Conservative Vector Fields

Lecture for 7/2

General Idea

Recall Fundamental Theorem of Calculus: $\int_a^b f'(x)\,dx = f(b) - f(a)$

- Do we have a similar theorem for line integrals?
- Use FTOC by trying to express $f(\mathbf{r}(t))$ as a derivative
- Will end up having gradient in place of derivative

How do we tell if a vector field is conservative?

- Will explain and derive method

Fundamental Theorem of Line Integrals

Consider some path C parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$

- Recall $\int_C (\mathbf{F} \cdot d\mathbf{r}) = \int_C (\mathbf{F} \cdot d\mathbf{r}/dt)\, dt = \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t))\, dt$
- Now suppose $\mathbf{F}$ is conservative and $\mathbf{F} = \nabla f$
- Then $\int_C (\mathbf{F} \cdot d\mathbf{r}) = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Fundamental Theorem Derivation

Extra Space

Conservation Efforts

Suppose we have 2 variables and $\mathbf{F} = \nabla f = \langle f_x, f_y \rangle$

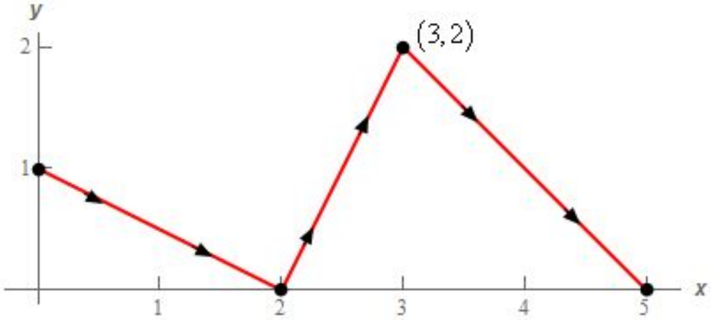
- Then $(F_1)_y = (f_x)_y = f_{xy} = f_{yx} = (F_2)_x$
- So if $(F_1)_y \neq (F_2)_x$, $\mathbf{F}$ is not conservative
- If they are equal, then $f = \int f_x \, dx = \int F_1 \, dx = G(x,y) + h(y)$
  - Evaluate integral to get G, then find function of integration h

Be careful if f is not defined or not differentiable everywhere

Practice Problems

Evaluate $\int_C (\mathbf{F} \cdot d\mathbf{r})$ for the following functions and curves

- $\mathbf{F} = \nabla f$ with $f = ye^{(x^2 - 1)} + 4xy^{1/2}$, $\mathbf{r}(t) = (1-t, 2t^2-2t)$, $0 \leq t \leq 2$
- $\mathbf{F} = \nabla f$ with $f(x,y,z) = x+\sin(y+\cos(x+e^{x-y})+\ln(3+\sec(y)+11x^{1/2}))$ and C the ellipse $(x-4)^2/711+(y-3)^2/1215 = 1$ oriented clockwise
- $\mathbf{F} = \langle 2z^4-y^3-2y, z-2x-3xy^2, 6+y+8xz^3\rangle$, C the curve shown below



Scratchwork