No pre-lec content for today

Quiz 2 graded later today

Basic 3D integrals

Lecture for 6/27

R/E are standard notation in other calc 3 sources

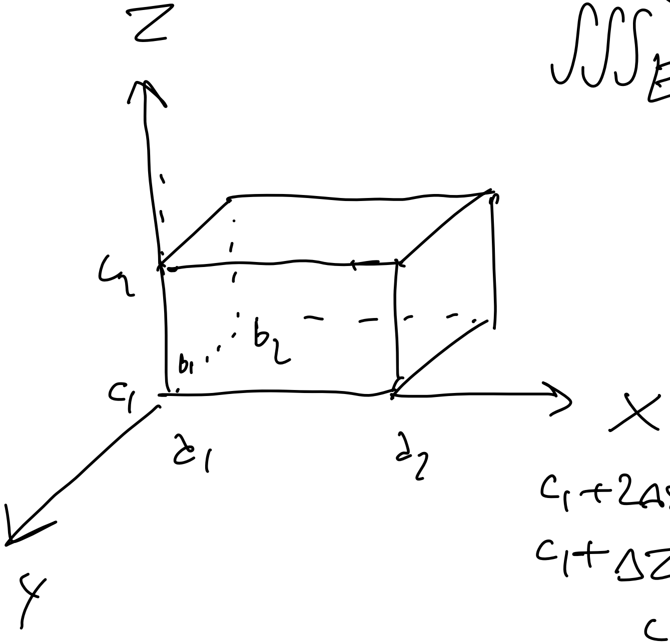
General Idea

we already did all this for 2D integrals

- We have seen $\int f(x)\, dx$ and $\iint_R f(x, y)\, dA$
- Next step forward is $\iiint_E f(x, y)\, dV$
- Riemann sums can be done just as for 2D integrals
- If R is a prism, new version of Fubini's Theorem applies
- Can integrate with general regions
- Can swap order of integration with some care
- Center of mass & average value generalize

Riemann Sums

- Suppose E is a prism $a_1 \leq x \leq a_2$, $b_1 \leq y \leq b_2$, $c_1 \leq z \leq c_2$
- Can approximate $\cancel{\iint_V}$ $\iiint_E$ f(x, y, z) dV with Riemann sums

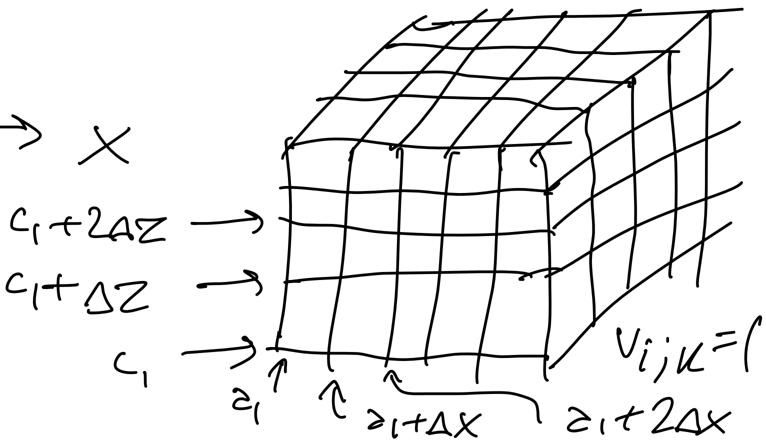


In 2D, rectangle into small rectangles.

In 3D, break prism into small prisms

Each tiny prism has a volume of $\Delta x \Delta y \Delta z$,

There will be n^3 prisms with corners given by



$V_{ijk} = (a_1 + i\Delta x, b_1 + j\Delta y, c_1 + k\Delta z)$

$0 \leq i, j, k < n$

Let E_{ijk} be tiny prism with corner v_{ijk}.

$$\iiint_E f\,dV = \sum_{i,j,k} \iiint_{E_{ijk}} f\,dV \approx \sum_{i,j,k} \iiint_{E_{ijk}} f(v_{ijk})\,dV$$

If E_{ijk} is small enough, f is almost constant on it.

