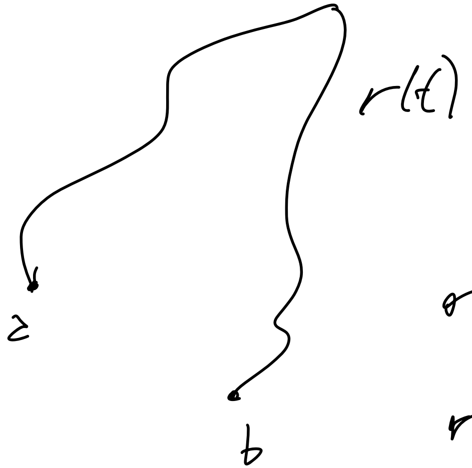
Surface Integrals

Lecture for 7/8

Parameterizing Surfaces

1D curves needed 1 parameter, 2D surfaces will need 2 parameters

- Just as with curves, must include bounds on parameters



$\Rightarrow \quad \int F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt$

$F = \nabla f, \quad \int = f(r(t))\Big|_a^b$

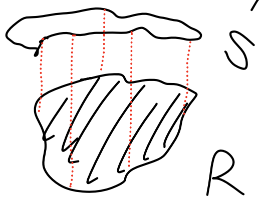
$r(t), \quad a \le t \le b \Rightarrow$ curve

$r(s,t), \quad a \le s \le b, \quad c \le t \le d \Rightarrow$ 2D surface

But how do we write down LHS for $r(s,t) = \ldots$

and how do we find the bounds on s & t?

Common scenario: surface can be given as $z = f(x,y)$ for (x,y) in some region R.



In the process of analyzing R, you may have found bounds on x & y, such as $0 \le y \le 1$ and $y^2 \le x \le y$. Then:

$r(s,t) = f(s,t)$, bound on s is going to be your bound on x, bound on t is going to be your bound on y.

All standard disclaimers in general remarks about freedom of choice of letters, make sure work is clear, don't leave out bounds when stating your answer, define anything you use etc. apply

Example: surface is given by $z = x^2+y^2$ over ellipse $x^2 + \frac{y^2}{4} = 1$.

Use polar: $x = r\cos\theta$, $y = 2r\sin\theta$, then bounds are $0 \le r \le 1$, $0 \le \theta < 2\pi$

Then $r(s,t) = s^2+t^2$ is a good start, but we can convert to new variables.

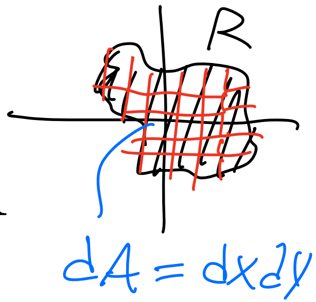
$s = u\cos v$, $t = 2u\sin v$, $\underline{0 \le u \le 1,\ 0 \le v < 2\pi}$

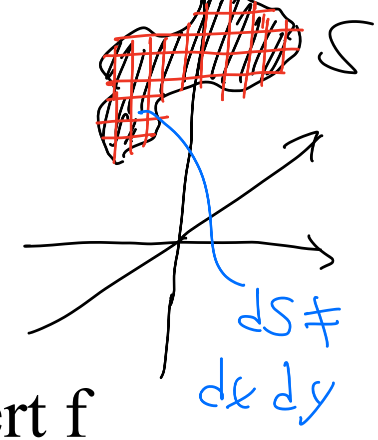
Plug in: $\underset{z \text{ coordinate of}}{\underline{r(u,v)}} = u^2\cos^2 v + 4u^2\sin^2 v = \underline{u^2(1+3\sin^2 v)}$.

So $r(u,v) = (u, v, u^2(1+3\sin^2 v))$.

Note: you may also have the same common scenario, but for $y = f(x,z)$ or $x = f(y,z)$

Surface Integral





Suppose S is parametrized by r(u, v)

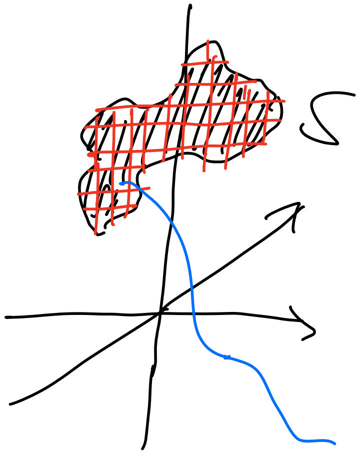
- To calculate $\iint_S f\, dS$, we need bounds, convert dS, convert f
- If $(x_0, y_0, z_0) = r(u_0, v_0)$, then $f(x_0,y_0,z_0) = f(r(u_0, v_0))$
- Bounds will be the bounds on r, let's suppose $0 \leq u, v \leq 1$
- $dS = \|r_u \times r_v\|\, du\, dv$
- In the end, we get $\int_0^1 \int_0^1 f(r(u,v))\, \|r_u \times r_v\|\, du\, dv$

Note: you will get some starting surface integral after plugging in your parametrization for r, your function f, and the calculation for dS. But you still have the option of coordinate conversions as in prior content on order switching, polar, Jacobian etc.

