Topics: 12.5 Equations of Lines and Planes; 13.1 Vector Functions and Space Curves
Please put away all electronic devices, including cell phones and calculators.

1. Find the following equations of the line through the point $P(2, 2, 4)$ that is parallel to the vector $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k}$. For parts (a) and (b), use the parameter t.
   (a) Vector equation
   (b) Parametric equations
   (c) Symmetric equations

The position vector of the point P is $\overrightarrow{OP} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Since the line is parallel to $\mathbf{a}$, we may take the direction vector to be $\mathbf{v} = \mathbf{a} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k}$. Any point on the line has position vector $\mathbf{r}$.

(a) A vector equation of the line through P with direction vector $\mathbf{v}$ is

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v} = (2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) + t(4\mathbf{i} - \mathbf{j} + 7\mathbf{k}).$$

Equivalently,

$$\mathbf{r}(t) = (2 + 4t)\mathbf{i} + (2 - t)\mathbf{j} + (4 + 7t)\mathbf{k}.$$

(b) Parametric equations are

$$x(t) = 2 + 4t, \quad y(t) = 2 - t, \quad z(t) = 4 + 7t.$$

(c) Symmetric equations are

$$\frac{x-2}{4} = \frac{y-2}{-1} = \frac{z-4}{7}.$$

2. (a) Find symmetric equations of the line through the point $P(3, 4, 0)$ that is perpendicular to both vectors $2\mathbf{i} + 2\mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

A direction vector $\mathbf{v}$ for the desired line must be perpendicular to both given vectors, so we can take the cross product:

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

Thus a direction vector is $\langle 2, -2, 2 \rangle$. The symmetric equations are

$$\frac{x-3}{2} = \frac{y-4}{-2} = \frac{z-0}{2}.$$

(b) Find the point where the line from part (a) intersects the yz-plane.

The yz-plane is given by $x = 0$. Using the symmetric equations,

$$\frac{0-3}{2} = \frac{y-4}{-2} = \frac{z-0}{2}.$$

So

$$-\frac{3}{2} = \frac{y-4}{-2} \quad \text{and} \quad -\frac{3}{2} = \frac{z}{2}.$$

Solving gives $y = 7$ and $z = -3$. Therefore, the line intersects the yz-plane at

$$(0, 7, -3).$$

3. Determine whether the lines L_1 and L_2 are parallel, intersecting, or skew.

(a) $L_1 : \mathbf{r}(t) = \langle -1+3t,\ 2+4t,\ 3-2t \rangle$, $\qquad L_2 : \dfrac{x-1}{2} = \dfrac{y}{-3} = \dfrac{z+1}{-3}$

Write L_2 in parametric form by letting

$$\frac{x-1}{2} = \frac{y}{-3} = \frac{z+1}{-3} = s.$$

Then

$$x = 1+2s, \quad y = -3s, \quad z = -1-3s,$$

so a direction vector for L_2 is $\mathbf{v}_2 = \langle 2, -3, -3 \rangle$. A direction vector for L_1 is $\mathbf{v}_1 = \langle 3, 4, -2 \rangle$. Since $\mathbf{v}_1$ is not a scalar multiple of $\mathbf{v}_2$, the lines are not parallel. To check whether they intersect, solve

$$-1+3t = 1+2s, \quad 2+4t = -3s, \quad 3-2t = -1-3s.$$

From the first two equations, we obtain $t = \frac{2}{17}$ and $s = -\frac{14}{17}$. Substituting into the third equation gives

$$3 - 2\left(\frac{2}{17}\right) \neq -1 - 3\left(-\frac{14}{17}\right),$$

so the system is inconsistent and the lines do not intersect. Therefore, the lines are **skew**.

(b) $L_1 : x = 2t,\ y = -3+t,\ z = 5-t$, $\qquad L_2 : x = 3-3s,\ y = 2 - \dfrac{3}{2}s,\ z = \dfrac{3}{2}s$

Direction vectors are $\mathbf{v}_1 = \langle 2, 1, -1 \rangle$ and $\mathbf{v}_2 = \left\langle -3, -\frac{3}{2}, \frac{3}{2} \right\rangle$. Since

$$\mathbf{v}_1 = -\frac{2}{3}\,\mathbf{v}_2,$$

the lines are **parallel**. Note that they are not the same line (for example, $(0, -3, 5)$ lies on L_1 when $t = 0$, but it does not satisfy the equations of L_2 for any s).

