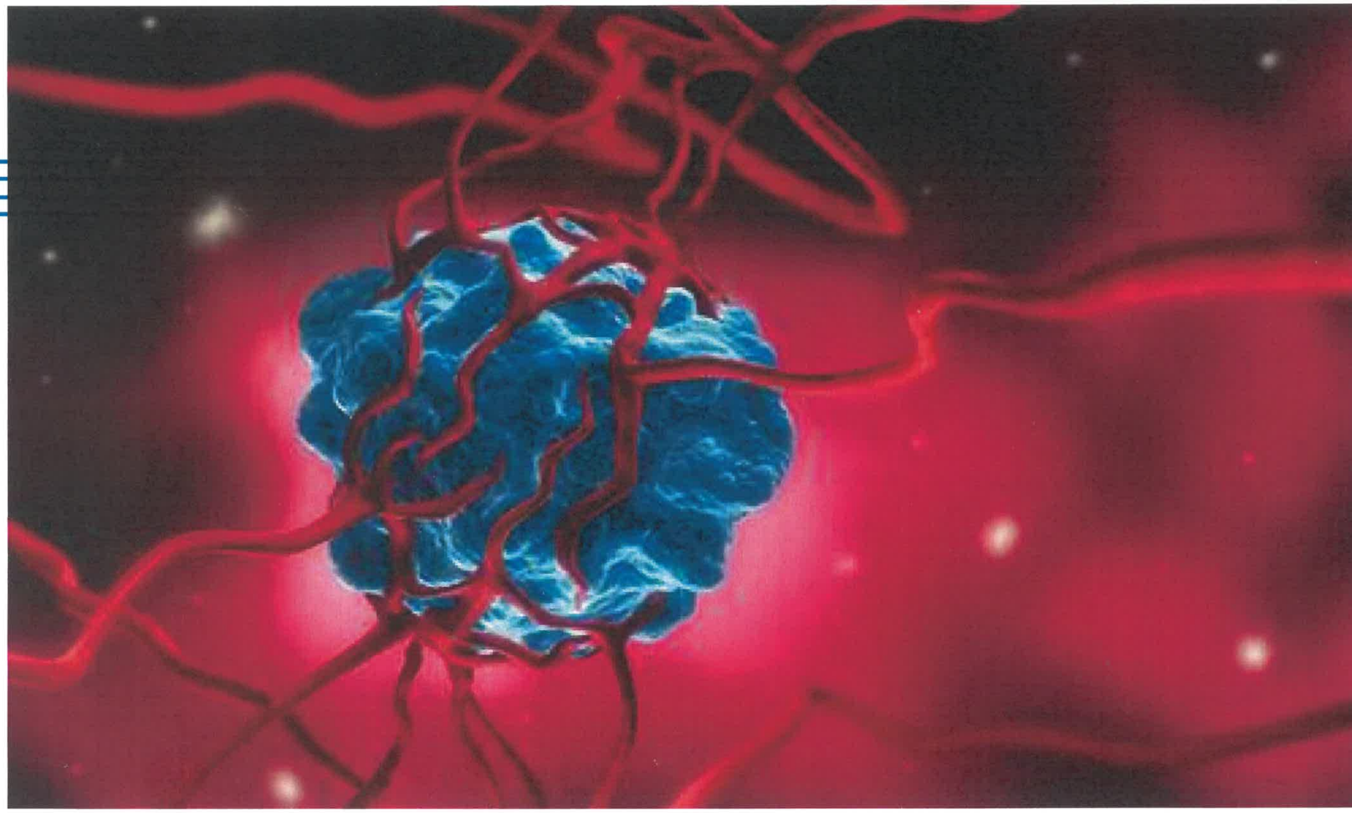
15 Multiple Integrals



15.3 Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates (1 of 1)

Suppose that we want to evaluate a double integral $\iint_R f(x,y)\, dA$,
Where the region R is a circular disk centered at the origin.

Review of Polar Coordinates

Review of Polar Coordinates (1 of 2)

We know that from Figure 1 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

$\tan\theta = \frac{y}{x}$

$\Leftrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

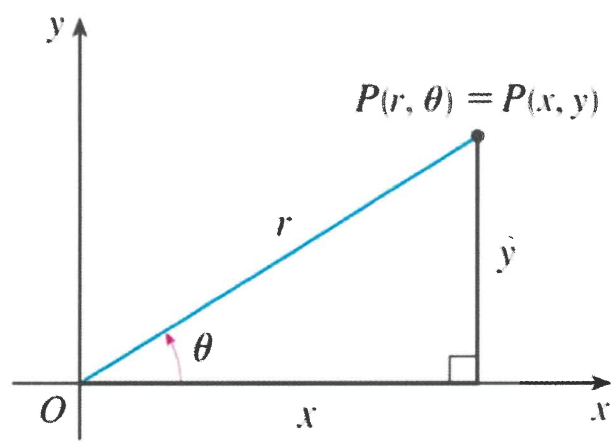
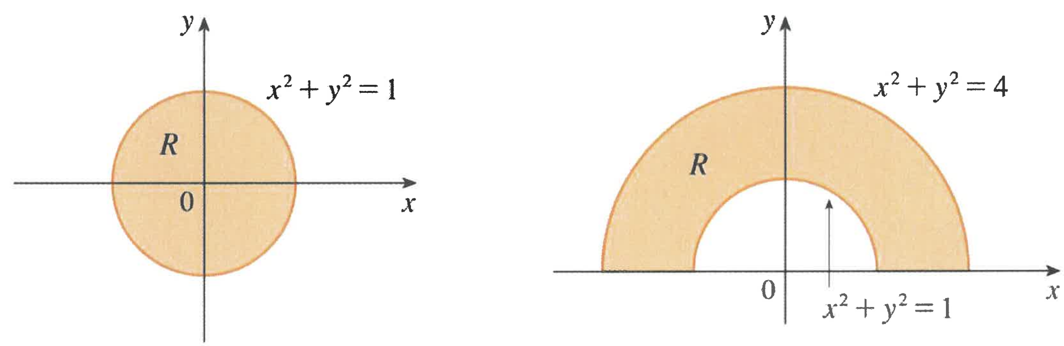


Figure 1

Review of Polar Coordinates (2 of 2)

Equations of circles centered at the origin are particularly simple in polar coordinates. The unit circle has equation $r = 1$; the region enclosed by this circle is shown in Figure 2(a). Figure 2(b) illustrates another region that is conveniently described in polar coordinates.



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Figure 2

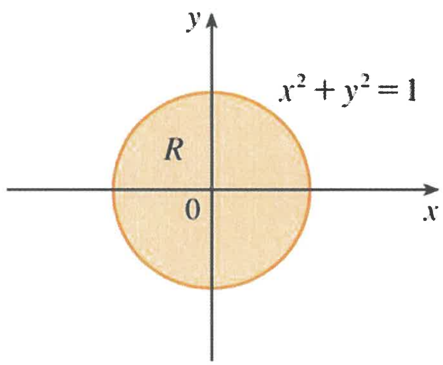
Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates (1 of 9)

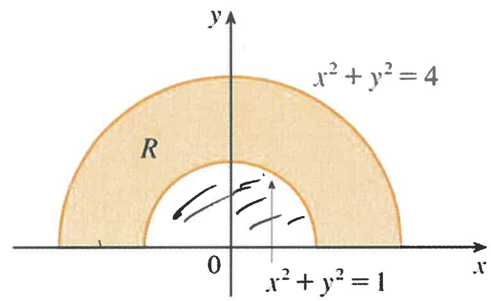
The regions in Figure 2 are special cases of a **polar rectangle**

$$R = \{(r,\ \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$$

which is shown in Figure 3.

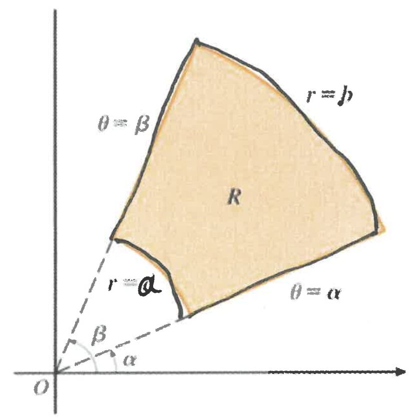


(a) $R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$



(b) $R = \{(r,\theta) \mid \underline{1 \le r \le 2},\ 0 \le \theta \le \pi\}$

Figure 2



Polar rectangle

Figure 3

$x^2 + y^2 = r^2$

$x = r\cos\theta$

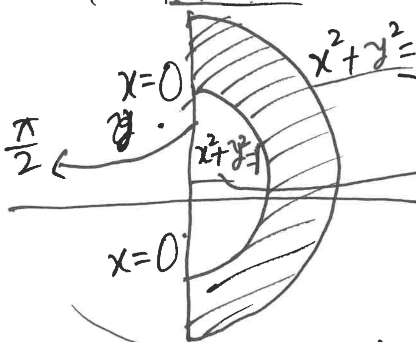
$y = r\sin\theta$

$\tan\theta = \dfrac{y}{x}$

$\Leftrightarrow \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

∞



$\theta = \dfrac{\pi}{2}$

$r^2 = 4 \Rightarrow r = 2$

$r^2 = 1 \Rightarrow r = 1$

$1 \le r \le 2$

$R =$

$\tan\theta = \dfrac{y}{x} = \dfrac{y}{0} = \infty$

8

$\theta = \dfrac{\pi}{2}$

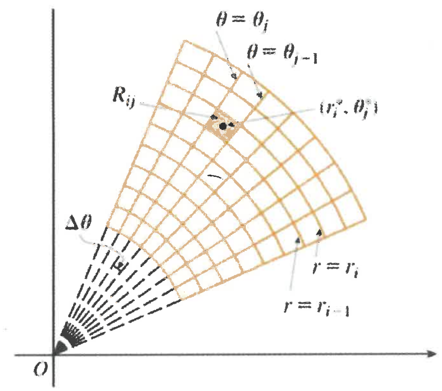
$\tan\theta = \dfrac{y}{0} = -\infty$

$\theta = -\dfrac{\pi}{2}$, $\cancel{\dfrac{3\pi}{2}}$

Double Integrals in Polar Coordinates (2 of 9)

In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = \dfrac{(b-a)}{m}$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = \dfrac{(\beta - \alpha)}{n}$.

Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij} shown in Figure 4.



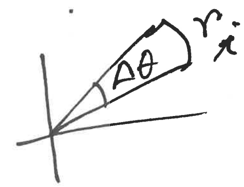
Dividing R into polar subrectangles

Figure 4

Double Integrals in Polar Coordinates (3 of 9)

The “center” of the polar subrectangle

$$R_{ij} = \{(r,\ \theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$



has polar coordinates

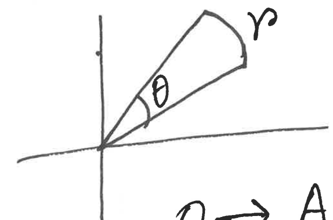
$$r_i^* = \tfrac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \tfrac{1}{2}(\theta_{j-1} + \theta_j)$$

$$A = \frac{r_i^2 \Delta\theta}{2}$$

$$\Delta\theta = (\theta_j - \theta_{j-1})$$

$$A = r_{i-1}^2 \Delta\theta/2$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$.



$$\theta \Rightarrow A = r^2\theta/2$$

Double Integrals in Polar Coordinates (4 of 9)

Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

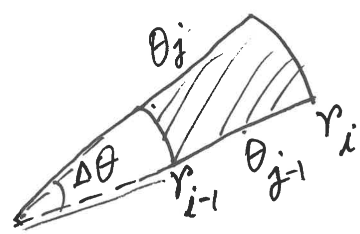
$$\Delta A_i = \tfrac{1}{2} r_i^2 \Delta\theta - \tfrac{1}{2} r_{i-1}^2 \Delta\theta = \tfrac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

$$= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\,\Delta\theta = r_i^*\,\Delta r\Delta\theta$$

Although we have defined the double integral $\iint_R f(x, y)\, dA$ in terms of ordinary rectangles, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles.

$A_{In} = \Delta\theta\, r_{i-1}^2/2$

$A_{out} = \Delta\theta\, r_i^2/2$



Double Integrals in Polar Coordinates (5 of 9)

The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$, so a typical Riemann sum is

$$\textbf{1} \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*)\ \Delta A_i = \underline{\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*)\ \underline{r_i^*}\ \underline{\Delta r}\ \underline{\Delta\theta}}$$

If we write $g(r, \theta) = rf(r\cos\theta, r\sin\theta)$, then the Riemann sum in Equation 1 can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*,\ \theta_j^*)\ \Delta r\ \Delta\theta$$

$f(x, y) = f(r\cos\theta, r\sin\theta)$

which is a Riemann sum for the double integral

$$\int_{\alpha}^{\beta}\int_{a}^{b} \underline{g(r,\ \theta)}\ dr\ d\theta$$

$\int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)\,\underline{r\,dr\,d\theta}$

$dA = r\,dr\,d\theta$

Double Integrals in Polar Coordinates (6 of 9)

Therefore we have

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*)\ \Delta A_i$$

$$= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*,\ \theta_j^*)\ \Delta r\ \Delta\theta = \int_\alpha^\beta \int_a^b g(r,\ \theta)\ dr\ d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\ r\ dr\,d\theta$$

2 Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\ r\ dr\,d\theta$$

Double Integrals in Polar Coordinates (7 of 9)

The formula in (2) says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\,dr\,d\theta$.

Be careful not to forget the additional factor r on the right side of Formula 2.

A classical method for remembering this is shown in Figure 5, where the "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $r\,d\theta$ and dr and therefore has "area" $dA = r\,dr\,d\theta$.

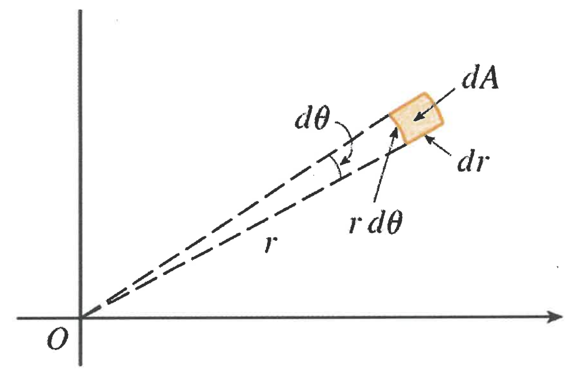


Figure 5

$$x^2+y^2=r^2, \quad x=r\cos\theta,\ y=r\sin\theta,\quad \theta=\tan^{-1}\left(\frac{y}{x}\right) \iff \frac{y}{x}=\tan\theta$$

$$dA = r\,dr\,d\theta$$

Example 1

Evaluate $\int\int_R (2x-y)dA$ where R is the region in the first quadrant enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=4$ and the lines $x=y$ and $y=0$.

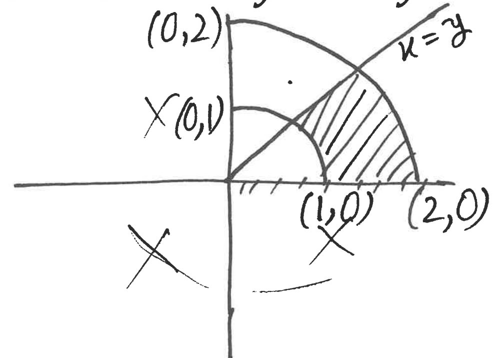
$x^2+y^2=1$

$\Rightarrow r^2=1$

$\Rightarrow r=1$

$x^2+y^2=4$

$\Rightarrow r=2$



$1 \le r \le 2$

bound for θ:

$y=0$ and $y=x$

$\Rightarrow \theta=0$ and $\theta=\tan^{-1}(1)=\frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}}\int_1^2 (2r\cos\theta - r\sin\theta)\, r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\int_1^2 (2\cos\theta-\sin\theta)\, r^2\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{4}} \left[(2\cos\theta-\sin\theta)\frac{r^3}{3}\right]_{r=\cancel{0}1}^{2} d\theta$$

$$=\int_0^{\frac{\pi}{4}}\left[(2\cos\theta-\sin\theta)\left(\frac{8}{3}-\frac{1}{3}\right)\right]d\theta = \frac{7}{3}\int_0^{\frac{\pi}{4}}(2\cos\theta-\sin\theta)\,d\theta =$$

Example 1

Evaluate $\int\int_R (2x - y)dA$ where R is the region in the first quadrant enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the lines $x = y$ and $y = 0$.

