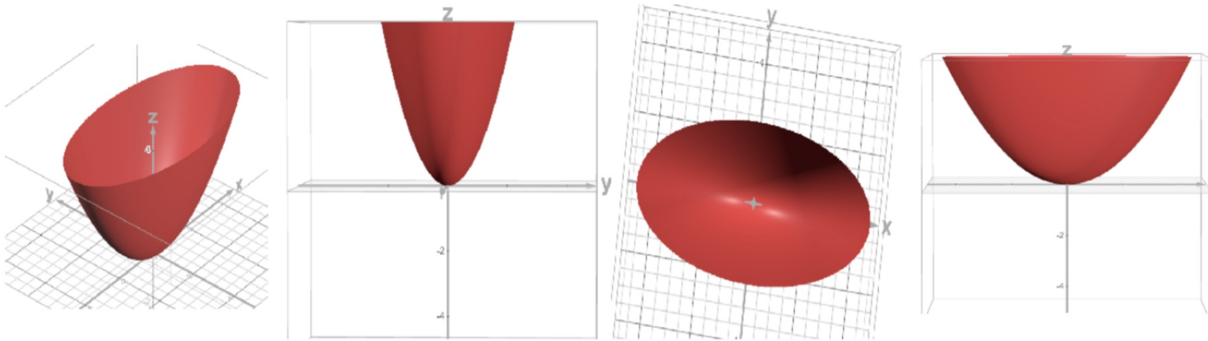
1. Observe the following graphs:



Which equation below gives the surface shown above?

A. $z = \frac{x^2}{4} + y^2$

B. $\frac{z^2}{9} = \frac{x^2}{4} - y^2$

C. $\cancel{1 - \frac{z^2}{9} = \frac{x^2}{4} + y^2}$

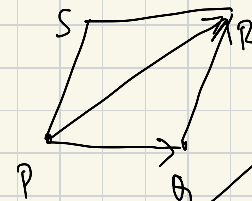
D. $z = \frac{x^2}{4} - y^2$

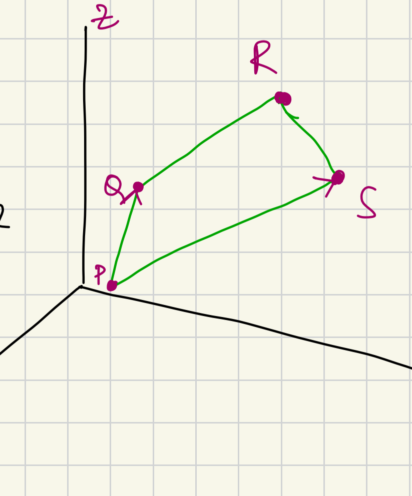
5.

(a) Find the area of the parallelogram with vertices $P(1,2,1)$, $Q(2,5,4)$, $R(6,9,12)$ and $S(5,6,9)$.

(b) Find the area of the triangle PQS.

(c) Show that the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are coplanar.





(a) Take $\overrightarrow{PQ} = \langle 1, 3, 3 \rangle$, $\overrightarrow{PS} = \langle 4, 4, 8 \rangle$

$$\text{Area} = \|\overrightarrow{PQ} \times \overrightarrow{PS}\|$$

$$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} i & j & k \\ 1 & 3 & 3 \\ 4 & 4 & 8 \end{vmatrix} \overset{\text{DIY}}{=} \langle 12, 4, -8 \rangle$$

$$\Rightarrow \text{Area} = \sqrt{12^2 + 4^2 + (-8)^2} = 4\sqrt{14}.$$

(b) Area of triangle $= \frac{1}{2}(4\sqrt{14})$

(c) $\overrightarrow{PQ} = \langle 1, 3, 3 \rangle$, $\overrightarrow{PR} = \langle 5, 7, 11 \rangle$, $\overrightarrow{PS} = \langle 4, 4, 8 \rangle$

$$\text{Volume} = |\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})| = \begin{vmatrix} 1 & 3 & 3 \\ 5 & 7 & 11 \\ 4 & 4 & 8 \end{vmatrix} \overset{\text{DIY}}{=} 0$$

Correction: In class I mentioned that you want to avoid taking cross products of diagonals. But this misleading! Apologies and thanks to the student who brought this to my attention!

