MATH 243: Worksheet 7 **Answer Key**

This worksheet covers Chap 14.3 Partial Derivatives, Chap 14.4 Tangent planes and linear approximations and Chap14.5 The Chain Rule

PARTIAL DERIVATIVES

1. Find all the first order partial derivatives of the following function.

$$f(x,y,z) = 4x^3y^2 - e^z y^4 + \frac{z^3}{x^2} + 4y - x^5$$

$$f_x(x,y,z) = 12x^2y^2 - \frac{2z^3}{x^3} - 5x^4, \quad f_y(x,y,z) = 8x^3y - 4e^z y^3 + 4, \quad f_z(x,y,z) = -e^z y^4 + \frac{3z^2}{x^2}$$

2. Consider $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$.

   (a) Find all of the first order partial derivatives for the function $g(x,y,z)$.

$$g_x(x,y,z) = \frac{\sin(y)}{z^2}, \quad g_y(x,y,z)) = \frac{x\cos(y)}{z^2}, \quad g_z(x,y,z) = -2x\sin(y)z^{-3} = -\frac{2x\sin(y)}{z^3}$$

   (b) Find all of the second order partial derivatives for the function $g(x,y,z)$.

$$g_{xx}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\sin(y)}{z^2}\right) = 0, \qquad g_{xy}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\sin(y)}{z^2}\right) = \frac{\cos(y)}{z^2}$$

$$g_{xz}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\sin(y)}{z^2}\right) = -\frac{2\sin(y)}{z^3}, \qquad g_{yx}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{x\cos(y)}{z^2}\right) = \frac{\cos(y)}{z^2}$$

$$g_{yy}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{x\cos(y)}{z^2}\right) = -\frac{x\sin(y)}{z^2}, \qquad g_{yz}(x,y,z)) = \frac{\partial}{\partial z}\left(\frac{x\cos(y)}{z^2}\right) = -\frac{2x\cos(y)}{z^3}$$

$$g_{zx}(x,y,z) = \frac{\partial}{\partial x}\left(-\frac{2x\sin(y)}{z^3}\right) = -\frac{2\sin(y)}{z^3}, \qquad g_{zy}(x,y,z) = \frac{\partial}{\partial y}\left(-\frac{2x\sin(y)}{z^3}\right) = -\frac{2x\cos(y)}{z^3}$$

$$g_{zz}(x,y,z) = \frac{\partial}{\partial z}\left(-\frac{2x\sin(y)}{z^3}\right) = \frac{6x\sin(y)}{z^4}$$

TANGENT PLANES AND LINEAR APPROXIMATIONS

3. Find an equation of the tangent plane to the given surface at the specified point.

(a) $z = 4x^2 + y^2 - 9y$ at the point $(1, 4)$

We have $z = f(x, y) = 4x^2 + y^2 - 9y$, so $f(1, 4) = -16$. Also,

$$f_x(x, y) = 8x, \quad f_y(x, y) = 2y - 9,$$

Then, $f_x(1, 4) = 8$ and $f_y(1, 4) = -1$. By the equation

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

an equation of the tangent plane is $z - (-16) = f_x(1, 4)(x - 1) + f_y(1, 4)(y - 4)$. So,

$$z + 16 = 8(x - 1) - 1(y - 4) \quad \text{OR} \quad z = 8x - y - 20.$$

(b) $z = y \tan(x)$ at the point $\left(\frac{\pi}{4}, 6\right)$

We have $f(x, y) = y \tan(x)$, so $f\left(\frac{\pi}{4}, 6\right) = 6$.
Also, the partial derivatives are $f_x(x, y) = y \sec^2(x)$ and $f_y = \tan(x)$, so

$$f_x\left(\frac{\pi}{4}, 6\right) = 12, \quad f_y\left(\frac{\pi}{4}, 6\right) = 1.$$

Since both f_x and f_y are continuous at the point $\left(\frac{\pi}{4}, 6\right)$, then $f(x, y)$ is differentiable at $\left(\frac{\pi}{4}, 6\right)$. So, an equation of the tangent plane is

$$z - 6 = 12\left(x - \frac{\pi}{4}\right) + 1(y - 6) \quad \text{OR} \quad z = 12x + y - 3\pi$$

4. Find the (linear) approximation (tangent plane approximation) of each function at the specified point.

(a) $f(-0.99, 1.01)$, where $f(x, y) = \dfrac{5\sqrt{y}}{x}$ at the point $(-1, 1)$.

