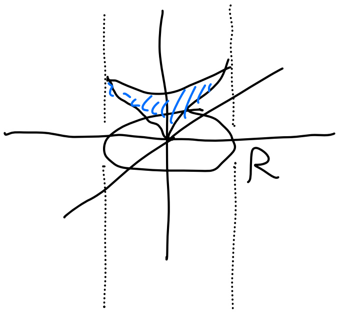
Setting up integrals for: 2D integrals in polar, surface area

Find surface area of $z=xy$ in cylinder given by $x^2+y^2=1$.



Integrating over $R=\{x^2+y^2\le 1\}$.

$x=r\cos\theta$
$y=r\sin\theta$ $\Rightarrow$ $r^2\le 1 \Rightarrow 0\le r\le 1$.

No restriction on $\theta \Rightarrow 0\le\theta<2\pi$

surface area $=\iint_R dS = \underbrace{\iint_R \sqrt{1+x^2+y^2}\,dx\,dy}$ $=$

$dS=\sqrt{1+f_x^2+f_y^2}=\underbrace{\sqrt{1+y^2+x^2}}_{=\sqrt{1+r^2}}$, $\quad dx\,dy = r\,dr\,d\theta$

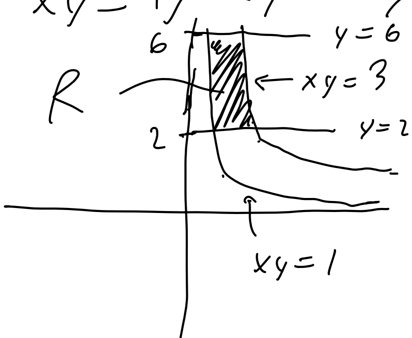
$z=f(x,y)=xy$

$$=\int_0^{2\pi}\int_0^1 r\sqrt{1+r^2}\,dr\,d\theta = 2\pi\int_0^1 r(1+r^2)^{1/2}\,dr$$

$$=2\pi\cdot\frac{1}{3}(1+r^2)^{3/2}\Big|_0^1 = \frac{2\pi}{3}\left[2^{3/2}-1^{3/2}\right]=\frac{2\pi}{3}(2\sqrt{2}-1).$$

Jacobian review: $\iint_R xy^3\,dA$ where R bounded by

$xy=1$, $xy=3$, $y=2$, $y=6$ using $x=\frac{v}{6u}$, $y=2u$



$2\le y\le 6 \Rightarrow 1\le u\le 3$

$1\le xy\le 3 \Rightarrow 1\le \frac{v}{3}\le 3 \Rightarrow 3\le v\le 9$

Compute $J=\begin{bmatrix} x_u & x_v \\ y_u & y_v\end{bmatrix}=\begin{bmatrix} -v/6u^2 & 1/6u \\ 2 & 0\end{bmatrix}$

$|\det J| = \left|-\frac{1}{3u}\right| = \frac{1}{3u}$

To come up with substitution yourself:
keep $y=u$, consider $x=\frac{xy}{y}=\frac{xy}{u}$, so let $xy=v \Rightarrow (y,x)=\left(u,\frac{v}{u}\right)$

$\Rightarrow$ new bounds on u & v.

$1 \le u \le 3$, $3 \le v \le 9$, $x = \frac{v}{6u}$, $y = 2u$, $|\det J| = \frac{1}{3u}$.

$$xy^3 = \frac{v}{6u}(2u)^3 = \frac{v}{6u} \cdot 8u^3 = \frac{4}{3}vu^2$$

$$\iint_R xy^3\,dx\,dy = \int_3^9\int_1^3 \frac{4}{3}vu^2\left(\frac{1}{3u}\right)du\,dv = \int_3^9\int_1^3 \frac{4}{9}uv\,du\,dv$$

$= \ldots$, rest is evaluating normal 2D integral

Week 3 : 6\23 – 6\27] 6\23, 24, 25, 26, 27, 30
Week 4 : 6\30 – 7\4] 7\1, 2

Anything from these days is on the midterm

Week 1–2 topics won't be asked on midterm, but because week 3–4 build on top of prior topics, you may see them indirectly within the algebra for solving the problems.

For example: "Find f_x, f_y, f_{xxy} for $f = \ldots$" won't be on it because that's a week 2 question, but nonetheless many topics & questions require finding partial derivatives during the course of computation.

Find center of mass of square $\underbrace{0 \le x, y \le \pi}_{R}$ with weight function $f(x,y) = x\sin(x)y^3$.

