Try HW 4, 5

Quiz 2 graded, solutions, mistakes document by end of today

TBD on future materials, so if you want to get a head start, trying the homeworks is your best bet

Integrals in Polar

Pre-lecture for 6/26

General Idea

What if we convert f(x, y) into polar coordinates?

- Things can simplify if we have terms like x^2+y^2 and y/x
- Recall: x = rcos(θ), y = rsin(θ)
- But we need to figure out what dA becomes
  - Normally, dA = dx dy

$dA \neq dr\, d\theta$

$0 \le x \le 1$

$\sqrt{3-x^2} \le y \le \sqrt{5-x^2}$

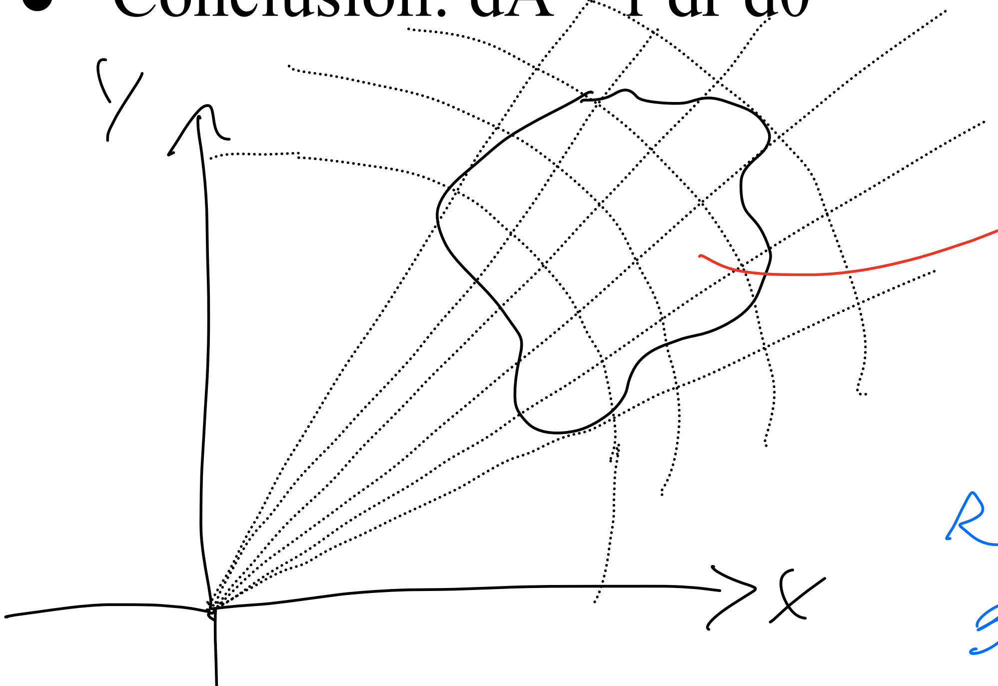
→ outer integral might be messy after finding antiderivative & plugging in bounds $\sqrt{3-x^2} \to \sqrt{5-x^2}$

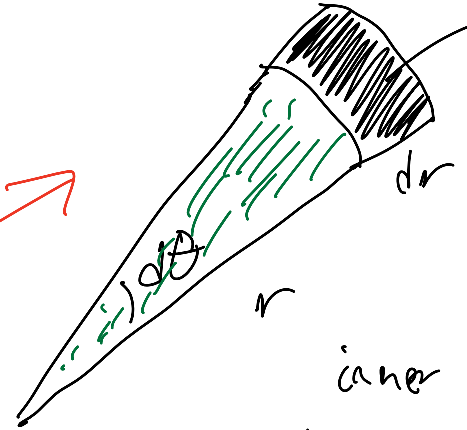
because many 1D integrals with square roots & non-linear polynomials are tricky to compute.

Example: $\int \sqrt{2+\sin^2 x}\, dx$ is impossible to express in elementary functions, would need elliptic integral of the 1st, 2nd, ... kind

Converting dA

- Split the region R we're integrating into radial slices
- Find the area of each slice
- Conclusion: dA = r dr dθ





area of this will be dA

inner radius: r

outer radius: r+dr

angle: dθ

Recall from high school geometry, the area of a circular sector of angle θ (in radians) & radius R is $\frac{1}{2}\theta R^2$. Ex: for whole circle, $\theta = 2\pi$ & you get classic πR^2

dA = (area larger wedge) − (area smaller wedge)

$$= \frac{1}{2}d\theta\,(r+dr)^2 - \frac{1}{2}d\theta\, r^2$$

$$= \frac{1}{2}d\theta\left[2r\,dr + (dr)^2\right] = r\,dr\,d\theta + \frac{1}{2}d\theta\,(dr)^2.$$

But $(dr)^2 = 0$, so $dA = r\,dr\,d\theta$.

