14 Partial Derivatives

14.2 Limits and Continuity

Limits of Functions of Two Variables

Limits of Functions of Two Variables (1 of 7)

Let’s compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and

$$g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as *x* and *y* both approach 0 [and therefore the point (*x*, *y*) approaches the origin].

Limits of Functions of Two Variables (2 of 7)

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Values of $f(x, y)$

Table 1

Limits of Functions of Two Variables (3 of 7)

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Values of $g(x, y)$

Table 2

Limits of Functions of Two Variables (4 of 7)

It appears that as (x, y) approaches (0, 0), the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any particular number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1$$ and

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$ does not exist

Limits of Functions of Two Variables (5 of 7)

In general, we use the notation

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) (staying within the domain of f).

Limits of Functions of Two Variables (6 of 7)

In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). A more precise definition follows.

1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of *f*(*x*, *y*) as (*x*, *y*) approaches (*a*, *b*)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x,y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x,y) - L| < \varepsilon$$

Limits of Functions of Two Variables (7 of 7)

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L$$

and

$$f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

Showing That a Limit Does Not Exist

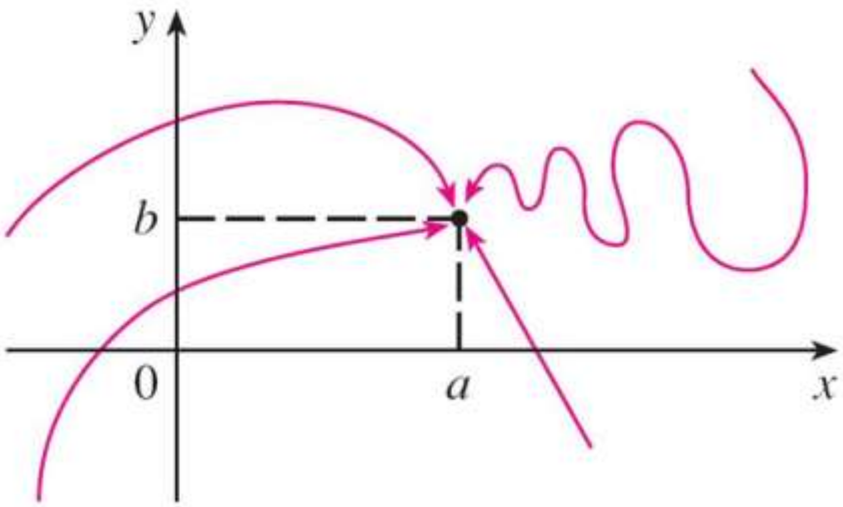
Showing That a Limit Does Not Exist (1 of 4)

For functions of a single variable, when we let *x* approach *a*, there are only two possible directions of approach, from the left or from the right.

We know that if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

Showing That a Limit Does Not Exist (2 of 4)

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f.



Different paths approaching (a, b)

Figure 3

Showing That a Limit Does Not Exist (3 of 4)

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0).

The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach.

Therefore, if the limit exists, then $f(x, y)$ must approach the same limit *no matter how* (x, y) approaches (a, b).

Showing That a Limit Does Not Exist (4 of 4)

Thus one way to show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist is to find different paths of approach along which the function has different limits.

If $f(x,y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 1

Show that $\lim_{(x,y)\to(0,\ 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution:

Let $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$.

First let's approach (0, 0) along the *x*-axis. On this path *y* = 0 for every point (*x*, *y*), so the function becomes $f(x, 0) = \frac{x^2}{x^2} = 1$ for all $x \neq 0$ and thus

$$f(x, y) \to 1 \quad \text{as} \quad (x, y) \to (0,\ 0) \text{ along the } x - axis$$

Example 1 – Solution (1 of 2)

We now approach along the *y*-axis by putting *x* = 0.

Then $f(0, y) = \dfrac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \quad \text{as} \quad (x, y) \to (0,\ 0) \text{ along the } y - \text{axis}$$

(See Figure 4.)

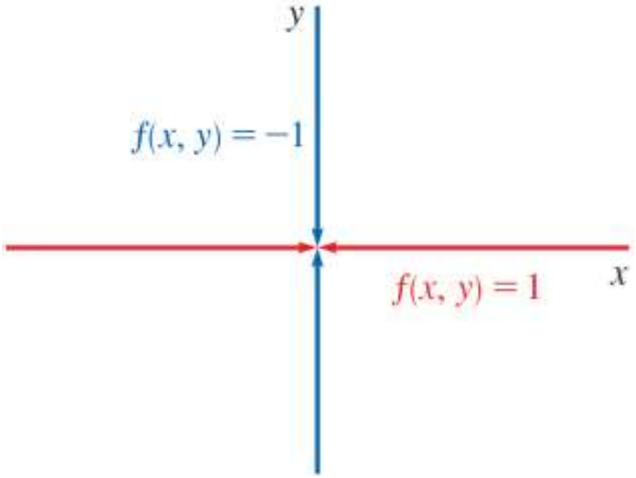


Figure 4

Example 1 – Solution (2 of 2)

Since f has two different limits as (x, y) approaches (0, 0) along two different lines, the given limit does not exist. (This confirms the conjecture we made on the basis of numerical evidence at the beginning of this section.)

