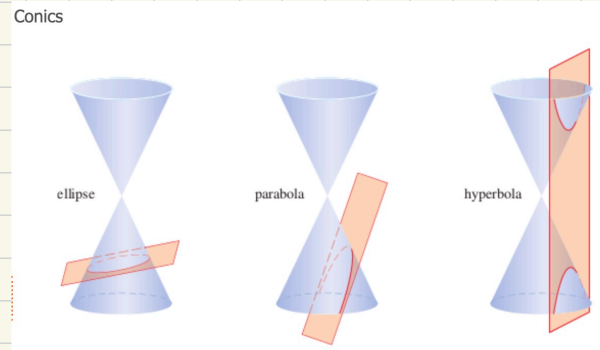
Midterm on 9/29. Syllabus chapters 12 and 13. (Of course not everything, see discussion worksheets)

Time: 6:30 - 8:00 pm. Show up around 6-6:15 if possible. Location: TBA.

Section 12.6: Quadric Surfaces.



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\qquad$ $y^2 = 4ax$ $\qquad$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Def: A quadric surface is the graph of a second degree equation in three variables x, y, z.

- In general: the equation is of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

- In standard form: (after rotation and translation)

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{OR} \quad Ax^2 + By^2 + Iz = 0$$

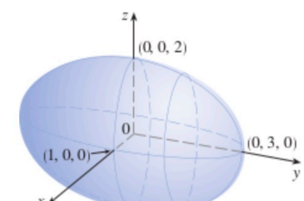
- They form the three dimensional counterparts to conic sections.

Example 3

Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$



A: Check traces when $x=0, y=0, z=0$. Eg $x=0 \to \frac{y^2}{9}+\frac{z^2}{4}=1$ which is an ellipse in the y-z plane

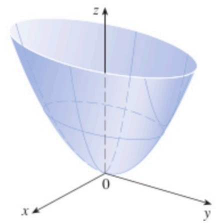
Similarly, $y=0, z=0$ will give you ellipses on the xz-plane and xy plane resp.

in general, $x=k$ will also give you an ellipse but k values must be restricted.

Example 4

Use traces to sketch the surface $z = 4x^2 + y^2$.

The surface $z = 4x^2 + y^2$ is an elliptic paraboloid. Horizontal traces are ellipses; vertical traces are parabolas.

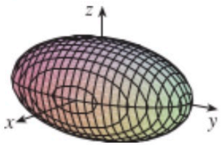
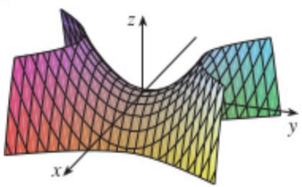
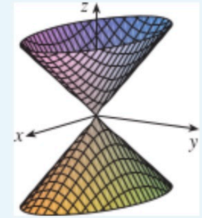
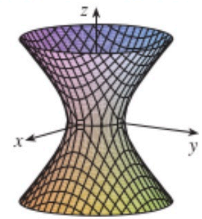
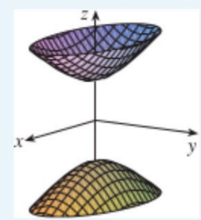


A: $x=0 \Rightarrow z=y^2$ (parabola on the yz plane)

$y=0 \Rightarrow z=4x^2$ (parabola on the xz plane)

$z=k \Rightarrow k=4x^2+y^2 \Rightarrow x^2+\frac{y^2}{4}=\frac{k}{4} \Rightarrow \frac{x^2}{k/4}+\frac{y^2}{k}=1$ (ellipse on the plane $z=k$)

Horizontal traces are ellipses and vertical traces are parabolas.

