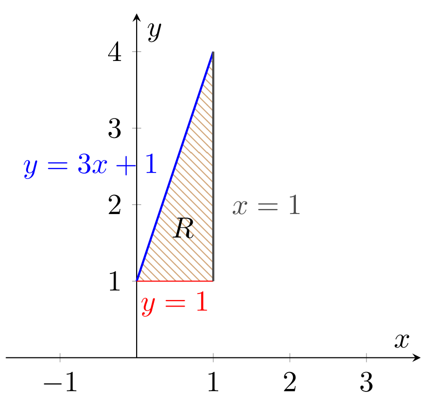
Textbook Sections: 15.2 and 15.3
Topics: Double integrals over general domains, and double integrals in polar coordinates.
Instructions: Try each of the following problems, show the detail of your work.
Clearly mark your choices in multiple choice items. Justify your answers.
Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

Double Integrals over general Domains:

1. Given the region R bounded by the lines $y = 3x + 1$, $x = 1$ and $y = 1$, draw the region R and iterate the double integral (Do NOT evaluate).

$$\iint_R f(x, y)\, dA$$



Type I

$R = \{(x, y) \mid 0 \le x \le 1,\ 1 \le y \le 3x + 1\}$

$$\int_0^1 \int_1^{3x+1} f(x, y)\, dydx$$

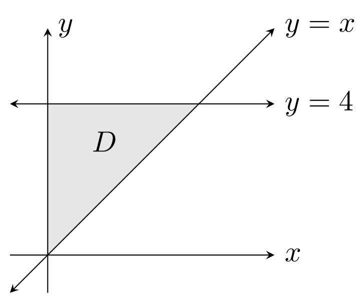
OR

Type II

$$R = \left\{(x, y) \mid 1 \le y \le 4,\ \frac{y-1}{3} \le x \le 1\right\}$$

$$\int_1^4 \int_{\frac{y-1}{3}}^1 f(x, y)\, dxdy$$

2. Evaluate $\displaystyle\iint_D y^2 e^{xy}\, dA$, where D is the region bounded by $y = x,\ y = 4,\ x = 0$.



Taking D as a type I region, we get $D = \{(x, y) \mid 0 \le x \le 4,\ x \le y \le 4\}$ and

$$\iint_D y^2 e^{xy}\, dA = \int_0^4 \int_x^4 y^2 e^{xy}\, dy\, dx.$$

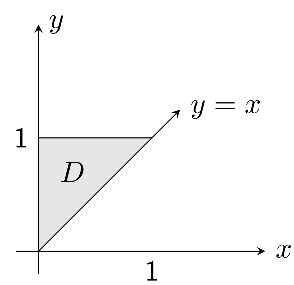
Taking D as a type II region, we get $D = \{(x, y) \mid 0 \le y \le 4,\ 0 \le x \le y\}$ and

$$\iint_D y^2 e^{xy}\, dA = \int_0^4 \int_0^y y^2 e^{xy}\, dx\, dy.$$

Note that integrating with respect to y first requires integration by parts, and integrating with respect to x first requires u-substitution. So, if we evaluate the integral using D as a type II plane region, after integrating with respect to x, we need to integrate the function ye^{y^2} which can be done with the u-substitution $u = y^2$.Thus, we get

$$\int_0^4 \int_0^y y^2 e^{xy}\, dx\, dy = \int_0^4 \left(ye^{y^2} - y\right)\, dy = \frac{1}{2}[e^{16} - 17].$$

3. (a) Sketch the region of integration for $\displaystyle\int_0^1 \int_0^y f(x, y)\, dx\, dy$



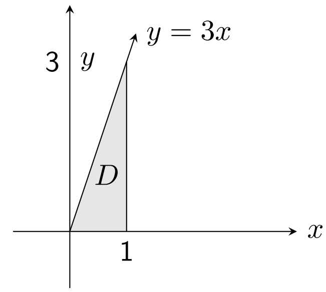
(b) Switch the order of integration for the integral in part a.

$$\int_0^1 \int_x^1 f(x, y)\, dy\, dx$$

(c) Calculate the integral from part b, for the function $f(x, y) = \sqrt{x} + 3y$.

