MATH 243: Discussion Worksheet 1 **Answer Key**

Topics: Review of differentiation and integration; 12.1 Three-Dimensional Coordinate Systems
Please put away all electronic devices, including cell phones and calculators.

1. Find the derivative of $f(x) = \cos(x^2 + e^x)$.

By the Chain Rule,
$$f'(x) = -\sin(x^2 + e^x)\,(2x + e^x).$$

2. Given that $x\ln(y) + e^y = 3$, find $\frac{dy}{dx}$ by implicit differentiation.

Differentiate both sides with respect to x:
$$\frac{d}{dx}\big(x\ln(y)\big) + \frac{d}{dx}\big(e^y\big) = 0.$$
Using the Product Rule and Chain Rule,
$$\ln(y) + x \cdot \frac{1}{y}\frac{dy}{dx} + e^y\frac{dy}{dx} = 0.$$
Collect the $\frac{dy}{dx}$ terms:
$$\left(\frac{x}{y} + e^y\right)\frac{dy}{dx} = -\ln(y).$$
Therefore,
$$\frac{dy}{dx} = \frac{-\ln(y)}{\frac{x}{y} + e^y} = \frac{-y\ln(y)}{x + ye^y}.$$

3. Suppose that $f(x)$ is a differentiable function such that $f(1) = 7$ and $f'(1) = 4$. Find an equation of the tangent line to $y = \sqrt{4 + 3f(x)}$ at $x = 1$.

Let $h(x) = \sqrt{4 + 3f(x)} = (4 + 3f(x))^{1/2}$.
By the Chain Rule, we get
$$h'(x) = \frac{1}{2}(4 + 3f(x))^{-1/2} \cdot 3f'(x) = \frac{3f'(x)}{2\sqrt{4 + 3f(x)}}.$$
Evaluate at $x = 1$:
$$h(1) = \sqrt{4 + 3f(1)} = \sqrt{4 + 3 \cdot 7} = \sqrt{25} = 5, \qquad h'(1) = \frac{3f'(1)}{2\sqrt{4 + 3f(1)}} = \frac{3 \cdot 4}{2 \cdot 5} = \frac{6}{5}.$$
Therefore the tangent line at $x = 1$ is
$$y - 5 = \frac{6}{5}(x - 1) \qquad \text{OR} \qquad y = \frac{6}{5}x + \frac{19}{5}.$$

4. Evaluate the indefinite integrals.

(a) $\int e^{\tan x} \sec^2(x)\, dx$

We use u-substitution. Let $u = \tan x$, then $du = \sec^2(x)dx$. Now,

$$\int e^{\tan x} \sec^2(x)\, dx = \int e^u\, du = e^u + C = e^{\tan x} + C$$

(b) $\int (2x^4 + 1) \ln(x)\, dx$

We use integration by parts:

$$\begin{aligned} u &= \ln(x) & du &= \frac{1}{x}\, dx \\ dv &= (2x^4+1)\, dx & v &= \frac{2x^5}{5} + x \end{aligned}$$

Now, applying integration by parts:

$$\begin{aligned} \int (2x^4+1)\ln(x)\, dx &= \left(\frac{2x^5}{5} + x\right)\ln(x) - \int \left(\frac{2x^5}{5} + x\right) \cdot \frac{1}{x}\, dx \\ &= \left(\frac{2x^5}{5} + x\right)\ln(x) - \int \left(\frac{2x^4}{5} + 1\right) dx \\ &= \left(\frac{2x^5}{5} + x\right)\ln(x) - \left(\frac{2}{5} \cdot \frac{x^5}{5} + x\right) + C \end{aligned}$$

5. Evaluate the definite integrals.

(a) $\int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta$

We use u-substitution. Let $u = \sin\theta$, then $du = \cos\theta\, d\theta$.
Change the limits: when $\theta = 0$, $u = \sin 0 = 0$; when $\theta = \pi/2$, $u = \sin(\pi/2) = 1$.

$$\int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta = \int_0^1 u^2\, du = \frac{1}{3}u^3\bigg|_0^1 = \frac{1}{3}.$$

(b) $\int_0^3 \frac{x}{\sqrt{x^2+1}}\, dx$

We use u-substitution. Let $u = x^2 + 1$, then $du = 2x\, dx$ so $\frac{1}{2}du = x\, dx$.
Change the limits: when $x = 0$, $u = 1$; when $x = 3$, $u = 10$.

$$\int_0^3 \frac{x}{\sqrt{x^2+1}}\, dx = \frac{1}{2}\int_1^{10} u^{-1/2}\, du = \frac{1}{2}\, 2u^{1/2}\bigg|_1^{10} = \sqrt{u}\Big|_1^{10} = \sqrt{10} - 1.$$

6. A particle is moving with acceleration at time t given by

$$a(t) = 3\cos t - 2\sin t.$$

Given that $s(0) = 0$ and $v(0) = 4$, determine the position of the particle $s(t)$ at any time t.

$$\begin{aligned}
v(t) &= \int a(t)\,dt = \int 3\cos t - 2\sin t\,dt = 3\sin t + 2\cos t + C \\
v(0) &= 2\cos(0) + C = 4 \Rightarrow C = 2 \\
v(t) &= 3\sin t + 2\cos t + 2 \\
s(t) &= \int v(t)\,dt = \int 3\sin t + 2\cos t + 2\,dt = -3\cos t + 2\sin t + 2t + C' \\
s(0) &= -3\cos(0) + C' = 0 \Rightarrow C' = 3 \\
s(t) &= -3\cos(t) + 2\sin(t) + 2t + 3
\end{aligned}$$

7. Consider the point $P(3, 4, 5)$.

   (a) What is the projection of the point onto the xy-plane?

