Textbook Sections: 16.2, 16.3, 16.4

Topics: Line integrals, The Fundamental Theorem for Line Integrals, conservative vector field theorems, Green's Theorem

Instructions: Try each of the following problems, show the detail of your work.

Clearly mark your choices in multiple choice items. Justify your answers.

Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

1. Determine whether or not $\mathbf{F}$ is a conservative vector field, where $\mathbf{F}(x, y) = (xy + y^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$. If it is, find a function f such that $\mathbf{F} = \nabla f$.

Let $P(x, y) = xy + y^2$ and $Q(x, y) = x^2 + 2xy$.
$P_y = x + 2y$ and $Q_x = 2x + 2y$.
Since $P_y \neq Q_x$, by the contrapositive of Theorem 5, we have that $\mathbf{F}(x, y)$ is not a conservative vector field.

2. Determine whether or not $\mathbf{F}$ is a conservative vector field, where $\mathbf{F}(x, y) = ye^x\mathbf{i} + (e^x + e^y)\mathbf{j}$. If it is, find a function f such that $\mathbf{F} = \nabla f$.

Let $P(x, y) = ye^x$ and $Q(x, y) = e^x + e^y$, with $P_y = e^x$ and $Q_x = e^x$.
The domain of F(x,y) is the real plane which is an open simply connected region, and the functions P and Q have continuous first-order partial derivatives on this domain. By Theorem 6, since $P_y = Q_x$, we have that $\mathbf{F}(x, y)$ is a conservative vector field, i.e. there exists a scalar function f such that $\mathbf{F} = \nabla f$.
This is equivalent to $< P, Q >=< f_x, f_y >$
If we integrate $f_x = P$ with respect to x, we get $f(x, y) = \int P(x, y)dx = \int ye^x dx = ye^x + g(y)$, where $g(y)$ is a function that does not contain x, so g is constant with respect to x. This function f needs to have the partial derivative with respect to y equal to Q.
$f_y = Q$ gives us the equation $e^x + g'(y) = e^x + e^y$, from which we conclude that $g'(y) = e^y$. To find $g(y)$, we integrate $g'(y) = e^y$ with respect to y. Then, $g(y) = e^y + c$, where c is a constant that does not contain x nor y. Then, $f(x, y) = ye^x + e^y + c$. Thus, a potential function is $f(x, y) = ye^x + e^y$, another potential function is $f(x, y) = ye^x + e^y - 150$, and so on.

3. (a) Find a function f such that $\mathbf{F} = \nabla f$, where $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$.

a) The domain of F(x,y) is the real plane which is an open simply connected region, and the functions P and Q have continuous first-order partial derivatives on this domain. We first check whether $\mathbf{F}(x, y)$ is a conservative vector field. Let $P(x, y) = (1 + xy)e^{xy}$ and $Q(x, y) = x^2e^{xy}$. Then $P_y = e^{xy}(2x + x^2y)$ and $Q_x = e^{xy}(2x + x^2y)$. By Theorem 6, since $P_y = Q_x$, we have that $\mathbf{F}(x, y)$ is a conservative vector field, i.e. there exists a scalar function f such that $\mathbf{F} = \nabla f$.
This is equivalent to $< P, Q >=< f_x, f_y >$
Method 1: We integrate $f_x = P$ with respect to x. Using integration by parts, we get
$f(x, y) = \int P(x, y)dx = \int (1 + xy)e^{xy}dx = \frac{1+xy}{y}e^{xy} - \frac{y}{y^2}e^{xy} + g(y) = xe^{xy} + g(y)$, where $g(y)$ is a function that does not contain x, so g is constant with respect to x. This function f needs to have the partial derivative with respect to y equal to Q.

$f_y = Q$ gives us the equation $x^2e^{xy} + g'(y) = x^2e^{xy}$ from which we conclude that $g'(y) = 0$. Then, $g(y) = c$, where c is a constant that does not contain x nor y. Then, $f(x, y) = xe^{xy} + c$. Thus, a potential function is $f(x, y) = xe^{xy}$, another potential function is $f(x, y) = xe^{xy} - 20$, and so on.

Method 2: If we integrate $f_y = Q$ with respect to y, we do not need to use integration by parts. We get $f(x, y) = \int Q(x, y)dy = \int x^2e^{xy}dy = xe^{xy} + h(x)$, where $h(x)$ is a function that does not contain y, so h is constant with respect to y. This function f needs to have the partial derivative with respect to x equal to P.

