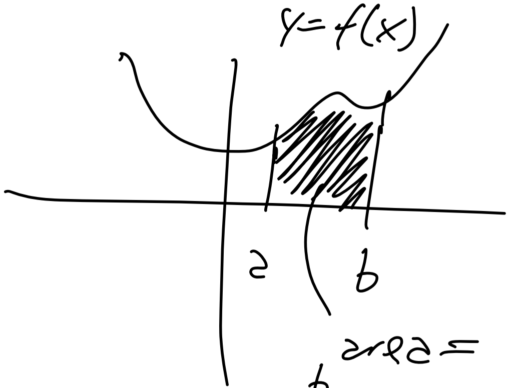
Rectangular 2D Integrals

Pre-lecture for 6/25

Volume under a Curve

Note! $dA = dx\,dy$

- In Calc 1, we have area under curve y = f(x)
- Concept is now volume under curve z = f(x, y)
- If region is R, let $\iint_R f(x, y)\, dA$ be this volume

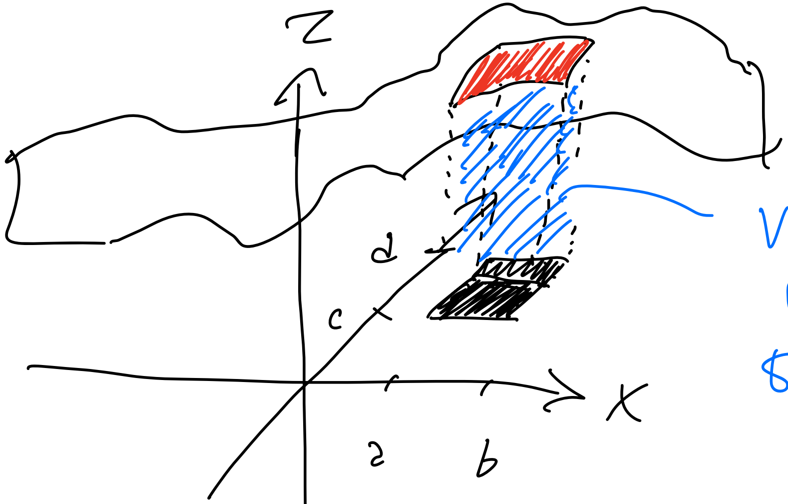


area $= \int_a^b f(x)\,dx$

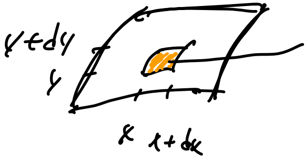
$R = \{(x,y): a \le x \le b,$

$c \le y \le d\},$

volume $= \iint_R f(x,y)\,dx\,dy$



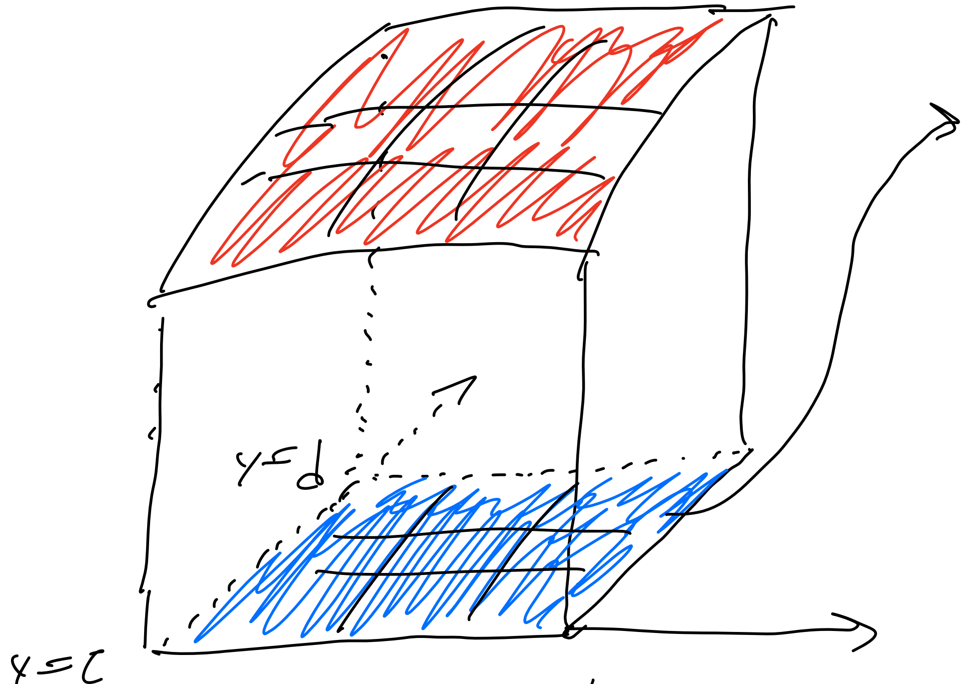
Volume is volume of this blue solid

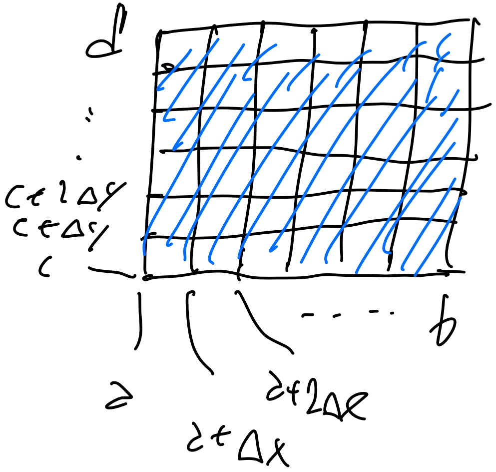


area of this infinitesimal area patch is $((x+dx)-x)((y+dy)-y) = dx\,dy$

Riemann Sums

- Suppose R is a rectangle $a \le x \le b$, $c \le y \le d$
- Can approximate $\iint_R f(x, y)\, dx\, dy$ with Riemann sums





split blue rectangle into small rectangles with length Δx and height Δy

Let $[a, b]$ be split into n segments by a, $a+\Delta x$, $a+2\Delta x$, $\dots$, $a+n\Delta x = b$

$x=a$ $\qquad$ $x=b$

Let $[c,d]$ be split into n segments by $c, c+\Delta y,$
$c+2\Delta y, \ldots, c+n\Delta y = d$

We are going to split R into n^2 smaller rectangles
$R_{0,0}, R_{0,1}, \ldots, R_{n-1,n-1}$. As a result of volume def.,

$$\iint_R f(x,y)\,dx\,dy = \iint_{R_{0,0}} f(x,y)\,dx\,dy + \ldots + \iint_{R_{n-1,n-1}} f(x,y)\,dx\,dy$$

because LHS is volume under curve for R, RHS is volume under curve for individual R_i, and R is the union of all R_i.

Let $R_{i,j} = [a+i\Delta x, a+(i+1)\Delta x] \times [c+j\Delta y, c+(j+1)\Delta y]$

Then $\iint_R f(x,y)\,dx\,dy = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\iint_{R_{i,j}} f(x,y)\,dx\,dy.$

