MATH 243: Discussion Worksheet 4

Topics: 13.2 Derivatives and Integrals of Vector Functions; 13.3 Arc Length and Curvature; 13.4 Motion in Space: Velocity and Acceleration

Please put away all electronic devices, including cell phones and calculators.

1. Find the derivative of the vector function
$$\mathbf{r}(t) = \left\langle e^{t^2+2t},\, \ln(\cos(t)),\, t\arctan(t) \right\rangle.$$

2. Evaluate the definite integral
$$\int_0^1 \left(\frac{1}{t+1}\mathbf{i} + \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k} \right) dt.$$

3. Find a vector equation of the tangent line to the curve
$$\mathbf{r}(t) = t\cos(2t)\,\mathbf{i} + 4t\,\mathbf{j} + \sin(t)\,\mathbf{k}$$
at the point corresponding to $t = \frac{\pi}{4}$.

4. Find the length of the curve $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}$ for the interval $0 \le t \le 1$.

5. A particle starts at the point $(0, 0, 3)$ and moves along the curve $\mathbf{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle$ in the direction of increasing t. Find the position of the particle after it has traveled a distance of 5π units.

6. Consider the vector function $\mathbf{r}(t) = \langle t, t^2, 4\rangle$.
(a) Find the unit tangent vector $\mathbf{T}(t)$.
(b) Find the unit normal vector $\mathbf{N}(t)$.
(c) Find the curvature $\kappa(t)$.

7. Find the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the point $(1, \frac{2}{3}, 1)$ for the vector function
$$\mathbf{r}(t) = \langle t^2, \tfrac{2}{3}t^3, t\rangle.$$

8. A particle moves in space with position function
$$\mathbf{r}(t) = \langle t\ln t, t, e^{-t}\rangle.$$
Find the velocity, speed, and acceleration of the particle.

9. The acceleration of an object is given by $\mathbf{a}(t) = t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$. Find the object's position function $\mathbf{r}(t)$ if the initial velocity is $\mathbf{v}(0) = \mathbf{k}$ and the initial position is $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$.

10. At a point P, the velocity and acceleration vectors of a particle moving in space are $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$, respectively. Determine the curvature of the particle's path at P.

SOME USEFUL DEFINITIONS, THEOREMS, AND NOTATION

Arc Length. If $\mathbf{r}$ is a smooth vector function on the interval $[a, b]$, then the arc length of the curve described by $\mathbf{r}$ between the points with position vectors $\mathbf{r}(a)$ and $\mathbf{r}(b)$ is

$$L = \int_a^b |\mathbf{r}'(t)|\,dt.$$

Unit Tangent, Unit Normal, and Binormal Vectors. Let $\mathbf{r}$ be a smooth vector function. The **unit tangent vector** at time t is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

The **(principal) unit normal vector** is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

The **binormal vector** is defined by

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

The vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are orthogonal unit vectors.

Curvature. The **curvature** of a smooth curve is given by

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Curvature is a scalar quantity that measures how rapidly a smooth curve changes direction at a given point. Intuitively, it describes how sharply the curve bends.

Position, Velocity, and Acceleration. If a particle moves in space so that its position vector at time t is $\mathbf{r}(t)$, then the **velocity vector** of the particle is $\mathbf{v}(t) = \mathbf{r}'(t)$, its **speed** is $|\mathbf{v}(t)|$, and its **acceleration vector** is $\mathbf{a}(t) = \mathbf{r}''(t)$.
In terms of velocity and acceleration, the third curvature formula may be written as

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}.$$

Suggested Textbook Problems
Section 13.2: 1-44, 49-52
Section 13.3: 1-30, 40-45, 51, 52
Section 13.4: 1-32, 36