Multivariable Chain Rule

Pre-lecture for 6/18

Quizzes graded. Talk to me about any questions on grading or about how to improve, email is best.

was generous with issues, make sure to address these issues by next quiz

Pre-lec video was posted yesterday ~ 7pm, next videos will be posted even earlier.

common mistakes doc.

Plan to post WA HW2,3, Quiz solutions, DW 3 solutions, recap video for midterm, recap slides, survey all by end of today

Reminder: no class & discussion tomorrow (6/19)

HW2: 6\13, 6\16, 6\17 → due by: 6\19 ~~6\18 orig~~

HW3: 6\18, 6\23 → due by: 6\24 ~~6\21 orig~~

HW4: 6\24, 6\25

HW5: 6\26, 6\27

Ordinary Chain Rule

all you saw this

- Recall: h(t) = f(g(t)) implies h'(t) = f'(g(t))g'(t)
- If y = f(x), x = g(t), then dy/dt = (df/dx)(dx/dt)
- We will extend to more variables by using fraction form

most have seen this

$$y = f(x) = f(g(t)) = h(t)$$

$$\frac{dy}{dt} = \frac{d}{dt}h(t) = h'(t) = f'(g(t))\,g'(t) = f'(x)\,g'(t)$$

$$= \frac{df}{dx}\,\frac{dg}{dt} = \frac{df}{dx}\,\frac{dx}{dt}$$

Multivariable Chain Rule

- Let z = f(x, y), x = g(t), y = h(t)
- dz/dt = (∂f/∂x)(dx/dt) + (∂f/∂y)(dy/dt)

Recall: $dz = f_x\, dx + f_y\, dy$

Divide by dt: $\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$

Seems illegal, why divide.

$z = f(x,y) = f(g(t), h(t))$

$z + dz = f(g(t+dt), h(t+dt))$

$$dZ = \underline{f(g(t+dt), h(t+dt)) - f(g(t), h(t))}$$

Recall: dt is infintes, $\frac{g(t+dt) - g(t)}{dt} = g'(t) \Rightarrow$

$$g(t+dt) = \underline{g(t)} + g'(t)dt, \quad \cancel{f(t+dt)} = f(t) + f'(t)dt$$

$$= \underline{f(g(t) + g'(t)dt, h(t) + h'(t)dt)} - f(g(t), h(t))$$

Recall: $f(x+dx, y+dy) = f(x,y) + f_x(x,y)dx + f_y(x,y)dy$

Taking $dx = g'(t)dt$, $dy = h'(t)dt$

$$f(\cdots) = f(g(t), h(t)) + f_x(g(t), h(t))\, g'(t)dt + f_y(\ldots)\, h'(t)dt$$

$$\text{orange} = f_x(\ldots)\, g'(t)dt + f_y(\ldots)\, h'(t)dt$$

$$\frac{dZ}{dt} = \frac{\partial f}{\partial x} g'(t) + \frac{\partial f}{\partial y} h'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$\frac{d}{dt}$ $\quad \partial x \quad \partial y \quad \partial x \quad dt \quad \partial y \quad dt$

$x = g(t) \Rightarrow \frac{dx}{dt} = g'(t)$

Even More Variables

- Let $z = f(x_1,x_2,\ldots)$, x_i a function of $t_1, t_2, \ldots, t_m$
- $\partial z/\partial t_i = (\partial z/\partial x_1)(\partial x_1/\partial t_i) + (\partial z/\partial x_2)(\partial x_2/\partial t_i) + \ldots$

Ʒ doing this for $i=1$ for sake of clarity

$$z = f(\underline{x_1(t_1,\ldots), x_2(\ldots), \ldots})$$

$$dz = f(\underline{x_1(t_1+dt_1,\ldots)}, x_2(\ldots),\ldots) - f(x_1, x_2, \ldots)$$

$$x_1(t_1+dt_1, t_2, \ldots) = x_1(t_1,t_2,\ldots) + x_{1_{t_1}}(t_1,\ldots)\,dt_1$$

https://www.arizonadiamondcenter.com/blog/2023/Jan/30/gold-or-silver-chain-what-should-you-wear/

$$\text{green} = f\Big(\underbrace{(x_1, x_2, \dots)}_{x} + \underbrace{\big({x_1}_{t_1} dt_1,\ {x_2}_{t_1} dt_1, \dots\big)}_{dx}\Big)$$

$$f(x_1 + dx_1, x_2 + dx_2, \dots) = f(x_1, x_2, \dots) + f_{x_1}\, dx_1 + f_{x_2}\, dx_2 + \dots$$

