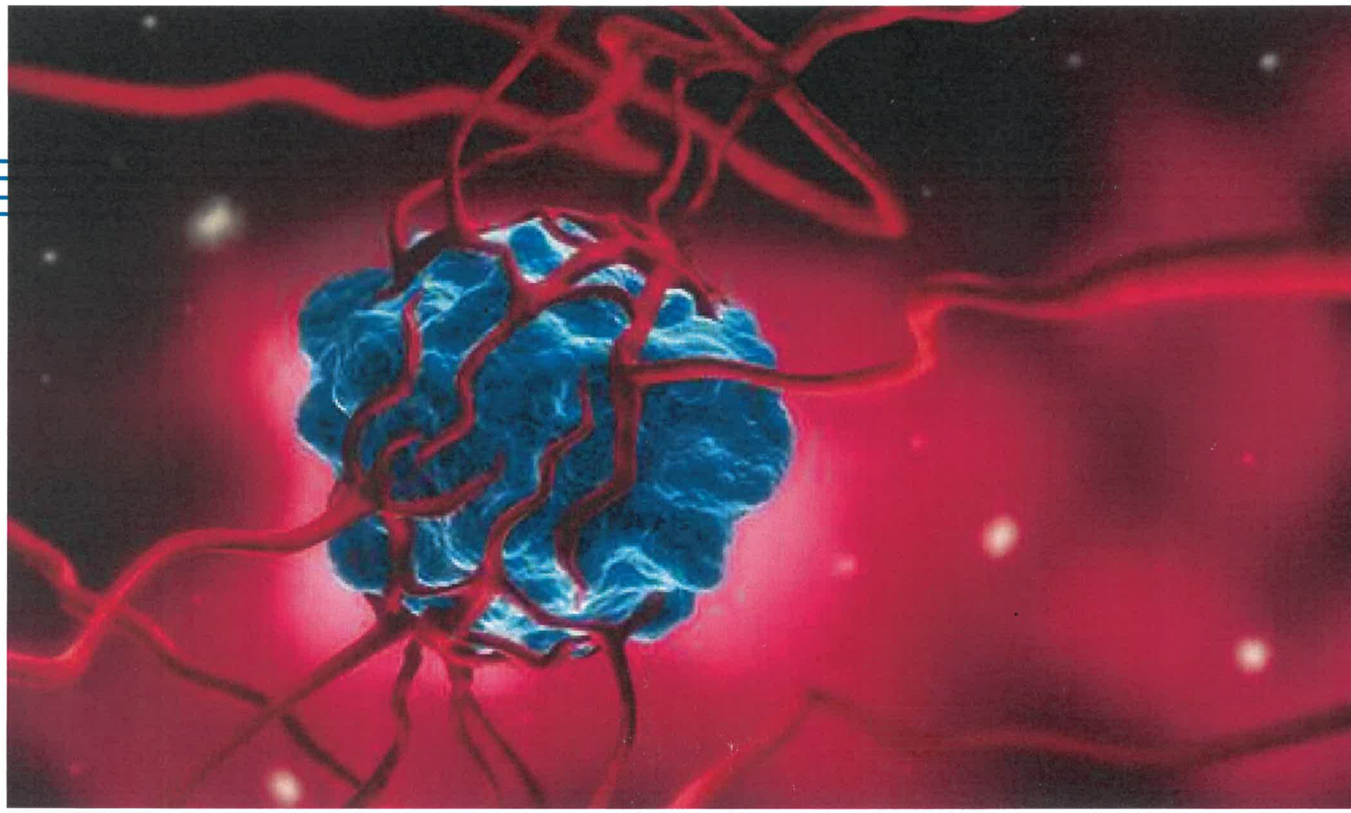
15 Multiple Integrals



15.7 Triple Integrals in Cylindrical Coordinates

Triple Integrals in Cylindrical Coordinates (1 of 2)

In plane geometry the polar coordinate system is used to give a convenient description of certain curves and regions.

Figure 1 enables us to recall the connection between polar and Cartesian coordinates.

