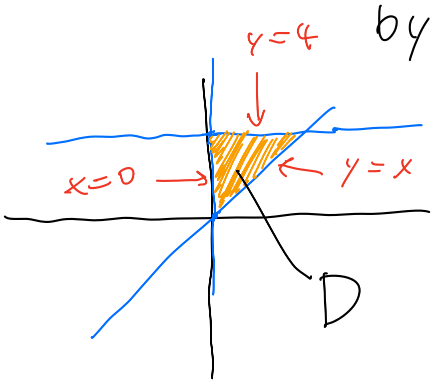
Discussion Attendance entered later today

DW 7-10 now have challenge problems marked

DW 10 Q2: Evaluate $\iint_D y^2 e^{xy}\,dA$, D is region bounded by $y=x$, $y=4$, $x=0$



As seen in graph, $0 \le y \le 4$ for R. Also, $0 \le x \le y$. Plug these bounds in:

$$\int_0^4 \left(\int_0^y y^2 e^{xy}\,dx \right) dy = \int_0^4 y^2 \left. \frac{e^{xy}}{y} \right|_{x=0}^{x=y} dy =$$

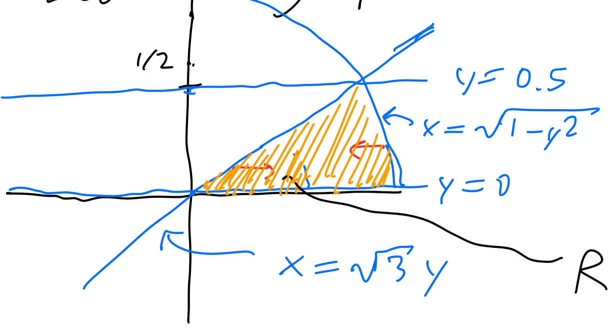
$$\int_0^4 y\left(e^{y^2} - e^0\right) dy = \int_0^4 \left(ye^{y^2} - y\right) dy = \left. \frac{1}{2}e^{y^2} - \frac{y^2}{2} \right|_0^4 =$$

$$\frac{1}{2}e^{16} - 8 - \left(\frac{1}{2} - 0\right) = \frac{e^{16} - 17}{2}.$$

Note: If you choose to put x on the outside so that $0 \le x \le 4$ & $x \le y \le 4$ are your bounds, your inner integral will be $\int^4 y^2 e^{xy}\,dy$, which is harder to integrate & requires IBP. Be careful which order you choose; check which is simpler

Q6: Find $\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\,dx\,dy$ using polar.

Let's graph the bounds $0 \le y \le \frac{1}{2}$ & $\sqrt{3}y \le x \le \sqrt{1-y^2}$.



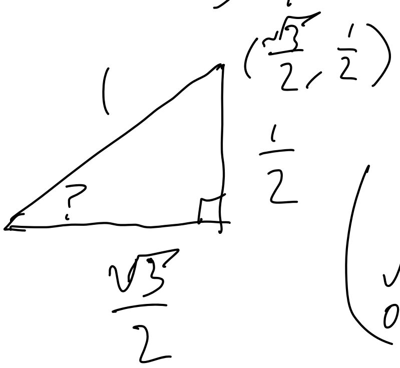
Let's use constant bounds for y & have dx on the inside since otherwise we'd have a piecewise upper bound in the inner integral, making the double integral split into 2 separate double integrals.

$y_{min}=0$, $y_{max}=$ green intersection point, where $x=\sqrt{1-y^2}$ & $x=\sqrt{3}\,y$ intersect. Solve $\sqrt{3}\,y=\sqrt{1-y^2} \Rightarrow$ $3y^2=1-y^2 \Rightarrow 4y^2=1 \Rightarrow y=\pm\frac{1}{2}$, but y is positive, so $y=\frac{1}{2}$.

Bounds for x already given. Now convert to polar:

$xy^2\,dx\,dy = r^3\cos\theta\sin^2\theta\,(r\,dr\,d\theta) = r^4\cos\theta\sin^2\theta\,dr\,d\theta$.

