Apologies for the delay in materials this & also last week
WA HW 6&7 posted ≈2:30pm today, due 7/11 11:59pm
Midterm 2 graded this evening, solutions to Q3 & M2
posted by end of day, optional Q4 up by next morning & due

Stokes' & Divergence Theorem

7/10 11:59pm, any missing solution documents for

Lecture for 7/9

worksheets (& even sol. for random things like prereq quiz)
up 7/10, EC bug bounty program starts 7/11 12am &
ends 7/14 11:59pm, extra practice added to DW5–8
on 7/10, survey up 7/10, more random features by 7/11
Note: final still on time of course, 7/12 2pm

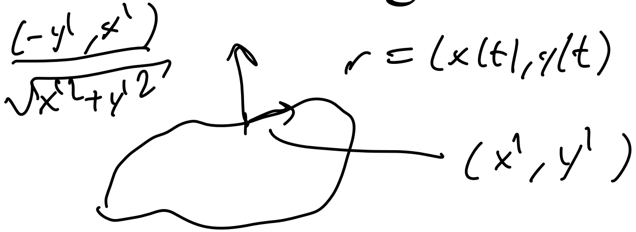
Asked registrar if students who need makeup can take on 7/13, respond TBD. Must be 7/12 or 7/13.

On 7/14, summer session 2 starts.

Motivation

Recall how we can write line integrals in terms of normal vectors

- $\int_C (\mathbf{F} \cdot \mathbf{n})\, ds = \int_C (A\, dy - B\, dx)$ if $\mathbf{F} = (A, B)$ and C is closed curve
  - n is unit normal pointing out of D if $C = \partial D$
  - If n points inside D, add minus sign
- This formulation is useful when normals are easy to find
- Can we do the same in higher dimensions?
  - Can we get a line ↔ surface integral conversion?
  - Can we get a Green's Theorem for 2D ↔ 3D conversions?



$\vec{n} = \frac{(-y', x')}{ds} \Rightarrow \vec{n}\, ds = (-y', x')$

$F \cdot \vec{n}\, ds = -Ay' + Bx' \rightarrow$

In many cases, $n=(1,0)$, $(0,-1)$, $n=(\cos\theta, \sin\theta)$ → $(F\cdot n)\,ds$ is easy to expand & simplify

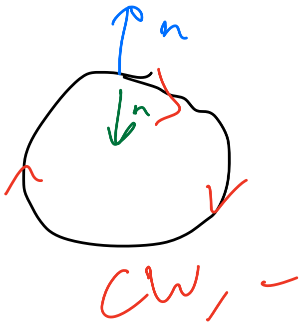
another − sign → $A\,dy - B\,dx$ & diff

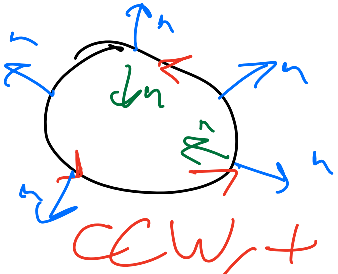
Oriented Surfaces

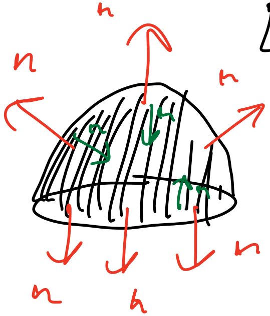
We have seen each curve has 2 possible direction

- Let S be a closed surface so that $S = \partial E$ for some solid E
- Further assume S has 2 sides, so Mobius strips banned
- At any point, we may pick unit normal pointing in or out of E
- Call S +, - oriented resp if we pick normals pointing in, out of E resp
    - Unless otherwise mentioned, we'll assume S is positive

look this up if you are interested
Has only 1 side, no inside or outside.







E = solid hemisphere

$S = \partial E$ = surface of E

Note! We say S is continuous, diff, smooth etc. if some $r(u,v)$

defining S is cont, diff, smooth etc. resp.

