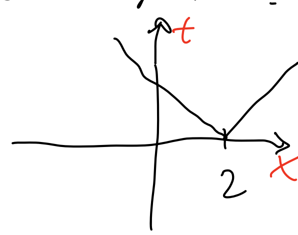
Check site for worksheet
HW 4 & 5, midterm sol & mistakes, 6/25 materials up by end of today.
Materials past that TBD
HW 4 due 6\26, HW5 6\28, quiz 3 6\29

Selected problems for today: 1d, 1e, 2d
Student suggestions:

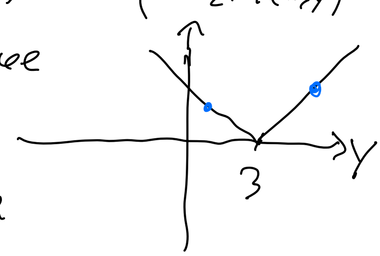
1d: Find & classify critical points of
$f(x,y) = |x-2| + |y-3|$.

∇f DNE is also condition for critical point.



If $x \neq 2$, then $|x-2|$ is differentiable
similarly, $y \neq 3 \Rightarrow |y-3|$ diff.
So $x \neq 2$ & $y \neq 3 \Rightarrow \nabla f = \langle \frac{d}{dx}|x-2|, \frac{d}{dy}|y-3| \rangle$
$= \langle -1 \text{ or } 1, -1 \text{ or } 1 \rangle \neq \langle 0, 0 \rangle \Rightarrow$
no critical points in this case.

$x = 2$ & $y \neq 3 \Rightarrow f(x,y) = f(2,y) = |y-3|$.



If $y \neq 3$, we are at one of the blue points to the left/right of the corner.
f increases in one direction & dec. in the other direction, so $(2,y)$ can't be a local min or max

$x \neq 2$ & $y = 3 \Rightarrow (x,3)$ can't be a local min or max because f gets smaller along 1 direction and larger in another direction.

Last case: $x=2$ & $y=3$, then $f(2,3)=|2-2|+|3-3|=0$

If $(x,y)\neq(2,3)$, then $f(x,y)=|x-2|+|y-3|>0$

because $x\neq 2 \Rightarrow |x-2|>0$ and $y\neq 3 \Rightarrow |y-3|>0$.

So for all (x,y), $f(x,y)\geq f(2,3)=0$.

So $(2,3)$ is a local minimum (in fact, it's a global min & the global max is ∞)

$(2,y)$, $y\neq 3$ is saddle

$(x,3)$, $x\neq 2$ is saddle

Ie: $f(x,y,z)=x^2+y^2+z^2+xy+yz+zx+x+y+z+1$, find & classify critical points.

$$0=\nabla f=(2x+y+z+1,\ 2y+x+z+1,\ 2z+x+y+1).$$

$$\begin{cases} 2x+y+z=-1 \\ x+2y+z=-1 \\ x+y+2z=-1 \end{cases} \Rightarrow \begin{array}{l} x+y+z=\underline{-1-x} \\ x+y+z=\underline{-1-y} \\ x+y+z=\underline{-1-z} \end{array}$$

All 3 red things equal: $-1-x=-1-y=-1-z$

$\Rightarrow -x=-y=-z \Rightarrow x=y=z$

$\Rightarrow 4x=-1 \Rightarrow x=y=z=-\frac{1}{4}$

$$f\left(-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4}\right)=\frac{3}{16}+\frac{3}{16}-\frac{3}{4}+1=\frac{3}{8}+\frac{1}{4}=\frac{5}{8}$$

Let's move on to Hessian test. One tip is to see what you expect before applying. $f\to\infty$ as $x,y,z\to\infty$ and $f\geq 1$ at various points nearby like $(0,0,0)$, so we don't expect a max.

Also, since $f\to\infty$ for large (x,y,z) and f is small for small (x,y,z), we expect a global minimum to exist.

Minimum will be inside red circle by this logic, and min on red disk exists since f continuous, D bounded, D closed.

Since we only have 1 crit pt., we expect it to be min. By process of elimination, expect no saddles.

$$(f_x, f_y, f_z) = (2x+y+z+1,\ 2y+x+z+1,\ 2z+x+y+1)$$

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

So $H(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$ also

Now we find the eigenvalues of H. Recall that we need to solve $\det(H - \lambda I) = 0$.

$$0 = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} \underset{R_1 - R_2}{=} \begin{vmatrix} 1-\lambda & \lambda-1 & 0 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$- (\lambda - 1)\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)((2-\lambda)^2 - 1) - (\lambda - 1)(2 - \lambda - 1) = (\lambda-1)[(1-(2-\lambda)^2) - (1-\lambda)] =$$

$$(\lambda-1)(1 - 4 - \lambda^2 + 4\lambda - 1 + \lambda) = -(\lambda-1)(\lambda^2 - 5\lambda + 4)$$

$$= (\lambda-1)^2(\lambda-4) \Rightarrow \lambda = \underbrace{1, 1}_{2}, 4$$

2 eigenvectors will be obtained

All eigenvalues are positive, so H pos def, $(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$ is a local minimum.

