HW1–5 last chance: turn in by 6/11 11:59pm

Optional Quiz 4 up, due 11:59pm today

Discussion worksheets have extra practice now

Even more practice: DW6 bonus on spherical & cylindrical

Solutions to be posted: DW5–8, Q0 2 3, M2

More features TBA

Final! registrar has allowed async, due by 7/13 (as long as course grades are submitted by deadline, they don't care).

So final opens 7/12 2pm, closes 7/13 11:59pm

You may take the extra day to practice & prepare more, but do not take it for granted and wait until last minute to do things.

Lecture today: 3rd review session for final

Student suggested topics | subtopics: surface integrals of vector fields & related theorems

Student suggested problems:

Selected: DW8 1abcd recycled for surf int of vec fields, also do DW8 3a

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DW8 3a: Find $\iint_S F \cdot dS$, $F = (\sin(\pi x), zy^3, z^2+4x)$, S is surface of box $-1<x<2$, $0<y<1$, $1<z<4$ oriented pointing out of the box.

Let B be the box, then $S = \partial B$ by definition, and $\vec{n}$ out of box $\Rightarrow$ S pos orient. as a closed surface, so $\iint_S F \cdot dS = \iiint_B (\nabla \cdot F)\, dV$.

$\nabla \cdot F = \pi\cos(\pi x) + 3y^2z + 2z$, so now we have prismatic triple integral

$$\int_1^4\int_0^1\int_{-1}^2 (\pi\cos\pi x + 3y^2z + 2z)\,dx\,dy\,dz =$$

$$\int_1^4\int_0^1 \Big(\underbrace{\sin\pi x \Big|_{x=-1}^{x=2}}_{0} + 9y^2z + 6z\Big)\,dy\,dz =$$

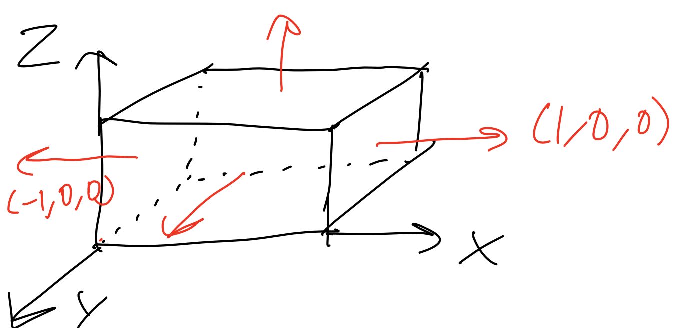
$$\int_1^4 (3y^3z + 6yz)\Big|_{y=0}^{y=1} dz = \int_1^4 (3z+6z)\,dz =$$

$$\frac{9}{2}z^2\Big|_1^4 = \frac{9}{2}\cdot 15 = \frac{135}{2}$$

Now we do it without divergence theorem.

To save time, observe / recall $F \cdot d\vec{S} = (F \cdot \vec{n})\,dS$.

This is more convient as the $\vec{n}$ will be simple.



Specifically, there are 6 different faces for S, and $\vec{n} = (\pm 1, 0, 0)$ $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ for them.

Faces for S: $x=-1$, $x=2$, $y=0$, $y=1$, $z=1$, $z=4$

$x = -1 \Rightarrow \vec{n} = (-1, 0, 0)$, so $\vec{F} \cdot \vec{n} = \sin\pi x =$

$$\boxed{\begin{array}{l} F = (\sin(\pi x), zy^3, z^2+4x), \\ -1<x<2,\ 0<y<1,\ 1<z<4 \end{array}}$$

$\sin(-\pi) = 0$, so

$$\iint_{S\cap\{x=-1\}} F \cdot d\vec{S} = 0.$$

As $\sin(2\pi) = 0$, $\iint_{x=2} F \cdot d\vec{S} = 0$ as well.

$zy^3 = 0$ & $\vec{n} = (0,-1,0)$ for $y=0$, so $\iint_{y=0} = 0$.

$$\iint_{y=1} F \cdot d\vec{S} = \iint_{y=1} zy^3 = \iint_{y=1} z := A$$

$$\iint_{z=1} F \cdot d\vec{S} = \iint_{z=1} -z^2 - 4x = -\left(\iint_{z=1} 1+4x\right) = -B.$$

$$\iint_{z=4} F \cdot dS = \iint_{z=4} 16+4x = 4\left(\iint_{z=4} 4+x\right) = 4C.$$

Face $y=1 \Rightarrow -1<x<2$ & $1<z<4$, so

$$A = \int_{-1}^{2}\int_{1}^{4} z\,dz\,dx = \frac{4^2-1^2}{2}\cdot 3 = \underline{\frac{45}{2}}.$$

Similarly, $B = \int_0^1\int_{-1}^{2}(1+4x)\,dx\,dy = \left(x+2x^2\right)\Big|_{-1}^{2} =$

$3+2\cdot 3 = \underline{9}$

$$C = \int_0^1\int_{-1}^{2}(4+x)\,dx\,dy = \left(4x+\frac{x^2}{2}\right)\Big|_{-1}^{2} = 12+\frac{3}{2} = \underline{\frac{27}{2}}$$

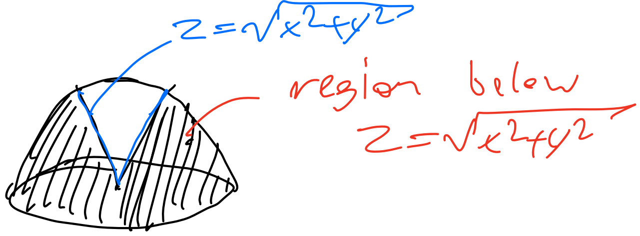
So $\iint_S = A - B + 4C = \frac{45}{2} - 9 + 54 = \frac{135}{2}$.

DW 6 bonus Q5: find volume of solid within $x^2+y^2+z^2 = 4$, above xy-plane, below $z=\sqrt{x^2+y^2}$.

Use spherical: within $\ldots \Rightarrow 4 \geq x^2+y^2+z^2 = \rho^2$

$\Rightarrow 2 \geq \rho$. Also, $\rho \geq 0$. Above xy-plane $\Rightarrow$

$0 \leq z = \rho\cos\varphi \Rightarrow 0 \leq \cos\varphi \Rightarrow 0 \leq \varphi \leq \frac{\pi}{2}$.



So below $\Rightarrow$ $z \le \sqrt{x^2+y^2}$

$\rho\cos\varphi \le \rho\sin\varphi \Rightarrow$

$\cos\varphi \le \sin\varphi \Rightarrow \varphi \ge \frac{\pi}{4}$,

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \\ dV = \rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta \end{cases}$$

So our bounds are $\varphi \in [\frac{\pi}{4}, \frac{\pi}{2}]$. $\rho \in [0, 2]$, $\theta \in [0, 2\pi]$.

$$V = \iiint\limits_{\text{region}} dV = \int_0^{2\pi}\int_{\pi/4}^{\frac{\pi}{2}}\int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta =$$

$$2\pi\left(\int_{\pi/4}^{\pi/2} \sin\varphi\right)\left(\int_0^2 \rho^2\right) = 2\pi \cdot \frac{2^3}{3} \cdot (-\cos\varphi)\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{16\pi}{3}\left(\cos\frac{\pi}{4}\right) = \frac{8\pi}{3}\sqrt{2}.$$

Makes sense, because it's $\frac{\sqrt{2}}{2} \approx .707 \approx 70\%$ of the volume of the whole hemisphere

Only use this method for checking your work if you trust your estimation and/or graphing skills are good enough