# 13 Vector Functions

# 13.4 Motion in Space: Velocity and Acceleration

# Motion in Space: Velocity and Acceleration (1 of 1)

In this section we show how the ideas of tangent and normal vectors and curvature can be used in physics to study the motion of an object —including its velocity and acceleration—along a space curve.

# Velocity, Speed, and Acceleration

# Velocity, Speed, and Acceleration (1 of 4)

Suppose a particle moves through space so that its position vector at time $t$ is $\mathbf{r}(t)$. Notice from Figure 1 that, for small values of $h$, the vector

$$\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} \qquad (1)$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$.

Its magnitude measures the size of the displacement vector per unit time.

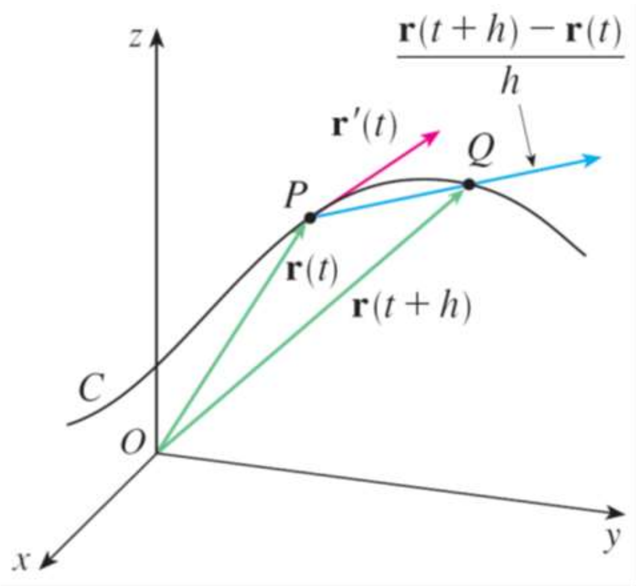


Figure 1

# Velocity, Speed, and Acceleration (2 of 4)

The vector (1) gives the average velocity over a time interval of length *h* and its limit is the **velocity vector v**(*t*) at time *t*:

$$2 \quad \boldsymbol{v}(t) = \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h} = \boldsymbol{r}'(t)$$

Thus the velocity vector is also the tangent vector and points in the direction of the tangent line.

The **speed** of the particle at time *t* is the magnitude of the velocity vector, that is, $|\boldsymbol{v}(t)|$.

# Velocity, Speed, and Acceleration (3 of 4)

This is appropriate because, from (2), we have

$$|\boldsymbol{v}(t)| = |\boldsymbol{r}'(t)| = \frac{ds}{dt}$$

$$= \text{rate of change of distance with respect to time}$$

As in the case of one-dimensional motion, the **acceleration** of the particle is defined as the derivative of the velocity:

$$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t)$$

# Example 1

The position vector of an object moving in a plane is given by $\mathrm{r}(t) = t^3\ \mathrm{i} + t^2\ \mathrm{j}$.

Find its velocity, speed, and acceleration when $t$ = 1 and illustrate geometrically.

# Example 1

The position vector of an object moving in a plane is given by $\mathrm{r}(t) = t^3\ \mathrm{i} + t^2\ \mathrm{j}$.

Find its velocity, speed, and acceleration when $t$ = 1 and illustrate geometrically.

Solution:

The velocity and acceleration at time $t$ are

$$\mathrm{v}(t) = \mathrm{r}'(t) = 3t^2\ \mathrm{i} + 2t\ \mathrm{j}$$
$$\mathrm{a}(t) = \mathrm{r}''(t) = 6t\ \mathrm{i} + 2\ \mathrm{j}$$

and the speed is

$$|\mathrm{v}(t)| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$$

# Example 1 – Solution

When $t$ = 1, we have

$$\mathrm{v}(1) = 3\ \mathrm{i} + 2\ \mathrm{j} \quad \mathrm{a}(1) = 6\ \mathrm{i} + 2\ \mathrm{j} \quad |\mathrm{v}(1)| = \sqrt{13}$$

These velocity and acceleration vectors are shown in Figure 2.

