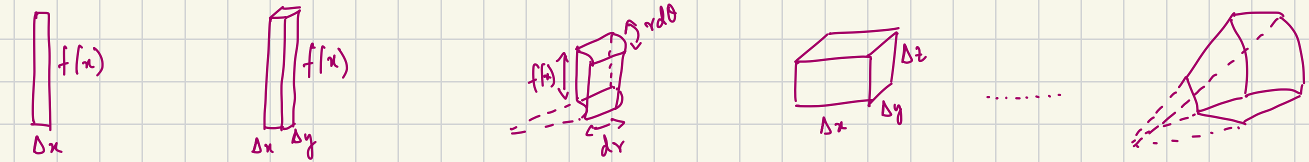
$\int f(x)\,dx$ $\qquad \iint_R f(x,y)\,dA$ $\qquad \iint_R f(r,\theta)\,r\,dr\,d\theta$ $\qquad \iiint_E f(x,y,z)\,dV$

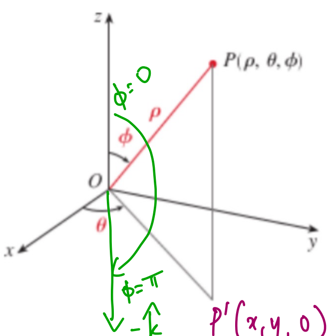
generalisation of polar coordinates



spherical wedge.

1. **Spherical Coordinates**

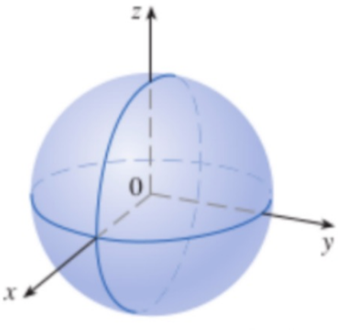
The spherical coordinates (ρ, θ, ϕ) of a point P in space, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, ϕ is the angle between the positive z-axis and the line segment OP.



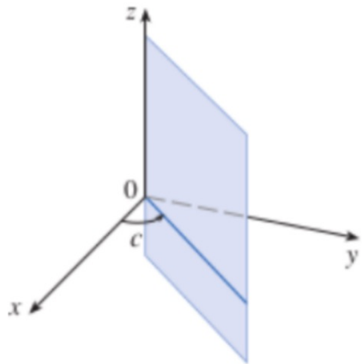
Note that

$$\rho \geq 0 \ , \ 0 \leq \phi \leq \pi \ , \ 0 \leq \theta \leq 2\pi$$

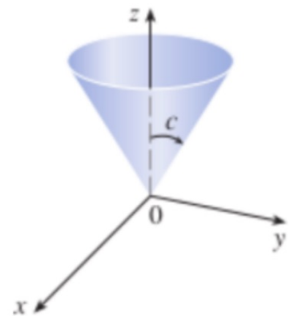
The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.



