Directional Derivatives

Directional Derivatives and the Gradient Vector (1 of 1)

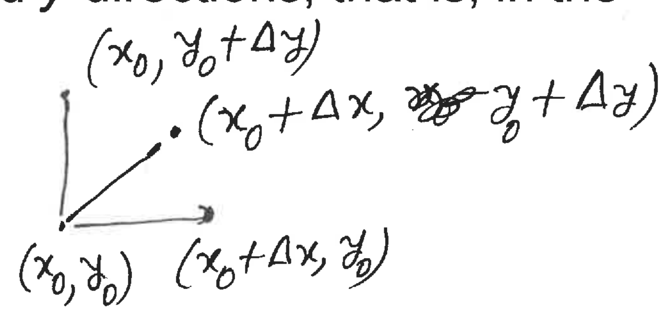
In this section we introduce a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction.

Directional Derivatives (1 of 8)

We know that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$
\begin{aligned}
f_x(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\
f_y(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}
\end{aligned} \qquad \mathbf{1}
$$

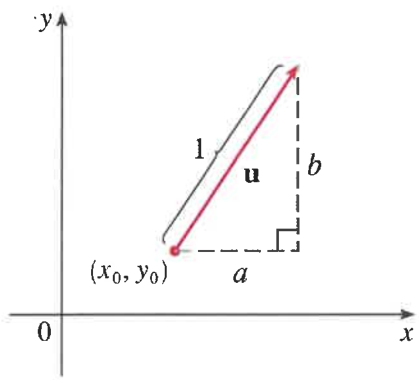
and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors **i** and **j**.



Directional Derivatives (2 of 8)

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.)

To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S.



A unit vector $\mathbf{u} = \langle a, b \rangle$

Figure 2

The vertical plane that passes through *P* in the direction of **u** intersects *S* in a curve *C*. (See Figure 3.)

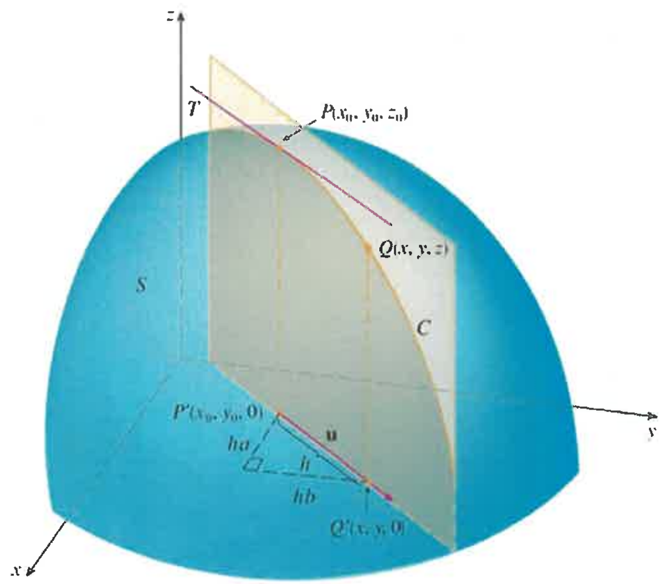


Figure 3

Directional Derivatives (4 of 8)

The slope of the tangent line *T* to *C* at the point *P* is the rate of change of *z* in the direction of **u**.

If *Q*(*x*, *y*, *z*) is another point on *C* and P', Q' are the projections of *P*, Q onto the *xy*-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to **u** and so

$$\overrightarrow{P'Q'} = h\mathrm{u} = \langle ha, hb \rangle$$

for some scalar *h*. Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Directional Derivatives (5 of 8)

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of **u**, which is called the directional derivative of f in the direction of **u**.

2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathrm{u} = \langle a, b \rangle$ is

$$D_{\mathrm{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Directional Derivatives (6 of 8)

By comparing Definition 2 with Equations 1, we see that if $\mathbf{u} = \mathbf{i} = \langle 1,0 \rangle$, then $D_{\mathbf{i}}f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0,1 \rangle$, then $D_{\mathbf{j}}f = f_y$.

In other words, the partial derivatives of *f* with respect to *x* and *y* are just special cases of the directional derivative.

