This worksheet covers selected topics in Sections 14.1, 14.2, 14.3, 14.4, 14.5, 14.6, 14.7, 14.8, 15.1, 15.2, 15.3. **This worksheet does NOT cover all problems and situations on the Exam 2.** Problems on the exam may not necessarily look exactly like problems on this worksheet. For more practice problems, please see the list of practice problems on the instructor Syllabus under "Suggested List of Textbook Problems"; on the lecture notes; on the discussion worksheets for weeks 6, 7, 8, 9, and 10; on previous quizzes, and on the WebAssign homework.

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1. Show that the limit
$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$
does not exist. Justify your answer.

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We first approach along the line $y = 0$,
$$L = \lim_{x\to 0} \frac{x^2 \cdot 0}{x^4 + 0^2} = \lim_{x\to 0} \frac{0}{x^4} = 0.$$

Now, we approach along the curve $y = x^2$,
$$L = \lim_{x\to 0} \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}.$$

Since the limiting value depends on the path we take to $(0,0)$, the limit does not exist.

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2. Show that the limit
$$\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2 + y^2}$$
does not exist. Justify your answer.

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We first approach along the line $y = 0$,
$$L = \lim_{x\to 0} \frac{4x \cdot 0}{3x^2 + 0^2} = \lim_{x\to 0} \frac{0}{3x^2} = 0.$$

Now, we approach along the line $y = x$,
$$L = \lim_{x\to 0} \frac{4x \cdot x}{3x^2 + x^2} = \lim_{x\to 0} \frac{4x^2}{4x^2} = 1.$$

Since the limiting value depends on the path we take to $(0,0)$, the limit does not exist.

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3. Find the first partial derivatives and second partial derivatives of $f(x,y) = x^3 - 2xy + xy^3 + 3y^2$.

$$f_x(x, y) = 3x^2 - 2y + y^3, \quad f_y(x, y) = -2x + 3xy^2 + 6y$$
$$f_{xx}(x, y) = 6x, \quad f_{xy}(x, y) = -2 + 3y^2 = f_{yx}(x, y), \quad f_{yy}(x, y) = 6xy + 6$$

4. Given an implicit function $e^z = xyz$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Let $F(x, y, z) = e^z - xyz$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-yz}{e^z - xy} = \frac{yz}{e^z - xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}$$

5. Consider the function $f(x, y) = \ln(x - 4y)$.
   (a) Find the linearization of $f$ at the point $(5, 1)$.

   We start by finding the first partial derivatives of $f$,

   $$f_x(x, y) = \frac{1}{x - 4y} \quad \text{and} \quad f_y(x, y) = \frac{-4}{x - 4y}$$

   Evaluating these at $(5, 1)$,

   $$f_x(5, 1) = \frac{1}{5 - 4 \cdot 1} = 1 \quad \text{and} \quad f_y(5, 1) = \frac{-4}{5 - 4 \cdot 1} = -4$$

   Note that $f(5, 1) = \ln(5 - 4) = \ln 1 = 0$, so the linearization of $f$ at $(5, 1)$ is

   $$f(4.9, 0.8) \approx L(4.9, 0.8) = 0 + 1 \cdot (x - 5) - 4 \cdot (y - 1) = (x - 5) - 4(y - 1).$$

   (b) Use your answer to part (a) to approximate $f(4.9, 0.8)$.

   Evaluating the linearization at $(4.9, 0.8)$, we get

   $$L(4.9, 0.8) = (4.9 - 5) - 4(0.8 - 1) = -0.1 + 0.8 = 0.7.$$

6. Consider the function $g(x, y, z) = y \arctan(x^2 + z)$.
   (a) Find the linearization of $g$ at the point $(1, 2, -1)$.

   We have that
   $$g(1, 2, -1) = 0.$$

Now, we determine the first order partial derivatives,

$$g_x(x, y, z) = \frac{2xy}{1 + (x^2 + z)^2}$$

$$g_y(x, y, z) = \arctan(x^2 + z)$$

$$g_z(x, y, z) = \frac{y}{1 + (x^2 + z)^2}.$$

Evaluating at $(1, 2, -1)$, we get that

$$g_x(1, 2, -1) = 4, \quad g_y(1, 2, -1) = 0, \quad g_z(1, 2, -1) = 2.$$

Hence, the linearization of $g$ at $(1, 2, -1)$ is

$$L(x, y, z) = 0 + 4(x - 1) + 0(y - 2) + 2(z + 1) = 4(x - 1) + 2(z + 1).$$

(b) Use your answer to part (a) to approximate $g(0.9, 2.1, -1.1)$.

