14 Partial Derivatives

14.7 Maximum and Minimum Values

$f'(x) = 0$

$f''(x) < 0$

Local Maximum and Minimum Values

Local Maximum and Minimum Values (1 of 7)

In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables.

Look at the hills and valleys in the graph of *f* shown in Figure 1.

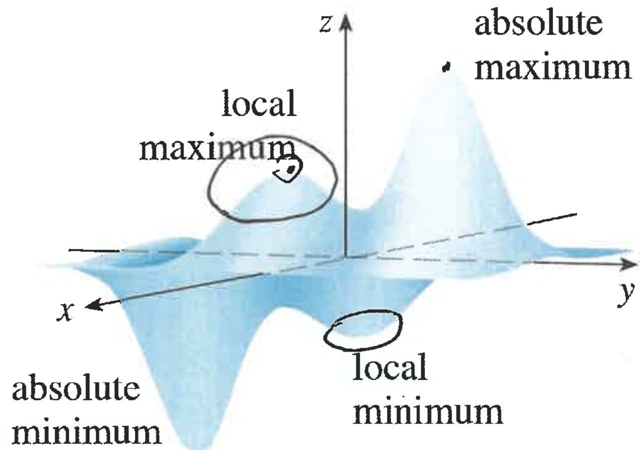


Figure 1

Local Maximum and Minimum Values (2 of 7)

There are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$.

Likewise, f has two *local minima*, where $f(a, b)$ is smaller than nearby values.

The largest value of $f(x, y)$ on the domain of f is the *absolute maximum*, and the smallest value is the *absolute minimum*.

Local Maximum and Minimum Values (3 of 7)

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b).

[This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).]

The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Local Maximum and Minimum Values (4 of 7)

Fermat's Theorem states that, for single-variable functions, if f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. The following theorem states a similar result for functions of two variables.

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

f_x , f_y

$f_x = 0$

$f_y = 0$

Local Maximum and Minimum Values (5 of 7)

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f.

However, as in single-variable calculus, not all critical points give rise to maxima or minima.

Example 1 (1 of 2)

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14.$

Then

$$f_x(x, y) = 2x - 2 \quad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is (1, 3).

Example

$$f(x, y) = e^x \cos y$$

$f_x(x, y) = e^x \cos y$

$f_y(x, y) = -e^x \sin y$

$\left.\begin{aligned} e^x \cos y = 0 \\ -e^x \sin y = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \cos y = 0 &\longrightarrow \frac{\pi}{2}, \frac{3\pi}{2} \\ -\sin y = 0 &\longrightarrow 0, \pi, 2\pi \end{aligned}$

Example

$$f(x,y) = x^3 + y^3 \cancel{+} -3x$$

$f_x(x,y) = 3x^2 \cancel{+} -3$, $3x^2 \cancel{+} -3 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$f_y(x,y) = 3y^2$ $\quad 3y^2 = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0$

$D(1,0) = 6 \times 0 - \cancel{0} \cdot (0)^2 = 0$.

$(1,0)$, $(-1,0)$ $\quad D(-1,0) = -6 \times 0 - (0)^2 = 0$

$$\begin{cases} 3x^2 - 3 = 0 \Rightarrow x = \pm 1 \\ \cancel{3x} \; 3y^2 = 3x^2 \Rightarrow 3y^2 = 3 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1 \end{cases}$$

$(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = 0$

$3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow x \neq -1$

$3y^2 = 3x \Rightarrow y^2 = x \Rightarrow y = \pm 1$

$(1,1)$, $(1,-1)$

Local Maximum and Minimum Values (6 of 7)

The following test, is analogous to the Second Derivative Test for functions of one variable.

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is a saddle point of f.

Note 1 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b) or (a, b) could be a saddle point of f.

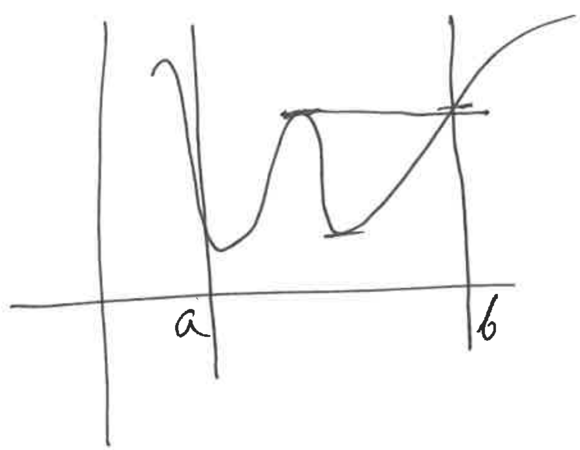
$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$$\Rightarrow f_{xx} f_{yy} = D + \underline{(f_{xy})^2} \geq D > 0$$

