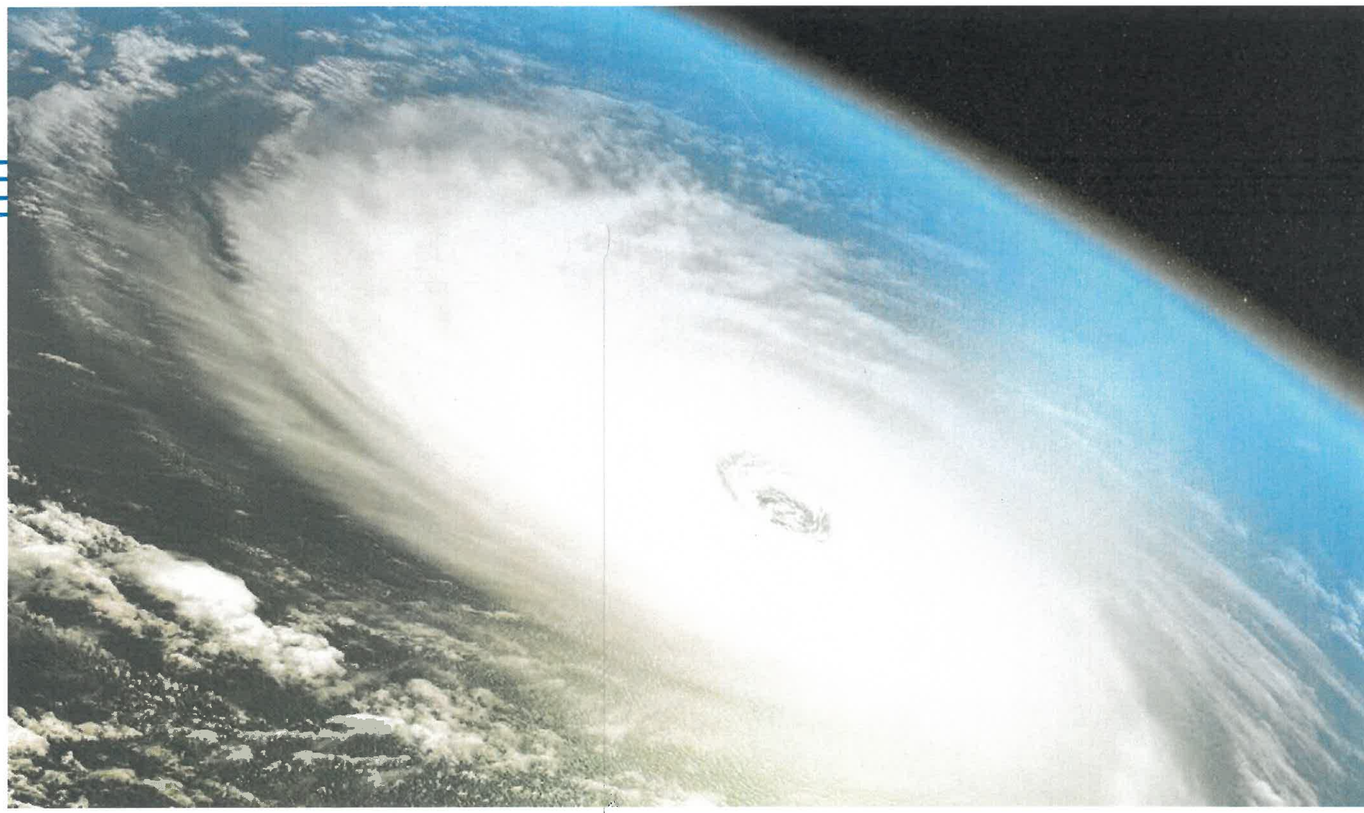
16 Vector Calculus



16.1 Vector Fields

Vector Fields in $\mathbb{R}^2 \text{and} \mathbb{R}^3$

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (4 of 8)

Another type of vector field, called a *force field*, associates a force vector with each point in a region. An example is the gravitational force field.
In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3).

