Logistical announcements

1: Lecture quiz 2 has been graded. Grades are not released yet, Arnab is waiting for makeups to fully complete.

2: Feedback survey coming end of week

DW5 #2: $r(t) = \langle t, t^2, 4\rangle$, find T, N, K, the unit tangent, unit normal, and curvature.

(a): $T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle 1, 2t, 0\rangle}{\sqrt{1+4t^2+0}} = \frac{1}{\sqrt{4t^2+1}}\langle 1, 2t, 0\rangle$.

(b): 2 approaches. Slower: bring sqrt inside components & differentiate. Faster: use product rule for $\frac{d}{dt}(f\vec{v})$

$T'(t) = \frac{-4t}{(1+4t^2)^{3/2}}\langle 1, 2t, 0\rangle + \frac{1}{\sqrt{4t^2+1}}\langle 0, 2, 0\rangle =$

$\boxed{\frac{d}{dt}(1+4t^2)^{-1/2} = -\frac{1}{2}\cdot 8t\,(1+4t^2)^{-3/2} = \frac{-4t}{(1+4t^2)^{3/2}}}$

$\frac{1}{(1+4t^2)^{3/2}}\left(-4t\langle 1, 2t, 0\rangle + (1+4t^2)\langle 0, 2, 0\rangle\right) =$

$\frac{1}{(1+4t^2)^{3/2}}\langle -4t, 2, 0\rangle = \frac{2}{(1+4t^2)^{3/2}}\langle -2t, 1, 0\rangle$

$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\cancel{\frac{2}{(1+4t^2)^{3/2}}}\langle -2t, 1, 0\rangle}{\cancel{\frac{2}{(1+4t^2)^{3/2}}}\|\langle -2t, 1, 0\rangle\|} = \frac{\langle -2t, 1, 0\rangle}{\sqrt{4t^2+1^2+0^2}}$

$= \frac{1}{\sqrt{4t^2+1}}\langle -2t, 1, 0\rangle$

Note: good way to check your work is check $\|T\| = \|N\| = 1$. If not, you made a mistake. If they do $= 1$, still could have mistake but less likely.

(c): Use $K = \dfrac{\|T'(t)\|}{\|r'(t)\|} = \dfrac{\frac{2}{(1+4t^2)^{3/2}} \cancel{\sqrt{4t^2+1^2+0^2}}}{\cancel{\sqrt{4t^2+1}}}$

Note: K is a scalar, but T, N, B, r, v, a are all vectors

$$= \frac{2}{(1+4t^2)^{3/2}}$$

Note: Don't try to gamble with assuming that your unsimplified form will be ok for quizzes and exams. Simplify as much as possible to be safe.

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

$\int fg \neq (\int f)(\int g)$

However, don't do fake simplifications.

$4t^2 + 4t^4 + 1 = (2t^2+1)^2$

3: $r(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$.

$r'(t) = \langle 2t, 2t^2, 1 \rangle$.

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{4t^2+4t^4+1}} =$$

$$\frac{1}{2t^2+1} \langle 2t, 2t^2, 1 \rangle.$$

$$T'(t) = \frac{-4t}{(2t^2+1)^2} \langle 2t, 2t^2, 1 \rangle + \frac{1}{2t^2+1} \langle 2, 4t, 0 \rangle =$$

$$\frac{1}{(2t^2+1)^2} \left(-4t \langle 2t, 2t^2, 1 \rangle + (2t^2+1) \langle 2, 4t, 0 \rangle \right)$$

$$= \frac{2}{(2t^2+1)^2}\langle 1-2t^2,\ 2t,\ -2t\rangle .$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\cancel{\frac{2}{(2t^2+1)^2}}\langle 1-2t^2, 2t, -2t\rangle}{\cancel{\frac{2}{(2t^2+1)^2}}\sqrt{(1-2t^2)^2+8t^2}}$$

$$(1-2t^2)^2+8t^2 = 4t^4+4t^2+1 = (2t^2+1)^2$$

$$= \frac{1}{2t^2+1}\langle 1-2t^2, 2t, -2t\rangle .$$

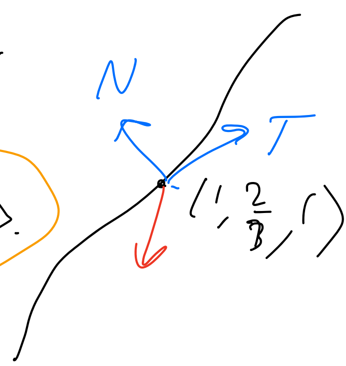
$$B(t) = T(t)\times N(t) = \frac{1}{(2t^2+1)^2}\left(\langle 2t, 2t^2, 1\rangle \times \langle 1-2t^2, 2t, -2t\rangle\right)$$

