DW6 on the site, only leftover problems from slides for now

Selected problems: ~~1d~~, 1f, 2, 4, 1e

Student suggestions:

Any questions from DW4,5:

~~1d: Find $\iiint_E f\,dV$ for $f = z$, E is region inside $y^2+z^2 = 1$, and between $x+y+z=2$, $x=0$~~

already done yesterday

1e: Find $\iiint_E f\,dV$ for $f = x^2+y^2$, E is portion

$x^2+y^2+z^2 = 4$, $y \geq 0$

E is a semisphere, so use spherical coordinates.

$x = \rho \sin\varphi \cos\theta$
$z = \rho \sin\varphi \sin\theta$
$y = \rho \cos\varphi$

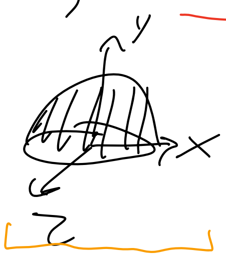
these 2 can be swapped

Since $y \geq 0$, $\cos\varphi \geq 0 \Rightarrow$

$0 \leq \varphi \leq \frac{\pi}{2}$

$4 = x^2+y^2+z^2$ sphere $\Rightarrow x^2+y^2+z^2 \leq 4$

$\Rightarrow 0 \leq \rho \leq 2$, $0 \leq \theta < 2\pi$



Now we plug in for everything and solve the integral.

$f = x^2+y^2 = \rho^2(\cos^2\varphi + \sin^2\varphi\cos^2\theta).$

$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi.$

$\int_0^2 \rho^4 = \frac{\rho^5}{5}\Big|_0^2 = 2^5/5$

$$\iiint_E f\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^4 \sin\varphi(c^2\varphi + s^2\varphi\, c^2\theta)\,d\rho\,d\theta\,d\varphi$$

$$= \frac{2^5}{5}\int_0^{\pi/2}\int_0^{2\pi}(\sin\varphi\, c^2\varphi + s^3\varphi\, c^2\theta)\,d\theta\,d\varphi$$

$$= \frac{32}{5}\int_0^{\pi/2}\left(2\pi s\varphi\, c^2\varphi + s^3\varphi\left(\int_0^{2\pi} c^2\theta\,d\theta\right)\right)d\varphi$$

$$\int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\frac{\cos 2\theta + 1}{2}\,d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta + 0 = \pi.$$

$$= \frac{32}{5}\pi\int_0^{\pi/2}(\underbrace{2s\varphi c^2\varphi + s^3\varphi}_{1}) = \frac{32\pi}{5}\int_0^{\pi/2}(s\varphi + s\varphi\, c^2\varphi)\,d\varphi$$

$= s\varphi(2c^2 + s^2) = s(1+c^2)$

$= s + sc^2$

$$= \frac{32\pi}{5}\left(-\cos\varphi - \frac{1}{3}\cos^3\varphi\right)\Big|_0^{\pi/2} = \frac{32\pi}{5}\cdot\frac{4}{3} = \frac{128\pi}{15}.$$

1f. $\iiint_E f\,dV$, $f = x^2$, $E =$ inside $x^2+y^2+z^2 = 36$ and $z = -\sqrt{3x^2+3y^2}$

$x = \rho\sin\varphi\cos\theta$
$y = \rho\sin\varphi\sin\theta$
$z = \rho\cos\varphi$

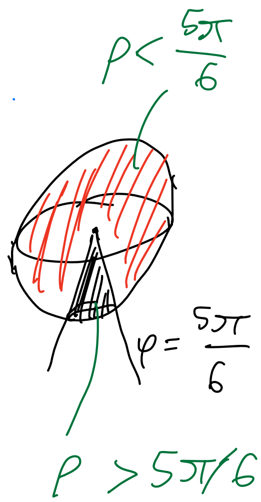
Inside sphere $\Rightarrow 36 \geq x^2+y^2+z^2 = \rho$

$\Rightarrow 0 \leq \rho \leq 6$

$\rho\cos\varphi = -\sqrt{3\rho^2\sin^2\varphi} = -\rho\sin\varphi\sqrt{3}$

$$\Rightarrow \cos\varphi = (-\sin\varphi)\sqrt{3} \Rightarrow -\frac{1}{\sqrt{3}} = \tan\varphi$$

$$\tan\varphi = -\frac{1}{\sqrt{3}} = \tan(150^\circ) = \tan\left(\frac{5\pi}{6}\right).$$



$$\Rightarrow \varphi = \frac{5\pi}{6} \Rightarrow \varphi \geq \frac{5\pi}{6}, \text{ so}$$

$$\underline{0 \leq \theta < 2\pi}, \quad \underline{\frac{5\pi}{6} \leq \varphi \leq \pi}$$

Bounds found, now plug in

$$f = x^2 = \rho^2 s^2\varphi\, c^2\theta, \quad dV = \rho^2 s\varphi\, d\rho\, d\theta\, d\varphi$$

$$\int_E f\,dV = \int_{5\pi/6}^{\pi}\int_0^{2\pi}\int_0^{6} \rho^4 c^2\theta\, s^3\varphi\, d\rho\, d\theta\, d\varphi =$$

$$\underbrace{\left(\int_{5\pi/6}^{\pi} s^3\varphi\, d\varphi\right)}_{?}\underbrace{\left(\int_0^{2\pi} c^2\theta\, d\theta\right)}_{\pi \text{ from 1st problem}}\underbrace{\left(\int_0^{6}\rho^4\, d\rho\right)}_{\frac{\rho^5}{5}\Big|_0^6 = \frac{6^5}{5}} =$$

$$\int \sin^3 x\,dx = \int s(1-c^2) \underset{u=\cos x}{=} \int (u^2-1)\,du = \frac{u^3}{3} - u$$

$$= \frac{\cos^3 x}{3} - \cos x$$

$$? = \left(\frac{\cos^3\varphi}{3} - \cos\varphi\right)\Bigg|_{5\pi/6}^{\pi} = \left(-\frac{1}{3}+1\right) - \left(\frac{-3\sqrt{3}/8}{3} + \frac{\sqrt{3}}{2}\right)$$

$$\cos 150^\circ = \sin(90 - 150^\circ) = \sin(-60^\circ) = -s60 = -\sqrt{3}/2$$

$$= \frac{2}{3} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{2} = \frac{2}{3} - \frac{3\sqrt{3}}{8}$$