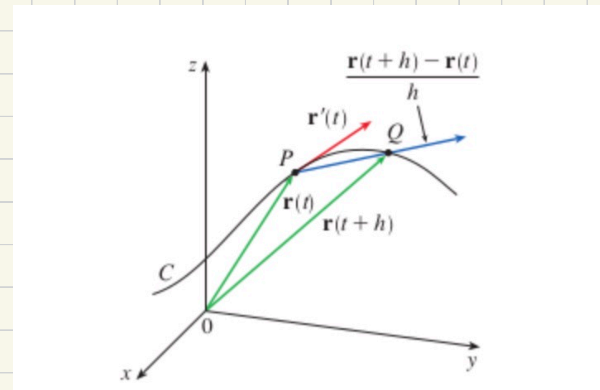
Surface	Equation
Ellipsoid  	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.
Elliptic Paraboloid  	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.
Hyperbolic Paraboloid  	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.
Cone  	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Hyperboloid of One Sheet  	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperboloid of Two Sheets  	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Section 13.4, Velocity, Speed, and Acceleration:

- Suppose that a particle moves through space so that it's position vector is $\vec{r}(t)$ at time t.
- $\dfrac{r(t+h) - r(t)}{h}$ is the average velocity over a time interval of length h

direction vector for the tangent line

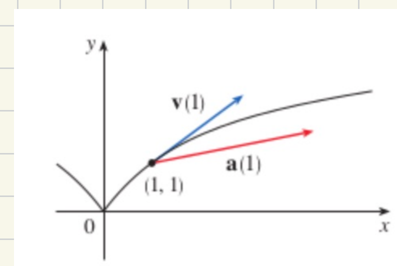
- velocity vector: $\vec{v}(t) = \lim_{h \to 0} \dfrac{r(t+h) - r(t)}{h} = r'(t)$



- speed: $\|\vec{v}(t)\| = \|\vec{r}\,'(t)\| = v$

Aside: $\underbrace{L = \int_a^b \|\vec{r}\,'(t)\|\, dt}_{\text{arc length}}$ is the distance travelled by the particle

- acceleration: $\vec{a}(t) = r''(t) = v'(t)$.



Example 1

The position vector of an object moving in a plane is given by $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$ and illustrate geometrically.

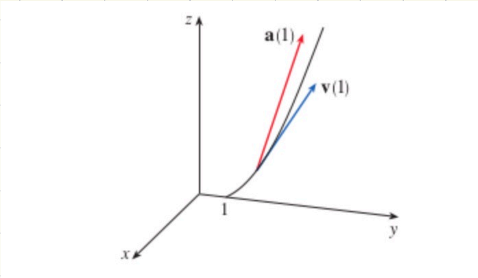
A: $\vec{r}(t) = \langle t^3, t^2 \rangle \Rightarrow \vec{v}(t) = \vec{r}'(t) = \langle 3t^2, 2t \rangle$, $\|\vec{r}'(t)\| = \sqrt{9t^4 + 4t^2} = t\sqrt{9t^2+4}$

$\Rightarrow \vec{a}(t) = \langle 6t, 2 \rangle$.

When $t=1$, $\vec{r}(1) = \langle 1,1 \rangle$, $\vec{v}(1) = \langle 3,2 \rangle$, $v(1) = \sqrt{13}$, $\vec{a}(1) = \langle 6,2 \rangle$.

Example 2

Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.



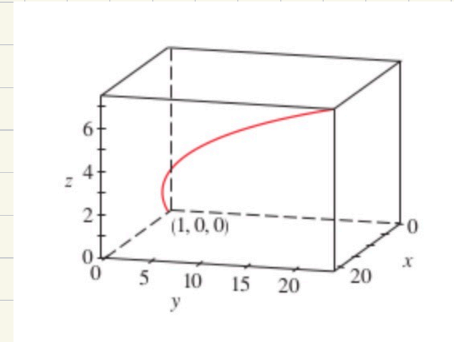
A: $\vec{r}(t) = \langle t^2, e^t, te^t \rangle$

$\vec{r}'(t) = \langle 2t, e^t, \underbrace{e^t + te^t}_{=e^t(t+1)} \rangle \Rightarrow v(t) = \sqrt{4t^2 + e^{2t} + e^{2t}(t+1)^2}$

$\Rightarrow \vec{a}(t) = \langle 2, e^t, e^t + e^t(t+1) \rangle$.

Example 3

A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\,\mathbf{i} + 6t\,\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.