2d: Find min & max, or show they DNE for $f(x,y,z) = xyz$ on $x^2+y^2+z^2=1$.

Let R be region $\{(x,y,z): x^2+y^2+z^2=1\}$. Then R is closed since it is $g^{-1}(\{0\})$ for the cont. function $g(x,y,z) = x^2+y^2+z^2-1$. R is also bounded since $\|v\|=1$ for any $v \in R$.

f is continuous, R closed bounded $\Rightarrow$ max & min exist.

$(yz, xz, xy) = \nabla f = \lambda \nabla g = \lambda(2x, 2y, 2z) = (\lambda x, \lambda y, \lambda z)$ ← absorb 2 into λ.

$$\begin{cases} yz = \lambda x \\ xz = \lambda y \\ yx = \lambda z \end{cases} \Rightarrow \lambda^3 xyz = (xyz)^2.$$

Case 1: $xyz \neq 0$. Then $\lambda^3 = xyz \Rightarrow \frac{\lambda^3}{x} = yz$

$\Rightarrow \lambda x = yz = \frac{\lambda^3}{x} \Rightarrow \lambda x^2 = \lambda^3$. If $\lambda = 0$, then $(xyz)^2 = 0 \Rightarrow xyz = 0$, impossible. So $\lambda \neq 0$, and $x^2 = \lambda^2 \Rightarrow x = \pm\lambda$. Similarly, $y = z = \pm\lambda$

$\Rightarrow g(x,y,z) = 3\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm\sqrt{\frac{1}{3}}$

$\Rightarrow f(x,y,z) = \left(\pm\sqrt{\frac{1}{3}}\right)\left(\pm\sqrt{\frac{1}{3}}\right)\left(\pm\sqrt{\frac{1}{3}}\right) = \pm\frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}, -\frac{\sqrt{3}}{9}$ at $(x,y,z) = \left(\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}\right)$

Case 2: $xyz = 0$. One of the variables must be 0, WLOG let $x=0$. Then

$\left\{\begin{array}{l} yz=0 \\ 0=\lambda y \\ 0=\lambda z \end{array}\right\}$. If $\lambda \neq 0$, then $y=z=0 \Rightarrow$

$g = x^2+y^2+z^2-1 = -1 \neq 0$.

So $\lambda=0 \Rightarrow yz=0 \Rightarrow y=0$ or $z=0$,

WLOG $y=0$. Then $0=g=z^2-1 \Rightarrow z=\pm 1$

$\Rightarrow f(x,y,z)= xyz = 0\cdot 0\cdot \pm 1 = 0$.

Similarly, $f=0$ if $y=0$ or if $z=0$.

All values in green, so

min $= -\frac{\sqrt{3}}{9}$ at $(x,y,z)=\left(\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}\right)$

with odd number of minus signs. (4 points)

max $= \frac{\sqrt{3}}{9}$ at $(x,y,z)=\left(\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}\right)$

with even number of minus signs. (4 points)

2d: find min & max or show they DNE for

$f(x,y,z)=|x|+|y|+|z|$ on $\underbrace{x^2+y^2+z^2<1}_{\text{not closed}}$.

$R=\{x^2+y^2+z^2<1\}$ is not closed because

it doesn't contain points in ∂R such as $(1,0,0)$.

So whether min or max exist is unclear.

$f=|x|+|y|+|z| \geq 0+0+0=0$ and $f(0,0,0)=0$.

So 0 is minimum & is attained at $(0,0,0)$.

$x^2+y^2+z^2<1 \Rightarrow x^2<1 \Rightarrow |x|<1$. So

$f < 1+1+1 = 3$, but we can do better.

Consider $S = R \cup \partial R = \{x^2+y^2+z^2 \leq 1\}$.

On S, max exists since S closed & bounded.

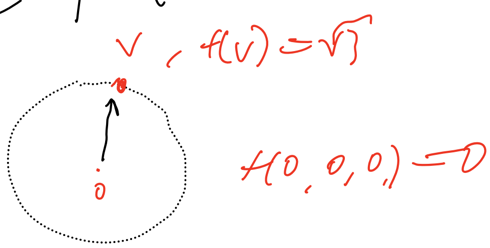
On S, $\max = \frac{3}{\sqrt{3}} = \sqrt{3}$ at $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

with any choice of $\pm$'s.

This means that anywhere in S outside these 8 points, $f < \sqrt{3}$.

These points are in ∂R but not R, so $f < \sqrt{3}$ on R.

Furthermore, we can approach $\sqrt{3}$ by approaching a point on ∂R since f is continuous.



By IVT, entire range $[0, \sqrt{3})$ is covered.

So range of f on R is $[0, \sqrt{3})$.

So maximum of f DNE on R.

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2 more methods for showing max/min DNE on R:

1: If R is unbounded

Show $f \to \infty$ or $f \to -\infty$ along some unbounded path, like a line or curve $r(t)$.

In this case, make sure to write $\max = \infty$

and/or $\min = -\infty$ instead of saying max/min DNE

2: If R bounded

a: Then $S = \bar{R} =$