1. (8 points) **Example:** Let T be the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$, $C(0,4,0)$
(*Note that the plane containing A, B, C has the equation $3x + 3y + 2z = 12$.*)

(a) Express T as a solid region of type 1.
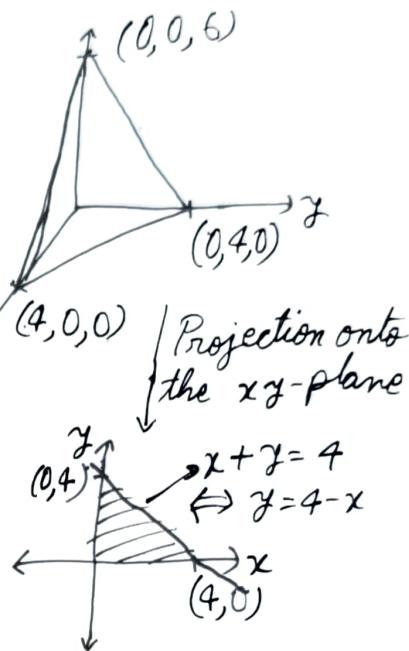
(b) Express $\iiint_T f(x,y,z)\, dV$ as an iterated integral.

Bounding surfaces: $x=0,\ y=0,\ z=0,\ 3x+3y+2z=12$

$$z = 6 - \frac{3}{2}x - \frac{3}{2}y$$

(a) $T = \left\{ (x,y,z): 0 \le x \le 4,\ 0 \le y \le 4-x,\ 0 \le z \le 6 - \frac{3}{2}x - \frac{3}{2}y \right\}$

(b) $$\int_0^4 \int_0^{4-x} \int_0^{6-\frac{3}{2}x-\frac{3}{2}y} f(x,y,z)\, dz\, dy\, dx$$

Projection onto the xy-plane

$x + y = 4 \Leftrightarrow y = 4 - x$

2. (8 points) Change the order of integration in

$$\int_{x=0}^{2} \int_{y=x}^{2} \int_{z=0}^{y} (x+y+z)\, dz\, dy\, dx$$

so that the order becomes $dz\, dx\, dy$, and sketch the projection of the region on the xy–plane.

In order to change from $dz\,dy\,dx$ to $dz\,dx\,dy$ we don't need to make any changes to the integral limit of z as z is the innermost integral for both orders.
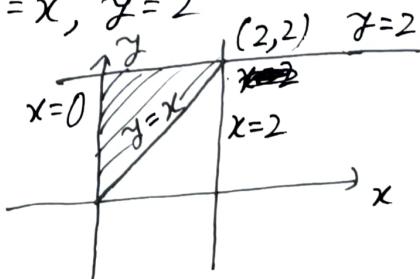
Bounding curves: $x=0,\ x=2,\ y=x,\ y=2,\ z=0,\ z=y.$

Projection onto xy-Plane: $x=0,\ x=2,\ y=x,\ y=2$

Since $y=x$ is the lower bound for y the domain must lie above $y=x$

Hence in $dz\,dx\,dy$ order –
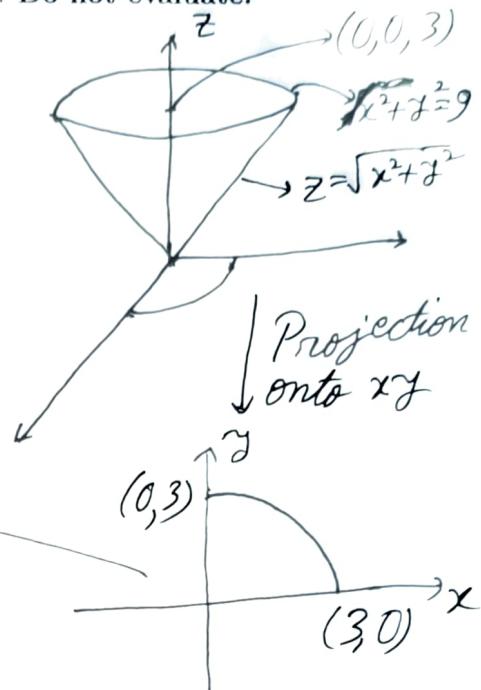
$$\int_{y=0}^{2} \int_{x=0}^{y} \int_{z=0}^{y} f(x+y+z)\, dz\, dx\, dy$$

