Answer Key

University of Delaware - Department of Mathematical Sciences
MATH 243 Midterm Exam 1 - Spring 2024

Tuesday 12$^{\text{th}}$ March, 2024

Instructions:

- The time allowed for completing this exam is **75** minutes in total.
- Check your examination booklet before you start. There should be **4** questions on **5** pages.
- Turn off your cell phone and put it away. Headsets, earbuds and any other electronic devices are prohibited.
- No calculators.
- Answer the questions in the space provided. If you need more space for an answer, continue your answer on the back of the page and/or the margins of the test pages. No extra paper. *Do not separate the pages from the exam booklet.*
- For full credit, sufficient work must be shown to justify your answer.
- Partial credit will not be given if appropriate work is not shown.
- Write legibly and clearly; indicate your final answer to every problem. Cross out any work that you do not want graded. If you produce multiple solutions for a problem, indicate clearly which one you want graded.
- **Any form of academic misconduct will result in a failing grade.**

Question:	1	2	3	4	Total
Points:	25	25	25	25	100
Score:					

1. Let the curve $\mathcal{C}$ be given by the vector function $\mathbf{r}(t) = \dfrac{1}{t^2+1}\mathbf{i} + \sin\left(2t + \dfrac{\pi}{3}\right)\mathbf{j} + (\sqrt[3]{t+8})\,\mathbf{k}$.

(a) (6 points) Find the coordinates of the point P on the curve $\mathcal{C}$ corresponding to $t = 0$.

We can find the position vector $\mathbf{r}(t)$ at $t = 0$

$$\mathbf{r}(0) = \left\langle \frac{1}{0+1},\ \sin\left(0 + \frac{\pi}{3}\right),\ \sqrt[3]{0+8} \right\rangle = \left\langle 1,\ \sin\left(\frac{\pi}{3}\right),\ \sqrt[3]{8} \right\rangle$$

The point P on the curve is

$$\left(1,\ \frac{\sqrt{3}}{2},\ 2\right)$$

(b) (6 points) Determine the vector function $\mathbf{r}'(t)$. Fully simplify your answer.

$$\mathbf{r}'(t) = \left\langle \frac{-2t}{(t^2+1)^2},\quad 2\cos\left(2t + \frac{\pi}{3}\right),\quad \frac{1}{3\sqrt[3]{(t+8)^2}} \right\rangle$$

(c) (6 points) Find a scalar equation of the **normal plane** to the curve $\mathcal{C}$ at the point where $t = 0$.

When $t = 0$, we have the point $\left(1, \dfrac{\sqrt{3}}{2}, 2\right)$.

Since a vector perpendicular to the normal plane is the vector

$$\mathbf{r}'(t) = \left\langle \frac{-2t}{(t^2+1)^2},\ 2\cos\left(2t + \frac{\pi}{3}\right),\ \frac{1}{3\sqrt[3]{(t+8)^2}} \right\rangle,$$

then we have the following

$$\mathbf{r}'(0) = \left\langle 0, 2\cos\left(\frac{\pi}{3}\right), \frac{1}{3\sqrt[3]{8^2}} \right\rangle \quad \longrightarrow \quad \mathbf{r}'(0) = \left\langle 0, 1, \frac{1}{12} \right\rangle$$

So, an equation of the normal plane is

$$0\,(x-1) + 1\left(y - \frac{\sqrt{3}}{2}\right) + \frac{1}{12}\,(z-2) = 0.$$

(d) (7 points) Write **parametric equations** of the tangent line to the curve $\mathcal{C}$ at the point $P\left(1, \dfrac{\sqrt{3}}{2}, 2\right)$.

The point $\left(1, \dfrac{\sqrt{3}}{2}, 2\right)$ corresponds to $t = 0$. The direction vector of the tangent line at $t = 0$ is

$$\mathbf{r}'(0) = \left\langle 0,\ 1,\ \frac{1}{12} \right\rangle$$

So, the parametric equations of the tangent line is

$$x(t) = 1 + 0 \cdot t, \quad y(t) = \frac{\sqrt{3}}{2} + 1 \cdot t, \quad z(t) = 2 + \frac{1}{12} \cdot t$$

2. Given that the **velocity** vector function of a particle is $\mathbf{v}(t) = 2\mathbf{i} + (te^{3t-3})\mathbf{j} + (4t^3 - 2t - 5)\mathbf{k}$, and that the **initial position** of the particle is $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$, find the following.

(a) (8 points) The speed of the particle at the point corresponding to $t = 1$.

$$speed = \|\mathbf{v}(t)\| \qquad \mathbf{v}(1) = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$
$$\|\mathbf{v}(1)\| = \|\langle 2, 1, -3\rangle\| = \sqrt{4+1+9} = \sqrt{14}$$

(b) (8 points) The acceleration of the particle at the point corresponding to $t = 1$.