From graph of R, $0\le r\le 1$ & $0\le\theta\le ? = \frac{\pi}{6}$



Then $\iint_R \cdots = \int_0^{1/2}\int_0^{\pi/6} r^4\cos\theta\sin^2\theta\,d\theta\,dr =$

$\left(\int_0^1 r^4\,dr\right)\left(\int_0^{\pi/6}\cos\theta\sin^2\theta\,d\theta\right) =$

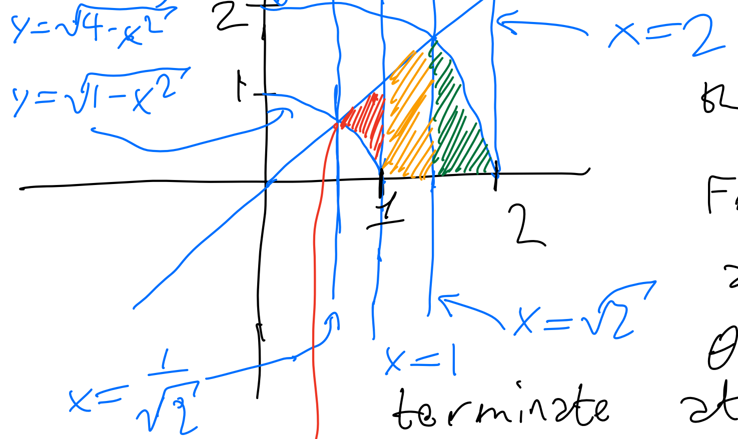
$\frac{r^5}{5}\Big|_0^1 \frac{\sin^3\theta}{3}\Big|_0^{\pi/6} = \frac{1}{5}\,\frac{(1/2)^3}{3} = \frac{1}{8\cdot5\cdot3} =$

$\frac{1}{120}$

Q7: Use polar coordinates to combine

$$\left(\underbrace{\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x}}_{R_1} + \underbrace{\int_{1}^{\sqrt{2}}\int_{0}^{x}}_{R_2} + \underbrace{\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}}}_{R_3}\right) xy\,dy\,dx$$

into one integral & evaluate.



Now that we've drawn everything, need to find r & θ bounds.

From graph, $1 \le r \le 2$ & we have a portion of circular sector. $\theta \ge 0$, and since $R_1 + R_2 + R_3$ terminate at $y=x$, $\theta \le 45^\circ = \pi/4$ since 45° is where $\cos\theta = \sin\theta$ or $y=x$.

Note! all 3 curves pass through one point here. However, do not assume that this happens when you are drawing a region & see 3 curves that appear to concur. You must double-check they really do concur.

$xy\,dy\,dx = r^2\cos\theta\sin\theta\,(r\,dr\,d\theta) = r^3\cos\theta\sin\theta\,dr\,d\theta$, so our integral is $\int_1^2\int_0^{\pi/4} r^3\cos\theta\sin\theta\,d\theta\,dr =$

$$\left(\int_1^2 r^3\,dr\right)\left(\int_0^{\pi/4}\frac{1}{2}\sin 2\theta\,d\theta\right) = \frac{r^4}{4}\Big|_1^2\cdot\left(-\frac{\cos 2\theta}{4}\right)\Big|_0^{\pi/4} =$$

$$\frac{15}{4}\cdot\frac{1}{4}\left(\cos 0-\cos\frac{\pi}{2}\right) = \frac{15}{4}\cdot\frac{1}{4} = \frac{15}{16}.$$

3b: switch order of integ. in $\int_0^1\int_0^y \cdots\,dx\,dy$

First, find original: $0 \le y \le 1$ & $0 \le x \le y$

Next, combine into 1 ineq: $0 \le x \le y \le 1$

Now break apart: $0 \le x \le 1$ & $x \le y \le 1$

Put back into int: $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$.

