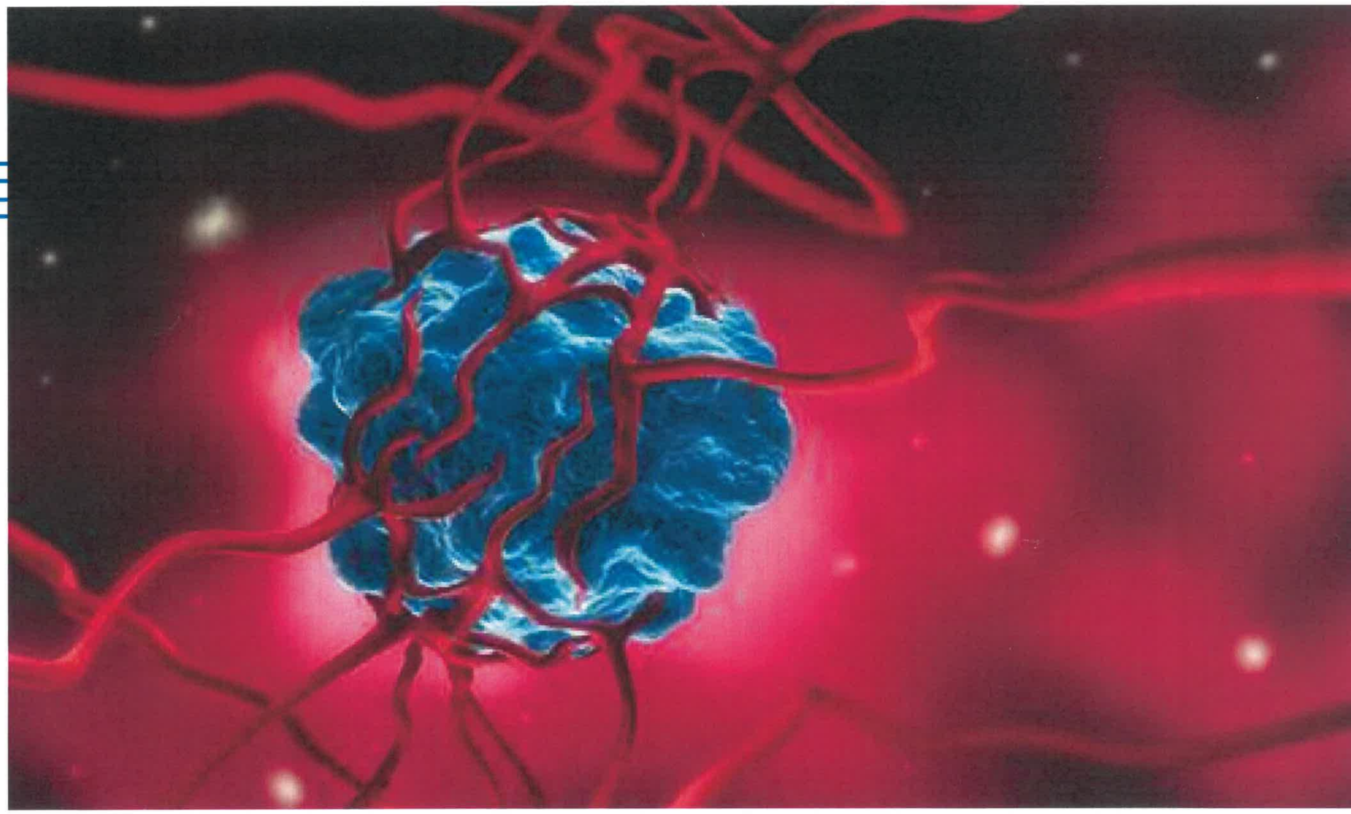
15 Multiple Integrals



15.2 Double Integrals over General Regions

Exercise for class

Do the partial integration $\int_0^x e^{x^2} dy$.

$$\int_0^2 e^{x^2} dy$$

$$\int_0^x e^{x^2} dy = \left[y e^{x^2} \right]_{y=0}^{x} = x e^{x^2} - 0 = x e^{x^2}$$

Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval.

But for double integrals, we want to be able to integrate a function not just over rectangles but also over regions of more general shape.

General Regions

General Regions (1 of 12)

Consider a general region D like the one illustrated in Figure 1.

