12 Vectors and the Geometry of Space

12.4 The Cross Product

The Cross Product of Two Vectors

The Cross Product of Two Vectors (1 of 9)

Given two nonzero vectors $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$, suppose that a nonzero vector $\boldsymbol{c} = \langle c_1, c_2, c_3 \rangle$ is perpendicular to both **a** and **b**. Then $\boldsymbol{a} \cdot \boldsymbol{c} = 0$ and $\boldsymbol{b} \cdot \boldsymbol{c} = 0$ and so

$$a_1 c_1 + a_2 c_2 + a_3 c_3 = 0 \qquad (1)$$

$$b_1 c_1 + b_2 c_2 + b_3 c_3 = 0 \qquad (2)$$

The Cross Product of Two Vectors (2 of 9)

To eliminate c_3 we multiply (1) by b_3 and (2) by a_3 and subtract:

$$3 \quad (a_1 b_3 - a_3 b_1) c_1 + (a_2 b_3 - a_3 b_2) c_2 = 0$$

Equation 3 has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So a solution of (3) is

$$c_1 = a_2 b_3 - a_3 b_2 \quad c_2 = a_3 b_1 - a_1 b_3$$

The Cross Product of Two Vectors (3 of 9)

Substituting these values into (1) and (2), we then get

$$c_3 = a_1 b_2 - a_2 b_1$$

This means that a vector perpendicular to both **a** and **b** is

$$\langle c_1, c_2, c_3 \rangle = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

The resulting vector is called the *cross product* of **a** and **b** and is denoted by **a** × **b**.

4 Definition of the Cross Product $\mathrm{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathrm{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of **a** and **b** is the vector

$$\boldsymbol{a} \times \boldsymbol{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Notice that the **cross product a × b** of two vectors **a** and **b** is a vector (whereas the dot product, product is a scalar). For this reason it is also called the **vector product**.

Note that **a** × **b** is defined only when **a** and **b** are *three-dimensional* vectors.

The Cross Product of Two Vectors (5 of 9)

In order to make Definition 4 easier to remember, we use the notation of determinants.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

The Cross Product of Two Vectors (6 of 9)

A **determinant of order 3** can be defined in terms of second-order determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{5}$$

Observe that each term on the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears.

The Cross Product of Two Vectors (7 of 9)

Notice also the minus sign in the second term. For example,

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} &= 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix} \\
&= 1(0-4) - 2(6+5) + (-1)(12-0) \\
&= -38
\end{aligned}$$

The Cross Product of Two Vectors (8 of 9)

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors **i**, **j**, and **k**, we see that the cross product of the vectors **a** = a_1 **i** + a_2 **j** + a_3 **k** and **b** = b_1 **i** + b_2 **j** + b_3 **k** is

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

6 $$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathrm{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathrm{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathrm{k}$$

The Cross Product of Two Vectors (9 of 9)

In view of the similarity between Equations 5 and 6, we often write

$$7 \qquad \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain Equation 6.

The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.

Example 1

If $\mathrm{a} = \langle 1, 3, 4 \rangle$ and $\mathrm{b} = \langle 2,\ 7,\ -5 \rangle$, then

$$\begin{aligned}
\mathrm{a} \times \mathrm{b} &= \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathrm{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathrm{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathrm{k} \\
&= (-15 - 28)\mathrm{i} - (-5 - 8)\mathrm{j} + (7 - 6)\mathrm{k} \\
&= -43\mathrm{i} + 13\mathrm{j} + \mathrm{k}
\end{aligned}$$

Properties of the Cross Product

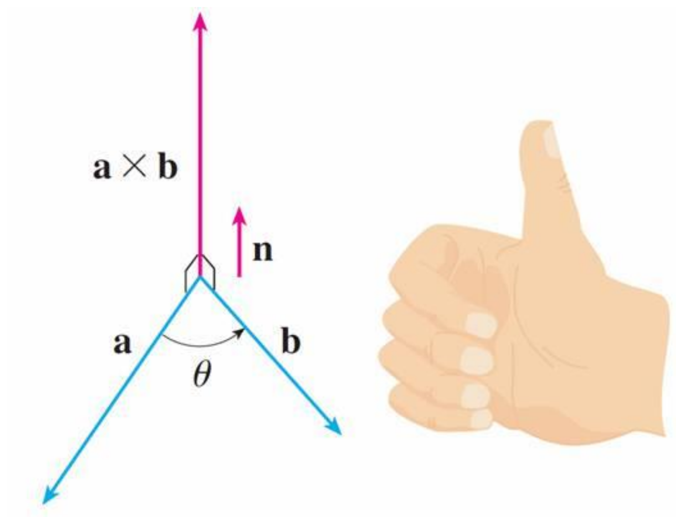
Properties of the Cross Product (1 of 10)

We constructed the cross product $\mathbf{a} \times \mathbf{b}$ so that it would be perpendicular to both **a** and **b**. This is one of the most important properties of a cross product.