$$\begin{aligned}
\int_0^1 \int_x^1 \left(\sqrt{x} + 3y\right)\, dy\, dx &= \int_0^1 \left[y\sqrt{x} + \frac{3}{2}y^2\right]_{y=x}^{y=1} dx \\
&= \int_0^1 \left(\sqrt{x} + \frac{3}{2} - x^{\frac{3}{2}} - \frac{3}{2}x^2\right) dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{3}{2}x - \frac{2}{5}x^{\frac{5}{2}} - \frac{1}{2}x^3\right]_0^1 \\
&= \frac{2}{3} + \frac{3}{2} - \frac{2}{5} - \frac{1}{2} = \frac{19}{15}
\end{aligned}$$

4. Find the average value of $f(x, y) = xy$ over the triangle D with the vertices $(0, 0), (1, 0), (1, 3)$.



The area of a triangle with base b and height h is $A = \frac{1}{2}bh$. So the area of the domain is $\frac{1}{2}(1)(3) = \frac{3}{2}$. Then

$$f_{\mathsf{avg}} = \frac{2}{3}\int_0^1 \int_0^{3x} xy\,dy\,dx = \int_0^1 3x^3\,dx = \frac{3}{4}$$

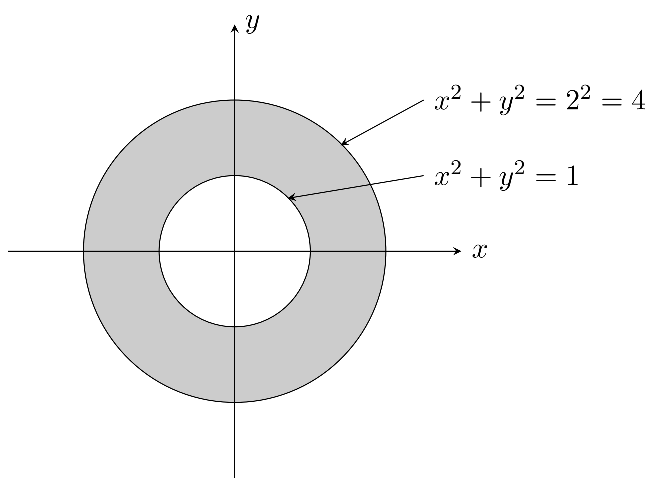
Double integrals in polar coordinates:

5. Use polar coordinates to find the volume of the solid $z = \sqrt{x^2+y^2}$ and above the ring $1 \le x^2+y^2 \le 4$.

The function $z = f(x,y) = \sqrt{x^2+y^2} \ge 0$, so

$$V = \int\int_D f(x,y)\,dA = \int\int_D \sqrt{x^2+y^2}\,dA$$

where $D = \{(x,y) \mid 1 \le x^2+y^2 \le 4\}$.

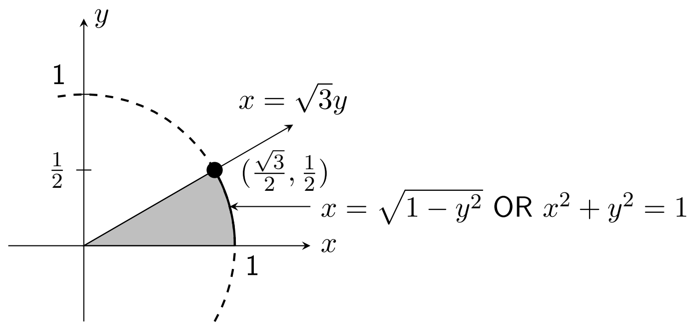


Converting to polar coordinates, $D = \{(r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le 2\pi\}$, and recalling $r^2 = x^2+y^2$, $dA = r dr d\theta$

$$V = \int_0^{2\pi}\int_1^2 \sqrt{r^2}\, r\ dr\ d\theta = \int_0^{2\pi}\int_1^2 r^2\ dr\ d\theta = \int_0^{2\pi} \frac{7}{3}\ d\theta = 2\pi\frac{7}{3} = \frac{14\pi}{3}$$

6. Calculate $\displaystyle\int_0^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\,dx\,dy$ by changing to polar coordinates.

