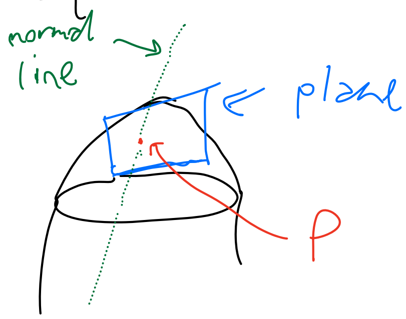
DW8 Challenge Problems: 3, 7, 10, 11

Analysis, responses, changes coming from survey TBA in Canvas annoucement later today

#3: For $P=(2,2,1)$ on surface $xy^2z^3=8$, find the equations of the tangent plane & normal line through P.



Recall: $f_x(\ldots)(x-x_0)+f_y(\ldots)(y-y_0)=z-f(x_0,y_0)$ is the tangent plane formula if the surface is of the form $z=f(x,y)$.

Thus, we need to solve for z: $z^3=\dfrac{8}{xy^2} \Rightarrow$

$$z=\frac{2}{x^{1/3}y^{2/3}}=2x^{-1/3}y^{-2/3}:=f(x,y)$$

$$f_x=-\frac{2}{3}x^{-4/3}y^{-2/3}=-\frac{2}{3}2^{-6/3}=-\frac{1}{6}$$

$$f_y=-\frac{4}{3}x^{-1/3}y^{-5/3}=-\frac{4}{3}2^{-6/3}=-\frac{1}{3}$$

$$f(2,2)=2\cdot 2^{-3/3}=1$$ $\rightarrow$ if this wasn't 1, then you made a mistake in finding f.

So tangent plane is $-\frac{1}{6}(x-2)-\frac{1}{3}(y-2)=z-1$

Notice that the normal line to the surface through P is just the normal to the tangent plane.

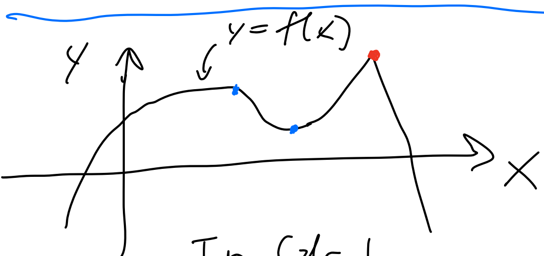
Rewrite tangent plane equation: $\frac{1}{6}(x-2)+\frac{1}{3}(y-2)+1(z-1)=0$, so a normal vector is $\langle \frac{1}{6}, \frac{1}{3}, 1\rangle$, which tells us the direction of the normal line.

So $\boxed{r(t) = (2,2,1) + t\left(\frac{1}{6},\frac{1}{3},1\right) = \left(2+\frac{t}{6},\, 2+\frac{t}{3},\, 1+t\right),\ t\in\mathbb{R}}$ is a parametrization of the normal line.

Note: you can also use the gradient to find the normal line. For a surface S described by $F=0$ for some differentiable $F(x,y,z)$, the vector $(\nabla F)(x_0,y_0,z_0)$ is normal to $(x_0,y_0,z_0)\in S$.

#7: Critical points of $f(x,y) = (y-2)x^2 - y^2$.

Critical points of f are where ∇f undefined or $\nabla f = \vec{0}$. In this case, ∇f exists everywhere (since f is a polynomial, it's infinitely differentiable, so f_x & f_y exist, so $\nabla f = \langle f_x, f_y\rangle$ exists), so we need to solve $\nabla f = 0$.



In Calc 1, you would need to check the red point because f' DNE there, similarly you need to check where ∇f DNE

$(0,0) = (f_x, f_y) = (2x(y-2),\, x^2-2y)$,

so we have $\begin{cases} x(y-2)=0 \\ x^2 = 2y \end{cases}$

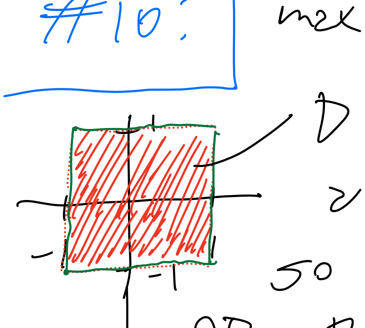
1st equation: $x=0$ or $y=2$.