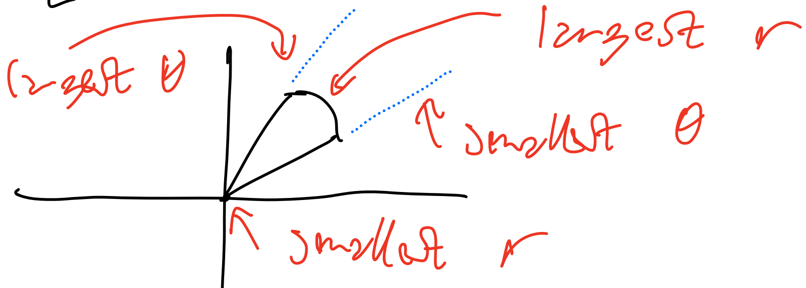
Note: if you want to derive completely rigorously, take Δr, $\Delta\theta$, find $\Delta A = r\Delta r\Delta\theta + \frac{1}{2}\Delta\theta(\Delta r)^2$, show Riemann sum for $\Delta\theta(\Delta r)^2$ is so small it converges to 0. There is a general method for showing sums arising from having $(\Delta u)^2$ for some variable u are small & tend to 0.

The Conversion Process

- $\iint_R f(x, y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$
  - Now we need to figure out the bounds on r, θ
- Draw R
- For r: closest and furthest points in R from origin
- For θ: smallest & largest angle for which angle θ line intersects R

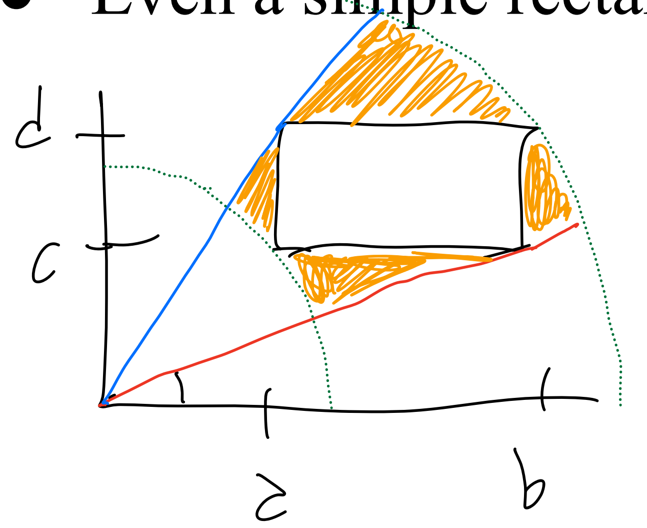
There is no formula for "description of R" → "bounds on R"

[R is bounded by $f(x,y)=0$, $g(x,y)=0$, ... → $? \le x \le ?$, $? \le y \le ?$ ↓ $? \le r \le ?$, $? \le \theta \le ?$]



Usage Warning

- Polar coordinates are useful on circular regions
- Converting bounds is messy or impossible for general regions
- Use your judgement to see if polar is worth it
- Even a simple rectangle becomes messy



$\tan^{-1}\frac{c}{b} \le \theta \le \tan^{-1}\frac{d}{a}$

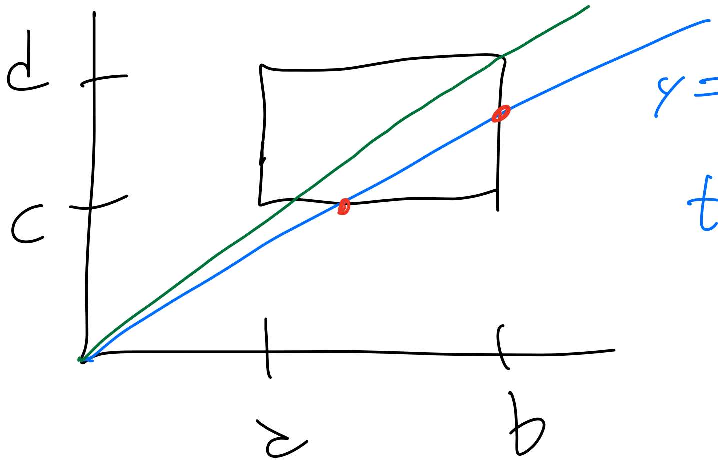
$\sqrt{a^2+c^2} \le r \le \sqrt{b^2+d^2}$

But not so fast! Not all r & θ meeting these bounds work

There are 4 orange areas which

satisfy the bounds but aren't part of the rectangle.

