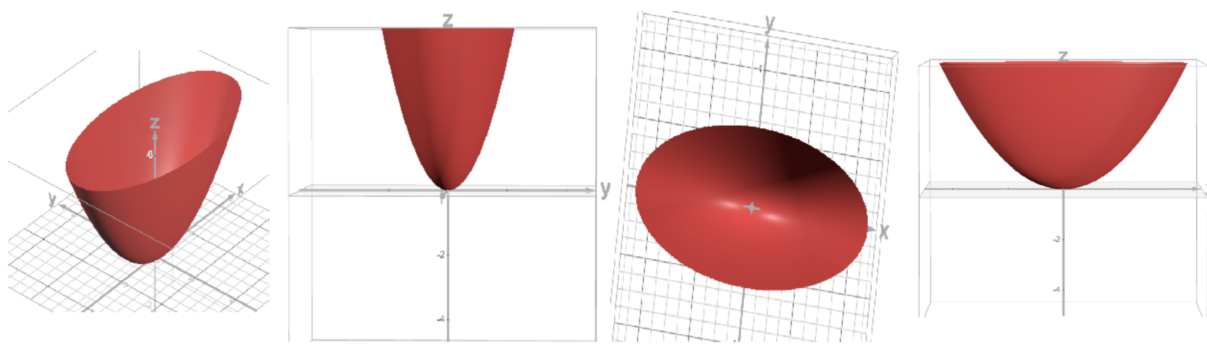
This worksheet covers the material of the textbook sections 12.1, 12.2, 12.3, 12.4, 12.5, 12.6, 13.1, 13.2, 13.3, 13.4. **This worksheet does NOT cover all topics/types of problems on the Exam 1 material.** Problems on the exam may not necessarily look exactly like the problems on this list. More practice problems can be found on the following resources: the instructor's Syllabus under **Suggested List of Textbook Problems**; the lecture notes; the discussion worksheets; the previous quizzes; the WebAssign homework.

1. Observe the following graphs:



Which equation below gives the surface shown above?

A. $z = \frac{x^2}{4} + y^2$

B. $\frac{z^2}{9} = \frac{x^2}{4} - y^2$

C. $1 - \frac{z^2}{9} = \frac{x^2}{4} + y^2$

D. $z = \frac{x^2}{4} - y^2$

ANSWER: (B)

2. Given 2 vectors $\mathbf{u} = \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{k}$.

(a) Compute $\mathbf{u} \cdot \mathbf{v}$

$\mathbf{u} \cdot \mathbf{v} = 0(1) + 1(0) + 1(1) = 1$

(b) Compute $\mathbf{u} \times \mathbf{v}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = (1-0)\mathbf{i} - (0-1)\mathbf{j} + (0-1)\mathbf{k} = \mathbf{i} + \mathbf{j} - \mathbf{k}$$

(c) Determine the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{1}{\sqrt{2}(\sqrt{2})} = \frac{1}{2}$$

So, $\theta = \frac{\pi}{3}$.

3. Consider the vectors $\mathbf{a} = \langle -1, 4, 8\rangle$ and $\mathbf{b} = \langle 18, 2, 1\rangle$.
   (a) Find the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$.

Since $||\mathbf{a}|| = \sqrt{1+16+64} = \sqrt{81} = 9$, then the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$comp_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||} = \frac{(-1)\cdot 18 + 4\cdot(2) + 8\cdot(1)}{9} = -\frac{2}{9}.$$

   (b) Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$proj_{\mathbf{a}}\mathbf{b} = (comp_{\mathbf{a}}\mathbf{b})\frac{\mathbf{a}}{||\mathbf{a}||} = -\frac{2}{9}\frac{\mathbf{a}}{||\mathbf{a}||} = -\frac{2}{9}\frac{\langle -1,4,8\rangle}{9} = \left\langle \frac{2}{81}, -\frac{8}{81}, -\frac{16}{81}\right\rangle$$

4. Determine whether the given vectors are orthogonal, parallel, or neither.
   (a) $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 12\mathbf{j} - 2\mathbf{k}$