Take $dx_i = (x_i)_{t_1} dt_1$

$$= f(x_1, x_2, \dots) + f_{x_1} (x_1)_{t_1} dt_1 + f_{x_2} (x_2)_{t_1} dt_1 + \dots$$

$$dz = \left[f_{x_1} (x_1)_{t_1} + (f_{x_2}) (x_2)_{t_1} + \dots \right] dt_1 \Longrightarrow$$

$$\frac{\partial z}{\partial t_1} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_1} + \frac{\partial f}{\partial x_3} \frac{\partial x_3}{\partial t_1} + \dots$$

$\frac{\partial}{\partial t_1}$ $\frac{\partial x_1}{\partial t_1}$ $\frac{\partial x_2}{\partial t_1}$ $\frac{\partial x_3}{\partial t_1}$

Implicit Differentiation Shortcut

$x^5 + y^3 x + e^{xy} = 7$

Recall trying to find dy/dx in Calc 1:

- Usually given a long equation in x and y
- Have to do a bunch of algebra to find dy/dx

$a = b \Rightarrow a - b = 0$

if everything moved to one side

Witness the power of Calc 3:

- Let equation be F(x, y) = 0, y depends on x
- $dy/dx = -F_x/F_y$

$0 = \frac{\partial}{\partial x} F = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} =$

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\frac{\partial F}{\partial y} = -\frac{\partial F}{\partial x} \Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y}$

Func. $f(x,y,z)$ & $x(r,s,t)$, $y(r,s,t)$, $z(r,s,t)$. Then $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$

$x = r+s+t$, $y = rst$, $z = r+t$,

$f(x,y,z) = xy + z$

$f_x = y$, $f_y = x$, $f_z = 1$

$x_r = 1$, $y_r = st$, $z_r = 1$

$f_r = f_x x_r + f_y y_r + f_z z_r = y + xst + 1 =$

$rst + (r+s+t)st = \underline{2rst + s^2t + st^2}$

Do it directly: $f = xy + z = (r+s+t)rst + r+t$

$= r^2st + rs^2t + rst^2 + r + t$

$f_r = \underline{2rst + s^2t + st^2}$

Try your own: pick some Imp Diff prob from Calc 1, do it normally, then do it with part deriv trick.

(For example, try $x^4y + \frac{x}{y} + \cos(x^y) = 0$

Practice Problems

Find all of the derivatives

- dz/dt when $z = xe^{xy}$, $x = t^2$, $y = 1/t$
- $\partial z/\partial s$, $\partial z/\partial t$ for $z = e^{2r}\sin(\theta)$, $r = st-t^2$, $\theta = (s^2+t^2)^{1/2}$
- $f_{\theta\theta}$ for $f(x,y)$ if $x = r\cos(\theta)$, $y = r\sin(\theta)$

Use same idea behind shortcut to find a formula for $\partial z/\partial x$, $\partial z/\partial y$ when z depends on x and y according to $F(x, y, z) = 0$

- Apply on $x^2\sin(y-z) = 1+y\cos(xz)$ to find $\partial z/\partial x$, $\partial z/\partial y$

Scratchwork

Step $\underline{1}$: for each problem on $\partial\ldots/\partial\ldots$'s: write out the formula to be applied on your funcs

Usual tactic: $z = f(x, y) = f(g(t), h(t))$, so we are in slide 3 when considering what formula to use.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt}$$

Tactic from general remark: draw tree & pretend to cancel fractions. Draw $u \longrightarrow v$ if the variable u depends on v.

$$\begin{tikzcd}[column sep=small]
 & z \arrow[dl, "\text{dep.}"'] \arrow[dr, "\text{dep.}"] & \\
x \arrow[d] & & y \arrow[d] \\
t & & t
\end{tikzcd}$$

