Recall:

- Equation for the tangent plane at (x_0, y_0, z_0) is

$$\boxed{-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + (z-z_0) = 0}$$

OR: $$\boxed{z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

will cover today.

Important: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

- The linear function $\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$ is called the $\boxed{\text{linearization}}$ of f at (a,b).

- The approximation $\boxed{f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$ is called the linear approximation or the tangent plane approximation of f at (a,b).

8 Theorem

If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

$\Rightarrow$ If f_x, f_y are CTS, linearizations "work" as approximations

- **Differential:**

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

From calc1
$y = f(x)$
$\Rightarrow dy = f'(x)\,dx$

- **Chain Rule:**

1 The Chain Rule (Case 1)

Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

3 The Chain Rule (General Version)

intermediate

Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

independent

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

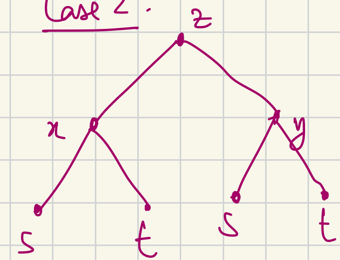
for each $i = 1, 2, \ldots, m$.

2 The Chain Rule (Case 2)

Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Case 2:



Implicit differentiation:

- $f(x,y)=0$ and "$y=f(x)$":

$$\boxed{\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-F_x}{F_y}}$$

- $f(x,y,z)=0$ and "$z=f(x,y)$":

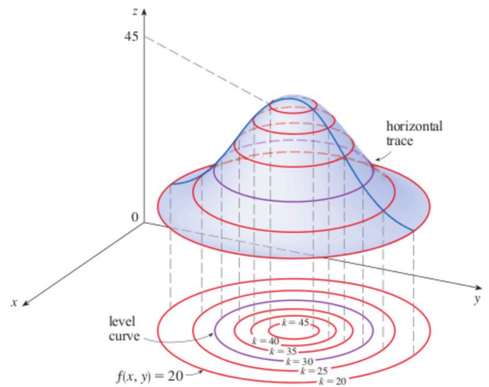
$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x = \frac{-F_x}{F_z}} \; , \; \boxed{\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y = \frac{-F_y}{F_z}}$$

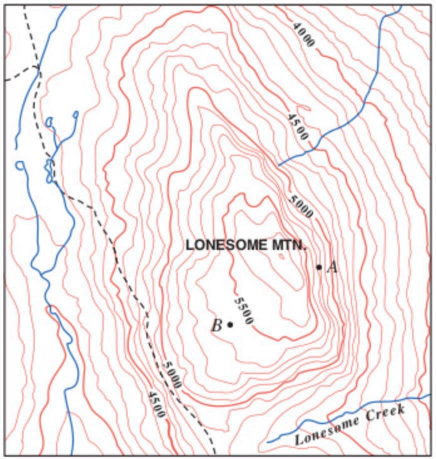
Level curves:

Definition

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Figure 11



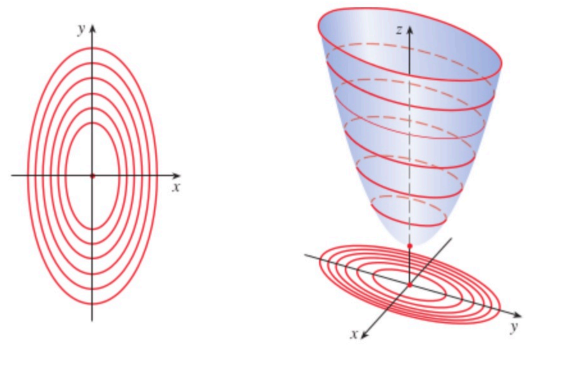


Example 12

Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

Figure 18

The graph of $h(x, y) = 4x^2 + y^2 + 1$ is formed by lifting the level curves.