Example 1

Use the weather map in Figure 1 to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

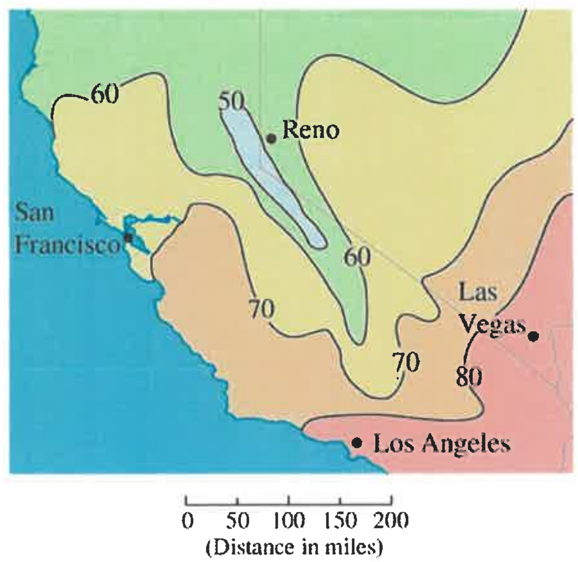


Figure 1

Example 1 – Solution (1 of 2)

We start by drawing a line through Reno toward the southeast [in the direction of $u = \frac{(i - j)}{\sqrt{2}}$; see Figure 4].

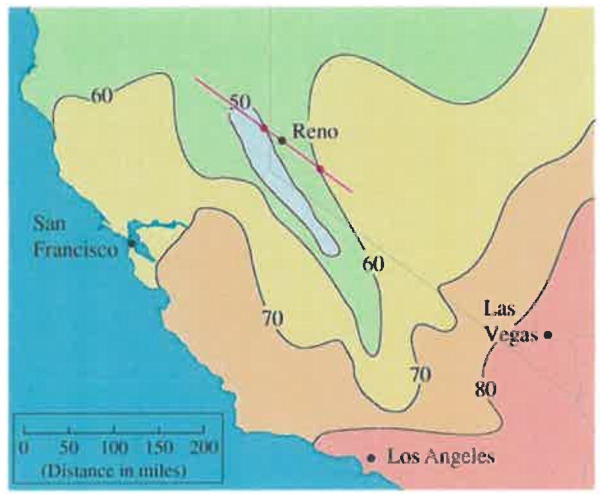


Figure 4

Example 1 – Solution (2 of 2)

We approximate the directional derivative $D_u T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and $T = 60$.

The temperature at the point southeast of Reno is $T = 60°F$ and the temperature at the point northwest of Reno is $T = 50°F$.

The distance between these points looks to be about 75 miles. So the rate of change of the temperature in the southeasterly direction is

$$D_u T \approx \frac{60 - 50}{75} = \frac{10}{75} \approx 0.13°\text{F/mi}$$

Directional Derivatives (7 of 8)

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

3 Theorem If *f* is a differentiable function of *x* and *y*, then *f* has a directional derivative in the direction of any unit vector $\mathrm{u} = \langle a, b \rangle$ and

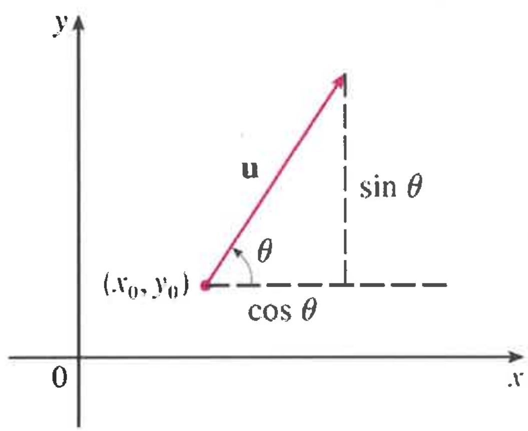
$$D_{\mathrm{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle$$

Directional Derivatives (8 of 8)

If the unit vector **u** makes an angle θ with the positive *x*-axis (as in Figure 5), then we can write $\mathrm{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

$$\textbf{6} \quad D_{\mathrm{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$



