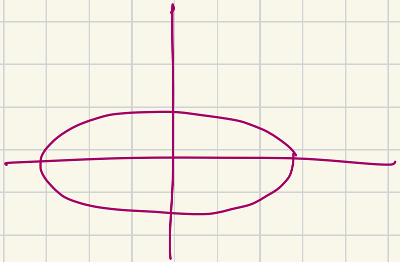
17. Find the extreme values of the function $f(x,y)=xy$ on the ellipse $36x^2+y^2=72$.

$36x^2+y-72=0$

A: ① $f(x,y) = xy$



$g(x,y) = 36x^2+y^2-72$

② $\nabla f = \langle y, x \rangle$

$\nabla g = \langle 72x, 2y \rangle$

③ (x,y) are extreme values $\Leftrightarrow$ $\nabla f(x,y) = \lambda \nabla g(x,y)$ for some $\lambda \in \mathbb{R}$.

So solve for x, y, λ in

$$f_x(x,y) = \lambda g_x(x,y)$$
$$f_y(x,y) = \lambda g_y(x,y)$$
$$g(x,y) = 0.$$

(4) $y = \lambda(72x) \Rightarrow x = 2\lambda(72\lambda x) \Rightarrow x = 144\lambda^2 x$

$x = \lambda(2y)$ $\Rightarrow x(1-144\lambda^2) = 0$

$36x^2 + y^2 - 72 = 0$

$(0, 6\sqrt{2})$ $\Leftarrow$ $\boxed{\text{Case 1}}$: $x = 0 \Rightarrow y^2 = 72 \Rightarrow y = \pm\sqrt{72} = \pm\sqrt{2\times 36} = \pm 6\sqrt{2}$

$(0, -6\sqrt{2})$

Case 2 : $1 - 144\lambda^2 = 0 \Rightarrow \lambda^2 = \frac{1}{144} \Rightarrow \lambda = \frac{1}{\pm\sqrt{144}} = \frac{1}{\pm 12}$

$\boxed{\text{Case 2a}}$ $\lambda = \frac{1}{12} \Rightarrow y = \frac{72}{12}x = 6x$

$\swarrow$

$(1, 6)$ $\Rightarrow 36x^2 + 36x^2 - 72 = 0 \Rightarrow 72x^2 - 72 = 0$

$\Rightarrow x = \pm 1$

$(-1, -6)$

$\boxed{\text{Case 2b}}$ $\lambda = \frac{-1}{12} \Rightarrow y = \frac{-72}{12}x = -6x$

$(1, -6)$ $\Rightarrow 36x^2 + (-6x)^2 - 72 = 0$

$(-1, 6)$ $\Rightarrow 36x^2 + 36x^2 - 72 = 0 \Rightarrow x = \pm 1$

	$(0, 6\sqrt{2})$	$(0, -6\sqrt{2})$	$(1, 6)$	$(-1, -6)$	$(1, -6)$	$(-1, 6)$
$f(x,y) = xy$	0	0	6	6	-6	-6

f has a maxima at $(1, 6)$ and $(-1, -6)$
and minima at $(1, -6)$ and $(-1, 6)$

4. Given $e^z = xyz$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

A: $\cdot\ F(x,y,z) = e^z - xyz \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

$$F_x = \frac{\partial F}{\partial x} = -yz \quad , \quad F_y = \frac{\partial F}{\partial y} = -xz \quad , \quad F_z = e^z - xy$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{yz}{e^z - xy} \quad , \quad \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

$$\cdot \quad e^z = xyz \Rightarrow e^z \frac{\partial z}{\partial x} = yz + xy \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$$

$$\Downarrow$$

$$\frac{de^z}{dz} \cdot \frac{dz}{dx} = y \left(\frac{dx}{dx} \cdot z + x \cdot \frac{dz}{dz} \cdot \frac{dz}{dx} \right)$$