$f(x,y) = e^{x+y}$

$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$

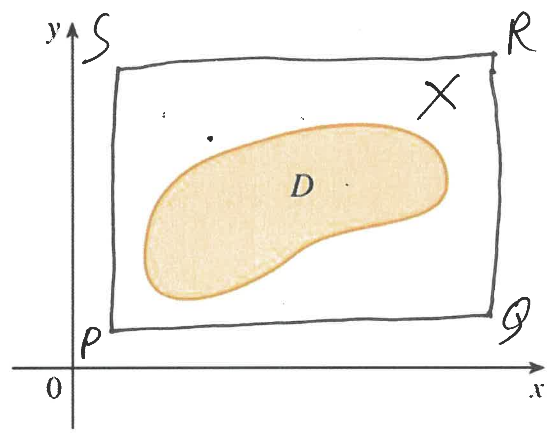


Figure 1

$\int_D f(x,y)\,dA = \int_{PQRS} F(x,y)\,dA$

General Regions (2 of 12)

We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2.

In order to integrate a function f over D we define a new function F with domain R by

1 $$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

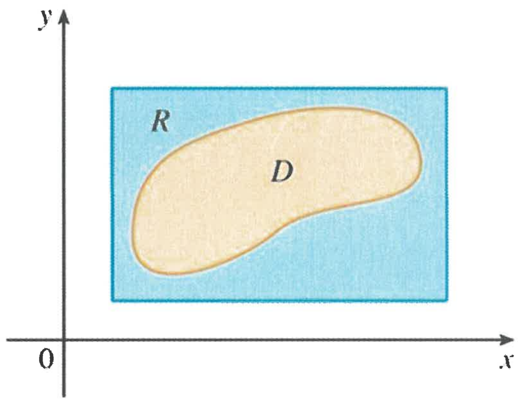


Figure 2

General Regions (3 of 12)

If F is integrable over R, then we define the **double integral of *f* over *D*** by

$$2 \quad \iint\limits_D f(x, y)\, dA = \iint\limits_R F(x, y)\, dA \quad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because R is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined.

General Regions (4 of 12)

The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral. This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geq 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f).

General Regions (5 of 12)

You can see that this is reasonable by comparing the graphs of *f* and *F* in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of *F*.

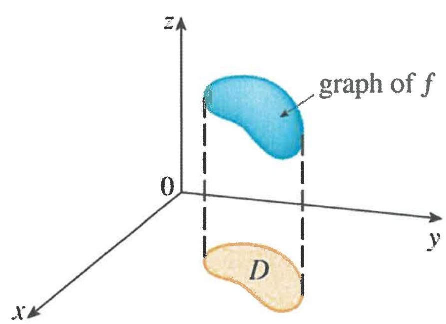


Figure 3

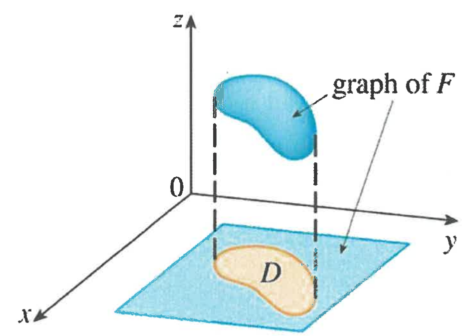


Figure 4

General Regions (6 of 12)

Figure 4 also shows that *F* is likely to have discontinuities at the boundary points of *D*.

Nonetheless, if *f* is continuous on *D* and the boundary curve of *D* is "well behaved", then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists.

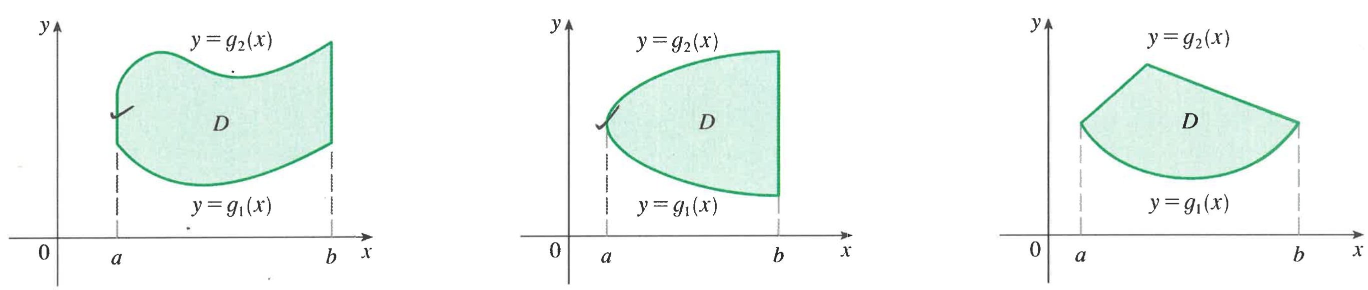
In particular, this is the case for **type I** and **type II** regions.