$$f_{xx} > 0, \quad f_{yy} > 0$$

$$f_{xx} < 0, \quad f_{yy} < 0$$

Absolute Maximum and Minimum Values



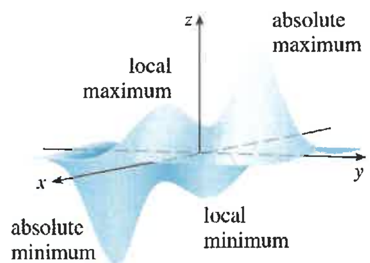
$f(x)$, $[a, b]$

Absolute Maximum and Minimum Values (1 of 6)

Just as for single-variable functions, the absolute maximum and minimum values of a function f of two variables are the largest and smallest values that f achieves on its domain.

7 Definition Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geq f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leq f(x, y)$ for all (x, y) in D.



Absolute Maximum and Minimum Values (2 of 6)

For a function *f* of one variable, the Extreme Value Theorem says that if *f* is continuous on a closed interval [*a*, *b*], then *f* has an absolute minimum value and an absolute maximum value.

According to the Closed Interval Method, we found these by evaluating *f* not only at the critical numbers but also at the endpoints *a* and *b*.

There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

Absolute Maximum and Minimum Values (3 of 6)

[A boundary point of *D* is a point (*a*, *b*) such that every disk with center (*a*, *b*) contains points in *D* and also points not in *D*.]

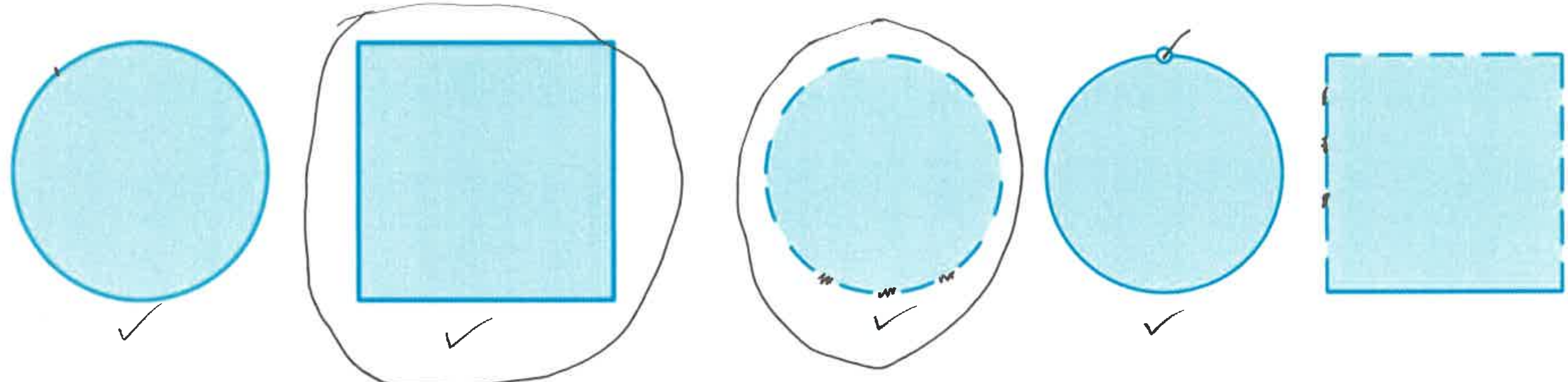
For instance, the disk

$$D = \{(x, y) | x^2 + y^2 \leq 1\}$$

which consists of all points on or inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$).