$0 \le y \le 1 \longrightarrow 0 \le ? = x \le y \le$ ~~?~~ ≤ 1

$0 \le x \le y \le 1$

$$\int_0^1 \left(\int_0^1 xy\,dx \right) dy = \int_0^1 \frac{1}{2} y\,dy = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$\left(\int_0^1 x\,dx \right)\left(\int_0^1 y\,dy \right) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$\int_a^b \int_c^d f(x)g(y)\,dx\,dy = \int_a^b \underbrace{\left(\cdots \right)}_{H(y)} dy$$

$$H(y) = \int_c^d f(x)g(y)\,dx = g(y)\int_c^d f(x)\,dx$$

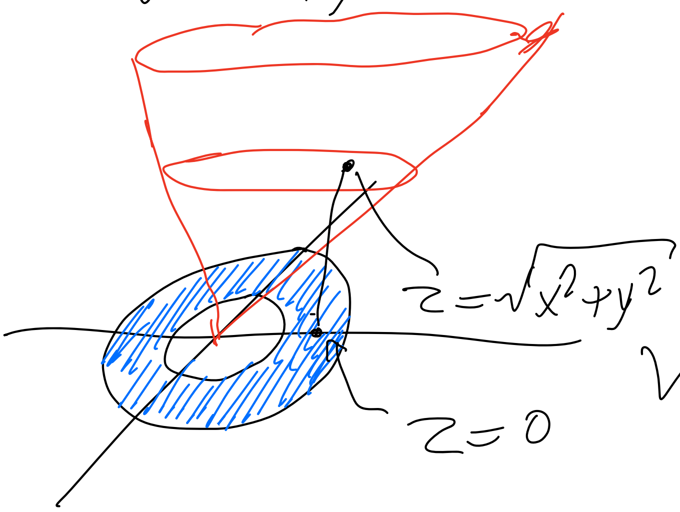
$= C g(y)$ where $C = \underbrace{\qquad\qquad\qquad}$

$$= \int_a^b C g(y)\,dy = C \int_a^b g(y)\,dy =$$

$$\left(\int_c^d f(x)\,dx\right)\left(\int_a^b g(y)\,dy\right).$$

---

5. Use polar to find volume under $z = \sqrt{x^2+y^2}$ above the ring $\underbrace{1 \le x^2+y^2 \le 4}_{\text{region } R}$



$$V = \iint_R \left(\sqrt{x^2+y^2} - 0\right) dA$$

Now let's find bounds for R in polar.

By symmetry, no restrictions on $\theta \Rightarrow$ $0 \le \theta \le 2\pi$. Also, $x^2+y^2 = r^2$, so $1 \le r^2 \le 4 \Rightarrow 1 \le r \le 2$.

$$\sqrt{x^2+y^2}\, dA = \sqrt{r^2}\, r\,dr\,d\theta = r^2\,dr\,d\theta$$

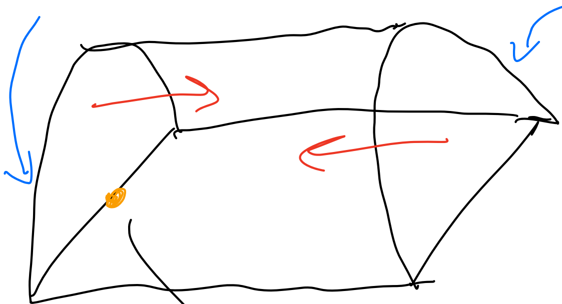
$$V=\int_1^2\int_0^{2\pi} r^2\,d\theta\,dr = \left(\int_1^2 r^2\,dr\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \frac{r^3}{3}\Big|_1^2\ \theta\Big|_0^{2\pi} = \frac{7}{3}\cdot 2\pi = \frac{14\pi}{3}$$

$$x^2+y = r^2\cos^2\theta + r\sin\theta = r(r\cos^2\theta+\sin\theta)$$

HW10 #15: (a) Express $\iiint_E f\,dV$ as an iterated integral for given f & solid E.

$z=1-x^2$



$y+z=2$

$(0,0,0)$ in region and $y+z=0+0=0\le 2$, so $y+z\le 2$ is correct bound for region.

