Recall: • A vector valued function is a function whose domain is a set of real #s and range is a set of vectors.

• $\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k} = \langle f(t), g(t), h(t) \rangle$

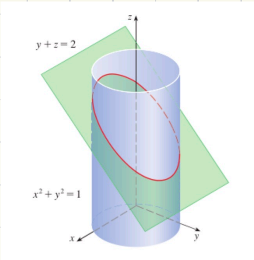
• $\boxed{\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} f(t)\,\hat{i} + \lim_{t\to a} g(t)\,\hat{j} + \lim_{t\to a} h(t)\,\hat{k}.}$

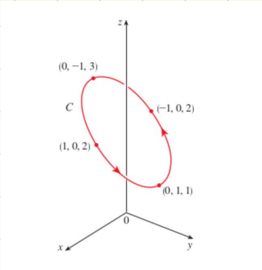
• $\vec{r}(t)$ is CTS on an interval $I \iff \lim_{t\to a} \vec{r}(t) = \vec{r}(a)$ for each $a \in I$.

$\boxed{\iff \text{components of } \vec{r}(t) \text{ are CTS on } I.}$

• Space curve: For CTS functions f, g, h on an interval I, the set of points $\boxed{C = \{(x,y,z) : x = f(t), y = g(t), z = h(t) \text{ for some } t \in I\}}$ is called a space curve.

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.





A: $x = \cos t$, $y = \sin t$, $0 \le t < 2\pi$

Plug $y = \sin t$ into $y + z = 2 \Rightarrow z = 2 - \sin t$

$\Rightarrow C$ is represented by $\vec{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$

- $$\boxed{\frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}}$$

$$= \left\langle \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}, \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}, \lim_{s \to 0} \frac{h(t+s) - h(t)}{s} \right\rangle$$

- If $\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$ where f, g, h are differentiable. Then

$$\boxed{\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = f'(t)\,\hat{i} + g'(t)\,\hat{j} + h'(t)\,\hat{k}} = \langle f'(t), g'(t), h'(t) \rangle$$

- $\boxed{\text{Unit Tangent Vector. } \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}$

3 Theorem

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

} linearity

3. $\frac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

} product rules

6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$ (Chain Rule)

4 Theorem

If $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

- $$\boxed{\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\hat{i} + \left(\int_a^b g(t)\,dt\right)\hat{j} + \left(\int_a^b h(t)\,dt\right)\hat{k}.}$$

- Suppose $\vec{R}'(t) = \vec{r}(t)$ then $\boxed{\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)}$

- Eg: $\vec{r}(t) = 2\cos t\,\hat{i} + \sin t\,\hat{j} + 2t\,\hat{k}$

$$\Rightarrow \int \vec{r}\,dt = (2\sin t + C_1)\hat{i} + (-\cos t + C_2)\hat{j} + (t^2 + C_3)\hat{k}$$

$$= 2\sin t\,\hat{i} - \cos t\,\hat{j} + t^2\,\hat{k} + \underbrace{\vec{C}}_{= C_1\hat{i} + C_2\hat{j} + C_3\hat{k}.}$$

$$\Rightarrow \int_0^{\pi/2} \vec{r}\,dt = \left[2\sin t\,\hat{i} - \cos t\,\hat{j} + t^2\,\hat{k}\right]\Big|_0^{\pi/2} = 2\hat{i} + \hat{j} + \frac{\pi^2}{4}\hat{k}.$$

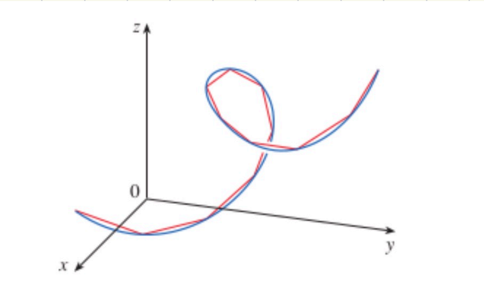
13.3: Arc Length and Curvature:

Definition: Suppose that $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ on the domain $[a,b]$ and

f', g', h' are continuous on $[a,b]$. Then the arc length of the curve is

$$\boxed{L = \int_a^b \|\underbrace{r'(t)}_{\langle f', g', h' \rangle}\| \, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \, dt}$$

<u>Intuition:</u> $\|r'(t)\|$ is the length of an infinitesimal matchstick along the curve.
$\Rightarrow$ $\Sigma \|r'(t)\|$ is the total length.



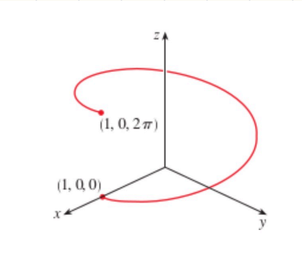
<u>Remark:</u> · If the curve is traversed exactly once as t increases from a to b then L is the length of the curve.

· The arc length is invariant under reparametrization.

