13 Vector Functions

13.3 Arc Length and Curvature

Arc Length
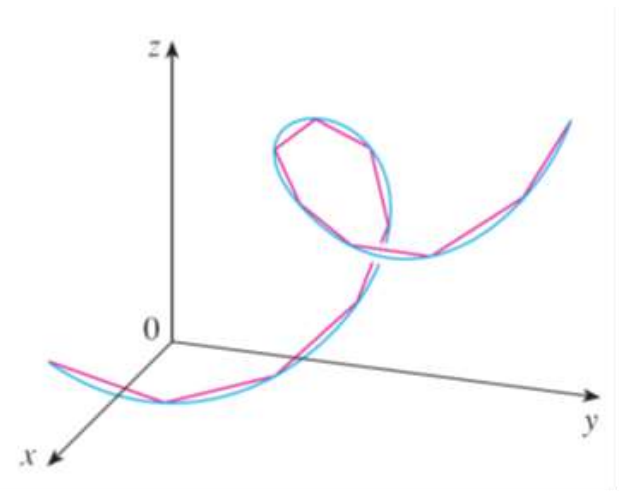
We have defined the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, as the limit of lengths of inscribed polygons and, for the case where f' and g' are continuous, we arrived at the formula

$$1 \quad L = \int_a^b \sqrt{[[f'(t)]^2 + [g'(t)]^2]}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The length of a space curve is defined in exactly the same way (see Figure 1).

The length of a space curve is the limit of lengths of approximating polygonal paths.

Figure 1

Suppose that the curve has the vector equation, $\mathrm{r}(t) = \langle f(t), g(t), h(t) \rangle,$ $a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where $f', g',$ and h' are continuous.

If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$\color{red}{2} \qquad \color{black}{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

$$3 \quad L = \int_a^b |r'(t)| \, dt$$

because, for plane curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$,

$$|r'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j}|$$
$$= \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$,

$$|r'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}|$$
$$= \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Example 1

Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$ from the point (1, 0, 0) to the point (1, 0, 2π).

Solution:

Since $\mathbf{r}'(t) = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + \mathbf{k}$, we have

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from (1, 0, 0) to (1, 0, 2π) is described by the parameter interval $0 \leq t \leq 2\pi$ and so, from Formula 3, we have

$$L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi$$

A single curve *C* can be represented by more than one vector function. For instance, the twisted cubic

$$4 \quad \mathrm{r}_1(t) = \langle t,\ t^2,\ t^3 \rangle \quad 1 \leq t \leq 2$$

could also be represented by the function

$$5 \quad \mathrm{r}_2(u) = \langle e^u,\ e^{2u},\ e^{3u} \rangle \quad 0 \leq u \leq \ln 2$$

where the connection between the parameters *t* and *u* is given by $t = e^u$.

We say that Equations 4 and 5 are **parametrizations** of the curve *C*.

Arc Length (5 of 5)

If we were to use Equation 3 to compute the length of *C* using Equations 4 and 5, we would get the same answer.

This is because arc length is a geometric property of the curve and hence is independent of the parametrization that is used.

The Arc Length Function

Now we suppose that *C* is a curve given by a vector function

$$\mathrm{r}(t) = f(t)\mathrm{i} + g(t)\mathrm{j} + h(t)\mathrm{k} \quad a \le t \le b$$

where r' is continuous and *C* is traversed exactly once as *t* increases from *a* to *b*.

We define its **arc length function** *s* by

$$6 \quad s(t) = \int_a^t |\mathrm{r}'(u)|\,du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\; du$$