Using part (a), we have the linear approximation

$$g(0.9, 2.1, -1.1) \approx L(0.9, 2.1, -1.1) = 4(0.9 - 1) + 2(-1.1 + 1) = -0.6.$$

7. Let $z = f(x, y)$ with $x = g(t)$ and $y = h(t)$. Find $z'(1)$, assuming all functions involved are differentiable, and

$$g(1) = 3, \quad g'(1) = 4, \quad h(1) = 7, \quad h'(1) = \frac{1}{3}, \quad f_x(3, 7) = \frac{1}{2}, \quad f_y(3, 7) = 3.$$
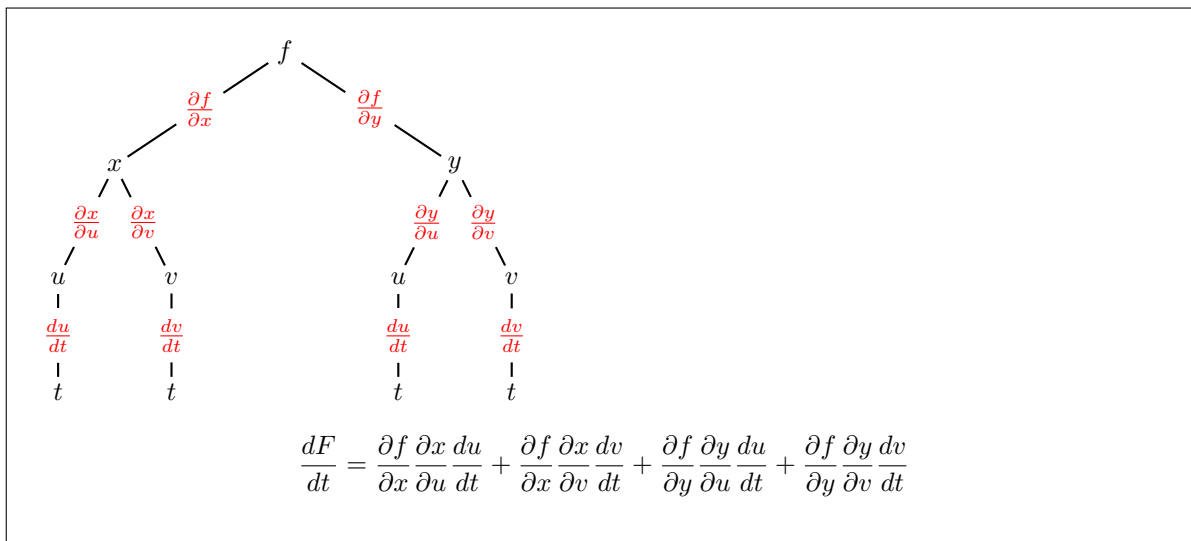
since

$$z'(t) = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= f_x(x, y)g'(t) + f_y(x, y)h'(t)$$

$$= f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t),$$

then

$$z'(1) = f_x(g(1), h(1))g'(1) + f_y(g(1), h(1))h'(1)$$

$$= f_x(3, 7)g'(1) + f_y(3, 7)h'(1)$$

$$= \left(\frac{1}{2}\right)(4) + (3)\left(\frac{1}{3}\right)$$

$$= 3.$$

8. Let $F(t) = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$, $u = u(t)$, and $v = v(t)$. We assume all functions involved are differentiable. What is an expression for $\dfrac{dF}{dt}$?

$$f$$

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y}$$

$$x \qquad\qquad y$$

$$\frac{\partial x}{\partial u} \quad \frac{\partial x}{\partial v} \qquad\qquad \frac{\partial y}{\partial u} \quad \frac{\partial y}{\partial v}$$

$$u \qquad v \qquad\qquad u \qquad v$$

$$\frac{du}{dt} \qquad \frac{dv}{dt} \qquad\qquad \frac{du}{dt} \qquad \frac{dv}{dt}$$

$$t \qquad t \qquad\qquad t \qquad t$$

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt}$$

9. The radius of a right circular cone is increasing at a rate of 2 m/s while its height is decreasing at a rate of 3 m/s.

(a) Identify the mathematical interpretation of each given quantity and write an expression for $\dfrac{dV}{dt}$, where $V = \dfrac{\pi}{3}r^2h$ is the volume of the cone.

The increasing rate of 2 m/s corresponds to $\dfrac{dr}{dt} = 2$ and the decreasing rate of 3 m/s corresponds to $\dfrac{dh}{dt} = -3$. We apply the Chain Rule-Case I to the equation $V = \dfrac{\pi}{3}r^2h$, so we have

$$\frac{\partial V}{\partial r} = \frac{2\pi}{3}rh, \qquad \frac{\partial V}{\partial h} = \frac{\pi}{3}r^2.$$

To find $\dfrac{dV}{dt}$,

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \left(\frac{2\pi}{3}rh\right)(2) - \left(\frac{\pi}{3}r^2\right)(3) = \pi\left(\frac{4}{3}rh - r^2\right)$$

(b) What is the physical interpretation of the quantity $\dfrac{dV}{dt}$?

The quantity $\dfrac{dV}{dt}$ represents the rate of change in volume over time, and shows how the volume is changing (increasing or decreasing) over time.

