# 12 Vectors and the Geometry of Space

# 12.5 Equations of Lines and Planes

# Lines

# Lines (1 of 12)

A line in the *xy*-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given.

The equation of the line can then be written using the point-slope form.

Likewise, a line $L$ in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on $L$ and the direction of $L$, which is conveniently described by a vector $\mathbf{v}$ parallel to the line.

# Lines (2 of 12)

Let $P(x, y, z)$ be an arbitrary point on $L$ and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of $P_0$ and $P$ (that is, they have representations $\overrightarrow{OP_o}$ and $\overrightarrow{OP}$).

If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_oP}$, as in Figure 1,then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$.

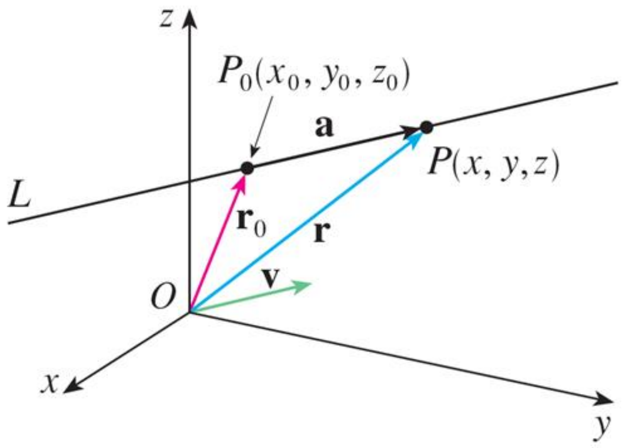


**Figure 1**

# Lines (3 of 12)

Since **a** and **v** are parallel vectors, there is a scalar *t* such that **a** = *t***v**. Thus

$$\textbf{1} \quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

which is a **vector equation** of *L*. Each value of the **parameter** *t* gives the position vector **r** of a point on *L*. In other words, as *t* varies, the line is traced out by the tip of the vector **r**.

As Figure 2 indicates, positive values of $t$ correspond to points on $L$ that lie on one side of $P_0$, whereas negative values of $t$ correspond to points that lie on the other side of $P_0$.

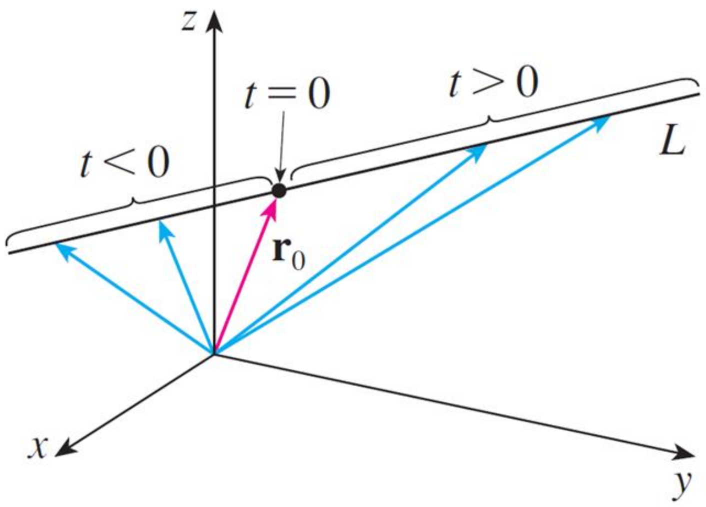


**Figure 2**

# Lines (5 of 12)

If the vector **v** that gives the direction of the line *L* is written in component form as $\mathrm{v} = \langle a, b, c \rangle$, then we have $t\mathrm{v} = \langle ta,\ tb,\ tc \rangle$.

We can also write $\mathrm{r} = \langle x, y, z \rangle$ and $\mathrm{r}_0 = \langle x_0, y_0, z_0 \rangle$, so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal.

# Lines (6 of 12)

Therefore we have the three scalar equations

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

where $t \in \mathbb{R}$.

These equations are called **parametric equations** of the line *L* through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathrm{v} = \langle a, b, c \rangle$.

Each value of the parameter *t* gives a point (*x*, *y*, *z*) on *L*.

# Lines (7 of 12)

**2** Parametric equations for a line through the point $(x_0, y_0, z_0)$ and parallel to the direction vector $\langle a, b, c \rangle$ are

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

# Example 1

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

(b) Find two other points on the line.

# Example 1

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector **i** + 4**j** − 2**k**.

(b) Find two other points on the line.

Solution:

(a) Here $\mathrm{r}_0 = \langle 5,1,3 \rangle = 5\mathrm{i} + \mathrm{j} + 3\mathrm{k}$ and $\mathrm{v} = \mathrm{i} + 4\mathrm{j} - 2\mathrm{k}$ so the vector equation (1) becomes

$$\mathrm{r} = (5\mathrm{i} + \mathrm{j} + 3\mathrm{k}) + t(\mathrm{i} + 4\mathrm{j} - 2\mathrm{k})$$

$$\text{or} \quad \mathrm{r} = (5 + t)\mathrm{i} + (1 + 4t)\mathrm{j} + (3 - 2t)\mathrm{k}$$

# Example 1 – Solution

Parametric equations are

$$x = 5 + t \quad y = 1 + 4t \quad z = 3 - 2t$$

(b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so (6, 5, 1) is a point on the line. Similarly, $t = -1$ gives the point (4, −3, 5).

