• Directional derivative of $z = f(x, y)$ at a point (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$

is $\boxed{D_u f(x_0, y_0) = f_x(x_0, y_0) \cdot a + f_y(x_0, y_0) \cdot b}$ $\Rightarrow$ $\boxed{D_u f(x_0, y_0) = f_x(x_0, y_0) \cdot \cos\theta + f_y(x_0, y_0) \cdot \sin\theta}$

• **8 Definition**

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\boxed{\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}} \Rightarrow \boxed{D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}}$$

• For functions of three variables:

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

OR (succinctly) $\nabla f = \langle f_x, f_y, f_z \rangle$.

$$\Rightarrow \boxed{D_u f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \vec{u}}$$

- **15 Theorem**

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

- Level surfaces : $F(x,y,z)=k$. Equation of the tangent plane at (x_0,y_0,z_0) on the surface. That is, $F(x_0,y_0,z_0)=k$, is

$$\boxed{\nabla F(x_0,y_0,z_0)\cdot\left(\langle x,y,z\rangle-\langle x_0,y_0,z_0\rangle\right)=0}$$

OR

$$\boxed{f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0}$$

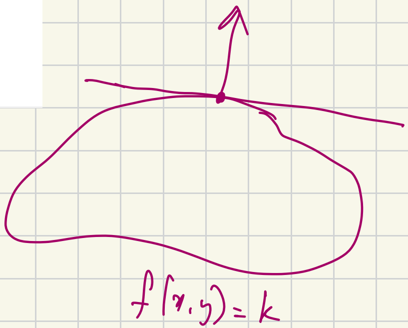
- Equation of a normal line (line through (x_0,y_0,z_0) in the direction $\nabla F(x_0,y_0,z_0)$):

$$\boxed{\frac{x-x_0}{f_x(x_0,y_0,z_0)}=\frac{y-y_0}{f_y(x_0,y_0,z_0)}=\frac{z-z_0}{f_z(x_0,y_0,z_0)}}$$

Properties of the Gradient Vector

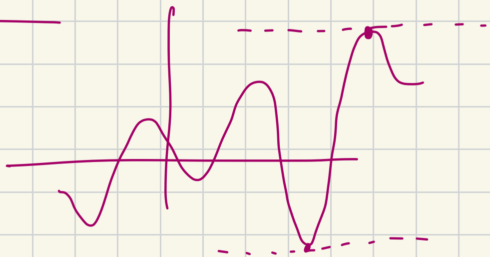
Let f be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x}) \neq \mathbf{0}$.

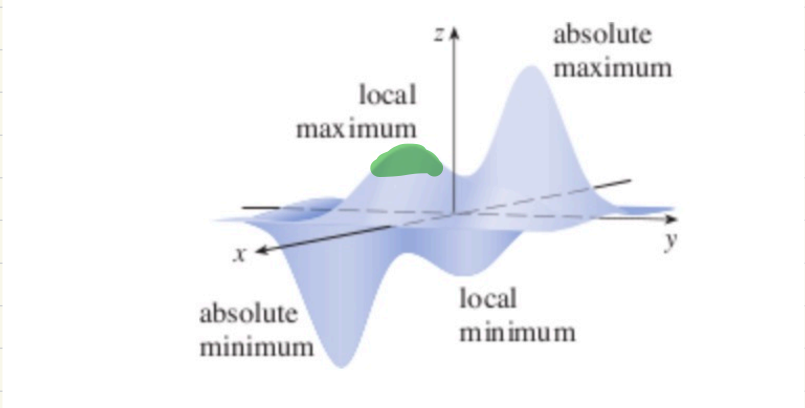
- The directional derivative of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_u f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.
- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that maximum rate of change is $|\nabla f(\mathbf{x})|$.
- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.



Section 14.7:

Recall:





local max or min has horizontal tangent line, that is, $f'(c) = 0$.

1 Definition

A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

2 Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Important concepts: Gradient, critical points, horizontal tangent plane

Remarks: • $f_x(a,b) = 0$, $f_y(a,b) = 0 \Rightarrow \nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle = \langle 0, 0\rangle = \vec{0}$.

• Points (a,b) where $f_x(a,b) = f_y(a,b) = 0$ are called critical points.

• Equation of the tangent plane at a critical point:

$(z = f(x,y))$ $\quad z - f(a,b) = \underbrace{f_x(a,b)}_{=0}(x-a) + \underbrace{f_y(a,b)}_{=0}(y-b) \Rightarrow z = z_0$

$\Rightarrow$ tangent plane is horizontal i.e parallel to the xy-plane.

Example 1

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$f_x = 2x - 2$

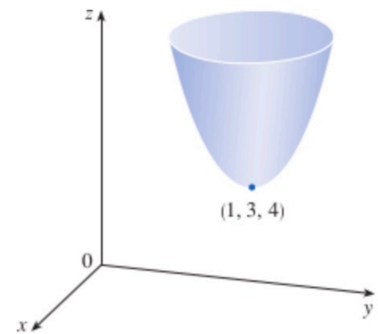
$f_y = 2y - 6$

$f_x = 0 \Rightarrow x = 1, \quad f_y = 0 \Rightarrow y = 3$

$\Rightarrow$ critical point is $(1, 3)$.

