MATH 243: Worksheet 2 **Answer Key**

Textbook Sections: 12.1, 12.2, 12.3
Topics: Regions in the 3 D space, the equation of the sphere, vectors and their properties, magnitude of a vector, operations with vectors, parallel vectors, the dot product, scalar and vector projections.
Instructions: Try each of the following problems, show the detail of your work. Clearly mark your choices in multiple choice items. Justify your answers. Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

1. Consider the point $P(3,4,5)$.

   (a) What is the projection of the point on the xy-plane?

   $$(x,y,z) = (3,4,0)$$

   (b) What is the projection of the point on the xz-plane?

   $$(x,y,z) = (3,0,5)$$

   (c) Find the length of $\overline{OP}$, which is a line from the origin $O(0,0,0)$ and the point $P(3,4,5)$.

   The length is the distance between the origin $O(0,0,0)$ and $P(3,4,5)$, given by the following.

   $$\sqrt{(3-0)^2+(4-0)^2+(5-0)^2} = \sqrt{50} = 5\sqrt{2}.$$

   (d) Find the position vector for the point $P(3,4,5)$ in **ijk**-form.

   The position vector is the origin O and the ending point is the point P we're working with. So, the vector $\overrightarrow{OP}$ from the origin $O(0,0,0)$ and the point $P(3,4,5)$ is

   $$(3-0)\mathbf{i} + (4-0)\mathbf{j} + (5-0)\mathbf{k} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}.$$

2. (a) Find an equation of the sphere that passes through the point $(1,8,5)$ and has center $(3,1,-3)$.

   The radius of the sphere is the distance between $(3,1,-3)$ and $(1,8,5)$:

   $$r = \sqrt{(1-3)^2+(8-1)^2+(5-(-3))^2} = \sqrt{117}.$$

   Thus, an equation of the sphere is

   $$(x-3)^2+(y-1)^2+(z+3)^2 = 117.$$

(b) Using your answer from (a), find an equation to describe its intersection with the yz-plane. If the sphere does not intersect with the yz-plane, write DNE.

To find the intersection with the yz-plane, we set $x = 0$;

$$(y-1)^2 + (z+3)^2 = 108, \quad x = 0,$$

which is a circle in the yz-plane with center $(0, 1, -3)$ and radius $\sqrt{108} = 6\sqrt{3}$.

3. Find an equation of a sphere if one of its diameters has the endpoints at $(1, 2, 4)$ and $(4, 3, 10)$.

First, we find the sphere's diameter by computing the distance between the two given points, $(1, 2, 4)$ and $(4, 3, 10)$. We have

$$d = \sqrt{(1-4)^2 + (2-3)^2 + (4-10)^2} = \sqrt{46}$$

Since the radius, r, of the sphere, is half the diameter, we obtain $r = \dfrac{d}{2} = \dfrac{\sqrt{46}}{2}$.

Now the center of the sphere, C, is located at the midpoint of its diameter. To find the coordinates of the midpoint of a segment, we take the average of the corresponding coordinates of the end points of the segment:

$$\left(\frac{4+1}{2}, \frac{2+3}{2}, \frac{4+10}{2}\right).$$

Hence, the center is located at

$$C\left(\frac{5}{2}, \frac{5}{2}, 7\right).$$

Therefore our sphere has radius $r = \dfrac{\sqrt{46}}{2}$ with a center at $\left(\dfrac{5}{2}, \dfrac{5}{2}, 7\right)$. Therefore its equation can be written as

$$\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 + (z-7)^2 = \frac{23}{2}$$

4. Given $\mathbf{a} = 9\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = \langle 7, 0, -9 \rangle$, find the followings. Simplify your vectors completely.

(a) $\mathbf{a} + \mathbf{b}$

$$\mathbf{a} + \mathbf{b} = \langle 9, -8, 7 \rangle + \langle 7, 0, -9 \rangle = \langle 16, -8, -2 \rangle$$

(b) $3\mathbf{a} - \mathbf{b}$

$$3\mathbf{a} - \mathbf{b} = 3\langle 9, -8, 7 \rangle - \langle 7, 0, -9 \rangle = \langle 27, -24, 21 \rangle - \langle 7, 0, -9 \rangle = \langle 20, -24, 30 \rangle.$$

(c) $||\mathbf{b}||$

$$||\mathbf{b}|| = \sqrt{7^2 + 0^2 + (-9)^2} = \sqrt{49 + 81} = \sqrt{130}.$$

(d) $||\mathbf{b}-\mathbf{a}||$

$$||\mathbf{b}-\mathbf{a}|| = ||\langle -2, 8, -16\rangle|| = \sqrt{(-2)^2+8^2+(-16)^2} = \sqrt{4+64+256} = \sqrt{324} = 18.$$

5. Find the vector that has the opposite direction as $\langle 9, -6, -2\rangle$ but has length 5.

Since $||\langle 9, -6, -2\rangle|| = \sqrt{9^2+(-6)^2+(-2)^2} = \sqrt{121} = 11$, then a unit vector in the opposite direction of $\langle 9, -6, -2\rangle$ is

$$\mathbf{u} = \frac{1}{11}\langle -9, 6, 2\rangle = \left\langle -\frac{9}{11}, \frac{6}{11}, \frac{2}{11}\right\rangle.$$

A vector in the opposite direction but with length 5 is

$$5\mathbf{u} = 5\cdot\left\langle -\frac{9}{11}, \frac{6}{11}, \frac{2}{11}\right\rangle = \left\langle -\frac{45}{11}, \frac{30}{11}, \frac{10}{11}\right\rangle.$$

6. Determine if $\mathbf{a} = 8\mathbf{i} - 12\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i} - 9\mathbf{j}$ are parallel vectors.