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.

Properties of the Cross Product (2 of 10)

If **a** and **b** are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product **a × b** points in a direction perpendicular to the plane through **a** and **b**.



The right-hand rule gives the direction of **a** × **b**.

Figure 1

Properties of the Cross Product (3 of 10)

It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule*: If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from to $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Now that we know the direction of the vector $\mathbf{a} \times \mathbf{b}$, the remaining thing we need to complete its geometric description is its length $|\mathrm{a} \times \mathrm{b}|$. This is given by following theorem.

9 Theorem If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leq \theta \leq \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by

$$|\mathrm{a} \times \mathrm{b}| = |\mathrm{a}||\mathrm{b}| \sin \theta$$

Properties of the Cross Product (4 of 10)

10 Corollary Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

Since a vector is completely determined by its magnitude and direction, we can now say that for nonparallel vectors **a** and **b, a** × **b** is the vector that is perpendicular to both **a** and **b**, whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}||\mathbf{b}| \sin \theta$.

In fact, that is exactly how physicists *define* **a** × **b**.

Properties of the Cross Product (5 of 10)

The geometric interpretation of Theorem 9 can be seen by looking at Figure 2.

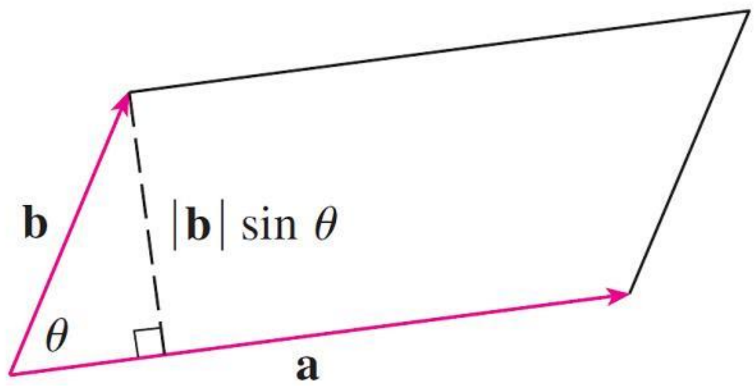


Figure 2

Properties of the Cross Product (6 of 10)

If **a** and **b** are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathrm{a}|$, altitude $|\mathrm{b}|\sin\theta$, and area

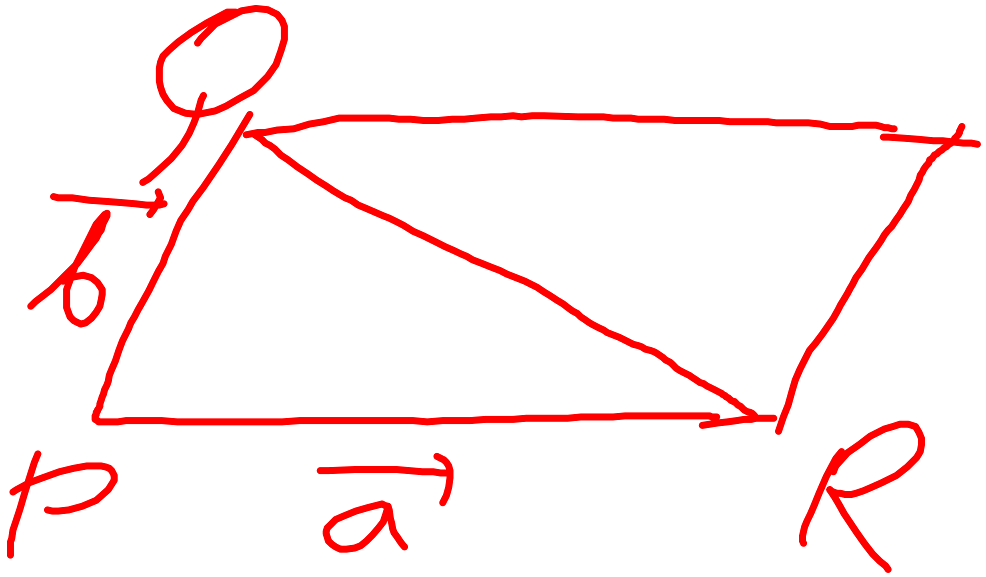
$$A = |\mathrm{a}|(|\mathrm{b}|\sin\theta) = |\mathrm{a}\times\mathrm{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product.

