

Topics: 12.2 Vectors; 12.3 The Dot Product; 12.4 The Cross Product

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1. Given $\mathbf{a} = 9\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = \langle 7, 0, -9 \rangle$, find the following. Simplify your answers completely.

(a) $\mathbf{a} + \mathbf{b}$

$$\mathbf{a} + \mathbf{b} = \langle 9, -8, 7 \rangle + \langle 7, 0, -9 \rangle = \langle 16, -8, -2 \rangle.$$

(b) $3\mathbf{a} - \mathbf{b}$

$$3\mathbf{a} - \mathbf{b} = 3\langle 9, -8, 7 \rangle - \langle 7, 0, -9 \rangle = \langle 27, -24, 21 \rangle - \langle 7, 0, -9 \rangle = \langle 20, -24, 30 \rangle.$$

(c) $|\mathbf{b}|$

$$|\mathbf{b}| = \sqrt{7^2 + 0^2 + (-9)^2} = \sqrt{49 + 81} = \sqrt{130}.$$

(d) $|\mathbf{b} - \mathbf{a}|$

$$|\mathbf{b} - \mathbf{a}| = |(-2, 8, -16)| = \sqrt{(-2)^2 + 8^2 + (-16)^2} = \sqrt{4 + 64 + 256} = \sqrt{324} = 18.$$

2. Find the vector that has the opposite direction as $\langle 9, -6, -2 \rangle$ and has length 5.

Since

$$|\langle 9, -6, -2 \rangle| = \sqrt{9^2 + (-6)^2 + (-2)^2} = \sqrt{121} = 11,$$

the unit vector in the opposite direction of $\langle 9, -6, -2 \rangle$ is

$$\mathbf{u} = -\frac{1}{11}\langle 9, -6, -2 \rangle = \left\langle -\frac{9}{11}, \frac{6}{11}, \frac{2}{11} \right\rangle.$$

The vector in the opposite direction and with length 5 is

$$5\mathbf{u} = 5 \left\langle -\frac{9}{11}, \frac{6}{11}, \frac{2}{11} \right\rangle = \left\langle -\frac{45}{11}, \frac{30}{11}, \frac{10}{11} \right\rangle.$$

3. Determine whether the given vectors are orthogonal, parallel, or neither.

(a) $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 12\mathbf{j} - 2\mathbf{k}$

Orthogonal: We have

$$\mathbf{a} \cdot \mathbf{b} = 4 \cdot 5 + (-1) \cdot 12 + 4 \cdot (-2) = 0,$$

so \mathbf{a} and \mathbf{b} are orthogonal.

- (b) $\mathbf{a} = \langle 6, 5, -2 \rangle$ and $\mathbf{b} = \langle 5, 0, 9 \rangle$

Neither: We have

$$\mathbf{a} \cdot \mathbf{b} = 6 \cdot 5 + 5 \cdot 0 + (-2) \cdot 9 \neq 0,$$

so the vectors are not orthogonal. Since \mathbf{a} is not a scalar multiple of \mathbf{b} , they are not parallel.

- (c) $\mathbf{a} = \langle -18, 15 \rangle$ and $\mathbf{b} = \langle 12, -10 \rangle$

Parallel: We see that $\mathbf{b} = -\frac{2}{3}\mathbf{a}$, so the vectors are parallel (with opposite direction).

4. Given vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} in V_3 , which of the following expressions are meaningful? Which are meaningless? Explain your reasoning.

- (a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$

The expression $\mathbf{a} \cdot \mathbf{b}$ is a scalar and \mathbf{c} is a vector. One cannot take the dot product of a scalar and a vector, so the expression is meaningless.

- (b) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}$

Both $\mathbf{a} + \mathbf{b}$ and \mathbf{c} are vectors, so the dot product $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}$ has meaning.

- (c) $|\mathbf{a}| \cdot |\mathbf{c}|$

Both $|\mathbf{a}|$ and $|\mathbf{c}|$ are scalars. If “.” is interpreted as the dot product of vectors in V_3 , the expression is meaningless. However, if it is interpreted as ordinary multiplication, then the expression has meaning. In the context of Math 243, the dot product interpretation is intended.

