- If $\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$ where f, g, h are differentiable. Then

$$\boxed{\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = f'(t)\,\hat{i} + g'(t)\,\hat{j} + h'(t)\,\hat{k}} = \langle f'(t), g'(t), h'(t) \rangle$$

A direction for the equation of the tangent line at t.

- $\boxed{\text{Unit Tangent Vector.} \quad \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}$

- Suppose that $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ on the domain $[a, b]$ and f', g', h' are continuous on $[a, b]$. Then the arc length of the curve is

$$\boxed{L = \int_a^b \|\underbrace{r'(t)}_{\langle f', g', h' \rangle}\|\, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt}$$

- Suppose that a space curve C is given by $\vec{r} = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$ where $\vec{r}(t)$ is CTS on $[a, b]$ and C is traversed exactly once as t increases from a to b.

<u>Definition</u>: Under the above hypothesis, it's arc length function is defined by

$$\boxed{s(t) = \int_a^t \|r'(u)\| \, du}$$

- Consequence of Fundamental Theorem of Calculus:

$$\boxed{\frac{ds}{dt} = \| \vec{r}\,'(t) \|}$$

- Reparametrization:

<u>Goal</u>: Reparametrize the curve, say with parameter s, so that as s increases from a to b $\vec{r}(s)$ is a position vector of the point 's' units along the curve from its starting point.

<u>Algorithm:</u>

- Find $s(t)$.
- Solve for t as a function of s.
- $r(t(s))$ is the desired reparametrization.

<u>Sidenote:</u> In the example from previous class, we had $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$ for $t \geq 0$.

We then got $s(t) = \int_0^t \|r'(u)\|\,du = \sqrt{2}\,t \Rightarrow t(s) = \frac{s}{\sqrt{2}} \Rightarrow r(s) = \cos\left(\frac{s}{\sqrt{2}}\right)\hat{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\hat{j} + \frac{s}{\sqrt{2}}\hat{k}$.

We want $g(s) = \int_0^s \|r'(u)\|\,du = s$ where $r(u) = \cos\left(\frac{u}{\sqrt{2}}\right)\hat{i} + \sin\left(\frac{u}{\sqrt{2}}\right)\hat{j} + \frac{u}{\sqrt{2}}\hat{k}$.

$$\Rightarrow r'(u) = \frac{-1}{\sqrt{2}}\sin\left(\frac{u}{\sqrt{2}}\right)\hat{i} + \frac{1}{\sqrt{2}}\cos\left(\frac{u}{\sqrt{2}}\right)\hat{j} + \frac{1}{\sqrt{2}}\hat{k} \Rightarrow \|r'(u)\| = \sqrt{\frac{1}{2}\sin^2(u/\sqrt{2}) + \frac{1}{2}\cos^2(u/\sqrt{2}) + \frac{1}{2}} = 1$$

$$\Rightarrow g(s) = \int_0^s du = s - 0 = s \checkmark.$$

- Recall: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

- <u>Definition</u>: The curvature of a curve is

will be proved later.

kappa

$$\boxed{\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{\| d\vec{T}/dt \|}{\| ds/dt \|} = \frac{\| \vec{T}\,'(t) \|}{\| \vec{r}\,'(t) \|}} \quad \boxed{= \frac{\| \vec{r}\,'(t) \times r''(t) \|}{\| \vec{r}\,'(t) \|^3}}$$

<u>The Normal and Binormal vectors:</u>

(Assume in this section that $r(t)$ is a smooth space curve).

<u>Recall:</u> If $|r(t)| = c$. Then $r(t) \cdot r'(t) = 0$.

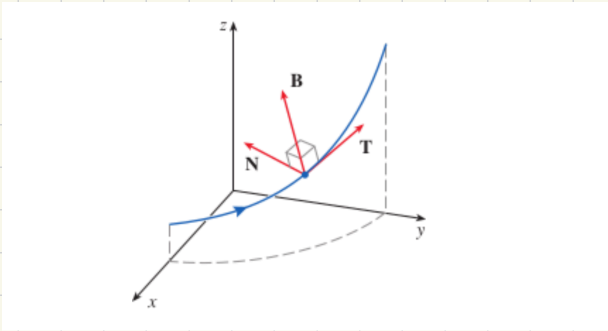
- So $|T(t)| = 1 \Rightarrow T(t) \cdot T'(t) = 0 \Rightarrow T'(t)$ is orthogonal to $T(t)$.

