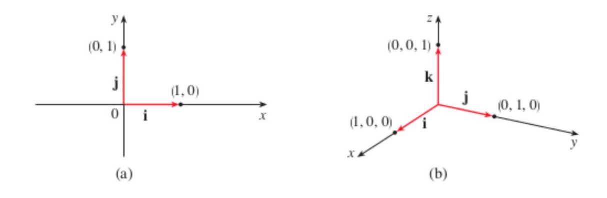
Review for LQ1: Calc 1 and 2 (differentiation and u-sub)

Defn: • n-dimensional vector $\vec{a} = \langle a_1, \ldots, a_n \rangle$ has "n directions" and length

$$\|\vec{a}\| = |\vec{a}| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

• V_n is the set of all n-dimensional vectors.

Defn. • V_3 has 3 standard basis vectors $i = \langle 1,0,0 \rangle$, $j = \langle 0,1,0 \rangle$, $k = \langle 0,0,1 \rangle$.

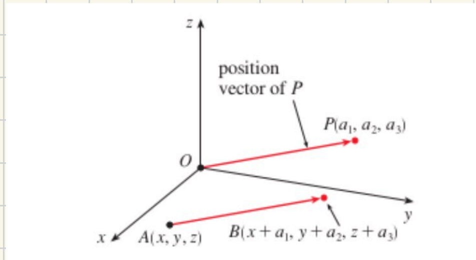


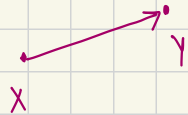
• Any vector in V_3 can be written as $\vec{a} = \langle a_1, a_2, a_3 \rangle = a_1 i + a_2 j + a_3 k$

and has length $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

<u>Important facts:</u> ① For points $X(x_1, x_2, x_3)$ and $Y(y_1, y_2, y_3)$ the vector $\overrightarrow{XY}$ is

such that $Y = X + \overrightarrow{XY}$.





$\Rightarrow \overrightarrow{XY} = \langle y_1 - x_1,\ y_2 - x_2,\ y_3 - x_3 \rangle$ " $= Y - X$ "

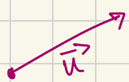
② • Unit vectors: for any vector $\vec{u} \in V_n$, the unit vector along $\vec{u}$ is the vector with length 1 and same direction as $\vec{u}$.

• So for any vector $\vec{u}$, $\vec{v} = \dfrac{1}{|\vec{u}|} \cdot \vec{u}$ is the unit vector along $\vec{u}$.

<u>Aside:</u> $|\vec{v}| = \left| \dfrac{1}{|\vec{u}|} \vec{u} \right|$

$= \dfrac{1}{\cancel{|\vec{u}|}} \cdot \cancel{|\vec{u}|}$

$= 1$



$\vec{v} = c \cdot \vec{u}$

$c = \dfrac{1}{|\vec{u}|}$

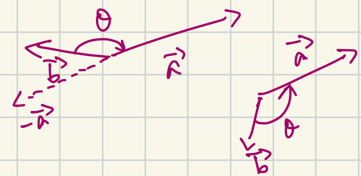
Webassign: HW 2 due 9/8

Hw 3 due 9/15

Section 12.3:

Def: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$. Then

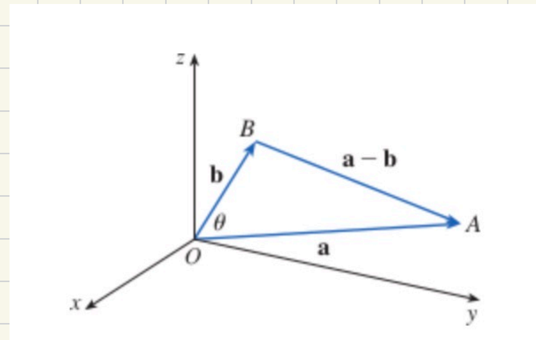
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$



2 Properties of the Dot Product

If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$



Def: θ is the angle in the triangle with sides $\vec{a}, \vec{b}, \vec{a} - \vec{b}$ that is opposite the side $\vec{a} - \vec{b}$.

Thm: If $0 \leq \theta \leq \pi$ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$$

Corollary: $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$

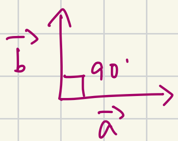
Example: Find the angle between the vectors $\vec{a} = \langle 2,2,-1 \rangle$ and $\vec{b} = \langle 5,-3,2 \rangle$.

$$\vec{a} \cdot \vec{b} = 2 \cdot 5 + 2 \cdot (-3) + (-1) \cdot 2 = 10 - 6 - 2 = 2$$

$$|\vec{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$|\vec{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{25+13} = \sqrt{38}$$

$$\Rightarrow \cos\theta = \frac{2}{3 \cdot \sqrt{38}} \Rightarrow \theta = \arccos\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \text{ or } 84^\circ$$



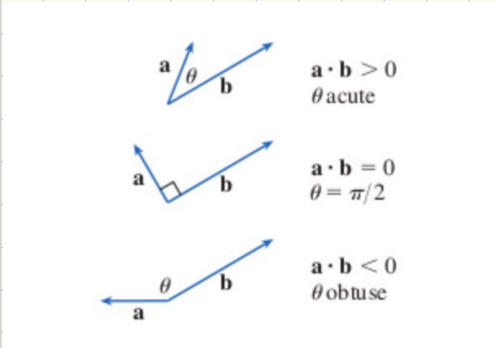
$$\Rightarrow \theta = 90^\circ \Rightarrow \cos\theta = 0 \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot 0 = 0$$

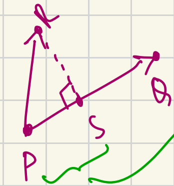
If $\vec{a} \cdot \vec{b} = 0$, $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0 \Rightarrow \cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = 0 \Rightarrow \theta = \pi/2$

Thm: $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

Example: Show that $2i + 2j - k$ is perpendicular to $5i - 4j + 2k$.