A: $a(t) = v'(t) = 4t\,\hat{i} + 6t\,\hat{j} + \hat{k}$

$\Rightarrow \int a(t)\,dt = v(t)$

$\Rightarrow v(t) = \int 4t\,dt\,\hat{i} + \int 6t\,dt\,\hat{j} + \int dt\,\hat{k}$

$= (2t^2 + C_1)\,\hat{i} + (3t^2 + C_2)\,\hat{j} + (t + C_3)\,\hat{k}$

$= 2t^2\,\hat{i} + 3t^2\,\hat{j} + t\,\hat{k} + \vec{C}$

$\Rightarrow v(0) = \hat{i} - \hat{j} + \hat{k} \Rightarrow \vec{C} = \hat{i} - \hat{j} + \hat{k} \Rightarrow v(t) = (2t^2 + 1)\,\hat{i} + (3t^2 - 1)\,\hat{j} + (t + 1)\,\hat{k}.$

$v(t) = r'(t) \Rightarrow r(t) = \left(\frac{2t^3}{3} + t\right)\hat{i} + (t^3 - t)\,\hat{j} + \left(\frac{t^2}{2} + t\right)\hat{k} + \vec{D}.$

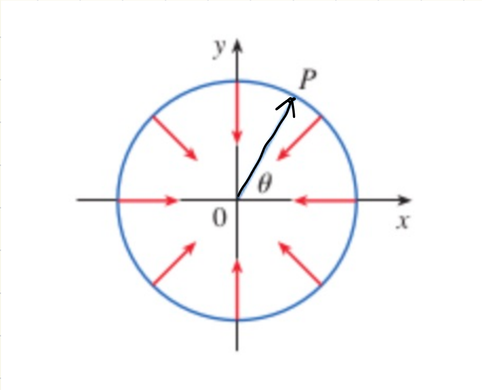
$r(0) = \hat{i} + 0\hat{j} + 0\hat{k} \Rightarrow \vec{D} = \hat{i} \Rightarrow r(t) = \left(\frac{2t^3}{3} + t + 1\right)\hat{i} + (t^3 - t)\,\hat{j} + \left(\frac{t^2}{2} + t\right)\hat{k}.$

Recall: Newton's 2nd law: $\vec{F}(t) = m \cdot \vec{a}(t)$

$\left(\omega = \dfrac{d\theta}{dt} \right)$

Example 4

An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = a \cos \omega t\, \mathbf{i} + a \sin \omega t\, \mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.



$$\vec{r}\,'(t) = -a\omega \sin(\omega t)\,\hat{i} + a\omega \cos(\omega t)\,\hat{j}$$

$$\Rightarrow \vec{r}\,''(t) = -a\omega^2 \cos(\omega t)\,\hat{i} - a\omega^2 \sin(\omega t)\,\hat{j}$$

$$= -\omega^2 \left(a\cos(\omega t)\,\hat{i} + a \sin(\omega t)\,\hat{j} \right) = -\omega^2\, \vec{r}(t)$$

$$\Rightarrow \vec{F} = -m\omega^2\, \vec{r}(t)$$ (so force points in the opposite direction of position)

(Such a force is called centripetal (centre-seeking) force).

Tangential and Normal Components of Acceleration.

<u>Recall</u>: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{\vec{v}(t)}{v(t)} \Rightarrow \vec{v}(t) = v(t) \cdot \vec{T}(t)$

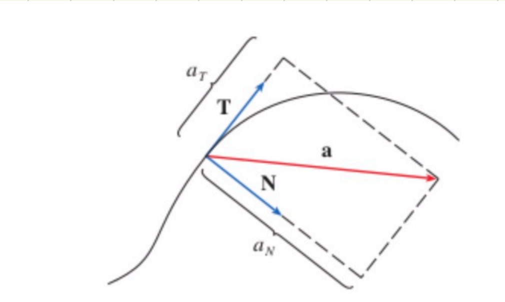
$$\Rightarrow \underset{\substack{\| \\ a(t)}}{\vec{v}\,'(t)} = v'(t) \cdot \vec{T}(t) + v(t) \cdot \vec{T}\,'(t)$$

<u>Recall</u>: · $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} \Rightarrow \vec{T}\,'(t) = \|\vec{T}\,'(t)\| \cdot \vec{N}(t)$