So take $f \approx f(v_{ijk})$ on all of E_{ijk}.

$$= \sum_{i,j,k} f(v_{ijk}) \cdot vol(E_{ijk}) = \sum_{i,j,k} f(v_{ijk})\,\Delta x \Delta y \Delta z =$$

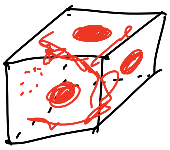
$$\sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1}\left(\sum_{i=0}^{n-1} f(v_{ijk})\,\Delta x\right)\Delta y\right)\Delta z$$

$$\iiint_E f\,dV$$

as $\Delta x, \Delta y, \Delta z \to 0$

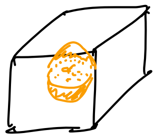
$$\int_e^f\left(\int_c^d\left(\int_a^b f(x,y,z)\,dx\right)dy\right)dz$$

$f \geq 0$ continous, contineous except at countably many points, or $D_f = \{(x_0, y_0, z_0) : f \text{ discont. at } (x_0, y_0, z_0)\}$ has volume 0 $\Rightarrow$ f is Riemann integrable.



Df in red

Riemann int since only discont. on points, curves, surfaces



Df in orange

Not Riemann int, discont. on entire orange ball

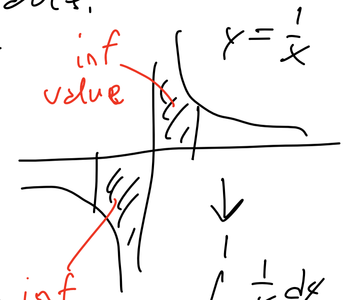
In general, write $f = f^+ - f^-$ where

$f^+ = \max(f, 0)$,

$f^- = \max(-f, 0)$

are pos & neg comp. of f.

Then f is Riemann integrable if f^+, f^- are RI & $\left(\int_V f^+, \int_V f^-\right) \neq (\infty, \infty)$.



$$\int_{-2}^{1} \frac{1}{x}\,dx$$

not RI

This is because $\infty - \infty$ is not defined.

New Fubini

Suppose E = [a, b] x [c,d] x [e, f] is a prism

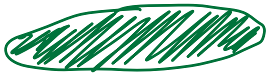
- Then $\iiint_E f(x, y, z)\, dV = \int_{\cancel{f}\; e}^{f\; \cancel{e}} (\int_c^d (\int_a^b f(x,y,z)\, dx)\, dy)\, dz$ = any of the other 5 orders for integrating over x, y, z
- Repeatedly apply standard Fubini to prove

$$= \int_e^f \left(\int_c^d \int_a^b \cdots \right) dz = \int_e^f \left(\int_a^b \int_c^d \cdots \right) dz = \int_e^f \int_a^b (\cdots)\, dx\, dz$$

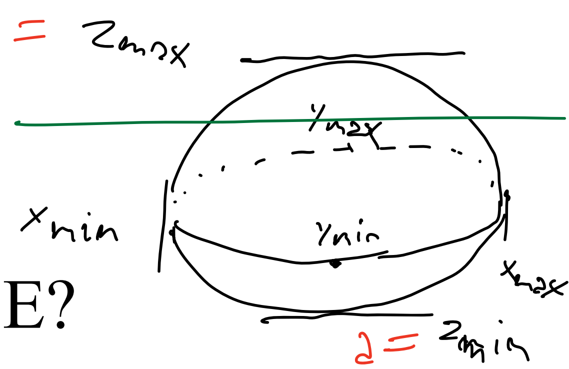
$$= \int_a^b \int_e^f (\cdots) = \int_a^b \int_e^f \int_c^d f(x,y,z)\, dy\, dz\, dx$$

only do this for a prism!

2D region R_c:



slice $z=\frac{1}{3}(a+2b)$



General 3D integrals

How do we find the integral over a general 3D region E?

