Tangent planes:

Recall: • On the xy plane, any line has equation $y = mx + b$. A vector equation for this line is

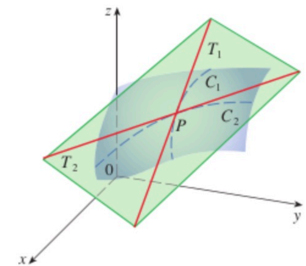
$L(t) = \langle 0, b \rangle + t\langle 1, m \rangle = \langle t, mt + b \rangle$ (parametric: $x = t$, $y = mt + b$ $\Rightarrow$ symmetric: $x = \frac{y-b}{m}$ $\Rightarrow y = mx + b$)

• If $f(x)$ is differentiable at a then $y - f(a) = f'(a)(x-a)$ is the equation for the tangent line $\Rightarrow y = f'(a)x + f(a) - af'(a)$

• In vector form: $\boxed{L(t) = \langle 0, f(a) - a \cdot f'(a) \rangle + t\langle 1, f'(a) \rangle}$

• So a direction vector for the tangent line is $\langle 1, f'(a) \rangle$.

The tangent plane contains the tangent lines T_1 and T_2.



• For a surface $z = f(x,y)$, fix a point $x = x_0$, $y = y_0$ and $z = z_0 = f(x_0, y_0)$, say this is the point (x_0, y_0, z_0)

• Consider the plane $y = y_0$, and let C_1 be the trace of the surface on this plane.

• C_1 has the equation $z = f(x, y_0)$. It can be thought of as a curve on the z-x plane.

- The equation to its tangent line at (x_0, y_0) is $z = z_0 + f_x(x_0,y_0)(x-x_0)$
- Vector equation for T_1 : $L_1(t) = \langle 0, 0, z_0 - f_x(x_0,y_0)x_0 \rangle + t\langle 1, 0, f_x(x_0,y_0)\rangle$
- Similarly, let C_2 be the trace along the $x = x_0$ plane, with tangent line T_2. $(z = z_0 + f_y(x_0,y_0)(y-y_0))$
- It has the vector equation T_2 : $L_2(s) = \langle 0, 0, z_0 - f_y(x_0,y_0)y_0 \rangle + t\langle 0, 1, f_y(x_0,y_0)\rangle$
- So T_1 has direction vector $\vec{v}_1 = \langle 1, 0, f_x(x_0,y_0)\rangle$ and T_2 has direction vector $\vec{v}_2 = \langle 0, 1, f_y(x_0,y_0)\rangle$.
- <u>Def</u>: Tangent plane at (x_0,y_0,z_0) is the plane spanned by $\vec{v}_1$ and $\vec{v}_2$.
- It has $\vec{n} = \vec{v}_1 \times \vec{v}_2$ as a normal vector.

where $$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x(x_0,y_0) \\ 0 & 1 & f_y(x_0,y_0) \end{vmatrix} = -f_x(x_0,y_0)\hat{i} - f_y(x_0,y_0)\hat{j} + \hat{k}$$

$\Rightarrow$ Equation for the tangent plane at (x_0, y_0, z_0) is

$$\boxed{-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + (z-z_0) = 0}$$

OR: $\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$

Linear approximations:

Def: The linear function $\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$ is called the linearization of f at (a,b).

- The approximation $\boxed{f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$ is called the linear approximation or the tangent plane approximation of f at (a,b).

Important: If f_x and f_y are not CTS at a point, linear approximation there need not work.

7 Definition

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\, \Delta x + f_y(a, b)\, \Delta y + \varepsilon_1\, \Delta x + \varepsilon_2\, \Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy such that ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

In other words: differentiable functions are those for which linear approximations work.

8 Theorem

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Differential:

Def: For a differentiable function $z = f(x,y)$, define the differentials dx and dy to be independent variables; that is they can be given any values.

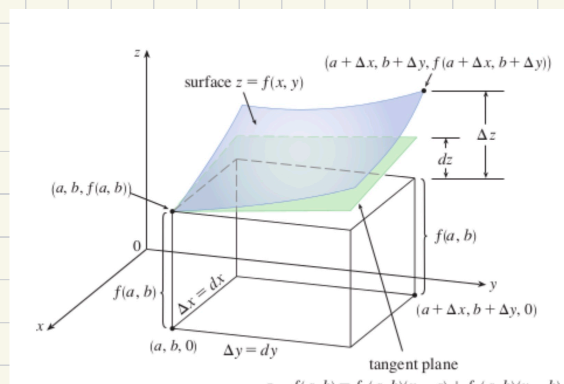
Then the differential dz (also called total differential), is defined by

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

Remarks: Let $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ then

$$dz = f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b).$$

- So that $f(x,y) \approx f(a,b) + dz$.
- dz represents the change in height of the tangent plane.
- Δz represents the change in height of the surface when (x,y) changes from (a,b) to $(a+\Delta x, b+\Delta y)$.



