Textbook Sections: 14.1, 14.2 and 14.3
Topics: Functions of several variables; limits and continuity; partial derivatives
Instructions: Try each of the following problems, show the detail of your work.

Clearly mark your choices in multiple choice items. Justify your answers.

Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

FUNCTIONS OF SEVERAL VARIABLES:

1. If $g(x,y) = x\sin y + y\sin x$, find

   (a) $g(\pi, 0)$.

   $$g(\pi,0) = \pi\sin(0) + 0\sin(\pi) = 0.$$

   (b) $g(\pi/2, \pi/4)$.

   $$g(\pi/2,\pi/4) = \frac{\pi}{2}\sin(\pi/4) + \frac{\pi}{4}\sin(\pi/2) = \frac{\pi}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\pi}{4} = \frac{\pi}{4}(1+\sqrt{2}).$$

   (c) $g(0, y)$.

   $$g(0,y) = 0\cdot\sin(y) + y\sin(0) = 0.$$

   (d) $g(x, y+h)$.

   $$g(x,y+h) = x\sin(y+h) + (y+h)\sin(x).$$

2. Let $F(x,y,z) = \sqrt{y} - \sqrt{x-2z}$.

   (a) Evaluate $F(3,4,1)$.

   $$F(3,4,1) = \sqrt{4} - \sqrt{3-2} = 2-1 = 1.$$

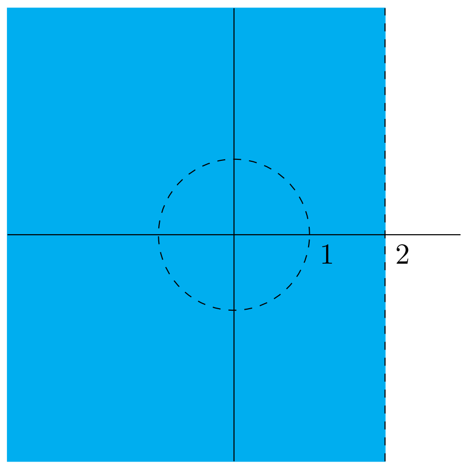
   (b) Find and describe the domain of F.

For each square root to be defined, y and $x - 2z$ must be nonnegative, so the domain of F is
$$\left\{(x, y, z) : y \geq 0, z \leq \frac{x}{2}\right\}.$$

3. Find and sketch the domain of the function $g(x, y) = \dfrac{\ln(2 - x)}{1 - x^2 - y^2}$.

For $g(x, y)$ to be defined, $\ln(2 - x)$ must be defined, so $2 - x > 0$. Also, the denominator must be nonzero, so $1 - x^2 - y^2 \neq 0$. Thus, the domain is

$$D = \{(x, y) : x < 2, x^2 + y^2 \neq 1\}.$$



LIMITS AND CONTINUITY:

4. Evaluate the following limits.

(a) $\displaystyle\lim_{(x,y)\to(3,2)} e^{\sqrt{2x-y}}$.

$$\lim_{(x,y)\to(3,2)} e^{\sqrt{2x-y}} = e^{\sqrt{6-2}} = e^2.$$

(b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2y^3 - x^3y^2}{x^2 - y^2}$.

$$\lim_{(x,y)\to(1,1)} \frac{x^2y^3 - x^3y^2}{x^2 - y^2} = \lim_{(x,y)\to(1,1)} \frac{x^2y^2(y - x)}{(x - y)(x + y)} = \lim_{(x,y)\to(1,1)} -\frac{x^2y^2}{x + y} = -\frac{1}{2}$$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2}$$
$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1 = 2$$

5. Show that the following limits do not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x-2y}{4x^2-y^2}$.

   $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x-2y}{4x^2-y^2}$ does not exist because along the path given by the line $y=x$ passing through $(0,0)$, we have $\displaystyle\lim_{x\to0}\frac{3x-2x}{4x^2-x^2} = \lim_{x\to0}\frac{x}{3x^2} = \lim_{x\to0}\frac{1}{3x}$ does not exist.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$

   Let $y=ax$. Then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2} = \lim_{x\to0}\frac{2ax^2}{(1+3a^2)x^2} = \lim_{x\to0}\frac{2a}{1+3a^2} = \frac{2a}{1+3a^2}$.
   However, this value changes based on the value of a.
   Let $a=1$. Along the path C_1 given by the line $y=x$, $f(x,y)=\frac{2xy}{x^2+3y^2}\to L_1=\frac{2}{4}=\frac{1}{2}$ as $(x,y)\to(0,0)$.
   Let $a=2$.Along the path C_2 given by the line $y=2x$, $f(x,y)=\frac{2xy}{x^2+3y^2}\to L_2=\frac{4}{1+3(4)}=\frac{4}{13}$ as $(x,y)\to(0,0)$. Thus, on two different paths, the functions reaches two different numbers as $(x,y)\to(0,0)$.
   So, the limit does not exist.