The region is given as a type II plane region $D = \{(x, y) \mid 0 \le y \le 1/2, \sqrt{3}y \le x \le \sqrt{1-y^2}\}$



In polar coordinates, $D = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le \pi/6\}$, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so the line $x = \sqrt{3}y$ in polar coordinates is $r\cos(\theta) = \sqrt{3}r\sin(\theta)$. That is $\tan(\theta) = \frac{1}{\sqrt{3}}$ and $\theta = \frac{\pi}{6}$. Then,

$$\begin{aligned}
\int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\,dx\,dy &= \int_0^{\frac{\pi}{6}}\int_0^1 (r\cos(\theta))(r\sin(\theta))^2 r\;dr\;d\theta \\
&= \int_0^{\frac{\pi}{6}}\int_0^1 r^4\cos(\theta)\sin^2(\theta)\;dr\;d\theta \\
&= \frac{1}{5}\int_0^{\frac{\pi}{6}} \cos(\theta)\sin^2(\theta)\;d\theta \\
&= \frac{1}{120}
\end{aligned}$$

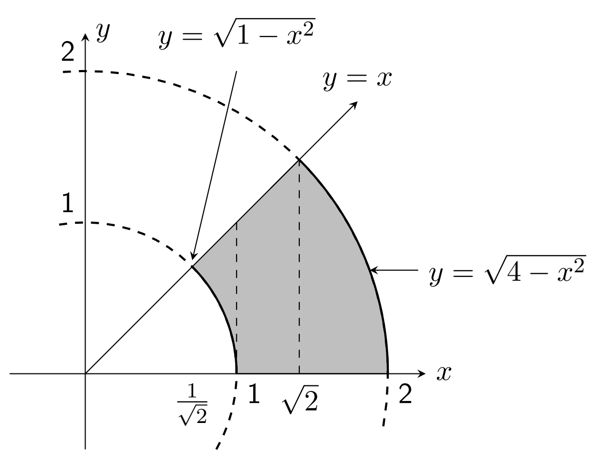
where the last integral is solved by the u-substitution $u = \sin(\theta)$.

7. Use polar coordinates to combine the sum

$$\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\;dy\;dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$

into one double integral and evaluate the double integral.

Answer:



Note that the domain of each integral in the problem statement above represents one of the vertical bands of the domain shaded in gray. Each region is written as a type II plane region. We can simplify the problem if we can represent the region D in polar coordinates as $D = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi/4\}$. Then, the integral in polar coordinates is

$$\begin{aligned}\int\int_D xy\, dA &= \int_0^{\frac{\pi}{4}} \int_1^2 r\cos(\theta)\, r\, \sin(\theta)\, r\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{4}} \int_1^2 r^3 \cos(\theta)\sin(\theta)\, dr\, d\theta \\ &= \frac{15}{4} \int_0^{\frac{\pi}{4}} \cos(\theta)\sin(\theta)\, d\theta \\ &= \frac{15}{16}\end{aligned}$$

where the final integral is solved by using the u-substitution $u = \sin(\theta)$.

Suggested Textbook Problems

Section 15.2	1-10, 13-32, 39-40, 45-57, 65, 68
Section 15.3	1-11, 13-16, 22-27, 29-35, 39-42, 49

SOME USEFUL DEFINITIONS, THEOREMS AND NOTATION:

The Average Formula

The average value of a function f of two variables defined on a region R with area $A(R)$, is given by

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

Integrals Over General Regions

If f is continuous on a type I region $D = \{\ (x, y)|\ a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$ where g_1 and g_2 are continuous functions, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

If f is continuous on a type II region $D = \{\ (x, y)|\ c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$, where h_1 and h_2 are continuous functions, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Properties of double integrals

Suppose the integrals of $f(x, y)$ and $g(x, y)$ exist, then

(1) $\displaystyle\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$

(2) $\displaystyle\iint_D cf(x, y)\, dA = c \iint_D f(x, y)\, dA$, where c is a constant

(3) If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then, $\displaystyle\iint_D f(x, y)\, dA \geq \iint_D g(x, y)\, dA$

(4) If $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except perhaps on their boundaries ,then,

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$

(5) If we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D, i. e. $\iint_D 1\, dA = Area(D)$

Change to Polar Coordinates in a Double Integral

(1) If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r dr\, d\theta$$

(2) If f is continuous on a polar region of the form $D = \{\ (r, \theta)|\ \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$ then,

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r dr\, d\theta$$