3. (8 points) Express $\iiint_E (x+z)\,dV$ in cylindrical coordinates, where E lies *above* the cone $z = \sqrt{x^2 + y^2}$ and *below* the plane $z = 3$ in the first octant. **Do not evaluate.**

First octant $\Rightarrow$ $x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0$

above $z = \sqrt{x^2 + y^2} \Rightarrow z \geqslant \sqrt{x^2 + y^2} \Rightarrow z \geqslant r$

below $z = 3 \Rightarrow z \leqslant 3$

$\Rightarrow r \leqslant z \leqslant 3$

$x \geqslant 0 \qquad ,\quad y \geqslant 0$

$\Rightarrow r\cos\theta \geqslant 0,\quad r\sin\theta \geqslant 0$

$\Rightarrow \cos\theta \geqslant 0,\quad \sin\theta \geqslant 0$
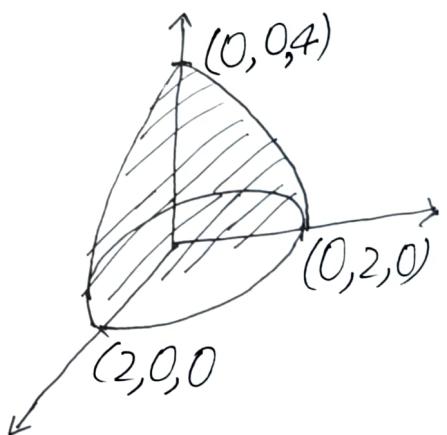
$0 \leqslant r \leqslant 3,\quad 0 \leqslant \theta \leqslant \dfrac{\pi}{2}$

$\displaystyle\int_0^{\frac{\pi}{2}} \int_0^3 \int_r^3 (r\cos\theta + z)\, r\,dz\,dr\,d\theta$

(0,0,3)

$x^2 + y^2 = 9$

$z = \sqrt{x^2 + y^2}$

Projection onto xy

(0,3)

(3,0)

4. (8 points) Evaluate $\iiint_E (x^2 + y^2)\,dV$, where E is the solid under the paraboloid $z = 4 - x^2 - y^2$ and above the xy–plane.

Cylindrical coordinate is the best a choice.

xy-plane $\Longleftrightarrow z = 0$

above xy-plane $\Rightarrow z \geqslant 0$.

below $z = 4 - x^2 - y^2 \Rightarrow z \leqslant 4 - x^2 - y^2$

$\qquad\qquad\qquad\qquad\Rightarrow z \leqslant 4 - (x^2 + y^2)$

$\qquad\qquad\qquad\qquad\quad z \leqslant 4 - r^2$

(0,0,4)

(0,2,0)

(2,0,0)

The integral becomes :–

$\displaystyle\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r^2 \cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \Big[r^3 z\Big]_{z=0}^{4-r^2} dr\,d\theta$

$\displaystyle = \int_0^{2\pi}\int_0^2 (4r^3 - r^5)\,dr\,d\theta = \int_0^{2\pi}\Big[r^4 - \frac{r^6}{6}\Big]_{r=0}^{2} d\theta = \int_0^{2\pi}\Big(16 - \frac{64}{6}\Big)d\theta$

