**Recall:** • Double integral of a CTS function $z=f(x,y)$ over a rectangle

$R=[a,b]\times[c,d]$ is

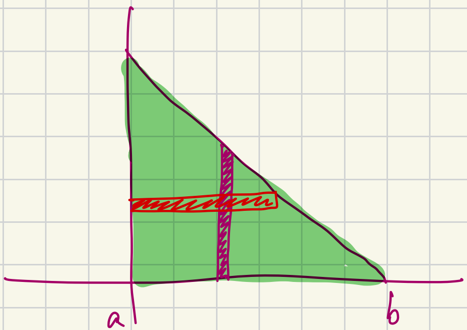
$$\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy=\iint_R f(x,y)\,dA$$

where, for example, you evaluate $\int_c^d f(x,y)\,dy$ by "integrating" in $y$, keeping $x$ "constant".

• Type 1: variable height in $y$

$D=\{(x,y): a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$

("Divide the region vertically")




• Type 2: variable "height" in $x$

$D=\{(x,y): c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$

**The Average Formula**

The average value of a function $f$ of two variables defined on a region $R$ with area $A(R)$, is given by

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

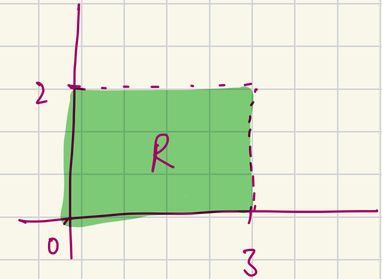
Recall: for a CTS function $f(x)$ on an interval $[a, b]$,

its average value is $\dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$.

**Question 6.** Find the average value of the function

$$f(x,y) = \frac{x}{1+xy}$$

on the rectangle $R = [\overset{x}{0,3}] \times [\overset{y}{0,2}]$.




$A(R) = 3 \times 2 = 6$

$$\cdot \iint_R \frac{x}{1+xy}\,dA = \int_0^3 \int_0^2 \frac{x}{1+xy}\,dy\,dx = J$$

$I = \int_0^2 \frac{x}{1+xy}\,dy$ take $u = 1+xy \Rightarrow du = x\,dy$, $y=0 \Rightarrow u = 1$, $y=2 \Rightarrow u = 1+2x$

$$\int \frac{1}{1+ay}\,dy \longrightarrow \int \frac{1}{u}\,du$$

$$\Rightarrow I = \int_1^{1+2x} \frac{du}{u} = \ln u \Big|_1^{1+2x} = \ln(1+2x) - \underbrace{\ln(1)}_{=0}$$

$$= \ln(1+2x)$$

$\Rightarrow J = \int_0^3 \ln(1+2x)\,dx$. Take $u = 1+2x \Rightarrow du = 2\,dx \Rightarrow \frac{du}{2} = dx$

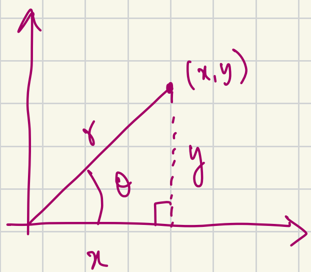
$$= \int_1^7 \frac{1}{2} \cdot \ln u \, du = \frac{1}{2} \int_1^7 \underbrace{\ln t}_{u} \, \underbrace{dt}_{dv} = \frac{1}{2} \left[ x \ln x \Big|_1^7 - \int_1^7 \frac{1}{x} \cdot x \, dx \right]$$

$$= \frac{1}{2} \left[ 7 \ln 7 - 0 - (7 - 1) \right]$$

$$= \frac{1}{2} [7 \ln 7 - 6]$$

Average value is $\frac{1}{6} \cdot \frac{1}{2} [7 \ln 7 - 6] = \boxed{\frac{1}{12} [7 \ln 7 - 6]}$.

# Polar Coordinates:



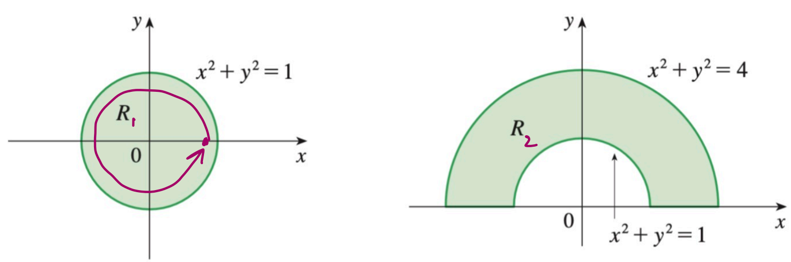

where $r = \sqrt{x^2 + y^2}$, $\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

and $\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$(x,y) \longrightarrow (r,\theta)$

Cartesian coordinates $\qquad$ Polar coordinates

Express the regions below using polar coordinates.




