

Textbook Sections: 15.6 and 15.7

Topics: Triple Integrals and Triple Integrals in Cylindrical Coordinates

Instructions: Try each of the following problems, show the detail of your work.

Clearly mark your choices in multiple choice items. Justify your answers.

Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

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Triple Integrals over rectangular boxes:

- Evaluate the integral $\iiint_E (xy + z^2) dV$ where $E = \{(x, y, z) | 0 \leq x \leq 2, 0 \leq z \leq 3, 0 \leq y \leq 1\}$ using three different orders of integration.

Among the six possible orders of integration we can choose the following three:

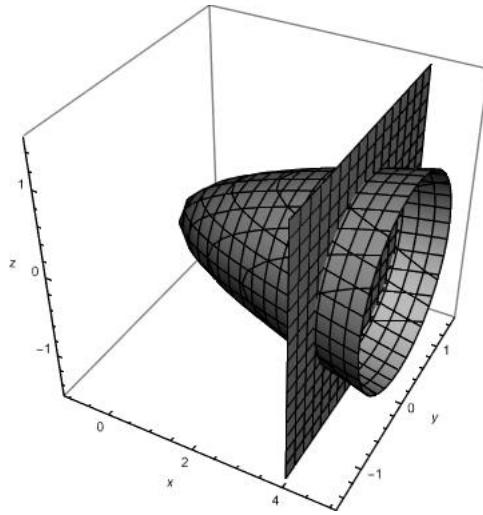
$$\begin{aligned} \int_0^2 \int_0^1 \int_0^3 (xy + z^2) dz dy dx &= \int_0^2 \int_0^1 (3xy + 9) dy dx \\ &= \int_0^2 \left(\frac{3}{2}x + 9 \right) dx = 21 \\ \int_0^1 \int_0^2 \int_0^3 (xy + z^2) dz dx dy &= \int_0^1 \int_0^2 (3xy + 9) dx dy \\ &= \int_0^1 (6y + 18) dy = 21 \\ \int_0^2 \int_0^3 \int_0^1 (xy + z^2) dy dz dx &= \int_0^2 \int_0^3 \left(\frac{1}{2}x + z^2 \right) dz dx \\ &= \int_0^2 \left(\frac{3}{2}x + 9 \right) dx = 21 \end{aligned}$$

Triple Integrals over general domains:

- Evaluate $\iiint_E \frac{1}{x^3} dV$ where $E = \{(x, y, z) | 0 \leq y \leq 1, 0 \leq z \leq y^2, 1 \leq x \leq z+1\}$.

$$\begin{aligned} \iiint_E \frac{1}{x^3} dV &= \int_0^1 \int_0^{y^2} \int_1^{z+1} x^{-3} dx dz dy \\ &= \int_0^1 \int_0^{y^2} \frac{1}{2} - \frac{1}{2}(z+1)^{-2} dz dy \\ &= \int_0^1 \left(\frac{1}{2}y^2 + \frac{1}{2} \frac{1}{1+y^2} - \frac{1}{2} \right) dy \\ &= \frac{3\pi - 8}{24} \end{aligned}$$

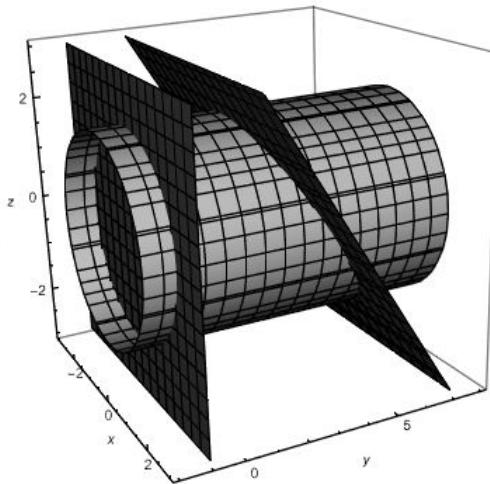
- Evaluate $\iiint_E x dV$ where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and $x = 4$.