Now let's try to treat $\frac{dz}{dt}$ as a fraction: if $\frac{a}{b}$ is an ordinary fraction with a, b real numbers, then we have

$$\frac{a}{b} = \frac{a}{c}\,\frac{c}{b} \quad \text{for any } c \neq 0$$

So let's introduce something to $\frac{dz}{dt}$. We have to introduce everything in the tree:

only things left are x & y

Introduce x: $$\frac{dZ}{dt} = \frac{dZ}{dx}\frac{dx}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt}$$

Introduce y: $$\frac{dZ}{dt} = \frac{dZ}{dy}\frac{dy}{dt} = \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Make each expression proper, done in blue above

Last step, add up contributions:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Step 2: calculate all terms in the equation you found

$z = xe^{xy}$, $x = t^2$, $y = 1/t$, 4 things to calc.

$$\frac{\partial z}{\partial x} = (xe^{xy})_x = e^{xy} + xye^{xy} = (1+xy)e^{xy}$$

$$z_y = (xe^{xy})_y = x^2e^{xy}$$

$$dx/dt = 2t, \quad dy/dt = -1/t^2 = -\frac{1}{t^2}$$

Step 3: plug in everything you found

$$\frac{dz}{dt} = (1+xy)e^{xy}\,2t + x^2e^{xy}\cdot\left(-\frac{1}{t^2}\right) =$$

$$e^{xy}\left[(1+xy)\cdot 2t - \frac{x^2}{t^2}\right]$$

Step 4: We are not done yet. Plug in for leftover variables so that the answer is fully simplified

What to plug in: $x = t^2$, $y = 1/t$

$$\frac{dz}{dt} = e^t\left[(1+t)2t - \frac{t^4}{t^2}\right] =$$

$$e^t\left[2t + 2t^2 - t^2\right] = \underbrace{e^t(t^2+2t)}_{\text{good}}$$

$$= \underbrace{t(t+2)e^t}_{\text{also good}}$$

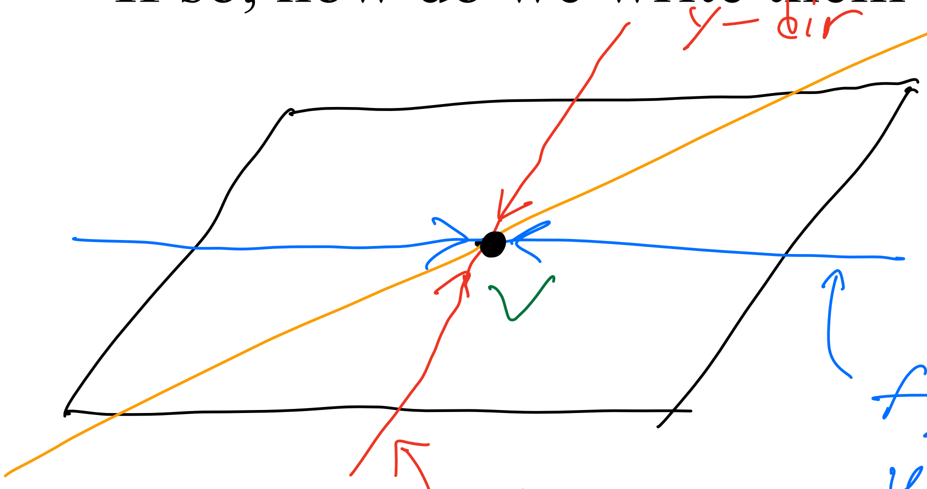
You can follow this step 1–4 recipe for other multivariable derivative problems