Nothing to do here but plug in all of the values and integrate.

$$x_{com} = \frac{\iint_R x f(x,y)\,dA}{\iint_R f(x,y)\,dA}, \quad y_{com} = \frac{\iint y f(x,y)\,dA}{\iint_R f(x,y)\,dA}$$

$$\iint_R f\,dA = \int_0^{\pi}\int_0^{\pi} x\sin(x)\,y^3\,dx\,dy = \int_0^{\pi} y^3\left(\int_0^{\pi} x\sin(x)\,dx\right)dy$$

$$= \frac{\pi^4}{4}\cdot\pi = \pi^5/4$$

$\int x\sin(x)\,dx = -x\cos(x) + \int \cos x$
$u = x$
$= -x\cos(x) + \sin x$

$\int x^2 \sin x = -x^2\cos x + \int 2x\cos x$
$= -x^2\cos x + 2x\sin x - \int 2\sin x$
$= -x^2\cos x + 2x\sin x + 2\cos x$

$\int_0^{\pi} x\sin x = (-x\cos x + \sin x)\Big|_0^{\pi}$
$= (\pi + \sin\pi) - (0+0) = \pi$

$\int_0^{\pi} x^2\sin x = (-x^2\cos x + 2x\sin x + 2\cos x)\Big|_0^{\pi}$
$= \pi^2 + 2\cos\pi - 2\cos 0 = \pi^2 - 4$

$$\iint_R x f\,dA = \int_0^{\pi} y^3\left(\int_0^{\pi} x^2\sin x\,dx\right)dy$$

$$= \frac{\pi^4}{4}\cdot(\pi^2-4) = \frac{1}{4}(\pi^6 - 4\pi^4)$$

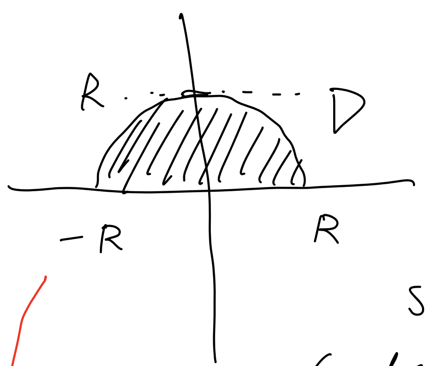
$$\iint y f\,dA = \int_0^{\pi} y^4\left(\int_0^{\pi} x\sin x\,dx\right)dy$$

$$= \frac{\pi^5}{5}\cdot\pi = \pi^6/5$$

$$x_{com} = \frac{\pi^2-4\cdot(\pi^4/4)}{\pi\cdot(\pi^4/4)} = \frac{\pi^2-4}{\pi}$$

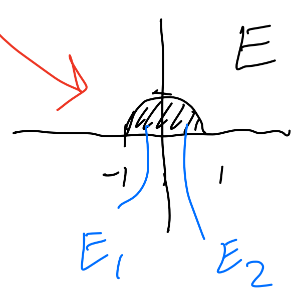
$$y_{com} = \frac{\pi^6/5}{\pi^5/4} = \frac{4}{5}\pi$$

So center of mass is $\left(\pi - \frac{4}{\pi}, \frac{4}{5}\pi\right)$.



Find center of mass of uniform semicircle of radius R, take half with $y \geq 0$

1st symmetry: D is similar to the semicircle of radius 1 by a dilation of factor R. So center of mass is also scaled by a factor of R.



Note: be careful using this trick when your weight is not uniform. $f(x,y)$ will transform to $f\left(\frac{x}{R}, \frac{y}{R}\right)$ instead.

So COM of $D = R(x_0, y_0)$ where (x_0, y_0) is the COM of E.

2nd symmetry: E is symmetrical about the y-axis. If $(x,y) \in E$, then $(-x,y) \in E$. So if you split E into $E_1 = E \cap \{x \le 0\}$, $E_2 = E \cap \{x \ge 0\}$,

then $\iint_E x\,dA = \iint_{E_1} x\,dA + \iint_{E_2} x\,dA \underset{--=+}{=} \iint_{E_1} x\,dA - \iint_{E_2} (-x)\,dA$

$= \iint_{E_1} x\,dA - \iint_{E_1} x\,dA = 0$, so $x_{com} = \frac{0}{\cdots} = 0$.

Still have to find y_{com}, but you saved $\frac{1}{2}$ of your work by recognizing $x_{com} = 0$.

You may also use other kinds of symmetries, such as the 4-fold symmetry used to justify $\iint_R xy\,dA = 0$ for 6/26 pre-lec practice problem #1.