Orthogonal: We have
$$\mathbf{a}\cdot\mathbf{b} = 4\cdot 5 + (-1)\cdot 12 + 4\cdot(-2) = 0,$$
so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

   (b) $\mathbf{a} = \langle 6, 5, -2\rangle$ and $\mathbf{b} = \langle 5, 0, 9\rangle$

Neither: We have
$$\mathbf{a}\cdot\mathbf{b} = 6\cdot 5 + 5\cdot 0 + (-2)\cdot 9 \neq 0,$$
so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

   (c) $\mathbf{a} = \langle -18, 15\rangle$ and $\mathbf{b} = \langle 12, -10\rangle$

Parallel: We have
$$\mathbf{a}\cdot\mathbf{b} = -18\cdot 12 + 15\cdot(-10) \neq 0,$$
so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Because the components of these two vectors are proportional by the same constant of proportionality, i.e., $\frac{-18}{12} = \frac{15}{-10} = \frac{-3}{2}$, then $\mathbf{b} = -\frac{2}{3}\mathbf{a}$, or $\mathbf{a} = -\frac{3}{a}\mathbf{b}$, so one vector is a constant multiple of the other. Hence, $\mathbf{a}$ and $\mathbf{b}$ are parallel (going in opposite directions).

5. Consider points $P(1,2,1), Q(2,5,4), R(6,9,12)$ and $S(5,6,9)$ in $\mathbb{R}^3$.

(a) Find the area of the parallelogram with vertices $P(1,2,1), Q(2,5,4), R(6,9,12)$ and $S(5,6,9)$.

By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{PQ} = \langle 1,3,3\rangle$ and $\overrightarrow{PS} = \langle 4,4,8\rangle$. Then, the cross product is

$$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 3 \\ 4 & 4 & 8 \end{vmatrix} = (24-12)\mathbf{i} - (8-12)\mathbf{j} + (4-12)\mathbf{k} = 12\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$$

So, the area of parallelogram PQRS is

$$||\overrightarrow{PQ} \times \overrightarrow{PS}|| = \sqrt{12^2 + 4^2 + (-8)^2} = 4\sqrt{14}.$$

(b) Find the area of the triangle PQS.

The area of the triangle determined by P, Q, and S is equal to half the area of the parallelogram PQRS. Using part (a), since the area of the parallelogram is $4\sqrt{14}$, then the area of triangle PQS is

$$\frac{1}{2}\text{(Area of PQRS)} = \frac{1}{2}4\sqrt{14} = 2\sqrt{14}.$$

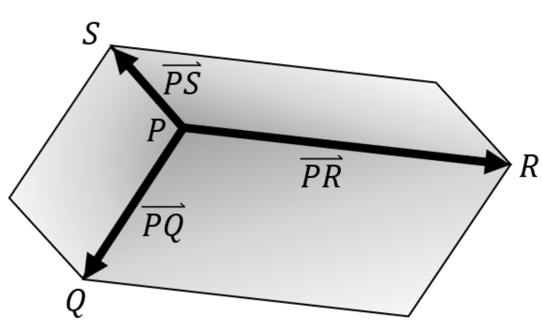
(c) Show that the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are coplanar.

We have $\overrightarrow{PQ} = \langle 1,3,3\rangle$, $\overrightarrow{PR} = \langle 5,7,11\rangle$, and $\overrightarrow{PS} = \langle 4,4,8\rangle$. Then,

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} 1 & 3 & 3 \\ 5 & 7 & 11 \\ 4 & 4 & 8 \end{vmatrix} = 1(56-44) - 3(40-44) + 3(20-28) = 12 + 12 - 24 = 0.$$

Since $||\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})|| = 0$, which means that the volume of the box determined by the three vectors is 0, i.e. there is no box. Then, the vectors must lie in the same plane; that is, they are coplanar.

6. Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS. The points are given by $P(3,0,1)$, $Q(-1,2,5)$, $R(5,1,-1)$, and $S(0,4,2)$.



$$\overrightarrow{PQ} = \langle -4, 2, 4\rangle, \qquad \overrightarrow{PR} = \langle 2, 1, -2\rangle, \qquad \overrightarrow{PS} = \langle -3, 4, 1\rangle.$$

$$\text{VOLUME} = \left|\overrightarrow{PQ}\cdot\left(\overrightarrow{PS}\times\overrightarrow{PR}\right)\right|.$$

We need $\overrightarrow{PS}\times\overrightarrow{PR}$.

$$\overrightarrow{PS}\times\overrightarrow{PR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 4 & 1 \\ 2 & 1 & -2 \end{vmatrix} = -9\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 11\hat{\mathbf{k}}.$$

Next, we compute the dot product $\overrightarrow{PQ}\cdot\left(\overrightarrow{PS}\times\overrightarrow{PR}\right) = \langle -4, 2, 4\rangle\cdot\langle -9, -4, -11\rangle = 36 - 8 - 44 = -16$. To find the volume, we take the absolute value of the number given by the dot product.

$$\text{VOLUME} = |-16| = 16 \text{ cubic units.}$$

7. Consider the following vectors.

$$\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 3\mathbf{k}, \quad \mathbf{v} = \mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = 3\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$$

(a) Find the scalar triple product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$.

(b) Find the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

(c) Are the given vectors coplanar?

(a) We have

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 3 & -3 \\ 1 & 2 & 0 \\ 3 & 8 & -3 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 8 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 3 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix}$$

$$= 1(-6-0) - 3(-3-0) - 3(8-6) = -6 + 9 + 6 = 9$$

(b) Using (a), $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{b})| = |9| = 9$.
(c) No, when $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$ is 0, we can say that these three vectors are coplanar.

8. Find the parametric equations for the line of intersection of the planes $2x + 3y + 5z = 7$ and $x - y + 2z = 3$.

The normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ to the respective planes can be found by reading off the coefficients from the equations:

$$\mathbf{n}_1 = \langle 2, 3, 5\rangle$$
$$\mathbf{n}_2 = \langle 1, -1, 2\rangle.$$

Now we compute

$$\mathbf{n}_1\times\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}\mathbf{k} = 11\mathbf{i} + \mathbf{j} - 5\mathbf{k}.$$

The line of intersection of the planes lies in both planes, so its direction vector $\mathbf{v}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. This condition is satisfied if we choose

$$\mathbf{v} = \mathbf{n}_1\times\mathbf{n}_2 = \langle 11, 1, -5\rangle.$$

To find a point on the line, we look for any simultaneous solution of the equations

$$
\begin{aligned}
2x + 3y + 5z &= 7 \\
x - y + 2z &= 3.
\end{aligned}
$$

If we set $y = 0$, these reduce to

$$
\begin{aligned}
2x + 5z &= 7 \\
x + 2z &= 3
\end{aligned}
$$

which we solve to obtain $x = 1, z = 1$. Thus $P_0(1, 0, 1)$ is a point on the line of intersection which lies on both planes, and has the position vector $\mathbf{r}_0 =< 1, 0, 1 >$. Finally, we can write the vector equation of the line of intersection of the planes given by the point P_0 and the vector $\mathbf{v}$ using the form $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ where t ranges over all real numbers, i.e.

$$
\langle x, y, z\rangle = \langle 1, 0, 1\rangle + t\langle 11, 1, -5\rangle \Longrightarrow \begin{cases} x = 1 + 11t \\ y = t \\ z = 1 - 5t \end{cases} .
$$

9. Find the vector equation, parametric equations and symmetric equations of the line passing through the points $A(2, 1, 1)$ and $B(3, 2, -2)$.

