Lecture quizzes graded, but grades not released yet due to makeups

Challenge Problems: 2, 6, 8

DW 9 #2: (a) Setup Lagrange for $f = 8x+8y+3z$ subject to $\underbrace{4x^2+4y^2+3z^2}_{g(x,y,z)}=35$. (b) Then find max/min of f.

In this case, $g(x,y,z)$ is the function above.

Now we $\nabla f = \langle 8, 8, 3\rangle = \lambda \nabla g = \lambda \langle 8x, 8y, 6z\rangle$

Match the components to get
$$\left\{\begin{array}{l} 8 = 8\lambda x \Rightarrow \lambda x = 1 \\ 8 = 8\lambda y \Rightarrow \lambda y = 1 \\ 3 = 6\lambda z \Rightarrow \lambda z = 1/2 \end{array}\right\}$$

(b) $\lambda = 0 \Rightarrow 0 = 1$, so $\lambda \neq 0 \Rightarrow x = \frac{1}{\lambda}$, $y = \frac{1}{\lambda}$, $z = \frac{1}{2\lambda}$.

Once you have exhausted all the equations from $\nabla f = \lambda \nabla g$ and still have 1 variable to solve for, you must plug back into the "$g=$" constraint.

Thus, $35 = \frac{4}{\lambda^2} + \frac{4}{\lambda^2} + \frac{3}{4\lambda^2} = \frac{1}{\lambda^2}\left(4+4+\frac{3}{4}\right) = \frac{35}{4}\frac{1}{\lambda^2}$.

$\Rightarrow 4\lambda^2 = 1 \Rightarrow \lambda = \pm \frac{1}{2}$.

$\lambda = \pm\frac{1}{2} \Rightarrow (x,y,z) = \pm(2,2,1)$

Now we have 2 possible extremal points, plug both back into f to see what values result.

$f = 8x+8y+3z = \pm(8\cdot 2 + 8\cdot 2 + 3\cdot 1) = \pm 35$, so $\max f = 35$ & $\min f = -35$.

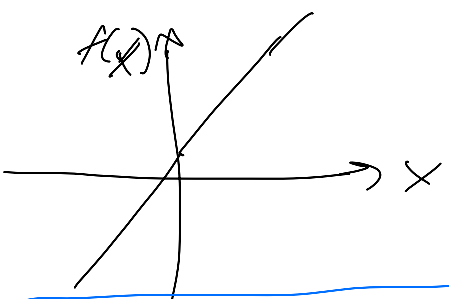
Note: for this problem, like many other problems

done with Lagrange multipliers, the min & max actually exist. This is because f is continuous & $g = 35$ is a closed + bounded, hence compact, region, and a continuous function on a compact domain always achieves a min & max.
This would not work if $g = x^2 - y^2 - z^2$ for example since the region would no longer be bounded.

$$35 = 35 \cdot \frac{1}{4\lambda^2} \Longrightarrow 1 = \frac{1}{4\lambda^2} \Longrightarrow 4\lambda^2 = 1$$

$$\frac{35}{35} = \frac{35}{35} \cdot \frac{1}{4\lambda^2}$$

Example with unbounded region: $f = 8x + 8y + 3z$, but now $4x^2 - 4y^2 + 3z^2 = 35$. In this case, region unbounded, so you can try disproving the existence of the min & max. Try $x = y$, then $3z^2 = 35$, set $z = \sqrt{\frac{35}{3}}$. So the red constraint is satisfied and $f = 8x + 8y + 3z = 16x + \sqrt{105}$. As $x \to \infty$, $f \to \infty$. As $x \to -\infty$, $f \to -\infty$, so f has no finite min/max. Alternatively, you can say that $\max f = \infty$, $\min f = -\infty$.



2nd example: But why can't you just always say the min/max don't exist if the region is unbou-nded? Consider $f = x^2 + y^2 + z^2$ and $4x^2 - 4y^2 + 3z^2 = 0$. The region described by red constraint is still unbounded, but the minimum of f still exists.

Note that $f = x^2+y^2+z^2 \geq 0+0+0 = 0$, and we can get 0 if $x=y=z=0$, which satisfies the constraint. So $\min f = 0$, achieved at $(0,0,0)$.

f has no finite max though as $f(x,x,0) = x^2+x^2+0^2 = 2x^2 \to \infty$ as $x \to \infty$ and $(x,x,0)$ satisfies the constraint.

#8: Evaluate $\iint_R \frac{1}{(2x+3y)^2}\,dA$, $R = [0,1]\times[1,2]$.

