instructions: Try each of the following problems, show the detail of your work. Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

ARC LENGTH AND CURVATURE:

1. Find the length of the curve $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3\rangle,\ 0 \le t \le 1$.

$$\begin{aligned} \text{Let} \quad \vec{r}\,(t) &= f\,(t)\,\mathbf{i} + g\,(t)\,\mathbf{j} + h\,(t)\,\mathbf{k}, \\ \text{then} \quad \vec{r}'\,(t) &= f'\,(t)\,\mathbf{i} + g'\,(t)\,\mathbf{j} + h'\,(t)\,\mathbf{k}, \\ \text{and} \quad ||\vec{r}'\,(t)\,|| &= \sqrt{(f'\,(t))^2 + (g'\,(t))^2 + (h'\,(t))^2}. \end{aligned}$$

So given that $\vec{r}\,(t) = 2t\mathbf{i} + t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$, then

$$\begin{aligned} |\vec{r}'\,(t)\,| &= \sqrt{(2)^2 + (2t)^2 + (t^2)^2} \\ &= \sqrt{4 + 4t^2 + t^4} \\ &= \sqrt{(2 + t^2)^2} \\ &= 2 + t^2 \end{aligned}$$

So using the arc length formula $L = \int_a^b ||\vec{r}'\,(t)\,||dt$, we get

$$\begin{aligned} L &= \int_0^1 \left(2 + t^2\right) dt \\ &= \left[2t + \frac{1}{3}t^3\right]_0^1 \\ &= \left[2\,(1) + \frac{1}{3}\,(1)^3\right] - \left[2\,(0) + \frac{1}{3}\,(0)^3\right] \\ &= \frac{7}{3} \end{aligned}$$

2. Consider the vector $\mathbf{r}(t) = \langle t, t^2, 4\rangle$.
   (a) Find the unit tangent vector $\mathbf{T}(t)$
   (b) Find the unit normal vector $\mathbf{N}(t)$
   (c) Find the curvature of the curve given by $\mathbf{r}(t)$ at any point t.

$$\vec{r}\,(t) = \langle t, t^2, 4\rangle \implies \vec{r}'\,(t) = \langle 1, 2t, 0\rangle$$
$$||\vec{r}'\,(t)\,|| = \sqrt{1^2 + (2t)^2 + 0^2} = \sqrt{1 + 4t^2}$$

(a) Then the unit tangent vector is given by

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{||\overrightarrow{r}'(t)||} = \frac{1}{\sqrt{1+4t^2}} \cdot \langle 1, 2t, 0\rangle.$$

(b) The unit normal vector is given by

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T'}(t)}{||\overrightarrow{T'}(t)||}$$

Now, by using the Product Rule for the product of a scalar function with a vector function, we have

$$\begin{aligned}
\overrightarrow{T}'(t) &= \frac{1}{\sqrt{1+4t^2}}\langle 0,2,0\rangle - \frac{1}{2}(1+4t^2)^{-\frac{3}{2}} \cdot 8t\langle 1,2t,0\rangle \\
&= \frac{1}{\sqrt{1+4t^2}}\langle 0,2,0\rangle - \frac{4t}{(1+4t^2)^{\frac{3}{2}}}\langle 1,2t,0\rangle \\
&= \frac{1}{(1+4t)^{\frac{3}{2}}}\left[(1+4t^2)\langle 0,2,0\rangle - 4t\langle 1,2t,0\rangle\right] \\
&= \frac{1}{(1+4t)^{\frac{3}{2}}}\langle -4t, 2, 0\rangle
\end{aligned}$$

Then, the magnitude of T' is given by

$$\begin{aligned}
||\overrightarrow{T}'(t)|| &= ||\frac{1}{(1+4t)^{\frac{3}{2}}}\langle -4t,2,0\rangle|| \\
&= \sqrt{\frac{(4t)^2}{(1+4t^2)^3} + \frac{2^2}{(1+4t^2)^3}} = \sqrt{\frac{16t^2+4}{(1+4t^2)^3}} \\
&= \sqrt{\frac{4(4t^2+1)}{(1+4t^2)^3}} = \sqrt{\frac{4}{(1+4t^2)^2}} = \frac{2}{1+4t^2}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \overrightarrow{N}(t) = \frac{\overrightarrow{T'}(t)}{||\overrightarrow{T'}(t)||} &= \frac{1}{(1+4t^2)^{\frac{3}{2}}}\langle -4t,2,0\rangle \cdot \frac{1}{\frac{2}{1+4t^2}} \\
&= \frac{1}{(1+4t^2)^{\frac{3}{2}}} \cdot \frac{(1+4t^2)}{2} \cdot \langle -4t,2,0\rangle \\
&= \frac{1}{(1+4t^2)^{\frac{1}{2}}} \cdot \frac{1}{2}\langle -4t,2,0\rangle \\
&= \frac{1}{(1+4t^2)^{\frac{1}{2}}} \cdot \langle -2t,1,0\rangle
\end{aligned}$$

