MATH 243 Worksheet 2 Solutions

0: Problems from lecture slides:
a. Let $A = (2, -1, 3), B = (1, 4, -3)$. The expression for the line is $A + t(B - A) = (2, -1, 3) + t(-1, 5, -6) = (2 - t, -1 + 5t, 3 - 6t)$, with no constraints on t.
b. Let $A = (1, -2, 0), B = (3, 1, 4), C = (0, -1, 2)$. We may take

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (2, -3, 4) \times (-3, -2, -2) = (2, -8, 5)$$

to be the normal vector and A to be the point on the plane, giving the equation

$$0 = 2(x - 1) - 8(y + 2) + 5(z - 0) \Rightarrow 2x - 8y + 5z = 18.$$

c. By rewriting the first line as $-x + 0y + 2z = 10$, we see its direction is $u = \langle -1, 0, 2 \rangle$. By reading off the t coefficients, the direction of the second line is $v = \langle 0, -1, 4$. We see that u, v aren't parallel, so the lines aren't parallel. Since $u \cdot v = 8 \neq 0$, the lines aren't perpendicular either. So the answer is $\boxed{\text{neither}}$.

1: Let $\mathbf{b} = \langle 2, 4, -1 \rangle$ and $\mathbf{a} = \langle 3, -3, 1 \rangle$.

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} = \frac{6 - 12 - 1}{(3)^2 + (-3)^2 + (1)^2} = -\frac{7}{19}.$$

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a} = \left(-\frac{7}{19}\right)\langle 3, -3, 1 \rangle = \left\langle -\frac{21}{19}, \frac{21}{19}, -\frac{7}{19} \right\rangle.$$

$$\frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{b}\|^2} = \frac{6 - 12 - 1}{(2)^2 + (4)^2 + (-1)^2} = -\frac{7}{21} = -\frac{1}{3}.$$

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{b}\|^2}\right)\mathbf{b} = \left(-\frac{1}{3}\right)\langle 2, 4, -1 \rangle = \left\langle -\frac{2}{3}, -\frac{4}{3}, \frac{1}{3} \right\rangle.$$

2:

$$\overrightarrow{PQ} = \langle -4, 2, 4 \rangle, \qquad \overrightarrow{PR} = \langle 2, 1, -2 \rangle, \qquad \overrightarrow{PS} = \langle -3, 4, 1 \rangle.$$

$$\text{VOLUME} = \left| \overrightarrow{PQ} \cdot \left(\overrightarrow{PS} \times \overrightarrow{PR} \right) \right|.$$

We need $\overrightarrow{PS} \times \overrightarrow{PR}$.

$$\overrightarrow{PS} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 4 & 1 \\ 2 & 1 & -2 \end{vmatrix} = -9\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 11\hat{\mathbf{k}}.$$

Next, we compute the dot product $\overrightarrow{PQ} \cdot \left(\overrightarrow{PS} \times \overrightarrow{PR} \right) = \langle -4, 2, 4 \rangle \cdot \langle -9, -4, -11 \rangle = 36 - 8 - 44 = -16$. To find the volume, we take the absolute value.

$$\text{Volume} = |-16| = 16.$$

3: The four points lie on the same plane if and only if the paralleleliped they form has volume 0, so we compute the volume formed by $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$.

$$\text{Volume} = \left| \overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD} \right) \right|.$$

$$\overrightarrow{AB} = \langle 0,6,0\rangle, \qquad \overrightarrow{AC} = \langle 1,0,4\rangle, \qquad \overrightarrow{AD} = \langle -1,6,4\rangle.$$

$$\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 4 \\ -1 & 6 & 4 \end{vmatrix} = -24\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

$$\text{Volume} = |\langle 0,6,0\rangle \cdot \langle -24,-8,6\rangle| = |0 - 48 + 0| = 48 \neq 0.$$

Hence, the points A, B, C, D are NOT coplanar (do not lie on the same plane.)
4: Clearly the two vectors are not parallel because they are not scalar multiples of each other. Hence, the angle between the two vectors is not zero degrees, regardless of the choice of x.

Next, note that

$$\langle x,-1,1\rangle \cdot \langle x,2,3\rangle = x^2 + 1$$

Since $x^2 + 1 > 0$ for all real values of x,it follows that the dot product is strictly positive for all values of x. The dot product is positive when the angle θ is less than 90 degrees, so the answer is $\boxed{B}$.

