14 Partial Derivatives

$y = f(x), \quad x = g(t)$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(g(t)) \cdot g'(t)$

14.5 The Chain Rule

The Chain Rule (1 of 1)

We know that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = f'(g(t))g'(t)$$

The Chain Rule: Case 1

The Chain Rule: Case 1 (2 of 2)

We know that this is the case when f_x and f_y are continuous.

1 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Since we often write $\frac{\partial z}{\partial x}$ in place of $\frac{\partial f}{\partial x}$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 1

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\dfrac{dz}{dt}$ when $t = 0$.

Solution:

The Chain Rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

It's not necessary to substitute the expressions for *x* and *y* in terms of *t*.

Example 1 – Solution

We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$.

Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6$$

Example

$$z = \sin(x)\cos(y)\,, x = \sqrt{t}, y = \frac{1}{t}, \frac{dz}{dt} = ?$$

$$\frac{dz}{dt} = \cancel{\frac{\partial x}{\partial y}} \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = \cos(x)\cos(y) = \cos(\sqrt{t})\cos\left(\frac{1}{t}\right)$$

$$\frac{\partial z}{\partial y} = \cancel{\cos} -\sin(x)\sin(y) = -\sin(\sqrt{t})\sin\left(\frac{1}{t}\right)$$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \quad \frac{dy}{dt} = -\frac{1}{t^2}$$

$$\frac{dz}{dt} = \cos(\sqrt{t})\cos\left(\frac{1}{t}\right)\frac{1}{2\sqrt{t}} + \left(-\sin(\sqrt{t})\sin\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)\right)$$

$$= \frac{1}{2\sqrt{t}}\cos(\sqrt{t})\cos\left(\frac{1}{t}\right) + \frac{1}{t^2}\sin(\sqrt{t})\sin\left(\frac{1}{t}\right)$$

Example

$$z = \sin(x)\cos(y)\,, x = \sqrt{t}, y = \frac{1}{t}, \frac{dz}{dt} = ?$$

The Chain Rule: Case 2

The Chain Rule: Case 2 (1 of 4)

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t: $x = g(s, t)$, $y = h(s, t)$.

Then z is indirectly a function of s and t and we wish to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

We know that in computing $\frac{\partial z}{\partial t}$ we hold s fixed and compute the ordinary derivative of z with respect to t.

Therefore we can apply Theorem 1 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The Chain Rule: Case 1 (1 of 2)

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

The first version (Theorem 1) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t. $x = g(t)$, $y = h(t)$

This means that z is indirectly a function of t, $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t. We assume that f is differentiable.

The Chain Rule: Case 2 (2 of 4)

A similar argument holds for $\frac{\partial z}{\partial s}$ and so we have proved the following version of the Chain Rule.

2 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable.

Example 3

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t, find\ \dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Solution:

Applying Case 2 of the Chain Rule, we get

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

Example 3 – Solution

If we wish, we can now express $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ solely in terms of *s* and *t* by substituting $x = st^2, y = s^2t,$ to get

$$\frac{\partial z}{\partial s} = t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t)$$
$$\frac{\partial z}{\partial t} = 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t)$$

The Chain Rule: Case 2 (3 of 4)

Notice that Theorem 2 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule in Equation 1.

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2.

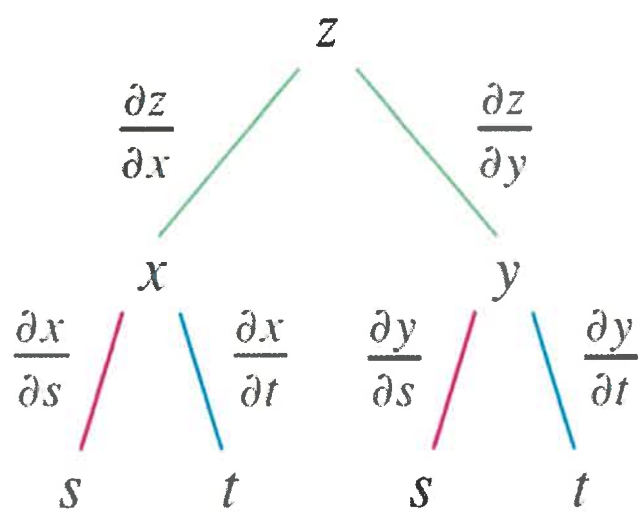


Figure 2

The Chain Rule: Case 2 (4 of 4)

We draw branches from the dependent variable *z* to the intermediate variables *x* and *y* to indicate that *z* is a function of *x* and *y*. Then we draw branches from *x* and *y* to the independent variables *s* and *t*.

