Answer Key

University of Delaware
Department of Mathematical Sciences
MATH 243 Midterm Exam 2
Fall 2025

Monday 3rd November, 2025

Instructions:

- The time allowed for completing this exam is **90** minutes in total.
- Check your examination booklet before you start. There should be **8** questions on **9** pages.
- Turn off your cell phone and put it away. Headsets, and any other electronic devices are prohibited.
- No calculators.
- Answer the questions in the space provided. If you need more space for an answer, continue your answer on the back of the page and/or the margins of the test pages. No extra paper. *Do not separate the pages from the exam booklet.*
- For full credit, sufficient work must be shown to justify your answer.
- Partial credit will not be given if appropriate work is not shown.
- Write legibly and clearly; indicate your final answer to every problem. Cross out any work that you do not want graded. If you produce multiple solutions for a problem, indicate clearly which one you want graded.
- **Any form of academic misconduct will result in a failing grade.**

Question:	1	2	3	4	5	6	7	8	Total
Points:	9	18	8	10	24	10	10	11	100
Score:									

1. (9 points) Let $W = xy\sqrt{z}$,

$$\text{where } x = \ln(st), \quad y = \tan(s+2t) \quad \text{and} \quad z = s^2e^{2t}.$$

Find $\dfrac{\partial W}{\partial t}$. Your final answer **must be** in terms of only s and t. Do NOT simplify.

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial W}{\partial z}\frac{\partial z}{\partial t} \text{ (2 points)}$$

$$= y\sqrt{z}\cdot\frac{1}{t} + x\sqrt{z}\cdot 2\sec^2(s+2t) + \frac{xy}{2\sqrt{z}}\cdot 2s^2e^{2t} \text{ (1 point) each partial derivative}$$

$$= \tan(s+2t)\sqrt{s^2e^{2t}}\cdot\frac{1}{t} + \ln(st)\sqrt{s^2e^{2t}}\cdot 2\sec^2(s+2t) + \frac{\ln(st)\tan(s+2t)}{2\sqrt{s^2e^{2t}}}\cdot 2s^2e^{2t} \text{ (1 pt)}$$

$$= \tan(s+2t)\frac{se^t}{t} + 2\ln(st)se^t\sec^2(s+2t) + \ln(st)\tan(s+2t)se^t$$

Answer : $\tan(s+2t)\sqrt{s^2e^{2t}}\cdot\dfrac{1}{t} + \ln(st)\sqrt{s^2e^{2t}}\cdot 2\sec^2(s+2t) + \dfrac{\ln(st)\tan(s+2t)}{2\sqrt{s^2e^{2t}}}\cdot 2s^2e^{2t}$

OR $\tan(s+2t)\dfrac{se^t}{t} + 2\ln(st)se^t\sec^2(s+2t) + \ln(st)\tan(s+2t)se^t$

2. Consider the function

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5.$$

(a) (5 points) Find the critical point(s) of f that lie in the region $x^2 + y^2 < 16$.

To find the critical points, we are taking the first order partial derivatives.

$$f_x = 4x - 4 = 0 \to x = 1 \text{ (2 point)}$$

$$f_y = 6y = 0 \to y = 0 \text{ (2 point)}$$

Since $1^2 + 0^2 = 1 < 16$, this point lies in the region. (1 point)
So, the only critical point is $(1, 0)$.

(b) (8 points) Use the method of Lagrange multipliers to find the extreme values of the function subject to the constraint

$$g(x, y) = x^2 + y^2 = 16.$$

$$\nabla f = \lambda \nabla g \text{ (2 points)} \tag{1}$$

$$\langle 4x - 4, 6y \rangle = \lambda \langle 2x, 2y \rangle \tag{2}$$

$$4x - 4 = 2\lambda x \text{ (1 point)} \to 2x - 2 = \lambda x \tag{3}$$

$$6y = 2\lambda y \text{ (1 point)} \to 3y - \lambda y = 0 \to y(3 - \lambda) = 0 \tag{4}$$

$$x^2 + y^2 = 16 \tag{5}$$

From Equation (4), either $y = 0$ or $\lambda = 3$ (1 point)

- If $y = 0$, $x^2 + 0^2 = 16$ implies that $x = \pm 4$ (1 point). We found 2 points $(x, y) = (\pm 4, 0)$. (0.5 point)
- If $\lambda = 3$, Equation 3 gives $2x - 2 = 3(x) \to x = -2$ and substituting this value to (5), we obtain $4 + y^2 = 16$, so $y = \pm\sqrt{12} = \pm 2\sqrt{3}$. (1 point) We found another two points $(x, y) = (-2, \pm 2\sqrt{3})$. (0.5 point)

(c) (5 points) Using the results from part (a) and (b), determine the absolute maximum and absolute minimum of the function

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5 \text{ on the disk } D = \{(x, y) \mid x^2 + y^2 \leq 16\}.$$

From part (a), we found the critical point in the region $(x, y) = (1, 0)$ and from part (b) we found 4 points: $(x, y) = (\pm 4, 0)$ and $(x, y) = (-2, \pm 2\sqrt{3})$

- $f(1, 0) = 2 + 0 - 4 - 5 = -7$ (1 point)
- $f(4, 0) = 32 + 0 - 16 - 5 = 11$ (1 point)
- $f(-4, 0) = 32 + 0 + 16 - 5 = 43$ (1 point)
- $f(-2, \pm 2\sqrt{3}) = 8 + 36 + 8 - 5 = 47$ (1 point)

Thus, the absolute maximum of $f(x, y)$ on the disk $x^2 + y^2 \leq 16$ is $f(-2, \pm 2\sqrt{3}) = 8 + 36 + 8 - 5 = 47$ and the absolute minimum is $f(1, 0) = 2 + 0 - 4 - 5 = -7$ (1 point)

3. (8 points) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + 2xy + y^2}{x^2 + y^2}$ does not exist. Justify your answer.

$f(x, y) = \dfrac{x^2 + 2xy + y^2}{x^2 + y^2}$. Let C_1 be the x-axis: $y = 0$ (1 point) choosing reasonable path

$$f(x, 0) = \frac{x^2}{x^2} = 1$$

So, on C_1,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + 2xy + y^2}{x^2 + y^2} = \lim_{x\to 0} f(x, 0) = 1 \text{ (2 points)}$$

Let C_2 be the y-axis, $x = 0$

$$f(0, y) = \frac{y^2}{y^2} = 1$$

So, on C_2,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + 2xy + y^2}{x^2 + y^2} = \lim_{y\to 0} f(0, y) = 1$$

Let C_3 be the $y = x$ (1 point) Choosing reasonable path

$$\lim_{x\to 0} f(x, x) = \frac{x^2 + 2x^2 + x^2}{2x^2} = \lim_{x\to 0} \frac{4x^2}{2x^2} = 2 \text{ (2 points)}$$

On two different paths $C_1 \neq C_3$ that pass through the point $(0, 0)$, $f(x, y)$ approaches two different numbers, 1 and 2, therefore, the limit does NOT exist. (2 points)

4. (10 points) Determine the **average value** of the function $f(x,y) = e^x + 2y$ over the rectangular region R,

$$R = \{(x,y) \,|\, 0 \le x \le 1,\ 0 \le y \le 3\}.$$

The average value is

$$f_{\text{avg}} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA. \text{ (2 points)}$$

Since R is a rectangle,

$$\text{Area}(R) = (1-0)(3-0) = 3. \text{ (1 point)}$$

We now compute the double integral.

Method 1.

R as a Type I plane region: Since $R = \{(x,y) \,|\, 0 \le x \le 1,\ 0 \le y \le 3\}$, then

$$\underbrace{\int_0^1 \int_0^3 (e^x + 2y)\ dydx}_{\text{(2 points)}} = \int_0^1 \underbrace{\left[e^x y + y^2\right]_0^3}_{\text{(1 point)}} dx = \int_0^1 (3e^x + 9) - (0+0)\ dx$$

$$\underbrace{\int_0^1 (3e^x + 9)\ dx}_{\text{(1 point)}} = \underbrace{\left[3e^x + 9x\right]_0^1}_{\text{(1 point)}} = \left(3e^1 + 9\right) - \left(3e^0 + 0\right) = 3e + 9 - 3 = 3e + 6 \text{ (1 point)}.$$