The length of the cross product **a** × **b** is equal to the area of the parallelogram determined by **a** and **b**.

Example 4

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.



$|a \times b|$

$\vec{b} = \langle -3, 1, -7 \rangle$

$\vec{a} = R - P = \langle 0, -5, -5 \rangle$

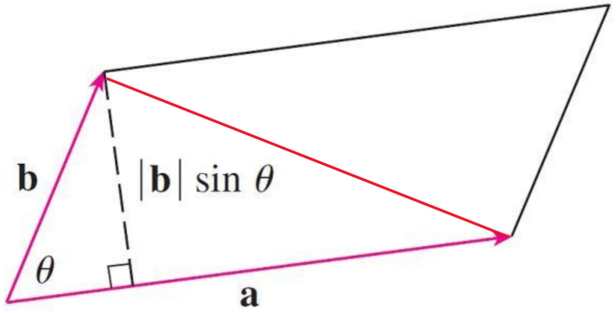
Example 4

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution:

We computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$\begin{aligned} \left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| &= \sqrt{(-40)^2 + (-15)^2 + 15^2} \\ &= 5\sqrt{82} \end{aligned}$$



The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$.

Properties of the Cross Product (7 of 10)

If we apply Theorems 8 and 9 to the standard basis vectors **i**, **j**, and **k** using $\theta = \frac{\pi}{2}$, we obtain

$$\begin{array}{ccc} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

Properties of the Cross Product (8 of 10)

Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0 \times \mathbf{j} = 0$$

So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products.

Properties of the Cross Product (9 of 10)

The following theorem summarizes the properties of vector products.

11 Properties of the Cross Product If **a**, **b**, and **c** are vectors and *c* is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Properties of the Cross Product (10 of 10)

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product.

If $\mathrm{a} = \langle a_1, a_2, a_3 \rangle, \mathrm{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathrm{c} = \langle c_1, c_2, c_3 \rangle$, then

$$
\begin{aligned}
12 \quad \mathrm{a} \cdot (\mathrm{b} \times \mathrm{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\
&= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\
&= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\
&= (\mathrm{a} \times \mathrm{b}) \cdot \mathrm{c}
\end{aligned}
$$

Triple Products

Triple Products (1 of 5)

The product **a** · (**b** × **c**) that occurs in Property 5 is called the **scalar triple product** of the vectors **a**, **b**, and **c**. Notice from Equation 12 that we can write the scalar triple product as a determinant:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{13}$$

Triple Products (2 of 5)

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**. (See Figure 3.)

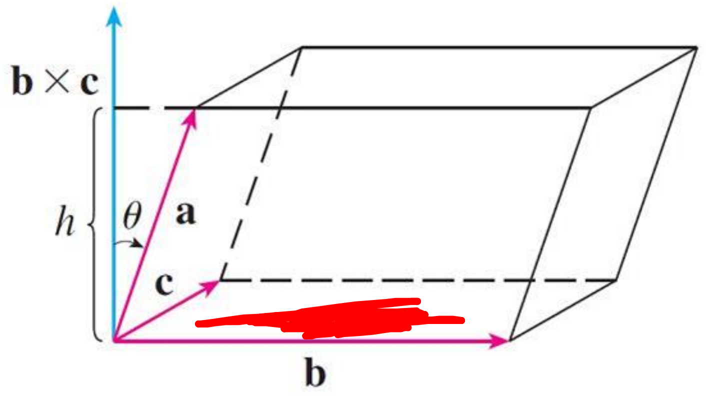


Figure 3

The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$.

