

Try each of the following problems, show the detail of your work. Partial credit will be given. No credit for unsupported results.

Cellphones, calculators, computers and any other electronic devices are prohibited.

Quiz time: 30 minutes.

1. (1 point) Which of the following paths is NOT an appropriate path to use for showing that $\lim_{(x,y) \rightarrow (0,0)} \frac{6x^2y^2}{3x^4 + y^4}$ does not exist.

- A. $y = \frac{1}{2}x$
- B. $x = 0$
- C. $y = x + 5$
- D. $y = 0$
- E. $y^2 = x$

C Because the line $y = x + 5$ does NOT pass through the point $(0, 0)$.

2. (3 points) Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + 6xy - 2y^2}{2x^2 + 4y^2}$ does not exist. Justify your answer.

$$f(x, y) = \frac{x^2 + 6xy - 2y^2}{2x^2 + 4y^2}. \text{ Let } C_1 \text{ be the } x\text{-axis: } y = 0$$

$$f(x, 0) = \frac{x^2}{2x^2} = \frac{1}{2}$$

So, on C_1 ,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + 6xy - 2y^2}{2x^2 + 4y^2} = \lim_{x \rightarrow 0} f(x, 0) = \frac{1}{2}$$

Let C_2 be the line y -axis, $x = 0$

$$f(0, y) = \frac{x^2 + 6xy - 2y^2}{2x^2 + 4y^2} = \frac{-2y^2}{4y^2} = -\frac{1}{2}$$

So, on C_2 ,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + 6xy - 2y^2}{2x^2 + 4y^2} = \lim_{y \rightarrow 0} f(0, y) = -\frac{1}{2}$$

On two different paths $C_1 \neq C_2$ that pass through the point $(0, 0)$, $f(x, y)$ approaches two different numbers, $\frac{1}{2}$ and $-\frac{1}{2}$, therefore, the limit does NOT exist.

3. Given the function $f(x, y) = x^2y^3$

- (a) (2 points) Find an equation of the tangent plane to the given surface $z = x^2y^3$ at the point $(2, 1, 4)$.

$$\begin{aligned}f_x(x, y) &= 2xy^3 \rightarrow f_x(2, 1) = 4 \\f_y(x, y) &= 3x^2y^2 \rightarrow f_y(2, 1) = 12\end{aligned}$$

Equation of tangent plane:

$$\begin{aligned}z - 4 &= 4(x - 2) + 12(y - 1) \\z &= 4x + 12y - 16\end{aligned}$$

- (b) (2 points) Use your answer from part (a) to estimate $f(2.1, 0.9)$.

$$\begin{aligned}L(x, y) &= 4x + 12y - 16 \\f(2.1, 0.9) &\approx 4(2.1) + 12(0.9) - 16 = 8.4 + 10.8 - 16 = 3.2\end{aligned}$$

4. (2 points) Given that $x^3 + y^2z - 3xyz = 4xy$, use implicit differentiation to find $\frac{\partial z}{\partial x}$.

$$\begin{aligned}F(x, y, z) &= x^3 + y^2z - 3xyz - 4xy \\F_x &= \frac{\partial F}{\partial x} = 3x^2 - 3yz - 4y \\F_z &= \frac{\partial F}{\partial z} = y^2 - 3xy \\\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{3x^2 - 3yz - 4y}{y^2 - 3xy}\end{aligned}$$

5. If $z = z(x, y, t) = yx^2 e^{t^2}$, where $x = x(u, w) = w \sin u$, $y = y(u, w) = u \tan(w)$ and $t = t(u, w) = u \ln(w)$, find the following.

(a) (1 point) $\frac{\partial y}{\partial u}$

$$\frac{\partial y}{\partial u} = \tan w$$

(b) (1 point) $\frac{\partial z}{\partial t}$

$$\frac{\partial z}{\partial t} = yx^2 e^{t^2} (2t) = 2tyx^2 e^{t^2}$$

(c) (2 points) $\frac{\partial z}{\partial u}$

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial u} \\ &= 2xye^{t^2} (w \cos u) + x^2 e^{t^2} (\tan w) + 2tyx^2 e^{t^2} (\ln w) \\ &= 2(w \sin u)(u \tan w)e^{(u \ln(w))^2} (w \cos u) + (w \sin u)^2 e^{(u \ln(w))^2} (\tan w) \\ &\quad + 2(u \ln w)(u \tan w)(w \sin u)^2 e^{(u \ln(w))^2} (\ln w)\end{aligned}$$

(d) (2 points) $\frac{\partial^2 z}{\partial y \partial x}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (2xye^{t^2}) = 2xe^{t^2}$$

6. Given a function $f(x, y) = x^2 \ln(y)$

- (a) (2 points) Find the gradient of f .

$$\nabla f = \left\langle 2x \ln y, \frac{x^2}{y} \right\rangle$$

- (b) (1 point) Evaluate the gradient at the point $P(3, e^2)$.

$$\nabla f(3, e) = \left\langle 2(3) \ln(e^2), \frac{3^2}{e^2} \right\rangle = \left\langle 12, \frac{9}{e^2} \right\rangle$$

- (c) (1 point) Find the rate of change of f at P in the direction of the unit vector $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$.

$$D_{\mathbf{u}}f(3, e^2) = \nabla f(3, e^2) \cdot \mathbf{u} = \left\langle 12, \frac{9}{e^2} \right\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle = -\frac{36}{5} + \frac{36}{5e^2} = \frac{-36e + 36}{5e^2}$$