**Textbook Sections:** 16.9
**Topics:** The Divergence Theorem
**Instructions:** Try each of the following problems, show the detail of your work.

Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.



## The Divergence Theorem

1. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x,y,z) = \langle x^2, -y, z\rangle$ where $E$ is the solid cylinder $y^2+z^2 \leq 9$, $0 \leq x \leq 2$, by computing the surface integrals on the boundary and by applying the divergence theorem.

First, we check the hypotheses of the Divergence Theorem:

i) The solid cylinder $E$ is a simple solid region.

ii) The surface $S$ of the solid region $E$ is a closed surface, so we choose the positive orientation to be given by the outward orientated normal vectors.

iii) The component functions $P(x,y,z) = x^2$, $Q(x,y,z) = -y$, and $R(x,y,z) = z$ of the vector field $\mathbf{F}$ have continuous partial derivatives on $R^3$ which is an open set that contains $E$.

By the Divergence theorem, we have

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \nabla\cdot\vec{F}\ dV,\ or\ \iint_S \vec{F}\cdot\vec{n}\ dS = \iiint_E\ div(\vec{F})\ dV$$

. To verify this theorem, we need to compute both sides of the equation above and show they are equal.

**Surface Integral Calculation:**

Notice here that the surface $S$ can be decomposed as $S = S_1 \cup S_2 \cup S_3$, where:

$$S_1 = \{(0,y,z) : y^2+z^2 = 9\} \implies \text{The circular cap at } x = 0.$$
$$S_2 = \{(x,y,z) : y^2+z^2 = 9,\ 0 \leq x \leq 2\} \implies \text{The cylindrical shell.}$$
$$S_3 = \{(2,y,z) : y^2+z^2 = 9\} \implies \text{The circular cap at } x = 2.$$

Then,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot\vec{n_1}\ dS + \iint_{S_2} \vec{F}\cdot\vec{n_2}\ dS + \iint_{S_3} \vec{F}\cdot\vec{n_3}\ dS.$$

From here we form parametrizations for each surface to compute their respective normal vectors, and then compute the surface integral.

I) On $S_1$, $x = 0$, and a normal vector to this back cap pointing outward is the vector $-\mathbf{i} =< -1, 0, 0 >$.

Then,

$$\iint_{S_1} \vec{F}\cdot\vec{n_1}\ dS = \iint_{S_1} < 0, -y, z > \cdot < -1, 0, 0 >\ dS = \iint_{S_1} 0\ dS = 0$$

. Another method is to parametrize the surface $S_1$ using polar coordinates with $r = v$ and $\theta = u$.

$$S_1 \implies \begin{cases} \vec{r}_1(u,v) = \langle 0, v\cos(u), v\sin(u)\rangle \\ 0 \leq u \leq 2\pi \\ 0 \leq v \leq 3 \end{cases} \implies (\vec{r}_1)_u \times (\vec{r}_1)_v = \langle -v, 0, 0\rangle$$

Then,

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{D_1} \vec{F}(\vec{r}_1(u,v))\cdot[(\vec{r}_1)_u \times (\vec{r}_1)_v]dA = \int_0^3\int_0^{2\pi} \langle 0, -v\cos(u), v\sin(u)\rangle \cdot \langle -v, 0, 0\rangle dudv = 0$$

II) For $S_2$, we write a parametrization using cylindrical coordinates, where $x = v$, $r = 3$, and $\theta = u$

$$S_2 \implies \begin{cases} \vec{r}_2(u,v) = \langle v, 3\cos(u), 3\sin(u)\rangle \\ 0 \le u \le 2\pi \\ 0 \le v \le 2 \end{cases} \implies (\vec{r}_2)_u \times (\vec{r}_2)_v = \langle 0, 3\cos(u), 3\sin(u)\rangle$$