(c) How fast is the volume of the cone changing when the radius is 1 m and the height is $\dfrac{3}{\pi}$m? Specify the unit measure.

$$\frac{dV}{dt} = \pi \left( \frac{4}{3}rh - r^2 \right) \quad \rightarrow \quad \frac{dV}{dt}\bigg|_{r=1,\,h=3/\pi} = \pi \left( \frac{4}{3}(1)\left(\frac{3}{\pi}\right) - (1)^2 \right) = (4 - \pi)\ m^3/s.$$

10. The gradient of a function of two variables is

    **A. a vector function**

    B. a line

    C. a negative number

    D. a positive number

A. The gradient of a function of two variables is a vector function.

11. The rate of change of a function of two variables at a point and in the direction of a unit vector is

    A. a function

    B. a line

    **C. a scalar**

    D. a vector

C. The rate of change of a function of two variables at a point and in the direction of a unit vector is a scalar, because it is the directional derivative of the function at the point in the direction of the unit vector, which is the dot product of the gradient vector at the point with the unit vector.

12. (a) Find the rate of change of the function $v(x, y) = y \cos x$ at the point $P(0, \pi)$, in the direction of the vector $\mathbf{w} = \mathbf{i} - \mathbf{j}$.

    The rate of change of the function $v(x, y) = y \cos x$ at the point $P(0, \pi)$ in the direction of the vector $\mathbf{w} = \mathbf{i} - \mathbf{j}$ is

    the directional derivative of the function $v(x, y)$ at P in the direction of the unit vector $\dfrac{\mathbf{w}}{||\mathbf{w}||}$,

    and is equal to

    $$D_{\mathbf{w}}v(0, \pi) = \langle v_x(0, \pi),\, v_y(0, \pi) \rangle \cdot \left\langle \frac{1}{\sqrt{2}},\, -\frac{1}{\sqrt{2}} \right\rangle = \langle 0,\, 1 \rangle \cdot \left\langle \frac{1}{\sqrt{2}},\, -\frac{1}{\sqrt{2}} \right\rangle = -\frac{1}{\sqrt{2}}.$$

    (b) In which direction does the function $v(x, y) = y \cos x$ change most rapidly at $P(0, \pi)$? Justify your answer.

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The function $v(x, y) = y \cos x$ changes most rapidly at $P(0, \pi)$ in the direction of the gradient vector at the point P, given by the vector

$$\nabla v(0, \pi) = \langle v_x(0, \pi), v_y(0, \pi) \rangle = \langle 0, 1 \rangle = \mathbf{j}.$$

(c) What is the maximum rate of change of $v$ at the point $P(0, \pi)$? Justify your answer.

The maximum rate of change of $v$ at the point $P(0, \pi)$ is

$$||\nabla v(0, \pi)|| = ||\langle v_x(0, \pi), v_y(0, \pi) \rangle|| = ||\langle 0, 1 \rangle|| = ||\mathbf{j}|| = 1,$$

which is the magnitude of the gradient vector at P.

13. Find the directional derivative of the function

$$f(x, y) = x \ln y$$

at the point $(2, e)$ in the direction making an angle of $\frac{\pi}{6}$ radians above the horizontal.

Computing the gradient of $f$,

$$\nabla f = \langle \ln y, \frac{x}{y} \rangle.$$

Evaluating at $(x, y) = (2, e)$, we get that

$$\nabla f(2, e) = \langle 1, \frac{2}{e} \rangle.$$

The unit vector in the direction of $\pi/6$ above the $x$-axis is

$$\mathbf{u} = \langle \cos \frac{\pi}{6}, \sin \frac{\pi}{6} \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle.$$

Now, the directional derivative is

$$D_{\mathbf{u}} f(2, e) = \nabla f(2, e) \cdot \mathbf{u} = \langle 1, \frac{2}{e} \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \frac{\sqrt{3}}{2} + \frac{1}{e}.$$

14. Let $P(1, 2, 1)$ be a point on the ellipsoid $4x^2 + y^2 + 4z^2 = 12$.
   (a) Find an equation of the tangent plane to the ellipsoid at the point $P(1, 2, 1)$.

Let $f(x, y, z) = 4x^2 + y^2 + 4z^2$. Then

$$\nabla f(x, y, z) = \langle 8x, 2y, 8z \rangle \quad \rightarrow \quad \nabla f(1, 2, 1) = \langle 8, 4, 8 \rangle$$

is a normal vector to the tangent plane at $P(1, 2, 1)$. An equation of the tangent plane to the ellipsoid at the point $P(1, 2, 1)$ is

$$8(x - 1) + 4(y - 2) + 8(z - 1) = 0.$$

(b) Find parametric equations of the normal line to the ellipsoid at the point $P(1, 2, 1)$.