Just like in 1D Riemann sums, let's imagine that Δx, Δy are small (not infinitesimal yet though) and try to find an approximation for each $\iint_{R_{i,j}} f(x,y)\,dx\,dy$.

When Δx & Δy get smaller & smaller, region under each $R_{i,j}$ resembles rectangular prism more & more.

Treat each region as a rectangular prism & find its volume to ~~get~~ an approximation for $\iint_{R_{i,j}} f(x,y)\,dA$.

don't worry about func. like Dirichlet's func, all functions in 243 will end up being Riemann integrable

Base: length · height $= \Delta x\,\Delta y$

Height: since $R_{i,j}$ is small, almost all values of f on $R_{i,j}$ are close together. If f is continuous, then in fact we know that all values of f on $R_{i,j}$ are close together.

Take any f value, like $f(a+i\Delta x,\ c+j\Delta y)$, for height.

Note: also take $f(a+(i+1)\Delta x,\ c+(j+1)\Delta y)$.

$\iint_{R_{i,j}} f(x,y)\,dA \approx$ volume of prism $= f(a+i\Delta x,\ c+j\Delta y)\Delta x\,\Delta y$

So $\iint_R f(x,y)\,dA \approx \underbrace{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(a+i\Delta x,\ c+j\Delta y)\Delta x\,\Delta y}_{\text{One possible Riemann sum for } \iint_R f(x,y)\,dA}$

f will be Riemann integrable on R no matter what $V_{i,j} = (x_{i,j}, y_{i,j})$ we pick in $R_{i,j}$ and no matter how $[a,b]$ & $[c,d]$ are split into segments (may even have unequal lengths like $[a, a+\Delta x], [a+\Delta x, a+3\Delta x], [a+3\Delta x, a+4\Delta x], \ldots \Rightarrow \Delta x, 2\Delta x, \Delta x, \sqrt{7}\Delta x$, whatever for the lengths of the segments), when the size of the largest $R_{i,j} \to 0$, then $\sum_{i,j} f(V_{i,j})$ converges to the same value L independent of the choice of rectangles and choice of $V_{i,j}$.

