MATH 243: Discussion Worksheet 2

Topics: 12.2 Vectors; 12.3 The Dot Product; 12.4 The Cross Product
Please put away all electronic devices, including cell phones and calculators.

1. Given $\mathbf{a} = 9\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = \langle 7, 0, -9 \rangle$, find the following. Simplify your answers completely.
   (a) $\mathbf{a} + \mathbf{b}$
   (b) $3\mathbf{a} - \mathbf{b}$
   (c) $|\mathbf{b}|$
   (d) $|\mathbf{b} - \mathbf{a}|$

2. Find the vector that has the opposite direction as $\langle 9, -6, -2 \rangle$ and has length 5.

3. Determine whether the given vectors are orthogonal, parallel, or neither.
   (a) $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 12\mathbf{j} - 2\mathbf{k}$
   (b) $\mathbf{a} = \langle 6, 5, -2 \rangle$ and $\mathbf{b} = \langle 5, 0, 9 \rangle$
   (c) $\mathbf{a} = \langle -18, 15 \rangle$ and $\mathbf{b} = \langle 12, -10 \rangle$

4. Given vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ in V_3, which of the following expressions are meaningful? Which are meaningless? Explain your reasoning.
   (a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$
   (b) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}$
   (c) $|\mathbf{a}| \cdot |\mathbf{c}|$

5. Find the angle θ between the vectors $\mathbf{a} = \langle 2, 0, -3 \rangle$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

6. Consider $\mathbf{a} = \langle -1, 4, 8 \rangle$ and $\mathbf{b} = \langle 18, 2, 1 \rangle$.
   (a) Find the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$.
   (b) Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.

7. Find the work (in joules) done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$ that moves an object from the point $(0, 6, 4)$ to the point $(4, 14, 22)$ along a straight line. The distance is measured in meters and the force in newtons.

8. Compute the dot product and cross product for the following pairs of vectors:
   (a) $\mathbf{u} = \langle -1, 1, 2 \rangle$, $\mathbf{v} = \langle 4, 5, -2 \rangle$
   (b) $\mathbf{u} = \langle 1, -1, 3 \rangle$, $\mathbf{v} = \langle 2, -2, 6 \rangle$
   (c) $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle -3, 0, 1 \rangle$
   (d) What is the significance of your answers to parts (b) and (c)?

9. Consider $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{j} + \mathbf{k}$.
   (a) Find the cross product $\mathbf{a} \times \mathbf{b}$.
   (b) Verify that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.

(c) Find two unit vectors orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

10. Consider points $P(1,2,1), Q(2,5,4), R(6,9,12)$, and $S(5,6,9)$ in $\mathbb{R}^3$.
   (a) Find the area of the parallelogram with vertices P, Q, R, and S.
   (b) Find the area of triangle PQS.
   (c) Show that the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are coplanar.

SOME USEFUL DEFINITIONS, THEOREMS, AND NOTATION:

Dot Product If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta.$$

Scalar and Vector Projections If $\mathbf{a} \neq \mathbf{0}$, then the *scalar projection* of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$

The *vector projection* of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right) \mathbf{a}.$$

Work by a Constant Force If a constant force $\mathbf{F}$ moves an object through displacement $\mathbf{D}$, then the work done is

$$W = \mathbf{F} \cdot \mathbf{D}.$$

Cross Product If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta.$$

Suggested Textbook Problems
Section 12.2: 1-33, 37-44
Section 12.3: 1-56
Section 12.4: 1-22, 27-38, 43-46