The paraboloid and the plane intersect in the plane $x = 4$ by the circle $4y^2 + 4z^2 = 4$. This solid region projects onto the yz plane into the disk $y^2 + z^2 = 1$. We use cylindrical coordinates (r, θ, x) , and the change of variables $y = r \cos \theta, z = r \sin \theta$. Then the region is bounded by $x = 4r^2$ and $x = 4$ which gives the new bounds of integration:

$$\begin{aligned} \iiint_E x dV &= \int_0^{2\pi} \int_0^1 \int_{4r^2}^4 x r dx dr d\theta \\ &= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^1 (8r - 8r^5) dr \right) \\ &= \frac{16\pi}{3} \end{aligned}$$

4. Find the volume of the solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$.



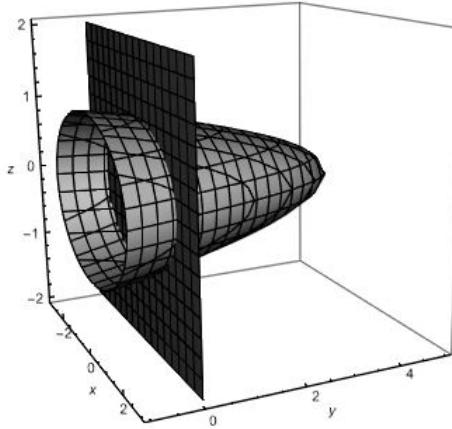
We use cylindrical coordinates (r, θ, y) , and the change of variables $x = r \cos \theta, z = r \sin \theta$. The solid projects on the xz -plane into the disk $x^2 + z^2 = 4$. Thus, $0 \leq r \leq 2$, and $0 \leq \theta \leq 2\pi$. Then we can write the bounds for y as

$$-1 \leq y \leq 4 - z = 4 - r \sin \theta.$$

Then:

$$\begin{aligned}\iiint_E 1 \, dV &= \int_0^{2\pi} \int_0^2 \int_{-1}^{4-r \sin \theta} r \, dy \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 (5r - r^2 \sin \theta) \, dr \, d\theta \\ &= \int_0^{2\pi} \left(10 - \frac{8}{3} \sin \theta\right) d\theta = 20\pi.\end{aligned}$$

5. Express the integral $\iiint_E f(x, y, z) \, dV$ as an iterated integral in six different ways, where E is bounded by $y = 4 - x^2 - 4z^2$ and $y = 0$.



We have

$$\begin{aligned}\iiint_E f(x, y, z) \, dV &= \int_{-1}^1 \int_{-2\sqrt{1-z^2}}^{2\sqrt{1-z^2}} \int_0^{4-x^2-4z^2} f(x, y, z) \, dy \, dx \, dz \\ &= \int_{-2}^2 \int_{-\sqrt{1-(x/2)^2}}^{\sqrt{1-(x/2)^2}} \int_0^{4-x^2-4z^2} f(x, y, z) \, dy \, dz \, dx \\ &= \int_{-1}^1 \int_0^{4-4z^2} \int_{-\sqrt{4-4z^2-y}}^{\sqrt{4-4z^2-y}} f(x, y, z) \, dx \, dy \, dz \\ &= \int_{-2}^2 \int_0^{4-x^2} \int_{-\sqrt{1-x^2/4-y/4}}^{\sqrt{1-x^2/4-y/4}} f(x, y, z) \, dz \, dy \, dx \\ &= \int_0^4 \int_{-\frac{1}{2}\sqrt{4-y}}^{\frac{1}{2}\sqrt{4-y}} \int_{-\sqrt{4-4z^2-y}}^{\sqrt{4-4z^2-y}} f(x, y, z) \, dx \, dz \, dy \\ &= \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-\sqrt{1-x^2/4-y/4}}^{\sqrt{1-x^2/4-y/4}} f(x, y, z) \, dz \, dx \, dy\end{aligned}$$

TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES:

6. What surface does the equation $z^2 + r^2 = 9$ describe in cylindrical coordinates?

Given $x = r \cos(\theta)$ and $y = r \sin(\theta)$, therefore, $x^2 + y^2 = r^2 (\cos^2(\theta) + \sin^2(\theta)) = r^2$. Therefore the given equation can be written as

$$\begin{aligned} z^2 + r^2 &= 9 \\ z^2 + x^2 + y^2 &= 3^2 \end{aligned}$$

Hence, the above equation represents a sphere centered at the origin, with radius 3.