5. $f(x,y) = \ln(x - 4y)$

(a) Find the linearization at $(5,1)$

A: $\cdot\ f_x = \frac{1}{x-4y}$, $f_y = \frac{1}{x-4y} \cdot (-4)$

- $f_x(5,1) = \dfrac{1}{5-4} = 1$, $f_y(5,1) = \dfrac{-4}{5-4} = -4$

- Equation for tangent plane of f at $(5,1)$ is

$$f_x(5,1)(x-5) + f_y(5,1)(y-1) = z - \underbrace{f(5,1)}_{= \ln(1) = 0}$$

$$\Rightarrow z = (x-5) + (-4)(y-1)$$

- $$\Rightarrow \boxed{L(x,y) = x - 5 - 4y + 4 = x - 4y - 1}$$

- $L(x,y) = f(5,1) + f_x(5,1)(x-5) + f_y(5,1)(y-1)$

(b) Use (a) to approximate $f(4.9, 0.8)$

$$f(4.9, 0.8) \approx L(4.9, 0.8)$$
$$= 4.9 - 3.2 - 1 = 0.7$$

32. $f(x,y) = 100\, e^{-(x^2+y^2)/4}$

D is the disk of radius 4 centered at $(0,0) = \{(r,\theta) : 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

$= \{(x,y) : x^2+y^2 \le 16\}$

(a) Set up $\underbrace{\iint_D f(x,y)\, dA}_{I}$ in polar coordinates.

$f(r\cos\theta, r\sin\theta) = 100\, e^{-\frac{1}{4}(r^2\cos^2\theta + r^2\sin^2\theta)} = 100\, e^{-r^2/4}$ (note: $x^2+y^2 = r^2$)

$$I = \int_0^{2\pi} \int_0^4 100\, e^{-r^2/4}\, r\, dr\, d\theta$$

(b) Compute I. Hint: take $u = -\frac{r^2}{4}$

1. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ does not exist. $\to f(x,y)$

A: ① Take $y = 0$ (x-axis). Then $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2 \cdot 0}{x^4 + 0^2} = \lim_{x\to 0} \frac{0}{x^4}$

$$= \lim_{x \to 0} 0 = 0.$$

② Take $x = 0$ (y-axis). Then $\lim_{y \to 0} f(0, y) = \lim_{y \to 0} \frac{0 \cdot y}{0 + y^2} = \lim_{y \to 0} \frac{0}{y^2}$

$$= \lim_{y \to 0} 0 = 0$$

③ Take $y = x$. Then $\lim_{x \to 0} f(x, x) = \lim_{x \to 0} \frac{x^2 \cdot x}{x^4 + x^2} = \lim_{x \to 0} \frac{\cancel{x^2} \cdot x}{\cancel{x^2}(x^2+1)}$

$$= \lim_{x \to 0} \frac{x}{x^2 + 1} = \frac{0}{1} = 0$$

④ Take $y = x^2$. Then $\lim_{x \to 0} f(x, x^2) = \lim_{x \to 0} \frac{x^2 \cdot x^2}{x^4 + x^4} = \lim_{x \to 0} \frac{\cancel{x^4}}{2\cancel{x^4}} = \lim_{x \to 0} \frac{1}{2}$

$$= \frac{1}{2}$$

Since the limits along $y = 0$ and $y = x^2$ are different, the limit does not exist. $\boxed{}$

21. The volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + \alpha$ and above the rectangle $R = \underset{x}{[-1,1]} \times \underset{y}{[0,1]}$ is $\frac{10}{3}$.

Find the value of the constant α, where $\alpha \geq 1$.

A: Let $E(\alpha) = \{(x,y,z) : (x,y) \in R,\ 0 \leq z \leq \underbrace{3y^2 - x^2 + \alpha}_{= f(x,y)}\}$

Find α such that Volume $(E(\alpha)) = \frac{10}{3}$.

$$V(E(\alpha)) = \iint_R f(x,y)\, dA = \int_{-1}^{1} \int_{0}^{1} 3y^2 - x^2 + \alpha \, dy\, dx = \frac{4}{3} + 2\alpha$$

Solve for $\frac{4}{3} + 2\alpha = \frac{10}{3} \iff 2\alpha = 2 \Rightarrow \alpha = 1$.