From the description of R, our bounds are $0 \leq x \leq 1$ and $1 \leq y \leq 2$. Also, $dA = dx\,dy$. So the integral is

$$\int_1^2 \left(\int_0^1 \frac{dx}{(2x+3y)^2} \right) dy = \int_1^2 \frac{1}{4}\left(\frac{1}{1.5y} - \frac{1}{1+1.5y} \right) dy =$$

$$\int_0^1 \frac{dx}{(2x+3y)^2} = \frac{1}{4}\int_0^1 \frac{dx}{(x+1.5y)^2} =$$

$$\frac{1}{4} \cdot -\frac{1}{(x+1.5y)}\Bigg|_{x=0}^{x=1} =$$

$$\frac{1}{4} \cdot \frac{1}{x+1.5y}\Bigg|_{x=1}^{x=0} =$$

$$\frac{1}{4}\left(\frac{1}{1.5y} - \frac{1}{1+1.5y} \right)$$

$$\frac{1}{4}\left(\frac{1}{1.5}\ln|y| - \frac{1}{1.5}\ln|1+1.5y| \right)\Bigg|_1^2$$

$$= \frac{1}{6}\left(\ln|y| - \ln|1+1.5y| \right)\Bigg|_1^2$$

$$= \frac{1}{6}\left(\ln 6 - \ln 4 - (\ln 1 - \ln 2.5) \right)$$

$$= \frac{1}{6}\left(\ln 6 - \ln 4 + \ln 2.5 \right) =$$

$$\frac{1}{6}\ln\frac{6\cdot 2.5}{4} = \frac{1}{6}\ln\frac{15}{4}$$

Extra Step: $\frac{1}{(2x+3y)^2} = \frac{1}{2^2(x+\frac{3}{2}y)^2} =$

$\frac{1}{4}\cdot\frac{1}{(x+1.5y)^2}$

Note: any techiques for Calc I\2 integrals will also apply to

Extra extra Step:
$(2x+3y)^2 = \left(2(x+\frac{3}{2}y)\right)^2 = 2^2(x+\frac{3}{2}y)^2$ using the fact $(ab)^2 = a^2 \cdot b^2$.

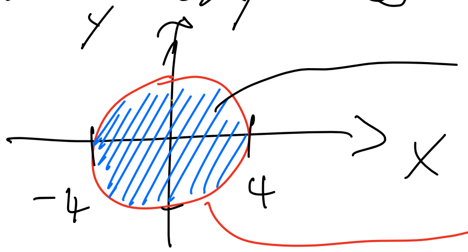
an inner\outer integral in Calc 3. Just treat all variables besides the one integrated as constant

Note: In Calc 1\2, you solve a variety of levels of 1 variable single integrals. For example, $\int \frac{1}{2x^3}\,dx$ easy, $\int \sin^2 x \cos x\, dx$ medium, $\int_0^\infty \frac{x^2+x+2}{1+x^3}\,dx$ hard in terms of amount of algebra & substitutions.

In this regard, you will almost always only see easy & medium level integrals for one of your outer or inner integrals in Calc 3.

#4: min/max $f = 2x^2 - y^2 + 6y$ on the disk $x^2+y^2 \leq 16$.

This is an optimization on closed 2D\3D region problem, for which the procedure is to find local extrema inside the region by solving $\vec{0} = \nabla f$, then check the boundary of the region manually.



the region

boundary: circle $x^2+y^2=16$

$\langle 0,0\rangle = \nabla f = \langle 4x, -2y+6\rangle \implies$

$4x = 0$ & $-2y+6 = 0 \implies x = 0$ & $y = 3$

$\implies (0,3)$ is only interior candidate.

$f(0,3) = 0 - 3^2 + 6\cdot 3 = 9$.

Check perimeter: we need to min/max $f = 2x^2 - y^2 + 6y$ subject to $g := x^2+y^2 = 16$, which we can do with Lagrange, or direct trig substitution. Since most other problems involve using Lagrange, let's try trig substitution.

Note: if the region is described by $g(x,y) \le C$, the boundary is given by $g(x,y) = C$

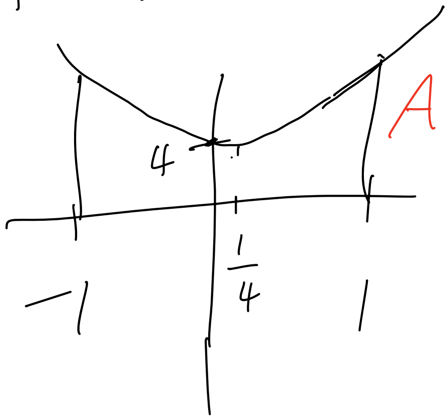
Let $x = 4\cos\theta$, $y = 4\sin\theta$, which parametrizes the circle.