$\rho = c$

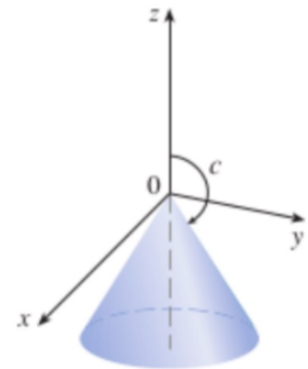


$\theta = c$



$0 < c < \pi/2$

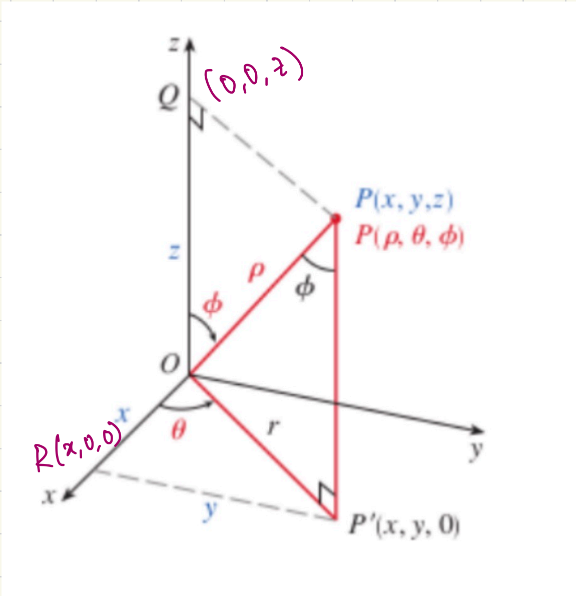
$\phi = c$

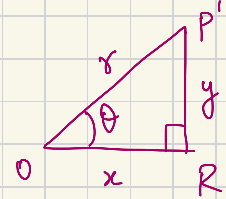


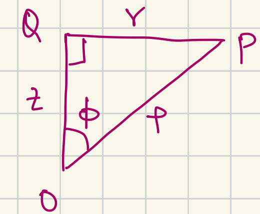
$\pi/2 < c < \pi$

$\phi = c$

Transforming Cartesian to Spherical and vice-versa.







spherical to cartesian

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \end{aligned}}$$

$$\left.\begin{aligned} \cos\phi &= \frac{z}{\rho} \\ \sin\phi &= \frac{r}{\rho} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} z &= \rho\cos\phi \\ r &= \rho\sin\phi \end{aligned}}$$

Cartesian to spherical: $\rho = \sqrt{x^2 + y^2 + z^2} \Rightarrow$

$$\boxed{\begin{aligned} \rho^2 &= x^2 + y^2 + z^2 \\ \theta &= \arctan\left(\frac{y}{x}\right) \\ \phi &= \arccos\left(\frac{z}{\rho}\right) \end{aligned}}$$

Example: Convert from Cartesian Coordinate $(x, y, z) = (-1, 1, \sqrt{\frac{2}{3}})$ to spherical coordinate (ρ, θ, ϕ).

$$A: \rho^2 = (-1)^2 + (1)^2 + \left(\sqrt{\frac{2}{3}}\right)^2 = 1 + 1 + \frac{2}{3} = \frac{8}{3} \Rightarrow \rho = \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}}$$

$$\theta = \arctan\left(\frac{1}{-1}\right) = \arctan(-1) = \frac{3\pi}{4}$$

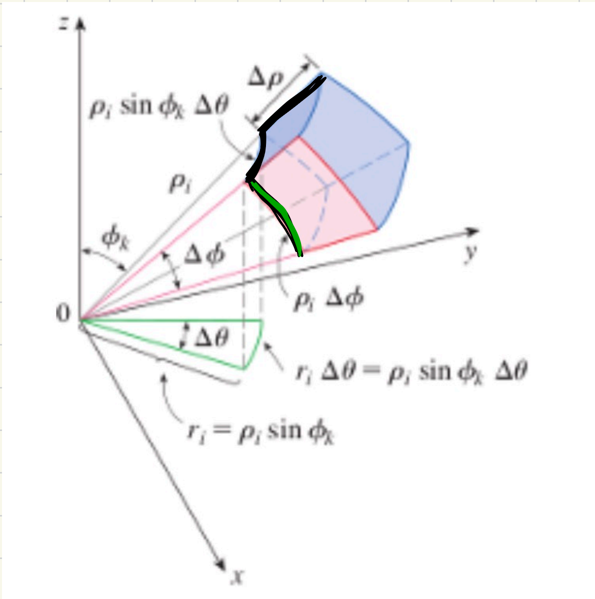
$$\phi = \arccos\left(\frac{\sqrt{\frac{2}{3}}}{2\sqrt{\frac{2}{3}}}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} \Rightarrow (-1, 1, \sqrt{\tfrac{2}{3}}) \text{ in spherical is}$$

$$\left(2\sqrt{\frac{2}{3}}, \frac{3\pi}{4}, \frac{\pi}{3}\right)$$

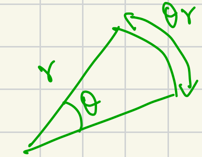
Example: Convert from Cartesian Coordinate $(\rho, \theta, \phi) = (2, \frac{\pi}{2}, \frac{\pi}{4})$ to spherical coordinate (x, y, z).