We test the normal vector at a point on the surface to see if it points inward or outward. Note: Do not use a point on the boundary because it might give skewed results. For example choose the point $A(1,0,3$ which is given by $v = 1$ and $u = \pi/2$. Then a normal vector to this surface at $A$ is the vector $< 0, 0, 3 >$ which points upward so outward. Then, the integral is given by

$$\begin{aligned}\iint_{S_2} \vec{F}\cdot d\vec{S} &= \iint_{D_2} \vec{F}(\vec{r}_2(u,v))\cdot[(\vec{r}_2)_u \times (\vec{r}_2)_v]dA \\ &= \int_0^2\int_0^{2\pi} \langle v^2, -3\cos(u), 3\sin(u)\rangle \cdot \langle 0, 3\cos(u), 3\sin(u)\rangle dudv \\ &= -9\int_0^2\int_0^{2\pi} \cos^2(u) - \sin^2(u) dudv = -9\int_0^2\int_0^{2\pi}\cos(2u)dudv \\ &= -9\int_0^2 \left[\frac{1}{2}\sin(2u)\right]_0^{2\pi} dv = 0\end{aligned}$$

III) On $S_3$, $x = 2$, and a normal vector to this front cap pointing outward is the vector $\mathbf{i} =< 1, 0, 0 >$.

Then,

$$\iint_{S_3} \vec{F}\cdot\vec{n_3}\; dS = \iint_{S_3} < 2^2, -y, z > \cdot < 1, 0, 0 > \; dS = \iint_{S_3} 4\; dS = 4Area(S_3) = 4(\pi 3^2) = 36\pi$$

. Another method is to parametrize the surface $S_3$ using polar coordinates with $r = v$ and $\theta = u$.

$$S_3 \implies \begin{cases} \vec{r}_3(u,v) = \langle 2, v\cos(u), v\sin(u)\rangle \\ 0 \le u \le 2\pi \\ 0 \le v \le 3 \end{cases} \implies (\vec{r}_3)_u \times (\vec{r}_3)_v = \langle -v, 0, 0\rangle$$

This normal vector points inward, and since we require an outward orientation on each surface with respect to the solid $E$, we use the normal vector $-[(\vec{r}_3)_u \times (\vec{r}_3)_v]$ for $S_3$. Then,

$$\begin{aligned}\iint_{S_3} \vec{F}\cdot d\vec{S} &= \iint_{D_3} \vec{F}(\vec{r}_3(u,v))\cdot[-(\vec{r}_3)_u \times (\vec{r}_3)_v]dA \\ &= \int_0^3\int_0^{2\pi} \langle 4, -v\cos(u), v\sin(u)\rangle \cdot \langle v, 0, 0\rangle dudv \\ &= \int_0^3\int_0^{2\pi} 4v\; dudv = \left(\int_0^3 4vdv\right)\left(\int_0^{2\pi} du\right) = (18)(2\pi) = 36\pi.\end{aligned}$$

So we have that:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S} = 0 + 0 + 36\pi = 36\pi.$$

**Divergence Calculation:**

$$div(\vec{F}) = \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial y}[-y] + \frac{\partial}{\partial z}[z] = 2x - 1 + 1 = 2x$$

Notice that $E$ can be represented in cylindrical coordinates using $x = x, y = r\cos(\theta)$ and $z = r\sin(\theta)$ yielding:

$$E = \{(r,\theta,x) : 0 \le r \le 3,\; 0 \le \theta \le 2\pi,\; 0 \le x \le 2\}.$$

Then we have that:

$$\iiint_E div(\vec{F})\ dV = \iiint_E (2x)\ dV$$
$$= \int_0^2 \int_0^{2\pi} \int_0^3 2xr\ dr\ d\theta\ dx$$
$$= \left(\int_0^2 2x\ dx\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^3 rdr\right) = (4)(2\pi)\left(\frac{9}{2}\right) = 36\pi.$$

Therefore, the Divergence theorem's equality holds.

2. Calculate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\ dS$ where $\mathbf{F}(x,y,z) = \langle xye^z, xy^2z^3, -ye^z\rangle$ and $S$ is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$.

