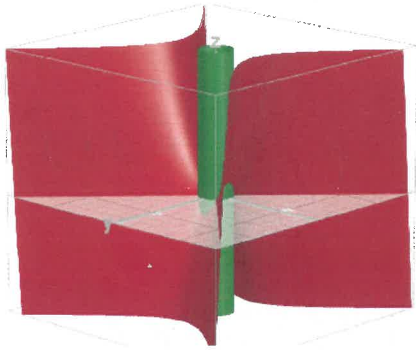
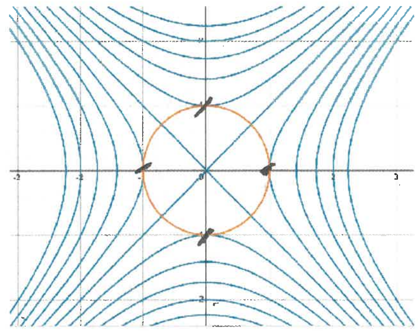
14.8 Lagrange Multipliers

In this section, we will discuss Lagrange's method for maximizing or minimizing a multivariable function f subject to a constraint (or side condition) of the form $g = k$.





$P(L, R)$ $\quad f(x,y) = x^2 - y^2$

$L + R = A$ $\quad g(x,y) = x^2 + y^2 = 1$

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming these extreme values exist and $\nabla g \neq \mathbf{0}$ on $g(x, y, z) = k$):

1. Find all values of x, y, z, and λ such that

$$\underline{\nabla f(x, y, z) = \lambda \nabla g(x, y, z)} \quad \text{and} \quad g(x, y, z) = k.$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example 5. *Find the extreme values of the function $f(x, y) = x^2 - y^2$ on the circle $x^2 + y^2 = 1$.*

$\nabla f(x, y, \cancel{z}) = \lambda \nabla g(x, y)$

$f(x, y) = x^2 - y^2 \Rightarrow f_x = 2x,\ f_y = -2y \Rightarrow \nabla f(x, y) = \langle 2x, -2y \rangle$

$g(x, y) = x^2 + y^2 \Rightarrow g_x = 2x,\ g_y = 2y \Rightarrow \nabla g(x, y) = \langle 2x, 2y \rangle$

$\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle = \langle 2\lambda x, 2\lambda y \rangle$

$\cancel{2}x = \cancel{2}\lambda x$

$\Rightarrow x(1 - \lambda) = 0$

$\Rightarrow x = 0$ or $\lambda = 1$

$-2y = 2\lambda y$

$\Rightarrow -2y = 2y$

$\Rightarrow 4y = 0$

$\Rightarrow y = 0$

$x^2 + y^2 = 1$

$x^2 = 1$

$\Rightarrow x = \pm 1$

$(1, 0)\ (-1, 0)$

$0^2 + y^2 = 1$

$\Rightarrow y^2 = 1$

$\Rightarrow y = \pm 1$

$(0, 1)\ (0, -1)$

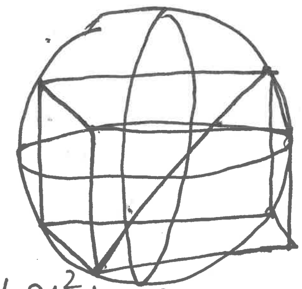
Crit. ~~f(x, y)~~	$f(x, y)$
$(1, 0)$	1
$(-1, 0)$	1
$(0, 1)$	-1
$(0, -1)$	-1

$f_{max} = 1$ for $(1, 0)$ and $(-1, 0)$

$f_{min} = -1$ for $(0, 1)$ and $(0, -1)$

Example 6. *Find the maximum volume of a rectangular box that is inscribed in a sphere of radius 3.*

$$V(x,y,z) = xyz$$



$$\sqrt{x^2+y^2+z^2} = 6, \quad x>0,\ y>0,\ z>0$$

$$\Rightarrow x^2+y^2+z^2 = 36$$

$$g(x,y,z) = x^2+y^2+z^2$$

$$\cancel{\nabla f(x) = \lambda \nabla g}$$

$$\nabla V(x,y,z) = \lambda \nabla g(x,y,z)$$

$$yz = \lambda 2x \qquad xz = 2\lambda y \qquad xy = 2\lambda z \qquad x^2+y^2+z^2=36$$

$$\Rightarrow 2\lambda = \frac{yz}{x} \qquad 2\lambda = \frac{xz}{y} \qquad 2\lambda = \frac{xy}{z}$$

$$\frac{yz}{x} = \frac{xz}{y} = \frac{xy}{z}$$

$$\frac{yz}{x} = \frac{xz}{y} \qquad\qquad \frac{xz}{y} = \frac{xy}{z}$$

$$\Rightarrow y^2\cancel{z} = x^2\cancel{z} \qquad\qquad \Rightarrow \cancel{x}z^2 = \cancel{x}y^2$$

