This worksheet covers Chap14.6 Directional Derivatives and the Gradient Vector, Chap14.7 Maximum and Minimum Values.

Directional Derivatives and the Gradient Vector

1. Find the directional derivative of $f(x,y) = xy^3 - x^2$ at the given point $(1,2)$ in the direction indicated by the angle $\theta = \pi/3$.

The unit vector in the direction of $\theta = \pi/3$ is given by $\mathbf{u} = \langle \cos(\pi/3), \sin(\pi/3) \rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$. The gradient of f is $\nabla f = \langle f_x, f_y \rangle = \langle y^3 - 2x, 3xy^2 \rangle$. Hence, we have that

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \langle 6, 12 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = 3 + 6\sqrt{3}.$$

2. Find the directional derivative of the function $f(x,y) = \dfrac{x}{y^2}$ at the point $P(3,-1)$ in the direction of the point $Q(-2,11)$.

We first need to form our direction vector $\overrightarrow{PQ} = \langle -2-3, 11-(-1) \rangle = \langle -5, 12 \rangle$. Next, we find the unit vector $\mathbf{u}$ in the direction of

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{||\overrightarrow{PQ}||} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle.$$

Next,

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \left\langle \frac{1}{y^2}, -\frac{2x}{y^3} \right\rangle \quad \rightarrow \quad \nabla f(3,-1) = \langle 1, 6 \rangle.$$

Thus the directional derivative is

$$D_{\mathbf{u}} f(3,-1) = \nabla f(3,-1) \cdot \mathbf{u} = \langle 1, 6 \rangle \cdot \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle = \frac{67}{13}.$$

3. Given a surface of equation $xy^2z^3 = 8$ and a point $P(2,2,1)$ on this surface, find the equations of the tangent plane and parametric equations of the normal line to the given surface at the specified point P.

Let's define $F(x,y,z) = xy^2z^3$. Then the surface $xy^2z^3 = 8$ is a level surface of F, and we know that $\nabla F(2,2,1)$ is a vector normal to the level surface $xy^2z^3 = 8$ at the point $(2,2,1)$.
We compute the gradient of F

$$\nabla F(x,y,z) = \langle f_x, f_y, f_z \rangle = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle \quad \rightarrow \quad \nabla F(2,2,1) = \langle 4, 8, 24 \rangle.$$

Then, an equation of the tangent plane is

$$4(x-2) + 8(y-2) + 24(z-1) = 0,$$

and parametric equations of the normal line are

$$x(t) = 2 + 4t, \quad y(t) = 2 + 8t, \quad z(t) = 1 + 24t$$

4. Determine the gradient of the function $f(x, y, z) = x^2y^3 - 4xz$ and use the gradient to determine the directional derivative of $f(x, y, z)$ at $(1, 1, 1)$ in the direction of $\mathbf{v} = \langle -1, 2, 0 \rangle$.

We compute the gradient $\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle = \langle 2xy^3 - 4z, 3x^2y^2, -4x \rangle$.

To find the directional derivative, we first need to find the unit vector in the direction of $\mathbf{v}$ with $||\mathbf{v}|| = \sqrt{1 + 4 + 0} = \sqrt{5}$, which is

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{\langle -1, 2, 0 \rangle}{\sqrt{5}} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \right\rangle.$$

Next, since $\nabla f(1, 1, 1) = \langle -2, 3, -4 \rangle$, then

$$D_{\mathbf{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = (-2) \cdot \left(-\frac{1}{\sqrt{5}} \right) + 3 \cdot \left(\frac{2}{\sqrt{5}} \right) + (-4) \cdot 0 = \frac{10}{\sqrt{5}}.$$

5. Find the maximum rate of change of $f(x, y) = \sin(xy)$ at the point $(1, 0)$ and the direction in which it occurs.

The maximum rate of change refers to the largest directional derivative with respect to some unit vector $\mathbf{u}$. The directional derivative of f is given by

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \|\mathbf{u}\| \cos(\theta) = \|\nabla f\| \cos(\theta),$$

where θ is the angle between ∇f and $\mathbf{u}$. Notice the expression on the right above is maximized when $\cos(\theta) = 1$ which occurs exactly when $\theta = 0$ (i.e. when ∇f and $\mathbf{u}$ point in the same direction). Hence, the maximum directional derivative of f is $\|\nabla f\|$ (i.e. the magnitude of the gradient) and it occurs in the direction of the gradient. The gradient of the given f is $\nabla f = \langle y \cos(xy), x \cos(xy) \rangle$.
Thus, the direction of the maximum rate of change of f at $(1, 0)$ is:

$$\nabla f(1, 0) = \langle 0, 1 \rangle,$$

and the maximum rate of change is:

$$\|\nabla f(1, 0)\| = 1.$$

6. Find the maximum rate of change of $f(x, y, z) = e^{2x} \cos(y - 2z)$ at the point $(1, \pi, 0)$ and the direction in which this maximum rate of change occurs.

