MATH 243: Worksheet 9 **Answer Key**

This worksheet covers Chap14.8 Lagrange Multiplier, Chap15.1 Double Integrals over Rectangles.

The Lagrange Multipliers

1. In order to maximize and minimize a function

$$f(x, y) \text{ subject to the given constraint } g(x, y) = k,$$

we can use the method of Lagrange multiplier.

(a) Using the method of Lagrange multipliers, set up a system of equations for finding the minimum and the maximum for the function $f(x, y) = 9xe^y$ subject to the given constraint $x^2 + y^2 = 2$. Do NOT evaluate.

We have $f(x, y) = 9xe^y$ and $g(x, y) = x^2 + y^2 = 2$. By the method of Lagrange multipliers, we have

$$\nabla f = \lambda \nabla g, \qquad g(x, y) = k = 2$$

From $\nabla f = \lambda \nabla g$, we get

$$f_x = \lambda g_x \qquad \to \qquad 9e^y = \lambda(2x).$$

$$f_y = \lambda g_y \qquad \to \qquad 9xe^y = \lambda(2y).$$

Therefore, a system of equations is

$$9e^2 = 2\lambda x, \quad 9xe^y = 2\lambda y, \quad x^2 + y^2 = 2 \quad \text{(the constraint equation)}.$$

(b) Using (a), find the extreme values of the function $f(x, y) = 9xe^y$ subject to the given constraint $x^2 + y^2 = 2$.

From a system of equations in (a), the first two equations imply

$$\lambda = \frac{9e^y}{2x} = \frac{9xe^y}{2y} \quad \to \quad 18ye^y = 18x^2e^y \quad \to \quad 18e^y(y - x^2) = 0 \quad \to \quad y = x^2 \ (e^y \neq 0)$$

and substituting into the third equation gives

$$x^2 + (x^2)^2 = 2 \quad \to \quad x^4 + x^2 - 2 = 0 \quad \to \quad (x^2 + 2)(x^2 - 1) = 0 \quad \to \quad x = \pm 1 \ (x^2 + 2 \neq 0).$$

(i) $x = 1$ and $y = x^2 = 1$: f has the extreme value

$$f(1, 1) = 9 \cdot 1 \cdot e^1 = 9e \quad \text{at the point } (1, 1).$$

(ii) $x = -1$ and $y = x^2 = 1$: f has the extreme value

$$f(-1, -1) = 9 \cdot (-1) \cdot e^1 = -9e \quad \text{at the point } (-1, 1).$$

Therefore, the maximum value of $f(1, 1)$ is $9e$; the minimum value of $f(-1, 1)$ is $-9e$.

2. In order to maximize and minimize a function

$$f(x, y, z) \text{ subject to the given constraint } g(x, y, z) = k,$$

we can use the method of Lagrange multiplier.

(a) Using the method of Lagrange multipliers, set up a system of equations for finding the minimum and the maximum for the function $f(x,y,z) = 8x+8y+3z$ subject to the given constraint $4x^2+4y^2+3z^2 = 35$. Do NOT evaluate.

> We have $f(x,y,z) = 8x+8y+3z$ and $g(x,y,z) = 4x^2+4y^2+3z^2 = 35$. By the method of Lagrange multipliers, we have
>
> $$\nabla f = \lambda \nabla g, \qquad g(x,y,z) = k = 35$$
>
> From $\nabla f = \lambda \nabla g$, we get
>
> $$f_x = \lambda g_x \qquad \rightarrow \qquad 8 = \lambda(8x).$$
>
> $$f_y = \lambda g_y \qquad \rightarrow \qquad 8 = \lambda(8y).$$
>
> $$f_z = \lambda g_z \qquad \rightarrow \qquad 3 = \lambda(6z).$$
>
> Therefore, a system of equations is
>
> $$8 = 8\lambda x, \quad 8 = 8\lambda y, \quad 3 = 6\lambda z, \quad 4x^2+4y^2+3z^2 = 35 \quad \text{(the constraint equation)}.$$

(b) Using (a), find the extreme values of the function $f(x,y,z) = 8x+8y+3z$ subject to the given constraint $4x^2+4y^2+3z^2 = 35$.

> From a system of equations in (a), the first three equations imply
>
> $$8 = 8\lambda x, \quad 8 = 8\lambda y, \quad 3 = 6\lambda z, \quad 4x^2+4y^2+3z^2 = 35 \quad \text{(the constraint equation)}.$$
>
> The first three equations imply
>
> $$x = \frac{1}{\lambda}, \quad y = \frac{1}{\lambda}, \quad \text{and} \quad z = \frac{1}{2\lambda}.$$
>
> But substitution into the constraint equation gives
>
> $$4\left(\frac{1}{\lambda}\right)^2 + 4\left(\frac{1}{\lambda}\right)^2 + 3\left(\frac{1}{2\lambda}\right)^2 = 35 \quad \rightarrow \quad \frac{35}{4\lambda^2} = 35 \quad \rightarrow \quad \lambda = \pm\frac{1}{2}.$$
>
> (i) $\lambda = \frac{1}{2}$: We have the extreme point $(2,2,1)$. So, f has the extreme value
>
> $$f(2,2,1) = 16+16+3 = 35.$$
>
> (ii) $\lambda = -\frac{1}{2}$: We have the extreme point $(-2,-2,-1)$. So, f has the extreme value
>
> $$f(-2,-2,-1) = -16-16-3 = -35.$$
>
> Therefore, the maximum value of $f(2,2,1)$ is 35; the minimum value of $f(-2,-2,-1)$ is -35.

