Keep doing pre-req. quiz
Keep bringing WS 1&2
Start HW ASAP

reminders

Velocity and Acceleration

Lecture for 6/12

solved

Lec rec. yesterday lost, issue couldn't be duplicated, but issue will be gone in future by checking view of Zoom after ending meeting after starting meeting

Video will also be placed in playlist, which is unlisted

Definition of Velocity and Acceleration

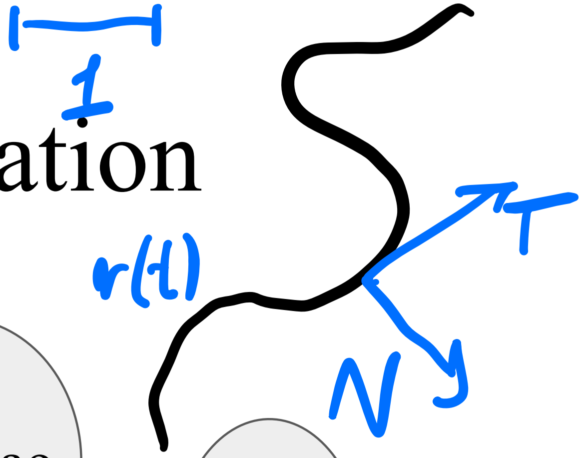
- Suppose **r**(t) is our vector function
- Then **v**(t) = **r**'(t) and **a**(t) = **r**''(t)

solved before discussion

$$\begin{tikzcd} r \arrow[r, bend left, "d"] & v \arrow[l, bend left, "\int"] \arrow[r, bend left, "d"] & a \arrow[l, bend left, "\int"] \end{tikzcd}$$

all vectors

Decomposition of Acceleration



- Recall unit tangent **T** and unit normal **N**
- We can express $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$
- Define $s = ||\mathbf{v}||$ as speed
- $a_T = s' = (r' \cdot r'')/||r'||$ 1st
- $a_N = \kappa s^2 = ||r' \times r''||/||r'||$ 2nd

Concl: a is hard to visualize for a vector r, but T & N are easy. So express a

via T & N

Formula Derivation Scratchwork

$$a = v' = \frac{d}{dt} v$$

$$v = \frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = T \frac{ds}{dt}$$

Why $T = \frac{dr}{ds}$: $T = \frac{r'}{\|r'\|} \Rightarrow r' = T\|r'\| =$

Recall $ds = \|r'(t)\|\,dt$ from arc

length lecture

Extra Space

$$\frac{dr}{dt} = r'(t)$$

$$\frac{dr}{ds} = \overbrace{\frac{r'(t)\,dt}{\|r'(t)\|dt}} = \frac{r'(t)}{\|r'(t)\|} = T(t)$$

$$a = \frac{d}{dt}\left(T\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}T + \frac{ds}{dt}\frac{dT}{dt}$$

$$= \underbrace{\frac{d^2s}{dt^2}T} + \underbrace{\frac{ds}{dt}\frac{dT}{ds}\frac{ds}{dt}} =$$

definition of K

Practice

$$\left(\frac{ds}{dt}\right)^2 \frac{dT}{ds} = \left(\frac{ds}{dt}\right)^2 \frac{dT/ds}{\underbrace{\|dT/ds\|}} \left\|\frac{dT}{ds}\right\| =$$

$$\left(\frac{ds}{dt}\right)^2 N \kappa = \|r'\|^2 N \kappa$$

another chain rule

$$\frac{dT/ds}{\|dT/ds\|} = \frac{dT/dt}{\|dT/dt\|} = \frac{T'(t)}{\ \ } =$$

One big problem

- If the acceleration is given by **a** = (1, 2, 6t), find the position r given that **v**(0) = (0, 1, -1) and **r**(0) = (1, -2, 3)
- Find the unit tangent and unit normal for **r**
- Find the tangential and normal components of acceleration