OR

R as Type II plane region: Since $R = \{(x,y) \,|\, 0 \le x \le 1,\ 0 \le y \le 3\}$, then

$$\int_0^3 \int_0^1 (e^x + 2y)\ dxdy = \int_0^3 \left[e^x + 2xy\right]_0^1 dy = \int_0^3 (e + 2y) - (1+0)\ dy$$

$$= \int_0^3 (e + 2y - 1)\ dy = \left[ey + y^2 - y\right]_0^3 = (3e + 9 - 3) - (0 + 0 - 0) = 3e + 6.$$

$$\textit{Answer:} \quad f_{\text{avg}} = \frac{1}{3}(3e+6) = e + 2. \text{ (1 point)}$$

5. Let $f(x, y) = xe^y + x^2\sin(3y)$.

(a) (4 points) Find all of the first partial derivatives of f.

$$f_x(x, y) = e^y + 2x\sin(3y), \quad f_y(x, y) = xe^y + 3x^2\cos(3y)$$

(1 point) for each term

(b) (4 points) Find the gradient of f at the point $(1, 0)$.

Since

$$\nabla f(x, y) = \left\langle e^y + 2x\sin(3y), \quad xe^y + 3x^2\cos(3y)\right\rangle,$$

then

$$\nabla f(1, 0) = \langle 1 + 0, \quad 1 + 3\rangle \text{ (2 points)} = \langle 1, 4\rangle \text{ (2 points)}.$$

(c) (4 points) Find the **rate of change** of f at the point $(1, 0)$ in the direction of the vector $3\mathbf{i} + 4\mathbf{j}$.

The unit vector $\mathbf{u}$ parallel the vector in the same direction is

$$\mathbf{u} = \frac{\langle 3, 4\rangle}{||\langle 3, 4\rangle||} = \frac{\langle 3, 4\rangle}{5} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle. \text{ (1 point)}$$

The rate of change of $f(x, y)$ at the point $(1, 0)$ in the direction of the vector $3\mathbf{i} + 4\mathbf{j}$ is

$$D_{\mathbf{u}}f(1, 0) = \underbrace{\nabla f(1, 0) \cdot \mathbf{u}}_{\text{(2 points)}} = \langle 1, 4\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = 1 \cdot \frac{3}{5} + 4 \cdot \frac{4}{5} = \frac{3}{5} + \frac{16}{5} = \frac{19}{5}. \text{ (1 point)}$$

$$\textit{Answer}: \quad \frac{19}{5}.$$

(d) (6 points) Find the **linearization** of $f(x,y) = xe^y + x^2\sin(3y)$ at the point $(1,0)$ and use it to **approximate** $f(0.9, 0.1)$.

The linearization of f at $(1,0)$ is given by

$$\begin{aligned} L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \text{ (2 points)} \\ &= 1 + 1(x-1) + 4(y-0). \text{ (2 points)} \end{aligned}$$

Now,

$$\underbrace{f(0.9,0.1) \approx L(0.9,0.1)}_{\text{(1 point)}} = 1 + 1(0.9-1) + 4(0.1-0) = 1 - 0.1 + 0.4 = 1.3. \text{ (1 point)}$$

(e) (6 points) Find **parametric equations of the normal line** to the surface

$$z = xe^y + x^2\sin(3y)$$

at the point $(1,0,1)$.

Let $F(x,y,z) = z - f(x,y) = z - (xe^y + x^2\sin(3y))$ (1 point). Then we get

$$\nabla F(x,y,z) = \langle F_x, F_y, F_z\rangle = \langle -e^y - 2x\sin(3y),\ -xe^y - 3x^2\cos(3y),\ 1\rangle. \text{ (2 points)}$$

So, at the point $(1,0,1)$,

$$\nabla F(1,0,1) = \langle -1,\ -4,\ 1\rangle. \text{ (1 point)}$$

Since the direction vector of the normal line is parallel to $\nabla F(1,0,1)$, let's choose the direction vector as $\nabla F(1,0,1)$. (1 point)

Therefore, parametric equations of the normal line to the surface $z = xe^y + x^2\sin(3y)$ at the point $(1,0,1)$ are

$$x = 1 - t, \quad y = 0 - 4t, \quad z = 1 + t, \quad -\infty < t < \infty.$$

$$\mathit{Answer}: \quad x = 1 - t, \quad y = -4t, \quad z = 1 + t. \text{ (1 point)}$$

OR

Let $F(x,y,z) = f(x,y) - z = xe^y + x^2\sin(3y) - z$. Then we get

$$\nabla F(x,y,z) = \langle F_x, F_y, F_z\rangle = \langle e^y + 2x\sin(3y),\ xe^y + 3x^2\cos(3y),\ -1\rangle.$$

So, at the point $(1,0,1)$,

$$\nabla F(1,0,1) = \langle 1,\ 4,\ -1\rangle.$$

Since the direction vector of the normal line is parallel with $\nabla F(1,0,1)$, let's choose the direction vector as $\nabla F(1,0,1)$.