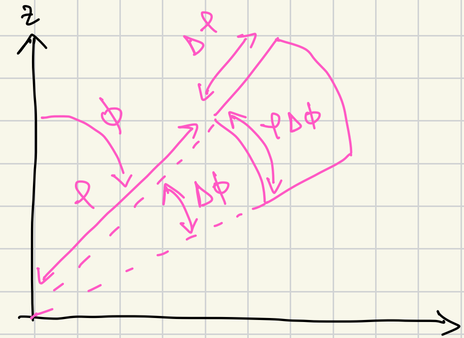
$$\left.\begin{aligned} x &= \rho \sin\phi \cos\theta = 2 \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{2}\right) = 0 \\ y &= \rho \sin\phi \sin\theta = 2 \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{2}\right) = \frac{2}{\sqrt{2}} = \sqrt{2} \\ z &= \rho \cos\phi = 2 \cos\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}. \end{aligned}\right\} (0, \sqrt{2}, \sqrt{2})$$

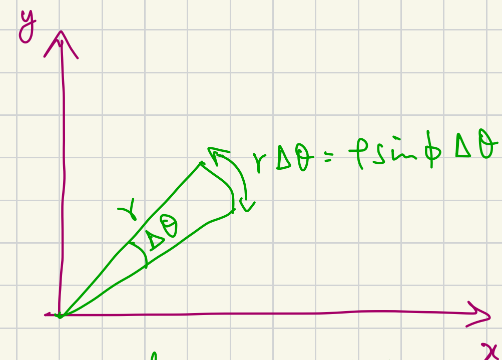
Triple integrals in spherical coordinates:



(a) A spherical wedge







where $r = \rho \sin\phi$

3 edges of lengths:
- $\rho\,\Delta\phi$
- $\Delta\rho$
- $\rho\sin\phi\,\Delta\theta$

$$\Rightarrow \quad dx\,dy\,dz \approx (\rho\,d\phi)(d\rho)(\rho\sin\phi\,d\theta) = \rho^2 \sin\phi \,d\phi\,d\rho\,d\theta$$

Let E be the spherical wedge $E = \{(\rho, \theta, \phi) : a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$

and $f(\rho, \theta, \phi) := f(\underbrace{\rho \sin\phi \cos\theta}_{x}, \underbrace{\rho \sin\phi \sin\theta}_{y}, \underbrace{\rho\cos\phi}_{z})$

Then
$$\boxed{\iiint_E f(x,y,z)\, \underbrace{dV}_{dx\,dy\,dz} = \int_{\phi=c}^{d} \int_{\theta=\alpha}^{\beta} \int_{\rho=a}^{b} f(\rho, \theta, \phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}$$

Example: Evaluate $\iiint_B e^{-(x^2+y^2+z^2)^{3/2}}\, dV$ where B is the ball $x^2+y^2+z^2 \le 1$.

A: $f(x,y,z) = \exp\left(-(x^2+y^2+z^2)^{3/2}\right) \Rightarrow f(\rho,\theta,\phi) = \exp\left(-(\rho^2)^{3/2}\right) = e^{-\rho^3}$

$B = \{(x,y,z) : \underbrace{x^2+y^2+z^2}_{\rho^2} \le 1\} = \{(\rho,\theta,\phi) : 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$

$$J = \iiint_B f(x,y,z)\, dV = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} e^{-\rho^3} \cdot \rho^2 \cdot \sin\phi \, d\rho\, d\theta\, d\phi$$

$$I = \int_{\rho=0}^{1} e^{-\rho^3} \rho^2 \, d\rho \quad \text{take } u = -\rho^3 \Rightarrow du = -3\rho^2 d\rho \Rightarrow \frac{-du}{3} = \rho^2 d\rho, \quad \rho = 0 \to u = 0$$

