Quiz graded by tomorrow's class

DW3 was posted overnight, pre-lec video a few hours ago. Goal: post 6/18 materials & WA HW2, HW3 today

Higher Order Partials

Coming by tomorrow: feedback survey, looking for ways to improve class quality further

Pre-lecture for 6/17

$\partial^2 f$

$f_{xxyxy} = ((((f_x)_x)_y)_x)_y$

$\frac{\partial^2}{\partial u \partial v}$

$\frac{\partial}{\partial v}\frac{\partial}{\partial u} f$

Definitions and Notations

- For any variables u & v, define f_{uv} as $(f_u)_v$
- Can also use operator and fraction notation
  - $\partial^2 f/(\partial u\, \partial v)$, $\partial_v(\partial_u f)$
- Extends to more than 2 variables

Function $f(x,y)$

$f\ ,\ \underbrace{f_x,\ f_y}_{\text{1st ord.}}$

2nd ord:

$f_{xx}, f_{xy}, f_{yx}, f_{yy}$

$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\frac{\partial}{\partial x} f$

$f(x,y) = x^2 + y^2$

$f_{xx} = (2x)_x = 2$

$f_{yy} = 2$

$f_{xy} = 0$

Switching Variables

Do we have $f_{xy} = f_{yx}$ for any function $f(x,y)$ where these exist?

- Clairaut's Theorem: yes, but need f_{xy}, f_{yx} continuous around the point
- Can also switch two variables for $f(x, y, z, \ldots)$
- Use it repeatedly to move around more than 2 variables

$(x,y) = (x_0, y_0)$

we also have $f_{xz} = f_{zx}$ for $f(x,y,z)$ w/ f_{xz} & f_{zx}

need f_{xy}, f_{yx} cont. in entire

red region

continuous

In practice, f_{xy} & f_{yx} will be continuous everywhere on their domains

$f_{xyxz} =$

$$(f_{xyx})_z = ((f_{xy})_x)_z = (f_{xy})_{xz} =$$

$(f_{xy})_{zx}$ using Clairaut

Repeat: $f_{xyzx} = f_{xzyx} = f_{zxyx}$

$$f(x,y) = yx^3 + y^3x \qquad (f_{xy}) = (f_x)_y =$$

$(f_y)_x = (x^3 + 3y^2x)_x$ $\qquad$ $(3x^2y + y^3)_y$

$= 3x^2 + 3y^2$ $\qquad$ $= 3x^2 + 3y^2$

Practice Problems

Find these higher order partial derivatives

- f_{xy}, f_{xx}, f_{yy} for $f(x, y) = e^{xy}$
- g_{xyz} for $g(x, y, z) = \cos(z+\sin(y+x))$
- h_{zzyzx} for $h(x,y,z) = z^3y^2\ln(x)$

Check the conclusion of Clairaut's Theorem for $f(x, y, z) = xe^{yz}$

https://upload.wikimedia.org/wikipedia/commons/a/a6/Zzyzx_Rd.jpg

Scratchwork

$f(x,y) = e^{xy}$

$f_x = (e^{xy})_x = \underline{(xy)_x}\, e^{xy}$

$= \underline{y} e^{xy}$, sim. $f_y = x e^{xy}$

$f_{xx} = (f_x)_x = (y e^{xy})_x =$

$y(e^{xy})_x = y^2 e^{xy}$

y is const.

Similarly, $f_{yy} = x^2 e^{xy}$

$f_{xy} = (f_x)_y =$

$(y e^{xy})_y = y_y e^{xy}$

$+ y (e^{xy})_y =$

$1 \cdot e^{xy} + y x e^{xy} =$

$(1+xy) e^{xy}$

$g(x,y,z) = \cos(z + \sin(y+x))$

Goal: $g_{xyz} = g_{zyx}$

We moved z over since it's more convenient to take first.

$$g_z = -\sin(z+\sin(y+x)) \cdot$$

$$(z+\sin(y+x))_z = -\sin(z+\sin(y+x))$$

$$g_{zy} = (g_z)_y = -\cos(\cdots) \cdot$$

$$(z+\sin(y+x))_y =$$

$$-\cos(z+\sin(y+x))\cos(y+x)(y+x)_y$$

$$= -\cos(z+\sin(y+x))\cos(y+x)$$

$$g_{zyx} = (g_{zy})_x =$$

$$\underline{-\cos(z+\sin(y+x))\cos(y+x)_x} - \overline{\cos(z+\sin(y+x))_x}\underline{\cos(y+x)}$$

Red: $-\cos(\cdots)\cdot -\sin(y+x)\cdot 1 =$

$$\cos(z+\sin(y+x))\sin(y+x)$$

O: $\cos(z+\sin(y+x))_x =$

$$-\sin(z+\sin(y+x))\cos(y+x)$$

G: $-\sin(z+\sin(y+x))\cos(y+x)^2$

Ans: $\cos(z+\sin(y+x))\sin(y+x)$

$$+\sin(z+\sin(y+x))\cos^2(y+x)$$

$h(x,y,z) = z^3y^2\ln x$, goal is

$h_{zzyzx} = h_{zzzyx}$

$h_z = 3z^2y^2\ln x$

$h_{zz} = 6zy^2\ln x$

$h_{zzz} = 6y^2\ln x$

$h_{zzzy} = 12y\ln x$

$h_{zzzyx} = \dfrac{12y}{x}$

$f(x,y,z) = xe^{yz}$

Need to check:

$f_{xy}=f_{yx}$, $f_{xz}=f_{zx}$, $f_{yz}=f_{zy}$

So calc. those 6 things

Calculate f_x, f_y, f_z first

for later use:

$f_x = e^{yz}$

$f_y = xze^{yz}$

$f_z = xye^{yz}$

$$(f_{xy}) = (f_x)_y = (e^{yz})_y = ze^{yz} = (xze^{yz}) = (f_y)_x$$

$$f_{xz} = (f_x)_z = (e^{yz})_z = ye^{yz} = (xye^{yz}) = (f_z)_x = f_{zx}$$

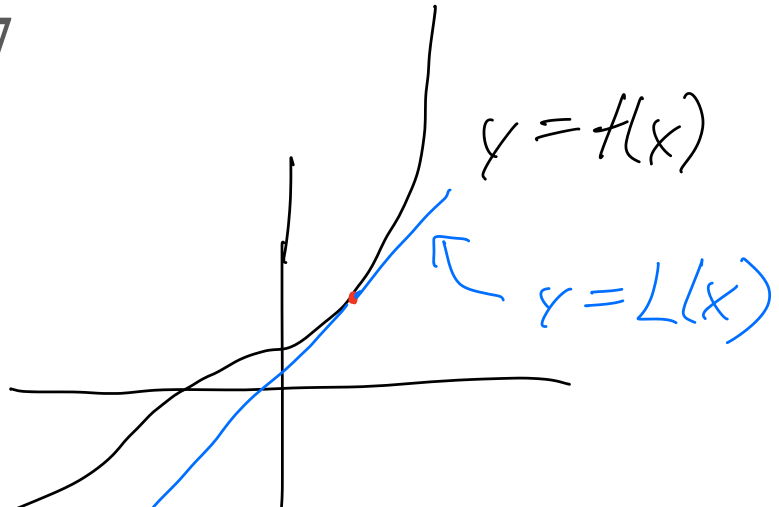
$$f_{yz} = (xze^{yz})_z = x(ze^{yz})_z = x(e^{yz} + zye^{yz})$$