By the product rule, $f_x = P$ gives us the equation $1 \cdot e^{xy} + xye^{xy} + h'(x) = (1+xy)e^{xy}$ from which we conclude that $h'(x) = 0$. Then, $h(x) = c$, where c is a constant that does not contain x nor y. Then, $f(x, y) = xe^{xy} + c$. Thus, a potential function is $f(x, y) = xe^{xy}$, another potential function is $f(x, y) = xe^{xy} - 20$, and so on.

(b) Then evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C, where $C : \mathbf{r}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j}$, $0 \le t \le \pi/2$

b) The curve C given by $\mathbf{r}(t)$ is smooth, and f is continuous with a continuous gradient vector on C. By the Fundamental Theorem for Line Integrals (Theorem 2), we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) = f(\mathbf{r}(a))$. Here, $a = 0$ and $b = \pi/2$, $\mathbf{r}(\pi/2) =< 0, 2 >$ and $\mathbf{r}(0) =< 1, 0 >$. Then, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 2) - f(1, 0) = 0 - 1 = -1$.

4. Show that the line integral $\int_C \sin y dx + (x \cos y - \sin y)dy$ is independent of the path, where C is any path from $(2, 0)$ to $(1, \pi)$. Then, evaluate the integral.

Letting $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \langle \sin(y), x\cos(y) - \sin(y)\rangle$, we want to know if $\mathbf{F}(x, y)$ is conservative. The domain of F(x,y) is the real plane which is an open simply connected region, and the functions P and Q have continuous first-order partial derivatives on this domain. $P_y = \cos(y)$ and $Q_x = \cos(y)$. By Theorem 6, since $P_y = Q_x$, we have that $\mathbf{F}(x, y)$ is a conservative vector field. By the Fundamental Theorem of Line Integrals, the line integral is independent of path. If $\mathbf{F}(x, y)$ is a conservative vector field, then, there exists a scalar function f such that **F** $= \nabla f$. This is equivalent to $< P, Q >=< f_x, f_y >$. We integrate $f_x = P$ with respect to x, and get $f(x, y) = \int P(x, y)dx = \int \sin(y)dx = x\sin(y) + g(y)$. where $g(y)$ is a function that does not contain x, so g is constant with respect to x. This function f needs to have the partial derivative with respect to y equal to Q. $f_y = Q$ gives us the equation $x\cos(y) + g'(y) = x\cos(y) - \sin(y)$ from which we conclude that $g'(y) = -\sin(y)$. Then, $g(y) = \cos(y) + c$, where c is a constant that does not contain x nor y. Then, $f(x, y) = x\sin(y) + \cos(y) + c$. Thus, a potential function is $f(x, y) = x\sin(y) + \cos(y)$

Let C be any path connecting $(2, 0)$ to $(1, \pi)$. By the Fundamental Theorem for line integrals, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, \pi) - f(2, 0) = -1 - 1 = -2$.

Green's Theorem:

5. Use Green's Theorem to evaluate the line integral

$$\int_C (y + e^{\sqrt[3]{x^5}})dx + (2x + \cos(y^2))dy$$

along the positively oriented curve C that is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

C is a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. P and Q have continuous partial derivatives on the real plane which is an open region that contains D. By Green's Theorem, we have:

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_D (Q_x - P_y)dA.$$

We can describe D as a Type 1 region given by:

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}.$$

Then the double integral above becomes:

$$\begin{aligned}
\iint_D (Q_x - P_y)dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-1)dydx \\
&= \int_0^1 \Big[y\Big]_{x^2}^{\sqrt{x}} dx \\
&= \int_0^1 (\sqrt{x} - x^2)dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 \\
&= \frac{1}{3}.
\end{aligned}$$

6. Use Green's Theorem to find the work done by the force $\mathbf{F}(x,y) = x(x+y)\mathbf{i} + xy^2\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1,0)$, then along the line segment to $(0,1)$, and then back to the origin along the y-axis.