A unit vector $\mathrm{u} = \langle \cos\theta, \sin\theta \rangle$

Figure 5

Example

$$f(x,y) = \sin\left(\frac{x}{y}\right), \boldsymbol{u} = \langle 3,4 \rangle, \qquad find\ D_{\hat{\mathbf{u}}} f(x,y)\ and D_{\hat{\mathbf{u}}} f(1,1).$$

$|\vec{u}| = \sqrt{3^2+4^2} = \sqrt{9+16} \cancel{= \sqrt{9}+\sqrt{16}} = \sqrt{25} = 5$

$\hat{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{y}\right) \cdot \frac{1}{y}$

$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{y}\right) \cdot \left(-\frac{x}{y^2}\right)$

$D_{\hat{u}} f(x,y) = \frac{\partial f}{\partial x}(x,y)\frac{3}{5} + \frac{\partial f}{\partial y}(x,y)\frac{4}{5}$

$= \frac{3}{5}\cos\left(\frac{x}{y}\right)\frac{1}{y} - \frac{4}{5}\cos\left(\frac{x}{y}\right)\frac{x}{y^2}$

$\cancel{D_{\hat{u}} f(x,y) = \frac{3}{}}$

$D_{\hat{u}} f(1,1) = \frac{3}{5}\cos(1) - \frac{4}{5}\cos(1)$

The Gradient Vector

$f(x,y)$, $\hat{u} = \langle a, b \rangle$, (x_0, y_0)

$$D_{\hat{u}} f(x_0, y_0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$

$$= \underbrace{\langle f_x(x_0,y_0),\ f_y(x_0,y_0) \rangle}_{\nabla f(x_0,y_0)} \cdot \langle a, b \rangle \rightarrow \hat{u}$$

$\nabla f(x_0, y_0) \rightarrow$ grad

The Gradient Vector (1 of 3)

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\textbf{7} \qquad \begin{aligned} D_{\mathbf{u}}f(x,y) &= f_x(x,y)a + f_y(x,y)b \\ &= \langle f_x(x,y),\ f_y(x,y)\rangle \cdot \langle a,b\rangle \\ &= \langle f_x(x,y),\ f_y(x,y)\rangle \cdot \mathbf{u} \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well.

So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f, which is read "del f").

The Gradient Vector (2 of 3)

8 Definition If *f* is a function of two variables *x* and *y*, then the **gradient** of *f* is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathrm{i} + \frac{\partial f}{\partial y}\mathrm{j} = f_x(x,y)\hat{i} + f_y(x,y)\hat{j}$$

$$\underline{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} \cancel{+ \frac{\partial}{\partial z}}$$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Example 3

If $f(x,y) = \sin x + e^{xy}$, then

$$\begin{aligned} \nabla f(x,y) &= \langle f_x, f_y \rangle \\ &= \langle \cos x + ye^{xy},\ \mathrm{x}e^{xy} \rangle \end{aligned}$$

and $$\nabla f(0,1) = \langle 2,0 \rangle$$

The Gradient Vector (3 of 3)

With the notation for the gradient vector, we can rewrite Equation 7 for the directional derivative of a differentiable function as

$$9 \quad D_{\mathrm{u}} f(x, y) = \nabla f(x, y) \cdot \mathrm{u}$$

This expresses the directional derivative in the direction of a unit vector **u** as the scalar projection of the gradient vector onto **u**.

Functions of Three Variables

Functions of Three Variables (1 of 4)

For functions of three variables we can define directional derivatives in a similar manner.

Again $D_{\mathbf{u}}f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector **u**.

10 Definition The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathrm{u} = \langle a, b, c \rangle$ is

$$D_{\mathrm{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Functions of Three Variables (2 of 4)

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

$$\mathbf{11} \quad D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$.

This is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + t\,\mathbf{u}$ and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

Functions of Three Variables (3 of 4)

If $f(x, y, z)$ is differentiable and $\mathrm{u} = \langle a, b, c \rangle$, then

$$\textbf{12} \quad D_{\mathrm{u}} f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c$$

For a function *f* of three variables, the **gradient vector**, denoted by ∇f or **grad** *f*, is

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

$$\nabla f = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

or, for short,

$$\textbf{13} \quad \nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathrm{i} + \frac{\partial f}{\partial y}\mathrm{j} + \frac{\partial f}{\partial z}\mathrm{k}$$

$$f = x^2 + y^2 + z^2$$

$$f(x,y) = x^2 + y^2 + 0.z$$

$$\nabla f(x,y,z) = 2x\hat{i} + 2y\hat{j} + 0 \cdot 1$$

Functions of Three Variables (4 of 4)

