Evaluate the double integral $\iint_R \sin x \cos y \, dA$ where $R = [0, \pi/2] \times [0, \pi/2]$.

$$\underbrace{\iint_R \sin x \cos y \, dA}_{\text{||}} \qquad \underbrace{R = [0,\pi/2]\times[0,\pi/2]}_{\text{Rectangle}}$$

$$\int_{y=0}^{\pi/2}\int_{x=0}^{\pi/2} \sin x \cos y \, dx\, dy$$

Important: When $f(x,y)$ is CTS $\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$

But one of those integrals might be way more annoying than the other. Worse, one of those might involve an integral that cannot be represented by elementary functions.

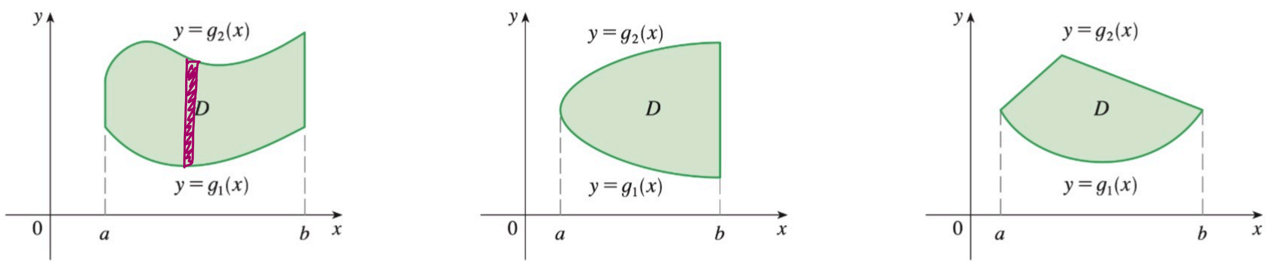
A: $\int_0^{\pi/2}\int_0^{\pi/2} \sin x \cos y \, dx\, dy = \int_0^{\pi/2} \left[-\cos x\right]_{x=0}^{\pi/2} \cos y \, dy = \int_0^{\pi/2} \left[-(0-1)\right] \cos y \, dy$

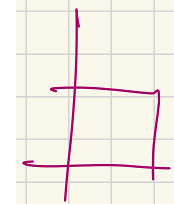
$$= \int_0^{\pi/2} \cos y \, dy = \sin y \Big|_0^{\pi/2} = 1 - 0 = \boxed{1}.$$

1. We say that D is of *Type I* if it has the form

$$D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}$$

for some continuous functions g_1 and g_2 on $[a, b]$.





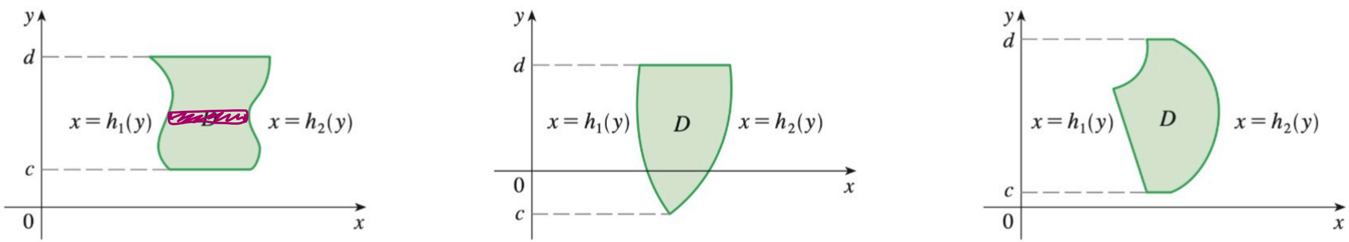
In this case, the double integral of f over the region D is

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \underbrace{f(x,y)}_{\text{height}}\, \underbrace{dy}_{\text{length}}\, \underbrace{dx}_{\text{width}}$$

2. We say that D is of *Type II* if it has the form

$$D = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } h_1(y) \leq x \leq h_2(y)\}$$

for some continuous functions h_1 and h_2 on $[c, d]$.



