Curl, Divergence, Green’s Theorem

Lecture for 7/7

Curl and Divergence

Consider ∇ as a operator and F = (A, B, C)

- Define div F = $\nabla \cdot F = A_x + B_y + C_z$
- Define curl F = $\nabla \times F = (C_y - B_z, A_z - C_x, B_x - A_y)$
  - Only for vector fields on R^3
- Let's see what happens to scalar functions
  - $\nabla \cdot fG = (\nabla f) \cdot G + f(\nabla \cdot G)$
  - $\nabla \times fG = \nabla f \times G + f(\nabla \times G)$https://en.wikipedia.org/wiki/Divergence

Properties of Curl and Divergence

- Let's see what happens to scalar functions
  - $\nabla \cdot fG = (\nabla f) \cdot G + f(\nabla \cdot G)$
  - $\nabla \times fG = \nabla f \times G + f(\nabla \times G)$
- Let's see how curl and div interact
  - $\nabla \cdot (F \times G) = (\nabla \times F) \cdot G - F \cdot (\nabla \times G)$
    - Consequence: $\nabla \cdot (\nabla \times F)$
  - $\nabla \times \nabla f = 0$
  - $\nabla \times (F \times G) = (\nabla \cdot G)F + (G \cdot \nabla)F - (\nabla \cdot F)G - (F \cdot \nabla)G$

Green’s Theorem General Idea

It’s often useful to switch between line and double integrals

- Double to Line: you’re reducing number of integrations
- Line to Double: function may be simpler to integrate

But how can we do this? Green’s Theorem will tell us

Green's Theorem

Suppose C is a simple closed curve oriented counterclockwise. Suppose

- Further suppose C encloses D
- Further suppose Q_x, P_y are Riemann integrable

Green's Theorem: $\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\, dA$

Decomposition Principle

- If GT holds on D_1 and D_2, we can consider $D = D_1 \cup D_2$
- To find integrals, break D into smaller regions where GT applies

Theorem Derivation

Vector Forms of Green's Theorem

Pretend F = (P, Q) is a vector field in R^3, with F = (P, Q, 0)

- $\int_C F \cdot dr = \iint_D (\nabla \times F) \cdot e_z \, dA$ where $e_z = (0, 0, 1)$
  - We shall use this later to build Stokes' Theorem
- $\int_C (F \cdot n)\, ds = \iint_D (\nabla \cdot F)\, dA$ where n is unit normal to r
  - We shall use this later to build Divergence Theorem

Practice Problems

Evaluate $\int_C (y^4-2y)\,dx - (6x-4xy^3)\,dy$ where C is the rectangle with coordinates (0,0), (6, 0), (6, 4), (0, 4) oriented clockwise

Let C be the triangle with vertices (-3, 0), (0,0), (0, 3) oriented clockwise. Verify Green's Theorem for $\int_C (xy^2+x^2)\,dx + (4x-1)\,dy$ by computing both the line integral and the corresponding double integral

Find a formula for $\nabla \times (\nabla \times F)$ and justify your claim

Scratchwork