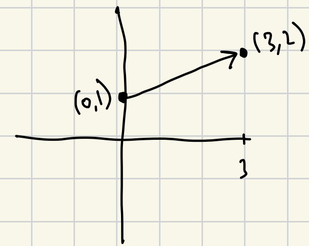
Section 12.2: Vectors

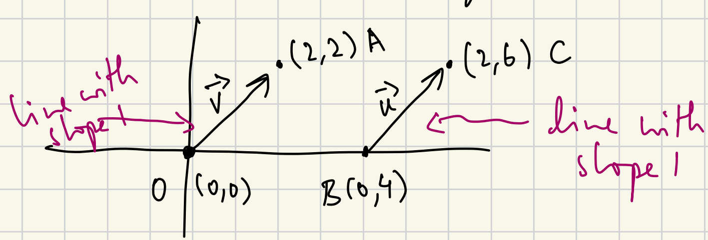
Definition: A vector $\vec{u}$ is a directed line segment from an initial point to a terminal point.

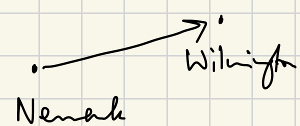
Eg: vector from point $(0,1)$ to $(3,2)$:



Remark: • A vector is a quantity with direction and magnitude / length.

• Two vectors are equivalent if they have the same direction and length.





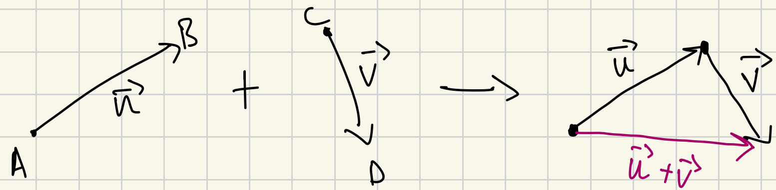
vectors $\vec{v} = \overrightarrow{OA}$ and $\vec{u} = \overrightarrow{BC}$ are equivalent

$$|\vec{v}| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$= \sqrt{(2-0)^2 + (2-0)^2}$$

$$|\vec{u}| = \sqrt{(2-0)^2 + (6-4)^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

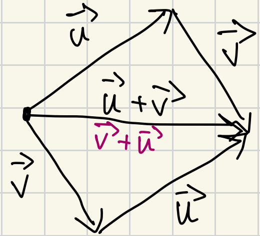
Addition of two vectors: To add vectors $\vec{u}$ and $\vec{v}$, bring the initial point of $\vec{v}$ to the terminal point of u. Then $\vec{u} + \vec{v}$ is the vector with same initial point of u and terminal point of v.



Triangle law

Parallelogram Law:

$$\vec{u}+\vec{v}=\vec{v}+\vec{u}$$

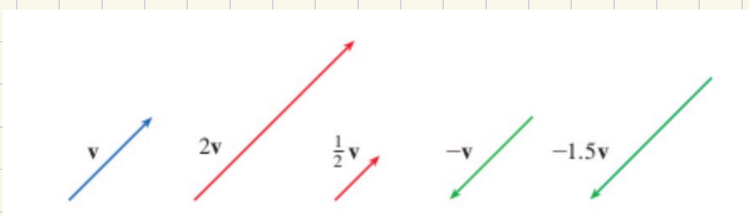


any real #

Scalar multiplication: If c is a scalar and $\vec{v}$ is a vector then $c\vec{v}$

$\left\{ \right.$ $\boxed{\text{if } c>0}$: is the vector with the same direction as $\vec{v}$ but with length c times that of $\vec{v}$.

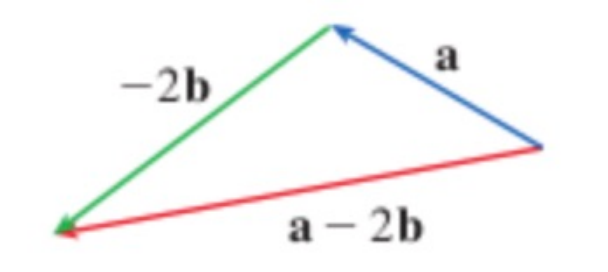
$\boxed{\text{if } c<0}$: is the vector with the opposite direction of $\vec{v}$ but with length c times that of $\vec{v}$



Example:

Q: what is $a - 2b$?

Fact: $\vec{u} - \vec{v}$ is defined as $\vec{u} + (-\vec{v})$.