General Regions (7 of 12)

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where g_1 and g_2 are continuous on [a, b]. Some examples of type I regions are shown in Figure 5.



Some type I regions

Figure 5

General Regions (8 of 12)

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D.

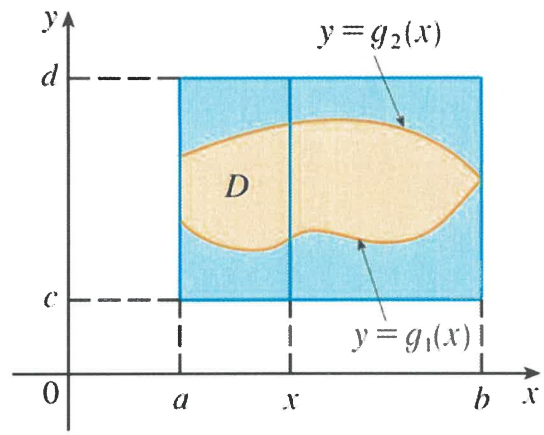


Figure 6

General Regions (9 of 12)

Then, by Fubini's Theorem,

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \le y \le g_2(x)$.

General Regions (10 of 12)

Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D described by

$$D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

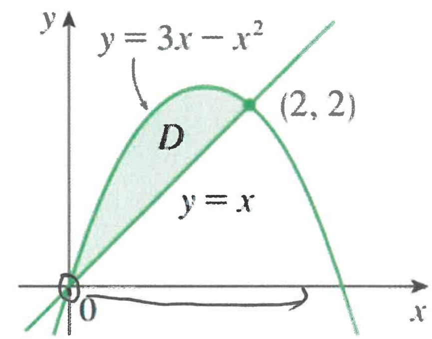
$$\iint_D f(x, y)\, dA = \int_a^b \boxed{\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy}\, dx$$

The integral on the right side of (3) is an iterated integral, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

Example: Type I region

Evaluate the integral of $f(x, y) = 2y$ in the scetched region D.

7. $f(x, y) = 2y$



$$D = \{(x, y) \mid 0 \leq x \leq 2, \; x \leq y \leq 3x - x^2\}$$

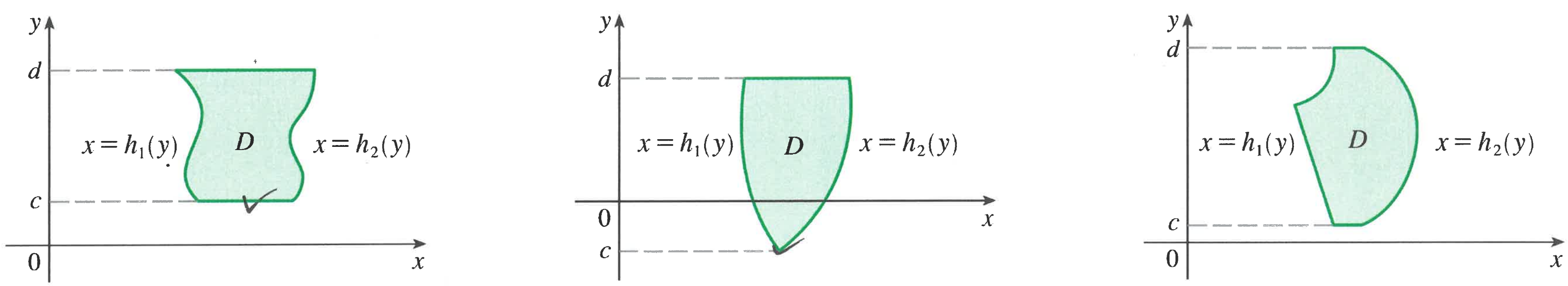
$$\int_0^2 \int_x^{3x-x^2} 2y \, dy \, dx$$

$$= \int_0^2 \left[y^2 \right]_{y=x}^{3x-x^2} dx = \int_0^2 \left((3x - x^2)^2 - x^2 \right) dx$$

We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.