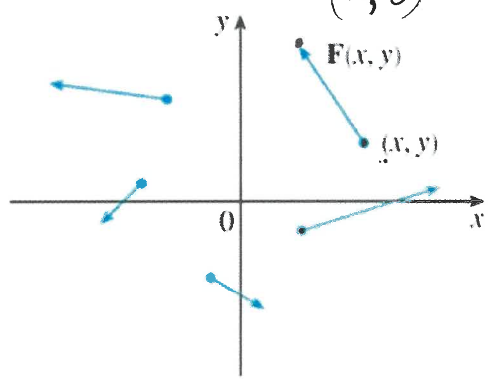
1 Definition Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function **F** that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (5 of 8)

The best way to picture a vector field is to draw the arrow representing the vector **F**(x, y) starting at the point (x, y).

Of course, it's impossible to do this for all points (x, y), but we can gain a reasonable impression of **F** by doing it for a few representative points in D as in Figure 3.

$(x, y) + F(x, y)$



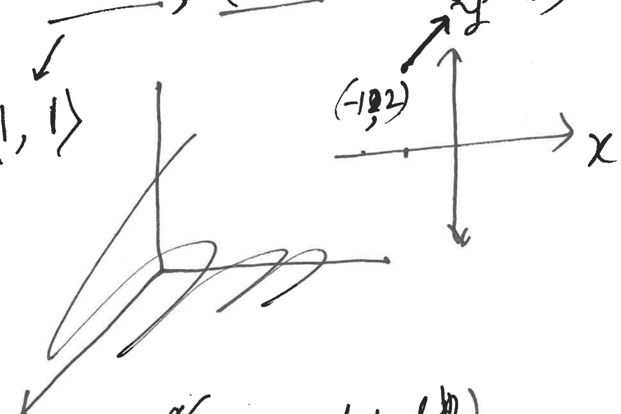
Vector field on $\mathbb{R}^2$

Figure 3

$\langle a, b \rangle$, $(c, d) \rightarrow (-1, 2)$

$\langle 1, 1 \rangle$

$(0, 3)$



$(a + c, b + d)$

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (6 of 8)

Since **F**(x, y) is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathrm{F}(x, y) = P(x, y)\mathrm{i} + Q(x, y)\mathrm{j} = \langle P(x, y), Q(x, y)\rangle$$

or, for short,

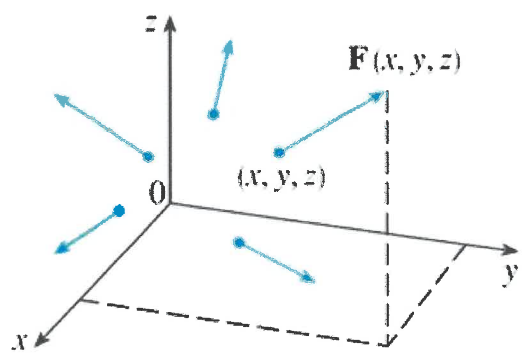
$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

2 Definition Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function **F** that assigns to each point (x, y, z) in E a three-dimensional vector **F**(x, y, z).

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (7 of 8)

A vector field **F** on $\mathbb{R}^3$ is pictured in Figure 4.



Vector field on $\mathbb{R}^3$

Figure 4

We can express it in terms of its component functions *P*, *Q*, and *R* as

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ (8 of 8)

As with the vector functions, we can define continuity of vector fields and show that **F** is continuous if and only if its component functions *P*, *Q*, and *R* are continuous.

We sometimes identify a point (x, y, z) with its position vector $\mathrm{x} = \langle x, y, z \rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$.

Then **F** becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector **x**.