$\displaystyle = \Big[\frac{16}{3}\theta\Big]_{\theta=0}^{2\pi} = \frac{32\pi}{3}$
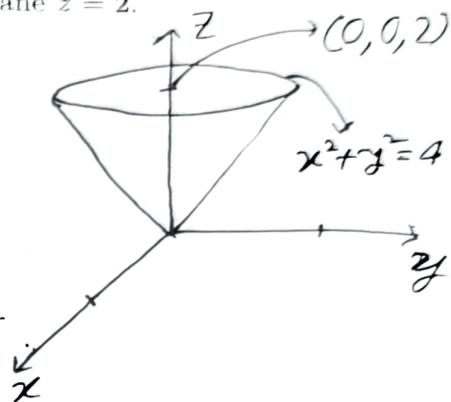
5. (6 points) Set up (but do not evaluate) a triple integral in cylindrical coordinates for the volume of the solid bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

Bounding surfaces: ~~~~~~~~ $z = \sqrt{x^2 + y^2} \Leftrightarrow z = r$

$$z = 2$$

$$\Rightarrow r \leq z \leq 2$$

$$\int_0^{2\pi} \int_0^2 \int_r^2 1 \cdot r \, dz \, dr \, d\theta$$

(0,0,2)

$x^2 + y^2 = 4$
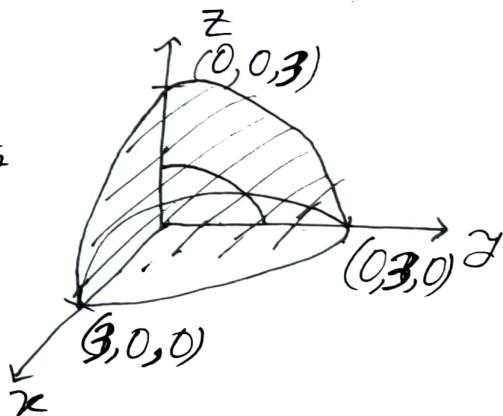
6. (8 points) Find the mass of the solid hemisphere $x^2 + y^2 + z^2 \leq 9$, $z \geq 0$, with density $\rho(x,y,z) = z$.

Best choice sperical coordinates

Geometrically:

$$x^2 + y^2 + z^2 \leq 9$$ $$z \geq 0$$

$$\Rightarrow \rho^2 \leq 9$$ $\Rightarrow \phi$ goes from z axis to

$$\Rightarrow \rho \leq 3$$ the xy-plane

$$\Rightarrow 0 \leq \rho \leq 3$$ $\Rightarrow$ for xy-plane $\phi = \frac{\pi}{2}$

$$\Rightarrow 0 \leq \phi \leq \frac{\pi}{2}$$

Algebraically:

$$z \geq 0$$

$$\Rightarrow \rho \cos\phi \geq 0$$

$$\Rightarrow \cos\phi \geq 0$$

$$\Rightarrow \phi \leq \frac{\pi}{2}$$

So our integral becomes:

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^3 \rho \cos\phi^2 \sin\phi \, d\rho \, d\theta \, d\phi \quad \text{(No space for evaluation)}$$

(0,0,3)

(0,3,0)

(3,0,0)

7. (8 points) Express the volume of the region *inside* the sphere $x^2 + y^2 + z^2 = 16$ but *outside* the cylinder $x^2 + y^2 = 4$ as a triple integral in spherical coordinates. **Do not evaluate.**

$$x^2 + y^2 + z^2 = 16 \quad \rightarrow \quad 4 + z^2 = 16 \Rightarrow z^2 = \cancel{12}\, 12 \Rightarrow z = \pm\sqrt{12}$$
$$x^2 + y^2 = 4$$
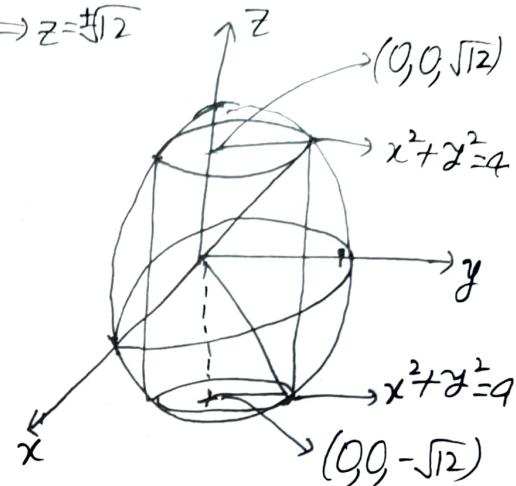
Inside: $x^2 + y^2 + z^2 \leq 16 \Rightarrow \rho^2 \leq 16 \Rightarrow \cancel{\rho \leq 4}\; \rho \leq 4$

