1st order of business: review past weeks

Next: actual worksheet, run as usual

Past DWs, lecs, HWs:

Selected for this discussion: 1a, 2, 4a, 4b

Student suggestions:

1a: Show $\nabla \times fG = \nabla f \times G + f(\nabla \times G)$

$G = (A, B, C) \Rightarrow \nabla \times fG = (\partial_x, \partial_y, \partial_z) \times (fA, fB, fC)$

$= (\partial_y(fC) - \partial_z(fB), \ldots, \ldots)$

$= (f_y C + f C_y - f_z B - f B_z, \ldots)$

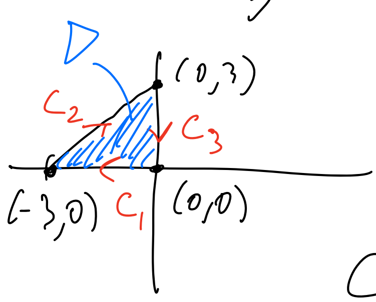
$= (f_y C - f_z B, \ldots) + (f C_y - f B_z, \ldots) =$

$= ((\partial_y f) C - (\partial_z f) B, \ldots) + f(C_y - B_z, A_z - C_x, B_x - A_y)$

$= (\partial_x f, \partial_y f, \partial_z f) \times (A, B, C) + f(\nabla \times G)$

$= \nabla f \times G + f(\nabla \times G)$.

2: Let C be triangle with vertices $(-3,0), (0,0), (0,3)$ oriented clockwise. Verify Green Theorem by computing both sides of the theorem separately for $\int_C (xy^2 + x)\,dx + (4x - 1)\,dy$.



Split up C into 3 curves for 3 sides C_1, C_2, C_3.

$C_1 = (-3t, 0), \quad 0 \le t \le 1$

$C_3 = (0, 3-3t), \quad 0 \le t \le 1$

Using formula for parametrizing line segments, we have $(-3,0) + ((0,3) - (-3,0))t$

$= (-3,0) + (3,3)t = (-3+3t, 3t), \ 0 \le t \le 1$

for parametrization of C_2.

Let $P = xy^2 + x$, $Q = 4x - 1$. Need to verify that $-\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dx\,dy$,

minus since we are going clockwise.

$Q_x - P_y = (4x-1)_x - (xy^2+x)_y = \underline{4 - 2xy}$.

$\int_{C_1} (\ldots) = \int_{C_1} P\,dx$ since y constant on C_1.

On C_1, $P = xy^2 + x = (-3t)\cdot 0^2 + (-3t) = -3t$,

so $\int_{C_1} P\,dx = \int_0^1 -3t\,dt = \underline{-\frac{3}{2}}$.

On C $Q = 4x - 1 = 4 \cdot 0 - 1 = -1$,

so $\int_{C_3} (P\,dx + Q\,dy) = \int_{C_3} Q\,dy = \int_0^1 -1\,dt = \underline{-1}$.

On C_2, $(-3+3t, 3t)$, $0 \le t \le 1$, so $Q = 4x - 1$

$= 4(-3+3t) - 1 = 12t - 13$, $P = x(y^2+1) =$

$(-3+3t)(9t^2+1) = -27t^2 + 27t^3 - 3 + 3t$,

so $P\,dx + Q\,dy = (27t^3 - 27t^2 + 15t - 16)\,dt$,

so $\int_{C_2} P\,dx + Q\,dy = \int_0^1 (27t^3 - 27t^2 + 15t - 16)\,dt$

$= \frac{27}{4} - 9 + \frac{15}{2} - 16 = \frac{57}{4} - 25 = -\frac{43}{4}$.

So $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = -\frac{3}{2} - \frac{43}{4} - 1 = -\frac{53}{4}$,

so $-\int_C = 53/4$.

Find bounds on D: $0 \le y \le 3$, $-3 \le x \le \left(\begin{smallmatrix}\text{line}\\ C_2\end{smallmatrix}\right)$.

C_2: $(-3,0) \to (0,3)$, so slope $= \frac{3-0}{0-(-3)} = 1$,

so line through C_2 is $y - 0 = 1(x-3) \Rightarrow$

$y = x - 3 \Rightarrow x = 3 + y$, so $-3 \le x \le 3+y$

$$\iint_D (Q_x - P_y)\,dx\,dy = \int_0^3 \int_{-3}^{3+y} (4 - 2xy)\,dx\,dy =$$

$$\int_0^3 \left(4(6+y) - y\left((3+y)^2 - (-3)^2\right)\right) dy =$$

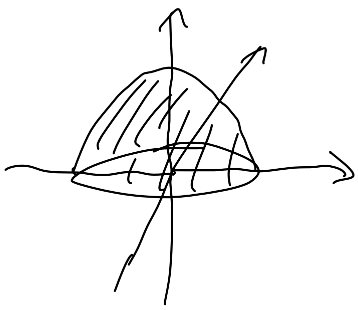
$$\int_0^3 \left[24 + 4y - y\left(y^2 + 6y\right)\right] dy = \int_0^3 (-y^3 - 6y^2 + 4y + 24)\,dy$$

$$= -\frac{3^4}{4} - \frac{6\cdot 3^3}{3} + 2\cdot 9 + 24\cdot 3 = -\frac{81}{4} - 54 + 18 + 72$$

$= -\frac{81}{4} + 36 = \frac{144-81}{4} = \frac{53}{4}$.

