(will also post rubric)

Midterm 2 graded, let me know any grading concerns.

HW 6 & 7 were up last afternoon

DW 8 for today's discussion is up, will add more problems

Since no more material, today's lecture used for practice problems and any review\questions students may want. This gives students extra chance (in addition to tomorrow) to review for final & get help on remaining tasks like HWs & quiz 4

Quiz 4 posted today, due 7/11 11:59pm, still optional, still benefit of lowest quiz dropped if you take it. Will keep quiz 4 on last week's material, so 6/30, 7/1 & 7/2. If you prefer quiz 4 to be covering this week too, let me know

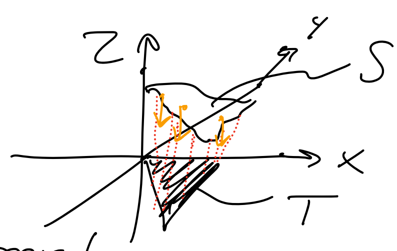
To do: post solution docs, add extra practice to latest discussions, make survey, make bug bounty program, add practice midterm 2 & final (not made by me), add common mistake documents, any random features, make quiz 4

Practice topics\subtopics: surface integral

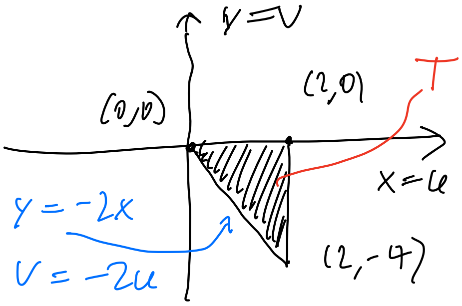
Practice problems: DW8 Q1a, DW8 Q2b

DW8 Q1a: Evaluate $\iint_S F \cdot d\vec{S}$, $F = (3x, 2z, 1-y^2)$, S portion of $z = 2 - 3y + x^2$ lies over triangle in xy-plane with vertices $(0,0), (2,0), (2,-4)$ oriented in negative z-axis direction

Let's first draw a graph.



Whatever normal vectors to S we find must have negative z-coord



Now we can plug in for everything and figure out the value of integral.

We are in $z = \dots$ scenario, so we can set $r(u,v) = (u, v, 2-3v+u^2)$, $(u,v) \in T$

Find bounds on T: $0 \le u \le 2$, $v \le 0$. From the fact $y=-2x$ bounds T, we have $v \ge -2u$ as well

Next step is find new differential.

Because in $z = \dots$ scenario, $\|r_u \times r_v\| =$

$$\sqrt{(2-3v+u^2)_u^2 + (2-3v+u^2)_v^2 + 1} = \sqrt{4u^2+9+1} = \sqrt{4u^2+10}$$

This approach with $r_u \times r_v$ does work but need to use $r_u \times r_v$. In our case, there is a faster approach: $\iint_S F \cdot d\vec{S} = \iint_T (-A g_x - B g_y + C)\, dA$.

$F = (3x, 2z, 1-y^2) = (3x, 2(2-3y+x^2), 1-y^2)$,

$z - g(x,y) = 0$ desc. S where $g(x,y) = 2-3y+x^2$.

$$-A g_x - B g_y + C = -3x \cdot 2x - 2(2-3y+x^2)(-3) + (1-y^2)$$

$$= -\cancel{6x^2} + 12 - 18y + \cancel{6x^2} + 1 - y^2 = -y^2 - 18y + 13$$

T bounds are $0 \le x \le 2$, $-2x \le y \le 0$, so

$$\iint_S F \cdot d\vec{S} = \iint_T (\dots)\, dA = \int_0^2 \int_{-2x}^0 (-y^2 - 18y + 13)\, dy\, dx$$

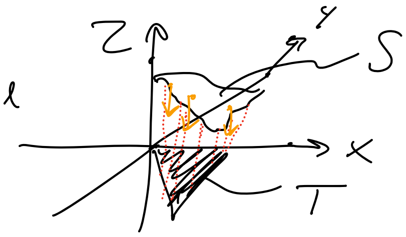
$$= - - \int_0^2 \int_0^{-2x} (y^2 + 18y - 13)\, dy\, dx = \int_0^2 \left[\frac{(-2x)^3}{3} + 9(-2x)^2 - 13(-2x)\right] dx$$

$$= \int_0^2 \left(-\frac{8}{3}x^3 + 36x^2 + 26x\right) dx = -\frac{2}{3} \cdot 2^4 + 12 \cdot 2^3 + 13 \cdot 2^2$$

$\leq -\frac{32}{3} + 96 + 52 = -\frac{32}{3} + 148 = \frac{412}{3}$

Note ... only true for S positively oriented.

