

Hessian Test

Lecture for 6/23

Review of Matrices: Eigenvectors

Let A be a matrix and v a vector

- If $Av = \lambda v$ and $v \neq 0$, then v is an eigenvector of A with eigenvalue λ
- Can find eigenvalues by solving $\det(A - \lambda I) = 0$ for λ
 - After finding the values, solve $(A - \lambda I)v = 0$ to get vectors

Review of Matrices: Definiteness

- Matrices with $A^T = A$ are called symmetric
- Symmetric matrices with $v^T A v > 0$ for all $v \neq 0$ are positive definite
 - Negative definite if $v^T A v < 0$ for all $v \neq 0$
 - Semidefinite if $>$ and $<$ are replaced with \geq and \leq
- If all eigenvalues positive, then A is positive definite
- If all eigenvalues are $<, \geq, \leq 0$, then neg, pos semi, neg semi resp.

Hessian Test

Consider a twice differentiable function $f(x_1, x_2, \dots, x_n)$

- Let H be the $n \times n$ matrix whose (i, j) entry is $f_{x_i x_j}$
- H is the Hessian of f

Suppose \mathbf{x} is a critical point of f

- If $H(\mathbf{x})$ is positive definite, \mathbf{x} is a local min
- If $H(\mathbf{x})$ is negative definite, \mathbf{x} is a local max
- If $H(\mathbf{x})$ has negative and positive eigenvalues, \mathbf{x} is a saddle point
- If $H(\mathbf{x})$ is positive or negative semidefinite, no info

Derivation of Test

Other Methods

What happens if Hessian or 2nd Derivative Test inconclusive?

- Try approaching point along different directions
 - Same strategies as figuring out 2D limits
 - Will suggest which kind of point your critical point is
- If local extrema, prove your guess via inequalities

Practice Problems

Find and classify all critical points by any means necessary

- $f(x, y, z) = x^2 + y^2 + z^2 + xy + xz + yz + x + y + z + 1$
- $f(x, y) = x^4 - y^4 - 4xy^2 - 2x^2$
- $f(x, y) = x^{2024} + y^{2026}$



The ends justifies the means.

~ Niccolo Machiavelli

Scratchwork