Gradient

Lecture for 6/18

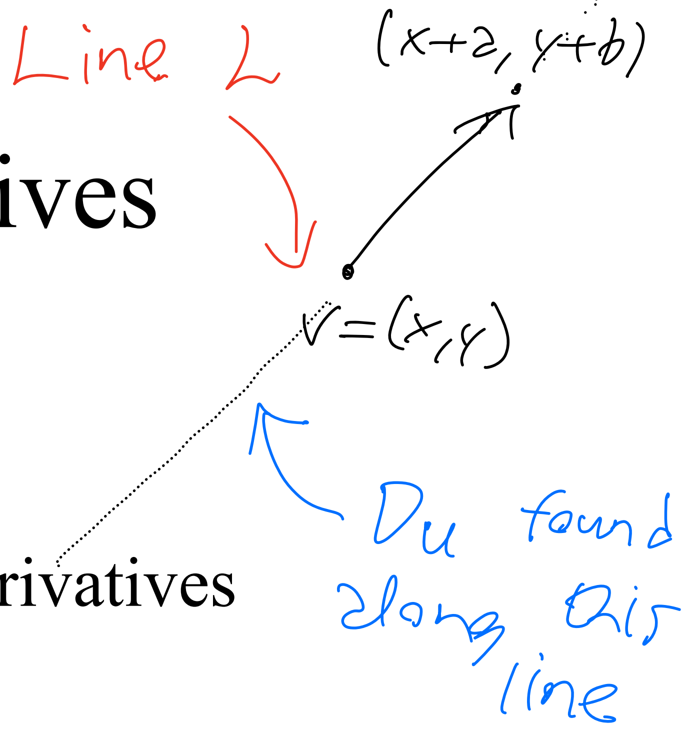
Motivation for Gradient

- For f(x, y), we have seen f_x and f_y
  - They represent x and y direction
- Can we take derivatives along other directions?
- If so, how do we write them and work with them?



along this line

Directional Derivatives



- Let **u** = ⟨a, b⟩ represent a direction, **v** = (x, y)
- Define $D_{\mathbf{u}}f(\mathbf{v}) = \lim_{h\to 0} [f(\mathbf{v}+h\mathbf{u})-f(\mathbf{v})]/h$
- We'll only consider $D_{\mathbf{u}}$ for unit vectors **u**
  - Bigger **u** in same direction give larger derivatives
- Generalizes to any number of variables

From 6/10 lecture, v+tu describes the equation of the line L. Get different points on the line as t varies.

Also consider v to be point, u to be on orange line in previous slide

So as $h \to 0$, $v + hu$ approaches v along the line L.

So derivative along L will be the diff quotient given earlier

If u is larger, derivative is larger.

$D_{2u} f(v) = 2 D_u f(v)$.

This does not make sense as u & $2u$ represent the same direction and same line since scaling a vector doesn't affect its direction.

So we impose condition that $\|u\| = 1$ to keep semblance of consistency.

More variables $\Rightarrow$ u has more coordinates.

Example for R^3: $u = (a, b, c)$, $\|u\| = 1$

$\Rightarrow a^2 + b^2 + c^2 = 1$, $v = (x, y, z)$, then

$$D_u f(v) = \lim_{h \to 0} \left[\frac{f(v + hu) - f(v)}{h} \right] =$$

$$\left[\lim_{h \to 0} \frac{f(x + ha, y + hb, z + hc) - f(x, y, z)}{h} \right]$$

Directional Derivatives Formula

- If $u = (a, b)$, then $D_u f(x,y) = af_x(x,y) + bf_y(x,y)$
- Define the operator ∇ by $\nabla f = (f_x, f_y)$
- $D_u f = (a, b) \cdot (f_x, f_y) = u \cdot \nabla f$
- Notice what happens if $u = (1,0)$ or $(0,1)$ → check this yourself, you get f_x & f_y

- Everything repeats in more dimensions

Let's show this formula for D_u:

$$D_u f = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$ and

the limit has only 1 variable.

How can we find a derivative with 1 variable? Need a 1 variable function.

The variable is h, so consider only h.

Define $g(h) = f(\underline{x_0 + ah}, y_0 + bh)$, then

$$D_u f = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0)$$

by limit def. of derivative

$$g'(h) = \frac{dg}{dh} = \frac{\partial f}{\partial h} = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

$$\boxed{\begin{aligned} x &= x_0 + ah \\ y &= y_0 + bh \end{aligned}}$$

$$= f_x \cdot a + f_y \cdot b$$

all constants

So $g'(h)$ is $af_x + bf_y = af_x(x_0+ah, y_0+bh) + \ldots$

Plug in $h=0 \Rightarrow g'(0) = af_x(x_0, y_0) + bf_y(x_0, y_0)$

So $D_u f = af_x + bf_y =$

$\langle a, b\rangle \cdot (f_x, f_y) = u \cdot \nabla f$

For $f(x,y,z)$, $\nabla f = \langle f_x, f_y, f_z \rangle$

Rates of Change

- Maximum value of $D_u f$ occurs when u, ∇f are parallel
- Minimum value occurs in opposite direction

