MATH 243 Worksheet 3 Solutions

0: Bonuses:
a. Use the Pythagorean theorem. The hypotenuse is represented by the vector $u - v$ while the legs are u, v. Thus,

$$\|u\|^2 + \|v\|^2 = \|u - v\|^2 = (u - v) \cdot (u - v) = u \cdot u - 2(u \cdot v) + (-v) \cdot (-v) = \|u\|^2 - 2(u \cdot v) + \|v\|^2$$

Cancel out terms to get $u \cdot v = 0$. Conversely, if $u \cdot v = 0$ then we can reverse all the steps to find the Pythagorean theorem holds, which means the triangle with vertices $0, u, v$ is right, which means u, v are perpendicular.
For the next part, recall the Law of Cosines says

$$c^2 = a^2 + b^2 - 2ab \cos C$$

We'll take c to be the length of $\overline{uv}$ and a, b to represent the other two sides. Then $c = \|u - v\|, a = \|u - 0\| = \|u\|, b = \|v - 0\| = \|v\|$, and C is the angle between u and v. Plug everything in:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{\|u\|^2 + \|v\|^2 - \|u - v\|^2}{2\|u\|\|v\|} = \frac{2(u \cdot v)}{2\|u\|\|v\|} = \frac{u \cdot v}{\|u\|\|v\|}$$

where we used $\|u - v\|^2 = \|u\|^2 - 2(u \cdot v) + \|v\|^2$ as shown in the previous part.

b. Let P be the suggested shape. P has volume $(a \times b) \cdot (a \times b) = \|a \times b\|$ by the parallelepiped volume formula. At the same time, since $a \times b$ is perpendicular to both a and b, we know P is a prism with base the parallelogram Q formed by a, b and height the length of $a \times b$. Since the volume of a prism is base times height, we get

$$\|a \times b\|^2 = \text{vol}(P) = \text{area}(Q)\|a \times b\| \Rightarrow \|a \times b\| = \text{area}(Q).$$

By law of sines, the area of the triangle formed by $0, a, b$ is $\frac{1}{2}\|a\|\|b\| \sin(\theta)$ where θ is the angle between a and b. The shape Q is made out of two copies of this triangle, so its area is $\|a\|\|b\| \sin(\theta)$.

1: Leftover problems from lecture slides:
a. $v = \int a\, dt = \int (1, 2, 6t)\, dt = (t, 2t, 3t^2) + (c_1, c_2, c_3)$. Since $v(0)$ is given to us, let's use it: $(0, 1, -1) = v(0) = (c_1, c_2, c_3)$. Thus, $v = (t, 2t + 1, 3t^2 - 1)$. Then

$$r = \int v\, dt = \int (t, 2t + 1, 3t^2 - 1)\, dt = (0.5t^2, t^2 + t, t^3 - t) + (c_1, c_2, c_3).$$

Finally, $(1, -2, 3) = r(0) = (c_1, c_2, c_3)$, so we obtain $r(0) = (0.5t^2 + 1, t^2 + t - 2, t^3 - t + 3)$.

To find the components of acceleration, let's find all the things we need to plug in. $r'(t) = (t, 2t + 1, 3t^2 - 1), r''(t) = (1, 2, 6t), r'(t) \cdot r''(t) = t + 2(2t + 1) + 6t(3t^2 - 1) = 18t^3 - t + 2$. With a great deal of care, we compute $r'(t) \times r''(t) = (6t^2 + 6t + 2, -3t^2 - 1, -1)$. The last things we need are

$$\|r'(t)\| = \sqrt{9t^4 - t^2 + 4t + 2},\ \|r'(t) \times r''(t)\| = \sqrt{45t^4 + 72t^3 + 66t^2 + 24t + 6}.$$

Let's plug everything in:

$$\mathbf{a}_T = \frac{r'(t) \cdot r''(t)}{\|r'(t)\|} = \frac{18t^3 - t + 2}{\sqrt{9t^4 - t^2 + 4t + 2}},\ \mathbf{a}_N = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|} = \sqrt{\frac{45t^4 + 72t^3 + 66t^2 + 24t + 6}{9t^4 - t^2 + 4t + 2}}$$

b. We have $r'(t) = \langle -2\sin(2t), -2\cos(2t), 4\rangle = -2\langle \sin(2t), \cos(2t), -2\rangle$ and $r''(t) = \langle -4\cos(2t), 4\sin(2t), 0\rangle = 4\langle -\cos(2t), \sin(2t), 0\rangle$. Compute

$$\|r'(t)\| = 2\sqrt{\sin^2(2t) + \cos^2(2t) + 4} = 2\sqrt{5}$$

and

$$r'(t) \cdot r''(t) = -8(-\sin(2t)\cos(2t) + \sin(2t)\cos(2t) + 0) = 0.$$

This means r', r'' are perpendicular, so

$$\|r' \times r''\| = \|r'\|\|r''\|\sin(90^\circ) = 2\sqrt{5}\|r''\| = 8\sqrt{5}.$$