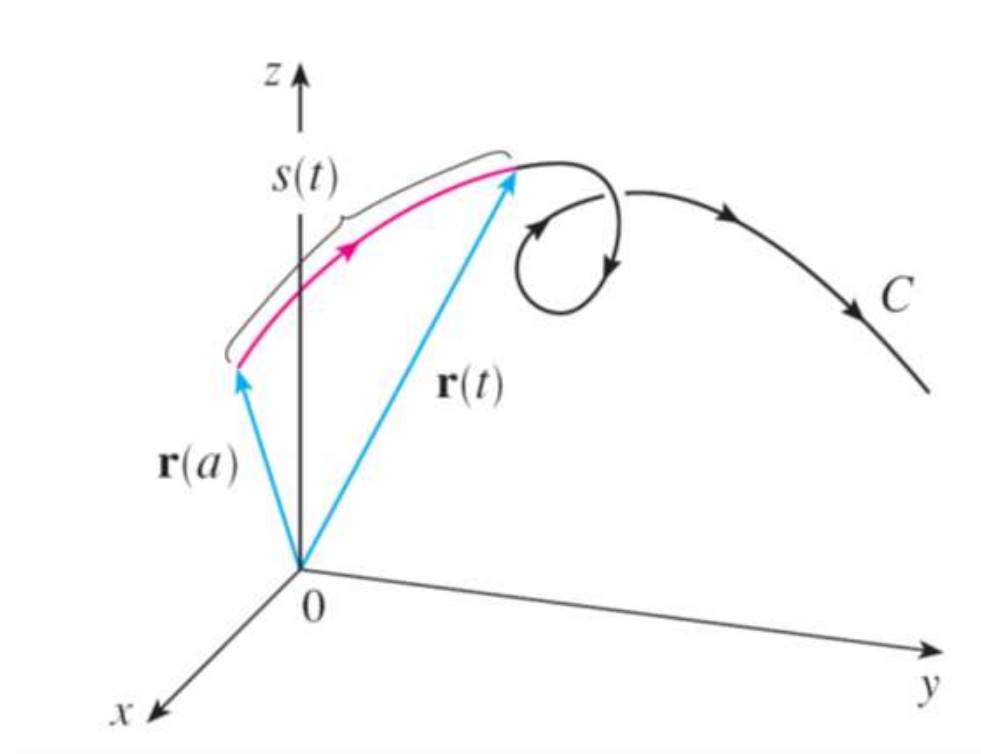Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. (See Figure 3.)

Figure 3

If we differentiate both sides of Equation 6 using Part 1 of the Fundamental Theorem of Calculus, we obtain

$$7 \quad \frac{ds}{dt} = |r'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system or a particular parametrization.

If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by Equation 6, then we may be able to solve for t as a function of s: $t = t(s)$.

Then the curve can be reparametrized in terms of s by substituting for t: $\mathbf{r} = \mathbf{r}(t(s))$.

Thus, if $s = 3$ for instance, $\mathbf{r}(t(3))$ is the position vector of the point 3 units of length along the curve from its starting point.

Example 2

Reparametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from (1, 0, 0) in the direction of increasing t.

Solution:

The initial point (1, 0, 0) corresponds to the parameter value $t = 0$. From Example 1 we have

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2}$$

and also

$$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}t$$

Example 2 – Solution

Therefore $t = \dfrac{s}{\sqrt{2}}$ and the required reparametrization is obtained by substituting for t:

$$\mathrm{r}(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right)\mathrm{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathrm{j} + \left(\frac{s}{\sqrt{2}}\right)\mathrm{k}$$

Curvature

A parametrization **r**(t) is called **smooth** on an interval I if r' is continuous and $\mathrm{r}'(t) \neq 0 \text{ on } I$.

A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.
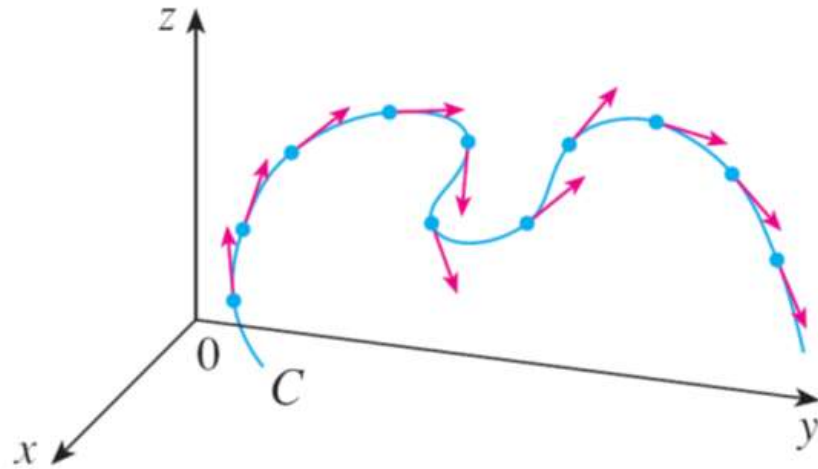
If C is a smooth curve defined by the vector function **r**, recall that the unit tangent vector **T**(t) is given by

$$\mathrm{T}(t) = \frac{\mathrm{r}'(t)}{|\mathrm{r}'(t)|}$$

and indicates the direction of the curve.

From Figure 4 you can see that **T**(t) changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.

Unit tangent vectors at equally spaced points on C

Figure 4

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point.

Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.) Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of **T.**

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathrm{T}}{ds} \right|$$

where **T** is the unit tangent vector.