$R_1 = \{(x,y) : x^2 + y^2 \leq 1\}$

$= \{(r,\theta) : 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi\}$

$R_2 = \{(x,y) : 1 \leq x^2 + y^2 \leq 4\}$

$= \{(r,\theta) : 1 \leq r \leq 2, \ 0 \leq \theta \leq \pi\}$

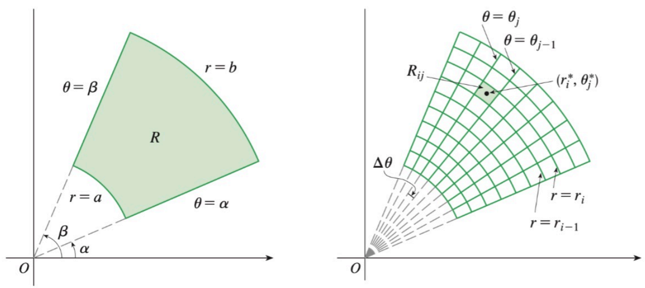
## Double Integrals in Polar Coordinates

The regions appearing in Question 2 are examples of *polar rectangles*,

$$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

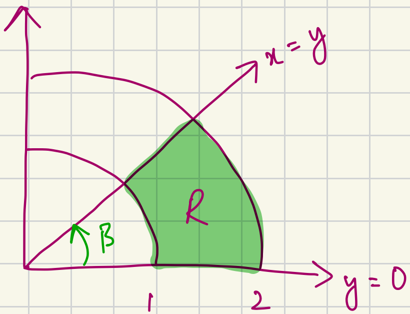
where $a \ge 0$ and $0 \le \beta - \alpha \le 2\pi$. If $f$ is continuous on $R$, then

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_{\theta=\alpha}^{\beta}\int_{r=a}^{b} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$




$$dA = dx\,dy \approx r\,dr\,d\theta$$

**Example:** Evaluate $\iint\limits_R (2x-y)\,dA$ where $R$ is the region in the first quadrant enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=4$ and the lines $x=y$ and $y=0$.




$R = \{(r,\theta) : 1 \leq r \leq 2 \;,\; 0 \leq \theta \leq \frac{\pi}{4}\}$

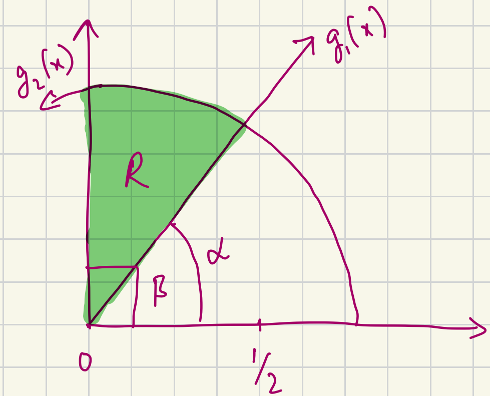
for bounds on $\theta$:

- lower bound is 0 since $R$ is bounded by $x$-axis
- upper bound, $\beta$, is the angle that the line $y=x$ makes with the $x$-axis.
- slope of $y=x$ is $m=1 \Rightarrow \tan\beta = 1 \Rightarrow \beta = \arctan 1 = \pi/4$.

- $\iint\limits_R 2x-y\,dA = \int_0^{\pi/4}\int_1^2 (2r\cos\theta - r\sin\theta)\,r\,dr\,d\theta \overset{\text{DIY}}{=} \boxed{\frac{7}{6}(3\sqrt{2}-2)}$

**Example:** Evaluate $\int_0^{1/2}\int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\,dy\,dx$ by switching to polar coordinates.

$R = \{(x,y) : 0 \leq x \leq 1/2 \ , \ \underbrace{\sqrt{3}x}_{g_1(x)=\sqrt{3}x} \leq y \leq \underbrace{\sqrt{1-x^2}}_{g_2(x)=\sqrt{1-x^2}} \}$




- $y = g_2(x) \Rightarrow y = \sqrt{1-x^2} \Leftrightarrow y^2 = 1-x^2 \Leftrightarrow x^2+y^2 = 1$

In polar coordinates, $R = \{(r,\theta) : 0 \leq r \leq 1 \ , \ \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}\}$

- $\tan\alpha$ = slope of the line $y = \sqrt{3}x$, which is $\sqrt{3}$.

$\Rightarrow \alpha = \arctan\sqrt{3} = \pi/3$

$$\Rightarrow \int_0^{1/2}\int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\,dy\,dx = \int_{\pi/3}^{\pi/2}\int_0^1 (r\cos\theta)\, r\,dr\,d\theta \overset{\text{DIY}}{=} \boxed{\frac{2-\sqrt{3}}{6}}$$

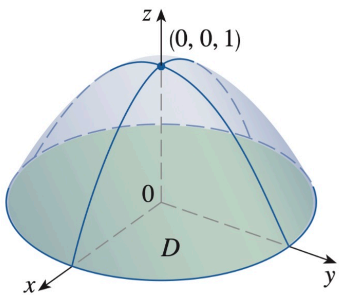
Evaluate the double integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

**Question 4.** Find the volume of the solid $E$ bounded below the paraboloid

$$z = 1 - x^2 - y^2$$

and above the plane $z = 0$.



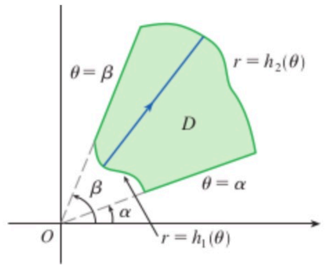
3 If $f$ is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$




Area of D, denoted by $A(D)$, is $A(D) = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \underline{1}\, r\, dr\, d\theta$

Eg: in 1-D, $\int_a^b 1\, dx = x \Big|_a^b = b - a =$ length of $[a, b]$.

**Example:** Find the area of $D$ where $D$ is the region inside the circle $(x-1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$.

circle of radius 1 with center at $(1,0)$

- $R_1 = \{(x,y) : (x-1)^2 + y^2 \le 1\}$

$$(r\cos\theta - 1)^2 + (r\sin\theta)^2 = r^2\cos^2\theta + \cancel{1} - 2r\cos\theta + r^2\sin^2\theta = \cancel{1}$$

$$\Leftrightarrow \quad r^2 - 2r\cos\theta = 0 \quad \Leftrightarrow \quad r = 2\cos\theta \quad (\text{when } r \neq 0)$$