$$D_u f = \nabla f \cdot u \Rightarrow |D_u f| = |\nabla f \cdot u|$$

$= \underline{\|\nabla f\|} \, \underline{\|u\|} \, |\cos\theta|$ where θ is angle between ∇f & u. Set $\cos\theta = \pm 1 \Rightarrow \theta = 0, \pi$

to maximize $\Rightarrow$ u & ∇f are parallel

$\Rightarrow \quad u = \pm \dfrac{\nabla f}{\|\nabla f\|}$ since u is unit vector

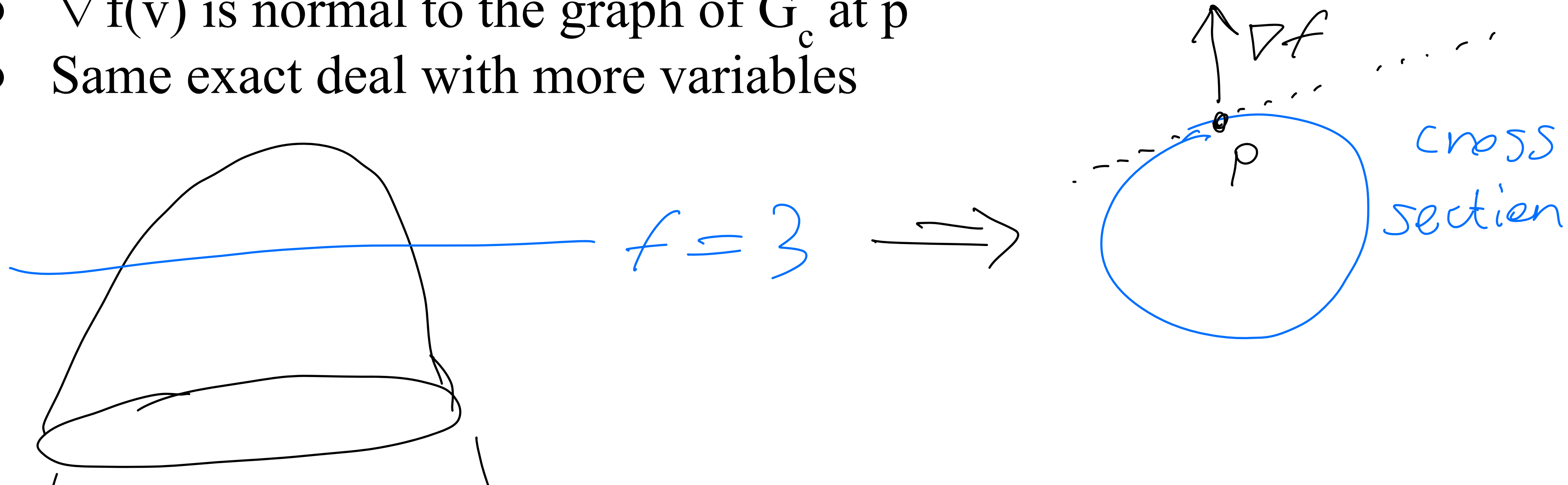
One will give maximum, other the minimum

Gradients vs Graphs

Since
$$\underbrace{D_{-u}(f) = -D_u f}_{\text{extra practice: check this}}$$

Can we say anything about ∇f graphically?