7. Evaluate $\iiint_E z \, dV$ where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

The equation of the paraboloid $z = x^2 + y^2$ in cylindrical coordinates is $z = r^2$. We can describe E using cylindrical coordinates.

$$E = \{(r, \theta, z) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, r^2 \leq z \leq 4\}$$

$$\begin{aligned} \iiint_E z \, dV &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 z \, r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \frac{z^2}{2} r \Big|_{r^2}^4 \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \left(8r - \frac{r^5}{2} \right) \, dr \, d\theta \\ &= \int_0^{2\pi} \left(4r^2 - \frac{r^6}{12} \Big|_0^2 \right) \, d\theta \\ &= \int_0^{2\pi} \left[\left(16 - \frac{16}{3} \right) - 0 \right] \, d\theta \\ &= \int_0^{2\pi} \frac{32}{3} \, d\theta \\ &= \frac{32}{3} \cdot 2\pi \\ &= \frac{64\pi}{3} \end{aligned}$$

8. Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

We can describe E using cylindrical coordinates.

Sphere: $r^2 + z^2 = 2$

$$z = \pm \sqrt{2 - r^2}$$

$$z = +\sqrt{2 - r^2} \quad (\text{Note that we only need the } + \text{ for this problem.})$$

Paraboloid: $z = r^2$

$$E = \{(r, \theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r^2 \leq z \leq \sqrt{2 - r^2}\}$$

Note that $0 \leq r \leq 1$ since

$$\begin{aligned} r^2 + (r^2)^2 &= 2 \\ r^4 + r^2 - 2 &= 0 \\ (r^2 + 2)(r^2 - 1) &= 0 \\ \implies r &= 1 \end{aligned}$$

$$\begin{aligned} \text{Volume}(E) &= \iiint_E dV \\ &= \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{z-r^2}} r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 z r \Big|_{r^2}^{\sqrt{z-r^2}} dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (r\sqrt{2-r^2} - r^3) dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r\sqrt{2-r^2} dr d\theta - \int_0^{2\pi} \int_0^1 r^3 dr d\theta \end{aligned}$$

To evaluate the first integral in the expression above, we proceed by u-substitution

$$\begin{cases} \text{Let } u = 2 - r^2 \rightarrow du = -2rdr \\ r = 0 \rightarrow u = 2 \\ r = 1 \rightarrow u = 1 \end{cases}$$

$$\begin{aligned} \text{Volume}(E) &= -\frac{1}{2} \int_0^{2\pi} \int_2^1 \sqrt{u} du d\theta - \int_0^{2\pi} \left. \frac{r^4}{4} \right|_0^1 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left. \frac{2}{3} u^{\frac{3}{2}} \right|_2^1 d\theta - \int_0^{2\pi} \frac{1}{4} d\theta \\ &= \frac{1}{3} \int_0^{2\pi} (2\sqrt{2} - 1) d\theta - \frac{1}{4} \cdot 2\pi \\ &= \left(\frac{2\sqrt{2}}{3} - \frac{1}{3} \right) 2\pi - \frac{\pi}{2} \\ &= \frac{4\sqrt{2}\pi}{3} - \frac{\pi}{3} - \frac{\pi}{2} \\ &= \frac{8\sqrt{2}\pi - 7\pi}{6} = \frac{\pi(8\sqrt{2} - 7)}{6} \end{aligned}$$

9. Find the volume of the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, the planes $z = 0$ and $z = 4$, and in the first octant.

In cylindrical coordinates, the region of integration E is bounded by the planes $z = 0$ and $z = 4$. The projection of the first cylinder in the first octant onto the xy -plane is a quarter circle in the first quadrant centered at the origin with radius $r = 1$, and the projection of the second cylinder in the first octant onto the xy -plane is a quarter

circle in the first quadrant centered at the origin with radius $r = 2$. Hence, E is given by

$$E = \{(r, \theta, z) : 0 \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 2, 0 \leq z \leq 4\}$$

$$\begin{aligned}\text{Volume}(E) &= \iiint_E dV \\ &= \int_0^{\frac{\pi}{2}} \int_1^2 \int_0^4 r dz dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_1^2 rz \Big|_0^4 dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_1^2 4r dr d\theta \\ &= \int_0^{\frac{\pi}{2}} 2r^2 \Big|_1^2 d\theta \\ &= \int_0^{\frac{\pi}{2}} (8 - 2) d\theta \\ &= \int_0^{\frac{\pi}{2}} 6 d\theta \\ &= 3\pi\end{aligned}$$

10. Evaluate the integral

$$\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2} dz dy dx$$

by changing to cylindrical coordinates.

