MATH 243 Final Exam

1. Victory Day for vectors. Select all valid statements

A. If the curvature κ is constant, then the arc length derivative $\frac{ds}{dt}$ is constant

B. If $\frac{ds}{dt}$ is constant, then the curvature κ is constant

C. The volume of the parallelepiped formed by $\mathbf{T}, \mathbf{N}, \mathbf{B}$ is 1

D. We can parametrize the acceleration components pair (a_T, a_N) as $(f(t)\cos(g(t)), f(t)\sin(g(t)))$ for some f and g

2. Which method of showing a limit exists **doesn't** work?

A. For $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, showing $\lim\limits_{r\to 0} f(r\cos\theta, r\sin\theta)$ exists

B. For $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$, showing $\lim\limits_{\rho\to 0} f(\rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi)$ exists

C. For $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$, showing $\lim\limits_{t\to 0} f(\mathbf{r}(t))$ exists for all continuous paths $\mathbf{r}$ with $\mathbf{r}(\mathbf{0}) = \mathbf{0}$

D. For $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, showing the nested limit $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)$ exists

3. General topic bonanza. Select all statements which are true

A. By cross product properties, the vector field $\mathbf{F}$ is perpendicular to $\nabla \times \mathbf{F}$

B. If p, q are 2 variable polynomials, $f(x,y) = \frac{p(x,y)}{q(x,y)}$ is continuous on its domain

C. If $f(a,b,c,\ldots,z) = (x-a)(x-b)(x-c)\cdots(x-z)$, then $f_{abyz}(1,2,3,\ldots,26) = 0$

D. If the components of $\mathbf{F}$ are everywhere positive and S is a positively oriented surface, then $\iint_S \mathbf{F}\cdot d\mathbf{S} > 0$

4. Let P_c be the plane with equation $4x - 3y + 5z = c$. The line L intersects $P_{10r+s}, P_{14r+s}, P_{20r+s}$ at the points A, B, C respectively for some r, s. It was revealed in a dream that $A = (1,2,4)$ and $B = (3,5,6)$. Treating C as a vector, find $4\|C\|^2$

5. Let M be the maximum of $xy + xz - yz^2$ on the solid box $B = [0,1]^3$. If $M = \frac{a}{b}$ for integers a, b with $b > 0$ minimal, find $10a + b$.

6. Let C be the unit circle taken counterclockwise. For $F = \langle \frac{-x^2+y-2xy}{(x^2+y)^2} + y, \frac{x^4+2x^2y+x^2+y^2-x}{(x^2+y)^2} \rangle$, let $L = \int_C \mathbf{F}\cdot d\mathbf{r}$. If $L = \frac{a}{b}\pi$ for integers a, b with $b > 0$ minimal, find $10a + b$.

7. Classify the critical points of $f(x,y,z) = x^2 + y^4 + y^2 + 2z^2 + 2xy^2 - 2yz$.

8. Today you will learn how the ultra rich dodge taxes. A shell company in the Cayman Islands creates spherical shells, drills through the surfaces by using cones, hides their assets in one of the spaces, and then seals everything back up. You found a hiding spot and want to know how much money there is.
E is the solid bounded by $x^2 + y^2 + z^2 = 1, x^2 + y^2 + z^2 = 4, z = \sqrt{x^2 + y^2}$. Knowing that more valuable bills are stored at the bottom, you model the monetary density by $f(x,y,z) = 2 - z$ after some appropriate

choice of units. Find $\iiint_E f\, dV$.

9. Let $\mathbf{F} = \langle xe^y, z\sin(\sin(x)) + xz^3 + e^y + 1, -2ze^y\rangle$. Let S be the surface described by $y = x^2 + z^2$ for $y \leq 4$ and assign S a positive orientation. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

10. Extra credit: Show that the integral of a conservative vector field from $\mathbf{a}$ to $\mathbf{b}$ is independent of the path taken from $\mathbf{a}$ to $\mathbf{b}$ by using Stokes' Theorem.

11. Extra credit: Consider the following method for computing line integrals in $\mathbf{R}^3$. Suppose $\mathbf{F}$ is a vector field and C is some simple closed curve. Draw a surface S with boundary C, making sure S has no self intersections, and endow S with positive orientation. Let E be the solid that S encloses. By Stokes' and Divergence Theorems, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\,\mathbf{S} = \iiint_E \nabla \cdot (\nabla \times F)\, dV$. By the curl and divergence identities from class, $\nabla \cdot (\nabla \times F) = 0$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.
Is there a mistake in this argument? If so, point out what is wrong.

12. Extra credit: You find a cave with endless gold, platinum, and rhodium. Gold, platinum, and rhodium cost $\$100, 40, 200$ per gram respectively. Your pouch would weigh $60, 65, 20$ kilograms respectively if filled to the brim with only gold, only platinum, only rhodium respectively.
Let M be the maximum value you can keep given that you can only carry away 50 kilograms. If M is x millions of dollars, find the value of x.
Note: You are not allowed to transport minerals in your hands, overfill the pouch, get someone else to carry the loot for you, or violate any other spirit of the problem.

13. Extra credit: Prove Clairaut's Theorem. For convenience, we assume 2 variables and shift to the origin so that the statement to prove is $f_{xy}(0,0) = f_{yx}(0,0)$ for f twice continuously differentiable.
Hint: It may be help to the lecture method to prove f_x and f_y continuous implies f is differentiable.