Logistical Announcements

1. Midterms graded by end of this week, but grades may not be released until next week due to makeups.
2. Survey released, may add extra credit & reformat
3. For Shashank: disc. attendance may be graded starting next week.

DW6 #8: Verify Clairaut conclusion for the function $u(x,y) = e^{xy}\sin(y)$.

Recall: Clairaut says $u_{xy} = u_{yx}$ under some conditions about the 2nd order derivatives u_{xy} & u_{yx}. It suffices xy & yx - der cont. everywhere, and even continuous around the point is enough.

To verify conclusion, need to check $u_{xy} = u_{yx}$.

$$u = e^{xy}\sin y$$

$$u_x = \sin y\,(e^{xy})_x = y\sin y\, e^{xy}$$

$$u_y = \cos y\, e^{xy} + x\sin y\, e^{xy} = (\cos y + x\sin y)e^{xy}$$

$$u_{xy} = (u_x)_y = \underline{(y\sin y)_y e^{xy} + y\sin y (e^{xy})_y} =$$

$$\underline{(\sin y + y\cos y + xy\sin y)e^{xy}}.$$

$$u_{yx} = (u_y)_x = (\cos y + x\sin y)_x e^{xy} + (\cos y + x\sin y)(e^{xy})_x$$

$$= \underline{(\sin y + y\cos y + yx\sin y)e^{xy}} = u_{xy}$$

$(u_y)_x = \left(\left(\cos y + x\sin y\right)e^{xy}\right)_x =$

$\left(\cos y + x\sin y\right)_x e^{xy} + \left(\cos y + x\sin y\right)\left(e^{xy}\right)_x$

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9. Given $f = x^4y^2 - x^3y$, find f_{xxx} & f_{xyx}.

Notice that f & all of its derivatives are continuous because f is a polynomial, so we can apply Clairaut to get

$$f_{xyx} = (f_x)_{yx} = (f_x)_{xy} = (f_{xx})_y$$

Note on notation: $f_{abcd} = (((f_a)_b)_c)_d$ and similarly for higher order derivatives.

Also note $f_\alpha = \frac{\partial}{\partial\alpha} f = \frac{\partial f}{\partial \alpha}$.

Our computational procedure: $f \to f_x \to f_{xx}$ $\nearrow f_{xxx}$ $\searrow f_{xxy}$

to get the 2 answers in red.

$f = x^4y^2 - x^3y$

$f_x = 4x^3y^2 - 3x^2y$

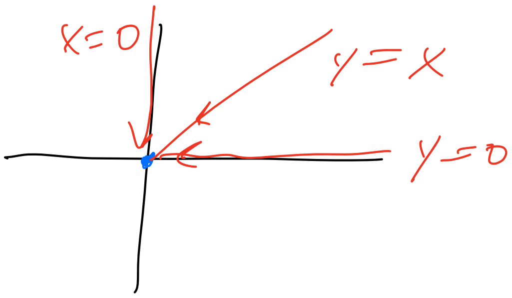
$f_{xx} = 12x^2y^2 - 6xy$

$f_{xxx} = 24xy^2 - 6y$

$f_{xyx} = f_{xxy} = 24x^2y - 6x$

4c: Evaluate if exists \ Show limit DNE: $\lim\limits_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$.

We don't know if limit DNE or not, so let's try taking limit along directions.



If $y=0$: $\frac{2xy}{x^2+3y^2} = \frac{2x\cdot 0}{x^2+3\cdot 0^2} = \frac{0}{x^2} = 0 \longrightarrow 0$, so limit $=0$.

If $y=x$: $\frac{2xy}{x^2+3y^2} = \frac{2x\cdot x}{x^2+3x^2} = \frac{2x^2}{4x^2} = \frac{1}{2} \longrightarrow \frac{1}{2}$, so limit $= \frac{1}{2}$ along this path.

$\frac{1}{2} \neq 0$, we have 2 different limit along 2 different directions, so limit DNE.

Note: for limit at (x_0,y_0) to exist, you must have the same value no matter which path you approach (x_0,y_0) along.

$$\int \ln t\,dt = \int \underbrace{\ln t}_{v} \cdot \underbrace{(1\,dt)}_{du} = t\ln t - \int t\cdot\frac{1}{t}\,dt$$

$v = \ln t$ $\quad du = dt$

$dv = \frac{1}{t}\,dt$

$$= t\ln t - \int dt$$

$$= t\ln t - t + C$$

2: Let $F(x,y,z) = \sqrt{y} - \sqrt{x-2z}$, find $F(3,4,1)$ & domain of F.

(a): $F(3,4,1) = \sqrt{4} - \sqrt{3-2\cdot 1} =$
$2 - \sqrt{3-2} = 2 - 1 = 1$.