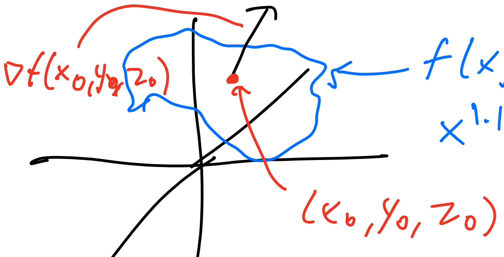
Unit Normals to Surfaces

Recall: smooth means ∞-diff.

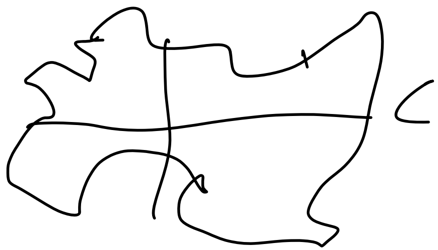
Suppose S is described by f(x, y, z) = 0 and param by **r**(u, v)

- Recall that ∇f is perpendicular to S
- Thus, **n** = $\nabla f/||\nabla f||$ is a unit normal to S
- As $\mathbf{r}_u$, $\mathbf{r}_v$ tangent to surface, $\mathbf{n} = (\mathbf{r}_u \times \mathbf{r}_v)/||\mathbf{r}_u \times \mathbf{r}_v||$ also works
- Define $d\vec{\mathbf{S}} = \vec{\mathbf{n}}\, dS$ for the sake of convenience
- Note: we are back to using bold for $\vec{\mathbf{v}}$ectors

Curve C can be desc. by $f(x,y)=0$



Sometimes, this $f(x,y,z)=0$ desc. S only exists abstractly.



$r(t) = (x(t), y(t))$

$x(t)y(t) - y(t)x(t) = 0$

$t = x^{-1}x(t) = y^{-1}y(t)$

Note: If S is described by $f(x,y,z)=0$ but also, for example, $x=g(y,z)$, don't confuse f with g. ∇g will only be a 2D vector and is NOT normal to S.

If we have a param $r(u,v)=(a(u,v),b(u,v),c(u,v))$ describing S, may be difficult or impossible to find some equation E in a,b,c such that $E(a,b,c)=0$.

$$\Rightarrow \underbrace{x^{-1}}_{g}A=\underbrace{y^{-1}}_{h}B$$

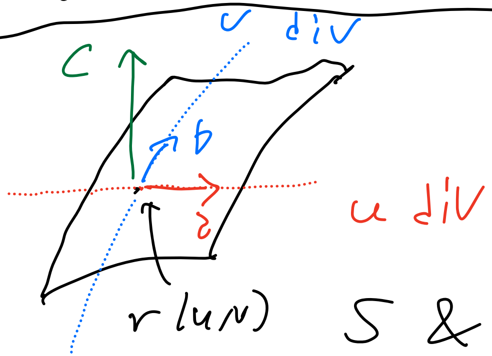
$$f(x,y)=g^{-1}x-h^{-1}y$$

$$0\cdot(a+b+c)=0 \Rightarrow 0=0 \quad \times$$

$r(\phi,\theta)=$ (spherical in θ & ϕ, ...)

$$\Rightarrow x^2+y^2+z^2=1 \Rightarrow f(x,y,z)=x^2+y^2+z^2-1,$$

but not all situations are this simple



As seen many times in many derivations, a & b are tangent to S & $a=r_u\,du$, $b=r_v\,dv$.

Scale a & b to be r_u & r_v instead.

So $c=a\times b=r_u\times r_v$ normal to S

Surface Integrals of Vector Fields

$\int (\text{vec} \cdot \text{vec})\, d(\text{scal})$

Suppose S, f, r are as before

$\int \text{scal} \cdot d(\text{sc})$

- Let's introduce a vector field $\mathbf{F} : S \to \mathbb{R}^3$
- Define flux as $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$

$\int \text{vec} \cdot d(\text{vec})$

- We have a nice formula when $f(x, y, z) = z - g(x,y)$
  - Suppose $\mathbf{F} = (A, B, C)$, and S lies over $D \subseteq \mathbb{R}^2$
  - Then $\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \iint_D (-Ag_x - Bg_y + C)\, dA$

$$d\vec{S} = \underbrace{n}\,\underbrace{dS} = \underbrace{\frac{r_u \times r_v}{\|r_u \times r_v\|}}\,\underbrace{\|r_u \times r_v\|\, du\, dv} = (r_u \times r_v)\, du\, dv$$