The work done by $\mathbf{F}(x,y)$ on the particle along this (closed) path C is given by $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P(x,y)dx + Q(x,y)dy$. C is a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. P and Q have continuous partial derivatives on the real plane which is an open region that contains D. Since all the conditions for Green's theorem are satisfied, we can apply Green's Theorem: $\oint_C (P(x,y)dx + Q(x,y)dy) = \iint_D (Q_x - P_y)dA$, where D is the region bounded by C.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D (Q_x - P_y)dA \\
&= \int_0^1 \int_0^{1-x} (y^2 - x)dydx \\
&= \int_0^1 \left[\frac{y^3}{3} - xy\right]_{y=0}^{y=1-x} dx \\
&= \int_0^1 \left(\frac{(1-x)^3}{3} - x(1-x)\right) dx \\
&= \left[\frac{-(1-x)^4}{12} - \frac{x^2}{2} + \frac{x^3}{3}\right]_{x=0}^{x=1} \\
&= -\frac{1}{2} + \frac{1}{3} + \frac{1}{12} \\
&= -\frac{1}{12}.
\end{aligned}$$

The curl and the divergence of a vector field:

7. Find the curl and the divergence of the following vector fields.

(a) $\mathbf{F}(x, y, z) = x^3yz^2\mathbf{j} + y^4z^3\mathbf{k}$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x^3yz^2 & y^4z^3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(y^4z^3) - \frac{\partial}{\partial z}(x^3yz^2)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(y^4z^3) - \frac{\partial}{\partial z}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(x^3yz^2) - \frac{\partial}{\partial y}(0)\right]\mathbf{k}$$

$$= (4y^3z^3 - 2x^3yz)\mathbf{i} - (0-0)\mathbf{j} + (3x^2yz^2 - 0)\mathbf{k}$$

$$= (4y^3z^3 - 2x^3yz)\mathbf{i} + 3x^2yz^2\mathbf{k}$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(y^4z^3) = 0 + x^3z^2 + 3y^4z^2 = x^3z^2 + 3y^4z^2.$$

(b) $\mathbf{F}(x, y, z) = \ln(2y + 3z)\mathbf{i} + \ln(x + 3z)\mathbf{j} + \ln(x + 2y)\mathbf{k}$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln(2y+3z) & \ln(x+3z) & \ln(x+2y) \end{vmatrix}$$

$$= \left(\frac{2}{x+2y} - \frac{3}{x+3z}\right)\mathbf{i} - \left(\frac{1}{x+2y} - \frac{3}{2y+3z}\right)\mathbf{j} + \left(\frac{1}{x+3z} - \frac{2}{2y+3z}\right)\mathbf{k}$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}[\ln(2y+3z)] + \frac{\partial}{\partial y}[\ln(x+3z)] + \frac{\partial}{\partial z}[\ln(x+2y)] = 0 + 0 + 0 = 0.$$

8. Consider the following vector field $\mathbf{F}(x, y, z) = (x^2\ln(y+1))\mathbf{i} + (y^2z^3)\mathbf{j} + \left(\frac{z}{y}\right)\mathbf{k}$.

(a) Show all calculations for finding the curl of $\mathbf{F}$.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{j}$$

$$\text{curl } \mathbf{F} = \left(-\frac{z}{y^2} - 3y^2z^2\right)\mathbf{i} - (0-0)\mathbf{j} + \left(0 - \frac{x^2}{y+1}\right)\mathbf{k}$$

$$\text{curl } \mathbf{F} = \left\langle -\frac{z}{y^2} - 3y^2z^2, 0, -\frac{x^2}{y+1} \right\rangle$$

(b) Show all calculations for finding the divergence of $\mathbf{F}$.

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\text{div } \mathbf{F} = 2x\ln(y+1) + 2yz^3 + \frac{1}{y}$$

Suggested Textbook Problems

Section 16.3	3-10, 13-24,29, 30
Section 16.4	1-14, 17, 18, 21
Section 16.5	1-20, 25-34

SOME USEFUL DEFINITIONS, THEOREMS AND NOTATION:

The Fundamental Theorem for Line Integrals – Theorem 2 in Section 16.3
Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, where $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Connections between Independence of path of line integrals, conservative vector fields and the partial derivatives of the components of F

Theorem 3. $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.

Theorem 4. Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Theorem 5. If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem 6. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ throughout } D.$$

Then $\mathbf{F}$ is conservative.

Green's Theorem
Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

Curl
If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, R all exist, then the Curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{j}$$