(c) The curvature $\kappa(t)$ is given by

$$\kappa(t) = \frac{||\overrightarrow{T}'(t)||}{||\overrightarrow{r}'(t)||} \Longrightarrow \kappa(t) = \frac{2/(1+4t^2)}{(1+4t)^{\frac{1}{2}}} = \frac{2}{(1+4t)^{\frac{3}{2}}}$$

3. Consider the position function $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t\rangle$. Find the vectors $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ at the point $(1, \frac{2}{3}, 1)$.

Because $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$, the point $P\left(1, \frac{2}{3}, 1\right)$ corresponds to $t = 1$. Then, $\mathbf{r}'(t) = \langle 2t, 2t^2, 1 \rangle$ The tangent unit vector is given

$$\begin{aligned}
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} &= \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{(2t)^2 + (2t^2)^2 + 1^2}} \\
&= \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{4t^2 + 4t^4 + 1}} \\
&= \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{(2t^2 + 1)^2}} \\
&= \frac{\langle 2t, 2t^2, 1 \rangle}{(2t^2 + 1)}
\end{aligned}$$

So,

$$\mathbf{T}(1) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle.$$

$$\begin{aligned}
\mathbf{T}'(t) &= \frac{\langle 2, 4t, 0 \rangle}{(2t^2 + 1)} - \left(2t^2 + 1\right)^{-2} \cdot 4t \cdot \langle 2t, 2t^2, 1 \rangle \\
&= \frac{\langle 2, 4t, 0 \rangle}{(2t^2 + 1)} - \frac{4t \langle 2t, 2t^2, 1 \rangle}{(2t^2 + 1)^2} \\
&= \frac{1}{(2t^2 + 1)^2} \left[\langle 4t^2 + 2, 8t^3 + 4t, 0 \rangle - \langle 8t^2, 8t^3, 4t \rangle \right] \\
&= \frac{1}{(2t^2 + 1)^2} \left[\langle -4t^2 + 2, 4t, -4t \rangle \right] \\
&= \frac{2}{(2t^2 + 1)^2} \left[\langle 1 - 2t^2, 2t, -2t \rangle \right].
\end{aligned}$$

Hence, the unit Normal vector is given by

$$\begin{aligned}
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} &= \frac{2\left(2t^2 + 1\right)^{-2} \langle 1 - 2t^2, 2t, -2t \rangle}{2\left(2t^2 + 1\right)^{-2} \sqrt{(1 - 2t^2)^2 + (2t)^2 + (-2t)^2}} \\
&= \frac{\langle 1 - 2t^2, 2t, -2t \rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}} \\
&= \frac{\langle 1 - 2t^2, 2t, -2t \rangle}{\sqrt{(1 + 2t^2)^2}} \\
&= \frac{\langle 1 - 2t^2, 2t, -2t \rangle}{(1 + 2t^2)} \\
\implies \mathbf{N}(1) &= \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle
\end{aligned}$$

Now,

$$
\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
\mathbf{B}(1) &= \mathbf{T}(1) \times \mathbf{N}(1) \\
&= \begin{vmatrix} i & j & k \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{vmatrix} \\
&= \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{vmatrix} i - \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{vmatrix} j + \begin{vmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix} k \\
&= -\frac{6}{9} i + \frac{3}{9} j + \frac{6}{9} k \\
&= \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle
\end{aligned}
$$

MOTION IN SPACE: VELOCITY and ACCELERATION

4. A particle moves with position function $\mathbf{r}(t) = \langle t \ln t, t, e^{-t} \rangle$. Find the velocity, speed, and acceleration of the particle.

The velocity vector $\mathbf{v}(t) = \mathbf{r}'(t) = \langle \ln t + 1, 1, -e^{-t} \rangle$

Acceleration vector $\mathbf{a}(t) = \mathbf{v}'(t) = \langle \frac{1}{t}, 0, e^{-t} \rangle$

And the speed of the particle is the magnitude of velocity vector function:

$s(t) = \|\mathbf{v}(t)\| = \sqrt{((\ln t + 1)^2 + 1^2 + (e^{-t})^2} = \sqrt{((\ln t + 1)^2 + 1 + (e^{-2t})}$

5. Find the tangential and normal components of the acceleration vector for the position function $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + t^3\mathbf{j}$, $t \geq 0$.