Find the tangential and normal components of acceleration for the object whose position is **r**(t) = (cos(2t), -sin(2t), 4t)

ds $\|r'(t)\|$

$$\frac{ds}{dt} = \|r'(t)\|$$

Scratch Work

$$\|T'(t)\|$$

$$N(t)$$

$$a = \frac{d^2s}{dt^2} T + \frac{ds}{dt}\frac{dT}{ds}\frac{ds}{dt}$$

$$= \|r'(t)\|' \, T + \|r'(t)\|^2 \cdot \kappa \cdot N$$

$$= \text{unfinished business} \; T + \|r'(t)\|^2 \frac{\|r' \times r''\|}{\|r'\|^3} N$$

$$\frac{r' \cdot r''}{\|r'\|} \qquad + \frac{\|r' \times r''\|}{\|r'\|} N$$

$\|r\|$

Extra Space

You can informally cancel out dt. But if you want to be more careful, use chain rule one more time

$$\frac{dr}{ds} = \frac{dr/dt}{ds/dt} \quad : \quad \overbrace{\underbrace{\frac{dr}{dt}}_{\text{vec}} = \underbrace{\frac{dr}{ds}}_{\text{vec}}\,\underbrace{\frac{ds}{dt}}_{\text{scal.}}}$$

divide by

which is allowed

ds/dt, which is greater

$$= \frac{r'(t)}{\|r'(t)\|} \quad \text{since } \frac{ds}{dt} = \|r'(t)\| \text{ from arc length yesterday}$$

$$\frac{d}{dt}(f \circ g) = g'(f' \circ g) =$$

Tangencies and Curvature

Pre-lecture for 6/12

$g = s$

$r = f$

$$\left.\frac{df}{dt}\right|_{t=g(u)} = \frac{dg}{dt}\left(\frac{df}{dt} \circ g\right)$$

$$\frac{dr}{dt} = \frac{ds}{dt}\left(\frac{dr}{dt} \circ s\right)$$

why is this $\frac{dr}{ds}$

$$\underbrace{\frac{dr}{ds} = \frac{dr}{dt} \circ s}_{\text{no algebraic proof, requires a limit proof}}$$

Tangent Vector

- Tangent to r(t) is r'(t)
- Unit tangent is T(t) = r'(t)/||r'(t)||

recall: unit vectors are just vectors with mag. 1

Source: https://heliooptometry.ca/blog/does-squinting-mean-you-need-glasses

For limit proof, express RHS as a limit and use limit rules to show they are equal

LHS

Normal Vector

ask me how

- Define $N(t) = T'(t)/||T'(t)||$
- Fact: If u(t) is a unit vector, then u' and u are orthogonal
- Fact: N is orthogonal to T

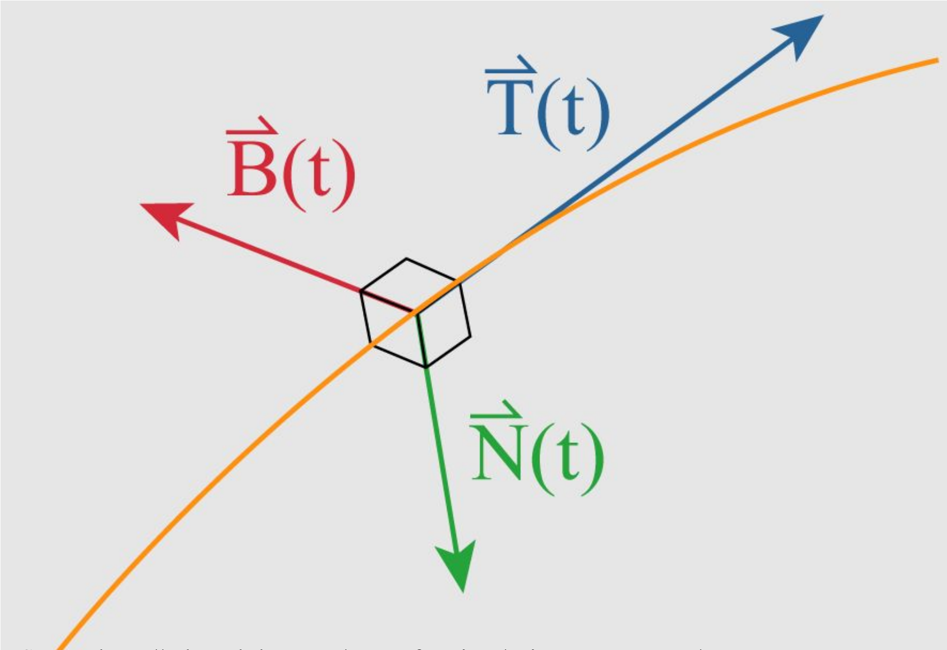
$$1 = ||u(t)|| \Rightarrow 1 = ||u(t)||^2 \Rightarrow 0 = \frac{d}{dt}||u(t)||^2 = \frac{d}{dt}(u(t)\cdot u(t)) = \underline{2u'(t)\cdot u(t)} \Rightarrow u'\cdot u = 0$$