(b): $\text{domain}(F) = \{\text{where } F \text{ defined}\}$
$= \{\sqrt{y} \text{ def.}\} \cap \{\sqrt{x-2z} \text{ is def.}\}$.

$\sqrt{y}$ defined $\iff y \geq 0$ because $\sqrt{\ }$ only def. for non-negative inputs.

Pedantic remark: $\sqrt{-1} = i$, but we are only concerned about real outputs since F is implicitly considered to be a function $F: R^3 \to R$.

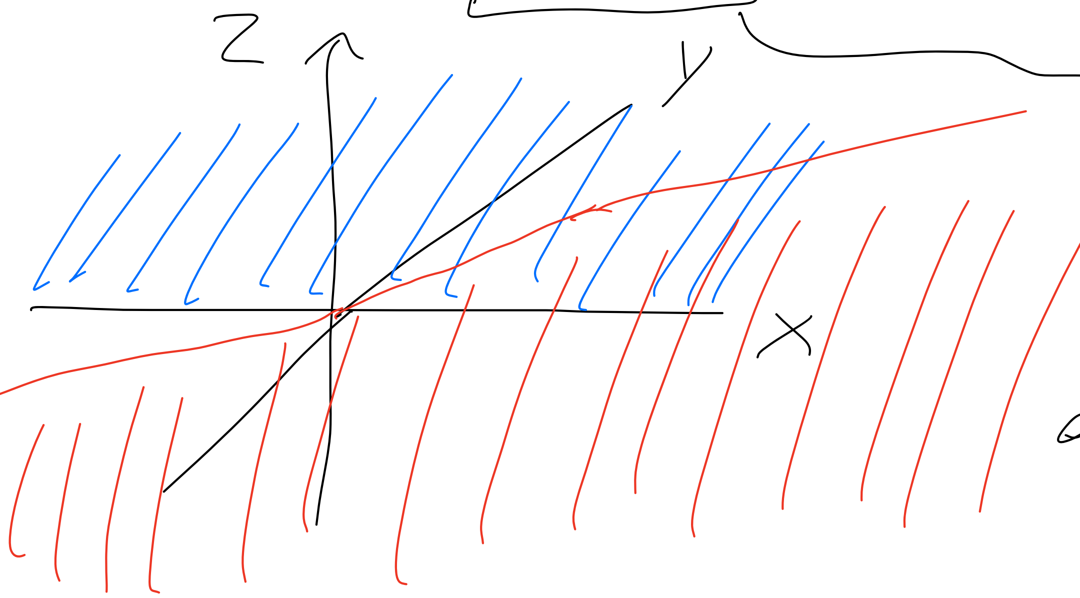
general remark: you will only see real numbers (& subsets thereof like N & Z) in this class. If you end up with $a+bi$ and there wasn't previous notice from instructor/TA/

Textbook about the unit relying on complex numbers, you made a mistake should check your work.

$\sqrt{x-2z}$ def. $\iff x-2z \geq 0 \iff x \geq 2z$.

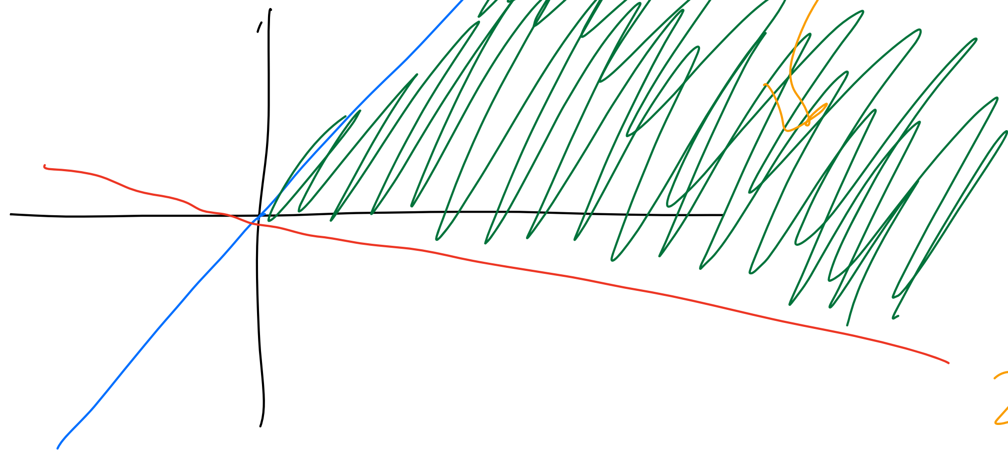
$$\mathrm{Dom}(F) = \{(x,y,z) : y \geq 0 \ \& \ x \geq 2z\}$$
$$= \underbrace{\{y \geq 0\} \cap \{x \geq 2z\}}_{\text{half spaces of } R^3} \subseteq R^3$$

half spaces of R^3, i.e. everything on one side of a plane.