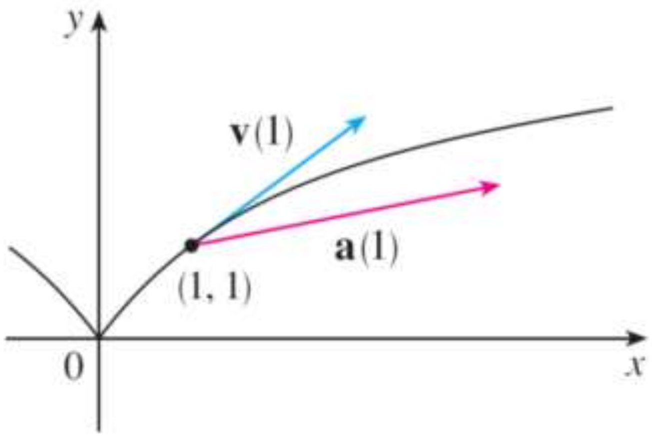


**Figure 2**

# Velocity, Speed, and Acceleration (4 of 4)

In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)\,du \qquad \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\,du$$

If the force that acts on a particle is known, then the acceleration can be found from **Newton's Second Law of Motion**.

The vector version of this law states that if, at any time *t*, a force **F**(*t*) acts on an object of mass *m* producing an acceleration **a**(*t*), then

$$\mathbf{F}(t) = m\mathbf{a}(t)$$

# Projectile Motion

# Example 5

A projectile is fired with angle of elevation $\alpha$ and initial velocity $\boldsymbol{v}_0$. (See Figure 6.)

Assuming that air resistance is negligible and the only external force is due to gravity, find the position function **r**$(t)$ of the projectile.

What value of $\alpha$ maximizes the range (the horizontal distance traveled)?

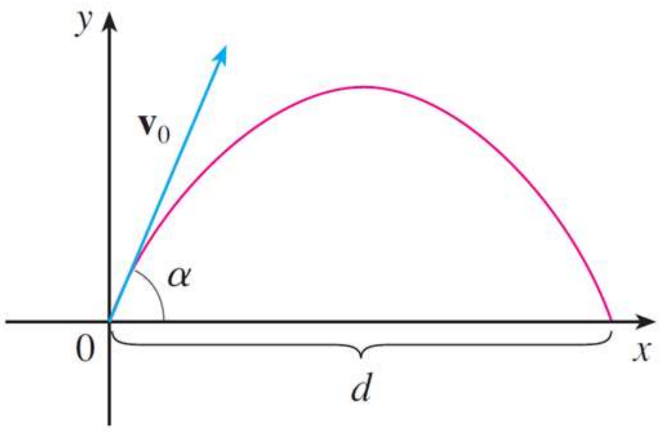


Figure 6

# Example 5 – Solution (1 of 4)

We set up the axes so that the projectile starts at the origin. Since the force due to gravity acts downward, we have

$$\boldsymbol{F} = m\boldsymbol{a} = -mg\boldsymbol{j}$$

Where $g = |\mathrm{a}| \approx 9.8\ \mathrm{m/s^2}$. Thus

$$\boldsymbol{a} = -g\ \boldsymbol{j}$$

Since $\mathrm{v}'(t) = \mathrm{a}$, we have

$$\boldsymbol{v}(t) = -gt\ \boldsymbol{j} + \boldsymbol{C}$$

# Example 5 – Solution (2 of 4)

where **C** = **v**(0) = $\mathbf{v}_0$. Therefore

$$\boldsymbol{r}'(t) = \boldsymbol{v}(t) = -gt\boldsymbol{j} + \boldsymbol{v}_0$$

Integrating again, we obtain

$$\boldsymbol{r}(t) = -\frac{1}{2}gt^2\,\boldsymbol{j} + t\ \boldsymbol{v_0} + \boldsymbol{D}$$

But **D** = **r**(0) = **0**, so the position vector of the projectile is given by

$$\boldsymbol{r}(t) = -\frac{1}{2}gt^2\,\boldsymbol{j} + t\ \boldsymbol{v_0} \tag{3}$$

