Curl, Divergence, Green’s Theorem

Lecture for 7/7

Curl and Divergence

Consider ∇ as a operator and $\mathbf{F} = \langle A, B, C \rangle$

- Define div $\mathbf{F} = \nabla \cdot \mathbf{F} = A_x + B_y + C_z$
- Define curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle C_y - B_z, A_z - C_x, B_x - A_y \rangle$
  - Only for vector fields on R^3

Properties of Curl and Divergence

- Let's see what happens to scalar functions
  - $\nabla \cdot f\mathbf{G} = (\nabla f) \cdot \mathbf{G} + f(\nabla \cdot \mathbf{G})$
  - $\nabla \times f\mathbf{G} = \nabla f \times \mathbf{G} + f(\nabla \times \mathbf{G})$
- Let's see how curl and div interact
  - $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
    - Consequence: $\nabla \cdot (\nabla \times \mathbf{F})$
  - $\nabla \times \nabla f = 0$
  - $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

Green’s Theorem General Idea

It’s often useful to switch between line and double integrals

- Double to Line: you’re reducing number of integrations
- Line to Double: function may be simpler to integrate

But how can we do this? Green’s Theorem will tell us

Green's Theorem

Suppose C is a simple closed curve oriented counterclockwise. Suppose

- Further suppose C encloses D
- Further suppose Q_x, P_y are Riemann integrable

Green's Theorem: $\int_C P dx + Q dy = \iint_D (Q_x - P_y)\, dA$

Decomposition Principle

- If GT holds on D_1 and D_2, we can consider $D = D_1$ u D_2
- To find integrals, break D into smaller regions where GT applies

Theorem Derivation

Vector Forms of Green's Theorem

Pretend $\mathbf{F} = \langle P, Q \rangle$ is a vector field in R^3, with $\mathbf{F} = \langle P, Q, 0 \rangle$

- $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{e}_z \, dA$ where $\mathbf{e}_z = (0, 0, 1)$
  - We shall use this later to build Stokes' Theorem
- $\int_C (\mathbf{F} \cdot \mathbf{n}) \, ds = \iint_D (\nabla \cdot \mathbf{F}) \, dA$ where $\mathbf{n}$ is unit normal to $\mathbf{r}$
  - We shall use this later to build Divergence Theorem

Practice Problems

Evaluate $\int_C (y^4-2y)\,dx - (6x-4xy^3)\,dy$ where C is the rectangle with coordinates (0,0), (6, 0), (6, 4), (0, 4) oriented clockwise

Let C be the triangle with vertices (-3, 0), (0,0), (0, 3) oriented clockwise. Verify Green's Theorem for $\int_C (xy^2+x^2)\,dx + (4x-1)\,dy$ by computing both the line integral and the corresponding double integral

Find a formula for $\nabla \times (\nabla \times \mathbf{F})$ and justify your claim

Scratchwork