12 Vectors and the Geometry of Space

12.2 Vectors

Vectors

The term **vector** is used in mathematics and the sciences to indicate a quantity that has both magnitude and direction.

Geometric Description of Vectors

Geometric Description of Vectors (1 of 10)

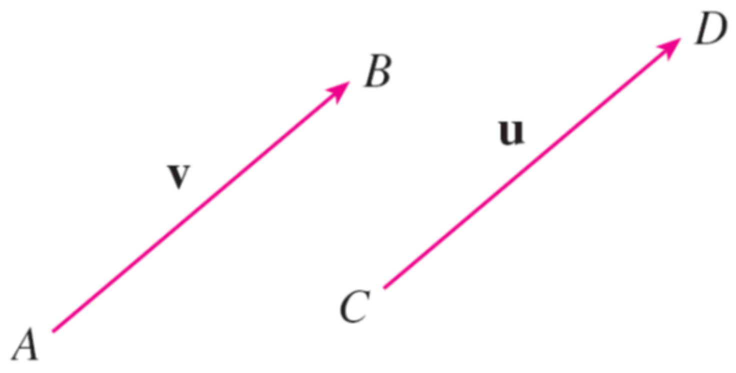
A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector.

We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter ($\vec{v}$).

Geometric Description of Vectors (2 of 10)

For instance, suppose a particle moves along a line segment from point *A* to point *B*.

The corresponding **displacement vector v**, shown in Figure 1, has **initial point** *A* (the tail) and **terminal point** *B* (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$.



Equivalent vectors

Figure 1

Geometric Description of Vectors (3 of 10)

Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as **v** even though it is in a different position.

We say that **u** and **v** are **equivalent** (or **equal**) and we write **u** = **v**.

The **zero vector**, denoted by **0**, has length 0. It is the only vector with no specific direction.

Geometric Description of Vectors (4 of 10)

Suppose a particle moves from A to B, so its displacement vector is $\overrightarrow{AB}$.

Then the particle changes direction and moves from B to C, with displacement vector $\overrightarrow{BC}$ as in Figure 2.

The combined effect of these displacements is that the particle has moved from A to C.

The resulting displacement vector $\overrightarrow{AC}$ is called the *sum* of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

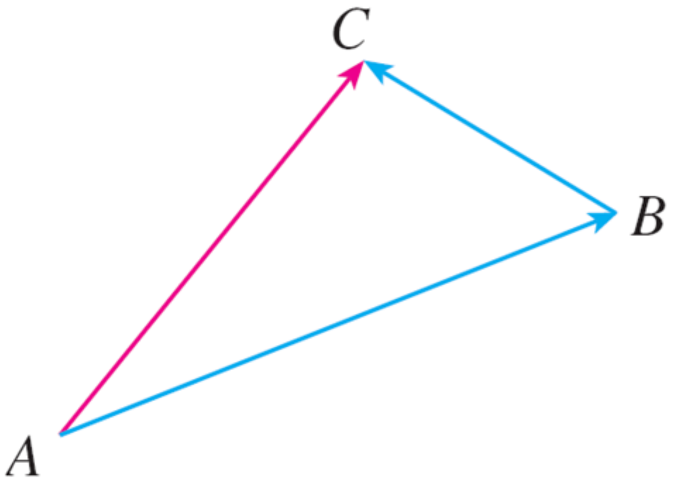


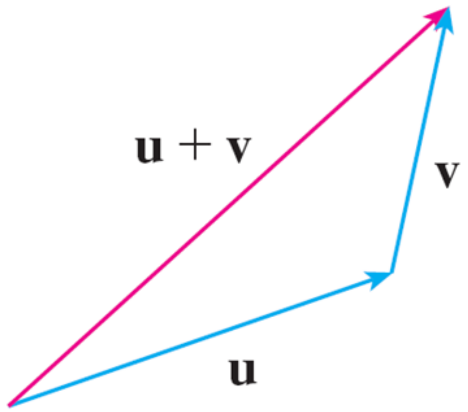
Figure 2

Geometric Description of Vectors (5 of 10)

In general, if we start with vectors **u** and **v**, we first move **v** so that its tail coincides with the tip of **u** and define the sum of **u** and **v** as follows.

Definition of Vector Addition If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u** + **v** is the vector from the initial point of **u** to the terminal point of **v**.

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.



The Triangle Law

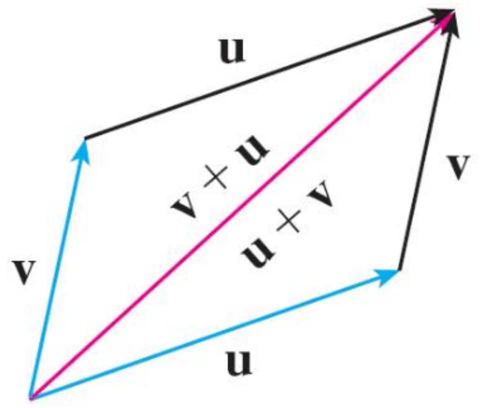
Figure 3

Geometric Description of Vectors (6 of 10)

In Figure 4 we start with the same vectors **u** and **v** as in Figure 3 and draw another copy of **v** with the same initial point as **u**.

Completing the parallelogram, we see that **u** + **v** = **v** + **u**.

This also gives another way to construct the sum: If we place **u** and **v** so they start at the same point, then **u** + **v** lies along the diagonal of the parallelogram with **u** and **v** as sides. (This is called the **Parallelogram Law**.)



The Parallelogram Law

Figure 4

Geometric Description of Vectors (7 of 10)

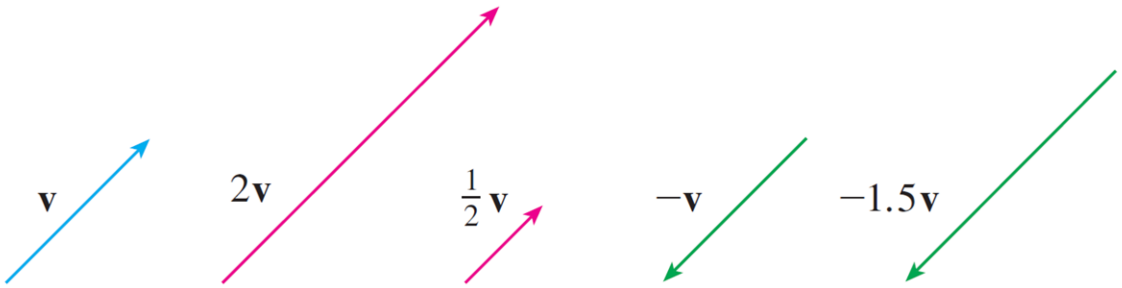
It is possible to multiply a vector **v** by a real number *c*. In this context we call the real number *c* a **scalar** to distinguish it from a vector.

For instance, we want the *scalar multiple* 2**v** to be the same vector as **v** + **v**, which has the same direction as **v** but is twice as long. In general, we multiply a vector by a scalar as follows.

Definition of Scalar Multiplication If *c* is a scalar and **v** is a vector, then the **scalar multiple** *c***v** is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Geometric Description of Vectors (8 of 10)

This definition is illustrated in Figure 7.



Scalar multiples of $\mathbf{v}$

Figure 7

We see that real numbers work like scaling factors here; that's why we call them scalars.

Geometric Description of Vectors (9 of 10)

Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another.

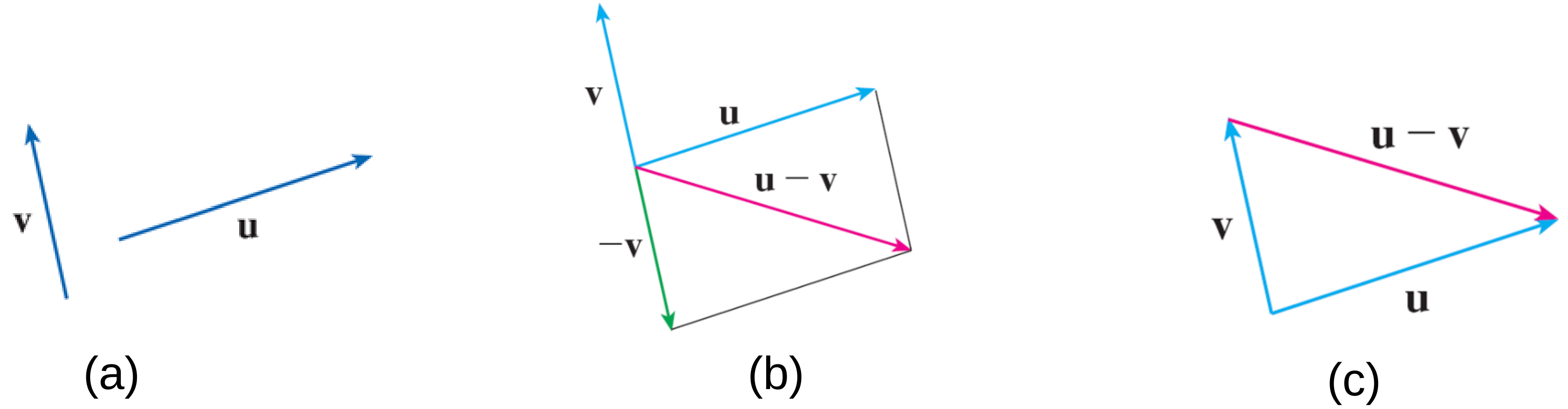
In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but points in the opposite direction. We call it the **negative** of $\mathbf{v}$.

By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

For the vectors **u** and **v** shown in Figure 8(a), we can construct the difference **u** − **v** by first drawing the negative of **v**, −**v**, and then adding it to **u** by the Parallelogram Law as in Figure 8(b).

Alternatively, since **v** + (**u** − **v**) = **u**, the vector **u** − **v**, when added to **v**, gives **u**. So we could construct **u** − **v** as in Figure 8(c) by means of the Triangle Law.



Drawing the difference **u** − **v**

Figure 8

Components of a Vector

For some purposes it's best to introduce a coordinate system and treat vectors algebraically.

If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (see Figure 11).

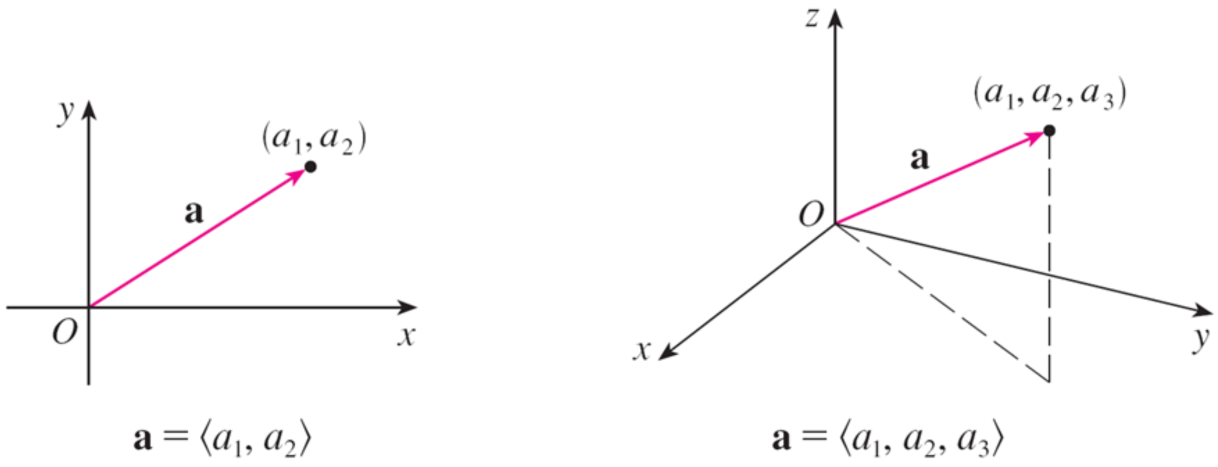


Figure 11

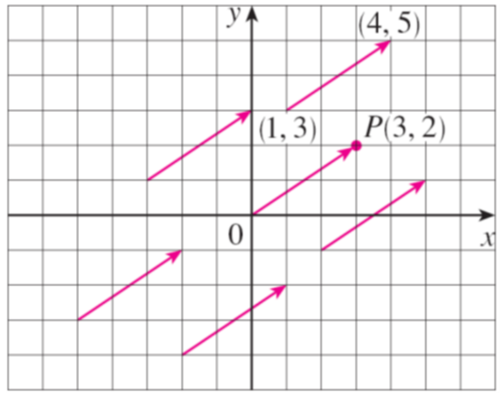
Components of a Vector (2 of 16)

These coordinates are called the **components** of **a** and we write

$$\mathbf{a} = \langle a_1,\ a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1,\ a_2,\ a_3 \rangle$$

We use the notation $\langle a_1,\ a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

For instance, the vectors shown in Figure 12 are all equivalent to the vector $\overrightarrow{OP} = \langle 3,\ 2 \rangle$ whose terminal point is $P(3, 2)$.



Representations of $\mathbf{a} = \langle 3,\ 2 \rangle$

Figure 12

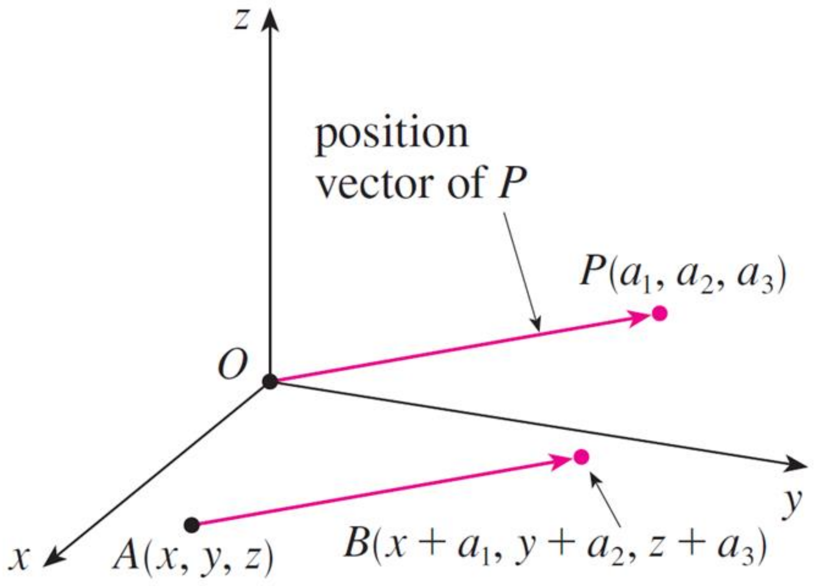
Components of a Vector (3 of 16)

What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward.

We can think of all these geometric vectors as **representations** of the algebraic vector $\mathbf{a} = \langle 3,\ 2 \rangle$.

The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. (See Figure 13.)



Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Figure 13

Components of a Vector (5 of 16)

Let's consider any other representation of **a** by a directed line segment $\overrightarrow{AB}$ with initial point $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$.

Then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$.

Thus we have the following result.

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$

Example 3

Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Solution:

By (1), the vector corresponding to $\overrightarrow{AB}$ is

$$\begin{aligned} \mathbf{a} &= \langle -2-2,\ 1-(-3),\ 1-4 \rangle \\ &= \langle -4,\ 4,\ -3 \rangle \end{aligned}$$

Components of a Vector (6 of 16)

The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment *OP*, we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1,\ a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1,\ a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Components of a Vector (7 of 16)

How do we add vectors algebraically? Figure 14 shows that if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive.

In other words, *to add algebraic vectors we add corresponding components*. Similarly, *to subtract vectors we subtract corresponding components*.

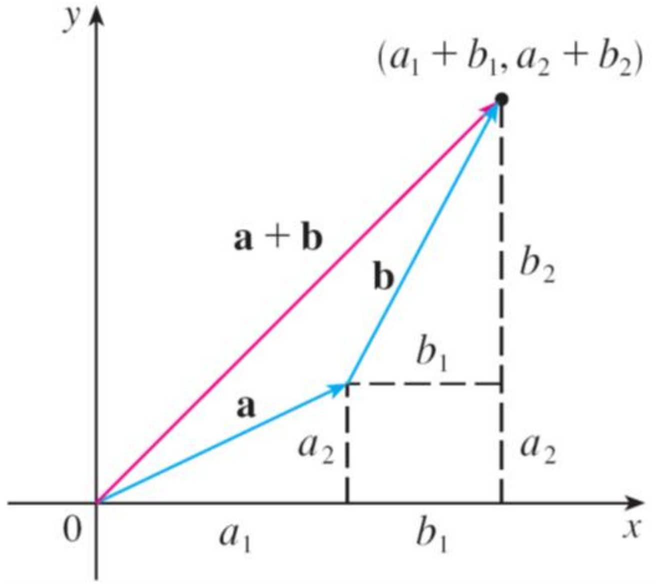


Figure 14

Components of a Vector (8 of 16)

From the similar triangles in Figure 15 we see that the components of $c\mathbf{a}$ are ca_1 and ca_2.

So *to multiply a vector by a scalar we multiply each component by that scalar*.

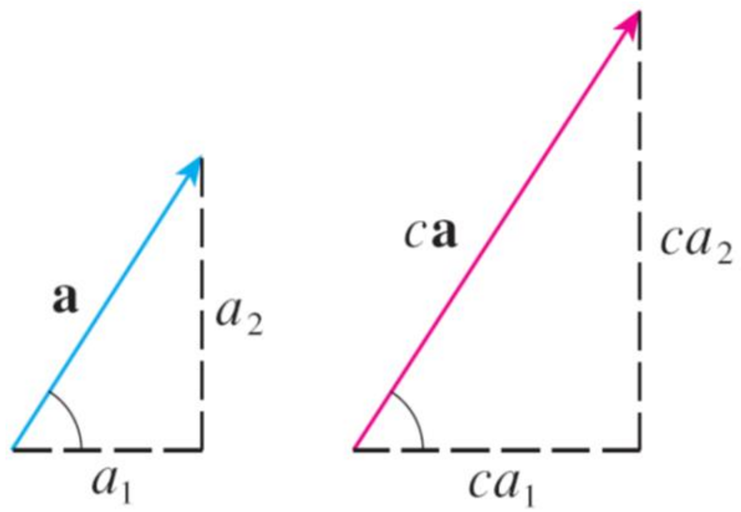


Figure 15

Components of a Vector (9 of 16)

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1,\ b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1,\ a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1,\ a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1,\ ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\begin{aligned}
\langle a_1,\ a_2,\ a_3 \rangle + \langle b_1,\ b_2,\ b_3 \rangle &= \langle a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3 \rangle \\
\langle a_1,\ a_2,\ a_3 \rangle - \langle b_1,\ b_2,\ b_3 \rangle &= \langle a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3 \rangle \\
c\langle a_1,\ a_2,\ a_3 \rangle &= \langle ca_1,\ ca_2,\ ca_3 \rangle
\end{aligned}$$

Components of a Vector (10 of 16)

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors.

More generally, we will consider the set V_n of all n-dimensional vectors.

An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of **a**.

Components of a Vector (11 of 16)

Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

Properties of Vectors If **a**, **b**, and **c** are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Components of a Vector (12 of 16)

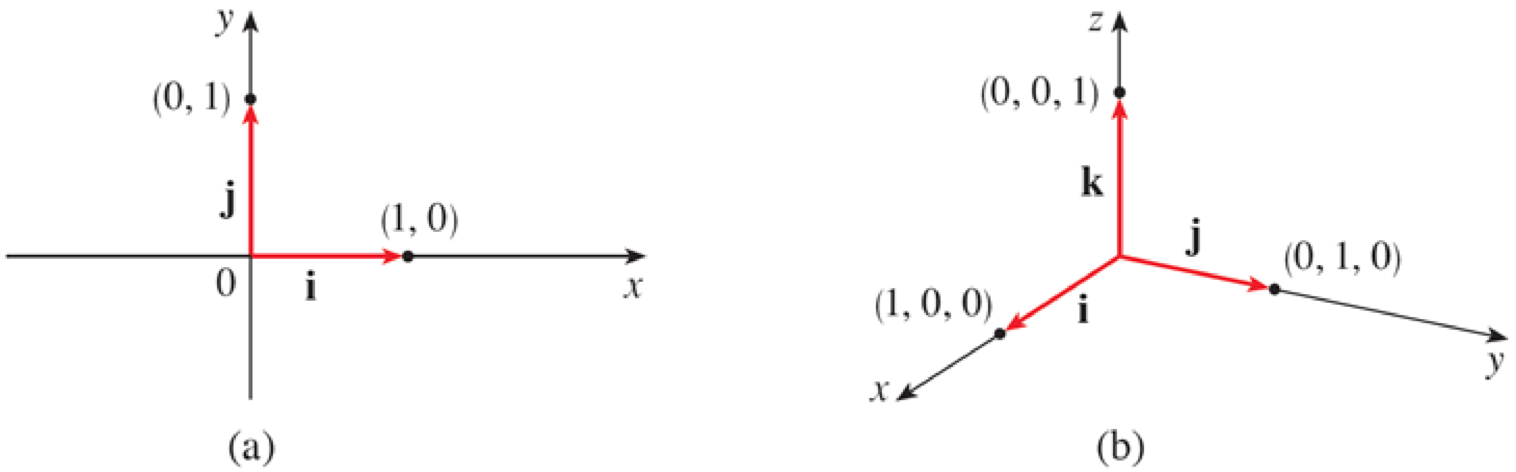
Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

These vectors **i**, **j**, and **k** are called the **standard basis vectors**.

They have length 1 and point in the directions of the positive x-, y-, and z-axes.

Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (See Figure 17.)



Standard basis vectors in V_2 and V_3

Figure 17

Components of a Vector (14 of 16)

If $\mathbf{a} = \langle a_1,\ a_2,\ a_3 \rangle$, then we can write

$$\begin{aligned}\mathbf{a} &= \langle a_1,\ a_2,\ a_3 \rangle = \langle a_1,\ 0,\ 0 \rangle + \langle 0,\ a_2,\ 0 \rangle + \langle 0,\ 0,\ a_3 \rangle \\ &= a_1 \langle 1,\ 0,\ 0 \rangle + a_2 \langle 0,\ 1,\ 0 \rangle + a_3 \langle 0,\ 0,\ 1 \rangle\end{aligned}$$

2 $\quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Thus any vector in V_3 can be expressed in terms of **i**, **j**, and **k**. For instance,

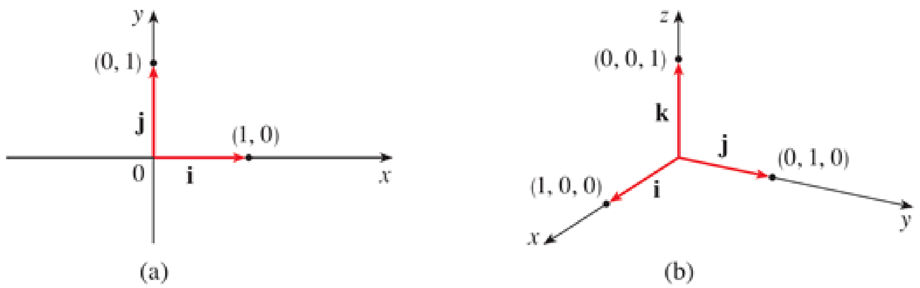
$$\langle 1,\ -2,\ 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

3 $\quad \mathbf{a} = \langle a_1,\ a_2 \rangle = a_1\mathbf{i} + a_1\mathbf{j}$

Components of a Vector (15 of 16)

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.



Standard basis vectors in V_2 and V_3

Figure 17

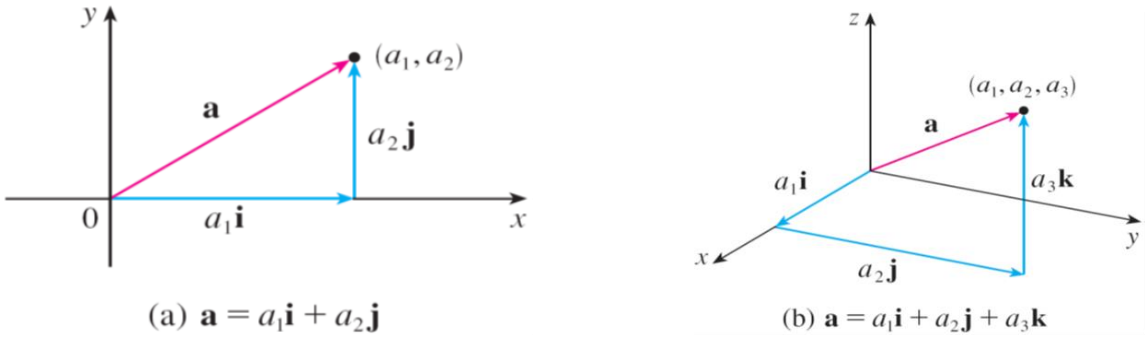


Figure 18

Components of a Vector (16 of 16)

A **unit vector** is a vector whose length is 1. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|} \qquad \text{4}$$

In order to verify this, we let $c = \dfrac{1}{|\mathbf{a}|}$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so **u** has the same direction as **a**. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

Applications

Applications (1 of 1)

Vectors are useful in many aspects of physics and engineering. Here we look at forces.

A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction.

If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

Example 7

A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.

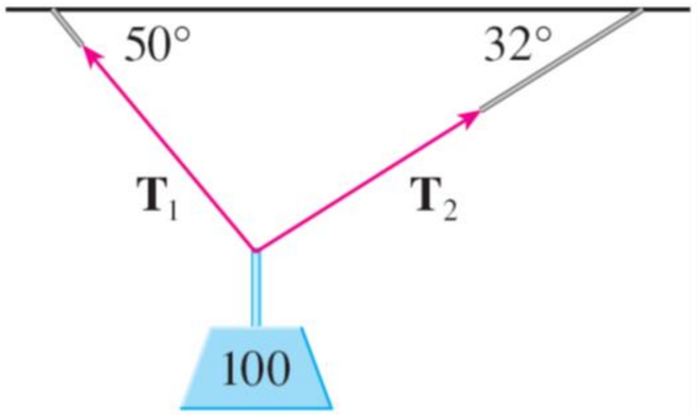


Figure 19

Example 7 – Solution (1 of 4)

We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components. From Figure 20 we see that

5 $\quad \mathbf{T}_1 = -|\mathbf{T}_1|\cos 50°\,\mathbf{i} + |\mathbf{T}_1|\sin 50°\,\mathbf{j}$

6 $\quad \mathbf{T}_2 = |\mathbf{T}_2|\cos 32°\,\mathbf{i} + |\mathbf{T}_2|\sin 32°\,\mathbf{j}$

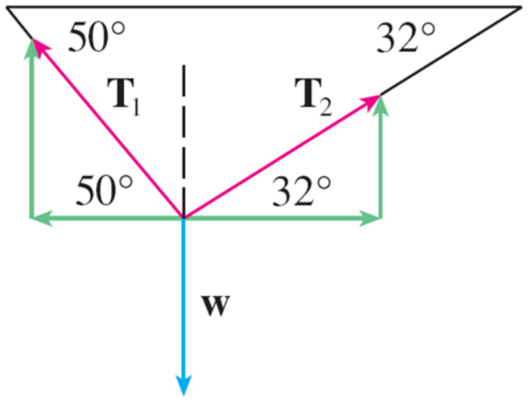


Figure 20

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w} = -100\,\mathbf{j}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\,\mathbf{j}$$

Example 7 – Solution (2 of 4)

Thus

$$\left(-\left|\mathbf{T}_1\right|\cos 50^\circ + \left|\mathbf{T}_2\right|\cos 32^\circ\right)\mathbf{i} + \left(\left|\mathbf{T}_1\right|\sin 50^\circ + \left|\mathbf{T}_1\right|\sin 32^\circ\right)\mathbf{j}$$
$$= 100\,\mathbf{j}$$

Equating components, we get

$$-\left|\mathbf{T}_1\right|\cos 50^\circ + \left|\mathbf{T}_2\right|\cos 32^\circ = 0$$
$$\left|\mathbf{T}_1\right|\sin 50^\circ + \left|\mathbf{T}_2\right|\sin 32^\circ = 100$$

Example 7 – Solution (3 of 4)

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1|\sin 50° + \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°}\sin 32° = 100$$

$$|\mathbf{T}_1|\left(\sin 50° + \cos 50° \frac{\sin 32°}{\cos 32°}\right) = 100$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50° + \tan 32° \cos 50°} \approx 85.64 \text{ lb}$$

and

$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°} \approx 64.91 \text{ lb}$$

Example 7 – Solution (4 of 4)

Substituting these values in (5) and (6), we obtain the tension vectors

$$\mathbf{T}_1 \approx -55.05\mathbf{i} + 65.60\mathbf{j} \qquad \mathbf{T}_2 \approx 55.05\mathbf{i} + 34.40\mathbf{j}$$