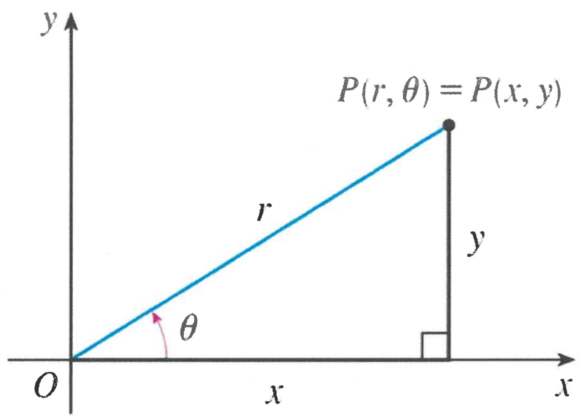


Figure 1

Triple Integrals in Cylindrical Coordinates (2 of 2)

If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure,

$$x = r\cos\theta \qquad y = r\sin\theta$$

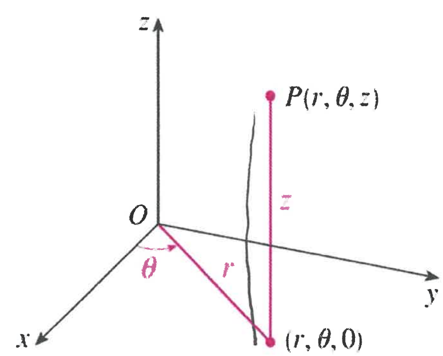
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

In three dimensions there is a coordinate system, called *cylindrical coordinates*, that is similar to polar coordinates and gives convenient descriptions of some commonly occurring surfaces and solids. As we will see, some triple integrals are much easier to evaluate in cylindrical coordinates.

Cylindrical Coordinates

Cylindrical Coordinates (1 of 2)

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P. (See Figure 2.)



The cylindrical coordinates of a point

Figure 2

Cylindrical Coordinates (2 of 2)

To convert from cylindrical to rectangular coordinates, we use the equations

$$1 \quad x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$2 \quad r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad z = z$$

Example 1

Represent the point $(1, \sqrt{3}, 5)_R$ in cylindrical coordinates.

$$r = \sqrt{x^2+y^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\left(2, \frac{\pi}{3}, 5\right)_C$$

Triple Integrals in Cylindrical Coordinates

Triple Integrals in Cylindrical Coordinates (1 of 5)

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates (see Figure 8).

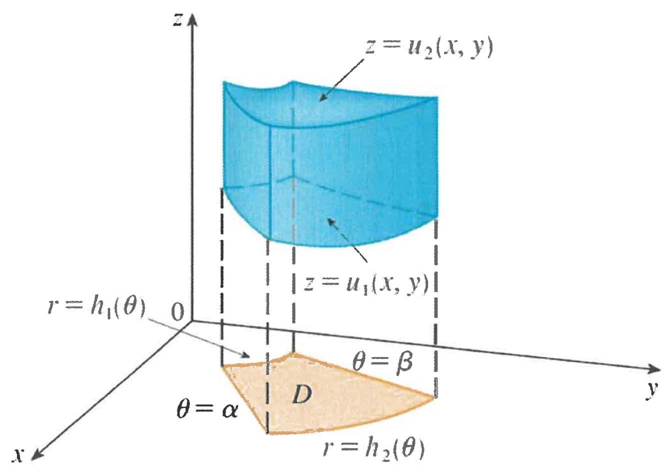


Figure 8

Triple Integrals in Cylindrical Coordinates (2 of 5)

In particular, suppose that *f* is continuous and

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where *D* is given in polar coordinates by

$$D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

We know

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA \tag{3}$$

Triple Integrals in Cylindrical Coordinates (3 of 5)

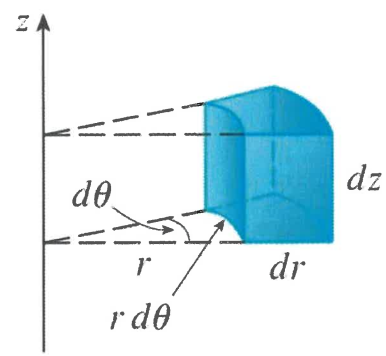
But to evaluate double integrals in polar coordinates, we have the formula

$$4 \quad \iiint\limits_E f(x, y, z)\, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\ \ )} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Formula 4 is the **formula for triple integration in cylindrical coordinates**.