Letting $x = r \cos(\theta)$ and $y = r \sin(\theta)$, and $x^2 + y^2 = r^2$, we obtain

$$\begin{cases} z = 9 - x^2 - y^2 \implies z = 9 - r^2 \cos^2(\theta) - r^2 \sin^2(\theta) = 9 - r^2(\cos^2(\theta) + \sin^2(\theta)) = 9 - r^2. \\ 0 \leq z \leq 9 - x^2 - y^2 \implies 0 \leq z \leq 9 - r^2 \end{cases}$$

Hence, the region of integration is the region above the plane $z = 0$, and below the paraboloid $z = 9 - x^2 - y^2$, or $z = 9 - r^2$. Also, given $-3 \leq x \leq 3$ and $0 \leq y \leq \sqrt{9 - x^2}$, the projection of the solid region onto the xy -plane is the upper half of the circle in the xy -plane, centered at the origin, with radius 3.

$$E = \{(r, \theta, z) : 0 \leq \theta \leq \pi, 0 \leq r \leq 3, 0 \leq z \leq 9 - r^2\}$$

The function $f(x, y, z) = \sqrt{x^2 + y^2}$ becomes $f(r, \theta, z) = r$. Therefore, the given integral becomes

$$\begin{aligned}
\int_0^\pi \int_0^3 \int_0^{9-r^2} \sqrt{r^2 \cos^2(\theta) + r^2 \sin^2(\theta)} r \, dz \, dr \, d\theta &= \int_0^\pi \int_0^3 \int_0^{9-r^2} r^2 \, dz \, dr \, d\theta \\
&= \int_0^\pi \int_0^3 r^2 z \Big|_0^{9-r^2} \, dr \, d\theta \\
&= \int_0^\pi \int_0^3 r^2 (9 - r^2 - 0) \, dr \, d\theta \\
&= \int_0^\pi d\theta \int_0^3 (9r^2 - r^4) \, dr \\
&= \pi \left(3r^3 - \frac{r^5}{5}\right) \Big|_0^3 \\
&= \pi \cdot \frac{162}{5}
\end{aligned}$$

Suggested Textbook Problems

Section 15.6	3-22, 27, 28, 31-38, 41
Section 15.7	1-13, 15-25a, 31,32

SOME USEFUL DEFINITIONS, THEOREMS AND NOTATION:

Fubini's Theorem for Triple Integrals

If f is continuous on the box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz.$$

Triple Integrals on a Type 1 Solid Region

If E is a region such that $E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$, then

$$\iiint_E f(x, y, z) dV = \iint_D \left(\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right) dA.$$

In particular, if D is a type I plane region, then $E = \{(x, y, z) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$ and

$$\iiint_E f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz dy dx,$$

whereas if D is a type II plane region, then $E = \{(x, y, z) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\}$ and

$$\iiint_E f(x, y, z) dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz dx dy.$$

Triple Integrals on a Type 2 Solid Region

If E is a region such that $E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$, then

$$\iiint_E f(x, y, z) dV = \iint_D \left(\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) dx \right) dA.$$

Triple Integrals on a Type 3 Solid Region

If E is a region such that $E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$, then

$$\iiint_E f(x, y, z) dV = \iint_D \left(\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) dy \right) dA.$$

Volume of a Solid Region

The volume of a solid region E is given by

$$V(E) = \iiint_E 1 dV.$$

Cylindrical Coordinates

To change from Cartesian coordinates to cylindrical coordinates, use the following transformation:

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z,$$

and to change from rectangular to cylindrical coordinates, use

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad z = z.$$

Here is the iteration of a triple integral in cylindrical coordinates over a type I solid region E :

$$\iiint_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta.$$