# Example 5 – Solution (3 of 4)

If we write $|\boldsymbol{v}_0| = \mathrm{v}_0$ (the initial speed of the projectile), then

$$\boldsymbol{v}_{\mathbf{0}} = \mathrm{v}_0 \cos\alpha \ \mathrm{i} + \mathrm{v}_0 \sin\alpha \ \ \mathrm{j}$$

and Equation 3 becomes

$$\boldsymbol{r}(t) = (\mathrm{v}_0 \cos\alpha)t \ \boldsymbol{i} + \left[(\mathrm{v}_0 \sin\alpha)t - \frac{1}{2}gt^2\right]\boldsymbol{j}$$

The parametric equations of the trajectory are therefore

$$4 \quad x = (\mathrm{v}_0 \cos\alpha)t \quad y = (\mathrm{v}_0 \sin\alpha)t - \frac{1}{2}gt^2$$

# Example 5 – Solution (4 of 4)

The horizontal distance *d* is the value of *x* when *y* = 0. Setting *y* = 0, we obtain

*t* = 0 or $t = \dfrac{(2v_0 \sin\alpha)}{g}$. This second value of *t* then gives

$$
\begin{aligned}
d = x\Big|_{t=\frac{2v_0 sin\alpha}{g}} &= (v_0 \cos\alpha)\frac{2v_0 \sin\alpha}{g} \\
&= \frac{v_0^2(2\sin\alpha\cos\alpha)}{g} \\
&= \frac{v_0^2 \sin 2\alpha}{g}
\end{aligned}
$$

Clearly, *d* has its maximum value when sin 2*α* = 1, that is, *α* = 45°.

# Tangential and Normal Components of Acceleration

# Tangential and Normal Components of Acceleration (1 of 6)

When we study the motion of a particle, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal.

If we write $v = |\boldsymbol{v}|$ for the speed of the particle, then

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \frac{\boldsymbol{v}(t)}{|\boldsymbol{v}(t)|} = \frac{\boldsymbol{v}}{v}$$

and so $\boldsymbol{v} = v\boldsymbol{T}$

If we differentiate both sides of this equation with respect to $t$, we get

$$5 \quad \boldsymbol{a} = \boldsymbol{v}' = v'\boldsymbol{T} + v\boldsymbol{T}'$$

# Tangential and Normal Components of Acceleration (2 of 6)

If we use the expression for the curvature, then we have

$$6 \quad \kappa = \frac{|\boldsymbol{T}'|}{|\boldsymbol{r}'|} = \frac{|\boldsymbol{T}'|}{v} \quad \text{so} \quad |\mathrm{T}'| = \kappa v$$

The unit normal vector was defined in the preceding section as $\boldsymbol{N} = \frac{\boldsymbol{T}'}{|\boldsymbol{T}'|}$,

so (6) gives

$$\mathrm{T}' = |\mathrm{T}'|\mathrm{N} = \kappa v \mathrm{N}$$

and Equation 5 becomes

$$7 \quad \mathrm{a} = v'\mathrm{T} + \kappa v^2 \mathrm{N}$$

Writing $a_T$ and $a_N$ for the tangential and normal components of acceleration, we have

$$\mathrm{a} = a_T\mathrm{T} + a_N\mathrm{N}$$

where

$$8 \qquad a_T = v' \quad \text{and} \quad a_N = v|\boldsymbol{T}'| = \kappa v^2$$

This resolution is illustrated in Figure 7.

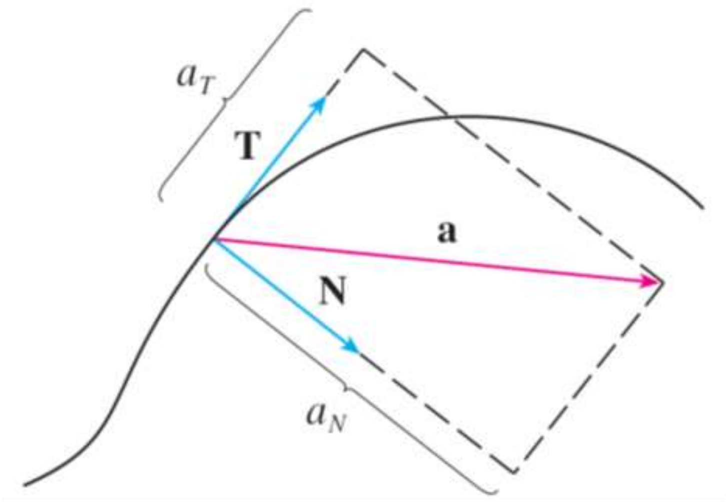


**Figure 7**

# Tangential and Normal Components of Acceleration (4 of 6)

Let's look at what Formula 7 says. The first thing to notice is that the binormal vector **B** is absent.