If $x=0$, 2nd equation $\Rightarrow y=0$.

If $y=2$, then $x^2=4 \Rightarrow x=\pm 2$.

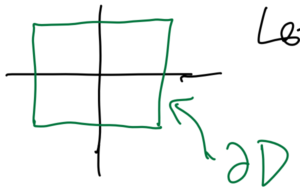
So there are 3 solutions, and the 3 critical points are $(0,0),\ (2,2),\ (-2,2)$.

#10: max & min of $f = x^2+y^2+x^2y+9$ on $D=\{|x|,|y|\le 1\}$.



Any max/min in interior of D must be a local min/max, so they'd be critical points, so they would satisfy $0=\nabla f$.

∂D, the boundary of D, has to be checked manually.



Let's look at interior: $\langle 0,0\rangle = \nabla f = \langle 2x+2xy,\ 2y+x^2\rangle$

$$\Rightarrow \left\{\begin{array}{l} 0 = x+xy = x(1+y) \\ y = -\frac{1}{2}x^2 \end{array}\right\}.$$

Note: make sure to exclude solutions outside region

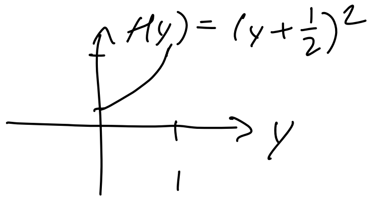
1st eq: $x=0$ or $y=-1$. $x=0 \Rightarrow y=0$. But if $y=-1$, then $-1 = -\frac{1}{2}x^2 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$.

So our candidates are $\cancel{(\pm\sqrt{2}, -1)}$, $(0,0)$.

$\cancel{f(\pm\sqrt{2},-1) = 2+1-2+9 = 10}$, $f(0,0) = 0+0+0+9 = 9$.

Now check boundary: note $\partial D \subseteq \{x=\pm 1\} \cup \{y=1\} \cup \{y=-1\}$.

If $x = \pm 1$: $f = 1+y^2+y+9 = y^2+y+10 = (y+\frac{1}{2})^2 + 9.75$



$\geq (\frac{1}{2})^2 + 9.75 = 10$, $\leq (1+\frac{1}{2})^2 + 9.75 = 2.25 + 9.75 = 12$, and $f(1,1) = 12$.

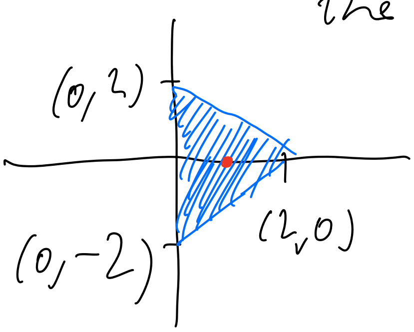
If $y=1$: $f = 2x^2+10$, between 10 and 12 since $0 \leq 2x^2 \leq 2$.

If $y=-1$: $f = x^2+1-x^2+9 = 10$, nothing new.

From all the values we found, 9 is lowest & 12 is highest, so max $= 12$, min $= 9$.

max is attained at $(1,1)$, $(-1,1)$
min is attained at $(0,0)$.

11. Absolute min/max of $f = x^2+y^2-2x$ on D, the triangle w/ vertices $(2,0)$, $(0,2)$, $(0,-2)$.



$(0,0) = \nabla f = (2x-2,\ 2y) \Rightarrow$

$$\left\{\begin{array}{l} 0 = 2x-2 \\ 0 = 2y \end{array}\right\} \Rightarrow x=1 \ \&\ y=0$$

Check $f(1,0) = 1^2+0^2-2\cdot 1 = -1$.