Vector direction of the line:

$$
\overrightarrow{AB} = \langle 3 - 2, 2 - 1, -2 - 1\rangle = \langle 1, 1, -3\rangle
$$

Point on the line: $A(2, 1, 1)$. Vector equation of the line:

$$
\langle x, y, z\rangle = \langle 2, 1, 1\rangle + t\langle 1, 1, -3\rangle
$$

The parametric equation of the line:

$$
x = 2 + t, \quad y = 1 + t, \quad z = 1 - 3t
$$

Symmetric equations of the line:

$$
x - 2 = y - 1 = \frac{z - 1}{-3}
$$

OR

Vector direction of the line:

$$
\overrightarrow{BA} = \langle 2 - 3, 1 - 2, 1 - (-2)\rangle = \langle -1, -1, 3\rangle
$$

Point on the line: $B(3, 2, -2)$. The vector equation of the line:

$$
\langle x, y, z\rangle = \langle 3, 2, -2\rangle + t\langle -1, -1, 3\rangle
$$

The parametric equation of the line:

$$
x = 3 - t, \quad y = 2 - t, \quad z = -2 + 3t
$$

The symmetric equation of the line

$$
\frac{x - 3}{-1} = \frac{y - 2}{-1} = \frac{z + 2}{3}
$$

10. Find an equation of the plane that passes through the point $P(1,1,3)$ and contains the line given by the symmetric equations $\dfrac{x+1}{2} = y+2 = \dfrac{z-3}{2}$.

The direction vector of the line $\mathbf{v} = \langle 2,1,2\rangle$.
A point on the line, take $t=0$: $(-1,-2,3)$.
In order to get the normal vector, we need another vector on the plane:

$$\overrightarrow{AP} = \langle 1-(-1), 1-(-2), 3-3\rangle = \langle 2,3,0\rangle$$

The normal vector $\mathbf{n} = \langle a,b,c\rangle$ is given by:

$$\mathbf{n} = \overrightarrow{AP} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 2 & 1 & 2 \end{vmatrix} = (6-0\mathbf{i} - (4-0\mathbf{j} + (2-6\mathbf{k} = 6\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} = \langle 6,-4,-4\rangle.$$

The equation of the plane:

$$\begin{aligned} a(x-x_0) + b(y-y_0) + c(z-z_0) &= 0 \\ 6(x-1) - 4(y-1) - 4(z-3) &= 0 \\ 6x - 4y - 4z + 10 &= 0 \end{aligned}$$

OR normal vector $\langle 3,-2,-2\rangle$.
The equation of the plane:

$$\begin{aligned} a(x-x_0) + b(y-y_0) + c(z-z_0) &= 0 \\ 3(x-1) - 2(y-1) - 2(z-3) &= 0 \\ 3x - 2y - 2z + 5 &= 0 \end{aligned}$$

11. Find an equation for the plane that passes through the points $(0,-2,5)$ and $(-1,3,1)$ and is perpendicular to the plane $2z = 5x+4y$.

The plane given by $2z = 5x + 4y$ (call it Plane 1) is also given by $5x + 4y - 2z = 0$ so it has normal vector $\mathbf{n_1} = \langle 5,4,-2\rangle$. Since the desired Plane 2 passes through the points $A(0,-2,5)$ and $B(-1,3,1)$, then the vector $\overrightarrow{AB} = \langle -1,5,-4\rangle$ is parallel to Plane 2. Thus, the normal vector $\mathbf{n_2}$ to Plane 2 is perpendicular to $\overrightarrow{AB} = \langle -1,5,-4\rangle$. When two planes are perpendicular, so are their normal vectors. So the normal vector $\mathbf{n_2}$ for Plane 2 is perpendicular to $\mathbf{n_1}$. Thus $\mathbf{n_2}$ is perpendicular to both $\overrightarrow{AB} = \langle -1,5,-4\rangle$ and $\mathbf{n_1} = \langle 5,4,-2\rangle$, so we can use

$$\mathbf{n} = \langle -1,5,-4\rangle \times \langle 5,4,-2\rangle = \langle 6,-22,-29\rangle.$$

We can choose P_0 to be the point $A(0,-2,5)$, so the scalar equation of the desired Plane 2 is

$$\begin{aligned} 6(x-0) - 22(y+2) - 29(z-5) &= 0 \\ 6x - 22y - 29z &= -101. \end{aligned}$$