$$T \cdot T' = 0$$

$$T \cdot N = T \cdot \frac{T'}{||T'||} = \frac{1}{||T'||}(T \cdot T') = 0$$

Binormal Vector

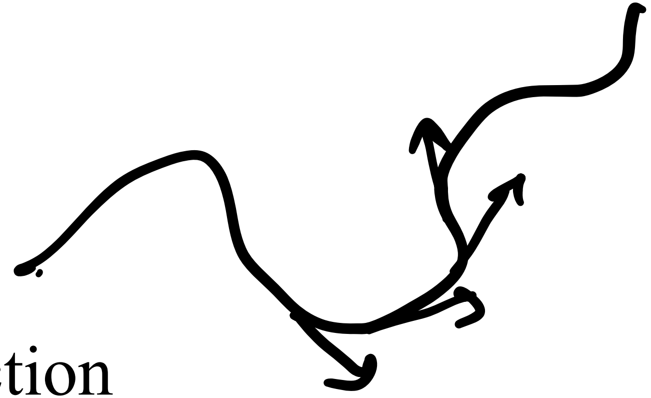
- Define B(t) = T(t) x N(t)
- Now T, N, B are pairwise orthogonal

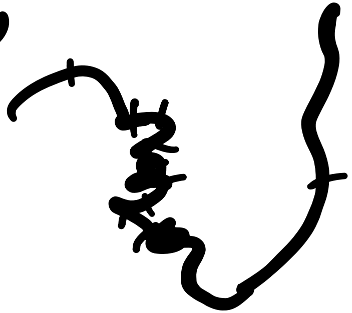


Source: https://calcworkshop.com/vector-functions/unit-tangent-vectors/

Curvature

rec: $ds = \|r'(t)\| dt$





- Measures how fast a curve is changing direction
- Defined by $\kappa = \|dT/ds\|$ where s is arc length
- Where this comes from:

$$\frac{dT}{ds} = \frac{dT}{dt}\,\frac{dt}{ds} = T'(t)\cdot\frac{1}{\|r'(t)\|}$$

$$\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{\|r'(t)\|dt/dt} = \frac{1}{\|r'(t)\|}$$

Reformulating κ for Calculations

To find κ, we need a convenient formula

- $\kappa = ||T'(t)||/||r'(t)||$ ✓
- $\kappa = ||r'(t) \times r''(t)||/||r'(t)||^3$

explain: $\|T'(t)\| = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^2}$

$T = \dfrac{r'(t)}{\|r'(t)\|}$ $\quad s = \|r'(t)\|$

$r'(t) = \underbrace{s}_{\text{ord}}\,\underbrace{T}_{\text{vec}} \Rightarrow r'' = s'T + sT'$

Scratch Work

$$r' \times r'' = r' \times (s'T + sT') =$$

$$\underline{(s')^2}\, T \times T'$$

$$\|r' \times r''\| = \underline{(s')^2}\, \|T'\|$$

$$\|T'\|^2 = \frac{\|r' \times r''\|}{\|r'(t)\|^2}$$

Practice Problems

$1 \quad 3\cos \quad -3\sin$

d

Let r(t) = (t, 3sin(t), 3cos(t)). Find the tangent, normal, and binormal vectors for r. Then determine the curvature of r.

Curvature of single-variable function

- Use one of the reformulations to show that the curvature of the graph of y = f(x) is $||f''(x)||/(1+f'(x)^2)^{3/2}$

$$T = \frac{r'(t)}{\|r'(t)\|} = \frac{(1, 3\cos(t), -3\sin(t))}{\| \cdots \|} =$$

$$\|r'(t)\| = \sqrt{1+(3\cos t)^2+(-3\sin t)^2} =$$

$$\sqrt{1+9\sin^2 t+9\cos^2 t} = \sqrt{1+9} = \sqrt{10}$$

1st $\rightarrow$ $\frac{1}{\sqrt{10}}(1, 3\cos(t), -3\sin(t))$

2nd $\rightarrow$ $= \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos(t), -\frac{3}{\sqrt{10}}\sin(t)\right)$

