Vector Functions

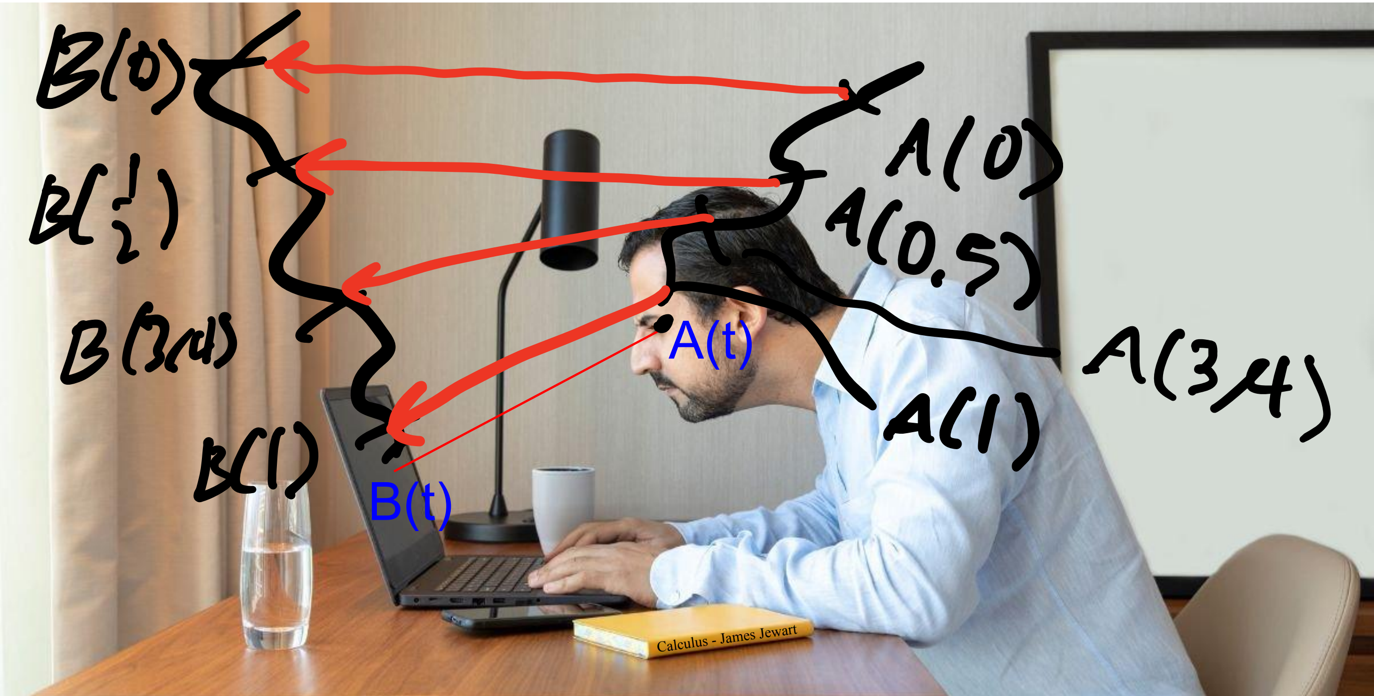
Lecture for 6/11

Definition of Vector Functions

$\langle a, b, c \rangle$
$\downarrow \quad \downarrow \quad \downarrow$
$a(t) \quad b(t) \quad c(t)$

- Write r(t) = (f(t), g(t)) or (f(t), g(t), h(t))
- Same B-A trick to figure out r when given 2 vectors
- Can restrict domain

Later on: we will see $\langle f(s,t), g(s,t), h(s,t) \rangle$



Source: https://heliooptometry.ca/blog/does-squinting-mean-you-need-glasses

1 — variable

Limits, derivatives, integrals

$\lim_{t \to 1} r(t)$ exist

- Limits are taken component-wise:
  - $\lim_{t \to 1} r(t) = (\lim_{t \to 1} f(t), \lim_{t \to 1} g(t), \lim_{t \to 1} h(t))$
- Vector function limit exists iff each component limit exists
- Derivatives and indefinite integrals also taken component-wise
- Constant of integration +C becomes vector $+\mathbf{c} = +(c_1,c_2,c_3)$
- Definite integrals evaluated using antiderivatives as usual