# Lines (8 of 12)

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change.

For instance, if, instead of (5, 1, 3), we choose the point (6, 5, 1) in Example 1, then the parametric equations of the line become

$$x = 6 + t \quad y = 5 + 4t \quad z = 1 - 2t$$

# Lines (9 of 12)

Or, if we stay with the point (5, 1, 3) but choose the parallel vector 2**i** + 8**j** − 4**k**, we arrive at the equations

$$x = 5 + 2t \quad y = 1 + 8t \quad z = 3 - 4t$$

In general, if a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line *L*, then the numbers *a*, *b*, and *c* are called **direction numbers** of *L*.

Since any vector parallel to **v** could also be used, we see that any three numbers proportional to *a*, *b*, and *c* could also be used as a set of direction numbers for *L*.

# Lines (10 of 12)

Another way of describing a line *L* is to eliminate the parameter *t* from Equations 2.

If none of *a*, *b*, or *c* is 0, we can solve each of these equations for *t*:

$$t = \frac{x - x_0}{a} \quad t = \frac{y - y_0}{b} \quad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{3}$$

These equations are called **symmetric equations** of *L*.

# Lines (11 of 12)

Notice that the numbers *a*, *b*, and *c* that appear in the denominators of Equations 3 are direction numbers of *L*, that is, components of a vector parallel to *L*.

If one of *a*, *b*, or *c* is 0, we can still eliminate *t*. For instance, if $a = 0$, we could write the equations of *L* as

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that *L* lies in the vertical plane $x = x_0$.

# Lines (12 of 12)

In general, we know from Equation 1 that the vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector $\mathbf{v}$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

If the line also passes through (the tip of) $\mathbf{r}_1$, then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$ and so its vector equation is

$$\mathrm{r} = \mathrm{r}_0 + t(\mathrm{r}_1 - \mathrm{r}_0) = (1 - t)\mathrm{r}_0 + t\mathrm{r}_1$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the parameter interval $0 \le t \le 1$.

**4** The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathrm{r}(t) = (1 - t)\mathrm{r}_0 + t\mathrm{r}_1 \quad 0 \le t \le 1$$

# Planes

# Planes (1 of 6)

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe.

A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify its direction.

Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector **n** that is orthogonal to the plane. This orthogonal vector **n** is called a **normal vector**.

# Planes (2 of 6)

Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of $P_0$ and $P$.

Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. (See Figure 6.)

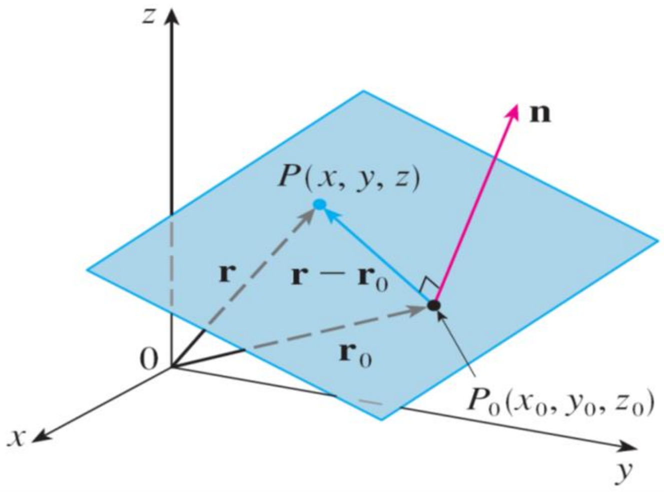


**Figure 6**

# Planes (3 of 6)

The normal vector **n** is orthogonal to every vector in the given plane. In particular, **n** is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \qquad (5)$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 \qquad (6)$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

# Planes (4 of 6)

To obtain a scalar equation for the plane, we write

$$\mathrm{n} = \langle a, b, c \rangle, \mathrm{r} = \langle x, y, z \rangle, \text{ and } \mathrm{r}_0 = \langle x_0, y_0, z_0 \rangle.$$

Then the vector equation (5) becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

**7** A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathrm{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

# Example 4

Find an equation of the plane through the point (2, 4, −1) with normal vector $\mathrm{n} = \langle 2,3,4 \rangle$. Find the intercepts and sketch the plane.

Solution:

Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 7, we see that an equation of the plane is

$$\begin{aligned} 2(x-2) + 3(y-4) + 4(z+1) &= 0 \\ \text{or} \qquad 2x + 3y + 4z &= 12 \end{aligned}$$

To find the $x$-intercept we set $y = z = 0$ in this equation and obtain $x = 6$.

# Example 4 – Solution

Similarly, the *y*-intercept is 4 and the *z*-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7).