> The direction vector of the normal line is $\nabla f(1, 2, 1) = \langle 8, 4, 8 \rangle$. So, an equation of the normal line to the ellipsoid at the point $P(1, 2, 1)$ is given by the parametric equations
>
> $$x = 1 + 8t, \quad y = 2 + 4t, \quad z = 1 + 8t.$$

(c) At what points does the normal line through the point $P$ on the ellipsoid intersect the sphere $x^2 + y^2 + z^2 = 18$?

> By plugging the parametric equations of the normal line through the point $P$ on the ellipsoid into the equation of the sphere $x^2 + y^2 + z^2 = 18$, we get an equation in terms of $t$ that we can solve by using the quadratic formula.
>
> $$x^2 + y^2 + z^2 = 18 \quad \rightarrow \quad (1 + 8t)^2 + (2 + 4t)^2 + (1 + 8t)^2 = 18 \quad \rightarrow \quad 144t^2 + 48t - 12 = 0$$
>
> $$\rightarrow \quad 12t^2 + 4t - 1 = 0 \quad \rightarrow (2t + 1)(6t - 1) = 0 \quad \rightarrow \quad t = -\frac{1}{2}, \frac{1}{6}.$$
>
> By plugging the $t$ values into the parametric equations, we get the coordinates of the two points
>
> $$t = -\frac{1}{2}: \quad \left( 1 + 8\left(-\frac{1}{2}\right), 2 + 4\left(-\frac{1}{2}\right), 1 + 8\left(-\frac{1}{2}\right) \right) = (-3, 0, -3)$$
>
> and
>
> $$t = \frac{1}{6}: \quad \left( 1 + 8\left(\frac{1}{6}\right), 2 + 4\left(\frac{1}{6}\right), 1 + 8\left(\frac{1}{6}\right) \right) = \left( \frac{7}{3}, \frac{8}{3}, \frac{7}{3} \right).$$

15. Let $f(x, y) = x^2 \ln(y)$ and $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

(a) Find the gradient vector field of $f$.

> $$\nabla f = \left\langle 2x \ln(y), \frac{x^2}{y} \right\rangle$$

(b) Find the gradient of $f$ at the point $(1, e)$.

> $$\nabla f(1, e) = \left\langle 2 \ln(e), \frac{1^2}{e} \right\rangle = \left\langle 2, \frac{1}{e} \right\rangle.$$

(c) Find the rate of change of $f$ at the point $(1, e)$ in the direction of the unit vector $\mathbf{u}$.

> The rate of change of $f$ at the point $(1, e)$ in the direction of the vector $\mathbf{u}$ is
>
> $$D_{\mathbf{u}}f(1, e) = \nabla f \cdot \mathbf{u} = \left\langle 2, \frac{1}{e} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = 2\left(\frac{3}{5}\right) + \frac{1}{e}\left(\frac{4}{5}\right) = \frac{6e + 4}{5e}$$

(d) Find an equation of the tangent plane to the surface $z = x^2 \ln(y)$ at the point $P(1, e, 1)$.

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Let $F(x, y, z) = z - x^2 \ln y$.

$$\nabla F = \left\langle -2x \ln(y), -\frac{x^2}{y}, 1 \right\rangle \quad \text{and} \quad \mathbf{n} = \nabla F(1, e, 1) = \left\langle -2, -\frac{1}{e}, 1 \right\rangle$$

An equation of the tangent plane to the surface $z = x^2 \ln(y)$ at the point $P(1, e, 1)$ is

$$-2(x - 1) - \frac{1}{e}(y - e) + (z - 1) = 0$$

OR

Let $F(x, y, z) = x^2 \ln y - z$

$$\nabla F = \langle 2x \ln(y), x^2/y, -1 \rangle$$

$$\mathbf{n} = \nabla F(1, e, 1) = \langle 2, \frac{1}{e}, -1 \rangle = \langle a, b, c \rangle$$

$$(x_0, y_0, z_0) = (1, e, 1)$$

An equation of the tangent plane to the surface $z = x^2 \ln y$ at the point $(1, e, 1)$ is

$$2(x - 1) + (1/e)(y - e) - (z - 1) = 0$$

(e) Find parametric equations of the normal line to the surface $z = x^2 \ln(y)$ at the point $P(1, e, 1)$.

Parametric equations of the normal line to the surface $z = x^2 \ln y$ at the point $(1, e, 1)$ are

$$x = 1 - 2t, \quad y = e - \frac{1}{e}t, \quad z = 1 - t, \quad -\infty < t < \infty.$$

OR

Parametric equations of the normal line to the surface $z = x^2 \ln y$ at the point $(1, e, 1)$ are

$$x = 1 + 2t, \ y = e + \frac{t}{e}, \ z = 1 - t, \quad -\infty < t < \infty.$$

16. Find the critical points of the function

$$g(x, y) = (x + y^2)e^x$$

Determine whether each critical point is a local maximum, local minimum, or saddle point.