Recall that the tangential component of acceleration is given by $\mathbf{a}_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}$, and the normal component is $\mathbf{a}_N = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$.
By taking derivatives we have: $\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$, and $\mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j}$

So, $\|\mathbf{r}'(t)\| = \sqrt{(2t)^2 + (3t^2)^2} = t\sqrt{4 + 9t^2}$, since $t \geq 0$

Computing the cross product: $\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 3t^2 & 0 \\ 2 & 6t & 0 \end{vmatrix} = (12t^2 - 6t^2)\mathbf{k} = 6t^2\mathbf{k}$

The dot product is: $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 4t + 18t^3$

Finally, putting the information above into the formulas of tangential and normal components yields:

$\mathbf{a}_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{4t + 18t^3}{t\sqrt{4+9t^2}} = \frac{4 + 18t^2}{\sqrt{4+9t^2}}$; $\mathbf{a}_N = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{0^2 + 0^2 + (6t^2)^2}}{t\sqrt{4+9t^2}} = \frac{6t^2}{t\sqrt{4+9t^2}} = \frac{6t}{\sqrt{4+9t^2}}$

6. If the acceleration of an object is given by $\mathbf{a}(t) = t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, find the object's position function given that the initial velocity is $\mathbf{v}(0) = \mathbf{k}$ and the initial position is $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

The acceleration is the derivative of the velocity: $\mathbf{a}(t) = \mathbf{v}'(t)$, and the velocity is the derivative of the position function: $\mathbf{v}(t) = \mathbf{r}'(t)$. Hence, we can find the velocity by computing the anti-derivative of the acceleration:

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)dt \\ &= \int (t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k})dt \\ &= \frac{t^2}{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C_1}\end{aligned}$$

However, it's given that $\mathbf{v}(0) = \mathbf{k}$, this implies $\mathbf{j} - \mathbf{k} + \mathbf{C_1} = \mathbf{k} \implies \mathbf{C_1} = 2\mathbf{k} - \mathbf{j}$

Thus, $\mathbf{v}(t) = \frac{t^2}{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + (2\mathbf{k} - \mathbf{j}) = \frac{t^2}{2}\mathbf{i} + (e^t - 1)\mathbf{j} + (2 - e^{-t})\mathbf{k}$

Similarly, by taking the anti-derivative of the velocity function gives the position function:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)dt \\ &= \int [\frac{t^2}{2}\mathbf{i} + (e^t - 1)\mathbf{j} - (2 - e^{-t})\mathbf{k}]dt \\ &= \frac{t^3}{6}\mathbf{i} + (e^t - t)\mathbf{j} + (2t + e^{-t})\mathbf{k} + \mathbf{C_2}\end{aligned}$$

But we know that: $\mathbf{r}(0) = \mathbf{j} + \mathbf{k} \implies \mathbf{j} + \mathbf{k} + \mathbf{C_2} = \mathbf{j} + \mathbf{k} \implies \mathbf{C_2} = \mathbf{0}$

Therefore, $\mathbf{r}(t) = \frac{t^3}{6}\mathbf{i} + (e^t - t)\mathbf{j} + (2t + e^{-t})\mathbf{k}$

7. Consider a particle with the position function $\mathbf{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$. Find the velocity, acceleration, and speed of the particle at any time t.

The velocity vector $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t + 1, 2t - 1, 3t^2 \rangle$

The acceleration vector $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 2, 6t \rangle$

And the speed $s(t) = \|\mathbf{v}(t)\| = \sqrt{(2t+1)^2 + (2t-1)^2 + (3t^2)^2} = \sqrt{9t^4 + 8t^2 + 2}$

Suggested Textbook Problems

Section 13.3	3-5, 7-9, 11, 13, 17, 19, 23, 24, 27, 29-33, 43, 47, 49, 50
Section 13.4	3, 5, 8, 9, 11, 15, 17-21, 23, 25, 28, 31, 32, 37, 39, 41

SOME USEFUL EQUATIONS:

The Arc Length Formula

The length L of a curve given by the vector function $\mathbf{r}(t)$, for $a \leq t \leq b$, can be computed using the formula

$$L = \int_a^b \|\mathbf{r}'(t)\| dt$$

Curvature

The *curvature* of the curve given by the vector function $\mathbf{r}$ is

$$\mathbf{k}(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

For a special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $r(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then

$$\mathbf{k}(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

Useful formula for curvature

$$\mathbf{k}(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Unit Tangent, Unit Normal and Unit Binormal Vectors

Unit tangent vector is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

Unit normal vector is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Unit binormal vector is given by

$$\mathbf{B}(t) = T(t) \times N(t)$$

The Tangential and Normal Components of Acceleration are denoted by $\mathbf{a_T}$ and $\mathbf{a_N}$ are respectively given by

$$a_T = v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(\mathbf{t})}{\|\mathbf{r}'(t)\|},$$

By using the curvature formula above, we have

$$a_N = \mathbf{k}v^2 = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|.\|\mathbf{r}'(t)\|^2}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$