We have

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2e^{2x} \cos(y - 2z),\ -e^{2x} \sin(y - 2z),\ 2e^{2x} \sin(y - 2z) \rangle.$$

The maximum rate of change at the point $(1, \pi, 0)$ is simply the gradient at the point $(1, \pi, 0)$. So, the direction in which the maximum rate of change of the function occurs is

$$\nabla f(1, \pi, 0) = \langle 2e^2 \cos(\pi),\ -e^2 \sin(\pi),\ 2e^2 \sin(\pi) \rangle = \langle -2e^2,\ 0,\ 0 \rangle.$$

Since the maximum rate of change is the magnitude of the gradient, then

$$||\nabla f(1, \pi, 0)|| = ||\langle -2e^2,\ 0,\ 0 \rangle|| = \sqrt{4e^4 + 0 + 0} = 2e^2$$

Maximum and Minimum Values

7. Find all the critical points of the following function.

$$f(x, y) = (y - 2)x^2 - y^2.$$

The first partial derivatives of $f(x, y)$ are

$$f_x(x, y) = 2(y - 2)x \quad \text{and} \quad f_y(x, y) = x^2 - y.$$

Let's solve $f_x(x, y) = 0$:

$$f_x = 0 \quad \to \quad 2(y - 2)x = 0 \quad \to \quad y = 2 \text{ or } x = 0.$$

As shown above we have two possible options. We can plug each into $f_y(x, y) = 0$ to get the critical points.

(1) $y = 2 : f_y(x, y) = x^2 - 2y = 0 \quad \to \quad x^2 - 4 = 0 \quad \to \quad x = \pm 2 \quad \to \quad (2, 2)$ and $(-2, 2)$

(2) $x = 0 : f_y(x, y) = x^2 - 2y = 0 \quad \to \quad 0 - 2y = 0 \quad \to \quad y = 0 \quad \to \quad (0, 0).$

So, we find three critical points

$$(2, 2),\ (-2, 2),\ \text{and } (0, 0).$$

8. Let $f(x, y) = 3x - x^3 - 2y^2 + y^4$.

(a) Check if the points $(1, -1)$ and $(2, 3)$ are critical points.

The first partial derivatives of $f(x, y)$ are

$$f_x(x, y) = 3 - 3x^2 \quad \text{and} \quad f_y(x, y) = -4y + 4y^3 = -4y(1 - y^2).$$

Solving for where $f_x = 0$ and $f_y = 0$, we find that the critical points are

$$(1, -1),\ (1, 0),\ (1, 1),\ (-1, -1),\ (-1, 0),\ \text{and } (-1, 1).$$

Therefore, $(1, -1)$ is the critical point, but not $(2, 3)$.

(b) Determine whether the points $(1,-1),(1,0),(1,1)$ are local extreme points or not. If yes, explain what kind of points and justify your answer.

Next, the second partial derivatives of $f(x,y)$ are

$$f_{xx}(x,y) = -6x, \quad f_{yy}(x,y) = -4+12y^2, \quad f_{xy} = 0, \quad f_{yx} = 0.$$

Evaluating D at the critical points, we find that

$$\begin{aligned} D(1,-1) &= f_{xx}(1,-1)\cdot f_{yy}(1,-1) - [f_{xy}(1,-1)]^2 = -48 < 0 \\ D(1,0) &= f_{xx}(1,0)\cdot f_{yy}(1,0) - [f_{xy}(1,0)]^2 = 48 > 0, \quad f_{xx}(1,0) = -6 < 0 \\ D(1,1) &= f_{xx}(1,1)\cdot f_{yy}(1,1) - [f_{xy}(1,1)]^2 = -48 < 0 \end{aligned}$$

By the Second Derivative Test, the points $(1,-1)$ and $(1,1)$ are saddle points; $(1,0)$ is the local maximum point.