Example 1

$$\mathbf{F}(x,y) = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle$$

(1, 0), (0, 1), (-1, 0), (0, -1)

$F(1,0) = \langle 1, 0 \rangle$ $(1,0)+(1,0)=(2,0)$

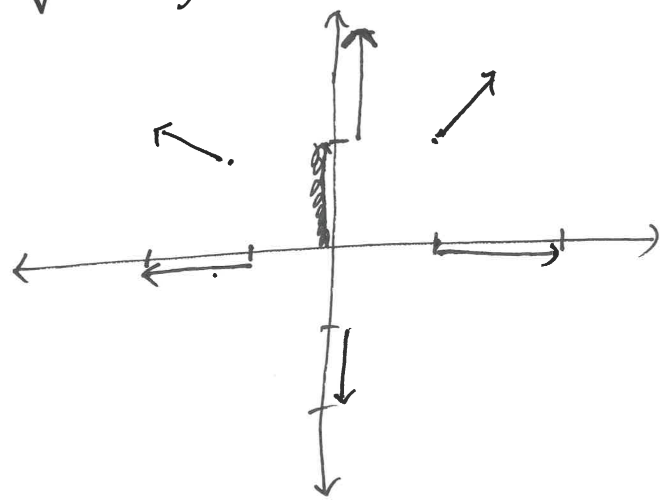
$F(0,1) = \langle 0, 1 \rangle$

$F(-1,0) = \langle -1, 0 \rangle$ $(-1,0)+(-1,0)=(-2,0)$

$F(0,-1) = \langle 0, -1 \rangle$

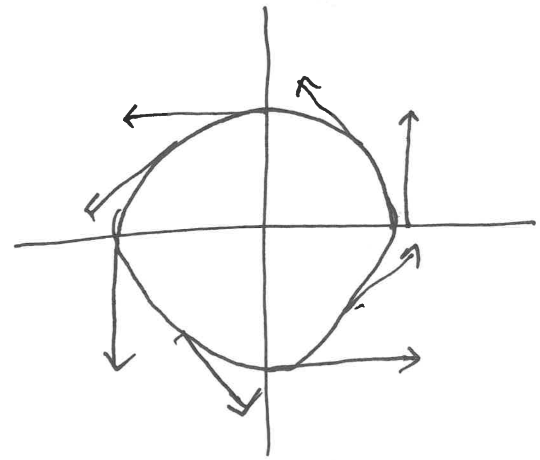
$F(1,1) = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ $(1, 1) + (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$F(1,-1) = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$ $(1,-1) + (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$



Exercise for class

Plot the vector field $\boldsymbol{F}(x,y) = \langle -y, x\rangle$ by drawing the output vectors of the points $(1,0), (0,1), (-1,0), (0,-1)$.



Gradient Fields

Gradient Fields (1 of 3)

If *f* is a scalar function of two variables, we know that its gradient ∇f (or grad *f*) is defined by

$$\nabla f(x, y) = f_x(x, y)\mathrm{i} + f_y(x, y)\mathrm{j}$$

Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if *f* is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathrm{i} + f_y(x, y, z)\mathrm{j} + f_z(x, y, z)\mathrm{k}$$

Example 2

Find the gradient vector field of $f(x, y) = ax + by$.

$f(x,y) = ax + by$

$\nabla f(x,y) = \langle a, b \rangle = a\hat{i} + b\hat{j}$

Gradient Fields (2 of 3)

A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function *f* such that $\mathrm{F} = \nabla f$.

In this situation *f* is called a **potential function** for **F**.

Not all vector fields are conservative, but such fields do arise frequently in physics.

Example 3

Every constant vector field is a conservative vector field.

Exercise for class: Find a potential function for the constant vector field $\boldsymbol{F}(x, y) = \langle a, b \rangle$.

$f(x,y) = ax + by + c$

$\nabla f(x,y) = \langle a, b \rangle = \vec{F}(x,y)$

Example 4

Not every vector field is a conservative vector field.

Example: Prove that $\boldsymbol{F}(x,y) = \langle x\text{^}2, xy\rangle$ *is not conservative.*

$\vec{F}(x,y) = (x^2, xy)$ is conservative

$\Rightarrow f(x,y)$ such that $\nabla f(x,y) = \langle x^2, xy\rangle$

$f_x = x^2 \Longrightarrow f_{xy} = 0$

$f_y = xy \Longrightarrow f_{yx} = y$

$f_{xy} = f_{yx}$

$\Rightarrow 0 = y$