Without knowing the second derivative test: complete the square.

$z = x^2 + y^2 - 2x - 6y + 14$



$$z = x^2 + y^2 - 2x - 6y + 14 = 4 + \underbrace{(x-1)^2}_{\geq 0} + \underbrace{(y-3)^2}_{\geq 0}$$

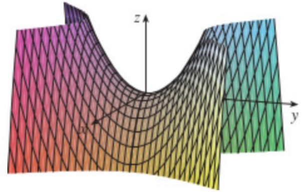
when $x \neq 1, y \neq 3$

Negative example: Critical point does not imply minima or maxima

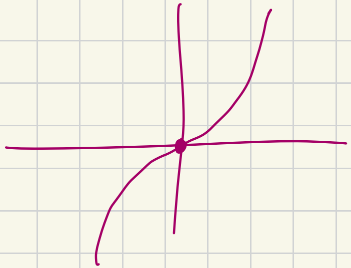
Example 2

Find the extreme values of $f(x, y) = y^2 - x^2$.

$z = y^2 - x^2$



Recall: $y = x^3 \Rightarrow y' = 0 \Leftrightarrow x = 0$ but $x = 0$ is an inflection point. That is



A: $f(x,y) = y^2 - x^2$, $f_x = -2x$, $f_y = 2y \Rightarrow$ critical pt is $(0,0)$.

$f(x,0) = -x^2$, $f(0,y) = y^2$ So $\lim\limits_{\substack{x \to \pm\infty \\ \text{along } x\text{-axis}}} f(x,y) = -\infty$, $\lim\limits_{\substack{y \to \pm\infty \\ \text{along } y\text{-axis}}} f(x,y) = +\infty$

3 Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2 = \det(\mathrm{Hess}(f)(a,b))$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is a saddle point of f.

(d) If $D = 0$, then test is inconclusive.

Def: Hessian of f at (a,b) is the matrix $\begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix} \leftarrow \mathrm{Hess}(f)(a,b)$

$\Rightarrow \det(\mathrm{Hess}(f)(a,b)) = f_{xx}(a,b) f_{yy}(a,b) - f_{yx}(a,b) f_{xy}(a,b)$ but CTS 2^{nd} order partials $\Rightarrow f_{xy} = f_{yx}$.

In 1-D: $f''(x) > 0 \Rightarrow$ local minimum $\cup$

$f''(x) < 0 \Rightarrow$ local maximum. $\cap$

Consider the following function.

$$f(x, y) = xy - 4x - 4y - x^2 - y^2$$

Q: Find local maxima / minima, if any.

$$\left.\begin{aligned} f_x &= y - 4 - 2x \\ f_y &= x - 4 - 2y \end{aligned}\right\} \Rightarrow \text{Solve for } (x,y) \text{ such that } \begin{aligned} y - 4 - 2x &= 0 \\ x - 4 - 2y &= 0 \end{aligned}$$

$$\rightarrow \begin{aligned} y - 4 - 2x &= 0 \\ 2x - 8 - 4y &= 0 \end{aligned}$$

$$-3y - 12 = 0 \rightarrow y = -4$$

$$\Rightarrow x = -4$$

$\Rightarrow$ $(-4,-4)$ is the only critical point.

$f_{xx} = -2 \qquad f_{xy} = 1 = f_{yx}$

$f_{yy} = -2$

$$\Rightarrow D(-4,-4) = \det \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = 4 - 1 = 3 > 0$$

and $f_{xx} = -2 < 0$ $\Rightarrow$ local maximum.

Example 3

Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

A: $f_x = 4x^3 - 4y \quad , \quad f_y = 4y^3 - 4x$

$$f_x = 0 \Rightarrow 4x^3 = 4y \Rightarrow x^3 = y \Leftrightarrow f_y = 4(x^9) - 4x = 0$$

$$\Leftrightarrow x(x^8 - 1) = 0$$

$$\Leftrightarrow x(x^4 - 1)(x^4 + 1) = 0$$

$$\Leftrightarrow x(x^2 - 1)\underbrace{(x^2 + 1)}_{\Rightarrow x = \pm i}\underbrace{(x^4 + 1)}_{\text{solutions are } \sqrt[4]{-1} \text{ which are not real}} = 0$$

Three critical points are $(0, 0), (1, 1), (-1, -1)$

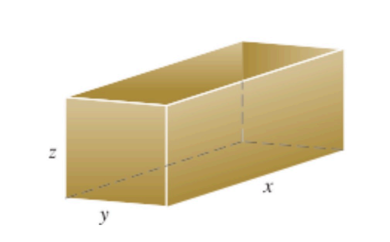
Figure out f_{xx}, f_{yy}, f_{xy} at these values and see 2^{nd} derivative test.

Example 5

Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Example 6

A rectangular box without a lid is to be made from $12\ \mathrm{m}^2$ of cardboard. Find the maximum volume of such a box.



$\boxed{7}$ Definition

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

Recall: Diff function on a closed interval $[a,b]$, check f-values where $f'(c)=0$ and $f(a)$ and $f(b)$

$\boxed{9}$ To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.