We compute that
$$g_x(x, y) = e^x(x + y^2 + 1) \quad \text{and} \quad g_y(x, y) = 2ye^x.$$

Setting both equal to $0$ and noting that $e^x \neq 0$, we get that

$$x + y^2 + 1 = 0 \quad \text{and} \quad y = 0.$$

By the second equation $y = 0$. Then the first equation gives $x = -1$. Hence, the only critical point of $g$ is the point $(-1, 0)$. Computing more derivatives,

$$g_{xx} = e^x(x + y^2 + 2), \quad g_{xy} = 2e^x y, \quad g_{yy} = 2e^x.$$

At $(-1, 0)$, these are

$$g_{xx}(-1, 0) = \frac{1}{e}, \quad g_{xy}(-1, 0) = 0, \quad g_{yy}(-1, 0) = \frac{2}{e}.$$

Recall that

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2.$$

Evaluating for our function:

$$D(-1, 0) = \frac{1}{e} \cdot \frac{2}{e} - 0^2 = \frac{2}{e^2} > 0.$$

Moreover, $g_{xx}(-1, 0) = \frac{1}{e} > 0$, so by the second derivative test, the point $(-1, 0)$ is a local minimum.

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17. Find the extreme values of the function $f(x, y) = xy$ on the ellipse $36x^2 + y^2 = 72$.

We'll use the method of Lagrange multipliers. Our objective function is $f(x, y) = xy$ and constraint is $g(x, y) = 36x^2 + y^2 = 72$. We compute that

$$\nabla f = \langle y, x \rangle \quad \text{and} \quad \nabla g = \langle 72x, 2y \rangle.$$

We must solve the system

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y) = 72$$

i.e.,

$$y = 72\lambda x \quad x = 2\lambda y \quad 36x^2 + y^2 = 72.$$

Using the first two,

$$y = 72\lambda x = 72\lambda(2\lambda y) = 144\lambda^2 y.$$

Rearranging,

$$y(1 - 144\lambda^2) = 0.$$

Hence $\lambda = \pm\frac{1}{12}$ or $\lambda \neq \pm\frac{1}{12}$.

Case 1: Assume $\lambda = \pm\frac{1}{12}$. Then $y = 72(\pm\frac{1}{12})x$, so

$$y = \pm 6x.$$

Plugging into $36x^2 + y^2 = 72$, we get

$$72 = 36x^2 + (\pm 6x)^2 = 72x^2,$$

so $x = \pm 1$. Thus, we get the points $(x, y) = (\pm 1, \pm 6)$.

Case 2: Assume $\lambda \neq \pm\frac{1}{12}$. Then $y = 0$, but $x = 2\lambda y$ implies $x = 0$. The point $(x, y) = (0, 0)$ does not satisfy $36x^2 + y^2 = 72$, so this case cannot occur.

Evaluating the function $f(x, y)$ at the points $(\pm 1, \pm 6)$ and $(\pm\sqrt{2}, 0)$, we find that the maximum value is 6 at $(1, 6)$ and $(-1, -6)$ and minimum value is $-6$ at $(1, -6)$ and $(-1, 6)$.

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18. Find the extreme values of
$$f(x, y) = x^2 + y^2 + 4x - 4y$$
on the region described by the inequality $x^2 + y^2 \leq 18$.

For the interior of the region, we find the critical points:

$$f_x = 2x + 4$$
$$f_y = 2y - 4.$$

Solving $f_x = 0$ and $f_y = 0$, we obtain only one critical point $(-2, 2)$ (which is inside the given region).

For the boundary, we use Lagrange multipliers:

$$\nabla f = \lambda \nabla g \tag{1}$$
$$\langle 2x + 4, 2y - 4 \rangle = \lambda \langle 2x, 2y \rangle \tag{2}$$
$$2x + 4 = 2\lambda x \rightarrow x + 2 = \lambda x \tag{3}$$
$$2y - 4 = 2\lambda y \rightarrow y - 2 = \lambda y \tag{4}$$

Adding (3) and (4), we obtain

$$(x + y) = \lambda(x + y) \rightarrow (x + y)(1 - \lambda) = 0$$

So, either $\lambda = 1$ or $x + y = 0 \rightarrow x = -y$

- If $\lambda = 1$, substituting to equation (3), we get $x + 2 = x$ which implies $0 = 2$ ( IMPOSSIBLE), so this leads to no solution.

- If $x = -y$, substituting to the constraint $x^2 + y^2 = 18$, we get $2y^2 = 18$, so $y = \pm 3$. If $y = 3$ then $x = -3$ and when $y = -3$ then $x = 3$. We obtain 2 points $(3, -3)$ and $(-3, 3)$.