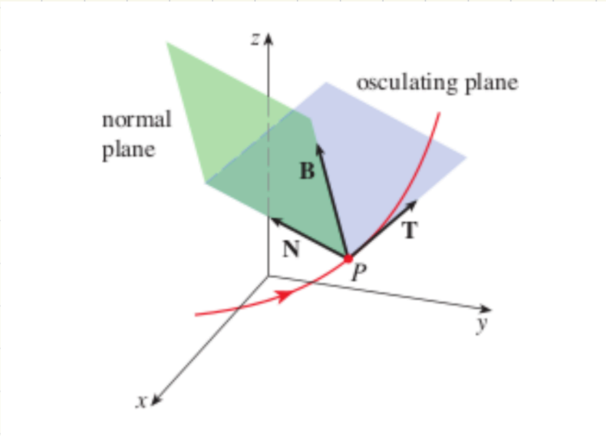
$$\boxed{\text{Principle Unit Normal Vector: } \vec{N}(t) = \frac{T'(t)}{|T'(t)|}}$$

(or just unit normal vector)

- Think of $\vec{N}(t)$ as the direction which the curve is turning at each point.

Binormal vector : $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$





Example 6

Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

A: $\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

$\|\vec{r}'(t)\| = \sqrt{2}$

$\Rightarrow \vec{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$

Recall: $\frac{d}{dt}\left(f(t)\cdot\vec{u}(t)\right) = f'(t)\cdot u(t) + f(t)\cdot u'(t)$

$\Rightarrow T'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle \quad \Rightarrow \|T'(t)\| = \sqrt{\left(\frac{-\cos t}{\sqrt{2}}\right)^2 + \left(\frac{-\sin t}{\sqrt{2}}\right)^2 + 0}$

$$= \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \frac{1}{\sqrt{2}} \Rightarrow \vec{N}(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\cancel{\frac{1}{\sqrt{2}}} \langle -\cos t, -\sin t, 0 \rangle}{\cancel{\frac{1}{\sqrt{2}}}}$$

$$= \langle -\cos t, -\sin t, 0 \rangle.$$

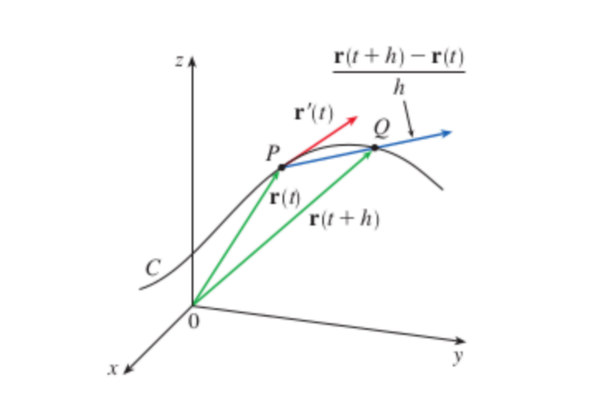
$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} i & j & k \\ \frac{-1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \hat{i} \begin{vmatrix} \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}} \\ -\sin t & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{-1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}} \\ -\cos t & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{-1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}}\cos t \\ -\cos t & -\sin t \end{vmatrix}$$

$$\boxed{\text{DIY}}$$

$$= \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$$

Section 13.4, Velocity, Speed, and Acceleration:

- Suppose that a particle moves through space so that its position vector is $\vec{r}(t)$ at time t.
- $\dfrac{r(t+h)-r(t)}{h}$ is the average velocity over a time interval of length h



- velocity vector: $\vec{v}(t) = \lim\limits_{h\to 0} \dfrac{r(t+h)-r(t)}{h} = r'(t)$
- speed: $\|\vec{v}(t)\| = \|\vec{r}'(t)\|$

Aside: $\underbrace{L = \int_a^b \|\vec{r}'(t)\|\,dt}_{\text{arc length}}$ is the distance travelled by the particle

- acceleration: $\vec{a}(t) = r''(t) = v'(t)$.