Challenge Problems in RED

MATH 243: Worksheet 12 **Discussion Section:** ____________

Textbook Sections: 15.8, 16.1 and 16.2

Topics: Triple Integrals in Spherical Coordinates, Vector Fields and Line Integrals

Instructions: Try each of the following problems, show the detail of your work.

Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

TRIPLE INTEGRALS IN SPHERICAL COORDINATES:

1. Evaluate $\iiint_E (x^2+y^2)\, dV$ where E is the region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

2. Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$.

3. Find the volume of the part of the ball $\rho \le a$ that lies between the cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$.

4. Evaluate the following integrals by switching to spherical coordinates.

   (a) $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z+y^2z+z^3)\, dz\, dx\, dy.$

   (b) $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\, dz\, dy\, dx.$

5. A solid lies inside the sphere $x^2+y^2+z^2=4z$ and outside the cone $z=\sqrt{x^2+y^2}$. Write a description of the solid in terms of inequalities involving spherical coordinates.

VECTOR FIELDS

6. Find the gradient vector ∇f of f.

   (a) $f(x,y) = 2y\sin(xy)$

   (b) $f(x,y,z) = x^3ye^{y/z}$

LINE INTEGRALS:

7. Evaluate $\int_C (x/y)ds$, where $C: x=t^3,\ y=t^4,\ 1\le t\le 2$.

8. Evaluate $\int_C ydx+zdy+xdz$, where $C: x=\sqrt{t},\ y=t,\ z=t^2,\ 1\le t\le 4$.

9. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y)=xy^2\mathbf{i}-x^2\mathbf{j}$, $\mathbf{r}(t)=t^3\mathbf{i}+t^2\mathbf{j}$, and $0\le t\le 1$.

10. Find the work done by the force field $\mathbf{F}(x,y)=x\mathbf{i}+(y+2)\mathbf{j}$ in moving an object along an arch of the cycloid $\mathbf{r}(t)=(t-\sin t)\mathbf{i}+(1-\cos t)\mathbf{j}$, $0\le t\le 2\pi$

Suggested Textbook Problems

Section 15.8	1-13, 15, 17-29, 32, 43-45
Section 16.1	25-28
Section 16.2	1-18, 21-24, 41-45

SOME USEFUL DEFINITIONS, THEOREMS AND NOTATION:

Spherical Coordinates

To change from Cartesian coordinates to spherical coordinates, use the following transformation:

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi,$$

and we note that the distance formula shows that $\rho^2 = x^2 + y^2 + z^2$. In particular, we have that

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \; d\rho \; d\theta \; d\phi,$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) | c \le \phi \le d, \alpha \le \theta \le \beta, a \le \rho \le b\}.$$

Definition of the Vector Field

Let E be a subset of $\mathbb{R}^3$. A **vector field** on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Definition of the Conservative Vector Field

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a potential function for $\mathbf{F}$.

Line Integral with Respect to Arc Length

If f is defined on a smooth curve C given by $x = x(t)$, $y = y(t)$, and $a \le t \le b$, then the *line integral of f along C* is

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Line Integral with Respect to x or y

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of $t : x = x(t)$, $y = y(t)$, $dx = x'(t)dt$, $dy = y'(t)dt$.

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))x'(t)dt \text{ and } \int_C f(x,y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

Line Integral Formula of a Continuous Vector Field

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the *line integral of* $\mathbf{F}$ *along* C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

The work W done by the force field $\mathbf{F}$ is the line integral with respect to arc length of the tangential component of the force.

Line Integral Formula for Vector Fields on $\mathbb{R}^2$

Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$, where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy$