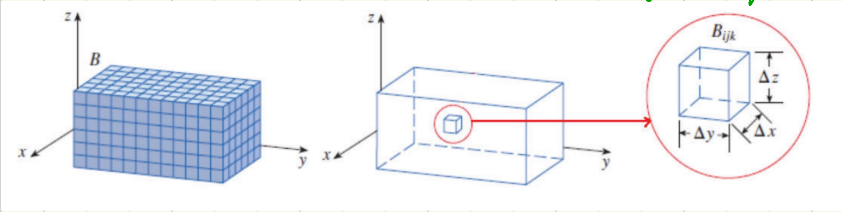
Section 15.6: Triple Integrals.

Volume of a box in 4-D: $\underbrace{f(x^*, y^*, z^*) \cdot \Delta x \cdot \Delta y \cdot \Delta z}_{\text{product of 4 lengths}}$

Definition The **triple integral** of f over the box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

if this limit exists.



Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Example: Evaluate $\underbrace{\iiint_B x\cos(y+z)\,dV}_{=I}$ where $B = \underset{x}{[0,1]} \times \underset{y}{[0,2]} \times \underset{z}{[0,3]}$.

$$I = \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} x\cos(y+z)\,dz\,dy\,dx = \int_0^1 \int_0^2 \left(x\sin(y+3) - x\sin y\right) dy\,dx$$

Let $I_1 = \int_0^3 x\cos(y+z)\,dz = x\sin(y+z)\Big|_{z=0}^{3} = x(\sin(y+3) - \sin(y+0))$

Let $I_2 = \int_0^2 (x\sin(y+3) - x\sin y)\,dy$

$= x\int_0^2 \sin(y+3)\,dy - x\int_0^2 \sin y\,dy = -x\cos(y+3)\Big|_0^2 + x\cos y\Big|_0^2$

$= -x(\cos(5) - \cos(3)) + x(\cos(2) - \cos(0))$

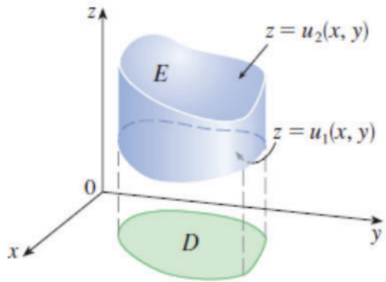
$I = \int_0^1 -x\cos(5) + x\cos(3) + x\cos(2) - x\,dx = (-\cos(5) + \cos(3) + \cos(2) - 1)\int_0^1 x\,dx$

$= (-\cos(5) + \cos(3) + \cos(2) - 1)\frac{1}{2}$

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

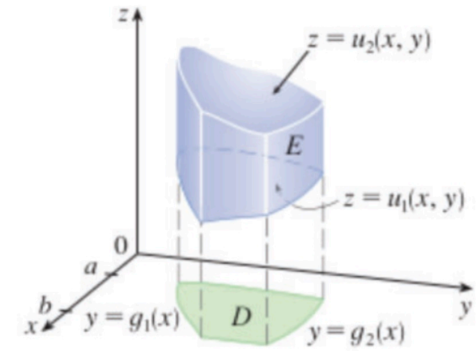
where D is the projection of E onto the xy-plane.

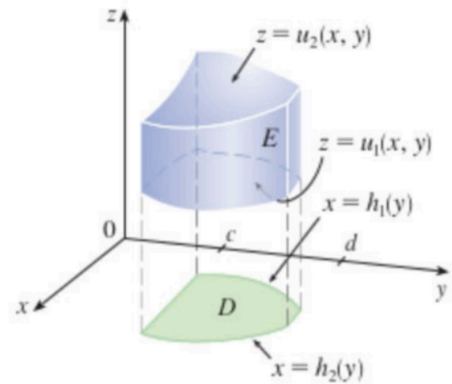


- Upper boundary of the solid E is the surface $z = u_2(x, y)$
- Lower boundary of the solid E is the surface $z = u_1(x, y)$.

