Quiz 2 due 6/24 11:59 pm now
Later today: video, DW4, HW4&5, midterm sol & mistakes document

Other materials for Q25+: TBD

Local Extrema

New features by tomorrow: stats added to Canvas grades, survey

Pre-lecture for 6/23 for feedback

see if bar chart exists for stats...

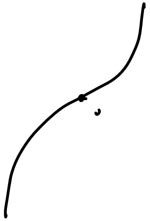
Types of Extrema

In Calc 1, we saw 2 types of extrema

- Local minimum
- Local maximum

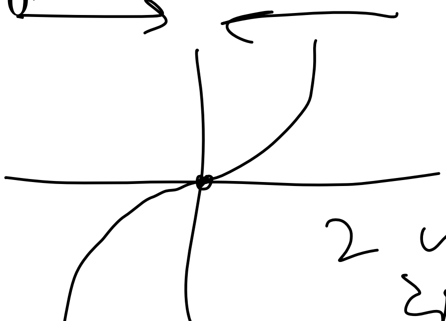
Definition for multivariable functions:

- If $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for $\mathbf{x}$ around $\mathbf{x}_0$, f has a local minimum at $\mathbf{x}_0$
- If $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for $\mathbf{x}$ around $\mathbf{x}_0$, f has a local maximum at $\mathbf{x}_0$



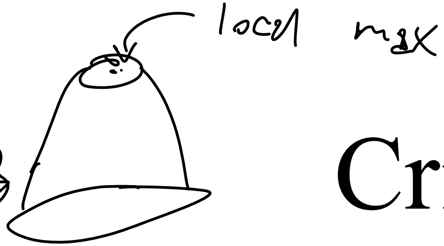
$f(x)=x^3$

$x=0 \Rightarrow$ not min or max



2 ways to approach c

$f(x,y) = -x^2 - y^2 - 3$



Critical Points

Calc 1 definition:

- c is a critical point if f'(c) = 0 or f' not defined at x = c

New definition:

- **v** is a critical point if ∇ f(**v**) = **0** or ∇ f not defined at **v**

recall $\nabla f = (f_x, f_y)$ or (f_x, f_y, f_z) or $(\ldots$ more var $)$

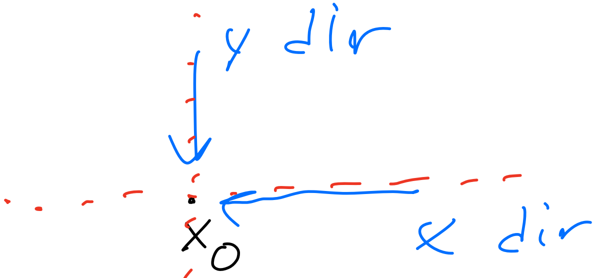
Just like Calc 1:

- If **v** is an extrema and ∇ f is defined at **v**, then ∇ f(**v**) = **0**

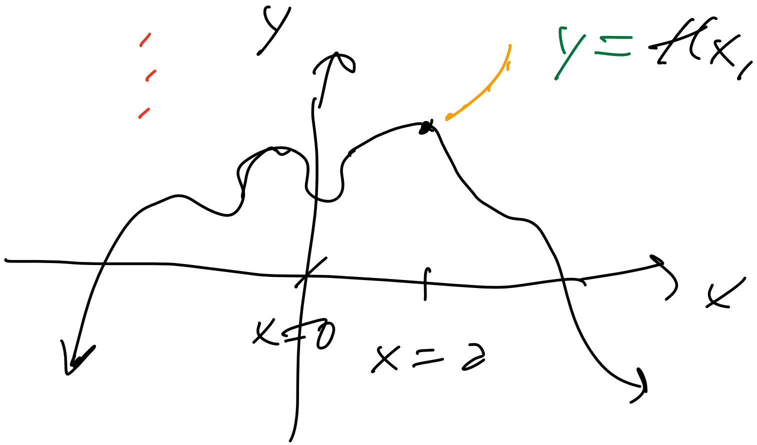
$$\nabla f(v) = 0 \iff 0 = f_x = f_y = \ldots$$

∇f not defined $\Longleftrightarrow$ f_x not defined OR
f_y not defined OR . . . .