12. Find an equation for the plane that passes through the points $(0,-2,5)$ and is parallel to the plane $2z = 5x+4y$.

The plane given by $2z = 5x + 4y$ (call it Plane 1) is also given by $5x + 4y - 2z = 0$ so it has normal vector $\mathbf{n_1} = \langle 5, 4, -2\rangle$. Since the desired Plane 2 is parallel to Plane 1, then the normal vector for plane 2 is also $\langle 5, 4, -2\rangle$. Since it passes through the points $A(0, -2, 5)$, the scalar equation of the desired Plane 2 is

$$5(x - 0) + 4(y + 2) - 2(z - 5) = 0$$
$$5x + 4y - 2z = 2.$$

13. Write an equation of the plane containing the points

$$P(4, -3, 1), \quad Q(-3, -1, 1), \quad R(4, -2, 8).$$

First, we'll need two vectors that lie in the plane and we can get two vectors determined by the three points we're given.

$$\overrightarrow{PQ} = \langle -7, 2, 0\rangle, \quad \overrightarrow{PR} = \langle 0, 1, 7\rangle.$$

These two vectors lie in the plane and we know that the cross product of any two vectors will be orthogonal to both of the vectors. Therefore, the cross product of these two vectors will also be orthogonal to the plane.

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & 2 & 0 \\ 0 & 1 & 7 \end{vmatrix} = \langle 14, 49, -7\rangle.$$

Using the point $P(4, -3, 1)$, an equation of the plane is

$$14(x - 4) + 49(y + 3) - 7(z - 1) = 0 \quad \text{OR} \quad 14x + 49y - 7z = -98.$$

14. Let $\mathcal{C}$ be the curve given by the vector function $\mathbf{r}(t) = \cos(t)\mathbf{i} + \ln(t)\mathbf{j} + \dfrac{1}{t-3}\mathbf{k}$.

(a) Find the the domain of $\mathbf{r}(t)$. Use the interval notation.

- x-component: Domain of $\cos(t)$

$$\text{All real numbers} = (-\infty, \infty)$$

- y-component: Domain of $\ln(t)$

$$t > 0 \quad \text{OR} \quad (0, \infty)$$

- z-component: Domain of $\dfrac{1}{t-3}$

$$\text{All real numbers except } 3 = (-\infty, 3) \cup (3, \infty)$$

So, the domain of $\mathbf{r}(t)$ is the intersection of the domains of the component functions, which is

$$(0, 3) \cup (3, \infty).$$

(b) Find $\lim_{t\to\pi} \mathbf{r}(t)$.

Since component functions are continuous at $t = \pi$, then we have

$$\lim_{t\to\pi} \mathbf{r}(t) = \left\langle \lim_{t\to\pi} \cos(t),\ \lim_{t\to\pi} \ln(t),\ \lim_{t\to\pi} \frac{1}{t-3} \right\rangle = \cos(\pi)\mathbf{i}+\ln(\pi)\mathbf{j}+\frac{1}{\pi-3}\mathbf{k} = -\mathbf{i}+\ln(\pi)\mathbf{j}+\frac{1}{\pi-3}\mathbf{k}$$

(c) Find the point P on the curve at $t = 1$.

Since we get

$$\mathbf{r}(1) = \left\langle \cos(1),\ \ln(1),\ \frac{1}{1-3} \right\rangle = \left\langle \cos(1),\ 0,\ -\frac{1}{2} \right\rangle,$$

then the point P on the curve is

$$\left(\cos(1),\ 0,\ -\frac{1}{2} \right).$$

15. Consider the vector function $\mathbf{r}(t) = 2\,\mathbf{i} + 2\sin(t)\,\mathbf{j} + 2\cos(t)\,\mathbf{k}$.

(a) Find the length of the curve of C with $\mathbf{r}(t)$, where $-2 \le t \le 2$.