Absolute Maximum and Minimum Values (4 of 6)

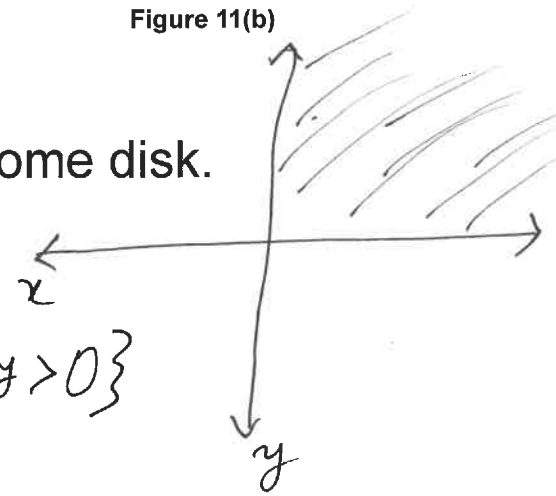
But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)



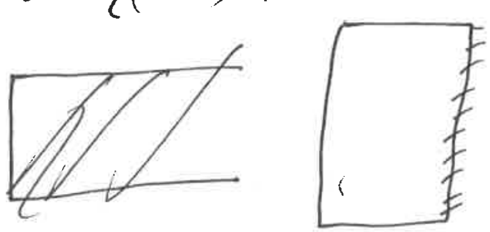
Closed sets
Figure 11(a)

Sets that are not closed
Figure 11(b)

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk.



$D = \{(x, y) \mid 0 \leq x < 2,\ 0 \leq y \leq 1\}$



$D = \{(x, y) \mid x > 0,\ y > 0\}$

Absolute Maximum and Minimum Values (5 of 6)

In other words, it is finite in extent.

Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Absolute Maximum and Minimum Values (6 of 6)

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if *f* has an extreme value at (x_1, y_1), then (x_1, y_1) is either a critical point of *f* or a boundary point of *D*.

Thus we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function *f* on a closed, bounded set *D*:

1. Find the values of *f* at the critical points of *f* in *D*.

2. Find the extreme values of *f* on the boundary of *D*.

3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) | 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Solution:

Since *f* is a polynomial, it is continuous on the closed, bounded rectangle *D*, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum.

According to step 1 in (9), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \quad f_y = -2x + 2 = 0$$

Example 7 – Solution (1 of 5)

So the only critical point is (1, 1). This point is in D and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12.

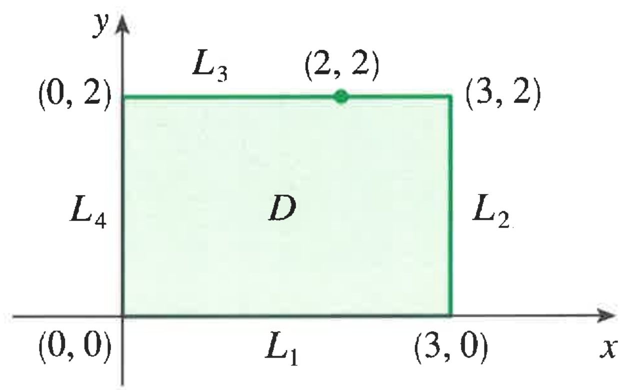


Figure 12

Example 7 – Solution (2 of 5)

On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \quad 0 \leq x \leq 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$.

On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \quad 0 \leq y \leq 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$.

Example 7 – Solution (3 of 5)

On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \quad 0 \leq x \leq 3$$

Simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$.

Example 7 – Solution (4 of 5)

Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \quad 0 \leq y \leq 2$$

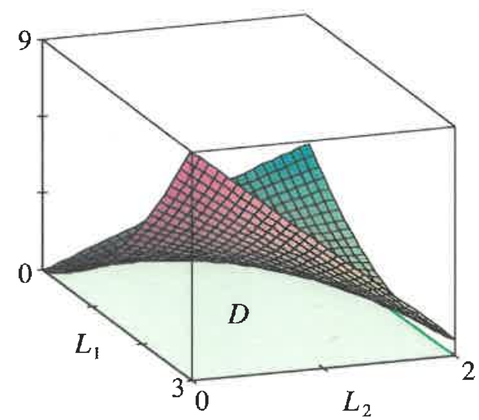
with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$.

Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

Example 7 – Solution (5 of 5)

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$.

Figure 13 shows the graph of f.



