MATH 243 - Fall 2025 - Worksheet 4 **Answer Key**

Textbook Sections: 12.5, 13.1 and 13.2

Topics: Lines and Planes; Vector Functions and Space Curves; Derivatives and Integrals of Vector Functions

Instructions: Try each of the following problems, show the detail of your work. Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

1. Find an equation for the plane that passes through the points $(0,-2,5)$ and $(-1,3,1)$ and is perpendicular to the plane $2z = 5x + 4y$.

The plane given by $2z = 5x + 4y$ (call it Plane 1) is also given by $5x + 4y - 2z = 0$ so it has normal vector $\mathbf{n_1} = \langle 5, 4, -2\rangle$. Since the desired Plane 2 passes through the points $A(0,-2,5)$ and $B(-1,3,1)$, then the vector $\overrightarrow{AB} = \langle -1, 5, -4\rangle$ is parallel to Plane 2. Thus, the normal vector $\mathbf{n_2}$ to Plane 2 is perpendicular to $\overrightarrow{AB} = \langle -1, 5, -4\rangle$. When two planes are perpendicular, so are their normal vectors. So the normal vector $\mathbf{n_2}$ for Plane 2 is perpendicular to $\mathbf{n_1}$. Thus $\mathbf{n_2}$ is perpendicular to both $\overrightarrow{AB} = \langle -1, 5, -4\rangle$ and $\mathbf{n_1} = \langle 5, 4, -2\rangle$, so we can use

$$\mathbf{n} = \langle -1, 5, -4\rangle \times \langle 5, 4, -2\rangle = \langle 6, -22, -29\rangle.$$

We can choose P_0 to be the point $A(0,-2,5)$, so the scalar equation of the desired Plane 2 is

$$\begin{aligned} 6(x-0) - 22(y+2) - 29(z-5) &= 0 \\ 6x - 22y - 29z &= -101. \end{aligned}$$

2. What is the angle between the vectors $x\vec{i} - \vec{j} + \vec{k}$ and $x\vec{i} + 2\vec{j} + 3\vec{k}$? Justify your answer.

   A. 0 degrees

   B. less than 90 degrees.

   C. greater than 90 degrees

   D. It can be any of the above depending on the value of x.

ANSWER: (B)

Clearly the two vectors are not parallel because they are not scalar multiples of each other. Hence, the angle between the two vectors is not zero degrees, regardless of the choice of x.

Next, note that

$$\langle x, -1, 1\rangle \cdot \langle x, 2, 3\rangle = x^2 + 1$$

Since $x^2 + 1 > 0$ for all real values of x,it follows that the dot product is strictly positive for all values of x. The dot product is positive when the angle θ is less than 90 degrees.
Note that if θ is greater than 90 degrees, then the dot product of the vectors is negative which is not possible here because $x^2 + 1$ is always positive for all real values of x.

3. Find the parametric equations for the line of intersection of the planes $2x + 3y + 5z = 7$ and $x - y + 2z = 3$.

The normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ to the respective planes can be found by reading off the coefficients from the equations:

$$\mathbf{n}_1 = \langle 2, 3, 5 \rangle$$
$$\mathbf{n}_2 = \langle 1, -1, 2 \rangle.$$

Now we compute

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \mathbf{k} = 11\mathbf{i} + \mathbf{j} - 5\mathbf{k}.$$

The line of intersection of the planes lies in both planes, so its direction vector $\mathbf{v}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. This condition is satisfied if we choose

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 11, 1, -5 \rangle.$$

To find a point on the line, we look for any simultaneous solution of the equations

$$2x + 3y + 5z = 7$$
$$x - y + 2z = 3.$$

If we set $y = 0$, these reduce to

$$2x + 5z = 7$$
$$x + 2z = 3$$

which we solve to obtain $x = 1, z = 1$. Thus $P_0(1, 0, 1)$ is a point on the line of intersection which lies on both planes, and has the position vector $\mathbf{r}_0 =< 1, 0, 1 >$. Finally, we can write the vector equation of the line of intersection of the planes given by the point P_0 and the vector $\mathbf{v}$ using the form $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ where t ranges over all real numbers, i.e.

$$\langle x, y, z \rangle = \langle 1, 0, 1 \rangle + t\langle 11, 1, -5 \rangle \Longrightarrow \begin{cases} x = 1 + 11t \\ y = t \\ z = 1 - 5t \end{cases}.$$

4. Find an equation of the plane through the $P(7, -2, -4)$ and parallel to the plane $z = 4x - 5y$.

Since the two planes are parallel, they will have the same normal vectors. If the equation of the given plane is $z = 4x - 5y$ or $4x - 5y - z = 0$, then we can take the normal vector $\mathbf{n} = \langle 4, -5, -1 \rangle$, and an equation of the parallel plane is

$$4(x-7) - 5(y+2) - 1(z+4) = 0 \quad \text{OR} \quad 4x - 5y - z = 42.$$

5. Find an equation of the plane that passes through the point $P(10, -1, 5)$ and contains the line with symmetric equations $\dfrac{x}{4} = y + 6 = \dfrac{z}{5}$.