If E is a type 1 region, then:

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$



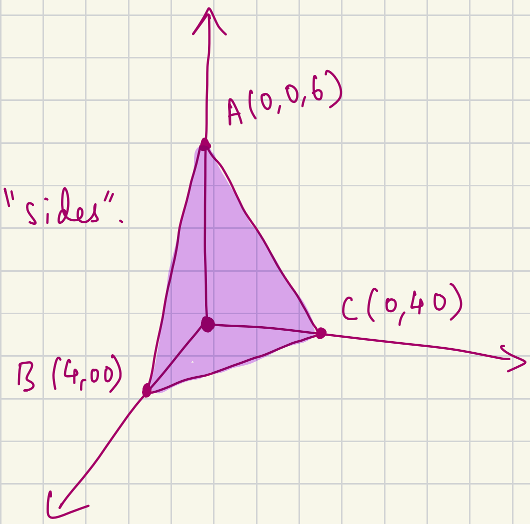


Example: Let T be the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$ and $C(0,4,0)$.
(Note that the plane containing the points A, B and C has the equation $3x+3y+2z=12$)

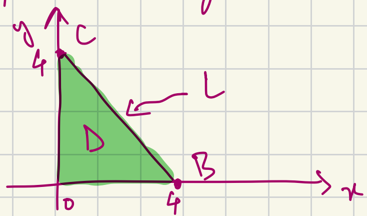
(a) Express T as a solid region type 1.

(b) Express $\iiint\limits_T f(x,y,z)\,dV$ as an iterated integral.

- Faces of the tetrahedron are parts of the following planes: $3x+3y+2z=12$ (in pink), xy plane (base), xz & yz planes are the "sides".



- Project the surface $3x+3y+2z=12$ onto the xy-plane to get the following region:



- D is the region in the xy plane bonded by the lines $x=0$, $y=0$, and L.
- L is the line between $\underbrace{B(4,0,0) \text{ and } C(0,4,0)}_{}$

think of these as $B(4,0)$ and $C(0,4)$

$\Rightarrow$ L has slope $\frac{0-4}{4-0} = -1$ and y-intercept 4 $\Rightarrow y = -x+4$

OR $x+y=4$

OR $x=-y+4$

- $E = \{(x,y,z) : 0 \le x \le 4,\ 0 \le y \le -x+4,\ 0 \le z \le 6 - \frac{3}{2}x - \frac{3}{2}y\}$

$3x+3y+2z=12 \Rightarrow z = 6 - \frac{3}{2}x - \frac{3}{2}y$

$= \{(x,y,z) : 0 \le y \le 4,\ 0 \le x \le -y+4,\ 0 \le z \le 6 - \frac{3}{2}x - \frac{3}{2}y\}$

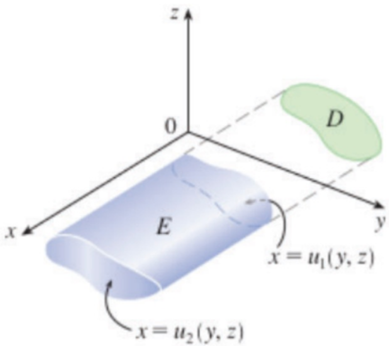
(b) $$\iiint_E f(x,y,z)\, dV = \int_0^4 \int_0^{-x+4} \int_0^{6-\frac{3}{2}x-\frac{3}{2}y} f(x,y,z)\, dz\, dy\, dx$$

$$= \int_0^4 \int_0^{-y+4} \int_0^{6-\frac{3}{2}x-\frac{3}{2}y} f(x,y,z)\, dz\, dx\, dy$$

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where D is the projection of E onto the yz-plane.



- The back surface is $x = u_1(y, z)$.
- The front surface is $x = u_2(y, z)$

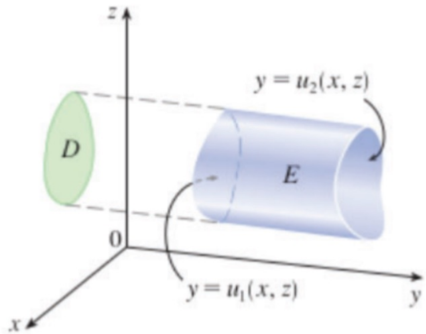
Then,

$$\boxed{\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx\right] dA}$$

A solid region E is of **type 3** if it is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the xz-plane.



- The left surface is $y = u_1(x, z)$.
- The right surface is $y = u_2(x, z)$

Then,

$$\boxed{\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\right] dA}$$

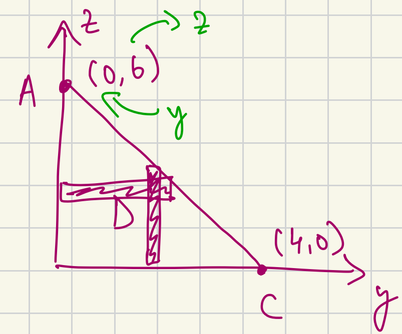
Note: In each of these equations, there may be two possible expressions for the integral depending on whether D is a type I or a type II plane region.

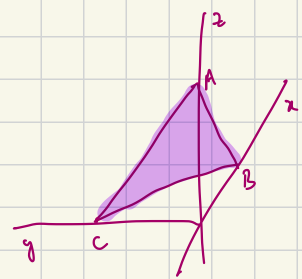
Example: Let T be the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$ and $C(0,4,0)$.
(Note that the plane containing the points A, B and C has the equation $3x+3y+2z=12$)

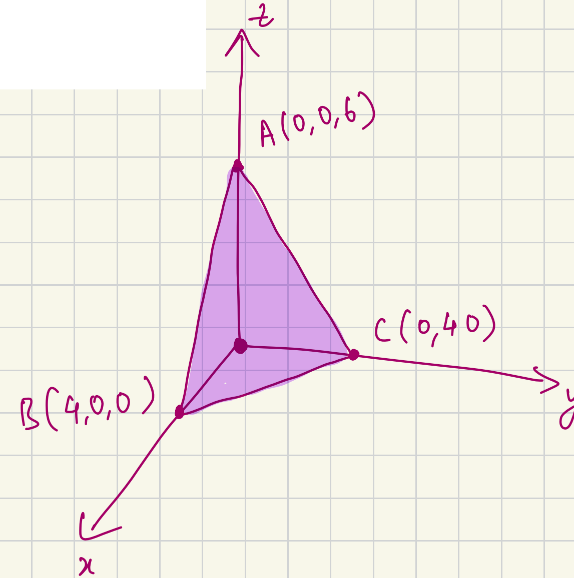
(a) Express T as a solid region type 2 and type 3.

(b) Express $\displaystyle\iiint_T f(x,y,z)\,dV$ as an iterated integral.

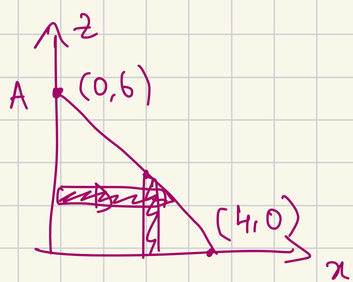
(a) Type 2:

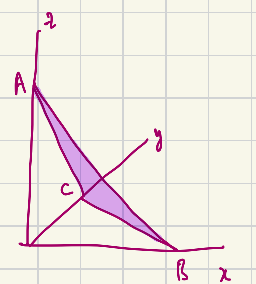






Type 3:





Example: Evaluate $\overbrace{\iiint\limits_T e^z \, dV}^{I}$ where T is the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$ and $C(0,4,0)$.

$$T = \left\{ (x,y,z) : \ 0 \le x \le 4 \ , \ 0 \le y \le -x+4 \ , \ 0 \le z \le 6 - \frac{3}{2}x - \frac{3}{2}y \right\}$$

$$I = \int_{x=0}^{4} \int_{y=0}^{4-x} \int_{z=0}^{6-\frac{3}{2}x-\frac{3}{2}y} e^z \, dz \, dy \, dx$$

$$= \int_0^4 \int_0^{4-x} e^z \Big|_{z=0}^{6-\frac{3}{2}x-\frac{3}{2}y} dy \, dx = \int_0^4 \int_0^{4-x} \left(e^{6-\frac{3}{2}x-\frac{3}{2}y} - 1 \right) dy \, dx \overset{\boxed{\text{DIY}}}{=} \boxed{\frac{4e^6 - 100}{9}}$$

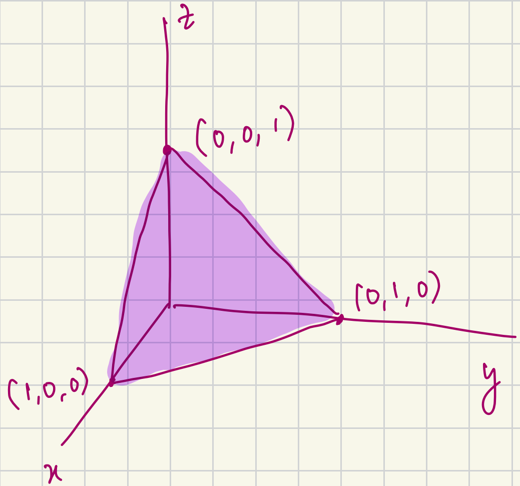
$$\int_{y=0}^{4-x} e^{6-\frac{3}{2}x-\frac{3}{2}y} - 1 \, dy = e^{6-\frac{3}{2}x} \int_0^{4-x} e^{-\frac{3}{2}y} \, dy - \int_0^{4-x} dy$$

Volume of E is given by

$$V(E) = \iiint_E dV$$

Example: Find the volume of the solid, E in the first octant bounded by $x + y + z = 1$ and $x + y + 2z = 1$.

$x \geq 0, y \geq 0, z \geq 0$

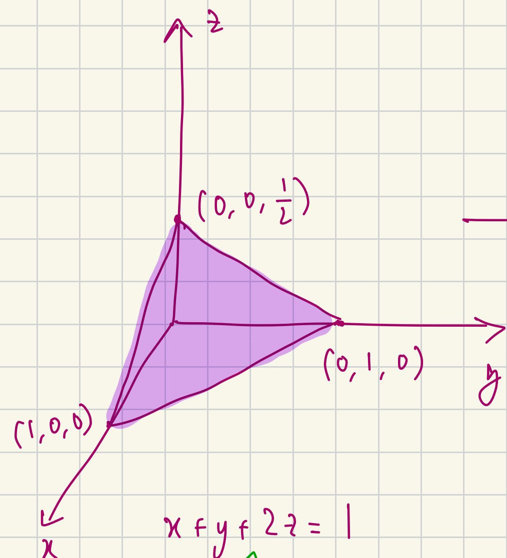


$x + y + z = 1$

$\Updownarrow$

$z = 1 - x - y$

$+$

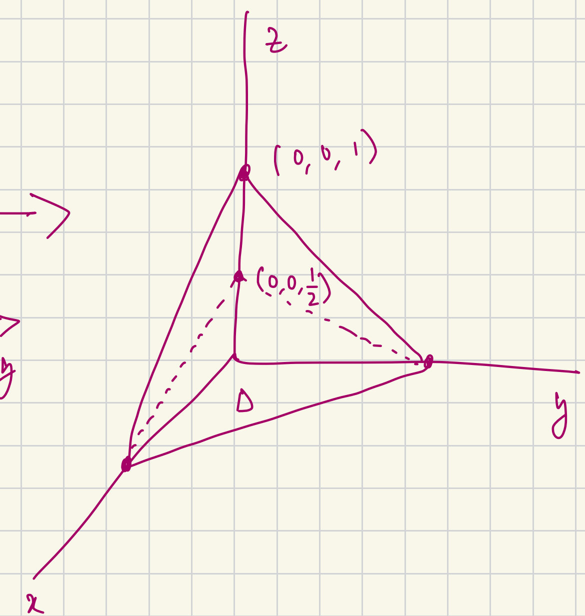


$x + y + 2z = 1$

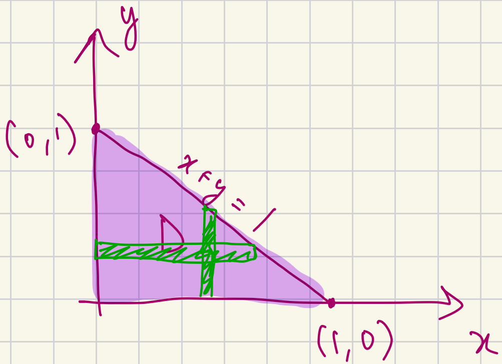
$\Updownarrow$

$2z = 1 - x - y$

$\longrightarrow$



$$E = \left\{ (x,y,z) : (x,y) \in D,\ \frac{1-x-y}{2} \le z \le 1-x-y \right\}$$



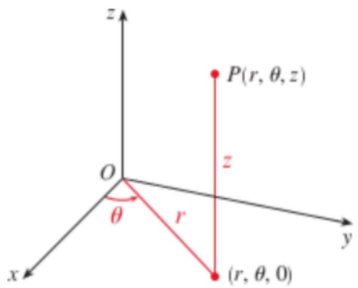
$$D = \{ (x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x \}$$

$$\Rightarrow E = \left\{ (x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1-x,\ \frac{1-x-y}{2} \le z \le 1-x-y \right\}$$

$$\Rightarrow V(E) = \int_0^1 \int_0^{1-x} \int_{\frac{1-x-y}{2}}^{1-x-y} dz\, dy\, dx \overset{\boxed{\text{DIY}}}{=} \boxed{\frac{1}{12}}$$

1. **Cylindrical Coordinates**

In the cylindrical coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.



To convert from cylindrical to rectangular coordinates, we use

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta \qquad z = z}$$

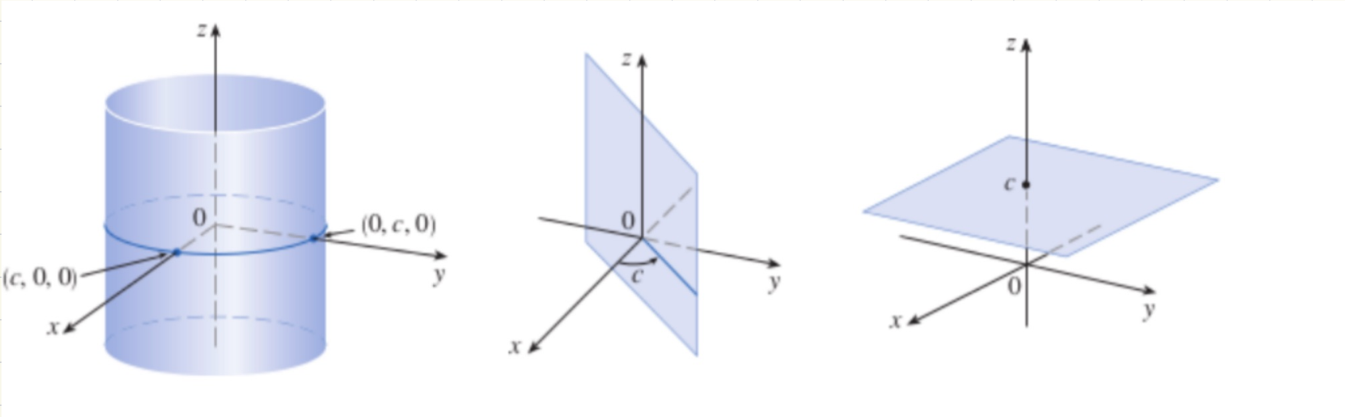
Example: $(r, \theta, z) = (2, \frac{\pi}{2}, 4) \leftrightarrow (x, y, z) = \left(2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}, 4\right) = (0, 2, 4)$

and to convert rectangular to cylindrical coordinates, we use:

$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z}$$

Example: $(x, y, z) = (1, \sqrt{3}, 5) \leftrightarrow (r, \theta, z) = \left(\sqrt{1+3}, \arctan\sqrt{3}, 5\right)$

$$= \left(2, \frac{\pi}{3}, 5\right)$$



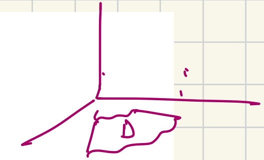
Cylindrical $\rightarrow$ $r = c$ $\qquad\qquad$ $\theta = c$ $\qquad\qquad$ $z = c$

i.e $\{(r, \theta, z) : r = c\}$

Cartesian $\rightarrow$ $x^2 + y^2 = c$

Suppose

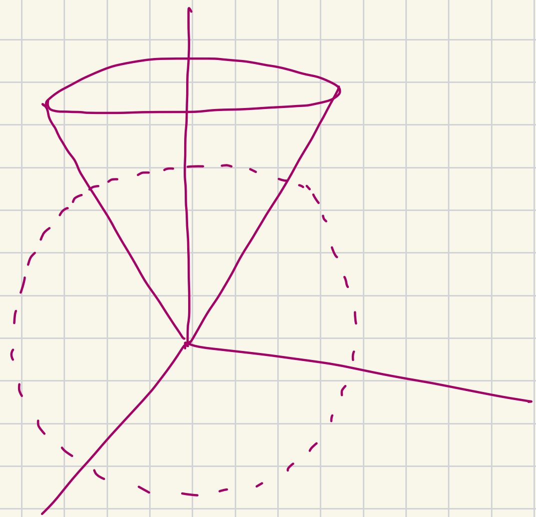
$$
\begin{aligned}
E &= \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \\
D &= \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\} \\
f &= f(x, y, z) \text{ is continuous on } E
\end{aligned}
$$