In reality: you can fix 1 variable, and the 2nd one will depend on the fixed one. For example, if you fix θ, here's what happens:



$y = tx$,

$t = \tan^{-1}\theta$

Largest & smallest r come from intersection of the angle θ line with the rectangle.

Closest: $\left(\frac{c}{t}, c\right)$

Furthest: (b, bt).

So $r_{min} = \left\|\left(\frac{c}{t}, c\right)\right\| = c\sqrt{1 + \frac{c}{(\tan^{-1}\theta)^2}}$

$r_{max} = \ldots = b\sqrt{1 + (\tan^{-1}\theta)^2}$

This calculation only applies to angles below the green line. For those above, "closest & furthest" has a different formula, so r_{max} & r_{min} different.

So we'd get $\int_a^b \int_c^d f(x,y)\,dy\,dx =$

$$\int_{\tan^{-1}\frac{c}{b}}^{\tan^{-1}\frac{d}{a}} \left(\int_{\frac{c}{(\tan^{-1}\theta)^2\sqrt{1+(\tan^{-1}\theta)^2}}}^{b\sqrt{1+(\tan^{-1}\theta)^2}} f(r\cos\theta, r\sin\theta)\, r\,dr \right) d\theta + \int_?^? \left(\int_?^? f(\ldots)\, r\,dr \right) d\theta$$

with similar complicated expressions for the 4 "?"s.

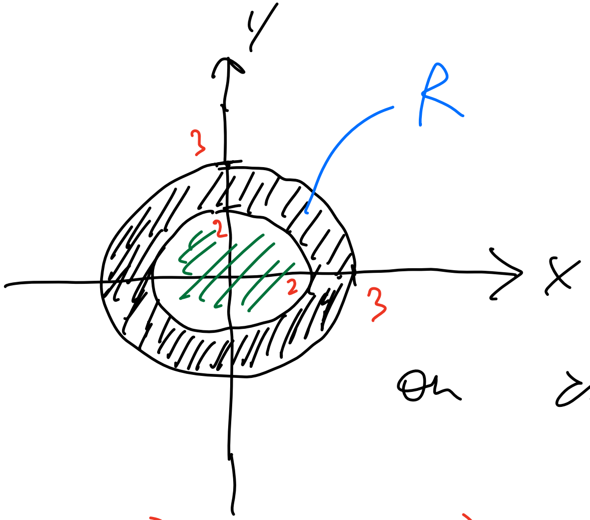
When finding bounds for converting, check for any unwanted orange areas to make sure you didn't make a mistake. Students may often include extra areas when finding new bounds due to missing dependencies between variables.

Practice Problems

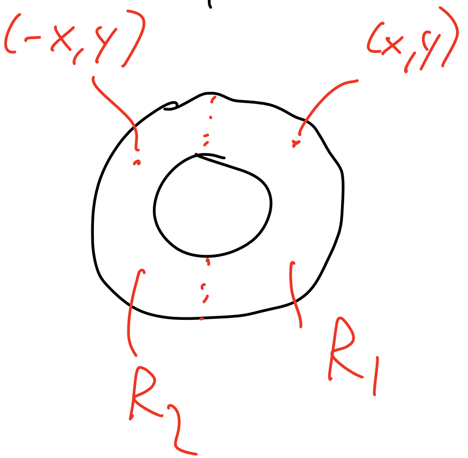
Problems

- Let R be the annulus centered at the origin with inner radius 2 and outer radius 3. Find $\iint_R (1+x+y+xy)\,dx\,dy$
- For the same R, find $\iint_R (|x|+|y|+x^2+y^2)\,dx\,dy$
- Find the volume inside $z = x^2+y^2$ and below $z = 16$
- Find the volume under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$, and inside the cylinder $x^2 + y^2 = 5$

Scratchwork



Notice that $2 \le r \le 3$ and R is rotationally symmetric, so no restrictions on angle $\implies 0 \le \theta < 2\pi$



