Stokes’ & Divergence Theorem

Lecture for 7/9

Motivation

Recall how we can write line integrals in terms of normal vectors

- $\int_C (\mathbf{F} \cdot \mathbf{n})\, ds = \int_C (A\, dy - B\, dx)$ if $\mathbf{F} = (A, B)$ and C is closed curve
  - n is unit normal pointing out of D if $C = \partial D$
  - If n points inside D, add minus sign
- This formulation is useful when normals are easy to find
- Can we do the same in higher dimensions?
  - Can we get a line ↔ surface integral conversion?
  - Can we get a Green's Theorem for 2D ↔ 3D conversions?

Oriented Surfaces

We have seen each curve has 2 possible direction

- Let S be a closed surface so that $S = \partial E$ for some solid E
- Further assume S has 2 sides, so Mobius strips banned
- At any point, we may pick unit normal pointing in or out of E
- Call S +, - oriented resp if we pick normals pointing in, out of E resp
    - Unless otherwise mentioned, we'll assume S is positive

Unit Normals to Surfaces

Suppose S is described by $f(x, y, z) = 0$ and param by $\mathbf{r}(u, v)$

- Recall that ∇f is perpendicular to S
- Thus, $\mathbf{n} = \nabla f/||\nabla f||$ is a unit normal to S
- As $\mathbf{r}_u$, $\mathbf{r}_v$ tangent to surface, $\mathbf{n} = (\mathbf{r}_u \times \mathbf{r}_v)/||\mathbf{r}_u \times \mathbf{r}_v||$ also works
- Define $d\mathbf{S} = \mathbf{n}\, dS$ for the sake of convenience
- Note: we are back to using bold for vectors

Surface Integrals of Vector Fields

Suppose S, f, r are as before

- Let's introduce a vector field $\mathbf{F} : S \to R^3$
- Define flux as $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$
- We have a nice formula when $f(x, y, z) = z - g(x,y)$
    - Suppose $\mathbf{F} = (A, B, C)$, and S lies over $D \subseteq R^2$
    - Then $\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \iint_D (-Ag_x - Bg_y + C)\, dA$

Space for Derivations

Stokes' Theorem

Suppose S, C positively oriented surface, curve resp, and $C = \partial S$

- Recall: C is positive, negative if it is CCW, CW respectively
- Also need condition that S is twice cont. differentiable
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$
- Add minus signs as necessary for S, C oriented differently

Derivation

Divergence Theorem

Let E be a solid, $S = \partial E$ is pos orient, and F is cont. diff vector field

- $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F}\, dV$
- Congratulations, this is the end of the course

Derivation

Practice Problems

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the following functions and surfaces

- $F = (3x, 2y, 1-y^2)$, S is portion of $z = 2-3y+x^2$ oriented downward and lying over triangle with vertices (0,0), (2,0), (2, -4)
- $F = (yz, x, 3y^2)$, S is surface of solid bounded by $x^2+y^2 = 4$, $z = x-3$, $z = x+2$ with negative orientation
- $F = \nabla \times G$, $G = (z^2-1, z+xy^3, 6)$, S is the portion of $x = 6-4y^2-4z^2$ in front of $x = -2$ with orientation in negative x-direction
  - For extra computation practice, prove without Stokes'

Oops! All practice problems

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following functions and curves

- F = (-yz, 4y+1, xy), C is circle of radius 3 centered at (0, 4, 0), perpendicular to y-axis, and oriented CW when looking above y>4
- F = ($3yx^2+z^3$, y^2, $4yx^2$), C is triangle with vertices (0, 0, 3), (0, 2, 0), (4, 0, 0) oriented CCW when looking above C towards origin

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for these with & without divergence theorem

- F = (sin(πx), zy^3, z^2+4x), S is surface of box with $-1 < x < 2$, $0 < y < 1$, $1 < z < 4$ oriented pointing out of the box
- F = (2xz, $1-4xy^2$, $2z-z^2$), S is surface of solid bounded by $z = 6-2x^2-2y^2$ and z = 0, oriented pointing inside the solid

Scratchwork