Curvature (4 of 6)

The curvature is easier to compute if it is expressed in terms of the parameter *t* instead of *s*, so we use the Chain Rule to write

$$\frac{d\mathrm{T}}{dt} = \frac{d\mathrm{T}}{ds}\frac{ds}{dt} \quad \text{and} \quad \kappa = \left|\frac{d\mathrm{T}}{ds}\right| = \left|\frac{\frac{d\mathrm{T}}{dt}}{\frac{ds}{dt}}\right|$$

But $\frac{ds}{dt} = |\mathrm{r}'(t)|$ from Equation 7, so

$$9 \quad \kappa(t) = \frac{|\mathrm{T}'(t)|}{|\mathrm{r}'(t)|}$$

Example 3

Show that the curvature of a circle of radius *a* is $\frac{1}{a}$.

Solution:

We can take the circle to have center the origin, and then a parametrization is

$$\mathbf{r}(t) = a \cos t\ \mathbf{i} + a \sin t\ \mathbf{j}$$

Therefore $\quad r'(t) = -a \sin\ t\ \mathrm{i} + a\ \cos\ t\ \mathrm{j}\quad$ and $\quad |r'(t)| = a$

so

$$T(t) = \frac{r'(t)}{|r'(t)|}$$
$$= -\sin\ t\ \mathrm{i} + \cos\ t\ \mathrm{j}$$

and

$$T'(t) = -\cos\ t\ \mathrm{i} - \sin\ t\ \mathrm{j}$$

Example 3 – Solution

This gives $|T'(t)| = 1,$ so using Equation 9, we have

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{1}{a}$$

Curvature (5 of 6)

The result of Example 3 shows that small circles have large curvature and large circles have small curvature, in accordance with our intuition.

We can see directly from the definition of curvature that the curvature of a straight line is always 0 because the tangent vector is constant.

Although Formula 9 can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

10 Theorem The curvature of the curve given by the vector function **r** is

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\ \mathbf{i} + f(x)\ \mathbf{j}$.

Then $\mathrm{r}'(x) = \mathrm{i} + f'(x)\mathrm{j}$ and $\mathrm{r}''(x) = f''(x)\mathrm{j}$.

Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathrm{r}'(x) \times \mathrm{r}''(x) = f''(x)\mathrm{k}.$

We also have $\dfrac{|\mathrm{r}'(x)|}{= \sqrt{1 + [f'(x)]^2}}$ and so, by Theorem 10,

$$11 \quad \kappa(x) = \frac{|f''(x)|}{\left[1 + \left(f'(x)\right)^2\right]^{\frac{3}{2}}}$$

The Normal and Binormal Vectors

The Normal and Binormal Vectors (1 of 5)

At a given point on a smooth space curve **r**(*t*), there are many vectors that are orthogonal to the unit tangent vector **T**(*t*).

We single out one by observing that, because $|\mathrm{T}(t)| = 1$ for all *t*, we have

$\mathrm{T}(t) \cdot \mathrm{T}'(t) = 0,$ so $\mathrm{T}'(t)$ is orthogonal to **T**(*t*).

Note that $\mathrm{T}'(t)$ is itself not a unit vector.

But at any point where $\kappa \neq 0$ we can define the **principal unit normal vector N**(*t*) (or simply **unit normal**) as

$$N(t) = \frac{\mathrm{T}'(t)}{|\mathrm{T}'(t)|}$$

The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector**.

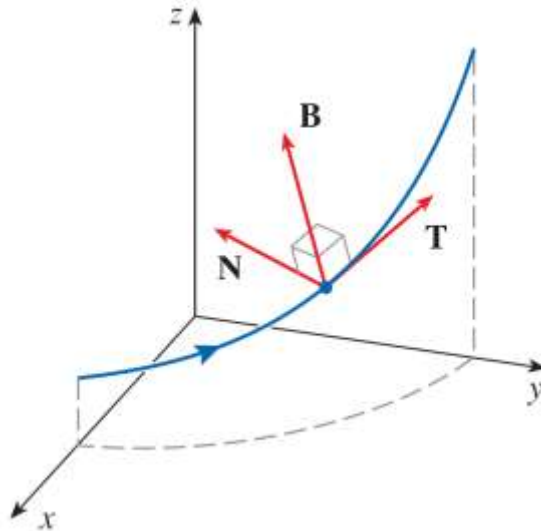It is perpendicular to both **T** and **N** and is also a unit vector. (See Figure 6.)