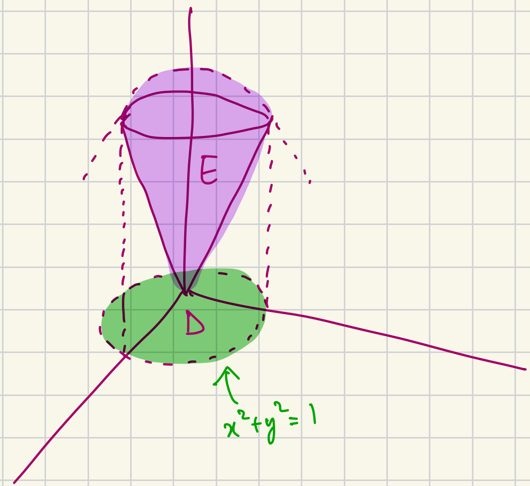


Let $g_1(r, \theta) = u_1(r\cos\theta, r\sin\theta)$ and $g_2(r, \theta) = u_2(r\cos\theta, r\sin\theta)$.

$$
\begin{aligned}
\iiint\limits_E f(x, y, z)\, dV &= \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA \\
&= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta
\end{aligned}
$$

Example: Let E be the region bounded above by the sphere $x^2+y^2+z^2=2$ and bounded below by the cone $z=\sqrt{x^2+y^2}$. Use a triple integral to find the volume of E.





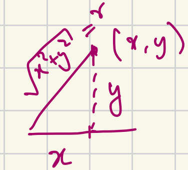
To find D, solve for (x,y,z) such that $z^2=x^2+y^2$ and $x^2+y^2+z^2=2$.

$$\Downarrow$$

$$x^2+y^2+x^2+y^2=2 \Longleftrightarrow x^2+y^2=1$$

$$E = \{(x,y,z) : x^2+y^2 \le 1 ,\ \sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}\}$$

$$= \{(r,\theta,z) : 0 \le \theta \le 2\pi ,\ 0 \le r \le 1 ,\ \underset{\sqrt{r^2}}{\underset{\parallel}{r}} \le z \le \sqrt{2-r^2}\}$$



$$\Rightarrow V(E) = \int_0^{2\pi}\int_0^{1}\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta \overset{\boxed{DIY}}{=} \frac{4\pi(\sqrt{2}-1)}{3}$$

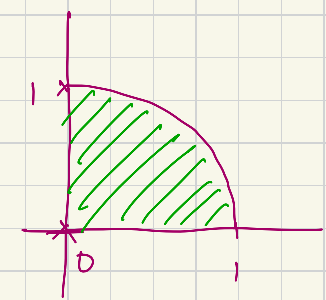
$$V(E) = \iiint_E dV = \iiint_E dz\,dy\,dx$$

Example: Rewrite the following integral using cylindrical coordinates

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_1^{\sqrt{9-x^2-y^2}} xyz\, dz\, dy\, dx$$

$$E = \{ (x,y,z) : 0 \le x \le 1,\ 0 \le y \le \underbrace{\sqrt{1-x^2}}_{},\ 1 \le z \le \sqrt{9-x^2-y^2} \}$$

$$y = \sqrt{1-x^2} \Leftrightarrow x^2+y^2=1$$



$$E = \{ (r,\theta,z) : 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1,\ 1 \le z \le \sqrt{9-r^2} \}$$

$$\Rightarrow I = \int_0^{\pi/2} \int_0^1 \int_1^{\sqrt{9-r^2}} (r\cos\theta)(r\sin\theta)\, z\, (r)\, dz\, dr\, d\theta .$$

Example: Evaluate $\displaystyle\iiint_E x\,dV$ where E is bounded by $\underbrace{x = 4y^2 + 4z^2}_{}$ and $x = 4$.

