14 Partial Derivatives

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14.4 Tangent Planes and Linear Approximations

Tangent Planes

Example 1

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

$z = f(x, y)$

Solution:

$f_x(x, y) = 4x, \quad f_y(x, y) = 2y$

Let $f(x, y) = 2x^2 + y^2$.

Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at (1, 1, 3) as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

Tangent Planes (4 of 8)

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$\textbf{1} \quad z - z_0 = a(x - x_0) + b(y - y_0)$$

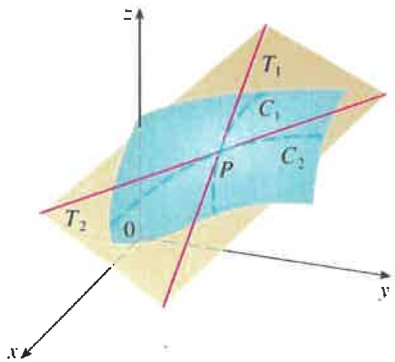
If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \qquad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a.

Tangent Planes (2 of 8)

Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)



The tangent plane contains the tangent lines T_1 and T_2.

Figure 1

Tangent Planes (3 of 8)

If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane.

Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Tangent Planes (1 of 8)

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

Tangent Planes (5 of 8)

But we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$. Therefore $a = f_x(x_0, y_0)$.

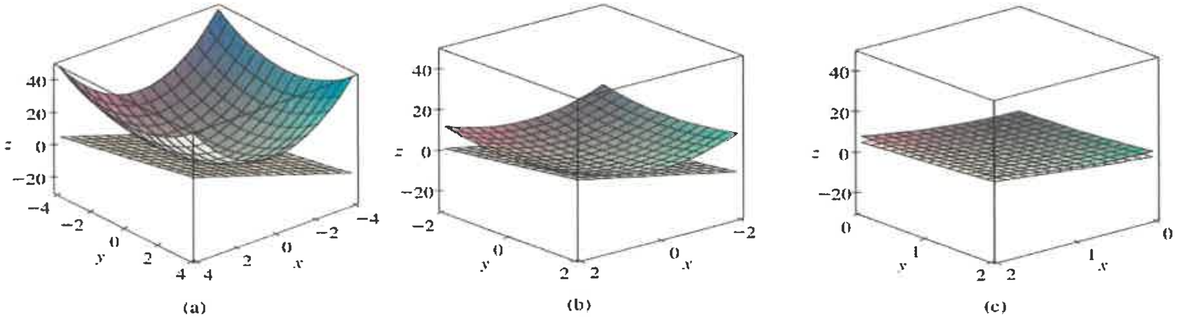
Similarly, putting $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2, so $b = f_y(x_0, y_0)$.

2 Equation of a Tangent Plane Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$\underline{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Tangent Planes (6 of 8)

Figure 2(a) shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1. In parts (b) and (c) we zoom in toward the point (1, 1, 3).



The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

Figure 2

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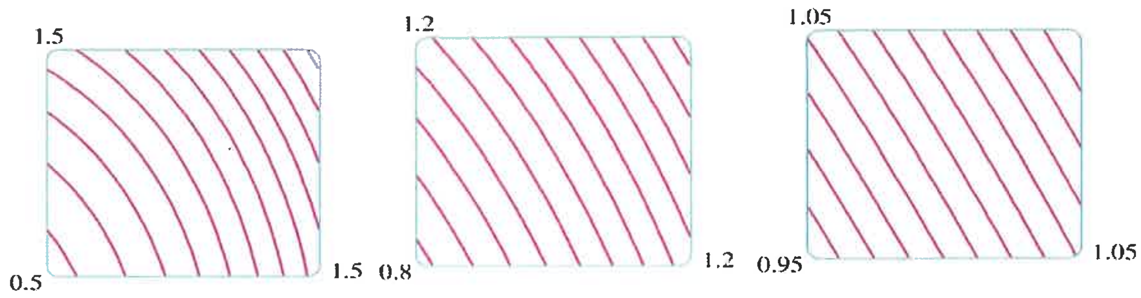
Tangent Planes (7 of 8)

Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane. In Figure 3 we corroborate this impression by zooming in toward the point (1, 1) on a contour map of the function $f(x, y) = 2x^2 + y^2$.



Zooming in toward (1, 1) on a contour map of $f(x, y) = 2x^2 + y^2$

Figure 3

Tangent Planes (8 of 8)

Notice that the more we zoom in, the more the level curves look like equally spaced parallel lines, which is characteristic of a plane.

