Quiz 4 releases 7/8, due 7/9, will be optional. However, if you take quiz 4 then your lowest quiz score is dropped.

HW 6 (on last week's material) & HW 7 on this week release today & are due 7/11 11:59pm

Quiz 3, midterm 2 graded by end of 7/8

Curl, Divergence, Green's Theorem

Bug bounty program: you can earn extra credit (typically 0.5-3% of final HW grade) per mistake in typed* materials found after 7/9 6pm. On 7/9, more info will be announced.

Lecture for 7/7

*writings with stylus excluded

Final must be 7/12. If you can't make it that day, let me know ASAP. There will not be any delay for

finals: Exam opens 7/12 2pm, format (Zoom or async) TBD.
(because registrar insists all finals for 5 week SS1 are on 7/12)

Curl and Divergence

Consider ∇ as a operator and $F = (A, B, C)$

- Define div $F = \nabla \cdot F = A_x + B_y + C_z$
- Define curl $F = \nabla \times F = (C_y - B_z, A_z - C_x, B_x - A_y)$
  - Only for vector fields on R^3
- ~~Let's see what happens to scalar functions~~
  - [redacted]
  - [redacted]

  ~~https://en.wikipedia.org/wiki/Divergence~~

$$\nabla f = (f_x, f_y, f_z), \quad \nabla = (\partial_x, \partial_y, \partial_z)$$

$$(\partial_x, \partial_y, \partial_z) \cdot (A, B, C) = \partial_x A + \partial_y B + \partial_z C$$

$$0 \xrightarrow{0} \text{scalar func}$$
$$\text{grad} \downarrow \nabla$$
$$\text{vector field}$$
$$\text{curl} \downarrow \nabla \times$$
$$\text{vector field}$$
$$\text{div} \downarrow \nabla \cdot$$
$$\text{scalar func}$$
$$0 \xleftarrow{0} \text{func}$$

$(\partial_x, \partial_y, \partial_z) \times (A, B, C) = (\partial_y C - \partial_z B, \dots)$

$\nabla \cdot (\nabla \times F) = 0$

Properties of Curl and Divergence

f = scalar func., F, G = vector func.

$(fg)' = f'g + fg'$

$(f \circ g)' = (f' \circ g) \cdot g'$

- Let's see what happens to scalar functions
  - $\nabla \cdot fG = (\nabla f) \cdot G + f(\nabla \cdot G)$
  - $\nabla \times fG = \nabla f \times G + f(\nabla \times G)$

implies

$\nabla \cdot (aF + bG) = a(\nabla \cdot F) + b(\nabla \cdot G)$

$\nabla \times (aF + bG) = a(\nabla \times F) + b(\nabla \times G)$

- Let's see how curl and div interact
  - $\nabla \cdot (F \times G) = (\nabla \times F) \cdot G - F \cdot (\nabla \times G)$
    - Consequence: $\nabla \cdot (\nabla \times F) = 0$
  - $\nabla \times \nabla f = 0$
  - $\nabla \times (F \times G) = (\nabla \cdot G)F + (G \cdot \nabla)F - (\nabla \cdot F)G - (F \cdot \nabla)G$

$\nabla \times (F \cdot G) =$ not defined because $F \cdot G$ is scalar

$\times \Rightarrow$ only have vec $\times$ vec, $\cdot \Rightarrow$ only have vec $\cdot$ vec

$$G = (A, B, C, \dots)$$

$$\nabla \cdot fG = \nabla \cdot (fA, fB, fC, \dots) =$$

$$\partial_x (fA) + \partial_y (fB) + \dots =$$

$$f_x A + f A_x + f_y B + f B_y + \dots =$$

$$(f_x A + f_y B + \dots) + f(A_x + B_y + \dots)$$

$$= (f_x, f_y, \dots) \cdot (A, B, \dots) + f(\nabla \cdot G)$$

$$= \nabla f \cdot G + f(\nabla \cdot G)$$

Green's Theorem General Idea

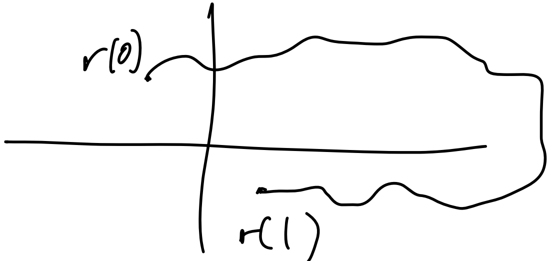
It's often useful to switch between line and double integrals