If $\lim_{t \to 1} f(t) = a$, $g \to b$, $h \to c$,

then $\lim_{t \to 1} r(t) = (a, b, c)$

$r'(t) = (f'(t), g'(t), h'(t))$

$\int r(t) =$

Derivative Rules

$(\int f, \int g, \int h) = (F + C_1, G + C_2, H + C_3)$

- Let r, s be vectors, f scalar, c constant $= (F, G, H) + \underline{(C_1, C_2, C_3)}$
- Basic properties still hold
  - Linearity: (cr)' = cr', (r+s)' = r'+s' $= R + C$

vector of integ.

vector antideriv.

- Product rules
  - (fr)' = f'r + fr' — vec & func
  - (r·s)' = r'·s + r·s' — 2 vec
  - (r x s)' = r' x s + r x s' — 2 vec
- Chain rule: [r(f(t))]' = f'(t)r'(f(t))

If $r(t) = (a(t), b(t), c(t))$, $fr = f(t)\,r(t) =$

$f(t)\,(a(t), b(t), c(t)) = (a(t)f(t), b(t)f(t), c(t)f(t))$

where you save the most time

r o f is complex

Arc Length

r', f' simpler

- Can’t reduce to components easily
- Call ds a tiny bit of the arc
- Line segment for ds is r(t) to r(t+dt)
- Use this to get ds = ||r’(t)|| dt
- L = $\int$ ds = $\int \sqrt{((f')^2+(g')^2+(h')^2)}\,dt$
- Now you have a basic integral

same exact bounds as your bounds on r(t)

$\|r'\| = \|(f', g', h')\| =$

$\sqrt{(f')^2 + (g')^2 + (h')^2}$

Practice problems

~~Understand your segments~~

- ~~Find a vector equation for the line segment between (a, b, c) and (d, e, f)~~

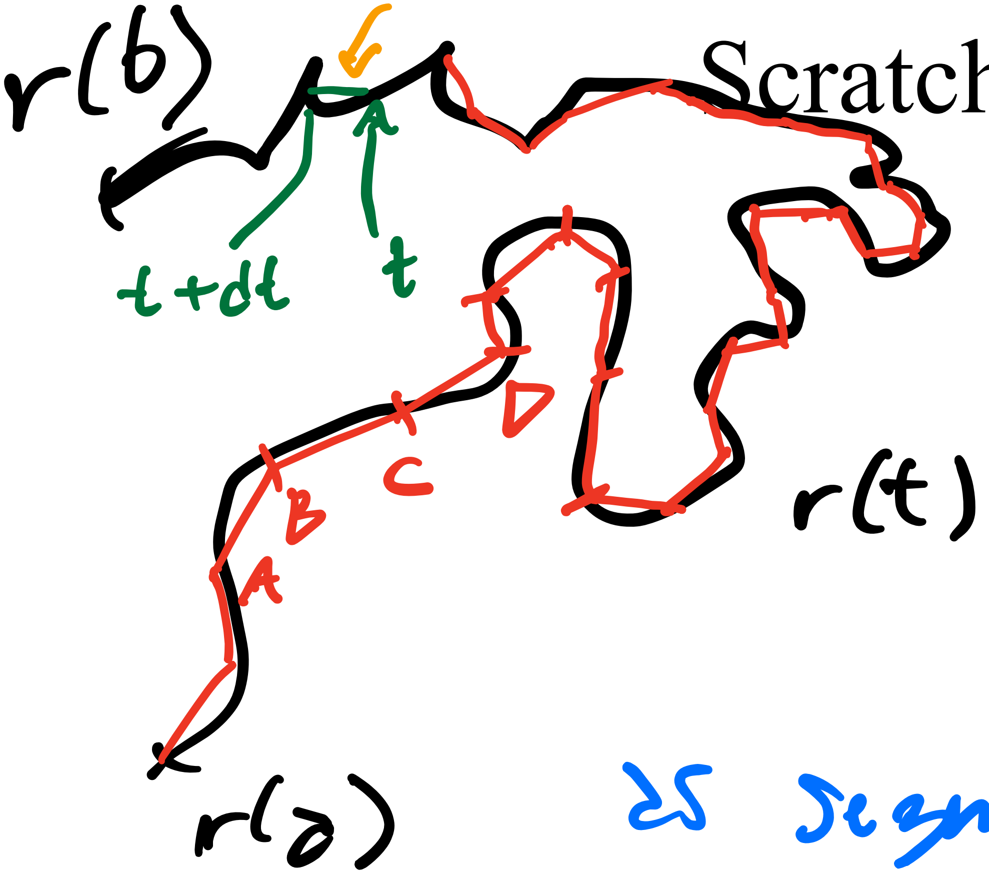
We already found the line segment formula yesterday

