Quiz 4 Solutions

1: If $P(A) = P(B)$ for two events A and B, does $P(A|B) = P(B|A)$?

No. To disprove this, we need to find an example where $P(A) = P(B)$ and $P(A|B) \neq P(B|A)$. Suppose that $P(A) = P(B) \neq 0$. By Bayes' Theorem, $P(B|A) = P(A|B)P(B)/P(A) = P(A|B)$. So we should look for an example with $P(A) = P(B) = 0$*.

Pick a point randomly from a square. Let A be the event of picking a corner and B the event of picking a point on the perimeter. There are infinitely many points in the square and only 4 corners. Since 4 points work and infinitely many don't work, $P(A) = 0$. The perimeter is 4 equal line segments and the square is made of infinitely many disjoint line segments, so similarly $P(B) = 0$.

To justify $P(B) = 0$ more carefully**, let N, E, S, W be the events of picking a point from the north, east, south, west perimeter edge respectively. Then $P(B) = P(N \text{ or } E \text{ or } S \text{ or } W) \leq P(N)+P(E)+P(S)+P(W) = 4P(N)$ since each segment is identical. But $P(N) = 0$ since the square is made of infinitely many disjoint identical horizontal segments and the north edge is just one of those. So $P(B) \leq 0$, which means $P(B) = 0$.

Every corner is on the perimeter, so if A happens then B always happens. This means $P(B|A) = 1$. There are 4 corners on the perimeter and infinitely many points on the perimeter which aren't corners, so if we know B happens then A still has probability 0 of happening. Thus, $P(A|B) = 0$.

*As mentioned in previous classes, X impossible -> $P(X) = 0$, but $P(X) = 0$ does not mean that X is impossible.

**You do not have to be this rigorous. This is for the sake of understanding.

2: Mesothelioma affects 0.01% of people. You tested positive with a test that is 99% accurate. What is the probability you actually have Mesothelioma?

Let +, M be the event of testing positive, having Mesothelioma respectively. We seek $P(M|+) = P(+|M)P(M)/P(+)$. Since the test is 99% accurate, it will be positive 99% of the time if you have it. This means $P(+|M) = 0.99$. Also, if you don't have Mesothelioma, the test will be positive 1% of the time because it's 1% inaccurate. Thus, $P(+|M^c) = 0.01$. Converting 0.01% to a probability, we get $P(M) = 0.0001$.

For the denominator, note P(+) = P(+ & M) + P(+ & M^c). We have P(+ & M) = P(+|M)P(M) = 0.99*0.0001 and P(+ & M^c) = P(+|M^c)P(M^c) = 0.01*0.9999.

Plugging in all the values, we have (.99*.0001)/(.99*.0001+.01*.9999). Multiplying by 100*10000 to clear the zeroes, then dividing by 9, then dividing by 11, we get (99*1)/(99*1+1*9999) = 99/(99+9999) = 11/(11+1111) = 1/(1+101) = 1/102 as our final answer.

Note how small this number is. You'd think since the test is so accurate that a positive result means you're likely to have it, but in reality it's still unlikely because of how rare the disease is.

3: Mark the following true or false **(1 point each)**

(a): A system of 2 linear equations in 3 variables can have a unique solution

False. With fewer equations than variables, you can't* have a unique solution.

Explanation of *: If you end up with a contradiction like 0 = 1, there are no solutions. If you manage to use all the equations, then there are still variables leftover. These will be free variables, can take infinitely many values, and will lead to infinitely many solutions.

(b): If a system of 3 linear equations in 3 variables has two solutions, it must have a 3rd solution

True. A system of linear equations has 0, 1, or ∞-many solutions*. If there's at least 2, then 0 and 1 are both impossible, so there's ∞-many, which means there's at least 3 since ∞ ≥ 3.

*You can explain or prove this as well, but it's better to already know this fact because it comes up a lot.

(c): In a row reduced matrix, the leading entry in any row is the largest entry in that row.

False. The one row matrix [1 2 0] is row reduced, has leading entry 1, but 1 is not the largest entry since 1 < 2.

Some students may say it's not fair to have a 1 row matrix, or that it feels like cheating. You can add the row [0 0 1] underneath. This is still row reduced and still serves as a counterexample.