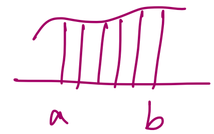
16.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C. Such integrals are called **line integrals.** They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

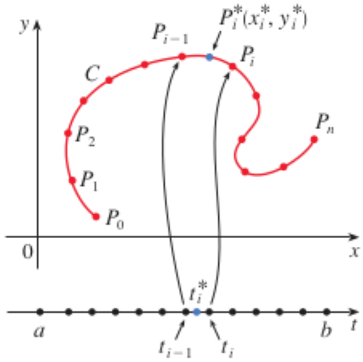


1. **Line Integrals in the Plane**

   We start with a plane curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ and we assume C is a smooth curve. (This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.)



We divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width. If f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

Definition If f is defined on a smooth curve C given by equations

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b,$$

then the line integral of f along C is

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

If f is a continuous function, then the limit always exists and the following formula can be used to evaluate the line integral:

$$\int_{t=a}^{b} f(x,y)\cdot\|r'(t)\|\, dt = \int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Recall: $r'(t) = \langle x'(t), y'(t) \rangle$ and arc length is $\int_a^b \|r'(t)\| dt$

Use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

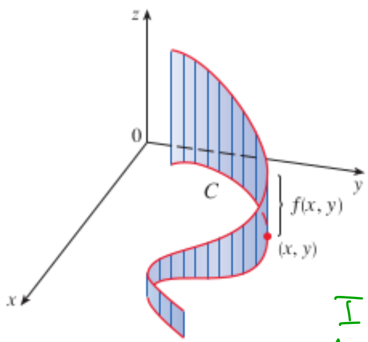
Note: in the special case where C is a line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as :

$$x = x \qquad y = 0 \qquad a \leq x \leq b$$

The formula becomes:

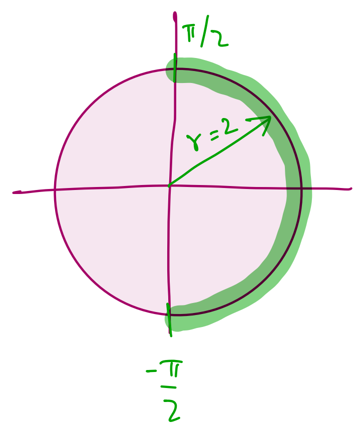
$$\int_C f(x,y)\, ds = \int_a^b f(x,0)\, dx \; = \int_a^b g(x)\, dx$$

and so the line integrals reduces to an ordinary single integral in this case.



If $f(x,y) \geq 0$, $\int_C f(x,y)\, ds$ represents the area of one side of the "fence" or "curtain", whose base is C and whose height above the point (x,y) is $f(x,y)$.

Example: Evaluate $\overbrace{\int_C \underbrace{xy^2}_{f(x,y)}\, ds}^{I}$ where C is the right half of the curve $x^2 + y^2 = 4$



$r(t) = \langle x(t), y(t) \rangle = \langle 2\cos t, 2\sin t \rangle$

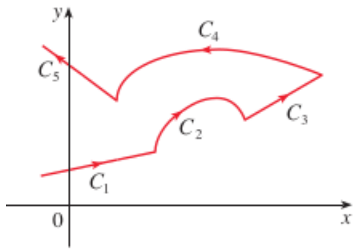
where $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

$$I = \int_C xy^2\, ds = \int_{-\pi/2}^{\pi/2} (2\cos t)(2\sin t)^2 \cdot \underbrace{\|r'(t)\|}_{=2}\, dt$$

where $r'(t) = \langle -2\sin t, 2\cos t \rangle \Rightarrow \|r'(t)\|^2 = 4\sin^2 t + 4\cos^2 t = 4$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} 16 \cos t \sin^2 t\, dt \overset{\boxed{DIY}}{=} \frac{32}{3}$$

$u = \sin t \Rightarrow du = -\cos t\, dt$



Suppose C is a piecewise-smooth curve; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots C_n$, where, the initial point of C_{i+1} is the terminal point of C_i.

Then, we define the integral of f along C as the sum of the integrals of f along each of the smooth piece of C:

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \ldots + \int_{C_n} f(x,y)\,ds$$

2. **Line Integrals with Respect to x or y**
   Two other types of line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$. They are called **the line integrals of f along C with respect to x and y**:

$$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i$$

The following formula say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t), y = y(t)$, $dx = x'(t)\,dt$, $dy = y'(t)\,dt$.