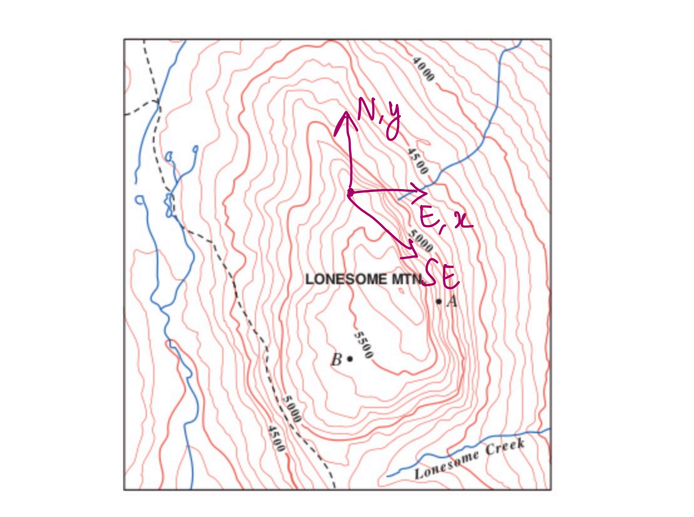


(a) Contour map

(b) Horizontal traces are raised level curves.

Directional Derivatives: (Section 14.6)

Say $f(x,y)$ = altitude on the map.



$f_x(x_0,y_0)$ = change in altitude travelling east.

$f_y(x_0,y_0)$ = change in altitude travelling north.

Problem: How to quantify change in altitude if you are travelling south east?

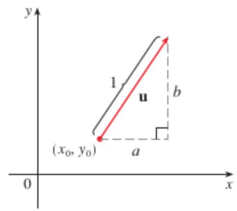
East $\to \hat{i}$, North $\to \hat{j}$ then southeast is $\hat{i}-\hat{j}$

Aside: unit vector in the SE direction: $\frac{1}{\sqrt{2}}(\hat{i}-\hat{j})$

Recall :

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

- 

- $\vec{u} = \langle a, b \rangle$ is a unit vector $\iff |\vec{u}| = 1$
  i.e $a^2 + b^2 = 1$

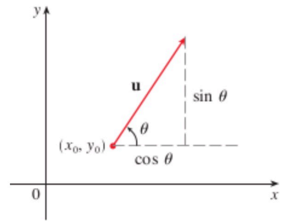
- Let θ be the angle subtended by $\vec{u}$ and $\hat{i}$ (x-axis)

- Then $\cos\theta = \dfrac{a}{|\vec{u}|} = a$

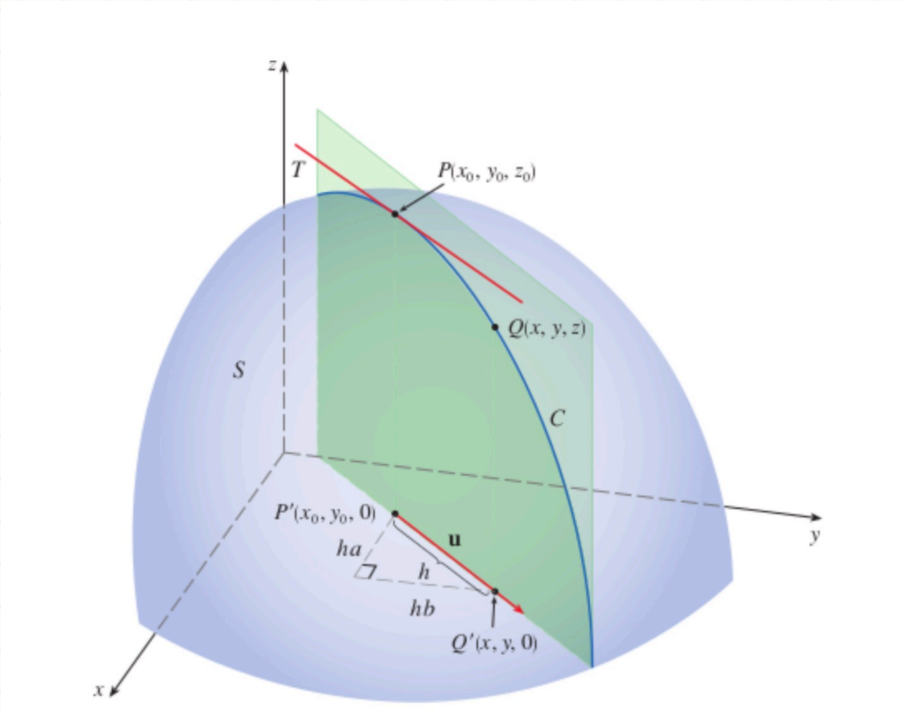
  and $\sin\theta = \dfrac{b}{|\vec{u}|} = b$

  Since $|\vec{u}| = 1$.

