Basic 3D integrals

Lecture for 6/27

General Idea

- We have seen $\int f(x)\, dx$ and $\iint_R f(x, y)\, dA$
- Next step forward is $\iiint_E f(x, y)\, dV$
- Riemann sums can be done just as for 2D integrals
- If R is a prism, new version of Fubini's Theorem applies
- Can integrate with general regions
- Can swap order of integration with some care
- Center of mass & average value generalize

Riemann Sums

- Suppose E is a prism $a_1 \leq x \leq a_2$, $b_1 \leq y \leq b_2$, $c_1 \leq z \leq c_2$
- Can approximate $\iint_V f(x, y, z)\, dV$ with Riemann sums

New Fubini

Suppose E = [a, b] x [c,d] x [e, f] is a prism

- Then $\iiint_E f(x, y, z)\, dV = \int_f^e(\int_c^d(\int_a^b f(x,y,z)\, dx)\, dy)\, dz$ = any of the other 5 orders for integrating over x, y, z
- Repeatedly apply standard Fubini to prove

General 3D integrals

How do we find the integral over a general 3D region E?

- Suppose $a \leq z \leq b$, but region isn't a prism
- For any c in [a, b], let R_c be the cross section of E with $z = c$
- R_c is a general 2D integral
  - For example, $d \leq y \leq e$ and $g(y) \leq x \leq h(y)$
- Combining dependencies: $g(z) \leq y \leq h(z)$, $r(y, z) \leq x \leq s(y, z)$
- Can try this with any order of x, y, and z

Switching Order of Integration

- Option 1: keep outer integral, switch inner 2
  - No issue, just use same tactics as in 2D switches
- Option 2: switch outermost integral
  - Must be very careful to make sure new bounds correct

Average Value & Center of Mass

Average value of function f over solid space E:

- Equal to $(1/V) \iiint_E f(x, y, z)\, dV$ where V is volume of E

Center of mass of E with weight function f is $(x_{COM}, y_{COM}, z_{COM})$

- $x_{COM} = (\iiint_E x\, f(x, y, z)\, dV)/(\iiint_E f(x, y, z)\, dV)$
- Similarly for y_{COM}, z_{COM}

Practice Problems

Evaluate $\iiint_E f(x, y, z)\, dV$ for these functions and regions:

- $f(x, y, z) = x$, E is region under $2x+3y+z = 6$ in the 1st octant
- $f(x,y, z) = (3x^2+3z^2)^{1/2}$, E is region bound by $y = 2x^2+2z^2$ and $y = 8$
- $f(x, y, z) = yz$, E is region bound by $x = 2y^2+2z^2-5$ and $x = 1$

Find the volume of the solid bound by $z = 8-x^2-y^2$, $z = -2(x^2+y^2)^{1/2}$, and $x^2+y^2 = 4$

Scratchwork