   The projection is $(3, 4, 0)$.

   (b) What is the projection of the point onto the xz-plane?

   The projection is $(3, 0, 5)$.

   (c) Find the length of the line segment $\overline{OP}$.

   The length is the distance between the origin $O(0, 0, 0)$ and $P(3, 4, 5)$, given by the following.

   $$\sqrt{(3-0)^2 + (4-0)^2 + (5-0)^2} = \sqrt{50} = 5\sqrt{2}.$$

   (d) Find the position vector for the point $P(3, 4, 5)$ in **ijk**-form.

   The position vector has initial point at the origin O and terminal point at P. So, the vector $\overrightarrow{OP}$ from the origin $O(0, 0, 0)$ to the point $P(3, 4, 5)$ is

   $$(3-0)\mathbf{i} + (4-0)\mathbf{j} + (5-0)\mathbf{k} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}.$$

8. (a) Find an equation of the sphere that passes through the point $(1, 8, 5)$ and has center $(3, 1, -3)$.

The radius of the sphere is the distance between $(3, 1, -3)$ and $(1, 8, 5)$:

$$r = \sqrt{(1-3)^2 + (8-1)^2 + (5-(-3))^2} = \sqrt{117}.$$

Thus, an equation of the sphere is

$$(x-3)^2 + (y-1)^2 + (z+3)^2 = 117.$$

(b) Using your answer from part (a), find an equation describing the intersection of the sphere with the yz-plane. If the sphere does not intersect the yz-plane, write DNE.

To find the intersection with the yz-plane, we set $x = 0$;

$$(y-1)^2 + (z+3)^2 = 108, \quad x = 0,$$

which is a circle in the yz-plane with center $(0, 1, -3)$ and radius $\sqrt{108} = 6\sqrt{3}$.

9. Find an equation of a sphere if one of its diameters has endpoints at $(1, 2, 4)$ and $(4, 3, 10)$.

First, we find the sphere's diameter by computing the distance between the two given points, $(1, 2, 4)$ and $(4, 3, 10)$. We have

$$d = \sqrt{(1-4)^2 + (2-3)^2 + (4-10)^2} = \sqrt{46}$$

Since the radius, r, of the sphere, is half the diameter, we obtain $r = \dfrac{d}{2} = \dfrac{\sqrt{46}}{2}$.

Now the center of the sphere, C, is located at the midpoint of its diameter. To find the coordinates of the midpoint of a segment, we take the average of the corresponding coordinates of the end points of the segment:

$$\left(\frac{4+1}{2}, \frac{2+3}{2}, \frac{4+10}{2}\right).$$

Hence, the center is located at

$$C\left(\frac{5}{2}, \frac{5}{2}, 7\right).$$

Therefore our sphere has radius $r = \dfrac{\sqrt{46}}{2}$ with a center at $\left(\dfrac{5}{2}, \dfrac{5}{2}, 7\right)$. Therefore its equation can be written as

$$\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 + (z-7)^2 = \frac{23}{2}$$

SOME USEFUL DEFINITIONS, THEOREMS, AND NOTATION:

Differentiation Formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(b^x) = b^x \ln b$$

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

The Product Rule $\quad \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

The Quotient Rule $\quad \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

The Chain Rule $\quad \frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

The Chain Rule for the power of a function

$$\frac{d}{dx}[f(x)^n] = n[f(x)]^{n-1} f'(x)$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}(\text{arccot}\, x) = -\frac{1}{1 + x^2}$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$$

Integration Formulas

a. If $n \neq -1$, $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$

b. $\int e^x\,dx = e^x + C$

c. $\int \sin x\,dx = -\cos x + C$

d. $\int \sec^2 x\,dx = \tan x + C$

e. $\int \sec x \tan x\,dx = \sec x + C$

f. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

g. $\int \tan x\,dx = \ln|\sec x| + C$

i. $\int \dfrac{1}{x}\,dx = \ln|x| + C$

j. $\int b^x\,dx = \dfrac{b^x}{\ln b} + C$

k. $\int \cos x\,dx = \sin x + C$

l. $\int \csc^2 x\,dx = -\cot x + C$

m. $\int \csc x \cot x\,dx = -\csc x + C$

n. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

o. $\int \cot x\,dx = \ln|\sin x| + C$

where C is any real constant.

The Substitution Rule: This rule is useful when the integrand contains a function and (a constant multiple of) its derivative. Let $u = g(x)$, so $du = g'(x)\,dx$. Then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$$

Integration by Parts: This rule is useful for integrals that look like a product of two functions. Choose u and dv from the integrand so that $du = u'\,dx$ and $v = \int dv$. Then

$$\int u\,dv = uv - \int v\,du.$$

Suggested Textbook Problems

Section 12.1: 1-46