# MATH 243 Midterm 1 Solutions

**1.** By the chain rule, the answer is $\boxed{A.}$
**2.** The tangent plane is actually $3x + 4y - z = 5$, so A is false.
For B, note we can choose our order of derivatives for each term by Clairaut's Theorem. For the 1st term, we do $\partial_x$ first and it vanishes. For the 2nd term, it vanishes under $\partial_z$. The last term is $g(x,z)h(y)$ where $g(x,z) = (2z+x)^4, h(y) = \sin(\sin(y)))$. Then $(g(x,z)h(y))_{zzzxy} = g_{zzzx}h_y = 192\cos(\sin(y))\cos(y)$, so B is true.
For C, first identify the curve goes from $t = 0$ to $t = 2$. Calculate $\|r'(t)\| = \sqrt{0 + (2t)^2 + (1)^2} = \sqrt{4t^2+1}$. Thus, C is true.
For D, note that a plane is determined by 3 non-collinear points, so we only have to plug the 3 points into the plane equation. The equation is true for each point, so D is true and the answer is $\boxed{B, C, D.}$

**3.** Let $P$ be the parallelepiped in question. The volume of $P$ is 0 if and only if it's completely flat, which happens if and only if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ lie in the same plane. Thus, A and B are equivalent. The volume of $P$ is given by $\|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\|$, so $\text{vol}(P) = 0 \Leftrightarrow \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0 \Leftrightarrow \mathbf{u}, \mathbf{v} \times \mathbf{w}$ are perpendicular, not parallel. So C is out of the loop.
Recall that we may also take a determinant to find volume: $|\det(M)| = \text{vol}(P)$. If $\text{vol}(P) = 0$, then swapping rows of $M$ doesn't affect $\det(M)$ because it goes from 0 to $-0$. Similarly, if swapping rows doesn't change $\det(M)$, then $\det(M) = 0 \Rightarrow \text{vol}(P) = 0$. So $A, B, D$ are equivalent and the answer is $\boxed{C.}$

**4.** Let $v_n$ be the vector where there are $n$ $a$'s following the $b$ so that $v_0 = a \times b$ and $v = v_{50}$.
By cross product properties, $a, v_0 = a \times b, v_1 = v_0 \times a$ are pairwise perpendicular. Thus, $v_2 = v_1 \times a$ is parallel to $v_0$ and we have $v_2 = kv_0$ for some constant $k$. By computing $v_0 = (-2, 3, 6), v_2 = (20, -30, -60)$, we find $k = -10$. We can guess that this pattern repeats to get $v_{2n} = (-10)^n v_0$, and prove it by writing $v_{2n+2} = (v_{2n} \times a) \times a = (-10)^n(v_0 \times a) \times a = (-10)^n v_2 = (-10)^{n+1}v_0$.
Thus, $v = (-10)^{25}(-2, 3, 6)$, making the sum of coordinates $10^{25} \cdot (-7)$, meaning $(r, s, t) = (10, 25, -7)$ and $r + s + t = 28$.
Note: strictly speaking, we could've stumbled into $v_2 = -10v_0$ without imagining the orientations of earlier vectors. However, the fact that $a, v_0, v_1$ are pairwise perpendicular gives us a good picture of why $v_2, v_3, \ldots$ keep alternating between pointing along $v_0$ or $v_1$.

**5.** $\rho^2 = x^2+y^2+z^2 = r^2+z^2 = 3+9 = 12$. We already know $\theta = \frac{\pi}{5} = 36°$, and we get $\cos\phi = \frac{z}{\rho} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$, so $\phi = 30° \Rightarrow \phi + \theta = 66°$.

**6.** The numerator and denominator are continuous, so the fraction is continuous and defined whenever the denominator is non-zero, so the limit exists anywhere except possibly when the denominator is 0. The terms $e^{\cdots}$ and $1 + x^2 + y^6$ are non-zero, so we can ignore them when analyzing the denominator, as they will neither make it 0 nor serve to cancel any terms.
Thus, we have to analyze $\frac{y^3 - x^2y}{8x^3 - 6x^2y - 3xy^2 + y^3}$. We can let $t = y/x$ and rewrite the denominator as $x^3(t^3 - 3t^2 - 6t + 8)$. In order to factor the cubic, we can notice $t = 1$ is a root, divide out $(t-1)$, obtain a quadratic in $t$, and factor the quadratic. However we do it, we arrive at $x^3(t-1)(t+2)(t-4) = (y-x)(y+2x)(y-4x)$. Meanwhile, the numerator factors as $y(y^2 - x^2) = y(y-x)(y+x)$.
From our factorizations, the denominator is 0 along the lines $y = x, y = -2x, y = 4x$. However, the line $y = x$ consists of removable singularities since there is also a $y - x$ factor in the numerator. The limit doesn't

exist among the other two lines, so $n = 2$.
**7.** The line through all the points is the same as the line through $B$ and $G$, for which one parametrization is $r(t) = B + t(E - B) = (1 - \frac{5}{3}t, 2 + t, 3 - 3t)$.
Notice that $t = 0$ gives us $B$ and $t = 1$ gives us $E$. The segment $BE$ consists of three equal segments $BC, CD, DE$, so $r(1/3) = C, r(2/3) = D$. Extrapolating forward and backward, we see that $r(-1/3) = A, r(5/3) = G$. So one parametrization of the segment $AG$ is $r(t) = (1 - \frac{5}{3}t, 2 + t, 3 - 3t)$, $-\frac{1}{3} \leq t \leq \frac{5}{3}$.

**8.** Acceleration is simple: $\mathbf{a}(t) = \mathbf{r}''(t) = (0, 1.5t^{-1/2}, 0)$. To compute its components, we use the formula $a_T = \frac{r' \cdot r''}{\|r'\|} = \frac{(1,3t^{1/2},-1)\cdot(0,1.5t^{-1/2},0)}{\|(1,3\sqrt{t},-1)\|} = \frac{4.5}{\sqrt{2+9t}}$.
Instead of computing a cross product to get $a_N$, notice that from $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ and the fact $\mathbf{T}, \mathbf{N}$ are perpendicular, we get $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a} = a_T^2 + a_N^2$, so

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{\frac{9}{4t} - \frac{81}{4(2+9t)}} = \frac{3}{2}\sqrt{\frac{1}{t} - \frac{9}{2+9t}} = \frac{3}{\sqrt{2t(9+2t)}}.$$

**9.** Let $a = f_x(p), b = f_y(p)$. The unit vector in the direction of $(3, -4)$ is $u = (3/5, -4/5)$, so we get $\nabla f(p) \cdot u = \frac{3a}{5} - \frac{4b}{5} = 9$. The unit vector in the direction of $\theta = 225°$ is $(\cos(225°), \sin(225°)) = -\frac{1}{\sqrt{2}}(1, 1)$, so we obtain another equation $-\frac{1}{\sqrt{2}}(a + b) = -4\sqrt{2} \Rightarrow a + b = 8$.
Solve the system of 2 linear equations to get $a = 11, b = -3$. The maximum rate of change occurs in the direction of $\nabla f$ and has value $\|\nabla f(p)\| = \|(a, b)\| = \sqrt{130}$. The unit vector along $(5, 12)$ is $(5/13, 12/13)$, so the desired directional derivative is $(5a + 12b)/13 = 19/13$.