Example 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

(a) $f_x = 2x + 3y$, $f_y = 3x - 2y$

$\Rightarrow dz = (2x+3y)\,dx + (3x-2y)\,dy$

(b) $(a, b) = (2, 3)$ and $\Delta x = 2.05 - 2 = 0.05$, $\Delta y = 2.96 - 3 = -0.04$

Actual change in height is $f(2.05, 2.96) - f(2,3) \simeq 0.6449$

Approximate change: $dz = f_x(2,3)\,\Delta x + f_y(2,3)\,\Delta y$

$= (4+9)(0.05) + (6-6)(-0.04)$

$= 13 \times 0.05 = 0.65$

Section 14.4 : Chain Rules

Recall: $y = f(x)$, $x = g(t) \Rightarrow \dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dg}{dt}$

$z = f(x, y) = f(g(t), h(t))$

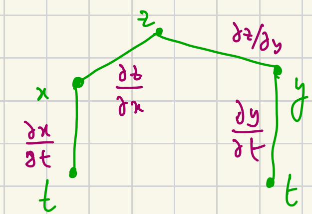
$$\Rightarrow \frac{\partial z}{\partial t} = f_x \cdot \frac{dg}{dt} + f_y \cdot \frac{dh}{dt}$$

1 The Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

z is the dependent var.
x, y are intermediate var.
t independent var.



Example 1

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

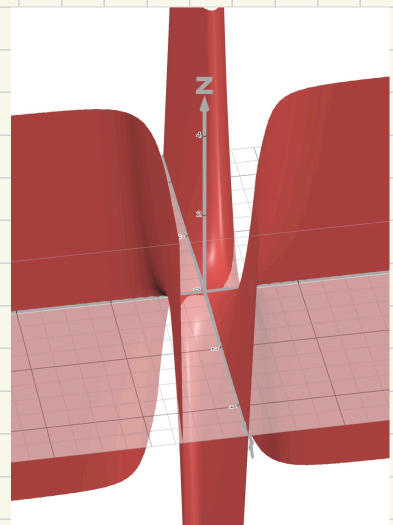
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

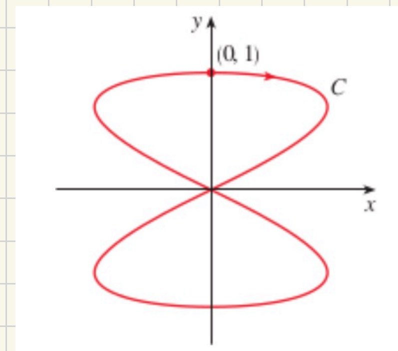
$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

$x(0) = 0$, $y(0) = 1$

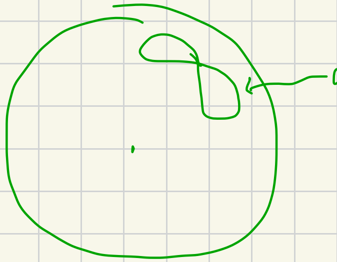
$$\Rightarrow z'(0) = (0 + 3)(2 \cdot 1) + (0 + 0)(0)$$

$$= 6.$$





Eg:

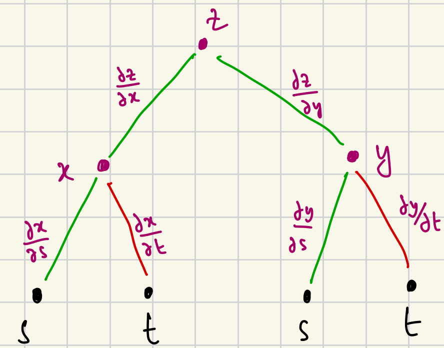


a curve "on" the sphere.

2 The Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Example 3

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

$$\frac{\partial z}{\partial x} = e^x \sin y, \qquad \frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial x}{\partial s} = t^2 \qquad \frac{\partial y}{\partial s} = 2st$$

$$\frac{\partial x}{\partial t} = 2st \qquad \frac{\partial y}{\partial t} = s^2$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (e^x \sin y)\cdot t^2 + (e^x \cos y)\, 2st$$

$$= \boxed{t^2 e^{st^2} \sin(s^2 t) + 2st\, e^{st^2} \cos(s^2 t)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = (e^x \sin y)\cdot(2st) + (e^x \cos y)\, s^2$$

$$= \boxed{2st\, e^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t)}$$

3 The Chain Rule (General Version)

intermediate

Suppose that u is a differentiable function of the $\overbrace{n \text{ variables}}$ $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m $\underbrace{\text{variables}}$ $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

independent

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

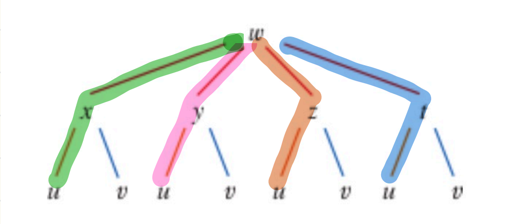
for each $i = 1, 2, \ldots, m$.