On each branch we write the corresponding partial derivative. To find $\frac{\partial z}{\partial s}$,

we find the product of the partial derivatives along each path from *z* to *s* and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

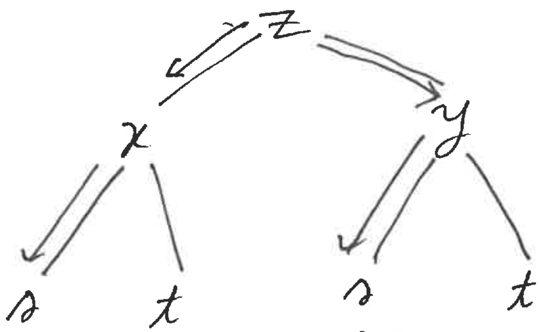
Similarly, we find $\frac{\partial z}{\partial t}$ by using the paths from *z* to *t*.

Example

$$z = x^3 e^{xy} + y^3 \ln(xy), \qquad x = s + t,\ y = s - t$$

$$\frac{\partial y}{\partial s} = \frac{\partial s}{\partial s} - \frac{\partial t}{\partial s} = 1$$

$$\frac{\partial z}{\partial s},\ \frac{\partial z}{\partial t}$$



$$\frac{d}{dx}\ln(cx) = \frac{c}{cx} = \frac{1}{x}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial y} = x^3 x e^{xy} + 3y^2\ln(xy) + \frac{y^3}{\cancel{xy}}$$

$$= x^4 e^{xy} + 3y^2\ln(xy) + \frac{y^3}{x}$$

$$= (s+t)^4 e^{s^2-t^2} + 3(s-t)^2\ln(s^2-t^2) + \cancel{\frac{(s-t)^3}{s+t}}\,(s-t)^2$$

$$\frac{\partial y}{\partial s} = 1$$

$$\frac{\partial z}{\partial x} = 3x^2 e^{xy} + yx^3 e^{xy} + y^3\frac{y^3}{xy}$$

$$= (3x^2 + yx^3)e^{xy} + \frac{y^3}{x}$$

$$= \left(3(s+t)^2 + (s-t)(s+t)^3\right)e^{s^2-t^2} + \frac{(s-t)^3}{s+t}$$

$$\frac{\partial x}{\partial s} = 1$$

$$\cancel{\frac{\partial z}{\partial s} = \left(3(s+t)^2 + (s-t)(s+t)^3\right)e^{s^2-t^2} + \frac{(s-t)^3}{s+t}}$$

$$\frac{\partial z}{\partial s} = \left(3(s+t)^2 + (s-t)(s+t)^3\right)e^{s^2-t^2} + \frac{(s-t)^3}{s+t} + (s+t)^4 e^{s^2-t^2} + 3(s^2-t^2)\ln(s^2-t^2)$$

The Chain Rule: General Version

The Chain Rule: General Version (1 of 2)

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, \ldots, x_n$, each of which is, in turn, a function of m independent variables $t_1, \ldots, t_m$.

Notice that there are n terms, one for each intermediate variable. The proof is similar to that of Case 1.

The Chain Rule: General Version (2 of 2)

3 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables x_1, x_2,..., x_n and each x_j is a differentiable function of the m variables t_1, t_2, ..., t_m. Then u is a function of t_1, t_2, ...t_m and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Example 4

Write out the Chain Rule for the case where $\omega = f(x, y, z, t)$ and
$x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$. $\frac{\partial \omega}{\partial u}$, $\frac{\partial \omega}{\partial v}$

Solutions:

We apply Theorem 3 with $n = 4$ and $m = 2$. Figure 3 shows the tree diagram. Although we haven't written the derivatives on the branches, it's understood that if a branch leads from y to u, then the partial derivative for that branch is $\frac{\partial y}{\partial u}$.