Let's use the equation of the linearization

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Let $(a, b) = (-1, 1)$. We have $z = f(x, y) = \dfrac{5\sqrt{y}}{x}$, so $f(-1, 1) = -5$. Also

$$f_x(x, y) = -\frac{5\sqrt{y}}{x^2}, \quad f_y(x, y) = \frac{5}{2x\sqrt{y}},$$

so $f_x(-1, 1) = -5$ and $f_y(-1, 1) = -\dfrac{5}{2}$. Since both f_x and f_y are continuous at the point $(-1, 1)$, then $f(x, y)$ is differentiable at $(-1, 1)$. So, the linearization of f at $(-1, 1)$

is given by

$$\begin{aligned} f(-0.99, 1.01) &\approx f(-1,1) + f_x(-1,1)(x-(-1)) + f_y(-1,1)(y-1) \\ &= -5 - 5(-0.99+1) - \frac{5}{2}(1.01-1) \\ &= -5.075 \end{aligned}$$

(b) $f(2.01, 0.99)$, where $f(x,y) = \ln(x+y^7)$ at the point $(2,1)$

Let's use the equation of the linearization

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Let $(a,b) = (2,1)$. We have $z = f(x,y) = \ln\big(x+y^7\big)$, so $f(2,1) = \ln(3)$. Also,

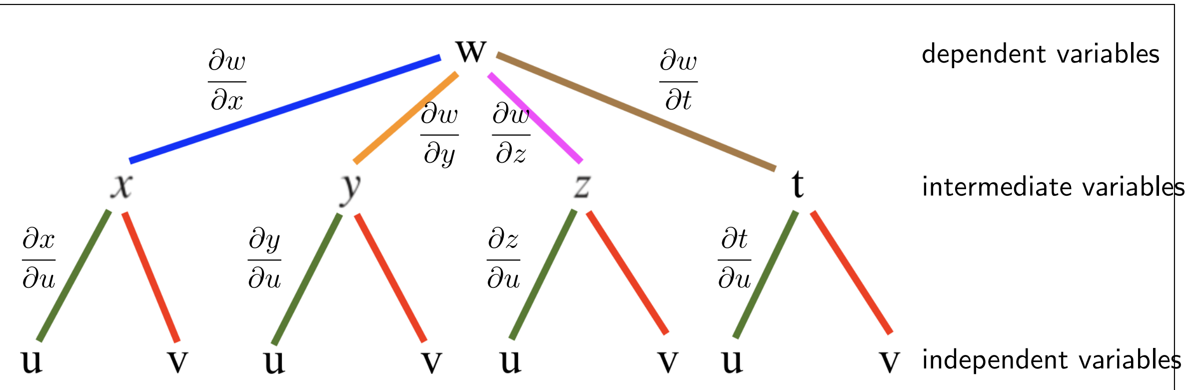
$$f_x(x,y) = \frac{1}{x+y^7}, \quad f_y(x,y) = \frac{7y^6}{x+y^7},$$

so $f_x(2,1) = \dfrac{1}{3}$ and $f_y(2,1) = \dfrac{7}{3}$. Since both f_x and f_y are continuous at the point $(2,1)$, then $f(x,y)$ is differentiable at $(2,1)$. So, the linearization of f at $(2,1)$ is given by

$$\begin{aligned} f(2.01, 0.99) &\approx f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= \ln(3) + \frac{1}{3}(2.01-2) + \frac{7}{3}(0.99-1) \\ &= \ln(3) - 0.02 \end{aligned}$$

THE CHAIN RULE

5. If $w = f(x,y,z,t)$ and $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $t = t(u,v)$, then use the Chain Rule to find the partial derivative of w with respect to u. Show a tree diagram of the dependent variable, the intermediate variables and the independent variables.



$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

OR

$$w_u = w_x \cdot x_u + w_y \cdot y_u + w_z \cdot z_u + w_t \cdot t_u$$

6. Use the chain rule to find $\dfrac{dz}{dt}$, where $z = \dfrac{x-y}{x+2y}$, $\quad x = e^{\pi t}$, $\quad$ and $\quad y = e^{-\pi t}$.