Example

$$z = x\,sin((x + y)\pi), (1,1,0)$$

$$z = f(x,y) \Rightarrow \quad z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(x_0, y_0, z_0)

$$f_x(x,y) = \sin((x+y)\pi) + x\cos((x+y)\pi)\pi$$

$$f_x(1,1) = 0 + 1\cos(2\pi)\pi = \pi$$

$$f_y(x,y) = x\cos((x+y)\pi)\pi$$

$$f_y(1,1) = \pi$$

$$z = \pi(x-1) + \pi(y-1)$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x\sin((x+y)\pi)\right)$$

$$= \frac{\partial}{\partial x}\left(x\sin(\pi x + \pi y)\right)$$

$$= \sin(\pi x + \pi y) + x\cos(\pi x + \pi y)\pi$$

Linear Approximations

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Linear Approximations (1 of 9)

In Example 1 we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is $z = 4x + 2y - 3$.

$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$

Therefore, the linear function of two variables

$L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near (1, 1). The function L is called the *linearization* of f at (1, 1) and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at (1, 1).

Linear Approximations (2 of 9)

For instance, at the point (1.1, 0.95) the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = \underline{3.3}$$

which is quite close to the true value of $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = \underline{3.3225}$.

But if we take a point farther away from (1, 1), such as (2, 3), we no longer get a good approximation.

In fact, $L(2, 3) = 11$ whereas $f(2, 3) = 17$.

Linear Approximations (3 of 9)

In general, we know from (2) that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$$\textbf{3} \quad L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b).

Linear Approximations (4 of 9)

The approximation

$$\textbf{4} \quad f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

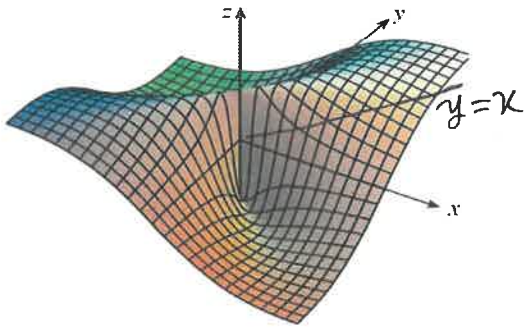
$$\frac{\partial}{\partial x} f(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

Linear Approximations (5 of 9)

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

You can verify that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous.



$f(x,y) = \dfrac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$,

$f(0,0) = 0$

Figure 4

$$L(x,y) = \underline{f_x(0,0)}(x-0) + \underline{f_y(0,0)}(y-0) + 0 = 0$$

Linear Approximations (6 of 9)

The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$.

So a function of two variables can behave badly even though both of its partial derivatives exist. To rule out such behavior, we formulate the idea of a differentiable function of two variables.

We know that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximations (7 of 9)

If f is differentiable at a, then

$$\mathbf{5} \quad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

$$\mathbf{6} \quad \Delta z = f(a + \Delta x,\ b + \Delta y) - f(a,\ b)$$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Linear Approximations (8 of 9)

By analogy with (5) we define the differentiability of a function of two variables as follows.

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a,b)\ \Delta x + f_y(a,b)\ \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy such that ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation (4) is a good approximation when (x, y) is near (a, b). In other words, the tangent plane approximates the graph of f well near the point of tangency.

Linear Approximations (9 of 9)

It's sometimes hard to use Definition 7 directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 2

Show that $f(x, y) = xe^{xy}$ is differentiable at (1, 0) and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution:

The partial derivatives are

$$f_x(x,y) = e^{xy} + xye^{xy} \qquad f_y(x,y) = x^2e^{xy}$$

$$f_x(1,0) = 1 \qquad f_y(1,0) = 1$$

Both f_x *and* f_y are continuous functions, so f is differentiable by Theorem 8. The linearization is

$$\begin{aligned} L(x,\ y) &= f(1,\ 0) + f_x(1,\ 0)(x-1) + f_y(1,\ 0)(y-0) \\ &= 1 + 1(x-1) + 1 \cdot y = x + y \end{aligned}$$

Example 2 – Solution

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

so $$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of

$$\begin{aligned} f(1.1-0.1) &= 1.1e^{-0.11} \\ &\approx 0.98542. \end{aligned}$$

Example

Show that $f(x, y) = 1 + \ln(xy - 5)$ is differentiable at $(2,3)$.

$$f_x(x,y) = 0 + \frac{y}{xy-5} = \frac{y}{xy-5}$$

$$f_x(2,3) = \frac{3}{2\times 3-5} = \frac{3}{6-5} = 3$$

$$f_y(x,y) = \frac{x}{xy-5}$$

$$f_x(2,3) = \frac{2}{2\times 3-5} = 2$$

Example

Show that $f(x,y) = 1 + \ln(xy - 5)$ is differentiable at $(2,3)$.

Differentials

Differentials (1 of 5)

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number.