Since $\mathbf{r}'(t) = \langle 0,\ 2\cos(t),\ -2\sin(t)\rangle$, then

$$||\mathbf{r}'(t)|| = \sqrt{0^2 + (2\cos(t))^2 + (-2\sin(t))^2} = \sqrt{0 + 4\cos^2(t) + 4\sin^2(t)} = \sqrt{4} = 2$$

Then we have

$$L = \int_{-2}^{2} ||\mathbf{r}'(t)||\,dt = \int_{-2}^{2} 2dt = \Big[2t\Big]_{-2}^{2} = 4 - (-4) = 8.$$

(b) Find the unit tangent vector $\mathbf{T}(t)$ at the point with the given value of the parameter $t = \pi/6$. Simplify the answer completely.

From (a), we have

$$\mathbf{r}'(t) = \langle 0,\ 2\cos(t),\ -2\sin(t)\rangle$$

$$||\mathbf{r}'(t)|| = \sqrt{0^2 + (2\cos(t))^2 + (-2\sin(t))^2} = \sqrt{0 + 4\cos^2(t) + 4\sin^2(t)} = \sqrt{4} = 2.$$

Then, the unit tangent vector $\mathbf{T}(t)$ is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{1}{2}\langle 0,\ 2\cos(t),\ -2\sin(t)\rangle = \langle 0,\ \cos(t),\ -\sin(t)\rangle \quad \text{OR} \quad \cos(t)\mathbf{j} - \sin(t)\mathbf{k}.$$

When $t = \pi/6$, we have

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)\mathbf{j} - \sin\left(\frac{\pi}{6}\right)\mathbf{k} = \frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$$

(c) Find the principal unit normal vector $\mathbf{N}(t)$ at the point with the given value of the parameter $t = \pi/6$. Simplify the answer completely.

From (b), we have

$$\mathbf{T}'(t) = \langle 0, -\sin(t), -\cos(t)\rangle \quad \text{and} \quad ||\mathbf{T}'(t)|| = \sqrt{0 + (-\sin(t))^2 + (-\cos(t))^2} = 1$$

Then, the principal unit normal vector $\mathbf{N}(t)$ is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \langle 0, -\sin(t), -\cos(t)\rangle = -\sin(t)\mathbf{j} - \cos(t)\mathbf{k}.$$

When $t = \pi/6$, we have

$$\mathbf{N}\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)\mathbf{j} - \cos\left(\frac{\pi}{6}\right)\mathbf{k} = -\frac{1}{2}\mathbf{j} - \frac{\sqrt{3}}{2}\mathbf{k}.$$

(d) Use the formula $\kappa = \dfrac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$ to find the curvature.

By the formula,

$$\kappa = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} = \frac{1}{2}$$

(e) Find the binormal vector $\mathbf{B}(t)$ at the point with the given value of the parameter $t = \pi/6$. Simplify the answer completely.

From (b) and (c),

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos(t) & -\sin(t) \\ 0 & -\sin(t) & -\cos(t) \end{vmatrix} = (-\cos^2(t) - \sin^2(t))\mathbf{i} - (0-0)\mathbf{j} + (0-0)\mathbf{k}$$

$$= -\mathbf{i} = \langle -1, 0, 0\rangle$$

(f) Find the tangential and normal components of acceleration $\mathbf{a}(t)$.

Using the formulas,

$$a_T = ||\mathbf{v}||' = ||\mathbf{r}'||' = (2)' = 0$$

$$a_N = \kappa||\mathbf{v}||^2 = \kappa||\mathbf{r}'||^2 = \frac{1}{2}(2)^2 = 2$$

16. Consider the position function $\mathbf{r}(t) = 8\sqrt{2}\,t\mathbf{i} + e^{8t}\mathbf{j} + e^{-8t}\mathbf{k}$.

(a) Find the velocity of a particle with the given position function $\mathbf{r}(t)$.

$$\mathbf{v}(t) = \mathbf{r}'(t) = 8\sqrt{2}\mathbf{i} + 8e^{8t}\mathbf{j} - 8e^{-8t}\mathbf{k}.$$

(b) Find the acceleration of a particle with the given position function $\mathbf{r}(t)$.

$$\mathbf{a}(t) = \mathbf{v}'(t) = 64e^{8t}\mathbf{j} + 64e^{-8t}\mathbf{k}.$$

(c) Find the speed of a particle with the given position function $\mathbf{r}(t)$.