5: To do this, we solve the system of equations. The second equation gives $x = 3 + y - 2z$. Substitute this into the first to get $5y + z + 6 = 7$, which rearranges to $z = 1 - 5y$.
Let $y = t$, then $z = 1 - 5t$ and $x = 3 + t - 2z = 1 + 11t$. Thus, the parametric equations for the line are $x = 1 + 11t, y = t, z = 1 - 5t$.

6: First we find the vectors

$$\overrightarrow{AB} = \langle 1,1,1\rangle,\ \overrightarrow{AC} = \langle 2,2,3\rangle.$$

To find a normal vector to the plane, we compute their cross product

$$\begin{aligned} \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} &= \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}\mathbf{k} \\ &= \mathbf{i} - \mathbf{j} = \langle 1,-1,0\rangle = \langle a,b,c\rangle. \end{aligned}$$

Since $(x_0, y_0, z_0) = A = (0,1,2)$ is in the plane, the scalar equation of the plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

becomes

$$\begin{aligned} x - (y - 1) + 0(z - 2) &= 0 \\ x - y + 1 &= 0 \\ x - y &= -1 \end{aligned}$$

7: The plane P_1 given by $2z = 5x+4y$ is also given by $5x+4y-2z = 0$, so it has normal vector $\mathbf{n_1} = \langle 5,4,-2\rangle$. Let P_2 be the plane we seek.
Since P_2 passes through $A = (0,-2,5)$ and $B = (-1,3,1)$, the vector $\overrightarrow{AB} = \langle -1,5,-4\rangle$ is parallel to P_2. Thus, the normal vector $\mathbf{n_2}$ to P_2 is perpendicular to $\overrightarrow{AB} = \langle -1,5,-4\rangle$.
When two planes are perpendicular, so are their normal vectors. So the normal vector $\mathbf{n_2}$ for P_2 is perpendicular to $\mathbf{n_1}$. Since $\mathbf{n_2}$ is perpendicular to $\overrightarrow{AB} = \langle -1,5,-4\rangle$ and $\mathbf{n_1} = \langle 5,4,-2\rangle$, we can use

$$\mathbf{n} = \langle -1,5,-4\rangle \times \langle 5,4,-2\rangle = \langle 6,-22,-29\rangle.$$

We can choose $A = (0,-2,5)$ for a point on the plane, so the scalar equation of P_2 is

$$\begin{aligned} 6(x - 0) - 22(y + 2) - 29(z - 5) &= 0 \\ 6x - 22y - 29z &= -101. \end{aligned}$$

8: To find the limit of a vector valued function, we find the limit of its components:

$$\lim_{t\to 1}\mathbf{r}(t)=\lim_{t\to 1}\left\langle \frac{t^2-1}{t^2-3t+2}, \frac{t-1}{\sqrt{t+3}-2}, \frac{\sin(t-1)}{t-1}\right\rangle = \left\langle \lim_{t\to 1}\frac{t^2-1}{t^2-3t+2}, \lim_{t\to 1}\frac{t-1}{\sqrt{t+3}-2}, \lim_{t\to 1}\frac{\sin(t-1)}{t-1}\right\rangle.$$

$$\lim_{t\to 1}\frac{t^2-1}{t^2-3t+2}=\lim_{t\to 1}\frac{(t+1)(t-1)}{(t-2)(t-1)}=\lim_{t\to 1}\frac{t+1}{t-2}=\frac{1+1}{1-2}=-2$$

$$\lim_{t\to 1}\frac{t-1}{\sqrt{t+3}-2}=\lim_{t\to 1}\frac{t-1}{\sqrt{t+3}-2}\left(\frac{\sqrt{t+3}+2}{\sqrt{t+3}+2}\right)=\lim_{t\to 1}\frac{(t-1)(\sqrt{t+3}+2)}{(t+3)-4}$$
$$=\lim_{t\to 1}\frac{(t-1)(\sqrt{t+3}+2)}{t-1}=\lim_{t\to 1}(\sqrt{t+3}+2)=\sqrt{4}+2=4$$

Using L'Hôpital's rule:

$$\lim_{t\to 1}\frac{\sin(t-1)}{t-1}=\lim_{t\to 1}\frac{\cos(t-1)}{1}=\frac{\cos(0)}{1}=1$$

Therefore,

$$\lim_{t\to 1}\mathbf{r}(t)=\langle -2,4,1\rangle$$

9: We can use $x=\cos(t)$, $y=\sin(t)$, and z arbitrary to get points on the cylinder. To get each point on the cylinder exactly once, let $0\le t<2\pi$. For a point on the cylinder to be on the plane as well, we need $z=y+2$. Substituting $y=\sin(t)$, we obtain $z=\sin(t)+2$. We then have the vector function

$$\mathbf{r}(t)=\langle x,y,z\rangle=\langle\cos(t),\sin(t),2+\sin(t)\rangle$$

with the constraint $t\in[0,2\pi)$.