Triple Products (3 of 5)

If θ is the angle between **a** and **b** × **c**, then the height h of the parallelepiped is $h = |\mathrm{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of cos θ in case $\theta > \frac{\pi}{2}$.) Therefore the volume of the parallelepiped is

$$V = Ah = |\mathrm{b} \times \mathrm{c}||\mathrm{a}||\cos\theta| = |\mathrm{a} \cdot (\mathrm{b} \times \mathrm{c})|$$ (by Theorem 12.3.3)

Thus we have proved the following formula.

14 The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathrm{a} \cdot (\mathrm{b} \times \mathrm{c})|$$

Triple Products (4 of 5)

If we use the formula in (14) and discover that the volume of the parallelepiped determined by **a**, **b**, and **c** is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

Example 5

Use the scalar triple product to show that the vectors
$\mathrm{a} = \langle 1,\ 4,\ -7\rangle, \mathrm{b} = \langle 2, -1, 4\rangle,$ and $\mathrm{c} = \langle 0, -9, 18\rangle,$ are coplanar.

Solution:

We use Equation 13 to compute their scalar triple product:

$$\mathrm{a} \cdot (\mathrm{b} \times \mathrm{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

Example 5 – Solution

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18)$$

$$= 0$$

Therefore, by (14), the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar.

Triple Products (5 of 5)

The product **a** × (**b** × **c**) that occurs in Property 6 is called the **vector triple product** of **a**, **b**, and **c**.

Application: Torque

Application: Torque (1 of 3)

The idea of a cross product occurs often in physics. In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**. (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.)

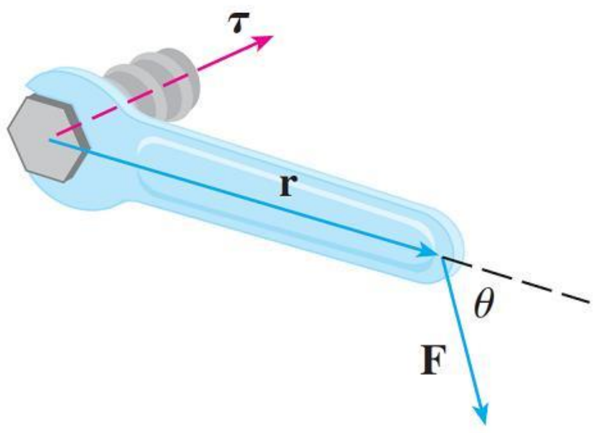


Figure 4

Application: Torque (2 of 3)

The **torque** τ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\tau = r \times F$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation.

Application: Torque (3 of 3)

According to Theorem 9, the magnitude of the torque vector is

$$|\tau| = |\mathrm{r} \times \mathrm{F}| = |\mathrm{r}||\mathrm{F}| \sin\theta$$

where θ is the angle between the position and force vectors. Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**, that is, $|\mathrm{F}| \sin\theta$.

The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

Example 6

A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5.

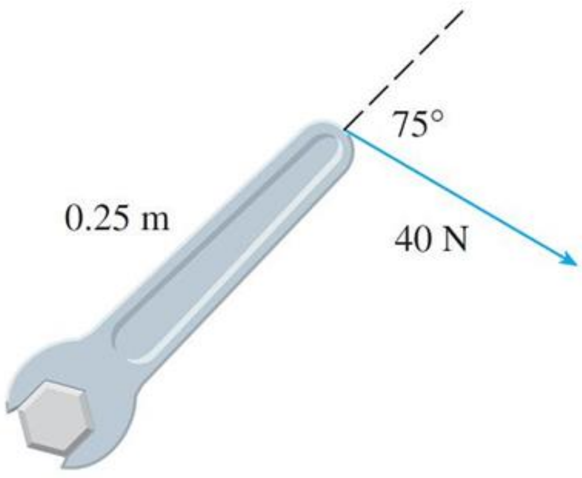


Figure 5

Find the magnitude of the torque about the center of the bolt.

Example 6 – Solution

The magnitude of the torque vector is

$$\begin{aligned} |\tau| = |\mathbf{r} \times \mathbf{F}| &= |\mathbf{r}||\mathbf{F}| \sin 75^\circ \\ &= (0.25)(40) \sin 75^\circ \\ &= 10 \sin 75^\circ \approx 9.66 \ \mathrm{N \cdot m} \end{aligned}$$

If the bolt is right-threaded, then the torque vector itself is

$$\tau = |\tau| \mathbf{n} \approx 9.66 \ \mathbf{n}$$

where **n** is a unit vector directed down into the page (by the right-hand rule).