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

$$\mathbf{14} \quad D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

Example 5

If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at (1, 3, 0) in the direction of $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

Solution:

(a) The gradient of f is

$$
\begin{aligned}
\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\
&= \langle \sin y\,z, xz \cos y\,z, xy \cos y\,z \rangle
\end{aligned}
$$

Example 5 – Solution

(b) At (1, 3, 0) we have $\nabla f(1,3,0) = \langle 0,0,3 \rangle$.

The unit vector in the direction of **v** = **i** + 2**j** − **k** is

$$u = \frac{1}{\sqrt{6}}i + \frac{2}{\sqrt{6}}j - \frac{1}{\sqrt{6}}k$$

Therefore Equation 14 gives

$$\begin{aligned} D_u f(1,3,0) &= \nabla f(1,3,0) \cdot u \\ &= 3k \cdot \left(\frac{1}{\sqrt{6}}i + \frac{2}{\sqrt{6}}j - \frac{1}{\sqrt{6}}k\right) \\ &= 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}} \end{aligned}$$

Example

Let $f(x,y,z) = \frac{e^{xyz}}{xyz}$. What is $\nabla f(x,y,z)$ and $\nabla f(1,1,1)$?

$$\nabla f(x,y,z) = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\rangle$$

$$f_x(x,y,z) = \cancel{yz\frac{\partial}{\partial x}e} \cdot \frac{e^{xyz}}{-x^2yz} + yz\frac{e^{xyz}}{xyz}$$

$f(x) = xyz$

$g(x) = \frac{1}{xyz}$

$$= \frac{e^{xyz}}{xyz}\left(yz - \frac{1}{x}\right)$$

$$f_y(x,y,z) = \frac{xyz(xze^{xyz}) - e^{xyz}xz}{(xyz)^2} = \frac{xyze^{xyz} - xze^{xyz}}{xy^2z}$$

$$= \frac{e^{xyz}}{xyz}\left(zx - \frac{1}{y}\right)$$

$$f_z(x,y,z) = \frac{e^{xyz}}{xyz}\left(xy - \frac{1}{z}\right)$$

$$\nabla f(x,y,z) = \frac{e^{xyz}}{xyz}\left\langle yz - \frac{1}{x},\ zx - \frac{1}{y},\ xy - \frac{1}{z}\right\rangle$$

$$\nabla f(1,1,1) = \langle 0,0,0\rangle$$

Maximizing the Directional Derivative

$$D_{\hat{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \hat{u}$$

$$= |\nabla f(x_0, y_0, z_0)| \, |\hat{u}| \cos\theta$$

$$= |\nabla f(x_0, y_0, z_0)| \cos\theta$$

$$\leq |\nabla f(x_0, y_0, z_0)|$$

$\theta = 0$ $$\hat{u} = \frac{\nabla f(x_0, y_0, z_0)}{|\nabla f(x_0, y_0, z_0)|}$$

Maximizing the Directional Derivatives (1 of 1)

Suppose we have a function *f* of two or three variables and we consider all possible directional derivatives of *f* at a given point.

These give the rates of change of *f* in all possible directions.

We can then ask the questions: In which of these directions does *f* change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

15 Theorem Suppose *f* is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x}_0)$ is $|\nabla f(\mathbf{x}_0)|$ and occurs when **u** has the same direction as the gradient vector $\nabla f(\mathbf{x}_0)$.