We then define $\iint_R f(x,y)\,dA := L$

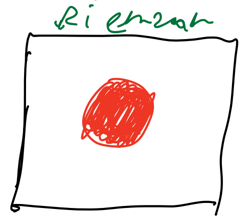
From this definition of $\iint_R f$ and how we found the Riemann sums, $\iint_R f$ indeed gives volume of $z=f(x,y)$ under rectangle R.

FACT: any continuous $f(x,y) \geq 0$ is Riemann integrable. Moreover, any f with countably many discontinuities is Riemann integrable. Moreover, any f where the set $D_f = \{(x_0,y_0) \mid f \text{ discontinuous at } (x_0,y_0)\}$ is a collection of curves and points is Riemann integrable. This means if D_f as in example below, then f still Riemann integrable.

Still Riemann int.

D_f in red
R in black

Probably won't be Riemann int.



D_f in red
R in black

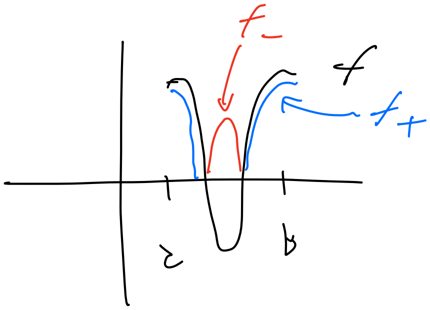
Blue: proven w/ enough time

Orange: proven w/ even more time

Purple: requires measure theory (at UD, math 602)

You can take this fact for granted.

If $f \geq 0$ is not true, write $f = f_+ - f_-$ where $f_+ = \max(0, f)$, $f_- = \min(0, -f)$.



Then define $\iint_R f\,dA = \iint_R f_+ - \iint_R f_-$ and we get that $\iint_R f\,dA$ exists if that f follows some conditions in previous FACT and $\underbrace{\left(\iint_R f_+, \iint_R f_-\right) \neq (\infty, \infty)}$.

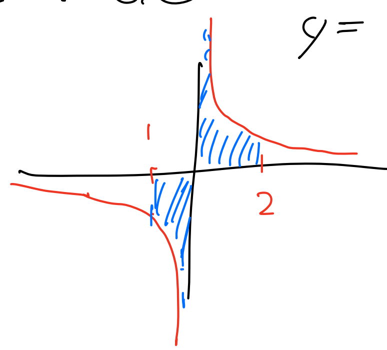
This will be the only thing stopping you from Riemann integrability in math 243.

if $= (\infty, \infty)$, Riemann int DNE

For example, $\int_{-1}^{2} \frac{1}{x}\,dx$ DNE, but $\int_0^2 \frac{1}{x}\,dx = \infty$ and

$$\int_{-1}^{0} \frac{1}{x}\,dx = -\infty$$

$y = \frac{1}{x}$



Fubini's Theorem

- $\iint_R f(x, y)\,dx\,dy = \int(\int f(x, y)\,dx)\,dy = \int(\int f(x, y)\,dy)\,dx$
  - If f is Riemann integrable

For example: $R = [0,2]\times[0,1] \iff 0 \le x \le 2,\ 0 \le y \le 1,$

then $\iint_R f(x,y)\,dx\,dy = \int_0^1 \left(\int_0^2 f(x,y)\,dx \right) dy = \int_0^2 \left(\int_0^1 f(x,y)\,dy \right) dx.$

Now we can actually find $\iint_R f$ by plug & chug

Plug in for f, get $g(y) = \int_0^2 f(x,y)\,dx$, then compute

$\int_0^1 g(y)\,dy$ with standard Calc 1 tactics.

Recall

$$\iint f(x,y)\,dA \approx \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(a+i\Delta x, c+j\Delta y)\Delta x\Delta y$$

$$= \underline{\sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} f(a+i\Delta x, c+j\Delta y)\Delta y\right)\Delta x}$$

$$= \underline{\sum_{j=0}^{n-1}\left(\sum_{i=0}^{n-1} f(a+i\Delta x, c+j\Delta y)\Delta x\right)\Delta y}$$

Take $\Delta y \to 0$, the <u>1st sum</u> is $\displaystyle\sum_{i=0}^{n-1}\left[\int_c^d f(a+i\Delta x, y)\,dy\right]\Delta x$.