Some type II regions

Figure 7

Example: Type II region

Evaluate the double integral: $\int_D e^{-y^2}\, dA\,, D = \{(x, y) : 0 \le y \le 3,\ 0 \le x \le y\}$.

$$\int_0^3 \int_0^y e^{-y^2}\, dx\, dy$$

$$= \int_0^3 \left[x e^{-y^2} \right]_{x=0}^{y} dy = \int_0^3 \left(y e^{-y^2} - 0 \right) dy = \int_0^3 y e^{-y^2} dy$$

$y^2 = u$

$\Rightarrow 2y\, dy = du$

$$= \frac{1}{2} \int_0^3 e^{-y^2}\, 2y\, dy = \frac{1}{2} \int_0^9 e^{-u}\, du = \frac{1}{2} \left[-e^{-u} \right]_0^9$$

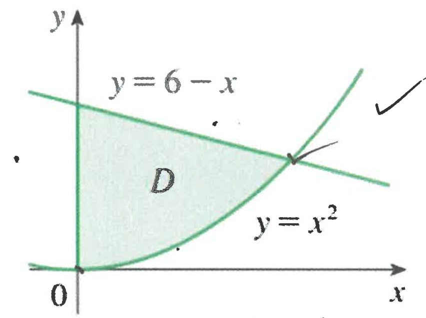
y	0	3
u	0	9

$$= \frac{1}{2} \left(-e^{-9} + e^{0} \right) = \frac{1}{2} \left(1 - e^{-9} \right)$$

Exercise of the class

Evaluate the integral of $f(x,y) = x$ in the scetched region D.

10. $f(x, y) = x$



$$\int_0^2 \int_{x^2}^{6-x} x \, dy \, dx$$

$$= \int_0^2 [xy]_{y=x^2}^{6-x} \, dx$$

$$\cancel{= \int_0^2 [x \cdot x^2 - x(6-x)]}$$

$$= \int_0^2 [x(6-x) - x \cdot x^2] \, dx$$

$$= \int_0^2 (6x - x^2 - x^3) \, dx$$

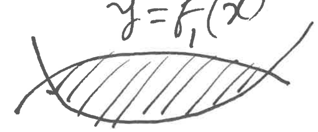
$$x^2 = 6 - x$$

$$\Rightarrow x^2 + x - 6 = 0$$

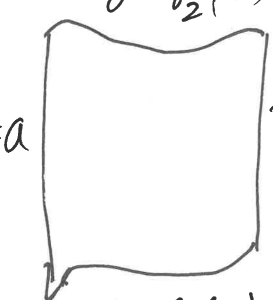
$$\Rightarrow x^2 + 3x - 2x - 6 = 0$$

$$\Rightarrow (x+3)(x-2) = 0$$

$$\Rightarrow x = \cancel{-3} \text{ or } 2$$



$$f_1(x) = f_2(x)$$



$$f_1(x) = f_2(x)$$

General Regions (12 of 12)

Using the same methods that were used in establishing (3), we can show that the following result holds.

4 If f is continuous on a type II region D described by

$$D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example 1

Evaluate $\iint_D (x + 2y)\, dA,$ where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution:

The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$.

We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$D = \{(x, y) | -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\}$

**In this example, notice that the limit of x is not explicitly given. In such situations we need to find out the point of intersection of the two curves $y = g_1(x)$ and $y = g_2(x)$ by solving $g_1(x) = g_2(x)$ and the x-coordinate of those points will give us the limit of x.

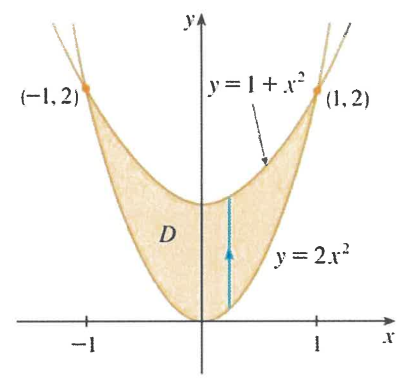


Figure 8

Example 1 – Solution (1 of 2)

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} [xy + y^2]_{y=2x^2}^{y=1+x^2}\, dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

Example 1 – Solution (2 of 2)