Suppose x_0 is a local extrema & $\nabla f(x)$ defined



Consider approaching $x_0 = (a, b)$ with y fixed & x varying. will get some sort of graph of $f(x, b)$



$y = f(x, b)$. Suppose x_0 a local max. Then $x = a$ a local max in the graph shown.

From Calc I, we know

if $g(x) = f(x, b)$, then $g'(a) = 0$ because $g' = 0$ for local extrema in 1D.

$$0 = g'(a) = \left(\frac{d}{dx} f(x,b)\right)\Big|_{x=a} = f_x(x, b)\Big|_{x=a} = f_x(a,b)$$
$$= f_x(x_0).$$

Similarly, if we consider x fixed & y varying, we get $f_y(x_0) = 0$.

So $\nabla f(x_0) = \langle f_x(x_0), f_y(y_0) \rangle = \langle 0, 0 \rangle = 0$.

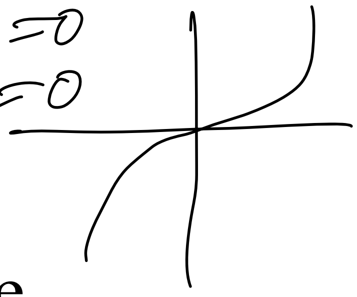
Logic is the same if we have more variables

Saddle Points

$f(x)=x^3$

$f'(0)=0$

$f''(0)=0$

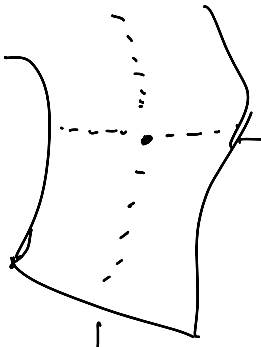


In Calc 1, critical points are almost always extrema

- If function continuously differentiable, “neither” is rare

Now, they are much more common → more variables means more space to approach x_0 → more likely to end up in "neither" category

- If $\mathbf{x}_0$ is a critical point but not an extrema, call it a saddle point
- Will have $f(\mathbf{x}) > f(\mathbf{x}_0)$ for some $\mathbf{x}$ near $\mathbf{x}_0$ and $f(\mathbf{x}) < f(\mathbf{x}_0)$ for others



→ height inc.

↓ height dec.

nature of

2nd Derivative Test

Suppose $\nabla f = 0$ solved & we have to figure out nature of solutions

How can we tell between local min, local max, or saddle?

- Define $D = f_{xx} f_{yy} - (f_{xy})^2$
- If $D(\mathbf{v}) > 0$ and $f_{xx}(\mathbf{v}) > 0$, then $\mathbf{v}$ is a local min
- If $D(\mathbf{v}) > 0$ and $f_{xx}(\mathbf{v}) < 0$, then $\mathbf{v}$ is a local max

red points

- If $D(\mathbf{v}) < 0$, $\mathbf{v}$ is a saddle point
- If $D(v) = 0$, test inconclusive; try other methods

We shall see explanation of test and "other methods" in lecture

$D > 0$ & $f_{xx} = 0$ is impossible case because $f_{xx} = 0$

means $D = 0 \cdot f_{yy} - (f_{xy})^2 = -f_{xy}^2 \le 0$

Remark: you can test $f_{yy} > 0$ or $f_{yy} < 0$ instead for the 2 red points

Practice Problems

why? $D > 0 \Rightarrow f_{xx} f_{yy} > 0 \Rightarrow$ $f_{xx}, f_{yy} > 0$ or $f_{xx}, f_{yy} < 0$

Find and classify all critical points

- $f(x, y) = x^2+xy+y^2+x+y+1$
- $f(x, y) = x^3+y^3-3xy + 06232025$
- $f(x, y) = y^3-3y^2+3x^2y-3x^2+1$

Find and classify the critical points of $f(x, y) = |x-2|+|y-3|$

Scratchwork

Hessian Test

Lecture for 6/23

Review of Matrices: Eigenvectors

Let A be a matrix and v a vector

- If $Av = \lambda v$ and $v \neq 0$, then v is an eigenvector of A with eigenvalue λ
- Can find eigenvalues by solving $\det(A-\lambda I) = 0$ for λ
  - After finding the values, solve $(A-\lambda I)v = 0$ to get vectors

assume familiarity with matrix multiplication, addition, subtraction, inverses & determinant

Example problem: find eigenvalues of $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Side note: if $Av = \lambda v$ with $v \neq 0$, then

$0 = Av - \lambda v = Av - \lambda I v = (A - \lambda I)v$. Set $B = A - \lambda I$. If B is invertible, then $0 = B^{-1} 0 = B^{-1}(Bv) = v$, contradiction. So B is not invertible, so $\det(B) = 0$.