We have $\dfrac{dx}{dt} = \pi e^{\pi t}$ and $\dfrac{dy}{dt} = -\pi e^{-\pi t}$. Also,

$$\frac{\partial z}{\partial x} = \frac{(x+2y)(1) - (x-y)(1)}{(x+2y)^2} = \frac{3y}{(x+2y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+2y)(-1) - (x-y)(2)}{(x+2y)^2} = \frac{-3x}{(x+2y)^2}$$

By The Chain Rule - Case 1,

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left(\frac{3y}{(x+2y)^2}\right)\pi e^{\pi t} + \left(\frac{-3x}{(x+2y)^2}\right)\left(-\pi e^{-\pi t}\right) \\
&= \frac{3\pi}{(x+2y)^2}\left(y e^{\pi t} + x e^{-\pi t}\right) = \frac{3\pi}{(e^{\pi t} + 2e^{-\pi t})^2}\left(e^{-\pi t}e^{\pi t} + e^{\pi t}e^{-\pi t}\right) \\
&= \frac{6\pi}{(e^{\pi t} + 2e^{-\pi t})^2}
\end{aligned}$$

7. Use the chain rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$,

$$\text{where} \quad z = \arctan\left(x^2 + y^2\right) \quad \text{with} \quad x(s,t) = s\ln(t), \quad y(s,t) = te^s.$$

We have

$$\frac{\partial x}{\partial s} = \ln(t), \quad \frac{\partial x}{\partial t} = \frac{s}{t}, \quad \text{and} \quad \frac{\partial y}{\partial s} = te^s, \quad \frac{\partial y}{\partial t} = e^s$$

Also,

$$\frac{\partial z}{\partial x} = \frac{2x}{1+(x^2+y^2)^2}, \quad \frac{\partial z}{\partial y} = \frac{2y}{1+(x^2+y^2)^2}$$

By The Chain Rule - Case 2,

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\frac{2x}{1+(x^2+y^2)^2}\right)(\ln(t)) + \left(\frac{2y}{1+(x^2+y^2)^2}\right)(te^s) \\ &= \left(\frac{2}{1+(x^2+y^2)^2}\right)(x\ln(t) + yte^s) = \frac{2\left(s(\ln(t))^2 + t^2e^{2s}\right)}{1+\left(s^2\ln^2 t + t^2e^{2s}\right)^2}\end{aligned}$$

We also know that

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{2x}{1+(x^2+y^2)^2}\right)\left(\frac{s}{t}\right) + \left(\frac{2y}{1+(x^2+y^2)^2}\right)(e^s) \\ &= \left(\frac{2}{1+(x^2+y^2)^2}\right)\left(\frac{xs}{t} + ye^s\right) = \frac{2\left(\frac{s^2}{t}\ln t + te^{2s}\right)}{1+(s^2(\ln t)^2 + t^2e^{2s})^2}\end{aligned}$$

8. Given that $yz + x\ln(y) = z^2$, use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

The given equation can be written as

$$F(x,y,z) = yz + x\ln(y) - z^2 = 0$$

We know the implicit differentiation formulas give us

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\ln(y)}{y-2z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z+\frac{x}{y}}{y-2z}$$

9. Suppose f is a differentiable function of x and y, and $f(x,y) = f(g(t),h(t))$, where

$$g(2) = 4,\ g'(2) = -3,\ h(2) = 5,\ h'(2) = 6,\ f_x(4,5) = 2,\ \text{and}\ f_y(4,5) = 8.$$

Find the derivative of $f(x,y)$ with respect to t at $t=2$.

Let $x = g(t)$ and $y = h(t)$ then

$$\frac{dx}{dt} = g'(t), \quad \text{and,} \quad \frac{dy}{dt} = h'(t), \quad \text{and,} \quad f(t) = f(g(t), h(t)) = f(x, y)$$

Thus, by the Chain rule- Case 1:

$$\begin{aligned} f'(t) = \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x, y)\, g'(t) + f_y(x, y)\, h'(t) \\ &= f_x(g(t), h(t))\, g'(t) + f_y(g(t), h(t))\, h'(t) \end{aligned}$$

When $t = 2$,

$$\begin{aligned} f'(2) &= f_x(g(2), h(2))\, g'(2) + f_y(g(2), h(2))\, h'(2) \\ &= f_x(4, 5)(-3) + f_y(4, 5)(6) = (2)(-3) + (8)(6) \\ &= -6 + 48 = 42 \end{aligned}$$

Suggested Textbook Problems
Chapter 14.4: 1-6, 11, 17
Chapter 14.5: 1-13, 15, 17, 18, 21-23, 25, 27-35, 39-43, 45, 50