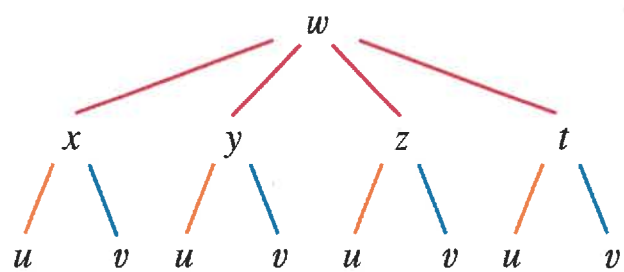
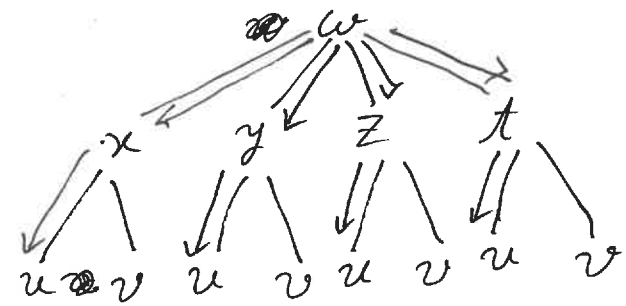


Figure 3



$$\frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \omega}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial \omega}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial \omega}{\partial v} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \omega}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial \omega}{\partial t}\frac{\partial t}{\partial v}$$

Example 4 – Solution

With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

Implicit Differentiation

$z = f(x, y)$

$e^z = xyz$

Implicit Differentiation (1 of 6)

The Chain Rule can be used to give a more complete description of the process of implicit differentiation. $\frac{dy}{dx}$

We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f.

If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x.

Since both x and y are functions of x, we obtain

$$\frac{dF}{dx} \quad \frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{dy}{dx}}_{\text{To find}} = 0$$

$$\Rightarrow \frac{\partial F}{\partial y}\frac{dy}{dx} = -\frac{\partial F}{\partial x} \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Implicit Differentiation (2 of 6)

But $\frac{dx}{dx} = 1$, so if $\frac{\partial F}{\partial y} \neq 0$ we solve for $\frac{dy}{dx}$ and obtain

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \tag{5}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x.

Implicit Differentiation (3 of 6)

The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid: it states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 5.

Example 8

Find y' if $x^3 + y^3 = 6xy$.

Solution:

The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

$F_x = 3x^2 + 0 - 6y$, $F_y = 0 + 3y^2 - 6x$

so Equation 5 gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

$x^3 + y^3 = 6xy$

$\Rightarrow \frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy)$

$\Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$

$\Rightarrow (3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$

Implicit Differentiation (4 of 6)

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$e^z = xyz \Rightarrow \frac{\partial}{\partial y} e^z = \frac{\partial}{\partial y}(xyz) \Rightarrow e^z \frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \qquad \Rightarrow \frac{\partial z}{\partial y}(e^z - xy) = xz \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

$$F(x,y,z) = e^z - xyz$$

$$\Rightarrow \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\underbrace{\frac{\partial z}{\partial x}}_{\text{To find}} = 0 \Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{F_x}{F_z}$$

Implicit Differentiation (5 of 6)

But $\frac{\partial}{\partial x}(x) = 1$ and $\frac{\partial}{\partial x}(y) = 0$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\frac{\partial F}{\partial z} \neq 0,$ we solve for $\frac{\partial z}{\partial x}$ and obtain the first formula in Equations 6.

The formula for $\frac{\partial z}{\partial y}$ is obtained in a similar manner.

Implicit Differentiation (6 of 6)

6 $$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid: if F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by (6).

Example

$$e^z = xyz$$

$$F(x,y,z) = e^z - xyz$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-yz}{e^z - xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-xz}{e^z - xy}$$