→ you can look up this fact if you were not previously aware

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{pmatrix}.$$

Choose to expand along 2nd row. Then

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} =$$

$$(1-\lambda)\left[(1-\lambda)(2-\lambda) + 1\right] = \underline{(1-\lambda)\left[\lambda^2 - 3\lambda + 3\right]}$$

so $\lambda = 1$ or $\lambda^2 - 3\lambda + 3 = 0$

$$\lambda = \frac{3 \pm \sqrt{-3}}{2} = \frac{1}{2}(3 \pm \sqrt{3}\, i).$$

$$\underline{(1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix}} - \cancel{0 \begin{vmatrix} 0 & 0 \\ 1 & 2-\lambda \end{vmatrix}} + \underline{(-1) \begin{vmatrix} 0 & 1-\lambda \\ 1 & 1 \end{vmatrix}}$$

2 2x2 det to check instead of 3

Note! you can choose any row or column to expand & will get same answer

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Review of Matrices: Definiteness

$$(A^T)_{ij} = A_{ji}$$

- Matrices with $A^T = A$ are called symmetric
- Symmetric matrices with $v^T A v > 0$ for all $v \neq 0$ are positive definite
  - Negative definite if $v^T A v < 0$ for all $v \neq 0$
  - Semidefinite if $>$ and $<$ are replaced with $\geq$ and $\leq$
- If all eigenvalues positive, then A is positive definite

proof requires a little linear algebra

- If all eigenvalues are $<$, $\geq$, ≤ 0, then neg, pos semi, neg semi resp.

Note: $v^T A v = \underbrace{v}_{\text{vec}} \cdot \underbrace{(Av)}_{\text{vec}}$ = some number

Orange proof: Let A be $n \times n$ real symmetric matrix, then by spectral theorem, A has n distinct real eigenvalue/eigenvector pairs.

beyond the scope of this course & would take too long to cover

Let $(v_1, \lambda_1), (v_2, \lambda_2), \ldots, (v_n, \lambda_n)$ be these pairs.

Also, spectral theorem says v_i are orthogonal and form a basis for $\mathbb{R}^n$.

$v_i \cdot v_j = 0$ if $i \neq j$

Suppose $v \neq 0$, write $v = c_1 v_1 + \ldots + c_n v_n$

possible because $\{v_1, \ldots, v_n\}$ is a basis

Then $Av = c_1 A v_1 + \ldots + c_n A v_n$

$= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n$

So $v^T A v = (c_1 v_1 + \dots + c_n v_n) \cdot (c_1 \lambda_1 v_1 + \dots + c_n v_n \lambda_n)$.

Every ~~term~~ with v_i & v_j for $i \neq j$ will be 0 because $v_i \cdot v_j = 0$

$c_1 v_1 \cdot c_1 \lambda_1 v_1 +$
$c_1 v_1 \cdot c_2 \lambda_2 v_2 + \dots$
$c_2 v_2 \cdot (c_1 \lambda_1 v_1 + \dots)$
$c_3 v_3 \cdot (c_1 \lambda_1 v_1 + \dots) +$
$\dots$

Only terms that survive are when $i = j$.

So $v^T A v = (c_1 v_1) \cdot (c_1 \lambda_1 v_1) + \dots$

$= c_1^2 \lambda_1 (v_1 \cdot v_1) + c_2^2 \lambda_2 (v_2 \cdot v_2) + \dots$

$= c_1^2 \lambda_1 \|v_1\|^2 + c_2^2 \lambda_2 \|v_2\|^2 + \dots$

We know that $\lambda_i > 0$ for all i. So

$c_i \lambda_i \|v_i\|^2 \geq 0$, so $v^T A v \geq 0$.

Is it possible $v^T A v = 0$? Then $c_i \lambda_i \|v_i\|^2 = 0 \Rightarrow$

$c_i \|v_i\|^2 = 0 \Rightarrow c_i = 0$ ~~for~~ all i.

because eigenvectors ~~are~~ $\neq 0$

But if $c_i = 0$ for all i, then $v = c_1 v_1 + \dots = 0$.

