13 Vector Functions

13.2 Derivatives and Integrals of Vector Functions

Derivatives

The derivative $\boldsymbol{r}'$ of a vector function **r** is defined in much the same way as for real-valued functions:

$$1 \quad \frac{d\mathrm{r}}{dt} = \mathrm{r}'(t) = \lim_{h \to 0} \frac{\mathrm{r}(t+h) - \mathrm{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1.

(a) The secant vector $\overrightarrow{PQ}$

(b) The tangent vector $\mathrm{r}'(t)$

Figure 1

Derivatives (2 of 4)

If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t + h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector.

If $h > 0$, the scalar multiple $\left(\frac{1}{h}\right)(\boldsymbol{r}(t+h) - \boldsymbol{r}(t))$ has the same direction as

$\mathbf{r}(t + h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line.

For this reason, the vector $\boldsymbol{r}'(t)$ is called the **tangent vector** to the curve defined by **r** at the point P, provided that $\mathrm{r}'(t)$ exists and $\mathrm{r}'(t) \neq 0$.

The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\boldsymbol{r}'(t)$.

Derivatives (3 of 4)

The following theorem gives us a convenient method for computing the derivative of a vector function **r**: just differentiate each component of **r**.

2 Theorem If $\mathrm{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathrm{i} + g(t)\mathrm{j} + h(t)\mathrm{k},$ where *f*, *g*, and *h* are differentiable functions, then

$$\mathrm{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle = f'(t)\ \mathrm{i} + g'(t)\ \mathrm{j} + h'(t)\ \mathrm{k}$$

A unit vector that has the same direction as the tangent vector is called the **unit tangent vector T** and is defined by

$$\mathrm{T}(t) = \frac{\mathrm{r}'(t)}{|\mathrm{r}'(t)|}$$

Example 1

(a) Find the derivative of $\mathrm{r}(t) = (1 + t^3)\ \mathrm{i} + te^{-t}\mathrm{j} + \sin\ 2t\ \mathrm{k}.$

(b) Find the unit tangent vector at the point where $t = 0$.

Solution:

(a) According to Theorem 2, we differentiate each component of **r**:

$$\mathrm{r}'(t) = 3t^2\mathrm{i} + (1 - t)e^{-t}\mathrm{j} + 2\ \cos\ 2t\ \mathrm{k}$$

Example 1 – Solution

(b) Since **r**(0) = **i** and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point (1, 0, 0) is

$$\begin{aligned} \mathbf{T}(0) &= \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} \\ &= \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1+4}} \\ &= \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k} \end{aligned}$$

Derivatives (4 of 4)

Just as for real-valued functions, the **second derivative** of a vector function **r** is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

For instance, the second derivative of the function, $\mathrm{r}(t) = \langle 2\cos t,\ \sin t,\ t\rangle$, is

$$\mathrm{r}''(t) = \langle -2\cos t,\ -\sin t,\ 0\rangle$$

Differentiation Rules

Differentiation Rules (1 of 3)

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose **u** and **v** are differentiable vector functions, *c* is a scalar, and *f* is a real-valued function. Then

1. $\frac{d}{dt}[\boldsymbol{u}(t)+\boldsymbol{v}(t)]=\boldsymbol{u}'(t)+\boldsymbol{v}'(t)$

2. $\frac{d}{dt}[c\boldsymbol{u}(t)]=c\boldsymbol{u}'(t)$

3. $\frac{d}{dt}[f(t)\boldsymbol{u}(t)]=f'(t)\boldsymbol{u}(t)+f(t)\boldsymbol{u}'(t)$

4. $\frac{d}{dt}[\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)]=\boldsymbol{u}'(t)\cdot\boldsymbol{v}(t)+\boldsymbol{u}(t)\cdot\boldsymbol{v}'(t)$

5. $\frac{d}{dt}[\boldsymbol{u}(t)\times\boldsymbol{v}(t)]=\boldsymbol{u}'(t)\times\boldsymbol{v}(t)+\boldsymbol{u}(t)\times\boldsymbol{v}'(t)$

6. $\frac{d}{dt}[\boldsymbol{u}(f(t))]=f'(t)\boldsymbol{u}'(f(t))$ (Chain Rule)

Differentiation Rules (2 of 3)

We use Formula 4 to prove the following theorem.

