Topics: A review of integration and differentiation

Instructions: Try each of the following problems, show the detail of your work.

Clearly mark your choices in multiple choice items. Justify your answers.

Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

1. Find the first derivative of each of the following functions.

(a) $v(t) = 6t^3 - \frac{2}{3}t^2 - t + 4$

$$v'(t) = 18t^2 - \frac{4}{3}t - 1$$

(b) $f(x) = 3^{3x} + \sin(x^2)$

By the Chain Rule, we get

$$(3^{3x})' = 3^{3x}\ln(3) * 3 = 3^{3x+1}\ln(3)$$

Also, by the Chain Rule, we get

$$(\sin(x^2))' = \cos(x^2) * 2x$$

$$f'(x) = 3^{3x+1}\ln(3) + 2x\cos(x^2)$$

(c) $g(x) = \dfrac{e^x}{\tan x}$

By the Quotient Rule, we get

$$g'(x) = \frac{e^x\tan(x) - e^x\sec^2 x}{\tan^2 x}$$

(d) $a(x) = \log_3(x^2 + e^x)$

By the Chain Rule, we get

$$a'(x) = \frac{1}{(x^2+e^x)\ln 3} * (2x + e^x) = \frac{2x+e^x}{(x^2+e^x)\ln 3}$$

(e) $h(x) = \sqrt{x}\,(\ln x)$

By the Product Rule, we get

$$h'(x) = \frac{1}{2}\left(x^{-1/2}\right)\ln x + \sqrt{x}\cdot\frac{1}{x} = \frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x}$$

(f) $k(u) = \sqrt{e^{2u} + 2 + e^{-2u}}$

Solution 1: By the Chain Rule, we get

$$k'(u) = \left(\sqrt{e^{2u} + 2 + e^{-2u}}\right)' = \frac{1}{2}(e^{2u} + 2 + e^{-2u})^{-1/2} * (2e^{2u} - 2e^{-2u})$$
$$= \frac{2e^{2u} - 2e^{-2u}}{2\sqrt{e^{2u} + 2 + e^{-2u}}} = \frac{e^{2u} - e^{-2u}}{\sqrt{e^{2u} + 2 + e^{-2u}}} = \frac{e^{2u} - e^{-2u}}{\sqrt{(e^u + e^{-u})^2}}$$
$$= \frac{e^{2u} - e^{-2u}}{e^u + e^{-u}} = \frac{(e^u - e^{-u})(e^u + e^{-u})}{e^u + e^{-u}} = e^u - e^{-u}$$

Solution 2: First, simplify the expression of the function

$$k(u) = \sqrt{e^{2u} + 2 + e^{-2u}} = \sqrt{(e^u + e^{-u})^2} = e^u + e^{-u},$$

and then, find the derivative.

$$k'(u) = (e^u + e^{-u})' = e^u - e^{-u}$$

(g) $h(t) = \sin(e^{2t}) + 2^{\sin(3t)}$

Derivation using chain rule.
Recall that $\frac{d}{dt}2^t = 2^t \cdot \ln(2)$

$$\begin{aligned} h'(t) &= \frac{d}{dt}\left(\sin(e^{2t}) + 2^{\sin(3t)}\right) = \frac{d}{dt}\left(\sin(e^{2t})\right) + \frac{d}{dt}\left(2^{\sin(3t)}\right) \\ &= \cos(e^{2t}) \cdot \frac{d}{dt}\left(e^{2t}\right) + 2^{\sin(3t)} \cdot \ln(2) \cdot \frac{d}{dt}\sin(3t) \\ &= \cos(e^{2t}) \cdot \left(2e^{2t}\right) + 2^{\sin(3t)} \cdot \ln(2) \cdot 3\cos(3t) \\ &= 2e^{2t}\cos\left(e^{2t}\right) + 3\ln(2)\cos(3t) \cdot 2^{\sin(3t)} \end{aligned}$$

2. Suppose that $f(x)$ is a differentiable function such that $f(1) = 7$, $f'(1) = 4$, and $h(x) = \sqrt{4 + 3f(x)}$. Find $\frac{d}{dx}(h(x))\Big|_{x=1}$

$h'(x) = \frac{3f'(x)}{2\sqrt{4+3f(x)}}$
$h'(1) = \frac{3f'(1)}{2\sqrt{4+3f(1)}}$
$h'(1) = \frac{12}{2\sqrt{4+3*7}} = 6/5$

3. Let f be a function given by $f(x) = \cos x - 2\sin x$. Determine the equation of the tangent line to the graph of the function at the point $x = \dfrac{\pi}{6}$.