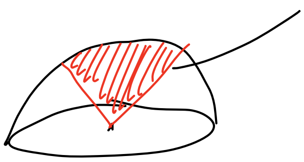
S looks like



, so the normals pointing outward point in positive z direction. In our case, normals pointing negative z-direction, hence pointing inward, hence S is neg orient.,

So $\int_{\partial S} = -\frac{412}{3}$ in reality.

Find $\iint_S yz\, d\sigma$ where S part of sphere $x^2+y^2+z^2=4$ lying above cone $z=\sqrt{x^2+y^2}$, $d\sigma$ = surface area differential.



So S is the boundary of the shaded portion of the sphere

So $\sqrt{x^2+y^2} \leq z \leq 2$ and $x^2+y^2 \leq 4$

Let's use spherical. $\rho = 2$, so

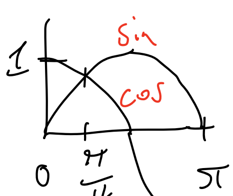
$x = 2\sin\varphi\cos\theta$
$y = 2\sin\varphi\sin\theta$
$z = 2\cos\varphi$.

Note that $d\sigma = dS = 4\sin\varphi\, d\varphi\, d\theta$ since ρ not present anymore.

No restriction on $\theta \Rightarrow 0 \leq \theta < 2\pi$.

$2 \geq 2\cos\varphi \geq \sqrt{x^2+y^2} = 2\sin\varphi \Rightarrow 1 \geq \cos\varphi \geq \sin\varphi$

$\cos\varphi \leq 1$ is always true. When is $\cos\varphi \geq \sin\varphi$.



$\cos \geq \sin$ on $[0, \frac{\pi}{4}] \Rightarrow 0 \leq \varphi \leq \frac{\pi}{4}$.

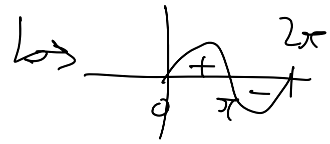
$yz\, d\sigma = (4\sin\varphi\sin\theta)(4\sin\varphi\, d\varphi\, d\theta) =$

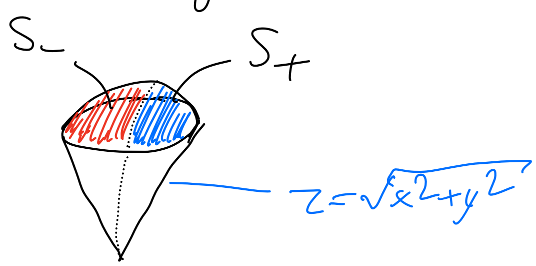
$16 \sin^2\varphi \sin\theta \, d\varphi \, d\theta$

$$\iint_S yz \, d\sigma = \int_0^{2\pi} \int_0^{\pi/4} 16 \sin^2\varphi \sin\theta \, d\varphi \, d\theta =$$

$16 \left(\int_0^{\pi/4} s^2\varphi \, d\varphi \right) \left(\int_0^{2\pi} \sin\theta \, d\theta \right) = 0$ because $\int_0^{2\pi} \sin\theta \, d\theta = 0$

b/c





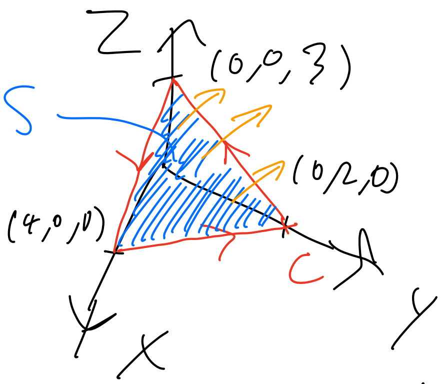
$S_+ = S \cap \{y \geq 0\}$

$S_- = S \cap \{y \leq 0\}$.

$$\iint_S yz = \iint_{S_+} yz - \iint_{S_-} (-yz) = \iint_{S_+} yz - \iint_{S_+} yz = 0$$

because the reflection $y \to -y$ send S_- to S_+ and send $-yz$ to $-(-y)z = yz$.

