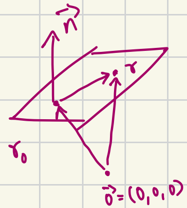
Question from class

Let P be a plane with point $P_0(x_0, y_0, z_0)$, and $\vec{n} = \langle a, b, c \rangle$.

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

then $P = \{ (x, y, z) \in \mathbb{R}^3 : \langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0 \}$

$$\Updownarrow$$



$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$$

$$\Leftrightarrow \quad ax + by + cz = ax_0 + by_0 + cz_0$$

That is $\langle a, b, c \rangle \cdot \langle x, y, z \rangle$ remains the same for each point on the plane.

So no matter which point is used to get the equation $ax + by + cz - d = 0$ the value d will be the same.

- Lecture quiz 2 syllabus: Sections 12.3 – 12.5. See discussion worksheets and Webassign for practice problems.

<u>Section 13.1:</u> Concepts from 1-D: limits, continuity, differentiation, integration.

<u>Definition:</u> A vector valued function has domain is a set of real numbers and whose range is a set of vectors.

- In 3-D: We are interested in functions of the form

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}.$$

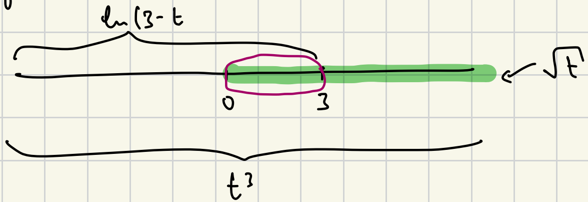
where f, g, h are real-valued functions.

<u>Example</u>. $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

$\Rightarrow$ domain of t^3 is $\mathbb{R}$

$\Rightarrow$ domain of $\ln(3-t)$ is $\{t : 3-t > 0\} = \{t : t < 3\} = (-\infty, 3)$

$\Rightarrow$ domain of $\sqrt{t}$ is $[0, +\infty)$



$\Rightarrow$ domain of $\vec{r}(t)$ is $[0, 3]$.

<u>Definition</u>: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ then

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

provided that each limit exists.

Example 2: Find $\lim_{t \to 0} r(t)$ where $r(t) = (1+t^3)\hat{i} + te^{-t}\hat{j} + \frac{\sin t}{t}\hat{k}$.

A: $\lim_{t \to 0} r(t) = \lim_{t \to 0} (1+t^3)\hat{i} + \lim_{t \to 0} te^{-t}\hat{j} + \lim_{t \to 0} \frac{\sin t}{t}\hat{k}$

$= 1 \cdot \hat{i} + (0 \cdot 1)\hat{j} + \lim_{t \to 0} \frac{\cos t}{1}\hat{k}$

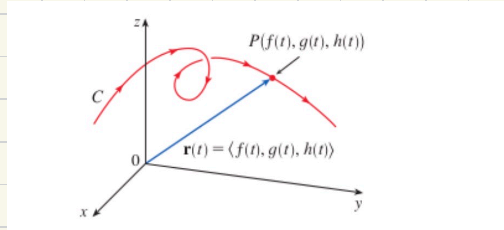
$= \hat{i} + \hat{k}$

<u>Definition</u>: A vector valued function is continuous if

$$\lim_{t \to a} r(t) = r(a)$$

This is equivalent to each component being continuous.

<u>Space Curves</u>: Suppose f, g, h are continuous on an interval I. Then the set C of all points (x, y, z) in space, $x = f(t)$, $y = g(t)$, $z = h(t)$ as t varies in I is called a space curve.