Space for Derivations

To derive $\iint_D (---)\,dA$ equation, let's use what we know

$$f = z - g(x,y) \Rightarrow$$

Thus, $\mathbf{n} = \nabla f/||\nabla f||$ is a unit normal to S

$$\nabla f = (-g_x, -g_y, 1) \Rightarrow$$

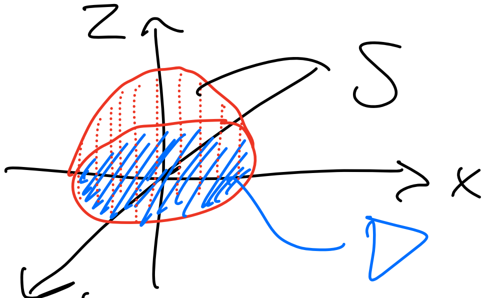
As $\mathbf{r}_u$, $\mathbf{r}_v$ tangent to surface, $\mathbf{n} = (\mathbf{r}_u \times \mathbf{r}_v)/||\mathbf{r}_u \times \mathbf{r}_v||$

Define $d\mathbf{S} = \mathbf{n}\, dS$ for the sake of convenience

$$\underbrace{\vec{n}}\,\underbrace{dS} = \underbrace{\frac{\nabla f}{||\nabla f||}}\,\underbrace{\sqrt{g_x^2+g_y^2+1}\,dx\,dy} = \frac{(-g_x, -g_y, 1)}{\sqrt{\cdots}}\sqrt{\cdots}\,dx\,dy$$

$$= (-g_x, -g_y, 1)\,dx\,dy \quad , \quad F = (A, B, C), \text{ so}$$

$$(F \cdot n)\,dS = (A,B,C)\cdot(-g_x, -g_y, 1)\,\underline{dx\,dy} = (-g_x A - g_y B + C)\,\underline{dA}$$



after converting everything to x & y, integral is over D and not S. We have

$$A(x,y,z) = A(x,y,g(x,y)), \text{ similarly}$$

for $B(x,y,z)$, $C(x,y,z)$. $g(x,y)$ remains $g(x,y)$

Note: If S has multiple points with same (x,y) coord but diff z coord, then we are not in the scenario $z = g(x,y)$ because $g(x_0,y_0)$ is one value but z_0 would take 2 values. So need to split S into 2 pieces and consider each separately, can also recombine via $\iint_S f = \iint_D f_1 + \iint_D f_2 = \iint_D (f_1+f_2) = \ldots = \iint_S f$ if certain symmetries exist.

Fun Fact: look up klein bottles

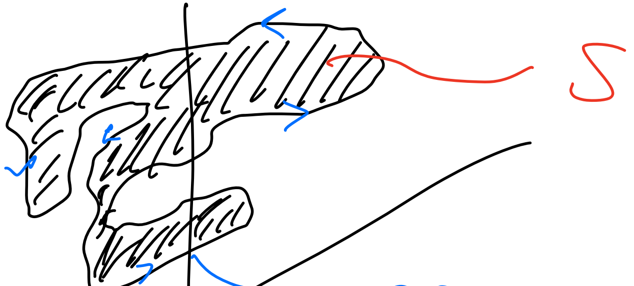
Example: unit sphere needs $z=\sqrt{1-x^2-y^2}$ & $z=-\sqrt{1-x^2-y^2}$

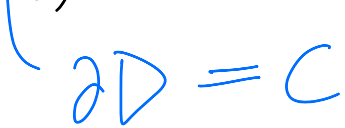
Stokes' Theorem

Suppose S, C positively oriented surface, curve resp, and $C = \partial S$

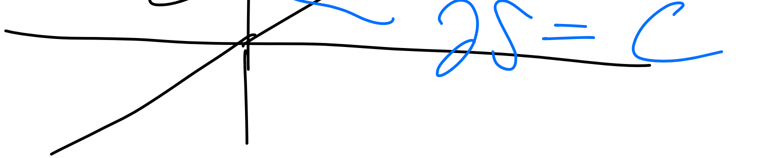
- Recall: C is positive, negative if it is CCW, CW respectively
- Also need condition that S is twice cont. differentiable
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$
- Add minus signs as necessary for S, C oriented differently