Answer: $\langle 2, 2, -1 \rangle \cdot \langle 5, -4, 2 \rangle = 10 - 8 - 2 = 0$.





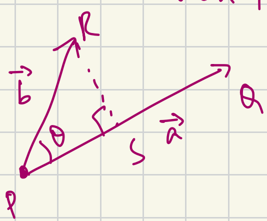
" shadow of $\vec{PR}$ onto $\vec{PQ}$ "

Projections:

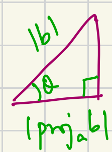
Problem statement: You are given two vectors $\vec{PR}$ and $\vec{PQ}$.

Find a representation for the vector $\vec{PS}$ in terms of $\vec{PR}$ and $\vec{PQ}$.

Answer:



$\Rightarrow$

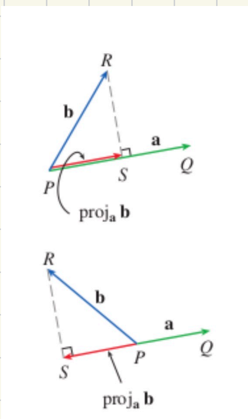


$\Rightarrow \cos\theta = \dfrac{|\vec{PS}|}{|\vec{b}|}$

$\vec{PS} = \text{proj}_a b$

$\Rightarrow |\text{proj}_a b| = |\vec{b}| \cdot \cos\theta$

Caution: this quantity can be negative.



<u>Definition</u>: • Scalar projection of $\vec{b}$ onto $\vec{a}$ (called the component of b along a) is defined as $|b| \cos\theta$ (signed magnitude of the projection)

$$\to \text{comp}_a b = |b| \cdot \cos\theta = |b| \cdot \frac{(a \cdot b)}{|a| \cdot |b|} = \frac{a \cdot b}{|a|}$$

• Vector projection of $\vec{b}$ onto $\vec{a}$:

$$\underbrace{\text{proj}_a b}_{\vec{PS}} = \text{comp}_a b \cdot \vec{a} = \underbrace{\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right)}_{\text{scala}} \cdot \underbrace{\left(\frac{\vec{a}}{|\vec{a}|}\right)}_{\text{unit vector along } \vec{a}}$$

<u>Example</u>: Find the scalar and vector projections of $\vec{b} = \langle 1, 1, 2 \rangle$ onto $\vec{a} = \langle -2, 3, 1 \rangle$.

<u>A</u>:

$$\vec{a} \cdot \vec{b} = -2 + 3 + 2 = 3$$

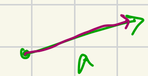
$$|\vec{a}| = \sqrt{4 + 9 + 1} = \sqrt{14}$$

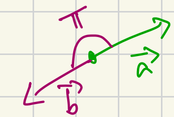
$$\text{comp}_a b = \frac{a \cdot b}{|\vec{a}|} = \frac{3}{\sqrt{14}}$$

$$\Rightarrow \text{proj}_a b = \frac{3}{\sqrt{14}} \cdot \frac{\vec{a}}{\sqrt{14}} = \frac{3}{14} \langle -2, 3, 1 \rangle$$

$$= \left\langle \frac{-6}{14}, \frac{9}{14}, \frac{3}{14} \right\rangle = \left\langle \frac{-2}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

Fact: $\vec{a}$ and $\vec{b}$ are parallel if and only if $\theta = 0$ or π



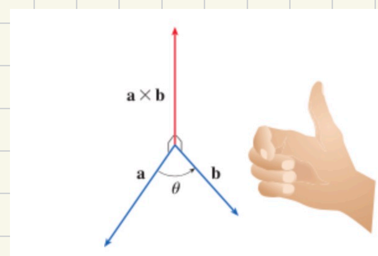


Aside: That is $\vec{a}$ and $\vec{b}$ are parallel

if and only if

$a \cdot b = |a| \cdot |b|$ OR $a \cdot b = -|a| \cdot |b|$

Section 12.4:



Cross product: $a \times b$ is a vector $\vec{c}$ that is

$\langle c_1, c_2, c_3 \rangle$

orthogonal to both $\vec{a}$ and $\vec{b}$: $\vec{b} \longrightarrow \langle b_1, b_2, b_3 \rangle$, $\vec{a} \longrightarrow \langle a_1, a_2, a_3 \rangle$

That is $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$.

So we need to solve the following system

$$a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$$

$$b_1 c_1 + b_2 c_2 + b_3 c_3 = 0$$

<u>Definition</u>: · Determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

· Determinant of $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

is $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

<u>Definition</u>: $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underbrace{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}}_{c_1} i - \underbrace{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}}_{c_2} j + \underbrace{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}_{c_3} k$$

$$\Rightarrow \vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\, i - (a_1 b_3 - a_3 b_1)\, j + (a_1 b_2 - a_2 b_1)\, k$$