$\vec{x}_0 = \langle x_0, y_0, z_0 \rangle$

Example 6

(a) If $f(x,y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(0.5, 2)$

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

$\overrightarrow{PQ} = \langle 0.5-2,\ 2-0\rangle = \langle -1.5, 2\rangle$

Solution:

(a) We first compute the gradient vector:

$$\begin{aligned} \nabla f(x,y) &= \langle f_x, f_y\rangle = \langle e^y, xe^y\rangle \\ \nabla f(2,0) &= \langle 1,2\rangle \end{aligned}$$

$$|\vec{PQ}| = \sqrt{\left(-\frac{3}{2}\right)^2 + (2)^2} = \sqrt{\frac{9}{4} + 4}$$
$$= \sqrt{\frac{9+16}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

Example 6 – Solution

The unit vector in the direction of $\vec{PQ} = \left\langle -\frac{3}{2}, 2 \right\rangle$ is $\mathrm{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$, so the rate of change of f in the direction from P to Q is

$$\begin{aligned} D_{\mathrm{u}} f(2,0) &= \nabla f(2,0) \cdot \mathrm{u} \\ &= \langle 1,2 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle \\ &= 1\left(-\frac{3}{5}\right) + 2\left(\frac{4}{5}\right) = 1 \end{aligned}$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2,0) = \langle 1,2 \rangle$. $\Rightarrow \hat{v} = \dfrac{\nabla f(2,0)}{|\nabla f(2,0)|} = \dfrac{\langle 1,2 \rangle}{\sqrt{1^2+2^2}} = \left\langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle$

The maximum rate of change is

$$|\nabla f(2,0)| = |\langle 1,2 \rangle| = \sqrt{5}$$

Tangent Planes to Level Surfaces

Tangent Planes to Level Surfaces (1 of 6)

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S.

Let C be any curve that lies on the surface S and passes through the point P. Recall that the curve C is described by a continuous vector function $\mathrm{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Let t_0 be the parameter value corresponding to P; that is, $\mathrm{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$\textbf{16} \quad F(x(t), y(t), z(t)) = k$$

$z = f(x, y)$ $\quad$ $e^{xyz} = xyz$

Tangent Planes to Level Surfaces (2 of 6)

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

$$\textbf{17} \quad \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathrm{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathrm{r}'(t) = 0$$

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

$$\textbf{18} \quad \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

Equation 18 says that *the gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P*. (See Figure 10.)

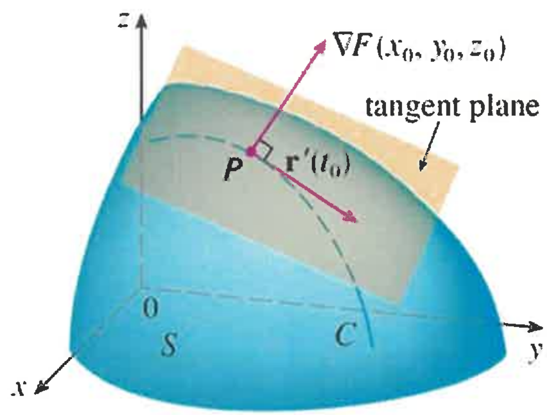


Figure 10

Tangent Planes to Level Surfaces (4 of 6)

If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

Using the standard equation of a plane, we can write the equation of this tangent plane as

$$\textbf{19} \quad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$F(x, y, z) = k$

~~$e^{xyz} - xyz = 0$, $P(0, 0, e)$~~

~~$e^{x+y+z} = xyz = 0$, $P(0, 0, 1)$ $P(1, 1, e)$~~

~~$F_x(x_0, y$~~

$\vec{n} = \langle F_x, F_y, F_z \rangle = \nabla F(x_0, y_0, z_0)$

Tangent Planes to Level Surfaces (5 of 6)

The **normal line** to *S* at *P* is the line passing through *P* and perpendicular to the tangent plane.

The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, its symmetric equations are

$$\textbf{20} \quad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Example 8

Find the equations of the tangent plane and normal line to ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

at the point $(-2, 1, -3)$.

Solution:

The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Example 8 – Solution

Therefore we have

$$F_x(x,y,z) = \frac{x}{2} \qquad F_y(x,y,z) = 2y \qquad F_z(x,y,z) = \frac{2z}{9}$$

$$F_x(-2,1,-3) = -1 \quad F_y(-2,1,-3) = 2 \quad F_z(-2,1,-3) = -\frac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at (−2, 1, −3) as

$$-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}$$

Tangent Planes to Level Surfaces (6 of 6)

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$\begin{aligned} F_x(x_0, y_0, z_0) &= f_x(x_0, y_0) \\ F_y(x_0, y_0, z_0) &= f_y(x_0, y_0) \\ F_z(x_0, y_0, z_0) &= -1 \end{aligned}$$

so Equation 19 becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Significance of the Gradient Vector

Significance of the Gradient Vector (1 of 7)

We first consider a function *f* of three variables and a point $P(x_0, y_0, z_0)$ in its domain.