First, we check the hypotheses of the Divergence Theorem:

i) The solid box $E$ is a simple solid region.

ii) The surface $S$ of the solid region $E$ is a closed surface, so we choose the positive orientation to be given by the outward orientated normal vectors.

iii) The component functions $P(x,y,z) = xye^z$, $Q(x,y,z) = xy^2z^3$, and $R(x,y,z) = -ye^z$ of the vector field $\mathbf{F}$ have continuous partial derivatives on $R^3$ which is an open set that contains $E$. By the Divergence theorem,

$$\iint_S \vec{F}\cdot\vec{n}\ dS = \iiint_E div(\vec{F})\ dV.$$

Here, $E$ is the solid box which can be represented in *rectangular* coordinates as:

$$E = \{(x,y,z) : 0 \le x \le 3,\ 0 \le y \le 2,\ 0 \le z \le 1\}.$$

$$div(\vec{F}) = \nabla\cdot\vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\rangle \cdot \langle xye^z, xy^2z^3, -ye^z\rangle = \frac{\partial}{\partial x}[xye^z] + \frac{\partial}{\partial y}[xy^2z^3] + \frac{\partial}{\partial z}[-ye^z] = ye^z + 2xyz^3 - ye^z = 2xyz^3$$

Hence, we have that:

$$\iiint_E div(\vec{F})\ dV = \int_0^3 \int_0^2 \int_0^1 \left(2xyz^3\right) dz\ dy\ dx$$
$$= 2\int_0^3 \int_0^2 \left[\frac{xyz^4}{4}\right]_{z=0}^{z=1} dydx = \frac{1}{2}\int_0^3 \int_0^2 xy\ dydx$$
$$= \frac{1}{2}\int_0^3 \left[\frac{xy^2}{2}\right]_{y=0}^{y=2} dx = \int_0^3 x\ dx = \left[\frac{x^2}{2}\right]_{x=0}^{x=3} = \frac{9}{2}.$$

3. Calculate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\ dS$ where $\mathbf{F}(x,y,z) = \langle xe^y, z - e^y, -xy\rangle$ and $S$ is the ellipsoid $x^2 + 2y^2 + 3z^2 = 4$.

First, we check the hypotheses of the Divergence Theorem:

i) The solid ellipsoid $E$ is a simple solid region.

ii) The surface $S$ of the solid region $E$ is a closed surface, so we choose the positive orientation to be given by the outward orientated normal vectors.

iii) The component functions $P(x,y,z) = xe^y$, $Q(x,y,z) = z - e^y$, and $R(x,y,z) = -xy$ of the vector field $\mathbf{F}$ have continuous partial derivatives on $R^3$ which is an open set that contains $E$.

By the Divergence theorem,

$$\iint_S \vec{F}\cdot\vec{n}\ dS = \iiint_E \nabla\cdot\vec{F}\ dV.$$

Notice that:

$$div(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}[xe^y] + \frac{\partial}{\partial y}[z - e^y] + \frac{\partial}{\partial z}[-xy] = 0.$$

Hence, we have that:

$$\iint_S \vec{F} \cdot \vec{n}\; dS = \iiint_E div(\vec{F})\; dV = 0.$$

4. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\; dS$ where

$$\mathbf{F}(x, y, z) = \langle xy + 2xz, x^2 + y^2, xy - z^2 \rangle$$

and $S$ is the surface of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = y - 2$ and $z = 0$.

First, we check the hypotheses of the Divergence Theorem:

i) The solid $E$ is a simple solid region.

ii) The surface $S$ of the solid region $E$ is a closed surface, so we choose the positive orientation to be given by the outward orientated normal vectors.

iii) The component functions $P(x, y, z) = xy + 2xz$, $Q(x, y, z) = x^2 + y^2$, and $R(x, y, z) = xy - z^2$ of the vector field $\mathbf{F}$ have continuous partial derivatives on $R^3$ which is an open set that contains $E$. By the Divergence theorem,

$$\iint_S \vec{F} \cdot \vec{n}\; dS = \iiint_E \nabla \cdot \vec{F}\; dV.$$

Here, $E$ is the solid enclosed by $x^2 + y^2 = 4$, $z = y - 2$, and $z = 0$, which can be represented in *cylindrical* coordinates as:

$$E = \{(r, \theta, z) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ r\sin(\theta) - 2 \le z \le 0\}.$$

Next, we compute the divergence of $\vec{F}$.