9. Find the local maximum and minimum values and saddle point(s) for the following functions.

(a) $f(x,y) = x^2+xy+y^2+y$

The first partial derivatives of $f(x,y)$ are

$$f_x(x,y) = 2x+y \quad \text{and} \quad f_y(x,y) = x+2y+1.$$

Solving for where $f_x = 0$ and $f_y = 0$, we find that the only critical point is $\left(\frac{1}{3},-\frac{2}{3}\right)$.
Next, the second partial derivatives of $f(x,y)$ are

$$f_{xx}(x,y) = 2, \quad f_{yy}(x,y) = 2, \quad f_{xy}(x,y) = 1, \quad f_{yx}(x,y) = 1.$$

Evaluating D at the critical point, we find that

$$D\left(\frac{1}{3},-\frac{2}{3}\right) = f_{xx}\left(\frac{1}{3},-\frac{2}{3}\right)\cdot f_{yy}\left(\frac{1}{3},-\frac{2}{3}\right) - \left[f_{xy}\left(\frac{1}{3},-\frac{2}{3}\right)\right]^2 = 3 > 0$$

and

$$f_{xx}\left(\frac{1}{3},-\frac{2}{3}\right) = 2 > 0.$$

By the Second Derivative Test, we have the local minimum value

$$f\left(\frac{1}{3},-\frac{2}{3}\right) = \frac{1}{9}-\frac{2}{9}+\frac{4}{9}-\frac{2}{3} = -\frac{1}{3} \quad \text{at} \quad \left(\frac{1}{3},-\frac{2}{3}\right).$$

(b) $f(x,y) = x^3+y^3+3xy$

The first partial derivatives of $f(x,y)$ are

$$f_x = 3x^2+3y \quad \text{and} \quad f_y(x,y) = 3y^2+3x.$$

Solving for where $f_x = 0$ and $f_y = 0$, we find that two critical points are $(0,0)$ and $(-1,-1)$.
Next, the second partial derivatives of $f(x,y)$ are:

$$f_{xx}(x,y) = 6x, \quad f_{yy}(x,y) = 6y, \quad f_{xy} = 3, \quad f_{yx} = 3.$$

(i) Point $(0,0)$: Evaluating D at the critical points, we find that

$$D(0,0) = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 = -9 < 0$$

By the Second Derivative Test, $(0,0)$ is the saddle point.

(ii) Point $(-1,-1)$:

$$D(-1,-1) = f_{xx}(-1,-1) \cdot f_{yy}(-1,-1) - [f_{xy}(-1,-1)]^2 = 27 > 0,$$

and $f_{xx}(-1,-1) = -6 < 0$. By the Second Derivative Test, we have the local maximum value

$$f(-1,-1) = (-1)^3 + (-1)^3 + 3(-1)(-1) = 1 \qquad \text{at} \quad (-1,-1).$$

(c) $f(x,y) = x^4 - 2x^2 + y^3 - 3y$

The first partial derivatives of $f(x,y)$ are

$$f_x(x,y) = 4x^3 - 4x \quad \text{and} \quad f_y(x,y) = 3y^2 - 3.$$

Solving for where $f_x = 0$ and $f_y = 0$, we find that the critical points are

$$(-1,-1),\ (-1,1),\ (0,1),\ (0,-1),\ (1,1),\ \text{and}\ (1,-1).$$

Next, the second partial derivatives of $f(x,y)$ are

$$f_{xx}(x,y) = 12x^2 - 4, \quad f_{yy}(x,y) = 6y, \quad f_{xy} = 0, \quad f_{yx} = 0.$$

Evaluating D at the critical points, we find that

$$\begin{aligned}
D(-1,-1) &= f_{xx}(-1,-1) \cdot f_{yy}(-1,-1) - [f_{xy}(-1,-1)]^2 = -48 < 0 \\
D(-1,1) &= f_{xx}(-1,1) \cdot f_{yy}(-1,1) - [f_{xy}(-1,1)]^2 = 48 > 0, \quad f_{xx}(-1,1) = 8 > 0 \\
D(0,1) &= f_{xx}(0,1) \cdot f_{yy}(0,1) - [f_{xy}(0,1)]^2 = -24 < 0 \\
D(0,-1) &= f_{xx}(0,-1) \cdot f_{yy}(0,-1) - [f_{xy}(0,-1)]^2 = 24 > 0, \quad f_{xx}(0,-1) = -4 < 0 \\
D(1,1) &= f_{xx}(1,1) \cdot f_{yy}(1,1) - [f_{xy}(1,1)]^2 = 48 > 0, \quad f_{xx}(1,1) = 8 > 0 \\
D(1,-1) &= f_{xx}(1,-1) \cdot f_{yy}(1,-1) - [f_{xy}(1,-1)]^2 = -48 < 0
\end{aligned}$$

