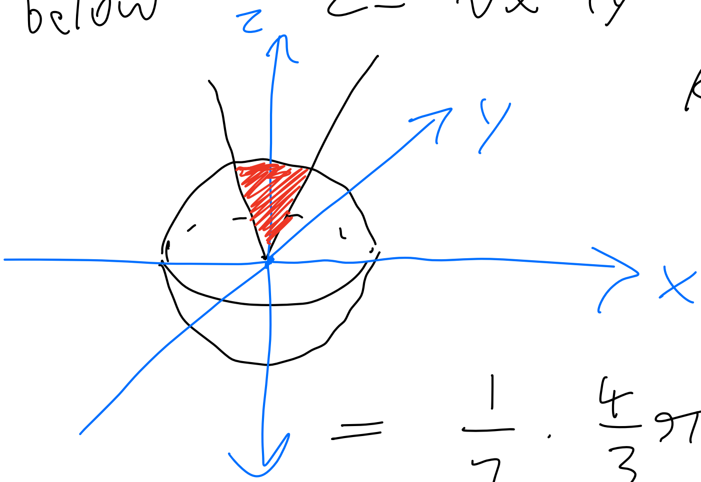
2. Volume of solid within $x^2+y^2+z^2=4$, above xy-plane, below $z=\sqrt{x^2+y^2}$



Region of interest is upper hemisphere minus the red region.

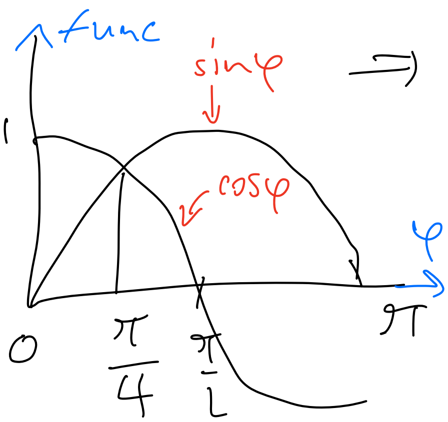
So $V = \text{vol}(\text{hemi}) - \text{vol}(\text{red})$

$= \frac{1}{2}\cdot\frac{4}{3}\pi\cdot 2^3 - \text{vol}(\text{red}) = \frac{16\pi}{3} - \text{vol}(\text{red})$.

$x=\rho\sin\varphi\cos\theta,\quad y=\rho\sin\varphi\sin\theta,\quad z=\rho\cos\varphi$

$4 = x^2+y^2+z^2=\rho^2 \Rightarrow \rho=2$, so inside of sphere is $0\le\rho\le 2$. Above cone $\Rightarrow z\ge\sqrt{x^2+y^2}$

$\Rightarrow \rho\cos\varphi \ge \sqrt{\rho^2\sin^2\varphi}=\rho\sin\varphi \Rightarrow \cos\varphi\ge\sin\varphi$



$\Rightarrow 0\le\varphi\le\frac{\pi}{4}$, no constraints on $\theta \Rightarrow 0\le\theta\le 2\pi$.

$$\text{vol}(\text{red}) = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

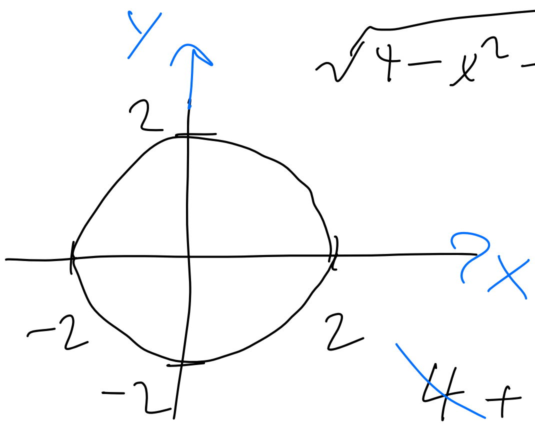
46: Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$

by switching to spherical.

$x = \rho\sin\varphi\cos\theta$,
$y = \rho\sin\varphi\sin\theta$,
$z = \rho\cos\varphi$.
Then $x^2+y^2+z^2=\rho^2$,
$dz\,dy\,dx = dV =$
$\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$,
and we need to find bounds on ρ, θ, φ.

Note: $z=\rho\cos\varphi$, $x=\cdots$, $y=\cdots$ won't always be the substitution to use for spherical. Much like cylindrical saw $x=x$ or $y=y$ sometimes, you may use $x=\rho\cos\varphi$, $y=\rho\cos\varphi$, or something similar depending on what sum of squares arise.