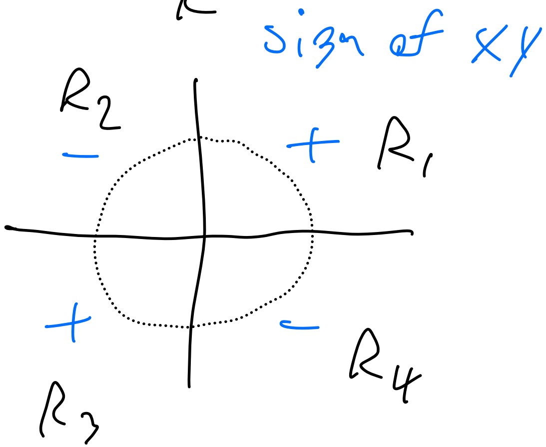
$$\iint_R x = \iint_{R_1} x + \iint_{R_2} x =$$

$$\iint_{R_1} x - \left(\iint_{R_2} (-x)\right) =$$

$$\iint_{R_1} x - \iint_{R_1} x = 0$$

By reflecting (x,y) over x-axis to $(x,-y)$,

$\iint_R y\,dA = 0$ as well.



By another symmetry, $\iint_R xy\,dA = 0$.

To do this properly, $R = R_1 + R_2 + R_3 + R_4$,

$\iint_{R_1} = -\iint_{R_2}$, $\iint_{R_3} = -\iint_{R_4}$,

$$\iint_R = \int_{R_1} + \int_{R_2} + \int_{R_3} + \int_{R_4} = \int_{R_1} - \int_{R_1} + \int_{R_3} - \int_{R_3} = 0$$

So $\iint_R (1+x+y+xy)\,dA = \iint_R 1\,dA + 0 + 0 + 0 =$

$$\iint_R dA = \text{area}(R) = \text{area}(\text{outer}) - \text{area}(\text{inner})$$

$$= \pi 3^2 - \pi 2^2 = 5\pi .$$

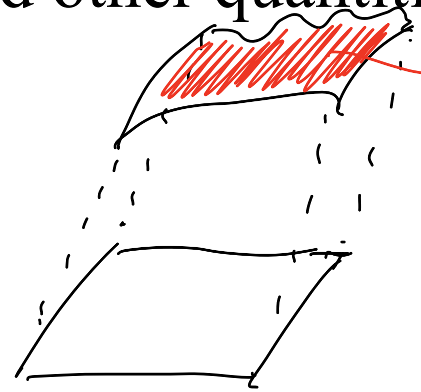
Surface Area

Lecture for 6/26

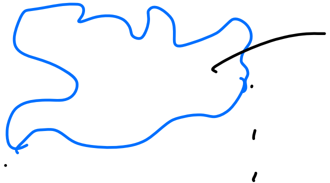
General Idea

Suppose we have some 2D region R and graph $z = f(x, y)$

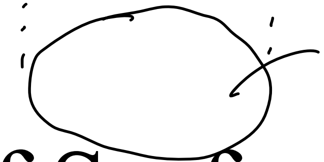
- Portion of the graph constrained by the region is a surface S
- If $f = c$, then S is just a translation of R
  - Area(S) = Area(R)
- Can we find the surface area if f is non-constant?
- Can we find other quantities like center of mass?



what is the area of this wavy section?



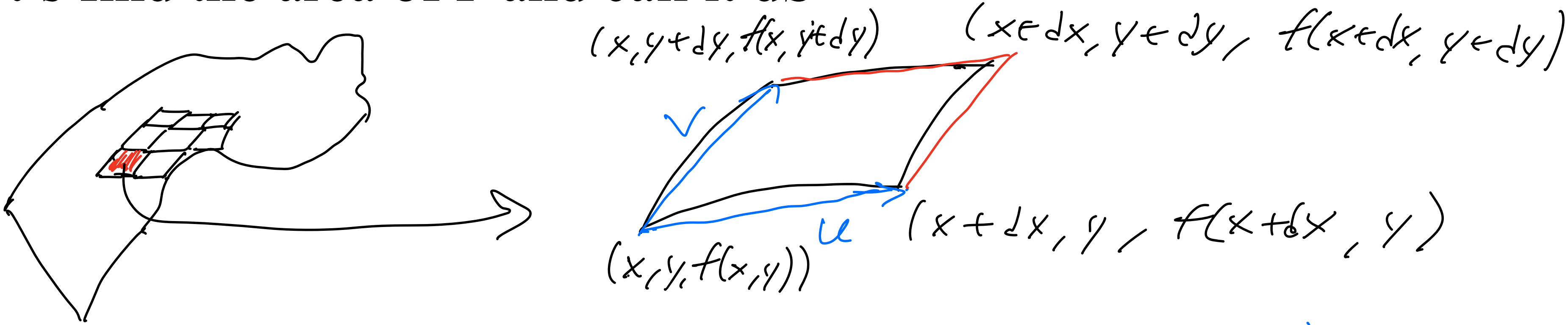
don't know area of this yet



Know the area of circle

Derivation of Surface Area

- Consider small patch of the surface from (x, y) to (x+dx, y+dy)
- For infinitesimal dx, dy, patch resembles a parallelogram P
  - In fact, P is a piece of the tangent plane
- Consider vectors u and v describing the parallelogram
- Let's find the area of P and call it dS