- A unit vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$



- So that $\boxed{\vec{u} = \langle \cos\theta, \sin\theta \rangle.}$



- Fix a differentiable function $f(x,y)$, a unit vector $\vec{u} = \langle a, b \rangle$, and a point $P(x_0, y_0, z_0)$ on the surface $z = f(x,y)$.
- Consider the plane with direction vectors $\vec{u}$ and $\vec{k}$ passing through P. (vertical plane in the $\vec{u}$ direction at P)
- Let C be the trace of the surface on the plane. We want the change in $f(x,y)$ along this curve.
- Let $g(h) = f(x_0 + ha, y_0 + hb)$ then $g(0) = f(x_0, y_0)$ so that $\frac{g(h) - g(0)}{h}$ captures the average change in f-values along C.

2 Definition

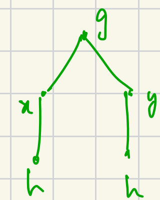
The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

- How do we compute $D_u f(x_0, y_0)$?

$g(h) = f(x_0 + ha, y_0 + hb)$



Chain rule: $D_u f(x_0, y_0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0)$

So we need $g'(h)$ and evaluate it at 0.

That is, let $x(h) = x_0 + ha$, $y(h) = y_0 + hb$

$$\Rightarrow \frac{dg}{dh} = \frac{dg}{dx} \cdot \frac{dx}{dh} + \frac{dg}{dy} \cdot \frac{dy}{dh}.$$

$$\Rightarrow g'(h) = f_x(x_0 + ha, y_0 + hb) \cdot a + f_y(x_0 + ha, y_0 + hb) \cdot b$$

Recall:

$\vec{u} = \langle a, b \rangle$

$= \langle \cos\theta, \sin\theta \rangle.$

$$\Rightarrow \boxed{D_u f(x_0, y_0) = f_x(x_0, y_0) \cdot a + f_y(x_0, y_0) \cdot b}$$

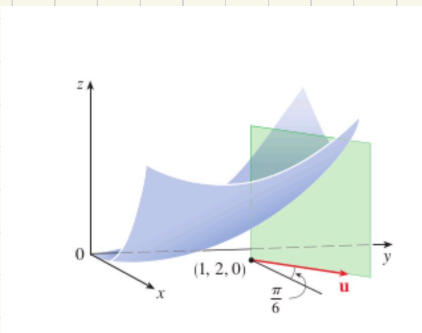
$$\Rightarrow \boxed{D_u f(x_0, y_0) = f_x(x_0, y_0) \cdot \cos\theta + f_y(x_0, y_0) \cdot \sin\theta}$$

Example 2

Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ the unit vector in the direction given by angle $\theta = \pi/6$ measured from the positive x-axis. What is $D_u f(1, 2)$?



$\underline{A:}$ $D_u f(1,2) = f_x(1,2) \cdot a + f_y(1,2)\, b$

where $u = \langle a, b \rangle$ subtends $\theta = \pi/6$ with $\hat{i}$.