You want to avoid taking cross products of parallel sides. But this is easily avoided by fixing a point and considering any two vectors emanating from this point.

Eg: $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$ so $\overrightarrow{PQ} \times \overrightarrow{PR} = \overrightarrow{PQ} \times (\overrightarrow{PQ} + \overrightarrow{QR})$

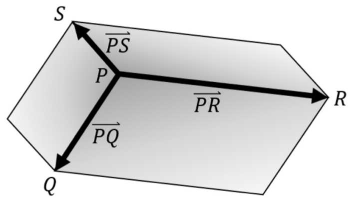
$$= \boxed{\overrightarrow{PQ} \times \overrightarrow{PQ}} + \overrightarrow{PQ} \times \overrightarrow{QR}$$

$0 =$

$$= \overrightarrow{PQ} \times \overrightarrow{QR} = \overrightarrow{PQ} \times \overrightarrow{PS}$$

since parallel sides have equal length and direction.

6. Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS. The points are given by $P(3,0,1)$, $Q(-1,2,5)$, $R(5,1,-1)$, and $S(0,4,2)$.



$$\text{Volume} = \left| \overrightarrow{PS} \cdot \left(\overrightarrow{PR} \times \overrightarrow{PQ} \right) \right| = \left| \det \begin{pmatrix} \overrightarrow{PS} \\ \overrightarrow{PR} \\ \overrightarrow{PQ} \end{pmatrix} \right|$$

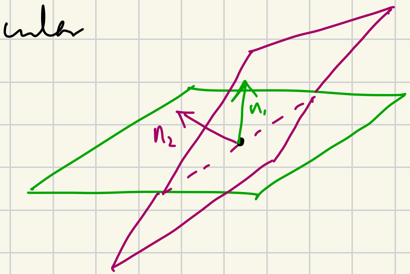
$$= \left| \overrightarrow{PR} \cdot \left(\overrightarrow{PS} \times \overrightarrow{PQ} \right) \right| = \left| \overrightarrow{PQ} \cdot \left(\overrightarrow{PS} \times \overrightarrow{PR} \right) \right|$$

$$\begin{vmatrix} -4 & 2 & 4 \\ -3 & 4 & 1 \\ 2 & 1 & -2 \end{vmatrix} = -9\,\underset{-4}{\hat{i}} - 4\,\underset{2}{\hat{j}} - 11\,\underset{4}{\hat{k}}$$

8. Find the parametric equations for the line of intersection of the planes $2x + 3y + 5z = 7$ (P_1) and $x - y + 2z = 3$ (P_2).

Direction vector for the desired line is perpendicular to the plane spanned by $\vec{n}_1$ and $\vec{n}_2$

(contains $\vec{n}_1$ and $\vec{n}_2$).



$\vec{n}_1 = \langle 2, 3, 5 \rangle$

$\vec{n}_2 = \langle 1, -1, 2 \rangle$

$$\Rightarrow \vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 2 & 3 & 5 \\ 1 & -1 & 2 \end{vmatrix} \overset{\text{DIY}}{=} \langle 11, 1, -5 \rangle$$

We need: $\vec{L}(t) = \vec{r}_0 + t\,\vec{v}$ → direction vector.

$\vec{r}_0$ ↑ position vector of a point on the line

$$\Rightarrow L(t) = \langle 1, 0, 1 \rangle + t\langle 11, 1, -5 \rangle$$
$$= \langle 1 + 11t,\ 0 + t,\ 1 - 5t \rangle$$

→ $x = 1 + 11t$
$y = t$
$z = 1 - 5t$

$2x + 3y + 5z = 7$
$x - y + 2z = 3$

Eg: $y = 0$

$\Rightarrow x = 1,\ z = 1$

Eg: $x = 0 \Rightarrow$ $3y + 5z = 7$
$-y + 2z = 3$

$\Rightarrow y = 2z - 3 \Rightarrow$ .....

Fun fact: you can take $x = 0, y = 0, z = 0$ ie any line in $\mathbb{R}^3$ passes through the xz, yz, xy-planes exactly once.