Outside: $x^2 + y^2 = 4 \Rightarrow \rho^2 \sin^2\phi \geq 4$
$$\Rightarrow \rho \sin\phi \geq 2$$
$$\Rightarrow \rho \geq \frac{2}{\sin\phi} = 2\csc\phi$$

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{0}^{2\pi} \int_{2\csc\phi}^{4} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$(0,0,\sqrt{12})$

$x^2 + z^2 = 4$

$x^2 + y^2 = 4$

$(0,0,-\sqrt{12})$

Upper ~~bis~~ bound for ϕ:
$$\phi = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{4}}{-\sqrt{12}}\right)$$
$$= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$$

Lower bound for ϕ:
$$\phi = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{4}}{\sqrt{12}}\right) = \frac{\pi}{6}$$

8. (3 points) Let $\mathbf{F}(x,y) = \langle 3x^2y - y^3,\ x^3 - 3xy^2 \rangle$. Determine whether $\mathbf{F}$ is conservative on $\mathbb{R}^2$.

$$\vec{F}(x,y) = \langle 3x^2y - y^3,\ x^3 - 3xy^2 \rangle$$

$$P(x,y) = 3x^2y - y^3 \qquad\qquad Q(x,y) = x^3 - 3xy^2$$

$$\frac{\partial P}{\partial y} = 3x^2 - 3y^2 \qquad\qquad \frac{\partial Q}{\partial x} = 3x^2 - 3y^2$$

Hence, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$$\Rightarrow \vec{F} \text{ is conservative.}$$

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9. Express the Triple integral as an iterated integral in cylindrical coordinates:

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{4} xy \, dz \, dx \, dy$$

Solution :-
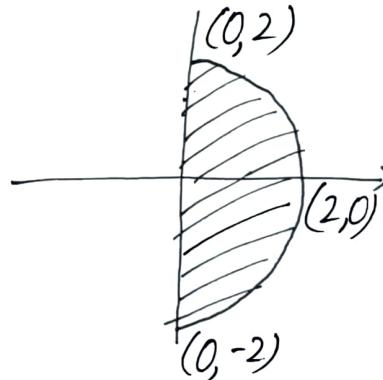
For this type of problem first find the bound for z in cylindrical coordinate :

$$\sqrt{x^2+y^2} \le z \le 4 \iff r \le z \le 4$$

Once bound for z is found we focus our attention to the projection of the domain onto the xy-plane which can be found by analyzing the bounds for x and y.

$$-2 \le y \le 2, \quad 0 \le x \le \sqrt{4-y^2}$$

$x \le \sqrt{4-y^2}$

$\Rightarrow x^2 \le 4 - y^2$

$\Rightarrow x^2 + y^2 \le 4$ (circle with radius 2)

$\Rightarrow r^2 \le 4$

$x \ge 0$ ensures it lies on the right half plane

(0,2)
(2,0)
(0,-2)

$$\Rightarrow r^2 \le 4 \Rightarrow r \le 2$$

Positive y-axis corresponds to $\theta = \frac{\pi}{2}$

Negative y-axis corresponds to $\theta = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$

So the integral becomes —

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2} \int_{r}^{4} r\cos\theta \cdot r\sin\theta \cdot r \, dz \, dr \, d\theta$$