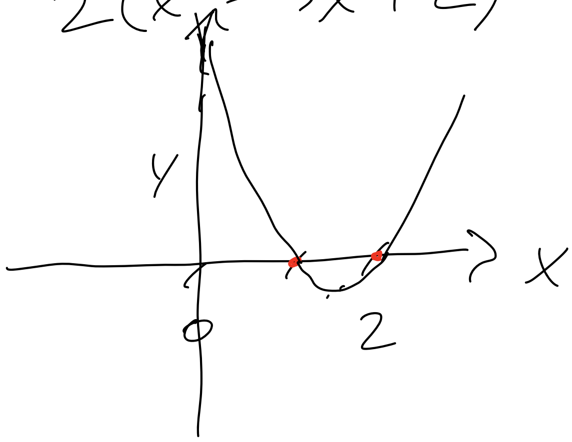
Now check $\partial D \subseteq \{x=0$ or $y=2-x$ or $y=x-2\}$

$= \{x=0\} \cup \{y=2-x\} \cup \{y=x-2\}$.

$y=2-x$ or $y=x-2$: In either case, $y^2=(x-2)^2$,

so $f = x^2-2x+(x-2)^2 = 2x^2-6x+4 =$

$2(x^2-3x+2) = 2(x-1)(x-2)$.



min/max for this case is

$x=1.5 \Rightarrow f=-\frac{1}{2}$

$x=0 \Rightarrow f=4$

$x=0$: then $f=0^2+y^2-2\cdot 0 = y^2 \in [0,4]$.

Combining all cases, max $=4$ & min $=-1$.

IIb: $z=x^2y+\sin y$, $x=u^2-v$, $y=e^{uv}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = 2xy\cdot 2u + (x^2+\cos y)ve^{uv}$$

$$\frac{\partial^2 z}{\partial v \partial u} = z_{vu} = z_{uv} \text{ by Clairaut's Theorem}$$

$$= (z_u)_v = \frac{\partial}{\partial v} z_u$$

1st way: ∂_v to the red expression directly

2nd way: plug in fully for x & y in terms of u & v, then take the partial derivative normally.

3rd way: v derivative of blue term using 2 more chain rules, then plug in for all of the terms.

5: Find f for which $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ along every path $y=mx$ but not 0 along some nonlinear path (e.g. $y=x^2$).

Try to make limit non-zero when $y=x^2$. To that end, let's place y & x^2 in the denominator. Placing $y-x^2$ in the denominator may work, but then we need to have some term going to 0 in the numerator to cancel this out.

Try placing $y+x^2$ in the denominator instead: $f(x,y) = \frac{?}{y+x^2}$. When we plug

in $y=x^2$, denominator will be $2x^2$.
We need to put something in the numerator to match $2x^2$ so that the limit is non-zero. Why not $? = 2x^2$.
Let's see if $f(x,y) = \frac{2x^2}{y+x^2}$ works.

Along $y=x^2$: $f = \frac{2x^2}{2x^2} = 1 \longrightarrow 1 \neq 0$.

Along $y=mx$: $f = \frac{2x^2}{x(m+x)} = \frac{2x}{m+x} \longrightarrow \frac{0}{m}$

$= 0$, so we get the desired behavior and this f works.

However, we may want to account for $m=0$, which is the line $y=0$.
To that end, we need to get something in the numerator that becomes 0 when $y=0$.
However, we don't want to destroy the fact that the numerator becomes $2x^2$ when $y=x^2$. So we need to do something that doesn't change when $y=x^2$,

but introduces y. So take one factor of x & replace it with $\sqrt{y}$. Then

$$f(x,y) = \frac{2x\sqrt{y}}{y+x^2}.$$

$$y = x^2 \rightarrow f = \frac{2x^2}{2x^2} = 1 \rightarrow 1 \checkmark$$

$$y = 0 \rightarrow \frac{2x \cdot 0}{0+x^2} = \frac{0}{x^2} = 0 \rightarrow 0 \checkmark$$

$$\begin{array}{l} y = mx \\ m \neq 0 \end{array} \rightarrow \frac{2x\sqrt{mx}}{x(m+x)} = \frac{2\sqrt{mx}}{m+x} \rightarrow \frac{0}{m} = 0$$

$1 \neq 0$, so 2 different values along 2 different paths, so overall limit DNE.