$\Rightarrow a = \cos \pi/6$, $b = \sin \pi/6 \Rightarrow a = \frac{\sqrt{3}}{2}$, $b = \frac{1}{2}$

$f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = -3x + 8y$

$\Rightarrow f_x(1,2) = 3 - 6 = -3$, $f_y(1,2) = -3 + 16 = 13$

$\Rightarrow D_u f(1,2) = (-3) \cdot \frac{\sqrt{3}}{2} + (13) \cdot \frac{1}{2} = \frac{\sqrt{13} - 3\sqrt{3}}{2}$

Gradient: $D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$

$$= \langle f_x(x,y), f_y(x,y)\rangle \cdot \underbrace{\langle a, b\rangle}_{=\vec{u}}$$

8 Definition

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\boxed{\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}} \Rightarrow \boxed{D_u f(x,y) = \nabla f(x,y)\cdot\vec{u}}$$

Example 3

If $f(x,y) = \sin x + e^{xy}$, then

Then $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$

$= \langle \cos x + y e^{xy}, x e^{xy}\rangle$.

and at $(0,1)$, $\nabla f(0,1) = \langle \cos 0 + 1\cdot e^{0\cdot 1}, 0\cdot e^{0\cdot 1}\rangle = \langle 2, 0\rangle$

Example 4

in the direction of a unit vector

Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\boxed{\mathbf{v} = 2\,\mathbf{i} + 5\mathbf{j}.}$ ← not a unit vector

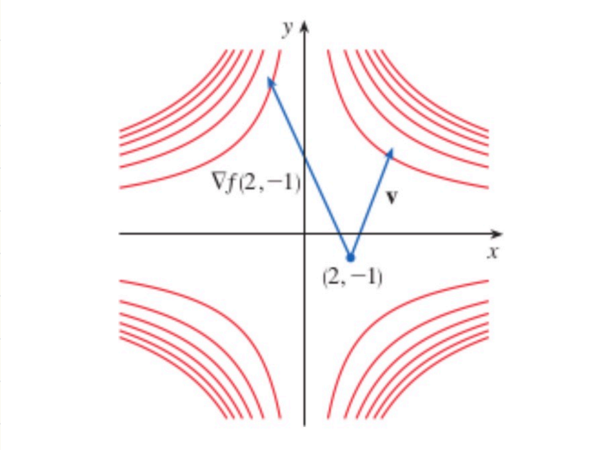
A: $|\vec{v}| = \sqrt{4+25} = \sqrt{29} \neq 1$ (not a unit vector!)

So consider $\vec{u} = \frac{1}{\sqrt{29}}\vec{v} = \frac{1}{\sqrt{29}}\langle 2, 5\rangle \Rightarrow \vec{u} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$

$f_x(x,y) = 2xy^3$, $f_y(x,y) = 3x^2y^2 - 4$

$\Rightarrow f_x(2,-1) = -4$, $f_y(2,-1) = 12 - 4 = 8$

$\Rightarrow D_u f(2,-1) = -4\left(\frac{2}{\sqrt{29}}\right) + 8\left(\frac{5}{\sqrt{29}}\right) = \frac{32}{\sqrt{29}}$



Functions of three variables:

10 Definition

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

$$\boxed{\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle \quad \text{OR (succinctly)} \quad \nabla f = \langle f_x, f_y, f_z \rangle.}$$

$$\Rightarrow \boxed{D_u f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \vec{u}}$$

Example 5

If $f(x, y, z) = x \sin yz$,

a. find the gradient of f and

b. find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

a. $f_x = \sin yz$, $f_y = xz\cos yz$, $f_z = xy\cos yz$

$\Rightarrow \nabla f = \langle \sin yz, xz\cos yz, xy\cos yz \rangle$

b. $\vec{v} = \hat{i} + 2\hat{j} - \hat{k} \Rightarrow \|\vec{v}\|^2 = 1 + 4 + 1 = 6 \Rightarrow \|\vec{v}\| = \sqrt{6}$

so take $\vec{u} = \dfrac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle$

$$\Rightarrow D_u f(1,3,0) = \underbrace{\nabla f(1,3,0)}_{\substack{\shortparallel \\ \langle \sin(3\cdot 0),\ 1\cdot 0\cdot\cos(3\cdot 0),\ 1\cdot 3\cdot\cos(3\cdot 0)\rangle}} \cdot \vec{u} = \langle 0, 0, 3 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle = \frac{-3}{\sqrt{6}} = -\sqrt{\frac{3}{2}}$$

