12 Vectors and the Geometry of Space

12.3 The Dot Product

The Dot Product (1 of 1)

So far we have added two vectors and multiplied a vector by a scalar. The question arises: Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows.

The Dot Product of Two Vectors

The Dot Product of Two Vectors (1 of 10)

To find the dot product of vectors **a** and **b** we multiply corresponding components and add.

1 Definition of the Dot Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of **a** and **b** is the number a • b given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The Dot Product of Two Vectors (2 of 10)

The dot product of two vectors is a real number, not a vector. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**).

Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1,\ a_2 \rangle \cdot \langle b_1,\ b_2 \rangle = a_1 + b_1 + a_2 b_2$$

Example 1

$$\langle 2,4\rangle \cdot \langle 3,-1\rangle = 2(3)+4(-1)$$
$$= 2$$

$$\langle -1,7,4\rangle \cdot \left\langle 6,2,-\frac{1}{2}\right\rangle = (-1)(6)+7(2)+4(-\frac{1}{2})$$
$$= 6$$

$$(\mathbf{i}+2\mathbf{j}-3\mathbf{k})\cdot(2\mathbf{j}-\mathbf{k}) = 1(0)+2(2)+(-3)(-1)$$
$$= 7$$

The Dot Product of Two Vectors (3 of 10)

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

2 Properties of the Dot Product If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

The Dot Product of Two Vectors (4 of 10)

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$

3.
$$
\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3 \\
&= (a_1 b_1 + a_2 b_2 + a_3 b_3) + (a_1 c_1 + a_2 c_2 + a_3 c_3) \\
&= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}
\end{aligned}
$$

The Dot Product of Two Vectors (5 of 10)

The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle** θ **between a and b**, which is defined to be the angle between the representations of **a** and **b** that start at the origin, where $0 \le \theta \le \pi$.

In other words, θ is the angle between the line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Figure 1. Note that if **a** and **b** are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

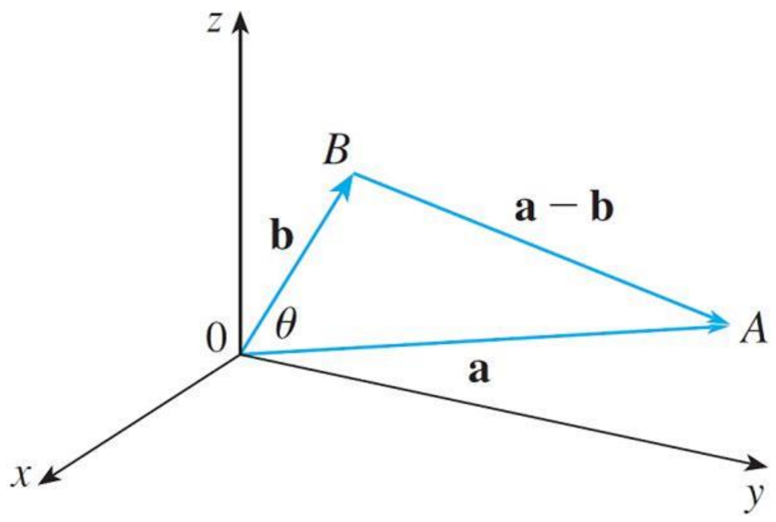


Figure 1

The Dot Product of Two Vectors (6 of 10)

The formula in the following theorem is used by physicists as the *definition* of the dot product.

3 Theorem If θ is the angle between the vectors **a** and **b**. then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Example 2

If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\frac{\pi}{3}$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution:

Using Theorem 3, we have

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\left(\frac{\pi}{3}\right) \\ &= 4 \cdot 6 \cdot \tfrac{1}{2} \\ &= 12\end{aligned}$$

The Dot Product of Two Vectors (7 of 10)

The formula in Theorem 3 also enables us to find the angle between two vectors.

6 Corollary If θ is the angle between the nonzero vectors **a** and **b,** then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example 3

Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

Solution:

Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \quad \text{and} \quad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

Example 3 – Solution

We have, from Corollary 6,

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right)$$

$$\approx 1.46 \quad (\text{or } 84^\circ)$$

The Dot Product of Two Vectors (8 of 10)

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \frac{\pi}{2}$. Then Theorem 3 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\left(\frac{\pi}{2}\right) = 0$$

and conversely if **a** • **b** = 0, then cos θ = 0, so $\theta = \frac{\pi}{2}$. The zero vector **0** is considered to be perpendicular to all vectors.