To find the absolute minimum and the maximum, we check the value of $f$ at these points:

- $f(-2, 2) = 4 + 4 - 8 - 8 = -8$

- $f(3, -3) = 9 + 9 + 12 + 12 = 42$

- $f(-3, 3) = 9 + 9 - 12 - 12 = -6$

So, the maximum value of $f$ on the disk $x^2 + y^2 \leq 18$ is $f(3, -3) = 42$ and the minimum value is $f(-2, 2) = -8$

19. Evaluate $\int_0^1 \int_0^1 (x - y)^2 \, dx \, dy$

We have

$$\int_0^1 \int_0^1 (x - y)^2 \, dx \, dy = \int_0^1 \int_0^1 (x^2 + y^2 - 2xy) \, dx \, dy = \int_0^1 \left[ \frac{x^3}{3} + y^2 x - x^2 y \right]_0^1 dy$$
$$= \int_0^1 \frac{1}{3} + y^2 - y \, dy = \left[ \frac{1}{3}y + \frac{y^3}{3} - \frac{y^2}{2} \right]_0^1 = \frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$$

20. Evaluate $\iint_R e^{-y} \cos(x) \, dA$, where $R = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2},\ 0 \leq y \leq 1\}$

We can iterate the double integral by describing the region $R$ as a type II plane region.

$$\iint_R e^{-y}\cos(x)\, dA = \int_0^1 \int_0^{\frac{\pi}{2}} e^{-y}\cos(x)\, dx\, dy = \int_0^1 \left[ e^{-y}\sin(x) \right]_0^{\frac{\pi}{2}} dy$$

$$= \int_0^1 e^{-y}\, dy = \left[ -e^{-y} \right]_0^1 = -e^{-1} + 1 = 1 - e^{-1}$$

21. The volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + \alpha$ and above the rectangle $R = [-1,1] \times [0,1]$ is $\dfrac{10}{3}$. Find the value of the constant $\alpha$, where $\alpha \geq 1$.

Since $z = f(x,y) = 3y^2 - x^2 + \alpha$

$$V = \iint_R f(x,y)\, dA = \iint_R (3y^2 - x^2 + \alpha)\, dA = \int_0^1 \int_{-1}^1 (3y^2 - x^2 + \alpha)\, dx\, dy = \int_0^1 \left[ 3y^2 x - \frac{x^3}{3} + \alpha x \right]_{-1}^1 dy$$

$$= \int_0^1 (3y^2 - \frac{1}{3} + \alpha) - \left( -3y^2 + \frac{1}{3} - \alpha \right) dy$$

$$= \int_0^1 \left( 6y^2 - \frac{2}{3} + 2\alpha \right) dy = \left[ 2y^3 - \frac{2}{3}y + 2\alpha y \right]_0^1$$

$$= 2 - \frac{2}{3} + 2\alpha = \frac{4}{3} + 2\alpha.$$

We are given that the volume of the solid is $V = \dfrac{10}{3}$. So we solve the equation for $\alpha$, which yields

$$\frac{4}{3} + 2\alpha = \frac{10}{3} \quad \rightarrow \quad 2\alpha = \frac{6}{3} \quad \rightarrow \quad \alpha = 1.$$

22. Find the average value of $f(x,y) = xe^y$ over the rectangle $R$ with the vertices $(0,0), (4,0), (4,1)$, and $(0,1)$.

The average value of a function of two variables $f$ defined on a region $R$ with area $A(R)$, is given by

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x,y)\, dA.$$

Since the region $R$ is given by $R = \{(x,y)|\, 0 \leq x \leq 4,\, 0 \leq y \leq 1\}$, then the rectangle of length 4 and width 1, has area $A(R) = 4 \times 1 = 4$. So, we have

$$f_{avg} = \frac{1}{4} \int_0^1 \int_0^4 xe^y\, dx\, dy = \frac{1}{4} \int_0^1 \left[ \frac{x^2}{2} e^y \right]_0^4 dy$$

$$= \frac{1}{4} \int_0^1 8e^y\, dy = \frac{1}{4} \left[ 8e^y \right]_0^1 = 2(e^1 - 1)$$

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23. Evaluate the double integral $\iint_D e^{x^3} \, dA$, where $D = \{(x, y) \mid 0 \le y \le x^2 \text{ and } 0 \le x \le 4\}$.

$$\int_0^4 \int_0^{x^2} e^{x^3} \, dy \, dx = \int_0^4 \left[ y e^{x^3} \right]_0^{x^2} dx = \int_0^4 x^2 e^{x^3} \, dx$$
$$= \frac{1}{3} \int_0^{64} e^u \, du, \text{ by letting } u = x^3, \ du = 3x^2 dx$$
$$= \frac{1}{3} \left( e^{64} - 1 \right)$$

24. Evaluate the double integral

$$\iint_D x\sqrt{y^2 - x^2} \, dA, \text{ where } D = \{(x, y) \mid 0 \le x \le y \text{ and } 0 \le y \le 3\}.$$

Use the $u$-substitution $u = y^2 - x^2$ with $du = -2x \, dx$ $\quad \rightarrow \quad -\dfrac{1}{2} du = x \, dx$.