No matter how an object moves through space, its acceleration always lies in the plane of **T** and **N** (the osculating plane). (Recall that **T** gives the direction of motion and **N** points in the direction the curve is turning.)

Next we notice that the tangential component of acceleration is $v'$, the rate of change of speed, and the normal component of acceleration is $\kappa v^2$, the curvature times the square of the speed.

# Tangential and Normal Components of Acceleration (5 of 6)

This makes sense if we think of a passenger in a car—a sharp turn in a road means a large value of the curvature $\kappa$ so the component of the acceleration perpendicular to the motion is large and the passenger is thrown against a car door.

High speed around the turn has the same effect; in fact, if you double your speed, $a_N$ is increased by a factor of 4.

Although we have expressions for the tangential and normal components of acceleration in Equations 8, it's desirable to have expressions that depend only on $\mathbf{r}, \mathbf{r}', \text{ and } \mathbf{r}''$.

# Tangential and Normal Components of Acceleration (6 of 6)

To this end we take the dot product of **v** = *v***T** with **a** as given by Equation 7:

$$\boldsymbol{v} \cdot \boldsymbol{a} = v\mathrm{T} \cdot (v'\mathrm{T} + \kappa v^2 \mathrm{N})$$

$$= vv'\mathrm{T} \cdot \mathrm{T} + \kappa v^3 \mathrm{T} \cdot \mathrm{N}$$

$$= vv' \qquad (\text{since } \boldsymbol{T} \cdot \boldsymbol{T} = 1 \text{ and } \boldsymbol{T} \cdot \boldsymbol{N} = 0)$$

Therefore

$$9 \qquad a_T = v' = \frac{\boldsymbol{v}(t) \cdot \boldsymbol{a}(t)}{v(t)} = \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|}$$

Using the formula for curvature, we have

$$10 \qquad a_N = \kappa v^2 = \frac{|\mathrm{r}'(t) \times \mathrm{r}''(t)|}{|\mathrm{r}'(t)|^3} |\mathrm{r}'(t)|^2 = \frac{|\mathrm{r}'(t) \times \mathrm{r}''(t)|}{|\mathrm{r}'(t)|} = \frac{|\boldsymbol{v}(t) \times \boldsymbol{a}(t)|}{v(t)}$$

# Example 7

A particle moves with position function $\mathrm{r}(t) = \langle t^2, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

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A particle moves with position function $\mathrm{r}(t) = \langle t^2, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

Solution:

$$
\begin{aligned}
\mathrm{r}(t) &= t^2\,\mathrm{i} + t^2\,\mathrm{j} + t^3\,\mathrm{k} \\
\mathrm{r}'(t) &= 2t\,\mathrm{i} + 2t\,\mathrm{j} + 3t^2\,\mathrm{k} \\
\mathrm{r}''(t) &= 2\,\mathrm{i} + 2\,\mathrm{j} + 6t\,\mathrm{k} \\
|\mathrm{r}'(t)| &= \sqrt{8t^2 + 9t^4}
\end{aligned}
$$

# Example 7 – Solution (1 of 2)

Therefore Equation 9 gives the tangential component as

$$\begin{aligned} a_T &= \frac{\mathrm{r}'(t) \cdot \mathrm{r}''(t)}{|\mathrm{r}'(t)|} \\ &= \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}} \end{aligned}$$

Since

$$\begin{aligned} \mathrm{r}'(t) \times \mathrm{r}''(t) &= \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} \\ &= 6t^2\mathrm{i} - 6t^2\mathrm{j} \end{aligned}$$

# Example 7 – Solution (2 of 2)

Equation 10 gives the normal component as

$$a_N = \frac{|\mathrm{r}'(t) \times \mathrm{r}''(t)|}{|\mathrm{r}'(t)|}$$

$$= \frac{6\sqrt{2}t^2}{\sqrt{8t^2 + 9t^4}}$$

# Kepler’s Laws of Planetary Motion

# Kepler's Laws of Planetary Motion (1 of 12)

## Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Since the gravitational force of the sun on a planet is so much larger than the forces exerted by other celestial bodies, we can safely ignore all bodies in the universe except the sun and one planet revolving about it.