$$u = (x+dx, y, f(x+dx, y)) - (x, y, f(x, y))$$

$= (dx, 0, f(x+dx,y) - f(x,y)) = dx\left(1, 0, \frac{f(x+dx,y)-f(x,y)}{dx}\right)$

Since dx is infinitesimal, $\frac{f(x+dx,y)-f(x,y)}{dx} = f_x$

Similarly, $v = dy(0, 1, f_y)$

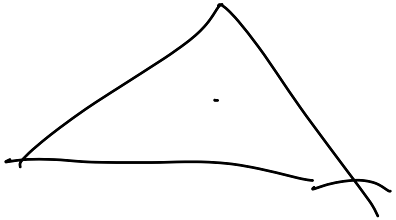
Area $= dS \approx$ area of parallelogram $= \|u \times v\| =$

$\|dx\,\underline{(1, 0, f_x)} \times dy\,\underline{(0, 1, f_y)}\| =$

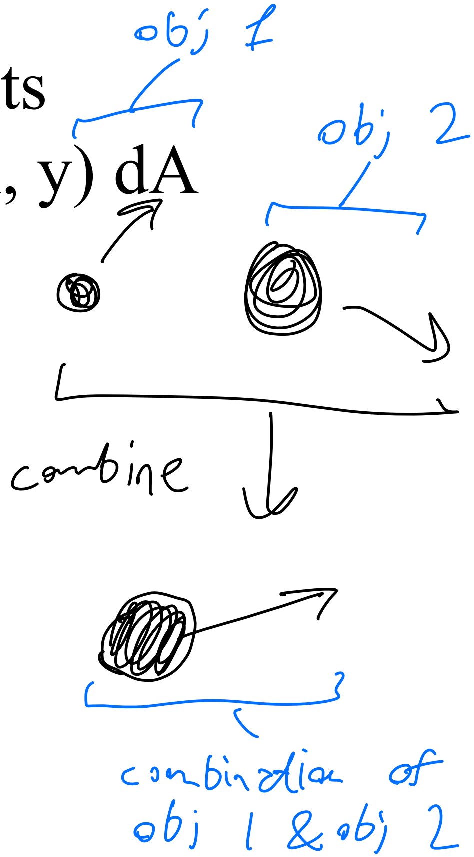
$dx\,dy\,\|\underline{(-f_x, -f_y, 1)}\| = \sqrt{f_x^2 + f_y^2 + 1}\; dx\,dy$

In order to do this properly, do same analysis with Δx, Δy and take $\Delta x, \Delta y \to 0$.

Center of Mass



- Assign a weight $f(x, y)$ to every point (x,y) in R
- To find center of mass, let's try to average the weights
- Recall: average value of some g is $(\text{area}(R))^{-1} \iint_R g(x, y)\, dA$



Motivation: important in physics, also makes for a cool balancing trick. If you have a flat sheet in the shape of R, mark the center of mass, and place some rod or pencil on the center, it will balance under gravity.

To ask a question about system with many different objects, you can, in many cases instead analyze the combined object found by taking averages & centers

Let's find average x coordinate. Imagine a mass of $f(x,y)$ attached to each point $p=(x,y)$. Each p has x-coord x, but is weighted by (x,y), so we compute $\iint_R x f(x,y)\,dA$ to find the sum of the weights. But we have to divide by total weight, which is $\iint_R f(x,y)\,dA$

So $x_{com} = \dfrac{\iint x f\,dA}{\iint f\,dA}$. Similarly, we find y_{com}.

Note! To do this properly, take Riemann sum partitioned by $v_{ij} = (a+i\Delta x, b+j\Delta y)$, $0 \le i,j \le n$, attach a mass $f(a+i\Delta x, b+j\Delta y)$ to each v_{ij}, and find the average x by using the formula for finite averages.