Since
$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, \quad e^{3t-3} + 3te^{3t-3}, \quad 12t^2 - 2\rangle,$$
then, we have the following
$$\mathbf{a}(1) = \langle 0, 4, 10\rangle$$
OR
$$\mathbf{a}(1) = 4\mathbf{j} + 10\mathbf{k}$$

(c) (9 points) The position of the particle at any time t, given that $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Since $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt$, then

$$\mathbf{r}(t) = \left\langle 2t + C_1,\ \frac{te^{3t-3}}{3} - \frac{e^{3t-3}}{9} + C_2,\ t^4 - t^2 - 5t + C_3 \right\rangle$$

Also we have $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\mathbf{r}(0) = \left\langle C_1,\ -\frac{e^{-3}}{9} + C_2,\ C_3 \right\rangle = \langle 1, 1, 1\rangle \quad \to \ C_1 = 1,\ C_2 = \frac{e^{-3}}{9} + 1,\ C_3 = 1$$

Therefore, the position vector is

$$\mathbf{r}(t) = \left\langle 2t + 1, \quad \frac{te^{3t-3}}{3} - \frac{e^{3t-3}}{9} + \frac{e^{-3}}{9} + 1, \quad t^4 - t^2 - 5t + 1 \right\rangle$$

NOTE: Integration by Part

Let $u = t$ and $dv = e^{3t-3}dt \quad \longrightarrow \quad du = dt$ and $v = \frac{1}{3}e^{3t-3}$

$$\int te^{3t-3}dt = uv - \int vdu = \frac{t}{3}e^{3t-3} - \int \frac{1}{3}e^{3t-3}dt = \frac{t}{3}e^{3t-3} - \frac{1}{9}e^{3t-3} + C$$

3. Given the points

$$A(2,0,1), \quad B(-1,1,2), \quad C(4,1,-1),$$

find the following.

(a) (3 points) The vector $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \langle -3, 1, 1\rangle$$

(b) (3 points) The vector $\overrightarrow{AC}$.

$$\overrightarrow{AC} = \langle 2, 1, -2\rangle$$

(c) (6 points) A unit vector that has the same direction as the vector $\overrightarrow{AC}$.

$$\mathbf{u} = \frac{\overrightarrow{AC}}{||\overrightarrow{AC}||} = \frac{\langle 2,1,-2\rangle}{\sqrt{(2)^2+(1)^2+(-2)^2}} = \frac{\langle 2,1,-2\rangle}{\sqrt{9}} = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle$$

(d) (7 points) The vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$. Fully simplify your answer.

The Scalar and Vector Projections of $\mathbf{b}$ onto $\mathbf{a}$ are respectively given by

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}; \qquad \text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right) = (\text{comp}_{\mathbf{a}}\mathbf{b})\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right)$$

$$\text{proj}_{\overrightarrow{AC}}\overrightarrow{AB} = \left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{||\overrightarrow{AC}||^2}\right)\overrightarrow{AC} = \left(\frac{(-3)\cdot 2 + 1\cdot 1 + 1\cdot(-2)}{(3)^2}\right)\langle 2,1,-2\rangle$$

$$= \frac{-7}{9}\langle 2,1,-2\rangle = \left\langle -\frac{14}{9}, -\frac{7}{9}, \frac{14}{9}\right\rangle$$

(e) (6 points) The area of the triangle ABC.

Since the cross product of $\overrightarrow{AB}\times\overrightarrow{AC}$ is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = (-2-1)\mathbf{i} - (6-2)\mathbf{j} + (-3-(2))\mathbf{k} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k} = \langle -3,-4,-5\rangle,$$

then the magnitude $||\overrightarrow{AB}\times\overrightarrow{AC}||$ is

$$||\overrightarrow{AB}\times\overrightarrow{AC}|| = ||\langle -3,-4,-5\rangle|| = \sqrt{(-3)^2+(-4)^2+(-5)^2} = \sqrt{50} = 5\sqrt{2}$$

So, the area of the triangle ABC is

$$\frac{1}{2}||\overrightarrow{AB}\times\overrightarrow{AC}|| = \frac{1}{2}||\overrightarrow{AC}\times\overrightarrow{AB}|| = \frac{\sqrt{50}}{2} = \frac{5\sqrt{2}}{2}$$

4. Let $z = f(x, y) = 2x \ln\big(x + y^2\big)$ be a function of two variables. Find the following.

(a) (5 points) $f(e, 0)$

$$f(e, 0) = 2e \ln\big(e + 0^2\big) = 2e \ln(e) = 2e$$

(b) (5 points) $\dfrac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial y} = z_y = 2x \frac{2y}{x + y^2} = \frac{4xy}{x + y^2}$$

(c) (5 points) The rate of change of $f(x, y)$ with respect to x when y is held fixed.

$$\frac{\partial f}{\partial x} = f_x(x, y) = 2\ln\big(x + y^2\big) + 2x \frac{1}{x + y^2} = 2 \ln\big(x + y^2\big) + \frac{2x}{x + y^2}$$

(d) (5 points) $\dfrac{\partial^2 z}{\partial y \partial x}$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{xy} = \frac{\partial}{\partial y}\left(2 \ln\big(x + y^2\big) + \frac{2x}{x + y^2}\right) = 2\frac{2y}{x + y^2} + \frac{0 - 2x(2y)}{(x + y^2)^2}$$
$$= \frac{4y(x + y^2) - 4xy}{(x + y^2)^2} = \frac{4y^3}{(x + y^2)^2}$$

OR

Since $z = f(x, y)$ have the first partial derivatives and they are differentiable, then

$$z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{4xy}{x + y^2}\right) = \frac{4y(x + y^2) - 4xy(1)}{(x + y^2)^2} = \frac{4y^3}{(x + y^2)^2}$$

(e) (5 points) The **slope** of the tangent line to the curve of intersection of the surface $z = f(x, y) = 2x \ln(x + y^2)$ with the vertical plane $y = 0$, at the point $P(e, 0, 2e)$.

$$\frac{\partial z}{\partial x} = 2 \ln\big(x + y^2\big) + 2x \frac{1}{x + y^2}$$

$$\left.\frac{\partial z}{\partial x}\right|_{x=e, y=0} = 2 \ln\big(e + 0^2\big) + 2e \frac{1}{e + 0^2} = 2 \ln(e) + 2e \frac{1}{e} = 2 + 2 = 4$$