3. Find the maximum and minimum values of $f(x,y) = 81x^2+y^2$ subject to the constraint $4x^2+y^2 = 9$.

We have $f(x,y) = 81x^2 + y^2$ and $g(x,y) = 4x^2 + y^2 = 9$. By the method of Lagrange multipliers, we have

$$\nabla f = \lambda \nabla g, \qquad g(x,y) = k = 9$$

From $\nabla f = \lambda \nabla g$, we get

$$f_x = \lambda g_x \qquad \to \qquad 162x = \lambda(8x).$$

$$f_y = \lambda g_y \qquad \to \qquad 2y = \lambda(2y).$$

Therefore, a system of equations is

$$162x = 8\lambda x, \quad 2y = 2\lambda y, \quad 4x^2 + y^2 = 9 \quad \text{(the constraint equation)}.$$

The second equation implies

$$2y(1-\lambda) = 0 \quad \to \quad y = 0 \text{ or } \lambda = 1$$

(i) $y = 0$: Substituting into the third equation gives

$$4x^2 + 0^2 = 9 \quad \to \quad x^2 = \frac{9}{4} \quad \to \quad x = \pm\frac{3}{2}.$$

We have two extreme points $\left(\frac{3}{2}, 0\right)$ and $\left(-\frac{3}{2}, 0\right)$.

(ii) $\lambda = 1$: Substituting into the first equation gives

$$162x = 8x \quad \to \quad 162x - 8x = 154x = 0 \quad \to \quad x = 0.$$

Then, substituting into the third equation gives

$$4 \cdot 0^2 + y^2 = 9 \quad \to \quad y^2 = 9 \quad \to \quad y = \pm 3.$$

We have another two extreme points $(0,3)$ and $(0,-3)$.

We have all the extreme values

$$f\left(\frac{3}{2}, 0\right) = 81\left(\frac{3}{2}\right)^2 + 0^2 = \frac{729}{4} \quad \text{and} \quad f\left(-\frac{3}{2}, 0\right) = 81\left(-\frac{3}{2}\right)^2 + 0^2 = \frac{729}{4}$$

$$f(0,3) = 81(0)^2 + 3^2 = 9 \quad \text{and} \quad f(0,-3) = 81(0)^2 + (-3)^2 = 9$$

Therefore, the maximum value of $f\left(\pm\frac{3}{2}, 0\right)$ is $\frac{729}{4}$; the minimum value of $f(0,\pm 3)$ is 9.

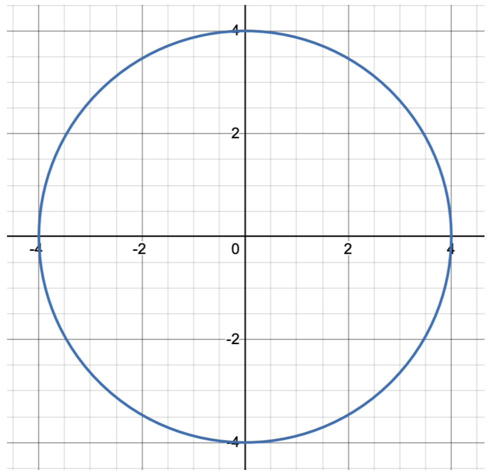
4. Find the absolute minimum and absolute maximum of $f(x,y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \le 16$.

(i) We have

$$f(x,y) = 2x^2 - y^2 + 6y \quad \to \quad f_x(x,y) = 4x, \quad f_y(x,y) = -2y + 6,$$

and setting $f_x = f_y = 0$ gives $(0,3)$ as the only critical point in D, with $f(0,3) = 9$.

(ii) On the boundary,



Since $x^2 = 16 - y^2$, then

$$g(y) = f(16 - y^2, y) = 2(16^2 - y^2) - y^2 + 6y = -3y^2 + 6y + 32, \quad \text{where } -4 \le y \le 4.$$

So, we have the endpoints $y = \pm 4$ and also

$$g'(y) = -6y + 6 \quad \to \quad y = 1.$$

Then,

$$\begin{aligned} y = \pm 4: &\quad g(\pm 4) = 16 - 16 = 0 \quad \to \quad x = 0 \\ y = 1: &\quad g(1) = 16 - 1 = 15 \quad \to \quad x = \pm\sqrt{15} \end{aligned}$$

We have $f(0,-4) = -40$, $f(0,4) = 8$, and $f(\pm\sqrt{15}, 1) = 35$.