Then take $\Delta x \to 0$ to get $\displaystyle\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$.

Similarly, take $\Delta x \to 0$, and then take $\Delta y \to 0$ to get the <u>2nd sum</u> $\longrightarrow \displaystyle\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$.

Practice Problems

Evaluate $\iint_R f(x, y)\, dA$ for these functions and regions:

- $f(x, y) = x\cos^2(y)$, $R = [0, 3] \times [0, \pi/2]$
- $f(x, y) = 2x - 4y^3$, $R = [4, 5] \times [0, 3]$
- $f(x, y) = xy + \cos(x) + \sin(y)$, $R = [0, 1] \times [0, 1]$

also on tomorrow's worksheet

Scratchwork

General 2D integrals

Lecture for 6/25

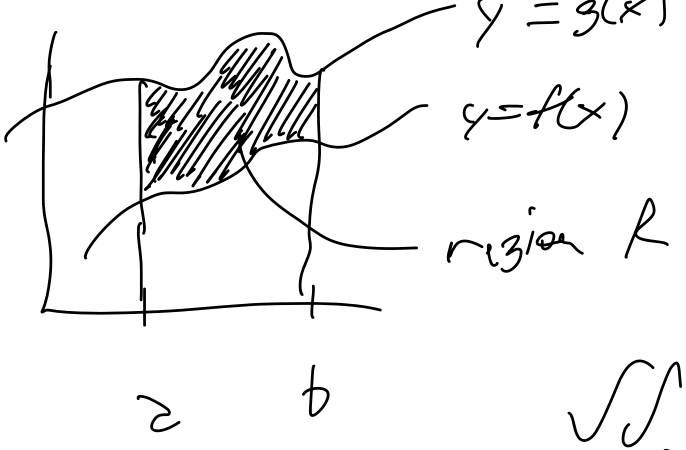
General 2D integrals

We can find the integral for a general 2D region

region type 1

- How to exactly describe?

Examples: let's say $a \le x \le b$, $f(x) \le y \le g(x)$. Then



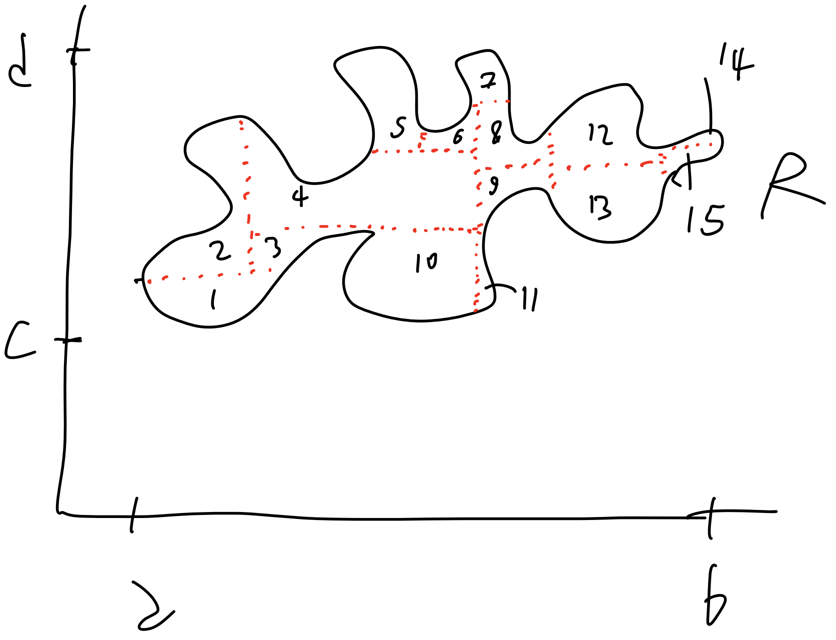
$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{f(x)}^{g(x)} f(x,y)\,dy \right) dx.$$

region type 2

If $a \le y \le b$, $f(y) \le x \le g(y)$, then

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{f(y)}^{g(y)} f(x,y)\,dx \right) dy.$$

What if we can't write R viz either of the 2 previous cases? We need to write equations for R & draw the boundaries of R.

