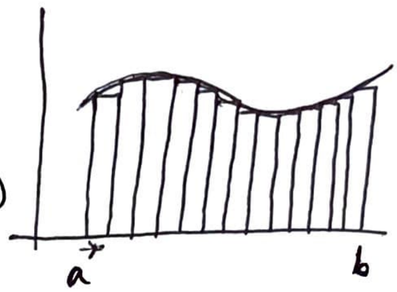
Section 15.1: Double Integrals Over Rectangles.

Recall: $\displaystyle\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum f(x_i^*)\,\Delta x$

$\displaystyle = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$ (where $[a,b]$ is divided into n intervals.....)

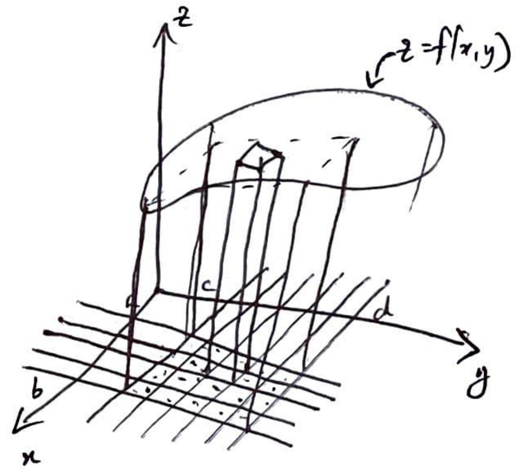


Volumes and Double integrals:

Let $R = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\}$

$= [a,b] \times [c,d]$.

and $S = \{(x,y,z) \in \mathbb{R}^3 : 0 \le z \le f(x,y)\}$



What is the volume of S?

Answer: $\displaystyle\int_c^d \int_a^b f(x,y)\,dx\,dy$

Definition: $\displaystyle\int_c^d \int_a^b f(x,y)\,dx\,dy = \lim_{\Delta A \to 0} \sum \underline{f(x_i^*, y_i^*)}\,\Delta A$

where ΔA is the size of a small square in R, and x_i^*, y_i^* are points in each square.

Example 4: Evaluate a) $\displaystyle\int_0^3 \int_1^2 x^2 y\,dy\,dx$ b) $\displaystyle\int_1^2 \int_0^3 x^2 y\,dx\,dy$.

A: a) $\displaystyle\int_0^3 \int_1^2 x^2 y\,dy\,dx = \int_0^3 \Big[\underline{\int_1^2 x^2 y\,dy}\Big]\,dx$

$\displaystyle\int_1^2 x^2 y\,dy = \frac{x^2 y^2}{2}\Big|_{y=1}^{2} = x^2\left(2 - \frac{1}{2}\right) = \frac{3x^2}{2}$

$$\Rightarrow \int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \frac{3x^2}{2} dx = \frac{x^3}{2}\Big|_0^3 = \frac{1}{2}(27-0) = \boxed{\frac{27}{2}}$$

b) $$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \frac{27y}{3} dy = \frac{27}{3}\int_1^2 y \, dy = \frac{27}{3}\frac{y^2}{2}\Big|_{y=1}^2 = \frac{27}{3}\cdot\frac{1}{2}(4-1)$$

$$\int_0^3 x^2 y \, dx = \frac{x^3 y}{3}\Big|_{x=0}^3 = \frac{y}{3}(27-0) = \frac{27y}{3} \qquad = \boxed{\frac{27}{2}}$$

Fubini's theorem: If f is CTS on $[a,b]\times[c,d]$ then

$$\int_a^b \int_c^d f(x,y)\, dy \, dx = \int_c^d \int_a^b f(x,y)\, dx \, dy.$$

Example 6: $\iint_R y \sin(xy)\, dx \, dy$ where $R = [1,2]\times[0,\pi]$.

