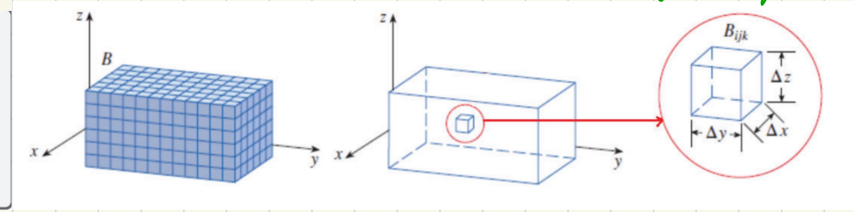
# Section 15.6: Triple Integrals.

Volume of a box in 4-D: $\underbrace{f(x^*, y^*, z^*) \cdot \Delta x \cdot \Delta y \cdot \Delta z}_{\text{product of 4 lengths}}$

**Definition** The **triple integral** of $f$ over the box $B$ is

$$\iiint_B f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta V$$

if this limit exists.




**Fubini's Theorem for Triple Integrals** If $f$ is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

**Example:** Evaluate $\underbrace{\iiint_B x\cos(y + z)\, dV}_{= I}$ where $B = \underset{x}{[0, 1]} \times \underset{y}{[0, 2]} \times \underset{z}{[0, 3]}$.

$$I = \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} x\cos(y+z)\, dz\, dy\, dx = \int_0^1 \int_0^2 \left(x\sin(y+3) - x\sin y\right) dy\, dx$$

Let $I_1 = \int_0^3 x\cos(y+z)\,dz = x\sin(y+z)\Big|_{z=0}^{3} = x\left(\sin(y+3) - \sin(y+0)\right)$

Let $I_2 = \int_0^2 \left(x\sin(y+3) - x\sin y\right) dy$

$= x\int_0^2 \sin(y+3)\,dy - x\int_0^2 \sin y\,dy = -x\cos(y+3)\Big|_0^2 + x\cos y\Big|_0^2$

$= -x\left(\cos(5) - \cos(3)\right) + x\left(\cos(2) - \cos(0)\right)$

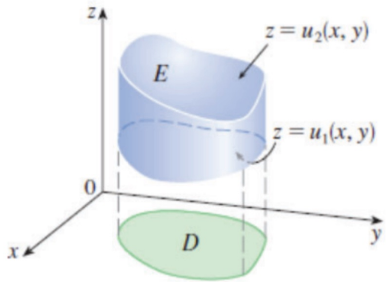
$I = \int_0^1 -x\cos(5) + x\cos(3) + x\cos(2) - x\,dx = \left(-\cos(5) + \cos(3) + \cos(2) - 1\right)\int_0^1 x\,dx$

$= \left(-\cos(5) + \cos(3) + \cos(2) - 1\right)\frac{1}{2}$

A solid region $E$ is said to be of **type 1** if it lies between the graphs of two continuous functions of $x$ and $y$, that is,

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

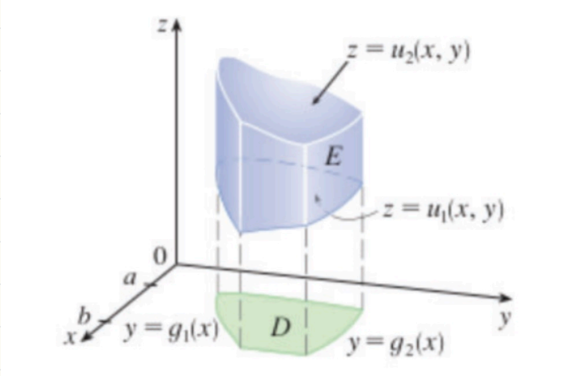
where $D$ is the projection of $E$ onto the $xy$-plane.

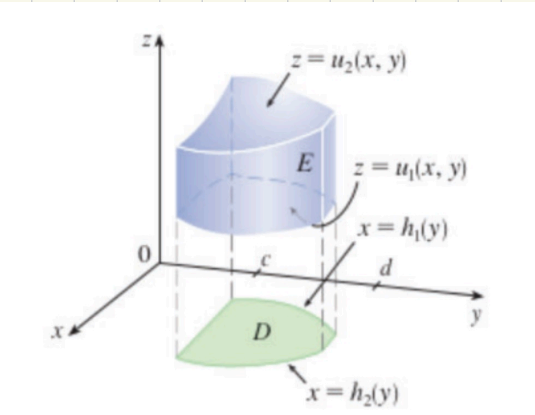



- Upper boundary of the solid $E$ is the surface $z = u_2(x, y)$
- Lower boundary of the solid $E$ is the surface $z = u_1(x, y)$.

