Hessian Test

Lecture for 6/23

Review of Matrices: Eigenvectors

Let A be a matrix and **v** a vector

- If A**v** = λ**v** and **v** ≠ **0**, then **v** is an eigenvector of A with eigenvalue λ
- Can find eigenvalues by solving det(A-λI) = 0 for λ
    - After finding the values, solve (A-λI)**v** = 0 to get vectors

Review of Matrices: Definiteness

- Matrices with $A^T = A$ are called symmetric
- Symmetric matrices with $\mathbf{v}^T A \mathbf{v} > 0$ for all $\mathbf{v} \neq \mathbf{0}$ are positive definite
  - Negative definite if $\mathbf{v}^T A \mathbf{v} < 0$ for all $\mathbf{v} \neq \mathbf{0}$
  - Semidefinite if $>$ and $<$ are replaced with $\geq$ and $\leq$
- If all eigenvalues positive, then A is positive definite
- If all eigenvalues are $<, \geq, \leq 0$, then neg, pos semi, neg semi resp.

Hessian Test

Consider a twice differentiable function $f(x_1, x_2, \ldots, x_n)$

- Let H be the n x n matrix whose (i, j) entry is $f_{x\_i\, x\_j}$
- H is the Hessian of f

Suppose **x** is a critical point of f

- If H(**x**) is positive definite, **x** is a local min
- If H(**x**) is negative definite, **x** is a local max
- If H(**x**) has negative and positive eigenvalues, **x** is a saddle point
- If H(**x**) is positive or negative semidefinite, no info

Derivation of Test

Other Methods

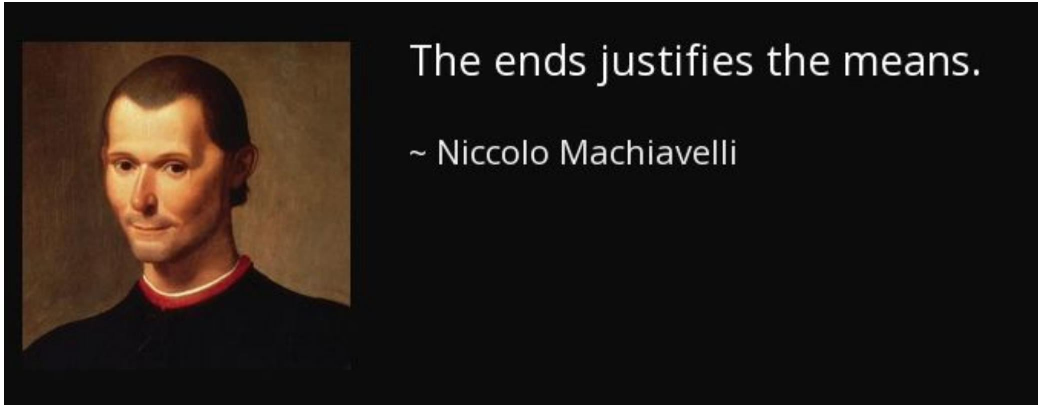
What happens if Hessian or 2nd Derivative Test inconclusive?

- Try approaching point along different directions
  - Same strategies as figuring out 2D limits
  - Will suggest which kind of point your critical point is
- If local extrema, prove your guess via inequalities

Practice Problems

Find and classify all critical points by any means necessary

- $f(x, y, z) = x^2+y^2+z^2+xy+xz+yz+x+y+z+1$
- $f(x, y) = x^4 - y^4 - 4xy^2 - 2x^2$
- $f(x, y) = x^{2024} + y^{2026}$



Scratchwork