Since the line $\dfrac{x}{4} = y + 6 = \dfrac{z}{5}$, lies in the plane, its direction vector $\mathbf{v}_1 = \langle 4, 1, 5 \rangle$ is parallel to the plane. The point $A(0, -6, 0)$ is on the line, and we can verify that the given point $P(10, -1, 5)$ in the plane is not on the line. The vector connecting these two points A and P is $\mathbf{v}_2 = \langle 10 - 0, -1 + 6, 5 - 0 \rangle = \langle 10, 5, 5 \rangle$, and is parallel to the plane, but not parallel to $\mathbf{v}_1$. Then $\mathbf{v}_1 \times \mathbf{v}_2$ is a normal vector $\mathbf{n}$ to the plane,

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & 5 \\ 10 & 5 & 5 \end{vmatrix} = (5 - 25)\mathbf{i} - (20 - 50)\mathbf{j} + (20 - 10)\mathbf{k} = \langle -20, 30, 10 \rangle$$

An equation of the plane through the point $P(10, -1, 5)$ and containing the given line is

$$-20(x - 10) + 30(y + 1) + 10(z - 5) = 0 \quad \text{OR} \quad -20x + 30y + 10z = -180.$$

6. Determine if the line L given by $\mathbf{r}(t) = \langle -2t, 2 + 7t, -1 - 4t \rangle$, intersects the plane given by $4x + 9y - 2z + 8 = 0$.

Intersecting; We can write the parametric equations of line L

$$x(t) = -2t, \quad y(t) = 2 + 7t, \quad z = -1 - 4t \quad \text{for all values of } t.$$

Then, let's plug the x, y, z coordinates of the points on the line L into the equation of the plane to get,

$$4(-2t) + 9(2 + 7t) - 2(-1 - 4t) + 8 = 0 \quad \rightarrow \quad 63t + 28 = 0 \quad \rightarrow \quad t = -\frac{4}{9}.$$

We were able to find a t value from this equation. Once we plug $t = -\dfrac{4}{9}$ into the parametric equations of the line, we get the coordinates of the point of intersection of the line and the plane. So, the line and plane do intersect at the point $\left(\dfrac{8}{9}, -\dfrac{10}{9}, \dfrac{7}{9}\right)$.

VECTOR FUNCTIONS and SPACE CURVES:

7. Find the domain of the vector function $\mathbf{r}(t) = \left\langle \ln(t+1), \dfrac{t}{4-t^2}, \sqrt{4-t} \right\rangle$

We first find the domain of each component of the vector function.

The domain of $\ln(t+1)$ is given by $t+1>0$, $(t>-1)$, so it is the set $(-1,\infty)$.

The domain of $\dfrac{t}{4-t^2} = \dfrac{t}{(2+t)(2-t)}$ is given by $t \neq 2$ and $t \neq -2$. In interval notation, that is
$(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

The domain of $\sqrt{4-t}$ is given by $4-t \geq 0$ ($t \leq 4$), so it is the set $(-\infty, 4]$.

The domain of $\mathbf{r}(t)$ is the intersection of the domains of its components. So the domain is $(-1,2)\cup(2,4]$.

8. Evaluate the limit:

$$\lim_{t\to 1}\mathbf{r}(t) = \lim_{t\to 1}\left\langle \frac{t^2-1}{t^2-3t+2}, \frac{t-1}{\sqrt{t+3}-2}, \frac{\sin(t-1)}{t-1} \right\rangle$$

To find the limit of a vector valued function, we first find the limit of its components, since

$$\lim_{t\to 1}\mathbf{r}(t) = \lim_{t\to 1}\left\langle \frac{t^2-1}{t^2-3t+2}, \frac{t-1}{\sqrt{t+3}-2}, \frac{\sin(t-1)}{t-1} \right\rangle = \left\langle \lim_{t\to 1}\frac{t^2-1}{t^2-3t+2}, \lim_{t\to 1}\frac{t-1}{\sqrt{t+3}-2}, \lim_{t\to 1}\frac{\sin(t-1)}{t-1} \right\rangle.$$

$$\lim_{t\to 1}\frac{t^2-1}{t^2-3t+2} = \lim_{t\to 1}\frac{(t+1)(t-1)}{(t-2)(t-1)} = \lim_{t\to 1}\frac{t+1}{t-2} = \frac{1+1}{1-2} = -2$$

$$\lim_{t\to 1}\frac{t-1}{\sqrt{t+3}-2} = \lim_{t\to 1}\frac{t-1}{\sqrt{t+3}-2}\left(\frac{\sqrt{t+3}+2}{\sqrt{t+3}+2}\right) = \lim_{t\to 1}\frac{(t-1)(\sqrt{t+3}+2)}{(t+3)-4}$$
$$= \lim_{t\to 1}\frac{(t-1)(\sqrt{t+3}+2)}{t-1} = \lim_{t\to 1}(\sqrt{t+3}+2) = \sqrt{(4)}+2 = 4$$