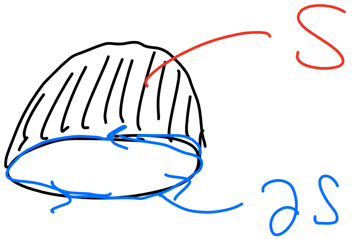
In Green's, only worried about $\pm$ for $\int_C$ in the equation $\int_C \cdots = \iint_D$. Now consider $\pm$ for both sides.





Derivation





To see what's going on, let's expand both sides for $F = (P, Q, R)$. Then

$$\int_C F \cdot dr = \int_C P\,dx + Q\,dy + R\,dz,$$ and

$$\nabla \times F = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y).$$ IF $z = g(x,y)$, $(x,y) \in D$ describes S, then $d\vec{S} = (-g_x, -g_y, 1)\,dA$

So $$\iint_S (\nabla \times F)\, d\vec{S} = \iint_D \left[(Q_z - R_y) g_x + g_y (R_x - R_z) + (Q_x - P_y)\right] dA$$

We have seen $P\,dx + Q\,dy$ & $(Q_x - P_y)\,dA$ in Green's Theorem. So let's try proving Stokes' with Green's.

But we can't match up both sides directly because Green is $\geq$ 2 for 1 match so using 3x Green would be $\geq$ 6 for 3, but we need 3 for 3.

Let's look at $R\,dz$ and see what happens when we plug in for z since we haven't used the assumption that $z = g(x,y)$ yet.

Then $R = R(x,y,g(x,y))$, $dz = g_x\,dx + g_y\,dy$ using differentials or chain rule, so $R\,dz = Rg_x\,dx + Rg_y\,dy$.

This is now fully in terms of x & y, so $\int_C R\,dz$ written this way becomes a 2D integral. So now we can use Green's Theorem properly to say

$$\int_C R\,dz = \int_C R g_x\,dx + R g_y\,dy = \iint_D \left[(R g_y)_x - (R g_x)_y\right] dA =$$

$$\iint_D \left[(R_x g_y - R_y g_x) + R(g_{yx} - g_{xy})\right] dA = \iint_D (R_x g_y - R_y g_x)\,dA$$

by Clairaut's Theorem.

So we have matched up green terms on both sides by fittingly using Green's Theorem. So now the terms are being matched up properly, we can do the same with Pdx & Qdy, and conclude.

But we can't do the same with Pdx & Qdy. Rdz was special because of $z = g(x,y) \longrightarrow dz = g_x dx + g_y dy$, but $dx = dx$ and $dy = dy$, so nothing gets expanded.

So let's pretend x & y components don't exist. We have seen breaking up functions & regions before, as in

$$\int_C (f+g) = \int_{C_1} f+g + \int_{C_2} f+g = \int_{C_1} f + \int_{C_1} g + \int_{C_2} f + \int_{C_2} g.$$

Consider $F = (P, Q, R)$ as $(P,0,0) + (0,Q,0) + (0,0,R)$.

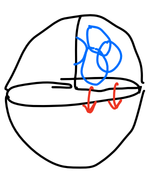
When $F = (0,0,R)$, Stokes' becomes $\int R\,dz = \int (R_x g_y - R_y g_x)\,dA$, which we saw was true if $z = g(x,y)$ describes S.

So ST is true for $(0,0,R)$ & $z = g$.

Around some ordinary point, we can write $x = g_1(y,z)$, $y = g_2(x,z)$, $z = g_3(x,y)$ for some g_1, g_2, g_3.

Example: for sphere $x^2+y^2+z^2=1$, we have around $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ that $g_1 = \sqrt{1-y^2-z^2}$, $g_2 = \sqrt{1-x^2-z^2}$, $g_3 = \sqrt{1-x^2-y^2}$.

If S can be written in all $x = \ldots$, $y = \ldots$, $z = \ldots$ formats for one choice of (g_1, g_2, g_3), then by breaking up F as $P00 + 0Q0 + 00R$, converting each piece, and recombining, we get ST is true for S.