Mixing vector products and derivatives

- Let r(t) = (cos(t), sin(t), 0) and s(t) = (sin(t), -cos(t), 1). Compute (r x s)’, (r · s)’ with and without the product rule

Δs $\qquad L = \sum \Delta s \rightarrow \int ds$

Scratch Work



$a \le t \le b$

L = arc length

$L \approx |\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}| + \ldots$

as segments become smaller and smaller, approximation goes to L

length of green seg. is $\|r(t+dt) - r(t)\|$

$$= \left\| \frac{r(t+dt) - r(t)}{dt} \right\| dt \xrightarrow{} \|r'(t)\| dt$$

by limit def. of derivative

Extra Problem

Arc length of helix

- Let r(t) = (cos(t), sin(t), t), 0 ≤ t ≤ 2π represent one curl of a helix. Find the arc length of this curve

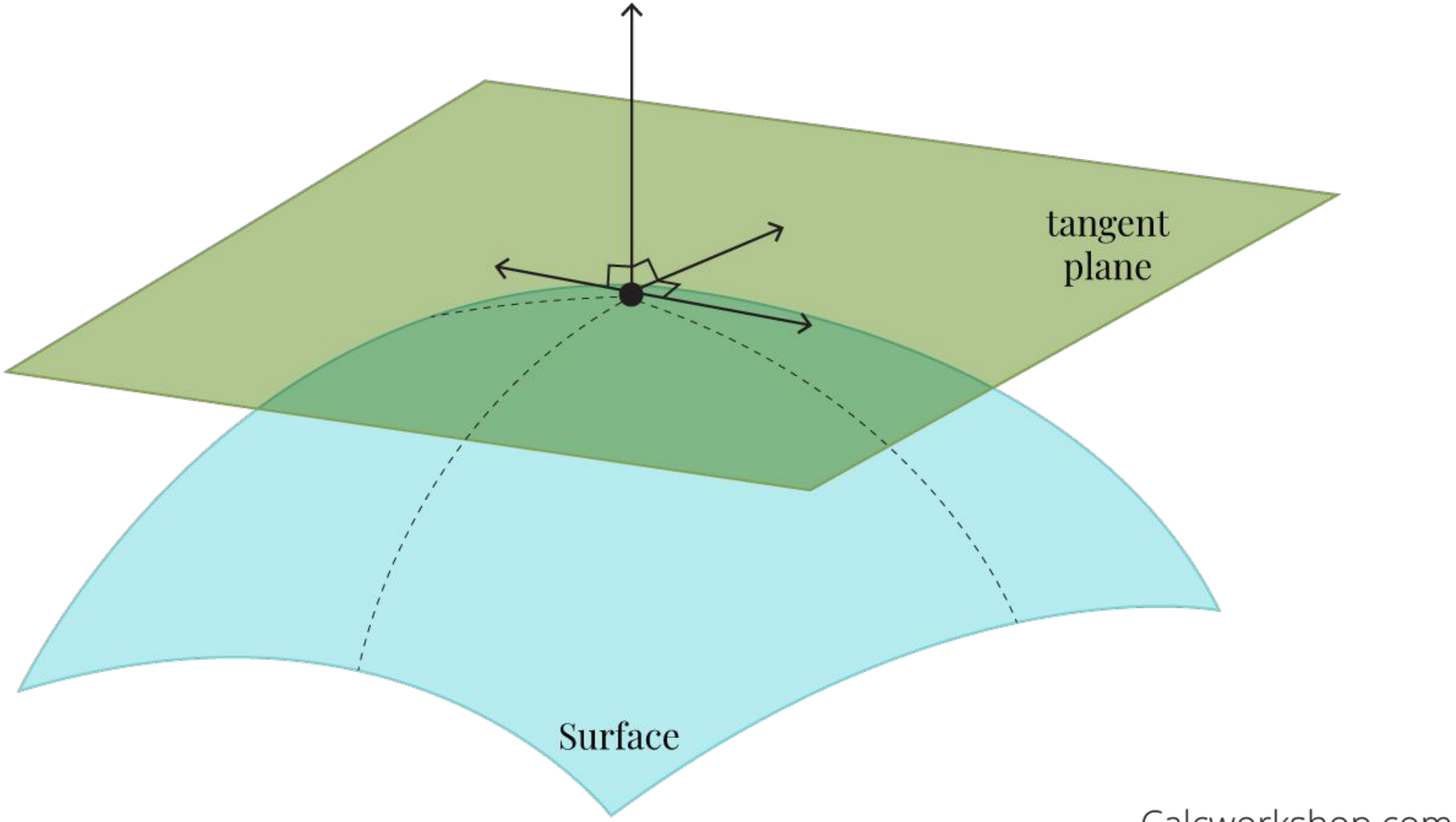
Before next class: try these 2 practice problems, try also discussion WS 2, try disc. WS1 if you haven't already try prereq. quiz IYHA

