MATH 243: Discussion Worksheet 4 **Answer Key**

Topics: 13.2 Derivatives and Integrals of Vector Functions; 13.3 Arc Length and Curvature; 13.4 Motion in Space: Velocity and Acceleration

Please put away all electronic devices, including cell phones and calculators.

1. Find the derivative of the vector function

$$\mathbf{r}(t) = \left\langle e^{t^2+2t},\, \ln(\cos(t)),\, t\arctan(t) \right\rangle.$$

We differentiate each component:

$$\frac{d}{dt}\left(e^{t^2+2t}\right) = (2t+2)e^{t^2+2t},$$

$$\frac{d}{dt}(\ln(\cos t)) = \frac{-\sin t}{\cos t} = -\tan t,$$

$$\frac{d}{dt}(t\arctan t) = \arctan t + \frac{t}{t^2+1}.$$

Therefore,

$$\mathbf{r}'(t) = \left\langle (2t+2)e^{t^2+2t},\, -\tan(t),\, \arctan(t) + \frac{t}{t^2+1} \right\rangle.$$

2. Evaluate the definite integral

$$\int_0^1 \left(\frac{1}{t+1}\mathbf{i} + \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k}\right) dt.$$

We integrate each component:

$$\int_0^1 \frac{1}{t+1}\,dt = \big[\ln|t+1|\big]_0^1 = \ln(2) - \ln(1) = \ln(2),$$

$$\int_0^1 \frac{1}{t^2+1}\,dt = \big[\arctan(t)\big]_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4},$$

$$\int_0^1 \frac{t}{t^2+1}\,dt = \frac{1}{2}\int_0^1 \frac{2t}{t^2+1}\,dt = \frac{1}{2}\big[\ln(t^2+1)\big]_0^1 = \frac{1}{2}\big(\ln(2) - \ln(1)\big) = \frac{1}{2}\ln(2).$$

Therefore,

$$\int_0^1 \left(\frac{1}{t+1}\mathbf{i} + \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k}\right) dt = \ln(2)\,\mathbf{i} + \frac{\pi}{4}\,\mathbf{j} + \frac{1}{2}\ln(2)\,\mathbf{k}.$$

3. Find a vector equation of the tangent line to the curve

$$\mathbf{r}(t) = t\cos(2t)\,\mathbf{i} + 4t\,\mathbf{j} + \sin(t)\,\mathbf{k}$$

at the point corresponding to $t = \frac{\pi}{4}$.

First, evaluate $\mathbf{r}(t)$ at $t = \frac{\pi}{4}$:

$$\mathbf{r}\left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}\cos\left(2\cdot\tfrac{\pi}{4}\right)\mathbf{i} + 4\cdot\tfrac{\pi}{4}\mathbf{j} + \sin\left(\tfrac{\pi}{4}\right)\mathbf{k} = \pi\,\mathbf{j} + \frac{\sqrt{2}}{2}\,\mathbf{k}.$$

Next, differentiate $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = \big(\cos(2t) - 2t\sin(2t)\big)\mathbf{i} + 4\mathbf{j} + \cos(t)\mathbf{k}.$$

Evaluating at $t = \frac{\pi}{4}$ gives

$$\mathbf{r}'\left(\tfrac{\pi}{4}\right) = \left(\cos\left(\tfrac{\pi}{2}\right) - 2\cdot\tfrac{\pi}{4}\sin\left(\tfrac{\pi}{2}\right)\right)\mathbf{i} + 4\mathbf{j} + \cos\left(\tfrac{\pi}{4}\right)\mathbf{k} = -\tfrac{\pi}{2}\mathbf{i} + 4\mathbf{j} + \tfrac{\sqrt{2}}{2}\mathbf{k}.$$

Therefore, a vector equation for the tangent line is

$$\mathbf{L}(t) = \left(\pi\,\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}\right) + t\left(-\tfrac{\pi}{2}\mathbf{i} + 4\mathbf{j} + \tfrac{\sqrt{2}}{2}\mathbf{k}\right).$$

4. Find the length of the curve $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}$ for the interval $0 \le t \le 1$.

First compute the derivative:

$$\mathbf{r}'(t) = 2\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}.$$

Now compute its magnitude:

$$\begin{aligned} \left|\mathbf{r}'(t)\right| &= \sqrt{(2)^2 + (2t)^2 + (t^2)^2} \\ &= \sqrt{4 + 4t^2 + t^4} \\ &= \sqrt{(2 + t^2)^2} \\ &= 2 + t^2. \end{aligned}$$

Using the arc length formula $L = \int_0^1 \left|\mathbf{r}'(t)\right|\,dt$, we get

$$\begin{aligned} L &= \int_0^1 (2 + t^2)\,dt \\ &= \left[2t + \frac{1}{3}t^3\right]_0^1 \\ &= \left(2 + \frac{1}{3}\right) - 0 \\ &= \frac{7}{3}. \end{aligned}$$

5. A particle starts at the point $(0, 0, 3)$ and moves along the curve $\mathbf{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle$ in the direction of increasing t. Find the position of the particle after it has traveled a distance of 5π units.