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

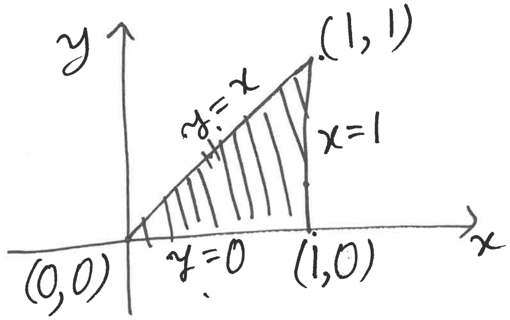
$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \Big]_{-1}^{1}$$

$$= \frac{32}{15}$$

Changing the Order of Integration (1 of 1)

Fubini's Theorem tells us that we can express a double integral as an iterated integral in two different orders. Sometimes one order is much more difficult to evaluate than the other—or even impossible. The next example shows how we can change the order of integration when presented with an iterated integral that is difficult to evaluate.

Changing the Order of Integration



$$D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\}$$

$$D = \{(x, y) \mid 0 \leq y \leq 1,\ y \leq x \leq 1\}$$

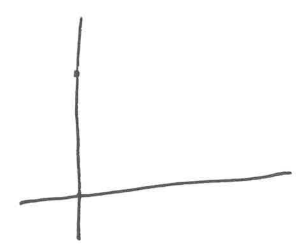
Example 5

Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

Solution:

If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\,dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\,dy$ is not an elementary function. So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using (3) backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA$$

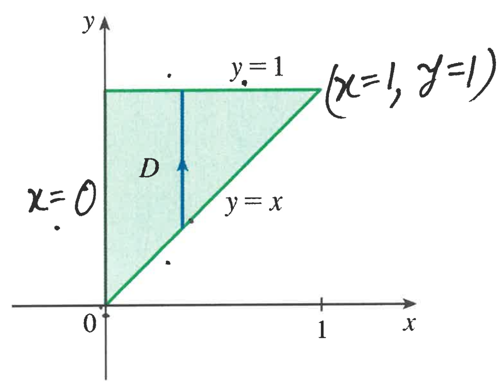


where $D = \{(x, y) \mid 0 \le x \le 1, x \le y \le 1\}$

Example 5 – Solution (1 of 2)

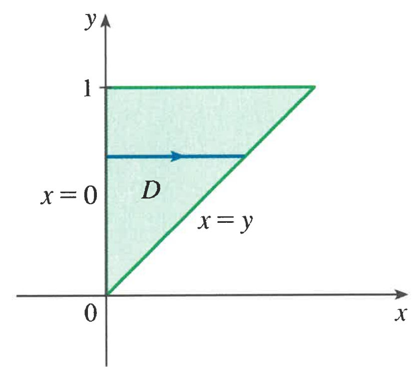
We sketch this region D in Figure 15. Then from Figure 16 we see that an alternative description of D is

$$D = \{(x, y) | 0 \leq y \leq 1, 0 \leq x \leq y\}$$



D as a type I region

Figure 15



D as a type II region

Figure 16

Example 5 – Solution (2 of 2)

This enables us to use (4) to express the double integral as an iterated integral in the reverse order:

$$\int_0^1 \int_x^1 \sin(y^2)dy\,dx = \iint_D \sin(y^2)dA$$

$$= \int_0^1 \int_0^y \sin(y^2)dx\,dy = \int_0^1 [x\sin(y^2)]_{x=0}^{x=y}\,dy$$

$$= \int_0^1 y\sin(y^2)dy = -\frac{1}{2}\cos(y^2)]_0^1 = \frac{1}{2}(1-\cos 1)$$

Example: Changing the order of the integral

Evaluate the integral by reversing the order of integration:

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dxdy$$

$D = \{(x,y) : \underline{0 \le y} \le 1,\ 3y \le x \le 3\}$

$\underline{3y} \le \underline{x \le 3}$

$0 \le x \le 3$

$0 \le y$

$\Rightarrow 0 \le 3y \le x$

$\Rightarrow 0 \le x$

$0 \le y$

$y \le \cancel{1}x,\ 3y \le x$

$\Rightarrow y \le \frac{x}{3}$

$3y \le x \le 3$

$\Rightarrow \cancel{3}y \le \cancel{3}$

$\Rightarrow y \le 1$

$0 \le y \le \frac{x}{3}$

$\int_0^3 \int_0^{\frac{x}{3}} e^{x^2} dy dx$

$= \int_0^3 \left[y e^{x^2} \right]_{y=0}^{y=\frac{x}{3}} dx = \int_0^3 \frac{x}{3} e^{x^2} dx$