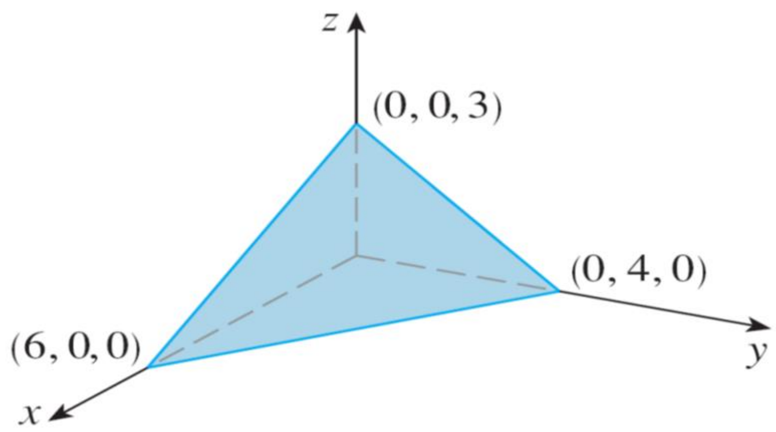


**Figure 7**

# Planes (5 of 6)

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

$$\mathbf{8} \quad ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$.

Equation 8 is called a **linear equation** in *x*, *y*, and *z*. Conversely, it can be shown that if *a*, *b*, and *c* are not all 0, then the linear equation (8) represents a plane with normal vector $\langle a, b, c \rangle$.

# Planes (6 of 6)

Two planes are **parallel** if their normal vectors are parallel. For instance, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors are $\mathbf{n}_1 = \langle 1, 2, -3 \rangle$ and $\mathbf{n}_2 = \langle 2, 4, -6 \rangle$ and $\mathbf{n}_2 = 2\mathbf{n}_1$.

If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle $\theta$ in Figure 9).

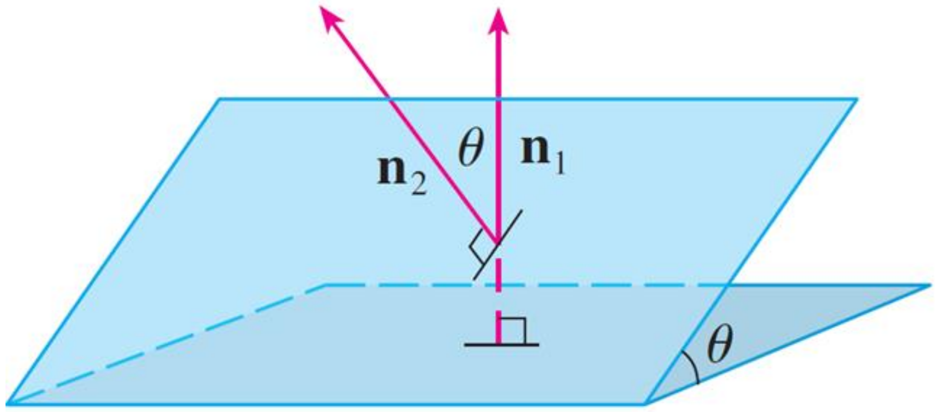


**Figure 9**

# Distances

# Distances (1 of 3)

Find a formula for the distance $D$ from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let **b** be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathrm{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

From Figure 12 you can see that the distance $D$ from $P_1$ to the plane is equal to the absolute value of the scalar projection of **b** onto the normal vector $\mathrm{n} = \langle a, b, c \rangle$.

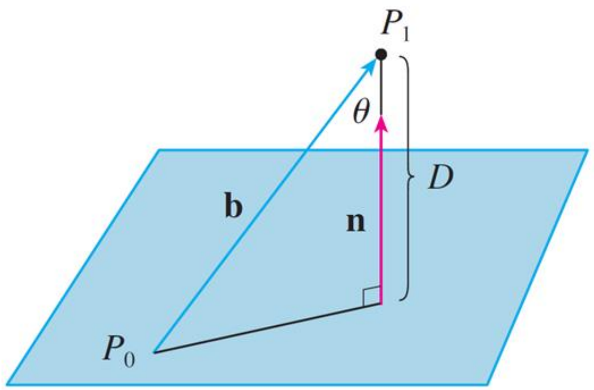


Figure 12

# Distances (2 of 3)

Thus

$$\begin{aligned} D &= |\text{comp}_{\mathrm{n}}\mathrm{b}| = \frac{|\mathrm{n} \cdot \mathrm{b}|}{|\mathrm{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

# Distances (3 of 3)

Since $P_0$ lies in the plane, its coordinates satisfy the equation of the plane and so we have

$$ax_0 + by_0 + cz_0 + d = 0.$$

Thus we have the following formula.

**9** The distance $D$ from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

# Example 8

Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Solution:

First we note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel. To find the distance $D$ between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$ and so $\left(\frac{1}{2}, 0, 0\right)$ is a point in this plane.

# Example 8 – Solution

By Formula 9, the distance between $\left(\frac{1}{2}, 0, 0\right)$ and the plane $5x + y - z - 1 = 0$. is

$$D = \frac{\left|5\left(\frac{1}{2}\right) + 1(0) - 1(0) - 1\right|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So the distance between the planes is $\frac{\sqrt{3}}{6}$.