Then $f = 32\cos^2\theta - 16\sin^2\theta + 24\sin\theta =$

$32(1-s^2) - 16s^2 + 24s = -48s^2 + 24s + 32$

$= -8(6s^2 - 3s - 4)$ where $s = \sin\theta$

for the sake of convenience. Motivation for replacing $\cos^2\theta$ is so we can reduce to a quadratic in s, which is easy to optimize viz various methods.



$A = \boxed{6s^2 - 3s - 4} = 6(s^2 - 0.5s - \frac{2}{3})$

$= 6((s - \frac{1}{4})^2 - \frac{2}{3} - \frac{1}{16}) =$

$6(s - \frac{1}{4})^2 + C$ for some C.

Since vertex is at $s = \frac{1}{4}$, the minimum of A is at $s = \frac{1}{4}$, max at $s = 1$ or -1.

$A(\frac{1}{4}) = \frac{6}{16} - \frac{3}{4} - 4 = -\frac{3}{8} - 4 = -\frac{35}{8}$.

$A(1) = 3 - 4 = -1$, $A(-1) = 5$. So

$\min A = -\frac{35}{8}$, $\max A = 5$.

So $\max f = -8 \cdot \frac{-35}{8} = 35$ on perim,

$\min f = -8 \cdot 5 = -40$, so

from the 3 orange values, largest is 9 & smallest is -104, so

$\max f = 35$, $\min f = -40$

3rd solution for perimeter: notice that

$2x^2 - y^2 + 6y = 2x^2 + 2y^2 - (3y^2 - 6y) =$

$32 - 3((y-1)^2 - 1) = 35 - 3(y-1)^2.$

Let this be $h(y)$.

Then $h(y) \leq 35$ since $3(y-1)^2 \geq 0$.

But also, $(y-1)^2 \leq (-4-1)^2 = 25$, so

$h(y) \geq 35 - 3 \cdot 25 = 35 - 75 = -40.$

But also $h(1) = 35$, $h(-4) = -40$,

so these mins & maxes are attainable.

$f = 2x^2 - y^2 + 6y$, $g = x^2 + y^2 = 16$

$\langle 4x, -2y+6 \rangle = \nabla f = \lambda \nabla g = \lambda \langle 2x, 2y \rangle.$

$$\Rightarrow \left\{ \begin{array}{l} 2x = \lambda x \\ -y+3 = \lambda y \end{array} \right\}.$$

1st eq $\Rightarrow$ $\lambda = 2$ or $x = 0$.

$\lambda = 2$: then $-y+3 = 2y \Rightarrow y = 1, x = 0$

$x = 0$: $y^2 = 16 \Rightarrow y = \pm 4$

#3: max/min $\underbrace{81x^2+y^2}_{f}$ if $\underbrace{4x^2+y^2}_{g} = 9$.

If we are to use Lagrange, f & g are as above. Then

$$\langle 162x, 2y\rangle = \nabla f = \lambda \nabla g = \lambda\langle 8x, 2y\rangle.$$

$$\Longrightarrow \left\{\begin{array}{l} 81x = 4\lambda x \\ y = \lambda y \end{array}\right\}.$$

2nd equation is simpler, so start with it. Either $y=0$ or $\lambda = 1$.

$y=0$: $4x^2 = 9 \Rightarrow x = \pm 1.5 \Rightarrow (\pm 1.5, 0)$.

$\lambda = 1$: $81x = 4x \Longrightarrow x = 0$. Plug back into constraint $\Rightarrow y^2 = 9 \Rightarrow y = \pm 3 \Rightarrow (0, \pm 3)$

Now let's check all the candidates.

$$f(0, \pm 3) = 81 \cdot 0^2 + (\pm 3)^2 = 9$$

$$f(\pm 1.5, 0) = 81 \cdot \frac{9}{4} = \frac{729}{4}.$$

So $\min f = 9$, $\max f = \frac{729}{4}$.

Note: Can also solve by noticing that $81x^2 + y^2 = 77x^2 + 9 \geq 9$, so $\min = 9$ at $x = 0$. Also, $x^2 = \frac{9-y^2}{4} \leq \frac{9}{4}$, so

$$77x^2+9 \leq 77\cdot\frac{9}{4}+9=\frac{729}{4}.$$