Using L'Hôpital's rule:

$$\lim_{t\to 1}\frac{\sin(t-1)}{t-1} = \lim_{t\to 1}\frac{\cos(t-1)}{1} = \frac{\cos(0)}{1} = 1$$

Therefore,

$$\lim_{t\to 1}\mathbf{r}(t) = \langle -2, 4, 1\rangle$$

9. Find a vector function that represents the curve of intersection of the surfaces $x^2+y^2=1$ and $z=y+2$. Specify the domain of the vector function so that the curve is covered exactly once. Sketch the curve.

We want a vector function for the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $z=y+2$. We can use $x=\cos(t)$, $y=\sin(t)$, and z arbitrary to get points on the cylinder. For a point on the cylinder to be on the plane as well, we need $z=y+2$. Substituting $y=\sin(t)$, we obtain $z=\sin(t)+2$. We then have the vector equation for the line

$$\mathbf{r}(t)=\langle x,y,z\rangle=\langle\cos(t),\sin(t),2+\sin(t)\rangle$$

which gives points on the curve of intersection. To get each point on the curve exactly once, let $0\le t<2\pi$.

DERIVATIVES and INTEGRALS of VECTOR FUNCTIONS:

10. Evaluate the derivative of the vector function $\mathbf{r}(t)=\left\langle e^{t^2+2t},\ln(\cos(t)),t\arctan(t)\right\rangle$.

$$\frac{d}{dt}e^{t^2+2t}=(2t+2)e^{t^2+2t}$$

$$\frac{d}{dt}\ln(\cos(t))=\frac{-\sin(t)}{\cos(t)}=-\tan(t)$$

$$\frac{d}{dt}t\arctan(t)=\arctan(t)+\frac{t}{t^2+1}$$

Therefore,

$$\mathbf{r}'(t)=\left\langle(2t+2)e^{t^2+2t},-\tan(t),\arctan(t)+\frac{t}{t^2+1}\right\rangle$$

11. Evaluate the integrals:

$$\int\left(\frac{1}{t+1}\mathbf{i}+\frac{1}{t^2+1}\mathbf{j}+\frac{t}{t^2+1}\mathbf{k}\right)\,dt$$

and

$$\int_0^1\left(\frac{1}{t+1}\mathbf{i}+\frac{1}{t^2+1}\mathbf{j}+\frac{t}{t^2+1}\mathbf{k}\right)\,dt$$

$$\int \frac{1}{t+1}\,dt = \ln|t+1| + C_1$$

$$\int \frac{1}{t^2+1}\,dt = \arctan(t) + C_2$$

$$\int \frac{t}{t^2+1}\,dt = \frac{1}{2}\int \frac{2t}{t^2+1}\,dt = \frac{1}{2}\ln\big(t^2+1\big) + C_3$$

Therefore,

$$\int \left(\frac{1}{t+1}\mathbf{i} + \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k}\right)\,dt = \left\langle \ln|t+1|,\ \arctan(t),\ \frac{1}{2}\ln\big(t^2+1\big)\right\rangle + \langle C_1, C_2, C_3\rangle$$

And,

$$\int_0^1 \left(\frac{1}{t+1}\mathbf{i} + \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k}\right)\,dt = \left\langle \ln(2), \arctan(1), \frac{1}{2}\ln(2)\right\rangle - \left\langle \ln(1), \arctan(0), \frac{1}{2}\ln(1)\right\rangle$$

$$= \left\langle \ln(2), \frac{\pi}{4}, \frac{1}{2}\ln(2)\right\rangle - \langle 0,0,0\rangle = \left\langle \ln(2), \frac{\pi}{4}, \frac{1}{2}\ln(2)\right\rangle.$$

12. For the vector function $\mathbf{r}(t) = \langle 3\sin(t), 2\cos(t)\rangle$, find the unit tangent vector $\mathbf{T}(\pi/3)$.

