MATH 243 Quiz 1 Solutions

1. From the cross product product rule, We have $\mathbf{w} = \mathbf{r}' \times \mathbf{T} + \mathbf{r} \times \mathbf{T}'$. From $\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|}$, the vectors $\mathbf{r}', \mathbf{T}$ are parallel, so the 1st term is $\mathbf{0}$.
From $\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$, we can rewrite the last term as $\|\mathbf{T}'\|(\mathbf{r} \times \mathbf{N})$. So $\mathbf{w} = \|\mathbf{T}'\|(\mathbf{r} \times \mathbf{N})$ is parallel to $\mathbf{r} \times \mathbf{N}$, which is perpendicular to $\mathbf{N}$ since the cross product of two vectors is perpendicular to both of the original vectors. Thus, the answer is $\boxed{D.}$
Since $\mathbf{w}$ is perpendicular to $\mathbf{r}$, B is false. If $\mathbf{r} \neq \mathbf{0}$, then $\mathbf{w} \neq 0$, so it can't be both perpendicular and parallel to $\mathbf{r}$, meaning C is false. Finally, A is false because the unit tangent to $\mathbf{v}$ is $\mathbf{z} = \frac{\mathbf{v}'}{\|\mathbf{v}'\|}$, which has no relation to $\mathbf{w}$. You can see that $\mathbf{w} \neq \mathbf{z}$ by choosing some $\mathbf{r}$ and calculating both vectors.

2. $\int \mathbf{r}(t)\,dt = \langle t^2, \frac{t^3}{3}, 2t\rangle + \mathbf{c}$, so A is false since the vector of integration is not present. B is true by taking the derivative. We compute $\|\mathbf{r}(t)\| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$ to see that C is true.
We can compute $\|\mathbf{s}'(t)\| = \|\langle 0, 2, 0\rangle\| = 2$ to see that the arc length represented by $\mathbf{s}$ from $t = 0$ to $t = 1$ is $\int_0^1 \|\mathbf{s}'(t)\|\,dt = 2 \neq 4$, so D is false. Thus, the correct options are $\boxed{B, C}$.

3. Recall that $\|\mathbf{u} \cdot \mathbf{v}\| = \|u\|\|v\|\cos(\theta)$ and $\|\mathbf{u} \times \mathbf{v}\| = \|u\|\|v\|\sin(\theta)$. By dividing, $\tan(\theta) = \frac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{u}\cdot\mathbf{v}\|} = \frac{6+3\sqrt{3}}{3} = 2 + \sqrt{3}$, so $\theta = \tan^{-1}(2+\sqrt{3}) = 75°$ and the final answer is $\boxed{75.}$
To see that $\tan(75°) = 2 + \sqrt{3}$, you can remember the value from your classes on trigonometry, remember the value $\tan(15°) = 2 - \sqrt{3}$ and do $\tan(75°) = \frac{1}{\tan 15°}$. by remembering/using $\tan(90 - x) = \frac{\sin(90-x)}{\cos(90-x)} = \frac{\cos(x)}{\sin(x)} = \cot(x)$, or use the sum of angles formula for tan :

$$\tan(30 + 45) = \frac{\tan(30) + \tan(45)}{1 - \tan(30)\tan(45)} = \frac{1/\sqrt{3} + 1}{1 - 1/\sqrt{3}} = 2 + \sqrt{3}.$$

There are even more ways to do this, but these are the easiest.

4. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. By trying to match the points up, we notice that $\mathbf{r}(0) = (0, 1, 12)$, $\mathbf{r}(1) = (e^4\sin(3), e^4\cos(3), 12)$, so the arc length we seek is $L = \int_0^1 \|\mathbf{r}'(t)\|\,dt$.
Now we compute the magnitude: $\mathbf{r}'(t) = \langle e^{4t}(3\cos(3t) + 4\sin(3t)), e^{4t}(-3\sin(3t) + 4\cos(3t)), 0\rangle = e^{4t}\langle 3\cos(3t) + 4\sin(3t), -3\sin(3t) + 4\cos(3t), 0\rangle$, so

$$\|\mathbf{r}'(t)\| = e^{4t}\sqrt{(3\cos(3t) + 4\sin(3t))^2 + (-3\sin(3t) + 4\cos(3t))^2} = e^{4t}\sqrt{25\sin^2(3t) + 25\cos^2(3t)} = 5e^{4t}.$$

Plug this in to get $L = \int_0^1 5e^{4t}\,dt = \frac{5}{4}(e^4 - 1)$, so $(p, q, r, s) = (5, 4, 4, 1)$ and the answer is $\boxed{5441.}$