- Suppose $a \le z \le b$, but region isn't a prism
- For any c in [a, b], let R_c be the cross section of E with $z = c$
- R_c is a general 2D integral
  - For example, $d \le y \le e$ and $g(y) \le x \le h(y)$
- Combining dependencies: $g(z) \le y \le h(z)$, $r(y, z) \le x \le s(y, z)$
- Can try this with any order of x, y, and z

At the end, we'll have

$$\iiint_E f\,dV = \int_a^b \left(\int_{g(z)}^{h(z)} \left(\int_{r(y,z)}^{s(y,z)} f(x,y,z)\,dx \right) dy \right) dz$$

Start with x_{min} & x_{max} for region, then analyze bounds for z in terms of x, then end with y, we get:

$$\int_a^b \left(\int_{g(x)}^{h(x)} \left(\int_{r(x,z)}^{s(x,z)} f(x,y,z)\, dy \right) dz \right) dx.$$

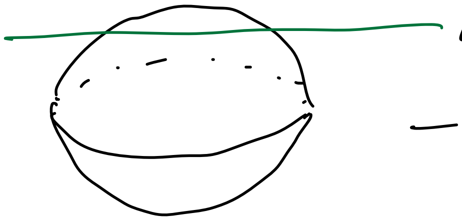
Choose the order which leads to the least messy expressions for g, h, r, s.

If you have constraints like $\overbrace{y = r(x,z)}^{\text{surface}}$, $\overbrace{y = s(x,z)}^{\text{surface}}$.

Then consider x & z first, y last $\Rightarrow \iiint \cdots \, dy\, dx\, dz$.

In general: when you have surfaces $S_1, S_2, S_3, \ldots$ bounding your solid region E, figure out which variable it is easiest to isolate in the surface equations.

Simple example: find volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

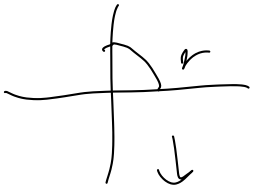


No variable stands out, so pick z.

$-c\le z\le c$ are bounds on z. Also notice symmetry: (x,y,z) in ellipsoid $\Longleftrightarrow$ $(\pm x,\pm y,\pm z)$ in ellipsoid for any of the 8 choices of signs. So volume $=$ $8\cdot$(volume in 1st octant).

In 1st octant: $x,y,z\ge 0$. Now suppose z is fixed, then $0\le \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}$. Let $f(z)=1-\frac{z^2}{c^2}$

$\Rightarrow$ $0\le \frac{x^2}{a^2}+\frac{y^2}{b^2}\le f(z)$. Recall from ls 2 & trig that $x=ar\sqrt{f(z)}\cos t$, $y=br\sqrt{f(z)}\sin t$,



$x = r\cos\theta$
$y = r\sin\theta$

$0 \le r \le 1, \quad 0 \le t < \frac{\pi}{2}$

Check x & y parametrization: $\frac{x^2}{a^2} = r^2 f(z)\cos^2 t$, $\frac{y^2}{b^2} = r^2 f(z)\sin^2 t$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2 f(z)$,

and $0 \le r^2 f(z) \le f(z) \Rightarrow 0 \le r \le 1$.

Keep this in mind for the future, but for this problem, we won't need this parametrization.

Let R_z be the resulting $\frac{1}{4}$ ellipse cross section (only $\frac{1}{4}$ since we are considering 1st quadrant).

$$\text{Volume} = 8 \cdot \int_{E \text{ in 1st octant}} 1\, dV = 8 \int_0^c \left(\iint_{R_z} 1\, dA \right) dz =$$

$$8 \int_0^c \text{area}(R_z)\, dz = 8 \int_0^c \frac{1}{4}\pi ab \sqrt{f(z)}^{\,2}\, dz$$

recall area of ellipse with semiminor, semimajor axes a, b is πab.