If $E$ is a type 1 region, then:

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$$



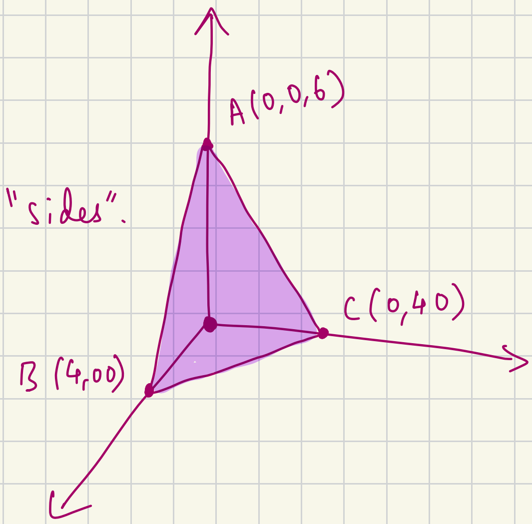



**Example:** Let $T$ be the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$ and $C(0,4,0)$.
(Note that the plane containing the points $A$, $B$ and $C$ has the equation $3x+3y+2z=12$)

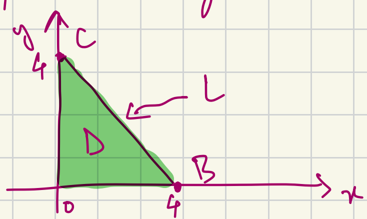
(a) Express $T$ as a solid region type 1.

(b) Express $\iiint_T f(x,y,z)\,dV$ as an iterated integral.

- Faces of the tetrahedron are parts of the following planes: $3x+3y+2z=12$ (in pink), $xy$ plane (base), $xz$ & $yz$ planes are the "sides".




- Project the surface $3x+3y+2z=12$ onto the $xy$-plane to get the following region:



- $D$ is the region in the $xy$ plane bounded by the lines $x=0$, $y=0$, and $L$.
- $L$ is the line between $\underbrace{B(4,0,0) \text{ and } C(0,4,0)}$

think of these as $B(4,0)$ and $C(0,4)$

$\Rightarrow$ $L$ has slope $\frac{0-4}{4-0} = -1$ and $y$-intercept $4$ $\Rightarrow$ $y = -x+4$

OR $x+y=4$

OR $x = -y+4$

- $E = \{(x,y,z) : 0 \le x \le 4,\ 0 \le y \le -x+4,\ 0 \le z \le 6 - \frac{3}{2}x - \frac{3}{2}y\}$

$3x+3y+2z=12 \Rightarrow z = 6 - \frac{3}{2}x - \frac{3}{2}y$

$= \{(x,y,z) : 0 \le y \le 4,\ 0 \le x \le -y+4,\ 0 \le z \le 6 - \frac{3}{2}x - \frac{3}{2}y\}$

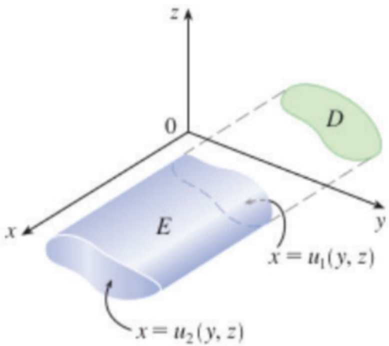
(b) $\iiint_E f(x,y,z)\,dV = \int_0^4 \int_0^{-x+4} \int_0^{6-\frac{3}{2}x-\frac{3}{2}y} f(x,y,z)\,dz\,dy\,dx$

$$= \int_0^4 \int_0^{-y+4} \int_0^{6-\frac{3}{2}x-\frac{3}{2}y} f(x,y,z)\, dz\, dx\, dy$$

A solid region $E$ is of **type 2** if it is of the form

$$E = \{(x,y,z) \mid (y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}$$

where $D$ is the projection of $E$ onto the $yz$-plane.