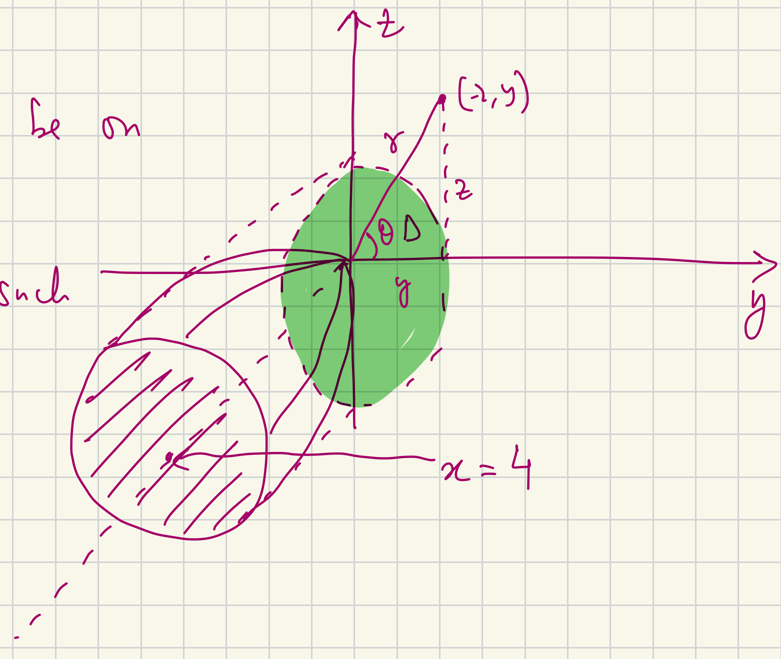
paraboloid

- Switch cylindrical coordinates:
  Take height to be in x, r and θ to be on the yz plane
- For the region D: solve for (z,y) such that $x = 4 = 4y^2 + 4z^2$

$$\Longleftrightarrow \quad y^2 + z^2 = 1$$

$$\Rightarrow D = \{(y,z) : y^2 + z^2 \leq 1\}$$

$$= \{(r,\theta) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1\}$$



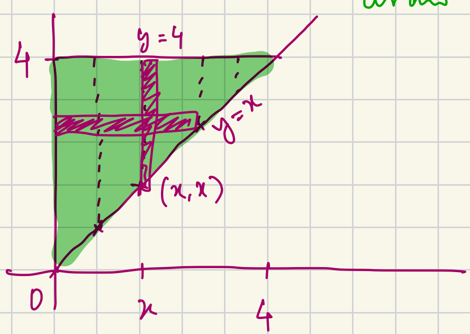
$\Rightarrow E = \{ (x,y,z) : (y,z) \in D, \ 4(y^2+z^2) \le x \le 4 \}$

$= \{ (r,\theta,x) : 0 \le \theta \le 2\pi, \ 0 \le r \le 1, \ 4r^2 \le x \le 4 \}$

$$\Rightarrow \iiint_E x \, dV = \int_0^{2\pi} \int_0^1 \int_{4r^2}^4 x \, r \, dx \, dr \, d\theta \overset{\boxed{\text{DIY}}}{=} \frac{16}{3}\pi$$

PQ 4 : $R = \{ (x,y) : 0 \le x \le 4, \ \underbrace{x \le y}_{\text{draw } y=x} \le 4 \} \leftarrow$ Type 1

1. a



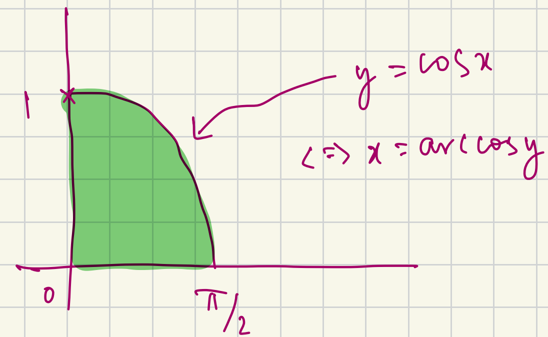
1.b $R = \{(x,y) : 0 \le y \le 4, 0 \le x \le y\}$ ⟵ Type 2

1.c $$\iint_R f(x,y)\,dA = \int_0^4 \int_x^4 f(x,y)\,dy\,dx = \int_0^4 \int_0^y f(x,y)\,dx\,dy.$$

MIDTERM 2 PRACTICE:

25. D is bounded by $y = \cos x$, $0 \le x \le \frac{\pi}{2}$, $y = 0$, $x = 0$.

(a) $$\iint_D \sin^2(x)\,dA = I$$



$$D = \{(x,y) : 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\}$$

$$I = \int_0^{\pi/2} \int_0^{\cos x} \sin^2(x)\,dy\,dx$$

(b) $D = \{(x,y) : 0 \le y \le 1 ,\ 0 \le x \le \arccos(y)\}$

$$I = \int_0^1 \int_0^{\arccos y} \sin^2(x)\, dx\, dy$$

$$\int_0^{\pi/2} \left[\int_0^{\cos x} \sin^2(x)\, dy \right] dx \qquad \int_0^{\pi/2} \sin^2(x)\, y \Big|_0^{\cos x} dx$$