$$||\mathbf{v}(t)|| = \sqrt{128 + 64e^{16t} + 64e^{-16t}} = \sqrt{64(2 + e^{16t} + e^{-16t})} = \sqrt{64(e^{8t} + e^{-8t})^2} = 8(e^{8t} + e^{-8t}).$$

17. Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

$$\mathbf{a}(t) = 2\mathbf{i} + 2t\mathbf{k}, \quad \mathbf{v}(0) = 5\mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}.$$

(1) We have the following.

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt = \int \langle 2, 0, 2t \rangle dt = \langle 2t + C_1, C_2, t^2 + C_3 \rangle$$

But we were given that $\mathbf{v}(0) = 5\mathbf{i} - \mathbf{j} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$, so $C_1 = 5$, $C_2 = -1$, and $C_3 = 0$. So,

$$\mathbf{v}(t) = (2t + 5)\mathbf{i} - 1\mathbf{j} + t^2\mathbf{k}.$$

(2) The position function is

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = \int \left[(2t + 5)\mathbf{i} - 1\mathbf{j} + t^2\mathbf{k}\right]dt = (t^2 + 5t + C_4)\mathbf{i} + (-t + C_5)\mathbf{j} + \left(\frac{t^3}{3} + C_6\right)\mathbf{k}$$

But we were given that $\mathbf{r}(0) = \mathbf{j} + \mathbf{k} = C_4\mathbf{i} + C_5\mathbf{j} + C_6\mathbf{k}$, so $C_4 = 0$, $C_5 = 1$, and $C_6 = 1$. So,

$$\mathbf{r}(t) = (t^2 + 5t)\mathbf{i} + (-t + 1)\mathbf{j} + \left(\frac{t^3}{3} + 1\right)\mathbf{k}$$

18. Given a vector function

$$\mathbf{r}(t) = (\arctan t)\,\mathbf{i} + 2t^2\,\mathbf{j} + t\ln(t)\,\mathbf{k}$$

(a) Find a vector equation of the line tangent to the vector function at the point $\left(\frac{\pi}{4}, 2, 0\right)$

The point $\left(\frac{\pi}{4}, 2, 0\right)$ corresponds to $t = 1$
Now, we take the derivative of $\mathbf{r}(t)$,

$$\mathbf{r}'(t) = (\frac{1}{1+t^2})\,\mathbf{i} + 4t\,\mathbf{j} + \left(\ln(t) + t\left(\frac{1}{t}\right)\right)\,\mathbf{k} = (\frac{1}{1+t^2})\,\mathbf{i} + 4t\,\mathbf{j} + (\ln(t) + 1)\,\mathbf{k}.$$

Evaluating the derivative at $t = 1$,

$$\mathbf{r}'(1) = \frac{1}{2}\,\mathbf{i} + 4\,\mathbf{j} + \,\mathbf{k}.$$

Now, the tangent line is given by

$$\mathbf{L}(t) = (\frac{1}{2}t + \frac{\pi}{4})\,\mathbf{i} + (4t+2)\,\mathbf{j} + t\,\mathbf{k}.$$

(b) Find the unit tangent vector $\mathbf{T}(t)$ at the point $\left(\frac{\pi}{4}, 2, 0\right)$.

$$\begin{aligned}
\mathbf{r}'(1) &= \frac{1}{2}\,\mathbf{i} + 4\,\mathbf{j} + \,\mathbf{k}. \\
|\mathbf{r}'(1)| &= \sqrt{\frac{1}{4} + 16 + 1} = \sqrt{\frac{69}{4}} = \frac{1}{2}\sqrt{69} \\
\mathbf{T}(1) &= \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\langle \frac{1}{2}, 4, 1\rangle}{\frac{1}{2}\sqrt{69}} = \left\langle \frac{1}{\sqrt{69}}, \frac{8}{\sqrt{69}}, \frac{2}{\sqrt{69}} \right\rangle
\end{aligned}$$