- The back surface is $x = u_1(y,z)$.
- The front surface is $x = u_2(y,z)$

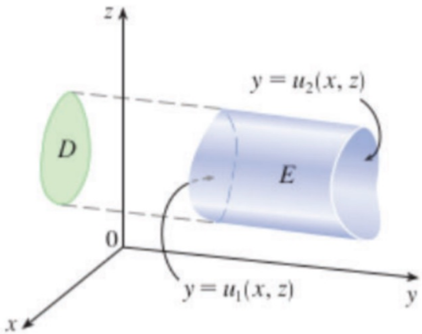
Then,

$$\boxed{\iiint_E f(x,y,z)\, dV = \iint_D \left[ \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA}$$

A solid region $E$ is of **type 3** if it is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where $D$ is the projection of $E$ onto the $xz$-plane.




- The left surface is $y = u_1(x, z)$.
- The right surface is $y = u_2(x, z)$

Then,

$$\boxed{\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\right] dA}$$

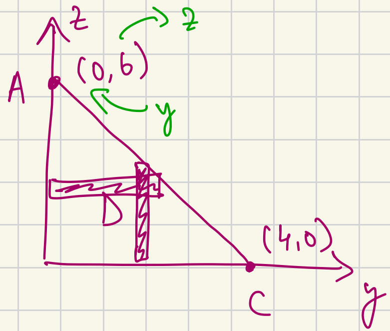
**Note:** In each of these equations, there may be two possible expressions for the integral depending on whether $D$ is a type I or a type II plane region.

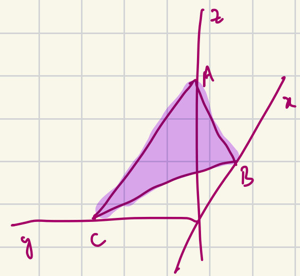
**Example:** Let $T$ be the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$ and $C(0,4,0)$.
(Note that the plane containing the points $A, B$ and $C$ has the equation $3x+3y+2z=12$)

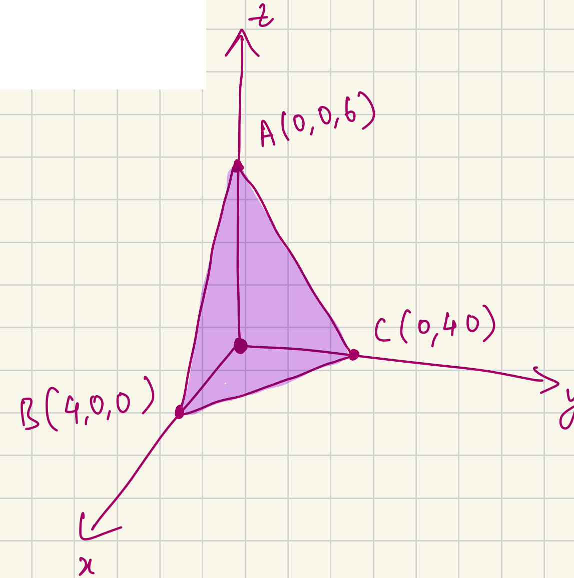
(a) Express $T$ as a solid region type 2 and type 3.

(b) Express $\displaystyle\iiint_T f(x,y,z)\,dV$ as an iterated integral.

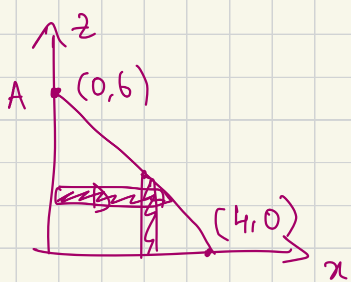
(a) Type 2:

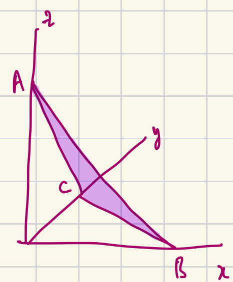



Type 3:



**Example:** Evaluate $\overbrace{\iiint_T e^z \, dV}^{I}$ where $T$ is the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$ and $C(0,4,0)$.