Start WebA HW IYHA

Tangent Plane to Graph

Lecture video for 6/11

What is the tangent plane? 🤔



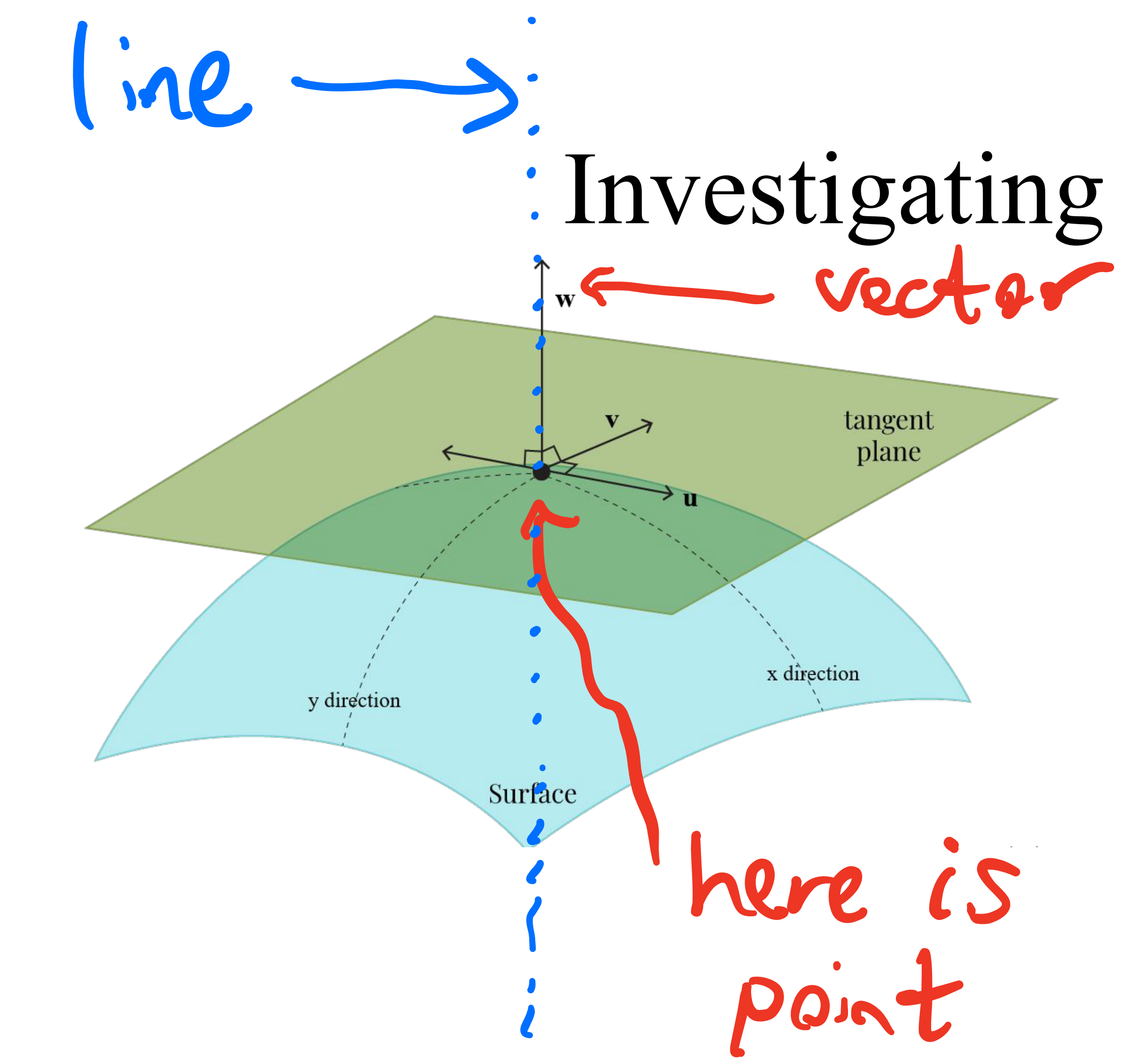
Sourced from https://calcworkshop.com/partial-derivatives/tangent-plane/