In this case, the double integral of f over the region D is

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy.$$

Find the volume of the tetrahedron bounded by the planes

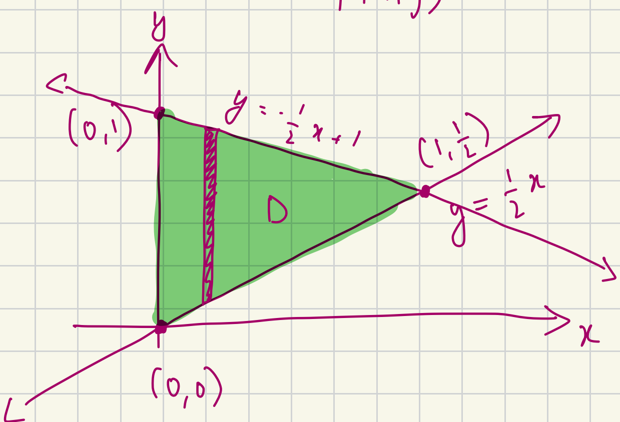
$$\underbrace{x + 2y + z = 2}, \quad \overbrace{x = 2y}^{\iff y = \frac{1}{2}x}, \quad x = 0, \quad z = 0.$$

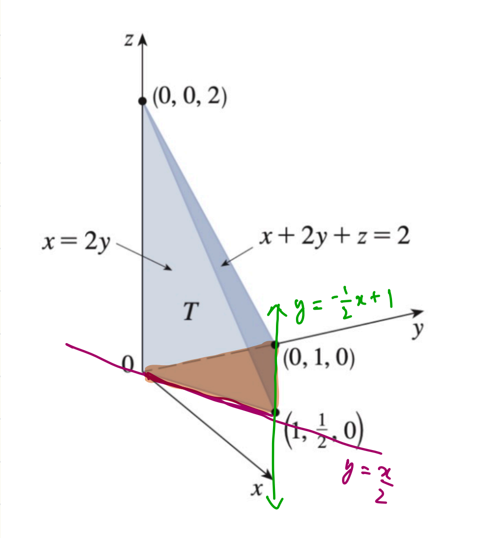
$$\Rightarrow z = f(x,y) = 2 - x - 2y \Rightarrow \text{at } z = 0, \; x + 2y = 2$$

$$\Updownarrow$$

$$2y = -x + 2$$

$$\iff y = \frac{-1}{2}x + 1$$





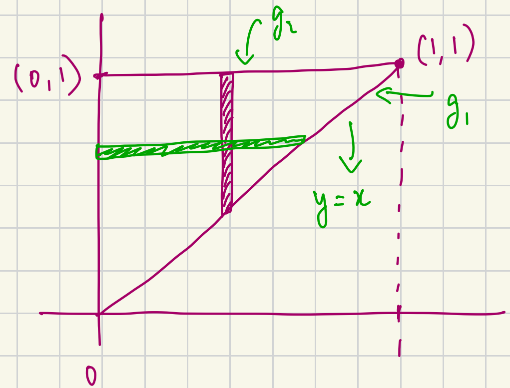
$$\cdot \quad y = \frac{1}{2}x = \frac{-1}{2}x + 1 \iff x = 1 \Rightarrow y = \frac{1}{2}$$

$$\iint_D f(x,y)\, dA = \int_0^1 \int_{\frac{1}{2}x}^{-\frac{1}{2}x+1} 2 - x - 2y \; dy\, dx \overset{\boxed{\text{DIY!}}}{=} \boxed{\frac{1}{3}}$$

Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

$\int \sin(y^2)\,dy$ is not an elementary function.

$$D = \{ (x,y) : 0 \le x \le 1 ,\ \underbrace{x}_{= g_1(x)=x} \le y \le \underbrace{1}_{= g_2(x)=1} \}$$



$$D = \{ (x,y) : 0 \le y \le 1 ,\ 0 \le x \le y \}$$

For any fixed y, (x,y) is in the region $\Leftrightarrow$ $0 \le x \le y$

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 \sin(y^2)\, [x]_0^y\, dy$$

$$= \int_0^1 \sin(y^2)\,(y-0)\,dy = \int_0^1 y \sin(y^2)\,dy$$

$$u = y^2 \Rightarrow \frac{du}{2} = y\,dy \qquad y=0 \Rightarrow u=0,\ y=1 \Rightarrow u=1$$