We use a coordinate system with the sun at the origin and we let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of the planet. (Equally well, $\mathbf{r}$ could be the position vector of the moon or a satellite moving around the earth or a comet moving around a star.)

# Kepler's Laws of Planetary Motion (2 of 12)

The velocity vector is $\mathbf{v} = \mathbf{r}'$ and the acceleration vector is $\mathbf{a} = \mathbf{r}''$.

We use the following laws of Newton:

$$\text{Second Law of Motion: } \mathbf{F} = m\mathbf{a}$$

$$\text{Law of Gravitation: } \mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\mathbf{u}$$

where **F** is the gravitational force on the planet, *m* and *M* are the masses of the planet and the sun, *G* is the gravitational constant, $r = |\mathbf{r}|$, and $\mathbf{u} = \left(\frac{1}{r}\right)\mathbf{r}$

is the unit vector in the direction of **r**.

# Kepler's Laws of Planetary Motion (3 of 12)

We first show that the planet moves in one plane.

By equating the expressions for **F** in Newton's two laws, we find that

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

and so **a** is parallel to **r**.

It follows that **r** × **a** = **0**.

# Kepler’s Laws of Planetary Motion (4 of 12)

We use the formula

$$\frac{d}{dt}[\mathrm{u}(t) \times \mathrm{v}(t)] = \mathrm{u}'(t) \times \mathrm{v}(t) + \mathrm{u}(t) \times \mathrm{v}'(t)$$

to write

$$\begin{aligned} \frac{d}{dt}(\mathrm{r} \times \mathrm{v}) &= \mathrm{r}' \times \mathrm{v} + \mathrm{r} \times \mathrm{v}' \\ &= \mathrm{v} \times \mathrm{v} + \mathrm{r} \times \mathrm{a} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

# Kepler's Laws of Planetary Motion (5 of 12)

Therefore

$$\mathbf{r} \times \mathbf{v} = \mathbf{h}$$

where **h** is a constant vector. (We may assume that $\mathbf{h} \neq \mathbf{0}$; that is, **r** and **v** are not parallel.)

This means that the vector $\mathbf{r} = \mathbf{r}(t)$ is perpendicular to **h** for all values of *t*, so the planet always lies in the plane through the origin perpendicular to **h**.

Thus the orbit of the planet is a plane curve.

# Kepler's Laws of Planetary Motion (6 of 12)

To prove Kepler's First Law we rewrite the vector **h** as follows:

$$\begin{aligned}\mathrm{h} &= \mathrm{r} \times \mathrm{v} = \mathrm{r} \times \mathrm{r}' = r\mathrm{u} \times (r\mathrm{u})' \\ &= r\mathrm{u} \times (r\mathrm{u}' + r'\mathrm{u}) = r^2(\mathrm{u} \times \mathrm{u}') + rr'(\mathrm{u} \times \mathrm{u}) \\ &= r^2(\mathrm{u} \times \mathrm{u}')\end{aligned}$$

Then

$$\begin{aligned}\mathrm{a} \times \mathrm{h} &= \frac{-GM}{r^2}\, \mathrm{u} \times (r^2\mathrm{u} \times \mathrm{u}') = -GM\, \mathrm{u} \times (\mathrm{u} \times \mathrm{u}') \\ &= -GM[(\mathrm{u} \cdot \mathrm{u}')\mathrm{u} - (\mathrm{u} \cdot \mathrm{u})\mathrm{u}'] \quad \text{by Formula } a \times (b \times c) = (a \cdot c)b - (a \cdot b)c\end{aligned}$$

# Kepler's Laws of Planetary Motion (7 of 12)

But $\mathrm{u} \cdot \mathrm{u} = |\mathrm{u}|^2 = 1$ and since $|\mathrm{u}(t)| = 1$, it follows that

$$\mathrm{u} \cdot \mathrm{u}' = 0$$

Therefore

$$\mathrm{a} \times \mathrm{h} = GM\,\mathrm{u}'$$

and so

$$\begin{aligned} (\mathrm{v} \times \mathrm{h})' &= \mathrm{v}' \times \mathrm{h} + \mathrm{v} \times \mathrm{h}' \\ &= \mathrm{a} \times \mathrm{h} \\ &= GM\,\mathrm{u}' \end{aligned}$$

# Kepler's Laws of Planetary Motion (8 of 12)

Integrating both sides of this equation, we get

$$\mathbf{v} \times \mathbf{h} = GM\,\mathbf{u} + \mathbf{c} \tag{11}$$

where **c** is a constant vector.

At this point it is convenient to choose the coordinate axes so that the standard basis vector **k** points in the direction of the vector **h**.

Then the planet moves in the *xy*-plane. Since both **v** × **h** and **u** are perpendicular to **h**, Equation 11 shows that **c** lies in the *xy*-plane.

# Kepler’s Laws of Planetary Motion (9 of 12)

This means that we can choose the *x*- and *y*-axes so that the vector **i** lies in the direction of **c**, as shown in Figure 8.

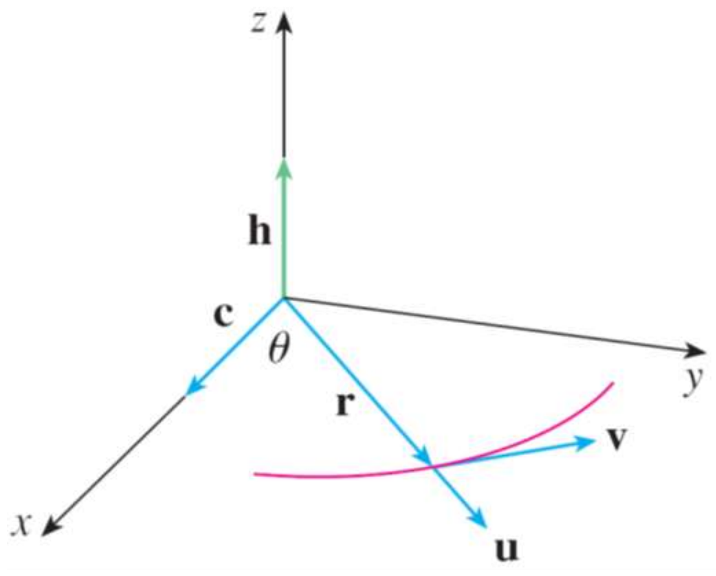


**Figure 8**

# Kepler’s Laws of Planetary Motion (10 of 12)

If $\theta$ is the angle between **c** and **r**, then $(r, \theta)$ are polar coordinates of the planet.

From Equation 11 we have

$$\begin{aligned}
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= \mathbf{r} \cdot (GM\,\mathbf{u} + \mathbf{c}) = GM\,\mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{c} \\
&= GMr\mathbf{u} \cdot \mathbf{u} + |\mathbf{r}||\mathbf{c}| \cos\theta \\
&= GMr + rc\cos\theta
\end{aligned}$$

where $c = |\mathbf{c}|$.

# Kepler's Laws of Planetary Motion (11 of 12)

Then

$$r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + c\cos\theta} = \frac{1}{GM}\frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e\cos\theta}$$

where $e = \dfrac{c}{(GM)}$.

But

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = |\mathbf{h}|^2 = h^2$$

where $h = |\mathbf{h}|$.

# Kepler's Laws of Planetary Motion (12 of 12)

So

$$r = \frac{\frac{h^2}{(GM)}}{1 + e\cos\theta} = \frac{\frac{eh^2}{c}}{1 + e\cos\theta}$$

Writing $d = \frac{h^2}{c}$, we obtain the equation

$$12 \quad r = \frac{ed}{1 + e\cos\theta}$$

we see that Equation 12 is the polar equation of a conic section with focus at the origin and eccentricity *e*. We know that the orbit of a planet is a closed curve and so the conic must be an ellipse.