Therefore we have the following method for determining whether two vectors are orthogonal.

7 Two vectors a and b are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 4

Show that 2**i** + 2**j** – **k** is perpendicular to 5**i** – 4**j** + 2**k**.

Solution:

Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by (7).

The Dot Product of Two Vectors (9 of 10)

Because cos $\theta > 0$ if $0 \leq \theta < \frac{\pi}{2}$ and cos $< \theta < 0$ if $\frac{\pi}{2} < 0 \leq \pi$, we see that **a** · **b** is positive for $\theta < \frac{\pi}{2}$ and negative for $\theta > \frac{\pi}{2}$.

We can think of **a** · **b** as measuring the extent to which **a** and **b** point in the same direction.

The dot product **a** · **b** is positive if **a** and **b** point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2).

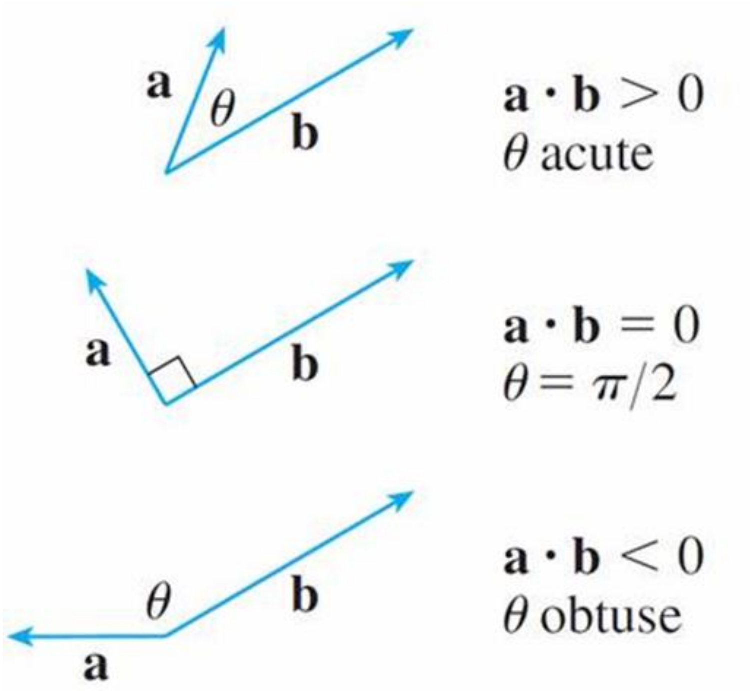


Figure 2

The Dot Product of Two Vectors (10 of 10)

In the extreme case where **a** and **b** point in exactly the same direction, we have $\theta = 0$, so $\cos\theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

If **a** and **b** point in exactly opposite directions, then we have $\theta = \pi$ and so $\cos\theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

Direction Angles and Direction Cosines

Direction Angles and Direction Cosines (1 of 4)

The **direction angles** of a nonzero vector **a** are the angles α, β, and γ (in the interval $[0, \pi]$) that **a** makes with the positive *x*-, *y*-, and *z*-axes, respectively. (See Figure 3.)

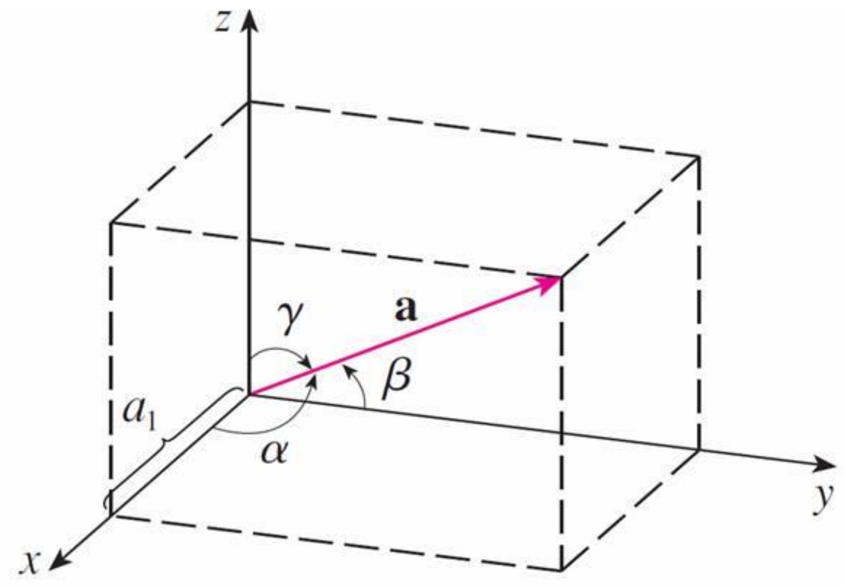


Figure 3

Direction Angles and Direction Cosines (2 of 4)