- For constant c, let G_c be graph of $f(x, y) = c$
- Suppose $p = (x_0,y_0)$ lies on G_c
- $\nabla f(v)$ is normal to the graph of G_c at p
- Same exact deal with more variables

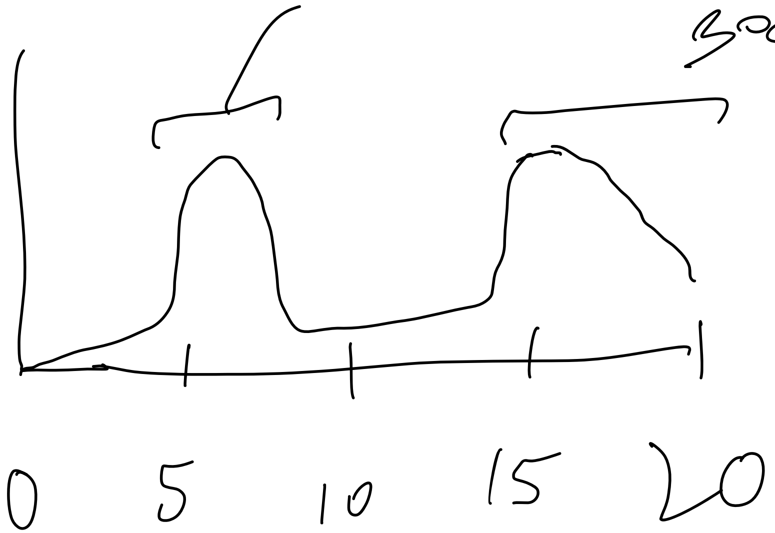


Remarks about midterm: joining Zoom is required, midterm is on class time, midterm appears as locked Canvas quiz, unlocks 5 minutes after class starts, continues for length of class + some extra time for upload, read instructions before starting, covers all material from this and last week.

Remarks on quiz results: score distribution was bimodal

try to improve, if you know how to, then you know how to; if you don't know how to, email me to ask for advice

good



For constant c, let G_c be graph of $f(x, y) = c$

Suppose $p = (x_0, y_0)$ lies on G_c

$\nabla f(v)$ is normal to the graph of G_c at p

Since p on G_c, $c = f(p) = f(x_0, y_0)$.

∇f at p is $\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$

Last thing to figure out

What does graph look like? It's a curve, so it can be parametrized by one variable.

Let's take a param. of the curve:

$$x = x(t) \quad \Rightarrow \quad c = f(x(t), y(t))$$
$$y = y(t)$$

Now we're stuck, so let's take derivative:

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} =$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right) = \nabla f \cdot (x'(t), y'(t))$$

Let $r(t) = (x(t), y(t))$ describe G_C

$$0 = \nabla f \cdot r'(t)$$

So this means ∇f is perpendicular to tangent vector of curve, so it's perp. to tangents to the graph, so it's normal to the graph G_C

Practice Problems

Find these directional derivatives

- $f(x, y) = x\cos(y)$ in the direction of (2, -1)
- $f(x, y, z) = e^z + \sin(xy)$ in direction of (3, 4) 0)

You are sitting on a hill whose height is $100 - x^2 - y^2$ at any point (x,y). If you are at (4, 7), which direction is the descent steepest? Which direction would you be climbing the steepest?

Scratchwork

$f(x,y,z) = e^z + \sin(xy)$ in direction of $(3,4)$

Step 1: Find unit vector in direction.

$$u = \frac{(3,4,0)}{\|(3,4,0)\|} = \frac{(3,4)}{\sqrt{3^2+4^2+0^2}} = \frac{1}{5}(3,4,0) = \left(\frac{3}{5}, \frac{4}{5}, 0\right)$$

Step 2: Find the gradient

$$f_x = (e^z + \sin(xy))_x = 0 + (xy)_x \cos(xy)$$

$$= 0 + y\cos(xy) = y\cos(xy)$$

$$f_y = (e^z + \sin(xy))_y = x\cos(xy)$$

$$f_z = (e^z + \sin(xy))_z = (e^z)_z = e^z$$

So $\nabla f = \langle f_x, f_y, f_z \rangle = \langle y\cos(xy), x\cos(xy), e^z \rangle$

That was 2(a): find partial derivatives &

2b: plug in to get gradient

Step 3: Plug in ∇f & u to get $D_u f$

$$D_u f = \nabla f \cdot u =$$

$$\langle y\cos(xy), x\cos(xy), e^z\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5}, 0\right\rangle =$$

$$\frac{3}{5}y\cos(xy) + \frac{4}{5}x\cos(xy) + 0\cdot e^z =$$

$$\left(\frac{3}{5}y + \frac{4}{5}x\right)\cos(xy)$$

final ans.

If you try practice problems and get stuck, you can ask me for help.