Break up S into small pieces $S_1, S_2, S_3, \ldots$ where ST is true. It seems we are now done, but unfortunately not yet. There may be barriers through which there is no piece that connects both sides so that we can pass through, like the plane $z=0$ for unit sphere.

So we need to show that it is possible to combine every small piece and get all of S without missing anything.

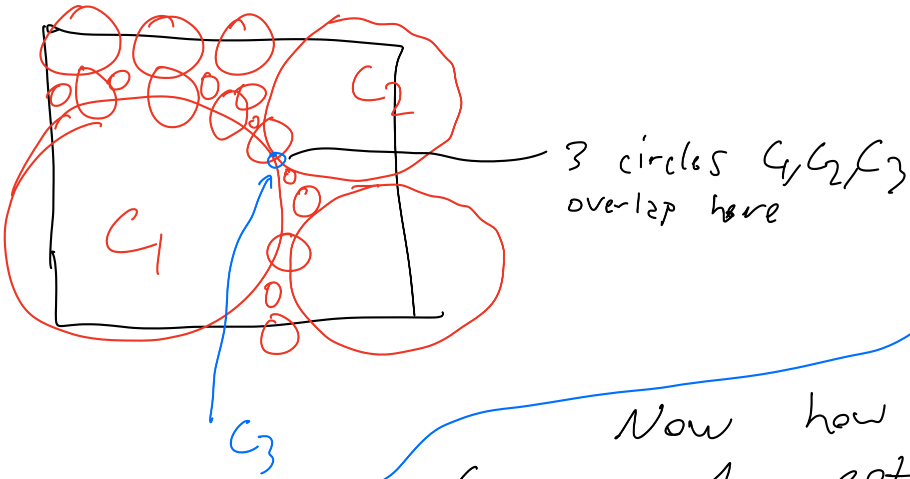
Let S_1, S_2, S_3 be the regions where we can express S with x, y, z resp. S_1, S_2, S_3 may overlap, but $S_1 \cup S_2 \cup S_3$ because [around any point, at least one of x, y, z work.]

[...] is true by implicit function theorem. S is described by $f(x,y,z)=0$ for some f, and ∇f is never $\vec{0}$ since you can't move at any point on the surface without changing at least one of x, y, z (otherwise you'd be at the same point). So at any point, $(f_x, f_y, f_z) \neq (0,0,0)$. WLOG $f_x \neq 0$ at $v_0 = (x_0, y_0, z_0)$. By implicit function theorem applied to $0 = \frac{df}{dx} = f_x + \frac{\partial y}{\partial x} f_y + \frac{\partial z}{\partial x} f_z \Rightarrow f_x = -\frac{\partial x}{\partial x} f_y - \frac{\partial z}{\partial x} f_z$, there exists some $g(y,z)$ such that $f(g(y,z), y, z) = 0$ in a neighborhood of v_0 in S.

Let $S_i = \bigcup_j V_{i,j}$ where $V_{i,j}$ is an open set on which S expressed in coordinate i. Then $\bigcup_{i=1}^{3} \bigcup_j V_{i,j}$ is an open covering of S. Assume S is bounded, then $\bar{S}$ compact $\Rightarrow \exists$ finite subcovering $\begin{Bmatrix} V_{1,1} & \cdots & V_{1,K} \\ V_{2,1} & \cdots & V_{2,L} \\ V_{3,1} & \cdots & V_{3,M} \end{Bmatrix}$.

By Lebesgue's result on topological dimension of nD surfaces, there exists at least one point where $n+1$ open sets overlap. So in 2D, there must be some $u \in S_1 \cap S_2 \cap S_3 \Rightarrow u \in V_{1,a} \cap V_{2,b} \cap V_{3,c}$.

Ex: if you cover graph paper with circles (not just 1 circle covering everything), there must be some point where 3 circles intersect.



Let $U = S_1 \cap S_2 \cap S_3$, then Stokes Theorem is true on U with no disclaimers or restrictions on types of S.

