Jacobian and Vector Fields

Lecture for 7/1

Jacobian General Idea

- Suppose $x = f(u, v)$, $y = g(u, v)$
- Can we make the substitution and convert $dx\ dy$?
- Yes, provided the substitution is invertible
- Cylindrical, spherical, polar become special cases of Jacobian

Jacobian

Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in R^n and each x_i depends on $t_1, \ldots, t_m$

- Define J to be n x m matrix with (i, j) entry $\partial x_i / \partial t_j$
- Suppose n = m and $M(\mathbf{t}) := \det(J(\mathbf{t})) \neq 0$
- Then $d\mathbf{x} = dx_1 \ldots dx_n = M(\mathbf{t})\, dt_1 \ldots dt_n$
    - Thus, $\int_A f(x_1, \ldots)\, d\mathbf{x} = \int_A f(x_1(t_1, \ldots), \ldots)\, M(\mathbf{t})\, d\mathbf{t}$

Jacobian Derivation

Vector Fields

A vector field on S assigns a vector to each point in S

- Any f: $R^n \to R^n$ can be considered a vector field for R^n
- We've already seen the field $\nabla f = \langle f_x, f_y, f_z \rangle$
- Recall ∇f is perpendicular to the graph of any cross section $f = c$

Is there anything the gradient can't do?

- Call $\mathbf{F}$ conservative if $\mathbf{F} = \nabla f$ for some function f

Line Integrals

We've integrated over intervals, rectangles, prisms, and general solids. What if we stretch an interval inside higher dimensions?

- Let $\mathbf{r}(t)$ with $a \leq t \leq b$ parametrize a curve C
- Can define $\int_C f(\mathbf{x})\, ds$ to be integral of f along C
- The previous integral expands to $\int_a^b f(\mathbf{r}(t))\, \|\mathbf{r}'(t)\|\, dt$
- Note: Value of integral depends on orientation of C

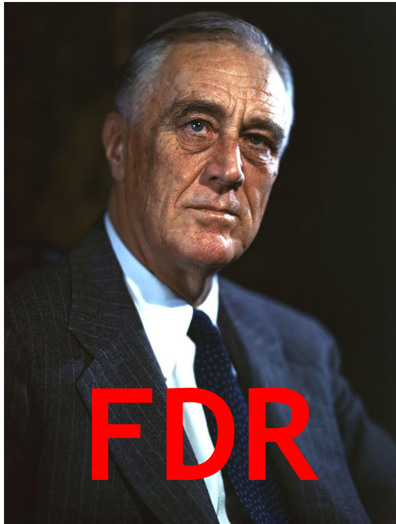
More Line Integrals

What if we only care about x or y when traveling along the curve?

- Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$ parametrize C
- $\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$
- Similarly for y, similarly for more variables

What if we want to mix line integrals and vector fields?

- Consider $\int_C (\mathbf{F} \cdot d\mathbf{r}) = \int_C (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t))\, dt$
- We have $\int_C (\mathbf{F} \cdot d\mathbf{r}) = \int_C (\mathbf{F} \cdot \mathbf{T})\, ds$



Practice Problems

Evaluate $\int_C f\, ds$ for the following functions and curves:

- $f(x, y) = 3x^2 - 2y$, C is line segment from (3, 6) to (1,-1)
- $f(x, y) = 6x$, C is portion of $y = x^2$ from $x = -1$ to $x = 2$
- $f(x, y) = 16y^5$, C is $x = y^4$ from $y = 0$ to $y = 1$, followed by segment from (1,1) to (1, -2), followed by segment from (1, -2) to (2,0)

Evaluate $\int_C (x^2\, dy - yz\, dz)$ where C is segment from (4, -1, 2) to (1, 7, -1)

Scratchwork