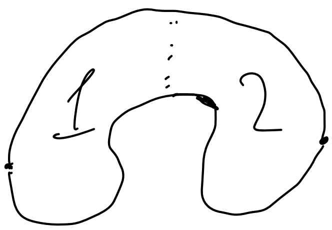


If R is bounded, let R lie inside $[a,b] \times [c,d]$. We must split R up into regions of type 1 & type 2.

Later on, we will see a nice formula for $\iint_R f\,dx\,dy$ if ∂R is parametrized by $r(t)$, $a \le t \le b$. But for now, splitting into type 1 & type 2 is the only way.

$\iint_R f\,dx\,dy$ is the sum of the integral over those 15 labeled regions, and each one is type 1 or type 2 since they either pass vertical line test or horizontal line test.

In practice, you will have 4 regions maximum, and if there's 3-4, some symmetry is involved that makes your life easier. If no symmetry, almost all examples will have 1-2 pieces.

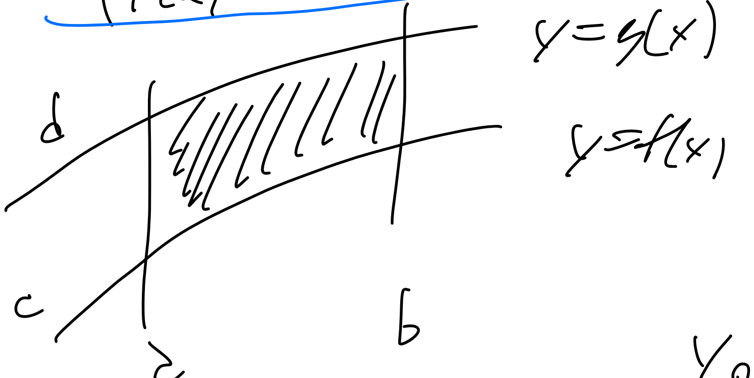


Switching Order of Integration

- We can switch order of integration

If both horizontal & vertical line test passed

$$\int_a^b \left(\int_{f(x)}^{g(x)} f(x,y)\, dy \right) dx = \int_c^d \left(\int_{g^{-1}(y)}^{f^{-1}(y)} f(x,y)\, dx \right) dy$$



$f(x) \le y \le g(x)$

$x \le f^{-1}(y)$

$g^{-1}(y) \le x$

You can draw out blue integral &

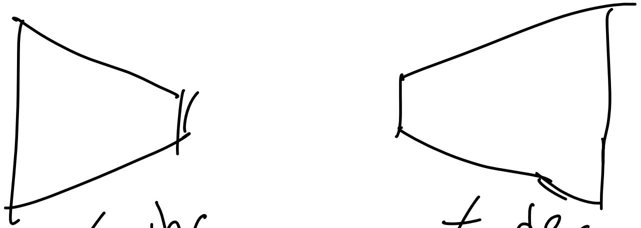
red integral to see they represent the same region.

when can we invert f & g?

1: f, g both inc

2: f, g both dec

in general: f strictly monotone, g is strictly monotone

↳  are 2 more scenarios

f inc
g dec

f dec
g inc

If f, g don't fall into these cases, you will have to split R into multiple regions and the switching of orders will look like

$$\int_R f = \int_{R_1} f + \int_{R_2} f + \ldots$$

Average Value

Average value of function f over region R:

- Equal to (1/A) $\iint_R f(x, y)\,dx\,dy$ where A is area of R

$$\text{average of } f \text{ over } R = \frac{\iint_R f(x,y)\,dx\,dy}{\text{area}(R)}.$$

Only defined if area(R) > 0.

R is made of points or line segments, then we use sums & 1D integrals to find average

$R = \{P_1, P_2, P_3\} \Rightarrow avg = \frac{1}{3}(f(P_1) + f(P_2) + f(P_3))$

$R = \{[0,2] \times \{4\}\} \Rightarrow avg = \frac{1}{2}\int_0^2 f(x,4)\,dx$

Practice Problems

Evaluate $\iint_D f(x, y)\,dA$ for these functions and regions:

- $f(x, y) = 4xy - y^3$, D is region bound by $y = x^{1/2}$ and $y = x^3$
- $f(x,y) = x^2 - 2y$, D is triangle with vertices (0,3), (1,1), (5,3)
- $f(x, y) = e^{x/y}$, $D = \{(x, y): 1 \leq y \leq 2,\ y \leq x \leq y^3\}$

these will be placed on tomorrow's worksheet

Try these 6 example problems before tomorrow's discussion.

Scratchwork