So both $-\int_C$ & $\iint_D$ are $53/4$ & the theorem has been verified.

4a: find $\iint_S f\,dS$ for $f = 6xy$, S upper half of sphere of radius 1.



Because it's upper half, $z \geq 0$

sphere is $x^2+y^2+z^2=1 \implies$

$z = \sqrt{1-x^2-y^2}$, $0 \leq x^2+y^2 \leq 1$

Now use spherical coordinates:

$x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$,

$0 \leq \varphi \leq \frac{\pi}{2}$ since only upper half,

$0 \leq \theta < 2\pi$ as usual. Now notice that

$\rho = 1$ exactly since only points on sphere are part of S, not points inside.

$x = \sin\varphi \cos\theta$, $y = \sin\varphi \sin\theta$, $z = \cos\varphi$, 2 bounds

$z = \sqrt{1-x^2-y^2} = \sqrt{1- \ldots} = \cos\varphi$

was not necessary, but you can replug in $(x,y,z) = (\ldots, \ldots, \ldots)$ parameters like this to check for mistakes in more complicated problems.

From our algebra ~~& coordinate~~ ~~conversion~~,

$r(u,v) = (u, v, \underbrace{\sqrt{1-u^2-v^2}}_{g(u,v)})$, so

$$\|r_u \times r_v\| = \sqrt{g_u^2 + g_v^2 + 1} = \sqrt{\left(\frac{-2u}{2\sqrt{1-u^2-v^2}}\right)^2 + \cdots}$$

$$= \sqrt{\frac{u^2}{1-u^2-v^2} + \frac{v^2}{1-u^2-v^2} + 1} = \sqrt{\frac{u^2+v^2+1-u^2-v^2}{1-u^2-v^2}}$$

$$= \frac{1}{\sqrt{1-u^2-v^2}}.$$

Now we can plug in the bounds & use all of the work on coordinate conversion.

Note! If you did conversion before finding $dS = \|r_u \times r_v\| \, du \, dv$, then you would spend more time computing cross product since r wouldn't be in the form $(u, v, g(u,v))$ to apply cross product formula.

Put $u = \sin\varphi\cos\theta$, $v = \sin\varphi\sin\theta$, notice how no w & no ρ (as expected, since there are only 2 variables in a surface integral).

Note: if you use spherical, cylindrical, or Jacobian to setup a surface integral, you may have 3+ variables initially. All but 2 of the variables should disappear by the time you calculate r & $\|d_u \times d_v\|$, otherwise you have made a mistake.

$$dS = \frac{1}{\sqrt{1-u^2-v^2}}\, du\, dv = \frac{\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}{\sqrt{1 - s^2\varphi\, c^2\theta - s^2\varphi\, s^2\theta}}$$

$$= \frac{\sin\varphi\, d\varphi\, d\theta}{\sqrt{1 - s^2\varphi}} = \tan\varphi\, d\varphi\, d\theta.$$

since $\rho = 1$ & ρ doesn't change on S, $\rho^2 d\rho = 1^2 = 1$

$$f = 6xy = 6\sin^2\varphi \cos\theta \sin\theta,$$

so $$f\, dS = 6\, \frac{\sin^3\varphi}{\cos\varphi} \cdot \frac{1}{2} \sin 2\theta\, d\varphi\, d\theta.$$

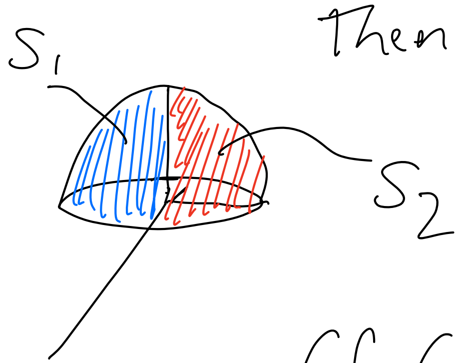
$$0 \le \theta < 2\pi, \quad 0 \le \varphi \le \frac{\pi}{2}.$$

$$\iint_S f\, dS = \int_0^{2\pi}\int_0^{\pi/2} 3\, \frac{\sin^3\varphi}{\cos\varphi} \sin 2\theta\, d\varphi\, d\theta =$$

$$3\left(\underline{\int_0^{2\pi} \sin 2\theta\, d\theta}\right)\left(\underline{\int_0^{\pi/2} \frac{\sin^3\varphi}{\cos\varphi}\, d\varphi}\right) = 3 \cdot A \cdot B$$

$$A = -\frac{1}{2}\cos 2\theta \Big|_0^{2\pi} = 0, \quad \text{so } 3AB = 0,$$

so $\iint_S f dS$. Note that we can also use the symmetry of $6xy$. Let S_1 be left half of S_1, S_2 be right half.



