Recall: (16.2)

- Integration with respect to arc length.

$$\boxed{C : \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \ , \ a \le t \le b \Rightarrow \int_C f(x,y)\, ds = \int_{t=a}^{b} f(x(t),y(t))\, \|r'(t)\|\, dt}$$

- Important parametrisations:
  - circle: $\langle r\cos t, r\sin t\rangle$ (t is angle θ)
  - $y=f(x)$: $\langle t, f(t)\rangle$ $(a \le t \le b)$
  - segment $[\langle x_1,y_1\rangle, \langle x_2,y_2\rangle]$
    - $\langle tx_1 + (1-t)x_2, ty_1 + (1-t)y_2\rangle$
    $= t\langle x_1,y_1\rangle + (1-t)\langle x_2,y_2\rangle$ $(0 \le t \le 1)$

- Integration with respect to x or y: $(C: \vec{r}(t) = \langle x(t), y(t)\rangle, a \le t \le b)$

$$\boxed{- \int_C f(x,y)\, dx = \int_a^b f(x(t),y(t))\, x'(t)\, dt \ , \ \int_C f(x,y)\, dy = \int_a^b f(x(t),y(t))\, y'(t)\, dt}$$

- Notation : $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$.

- Remarks : • $\int_C f(x,y)\,dx$ and $\int_C f(x,y)\,dy$ depend on the orientation.

$$\int_C f(x,y)\,dx = -\int_{-C} f(x,y)\,dx \qquad \int_C f(x,y)\,dy = -\int_{-C} f(x,y)\,dy.$$

• $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$.

- Work done by a force field $\vec{F} = \langle P, Q, R \rangle$ along $C : \vec{r}(t) = \langle x, y, z \rangle$

$$\boxed{\begin{aligned} W &= \int_C F \cdot T\,ds \quad \text{where } T = \frac{r'(t)}{\|r'(t)\|}. \\ &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt. \end{aligned}}$$

- Notation: The integral $\int_C \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{T}\,ds$

· **Recall** : (16.3)

· A vector field $\vec{F} = \langle P, Q, R \rangle$ is conservative if there exists a scalar function $f(x,y,z)$ such that $\nabla f = \langle f_x, f_y, f_z \rangle = \vec{F}$.

Theorem: $\left.\begin{array}{l} C : r(t) \ , \ a \leq t \leq b \\ \nabla f \text{ is continuous on } C \end{array}\right\} \Rightarrow \int_C \nabla f \cdot dr = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

· Definition: A vector field $\vec{F}$ is independent of path if for any two curves C_1 and C_2 with the same initial and terminal points,

$$\int_{C_1} \vec{F} \cdot dr = \int_{C_2} \vec{F} \cdot dr$$

- Remarks: Conservative vector fields are independent of path since

$$\int_{C_1} \vec{F} \cdot dr = \int_{C_1} \nabla f \cdot dr = f(r(b)) - f(r(a)) = \int_{C_2} \nabla f \, dr = \int_{C_2} \vec{F} \cdot dr$$

where C_1 and C_2 have the same endpoints $\vec{r}(a)$ and $\vec{r}(b)$.

A curve is called **closed** if its terminal point coincides with its initial point, that is $\mathbf{r}(b) = \mathbf{r}(a)$.

Theorem 3 $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

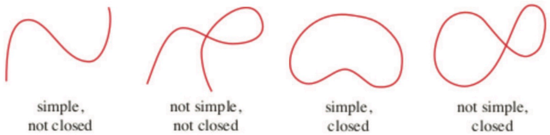
Theorem Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

- **Simple curve**: doesn't intersect itself



- D **simply connected**:
  (a) D is connected
  (b) Every simple closed curve in D encloses only points that belong to D.

Example: Determine whether the following vector fields are conservative or not.