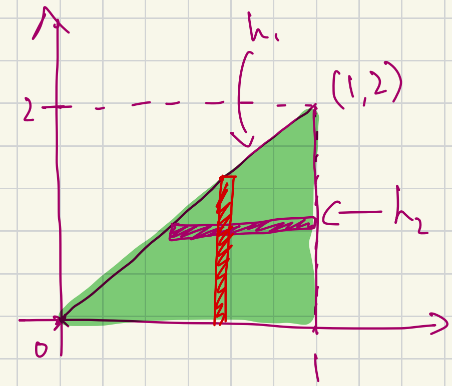
$$= \int_0^1 \sin(u)\,\frac{du}{2} = -\frac{\cos(u)}{2}\Big|_0^1 = -\frac{1}{2}(\cos 1 - \cos 0)$$

$$= \frac{1}{2}(1-\cos 1)$$

Evaluate the iterated integral $\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$.

$D = \{ (x,y) : 0 \le y \le 2, \underbrace{\frac{y}{2}}_{h_1(y) = \frac{y}{2}} \le x \le \underbrace{1}_{h_2(y)=1} \}$

i.e. $x = \frac{y}{2} \iff y = 2x$



$D = \{ (x,y) : 0 \le x \le 1, 0 \le y \le 2x \}$

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 e^{x^2} \left(y \Big|_0^{2x} \right) dx = \int_0^1 2x\, e^{x^2} dx$$

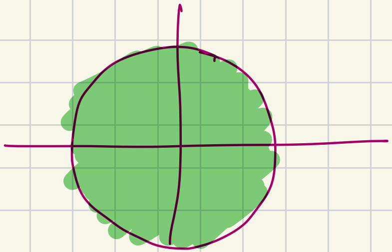
$u = x^2 \Rightarrow du = 2x\, dx$, $x = 0 \Rightarrow u = 0$

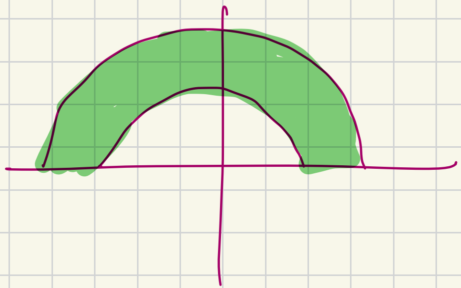
$x = 1 \Rightarrow u = 1$

$$= \int_0^1 e^u\, du = e^u \Big|_0^1 = e - 1$$

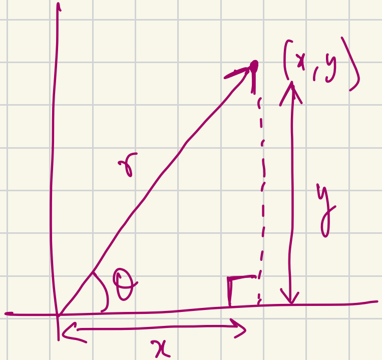
Section 15.3: Integrating in Polar coordinates.

Eg: Suppose you want to evaluate $\iint_D f(x,y)\,dA$ where D is like:





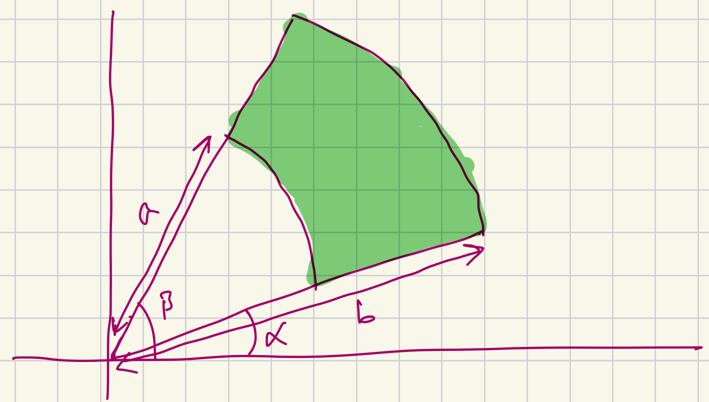
Definition: For any point $(x,y) \in \mathbb{R}^2$ it's polar coordinates are (r, θ) where $r = \|(x,y)\|$ and θ is the angle subtended by $\langle x,y \rangle$ and the x-axis