The bounds are $-2\le x\le 2$, $-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}$,
$2-\sqrt{4-x^2-y^2}\le z\le 2+\sqrt{4-x^2-y^2}$.



$\sqrt{4-x^2-y^2} = \sqrt{4-\rho^2\sin^2\varphi}$, so

$$\rho\cos\varphi \le 2+\sqrt{4-\rho^2\sin^2\varphi}$$
$$\rho\cos\varphi - 2 \le \sqrt{4-\rho^2\sin^2\varphi}$$
$$\cancel{4}+\rho^2\cos^2\varphi - 4\rho\cos\varphi \le \cancel{4}-\rho^2\sin^2\varphi$$
$$\Longrightarrow \rho^2\le 4\rho\cos\varphi \Rightarrow \rho\le 4\cos\varphi.$$

Upper bound: $z-2\le\sqrt{\cdots}$, lower: $2-z\le\sqrt{\cdots}$

Combine upper & lower bounds: $|z-2|\le\sqrt{\cdots}$, which is equivalent to $(z-2)^2 \leq \cdots$, so when squaring, we have taken care of both bounds.

So bounds on ρ are $0 \leq \rho \leq 4\cos\varphi$. Then

$4\cos\varphi \geq 0 \Rightarrow \cos\varphi \geq 0 \Rightarrow 0 \leq \varphi \leq \frac{\pi}{2}$. No restriction on θ anywhere, so $0 \leq \theta \leq 2\pi$. So

$(x^2+y^2+z^2)^{3/2}\, dz\,dy\,dx = \rho^3 \rho^2 \sin\varphi \cdots = \rho^5 \sin\varphi \cdots$,

so new integral is $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos\varphi} \rho^5 \sin\varphi \, d\rho\, d\varphi\, d\theta$

$$= 2\pi \int_0^{\pi/2} \sin\varphi \left. \frac{\rho^6}{6} \right|_{\rho=0}^{\rho=4\cos\varphi} d\varphi = \frac{4^6\pi}{3}\int_0^{\pi/2} s\varphi\, c^6\varphi\, d\varphi$$

$$= \frac{2^{12}\pi}{3} \cdot \left. -\frac{\cos^7\varphi}{7} \right|_0^{\pi/2} = \frac{4096\pi}{3}\cdot\frac{1}{7} = \frac{4096\pi}{21}$$

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7. Find $\displaystyle\int_C \frac{x}{y}\, ds$ where $C = (t^3, t^4)$, $1 \leq t \leq 2$.

Recall $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(3t^2)^2 + (4t^3)^2}\, dt$

$= t^2\sqrt{9+16t^2}\, dt$, $\quad \frac{x}{y} = \frac{t^3}{t^4} = \frac{1}{t} \Rightarrow$

$$\int_C \frac{x}{y}\,ds = \int_1^2 t(9+16t^2)^{1/2}\,dt = \frac{1}{48}(9+16t^2)^{3/2}\Big|_1^2$$

$$32t \cdot \frac{3}{2} = 48t \qquad = \frac{1}{48}\left(73^{3/2} - 25^{3/2}\right) = \frac{73\sqrt{73}-125}{48}$$

Note: Many problems in the spherical coordinates unit, and hence problems on LQ4 + the final, are solvable with only cylindrical because the regions are not complicated enough & only involve a $x^2+y^2+z^2$ term once. If you recognize this, you can try them with cylindrical first because for any problem that can be done with both, cylindrical will be faster because z & r are less annoying than φ and the dV for cylindrical is simpler. Just make sure the problem doesn't say you must use spherical. Also, make sure you've still mastered spherical in case cylindrical doesn't work out & your guess on feasibility was wrong.

4a: Find $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}(x^2z+y^2z+z^3)\,dz\,dx\,dy$

by switching to spherical coordinates.

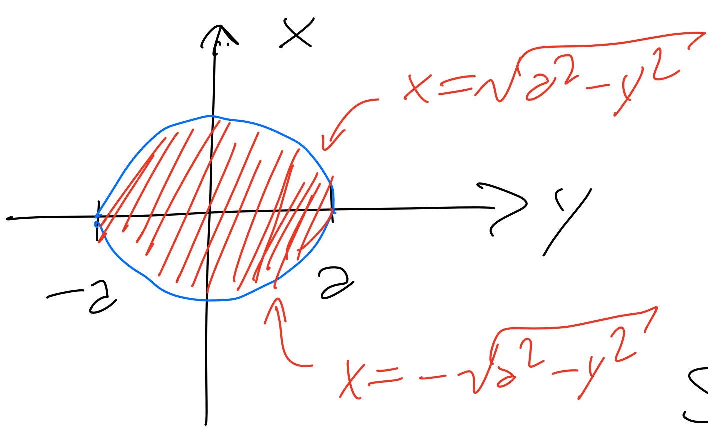
z is odd one out in innermost square root $\sqrt{a^2-x^2-y^2}$, so take $z=\rho\cos\varphi$, $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$. We need to convert integrand & find new bounds.