$$\rightarrow \int_0^\pi \int_1^2 y \sin(xy)\, dx\, dy = \int_0^\pi -\cancel{y}\frac{\cos(xy)}{\cancel{y}}\Big|_{x=1}^2 dy$$

$$= \int_0^\pi -[\cos 2y - \cos y]\, dy$$

$$= -\int_0^\pi \cos 2y \, dy + \int_0^\pi \cos y \, dy$$

$$\stackrel{\text{DIY}}{=} 0.$$

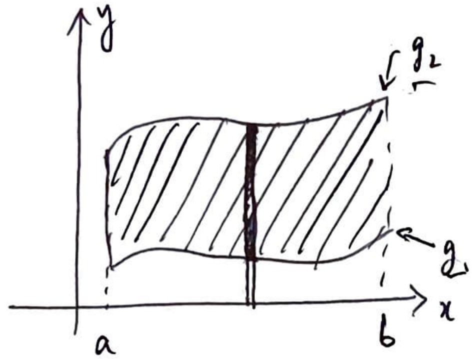
Remark: $\iint_R f(x,y)\, dx dy$ represents volume when $f(x,y) \geq 0$

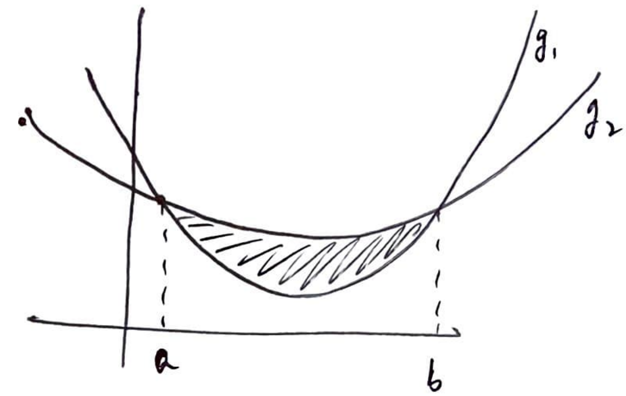
but not really when f can take negative values.

Section 15.2: Double Integrals over General Regions:

A plane region $D \subseteq \mathbb{R}^2$ is said to be of <u>type I</u> if it lies between the graphs of two CTS functions of x:

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$



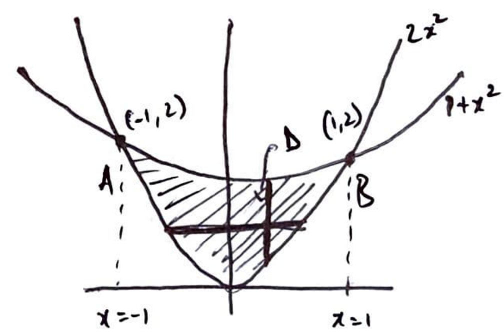


$$\iint_D \underbrace{f(x,y)}_{\text{(CTS on } D)}\, dx\, dy = \int_a^b \int_{g_1(x)}^{g_2(x)} \underbrace{f(x,y)}_{\text{height}}\, \underbrace{dy}_{\text{length}}\, \underbrace{dx}_{\text{width}}$$

Example:

Evaluate $\iint_D (x+2y)\, dA$ where D is the region between $y = 2x^2$ and $y = 1+x^2$.

$$D = \{(x,y) : -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\}$$



$$\iint_D (x+2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} xy + y^2 \Big|_{y=2x^2}^{1+x^2} dx$$

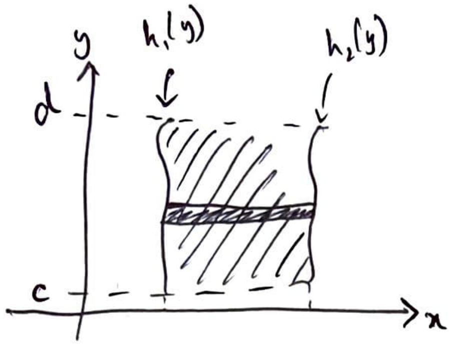
Solve $2x^2 = 1+x^2$ to find A, B.

$\Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$

$$= \int_{-1}^{1} x(1+x^2-2x^2) + (1+x^2)^2 - (2x^2)^2\, dx$$

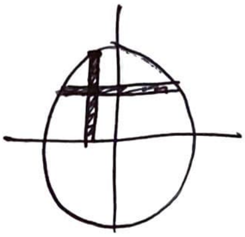
$$\overset{\text{DIY}}{=} \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1\, dx \overset{\text{DIY}}{=} \boxed{\frac{32}{15}}$$

type II regions: $D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

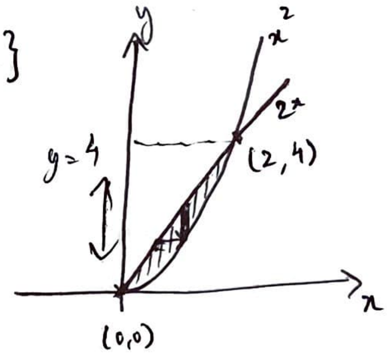


$$\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$

Remark:



Example 2: find the volume of the solid under the paraboloid $z = \overbrace{x^2+y^2}^{f(x,y)}$ and above the region $D = \{(x,y) : 0 \le x \le 2,\ x^2 \le y \le 2x\}$



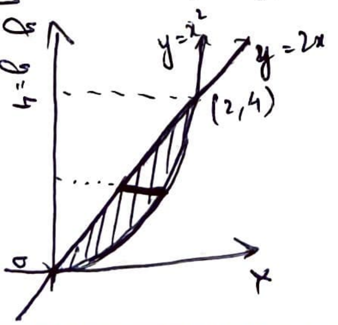
Solution 1:

$$\int_0^2 \int_{x^2}^{2x} x^2+y^2 \, dy\, dx$$

$$= \int_0^2 x^2 y + \frac{y^3}{3} \Big|_{y=x^2}^{y=2x} dx = \int_0^2 x^2 (2x) + \frac{(2x)^3}{3} - x^2(x^2) - \frac{(x^2)^3}{3}\, dx$$

$$\overset{DIY}{=} \int_0^2 \frac{-x^6}{3} - x^4 + \frac{14x^3}{3}\, dx \overset{DIY}{=} \frac{216}{35}.$$

Solution 2: $D = \{(x,y) : 0 \le y \le 4,\ \frac{y}{2} \le x \le \sqrt{y}\}$



$y = x^2 \Rightarrow x = \pm\sqrt{y}$

$y = 2x \Rightarrow \boxed{x = \frac{y}{2}}$

$$V = \int_0^4 \int_{y/2}^{\sqrt{y}} x^2 + y^2 \, dx \, dy = \int_0^4 \frac{x^3}{3} + y^2 x \Big|_{x=y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 \frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \, dy$$

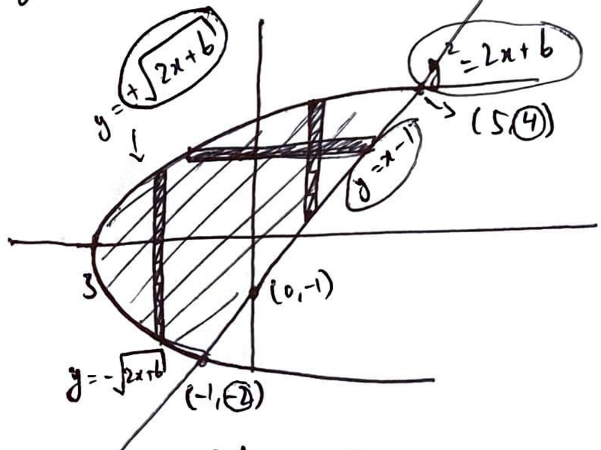
$$\overset{DIY}{=} \boxed{\frac{216}{35}}$$

<u>Example 3:</u> Evaluate $\iint_D \underbrace{xy}_{f(x,y)=z} \, dA$ where

D is the region bounded by the lines $y = x-1$ and the parabola

$\underbrace{y^2 = 2x+6}$.

$\frac{1}{2}y^2 - 3 = x$



$D = \{ (x,y) : -2 \le y \le 4,$
$\frac{1}{2}y^2 - 3 \le x \le y+1 \}$

Solve: $y^2 = 2x+6 = 2(y+1)+6$
DIY
$\Leftrightarrow y^2 - 2y - 8 = 0$
$\Leftrightarrow (y-4)(y+2) = 0$

$$V = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy \, dx \, dy$$

$$= \int_{-2}^{4} \frac{x^2 y}{2} \Big|_{x=\frac{1}{2}y^2-3}^{y+1} dy$$

$$= \int_{-2}^{4} \frac{y}{2} \left((y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2 \right) dy \overset{DIY}{=} \frac{1}{2} \left[-\frac{y^5}{24} + y^4 + \frac{2y^3}{3} - 4y^2 \right]_{-2}^{4} = \boxed{36}$$