9. Find the vector equation, parametric equations and symmetric equations of the line passing through the points $A(2, 1, 1)$ and $B(3, 2, -2)$.

direction vector : $\overrightarrow{AB} = \langle 3-2, 2-1, -2-1 \rangle = \langle 1, 1, -3 \rangle$ or $\overrightarrow{BA} = \langle -1, -1, 3 \rangle$

position vector of a point on the line : take A or B

Acceptable equations: $L(t) = \overrightarrow{OA} + t\overrightarrow{AB}$

$$L(t) = \overrightarrow{OA} + t\overrightarrow{BA}$$

$$L(t) = \overrightarrow{OB} + t\overrightarrow{AB}$$

$$L(t) = \overrightarrow{OB} + t\overrightarrow{BA}$$

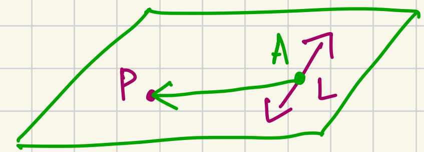
eg: $\langle x, y, z \rangle = \langle 2, 1, 1 \rangle + t\langle -1, -1, 3 \rangle$ ← vector

$\Rightarrow x = 2 - t, \; y = 1 - t, \; z = 1 + 3t$ ← parametric

Solve for t → $t = \dfrac{x-2}{-1} = \dfrac{y-1}{-1} = \dfrac{z-1}{3}$ ← symmetric

10. Find an equation of the plane that passes through the point $P(1,1,3)$ and contains the line given by the symmetric equations $\dfrac{x+1}{2} = y+2 = \dfrac{z-3}{2}$.

- $L(t) = \langle -1,-2,3\rangle + t\,\boxed{\langle 2,1,2\rangle}$ → vector "on" the plane, say $\vec{v}$.



- We need a $\underbrace{\text{normal vector}}_{\text{perp. to any vector on the plane}}$ and a point on the plane.

- Take $t=0$ which gives $A(-1,-2,3)$ as a point on the line.

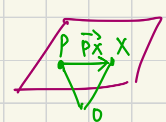
- $\vec{AP} = \langle 2,3,0\rangle$

- normal, $\vec{n} = \langle a,b,c\rangle = \vec{v}\times\vec{AP} = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 2 & 3 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 2 \\ 3 & 0\end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 2 \\ 2 & 0\end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 \\ 2 & 3\end{vmatrix}$

$= -6\hat{i} + 4\hat{j} + 4\hat{k}$

$\Rightarrow$ Equation is $-6(x-1)+4(y-1)+4(z-3)=0$

$\Leftrightarrow \vec{n}\cdot(\vec{OX}-\vec{OP}) = 0$



11. Find an equation for the plane that passes through the points $\overset{A}{(0, -2, 5)}$ and $\overset{B}{(-1, 3, 1)}$ and is perpendicular to the plane $2z = 5x + 4y$.

$\overrightarrow{AB} = \langle -1, 5, -4 \rangle$

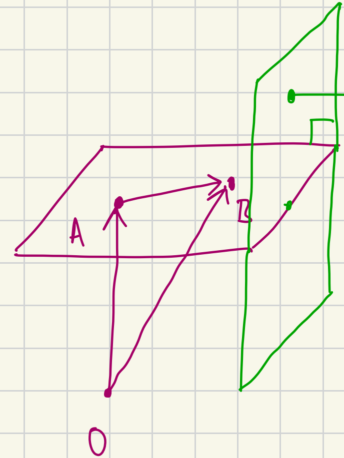
$\overrightarrow{n_1}$ = normal to the plane $2z = 5x + 4y$

$$\Updownarrow$$

$$5x + 4y - 2z = 0$$

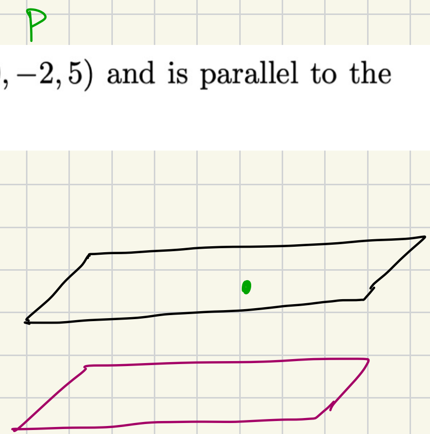
$\Rightarrow \overrightarrow{n_1} = \langle 5, 4, -2 \rangle$

$\Rightarrow$ a possible normal vector is $\overrightarrow{n_1} \times \overrightarrow{AB}$.



normal to this plane is parallel to the plane with points A and B.