Example 1

Evaluate $\int\int_R (2x - y)dA$ where R is the region in the first quadrant enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the lines $x = y$ and $y = 0$.

Example 2

Evaluate $\int_0^{\frac{1}{2}} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\, dydx$ by switching to polar coordinates.

$0 \le x \le \frac{1}{2}, \quad \sqrt{3}x \le y \le \sqrt{1-x^2}$

$y \le \sqrt{1-x^2}$

$\Rightarrow y^2 \le 1 - x^2$

$\Rightarrow x^2 + y^2 \le 1$

$\sqrt{3}x \le y$

Let us pick $x = \frac{1}{4}, \ y = \frac{\sqrt{3}}{4}$

$x^2 + y^2 \le 1$

$\Rightarrow r \le 1$

$x \ge 0$

$\cancel{\Rightarrow r \sin\theta \le 0}$

$\cancel{\Rightarrow r \cos\theta \ge 0}$

$\Rightarrow \cos\theta \ge \frac{0}{r} = 0$

$\Rightarrow \cos\theta \ge 0$

$\Rightarrow \theta \ge \cos^{-1}(0) = \frac{\pi}{2}$

$x = 0$

$\Rightarrow r\cos\theta = 0$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \frac{\pi}{2}$

$\sqrt{3}x = y$

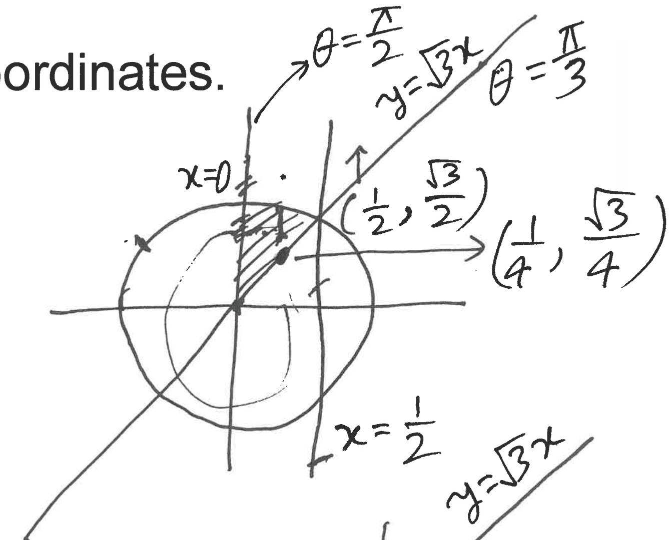
$\Rightarrow \frac{y}{x} = \sqrt{3}$

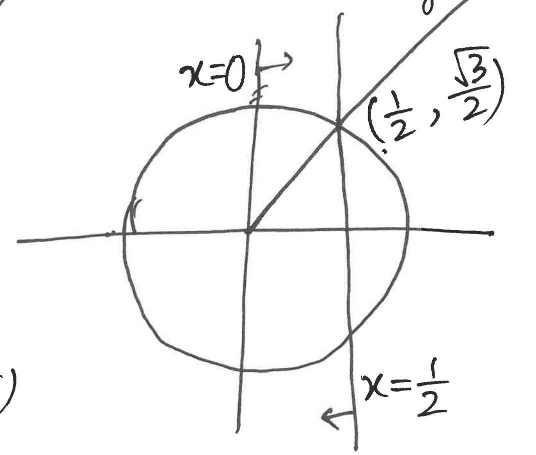
$\Rightarrow \tan(\theta) = \sqrt{3}$

$\Rightarrow \theta = \tan^{-1}(\sqrt{3})$

$\Rightarrow \theta = \frac{\pi}{3}$

$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$





Example 2

Evaluate $\int_0^{\frac{1}{2}} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} x \, dydx$ by switching to polar coordinates.