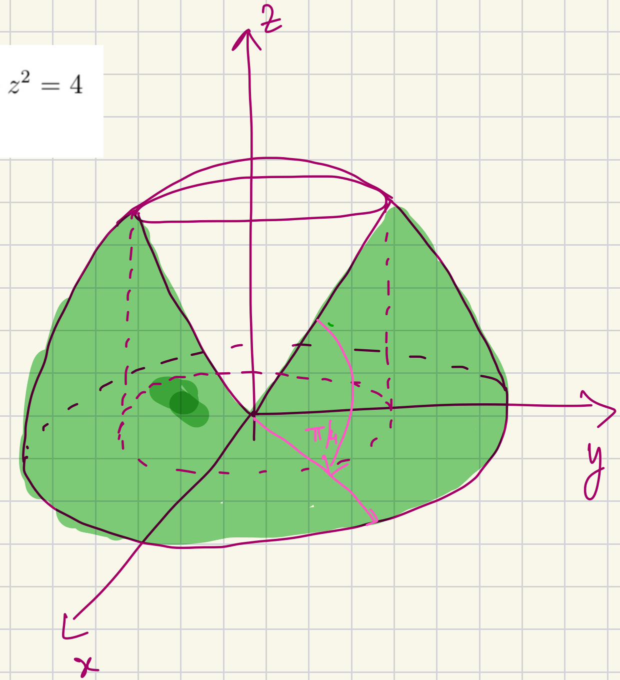
$$\rho = 1 \to u = -1$$

$$\Rightarrow \int_{0}^{-1} e^{u} \cdot \left(\frac{-du}{3}\right) = \frac{1}{3}\int_{-1}^{0} e^{u} \, du = \frac{1}{3}\left(1 - e^{-1}\right)$$

$$J = \frac{1}{3}\left(1 - e^{-1}\right) \int_{0}^{\pi} \int_{0}^{2\pi} \sin\phi \, d\theta \, d\phi = \frac{2\pi}{3}\left(1 - e^{-1}\right) \int_{0}^{\pi} \sin\phi \, d\phi$$

$$= \frac{2\pi}{3}\left(1 - e^{-1}\right) \left(-\cos\phi\right)_{0}^{\pi} = \frac{2\pi}{3}\left(1 - e^{-1}\right)\left(-\cos\pi + \cos 0\right) = \frac{4\pi}{3}\left(1 - e^{-1}\right)$$

Example: Find the volume of the region that lies above $z = 0$, below $x^2 + y^2 + z^2 = 4$ and outside the cone $z = \sqrt{x^2 + y^2}$.



$$z^2 = x^2 + y^2 \quad \text{and} \quad x^2 + y^2 + z^2 = 4$$

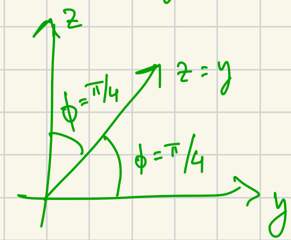
$$\Updownarrow$$

$$x^2 + y^2 = 2 \iff r = \sqrt{2}$$

$$E = \{(x,y,z) : 0 \le x^2+y^2 \le \sqrt{2},\ \underbrace{0 \le z \le \sqrt{x^2+y^2}}_{\text{bounded above by the cone}}\} \cup \{(x,y,z) : \sqrt{2} \le x^2+y^2 \le 2,\ \underbrace{0 \le z \le 2 - x^2 - y^2}_{\text{bounded above by sphere.}}\}$$

$$E = \{(\rho, \theta, \phi) : 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ \frac{\pi}{4} \le \phi \le \frac{\pi}{2}\}$$

To compute ϕ, project the cone onto the zy plane : $z^2 = y^2 \iff z = y$



$$\text{Volume}(E) = \iiint_E dV = \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \overset{\boxed{\text{DIY}}}{=} \boxed{\frac{8\pi\sqrt{2}}{3}}$$

Hints for WebAssign:

10. Write the equation x^2-y^2-z^2=1 in spherical coordinates

11. E lies above the cone \phi=\pi/3 and below the sphere \rho=1

14. E lies above the cone z=\sqrt{x^2+y^2} and between the spheres x^2+y^2+z^2=4 and x^2+y^2+z^2=16

15 Part of the ball \rho\leq a that lies between the cones \phi=\pi/6 and \phi=\pi/3

10. $x^2-y^2-z^2=1 \Rightarrow (\rho\sin\phi\cos\theta)^2-(\rho\sin\phi\sin\theta)^2-(\rho\cos\phi)^2=1$

11. $E=\{(\rho,\theta,\phi) : 0\leq\rho\leq 1,\ 0\leq\theta\leq 2\pi,\ 0\leq\phi\leq \pi/3\}$