$D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le \frac{x}{3}\}$

Exercise for class

Evaluate the integral by reversing the order of integration:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$$

$D = \{(x,y) : 0 \le y \le 8, \sqrt[3]{y} \le x \le 2\}$

$D = \{(x,y) : 0 \le x \le 2, 0 \le y \le x^3\}$

$$\int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$

$$= \int_0^2 \left[y e^{x^4} \right]_{y=0}^{x^3} dx$$

$$= \int_0^2 x^3 e^{x^4}\,dx$$

$$= \frac{1}{4} \int_0^2 e^{x^4}\, 4x^3\,dx = \frac{1}{4} \int_0^{16} e^u\,du$$

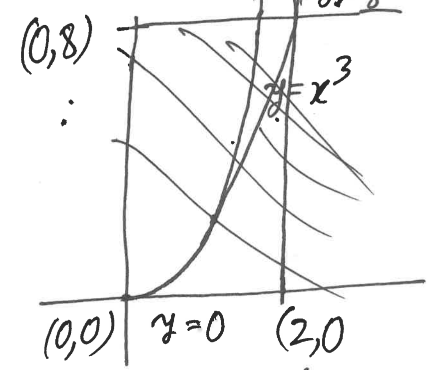
$x^4 = u$

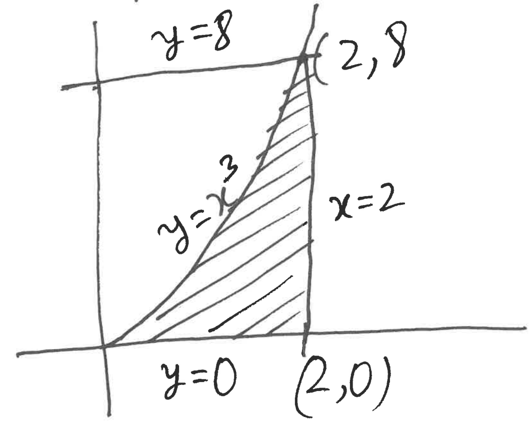
$\Rightarrow 4x^3\,dx = du$

x	0	2
u	0	16

$y=0$, $y=8$, $x=2$ and $x=\sqrt[3]{y}$

$y = x^3$





Exercise for class

Evaluate the integral by reversing the order of integration:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$$

$D = \{(x, y) \mid \underline{0 \leq y \leq 8}, \underline{\sqrt[3]{y} \leq x \leq 2}\}$

$\sqrt[3]{y} \leq x$

$\Rightarrow y \leq x^3$

$\Rightarrow 0 \leq x^3$

$\Rightarrow 0 \leq x$

$0 \leq y$ & $y \leq 8$

$0 \leq y \leq x^3$

$\sqrt[3]{y} \leq x$

$\Rightarrow y \leq x^3$

$0 \leq x \leq 2$

$D = \{(x, y) \mid 0 \leq x \leq 2, \; 0 \leq y \leq x^3\}$

Properties of Double Integrals

Properties of Double Integrals (1 of 6)

We assume that all of the following integrals exist. For rectangular regions *D* the first three properties can be proved in the same manner. And then for general regions the properties follow from Definition 2.

5 $$\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

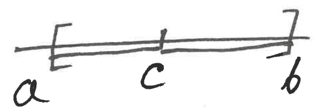
6 $$\iint_D cf(x,y)\, dA = c\iint_D f(x,y)\, dA \quad \text{where } c \text{ is a constant}$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then

7 $$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$$

Properties of Double Integrals (2 of 6)

The next property of double integrals is similar to the property of single integrals given by the equation



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

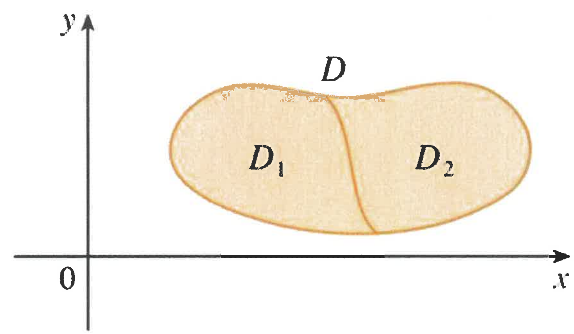
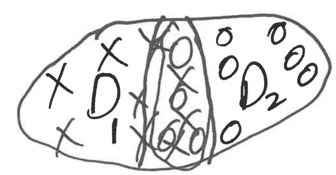