$(0,0,0)$ is in the region and $z=0$, $1-x^2=1$ at this point. So $z\le 1-x^2$.

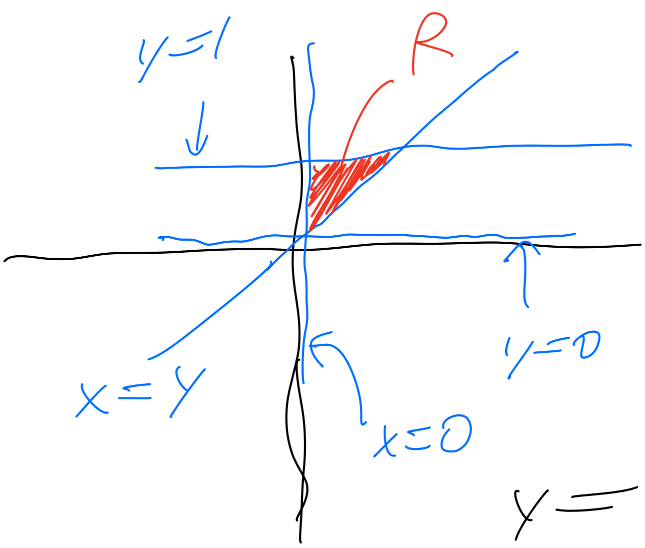
Note: This tactic works in general for figuring out integration bounds. Take a point in the region, plug into both sides of equation, and see which is larger. The larger one will have $\geq$.

The bounds for x are already forced to be $-1 \leq x \leq 1$ from the problem format, now continue with y & z. However, we already found those bounds. $z \leq 1 - x^2$ leave as is, and $y + z \leq 2$

$\Rightarrow \quad y \leq 2 - z$.

3a: Sketch region in $\int_0^1 \int_0^y f(x,y)\,dx\,dy$.

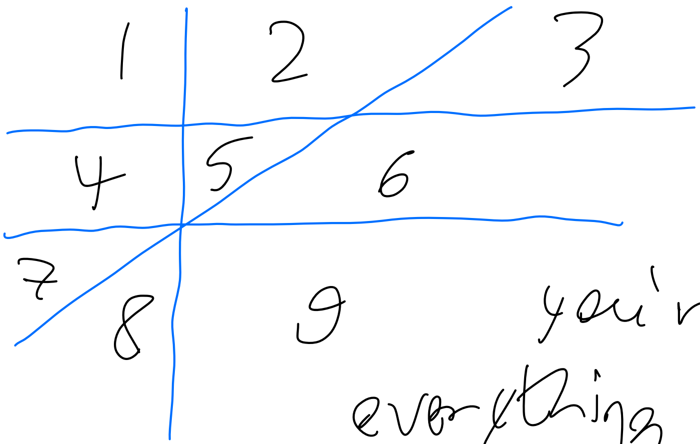
There are 3 steps: find the bounds, sketch curves by turning bounds into equations, and isolate the region from sketch.



Step 1: outer integral gives $0 \leq y \leq 1$, inner $0 \leq x \leq y$

Step 2: graph $y=0$, $y=1$, $x=0$, $x=y$

Step 3: only finite region is triangle



Note: If it is not clear upon an initial graph which region you're going after, redraw everything without labels, number the regions, and check each one to see whether it has finite area.

C: Find integral with $f=\sqrt{x}+3y$. Plug this into part 6 integral to get

$$\int_0^1\int_x^1 (\sqrt{x}+3y)\,dy\,dx =$$

$$\int_0^1 (1-x)\sqrt{x}+\frac{3}{2}(1^2-x^2)\,dx =$$

$$\int_0^1 \left(x^{1/2}-x^{3/2}+\frac{3}{2}-\frac{3}{2}x^2\right)dx =$$

$$\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}+\frac{3}{2}x-\frac{1}{2}x^3\Big|_0^1 =$$

$$\frac{2}{3}-\frac{2}{5}+\frac{3}{2}-\frac{1}{2}=1+\frac{4}{15}=\frac{19}{15}$$