Example 3

Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle$$

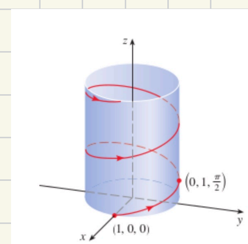
A: $r(t) = \langle 1, 2, -1 \rangle + t \langle 1, 5, 6 \rangle$

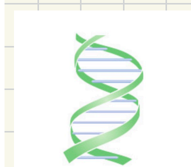
Take $C = \{ (x, y, z) : x = 1 + t,\ y = 2 + 5t,\ z = -1 + 6t,$
$t \in I = (-\infty, +\infty) \}$

Example 4

Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \overset{x}{\cos t}\,\mathbf{i} + \overset{y}{\sin t}\,\mathbf{j} + \overset{z}{t}\,\mathbf{k}$$





A: $x = \cos t$, $y = \sin t \Rightarrow x^2 + y^2 = 1$

For example: take $I = [0, 2\pi]$

Then $(0, 0, 0)$, $t = \frac{\pi}{3} \Rightarrow \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$. Space curve by $r(t)$ is called a helix.

Section 13.2: Differention and Integration.

2 Theorem

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

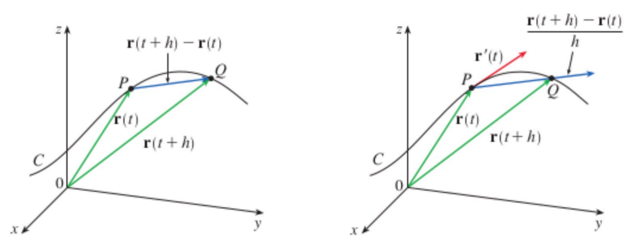
Example 1

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

A: (a) $r'(t) = 3t^2\,\hat{i} + (1-t)e^{-t}\,\hat{j} + 2\cos 2t\,\hat{k}$

Unit tangent vector:



(a) The secant vector $\overrightarrow{PQ}$ (b) The tangent vector $\mathbf{r}'(t)$

Def: $\cdot\ r'(t) = \lim_{h \to 0} \dfrac{r(t+h) - r(t)}{h}$ ← is a vector when the limit exists and is called the tangent vector at t.

· Unit tangent vector is the unit vector in the direction of $r(t)$.

$$T(t) = \frac{1}{\|r'(t)\|} \cdot r'(t)$$

Example problem (b) : $r'(t) = 3t^2\,\hat{i} + (1-t)e^{-t}\,\hat{j} + 2\cos 2t\,\hat{k}$

Question: Find $T(0)$. Answer: $T(0) = \dfrac{1}{\|r'(0)\|} \cdot r'(0)$

where $r'(0) = 0 \cdot \hat{i} + 1 \cdot \hat{j} + 2 \cdot 1\,\hat{k} = \hat{j} + 2\hat{k}$

and $\|r'(0)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

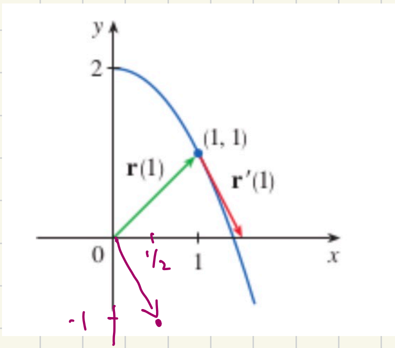
$$\Rightarrow T'(0) = \frac{1}{\sqrt{5}}\hat{j} + \frac{2}{\sqrt{5}}\hat{k}$$

Example 2

For the curve $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2-t)\,\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

A: $r'(t) = \dfrac{1}{2\sqrt{t}}\hat{i} - \hat{j}$

Note: $x = \sqrt{t}$, $y = 2-t$. Solve for x in term of y or y in terms of x: $x^2 = t \Rightarrow y = 2 - x^2$.



$r(1) = \sqrt{1}\,\hat{i} + (2-1)\hat{j} = \hat{i} + \hat{j}$

$= \langle 1, 1 \rangle$

$r'(1) = \dfrac{1}{2}\hat{i} - \hat{j} = \langle \tfrac{1}{2}, -1 \rangle$

Example 3

Find parametric equations for the tangent line to the helix with parametric equations

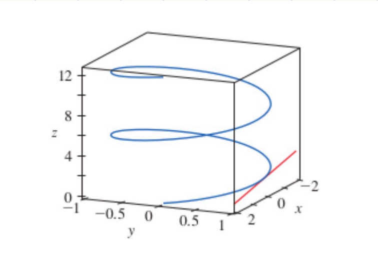
$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

A: For the equation of a line you need a point and a direction vector.