Properties of Limits

Properties of Limits (1 of 5)

Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits.

The Limit Laws can be extended to functions of two variables. Assuming that the indicated limits exist, we can state these laws verbally as follows:

Sum Law **1.** The limit of a sum is the sum of the limits.

Difference Law **2.** The limit of a difference is the difference of the limits.

Constant Multiple Law **3.** The limit of a constant times a function is the constant times the limit of the function.

Properties of Limits (2 of 5)

Product Law **4.** The limit of a product is the product of the limits.

Quotient Law **5.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

Properties of Limits (3 of 5)

A polynomial function of two variables (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where *c* is a constant and *m* and *n* are nonnegative integers. A **rational function** is a ratio of two polynomials. For instance,

$$p(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$q(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

Properties of Limits (4 of 5)

The special limits in (2) along with the limit laws allow us to evaluate the limit of any polynomial function *p* by direct substitution:

$$3 \qquad \lim_{(x,y)\to(a,b)} p\,(x,y) = p(a,b)$$

Similarly, for any rational function $q(x, y) = p(x, y) / r(x, y)$ we have

$$4 \qquad \lim_{(x,y)\to(a,b)} q\,(x,y) = \lim_{(x,y)\to(a,b)} \frac{p(x,y)}{r(x,y)} = \frac{p(a,b)}{r(a,b)} = q(a,b)$$

provided that (*a*, *b*) is in the domain of *q*.

Example 4

Evaluate $\lim_{(x,y)\to(1,\ 2)}(x^2y^3 - x^3y^2 + 3x + 2y)$.

Solution:

Let $f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,\ 2)}(x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$

Properties of Limits (5 of 5)

The Squeeze Theorem also holds for functions of two or more variables.

Theorem. Let $f, g, h\colon \mathbb{R}^2 \to \mathbb{R}$ and let $(x, y) \to (x_0, y_0)$ .Suppose there exists a punctured neighborhood $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_0$ on which

$$g(x, y) \;\leq\; f(x, y) \;\leq\; h(x, y).$$

If

$$\lim_{(x,y)\to(x_0,y_0)} g(x, y) \;=\; L \textit{ and } \lim_{(x,y)\to(x_0,y_0)} h(x, y) \;=\; L,$$

then the limit of f exists and

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \;=\; L.$$

Corollary: If $|\, f(x, y)\, | \leq g(x, y)$ near (x_0, y_0) and $\lim g = 0$,then $\lim f = 0$.

Continuity

Continuity (1 of 4)

We know that evaluating limits of *continuous* functions of a single variable is easy.

It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x \to a} f(x) = f(a)$.

Continuous functions of two variables are also defined by the direct substitution property.

6 Definition A function *f* of two variables is called **continuous at** (*a*, *b*) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say *f* is **continuous on** *D* if *f* is continuous at every point (*a*, *b*) in *D*.

Continuity (2 of 4)

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount.

This means that a surface that is the graph of a continuous function has no hole or break.

We have already seen that limits of polynomial functions can be evaluated by direct substitution. It follows by the definition of continuity that *all polynomials are continuous on* $\mathbb{R}^2$.

Continuity (3 of 4)

Likewise, Equation 4 shows that *any rational function is continuous on its domain*.

In general, using properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains.

Example 7

Where is the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

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Where is the function $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ continuous?

Solution:

The function *f* is discontinuous at (0, 0) because it is not defined there.

Since *f* is a rational function, it is continuous on its domain, which is the set

$D = \{(x, y) \mid (x, y) \neq (0, 0)\}$.

Continuity (4 of 4)

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third.

In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

Functions of Three or More Variables

Functions of Three or More Variables (1 of 4)

Everything that we have done in this section can be extended to functions of three or more variables.

The notation

$$\lim_{(x,y,z)\to(a,b,c)} f\,(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) (staying within the domain of f).

Functions of Three or More Variables (2 of 4)

Because the distance between two points (x, y, z) and (a, b, c) in $\mathbb{R}^3$ is given by $\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$, we can write the precise definition as follows: for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if (x, y, z) is in the domain of f and

$$\begin{aligned} 0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} &< \delta \\ \text{then } |f(x,y,z)-L| &< \varepsilon \end{aligned}$$

Functions of Three or More Variables (3 of 4)

The function *f* is **continuous** at (*a*, *b*, *c*) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,\ y,\ z) = f(a,\ b,\ c)$$

For instance, the function

$$f(x,\ y,\ z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1.$ In other words, it is discontinuous on the sphere with center the origin and radius 1.

Functions of Three or More Variables (4 of 4)

We can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

7 If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$