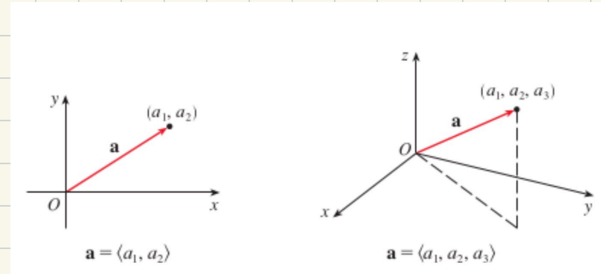


Components of a vector:

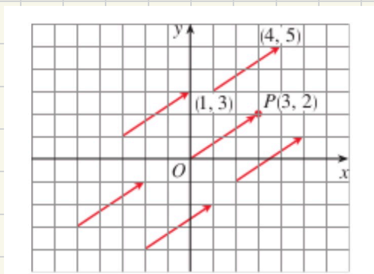
A vector with initial point as the origin and terminal point as $P(a_1, a_2, a_3)$ will be denoted by

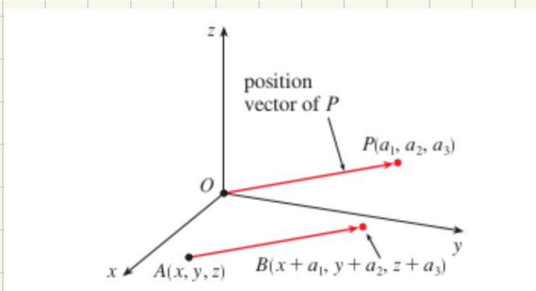
$$a = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$$

Also in $\mathbb{R}^2$, $a = \langle a_1, a_2 \rangle$.



Note: vector is a position vector if it starts at the origin.





Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ the vector $\vec{a}$ with representation $\overrightarrow{AB}$ is

$$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Eg: Find the vector that represents $\overrightarrow{AB}$ where

$A = (2, -3, 4)$, $B(-2, 1, 1)$

$$\Rightarrow \vec{AB} = \langle -2-2, 1-(-3), 1-4 \rangle$$
$$= \langle -4, 4, -3 \rangle$$

Definitions:

- The length of $\langle a_1, a_2 \rangle$ is $|a| = \|a\| = \sqrt{a_1^2 + a_2^2}$
- The length of $\langle a_1, a_2, a_3 \rangle$ is $|a| = \|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

- Addition of $\vec{a} = \langle x_1, y_1, z_1 \rangle$, $\vec{b} = \langle x_2, y_2, z_2 \rangle$.
  $$\vec{a} + \vec{b} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$
- Scalar multiplication: $c\vec{a} = \langle cx_1, cy_1, cz_1 \rangle$.

- V_2 is the 2 dimensional space of vectors
  $$V_2 = \{ \langle x_1, y_1 \rangle : x_1, y_1 \in \mathbb{R} \}$$
- $V_n = \{ \langle x_1, x_2, x_3, \dots, x_n \rangle : x_1, \dots, x_n \in \mathbb{R} \}$

  So $V_3 = \{ \langle x_1, x_2, x_3 \rangle : x_1, x_2, x_3 \in \mathbb{R} \}$

Properties of Vectors

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
2. $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$
3. $\mathbf{a}+\mathbf{0}=\mathbf{a}$
4. $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
5. $c(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}$
6. $(c+d)\mathbf{a}=c\mathbf{a}+d\mathbf{a}$
7. $(cd)\mathbf{a}=c(d\mathbf{a})$
8. $1\mathbf{a}=\mathbf{a}$

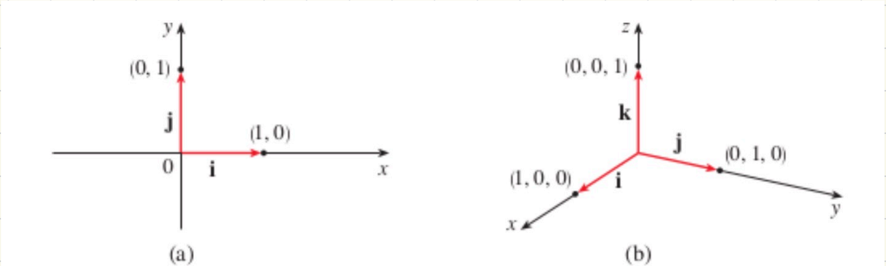
Three vectors in V_3 play a special role. Let

$$i = \langle 1, 0, 0 \rangle$$

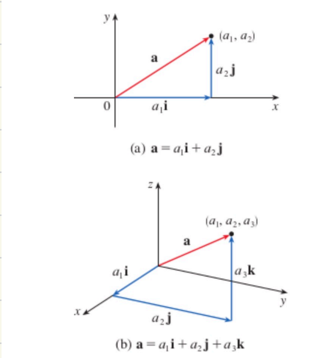
$$j = \langle 0, 1, 0 \rangle$$

$$k = \langle 0, 0, 1 \rangle.$$

They are called the standard basis vectors.