Triple Integrals in Cylindrical Coordinates (4 of 5)

It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r \cos \theta$, $y = r \sin \theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dz\,dr\,d\theta$. (Figure 9 shows how to remember this.)



Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$

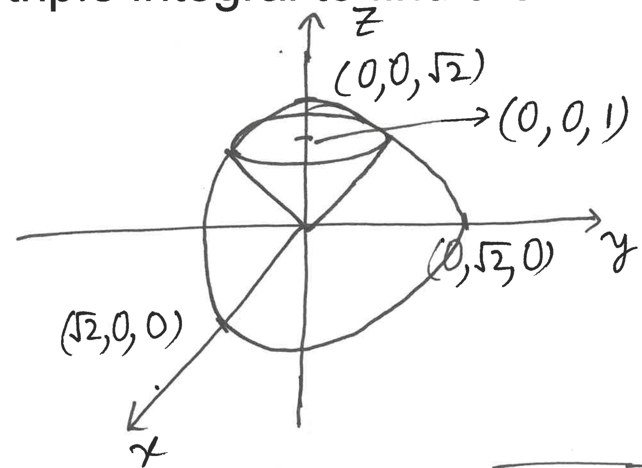
Figure 9

Triple Integrals in Cylindrical Coordinates (5 of 5)

It is worthwhile to use this formula when *E* is a solid region easily described in cylindrical coordinates, and especially when the function *f*(*x*, *y*, *z*) involves the expression $x^2 + y^2$.

Example 2

Let E be the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and bounded below by the cone $z = \sqrt{x^2 + y^2}$. Use a triple integral to find the volume of E.



$x^2+y^2+z^2=2$

$\Rightarrow z=\sqrt{2-x^2-y^2} \ldots\ldots (1)$

$z=\sqrt{x^2+y^2} \ldots\ldots (2)$

$\sqrt{2-x^2-y^2}=\sqrt{x^2+y^2}$

$\Rightarrow 2-x^2-y^2=x^2+y^2$

$\Rightarrow 2=2x^2+2y^2$

$\Rightarrow x^2+y^2=1$

$z=\sqrt{x^2+y^2}=\sqrt{1}=1$

$E=\{(x,y,z): x^2+y^2\le 1,\ \sqrt{x^2+y^2}\le z\le\sqrt{2-x^2-y^2}\}$

$=\{(r,\theta,z): 0\le\theta\le 2\pi,\ 0\le r\le 1,\ r\le z\le\sqrt{2-r^2}\}$

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} 1\, r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left[zr\right]_{z=r}^{\sqrt{2-r^2}}dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2}-r^2\right)dr\,d\theta$$

Example 2

Let E be the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and bounded below by the cone $z = \sqrt{x^2 + y^2}$. Use a triple integral to find the volume of E.

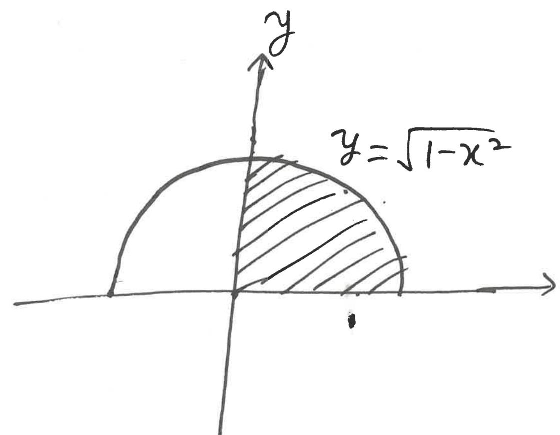
Example 3

Rewrite the following integral using cylindrical coordinates

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_1^{\sqrt{1-x^2-y^2}} xyz\,dzdydx$$

$1 \leq z \leq \sqrt{1-x^2-y^2} \Rightarrow 1 \leq z \leq \sqrt{1-r^2}$

$0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{1-x^2}$



$y = \sqrt{1-x^2}$

$\Rightarrow y^2 = 1 - x^2$

$\Rightarrow x^2 + y^2 = 1$

$0 \leq r \leq 1$

$0 \leq \theta \leq \frac{\pi}{2}$

$$\int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{1-r^2}} r\cos\theta \, r\sin\theta \, z \, r\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-r^2}} r^3 \cos\theta \sin\theta \cancel{z\,dr\,d\theta}\, z\,dz\,dr\,d\theta$$

