Textbook Sections: 16.6 and 16.7
Topics: Parametric Surfaces and Their Areas, Surface Integrals
Instructions: Try each of the following problems, show the detail of your work.
Clearly mark your choices in multiple choice items. Justify your answers.
Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

Parametric Surfaces and Their Areas:

1. Find parametric representations for the following surfaces:

   (a) The part of the hyperboloid $4x^2 - 4y^2 - z^2 = 4$ that lies in front of the yz-plane.

Solving for x in the given equation yields:

$$4x^2 - 4y^2 - z^2 = 4$$
$$\implies x^2 = y^2 + \frac{1}{4}z^2 + 4$$

We are only interested in the region in front of the y-z plane, hence, we only consider positive values of x. Therefore, x is given by

$$x = \sqrt{y^2 + \frac{1}{4}z^2 + 4}$$

Now, letting y and z be the parameters, the obtained parametric equations are

$$y = y, \quad z = z, \quad x = \sqrt{y^2 + \frac{1}{4}z^2 + 4}$$

Thus, $\mathbf{r}(y, z) =< \sqrt{y^2 + \frac{1}{4}z^2 + 4}, y, z >$

   (b) The part of the cylinder $x^2 + z^2 = 9$ that lies above the xy-plane and between the planes $y = -4$ and $y = 4$.

We can parametrize the given cylinder by letting $x = 3\cos(\theta)$, $y = y$, and $z = 3\sin(\theta)$. We are considering the part of the cylinder for y between -4 and 4, hence we restrict $-4 \leq y \leq 4$. Also, we are only considering part of the cylinder above the x-y plane, hence, we restrict the angle θ to $0 \leq \theta \leq \pi$. Hence the parametric equations are given by

$$x = 3\cos(\theta), y = y, z = 3\sin(\theta) \quad \text{with} \quad -4 \leq y \leq 4, \quad 0 \leq \theta \leq \pi$$

Thus, $\mathbf{r}(\theta, y) =< 3\cos(\theta), y, 3\sin(\theta) >$

2. Find an equation of the tangent plane to the surface S given by the parametric equations $x = u^2 + 1,\ y = v^3 + 1,\ z = u + v$ at the point $P(5, 2, 3)$.

The surface S is traced out by the vector function

$$\begin{aligned}\mathbf{r}(u, v) &= x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \\ &= (u^2 + 1)\mathbf{i} + (v^3 + 1)\mathbf{j} + (u + v)\mathbf{k}\end{aligned}$$

Next, we compute the tangent vectors r_u and r_v:

$$\mathbf{r}_u = 2u\mathbf{i} + \mathbf{k}$$
$$\mathbf{r}_v = 3v^2\mathbf{j} + \mathbf{k}$$

Next, we find a normal vector to the tangent plane, notably $\mathbf{r_u} \times \mathbf{r_u}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix} = -3v^2\mathbf{i} - 2u\mathbf{j} + 6uv^2\mathbf{k}$$

The given point $P(5,2,3)$ corresponds to $r(2,1)$, hence $u = 2$ and $v = 1$. Therefore a normal vector to the tangent plane at point P is $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v|_{(u=2,v=1)} = -3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$. Hence we have a point in the tangent plane, notably P(5,2,3), and a normal vector $\mathbf{n} = -3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$. We write the equation of the tangent plane as

$$-3(x-5) - 4(y-2) + 12(z-3) = 0 \quad \text{or} \quad 3x + 4y - 12z = 13$$

3. Find the area of the surface S that is the part of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 16$.

A parametric representation of the surface is $x = x$, $z = z$, and $y = g(x,z) = x^2 + z^2$, with $0 \le y \le 16 \implies 0 \le x^2 + z^2 \le 16$. We use formula 9 to compute the surface area, that is

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial g(x,z)}{\partial x}\right)^2 + \left(\frac{\partial g(x,z)}{\partial z}\right)^2} dA \\ &= \iint_D \sqrt{1 + \left(\frac{\partial (x^2+z^2)}{\partial x}\right)^2 + \left(\frac{\partial (x^2+z^2)}{\partial z}\right)^2} dA \\ &= \iint_D \sqrt{1 + 4x^2 + 4z^2} dA \end{aligned}$$

Now to compute the integral above, we use polar coordinates by letting $x = r\cos(\theta)$ and $z = r\sin(\theta)$. Since $0 \le x^2 + z^2 \le 16$, then $0 \le r \le 4$, $0 \le \theta \le 2\pi$ and $x^2 + z^2 = r^2$. Hence

$$\begin{aligned} A(S) &= \int_0^{2\pi} \int_0^4 \sqrt{1+4r^2}\ r\ dr d\theta \\ &= \int_0^{2\pi} 1\ d\theta \int_0^4 \sqrt{1+4r^2}\ r\ dr \\ &= 2\pi \int_0^4 \sqrt{1+4r^2}\ r\ dr \end{aligned}$$