To find the equation of the tangent line, we need to find its slope.

$$f'(x) = -\sin(x) - 2\cos x, \qquad f'\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) - 2\cos\left(\frac{\pi}{6}\right) = -\frac{1}{2} - \sqrt{3}$$

The tangent line pass through a point $(\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2} - 1\right)$, hence the equation of tangent line is:

$$y - \frac{\sqrt{3}-2}{2} = (-\frac{1}{2} - \sqrt{3})(x - \frac{\pi}{6})$$

4. Compute the following integrals.

(a) $\displaystyle\int e^{\tan x} \cdot \sec^2(x)\, dx$

Integration using u-substitution.
Let $u = \tan x$, then $du = \sec^2(x)dx$.

$$\int e^{\tan x} \cdot \sec^2(x)\, dx = \int e^u\, du = e^u + C = e^{\tan x} + C$$

(b) $\displaystyle\int (5^x + 2\sinh x)\ dx$

Recall that $\displaystyle\int 5^x\, dx = \frac{5^x}{\ln(5)} + C$ and $\displaystyle\int \sinh x\, dx = \cosh x + C$. Therefore,

$$\begin{aligned}\int (5^x + 2\sinh x)\, dx &= \int 5^x dx + 2\int \sinh x\, dx \\ &= \frac{5^x}{\ln(5)} + 2\cosh x + C\end{aligned}$$

(c) $\displaystyle\int \left(u^{1/3} + \sin(u) - \frac{1}{u}\right)\, du$

$$\int \left(u^{1/3} + \sin(u) - \frac{1}{u}\right)\, du = \frac{3}{4}u^{4/3} - \cos(u) - \ln|u| + C$$

(d) $\displaystyle\int \left(3^x + \csc^2 x + e^{-x}\right)\, dx$

$$\int \left(3^x + \csc^2 x + e^{-x}\right)\, dx = \frac{3^x}{\ln 3} - \cot x - e^{-x} + C$$

(e) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$

Integration using trig-substitution:

$$\begin{aligned}
x^2 &= 9\sec^2\theta \\
x &= 3\sec\theta \to dx = 3\sec\theta\tan\theta\,d\theta \\
\sqrt{x^2-9} &= \sqrt{9\sec^2\theta-9} = \sqrt{9(\sec^2\theta-1)} = \sqrt{9\tan^2\theta} = 3\tan\theta \\
\int \frac{\sqrt{x^2-9}}{x}\,dx &= \int \frac{3\tan\theta}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta \\
&= \int 3\tan^2\theta\,d\theta = \int 3(\sec^2\theta-1)\,d\theta \\
&= 3\tan\theta - 3\theta + C \\
&= \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C
\end{aligned}$$

(f) $\displaystyle\int \frac{x^4+3x^2+x}{x^2}\;dx$

$$\int \frac{x^4+3x^2+x}{x^2}\;dx = \int x^2+3+\frac{1}{x}\;dx = \frac{x^3}{3}+3x+\ln|x|+C$$

(g) $\displaystyle\int 14x(\sqrt[3]{x})\;dx$

$$\int 14x(\sqrt[3]{x})\;dx = \int 14x^{4/3}\;dx = 6x^{7/3}+C$$

(h) $\displaystyle\int_0^1 \left(10^x + \frac{3}{1+x^2}\right)\;dx$

$$\int_0^1 \left(10^x + \frac{3}{1+x^2}\right)\;dx = \frac{9}{\ln(10)} + \frac{3\pi}{4}$$

(i) $\displaystyle\int (\cos x)^3 \cdot (\sin x)\;dx$

$$\int (\cos x)^3 \cdot (\sin x)\;dx = -\frac{(\cos(x))^4}{4} + C$$

(j) $\displaystyle\int_1^2 x\;e^{x^2}\;dx$

$$\int_1^2 x\;e^{x^2}\;dx = \frac{1}{2}\int_1^4 e^u\,du = \frac{1}{2}\left(e^u\big|_{u=1}^4\right) = \frac{e^4-e}{2}$$

(k) $\displaystyle\int_0^2 x\ e^x\ dx$

$$\int_0^2 x\ e^x\ dx = xe^x\Big|_{x=0}^{2} - \int_0^2 e^x\,dx = (x-1)e^x\Big|_{x=0}^{2} = e^2+1$$

5. Evaluate each of the following integrals by making the given substitution:

(a) $\displaystyle\int \cos(2x)\,dx$, where $u = 2x$

$$u = 2x$$
$$du = 2dx \Rightarrow dx = \frac{du}{2}$$
$$\int \cos(2x)\,dx = \int \cos u\,\frac{du}{2} = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(2x) + C$$

(b) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$, where $u = x^3+1$

$$\begin{aligned} u &= x^3+1 \\ du &= 3x^2dx \Rightarrow x^2dx = \frac{du}{3} \\ \int x^2\sqrt{x^3+1}\,dx &= \int \sqrt{u}\,\frac{du}{3} \\ &= \frac{1}{3}\left(\frac{2}{3}u^{3/2}\right) + C \\ &= \frac{2}{9}(x^3+1)^{3/2} + C \end{aligned}$$

(c) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$, where $u = x^2+1$

Let $u = x^2+1$ and $du = 2x\,dx \to \frac{du}{2} = x\,dx$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{2}\frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+1} + C$$