The cosines of these direction angles, cos *α*, cos *β*, and cos *γ*, are called the **direction cosines** of the vector **a**. Using Corollary 6 with **b** replaced by **i**, we obtain

8 $$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

(This can also be seen directly from Figure 3.) Similarly, we also have

9 $$\cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

Direction Angles and Direction Cosines (3 of 4)

By squaring the expressions in Equations 8 and 9 and adding, we see that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{10}$$

We can also use Equations 8 and 9 to write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma \rangle \\ &= |\mathbf{a}|\langle \cos\alpha, \cos\beta, \cos\gamma \rangle\end{aligned}$$

Direction Angles and Direction Cosines (4 of 4)

Therefore

11 $$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

Example 5

Find the direction angles of the vector $a = \langle 1, 2, 3 \rangle$.

Solution:

Since $|a| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, Equations 8 and 9 give

$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

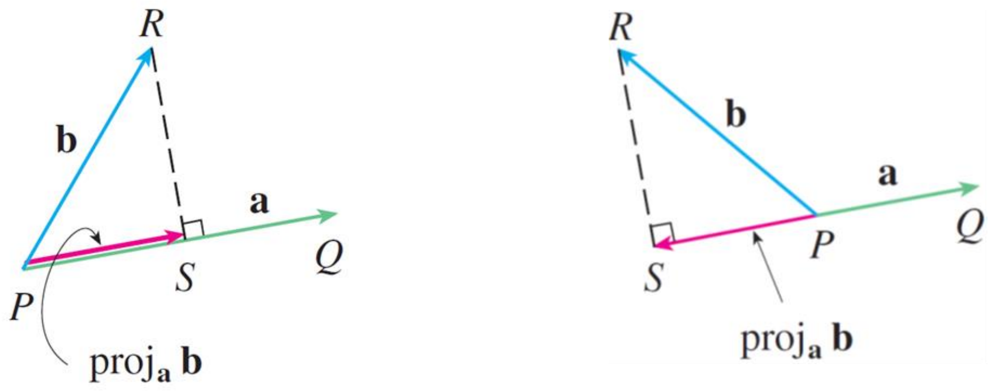
and so

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ \quad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ \quad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$$

Projections

Projections (1 of 4)

Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$. (You can think of it as a shadow of **b**).

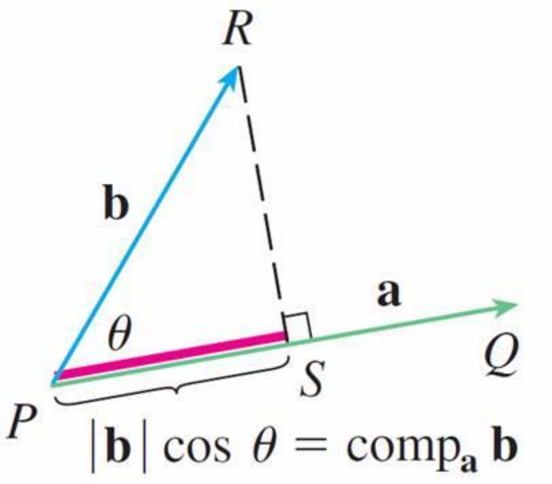


Vector projections

Figure 4

Projections (2 of 4)

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b**. (See Figure 5.)



Scalar projection

Figure 5

Projections (3 of 4)

This is denoted by $\text{comp}_{\mathbf{a}}\, \mathbf{b}$. Observe that it is negative if $\frac{\pi}{2} < \theta \leq \pi$.

The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**.

Projections (4 of 4)

We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

Example 6

Find the scalar projection and vector projection of $\mathbf{b} = \langle 1,\ 1,\ 2 \rangle$ and $\mathbf{a} = \langle -2, 3, 1 \rangle$.