So $\frac{1}{4}$th ellipse is $\frac{1}{4}\pi ab$

$0 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} \le f(z)$

$\Rightarrow 0 \le x \le a\sqrt{f(z)}$
$\quad\;\; 0 \le y \le b\sqrt{f(z)}$

so semimajor & semiminor axes are $a\sqrt{f(z)}$, $b\sqrt{f(z)}$

$$= 2\pi \int_0^c ab\, f(z)\, dz =$$

$$2ab\pi \int_0^c \left(1 - \frac{z^2}{c^2}\right) dz =$$

$$2ab\pi \left(z - \frac{z^3}{3c^2} \right)\Big|_0^c =$$

$$2ab\pi \left(c - \frac{c^3}{3c^2} \right) = 2ab\pi \cdot \frac{2c}{3}$$

$$= \frac{4}{3}\pi abc$$

Notice $a = b = c$, get ellipsoid is a sphere of radius a & volume $= \frac{4}{3}\pi a^3$ as expected.

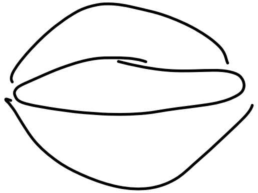
Switching Order of Integration

- Option 1: keep outer integral, switch inner 2
  - No issue, just use same tactics as in 2D switches
- Option 2: switch outermost integral
  - Must be very careful to make sure new bounds correct

$$\int_{\eta}^{\theta}\left(\underbrace{\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}}\cdots\right) = \int_{a}^{b}\left(\underbrace{\int_{\varepsilon}^{f}\int_{g}^{h}}\cdots\right)$$

change 2D order with 2D tactics

where $\alpha, \beta, \gamma, \ldots, h$ can be constants or functions



$$-c \le z \le c \longrightarrow 0 \le z \le c$$

$$0 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 - \frac{z^2}{c^2} \Longrightarrow$$

$$0 \le y \le b\sqrt{1 - \frac{z^2}{c^2}} = b\,f(z), \quad 0 \le x \le a\underbrace{\sqrt{1 - \frac{z^2}{c^2} - \frac{y^2}{b^2}}}_{g(y,z)}$$

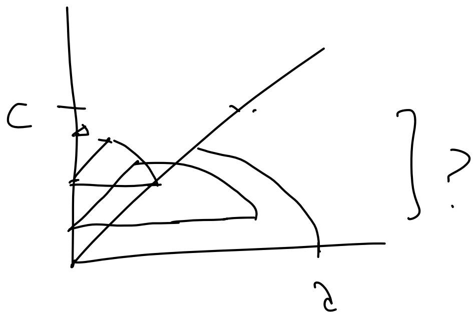
$$8 \int_0^c \int_0^{bf(z)} \int_0^{ag(y,z)} 1 \, dx \, dy \, dz = \dots$$

Let's suppose you somehow ended up with this triple integral, lost the context, and want to switch orders.

Step 1: write down all of the bounds

$$0 \le z \le c, \quad 0 \le y \le bf(z), \quad 0 \le x \le ag(y,z)$$

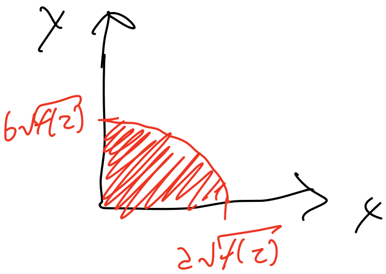
Step 2: graph the region (optional step)



i. Graph z first

ii. Fix some z, graph x & y.

