Challenge Problems in RED

MATH 243: Worksheet 11 **Discussion Section:** ____________

Textbook Sections: 15.6 and 15.7
Topics: Triple Integrals and Triple Integrals in Cylindrical Coordinates
Instructions: Try each of the following problems, show the detail of your work.
Clearly mark your choices in multiple choice items. Justify your answers.
Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

Triple Integrals over rectangular boxes:

1. Evaluate the integral $\iiint_E (xy+z^2)dV$ where $E = \{(x,y,z)|0 \le x \le 2, 0 \le z \le 3, 0 \le y \le 1\}$ using three different orders of integration.

Triple Integrals over general domains:

2. Evaluate $\displaystyle\iiint_E \frac{1}{x^3}dV$ where $E = \{(x,y,z)|0 \le y \le 1, 0 \le z \le y^2, 1 \le x \le z+1\}$.
3. Evaluate $\iiint_E x\,dV$ where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and $x = 4$.
4. Find the volume of the solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$.
5. Express the integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral in six different ways, where E is bounded by $y = 4 - x^2 - 4z^2$ and $y = 0$.

TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES:

6. What surface does the equation $z^2 + r^2 = 9$ describe in cylindrical coordinates?
7. Evaluate $\iiint_E z\,dV$ where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.
8. Find the volume of the solid that lies between the paraboloid $z = x^2+y^2$ and the sphere $x^2+y^2+z^2 = 2$.
9. Find the volume of the solid that lies between the cylinders $x^2+y^2 = 1$ and $x^2+y^2 = 4$, the planes $z = 0$ and $z = 4$, and in the first octant.
10. Evaluate the integral
$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$$
by changing to cylindrical coordinates.

Suggested Textbook Problems

Section 15.6	3-22, 27, 28, 31-38, 41
Section 15.7	1-13, 15-25a, 31,32

SOME USEFUL DEFINITIONS, THEOREMS AND NOTATION:

Fubini's Theorem for Triple Integrals
If f is continuous on the box $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)dV = \int_r^s\int_c^d\int_a^b f(x,y,z)dxdydz.$$

Triple Integrals on a Type 1 Solid Region
If E is a region such that $E = \{(x,y,z)|(x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$, then

$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right) dA.$$

In particular, if D is a type I plane region, then $E = \{(x,y,z)|a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\}$ and

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx,$$

whereas if D is a type II plane region, then $E = \{(x,y,z)|c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x,y) \le z \le u_2(x,y)\}$ and

$$\iiint_E f(x,y,z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy.$$

Triple Integrals on a Type 2 Solid Region
If E is a region such that $E = \{(x,y,z)|(y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}$, then

$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx \right) dA.$$

Triple Integrals on a Type 3 Solid Region
If E is a region such that $E = \{(x,y,z)|(x,z) \in D, u_1(x,z) \le y \le u_2(x,z)\}$, then

$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy \right) dA.$$

Volume of a Solid Region
The volume of a solid region E is given by

$$V(E) = \iiint_E 1 \; dV.$$

Cylindrical Coordinates
To change from Cartesian coordinates to cylindrical coordinates, use the following transformation:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z,$$

and to change from rectangular to cylindrical coordinates, use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z.$$

Here is the iteration of a triple integral in cylindrical coordinates over a type I solid region E:

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) \; r \; dz \; dr \; d\theta.$$