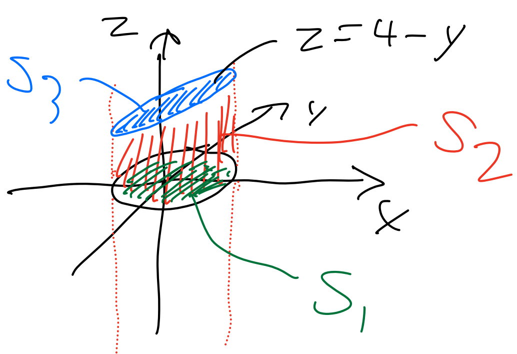
Then $\iint_S 6xy = \iint_{S_1} 6xy + \iint_{S_2} 6xy$

$= \iint_{S_1} 6xy - \iint_{S_2} (-6xy) =$

$\iint_{S_1} 6xy - \iint_{S_1} 6xy = 0$ since we can reflect a point (x,y) in S_2 to S_1 and $-6xy$ will become $-6(-x)y = 6xy$.

Note: we are relying on the fact that $f(x,y) = 6xy$ has some symmetry, specifically that $f(-x,y) = -6xy = -f(x,y)$. Don't assume this for every f.

4b: $f = y+z$, S is surface with sides given by $x^2+y^2=3$, bottom $x^2+y^2 \le 3$, top $z=4-y$



$\iint_S f = \iint_{S_1} f + \iint_{S_2} f + \iint_{S_3} f$

For S_1, note that $z=0$, so

$\iint_{S_1} f\,dS = \iint_{S_1} y\,dS = 0$ by reflection symmetry with y.

For S_3, $x = r\cos\theta$, $y = r\sin\theta$, $0 \le r \le \sqrt{3}$, $0 \le \theta \le 2\pi$

$z = 4 - y = 4 - r\sin\theta$

$r(u,v) = (u, v, 4-v)$, $\|r_u \times r_v\| =$

$\sqrt{(4-v)_u^2 + (4-v)_v^2 + 1} = \sqrt{0^2 + (-1)^2 + 1} =$

$\sqrt{2}$, so $f = y + z = 4$, so $f\,dS = 4\sqrt{2}\,du\,dv$

so $\iint_{S_3} f\,dS = \iint_{\text{disk}} 4\sqrt{2}\,du\,dv = 4\sqrt{2} \cdot$

(area of circle w/ radius $\sqrt{3}$) $= 4\sqrt{2} \cdot \pi\sqrt{3}^2$

$= 12\pi\sqrt{2}$.

For S_2, x & y are on the perimeter of disk,

so $x = \sqrt{3}\cos\theta$, $y = \sqrt{3}\sin\theta$, and for z,

$0 \le z \le 4 - y = 4 - \sqrt{3}\sin\theta$, so

our variables are z & θ with bounds for z

and bound $0 \le \theta < 2\pi$ for θ.

$r(\theta, z) = (\sqrt{3}c\theta, \sqrt{3}s\theta, z)$.

$r_\theta \times r_z = (-\sqrt{3}s\theta, \sqrt{3}c\theta, 0) \times (0, 0, 1) =$

$$\begin{matrix} i & j & k \\ -s & c & 0 \\ 0 & 0 & 1 \end{matrix}$$

$\sqrt{3}(\cos\theta, -\sin\theta, 0) \Longrightarrow$

$\|r_\theta \times r_z\| = \sqrt{3}$.

$f = y + z = \sqrt{3}\sin\theta + z$, $dS = \sqrt{3}\,d\theta\,dz$

$$\iint_{S_2} f\,dS = \int_0^{2\pi}\int_0^{4-\sqrt{3}\sin\theta} (\sqrt{3}\sin\theta + z)\,dz\,d\theta$$

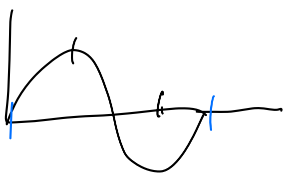
$$= \int_0^{2\pi}\left[\sqrt{3}\,s\,(4-\sqrt{3}\,s) + \frac{1}{2}(4-\sqrt{3}\,s)^2\right]d\theta$$

$$= \int_0^{2\pi}\left(\cancel{4\sqrt{3}\sin\theta} - 3s^2 + \frac{3}{2}s^2 - \cancel{4\sqrt{3}\,s} + 8\right) =$$

$$\int_0^{2\pi}\left(8 - \frac{3}{2}\sin^2\theta\right)d\theta = 8\cdot 2\pi - \frac{3}{2}\cdot\pi =$$

$$16\pi - 1.5\pi = 14.5\pi.$$

$$\boxed{\int_0^{2\pi}\sin^2\theta\,d\theta = \int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta = \pi}$$



So $\iint_S f\,dS = 0 + 12\sqrt{2}\,\pi + 14.5\pi$

$$= \left(12\sqrt{2} + \frac{29}{2}\right)\pi.$$