4. Find an equation of the plane through the points $A(0, 1, 2)$, $B(1, 2, 3)$, and $C(2, 3, 5)$.

First find direction vectors in the plane:

$$\overrightarrow{AB} = \langle 1, 1, 1 \rangle, \qquad \overrightarrow{AC} = \langle 2, 2, 3 \rangle.$$

A normal vector is their cross product:

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} = (1 \cdot 3 - 1 \cdot 2)\,\mathbf{i} - (1 \cdot 3 - 1 \cdot 2)\,\mathbf{j} + (1 \cdot 2 - 1 \cdot 2)\,\mathbf{k} = \langle 1, -1, 0 \rangle.$$

Using point $A(0, 1, 2)$, the plane equation is

$$1(x - 0) + (-1)(y - 1) + 0(z - 2) = 0.$$

This can be rewritten as:

$$x - y = -1.$$

5. Find an equation of the plane through the points $(0, -2, 5)$ and $(-1, 3, 1)$ that is perpendicular to the plane $2z = 5x + 4y$.

Rewrite the given plane as $5x + 4y - 2z = 0$, so a normal vector is $\mathbf{n}_1 = \langle 5, 4, -2 \rangle$.

Let $A(0, -2, 5)$ and $B(-1, 3, 1)$. Then

$$\overrightarrow{AB} = \langle -1, 5, -4 \rangle$$

lies in the desired plane, so it is perpendicular to the desired plane's normal vector $\mathbf{n}_2$. Also, since the planes are perpendicular, $\mathbf{n}_2$ is perpendicular to $\mathbf{n}_1$. Hence $\mathbf{n}_2$ is perpendicular to both $\overrightarrow{AB}$ and $\mathbf{n}_1$, so we can take

$$\mathbf{n}_2 = \overrightarrow{AB} \times \mathbf{n}_1 = \langle -1, 5, -4 \rangle \times \langle 5, 4, -2 \rangle = \langle 6, -22, -29 \rangle.$$

Using point $A(0, -2, 5)$, the plane equation is

$$6(x - 0) - 22(y + 2) - 29(z - 5) = 0.$$

This can be rewritten as:

$$6x - 22y - 29z = -101.$$

6. Find parametric equations for the line of intersection of the planes

$$2x + 3y + 5z = 7 \quad \text{and} \quad x - y + 2z = 3.$$

Normal vectors to the planes are

$$\mathbf{n}_1 = \langle 2, 3, 5 \rangle, \qquad \mathbf{n}_2 = \langle 1, -1, 2 \rangle.$$

A direction vector for the line of intersection is

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 1 & -1 & 2 \end{vmatrix} = 11\mathbf{i} + \mathbf{j} - 5\mathbf{k} = \langle 11, 1, -5 \rangle.$$

To find a point on the line, solve the system: Setting $y = 0$ gives

$$2x + 5z = 7, \qquad x + 2z = 3,$$

which yields $x = 1$ and $z = 1$. Thus $P_0(1, 0, 1)$ is on the line. Therefore,

$$\langle x, y, z\rangle = \langle 1, 0, 1\rangle + t\langle 11, 1, -5\rangle$$

so parametric equations are

$$x = 1 + 11t, \quad y = t, \quad z = 1 - 5t.$$

7. Find an equation of the plane through $P(7, -2, -4)$ that is parallel to the plane $z = 4x - 5y$.

Rewrite the equation of the given plane as $4x - 5y - z = 0$, so a normal vector is $\mathbf{n} = \langle 4, -5, -1\rangle$. A parallel plane through $P(7, -2, -4)$ has the same normal vector:

$$4(x - 7) - 5(y + 2) - (z + 4) = 0.$$

This can be rewritten as:

$$4x - 5y - z = 42.$$

8. Find an equation of the plane that passes through the point $P(10, -1, 5)$ and contains the line with symmetric equations $\dfrac{x}{4} = y + 6 = \dfrac{z}{5}$.

Setting $\frac{x}{4} = y + 6 = \frac{z}{5} = t$, we get a parametric form of the line:

$$x = 4t, \quad y = t - 6, \quad z = 5t,$$

so a direction vector is $\mathbf{v}_1 = \langle 4, 1, 5\rangle$. Taking $t = 0$ gives a point on the line: $A(0, -6, 0)$. The vector from A to P is

$$\mathbf{v}_2 = \overrightarrow{AP} = \langle 10 - 0,\ -1 - (-6),\ 5 - 0\rangle = \langle 10, 5, 5\rangle,$$

which also lies in the plane. Therefore a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & 5 \\ 10 & 5 & 5 \end{vmatrix} = \langle -20, 30, 10\rangle.$$

Using point $P(10, -1, 5)$, the plane equation is

$$-20(x - 10) + 30(y + 1) + 10(z - 5) = 0.$$

This can be rewritten as:

$$-20x + 30y + 10z = -180.$$

9. Determine whether the line L given by $\mathbf{r}(t) = \langle -2t, 2 + 7t, -1 - 4t\rangle$ intersects the plane given by $4x + 9y - 2z + 8 = 0$.