Solution:

Since $|\mathbf{a}| = \sqrt{-2^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is

$$\text{comp}_a \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

$$= \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}}$$

$$= \frac{3}{\sqrt{14}}$$

Example 6 – Solution

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|}$$

$$= \frac{3}{14}\mathbf{a}$$

$$= \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

Application: Work

Application: Work (1 of 2)

The work done by a constant force *F* in moving an object through a distance *d* as *W* = *Fd*, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 6.

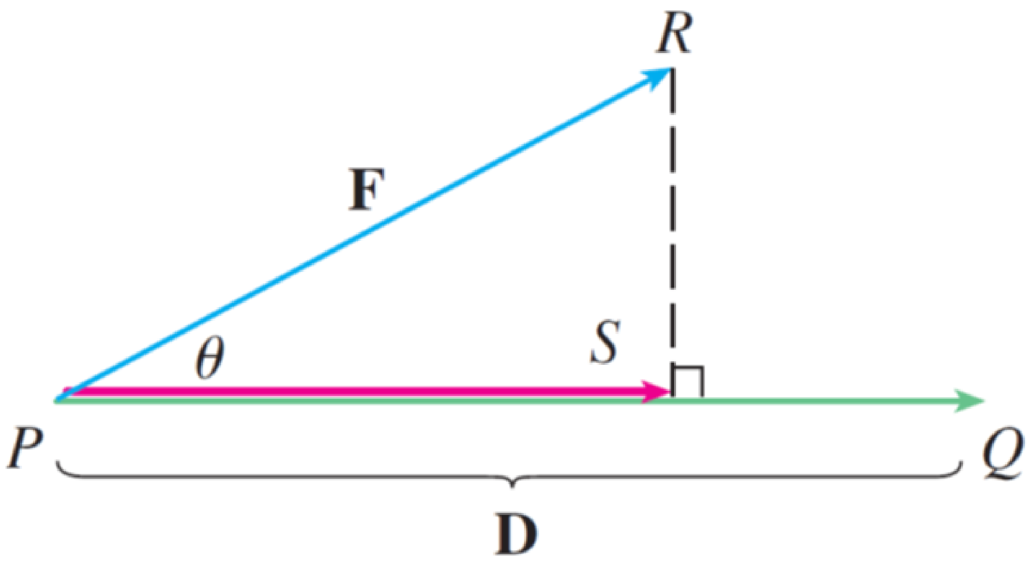


Figure 6

Application: Work (2 of 2)

If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. The **work** done by this force is defined to be the product of the component of the force along **D** and the distance moved:

$$W = (|\mathbf{F}|\cos\theta)|\mathbf{D}|$$

But then, from Theorem 3, we have

$$W = |\mathbf{F}||\mathbf{D}|\cos\theta = \mathbf{F}\cdot\mathbf{D} \qquad (12)$$

Thus the work done by a constant force **F** is the dot product **F** • **D**, where **D** is the displacement vector.

Example 7

A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

Solution:
If **F** and **D** are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$W = \mathbf{F} \cdot \mathbf{D}$$

$$= |\mathbf{F}||\mathbf{D}| \cos 35°$$

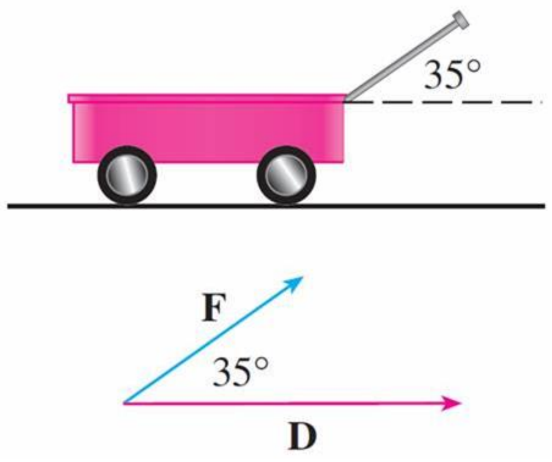


Figure 7

Example 7 – Solution

$$= (70)(100)\cos 35^\circ$$

$$\approx 5734 \text{ N} \cdot \text{m}$$

$$= 5734 \text{ J}$$