- Double to Line: you're reducing number of integrations
- Line to Double: function may be simpler to integrate

↳ usually in rigged scenarios

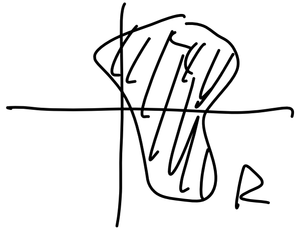
But how can we do this? Green's Theorem will tell us

$$\int_C (A\,dx + B\,dy) = \int_0^1 A(x(t), y(t))\,x'(t)\,dt + B(\dots)\,y'(t)\,dt$$



$$= \int_0^1 (\cdot \quad \cdot \quad - \quad \cdot)\,dt = \dots = \text{ans}$$

? >

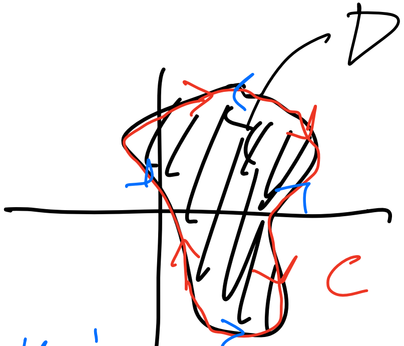


$\iint_R f\,dA = \int_?\int_? f(\text{polar or just } xy \text{ or Jacobian})\,\underbrace{dx\,dy}_{\text{convert}}$

Green's Theorem

Suppose C is a simple closed curve oriented counterclockwise. Suppose

- Further suppose C encloses D
- Further suppose Q_x, P_y are Riemann integrable



Green's Theorem: $\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$

counterclockwise

clockwise

To convert from line → double is easy, just compute Q_x & P_y

To convert from double → line is trickier. To convert $\iint_D f\,dA$, choose P & Q such that $Q_x - P_y = f$

Simple curve = no self-intersection

closed curve = start & end the same

$P=0$, $Q=\int f\,dx + g(y)$

$Q=0$, $P=-\int f(x,y)\,dy + g(x)$

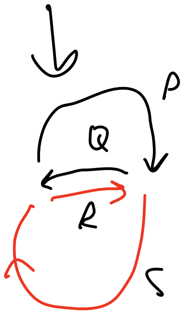
Decomposition Principle

- If GT holds on D_1 and D_2, we can consider $D = D_1 \cup D_2$
- To find integrals, break D into smaller regions where GT applies

$$-\int_{C_1} \cdots = \iint_{D_1} \cdots dA$$

$$-\int_{C_2} \cdots = \iint_{D_2} \cdots dA$$

$$-\int_{C_1 \cup C_2} \cdots = \iint_{D_1 \cup D_2} \cdots dA$$



$C_1 \cup C_2 = P \cup Q \cup R \cup S$ but $\int_Q \cdots = -\int_R \cdots$

because P & Q in opposite directions, so

$$\iint_D \cdots = -\int \cdots = -\int \cdots$$

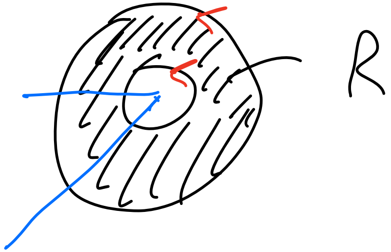
$D_1 \cup D_2$ $\quad PUS \quad \partial(D_1 \cup D_2)$