Example 4

intermediate

Write out the Chain Rule for the case where $w = f\,\overbrace{(x, y, z, t)}$ and $x = x\,\underbrace{(u, v)}$, $y = y\,(u, v)$, $z = z\,(u, v)$, and $t = t\,(u, v)$.

independent

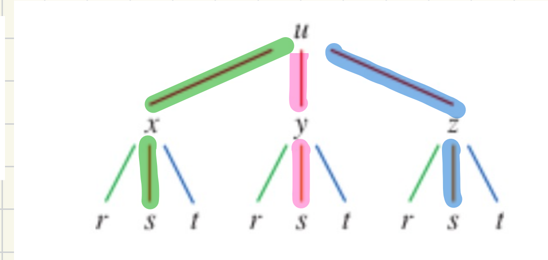


$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial v}.$$

Example 5

If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$, find the value of $\partial u / \partial s$ when $r = 2$, $s = 1$, $t = 0$.



$$\frac{\partial u}{\partial x} = 4x^3 y \quad , \frac{\partial u}{\partial y} = x^4 + 2yz^3 \quad , \frac{\partial u}{\partial z} = 3y^2 z^2$$

$$\frac{\partial x}{\partial s} = re^t \quad , \frac{\partial y}{\partial s} = 2rse^{-t} \quad , \frac{\partial z}{\partial s} = r^2 \sin t$$

$$x(2,1,0) = 2 \quad , y(2,1,0) = 2 \quad , z(2,1,0) = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}(2,1,0) = 4 \cdot 2^3 \cdot 2 = 64 \quad , \frac{\partial u}{\partial y}(2,1,0) = 16 \quad , \frac{\partial u}{\partial z}(2,1,0) = 0$$

$$\frac{\partial x}{\partial s}(2,1,0) = 2 \quad , \frac{\partial y}{\partial s}(2,1,0) = 4 \quad , \frac{\partial z}{\partial s} = 0$$

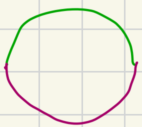
$$\Rightarrow \frac{\partial u}{\partial s}(2,1,0) = (64)(2) + (16)(4) + (0)(0) = \boxed{192}$$

Implicit Differentiation

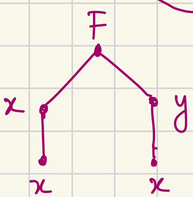
Context: • Suppose that $F(x,y)=0$ defines y implicitly as a differentiable function of x.

• That is $y=f(x)$ where $F(x, f(x))=0$ for all x in the domain of f.

Eg: $F(x,y)=x^2+y^2-r^2$ and $f(x)=\sqrt{r^2-x^2}$ or $f(x)=-\sqrt{r^2-x^2}$.



From the chain rule: (think of x as a function x as well)



$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\cdot\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

So if $\frac{\partial F}{\partial y}\neq 0$:

$$\boxed{\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-F_x}{F_y}}$$

Example 8

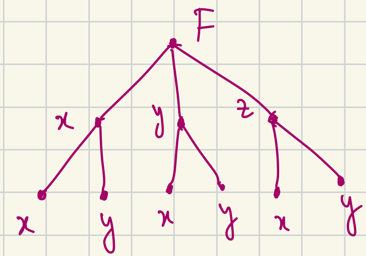
Find y' if $x^3 + y^3 = 6xy$.

A: $F(x,y) = x^3 + y^3 - 6xy \Rightarrow F_x = 3x^2 - 6y \quad , F_y = 3y^2 - 6x$

$$\Rightarrow \frac{dy}{dx} = y' = \frac{-F_x}{F_y} = \frac{-(3x^2-6y)}{(3y^2-6x)} = -\frac{(x^2-2y)}{(y^2-2x)}$$

- Now, suppose that $z = f(x,y)$ is given implicitly by $F(x,y,z) = 0$. That is $F(x,y,f(x,y))$ in the domain of f. (x and y are independent so $\frac{dy}{dx} = 0$ and $\frac{dx}{dy} = 0$)

- Suppose that F and f are differentiable, then by chain rule:



$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\boxed{\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x = \frac{-F_x}{F_z}}$$

- Similarly, $\frac{\partial F}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial y}}_{=0} + \frac{\partial F}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial y}}_{=1} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$

$$\boxed{\Rightarrow \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y = \frac{-F_y}{F_z}}$$

Example 9

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz + 4 = 0$.

$F(x,y,z) = x^3 + y^3 + z^3 + 6xyz + 4$

$F_x = 3x^2 + 6yz \Rightarrow \frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2+6yz)}{(3z^2+6xy)}$

$F_y = 3y^2 + 6xz$

$\Rightarrow \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(3y^2+6xz)}{(3z^2+6xy)}$

$F_z = 3z^2 + 6xy$