14 Partial Derivatives

14.3 Partial Derivatives

Partial Derivatives of Functions of Two Variables

Partial Derivatives of Functions of Two Variables (1 of 14)

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates.

The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity.

The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H.

So I is a function of T and H and we can write $I = f(T, H)$.

The following table of values of *I* is an excerpt from a table compiled by the National Weather Service.

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Heat index *I* as a function of temperature and humidity

Table 1

Partial Derivatives of Functions of Two Variables (3 of 14)

If we concentrate on the highlighted column of the table, which corresponds to a relative humidity of H = 70%, we are considering the heat index as a function of the single variable T for a fixed value of H. Let's write $g(T) = f(T, 70)$.

Then $g(T)$ describes how the heat index I increases as the actual temperature T increases when the relative humidity is 70%. The derivative of g when T = 96°F is the rate of change of I with respect to T when T = 96°F:

$$g'(96) = \lim_{h \to 0} \frac{g(96+h) - g(96)}{h} = \lim_{h \to 0} \frac{f(96+h, 70) - f(96, 70)}{h}$$

Partial Derivatives of Functions of Two Variables (4 of 14)

We can approximate $g'(96)$ using the values in Table 1 by taking h = 2 and −2:

$$g'(96) \approx \frac{g(98) - g(96)}{2} = \frac{f(98,70) - f(96,70)}{2} = \frac{133 - 125}{2} = 4$$

$$g'(96) \approx \frac{g(94) - g(96)}{-2} = \frac{f(94,70) - f(96,70)}{-2} = \frac{118 - 125}{-2} = 3.5$$

Averaging these values, we can say that the derivative $g'(96)$ is approximately 3.75.

Partial Derivatives of Functions of Two Variables (5 of 14)

This means that, when the actual temperature is 96°F and the relative humidity is 70%, the apparent temperature (heat index) rises by about 3.75°F for every degree that the actual temperature rises.

Partial Derivatives of Functions of Two Variables (6 of 14)

Now let's look at the highlighted row in Table 1, which corresponds to a fixed temperature of T = 96°F.

		Relative humidity (%)								
	T \ H	50	55	60	65	70	75	80	85	90
Actual temperature (°F)	90	96	98	100	103	106	109	112	115	119
	92	100	103	105	108	112	115	119	123	128
	94	104	107	111	114	118	122	127	132	137
	96	109	113	116	121	125	130	135	141	146
	98	114	118	123	127	133	138	144	150	157
	100	119	124	129	135	141	147	154	161	168

Heat index I as a function of temperature and humidity

Table 1

Partial Derivatives of Functions of Two Variables (7 of 14)

The numbers in this row are values of the function $G(H) = f(96, H)$, which describes how the heat index increases as the relative humidity H increases when the actual temperature is $T = 96°$F.

The derivative of this function when $H = 70\%$ is the rate of change of I with respect to H when $H = 70\%$:

$$G'(70) = \lim_{h \to 0} \frac{G(70+h) - G(70)}{h} = \lim_{h \to 0} \frac{f(96, 70+h) - f(96, 70)}{h}$$

Partial Derivatives of Functions of Two Variables (8 of 14)

By taking h = 5 and −5, we approximate $G'(70)$ using the tabular values:

$$G'(70) \approx \frac{G(75) - G(70)}{5} = \frac{f(96,75) - f(96,70)}{5} = \frac{130 - 125}{5} = 1$$

$$G'(70) \approx \frac{G(65) - G(70)}{-5} = \frac{f(96,65) - f(96,70)}{-5} = \frac{121 - 125}{-5} = 0.8$$

By averaging these values we get the estimate $G'(70) \approx 0.9$. This says that, when the temperature is 96°F and the relative humidity is 70%, the heat index rises about 0.9°F for every percent that the relative humidity rises.

Partial Derivatives of Functions of Two Variables (9 of 14)

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant.

Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of *f* with respect to *x* at (*a*, *b*)** and denote it by $f_x(a, b)$. Thus

1 $$f_x(a,b) = g'(a) \quad \text{where} \quad g(x) = f(x,b)$$

Partial Derivatives of Functions of Two Variables (10 of 14)

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

and so Equation 1 becomes

$$\textbf{2} \qquad f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,\, b) - f(a,\, b)}{h}$$

Similarly, the **partial derivative of *f* with respect to *y* at (*a*, *b*)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3 $$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

With this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 96$°F and $H = 70\%$ as follows:

$$f_T(96,70) \approx 3.75 \qquad f_H(96,70) \approx 0.9$$

Partial Derivatives of Functions of Two Variables (12 of 14)

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

4 Definition If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

There are many alternative notations for partial derivatives. For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f / \partial x$.