Theorem Derivation

Note: general form of Green following decomposition principle is that

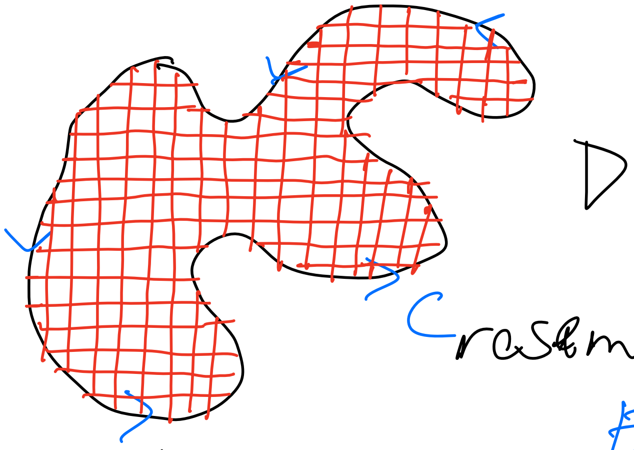
$$\int_{\partial D} \cdots = \iint_D \cdots \quad \text{for any } D$$

with ∂D a union of finitely many simple closed curves each CCW.

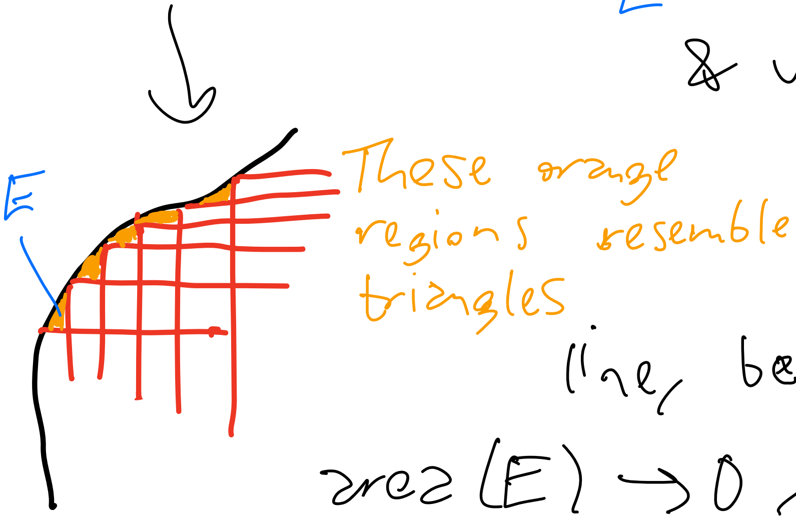


Slice R into 2 pieces with /, apply Green, combine to find $\iint_R f\,dA$

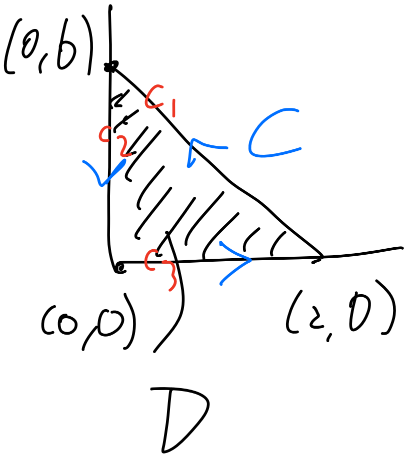
Proof Idea: break up D into smaller & smaller pieces. Prove Green on each piece to conclude Green is true on D.



Split D via very close horizontal & vertical lines. Most regions will be rectangles. Regions touching C will resemble right triangles since for any region E touching C, there are 2 horizontal & vertical lines forming the legs of the triangle, and the portion of C in ∂E is nearly a straight line, becoming a line in the limit as $\text{area}(E) \to 0$, since some parametrization of C is continuously differentiable. So by decomp princ, suffices to prove Green for rect & right triangles.



A rectangle is just 2 right triangles, so in fact we only have to consider some right triangle. WLOG the right triangle has vertices at $(0,0)$, $(a,0)$, $(0,b)$.