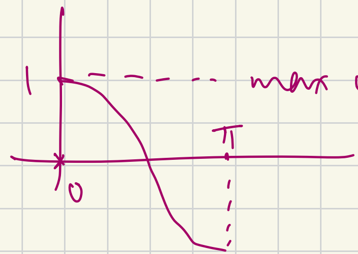
15 Theorem

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

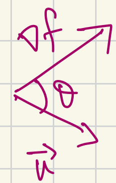
Pf: $|D_u f(\vec{x})| = |\nabla f(\vec{x}) \cdot \vec{u}|$

$= |\nabla f(\vec{x})| \cdot \underbrace{|\vec{u}|}_{=1} \cdot \cos\theta$ where

θ is the angle between 0 and π.



max of $\cos\theta = 1$ at $\theta = 0$



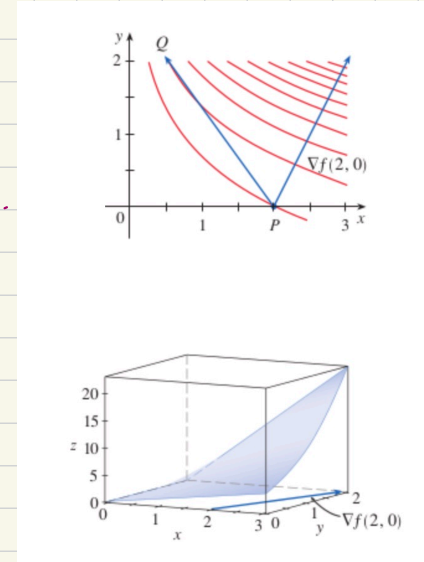
$\Rightarrow |D_u f(\vec{x})|$ is maximum when $\theta = 0$ i.e it has

maximum value $|\nabla f(\vec{x})| \cdot 1 \cdot 1 = |\nabla f(\vec{x})|$.

Example 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?



(a) $\cdot\ \overrightarrow{PQ} = \langle \frac{1}{2} - 2, 2 - 0 \rangle = \langle -\frac{3}{2}, 2 \rangle \Rightarrow \|\overrightarrow{PQ}\|^2 = \frac{9}{4} + 4 = \frac{17}{4} \Rightarrow \|\overrightarrow{PQ}\| = \frac{\sqrt{17}}{2}$.

so take $\vec{u} = \frac{2}{\sqrt{17}} \langle -\frac{3}{2}, 2 \rangle = \frac{1}{\sqrt{17}} \langle -3, 4 \rangle$. $= \frac{1}{\sqrt{17}/2} \langle -\frac{3}{2}, 2 \rangle$

$\cdot\ f_x = e^y,\ f_y = xe^y \Rightarrow \nabla f(2,0) = \langle 1, 2 \rangle$

$\Rightarrow D_u f(2,0) = \langle 1, 2 \rangle \cdot \langle \frac{-3}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle = \frac{-3+8}{\sqrt{17}} = \frac{5}{\sqrt{17}}$.

(b) Maximum change is along $\nabla f(2,0) = \langle 1, 2 \rangle$, with maximum rate of change being $\|\nabla f(2,0)\| = \sqrt{1+4} = \sqrt{5}$.

Example 7

Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

A: We need $|\nabla T(1,1,-2)|$.

$$T(x,y,z) = 80(1+x^2+2y^2+3z^2)^{-1}$$

$$T_x(x,y,z) = 80(-1)(1+x^2+2y^2+3z^2)^{-2}(2x)$$

$$= \frac{-160x}{(1+x^2+2y^2+3z^2)^2}$$

$$T_y(x,y,z) = \frac{-320y}{(1+x^2+2y^2+3z^2)^2}$$

$$T_z(x,y,z) = \frac{-480z}{(1+x^2+2y^2+3z^2)^2}$$

$$\Rightarrow \nabla T(x,y,z) = \frac{1}{(1+x^2+2y^2+3z^2)^2}\langle -160x, -320y, -480z \rangle$$

$$\Rightarrow \nabla T(1,1,-2) = \frac{160}{256}(-i-2j+6k)$$

and $|\nabla T(1,1,-2)| = \frac{5}{8}\sqrt{41} \approx 4°C/m$.