$$\int_0^3 \int_0^y x\sqrt{y^2 - x^2} dx \, dy = \int_0^3 \int_{y^2}^0 -\frac{1}{2}\sqrt{u} \, du \, dy = \int_0^3 \left(-\frac{1}{2}\right) \left( \int_{y^2}^0 u^{1/2} \, du \right) \, dy = \int_0^3 \left[ -\frac{1}{3} (u)^{3/2} \right]_{y^2}^0 dy$$
$$= \int_0^3 \frac{1}{3} \left( y^2 \right)^{3/2} dy = \frac{1}{3} \int_0^3 y^3 dy = \frac{1}{12} \left[ y^4 \right]_0^3 = \frac{27}{4}$$

25. Consider the region $D$, where $D$ is bounded by $y = \cos x$ where $0 \le x \le \pi/2$, $y = 0$, and $x = 0$.

    (a) Iterate the double integral $\iint_D \sin^2 x \, dA$ over a type I region $D$. DO NOT EVALUATE.

    Note that the region $D$ is the plane region below the graph of the cosine function from $x = 0$ to $x = \frac{\pi}{2}$.
    $$\iint_D \sin^2 x \, dA = \int_0^{\pi/2} \int_0^{\cos x} \sin^2 x \, dy \, dx.$$

    (b) Iterate the double integral $\iint_D \sin^2 x \, dA$ over a type II region $D$. DO NOT EVALUATE.

    $$\iint_D \sin^2(x) \, dA = \int_0^1 \int_0^{\arccos y} \sin^2 x \, dx \, dy.$$

    (c) Evaluate the integral you iterated in part (a).

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$$\int_0^{\frac{\pi}{2}} \int_0^{\cos x} \sin^2 x dy \, dx = \int_0^{\frac{\pi}{2}} \left[ y \sin^2 x \right]_0^{\cos x} dx = \int_0^{\frac{\pi}{2}} \cos x \cdot \sin^2 x dx$$

Let $u = \sin x$ and $du = \cos x dx$.

$$\int_0^1 u^2 du = \frac{1}{3} \left[ u^3 \right]_0^1 = \frac{1}{3}$$

26. Change the order of integration for the iterated integral (Do NOT evaluate):

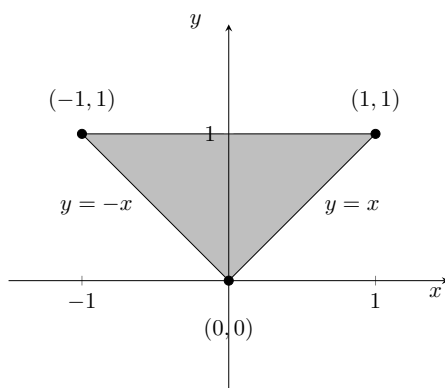$$\int_1^e \int_0^{\ln x} f(x, y) \, dy dx.$$

$$\int_0^1 \int_{e^y}^e f(x, y) \, dx \, dy$$

27. Evaluate the double integral

$$\iint_R 3x^2 y \, dA$$

where $R$ is the region enclosed by the triangle with vertices $(0, 0), (-1, 1)$, and $(1, 1)$.

We first sketch the region:



From the sketch, we see that the double integral can be expressed as

$$\iint_R 3x^2 y \, dA = \int_0^1 \int_{-y}^y 3x^2 y \, dx \, dy.$$
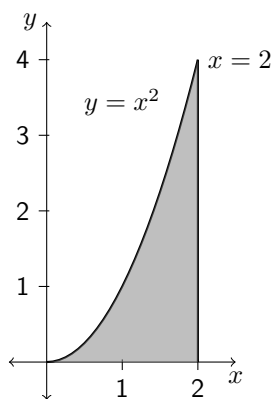
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Now, we evaluate the iterated integral

$$\int_0^1 \int_{-y}^y 3x^2 y \, dx \, dy = \int_0^1 x^3 y \Big|_{-y}^y \, dy$$
$$= \int_0^1 2y^4 \, dy$$
$$= \frac{2}{5} y^5 \Big|_0^1$$
$$= \frac{2}{5}.$$

Thus, the value of the double integral is $\frac{2}{5}$.

28. Evaluate the iterated integral by changing the order of integration:

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3} \, dx \, dy.$$

The equation $x = \sqrt{y}$ can be written as $y = x^2$ (with $x \geq 0$), so we see that the region is:



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We change the order of integration to evaluate,

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3} \, dx \, dy = \int_0^2 \int_0^{x^2} e^{x^3} \, dy \, dx$$

$$= \int_0^2 y e^{x^3} \Big|_0^{x^2} \, dx$$

$$= \int_0^2 x^2 e^{x^3} \, dx$$

$$= \frac{1}{3} e^{x^3} \Big|_0^2$$

$$= \frac{1}{3} e^8 - \frac{1}{3} e^0$$

$$= \frac{1}{3} e^8 - \frac{1}{3}.$$

29. Which of the following answers correctly rewrites the iterated integral $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2} x^3 \, dy \, dx$ in polar coordinates?