$$\Rightarrow x = y \qquad\qquad \Rightarrow z = y$$

$$x = y = z$$

$$x^2+x^2+x^2 = 36$$

$$\Rightarrow 3x^2 = 36$$

$$\Rightarrow x^2 = 12$$

$$\Rightarrow x = 2\sqrt{3}$$

$$y = 2\sqrt{3}, \quad z = 2\sqrt{3}$$

$$V_{max}(x,y,z) = (2\sqrt{3})^3 = 24\sqrt{3}$$

$$(2\sqrt{3}, 2\sqrt{3}, 2\sqrt{3})$$

Example 7. *Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point* $(4, 2, 0)$.

$(x, y, z) \longrightarrow d(x,y,z) = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$

$f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$

$g(x,y,z) = x^2 + y^2 - z^2 = 0$

$\nabla f = \langle 2(x-4), 2(y-2), 2z \rangle$

$\nabla g = \langle 2x, 2y, -2z \rangle$

$2(x-4) = 2\lambda x$

$\Rightarrow 2x - 8 = 2\lambda x$

$\Rightarrow \cancel{2} x(1-\lambda) = \cancel{8}\,4$

$\Rightarrow x(1-\lambda) = 4$

$\Rightarrow 1 - \lambda = \frac{4}{x}$

$\Rightarrow \lambda = 1 - \frac{4}{x}$

$\cancel{2}(y-2) = \cancel{2}\lambda y$

$y - 2 = \lambda y$

$\Rightarrow y(1-\lambda) = 2$

$\Rightarrow 1 - \lambda = \frac{2}{y}$

$\Rightarrow \lambda = 1 - \frac{2}{y}$

$\cancel{2}z = -\cancel{2}\lambda z$, $x^2 + y^2 - z^2 = 0$

$\Rightarrow z(1+\lambda) = 0$

$\cancel{\Rightarrow z = 0}$

$\Rightarrow z = 0$ or $\lambda = -1$

$\cancel{x^2 + y^2 = 0}$

$x^2 + y^2 = 0$

$x = 0, y = 0$

$\Rightarrow \lambda = -1$

$\Rightarrow -1 = 1 - \frac{2}{y}$

$\Rightarrow -\cancel{2} = -\frac{\cancel{2}}{y}$

$\Rightarrow \cancel{-}y = \cancel{-}1$

$\Rightarrow -1 = 1 - \frac{4}{x}$

$\Rightarrow -\frac{\cancel{4}\,2}{x} = -\cancel{2}$

$\Rightarrow \cancel{-}x = \cancel{-}2$

$x = 2, \; y = 1$

$2^2 + 1^2 - z^2 = 0$

$\Rightarrow z^2 = 4 + 1 = 5$

$\Rightarrow z = \pm\sqrt{5}$

$(2, 1, \sqrt{5}), \; (2, 1, -\sqrt{5})$

crit	$d(x,y,z)$
$(2,1,\sqrt{5})$	$\sqrt{10}$
$(2,1,-\sqrt{5})$	$\sqrt{10}$

Example 8. *Find the extreme values of the function* $f(x, y) = x^2 - y^2$ *on the closed disk* $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

This looks like a absolute maximum and minimum value problem on a closed and bounded ~~se~~ region; but the region in discussion is a disc instead of a rectangle. So we can't use the methods of 14.7 for this problem. However, we can convert this problem into a constrained optimization problem. We start with our usual first derivative test to determine the critical values of f.

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 2x, -2y \rangle$$

By first derivative test, at critical points $f_x = 0$ and $f_y = 0$ $\Rightarrow 2x = 0$ and $-2y = 0 \implies x = 0, y = 0 \Rightarrow$ only critical points is $(0,0)$. $(0,0)$ lies inside the region D.

With this we have taken care of the interior of D and need to only focus or attention on the boundary of D which is the circle $x^2 + y^2 = 1$.

~~So~~ So our problem now becomes finding out potential candidates for absolute maximum and minimum ~~vales~~ values of ~~f on the~~ $f(x, y) = x^2 - y^2$ on $g(x, y) = x^2 + y^2 = 1$. Notice that this is the exact same problem ~~in~~ as in Example 5.

Following similar steps we ~~f~~ find the candidates $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. At the end our table would look like

crit	$f(x, y)$
$(1, 0)$	1
$(-1, 0)$	1
$(0, 1)$	-1
$(0, -1)$	-1
$(0, 0)$	0

So $f_{max} = 1$ at $(\pm 1, 0)$

$f_{min} = -1$ at $(\cancel{\pm}0, \pm 1)$

[one extra point due to the critical point at the interior of D]