Now how are we going to, for example, get Stokes theorem on an entire square when we only have it on some small patch $C_1 \cap C_2 \cap C_3$ like in the drawing above?

Idea to continue! Stokes' is true on $\overline{U}$ as well since $\int_X f = \int_{\overline{X}} f$, as $\overline{X} \setminus X \subseteq \partial X$ and $\int_{\partial X} f = 0$ since ∂X is too small to affect integral (for $\iint_S dA$, we have $\iint_{\partial S} dA = 0$ since region & dA are 2D but ∂S is only 1D). For example, integral of $f(x)$ over a point $x=a$ is $\int_a^a f(x)\,dx = 0$ and $\partial(a,b) = \{a\} \cup \{b\} = 2$ points $\Rightarrow \int_{\partial(a,b)} f = 0$.

Thus, we can cover $T = S \setminus \overline{U}$ by open sets and get a new open set $U_2 \subseteq T$ on which Stokes is true. So Stokes is true on $U \cup U_2$. Can keep adding more ~~&~~ more to get Stokes true on U & U_2 & U_3 & U_4 & ...

But we might get stuck like how the sequence $0, 0.9, 0.99, 0.999, \ldots$ gets stuck below 1 and will never reach 2.

So how do we know we don't get stuck?

Final idea: use maximality principle. Let $V \subseteq S$ be the maximal open set on which Stokes' is true.

If $V \neq S$, we are not done yet. Stokes' also true on $\bar{V}$, but what if also $\bar{V} \neq S$?

Then let $T = S \setminus \bar{V}$ and find open $W \subseteq S \setminus \bar{V}$ with $W \neq \phi$ (non-empty) with Stokes' true on W, so Stokes' is true on $V \cup W$, which is larger than V, contradicting maximality of V.

So $V = S$ or $\bar{V} = S$, and we are done.

Finally, Stokes Theorem is true for any S and any F and any description of S by f's as long as the hypotheses on differentiability hold.

Note: since we used proof by contradiction, there is no concrete expression of how each open set will look in the final covering.

We know $S = S_1 \cup \dots \cup S_N$ for some pieces S_i where (g_1, g_2, g_3) choice exists on each S_i, but we may not know what these pieces are.

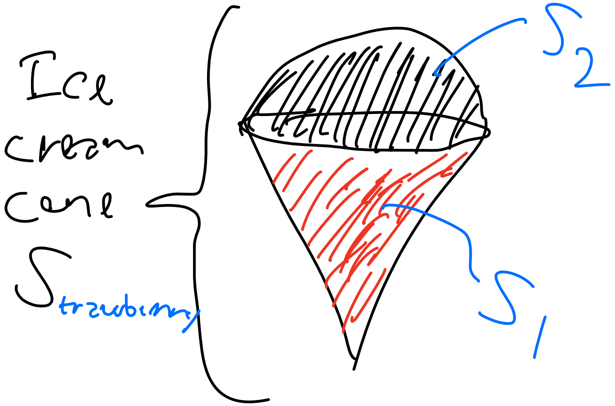
Note: we used that S is bounded. If S is not bounded, write $S = S_1 \cup S_2 \cup \dots$ where S_i is bounded and consider $\iint_S$ as $\sum_{i=1}^{\infty} \iint_{S_i}$ to get that Stokes' Theorem is true on S.

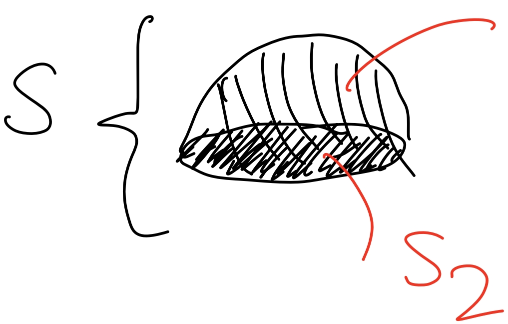
For example, $\mathbb{R}^2 = (-1,1)^2 \cup (-2,2)^2 \cup (-3,3)^2 \cup \dots$