Problem Statement

Given a continuous function f: R^2 -> R whose partial derivatives exist everywhere, find the tangent plane to the graph of f at the point when $x = x_0$ and $y = y_0$

Recall: the graph of f is given by $z = f(x,y)$

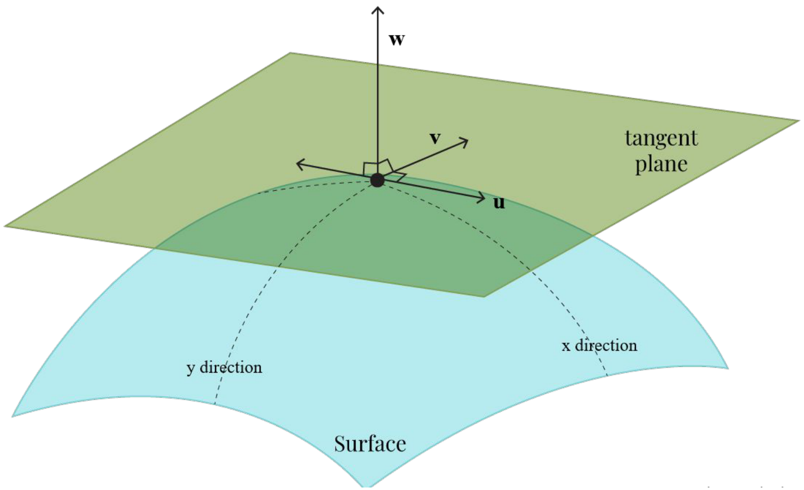
Investigating the Plane



- Let u, v be tangent vectors in plane in x, y direction respectively
- Let w be the normal to plane
- Let a, b be slope of u, v resp.
- In x direction, x changes while y is held constant
- y is constant while traveling along u
- If x changes by dx, then z = f(x,y) changes by ≈ a*dx
- So u = (1, 0, a)
- Similarly, v = (0, 1, b)

Continuing the Investigation

we found w



- Note: w is perpendicular to u and v
- Also, **u** x **v** is perpendicular to u and v
- We may take w = **u** x **v**
- Compute w = (1,0,a) x (0,1,b) = (-a, -b, 1)
- If t is perpendicular to w, then z·w = 0
- Let c = (x_0,y_0,f(x_0,y_0)) be point of tangency
- If t on tangent plane, vector from t to c is perpendicular to w
- We get 0 = -0 = -w·(t-c) = -w · (t-c)
- Let t = (x, y, z)
- a(x-x_0)+b(y-y_0) - (z-f(x_0,y_0)) = 0
- But what are a and b?