$0 \leq r \leq 1, \quad \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^1 r\cos\theta \, r \, dr \, d\theta$$

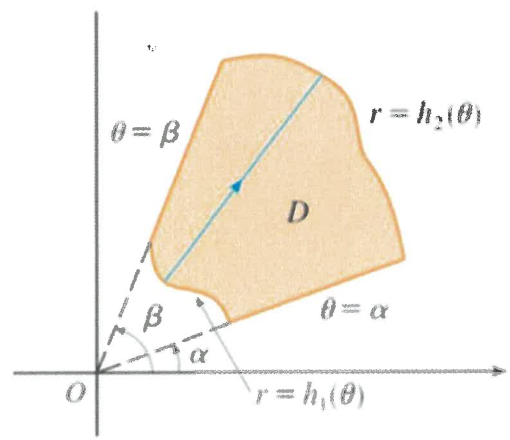
$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\frac{r^3}{3} \cos\theta \right]_{r=0}^{1} d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3} \cos\theta \, d\theta$$

Double Integrals in Polar Coordinates (8 of 9)

What we have done so far can be extended to the more complicated type of region shown in Figure 8. In fact, by combining Formula 2 with

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region, we obtain the following formula.



$D = \{(r, \theta) | \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$

Figure 8

3 If f is continuous on a polar region of the form

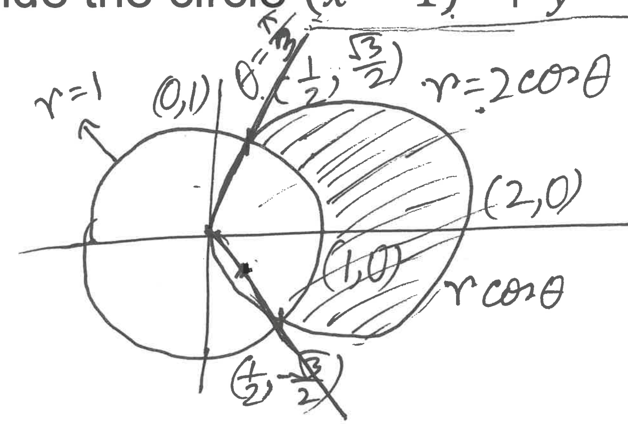
$$D = \{(r,\ \theta) | \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta$$

Example 3

Find the area of D where D is the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.



$f(x,y)=1$

$f(r\cos\theta, r\sin\theta)=1$

$x^2+y^2=1$

$\Rightarrow r=1$

$(x-1)^2+y^2=1$

$\Rightarrow x^2-2x+\cancel{1}+y^2=\cancel{1}$

$\Rightarrow r^2-2r\cos\theta=0$

$\Rightarrow r^{\cancel{2}}=2\cancel{r}\cos\theta$

$\Rightarrow r=2\cos\theta$

$1\le r\le 2\cos\theta$

$-\frac{\pi}{3}\le\theta\le\frac{\pi}{3}$

$y^2=\cancel{(x-1)}\ 1-(x-1)^2$

$y^2=1-x^2$

$1-(x-1)^2=1-x^2$

$\Rightarrow -\cancel{x^2}+2x-1=-\cancel{x^2}$

$\Rightarrow x=\frac{1}{2}$

$\Rightarrow 1-x^2=1-\frac{1}{4}=\frac{3}{4}$

$\Rightarrow y^2=\frac{3}{4}\Rightarrow y=\pm\frac{\sqrt{3}}{2}$

$(0,0)$ and $(\frac{1}{2},\frac{\sqrt{3}}{2})$

(a,b) and (c,d)

$m=\frac{d-b}{c-a}$

$m=\frac{\sqrt{3}/2}{1/2}=\sqrt{3}$

$\tan\theta=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$

$m=\frac{-\sqrt{3}/2}{1/2}=-\sqrt{3}$

$\tan\theta=-\sqrt{3}$

$\theta=-\frac{\pi}{3}$

Example 3

Find the area of D where D is the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{1}^{2\cos\theta} 1\, r\,dr\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{r^2}{2}\right]_{r=1}^{2\cos\theta} d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{4\cos^2\theta}{2} - \frac{1}{2}\right] d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(2\cos^2\theta - \frac{1}{2}\right) d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\cos 2\theta + \frac{1}{2}\right) d\theta = \left[\frac{1}{2}\sin 2\theta + \frac{\theta}{2}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$