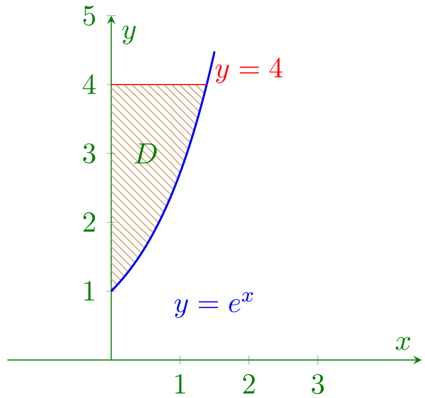
Therefore, parametric equations of the normal line to the surface $z = xe^y + x^2\sin(3y)$ at the point $(1,0,1)$ are

$$x = 1 + t, \quad y = 0 + 4t, \quad z = 1 - t, \quad -\infty < t < \infty.$$

$$\mathit{Answer}: \quad x = 1 + t, \quad y = 4t, \quad z = 1 - t.$$

6. (a) (5 points) Change the order of integration for the iterated integral

$$\int_0^{\ln(4)} \int_{e^x}^{4} f(x,y)\,dydx.$$



Type I region:

$$D = \{(x,y)\,|\,0 \le x \le \ln(4),\ e^x \le y \le 4\}$$

Let's write x in terms of y.

$$y = e^x \quad \to \quad x = \ln(y).$$

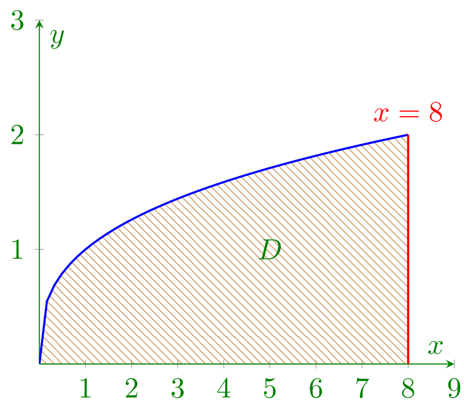
Then, we can write D as the Type II region:

$$D = \{(x,y)\,|\,0 \le x \le \ln(y),\ 1 \le y \le 4\}$$

Answer : $$\int_1^4 \int_0^{\ln(y)} f(x,y)\,dxdy$$

(b) (5 points) Change the order of integration for the iterated integral

$$\int_0^2 \int_{y^3}^{8} g(x,y)\,dxdy.$$



Type II region:

$$D = \{(x,y)\,|\,y^3 \le x \le 8,\ 0 \le y \le 2\}$$

Let's write y in terms of x.

$$x = y^3 \quad \to \quad y = \sqrt[3]{x}.$$

Then, we can write D as the Type I region:

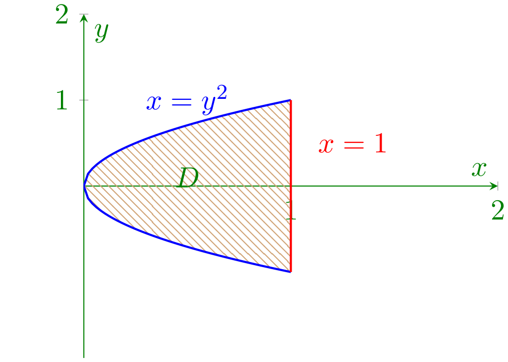
$$D = \{(x,y)\,|\,0 \le x \le 8,\ 0 \le y \le x^3\}$$

Answer : $$\int_0^8 \int_0^{\sqrt[3]{x}} g(x,y)\,dydx$$

7. (10 points) Evaluate the double integral

$$\iint_D (2xy+1)\,dA,$$

where the plane region D is bounded by the parabola $x = y^2$ and the vertical line $x = 1$.