We show $\int_C P\,dx = -\iint_D P_y\,dA$, the proof of $\int_C Q\,dy = \iint_D Q_x\,dA$ is similar.

$$\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx$$

$\int_{C_2} P\,dx = 0$ since x is constant on C_2.

$$\int_{C_3} P\,dx = \underline{\int_0^a P(t,0)\,dt}$$

C_1 param by $(1-t)(a,0) + t(0,b) = ((1-t)a, tb)$, $\underline{0 \le t \le 1}$.

$x = (1-t)a \Rightarrow dx = -a\,dt$

$$\int_C P\,dx = \int_0^1 P((1-t)a, tb) \cdot -a\,dt = \underline{-a \int_0^1 P((1-t)a, tb)}.$$

Now let's solve for $\iint P_y$. Bounds on D are $0 \le y \le b$,

$0 \le x \le a$, $y \le b - \frac{b}{a}x \Rightarrow$ $0 \le x \le a$, $0 \le y \le b - \frac{b}{a}x$

$$-\iint P_y\,dA = \int_0^a \int_0^{b-\frac{b}{a}x} -P_y\,dy\,dx = \int_0^a -P \Big|_{y=0}^{y=b-\frac{b}{a}x} dx =$$

$$\underline{\int_0^a P(t, 0)\,dt} - \underline{\int_0^a P(t, b - \tfrac{b}{a}t)\,dt}$$

Green integrals will be equal after u-sub $t = (1-u)a$

So $\int_C P\,dx = \iint_D -P_y\,dA$ for this right triangle

since red & green match up, concluding the proof

Vector Forms of Green's Theorem

Pretend $F = (P, Q)$ is a vector field in R^3, with $F = (P, Q, 0)$

- $\int_C F \cdot dr = \iint_D (\nabla \times F) \cdot e_z \, dA$ where $e_z = (0, 0, 1)$
  - We shall use this later to build Stokes' Theorem
- $\int_C (F \cdot n) \, ds = \iint_D (\nabla \cdot F) \, dA$ where n is unit normal to r
  - We shall use this later to build Divergence Theorem

Practice Problems

Evaluate $\int_C (y^4-2y)\,dx - (6x-4xy^3)\,dy$ where C is the rectangle with coordinates (0,0), (6, 0), (6, 4), (0, 4) oriented clockwise

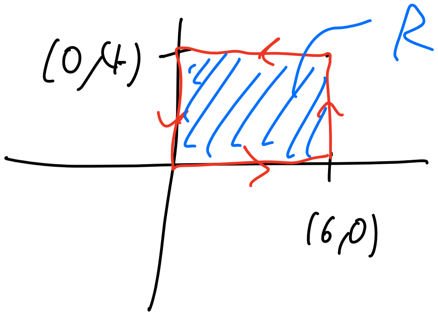
Let C be the triangle with vertices (-3, 0), (0,0), (0, 3) oriented clockwise. Verify Green's Theorem for $\int_C (xy^2+x^2)\,dx + (4x-1)\,dy$ by computing both the line integral and the corresponding double integral

Find a formula for $\nabla \times (\nabla \times F)$ and justify your claim

Scratchwork

Evaluate $\int_C (y^4-2y)\,dx - (6x-4xy^3)\,dy$ where C is the rectangle with coordinates (0,0), (6, 0), (6, 4), (0, 4) oriented clockwise

The integral is $\int_C P\,dx + Q\,dy$ with $P = y^4 - 2y$, $Q = -6x + 4xy^3$



C is a simple closed curve, P & Q are differentiable, in fact cont. diff, so Green's Theorem applies, with minus sign since C is clockwise

$Q_x - P_y = (-6 + 4y^3) - (4y^3 - 2) = -4$, so

$\int_C P\,dx + Q\,dy = -\int_R -4\,dA = 4\,\text{area}(R) = 4 \cdot 4 \cdot 6 = 96$