(a) $$\int_0^{\pi/2}\int_0^{\cos x} \sin^2(x)\, dy\, dx = \int_0^{\pi/2} \sin^2(x)\,(\cos x - 0)\, dx = \int_0^{\pi/2} \sin^2(x) \cos x\, dx$$

$$u = \sin x \Rightarrow du = \cos x\, dx \Rightarrow I = \int_0^1 u^2\, du = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}$$

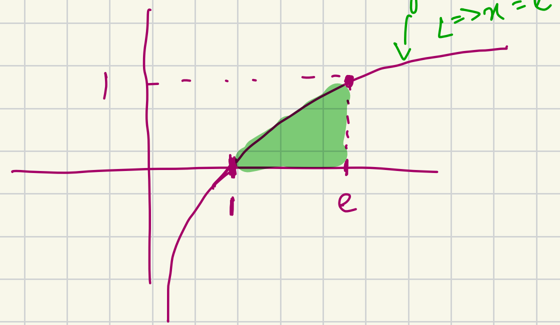
$x = 0,\ u = 0$

$x = \pi/2,\ u = 1$

26. Change the order of integration for

$$I = \int_1^e \int_0^{\ln x} f(x,y)\, dy\, dx$$

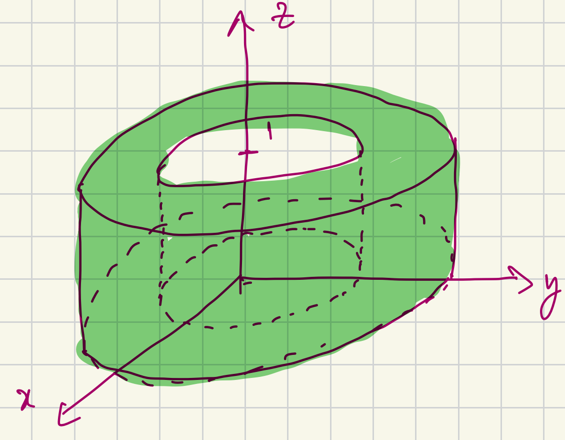
$D = \{ (x,y) : 1 \le x \le e,\ 0 \le y \le \ln x \}$



$= \{ (x,y) : 0 \le y \le 1,\ e^y \le x \le e \}$

$$\Rightarrow I = \int_0^1 \int_{e^y}^{e} f(x,y)\, dx\, dy$$

30. Find the volume of the solid region bounded by $z=0$, $z=1$, $x^2+y^2=1$, $x^2+y^2=9$.



$E = \{ (x,y,z) : 1 \le x^2+y^2 \le 9,\ 0 \le z \le 1 \}$

$= \{ (x,y,z) : (x,y) \in D,\ 0 \le z \le 1 \}$

where $D = \{ (x,y) : 1 \le x^2+y^2 \le 9 \}$

$1 \le r^2 \le 9 \Leftrightarrow 1 \le r \le 3$

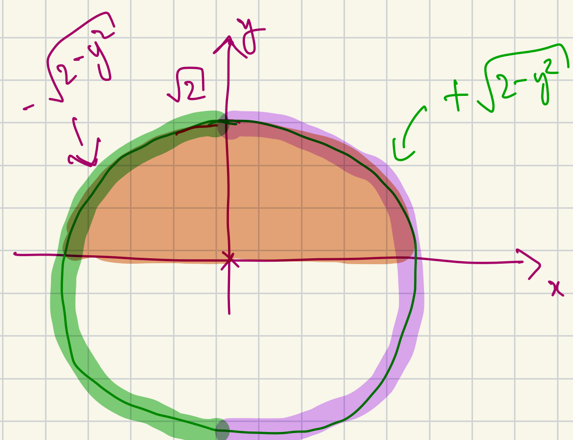
$$V(E) = \iint_D 1 \, dA = \int_0^{2\pi} \int_1^3 r \, dr \, d\theta \overset{\boxed{DIY}}{=} 8\pi$$

31. Compute $I = \int_0^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} (x^2+y^2) \, dx \, dy$ by converting to polar coordinates.

$x^2 = 2 - y^2 \iff x^2 + y^2 = 2$

A: $D = \{ (x,y) : 0 \le y \le \sqrt{2}, \ -\sqrt{2-y^2} \le x \le \underbrace{\sqrt{2-y^2}} \}$

$= D = \{ (r,\theta) : 0 \le \theta \le \pi, \ 0 \le r \le \sqrt{2} \}$



$$I = \int_0^{\pi} \int_0^{\sqrt{2}} r^2 \cdot (r) \cdot dr \, d\theta \overset{DIY}{=} \pi$$