why $\frac{\partial}{\partial x}(y^2)=0$

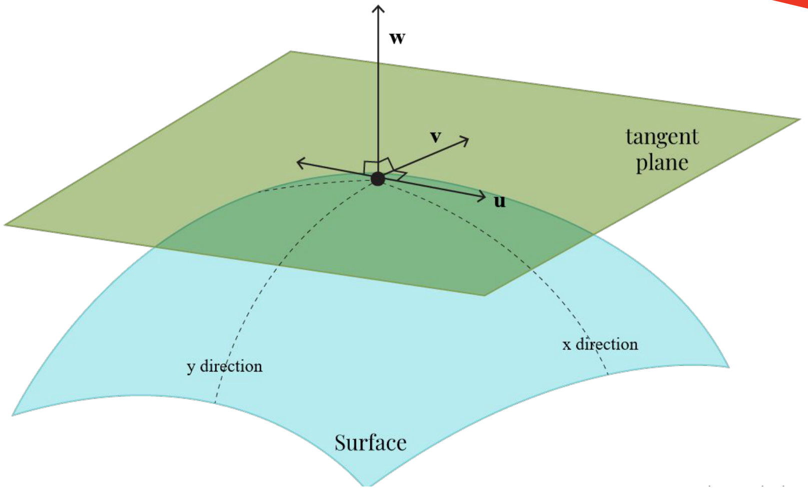
$6(x-3)+8(y-4)=z-25$

needs explanation after partial derivatives covered

$6x-18+8y-32=z-25$

Concluding the Investigation

$6x+8y-z=18+32-25=25$



- Recall that a is defined to be the slope of u
- As x changes by dx, z changes by a dx
- Thus, dz = a dx
- So a = dz/dx = d/dx f(x,y) = $f_x(x,y)$
- Thus, a = $f_x(x_0,y_0)$ $= a$
- Similarly, b = $f_y(x_0,y_0)$ $= b$
- Combine everything together:

$$\underset{6}{f_x(x_0,y_0)}(x-\underset{3}{x_0})+\underset{8}{f_y(x_0,y_0)}(y-\underset{4}{y_0}) = z-\underset{25}{f(x_0,y_0)}$$

is the equation of the tangent plane to the graph of f when $(x,y) = (x_0,y_0)$

$(x_0,y_0)=(3,4)$

$f(x_0,y_0)=25$

$f_x = \frac{\partial}{\partial x}(x^2+y^2) = \frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial x}(y^2)$ $\quad f(x,y)=x^2+y^2$

$f_x = \frac{\partial}{\partial x}(x^2+y^2) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2)$ $\quad f(x,y) = x^2+y^2$

$= 2x + 0 = 2x \qquad f_x(3,4) = 3\cdot 2 = 6$

Review for Understanding

$f_y = 2y \qquad f_y(3,4) = 4\cdot 2 = 8$

sub. x_0 & y_0 here

Use this formula to find the equation of the tangent plane to $z = x^2+y^2$ at the point (3, 4, 25)

x_0 y_0 z-coordinate

Next week

We know how to get tangent lines from Calculus 1. Explain how you can also find b by viewing v as a tangent line to a restricted graph of f

Next week, will cover an extra tangent plane problem, most

Find the formula for normal line to graph of f(x,y) (likely on Monday

does not require partial derivatives

Knew: if you have point P & vector v on a line, you can write down the line

In fact, $P + tv$ gives equation of the line.

same

same

On previous slides: $\underbrace{A}_{\text{start}} + t\underbrace{(B-A)}_{\text{direction}}$

Goal: find point & vector on the normal line, plug them in

By definition, tangency occurs at $x = x_0, y = y_0 \Rightarrow P = (x_0, y_0, f(x_0, y_0))$

Normal vector is $w = (-a, -b, 1)$

$$P + tw = (x_0, y_0, f(x_0, y_0)) + t(-a, -b, 1) =$$

$(x_0 - \underset{=?}{a}t,\ y_0 - \underset{=?}{b}t,\ f(x_0,y_0)+t)$,

$-\infty < t < \infty$, t can be any value, $t \in \mathbb{R}$, no constraint

Plug $a = f_x(x_0,y_0)$

$b = f_y(x_0,y_0)$

Actual formula:

$(x_0 - f_x(x_0,y_0)t,\ y_0 - f_y(x_0,y_0)t,\ f(x_0,y_0)+t)$

We have solved the problem even though we may not know $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ yet

ans. to student Q2: w is a vector, not a line. So we needed to do more work after finding w ans. to Q1: the video gives each step for why $w = \langle -a, -b, 1 \rangle$