DW8 Q2b: Find $\int_C F \cdot dr$ for $F = (3yx^2 + z^3, y^2, 4yx^2)$, C triangle w/ vertices $(0,0,3)$, $(0,2,0)$, $(4,0,0)$ orient CCW when looking above C towards origin



Could parametrize each edge of triangle, plug in for F, split into 3 integrals, and evaluate manually, but Stokes' Theorem saves lots of effort.

In order to do that, find S, dS & $\nabla \times F$.

$$\begin{array}{ccc} i & j & k \\ \partial x & \partial y & \partial z \\ A & B & C \end{array}$$

$\nabla \times F = (4x^2 - 0, 8xy - 3z^2, 0 - 3x^2) =$

$(4x^2, 8xy - 3z^2, -3x^2)$.

Draw S with normal vectors pointing upward as shown. Then C is CCW $\Rightarrow$ C is pos, S is pos, so $\int_C F \cdot dr = \iint_S (\nabla \times F) \cdot dS$.

Now we need to find dS & param S.

Note S is portion of plane $\frac{x}{4} + \frac{y}{2} + \frac{z}{3} = 1$ in 1st octant.

Solve for z: $z = \frac{3}{4}(4 - x - 2y) := g(x,y)$

We are in scenario S described by $z - g(x,y) = 0$,

so if $\nabla \times F = (A, B, C)$,

$$\iint_S F \cdot d\vec{S} = \iint_D (-Ag_x - Bg_y + C) =$$

$$\iint -4x^2 \cdot -\frac{3}{4} - (8xy - 3z^2) \cdot \frac{3}{4}(-2) + (-3x^2)\, dA =$$

$$\iint \cancel{3x^2} + \frac{3}{2}(8xy - 3z^2) \cancel{-3x^2} = \iint_D \left(12xy - \frac{9}{2}z^2\right) dA$$

where D is projection of S onto xy-plane

Bounds for D are $x, y \geq 0$ since 1st octant becomes 1st quadrant, and $x + 2y \leq 4 \Rightarrow x \leq 4 - 2y$,

$0 \leq y \leq 2$. $z = \frac{3}{4}(4 - x - 2y) \Rightarrow \frac{9}{2}z^2 =$

$$\frac{9}{2}\frac{9}{16}(4 - x - 2y)^2 = \frac{81}{32}(4y^2 + 2x + 16 - 8x - 16y + 4xy)$$

$$\Rightarrow 12xy - \frac{9}{2}z^2 = 12xy - \frac{81}{32}(\cdots)$$

$$= \int_0^2 \int_0^{4-2y} \left[12xy - \frac{81}{32}(4 - x - 2y)^2\right] dx\, dy =$$

$$\int_0^2 12(4 - 2y)y + \left[\frac{81}{32}\frac{1}{3}(4 - x - 2y)^3\right]\Bigg|_{x=0}^{x=4-2y} dy$$

$$\int_0^2 \left[12(4-2y)\,y - \frac{27}{32}(4-2y)^2\right] dy =$$

$$\int_0^2 \left(48y - 48y^2 - \frac{27}{8}\left(y^2 - 4y + 4\right)\right) dy =$$

$$\int_0^2 \left(48y - 48y^2 - \frac{27}{8}y^2 + \frac{27}{2}y - \frac{27}{2}\right) dy$$

$$= 24 \cdot 4 - 16 \cdot 27 - \frac{9}{8} \cdot 27 + \frac{27}{4} \cdot 4 - \frac{27}{2}$$

$$= 96 - 432 - \frac{243}{8} + \frac{27}{2} = -336 + \frac{108}{8} - \frac{243}{8}$$

$$= -336 - \frac{135}{8} = \frac{-2688 - 135}{8} = -\frac{2823}{8}$$

$$\begin{array}{r} \scriptstyle 4 \\ 27 \\ \cdot\ 16 \\ \hline 162 \\ \cancel{27\ } \\ \hline 432 \end{array} \qquad \begin{array}{r} \scriptstyle 24 \\ 336 \\ \cdot\ 8 \\ \hline 2688 \\ \hline \end{array}$$