(a) (b)



(a) $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}$

(b) $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$

If $a = \langle a_1, a_2, a_3 \rangle \Rightarrow a = a_1 i + a_2 j + a_3 k$

eg: $\langle 1, -2, 6 \rangle = i - 2j + 6k$

eg: $\vec{a} = i + 2j - 3k$, $\vec{b} = 4i + 7k$ then $\vec{a} + \vec{b} = 5i + 2j + 4k$

$= (1+4)i + (2+0)j + (-3+7)k$

<u>Definition:</u> A unit vector is a vector with length 1.

eg: i, j, k are unit vectors.

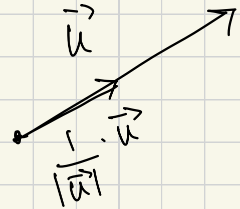
<u>Question:</u> For any vector $\vec{u}$ find a vector in the same direction as $\vec{u}$ but with length 1.

<u>Answer:</u> $\vec{v} = \dfrac{1}{|\vec{u}|} \cdot \vec{u}$. Verify:

$$\boxed{\begin{aligned} &\text{Take } c = \frac{1}{|\vec{u}|} \text{ then} \\ &|c\vec{u}| = |c| \cdot |\vec{u}| = \frac{1}{|\vec{u}|} \cdot |\vec{u}| = 1 \end{aligned}}$$

$$\begin{aligned} |c\vec{u}| &= |c| \cdot |\vec{u}| \\ &\| \\ |c\langle u_1, u_2, u_3 \rangle| & \\ &\| \\ |\langle cu_1, cu_2, cu_3 \rangle| & \\ &= \sqrt{(cu_1)^2 + (cu_2)^2 + (cu_3)^2} \end{aligned}$$

$$= |c|\sqrt{u_1^2+u_2^2+u_3^2} = |c|\cdot|\vec{u}|$$



Section 12.3: Dot product

Definition: If $a = \langle a_1, a_2, a_3\rangle$ and $b = \langle b_1, b_2, b_3\rangle$

then dot product of a and b is

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

eg: $\langle 2,4\rangle\cdot\langle 3,-1\rangle = 2\cdot 3 + 4(-1) = 6-4 = 2$

$\langle -1,7,4\rangle\cdot\langle 6,2,-\frac{1}{2}\rangle = -6+14-2 = 6$

2 Properties of the Dot Product

If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

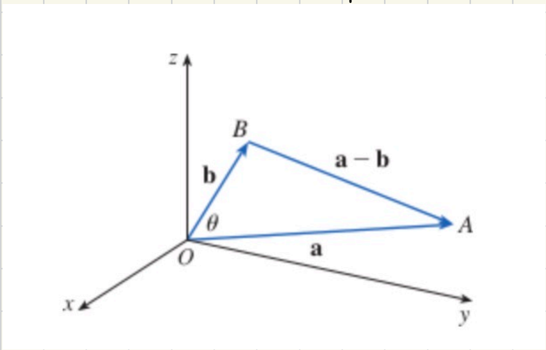
1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

eg: $\vec{a} \cdot \vec{a} = a_1 \cdot a_1 + a_2 \cdot a_2 + a_3 \cdot a_3$

$$= a_1^2 + a_2^2 + a_3^2$$

$$= |\vec{a}|^2$$

Geometric interpretation:



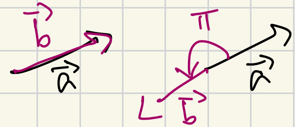
Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$ then

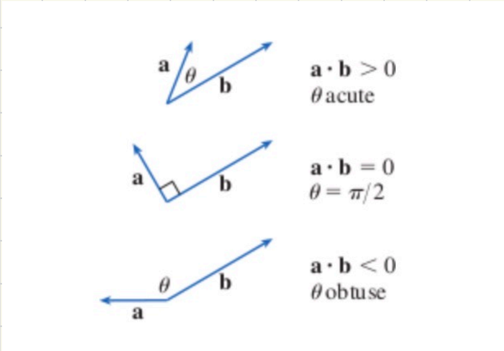
$$\vec{a} \cdot \vec{b} = |a| \cdot |b| \cos\theta$$

Remark: • θ is between 0 and π

• If $\theta = \frac{\pi}{2}$ then $\cos\theta = 0 \longrightarrow \vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

- $\vec{a}$ and $\vec{b}$ are parallel if and only if $\theta = 0$ or $\theta = \pi$.





$A = (2, 3, 4)$

$B = (-2, 1, 1)$

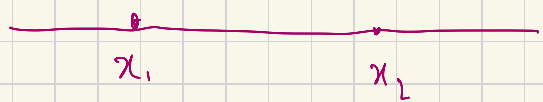
we need $\vec{v} = \langle v_1, v_2, v_3 \rangle$

such that $A + \vec{v} = B$

$\vec{v} = B - A$

$= \langle -2-2, 1-3, 1-4 \rangle$

$= \langle -4$



$x_2 = x_1 + (x_2 - x_1)$