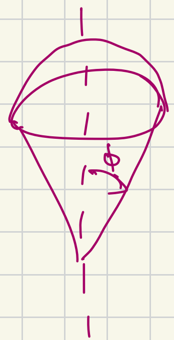
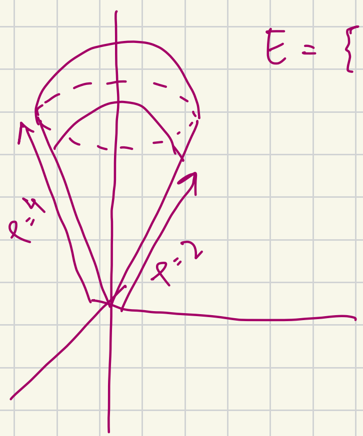
Hints for WebAssign:
10. Write the equation x^2-y^2-z^2=1 in spherical coordinates
11. E lies above the cone \phi=\pi/3 and below the sphere \rho=1
14. E lies above the cone z=\sqrt{x^2+y^2} and between the spheres x^2+y^2+z^2=4 and x^2+y^2+z^2=16
15 Part of the ball \rho\leq a that lies between the cones \phi=\pi/6 and \phi=\pi/3

10. $x^2-y^2-z^2=1 \Rightarrow (\rho\sin\phi\cos\theta)^2-(\rho\sin\phi\sin\theta)^2-(\rho\cos\phi)^2=1$

11. $E=\{(\rho,\theta,\phi) : 0\le\rho\le 1,\ 0\le\theta\le 2\pi,\ 0\le\phi\le \pi/3\}$



14.

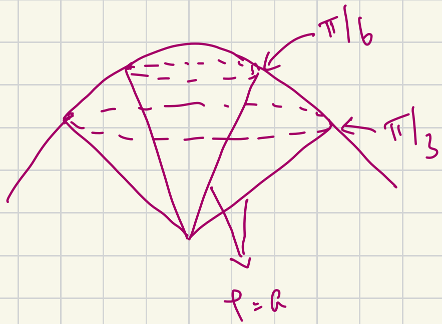


$E=\{(\rho,\theta,\phi) : 2\le\rho\le 4,\ 0\le\theta\le 2\pi,\ \underbrace{0\le\phi\le\pi/4}\}$

$y=0 \Rightarrow z^2=x^2$

$\Rightarrow z=x$

15.



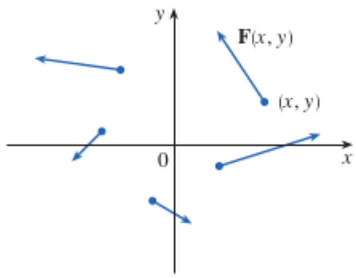
$$E = \{ (\rho, \theta, \phi) ;\ 0 \le \rho \le a,\ 0 \le \theta \le 2\pi,\ \frac{\pi}{6} \le \phi \le \frac{\pi}{3} \}$$

16. Vector Calculus

Vector Fields

1. **Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$**

> **Definition** Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field** on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$



To picture a vector field: to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y).

Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

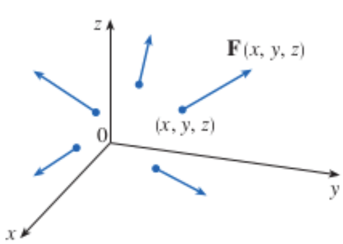
or

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$$

Note: both P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

Vector fields are used to model velocity vector fields, force fields, electric fields, fluid flow, etc.

> **Definition** Let E be a subset of $\mathbb{R}^3$. A **vector field** on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.



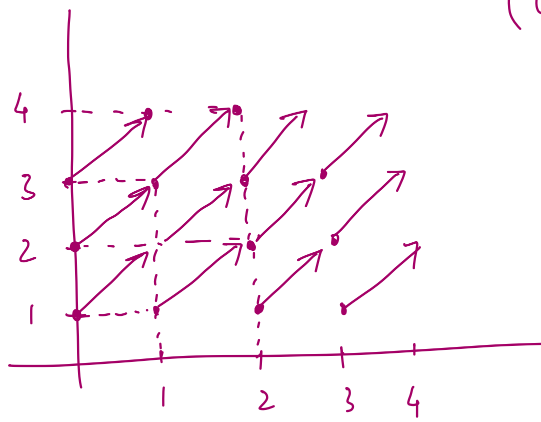
A vector field in $\mathbb{R}^3$ is a function $\mathbf{F}$ of the form

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

$\mathbf{F}$ is continuous if and only if its component functions P, Q, and R are continuous.