$\downarrow$

tangent vector!



$$r(t) = 2\cos t\ \hat{i} + \sin t\ \hat{j} + t\ \hat{k}$$

$$r'(t) = -2\sin t\ \hat{i} + \cos t\ \hat{j} + \hat{k}$$

we also need t value for $(0, 1, \pi/2)$ but $z = t$ so $t = \pi/2$!

$$\Rightarrow r'(\pi/2) = -2\sin\left(\tfrac{\pi}{2}\right)\hat{i} + \cos\left(\tfrac{\pi}{2}\right)\hat{j} + \hat{k}$$

$$= -2\hat{i} + 0\cdot\hat{j} + \hat{k} = -2\hat{i} + \hat{k}$$

$\Rightarrow$ Equation of tangent line is $r(s) = \langle 0, 1, \tfrac{\pi}{2} \rangle + s\, r'(\pi/2)$

$$\Rightarrow \quad x = 0 + (-2)s = -2s$$
$$y = 1 + (0)\cdot s = 1$$
$$z = \frac{\pi}{2} + (1)\cdot s = \frac{\pi}{2} + s.$$

<u>Definition:</u>
$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

3 Theorem

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

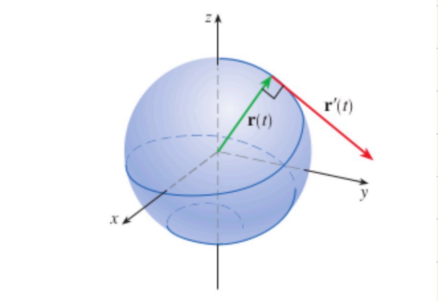
1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

$\Big\}$ linearity of $\frac{d}{dt}$

3. $\frac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$ (Chain Rule)

4 Theorem

If $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

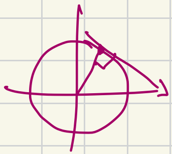


<u>pf:</u> $\|r(t)\| = c \Rightarrow \sqrt{r(t)\cdot r(t)} = c$

$$\Rightarrow r(t)\cdot r(t) = c^2 \Rightarrow \frac{d}{dt}\big(r(t)\cdot r(t)\big) = 0$$

$$\Rightarrow \quad r'(t)\cdot r(t) + r(t)\cdot r'(t) = 0$$

$$\Rightarrow \quad 2\, r'(t)\cdot r(t) = 0 \quad \Rightarrow r'(t)\cdot r(t) = 0$$



Section 13.1,

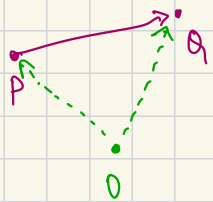
Example 5

Find a vector equation and parametric equations for the line segment that joins the point $P\,(1, 3, -2)$ to the point $Q\,(2, -1, 3)$.

A line segment between points P and Q is the following set of points : $\langle x,y,z\rangle = \overrightarrow{OP} + t\,\overrightarrow{PQ}$ where $t \in [0,1]$

at $t=0$, $\langle x,y,z\rangle = \overrightarrow{OP}$

at $t=1$, $\langle x,y,z\rangle = \overrightarrow{OP} + \overrightarrow{PQ}$
$= \overrightarrow{OQ}$



<u>Answer:</u> Line segment is

$$C = \{ (x, y, z) : \langle x, y, z\rangle = \langle 1, 3, -2\rangle + t\langle 1, -4, 5\rangle \text{ where } t \in [0,1] \}$$

i.e. $x = 1 + t$, $y = 3 - 4t$, $z = -2 + 5t$, $I = [0,1]$.