$$f_{zy} = (xye^{yz})_y = x(1 \cdot e^{yz} + yze^{yz})$$

So $\quad f_{yz} = f_{zy} = x(1+yz)e^{yz}$

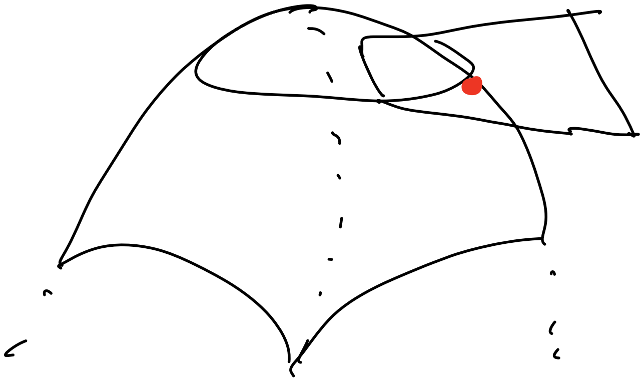
Linear Approximations

Lecture for 6/17



Idea Behind Linear Approximation

- Recall tangent line at x_0 good approximation for $y = f(x)$
- If line equation is $y = L(x)$, then $L(x) \approx f(x)$ for $x \approx x_0$
- So let's use tangent plane to $z = f(x,y)$
- Express plane equation as $z = P(x, y)$ and use it



2D graph becomes flatter until it looks like line

3D graph becomes flatter until it looks like plane

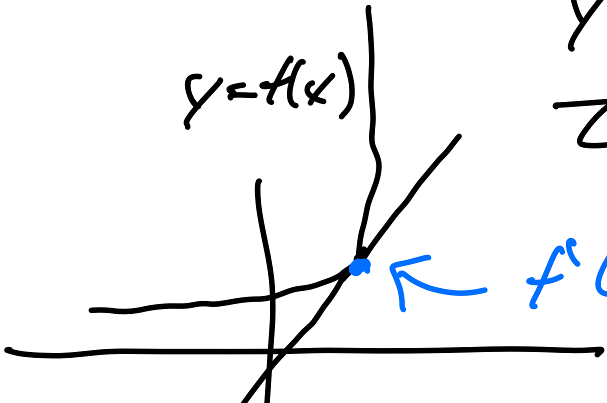
The Approximation

- Recall plane equation: $z = \overbrace{f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)}^{P(x,y)}$
- So $f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0)+ f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y$
- What happens if f_x and f_y not continuous?

plug in $x = x_0 + \Delta x$

$y = y_0 + \Delta y$

$z = f(x,y) = f(x_0+\Delta x, y_0+\Delta y)$



$f'(x)$ not continuous there

Differentiability

Let's define differentiability:

- Let $\mathbf{v}$ = (x,y), write plane equation as z = P($\mathbf{v}$)
- f($\mathbf{v}$) is differentiable at $\mathbf{v} = \mathbf{c}$ if $\lim_{\mathbf{v}\to\mathbf{c}} \underbrace{[f(\mathbf{v}) - P(\mathbf{v})]/\|\mathbf{v}-\mathbf{c}\|}_{\text{diff quotient}} = 0$
  - Same when f has more than 2 variables

Relation to partial derivatives:

- If f_x, f_y exist and are continuous, then f differentiable
- If f diff, then f_x and f_y exist
- But f diff. does not mean f_x and f_y ~~exist~~ continuous

$[f_x \,\&\, f_y \text{ cont.}] \geq [f \text{ diff.}]$

$P(x_0+\Delta x, y_0+\Delta y)$

So $f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + \underbrace{f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y}$

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) - f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y \approx 0$$

$$f(x_0+\Delta x, y_0+\Delta y) - P(x_0+\Delta x, y_0+\Delta y) \approx 0$$

$c = (x_0, y_0)$, then $f(v) - P(v) \approx 0$

when $v \approx c$ (i.e., Δx & Δy are small)

How well is the approximation?