Weighted Average of $z_1, z_2, \dots, z_k$ w/ weights $w_1, \dots, w_k$ respectively is
$$\frac{\sum_{i=1}^{k} w_i z_i}{\sum_{i=1}^{k} w_i}.$$

When you take $\Delta x, \Delta y \to 0$, you recover the x_{com} formula already derived.

Summary

Let R be the 2D region, S the portion of $z = f(x,y)$ above R

- $\text{area}(S) = \iint_R f\, dS = \iint_R (f_x^{\,2}+f_y^{\,2}+1)^{1/2}\, dA$
- $x_{COM} = (\iint_R x\, f(x, y)\, dA)/(\iint_R f(x, y)\, dA)$
- $y_{COM} = (\iint_R y\, f(x, y)\, dA)/(\iint_R f(x, y)\, dA)$

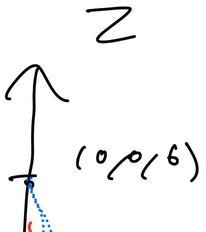
Practice Problems

Find the surface area of the following regions

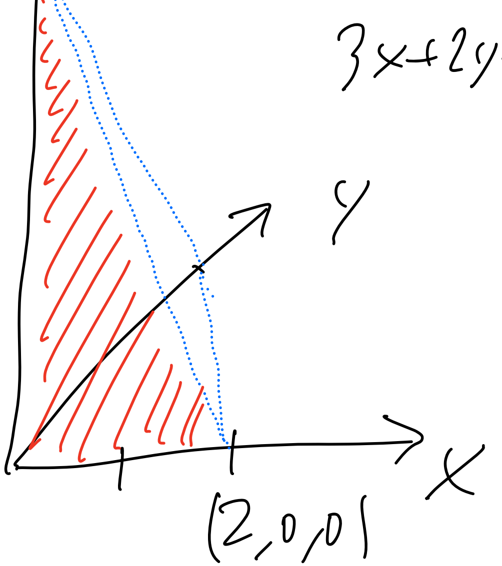
- Portion of plane $3x+2y+z = 6$ lying in the 1st octant
- Portion of $z=xy$ in the cylinder given by $x^2+y^2 = 1$

Find the center of mass of the following regions

- Portion of the unit disk lying in the 1st quadrant
- Square $0 \leq x, y \leq \pi$ with weight function $f(x,y) = x\sin(x)y^3$



Scratchwork



To draw a plane, pick 3 points and then connect them.

$$z = 6 - 3x - 2y := f(x,y).$$

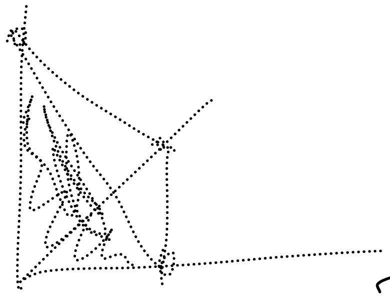
What are bounds on x & y?

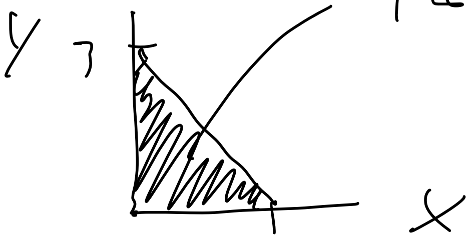
$$x, y \geq 0, \quad x \leq 2, \quad y \leq 3, \quad z \geq 0$$

$$\Rightarrow \quad 3x + 2y \leq 6 \quad \Rightarrow$$

$$2y \leq 6 - 3x$$

$$y \leq 3 - 1.5x$$





The region R which S sits above

Actual ^{2} bounds are not just in blue, they are in red:

$0 \leq x \leq 2$, $0 \leq y \leq 3 - 1.5x$

$f(x,y) = z = 6 - 3x - 2y \Rightarrow f_x = -3$, $f_y = -2$,

$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1+9+4} = \sqrt{14}$,

surface area $=$ area$(S) = \iint_R \sqrt{1+f_x^2+f_y^2}\, dx\, dy$

$= \int_0^2 \left(\int_0^{3-1.5x} \sqrt{14}\, dy \right) dx = \sqrt{14} \int_0^2 (3 - 1.5x)\, dx =$

$\sqrt{14} \left(3x - \frac{3}{4}x^2\right)\Big|_0^2 = \sqrt{14}(6-3) = \underline{3\sqrt{14}}$

Before discussion, try the practice problems in today's slides.

Worksheet 5 ~~for~~ today disc. will be available before disc starts. Includes uncovered problems from these slides + some more problems.