5. Find the angle θ between the vectors $\mathbf{a} = \langle 2, 0, -3 \rangle$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

We first compute the dot product:

$$\mathbf{a} \cdot \mathbf{b} = \langle 2, 0, -3 \rangle \cdot \langle -2, 3, 1 \rangle = 2(-2) + 0(3) + (-3)1 = -7.$$

Therefore

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{-7}{\sqrt{2^2 + 0^2 + (-3)^2} \sqrt{(-2)^2 + 3^2 + 1^2}} = \frac{-7}{\sqrt{13} \sqrt{14}}.$$

So, the answer is $\theta = \arccos\left(\frac{-7}{\sqrt{13} \sqrt{14}}\right)$.

6. Consider $\mathbf{a} = \langle -1, 4, 8 \rangle$ and $\mathbf{b} = \langle 18, 2, 1 \rangle$.

- (a) Find the scalar projection of \mathbf{b} onto \mathbf{a} .

Since $|\mathbf{a}| = \sqrt{1 + 16 + 64} = \sqrt{81} = 9$, the scalar projection of \mathbf{b} onto \mathbf{a} is

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-1) \cdot 18 + 4 \cdot 2 + 8 \cdot 1}{9} = -\frac{2}{9}.$$

- (b) Find the vector projection of \mathbf{b} onto \mathbf{a} .

The vector projection of \mathbf{b} onto \mathbf{a} is

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \text{comp}_{\mathbf{a}} \mathbf{b} \frac{\mathbf{a}}{|\mathbf{a}|} = -\frac{2}{9} \frac{\mathbf{a}}{|\mathbf{a}|} = -\frac{2}{9} \frac{\langle -1, 4, 8 \rangle}{9} = \left\langle \frac{2}{81}, -\frac{8}{81}, -\frac{16}{81} \right\rangle$$

7. Find the work (in joules) done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$ that moves an object from the point $(0, 6, 4)$ to the point $(4, 14, 22)$ along a straight line. The distance is measured in meters and the force in newtons.

The displacement vector is

$$\mathbf{D} = (4 - 0)\mathbf{i} + (14 - 6)\mathbf{j} + (22 - 4)\mathbf{k} = 4\mathbf{i} + 8\mathbf{j} + 18\mathbf{k}.$$

Since $W = \mathbf{F} \cdot \mathbf{D}$, the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = 8 \cdot 4 + (-6) \cdot 8 + 5 \cdot 18 = 32 - 48 + 90 = 74 \text{ joules.}$$

8. Compute the dot product and cross product for the following pairs of vectors:

- (a) $\mathbf{u} = \langle -1, 1, 2 \rangle$, $\mathbf{v} = \langle 4, 5, -2 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = \langle -1, 1, 2 \rangle \cdot \langle 4, 5, -2 \rangle = -4 + 5 - 4 = -3.$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 4 & 5 & -2 \end{vmatrix} = -12\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}.$$

- (b) $\mathbf{u} = \langle 1, -1, 3 \rangle$, $\mathbf{v} = \langle 2, -2, 6 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = \langle 1, -1, 3 \rangle \cdot \langle 2, -2, 6 \rangle = 2 + 2 + 18 = 22.$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ 2 & -2 & 6 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}.$$

- (c) $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle -3, 0, 1 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = \langle 1, 2, 3 \rangle \cdot \langle -3, 0, 1 \rangle = -3 + 0 + 3 = 0.$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -3 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}.$$

- (d) What is the significance of your answers to parts (b) and (c)?

For part (b), the cross product is the zero vector so the vectors are parallel. Indeed,

$$\langle 2, -2, 6 \rangle = 2\langle 1, -1, 3 \rangle.$$

For part (c), the dot product is zero, so the vectors are orthogonal.