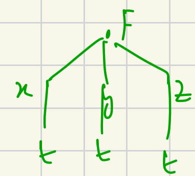
Tangent planes to level surfaces:

- Suppose that S is the surface $F(x,y,z)=k$.

eg: $F(x,y,z)=x^2+y^2+z^2$, $F(x,y,z)=k$ is the sphere with radius $\sqrt{k}$.

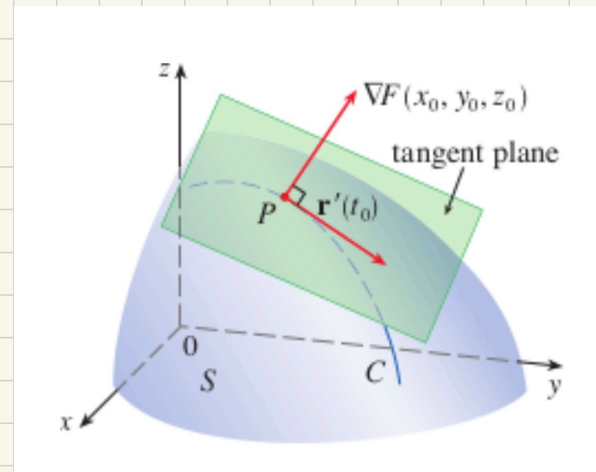
- Let $P(x_0,y_0,z_0)$ be a point on the surface.
- Let C be any curve $r(t)=\langle x(t),y(t),z(t)\rangle$ on S, passing through P.
- Let t_0 be the parameter such that $r(t_0)=\langle x_0,y_0,z_0\rangle$.
- Since C lies on S, $F(r(t))=F(x(t),y(t),z(t))=k$

$$\Rightarrow \frac{dF}{dt} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial t}=0$$



$$\Rightarrow \left\langle \frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right\rangle\cdot\left\langle\frac{\partial x}{\partial t},\frac{\partial y}{\partial t},\frac{\partial z}{\partial t}\right\rangle=0$$

$$\Rightarrow \boxed{\nabla F(x,y,z)\cdot r'(t)=0}$$



In particular, $\nabla F(x_0,y_0,z_0)\cdot r'(t_0)=0$ for $\boxed{\text{any curve}}$ C passing through P.

$\Leftarrow$ tangent contains $r'(t_0)$ for all such curves

$\Rightarrow$ $\nabla F(x_0, y_0, z_0)$ is the normal vector for the tangent plane to the level surface $F(x,y,z) = k$ at the point $P(x_0, y_0, z_0)$. The plane has the equation:

$$\boxed{\nabla F(x_0, y_0, z_0) \cdot \left(\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle \right) = 0}$$

OR

$$\boxed{f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0}$$

Definition: The normal line to $F(x,y,z) = k$ at $P(x_0, y_0, z_0)$ is the line through P with direction vector as $\nabla F(x_0, y_0, z_0)$.

- That is, it has the symmetric equation:

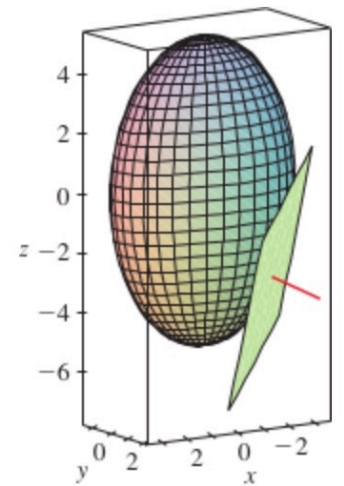
$$\boxed{\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}}$$

Example 8

Find the equations of the tangent plane and normal line to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

at the point $(-2, 1, -3)$



Let $f(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$. Then

$f_x = \frac{x}{2}$, $f_y = 2y$, $f_z = \frac{2z}{9}$

$\Rightarrow \nabla f(-2,1,-3) = \langle \frac{-2}{2}, 2\cdot 1, \frac{2\cdot(-3)}{9} \rangle = \langle -1, 2, \frac{-2}{3} \rangle$

$\Rightarrow$ Equation for tangent plane is $\langle -1, 2, \frac{-2}{3} \rangle \cdot \langle x+2, y-1, z+3 \rangle = 0$

$\Rightarrow$ Equation for normal line is $\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-2/3}$

Special case: surface is of the form $z = f(x,y)$, take $F(x,y,z) = f(x,y) - z = 0$.