By the Second Derivative Test, the points $(-1,-1)$, $(0,1)$ and $(1,-1)$ are saddle points; we have the local maximum $f(0,-1) = 2$; and have the local minimums $f(-1,1) = -3$ and $f(1,1) = -3$.

10. Find the absolute maximum and minimum values of f on the set D.

$$f(x,y) = x^2 + y^2 + x^2y + 9, \quad D = \{(x,y) \,|\, |x| \le 1, \quad |y| \le 1\}$$

(i) We have

$$f(x,y) = x^2 + y^2 + x^2y + 9 \quad \to \quad f_x(x,y) = 2x + 2xy, \quad f_y(x,y) = 2y + x^2,$$

and setting $f_x = f_y = 0$ gives $(0,0)$ as the only critical point in D, with $f(0,0) = 9$.

(ii) On the boundary,

- On L_1: $y = -1$, $f(x,-1) = 10$, a constant.
- On L_2: $x = 1$, $f(1,y) = y^2 + y + 10$, a quadratic in y which attains its maximum at $(1,1)$, $f(1,1) = 12$, and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{39}{4}$.
- On L_3: $f(x,1) = 2x^2 + 10$ which attains its maximum at $(-1,1)$ and $(1,1)$ with $f(\pm 1, 1) = 12$ and its minimum at $(0,1)$, $f(0,1) = 10$.
- On L_4: $f(-1,y) = y^2 + y + 10$ with maximum at $(-1,1)$, $f(-1,1) = 12$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{39}{4}$.

Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 12$ and the absolute minimum on D is attained at $(0,0)$ with $f(0,0) = 9$.

11. Find the absolute maximum and minimum values of $f(x,y) = x^2 + y^2 - 2x$ on the set D, where D is the closed triangular region with vertices $(2,0)$, $(0,2)$, and $(0,-2)$.

(i) We have

$$f(x,y) = x^2 + y^2 - 2x \quad \rightarrow \quad f_x(x,y) = 2x - 2, \quad f_y(x,y) = 2y,$$

and setting $f_x = f_y = 0$ gives $(1,0)$ as the only critical point in D, with $f(1,0) = -1$.

(ii) On the boundary,

- On the line L_1 from $(0,2)$ to $(2,0)$: $y = 2 - x$, where $0 \le x \le 2$.
  Evaluating $f(x,y)$ along Line 1, we find

  $$f(x, 2-x) = x^2 + (2-x)^2 - 2x = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}.$$

  This is just a parabola with positive concavity, so the minimum value is $f(\frac{3}{2}, \frac{1}{2}) = -\frac{1}{2}$ and the maximum value is $f(0,2) = 4$.
- On the line L_2 from $(0,2)$ to $(0,-2)$: $x = 0$, $-2 \le y \le 2$.
  Evaluating $f(x,y)$ along Line 2, we find $f(0,y) = y^2$. This is just a parabola with positive concavity, so the minimum value is $f(0,0) = 0$ and the maximum value is $f(0,-2) = f(0,2) = 4$.
- On the line L_3 from $(0,-2)$ to $(2,0)$: $y = x - 2$, $0 \le x \le 2$.
  Evaluating $f(x,y)$ along Line 3, we find

  $$f(x, x-2) = x^2 + (x-2)^2 - 2x = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}.$$

This is the same as what we got for Line 1, so the minimum value is $f(\frac{3}{2}, -\frac{1}{2}) = -\frac{1}{2}$ and the maximum value is $f(0, -2) = 4$.

Thus the absolute maximum is attained at $(1, 0)$ with $f(1, 0) = -1$ and the absolute minimum on D is attained at $(0, \pm 2)$ with $f(0, \pm 2) = 4$.

Suggested Textbook Problems

Chapter 14.6: 5, 7-17, 19-26, 28-35, 41-46, 54-61, 63, 64a

Chapter 14.7: 1-5, 7, 11, 13, 25, 27, 30, 31, 33-36, 41, 43, 45, 46, 51-53