Limits: Possible questions: Find $\lim_{(x,y)\to(a,b)} f(x,y)$ or check if it exists.

① Check if (a,b) is in the domain of f and if f is CTS at (a,b). If so, answer is $f(a,b)$.

② If not use calc 1 tricks to factor and cancel, find conjugates, squeeze theorem and use step ① again.

③ If step ② fails, check limits along $y=0$, $x=0$, $y=mx$, $y=x^2$ in that order. You should get different limit values along at least two of these curves.

Example 4

polynomial

Evaluate $\lim_{(x,\,y)\to(1,\,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

$$\Rightarrow \lim_{(x,y)\to(1,2)} x^2y^3 - x^3y^2 + 3x + 2y = (1)(8) - (1)(4) + 3 + 4$$
$$= \boxed{3}$$

Example 5

Evaluate $\lim_{(x,\,y)\to(-2,\,3)} \frac{x^2y+1}{x^3y^2-2x}$ if it exists.

$/\!/ L$

A: Check $x^3y^2 - 2x$ at $(-2,3) \neq 0$.

i.e $(-2)^3(3)^2 - 2(-2) = -72 + 4 \neq 0$

$$\Rightarrow L = \frac{(-2)^2(3)+1}{-68} = \frac{-13}{68}.$$

Example 6

Find $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$ if it exists.

(For HW problems, plotting on Desmos would be useful)

We want to show that $\dfrac{3x^2y}{x^2+y^2} \to 0$ as $(x,y) \to (0,0)$

Two facts:

① $x \geq y \Rightarrow \frac{1}{x} \leq \frac{1}{y}$

② $|x| \leq a \Rightarrow -a \leq x \leq a$

First: $\left|\dfrac{3x^2y}{x^2+y^2}\right| = \dfrac{3x^2|y|}{\underbrace{x^2+y^2}_{\geq x^2}} \leq \dfrac{3\cancel{x^2}|y|}{\cancel{x^2}} = 3|y|$

Second: $-3|y| \leq \dfrac{3x^2y}{x^2+y^2} \leq 3|y| \Rightarrow \overbrace{\lim_{(x,y)\to(0,0)} -3|y|}^{=0} \leq \lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} \leq \overbrace{\lim_{(x,y)\to(0,0)} 3|y|}^{=0}$

$\Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0.$

6 Definition

A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say that f is **continuous on** D if f is continuous at every point (a, b) in D.

Section 14.3: Partial derivatives

4 Definition

If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Notations for Partial Derivatives

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

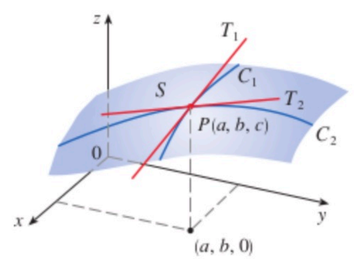
1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Take $g(a) = f(a,b)$ then the partial derivative with resp. to x is $g'(a) = f_x(a,b)$

$$= \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C



Example 1

If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

$f_x(x,y) = 3x^2 + 2xy^3 \Rightarrow f_x(2,1) = 12 + 4 = 16$

$f_y(x,y) = 3x^2y^2 - 4y \Rightarrow f_y(2,1) = 12 - 4 = 8$

Example 2

If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$\Rightarrow f_x(x,y) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$

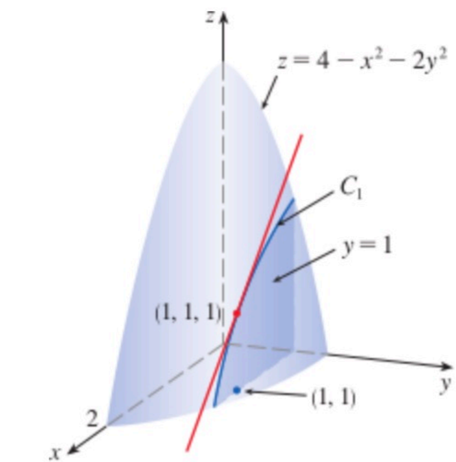
$\Rightarrow f_y(x,y) = \cos\left(\frac{x}{1+y}\right) \cdot x \cdot \frac{(-1)}{(1+y)^2}$