Ans: for linear approx, we expect at worst linear error

So if $v - c \approx t$ units approx off by Kt

So if $\|v-c\| \approx t$, $f(v)-P(v) \le Kt$

Sub in $\|v-c\|=t$: $f(v)-P(v) \le K\|v-c\|$

divide by $\|v-c\|$: $\dfrac{f(v)-P(v)}{\|v-c\|} \le K$

We want approximation to be better than any linear scaling, so we want $\dfrac{f(v)-P(v)}{\|v-c\|} \le K$

eventually for any $K>0$

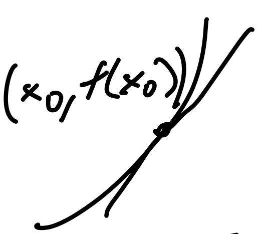
So by def. of limit, we want $\lim_{v \to c} \dfrac{f(v)-P(v)}{\|v-c\|} \to 0$

Extra explanation of limit:

Recall $f'(x_0) = \lim_{h \to 0} \dfrac{f(x_0+h)-f(x_0)}{h} \Rightarrow$

$$\lim_{h \to 0} \frac{f(x_0+h)-[f(x_0)+hf'(x_0)]}{h} = 0 = \lim_{h \to 0} \frac{f(x_0+h)-L(x_0+h)}{h}$$

Let $L(x)$ be tangent line



Slope $= f'(x_0)$

$y - f(x_0) = f'(x_0)(x-x_0)$

$y = f'(x_0)(x-x_0) + f(x_0)$

$L(x) = f'(x_0)(x-x_0) + f(x_0)$

$L(x_0+h) = hf'(x_0) + f(x_0)$

$$= \underbrace{\lim_{x \to x_0} \frac{f(x)-L(x)}{x-x_0}}_{\text{Diff of 1 var}}$$

So all we are doing is replacing L with P and $x-x_0$ w/ $\|v-c_0\|$

Diff of mult. var

Relation to Partial Derivatives

- If f_x, f_y exist and are continuous, then f differentiable
- If f diff, then f_x and f_y exist
- But f diff. does not mean f_x and f_y exist

] - exercise

f diff at $x=c$ $\Rightarrow$ $\dfrac{f(v)-p(v)}{\|v-c\|} \to 0$

must be defined

In order for — to be defined, $p(v)$ needs to be defined

In order for $p(v)$ to be defined, need f_x & f_y def.

because they are used in equation for $P(v)$

Suppose f_x, f_y cont. We may assume we are talking about f at $(0,0)$ by translation of

$f: f(x,y) \longrightarrow f(x-c_1, y-c_2)$ translates (c_1, c_2) to $(0,0)$

When $c=0$: $\lim_{v\to c} \dfrac{f(v)-P(v)}{\|v-c\|} = \lim_{v\to 0} \dfrac{f(v)-P(v)}{\|v\|}$

$$= \lim_{(x,y)\to(0,0)} \frac{f(v)-[f(0)+f_x(\vec{0})x+f_y(\vec{0})y]}{\|v\|} =$$

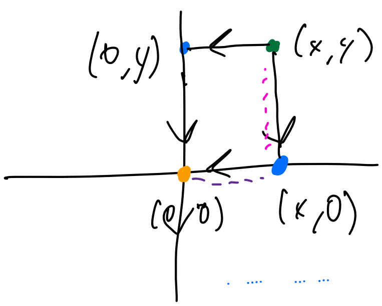
shift f

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-[f_x(0,0)x+f_y(0,0)y]}{\sqrt{x^2+y^2}}$$

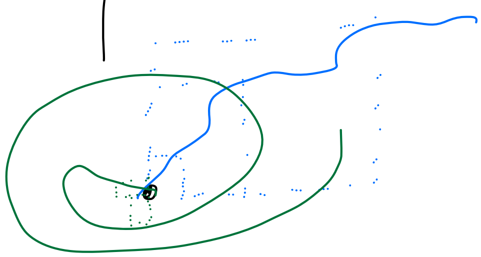
so that $f(0,0)=0$

Replace f with $f-c$ if $f(0)=c$

But no tactic we know about from yesterday's pre-lecture applies here. No factoring, no cancelling, polar doesn't seem to do anything, directions can only show non-existence but we want existence, nothing to squeeze w/ sq. lem.



This rectangular path can vary depending on how x & y vary



We can't just take the rectangle, but observe that

$$f(x,y)=\underline{f(x,y)-f(x,0)}+\underline{f(x,0)-f(0,0)}+f(0,0)$$

So $f(x,y)=\underline{[f(x,y)-f(x,0)]}+\underline{[f(x,0)-f(0,0)]}$

$f(x,0)-f(0,0)=\dfrac{f(x,0)-f(0,0)}{x-0}\,x$, so let's turn this into ordinary diff quot.