$\mathbf{r}(t) = \langle 3\sin(t), 2\cos(t)\rangle$

$\mathbf{r}'(t) = \langle 3\cos(t), -2\sin(t)\rangle$

$\mathbf{r}'(\pi/3) = \langle 3\cos(\pi/3), 2\sin(\pi/3)\rangle = \left\langle \frac{3}{2}, \frac{2\sqrt{3}}{2}\right\rangle = \left\langle\ \frac{3}{2}, \sqrt{3}\right\rangle$ The magnitude of $\mathbf{r}'(\pi/3)$ is

$||\left\langle\ \frac{3}{2}, \sqrt{3}\right\rangle|| = \sqrt{\frac{9}{4}+3} = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$ Dividing $\mathbf{r}'(\pi/3)$ by its magnitude gives us the unit tangent vector

$\mathbf{T}(\pi/3) = \frac{2}{\sqrt{21}}\left\langle\ \frac{3}{2}, \sqrt{3}\right\rangle = \left\langle \frac{3}{\sqrt{21}}, \frac{2}{\sqrt{7}}\right\rangle.$

Suggested Textbook Problems

Section 12.5	1-13,15,19-41, 45,48-51,53,57-59,61,63,65-69, 71,73,76-79
Section 13.1	1-6, 9, 11-13, 17, 19, 21-32, 41-46, 49-50
Section 13.2	3, 4, 8, 9, 13, 15, 18, 19, 21, 22, 24-28, 33-42

SOME USEFUL FACTS and EQUATIONS:

Equations for Lines

The **vector equation** of a line is: $\mathbf{r}(t) = \mathbf{r}_0(t) + t\mathbf{v}$
where $\mathbf{r}(t) = \langle x, y, z\rangle$ is the position vector of an arbitrary point on the line, $\mathbf{r}_0$ is the position vector of a specific point on the line, and $\mathbf{v}$ is the direction vector. Assuming that $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ and $\mathbf{v} = \langle a, b, c\rangle$, the component equations of the vector equation give the **parametric equations** for the line:

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct, \qquad -\infty < t < \infty$$

Manipulating each of the component equations for the line to solve for the parameter t (when a, b, c are non-zero), and equating the results give the **symmetric equations** for the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Equations for Planes

Say that $\mathbf{r} = \langle x, y, z\rangle$ is the position vector of an arbitrary point in the plane, $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ is the position vector of some specific point on the plane, and $\mathbf{n} = \langle a, b, c\rangle$, the **normal vector**, is orthogonal to the plane. Then a **vector equation** of the plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Plugging in the components of $\mathbf{n}$, $\mathbf{r}$, and $\mathbf{r}_0$ gives a **scalar equation of the plane**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Multiplying out and collecting terms in the scalar equation of the plane gives a **linear equation** in x, y, and z:

$$ax + by + cz + d = 0$$

Domain of a Vector Function

The domain of a vector function is the intersection of the domains of all of its component functions.

Limit and Continuity of Vector Functions

Limits of vector functions are evaluated component-wise: If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

A vector function is continuous at a point a if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$. A vector function is continuous on some domain if and only if all of its component functions are continuous on that domain.

Derivative of a Vector Function

The derivative of a vector function is defined as: $\dfrac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \dfrac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$

Since limits of vector functions are evaluated component-wise, derivatives are evaluated component-wise as well.

Tangent Vector, Unit Tangent Vector, Tangent Line

Say that the point P corresponds to $\mathbf{r}(a)$ on the curve $\mathcal{C}$ given by $\mathbf{r}(t)$. Then the vector $\mathbf{r}'(a)$ is tangent to $\mathcal{C}$ at the point P, and $\mathbf{r}'(a)$ is the **tangent vector** to $\mathcal{C}$ at P. The **tangent line** to the curve $\mathcal{C}$ at the point P passes through the point P given by $\mathbf{r}(a)$ and is parallel to the tangent vector $\mathbf{r}'(a)$. The **unit tangent vector** to the curve $\mathcal{C}$ at point P is given by

$\mathbf{T}(a) = \frac{\mathbf{r}'(a)}{|\mathbf{r}'(a)|}$, as long as $|\mathbf{r}'(a)| \neq 0$.

Differentiation Rules

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then:

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Integral of a Vector Function

Integrals of vector functions are evaluated component-wise: If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then we have the definite integral:

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

and the indefinite integral:

$$\int \mathbf{r}(t)\,dt = \left\langle \int f(t)\,dt,\ \int g(t)\,dt, \int h(t)\,dt \right\rangle = \left(\int f(t)\,dt\right)\mathbf{i} + \left(\int g(t)\,dt\right)\mathbf{j} + \left(\int h(t)\,dt\right)\mathbf{k}$$

There is a scalar constant of integration associated to each scalar indefinite integral, so the indefinite integral of a vector function requires a vector constant of integration C. If $\mathbf{r}(t)$ has the particular antiderivative $\mathbf{R}(t)$, then the most general antiderivative of $\mathbf{r}(t)$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$