$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))y'(t)\,dt$$

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

Some common parametrizations

- Circle: $x^2 + y^2 = r^2$ $\quad\langle r\cos t, r\sin t\rangle$, $0 \le t \le 2\pi$ (counterclockwise)
- line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, \qquad 0 \le t \le 1$$

Example: Evaluate $\int_C y\,dx + xy\,dy$ where C are given by the following

(a) $C_1 : \langle x(t), y(t)\rangle = \langle t, t^2\rangle, \quad 0 \le t \le 2$

(b) $C_2 : \langle x(t), y(t)\rangle = \langle t, 2t\rangle, \quad 0 \le t \le 2$

(c) $C_3 : \langle x(t), y(t)\rangle = \langle 2-t, 4-2t\rangle, \quad 0 \le t \le 2$

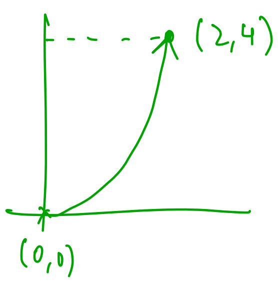
(a) $\int_{C_1} y\,dx + xy\,dy$ where $C_1 : \langle t, t^2\rangle$, $0 \le t \le 2$ $\Rightarrow x'(t) = 1$, $y'(t) = 2t$

$$= \int_0^2 t^2 (1 \cdot dt) + \int_0^2 t(t^2)(2t\,dt) = \int_0^2 t^2\,dt + 2\int_0^2 t^5\,dt \overset{\boxed{\text{DIY}}}{=} \frac{232}{15}$$

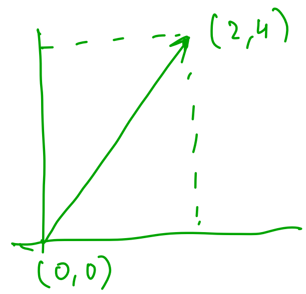
(b) $C_2 : \langle t, 2t\rangle$, $0 \le t \le 2$ $\Rightarrow x'(t) = 1$, $y'(t) = 2$

$$\Rightarrow \int_{C_2} y\,dx + xy\,dy = \int_0^2 t\,(1\,dt) + \int_0^2 (t)(2t)(2\,dt) \overset{\boxed{\text{DIY}}}{=} \frac{44}{3}$$

C_1 :



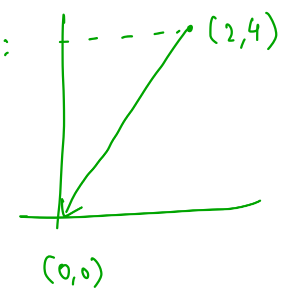
C_2 :



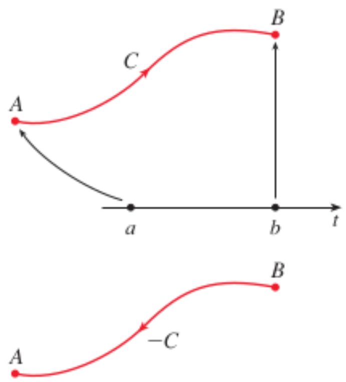
(c) $C_3 : \langle 2-t, 4-2t\rangle$, $0 \le t \le 2$

where $\int_{C_3} y\,dx + xy\,dy = -\frac{44}{3}$

C_3 :



In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t.



$$\int_C f(x,y)\,dx = -\int_{-C} f(x,y)\,dx$$

$$\int_C f(x,y)\,dy = -\int_{-C} f(x,y)\,dy$$

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

3. **Line Integrals in Space**

C: a smooth curve given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$

Write $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$ and

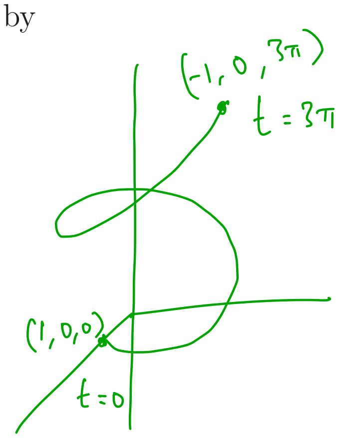
$$\|\mathbf{r}'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,dt$$
$$\int_C f(x,y,z)\,dx = \int_a^b f(\mathbf{r}(t))x'(t)\,dt$$
$$\int_C f(x,y,z)\,dy = \int_a^b f(\mathbf{r}(t))y'(t)\,dt$$
$$\int_C f(x,y,z)\,dz = \int_a^b f(\mathbf{r}(t))z'(t)\,dt$$

Example: Calculate $\int_C (x+y+z)\,ds$ where C is the circular helix given by

$$\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$$

for $0 \le t \le 3\pi$.