On the one hand, we know from Theorem 15 that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of *f*.

15 Theorem Suppose *f* is a differentiate function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when **u** has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Significance of the Gradient Vector (2 of 7)

On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P. (Refer to Figure 10.)

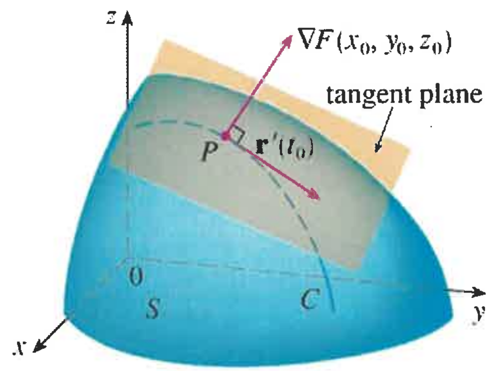


Figure 10

These two properties are quite compatible intuitively because as we move away from P on the level surface S, the value of f does not change at all.

Significance of the Gradient Vector (3 of 7)

So it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain.

Again the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f. Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P.

Significance of the Gradient Vector (4 of 7)

Again this is intuitively plausible because the values of *f* remain constant as we move along the curve. (See Figure 12.)

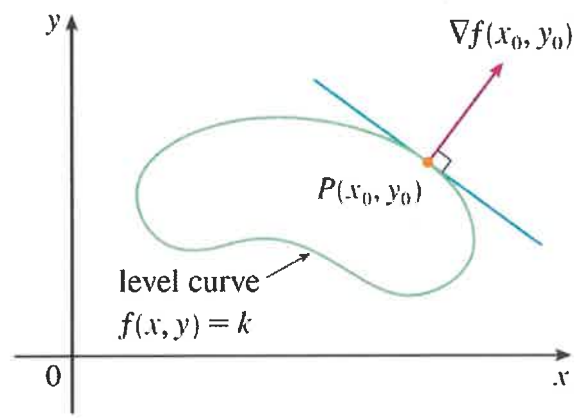


Figure 12

Significance of the Gradient Vector (5 of 7)

Properties of the Gradient Vector Let f be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x}) \neq \mathbf{0}$.

- The directional derivative of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_u f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.

- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that maximum rate of change is $|\nabla f(\mathbf{x})|$.

- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.

Significance of the Gradient Vector (6 of 7)

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn as in Figure 13 by making it perpendicular to all of the contour lines.

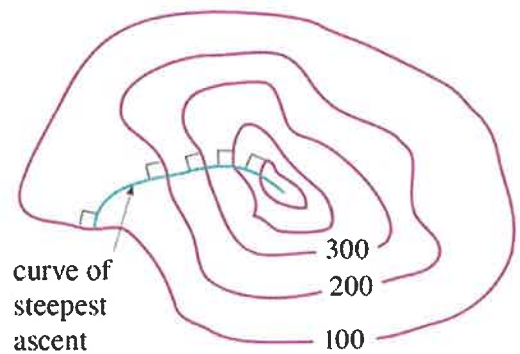


Figure 13

Significance of the Gradient Vector (7 of 7)

Mathematical software can plot sample gradient vectors, where each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b). Figure 14 shows such a plot (called a *gradient vector field*) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f.

As expected, the gradient vectors point "uphill" and are perpendicular to the level curves.

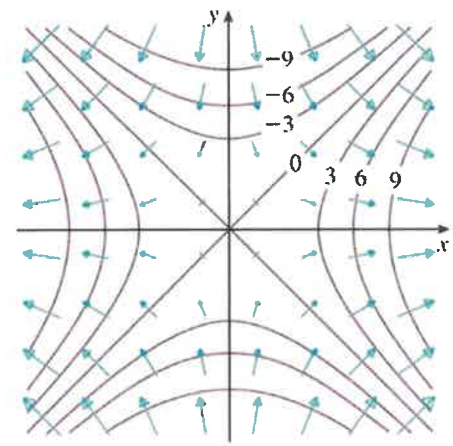


Figure 14