MATH230 Week 14 Worksheet - Markov Chains

1: Let this matrix be A. By definition, a Markov chain is regular if A^n has all positive entries for some n. You can calculate A^2 to see that n = 2 works. You can also use the equivalent definition that a Markov chain is regular if you can reach any state from any other state. Since column 3 has positive entries, you can reach 1 or 2 from 3. From positive entries in columns 1 and 2, you can reach 3 from 1 and 3 from 2.

By definition, a steady state vector v's entries sum to 1 and it satisfies Av = v. This becomes Bv = 0 for B = A-I, which you can solve by row reducing B. We get Bw = 0 for w = (3, 8, 8). The sum of entries of w is 19, so divide by 19 to find the steady-state vector v = (3/19, 8/19, 8/19).

2: To figure this out, we use the fact that the (i,j)th entry of A^n is the probability of being at state i given you started at state j and n transitions have happened. Note that for n = 1, this is just the definition of what happens after one step for the Markov chain that A represents.

Once you're in state 2 you're stuck, so the middle column is (0, 1, 0) for any n. We now need to find $\lim_n A^n$ as this is the steady-state matrix of A by definition. Since A is absorbing, you will eventually reach state 2 starting from any other state, at which point you stay there. This means the (2,1) and (2,3) entries of A^n tend to 1 as n -> ∞.

What about the other entries? The fact that powers of a stochastic matrix are still stochastic comes in clutch. In particular, the sum of column i of A^n is 1 for any choice of i and n. Considering i = 1, we get $(A^n)_{(1,1)}+(A^n)_{(3,1)} = 1 - (A^n)_{(2,1)}$ -> 0. The entries of A^n are positive, so $(A^n)_{(1,1)}$, $(A^n)_{(3,1)}$ -> 0. Similarly, the (1,3), (3,3) entries vanish.

Putting everything together, the steady state matrix has 1st, 2nd, 3rd row (0,0,0), (1,1,1), (0,0,0). While this may seem daunting compared to row reduction and inverse computations, for larger matrices with few absorbing states, this method will pay off once you understand it.

3: Since the rat starts at A, it's always there after 0 seconds, which means $p_0 = 1$. Consider what happens after n+1 seconds. For the rat to be at A, it either was at A previously and chose to stay, or it was at B previously and chose to switch. The probability of being at A, B after n seconds is p_n, $1-p_n$ respectively. Thus, the 1st possibility has probability p_n*0.6 and the 2nd has probability $(1-p_n)$*0.4. We obtain the recursion $p_{n+1} = 0.6p_n + 0.4(1-p_n) = 0.4 + 0.2p_n$.

But how do we solve this? From the perspective of movement probabilities, A and B have no difference between them. So let's treat them equally and set $p_n = q_n + ½$. Perhaps this extra ½ term will do something. Indeed, after making this substitution, we get $q_{n+1} = 0.2q_n$. Since $q_0 = p_0 - 0.5 = 0.5$, we obtain $q_n = 0.2q_{n-1} = 0.2*0.2q_{n-2} = … = 0.2^n q_0 = 0.5*0.2^n$ and $p_n = (1+0.2^n)/2$.

Since $0.2^n \to 0$, we get $\lim_n p_n = (1+0)/2 = ½$. You can also find the limit by setting up a Markovka chain for the rat and finding the steady-state matrix, but this won't tell you p_n itself.

4: The claim is false. Consider the Markov chain M on states 1,2,3,4 where from state k you always move to state 5-k. There are 2 loops 1<->4 and 2<->3, so M has no absorbing state. Furthermore, M is not regular because you can't reach state 1 from state 3.

5: The matrix is 4x3, so player 1's choices correspond to rows and player 2's choices correspond to columns. Once P1 chooses a row, P2 will pick the smallest entry in it. Why? Since the game is zero-sum, P2's payoff is the opposite of P1's payoff. If P1 gains x, then P2 loses x. So in order for P2 to maximize their score, they want to minimize P2's score.

The minimum entries in rows 1, 2, 3, 4 are -1, 1, -1, -1 respectively. Whatever row P1 chooses, they will receive that respective score. Hence they should choose row 2 and receive +1.