But here $\partial f / \partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Partial Derivatives of Functions of Two Variables (14 of 14)

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed.

Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Example 1

If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution:

Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

$$\text{and so } f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

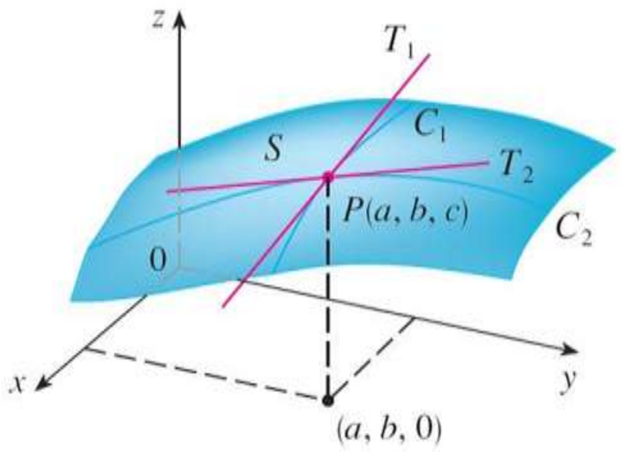
$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Interpretations of Partial Derivatives

Interpretations of Partial Derivatives (1 of 4)

To give a geometric interpretation of partial derivatives, we know that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S.

By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.)

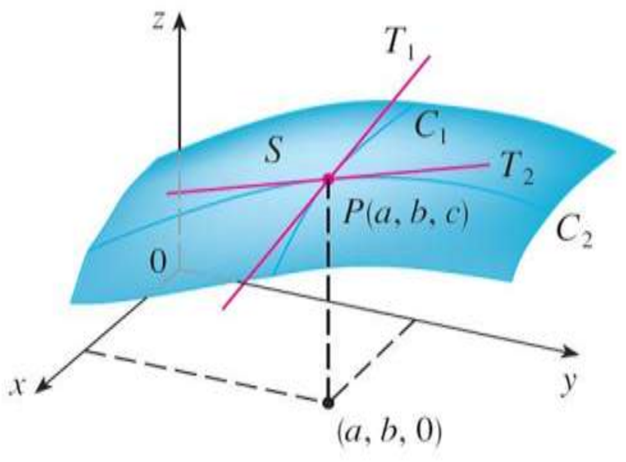


Interpretations of Partial Derivatives (2 of 4)

Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

Note that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$.

The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.



The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Figure 1

Interpretations of Partial Derivatives (3 of 4)

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the case of the heat index function at the beginning of this section, partial derivatives can also be interpreted as *rates of change*.

If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

Example 3

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Solution:

We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

Example 3 – Solution (1 of 2)

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.)

The slope of the tangent line to this parabola at the point (1, 1, 1) is $f_x(1, 1) = -2$.

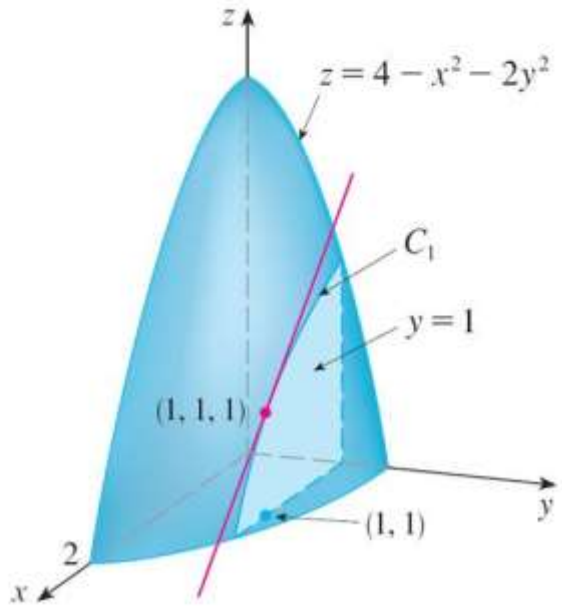


Figure 2

Example 3 – Solution (2 of 2)

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at (1, 1, 1) is $f_y(1, 1) = -4$. (See Figure 3.)