Type I region: (2 points)

$$D = \{(x,y)\,|\,0 \le x \le 1,\ -\sqrt{x} \le y \le \sqrt{x}\}$$

OR

Type II region:

$$D = \{(x,y)\,|\,y^2 \le x \le 1,\ -1 \le y \le 1\}$$

Type I:

$$\underbrace{\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (2xy+1)\,dydx}_{\text{(2 points)}} = \int_0^1 \underbrace{\Big[xy^2 + y\Big]_{-\sqrt{x}}^{\sqrt{x}}}_{\text{(2 points)}}\,dx = \int_0^1 \Big(x^2 + \sqrt{x}\Big) - \Big(x^2 - \sqrt{x}\Big)\,dx$$

$$= \int_0^1 \underbrace{2\sqrt{x}}_{\text{(1 point)}}\,dx = \Big[\frac{4}{3}x^{3/2}\Big]_0^1 \text{ (2 points)} = \frac{4}{3} - 0 = \frac{4}{3} \text{ (1 point)}.$$

Type II:

$$\int_{-1}^1 \int_{y^2}^1 (2xy+1)\,dxdy = \int_{-1}^1 \Big[x^2y + x\Big]_{y^2}^1\,dy = \int_{-1}^1 \Big(y+1\Big) - \Big(y^5 + y^2\Big)\,dy$$

$$= \int_{-1}^1 y + 1 - y^5 - y^2\,dy = \Big[\frac{y^2}{2} + y - \frac{y^6}{6} - \frac{y^3}{3}\Big]_{-1}^1$$

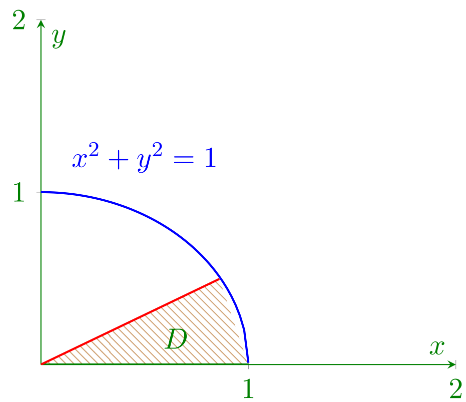
$$= \Big(\frac{1}{2} + 1 - \frac{1}{6} - \frac{1}{3}\Big) - \Big(\frac{1}{2} - 1 - \frac{1}{6} + \frac{1}{3}\Big) = 2 - \frac{2}{3} = \frac{4}{3}.$$

$$Answer: \quad \frac{4}{3}$$

8. (11 points) Use **polar coordinates** to **evaluate** the double integral

$$\iint_D (x^2+y^2)^{\frac{3}{2}}\, dA,$$

where D is the plane region in the first quadrant bounded by the circle $x^2+y^2=1$, the line $y=\dfrac{x}{\sqrt{3}}$ and the x-axis.



From the line $y=\dfrac{x}{\sqrt{3}}$, we have

$$\frac{y}{x}=\frac{1}{\sqrt{3}}=\tan(\theta) \quad \rightarrow \quad \theta=\frac{\pi}{6}.$$

then D is described as a polar region:

$$D=\{(r,\theta)\,|\,0\le r\le 1,\ 0\le\theta\le\frac{\pi}{6}\}.$$

Also, the function is

$$(x^2+y^2)^{\frac{3}{2}}=(r^2)^{\frac{3}{2}}=r^3$$

Then, the double integral is

$$\iint_D (x^2+y^2)^{\frac{3}{2}}\, dA=\int_0^{\pi/6}\int_0^1 (r^2)^{\frac{3}{2}}\, r dr d\theta=\int_0^{\pi/6}\int_0^1 r^3\, r dr d\theta=\int_0^{\pi/6}\int_0^1 r^4\, dr d\theta$$

So, we get

$$\int_0^{\pi/6}\int_0^1 r^4\, dr d\theta=\int_0^{\pi/6}\Big[\frac{r^5}{5}\Big]_0^1 d\theta=\int_0^{\pi/6}\frac{1}{5}\, d\theta=\Big[\frac{1}{5}\theta\Big]_0^{\pi/6}=\frac{\pi}{30}.$$

$$Answer:\ \frac{\pi}{30}$$