A. $\displaystyle\int_0^{\pi/2} \int_0^2 e^{-r^2} r^3 \cos^3(\theta) \, dr \, d\theta$

**B.** $\displaystyle\int_0^{\pi/2} \int_0^2 e^{-r^2} r^4 \cos^3(\theta) \, dr \, d\theta$

C. $\displaystyle\int_0^{\pi} \int_{-2}^2 e^{-r^2} r^3 \cos^3(\theta) \, dr \, d\theta$

D. $\displaystyle\int_0^2 \int_0^{\sqrt{4-r^2}} e^{-r^2} r^4 \cos^3(\theta) \, dr \, d\theta$

**B**. In polar coordinates, we have

$$x^2 + y^2 = r^2, \quad x = r\cos(\theta), \quad dA = r \, dr \, d\theta$$

The bounds for $x$ and $y$ describe the quarter disk of radius 2 residing in the first quadrant. We have that $0 \le r \le 2$ and $0 \le \theta \le \pi/2$. The function is $f(x,y) = e^{-x^2-y^2} x^3$, which in polar coordinates becomes $f(r,\theta) = e^{-r^2} r^3 \cos^3(\theta)$. Hence, the answer is

$$\int_0^{\pi/2} \int_0^2 e^{-r^2} r^3 \cos^3(\theta) \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 e^{-r^2} r^4 \cos^3(\theta) \, dr \, d\theta$$

30. Find the volume of the solid region bounded by $z = 0$, $z = 1$, $x^2 + y^2 = 1$, and $x^2 + y^2 = 9$.

The volume of the solid region $E$ bounded by the xy- plane $z = 0$ and the surface given by $z = f(x, y) \geq 0$ over a region $D$ can be computed using the double integral $\iint_D f(x, y)dA$. Here, $f(x, y) = z = 1$, and $D$ represents the ring between the circle of radius 1 and the circle of radius 3. $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 9\} = \{(r, \theta) : 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$. Then, the volume of $E$ is given by

$$V = \iint_D f(x, y)dA = \iint_D 1dA = \int_0^{2\pi} \int_1^3 r\,dr d\theta = (2\pi)\left(\frac{9-1}{2}\right) = 8\pi.$$

31. Compute the value of the iterated integral

$$\int_0^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} (x^2 + y^2)dx\,dy$$

by converting to polar coordinates.

The region $D$ described in rectangular coordinates as

$$D = \{(x, y) \mid -\sqrt{2 - y^2} \leq x \leq \sqrt{2 - y^2},\ 0 \leq y \leq \sqrt{2}\}$$

is the half disk of radius $\sqrt{2}$ residing in the upper half-plane. In polar coordinates

$$D = \{(r, \theta) \mid 0 \leq r \leq \sqrt{2},\ 0 \leq \theta \leq \pi\}$$

Also the function $f(x, y) = x^2 + y^2$ becomes $f(r, \theta) = r^2$. So, the iterated integral converted to polar coordinates is

$$\int_0^{\pi} \int_0^{\sqrt{2}} r^2\ r\ dr\ d\theta = \int_0^{\pi} \int_0^{\sqrt{2}} r^3\ dr\ d\theta = \pi.$$

32. Let $f(x, y) = 100e^{-(x^2+y^2)/4}$, and $D$ is the disk of radius 4 centered at the origin $(0, 0)$.

(a) Set up an iterated integral to compute $\iint_D f(x, y)\,dA$ in polar coordinates.

$$\iint_D f(x, y)\,dA = \int_0^{2\pi} \int_0^4 100e^{-r^2/4} r\,dr\,d\theta.$$

(b) Compute the iterated integral you set up in part (a).

$$\int_0^{2\pi} \int_0^4 100e^{-\frac{r^2}{4}} r\,dr\,d\theta = 100 \int_0^{2\pi} \left[-2e^{-\frac{r^2}{4}}\right]_0^4 d\theta = 100 \int_0^{2\pi} \left(-2e^{-4} + 2\right) d\theta = 400\pi(-e^{-4}+1).$$

33. Let $D$ be the region in the first quadrant of the $xy$-plane enclosed by the lines $x = 0$, $y = x$, and the curve $y = \sqrt{4 - x^2}$. Use a double integral in polar coordinates to compute the area of this region $D$.

In polar coordinates, we have
$$D = \{(r, \theta) \,|\, 0 \le r \le 2, \ \pi/4 \le \theta \le \pi/2 \,\}.$$

So,
$$Area = \iint_D 1 \, dA = \int_{\pi/4}^{\pi/2} \int_0^2 1 \cdot r \ drd\theta = \int_{\pi/4}^{\pi/2} 2 \, d\theta = \frac{\pi}{2}.$$