Let $g(x)=f(x,0)$, it's $\dfrac{g(x)-g(0)}{x-0}\cdot x$

Now we are stuck, but where else have we seen diff. quot before? Ans: mean value theorem from Calc I

$$\frac{g(x)-g(0)}{x-0} = g'(c_x) \text{ for some } c_x \in (0,x)$$

By lec exercise, $g' = f_x$, so $g'(c_x) = f_x(c_x, 0)$

$$\frac{g(x)-g(0)}{x-0}x = g'(c_x)x = \underline{f_x(c_x,0)x}$$

$$h_x(y) = f(x,y) \Rightarrow f(x,y) - f(x,0) = \frac{f(x,y)-f(x,0)}{y-0} \cdot y$$

$$= \frac{h_x(y)-h_x(0)}{y-0} \cdot y = h_x'(b_y)y \text{ for some}$$

$b_y \in (0,y)$.

Notice! c_x depends on x, b_y depends on y

and $h_x'(y) = \frac{d}{dy}h_x(y) = \frac{\partial f}{\partial y}(x,y) = f_y(x,y)$

because of practice problem from yestdy

$$h_x'(b_y)y = \underline{f_y(x,b_y)\,y}$$

So $f(x,y) = f_x(c_x,0)x + f_y(x,b_y)y$

$$f(x,y) - [f_x(0,0)x + f_y(0,0)y] =$$

$$\underline{(f_x(c_x,0) - f_x(0,0))x + (f_y(x,b_y) - f_y(0,0))y} =$$

Since $\sqrt{x^2+y^2}$ is in denominator, let's try

$x = r\cos\theta$, $y = r\sin\theta$ because $\sqrt{x^2+y^2} = \sqrt{r^2(\cos^2\theta+\sin^2\theta)} = \sqrt{r^2} = r$ in pol.

Meanwhile, green $= (\cdots)\, r\cos\theta + (\cdots)\, r\sin\theta =$ $r[(\cdots)\cos\theta + (\cdots)\sin\theta]$, so entire expression is

$$\underbrace{(f_x(c_x,0)-f_x(0,0))}_{\lim = 0 \text{ by continuity of } f_x}\cos\theta + \underbrace{(f_y(x,b_y)-f_y(0,0))}_{\lim = 0 \text{ by continuity of } f_y}\sin\theta$$

As $(x,y)\to(0,0)$, $(x,b_y)\to(0,0)$ since $b_y \le y$, so $f_y(x,b_y)\to f_y(0,0)$ by continuity & composition rule of limits

when we have fg with $f\to 0$ but g is moving around, can use squeeze thm.

$|(\cdots)\cos\theta + (\cdots)\sin\theta| \le |\cdots||\cos\theta| + |\cdots||\sin\theta| \le$ $|\cdots| + |\cdots| = |f_x(c_x,0)-f_x(0,0)| + |f_y(x,b_y)-f_y(0,0)|$ $\to 0+0 = 0$ by cont. of f_x & f_y

So original limit exists & equals 0 by squeeze theorem.

There is a similar, but slightly even

longer proof of Clairaut's theorem which I will not cover. It also relies on mean value theorem. I could make a bonus video on its proof if requested

CL Thm. is the first hard theorem in Multi-variable calc, which is why it's named

Clairaut didn't actually prove it. His proof failed, and then 10 mathematicians after him from ≈ 1760 to ≈ 1860 all failed as well. In ≈ 1865, someone found a mistake in all 10, then ≈ 1890 it finally got proven with MVT (but I forgot who, you can look it up)

$$\int f'(g(x))\, g'(x)\, dx = \int d(f\circ g) = (f\circ g)(x) + C$$

$\int dy = \int f'(x)\,dx = f(x)+C$ by FTOC

Differentials

- Recall dy = f'(x) dx when y = f(x)
- If x and y slightly change, how does z = f(x,y) change?
- $dz = df = f_x\,dx + f_y\,dy$

How did we get $\frac{dy}{dx} = f'(x)$ & $dy = dx\,f'(x)$?