$$T = \left\{ (x,y,z) : 0 \le x \le 4 , \; 0 \le y \le -x+4 , \; 0 \le z \le 6 - \frac{3}{2}x - \frac{3}{2}y \right\}$$

$$I = \int_{x=0}^{4} \int_{y=0}^{4-x} \int_{z=0}^{6-\frac{3}{2}x-\frac{3}{2}y} e^z \, dz \, dy \, dx$$

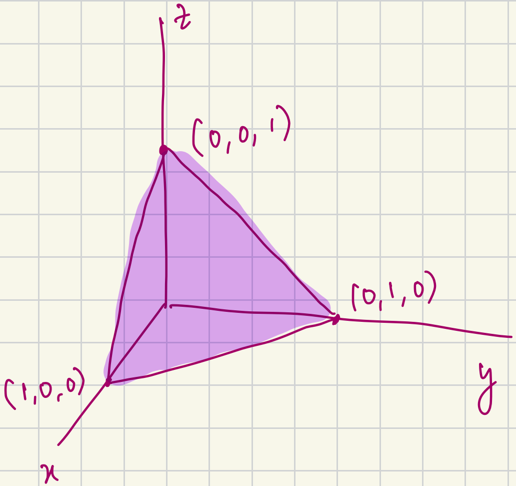
$$= \int_0^4 \int_0^{4-x} e^z \Big|_{z=0}^{6-\frac{3}{2}x-\frac{3}{2}y} dy \, dx = \int_0^4 \int_0^{4-x} \left( e^{6-\frac{3}{2}x-\frac{3}{2}y} - 1 \right) dy \, dx \overset{\boxed{\text{DIY}}}{=} \boxed{\frac{4e^6 - 100}{9}}$$

$$\int_{y=0}^{4-x} e^{6-\frac{3}{2}x-\frac{3}{2}y} - 1 \, dy = e^{6-\frac{3}{2}x} \int_0^{4-x} e^{-\frac{3}{2}y} \, dy - \int_0^{4-x} dy$$

**Volume of** $E$ is given by

$$V(E) = \iiint_E dV$$

**Example:** Find the volume of the solid, $E$ in the $\underbrace{\text{first octant}}_{x \geq 0,\, y \geq 0,\, z \geq 0}$ bounded by $x + y + z = 1$ and $x + y + 2z = 1$.