Figure 6

Example 6

Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

Solution:

We first compute the ingredients needed for the unit normal vector:

$$\mathbf{r}'(t) = -\sin t\,\mathrm{i} + \cos t\,\mathrm{j} + \mathrm{k} \quad |\mathbf{r}'(t)| = \sqrt{2}$$

$$
\begin{aligned}
\mathrm{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\mathrm{i} + \cos t\,\mathrm{j} + \mathrm{k}) \\
\mathrm{T}'(t) &= \frac{1}{\sqrt{2}}(-\cos t\,\mathrm{i} - \sin t\,\mathrm{j}) \quad |\mathrm{T}'(t)| = \frac{1}{\sqrt{2}}
\end{aligned}
$$

Example 6 – Solution

$$N(t) = \frac{T'(t)}{|T'(t)|} = -\cos t \; i - \sin t \; j$$
$$= \langle -\cos t, -\sin t, 0 \rangle$$

This shows that the normal vector at any point on the helix is horizontal and points toward the *z*-axis.

The binormal vector is

$$B(t) = T(t) \times N(t) \quad = \quad \frac{1}{\sqrt{2}} \begin{bmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix}$$
$$= \quad \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$$

The plane determined by the normal and binormal vectors **N** and **B** at a point *P* on a curve *C* is called the **normal plane** of *C* at *P*.

It consists of all lines that are orthogonal to the tangent vector **T**.

The plane determined by the vectors **T** and **N** is called the **osculating plane** of *C* at *P*.

The name comes from the Latin *osculum*, meaning "kiss." It is the plane that comes closest to containing the part of the curve near *P*. (For a plane curve, the osculating plane is simply the plane that contains the curve.)

The **circle of curvature**, or the **osculating circle**, of *C* at *P* is the circle in the osculating plane that passes through *P* with radius $\frac{1}{\kappa}$ and center a distance $\frac{1}{\kappa}$ from *P along* the vector **N**. The center of the circle is called the **center of curvature** of *C* at *P*.

We can think of the circle of curvature as the circle that best describes how *C* behaves near *P*— it shares the same tangent, normal, and curvature at *P.*

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathrm{T}(t) = \frac{\mathrm{r}'(t)}{|\mathrm{r}'(t)|} \quad \mathrm{N}(t) = \frac{\mathrm{T}'(t)}{|\mathrm{T}'(t)|} \quad \mathrm{B}(t) = \mathrm{T}(t) \times \mathrm{N}(t)$$

$$\kappa = \left|\frac{d\mathrm{T}}{ds}\right| = \frac{|\mathrm{T}'(t)|}{|\mathrm{r}'(t)|} = \frac{|\mathrm{r}'(t) \times \mathrm{r}''(t)|}{|\mathrm{r}'(t)|^3}$$

Torsion

Curvature $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$ at a point P on a curve C indicates how tightly the curve "bends." Since **T** is a normal vector for the normal plane, $d\mathbf{T}/ds$ tells us how the normal plane changes as P moves along C.

Since **B** is normal to the osculating plane, $d\mathbf{B}/ds$ gives us information about how the osculating plane changes as P moves along C. (See Figure 12.)
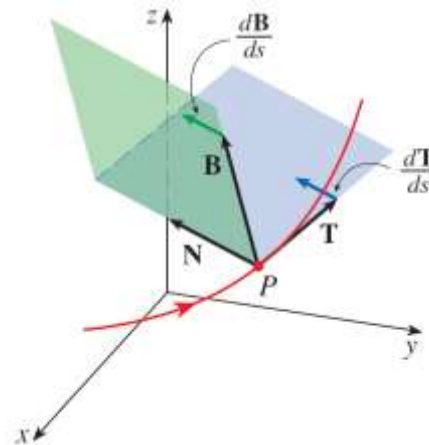

Figure 12

Torsion (2 of 3)

Thus there is a scalar τ such that

12
$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

The number τ is called the torsion of C at P. If we take the dot product with **N** of each side of Equation 12 and note that $\mathbf{N} \cdot \mathbf{N} = 1$, we get the following definition.

13 Definition The **torsion** of a curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Now from Definition 13 we have

14 $$\tau(t) = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|}$$

Example 10

Find the torsion of the helix $r(t) = \langle \cos t, \sin t, t \rangle.$

Solution:

We computed $\dfrac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2},$

$$N(t) = \langle -\cos t, -\sin t, 0 \rangle, \text{ and } B(t) = \left(\frac{1}{\sqrt{2}} \right) \langle \sin t, -\cos t, 1 \rangle.$$

Then

$$\mathbf{B}'(t) = \left(1/\sqrt{2} \right) \langle \cos t, \sin t, 0 \rangle$$

and Formula 14 gives

$$\tau(t) = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|} = -\frac{1}{2} \langle \cos t, \sin t, 0 \rangle \cdot \langle -\cos t, -\sin t, 0 \rangle = \frac{1}{2}$$