Write parametric equations for L:

$$x = -2t, \quad y = 2 + 7t, \quad z = -1 - 4t.$$

Substitute into the plane equation:

$$4(-2t) + 9(2 + 7t) - 2(-1 - 4t) + 8 = 0$$

$$-8t + 18 + 63t + 2 + 8t + 8 = 0 \implies 63t + 28 = 0 \implies t = -\frac{4}{9}.$$

Since a real value of t exists, the line intersects the plane. The intersection point is

$$(x, y, z) = \left(-2\left(-\frac{4}{9}\right), 2 + 7\left(-\frac{4}{9}\right), -1 - 4\left(-\frac{4}{9}\right)\right) = \left(\frac{8}{9}, -\frac{10}{9}, \frac{7}{9}\right).$$

10. Find the domain of the vector function

$$\mathbf{r}(t) = \left\langle \ln(t + 1), \frac{t}{4 - t^2}, \sqrt{4 - t} \right\rangle.$$

We first find the domain of each component of the vector function.

The domain of $\ln(t + 1)$ is given by $t + 1 > 0$, so it is the set $(-1, \infty)$.

The domain of $\dfrac{t}{4 - t^2} = \dfrac{t}{(2 + t)(2 - t)}$ is given by $t \neq 2$ and $t \neq -2$. In interval notation, that is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

The domain of $\sqrt{4 - t}$ is given by $4 - t \geq 0$, so it is the set $(-\infty, 4]$.

The domain of $\mathbf{r}(t)$ is the intersection of the domains of its components. So the domain of $\mathbf{r}(t)$ is

$$(-1, 2) \cup (2, 4].$$

11. Evaluate the limit $\lim_{t \to 1} \mathbf{r}(t)$, where

$$\mathbf{r}(t) = \left\langle \frac{t^2 - 1}{t^2 - 3t + 2}, \frac{t - 1}{\sqrt{t + 3} - 2}, \frac{\sin(t - 1)}{t - 1} \right\rangle.$$

To find the limit of a vector-valued function, we must compute

$$\lim_{t \to 1} \mathbf{r}(t) = \left\langle \lim_{t \to 1} \frac{t^2 - 1}{t^2 - 3t + 2}, \lim_{t \to 1} \frac{t - 1}{\sqrt{t + 3} - 2}, \lim_{t \to 1} \frac{\sin(t - 1)}{t - 1} \right\rangle.$$

For the first component,

$$\lim_{t \to 1} \frac{t^2 - 1}{t^2 - 3t + 2} = \lim_{t \to 1} \frac{(t + 1)(t - 1)}{(t - 1)(t - 2)} = \lim_{t \to 1} \frac{t + 1}{t - 2} = \frac{2}{-1} = -2.$$

For the second component, rationalize:

$$\lim_{t\to 1} \frac{t-1}{\sqrt{t+3}-2} = \lim_{t\to 1} \frac{(t-1)(\sqrt{t+3}+2)}{(t+3)-4} = \lim_{t\to 1} \left(\sqrt{t+3}+2\right) = \sqrt{4}+2 = 4.$$

For the third component, use L'Hôpital's rule:

$$\lim_{t\to 1} \frac{\sin(t-1)}{t-1} = \lim_{t\to 1} \frac{\cos(t-1)}{1} = \frac{\cos(0)}{1} = 1$$

Therefore,

$$\lim_{t\to 1} \mathbf{r}(t) = \langle -2, 4, 1\rangle.$$

SOME USEFUL DEFINITIONS, THEOREMS, AND NOTATION

Equations of a line. The line through the point (x_0, y_0, z_0) and parallel to the vector $\langle a, b, c\rangle$ has **vector equation**

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle.$$

The **parametric equations** for this line are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct.$$

We can eliminate t to obtain the **symmetric equations**

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

provided a, b, and c are all nonzero.

Equations of a plane. The plane through (x_0, y_0, z_0) with normal vector $\langle a, b, c\rangle$ is given by

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0.$$

By collecting terms, we can rewrite this equation in the form

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$ is a constant. Two planes are parallel if their normal vectors are parallel, and they are perpendicular if their normal vectors are orthogonal.

Vector functions. A **vector-valued function** (or **vector function**) is a function whose domain is a set of real numbers and whose range is a set of vectors. In three dimensions, a vector-valued function is written as

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}.$$

where f, g, and h are real-valued **component functions**. The domain of $\mathbf{r}(t)$ is the set of all values t for which $f(t)$, $g(t)$, and $h(t)$ are all defined. Limits of vector functions are defined component-wise: For a real number a,

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right\rangle,$$

provided the three component limits exist.

Suggested Textbook Problems
Section 12.5: 1-72
Section 12.6: 1-49
Section 13.1: 1-58