The differential of y is then defined as

$$dy = f'(x)\,dx \tag{9}$$

Differentials (2 of 5)

Figure 6 shows the relationship between the increment Δy and the differential dy: Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

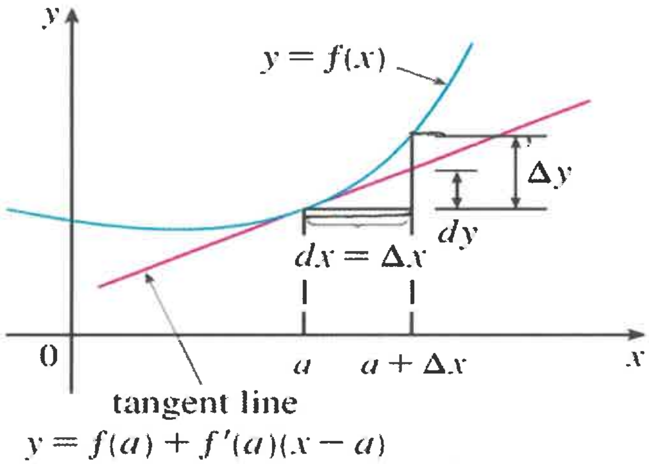


Figure 6

$f(a + \Delta x) - f(x) = \Delta y$

$L(a + \Delta x) - f(x) = dy$

Differentials (3 of 5)

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

$$\textbf{10} \qquad dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Sometimes the notation df is used in place of dz.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Differentials (4 of 5)

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation (4) can be written as

$$f(x, y) \approx f(a, b) + dz$$

Differentials (5 of 5)

Figure 7 is the three-dimensional counterpart of Figure 6 and shows the geometric interpretation of the differential *dz* and the increment Δ*z*: *dz* represents the change in height of the tangent plane, whereas Δ*z* represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

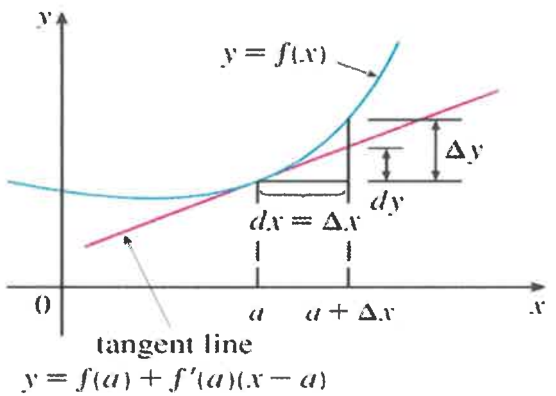


Figure 6

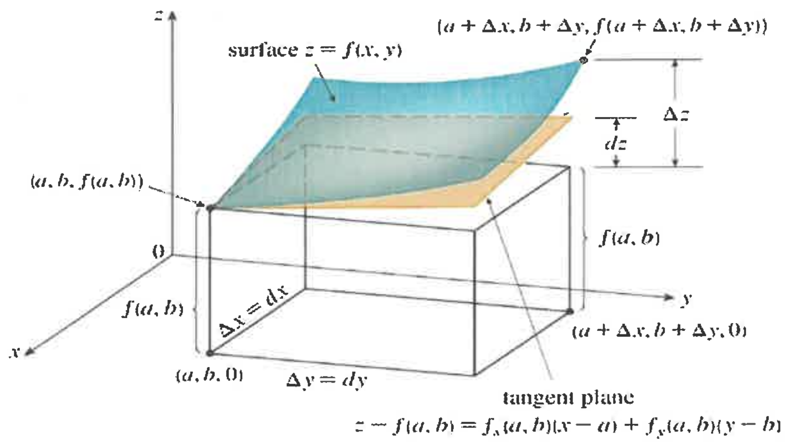


Figure 7

Example 4 – Solution

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = \underline{0.65}$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05,\ 2.96) - f(2,3) \\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 - 3(2)(3) - 3^2] \\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

Example 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Solution:

(a) Definition 10 gives

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= (2x + 3y)\, dx + (3x - 2y)\, dy$$

Functions of Three or More Variables

Functions of Three or More Variables (1 of 2)

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 7. For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

Functions of Three or More Variables (2 of 2)

If $w = f(x, y, z)$, then the **increment** of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Example 6

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within ε cm.

(a) Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

(b) What is the estimated maximum error in the calculated volume if the measured dimensions are correct to within 0.2 cm.

Solution:

(a) If the dimensions of the box are x, y, and z, its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\ dx + xz\ dy + xy\ dz$$

Example 6 – Solution (1 of 2)

We are given that $|\Delta x| \le \varepsilon$, $|\Delta y| \le \varepsilon$, and $|\Delta z| \le \varepsilon$.

To estimate the largest error in the volume, we therefore use $dx = \varepsilon$, $dy = \varepsilon$, and $dz = \varepsilon$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\begin{aligned} \Delta V \approx dV &= (60)(40)\varepsilon + (75)(40)\varepsilon + (75)(60)\varepsilon \\ &= 9900\varepsilon \end{aligned}$$

Thus the maximum error in the calculated volume is about 9900 times larger than the error in each measurement taken.

Example 6 – Solution (2 of 2)

(b) If the largest error in each measurement is ε = 0.2 cm, then

$dV = 9900(0.2) = 1980$, so an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 cm^3 in the calculated volume. (This may seem like a large error, but you can verify that it's only about 1% of the volume of the box.)