So $v \neq 0 \Rightarrow v^T A v > 0$

Suppose $f(x,y,z)$

Hessian Test

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

Consider a twice differentiable function $f(x_1, x_2, \ldots, x_n)$

- Let H be the n x n matrix whose (i, j) entry is $f_{x\_i\, x\_j}$
- H is the Hessian of f

Suppose **x** is a critical point of f

- If H(**x**) is positive definite, x is a local min
- If H(**x**) is negative definite, x is a local max
- If H(**x**) has negative and positive eigenvalues, x is a saddle point
- If H(**x**) is positive or negative semidefinite, no info
- How we know there's no 5th bullet point? Ans =

Derivation of Test

These 4 cases cover everything. If all λ's are >0, then pos def. If all λ's neg, then neg. def. If at least 1 pos & 1 neg, then 3rd bullet. So only remaining scenario is all λ's ≥ 0 or all λ's ≤ 0, but some are 0. All ≥ 0 & some $0 \Rightarrow$ pos semi, all ≤ 0 & some $0 \Rightarrow$ neg semi.

To explain Hessian Test, recall Taylor series.

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

There's also Taylor series for 2 variables:

$$f(x,y) = \sum_{n\geq 0} \sum_{a+b=n} \frac{f_{x^a y^b}(x-x_0, y-y_0)}{n!} (x-x_0)^a (y-y_0)^b$$

This generalizes to any # of variables. For simplicity, let's show how to derive Hessian test for 2 variables and $(x_0, y_0) = (0,0)$. You can always translate f so that your point of concern is at the origin.

Suppose $(0,0)$ is a critical point. Then $f_x(0,0) = f_y(0,0) = 0$. So then:

$=0$

$$f(x,y) = f(0,0) + \frac{1}{1!}\left(f_x(0,0)\cdot x + f_y(0,0)\cdot y\right) +$$

$$\frac{1}{2!}\left(f_{xy}(0,0)\cdot xy + f_{yx}(0,0)\cdot yx + f_{xx}(0,0)\cdot x^2 + f_{yy}(0,0)\cdot y^2\right) + \dots$$

$$\Rightarrow f(x,y) - f(0,0) = \frac{1}{2}\left(f_{xx}\cdot x^2 + 2f_{xy}\,xy + f_{yy}\,y^2\right) + \underbrace{\dots}$$

higher order terms like f_{xxx}, f_{yyx} etc.

Hessian test comes from definiteness, which comes from $v^T A v$. So let's rewrite by using a dot product:

$$= \frac{1}{2}(x,y)\cdot\underline{\left(f_{xx}x + f_{xy}y,\; f_{yy}y + f_{yx}x\right)}$$

$$= \frac{1}{2}(x,y)\cdot\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix}$$

$= \frac{1}{2}(x,y)^T H_f(0,0)\begin{bmatrix} x \\ y \end{bmatrix} + \ldots$ Let $v = \langle x, y \rangle$.

So $f(x,y) - f(0,0) = \frac{1}{2} v^T H v + \ldots$

So H pos def $\Rightarrow$ $v^T H v > 0$ for $(x,y) \neq (0,0)$

So $f(x,y) - f(0,0) > 0 \Rightarrow f(x,y) > f(0,0)$

for (x,y) NEAR $(0,0)$ because "$\ldots$" higher order terms are small enough.

$\boxed{\text{To properly do this, use Taylor series with remainder.}}$

So $(0,0)$ is a local minimum.

Idea for why "$\ldots$" doesn't matter: x^3 much smaller than x^2 when x near 0.

Next 2 bullet points can be shown similarly.

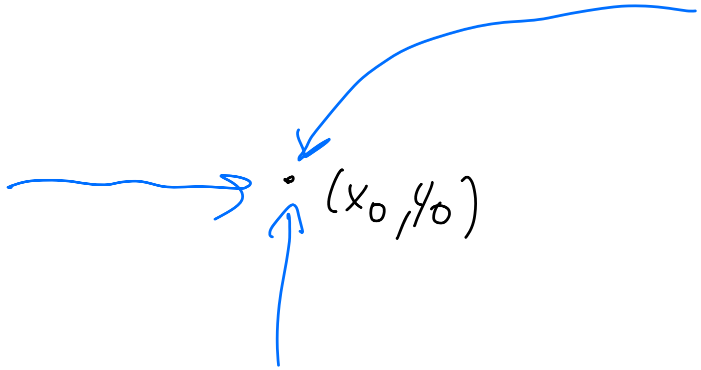
Other Methods

Historical remark: in 1800s when this test was devised by Hesse, he worked backwards from Taylor series & then defined H.