Intersecting 2 half-spaces gives a quarter space, the result of taking everything on one quadrant cut out by 2 planes.

A quarter space is a space whose



generic 2D cross sections are the infinite region between 2 intersecting lines.

5: Find where $F = \frac{1+x^2+y^2}{1-x^2-y^2}$, $G = \ln(1+x-y)$ are continuous.

Generally speaking: functions appearing in class that aren't piecewise will be continuous everywhere on their domain, so it suffices to find the domain of F & G.

$1, x^2, y^2$ are continuous

$1+x^2+y^2$, $1-x^2-y^2$ are both continuous as a sum/diff of cont. functions

$\Rightarrow F = \frac{1+x^2+y^2}{1-x^2-y^2}$ will be continuous everywhere $1-x^2-y^2 \neq 0$ since "f & g continuous, $g \neq 0 \Rightarrow \frac{f}{g}$ continuous"

Similarly, $1+x-y$ cont. $\Rightarrow \ln(1+x-y)$ cont. on its domain.

So now, let's find the domain.

$\ln(1+x-y)$ defined $\Longleftrightarrow 1+x-y > 0$

Note: if you are interested, read about the complex-valued logarithm for arbitrary nonzero inputs.

$\mathrm{Log}: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is a multivalued function with many of the same properties as ordinary $\log: \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

$\Longleftrightarrow y < 1+x.$

f defined $\Longleftrightarrow$ denom $\neq 0 \Longleftrightarrow$ $1-x^2-y^2 \neq 0 \Longleftrightarrow x^2+y^2 \neq 1.$

So $cont(F) = \{(x,y) : y < 1 + x\}$,

$cont(G) = \{(x,y) : x^2 + y^2 \neq 1\}$.

6: Find 1st partial derivatives of

$g(u,v) = (u^2 v - v^3)^5$.

This will be $g_u = \frac{\partial g}{\partial u}$ & $g_v = \frac{\partial g}{\partial v}$.

We will use the chain rule.

$g_u = 5(u^2 v - v^3)^4 \cdot (u^2 v - v^3)_u$

$= 5(u^2 v - v^3)^4 (2uv - 0) =$

$10uv(u^2 v - v^3)^4$

$g_v = 5(u^2 v - v^3)^4 \cdot (u^2 v - v^3)_v$

$= 5(u^2 v - v^3)^4 (u^2 - 3v^2)$

Among the harder ones: 4b, 4c, 4f, , 9, sketching part of 3

4b: $\lim\limits_{(x,y)\to(1,1)} \frac{x^2 y^3 - x^3 y^2}{x^2 - y^2} = \lim\limits_{(x,y)\to(1,1)} \frac{x^2 y^2 (y - x)}{(x-y)(x+y)}$

If we plug in $(x,y)=(1,1)$ directly, the fraction is $\frac{0}{0}$, indeterminate forms. Recall the Calc 1&2 methods for indet forms: $\underbrace{\text{cancel out terms}}$, rationalize square roots, L'Hopital, Taylor series expansion of numerator & denominator.

L'Hopital no longer applies since the limit is 2 dimensional. Taylor series don't do anything since numerator & denominator are already polynomials. Finally, there's no square roots to rationalize.

$$\lim_{(x,y)\to(1,1)} \frac{x^2y^2\cancel{(y-x)}}{\cancel{(x-y)}(x+y)} = -\lim_{(x,y)\to(1,1)} \frac{x^2y^2}{x+y}$$

because $x-y\neq 0$ in the domain of the function. When we take a limit, especially in 2D, you are only telling the limit along where the function is defined. $x=y \Rightarrow$ function is $\frac{0}{0}$

$\rightarrow$ function is undefined.

Now $\frac{x^2y^2}{x+y}$ is continuous around $(1,1)$

since x^2y^2, $x+y$ are continuous, and

$x+y=2 \neq 0$ at $(1,1)$. Thus, the

limit is $-\frac{1^2 1^2}{1+1} = -\frac{1}{2}$, and

in particular the limit exists.

4f: $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$, exists or not & what is its value?

Recall all the methods mentioned before. Rationalizing the square root stands out.

$$= \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{\sqrt{x^2+y^2+1}^2 - 1^2}$$ by multip-

Mulying numerator & denominator by conjugate.

$$= \frac{\cancel{(x^2+y^2)}(\sqrt{\ldots}+1)}{\cancel{x^2+y^2}} = \sqrt{x^2+y^2+1}+1.$$

This is continuous at $(0,0)$, so we can just plug in $(0,0)$ to get the limit is $\sqrt{0^2+0^2+1}+1 = 2$.