(a) $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

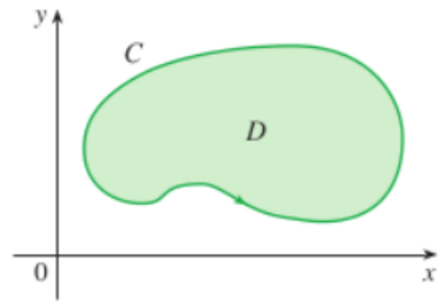
(b) $\mathbf{G}(x, y) = \langle 3x^2y + 2y - y^2, x^3 + 2x - 2yx + 9y^2 \rangle$

16.4 Green's Theorem

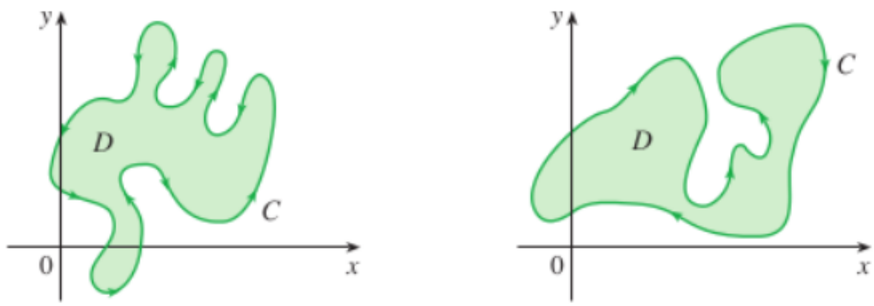
Green's Theorem gives the relationship between a line integral around a simple closed curve and a double integral over the plane region bounded by the curve.

1. **Green's Theorem**

   Let C be a simple closed curve and let D be the region bounded by C. (We assume that D consists of all points inside C as well as all points on C.)



In stating Green's Theorem, we use the convention that the **positive orientation** of a simple closed curve C refers to a **single counterclockwise traversal of** C.



(a) Positive orientation

(b) Negative orientation

> **Green's Theorem** Let C be a **positively oriented**, **piecewise-smooth**, **simple closed curve** in the plane and let D be the region bounded by C. If P **and** Q **have continuous partial derivatives** on an open region that contains D, then
>
> $$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Note:

- $\int_C P\,dx + Q\,dy = \int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle P, Q \rangle$.
- $\oint_C P\,dx + Q\,dy$ indicates positive orientation.
- Green's Theorem should be viewed as a kind of FTC for double integrals: integral involving derivatives, value of the original function (F, Q, and P) only on the boundary of the domain.
- ∂D: positively oriented boundary of D.

Example: Evaluate $\oint y\,dx + xy\,dy$ where C is the triangle with vertices $(0,0),(1,0)$ and $(1,2)$.

Without using Green's Theorem:

Using Green's Theorem

Remark:

- The double integral was easier to evaluate than the line integral;
- Sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction.
  For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C, then Green's Theorem gives
  $$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy = 0$$
  no matter what values P and Q assume in the region D.

2. **Finding Areas with Green's Theorem**

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint\limits_D 1\,dA$, we wish to choose P and Q such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities

$$P(x,y) = 0 \qquad P(x,y) = -y \qquad P(x,y) = -\frac{1}{2}y$$
$$Q(x,y) = x \qquad Q(x,y) = 0 \qquad Q(x,y) = \frac{1}{2}x$$

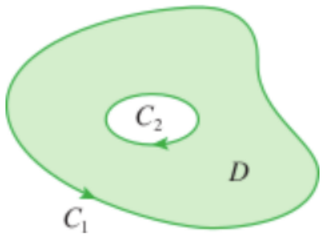
Then Green's Theorem gives the following formulas for the area of D:

$$\boxed{A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

Example: Use Green's Theorem to find the area of the circle $x^2 + y^2 = a^2$.

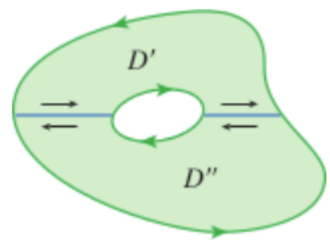
3. **Extended Versions of Green's Theorem**

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected.



Observe that the boundary C of the region D consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus, the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2.

If we divide D into two regions D' and D'' by means of the lines shown in the figure.



Then, apply Green's Theorem to each of D' and D'':

Example: Given $\mathbf{F}(x, y) = \dfrac{\langle -y, x\rangle}{x^2 + y^2}$. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

Note: $\int \mathbf{F} \cdot d\mathbf{r}$ is path dependent despite $P_y = Q_x$.