Conservative Vector Fields

Lecture for 7/2

General Idea

Recall Fundamental Theorem of Calculus: $\int_a^b f'(x)\, dx = f(b) - f(a)$
- Do we have a similar theorem for line integrals?
- Use FTOC by trying to express $f(r(t))$ as a derivative
- Will end up having gradient in place of derivative

How do we tell if a vector field is conservative?
- Will explain and derive method

Fundamental Theorem of Line Integrals

Consider some path C parametrized by r(t), $a \leq t \leq b$

- Recall $\int_C (F \cdot dr) = \int_C (F \cdot dr/dt) \, dt = \int_a^b (F(r(t)) \cdot r'(t)) \, dt$
- Now suppose F is conservative and $F = \nabla f$
- Then $\int_C (F \cdot dr) = f(r(b)) - f(r(a))$

Fundamental Theorem Derivation

Extra Space

Conservation Efforts

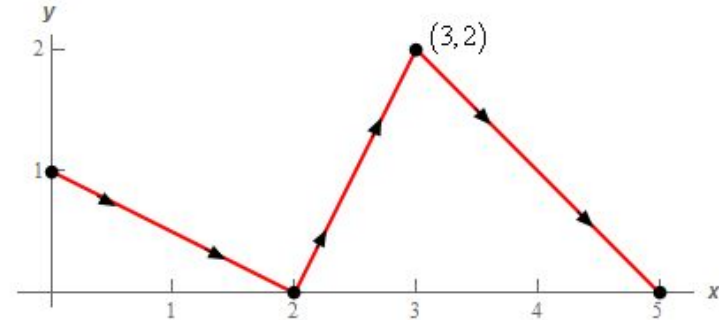Suppose we have 2 variables and $F = \nabla f = (f_x, f_y)$
- Then $(F_1)_y = (f_x)_y = f_{xy} = f_{yx} = (F_2)_x$
- So if $(F_1)_y \neq (F_2)_x$, F is not conservative
- If they are equal, then $f = \int f_x \, dx = \int F_1 \, dx = G(x,y) + h(y)$
  - Evaluate integral to get G, then find function of integration h

Be careful if f is not defined or not differentiable everywhere

Practice Problems

Evaluate $\int_C (F \cdot dr)$ for the following functions and curves
- $F = \nabla f$ with $f = ye^{(x^2 - 1)} + 4xy^{1/2}$, $r(t) = (1\text{-}t, 2t^2\text{-}2t)$, $0 \leq t \leq 2$
- $F = \nabla f$ with $f(x,y,z) = x+\sin(y+\cos(x+e^{x\text{-}y})+\ln(3+\sec(y)+11x^{1/2}))$ and C the ellipse $(x\text{-}4)^2/711+(y\text{-}3)^2/1215 = 1$ oriented clockwise
- $F = (2z^4\text{-}2y\text{-}y^3, z\text{-}2x\text{-}3xy^2, 6+y+8xz^3)$, C the curve shown below

Scratchwork