Recip: change in x is $(x+dx) - x = dx$,

then change in y is $y|_{x+dx} - y|_x = f(x+dx) - f(x)$,

so $dy = f(x+dx) - f(x)$,

So $\dfrac{dy}{dx} = \dfrac{f(x+dx)-f(x)}{dx} = f'(x)$

since dx infinitesimal

If x changes by dx, y by dy, then

z changes by $f(x+dx, y+dy) - f(x,y)$

But: $f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$

As Δx becomes smaller & smaller, this becomes more accurate. So it's exact for dx & dy:

$$f(x+dx, y+dy) - f(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

since that is what $z = f(x,y)$ changes by

Try these before discussion

Practice Problems

Explain why $(x+y)/(x^2+y^4+1)^{1/2}$ is differentiable everywhere

A sphere has center (2, 1, -1) and contains (4.01, 3.02, 0.99). Find a good approximation for the radius of the sphere without using a calculator

Find a function f where f_x, f_y always exist, but f is not differentiable

Goal: next lecture should go faster so that there's time for practice problems instead of moving them to discussion

Scratchwork

Explain why

$$\frac{x+y}{\sqrt{x^2+y^4+1}}$$ is diff.

$x+y$ is diff. as the sum of 2 diff. func.

x^2+y^4+1 is similarly differentiable & then

$\sqrt{x^2+y^4+1}$ is diff. by chain rule since x^2+y^4+1 is diff. & $t \to \sqrt{t}$ is diff except at $t=0$.

So check ~~that~~ $x^2+y^4+1 = x^2+(y^2)^2+1 \geq 0+0+1 = 1 > 0$, so it's not 0.

Now we have quotient of 2 diff. functions and denom is $\neq 0$, so quotient is diff.

$(2,1,-1) =$ center

point on sphere is $(4.01, 2.02, 0.99)$.

so radius $= \sqrt{(4.01-2)^2 + (2.02-1)^2 + (0.99+1)^2}$

$= \sqrt{2.01^2 + 1.02^2 + 1.99^2}$ but you can't use a calculator! So what can we do?

Ans: try a linear approximation

Let $f(x,y,z)=\sqrt{x^2+y^2+z^2}$, we seek $f(2.01, 1.02, 1.99)$. Notice we have a nice value: $f(2,1,2)=\sqrt{2^2+1^2+2^2}$

$=\sqrt{9}=3$.

Set $\Delta x=0.01$, $\Delta y=0.02$, $\Delta z=-0.01$

$f(2.01, 1.02, 1.99)=f(2+\Delta x, 1+\Delta y, 2+\Delta z)\approx$

$f(2,1,2)+f_x\Delta x+f_y\Delta y+f_z\Delta z$

$f_x=\dfrac{(x^2+y^2+z^2)_x}{\quad}$ because $(\sqrt{t})'=\dfrac{1}{2\sqrt{t}}$

$f_x \quad \dfrac{\quad}{2\sqrt{x^2+y^2+z^2}}$

$= \dfrac{x}{\sqrt{x^2+y^2+z^2}}$ so $f_x(2,1,2) = \dfrac{2}{3} = \dfrac{2}{3}$

Similarly, $f_y = \dfrac{y}{\sqrt{\cdots}} \Rightarrow f_y(2,1,2) = \dfrac{1}{3}$

$f_z(2,1,2) = 2/3$

$f(2,1,2) + \dfrac{2}{3}\Delta x + \dfrac{1}{3}\Delta y + \dfrac{2}{3}\Delta z =$

$3 + \cancel{\dfrac{2}{3}\cdot 0.01} + \dfrac{1}{3}\cdot 0.02 - \cancel{\dfrac{2}{3}\cdot 0.01} =$

$3 + 0.00\overline{6} = 3.00666\ldots$

Test Calc: 3.00675