$$\begin{vmatrix} i & j & k \\ 2t & 2t^2 & 1 \\ 1-2t^2 & 2t & -2t \end{vmatrix} = \frac{1}{(2t^2+1)^2}\langle -4t^3-2t,\ 2t^2+1,\ 2t^2+4t^4\rangle$$

j comp: $-(-4t^2-(1-2t^2))$
$= -(-2t^2-1)$
$= 2t^2+1$

k comp: $4t^2-2t^2(1-2t^2) =$
$2t^2+4t^4$

To find B at $(1, \frac{2}{3}, 1)$, note that $r(1) = (1, \frac{2}{3}, 1)$, so we need to plug in $t=1$.



Ans $= B(1) = \frac{1}{9}\langle -6, 3, 6\rangle = \frac{1}{3}\langle -2, 1, 2\rangle.$

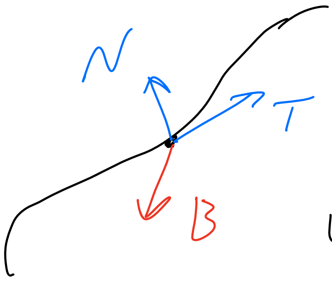
#4: Find normal & osculating planes of curve of intersection of $x=y^2$ & $z=x^2$ at point $(1,1,1)$.

Step 1: find the curve of intersection $r(t)$.

Set $y=t$ because there is nothing in terms of t and we need to get started with some variable.
Then $x=y^2=t^2$ and $z=x^2=(t^2)^2=t^4$. So $r(t)=(t^2, t, t^4)$. Note $(1,1,1)=r(1)$.

Step 2: find normal & osculating planes from curve.

Recall osculating plane is plane containing tangent & normal vector. Normal plane is the plane whose normal vector is the tangent to the curve.



So for osculating plane, one normal vector to plane is $T \times N$.

For normal plane, a normal vector is T, and in fact you can just take $r'(t)$.

$r(t)=(t^2, t, t^4)$
$r'(t)=(2t, 1, 4t^3)$
$r'(1)=(2,1,4)$

Normal plane: passes through $(1,1,1)$ & has $(2,1,4)$ as normal vector, so plane equation is $\boxed{2(x-1)+(y-1)+4(z-1)=0}$

Osculating plane: passes through $(1,1,1)$ & has $T(1)\times N(1) = \langle a,b,c\rangle$ (unknown, but you can find) as normal vector, so equation is $\boxed{a(x-1)+b(y-1)+c(z-1)=0}$

7: $a(t)=\langle t, e^t, e^{-t}\rangle$, find r given $v(0)=\langle 0,0,1\rangle$, $r(0)=\langle 0,1,1\rangle$.

$$\begin{tikzcd} r \arrow[d, bend left, "\frac{d}{dt}"] \\ v \arrow[u, bend left, "\int"] \arrow[d, bend left, "\frac{d}{dt}"] \\ a \arrow[u, bend left, "\int"] \end{tikzcd}$$

$$v(t) = \int \langle t, e^t, e^{-t} \rangle \, dt$$

$$= \left\langle \frac{t^2}{2}, e^t, -e^{-t} \right\rangle + \vec{c}$$

$$\langle 0, 0, 1 \rangle = v(0) =$$

$$\langle 0, 1, -1 \rangle + \vec{c}$$

$$\vec{c} = \langle 0,0,1 \rangle - \langle 0,1,-1 \rangle = \langle 0,-1,2 \rangle.$$

$$v(t) = \left\langle \frac{t^2}{2}, e^t - 1, -e^{-t} + 2 \right\rangle.$$

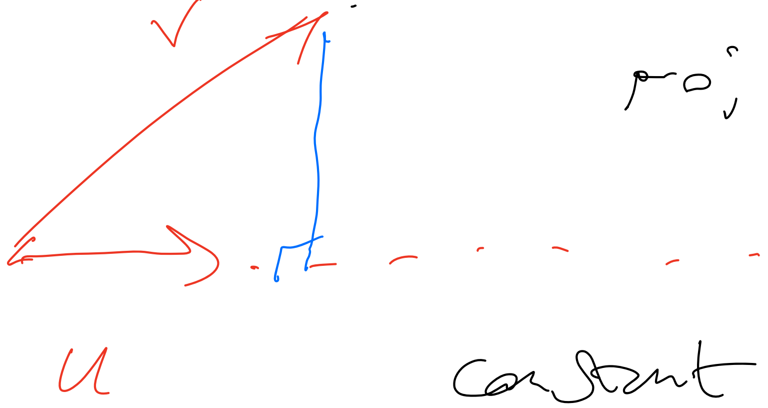
$$r(t) = \int \left\langle \frac{t^2}{2}, e^t - 1, -e^{-t} + 2 \right\rangle dt$$