So finally, $\int_E f\,dV = \left(\frac{2}{3} - \frac{3\sqrt{3}}{8}\right)\pi\left(\frac{6^5}{5}\right) =$

$$\frac{16-9\sqrt{3}}{\cancel{6\cdot 4}} \cdot \frac{6\cdot \cancel{24}\cdot 54}{5}\pi = \underbrace{\frac{324}{5}\pi\left(16-9\sqrt{3}\right)}$$

I will avoid answers like these for quiz or exam problems on triple integrals

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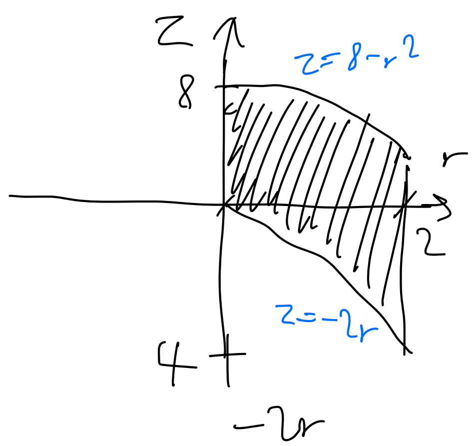
2. Volume of solid bound by $z = 8 - x^2 - y^2$, $x^2+y^2=4$, and $z = -2\sqrt{x^2+y^2}$.

x^2+y^2 every where, so use cylindrical.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$z = 8 - x^2 - y^2 \longrightarrow z = 8 - r^2$$
$$z = -2\sqrt{x^2+y^2} \longrightarrow z = -2r$$
?
$$x^2+y^2 = 4 \longrightarrow r^2 = 4 \to 0 \le r \le 2$$

$0 \le \theta < 2\pi$, $-2r \le z \le 8 - r^2$



$dV = r\,dr\,d\theta\,dz$, volume $=$

$$\int_0^{2\pi}\int_0^{2}\int_{-2r}^{8-r^2} r\,dz\,dr\,d\theta =$$

$$2\pi \int_0^2 r\left((8-r^2) - (-2r)\right) dr = 2\pi \int_0^2 \left(8 - r^2 + 2r\right) dr$$

$$= 2\pi\left(8r - \frac{r^3}{3} + r^2\right)\Big|_0^2 = 2\pi\left(16 - \frac{8}{3} + 4\right) = 2\pi\,\frac{52}{3} =$$

$$104\pi/3$$

4. $\int_C (x^2 dy - yz\,dz)$ where C seg $(4,-1,2) \to (1,7,-1)$.

$r(t) = (4,-1,2) + [(1,7,-1)-(4,-1,2)]t =$

$(4,-1,2) + (-3,8,-3)t = (4-3t, -1+8t, 2-3t),\ 0 \le t \le 1$

No ds to compute, so just plug in for everything.

Note: you may ask, is the following shortcut allowed? $\int_C (x^2 dy - yz\,dz) = \left[x^2 y - y\frac{z^2}{2}\right]\Big|_{C\text{ start}}^{C\text{ end}}$

$= (x^2 y - 0.5yz^2)\Big|_{(4,-1,2)}^{(1,7,-1)} = \dots = \text{ans}.$

$x = 4-3t$ $\quad x^2 dy = (4-3t)^2 8dt = 8(9t^2-24t+16)dt$

$y = -1+8t$ $\quad -yz\,dz = (8t-1)(3t-2)\cdot -3dt =$

$z = 2-3t$ $\quad -3(24t^2-19t+2)dt$

$dy = 8dt,\ dz = -3dt$

so $x^2 dy - yz\,dz = [(\cancel{72t^2} - 192t + 128) - (\cancel{72t^2} + 57t - 6)]dt$

$= (-249t + 122)dt$, so

$\text{ans} = \int_0^1 (-249t+122)\,dt = -\frac{249}{2} + 122 = -\frac{5}{2}.$

$C = r(t),\ 0 \le t \le 1,$ $\quad r(0) = (x_0, y_0, z_0)$, $r(1) = (x_1, y_1, z_1)$

$\int dy = y\Big|_{\text{start}}^{\text{end}} = y_1 - y_0$ using "shortcut"

$\int dy = \int_0^1 y'(t) = y(1) - y(0) = y_1 - y_0$ proper way

$$\int xy\,dy = \frac{1}{2}x\,y^2\Big|_{\text{start}}^{\text{end}} \quad \text{using shortcut}$$

$$\int xy\,dy = \int_0^1 x(t)\,y(t)\,y'(t)\,dt = \int_0^1 x(t)\left[\frac{y^2}{2}\right]' =$$

$$\frac{xy^2}{2}\Big|_{\text{start}}^{\text{end}} - \underbrace{\int_0^1 x'(t)\frac{y^2}{2}\,dt}_{\text{leftover term}}$$

Ans to note: Shortcut fails because x & z can't be treated as constants when finding $\int_C f(x,y,z)\,dy$ for example.