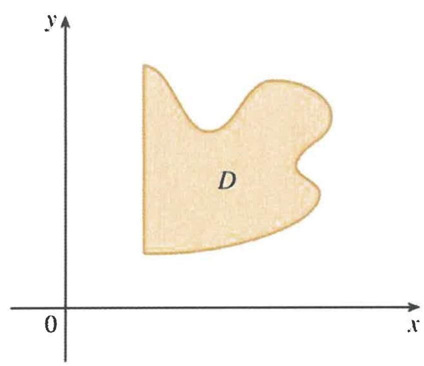
Figure 17

8 $$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

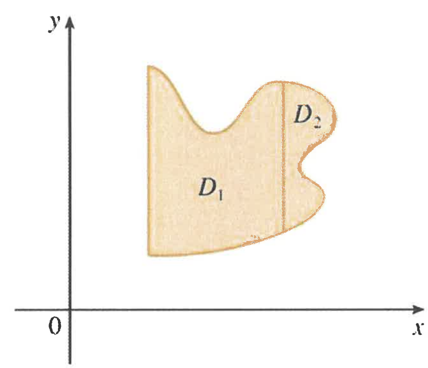


Properties of Double Integrals (3 of 6)

Property 8 can be used to evaluate double integrals over regions *D* that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure.



(a) *D* is neither type I nor type II.



(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

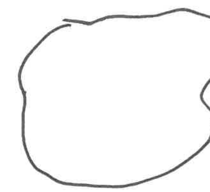
Figure 18

Properties of Double Integrals (4 of 6)

The next property of integrals says that if we integrate the constant function *f*(*x*, *y*) = 1 over a region *D*, we get the area of *D*:

$f(x, y) = 1$

$$\iint_D 1 \, dA = A(D) \tag{9}$$



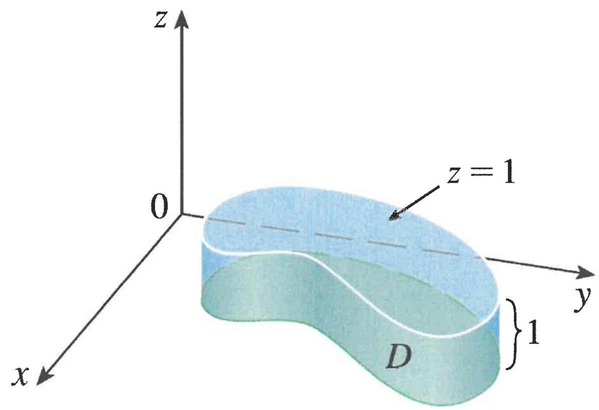
D

~~$f(x, y)$~~

$$f_{avg} = \frac{1}{A(D)} \iint_D f(x, y) \, dA$$

Properties of Double Integrals (5 of 6)

Figure 19 illustrates why Equation 9 is true: A solid cylinder whose base is *D* and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.



Cylinder with base *D* and height 1

Figure 19

Properties of Double Integrals (6 of 6)

Finally, we can combine Properties 6, 7, and 9 to prove the following property.

10 If $m \le f(x, y) \le M$ for all (x, y) in D, then

$$m \cdot A(D) \le \iint_D f(x,y)\, dA \le M \cdot A(D)$$

$$m \le f(x,y) \le M$$

$$\Rightarrow \iint_D m\, dA \le \iint_D f(x,y)\, dA \le \iint_D M\, dA$$

$$\Rightarrow m \iint_D 1\, dA \le \iint_D f\, dA \le M \iint_D 1\, dA$$

$$\Rightarrow m\, A(D) \le \iint_D f\, dA \le M\, A(D)$$

Example 6

Use Property 10 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA,$ where D is the disk with center the origin and radius 2.

Solution:

Since $-1 \le \sin x \le 1$ and $-1 \le \cos y \le 1$, we have $-1 \le \sin x \cos y \le 1$ and, because the natural exponential function is increasing, we have

$$e^{-1} \le e^{\sin x \cos y} \le e^1 = e$$

Thus, using $m = e^{-1} = \frac{1}{e}, M = e, \text{ and } A(D) = \pi(2)^2$ in Property 10, we obtain

$$\frac{4\pi}{e} \le \iint_D e^{\sin x \cos y}\, dA \le 4\pi e$$