Example: $\mathbf{F}(x, y) = \langle 1, 1 \rangle$

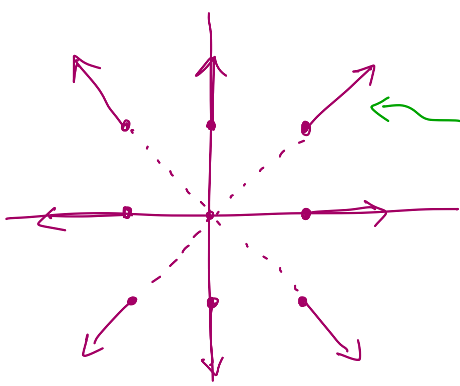
(Constant vector field)



Example: $\mathbf{F}(x, y) = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$

Recall: $\|\langle x,y\rangle\| = \sqrt{x^2+y^2}$

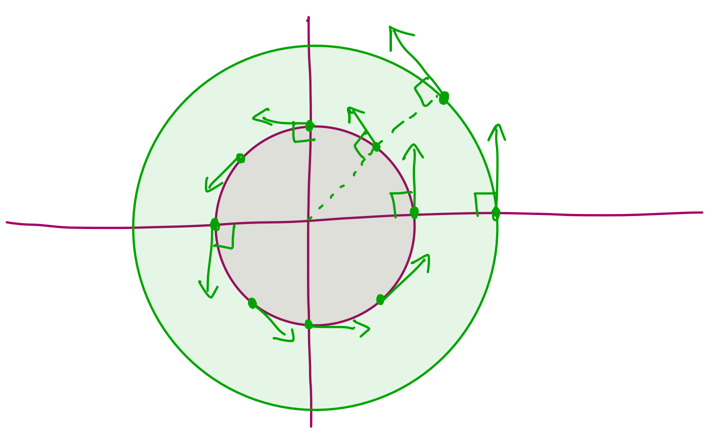
$\Rightarrow F(x,y) = \dfrac{1}{\|\langle x,y\rangle\|} \cdot \langle x,y\rangle$



unit vector pointing in the direction of $\langle x,y\rangle$.

Example: $\mathbf{F}(x, y) = \langle -y, x \rangle$

Note: $\langle -y, x\rangle \cdot \langle x,y\rangle = -xy + xy = 0 \Rightarrow F(x,y) \perp \langle x,y\rangle$



2. **Gradient Fields**

If f is a scalar function of two variables, its gradient ∇f (or grad f) is defined by

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

Therefore, ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**.

Similarly, if f is a scalar function of three variables, its gradient ∇f (or grad f) is defined by

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

is a vector field on $\mathbb{R}^3$.

Example: Find the gradient vector field of $f(x,y) = ax + by$.

$f_x = a$, $f_y = b \Rightarrow \nabla f = \langle f_x, f_y \rangle = \underbrace{\langle a, b \rangle}_{\text{constant}}$

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

Example: Every constant vector field is conservative.

Let $F(x,y) = \langle a,b \rangle$ then $F = \nabla f$ where $f(x,y) = ax + by$

Also, $g(x,y) = ax + by + c \Rightarrow F = \nabla f$.

Remark: Not all vector fields are conservative.
Example: $\mathbf{F}(x,y) = \langle x^2, xy \rangle$ is not conservative.

i.e, there is no scalar valued f such that $F = \nabla f$.

Proof by contradiction:

Suppose that there was. Then $f_x = x^2$ and $f_y = xy$.

$\Rightarrow f_{xy} = 0$ and $f_{yx} = y \Rightarrow f_{xy} \neq f_{yx}$ But f_{xy} and f_{yx} are continuous $\Rightarrow$ Highlighted assumption contradicts Clairaut's theorem.

Recall: Clairaut's theorem said that if f_{xy} and f_{yx} are CTS then $f_{xy} = f_{yx}$.