Thus the absolute maximum is attained at $(\pm\sqrt{15}, 1)$ with $f(\pm\sqrt{15}, 1) = 35$ and the absolute minimum on D is attained at $(0,-4)$ with $f(0,-4) = -40$.

Double Integrals over Rectangles

5. Calculate the iterated integral $\displaystyle\int_{-3}^{3} \int_{0}^{\frac{\pi}{2}} (y + y^2 \cos x)\, dx\, dy$

$$\int_{-3}^{3} \int_{0}^{\frac{\pi}{2}} (y + y^2 \cos x) dx dy = \int_{-3}^{3} \left[xy + y^2 \sin x \right]_0^{\frac{\pi}{2}} dy = \int_{-3}^{3} \left(\frac{\pi}{2} y + y^2 \right) dy = \left[\frac{\pi}{4} y^2 + \frac{y^3}{3} \right]_{-3}^{3} = 18.$$

6. Evaluate the double integral by first identifying it as the volume of a solid.

$$\iint_R (y + xy^{-2})\, dA, \quad R = \{\, (x, y) |\ 0 \le x \le 2,\ 1 \le y \le 2 \,\}$$

$$\iint_R (y + xy^{-2}) dA = \int_1^2 \int_0^2 (y + xy^{-2}) dx dy = \int_1^2 \left[xy + \frac{x^2 y^{-2}}{2} \right]_0^2 dy$$
$$= \int_1^2 (2y + 2y^{-2}) dy = \left[y^2 - 2y^{-1} \right]_1^2 = 4.$$

7. Evaluate $\iint_R x \cos^2(y)\ dA$, where $R = [-2, 3] \times [0, \pi/2]$.

$$\iint_R x\cos^2(y)\ dA = \int_{-2}^{3} \int_0^{\pi/2} x \cos^2(y)\, dy\, dx = \int_{-2}^{3} \int_0^{\pi/2} x \left(\frac{1 + \cos(2y)}{2} \right) dy\, dx$$
$$= \int_{-2}^{3} \left[\frac{x}{2} \left(y + \frac{1}{2} \sin(2y) \right) \right]_0^{\pi/2} dx = \int_{-2}^{3} \frac{x}{2} \left(\frac{\pi}{2} \right) dx = \int_{-2}^{3} \frac{\pi}{4} x dx = \left[\frac{\pi}{8} x^2 \right]_{-2}^{3} = \frac{9\pi}{8} - \frac{4\pi}{8} = \frac{5\pi}{8}$$

8. Evaluate $\iint_R \dfrac{1}{(2x + 3y)^2}\ dA$, where $R = [0, 1] \times [1, 2]$.

$$\iint_R \frac{1}{(2x+3y)^2}\ dA = \int_1^2 \int_0^1 (2x + 3y)^{-2}\, dx\, dy = \int_1^2 \left[-\frac{1}{2}(2x + 3y)^{-1} \right]_0^1 dy$$
$$= \int_1^2 -\frac{1}{2}(2 + 3y)^{-1} + \frac{1}{2}(3y)^{-1}\, dy = -\frac{1}{2} \int_1^2 \frac{1}{2 + 3y} - \frac{1}{3y}\, dy = -\frac{1}{2} \left[\frac{1}{3} \ln|2 + 3y| - \frac{1}{3} \ln|y| \right]_1^2$$
$$= -\frac{1}{2} \left[\left(\frac{1}{3} \ln|8| - \frac{1}{3} \ln|2| \right) - \left(\frac{1}{3} \ln|5| - \frac{1}{3} \ln|1| \right) \right] = -\frac{1}{6} \left(\ln(8) - \ln(2) - \ln(5) \right)$$

9. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

The solid lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ so we have the following

$$V = \int_{-1}^{1} \int_1^2 (3y^2 - x^2 + 2) dy dx = \int_{-1}^{1} \left[y^3 - yx^2 + 2y \right]_1^2 dx$$
$$= \int_{-1}^{1} (9 - x^2) dx = \left[9x - \frac{x^3}{3} \right]_{-1}^{1} = \frac{52}{3}$$

10. Find the average value of $f(x,y) = x^2y$ over the rectangle R with the vertices $(-1,0)$, $(-1,5)$, $(1,5)$, $(1,0)$.

We have the following

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x,y)dA.$$

So,

$$\begin{aligned} f_{avg} &= \frac{1}{10}\int_{-1}^{1}\int_{0}^{5} x^2y\,dydx = \frac{1}{10}\int_{-1}^{1}\left[\frac{x^2y^2}{2}\right]_0^5 dx = \frac{1}{10}\int_{-1}^{1}\frac{25}{2}x^2dx = \frac{1}{10}\left[\frac{25x^3}{6}\right]_{-1}^{1} \\ &= \frac{1}{10}\left(\frac{50}{6}\right) = \frac{5}{6}. \end{aligned}$$

Suggested Textbook Problems

Chapter 14.8: 3, 9, 11-17, 19-21, 31, 34, 36, 39, 42, 43

Chapter 15.1: 1a, 3, 9-43, 47