$$= \left\langle \frac{t^3}{6}, e^t - t, e^{-t} + 2t \right\rangle + \vec{c}$$

$$\langle 0,1,1 \rangle = r(0) = \langle 0, 1, 1 \rangle + \vec{c}$$

$$\Longrightarrow \vec{c} = \langle 0, 0, 0 \rangle,$$

$$r(t) = \left\langle \frac{t^3}{6}, e^t - t, e^{-t} + 2t \right\rangle$$



proj of v onto u = cu for some constant c.

proj of v onto u = $\dfrac{u \cdot v}{\|u\|^2} u$

$$\text{proj}_u (cu) = cu$$

$$\frac{u \cdot cu}{\|u\|^2} u = \frac{c \cancel{\|u\|^2}}{\cancel{\|u\|^2}} u = cu$$

$$u, v \xrightarrow{\text{scalar proj}} \frac{u \cdot v}{\|u\|}$$

direct

mult by unit vector

$$\frac{u \cdot v}{\|u\|^2} u = \frac{u \cdot v}{\|u\|} \frac{u}{\|u\|}$$

Note: $\langle i, 5j, 3k \rangle$ not proper syntax. $i + 5j + 3k$ and $\langle 1, 5, 3 \rangle$

are proper, but no mix & match.

$$\frac{1}{2}\langle 2,4,6\rangle \xrightarrow{\text{simplify}} \langle 1,2,3\rangle.$$

6: Find tangential & normal components of acceleration for $r(t) = \langle t^2+1, t^3, 0\rangle, t \geq 0$

$r'(t) = \langle 2t, 3t^2, 0\rangle$

$r''(t) = \langle 2, 6t, 0\rangle$

$\|r'(t)\| = \|t\langle 2,3t,0\rangle\| = t\sqrt{4+9t^2+0^2}$

$= t\sqrt{4+9t^2}$

Recall: $a_T = \dfrac{r'\cdot r''}{\|r'\|}$, $\boxed{a_N = \dfrac{\|r'\times r''\|}{\|r'\|}}$.

If you forget these formulas, note that

$a = a_T T + a_N N \Rightarrow \|a\|^2 = a_T^{\,2} + a_N^{\,2} \Rightarrow$

$\boxed{a_N = \sqrt{\|a\|^2 - a_T^{\,2}}}$ or $a_T = \sqrt{\|a\|^2 - a_N^{\,2}}$.

$r'\cdot r'' = \langle 2t, 3t^2, 0\rangle \cdot \langle 2, 6t, 0\rangle = 4t + 18t^3$

$\|r'\| = t\sqrt{4+9t^2}$

$a_T = \dfrac{r'\cdot r''}{\|r'\|} = \dfrac{2t(2+9t^2)}{t\sqrt{4+9t^2}} = \dfrac{2(2+9t^2)}{\sqrt{4+9t^2}}$.

2 ways of finding a_N (circled in red)

$$\begin{vmatrix} i & j & k \\ 2 & 3t & 0 \\ 1 & 3t & 0 \end{vmatrix}$$

$$r' \times r'' = 2t\left(\langle 2, 3t, 0\rangle \times \langle 1, 3t, 0\rangle\right)$$

$$= 2t\langle 0, 0, 3t\rangle = \langle 0, 0, 6t^2\rangle$$

$$\Rightarrow \|r' \times r''\| = 6t^2$$

$$\Rightarrow a_N = \frac{\|r' \times r''\|}{\|r'\|} = \frac{6t^2}{t\sqrt{4+9t^2}} = \frac{6t}{\sqrt{4+9t^2}}$$

Small Shortcut: $\left(\langle 1, 3t, 0\rangle + \langle 1, 0, 0\rangle\right) \times \langle 1, 3t, 0\rangle$

$$= \cancel{\langle 1, 3t, 0\rangle \times \langle 1, 3t, 0\rangle} + \langle 1, 0, 0\rangle \times \langle 1, 3t, 0\rangle$$

$$= 0 + 3t = 3t$$