Note: for triple integrals, you may need more than 1 graph to figure out how the region appears.



$$0 \le y \le b\sqrt{1-\frac{z^2}{c^2}}$$

$$0 \le x \le a\sqrt{1-\frac{z^2}{c^2}-\frac{y^2}{b^2}}$$

Rearrange these inequalities to graph x & y.

$$0 \le \underline{\frac{y^2}{b^2} \le 1-\frac{z^2}{c^2} = f(z)}$$

$$0 \le \underline{\frac{x^2}{a^2} \le 1-\frac{z^2}{c^2}-\frac{y^2}{b^2} = f(z)-\frac{y^2}{b^2}}.$$

$$\hookrightarrow \underline{\frac{x^2}{a^2}+\frac{y^2}{b^2} \le f(z)}, \quad \underline{x \,\&\, y \ge 0} \text{ also}$$

Step 3: pick new variable for outer bound

Pick y. $y \ge 0$ still, also $\underline{y \le b\sqrt{1-\frac{z^2}{c^2}}} \le b\sqrt{1} = b$,

So new 3D integral is $\int_0^b \left(\iint \cdots \right) dy$.

Step 4: rearrange inequalities to find new 2D bounds

$$y \le b\sqrt{1-\frac{z^2}{c^2}} \Longrightarrow \frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1 \Longrightarrow$$

$$\frac{z^2}{c^2} \le 1-\frac{y^2}{b^2} \Rightarrow \underline{z \le c\sqrt{1-\frac{y^2}{b^2}}},$$ Similarly, rearrange to find new bounds for x:

$0 \le x$ still, also $x \le a\,g(y,z) = a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}} \Longrightarrow$

$\frac{x^2}{a^2} \le 1 - \frac{y^2}{b^2} - \frac{z^2}{c^2} \Longrightarrow$ there would be more work

if y wasn't in terms of the right variables.

In this case, we are already done at ___ since

x is in terms of y & z and x needs to be

in terms of exactly y & z since it is the last

variable to solve for.

Step 5: check all constraints have been used & write

down final new integral

All constraints "$0 \le z \le c$, $0 \le y \le b f(z)$, $0 \le x \le a g(y,z)$"

have been used, new bounds are $0 \le y \le b$,

$0 \le z \le c\sqrt{1-\frac{y^2}{b^2}}$, $0 \le x \le a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$, so

new integral is

$$\int_0^b \left(\int_0^{c\sqrt{1-\frac{y^2}{b^2}}} \left(\int_0^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} 1\,dx \right) dz \right) dy.$$

Note: even though an ellipsoid is relatively simple and not all of steps 1-5 may be necessary

when finding new bounds, these steps come in

handy for more complicated 3D integrals involving

multiple surfaces & multiple original inequalities.

Also, step 2 is useful for setting up a triple

integral from scratch.

If you chose messy bounds for some originally described region, switch not by going through 1–5, but just going back to the start.

Average Value & Center of Mass

Average value of function f over solid space E:

- Equal to $(1/V) \iiint_E f(x, y, z)\, dV$ where V is volume of E

Center of mass of E with weight function f is $(x_{COM}, y_{COM}, z_{COM})$

- $x_{COM} = (\iiint_E x\, f(x, y, z)\, dV)/(\iiint_E f(x, y, z)\, dV)$
- Similarly for y_{COM}, z_{COM}

Only time to switch bounds the long way is if you're already given the triple integral

Orig surface

way 1 of setup

way 2 of setup

1st int $\iiint$

2nd int $\iiint$

don't take red path

Practice Problems

Evaluate $\iiint_E f(x, y, z)\, dV$ for these functions and regions:

- $f(x, y, z) = x$, E is region under $2x+3y+z = 6$ in the 1st octant
- $f(x,y, z) = (3x^2+3z^2)^{1/2}$, E is region bound by $y = 2x^2+2z^2$ and $y = 8$
- $f(x, y, z) = yz$, E is region bound by $x = 2y^2+2z^2-5$ and $x = 1$

Find the volume of the solid bound by $z = 8-x^2-y^2$, $z = -2(x^2+y^2)^{½}$, and $x^2+y^2 = 4$

Try these practice problems. Today's material is fair game ~~for~~ quiz 3 due Sunday.

Scratchwork

Quiz 3 covers M–F this week

Same format as quiz 2, may also

include an extra credit problem