9. Consider $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{j} + \mathbf{k}$.

- (a) Find the cross product $\mathbf{a} \times \mathbf{b}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 0 & -2 & 1 \end{vmatrix} = (1 - (6))\mathbf{i} - (2 - (0))\mathbf{j} + (-4 - (0))\mathbf{k} = -5\mathbf{i} - 2\mathbf{j} - 4\mathbf{k} = \langle -5, -2, -4 \rangle$$

- (b) Verify that $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{a} .

We calculate that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle -5, -2, -4 \rangle \cdot \langle 2, 1, -3 \rangle = -10 - 2 + 12 = 0.$$

Thus $(\mathbf{a} \times \mathbf{b})$ is orthogonal to \mathbf{a} .

- (c) Find two unit vectors orthogonal to both \mathbf{a} and \mathbf{b} .

The cross product of two vectors is orthogonal to both vectors. From part (a), $\mathbf{a} \times \mathbf{b} = \langle -5, -2, -4 \rangle$. Taking the magnitude,

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-5)^2 + (-2)^2 + (-4)^2} = \sqrt{45} = 3\sqrt{5}.$$

So two unit vectors orthogonal to both \mathbf{a} and \mathbf{b} are

$$\left\langle -\frac{5}{3\sqrt{5}}, -\frac{2}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}} \right\rangle \quad \text{and} \quad \left\langle \frac{5}{3\sqrt{5}}, \frac{2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}} \right\rangle$$

10. Consider points $P(1, 2, 1)$, $Q(2, 5, 4)$, $R(6, 9, 12)$, and $S(5, 6, 9)$ in \mathbb{R}^3 .

- (a) Find the area of the parallelogram with vertices P, Q, R , and S .

The parallelogram is determined by the vectors $\overrightarrow{PQ} = \langle 1, 3, 3 \rangle$ and $\overrightarrow{PS} = \langle 4, 4, 8 \rangle$. Their cross product is

$$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 3 \\ 4 & 4 & 8 \end{vmatrix} = (24 - 12)\mathbf{i} - (8 - 12)\mathbf{j} + (4 - 12)\mathbf{k} = 12\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}.$$

So, the area of parallelogram PQRS is

$$|\overrightarrow{PQ} \times \overrightarrow{PS}| = \sqrt{12^2 + 4^2 + (-8)^2} = 4\sqrt{14}.$$

- (b) Find the area of triangle PQS.

The area of the triangle PQS is half the area of the parallelogram PQRS. Using part (a), since the area of the parallelogram is $4\sqrt{14}$, the area of triangle PQS is therefore $2\sqrt{14}$.

- (c) Show that the vectors \overrightarrow{PQ} , \overrightarrow{PR} , and \overrightarrow{PS} are coplanar.

We have $\overrightarrow{PQ} = \langle 1, 3, 3 \rangle$, $\overrightarrow{PR} = \langle 5, 7, 11 \rangle$, and $\overrightarrow{PS} = \langle 4, 4, 8 \rangle$. Then,

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} 1 & 3 & 3 \\ 5 & 7 & 11 \\ 4 & 4 & 8 \end{vmatrix} = 1(56 - 44) - 3(40 - 44) + 3(20 - 28) = 12 + 12 - 24 = 0.$$

Since this scalar triple product is zero, the volume of the parallelepiped determined by the three vectors is zero. Therefore, the vectors are coplanar.

SOME USEFUL DEFINITIONS, THEOREMS, AND NOTATION:

Dot Product If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

If θ is the angle between \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta.$$

Scalar and Vector Projections If $\mathbf{a} \neq \mathbf{0}$, then the *scalar projection* of \mathbf{b} onto \mathbf{a} is

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$

The *vector projection* of \mathbf{b} onto \mathbf{a} is

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a}.$$

Work by a Constant Force If a constant force \mathbf{F} moves an object through displacement \mathbf{D} , then the work done is

$$W = \mathbf{F} \cdot \mathbf{D}.$$

Cross Product If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

If θ is the angle between \mathbf{a} and \mathbf{b} , then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta.$$

Suggested Textbook Problems

Section 12.2: 1-33, 37-44

Section 12.3: 1-56

Section 12.4: 1-22, 27-38, 43-46