10:

$$\frac{d}{dt}e^{t^2+2t}=(2t+2)e^{t^2+2t}$$
$$\frac{d}{dt}\ln(\cos(t))=\frac{-\sin(t)}{\cos(t)}=-\tan(t)$$
$$\frac{d}{dt}t\arctan(t)=\arctan(t)+\frac{t}{t^2+1}$$

Therefore,

$$\mathbf{r}'(t)=\left\langle (2t+2)e^{t^2+2t}, -\tan(t), \arctan(t)+\frac{t}{t^2+1}\right\rangle$$

11:

$$\int\frac{1}{t+1}\,dt=\ln|t+1|+C_1$$
$$\int\frac{1}{t^2+1}\,dt=\arctan(t)+C_2$$
$$\int\frac{t}{t^2+1}\,dt=\frac{1}{2}\int\frac{2t}{t^2+1}\,dt=\frac{1}{2}\ln(t^2+1)+C_3$$

Therefore,

$$\int\left(\frac{1}{t+1}\mathbf{i}+\frac{1}{t^2+1}\mathbf{j}+\frac{t}{t^2+1}\mathbf{k}\right)dt=\left\langle \ln|t+1|,\ \arctan(t),\ \frac{1}{2}\ln(t^2+1)\right\rangle+\langle C_1,C_2,C_3\rangle$$

12: We first calculate $\mathbf{u}'(t)$ and $\mathbf{v}'(t)$ as we will use them over and over again. We have

$$\mathbf{u}'(t) = \langle 1, 2\cos(t), -3/t \rangle$$
$$\mathbf{v}'(t) = \langle 0, -3t^2, 3e^{3t} \rangle.$$

a) $\frac{d}{dt}(\mathbf{u} - 2\mathbf{v}) = \mathbf{u}' - 2\mathbf{v}' = \langle 1, 2\cos(t) + 6t^2, -3/t - 6e^{3t} \rangle$
b) We use the product rule for cross products. We have $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = (\mathbf{u}' \times \mathbf{v}) + (\mathbf{u} \times \mathbf{v}')$. We need to calculate both of these cross products in order to get our answer. We find

$$\mathbf{u}' \times \mathbf{v} = \langle 2e^{3t}\cos(t) - 3t^2, -e^{3t}, -t^3 \rangle$$
$$\mathbf{u} \times \mathbf{v}' = \langle 6e^{3t}\sin(t) - 9t^2\ln(t), -3e^{3t}(t+1), -3t^2(t+1) \rangle.$$

Therefore we have

$$\begin{aligned}\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= \left\langle 2e^{3t}\cos(t) - 3t^2 + 6e^{3t}\sin(t) - 9t^2\ln(t), -e^{3t} - 3e^{3t}(t+1), -t^3 - 3t^2(t+1) \right\rangle \\ &= \left\langle 2e^{3t}\cos(t) - 3t^2 + 6e^{3t}\sin(t) - 9t^2\ln(t), -3te^{3t} - 4e^{3t}, -3t^2 - 4t^3 \right\rangle\end{aligned}$$

c) We use the product rule for dot products. That gives

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u}' \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{v}') = \left(-2t^3\cos(t) - \frac{3e^{3t}}{t}\right) + \left(-6t^2\sin(t) - 9e^{3t}\ln(t)\right).$$

d) Let $f(t) = t^2 + 2$. Then we have $f'(t) = 2t$. Finally by the chain rule we have

$$\frac{d}{dt}\mathbf{u}(f(t)) = f'(t)u'(f(t)) = 2t\left\langle 1, 2\cos(f(t)), \frac{-3}{f(t)} \right\rangle = \left\langle 2t, 4t\cos(t^2+2), \frac{-6t}{t^2+2} \right\rangle.$$