$x^2z+y^2z+z^3=z(x^2+y^2+z^2)=z\rho^2=\underline{\rho^3\cos\varphi}$.

$dz\,dx\,dy=dV=\underline{\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta}$.

To find bounds, same trick as cylindrical for x&y. Take x&y bounds & graph the resulting region.

Graph $-a\leq y\leq a$, $-\sqrt{a^2-y^2}\leq x\leq\sqrt{a^2-y^2}$



Bound for z becomes $|z|\leq\sqrt{a^2-x^2-y^2}$ by combining lower & upper.

Since we only have 1 inequality, both sides non-negative, we can square without worrying about introducing extraneous solutions.

$z^2 \leq a^2 - x^2 - y^2 \Rightarrow a^2 \geq x^2 + y^2 + z^2 = \rho^2$

$\Rightarrow \rho \leq a$ (Note: we assume $a \geq 0$ because the outer integral is $\int_{-a}^{a}$ and the problem would be janky otherwise. If you see a scenario like this on an exam, ask the instructor for clarification).

So $0 \leq \rho \leq a$. We never found any restrictions on θ & φ. The xy-graph illustrates this more since x&y can be anywhere within a great circle of the ball with radius a. So $0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$.

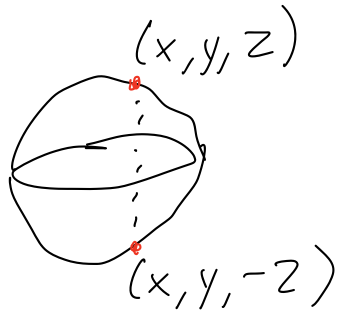
New integral is $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^5 \sin\varphi \cos\varphi \, d\rho \, d\varphi \, d\theta$

$$= 2\pi\left(\int_0^a \rho^5 d\rho\right)\left(\int_0^{\pi} \frac{1}{2}\sin 2\varphi\right) = 2\pi \cdot \frac{a^6}{6} \cdot 0 = 0.$$

However, there is a faster way. To see that the integral is 0, let $f = x^2z + y^2z + z^3 = z(x^2+y^2+z^2)$.

Let S^+, S^- be top ½ / bottom ½ of the sphere respectively. Then

$$\int_S f\,dV = \int_{S^+} f\,dV + \int_{S^-} f\,dV = \int_{S^+} f\,dV - \left(\int_{S^-} (-f)\,dV\right).$$



Now note that

$$f(x,y,-z) = -2(x^2+y^2+z^2) = -f,$$

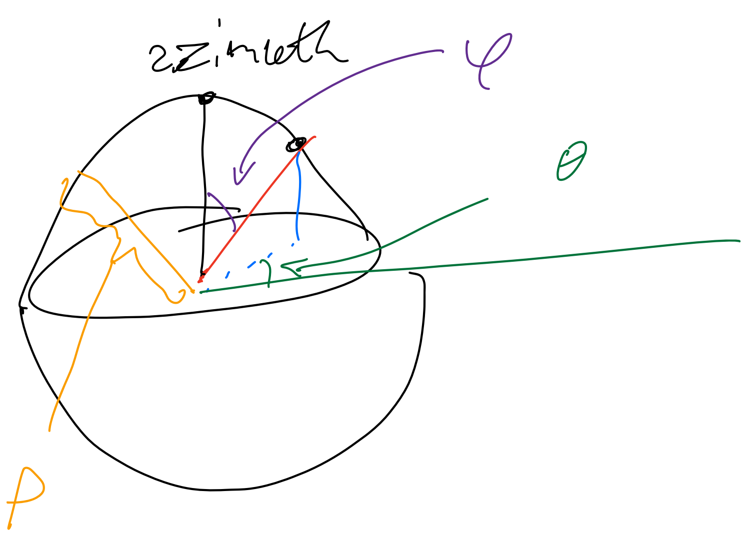
so

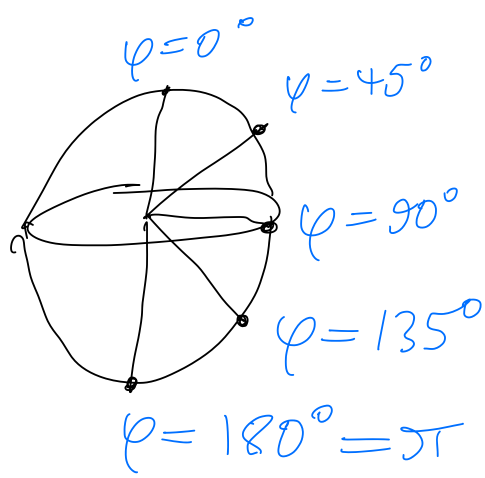
$$\int_{S^-} \left(-f(x,y,z)\,dV\right) = \int_{S^-} f(x,y,-z)\,dV$$