We proceed by u-substitution. Let $u = 1 + 4r^2 \implies du = 8rdr$. Also, $r = 0 \implies u = 1$ and $r = 4 \implies u = 65$. Hence, the integral becomes

$$\begin{aligned} A(S) &= 2\pi \int_1^{65} u^{\frac{1}{2}}\ \frac{du}{8} \\ &= \frac{\pi}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{65} \\ &= \frac{\pi}{6}(65^{\frac{3}{2}} - 1). \end{aligned}$$

Note: We could have used Definition 6 by considering the vector function $\mathbf{r}(x,z) = x\mathbf{i} + g(x,z)\mathbf{j} + z\mathbf{k}$ and computing the tangent vectors $\mathbf{r}_x$ and $\mathbf{r}_z$, which equal $\mathbf{i}+2x\mathbf{j}$ and $2z\mathbf{j}+\mathbf{k}$, respectively. Next compute the normal vector to the tangent plane, i.e., $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i}+2x\mathbf{j}) \times (2z\mathbf{j}+\mathbf{k}) = 2x\mathbf{i} - \mathbf{i} + 2z\mathbf{k}$. Then the surface Area is given by

$$\begin{aligned} A(S) &= \iint_D ||\mathbf{r}_u \times \mathbf{r}_v|| dA \\ &= \iint_D ||(2x,-1,2z)|| dA \\ &= \iint_D \sqrt{1+4x^2+4z^2} dA \end{aligned}$$

Proceeding in a similar manner as before, by switching to polar coordinates, yields the desired result $A(S) = \frac{\pi}{6}(65^{\frac{3}{2}} - 1)$.

Surface Integrals

4. Evaluate the following surface integrals.

(a) $\displaystyle\iint_S x^2yz\,dS$, where S is the part of the plane $z = 1+2x+3y$ that lies above the rectangle $[0,3]\times[0,2]$.

Let $z = 1+2x+3y = g(x,y)$, and $\frac{\partial z}{\partial x} = 2$ and $\frac{\partial z}{\partial y} = 3$.
The surface is the graph of a function $z = g(x,y)$, so we use the formula:

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA$$

This gives:

$$\begin{aligned} \iint_S x^2yz\,dS &= \iint_D x^2y(1+2x+3y)\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA \\ &= \int_0^3 \int_0^2 x^2y(1+2x+3y)\sqrt{4+9+1}\ dy\ dx \\ &= \sqrt{14}\int_0^3 \int_0^2 \left(x^2y + 2x^3y + 3x^2y^2\right) dydx = \sqrt{14}\int_0^3 \left[\frac{1}{2}x^2y^2 + x^3y^2 + x^2y^3\right]_{y=0}^{y=2} dx \\ &= \sqrt{14}\int_0^3 \left(10x^2 + 4x^3\right) dx = \sqrt{14}\left[\frac{10}{3}x^3 + x^4\right]_0^3 = 171\sqrt{14} \end{aligned}$$

(b) $\displaystyle\iint_S y^2z^2\ dS$,where S is the part of the cone $y = \sqrt{x^2+z^2}$ given by $0 \le y \le 5$.

Using x and z as parameters, we have $\mathbf{r}(x,z) = x\mathbf{i} + \sqrt{x^2+z^2}\mathbf{j} + z\mathbf{k}$, where $x^2+z^2 \le 25$. Then $\mathbf{r}_x \times \mathbf{r}_z = \left(\mathbf{i} + \frac{x}{\sqrt{x^2+z^2}}\mathbf{j}\right) \times \left(\frac{z}{\sqrt{x^2+z^2}}\mathbf{j} + \mathbf{k}\right) = \frac{x}{\sqrt{x^2+z^2}}\mathbf{i} - \mathbf{j} + \frac{z}{\sqrt{x^2+z^2}}\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{\frac{x^2}{x^2+z^2} + 1 + \frac{z^2}{x^2+z^2}} = \sqrt{\frac{x^2+z^2}{x^2+z^2} + 1} = \sqrt{2}$.
Thus, by Formula: $\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(x,z))\,|\mathbf{r}_x \times \mathbf{r}_z|\ dA$, and writing D in polar coordinates, as

follows, $D = \{(x,z)|0 \le x^2+z^2 \le 25\} = \{(r,\theta)|0 \le r \le 5, 0 \le \theta \le 2\pi\}$, we get

$$\iint_S y^2z^2 dS = \iint_D (x^2+z^2)\, z^2\sqrt{2}\; dA = \sqrt{2}\int_0^{2\pi}\int_0^5 r^2(r\sin\theta)^2 r\; dr\; d\theta$$
$$= \sqrt{2}(\int_0^{2\pi}\sin^2\theta\; d\theta)(\int_0^5 r^5 dr) = \sqrt{2}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\frac{1}{6}r^6\right]_0^5$$
$$= \sqrt{2}(\pi)\cdot\frac{1}{6}(15,625-0) = \frac{15,625\sqrt{2}}{6}\pi$$

5. Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S, for the vector field $\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$, and S is the hemisphere $x^2+y^2+z^2 = 4$, $z \ge 0$, oriented downward.

$\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}, z = g(x,y) = \sqrt{4-x^2-y^2}$ and

D is the disk $\{(x,y) \mid x^2+y^2 \le 4\} = \{(r,\theta)|0 \le r \le 2, 0 \le \theta \le 2\pi\}$.

Recall that if S has upward orientation, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

Here, S is given with downward orientation so we use the opposite normal vector, and thus

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-y\cdot\frac{1}{2}\left(4-x^2-y^2\right)^{-1/2}(-2x) - (-x)\cdot\frac{1}{2}\left(4-x^2-y^2\right)^{-1/2}(-2y) + 2z\right] dA$$
$$= -\iint_D \left(\frac{xy}{\sqrt{4-x^2-y^2}} - \frac{xy}{\sqrt{4-x^2-y^2}} + 2\sqrt{4-x^2-y^2}\right) dA$$
$$= -\iint_D 2\sqrt{4-x^2-y^2}dA = -2\int_0^{2\pi}\int_0^2\sqrt{4-r^2}rdrd\theta = -2(\int_0^{2\pi}d\theta)(\int_0^2 r\sqrt{4-r^2}dr)$$
$$= -2(2\pi)\left[-\frac{1}{2}\cdot\frac{2}{3}\left(4-r^2\right)^{3/2}\right]_0^2 = -4\pi\left[0+\frac{1}{3}(4)^{3/2}\right] = -4\pi\cdot\frac{8}{3} = -\frac{32}{3}\pi$$

6. Evaluate the surface integral $\iint_S z\,dS$, where S is the part of the paraboloid $z = x^2+y^2$ that lies under the plane $z = 4$.

Let $z = g(x,y) = x^2+y^2$, and D is the disk $\{(x,y) \mid x^2+y^2 \le 4\} = \{(r,\theta)|0 \le r \le 2, 0 \le \theta \le 2\pi\}$.

Again, by using Formula 4, we get:

$$\iint_S zdS = \iint_D (x^2+y^2)\sqrt{(2x)^2+(2y)^2+1}dA = \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\; rdrd\theta$$
$$= \int_0^{2\pi}\int_1^{17}\left(\frac{u-1}{4}\right)\sqrt{u}\left(\frac{1}{8}du\right)d\theta \quad \begin{bmatrix} u = 1+4r^2, \\ du = 8rdr \end{bmatrix} = \frac{1}{32}(2\pi)\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^{17}$$
$$= \frac{\pi}{16}\cdot\frac{2}{15}\left[\left(3u^2-5u\right)\sqrt{u}\right]_1^{17} = \frac{\pi}{120}[782\sqrt{17}+2] = \frac{1}{60}\pi(391\sqrt{17}+1)$$

Suggested Textbook Problems

Section 16.6	1-5, 13-26, 33-36, 39-49, 62
Section 16.7	5-27

SOME USEFUL DEFINITIONS, THEOREMS AND NOTATION:

Definition 6 If a smooth parametric surface $\mathbf{S}$ is given by the equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D ||\mathbf{r}_u \times \mathbf{r}_v|| dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

Formula 9

For the special case of a surface S with equation $z = f(x,y)$, where (x,y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The surface area formula in Definition 6 becomes

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Formula 2

The surface integral of f over the surface S is

$$\iint_S f(x,y,z)\ dS = \iint_D f(\mathbf{r}(u,v))||\mathbf{r}_u \times \mathbf{r}_v||\ dA$$

Formula 4

Any surface S with equation $z = g(x,y)$ can be regarded as a parametric surface with parametric equations $x = x, y = y, z = g(x,y)$, then

$$\iint_S f(x,y,z)\ dS = \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

Definition 8: If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of F over S is called the flux of $\mathbf{F}$ across S, and is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\ dS$$

Formula 9: If S is given by a vector function $\mathbf{r}(u,v)$, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA$$

Formula 10: If S is given by the graph of the function $z = g(x,y)$, and S has an upward orientation, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R)\ dA$$