To find the time T when the particle has traveled 5π units, we solve

$$5\pi = \int_0^T \left|\mathbf{r}'(t)\right| dt.$$

Differentiate $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = \langle 3\cos t,\ 4,\ -3\sin t\rangle.$$

Now compute the speed:

$$\begin{aligned}
\left|\mathbf{r}'(t)\right| &= \sqrt{(3\cos t)^2 + 4^2 + (-3\sin t)^2} \\
&= \sqrt{9\cos^2 t + 16 + 9\sin^2 t} \\
&= \sqrt{9(\sin^2 t + \cos^2 t) + 16} \\
&= \sqrt{9 + 16} \\
&= 5.
\end{aligned}$$

So the equation we are solving becomes

$$5\pi = \int_0^T 5\,dt = 5T,$$

which gives $T = \pi$. The position at that time is

$$\mathbf{r}(\pi) = \langle 3\sin\pi,\ 4\pi,\ 3\cos\pi\rangle = \langle 0,\ 4\pi,\ -3\rangle.$$

6. Consider the vector function $\mathbf{r}(t) = \langle t, t^2, 4\rangle$.
   (a) Find the unit tangent vector $\mathbf{T}(t)$.
   (b) Find the unit normal vector $\mathbf{N}(t)$.
   (c) Find the curvature $\kappa(t)$.

We begin by computing $\mathbf{r}'(t)$ and its magnitude:

$$\mathbf{r}'(t) = \langle 1, 2t, 0\rangle, \qquad \left|\mathbf{r}'(t)\right| = \sqrt{1 + (2t)^2} = \sqrt{1 + 4t^2}.$$

(a) The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 1, 2t, 0\rangle}{\sqrt{1 + 4t^2}}.$$

(b) The unit normal vector is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

Differentiate $\mathbf{T}(t)$:

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{\sqrt{1+4t^2}}\langle 0,2,0\rangle - \frac{1}{2}(1+4t^2)^{-3/2}\cdot 8t\,\langle 1,2t,0\rangle \\ &= \frac{1}{(1+4t^2)^{3/2}}\Big[(1+4t^2)\langle 0,2,0\rangle - 4t\langle 1,2t,0\rangle\Big] \\ &= \frac{\langle -4t,2,0\rangle}{(1+4t^2)^{3/2}}.\end{aligned}$$

Now compute its magnitude:

$$\begin{aligned}\big|\mathbf{T}'(t)\big| &= \frac{\sqrt{(-4t)^2+2^2}}{(1+4t^2)^{3/2}} = \frac{\sqrt{16t^2+4}}{(1+4t^2)^{3/2}} \\ &= \frac{2\sqrt{1+4t^2}}{(1+4t^2)^{3/2}} = \frac{2}{1+4t^2}.\end{aligned}$$

Therefore,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle -4t,2,0\rangle}{(1+4t^2)^{3/2}}\cdot\frac{1+4t^2}{2} = \frac{\langle -2t,1,0\rangle}{\sqrt{1+4t^2}}.$$

(c) The curvature is

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{2/(1+4t^2)}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}.$$

7. Find the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the point $(1,\frac{2}{3},1)$ for the vector function

$$\mathbf{r}(t) = \langle t^2, \tfrac{2}{3}t^3, t\rangle.$$

Since $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t\rangle$, the point $(1,\frac{2}{3},1)$ corresponds to $t=1$. First compute

$$\mathbf{r}'(t) = \langle 2t, 2t^2, 1\rangle.$$

Then the unit tangent vector is

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 2t,2t^2,1\rangle}{\sqrt{(2t)^2+(2t^2)^2+1}} \\ &= \frac{\langle 2t,2t^2,1\rangle}{\sqrt{4t^2+4t^4+1}} = \frac{\langle 2t,2t^2,1\rangle}{2t^2+1}.\end{aligned}$$

So,

$$\mathbf{T}(1) = \left\langle \frac{2}{3},\frac{2}{3},\frac{1}{3}\right\rangle.$$

Next, differentiate $\mathbf{T}(t)$:

$$\begin{aligned}\mathbf{T}'(t) &= \frac{\langle 2,4t,0\rangle}{2t^2+1} - (2t^2+1)^{-2}\cdot 4t\,\langle 2t,2t^2,1\rangle \\ &= \frac{1}{(2t^2+1)^2}\Big[\langle 4t^2+2,\ 8t^3+4t,\ 0\rangle - \langle 8t^2,\ 8t^3,\ 4t\rangle\Big] \\ &= \frac{2}{(2t^2+1)^2}\langle 1-2t^2,\ 2t,\ -2t\rangle.\end{aligned}$$

Thus,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle 1-2t^2, 2t, -2t\rangle}{\sqrt{(1-2t^2)^2+(2t)^2+(-2t)^2}}$$
$$= \frac{\langle 1-2t^2, 2t, -2t\rangle}{\sqrt{1+4t^2+4t^4}} = \frac{\langle 1-2t^2, 2t, -2t\rangle}{1+2t^2}.$$

At $t=1$,

$$\mathbf{N}(1) = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle.$$

Finally, the binormal vector is $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, so

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{vmatrix} = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle.$$

8. A particle moves in space with position function

$$\mathbf{r}(t) = \langle t \ln t, t, e^{-t} \rangle.$$

Find the velocity, speed, and acceleration of the particle.

The velocity is $\mathbf{v}(t) = \mathbf{r}'(t)$, so

$$\mathbf{v}(t) = \langle \ln(t)+1,\ 1,\ -e^{-t} \rangle.$$

The acceleration is $\mathbf{a}(t) = \mathbf{v}'(t)$, so

$$\mathbf{a}(t) = \left\langle \frac{1}{t},\ 0,\ e^{-t} \right\rangle.$$

Finally, the speed is the magnitude of the velocity:

$$\|\mathbf{v}(t)\| = \sqrt{(\ln(t)+1)^2 + 1^2 + (-e^{-t})^2} = \sqrt{(\ln(t)+1)^2 + 1 + e^{-2t}}.$$

9. The acceleration of an object is given by $\mathbf{a}(t) = t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$. Find the object's position function $\mathbf{r}(t)$ if the initial velocity is $\mathbf{v}(0) = \mathbf{k}$ and the initial position is $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$.

Since $\mathbf{a}(t) = \mathbf{v}'(t)$ and $\mathbf{v}(t) = \mathbf{r}'(t)$, we first find $\mathbf{v}(t)$ by integrating $\mathbf{a}(t)$:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int \left(t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k} \right)\, dt$$
$$= \frac{t^2}{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C}_1.$$

Use $\mathbf{v}(0) = \mathbf{k}$:

$$\mathbf{v}(0) = \langle 0, 1, -1 \rangle + \mathbf{C}_1 = \langle 0, 0, 1 \rangle \quad \Rightarrow \quad \mathbf{C}_1 = 2\mathbf{k} - \mathbf{j}.$$

Thus,

$$\mathbf{v}(t) = \frac{t^2}{2}\mathbf{i} + (e^t - 1)\mathbf{j} + (2 - e^{-t})\mathbf{k}.$$

Now integrate $\mathbf{v}(t)$ to get $\mathbf{r}(t)$:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt \\ &= \int \left(\frac{t^2}{2}\mathbf{i} + (e^t - 1)\mathbf{j} + (2 - e^{-t})\mathbf{k}\right) dt \\ &= \frac{t^3}{6}\mathbf{i} + (e^t - t)\mathbf{j} + (2t + e^{-t})\mathbf{k} + \mathbf{C}_2.\end{aligned}$$

Use $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$:

$$\mathbf{r}(0) = \langle 0, 1, 1\rangle + \mathbf{C}_2 = \langle 0, 1, 1\rangle \quad \Rightarrow \quad \mathbf{C}_2 = \mathbf{0}.$$

Therefore,

$$\mathbf{r}(t) = \frac{t^3}{6}\mathbf{i} + (e^t - t)\mathbf{j} + (2t + e^{-t})\mathbf{k}.$$

10. At a point P, the velocity and acceleration vectors of a particle moving in space are $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$, respectively. Determine the curvature of the particle's path at P.

To find the curvature κ, we use the formula

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

First compute the cross product:

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 0 & -3 \end{vmatrix} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

Now,

$$|\mathbf{v} \times \mathbf{a}| = \sqrt{3^2 + 5^2 + 2^2} = \sqrt{38} \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}.$$

Therefore, at point P,

$$\kappa = \frac{\sqrt{38}}{(\sqrt{3})^3}.$$

SOME USEFUL DEFINITIONS, THEOREMS, AND NOTATION

Arc Length. If $\mathbf{r}$ is a smooth vector function on the interval $[a, b]$, then the arc length of the curve described by $\mathbf{r}$ between the points with position vectors $\mathbf{r}(a)$ and $\mathbf{r}(b)$ is

$$L = \int_a^b |\mathbf{r}'(t)|\,dt.$$

Unit Tangent, Unit Normal, and Binormal Vectors. Let $\mathbf{r}$ be a smooth vector function. The **unit tangent vector** at time t is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

The **(principal) unit normal vector** is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

The **binormal vector** is defined by

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

The vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are orthogonal unit vectors.

Curvature. The **curvature** of a smooth curve is given by

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Curvature is a scalar quantity that measures how rapidly a smooth curve changes direction at a given point. Intuitively, it describes how sharply the curve bends.

Position, Velocity, and Acceleration. If a particle moves in space so that its position vector at time t is $\mathbf{r}(t)$, then the **velocity vector** of the particle is $\mathbf{v}(t) = \mathbf{r}'(t)$, its **speed** is $|\mathbf{v}(t)|$, and its **acceleration vector** is $\mathbf{a}(t) = \mathbf{r}''(t)$.
In terms of velocity and acceleration, the third curvature formula may be written as

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}.$$

Suggested Textbook Problems
Section 13.2: 1-44, 49-52
Section 13.3: 1-30, 40-45, 51, 52
Section 13.4: 1-32, 36