6. Determine the set of points at which the function is continuous.

   (a) $F(x,y)=\dfrac{1+x^2+y^2}{1-x^2-y^2}$.

   $\displaystyle\lim_{(x,y)\to(a,b)} \frac{1+x^2+y^2}{1-x^2-y^2} = \frac{1+a^2+b^2}{1-a^2-b^2}$ except when $1-a^2-b^2=0$, when the function is not defined. So F is continuous on the set of points $D=\{(x,y): x^2+y^2\neq1\}$.

   (b) $G(x,y)=\ln(1+x-y)$.

   As long as the function is defined (so $x-y>-1$) we have $\displaystyle\lim_{(x,y)\to(a,b)} \ln(1+x-y) = \ln(1+a-b)$. So G is continuous on the set of points $D=\{(x,y): x-y>-1\}$.

PARTIAL DERIVATIVES:

7. Find the first partial derivatives of the function $g(u, v) = (u^2v - v^3)^5$.

$g_u(u, v) = 5(u^2v - v^3)^4(2uv)$ $g_v(u, v) = 5(u^2v - v^3)^4(u^2 - 3v^2)$

8. Given the function $f(x, y) = y \arcsin(xy)$, find $f_y(1, \frac{1}{2})$.

By the Product Rule, we have

$$f_y(x, y) = y(\frac{1}{\sqrt{1-(xy)^2}})(x) + \arcsin(xy)$$

$$= \frac{xy}{\sqrt{1-(xy)^2}} + \arcsin(xy)$$

$$f_y(1, \frac{1}{2}) = \frac{(1)(\frac{1}{2})}{\sqrt{1-((1)(\frac{1}{2}))^2}} + \arcsin\left((1)(\frac{1}{2})\right)$$

$$= \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} + \arcsin\left(\frac{1}{2}\right)$$

$$= \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} + \frac{\pi}{6}$$

$$= \frac{1}{\sqrt{3}} + \frac{\pi}{6}$$

9. Verify that the conclusion of Clairaut's Theorem holds for the function $u(x, y) = e^{xy}\sin(y)$.

We need to show: $u_{xy} = u_{yx}$

$$u_x = ye^{xy}\sin(y)$$

$$u_{xy} = e^{xy}\sin(y) + y[xe^{xy}\sin(y) + e^{xy}\cos(y)]$$

$$= e^{xy}[\sin(y) + xy\sin(y) + y\cos(y)]$$

$$u_y = xe^{xy}\sin(y) + e^{xy}\cos(y)$$

$$u_{yx} = e^{xy}\sin(y) + xye^{xy}\sin(y) + ye^{xy}\cos(y)$$

$$= e^{xy}[\sin(y) + xy\sin(y) + y\cos(y)] = u_{xy}$$

Hence, the conclusion of Clairaut's Theorem holds

10. Given the function $f(x,y) = x^4y^2 - x^3y$, find $f_{xxx}(x,y)$ and $f_{xyx}(x,y)$.

Note:

$$f_x(x,y) = 4x^3y^2 - 3x^2y$$

So we have:

$$f_{xx}(x,y) = 12x^2y^2 - 6xy$$

$$f_{xy}(x,y) = 8x^3y - 3x^2$$

Using these, we find that:

$$f_{xxx}(x,y) = 24xy^2 - 6y$$

$$f_{xyx}(x,y) = 24x^2y - 6x$$

Suggested Textbook Problems

Section 14.1	9-22, 25, 27, 29, 30, 32-34, 36, 47, 49, 61-66
Section 14.2	5-22, 29, 33
Section 14.3	15-41, 47, 51-53, 56, 59, 63, 68, 73, 82, 83, 90

Some Useful Definitions and a Strategy:

A Function of Two Variables A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f, and its range is the set of values it takes on, namely $\{f(x, y) : (x, y) \in D\}$.

A Continuous Function A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

We say that f is continuous on D if it is continuous at every point of D.

Showing a Limit Does Not Exist If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, and $L_1 \neq L_2$, then the limit

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

does not exist.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f.$$

Rules For Finding Partial Derivatives

- To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
- To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Clairaut's Theorem Suppose f is defined on a disk D that contains the pouint (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$