12. Find an equation for the plane that passes through the point $(0, -2, 5)$ and is parallel to the plane $2z = 5x + 4y$.



Normal vector for the plane we want is parallel to the normal vector of

$$2z = 5x + 4y \iff 5x + 4y - 2z = 0$$

$\Rightarrow$ equation for the plane we want: $\langle 5, 4, -2\rangle \cdot \langle x - 0, y + 2, z - 5\rangle = 0$

$$\iff 5x + 4(y+2) - 2(z-5) = 0$$

13. Write an equation of the plane containing the points

$$P(4, -3, 1), \quad Q(-3, -1, 1), \quad R(4, -2, 8).$$

Take any 2 vectors from $\vec{PQ}$, $\vec{PR}$, $\vec{QR}$ and take their cross product for the normal to the plane cointaining P, Q, R.

14. Let $\mathcal{C}$ be the curve given by the vector function $\mathbf{r}(t) = \cos(t)\mathbf{i} + \ln(t)\mathbf{j} + \dfrac{1}{t-3}\mathbf{k}$.

(a) Find the domain of $\mathbf{r}(t)$. Use the interval notation.

(b) Find $\lim\limits_{t \to \pi} \mathbf{r}(t)$.

(c) Find the point P on the curve at $t = 1$.

15. Consider the vector function $\mathbf{r}(t) = 2\,\mathbf{i} + 2\sin(t)\,\mathbf{j} + 2\cos(t)\,\mathbf{k}$.

(a) Find the length of the curve of C with $\mathbf{r}(t)$, where $-2 \le t \le 2$.

(b) Find the unit tangent vector $\mathbf{T}(t)$ at the point with the given value of the parameter $t = \pi/6$. Simplify the answer completely.

(c) Find the principal unit normal vector $\mathbf{N}(t)$ at the point with the given value of the parameter $t = \pi/6$. Simplify the answer completely.

(d) Use the formula $\kappa = \dfrac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$ to find the curvature.

(e) Find the binormal vector $\mathbf{B}(t)$ at the point with the given value of the parameter $t = \pi/6$. Simplify the answer completely.

(f) Find the tangential and normal components of acceleration $\mathbf{a}(t)$.

(a) $r'(t) = \langle 0, 2\cos t, -2\sin t\rangle \Rightarrow ||r'(t)|| = 2$

$\Rightarrow L = \int_{-2}^{2} ||r'(t)||\,dt = 4 \cdot 2 = 8.$

(b) Find $r'(\pi/6)$ and then $||r'(\pi/6)||$

don't waste time with finding $||r'(t)||$ and then plugging in $t = \pi/6$.

$$T(\pi/6) = \frac{r'(\pi/6)}{||r'(\pi/6)||}$$

16. Consider the position function $\mathbf{r}(t) = 8\sqrt{2}\,t\mathbf{i} + e^{8t}\mathbf{j} + e^{-8t}\mathbf{k}$.

(a) Find the velocity of a particle with the given position function $\mathbf{r}(t)$.

(b) Find the acceleration of a particle with the given position function $\mathbf{r}(t)$.

(c) Find the speed of a particle with the given position function $\mathbf{r}(t)$.

17. Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

$$\mathbf{a}(t) = 2\mathbf{i} + 2t\mathbf{k}, \quad \mathbf{v}(0) = 5\mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}.$$

18. Given a vector function

$$\mathbf{r}(t) = (\arctan t)\,\mathbf{i} + 2t^2\,\mathbf{j} + t\ln(t)\,\mathbf{k}$$

(a) Find a vector equation of the line tangent to the vector function at the point $\left(\frac{\pi}{4}, 2, 0\right)$

(b) Find the unit tangent vector $\mathbf{T}(t)$ at the point $\left(\frac{\pi}{4}, 2, 0\right)$.

(a) direction vector is $r'(t)$

Find t-value that gives $r(t) = \left(\frac{\pi}{4}, 2, 0\right)$

but $t\ln t = 0$ when $t = 1$

Verify: $r(1) = \langle \arctan 1,\ 2(1)^2,\ t\ln 1\rangle = \langle \frac{\pi}{4}, 2, 0\rangle$.

$L(s) = \vec{r}(1) + s\,\vec{r}\,'(1)$

(b) $\vec{T}(1) = \dfrac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|}$