Example 3

Rewrite the following integral using cylindrical coordinates

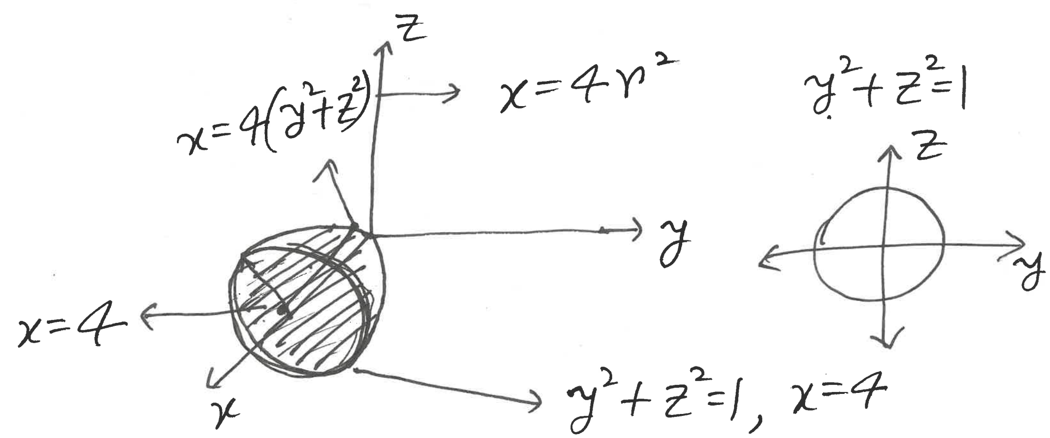
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_1^{\sqrt{1-x^2-y^2}} xyz \, dzdydx$$

Example 4

Evaluate $\int\int_E\int x\,dV$ where E is bounded by $x = 4(y^2+z^2)$ and $x = 4$.

$x = 4(y^2+z^2)$ $\boxed{x = r\cos\theta,\ y = r\sin\theta,\ z = z}$ $x = 4$

$\Rightarrow r\cos\theta = 4(r^2\sin^2\theta + z^2)$ $\Rightarrow r\cos\theta = 4$



$y = r\cos\theta$

$z = r\sin\theta$

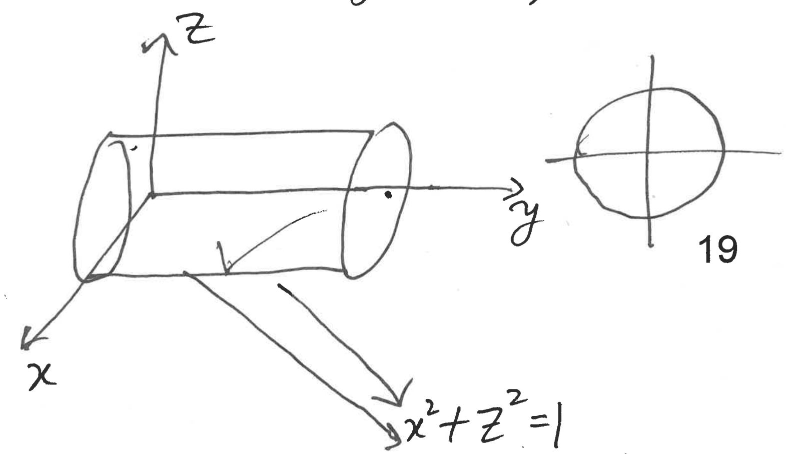
$x = x$

$x = 4(y^2+z^2)$ $x = 4$

$\Rightarrow x = 4r^2$

$$\int_0^{2\pi}\int_0^1\int_{4r^2}^4 x \quad r\,dx\,dr\,d\theta$$

$=$



Example 4

Evaluate $\int\int_E\int x\,dV$ where E is bounded by $x = 4(y^2+z^2)$ and $x = 4$.

$$\int_0^{2\pi}\int_0^{1}\int_{4r^2}^{4} x\,r\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} \left[r\frac{x^2}{2} \right]_{x=4r^2}^{4} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} [8r - 8r^5]\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[4r^2 - \frac{8r^6}{6} \right]_{r=0}^{1} d\theta$$

$$= \int$$