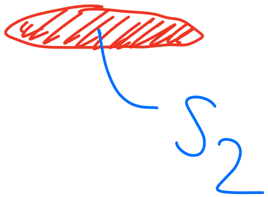
Divergence Theorem

Let E be a solid, S = ∂E is pos orient, and F is cont. diff vector field

- $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F}\, dV$
- Congratulations, this is the end of the course







$$\iint_S f = \iint_{S_1} f + \iint_{S_2} f$$

Derivation

By all the discussion on geometry of coverings of S by regions where $x = g(y,z)$, $y = g(x,z)$, $z = g(x,y)$ are possible, it suffices to show the theorem of each of those 3 cases. Each case is similar, so WLOG we just consider the case that S is described by $z = g(x,y)$ so that

$E = \{(x,y,z) : a(x,y) \leq z \leq b(x,y),\ (x,y) \in D\}$ for some 2D region D.

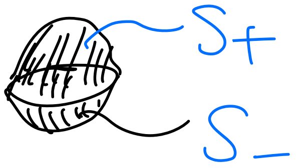
Now we plug everything in and match up both sides.

Furthermore, by all the same work on splitting up F, we only need to consider the case $F=(0,0,R)$.

Then $(\nabla \cdot F)\,dV = (0_x + 0_y + R_z)\,dV = R_z\,dx\,dy\,dz$,

so $$\iiint_E (\nabla\cdot F)\,dV = \iint_D \left(\int_{a(x,y)}^{b(x,y)} R_z(x,y,z) \right) dz\,dx\,dy$$

$$= \iint_D \left[R_z(x,y,b(x,y)) - R_z(x,y,a(x,y)) \right] dx\,dy.$$ Now split S as $S_+ \cup S_-$.



$\vec{n}\,dS = (-g_x, -g_y, 1)\,dA$ on S^+ from prior work,

so $F \cdot d\vec{S} = (0,0,R)\cdot(-g_x,-g_y,1)\,dA = R\,dA$, so

$$\iint_S F\cdot d\vec{S} = \iint_{S+} F\cdot d\vec{S} + \iint_{S-} F \cdot d\vec{S} =$$

$\iint_{S^+} R\,dA - \iint_{S^-} R\,dA$ since on S^-, $\vec{n}$ will point downward, picking up a minus sign.

Now $\iint_{S^+} R\,dA = \iint_D R(x,y,b(x,y))\,dA$ since $z_{max} = b(x,y)$ in E and S^+ is the upper boundary of $E \Rightarrow z$ is the largest value.

Similarly, $\iint_{S^-} R\,dA = \iint_D R(x,y,a(x,y))\,dA$ since S^- is lower boundary of E.

Hence both LHS & RHS of theorem are $\iint_D R(x,y,z)\Big|_{z=a}^{z=b} dA$.

So div theorem is true and we ~~are~~ done completely with this lecture.

Practice Problems

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the following functions and surfaces

- F = $(3x, 2y, 1-y^2)$, S is portion of $z = 2-3y+x^2$ oriented downward and lying over triangle with vertices (0,0), (2,0), (2, -4)
- F = $(yz, x, 3y^2)$, S is surface of solid bounded by $x^2+y^2 = 4$, $z = x-3$, $z = x+2$ with negative orientation
- F = $\nabla \times G$, G = $(z^2-1, z+xy^3, 6)$, S is the portion of $x = 6-4y^2-4z^2$ in front of $x = -2$ with orientation in negative x-direction
  - For extra computation practice, prove without Stokes'

Oops! All practice problems

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following functions and curves

- F = (-yz, 4y+1, xy), C is circle of radius 3 centered at (0, 4, 0), perpendicular to y-axis, and oriented CW when looking above y>4
- F = ($3yx^2+z^3$, y^2, $4yx^2$), C is triangle with vertices (0, 0, 3), (0, 2, 0), (4, 0, 0) oriented CCW when looking above C towards origin

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for these with & without divergence theorem

- F = (sin(πx), zy^3, z^2+4x), S is surface of box with $-1 < x < 2$, $0 < y < 1$, $1 < z < 4$ oriented pointing out of the box
- F = (2xz, $1-4xy^2$, $2z-z^2$), S is surface of solid bounded by $z = 6-2x^2-2y^2$ and $z = 0$, oriented pointing inside the solid

Scratchwork