$r'(t) = \langle -\sin t, \cos t, 1\rangle$

$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\Rightarrow \int_C (x+y+z)\,ds = \int_{t=0}^{3\pi} (\cos t + \sin t + t)(\sqrt{2})\,dt$

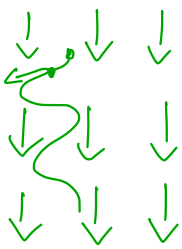
$\overset{\boxed{\text{DIY}}}{=} \sqrt{2}\left(2 + \frac{9\pi^2}{2}\right).$

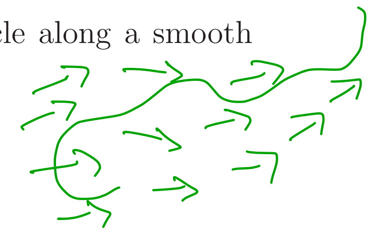
4. **Line Integrals of Vector Fields; Work**

We know that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \int_a^b f(x)\,dx$. Then we have found that the work done by a constant force $\mathbf{F}$ in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $D = \overrightarrow{PQ}$ is the displacement vector.

Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$. (A force field on $\mathbb{R}^2$ could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.)

We wish to compute the work done by this force in moving a particle along a smooth curve C.





We define the work W done by the force field $\mathbf{F}$ as

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\,ds = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

This equation says that *work is the line integral with respect to arc length of the tangential component of the force.*

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, so

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\,ds = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\right] \|\mathbf{r}'(t)\|\,dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

The integral is often abbreviated as $\int_C \mathbf{F}\,d\mathbf{r}$ and occurs in other areas of physics as well.

> **Definition** Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the line integral of $\mathbf{F}$ along C is
>
> $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

Note: $\mathbf{F}(\mathbf{r}(t)) = (x(t), y(t), z(t))$ and $d\mathbf{r} = \mathbf{r}'(t)dt$

The connection between line integrals of vector fields and line integrals of scalar field:

Suppose the vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\mathbf{r}'(t)dt$$

Work done : $\int_C F(x,y,z) \cdot T(x,y,z)\, ds = \int F(t) \cdot \frac{r'(t)}{\cancel{\|r'(t)\|}} \cdot \cancel{\|r'(t)\|}\, dt$

where $T = \frac{r'(t)}{\|r'(t)\|}$ and $ds = \|r'(t)\|\, dt$

$\Rightarrow$ Work done is $\boxed{\int_C F(t) \cdot r'(t)\, dt}$

$$= \int_C \langle P, Q, R \rangle \cdot \langle x', y', z' \rangle\, dt$$

$$= \int_C P x' + Q y' + R z'\, dt = \int_C P\,dx + Q\,dy + R\,dz$$

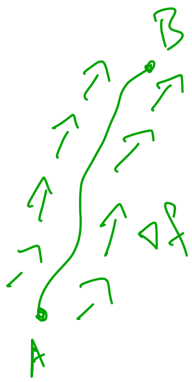
16.3 The Fundamental Theorem for Line Integrals

Fundamental Theorem of Calculus (part II)

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$. This equation is also called: Net Change Theorem: the integral of a rate of change is a net change.

1. **The Fundamental Theorem for Line Integrals**



If we think of gradient vector ∇f of a function f of two or three variables as a sort of a derivative of f, then:

> **Theorem 2** Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then
>
> $$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Proof. $\int_C \langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle \, dt$ □

$$= \int_C \left(\frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt} + \frac{df}{dz} \frac{dz}{dt} \right) dt$$

$$= \int_{t=a}^{b} \frac{d}{dt} \left(f(r(t)) \right) dt = f(r(b)) - f(r(a))$$

Example: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$ and $C : \mathbf{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$, $0 \le t \le 3\pi$.