· $k = \dfrac{\|\vec{T}\,'(t)\|}{\|r'(t)\|} = \dfrac{\|\vec{T}\,'(t)\|}{v(t)} \Rightarrow \|\vec{T}\,'(t)\| = v(t) \cdot k$

$$\Rightarrow \vec{T}\,'(t) = v(t) \cdot k \cdot \vec{N}(t)$$

$$\boxed{\Rightarrow \vec{a} = v' \cdot \vec{T} + v^2 k \vec{N}}$$



Let a_T and a_N be the tangential and normal components of acceleration. Then
$$\boxed{\begin{aligned} a_T &= v' \\ \text{and} \quad a_N &= v^2 \cdot \kappa \end{aligned}}$$

Remark: When κ is large, eg: sharp turns, a_N is high so "travelling at high speeds on a sharp curve pushes you towards the center of the curve."

Throwback: $\kappa = \dfrac{\|T'(t)\|}{\|r'(t)\|}$. Fact: $\kappa = \dfrac{\|r' \times r''\|}{\|r'\|^3}$

proof: $T = \dfrac{r'}{\|r'\|} \Rightarrow r' = \|r'\| \cdot T = \underbrace{\dfrac{ds}{dt}}_{\text{arc length function}} \cdot T$

$$\Rightarrow \gamma'' = \frac{d^2 s}{dt^2} \cdot T + \frac{ds}{dt} \cdot T'$$

$$\Rightarrow \gamma' \times \gamma'' = \gamma' \times \left(s'' \cdot T + s' \cdot T' \right) = s'' \underbrace{\left(\gamma' \times T\right)}_{=0 \text{ since } \gamma' \text{ and } T \text{ are parallel}} + s' \left(\gamma' \times T'\right)$$

$$\left(\gamma' \times T\right) \text{ term: } s'' \cdot \|\gamma'\| \left(\frac{\gamma'}{\|\gamma'\|} \times T \right) = s'' \cdot \|\gamma'\| \left(T \times T \right) = 0$$

$$\Rightarrow \gamma' \times \gamma'' = s' \left(\gamma' \times T' \right) \Rightarrow \| \gamma' \times \gamma'' \| = |s'| \cdot \| \gamma' \times T' \|$$

$$= |s'| \cdot \|\gamma'\| \cdot \| T \times T' \|$$

$$= |s'| \cdot \underbrace{\|\gamma'\|}_{=|s'|} \cdot \|T\| \cdot \|T'\| = (s')^2 \underbrace{\|T\|}_{=1} \cdot \|T'\|$$

$$\Rightarrow \|T'\| = \frac{\|\gamma' \times \gamma''\|}{(s')^2} = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^2} \Rightarrow \boxed{K = \frac{\|T'\|}{\|\gamma'\|} = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}}$$

Example 7: $r(t) = \langle t^2, t^2, t^3 \rangle$. Find a_T, a_N.

$\underline{A}$: $a_T = v'$, $a_N = v^2 \cdot \kappa$.

$r'(t) = \langle 2t, 2t, 3t^2 \rangle \Rightarrow v(t) = \sqrt{8t^2 + 9t^4} \Rightarrow a_T = \dfrac{1}{2\sqrt{8t^2+9t^4}} \cdot 16t + 36t^3$

$\Rightarrow r''(t) = \langle 2, 2, 6t \rangle$

$$= \frac{8 + 18t^2}{\sqrt{8+9t^2}}$$

$$\Rightarrow r' \times r'' = \begin{vmatrix} i & j & k \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} \overset{\text{DIY}}{=} 6t^2\,\hat{i} - 6t^2\,\hat{j}$$

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\sqrt{36t^4 + 36t^4}}{(8t^2 + 9t^4)^{3/2}} = \frac{6\sqrt{2}\,t^2}{(8t^2 + 9t^4)^{3/2}}$$

$$\Rightarrow a_N = (8t^2 + 9t^4) \cdot \frac{6\sqrt{2}\,t^2}{(8t^2+9t^4)^{3/2}} = \frac{6\sqrt{2}\,t^2}{\sqrt{8t^2+9t^4}}$$

In general, $\boxed{a_N = v^2 \kappa = v^2 \cdot \dfrac{\|r' \times r''\|}{v^3} = \dfrac{\|r' \times r''\|}{\|r'\|}}$