Since

$$\mathbf{b} = 6\mathbf{i} - 9\mathbf{j} = \frac{3}{4}(8\mathbf{i} - 12\mathbf{j}) = \frac{3}{4}\mathbf{a},$$

and $\mathbf{a}$ and $\mathbf{b}$ has the same direction, then $\mathbf{a} = 8\mathbf{i} - 12\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i} - 9\mathbf{j}$ are parallel.

Definition of the Dot Product If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the dot product of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a}\cdot\mathbf{b}$ given by:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Similarly, for two-dimensional vectors:

$$\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = a_1b_1 + a_2b_2$$

Theorem If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

7. Given two vectors, $\mathbf{a} = \langle 2, 5, 0\rangle$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$. Find $\mathbf{a}\cdot\mathbf{b}$.

$$\mathbf{a}\cdot\mathbf{b} = 2(2) + 5(0) + 0(1) = 4 + 0 + 0 = 4$$

8. Determine whether the given vectors are orthogonal, parallel, or neither.

(a) $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 12\mathbf{j} - 2\mathbf{k}$

(b) $\mathbf{a} = \langle 6, 5, -2\rangle$ and $\mathbf{b} = \langle 5, 0, 9\rangle$

(c) $\mathbf{a} = \langle -18, 15\rangle$ and $\mathbf{b} = \langle 12, -10\rangle$

(a) Orthogonal: We have

$$\mathbf{a}\cdot\mathbf{b} = 4\cdot 5 + (-1)\cdot 12 + 4\cdot(-2) = 0,$$

so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).
(b) Neither: We have

$$\mathbf{a}\cdot\mathbf{b} = 6\cdot 5 + 5\cdot 0 + (-2)\cdot 9 \neq 0,$$

so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.
(c) Parallel: We have

$$\mathbf{a}\cdot\mathbf{b} = -18\cdot 12 + 15\cdot(-10) \neq 0,$$

so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Because $\mathbf{b} = -\frac{2}{3}\mathbf{a}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel (in the opposite direction).

9. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(4, 16)$.

The slope of the tangent line to the graph of $y = x^2$ at the point $(4, 16)$ is

$$\frac{dy}{dx} = 2x \quad \rightarrow \quad \frac{dy}{dx} = 8 \quad \text{at } x = 4.$$

and a parallel vector is $\mathbf{i} + 8\mathbf{j}$, which has length $\sqrt{1^2 + 8^2} = \sqrt{65}$. So unit vectors parallel to the tangent line are

$$\left\langle \frac{1}{\sqrt{65}}, \frac{8}{\sqrt{65}} \right\rangle \quad \text{and} \quad \left\langle -\frac{1}{\sqrt{65}}, -\frac{8}{\sqrt{65}} \right\rangle$$

10. Given the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in V_2, which of the following expressions are meaningful? Which are meaningless? Justify your answer.

(a) $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$

(b) $(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}$

(c) $||\mathbf{a}||\cdot||\mathbf{c}||$

(a) The expression $\mathbf{a}\cdot\mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ has no meaning.
(b) Both $\mathbf{a}+\mathbf{b}$ and $\mathbf{c}$ are vectors, so the dot product $(\mathbf{a}+\mathbf{c})\cdot\mathbf{b}$ has meaning.
(c) The expression $||\mathbf{a}||$ is a scalar, and the expression $||\mathbf{c}||$ is also a scalar, if scalar is seen as one dimensional vector, then $||\mathbf{a}||\cdot||\mathbf{c}||$ is the scalar multiplication, has meaning.

11. Given $\mathbf{a} = \langle 2, 0, -3\rangle$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Find the following.

(a) $\mathbf{a}\cdot\mathbf{b}$

(b) the angle θ between $\mathbf{a}$ and $\mathbf{b}$.

(a)

$$\mathbf{a} \cdot \mathbf{b} = \langle 2, 0, -3 \rangle \cdot \langle -2, 3, 1 \rangle = 2(-2) + 0(3) + (-3)1 = -7$$

(b) By theorem, we have

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||\,||\mathbf{b}||} = \frac{-7}{\sqrt{2^2 + 0^2 + (-3)^2}\,\sqrt{(-2)^2 + 3^2 + 1^2}} = \frac{-7}{\sqrt{13}\,\sqrt{14}}.$$

So, the answer is $\theta = \arccos\left(\frac{-7}{\sqrt{13}\,\sqrt{14}}\right)$

Suggested Textbook Problems

Chapter 12.1: 1-42, 44-46

Chapter 12.2: 1-13, 15-19, 21, 23-32, 35, 37-41, 43-45, 47

Chapter 12.3: 1-13, 17, 19, 23, 25, 27, 28, 33, 40, 41, 43, 45, 47-52