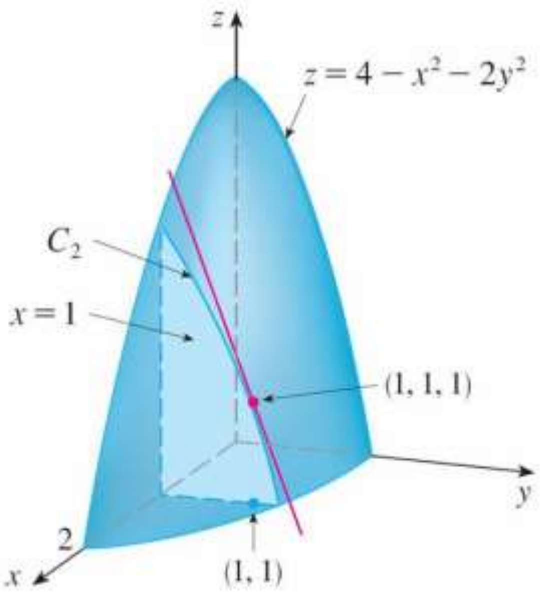


Figure 3

Interpretations of Partial Derivatives (4 of 4)

As we have seen in the case of the heat index function at the beginning of this section, partial derivatives can also be interpreted as *rates of change*.

If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

Implicit differentiation example

Implicit differentiation example

Functions of Three or More Variables

Functions of Three or More Variables (1 of 2)

Partial derivatives can also be defined for functions of three or more variables. For example, if *f* is a function of three variables *x*, *y*, and *z*, then its partial derivative with respect to *x* is defined as

$$f_x(x,y,z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

and it is found by regarding *y* and *z* as constants and differentiating *f*(*x*, *y*, *z*) with respect to *x*.

Functions of Three or More Variables (2 of 2)

If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the i th variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Example 6

Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Solution:

Holding y and z constant and differentiating with respect to x, we have

$$f_x = ye^{xy} \ln z$$

Similarly,

$$f_y = xe^{xy} \ln z \quad \text{and} \quad f_z = \frac{e^{xy}}{z}$$

Higher Derivatives

Higher Derivatives (1 of 4)

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f.

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Higher Derivatives (2 of 4)

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} $\left(\text{or } \partial^2 f / \partial y \partial x\right)$ means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example 7

Find the second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

Solution:

In Example 1 we found that

$$f_x(x,y) = 3x^2 + 2xy^3 \qquad f_y(x,y) = 3x^2y^2 - 4y$$

Therefore

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(3x^2 + 2xy^3) \\ &= 6x + 2y^3 \end{aligned}$$

Example 7 – Solution

$$f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3)$$

$$= 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y)$$

$$= 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y)$$

$$= 6x^2y - 4$$

Higher Derivatives (3 of 4)

Notice that $f_{xy} = f_{yx}$ in Example 7. This is not just a coincidence. It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice.

The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713−1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Higher Derivatives (4 of 4)

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Partial Differential Equations

Partial Differential Equations (1 of 4)

Partial derivatives occur in *partial differential equations* that express certain physical laws.

For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** after Pierre Laplace (1749–1827).

Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

Example 9

Show that the function $u(x,y) = e^x \sin y$ is a solution of Laplace's equation.

Solution:

We first compute the needed second-order partial derivatives:

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

So

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation.

Partial Differential Equations (2 of 4)

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

Partial Differential Equations (3 of 4)

For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as in Figure 5), then $u(x, t)$ satisfies the wave equation.

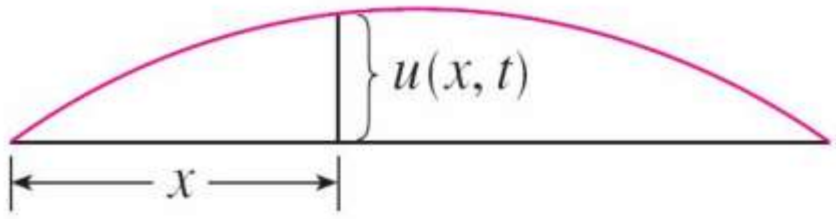


Figure 5

Here the constant a depends on the density of the string and on the tension in the string.

Partial Differential Equations (4 of 4)

Partial differential equations involving functions of three variables are also very important in science and engineering. The three-dimensional Laplace equation is

5 $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and one place it occurs is in geophysics. If $u(x, y, z)$ represents magnetic field strength at position (x, y, z), then it satisfies Equation 5. The strength of the magnetic field indicates the distribution of iron-rich minerals and reflects different rock types and the location of faults.