both are fine

$$N = \frac{T'}{\|T'\|} = \frac{\cancel{\frac{1}{\sqrt{10}}}(0, -3\sin(t), -3\cos(t))}{\cancel{\frac{1}{\sqrt{10}}}\|(0, -3\sin t, -3\cos t)\|}$$

$$\left\|\frac{1}{\sqrt{10}}v\right\| = \left|\frac{1}{\sqrt{10}}\right| \|v\| = \frac{1}{\sqrt{10}}\|v\|$$

$$\|(0, -3\sin t, -3\cos t)\| = \sqrt{0^2+9\sin^2+9\cos^2} =$$

$$\sqrt{9} = 3$$

$$N = \frac{(0, -3\sin t, -3\cos t)}{3} = \underline{(0, -\sin t, -\cos t)}$$

Check: $\|N\| = \sqrt{0+s^2+c^2} = \sqrt{1} = 1$

$$B = T \times N = \left(\frac{1}{\sqrt{10}}(1, 3c, -3s)\right) \times (0, s, -c)$$

$$= \frac{1}{\sqrt{10}}\left((1, 3c, -3s) \times (0, s, -c)\right)$$

$$\begin{vmatrix} i & j & k \\ 1 & 3c & -3s \end{vmatrix} \begin{matrix} i & j \\ 1 & 3c \end{matrix}$$

i: $3c \cdot -c - (-3s \cdot -s)$

$= -3c^2 - 3s^2 =$

$1 \quad 0 \quad -s \quad \cancel{-c} \quad 1 \quad 0 \quad s$

$-3(c^2+s^2) = -3$

$j: \ -3s\cdot 0 - (1\cdot -c) = 0 + c = c$

$k: \ 1\cdot s - (0\cdot 3c) = s$

$$= \frac{1}{\sqrt{10}}(-3, \cos t, \sin t)$$

put the t's back

κ = ||T'(t)||/||r'(t)||

$$\|T'(t)\| = \cancel{\frac{1}{\sqrt{10}}}\|(0, -3\sin t, -3\cos t)\|$$

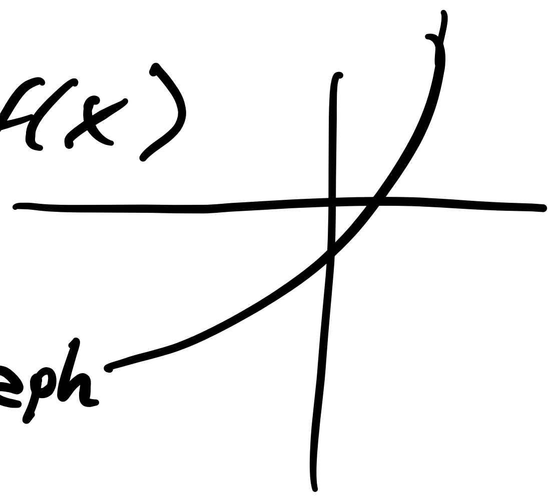
$$= \frac{1}{\sqrt{10}}\cdot 3 = \frac{3}{\sqrt{10}}$$

$$\|r'(t)\| = \sqrt{10}$$

$$\kappa = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$$

κ = ||r'(t) x r''(t)||/||r'(t)||^3

$y = f(x)$



$r(t) = (t, f(t))$

Trick: pretend the graph sits inside R^3

$r(t) = (t, f(t), 0)$

$$r'(t) = \langle 1, f'(t), 0 \rangle$$

$$r''(t) = \langle 0, f''(t), 0 \rangle$$

$$\|r'(t)\| = \sqrt{1 + f'(t)^2}$$

$$r' \times r'' = \langle 1, f', 0 \rangle \times \langle 0, f'', 0 \rangle$$

$$\begin{array}{ccc|cc} i & j & k & i & j \\ 1 & f' & 0 & 1 & f' \\ 0 & f'' & 0 & 0 & f'' \end{array} = \langle 0, 0, f'' \rangle$$

$$\|r' \times r''\| = |f''|$$

κ = ||r'(t) x r''(t)||/||r'(t)||³ $= \dfrac{|f''|}{\sqrt{1+f'^2}^{\,3}} =$

$$\frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$$