6. Evaluate the following indefinite integrals

(a) $\displaystyle\int xe^{-x^2}\,dx$

$$u = -x^2$$

$$du = -2xdx \Rightarrow xdx = \frac{du}{-2}$$

$$\int xe^{-x^2}\,dx = \int e^u \frac{du}{-2} = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

(b) $\displaystyle\int (2x^4+1)\ln(x)\,dx.$

$$u = \ln(x) \qquad du = \frac{1}{x}\,dx$$

$$dv = (2x^4+1)\,dx \qquad v = \frac{2x^5}{5} + x$$

$$\int (2x^4+1)\ln(x)\,dx = \left(\frac{2x^5}{5} + x\right)\ln(x) - \int \left(\frac{2x^5}{5} + x\right)\cdot\frac{1}{x}\,dx$$

$$= \left(\frac{2x^5}{5} + x\right)\ln(x) - \int \left(\frac{2x^4}{5} + 1\right)dx$$

$$= \left(\frac{2x^5}{5} + x\right)\ln(x) - \left(\frac{2}{5}\cdot\frac{x^5}{5} + x\right) + C$$

(c) $\displaystyle\int t\sec^2 t\,dt$

$$u = t \qquad du = dt$$

$$dv = \sec^2 t\,dt \qquad v = \tan t$$

$$\int t\sec^2 t\,dt = t\tan t - \int \tan t\,dt$$

$$= t\tan t - \int \frac{\sin t}{\cos t}\,dt$$

To find $\displaystyle\int \frac{\sin t}{\cos t}\,dt$, we use u-substitution: $u = \cos t$ and $du = -\sin t\,dt$.

$$\int \frac{\sin t}{\cos t}\,dt = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos t| + C = \ln|\sec t| + C$$

$$\int t\sec^2 t\,dt = t\tan t - \ln|\sec t| + C$$

(d) $\int \sin^2\theta \cos\theta \, d\theta$

$$\begin{aligned} u &= \sin\theta \\ du &= \cos\theta \, d\theta \\ \int \sin^2\theta \cos\theta \, d\theta &= \int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C \end{aligned}$$

7. A particle is moving with its acceleration at time t given by $a(t) = 3\cos t - 2\sin t$. Given that $s(0) = 0$ and $v(0) = 4$, determine the position of the particle $s(t)$ at any time t.

$$\begin{aligned} v(t) &= \int a(t) \, dt = \int 3\cos t - 2\sin t \, dt = 3\sin t + 2\cos t + C \\ v(0) &= 2\cos(0) + C = 4 \Rightarrow C = 2 \\ v(t) &= 3\sin t + 2\cos t + 2 \\ s(t) &= \int v(t) \, dt = \int 3\sin t + 2\cos t + 2 \, dt = -3\cos t + 2\sin t + 2t + C' \\ s(0) &= -3\cos(0) + C' = 0 \Rightarrow C' = 3 \\ s(t) &= -3\cos(t) + 2\sin(t) + 2t + 3 \end{aligned}$$

Textbook Sections: 3.1-3.6
Topics: Differentiation of real functions. Rules for differentiation.
Assume that b is a real number, $b > 0$, $b \neq 1$, c is a constant, n is a real number, and $f(x)$ and $g(x)$ are functions of x.

Differentiation Formulas

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(b^x) = b^x \ln b$

$\frac{d}{dx}(cf(x)) = cf'(x)$

$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

The Product Rule $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

The Quotient Rule $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

The Chain Rule $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Derivatives of Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

The Chain Rule for the power of a function

$\frac{d}{dx}[f(x)^n] = n[f(x)]^{n-1} f'(x)$

Derivatives of Inverse Trigonometric Functions

$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\text{arccot}\, x) = -\frac{1}{1+x^2}$

Derivatives of Logarithmic Functions

$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$

Textbook Sections: 4.9, 5.4, 5.5, 7.1-4.
Topics: Antiderivatives; indefinite integrals; definite integrals and the Fundamental Theorem of Calculus; integration by substitution; integration by parts.

Recall:

a. If $n \neq -1$, $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

b. $\displaystyle\int e^x\,dx = e^x + C$

c. $\displaystyle\int \sin x\,dx = -\cos x + C$

d. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

e. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

f. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

g. $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$

i. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

j. $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b} + C$

k. $\displaystyle\int \cos x\,dx = \sin x + C$

l. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

m. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

n. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

o. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

where C is any real constant.

Remark: (Part g.) Since $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$, then let $u = \cos x$ and $du = -\sin x\,dx$. So, we have

$$\int \frac{-du}{u} = -\ln|u| + C \quad \rightarrow \quad \int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

The Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Integration by Parts: If we let $u = f(x)$ and $v = g(x)$, then the differentials are $du = f'(x)dx$ and $dv = g'(x)dx$, so by the substitution rule, the **formula for integration by parts** is given by

$$\int u\ dv = uv\ - \int v\ du$$