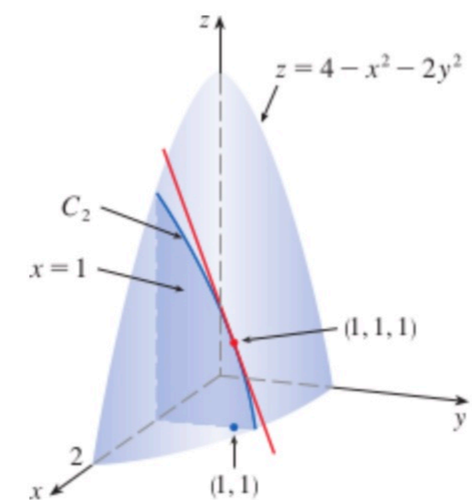
Example 3

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

$f_x(x,y) = -2x \qquad f_y(x,y) = -4y$

$\Rightarrow f_x(1,1) = -2 \quad , \quad f_y(1,1) = -4$





Example 5

Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz + 4 = 0$$

Then evaluate these partial derivatives at the point $(-1, 1, 2)$

A: find $\frac{\partial z}{\partial x}$: $3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$

$\Rightarrow \frac{\partial z}{\partial x}\left(3z^2 + 6xy\right) = -\left(3x^2 + 6yz\right) \Rightarrow \frac{\partial z}{\partial x} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy} = -\frac{(x^2 + 2yz)}{z^2 + 2xy}$

Similarly: $\frac{\partial z}{\partial y} \overset{\text{DIY}}{=} \frac{-(y^2 + 2xz)}{z^2 + 2xy}$

Plug in $x = -1$, $y = 1$, $z = 2$ into $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$: $\left.\frac{\partial z}{\partial x}\right|_{(-1,1,2)} = \frac{-\left((-1)^2 + 2(1)(2)\right)}{(2)^2 + 2(-1)(1)} = -\frac{5}{2}$

Functions of Three or more variables:

Def: $f_x(x,y,z) = \lim_{h \to 0} \dfrac{f(x+h,y,z) - f(x,y,z)}{h}$, and similarly for f_y, f_z.

Example 6

Find f_x, f_y, and f_z if $f(x,y,z) = e^{xy} \ln z$.

A: $f_x(x,y,z) = y \cdot e^{xy} \cdot \ln z$

$f_y(x,y,z) = x \cdot e^{xy} \cdot \ln z$

$f_z(x,y,z) = \dfrac{e^{xy}}{z}$

Higher derivatives:

Notation: $f_{xx} = (f_x)_x$, $f_{xy} = (f_x)_y$, ...

(f_x, f_y are also functions of two variables so f_{xx} is the partial derivative of f_x in the x-direction.)

Example 7

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2 y^3 - 2y^2$$

A: $f_x(x,y) = 3x^2 + 2xy^3$, $f_y(x,y) = 3y^2x^2 - 4y$

$f_{xy}(x,y) = 6xy^2$, $f_{yx}(x,y) = 6xy^2$

$f_{xx}(x,y) = 6y^2$, $f_{yy}(x,y) = 6yx^2 - 4$

Is it always the case that $f_{xy} = f_{yx}$? No, Clairaut's theorem.

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Eg: If the following functions are CTS then : $f_{xyy} = f_{yxy} = f_{yyx}$

Example 8

Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

A: $f_x = \cos(3x + yz) \cdot 3$

$f_{xx} = -9\sin(3x + yz)$

$f_{xxy} = -9z\cos(3x + yz)$

$f_{xxyz} = -9\cos(3x + yz) + 9zy\sin(3x + yz)$