What happens if Hessian or 2nd Derivative Test inconclusive?

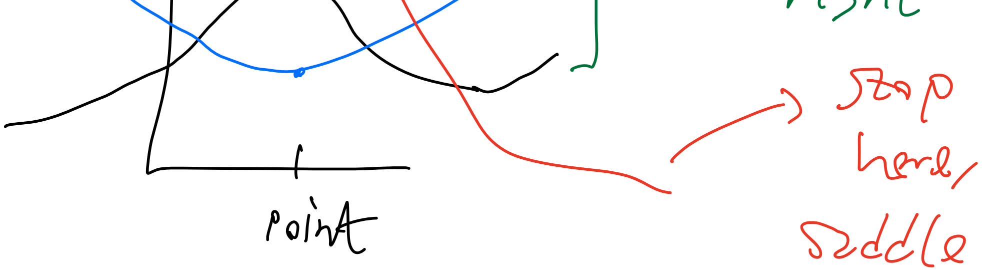
- Try approaching point along different directions
  - Same strategies as figuring out 2D limits
  - Will suggest which kind of point your critical point is
- If local extrema, prove your guess via inequalities

$H(x_0,y_0)$ is ≥ 0, some λ is 0.



See the 1 variable function you get along those blue curves

Show your guess is right



Example with inequalities:

$f(x,y) = x^2 + |y|$.

Then $(0,0)$ is a critical point. $f_x = 0$ & f_y not defined.

2nd derivative test won't help

Graphing & trying paths suggests $(0,0)$ is minimum

Let's see why it's a min:

$x^2 + |y| \geq 0 + 0 = 0 = f(0,0) \Rightarrow$

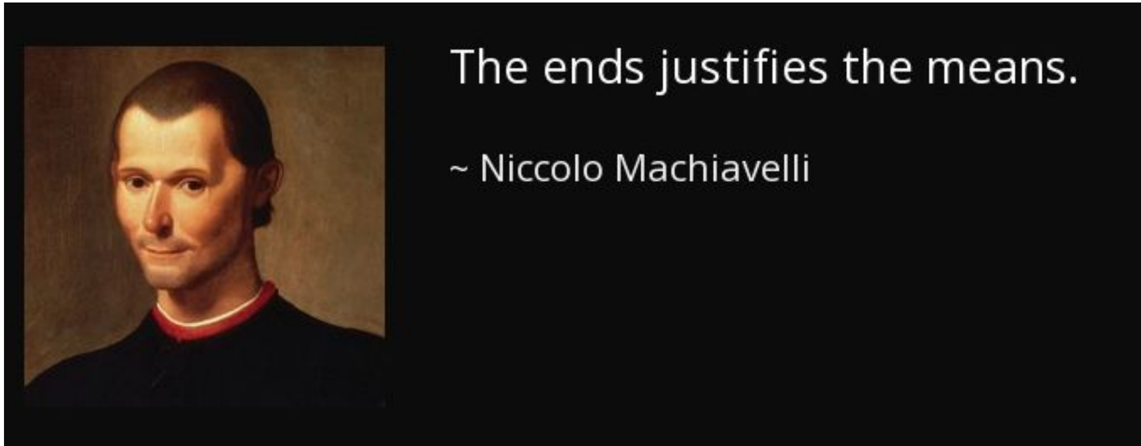
$f(x,y) \geq f(0,0)$ for all $(x,y) \Rightarrow (0,0)$ min.

Practice Problems

Find and classify all critical points by any means necessary

- $f(x, y, z) = x^2+y^2+z^2+xy+xz+yz+x+y+z+1$
- $f(x, y) = x^4 - y^4 - 4xy^2 - 2x^2$
- $f(x, y) = x^{2024} + y^{2026}$

These will be part of discussion worksheet as well as pre-lec worksheet problems



Scratchwork