Example: $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$, $1 \leq t \leq 2$ and $\vec{r}_2(t) = \langle e^u, e^{2u}, e^{3u} \rangle$, $0 \leq u \leq \ln 2$ have the same arc length.

Example 1

Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.



A: $r(t) = \langle \cos t, \sin t, t \rangle \Rightarrow r'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\Rightarrow \|r'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}.$

From $(1,0,0) \rightarrow (1,0,2\pi)$ t increases from 0 to 2π

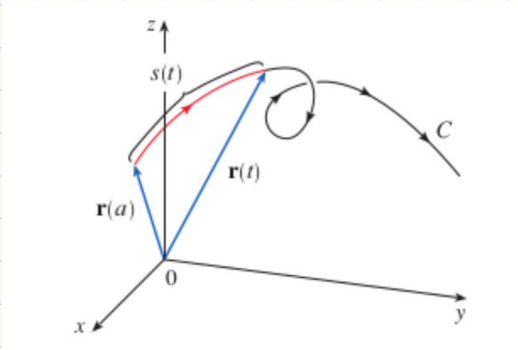
$\Rightarrow L = \int_0^{2\pi} \sqrt{2}\, dt = (2\pi - 0)\sqrt{2} = 2\sqrt{2}\pi.$

The Arc Length Function:

Suppose that a space curve C is given by $\vec{r} = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$ where $\vec{r}(t)$ is CTS on $[a,b]$ and C is traversed exactly once as t increases from a to b.

Definition: Under the above hypothesis, it's arc length function is defined by

$$\boxed{s(t) = \int_a^t \|r'(u)\|\, du}$$



· Consequence of Fundamental Theorem of Calculus:

$$\boxed{\frac{ds}{dt} = \|\vec{r}\,'(t)\|}$$

Parametrizing a curve with respect to arc length:

Goal: Reparametrize the curve, say with parameter s, so that as s increases from a to b $\vec{r}(s)$ is a position vector of the point 's' units along the curve from its starting point.

Algorithm:
- Find $s(t)$.
- Solve for t as a function of s.
- $r(t(s))$ is the desired reparametrization.

Eg: $r(t(3))$ is 3 units from the start in "time" and along the space curve.

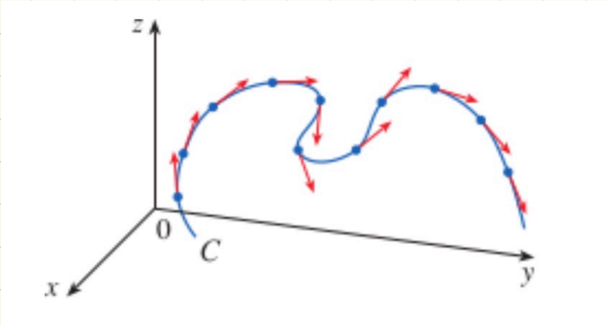
Example 2

Reparametrize the helix $\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

$$\Rightarrow \quad s(t) = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t \quad \Rightarrow \quad t = \frac{1}{\sqrt{2}} s.$$

$$\Rightarrow \text{Desired reparametrization is } \vec{r}(s) = \cos\left(\frac{s}{\sqrt{2}}\right)\hat{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\hat{j} + \left(\frac{s}{\sqrt{2}}\right)\hat{k}.$$

Curvature:



- A parametrization $\vec{r}(t)$ of a curve C on an interval I is called smooth if $\vec{r}'(t)$ is CTS on I and $\vec{r}'(t) \neq 0$ on I.

✓ (no sharp turns or cusps) (tangent vector always tells us where the "next" point is).

- A curve is called smooth if it has a smooth parametrization.

- Recall: $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$

- **Definition:** The curvature of a curve is

Greek letter kappa

$$\boxed{\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{\| d\vec{T}/dt \|}{\| ds/dt \|} = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}}$$

arc length

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} = \frac{d\vec{T}/dt}{ds/dt}$$

Example 3

Show that the curvature of a circle of radius a is $1/a$.

Take $C = \{(x,y) : x^2 + y^2 = a^2\}$ parametrize by $x = a\cos t$, $y = a\sin t$, $0 \leq t < 2\pi$

so we want $\kappa = \dfrac{\|T'(t)\|}{\|\vec{r}\,'(t)\|}$ where $r = \langle a\cos t, a\sin t \rangle$, $0 \leq t < 2\pi$.

- $\vec{r}\,'(t) = \langle -a\sin t, a\cos t \rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$
  $= \sqrt{a^2} = a$

- $T(t) = \dfrac{1}{\|\vec{r}\,'(t)\|} \cdot \vec{r}\,'(t) = \dfrac{1}{a}\langle -a\sin t, a\cos t \rangle = \langle \frac{1}{a}\cdot(-a\sin t), \frac{1}{a}(a\cos t) \rangle = \langle -\sin t, \cos t \rangle$

$\Rightarrow T'(t) = \langle -\cos t, -\sin t \rangle \Rightarrow \|T'(t)\| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\Rightarrow \kappa = \dfrac{1}{a}$