$\Rightarrow \nabla F(x_0, y_0, z_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle$ and the equation of the tangent is exactly as before: $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$.

Example 9

Find the tangent plane to the surface $z = 2x^2 + y^2$ at the point $(1,1,3)$.

A: $z = 2x^2+y^2 \rightarrow F(x,y,z) = 2x^2+y^2-z \Rightarrow F_x = 4x,\ F_y = 2y,\ F_z = -1$

$\Rightarrow \nabla F(1,1,3) = \langle 4, 2, -1 \rangle \Rightarrow$ Equation of tangent plane is $\langle 4,2,-1 \rangle \cdot \langle x-1, y-1, z-3 \rangle = 0$.

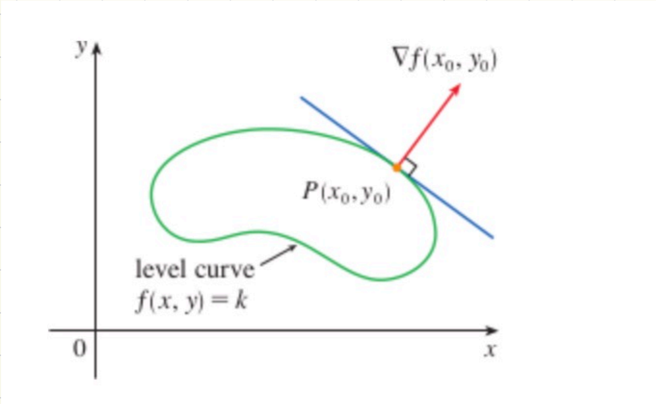
Significance of gradient vector:

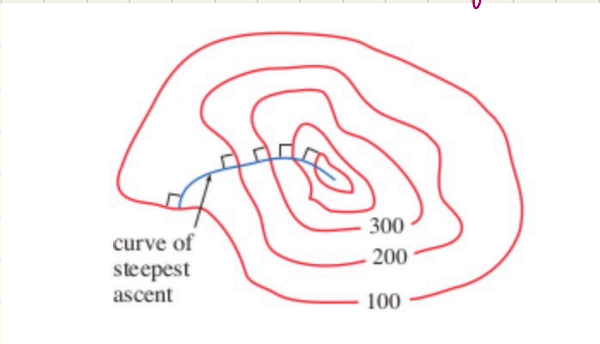
Properties of the Gradient Vector

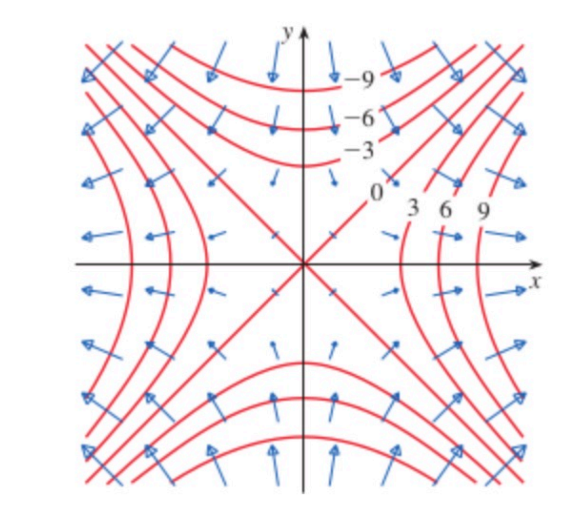
Let f be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x}) \neq \mathbf{0}$.

- The directional derivative of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_u f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.
- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that maximum rate of change is $|\nabla f(\mathbf{x})|$.
- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.

(∇f is perpendicular to the tangent line or tangent plane).







← $f(x,y) = x^2 - y^2$ with blue arrows pointing in the direction of steepest increase.