4 Theorem if $|\mathrm{r}(t)| = c$ (a constant), then $\mathrm{r}'(t)$ is orthogonal to **r**(t) for all t.

PROOF

Since

$$\mathrm{r}(t) \cdot \mathrm{r}(t) = |\mathrm{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathrm{r}(t) \cdot \mathrm{r}(t)] = \mathrm{r}'(t) \cdot \mathrm{r}(t) + \mathrm{r}(t) \cdot \mathrm{r}'(t) = 2\mathrm{r}'(t) \cdot \mathrm{r}(t)$$

Thus $\mathrm{r}'(t) \cdot \mathrm{r}(t) = 0,$ which says that $\mathrm{r}'(t)$ is orthogonal to **r**(t).

Differentiation Rules (3 of 3)

Geometrically, this result says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector **r**(*t*). (See Figure 4.)

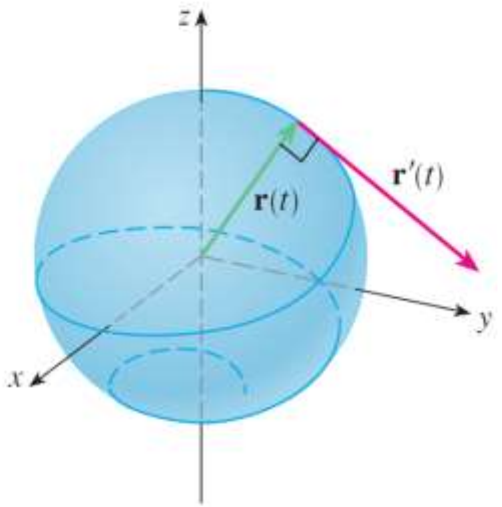

Figure 4

Integrals

Integrals (1 of 3)

The **definite integral** of a continuous vector function **r**(*t*) can be defined in much the same way as for real-valued functions except that the integral is a vector.

But then we can express the integral of **r** in terms of the integrals of its component functions *f*, *g*, and *h* as follows.

$$\begin{aligned}\int_a^b \mathbf{r}(t)\,dt &= \lim_{n\to\infty}\sum_{i=1}^{n}\mathbf{r}(t_i^*)\Delta t \\ &= \lim_{n\to\infty}\left[\left(\sum_{i=1}^{n} f(t_i^*)\Delta t\right)\mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\Delta t\right)\mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\Delta t\right)\mathbf{k}\right]\end{aligned}$$

Integrals (2 of 3)

So

$$\int_a^b \mathrm{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathrm{i} + \left(\int_a^b g(t)dt\right)\mathrm{j} + \left(\int_a^b h(t)dt\right)\mathrm{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

Integrals (3 of 3)

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where **R** is an antiderivative of **r**, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

We use the notation $\int \mathbf{r}(t)\ dt$ for indefinite integrals (antiderivatives).

Example 4

If **r**(*t*) = 2 cos *t* **i** + sin *t* **j** + 2*t* **k**, then

$$\begin{aligned}\int \mathrm{r}(t)dt &= \left(\int 2\cos t \ dt\right)\mathrm{i} + \left(\int \sin t \ dt\right)\mathrm{j} + \left(\int 2t \ dt\right)\mathrm{k} \\ &= 2\sin t \ \ \mathrm{i} - \cos t \ \ \mathrm{j} + t^2\mathrm{k} + \mathrm{C}\end{aligned}$$

where **C** is a vector constant of integration, and

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \mathrm{r}(t)dt &= [2\sin t \ \ \mathrm{i} - \cos t \ \ \mathrm{j} + t^2 \ \ \mathrm{k}]_0^{\frac{\pi}{2}} \\ &= 2\mathrm{i} + \mathrm{j} + \frac{\pi^2}{4}\mathrm{k}\end{aligned}$$