MATH 243: Analytic Geometry & Calculus C **Name:** __________________

Practice Problem Set **Date: Friday 17$^{\text{th}}$ October, 2025**

Instructions: Try each of the following problems carefully. Show all essential work for full credit. Partial credit may be awarded. No calculators or electronic devices are permitted.

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1. (2 points) Which of the following paths is **NOT** appropriate to use for showing that

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$

does not exist? Justify your answer briefly.

   A. $y = x$
   B. $y = x^2$
   C. $x = 0$
   D. $y = -x^2$
   E. $y = 2x$

2. (3 points) Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

does not exist. (Hint: Compare the limits along at least two different paths.)

3. (3 points) The limit exists. Find its value:

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2}.$$

Show your steps clearly.

4. (3 points) Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

by reasoning directly from the definition of limit and simple bounds (no polar coordinates).

5. (3 points) Find an example of a function $f(x,y)$ for which the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ is 0 along every line $y = mx$ but not 0 along some nonlinear path (e.g., $y = x^2$). Explain why the overall limit does not exist.

6. Let $f(x,y) = e^{xy^2}$.
   (a) (2 points) Find an equation of the tangent plane to $z = f(x,y)$ at the point $(1, 1, f(1,1))$.
   (b) (2 points) Determine the linearization $L(x,y)$ of f at the point $(1,1)$.
   (c) (2 points) Use $L(x,y)$ to approximate $f(1.02, 0.98)$.

7. Let $F(x, y, z) = x^2 + 2y^2 - 3z^2 = 9$ describe a surface.
   (a) (3 points) Find the equation of the tangent plane to this surface at the point $(2, 1, 1)$.
   (b) (3 points) Find the parametric equations of the normal line to the surface at the same point.

8. Let $z = x^3 - 3xy^2$.
   (a) (3 points) Find the equation of the tangent plane to this surface at the point $(1, 1, z_0)$, where $z_0 = f(1, 1)$.
   (b) (2 points) Interpret geometrically how the coefficients of the plane relate to $\nabla f(1, 1)$.

9. Let $f(x, y) = x^2y + 3y^2$.
   (a) (2 points) Compute the gradient $\nabla f(x, y)$.
   (b) (1 point) Evaluate ∇f at $P(1, 2)$.
   (c) (2 points) Find the rate of change of f at $P(1, 2)$ in the direction of $\mathbf{u} = \langle 3/5, 4/5\rangle$.
   (d) (1 point) In which direction does f increase most rapidly at P? What is the maximum rate of increase?

10. Let $f(x, y, z) = xyz + x^2z^2$.
    (a) (2 points) Compute $\nabla f(x, y, z)$.
    (b) (2 points) Find the directional derivative of f at the point $(1, -1, 2)$ in the direction of $\mathbf{v} = \langle 2, -1, 2\rangle$.
    (c) (2 points) Verify that the magnitude of ∇f at that point equals the maximum rate of change of f there.

11. Suppose $z = x^2y + \sin(y)$, where $x = u^2 - v$ and $y = e^{uv}$.
    (a) (2 points) Find $\dfrac{\partial z}{\partial u}$ using the Chain Rule.
    (b) (2 points) Find $\dfrac{\partial^2 z}{\partial v \partial u}$.

12. Let $w = x^2y + yz^3$, where $x = t^2$, $y = e^t$, and $z = \sin t$.
    (a) (3 points) Find $\dfrac{dw}{dt}$ using the multivariable Chain Rule.
    (b) (2 points) Evaluate $\dfrac{dw}{dt}$ at $t = \pi/4$.

13. (2 points) **Conceptual (Multiple Choice):** Which of the following statements about the Chain Rule is true?

    A. If $z = f(x, y)$ and x, y are functions of t, then $\dfrac{dz}{dt} = f_x\dfrac{dx}{dt} + f_y\dfrac{dy}{dt}$.

    B. If $z = f(x, y)$ and x, y are functions of u, v, then $\dfrac{\partial z}{\partial u} = f_x + f_y$.

    C. $\dfrac{dz}{dt}$ can be found only if z is a linear function.

    D. $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ are always equal.

14. (3 points) Warm-up: Given $x^2 + yz = 4$, find $\dfrac{\partial z}{\partial x}$ in terms of x, y, z.

15. (5 points) Given that $x^2 + y^2 + z^2 = 3xyz$, find $\dfrac{\partial z}{\partial x}$ using implicit differentiation. (Simplify your result as much as possible.)

16. Let $F(x, y, z) = x^2 + 2y^2 - 3z^2 = 3$ describe a surface S.

    (a) (2 points) Verify if the point $(2, 1, 1)$ lies on the surface.

    (b) (4 points) Find the equation of the tangent plane to this surface at the point $(2, 1, 1)$.

    (c) (4 points) Find the parametric equations of the normal line to the surface at the same point.