Notice how much time we saved by dragging out constants and using the formula for the magnitude of the cross product. Anyways, $\mathbf{a}_T = \frac{r'(t) \cdot r''(t)}{\|r'(t)\|} = 0$ and $\mathbf{a}_N = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|} = \frac{8\sqrt{5}}{2\sqrt{5}} = 4$.

c. Let $f(x,y), g(x,y), h(x,y,z)$ be the three functions. We have

$$f_x = (x^y)_x + (y^x)_x + (e^{xy})_x = yx^{y-1} + \ln(y)y^x + ye^{xy}.$$

Similarly, $f_y = \ln(x)x^y + xy^{x-1} + xe^{xy}$.
We have $g_x = 4x^4\sin(3y) - \frac{1}{y} - \frac{1}{y}\sin(\frac{x}{y})$ and $g_y = 3x^4\cos(3y) + \frac{x}{y^2} - \frac{x}{y^2}\sin(\frac{x}{y})$
For the last function, we need the quotient rule.

$$h_x = \frac{(xyz)_x(x+y+z) - (xyz)(x+y+z)_x}{(x+y+z)^2} = \frac{yz(x+y+z) - xyz}{(x+y+z)^2} = \frac{yz(y+z)}{(x+y+z)^2}.$$

By symmetry, we have $h_y = \frac{xz(x+z)}{(x+y+z)^2}$ and $h_z = \frac{xy(x+y)}{(x+y+z)^2}$.

d. Let's use the limit definition of the derivative:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \frac{f(x+h,c) - f(x,c)}{h} = f_x(x,c)$$

where the last equality is from the limit definition of partial derivatives.

1: The function is continuous at that point, so $\lim_{(x,y)\to(3,2)} e^{\sqrt{2x-y}} = e^{\sqrt{6-2}} = e^2$

2: $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^2y^3 - x^3y^2}{x^2 - y^2} = \lim\limits_{(x,y)\to(1,1)} \dfrac{x^2y^2(y-x)}{(x-y)(x+y)} = \lim\limits_{(x,y)\to(1,1)} -\dfrac{x^2y^2}{x+y} = -\dfrac{1}{2}$

3: Along the line $x = 0$, the limit is $\lim\limits_{y\to 0} \frac{2\cdot 0y}{0^2+3y^2} = \lim\limits_{y\to 0} \frac{0}{3y^2} \to 0$. Along $y = x$, we get $\frac{2x^2}{4x^2} = 0.5 \to 0.5$. Since we got two different values, the limit doesn't exist.

4: Show these limits exist and find them, or show they don't exist

(a) The function is continuous there, so $\lim\limits_{(x,y)\to(2,3)} \dfrac{3x-2y}{4x^2-y^2} = \dfrac{6-6}{16-9} = \dfrac{0}{7} = 0$

(b) Along $x = y$, the limit is $\lim\limits_{x\to 0} \frac{x}{3x^2} = \lim\limits_{x\to 0} \frac{1}{3x}$, which doesn't exist. So the limit doesn't exist.

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim\limits_{(x,y)\to(0,0)} \dfrac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2}$
$= \lim\limits_{(x,y)\to(0,0)} (\sqrt{x^2+y^2+1}+1) = 2$

5: The numerator and denominator are always defined and continuous. Thus, the function is defined and continuous anywhere the denominator isn't 0. The denominator is 0 precisely when $x^2 + y^2 = 1$. This means, the domain and the set of continuity points are both $\{(x,y) : x^2 + y^2 \neq 1\}$.

6: $g_u(u,v) = 5(u^2v - v^3)^4(2uv)$ and $g_v(u,v) = 5(u^2v - v^3)^4(u^2 - 3v^2)$
7: By the Product Rule, we have

$$f_y(x,y) = \frac{xy}{\sqrt{1-(xy)^2}} + \arcsin(xy)$$

$$f_y\left(1, \frac{1}{2}\right) = \frac{(1)(\frac{1}{2})}{\sqrt{1-((1)(\frac{1}{2}))^2}} + \arcsin(0.5) = \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} + \frac{\pi}{6} = \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} + \frac{\pi}{6} = \frac{1}{\sqrt{3}} + \frac{\pi}{6}.$$