$$\left(r = \sqrt{x^2 + y^2} \right)$$

Definition: Polar rectangle is of the form

$$R = \{ (r, \theta) : a \le r \le b , \alpha \le \theta \le \beta \}$$

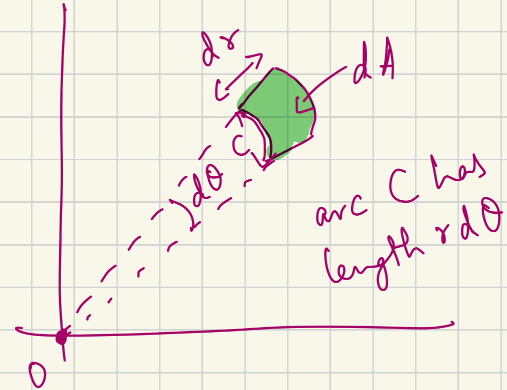


Example: $R = \{ (x, y) : x^2 + y^2 \le 1 \}$

$$= \{ (r, \theta) : 0 \le r \le 1 , 0 \le \theta \le 2\pi \}$$

Integrating in polar coordinates:

- Area of an infinitesimal polar rectangle



$\Rightarrow dA \approx r\,d\theta\,dr$

- $$\iint_R f(x,y)\,dA = \int_a^b \int_\alpha^\beta f(x,y)\,r\,d\theta\,dr$$

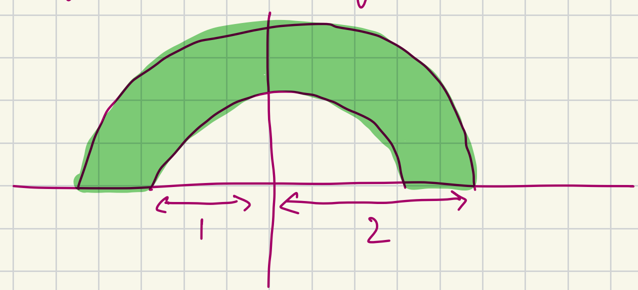
where $R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$.

Example: Evaluate $\iint_R (3x + 4y^2)\,dA$

$r = 1$ $\quad$ $r^2 \Rightarrow r = 2$

where R is the region between $x^2+y^2=1$ and $x^2+y^2=4$ in the upper half plane.

$R = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$



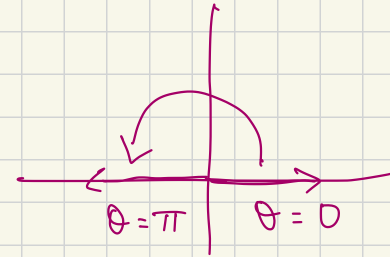
$$\Rightarrow \iint_R 3x + 4y^2 \, dA = \int_1^2 \int_0^{\pi} \left(3(r\cos\theta) + 4(r\sin\theta)^2\right) r \, dr \, d\theta$$

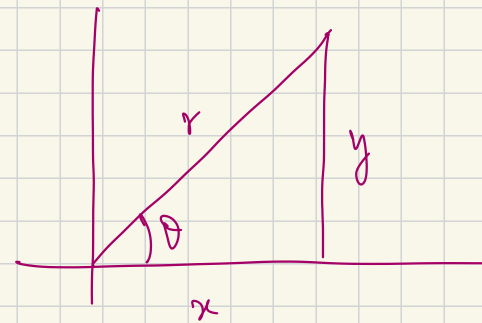
Fubini

$$= \int_0^{\pi} \int_1^2 3r^2 \cos\theta + 4r^3 \sin^2\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \int_1^2 3r^2 \cos\theta \, dr \, d\theta + \int_0^{\pi} \int_1^2 4r^3 \sin^2\theta \, dr \, d\theta$$

$$\overset{\boxed{\text{DIY}}}{=} \boxed{\frac{15\pi}{2}}$$





$$\Rightarrow \cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}$$

$$\Rightarrow x = r\cos\theta, \quad y = r\sin\theta.$$