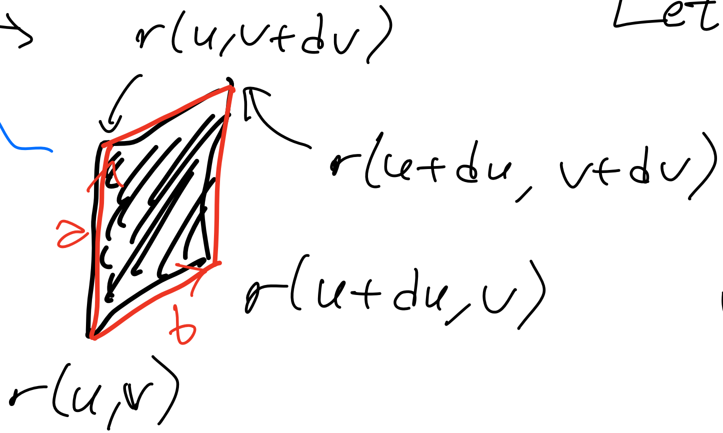
Differential Conversion Derivation

Note: for surface integrals requiring or desiring conversion of coordinates, you can convert 1st & then write down double integral, or you can write down double integral & then convert. Which option is best depends on the situation, use your judgement.

General Idea ~~for~~ Derivation: proceed along the same line as the Jacobian derivation, but with a surface instead of a nD region. Zoom into surface until it looks like a plane and each dS becomes a small parallelogram, which we can find area of with cross product.



Suppose each small piece of S is made by moving u slightly or v slightly.



Let a & b be the vectors of the parallelogram which approximates a small piece with corner $r(u,v)$.

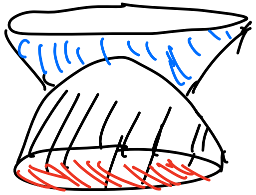
Recall area of parallelogram generated by a & b will be $\|a \times b\|$. $a = r(u+du,v) - r(u,v) = \frac{r(u+du,v)-r(u,v)}{du}du$ $= r_u du$, similarly $b = r_v dv$. So

$$dS = \|r_u du \times r_v dv\| = \|r_u \times r_v\| du dv$$

$$\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$\left.\begin{array}{l} \times \to \text{1st} \\ \times \to \text{2nd} \quad \times \to \text{3rd comp.} \end{array}\right\} \to \text{sum of squares} \xrightarrow{\sqrt{\ }} \text{magnitude} \to \text{ans}$$

Cross products can be annoying to compute, so let's see what happens in the scenario that S is described by one simple function $f(u,v)$ and



$$r(u,v) = (u, v, f(u,v)).$$

$$r_u = (1, 0, f_u), \quad r_v = (0, 1, f_v),$$

$$r_u \times r_v = (-f_u, f_v, 1) \Rightarrow \|r_u \times r_v\| = \sqrt{f_u^2 + f_v^2 + 1},$$

same formula we found for surface area

Practice Problems

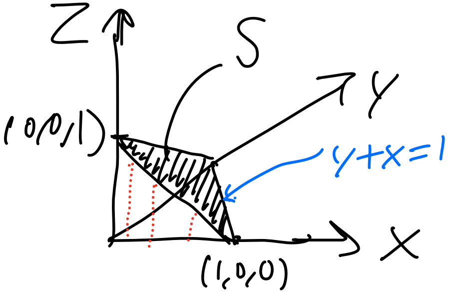
Evaluate $\iint_S f\, dS$ for the following functions and surfaces

- $f(x,y,z) = 6xy$, S is portion of $x+y+z = 1$ in 1st octant
- $f(x, y, z) = z$, S is upper half of sphere of radius 1
- $f(x,y,z) = y+z$, S is surface $x^2+y^2 \leq 3$ on the bottom, $z = 4-y$ on top, and the cylinder $x^2+y^2 = 3$ on the sides

Try these until 1:05

Scratchwork

f(x,y,z) = 6xy, S is portion of x+y+z = 1 in 1st octant



1st octant $\Rightarrow x,y,z \geq 0$.

Can we use a common scenario?

$x+y+z=1$ becomes $z = 1-x-y$, so

we are in situation $z = g(x,y)$ with bounds

on x & y and can proceed this way.

$\begin{matrix} x \\ y \\ z \end{matrix} \geq 0$ & $x+y+z=1 \Rightarrow 0 \leq y \leq x+y+z = 1$.

Now $x \geq 0$ & $x = 1-y-z \leq 1-y$.

So our bounds are $0 \leq y \leq 1$ & $0 \leq x \leq 1-y$.

Let $r(u,v) = (u, v, 1-u-v)$, $0 \le u \le 1-v$, $0 \le v \le 1$.

By shortcut for common scenario,

$$\|r_u \times r_v\| = \sqrt{1+g_u^2+g_v^2} = \sqrt{1+(-1)^2+(-1)^2} = \sqrt{3}$$

where $g(u,v) = 1-u-v$.

So $$\iint_S f\,dS = \iint_{\text{bounds}} f(u,v,1-u-v)\cdot\sqrt{3}\,du\,dv =$$

$$6\sqrt{3}\int_0^1\int_0^{1-v} uv\,du\,dv = 6\sqrt{3}\int_0^1 v\,\frac{u^2}{2}\Bigg|_{u=0}^{u=1-v} dv =$$

$$3\sqrt{3}\int_0^1 v(1-v)^2\,dv = 3\sqrt{3}\int_0^1 (v^3-2v^2+v) = 3\sqrt{3}\left(\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right)$$

$$= 3\sqrt{3}\left(\frac{3-8+6}{12}\right) = \frac{3\sqrt{3}}{12} = \sqrt{3}/4 = 0.25\sqrt{3}$$