$$= \int_{S^+} f(x,y,z)\,dV$$ by change of variables,

so

$$\int_S f\,dV = \int_{S^+} f\,dV - \int_{S^+} f\,dV = 0.$$





So $\varphi \in [0, \pi]$, not $0 \to 2\pi$.

8: $y\,dx + z\,dy + x\,dz = \left(y\frac{dx}{dt} + z\frac{dy}{dt} + x\frac{dz}{dt}\right)dt$

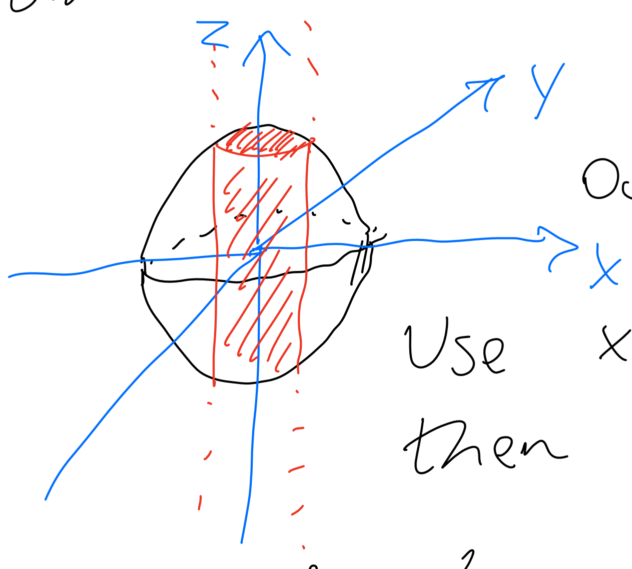
$= \left(t \cdot \frac{1}{2\sqrt{t}} + t^2 \cdot 1 + \sqrt{t} \cdot 2t\right)dt =$

$\left(0.5t^{-0.5} + t^2 + 2t^{3/2}\right)dt$

9: $F \cdot dr = F(r(t)) \cdot \frac{dr}{dt}\,dt =$

$F(t^3, t^2) \cdot \langle 3t^2, 2t \rangle\, dt$

PLQ4 #7: Volume inside $x^2+y^2+z^2=16$ but outside $x^2+y^2=4$ using cylindrical.



Inside sphere $\Rightarrow x^2+y^2+z^2 \le 16$

Outside cylinder $\Rightarrow x^2+y^2 \ge 4$

Use $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

then $4 \le x^2+y^2 = r^2 \Rightarrow 0 \le r \le 2$

$16 \ge z^2 + r^2 \Rightarrow z^2 \le 16 - r^2 \Rightarrow$

$|z| \le \sqrt{16-r^2} \Rightarrow -\sqrt{16-r^2} \le z \le \sqrt{16-r^2}$

No θ restrictions, so $0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{2}\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta$$

Now we do it with spherical as intended

$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$

$16 \geq x^2+y^2+z^2 = \rho^2 \implies \rho \leq 4$

always remember this

$4 \leq x^2+y^2 = \rho^2\sin^2\varphi = (\rho\sin\varphi)^2 \implies$

$\rho\sin\varphi \geq 2$ or $\rho\sin\varphi \leq -2$, so at first glance it looks like we will need 2 separate intervals and hence 2 separate integrals. But notice that $\sin\varphi \geq 0$ since $\varphi \in [0,\pi]$ & $\rho \geq 0$, so $\rho\sin\varphi \geq 0 \implies$ only $\rho\sin\varphi \geq 2$.

So $\dfrac{2}{\sin\varphi} \leq \rho \leq 4$. But it is possible for $2\csc\varphi > 4$ to happen since $\csc\varphi \to \infty$ as $\varphi \to 0$. So we need to ensure that $4 \geq \dfrac{2}{\sin\varphi} \implies \sin\varphi \geq \dfrac{1}{2}$

$\Longrightarrow \dfrac{\pi}{6} \leq \varphi \leq \dfrac{5\pi}{6}$. No restriction on

θ, so $0 \leq \theta \leq 2\pi$. Also, $dV = \rho^2 \sin\varphi \cdots$

So $$V = \int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{\frac{2}{\sin\varphi}}^{4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$