Take $f = \sqrt{x^2 + y^2} \Rightarrow f_x = \frac{1}{2\sqrt{x^2+y^2}} (2x) = \frac{x}{\sqrt{x^2+y^2}}$

$r(3\pi) = \langle -e^{3\pi}, 0 \rangle$ $\Rightarrow f_y = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow F = \nabla f$

$r(0) = \langle 1, 0 \rangle$

$\Rightarrow \int_C F \cdot dr = f(-e^{3\pi}, 0) - f(1, 0)$ $\Rightarrow \int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(3\pi)) - f(r(0))$

$\boxed{\text{DIY}}$ $= e^{3\pi} - 1$

2. **Independence of Path**

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and terminal point B. We know that, in general

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

but one implication of Theorem 2 is that whenever the gradient of f is continuous, we have

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

The line integral of a conservative vector field depends only on the initial point and terminal point of a curve.

In general, if $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points.

Note: *Line integrals for conservative vector fields are independent of path.*

A curve is called **closed** if its terminal point coincides with its initial point, that is $\mathbf{r}(b) = \mathbf{r}(a)$.

If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D and C is any closed path in D.

Conversely, if it is true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then:

Theorem 3 $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Remark: The line integral of any conservative vector field $\mathbf{F}$ is independent of path, so, it follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path.

Some terminology: Let $D \subseteq \mathbb{R}^2$

- D is **open** if for every point P in D, there is a disk centered at P that lies entirely in D.

- D is **connected** if any two points in D can be connected with a path in D.

Theorem Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Remark: The *only* vector fields that are independent of path are conservative.

3. **Conservative Vector Fields and Potential Functions**

Question: How is it possible to determine whether or not a vector $\mathbf{F}$ is conservative? And if we know that a field $\mathbf{F}$ is conservative, how can we find a potential function f?

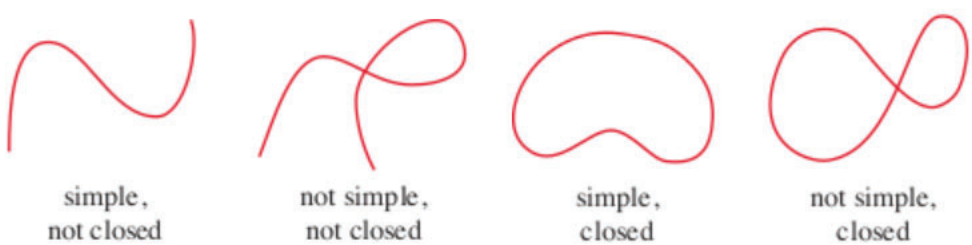
Suppose it is known that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is conservative where P and Q have continuous first-order partial derivatives.

Theorem If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The converse is true only for a special type of region (simply-connected region)

- **Simple curve**: doesn't intersect itself



- D **simply connected**:
  (a) D is connected
  (b) Every simple closed curve in D encloses only points that belong to D.

In terms of simply-connected regions, we can now state a partial converse to the theorem

Theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

Example: Determine whether the following vector fields are conservative or not.

(a) $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(b) $\mathbf{G}(x, y) = \langle 3x^2y + 2y - y^2, x^3 + 2x - 2yx + 9y^2 \rangle$

4. **Conservation of Energy**

- $\mathbf{F}$: a continuous force field that moves an object along a path C given $\mathbf{r}(t)$, $a \le t \le b$,
- $\mathbf{r}(a) = A$: the initial point of C.
- $\mathbf{r}(b) = A$: the initial point of C.

According to Newton's Second Law of Motion, the force $\mathbf{r}(\mathbf{t})$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation:

$$\mathbf{r}(\mathbf{t}) = m\mathbf{r}''(t)$$

So, the work done by the force on the object is:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\,dt$$

Kinetic Energy and Potential Energy

The quantity $\frac{1}{2}m\|\mathbf{v}(t)\|^2$ is called the **kinetic energy** of the object. We can rewrite:

$$W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Let's further assume that $\mathbf{F}$ is a conservative force field: $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have

$$\mathbf{F} = -\nabla P$$

Then, $W = \int_C \mathbf{F} \cdot d\mathbf{r} =$

$$P(A) + K(A) = P(B) + K(B)$$

If an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant.

This is called the **Law of Conservation of Energy** and it is the reason the vector field is called conservative.