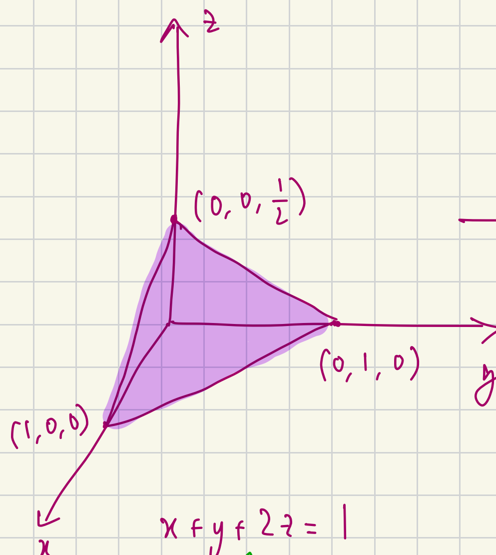



$x + y + z = 1$

$\Updownarrow$

$z = 1 - x - y$

$+$

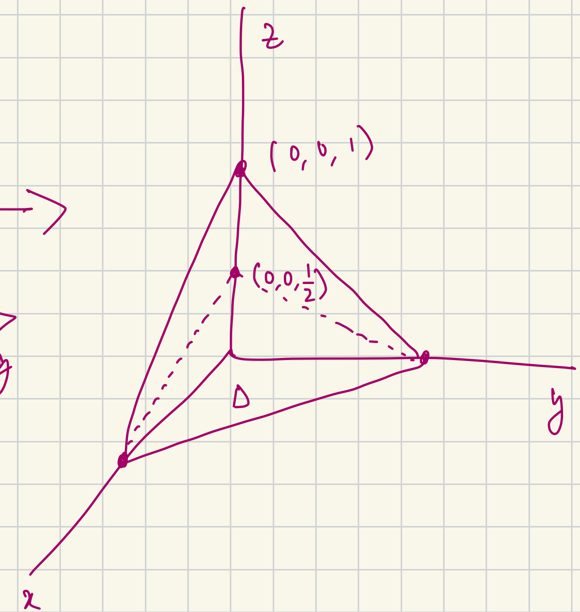



$x + y + 2z = 1$

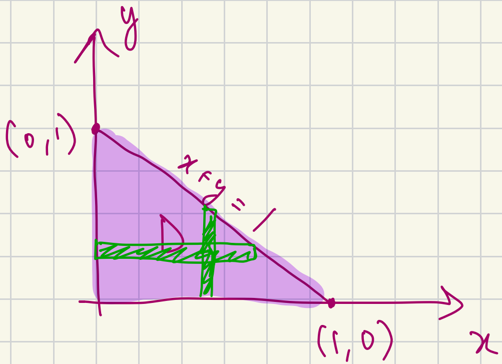
$\Updownarrow$

$2z = 1 - x - y$

$\longrightarrow$



$$E = \left\{ (x,y,z) : (x,y) \in D,\ \frac{1-x-y}{2} \le z \le 1-x-y \right\}$$



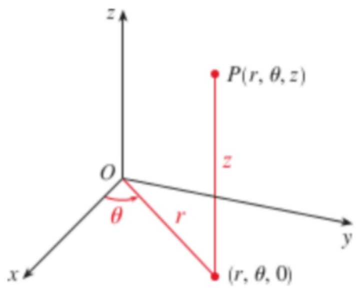

$$D = \{ (x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x \}$$

$$\Rightarrow E = \left\{ (x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1-x,\ \frac{1-x-y}{2} \le z \le 1-x-y \right\}$$

$$\Rightarrow V(E) = \int_0^1 \int_0^{1-x} \int_{\frac{1-x-y}{2}}^{1-x-y} dz\, dy\, dx \overset{\boxed{\text{DIY}}}{=} \boxed{\frac{1}{12}}$$

1. **Cylindrical Coordinates**

In the cylindrical coordinate system, a point $P$ in three-dimensional space is represented by the ordered triple $(r, \theta, z)$, where $r$ and $\theta$ are polar coordinates of the projection of $P$ onto the $xy$-plane and $z$ is the directed distance from the $xy$-plane to $P$.




To convert from cylindrical to rectangular coordinates, we use

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

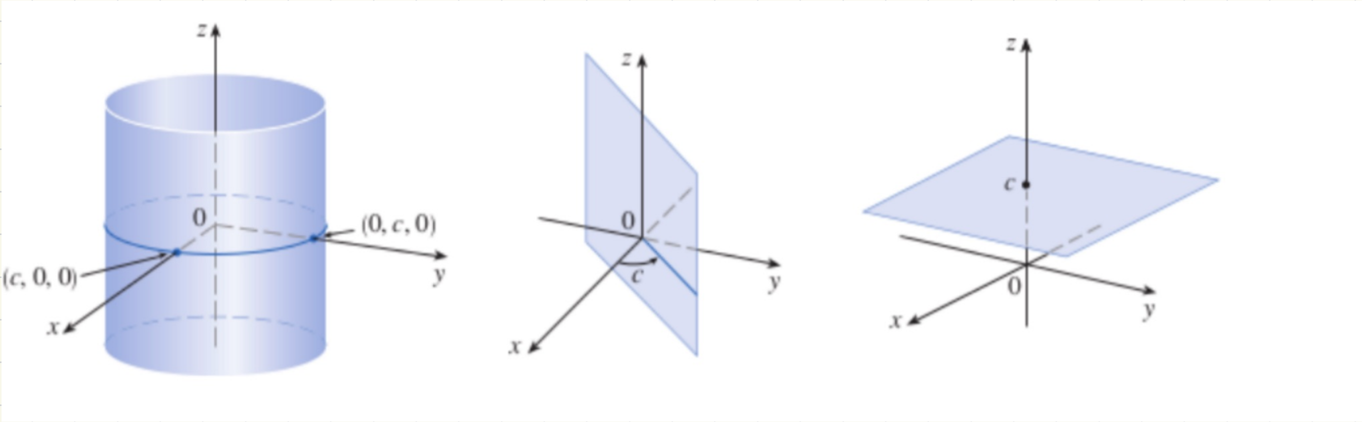
**Example:** $(r, \theta, z) = (2, \frac{\pi}{2}, 4) \leftrightarrow (x, y, z) = \left(2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}, 4\right) = (0, 2, 4)$

and to convert rectangular to cylindrical coordinates, we use:

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

**Example:** $(x, y, z) = (1, \sqrt{3}, 5) \leftrightarrow (r, \theta, z) = \left(\sqrt{1+3}, \arctan\sqrt{3}, 5\right)$

$= \left(2, \frac{\pi}{3}, 5\right)$



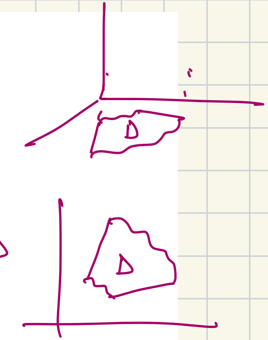

Cylindrical $\to$ $r = c$ $\qquad\qquad$ $\theta = c$ $\qquad\qquad$ $z = c$

i.e $\{(r, \theta, z) : r = c\}$

Cartesian $\to$ $x^2 + y^2 = c$

Suppose

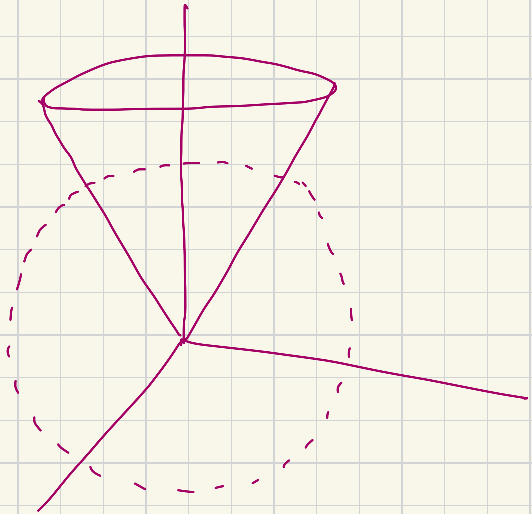
$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$
$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$
$$f = f(x, y, z) \text{ is continuous on } E$$



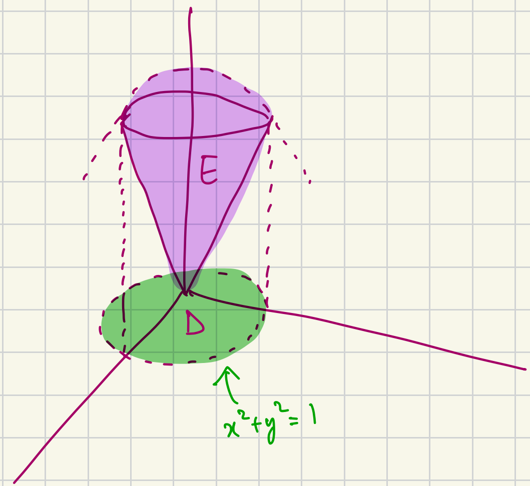

Let $g_1(r, \theta) = u_1(r\cos\theta, r\sin\theta)$ and $g_2(r, \theta) = u_2(r\cos\theta, r\sin\theta)$.

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$$
$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$$

**Example:** Let $E$ be the region bounded above by the sphere $x^2+y^2+z^2=2$ and bounded below by the cone $z=\sqrt{x^2+y^2}$. Use a triple integral to find the volume of $E$.



$\longrightarrow$




To find D, solve for $(x,y,z)$ such that $z^2=x^2+y^2$ and $x^2+y^2+z^2=2$.

$\Downarrow$

$x^2+y^2+x^2+y^2=2 \Leftrightarrow x^2+y^2=1$

$$E = \{(x,y,z) : x^2+y^2 \le 1 ,\ \sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}\}$$

$$= \{(r,\theta,z) : 0 \le \theta \le 2\pi ,\ 0 \le r \le 1,\ \underset{\substack{\| \\ \sqrt{r^2}}}{r} \le z \le \sqrt{2-r^2}\}$$

$$\Rightarrow V(E) = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta \overset{\boxed{DIY}}{=} \frac{4\pi(\sqrt{2}-1)}{3}$$

**Example:** Rewrite the following integral using cylindrical coordinates

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_1^{\sqrt{9-x^2-y^2}} xyz \, dz \, dy \, dx$$

**Example:** Evaluate $\iiint\limits_E x\,dV$ where $E$ is bounded by $x = 4y^2 + 4z^2$ and $x = 4$.