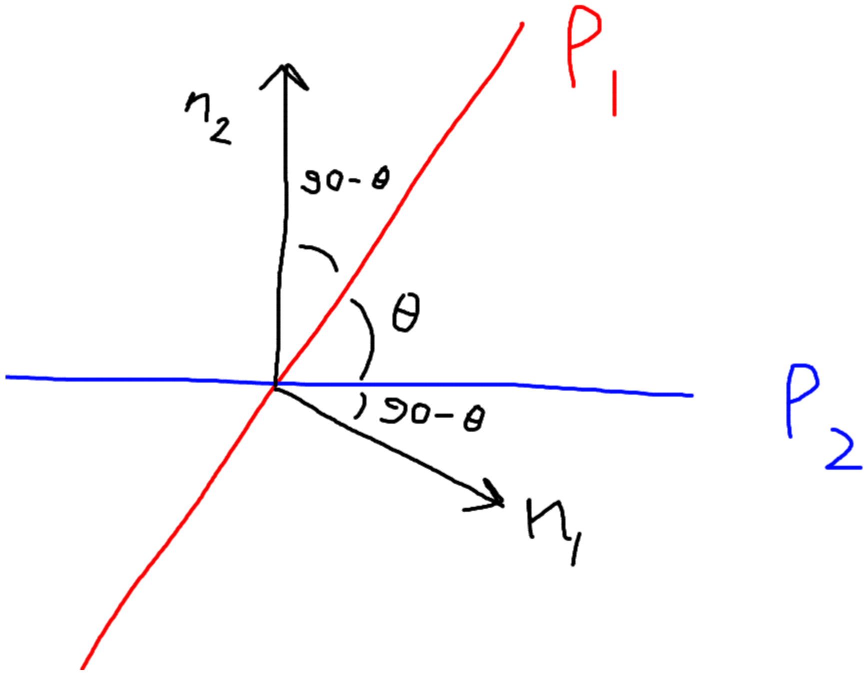
5. For shorthand, we write $s = \sin, c = \cos$. Compute $\mathbf{r}'(t) = (e^t(s + c), 0, e^t(c - s)) = e^t(c + s, 0, c - s)$ and

$$\|\mathbf{r}'(t)\| = e^t\|(c + s, 0, c - s)\| = e^t\sqrt{(c+s)^2 + (c-s)^2} = e^t\sqrt{2s^2 + 2c^2} = e^t\sqrt{2}.$$

By definition, $\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} = \frac{1}{\sqrt{2}}(\cos t + \sin t, 0, \cos t - \sin t)$. Compute $\mathbf{T}' = \frac{1}{\sqrt{2}}(-s + c, 0, -s - c)$ and $\|\mathbf{T}\| = 1$, so $\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{1}{\sqrt{2}}(-\sin(t) + \cos(t), 0, -\sin(t) - \cos(t))$. Finally, compute $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{2}((c + s, 0, c - s) \times (-s + c, 0, -s - c)) = \frac{1}{2}(0, 2, 0) = (0, 1, 0)$.

We saw one definition and two formulas for curvature. Since we have already found $\|\mathbf{r}'\|$ and $\|\mathbf{T}'\|$, the formula $\kappa = \frac{\|\mathbf{T}'\|}{\|\mathbf{r}'\|} = \frac{1}{e^t\sqrt{2}} = e^{-t}\sqrt{2}$ is most convenient.

6. Let θ be the angle between P_1 and P_2. Draw a 2D cross section of the planes intersecting and draw normal vectors $\mathbf{n}_1, \mathbf{n}_2$ to the planes sticking out the intersection point in the cross section. Your diagram may look something like the one below:



From the equations of the planes, we obtain $\mathbf{n}_1 = (2,-1,1), \mathbf{n}_2 = (1,1,2)$.
If $\mathbf{n}_1, \mathbf{n}_2$ are chosen to be on opposite sides of the angle like in the diagram, you get that the angle between $\mathbf{n}_1, \mathbf{n}_2$ is $180 - \theta$. Thus, $-\cos(\theta) = \cos(180-\theta) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{(2,-1,1)\cdot(1,1,2)}{\|(2,-1,1)\|\|(1,1,2)\|} = \frac{3}{6} = 0.5$, so $\theta = \cos^{-1}(-0.5) = 120°$. Since we want the smaller intersection angle, our answer is $180 - 120 = \boxed{60°.}$

To find a plane P perpendicular to P_1, P_2, note that by applying our diagram to perpendicular planes, we see that two planes are perpendicular if and only if their normal vectors are perpendicular. So we need P to have a normal vector perpendicular to $\mathbf{n}_1, \mathbf{n}_2$. We may take $\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = (-3,-3,3)$.
We're given that the origin $(0,0,0)$ lies on the plane, so one equation for P is $-3(x-0)-3(y-0)+3(z-0) = 0$, which simplifies to $\boxed{x+y=z.}$