$$div(\vec{F}) = \frac{\partial}{\partial x}[xy + 2xz] + \frac{\partial}{\partial y}[x^2 + y^2] + \frac{\partial}{\partial z}[xy - z^2] = y + 2z + 2y - 2z = 3y$$

Hence, we have that:

$$\begin{aligned}
\iiint_E \nabla \cdot \vec{F}\; dV &= \iiint_E (3y)\, dV \\
&= 3\int_0^{2\pi}\int_0^2\int_{r\sin(\theta)-2}^0 r\sin(\theta)\; r dz dr d\theta \\
&= 3\int_0^{2\pi}\int_0^2 r^2\sin(\theta)\Big[z\Big]_{r\sin(\theta)-2}^{z=0} dr d\theta \\
&= -3\int_0^{2\pi}\int_0^2 \left(r^3\sin^2(\theta) - 2r^2\sin(\theta)\right)\; dr d\theta \\
&= -3\int_0^{2\pi}\left[\frac{r^4}{4}\sin^2(\theta) - \frac{2r^3}{3}\sin(\theta)\right]_{r=0}^{r=2} d\theta \\
&= -3\int_0^{2\pi}\left(4\sin^2(\theta) - \frac{16}{3}\sin(\theta)\right)\; d\theta \\
&= -3\int_0^{2\pi}\left(2 - 2\cos(2\theta) - \frac{16}{3}\sin(\theta)\right)\; d\theta \\
&= -3\left[2\theta - \sin(2\theta) + \frac{16}{3}\cos(\theta)\right]_{\theta=0}^{\theta=2\pi} = -12\pi.
\end{aligned}$$

5. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$

(a) Compute *curl* **F**

Notice that:

$$curl\ \vec{F} = \nabla \times \vec{F} = \left\langle \frac{\partial}{\partial y}[x] - \frac{\partial}{\partial z}[y]\ ,\ \frac{\partial}{\partial z}[z] - \frac{\partial}{\partial x}[x]\ ,\ \frac{\partial}{\partial x}[y] - \frac{\partial}{\partial y}[z] \right\rangle = \langle 0, 0, 0 \rangle,$$

implying $\vec{F}$ is conservative.

(b) Compute *div* **F**

Notice that:

$$div\ \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}[z] + \frac{\partial}{\partial y}[y] + \frac{\partial}{\partial z}[x] = 1$$

(c) Calculate the **flux** of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the surface $S$ of the solid region $E$ which is the upper half of the ball of radius 1 given by the equations $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$.

We know that the flux of a vector field across a surface is given by:

$$\text{flux} = \iint_S \vec{F} \cdot \vec{n}\ dS,$$

where $\vec{n}$ is the outward unit normal to $S$ relative to $E$.

First, we check the hypotheses of the Divergence Theorem:

i) The solid region $E$ given by the top hemisphere is a simple solid region.

ii) The surface $S$ of the solid region $E$ is a closed surface, so we choose the positive orientation to be given by the outward orientated normal vectors.

iii) The component functions $P(x, y, z) = z$, $Q(x, y, z) = y$, and $R(x, y, z) = x$ of the vector field $\mathbf{F}$ have continuous partial derivatives on $R^3$ which is an open set that contains $E$. By the Divergence theorem,

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iiint_E \nabla \cdot \vec{F}\ dV = \iiint_E dV = \text{Volume}(E).$$

Since $E$ is the top hemisphere of the ball of radius 1, we have that:

$$\text{flux} = \frac{1}{2}\left(\frac{4}{3}\pi(1)^3\right) = \frac{2\pi}{3}.$$

## Suggested Textbook Problems

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|---|---|
| Section 16.9 | 1-12, 17-19 |
| Chapter 16 Concept Check | 1-13, 15, 16 |
| Chapter 16 True-False | 1-7, 11, 12 |
| Chapter 16 Review | 2-18, 27-30, 34, 35 |

## SOME USEFUL DEFINITIONS, THEOREMS AND NOTATION:

**The Divergence Theorem**

Let $E$ be a simple solid region and let $S$ be the boundary of $E$, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $E$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$