$f(x, y) = x^2 - 2xy + 2y$

Figure 13

Example

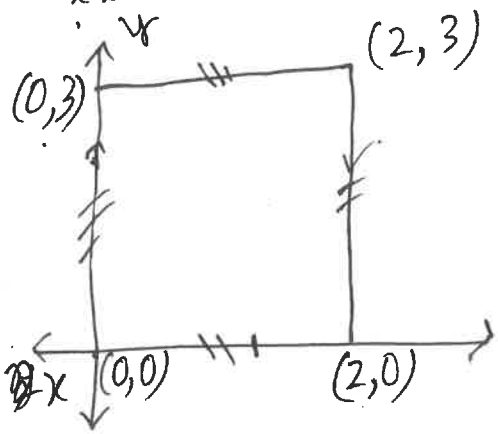
$f(x,y) = x^2 + 2y^2 - 2x - 4y + 1, \qquad D = \{(x,y): 0 \le x \le 2, 0 \le y \le 3\}$

$f_x = 2x - 2 \qquad 2x - 2 = 0 \Rightarrow x = 1$ only critical point $(1,1)$

$f_y = 4y - 4 \qquad 4y - 4 = 0 \Rightarrow y = 1$

$f_{xx} = 2, \; f_{yy} = 4, \; f_{xy} = 0 \Rightarrow D(1,1) = 2 \times 4 - (0)^2 = 8 > 0$

$f_{xx}(1,1) > 0 \Rightarrow (1,1)$ is local minima



$y = 0, \; 0 \le x \le 2$

$f(x,0) = x^2 - 2x + 1 = (x-1)^2$

$f'(x,0) = 2x - 2 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1, (1,0)$

$x = 2, \; 0 \le y \le 3$

$f(2,y) = \cancel{4} + 2y^2 - \cancel{4} - 4y + 1$

$= 2y^2 - 4y + 1$

$\Rightarrow f'(2,y) = 4y - 4 \Rightarrow 4y - 4 = 0 \Rightarrow y = 1, \; (2,1)$

Example

$$f(x,y) = x^2 + 2y^2 - 2x - 4y + 1, \qquad D = \{(x,y): 0 \le x \le 2, 0 \le y \le 3\}$$

$f(x,3) = x^2 + 18 - 2x - 12 + 1 = x^2 - 2x + 7$

$f'(x,3) = 2x - 2 \qquad f'(x,3) = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1 \qquad (1,3)$

$f(0,y) = 2y^2 - 4y + 1$

$f'(0,y) = 4y - 4 \qquad f'(0,y) = 0 \Rightarrow 4y - 4 = 0 \Rightarrow y = 1 \qquad (0,1)$

Crit.	$f(x,y)$
$(1,1)$	-2
$(1,0)$	0
$(2,1)$	-1
$(1,3)$	6
$(0,1)$	-1
$(0,0)$	1
$(0,3)$	7
$(2,3)$	7
$(2,0)$	1

$f_{max} = 7$ for $(0,3)$ and $(2,3)$

$f_{min} = -2$ for $(1,1)$

Proof of the Second Derivatives Test

Proof of the Second Derivatives Test (1 of 4)

We close this section by giving a proof of the first part of the Second Derivatives Test.

Part (b) has a similar proof.

Proof of Theorem 3, Part (a) We compute the second-order directional derivative of f in the direction of $\mathrm{u} = \langle h, k \rangle$.

The first-order derivative is given by Theorem 14.6.3:

$$D_{\mathbf{u}}f = f_x h + f_y k$$

Proof of the Second Derivatives Test (2 of 4)

Applying this theorem a second time, we have

$$\begin{aligned} D_u^2 f &= D_u(D_u f) = \frac{\partial}{\partial x}(D_u f)h + \frac{\partial}{\partial y}(D_u f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \qquad \text{(by Clairaut's Theorem)} \end{aligned}$$

If we complete the square in this expression, we obtain

10 $$D_u^2 f = f_{xx}\left(h^2 + 2 \times h \times \frac{f_{xy}}{f_{xx}}k + \left(\frac{f_{xy}}{f_{xx}}k\right)^2\right) - f_{xx}\left(\frac{f_{xy}}{f_{xx}}k\right)^2 + f_{yy}k^2 = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 - \frac{f_{xy}^2}{f_{xx}}k^2 + f_{yy}k^2 = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2) = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}D$$

Proof of the Second Derivatives Test (3 of 4)

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx}f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B. Therefore, by looking at Equation 10, we see that $D_u^2 f(x, y) > 0$ whenever (x, y) is in B.

This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of $\mathbf{u}$, then C is concave upward on an interval of length 2δ.

Proof of the Second Derivatives Test (4 of 4)

This is true in the direction of every vector **u**, so if we restrict (x, y) to lie in B, the graph of f lies above its horizontal tangent plane at P.

Thus $f(x, y) \geq f(a, b)$ whenever (x, y) is in B. This shows that $f(a, b)$ is a local minimum.