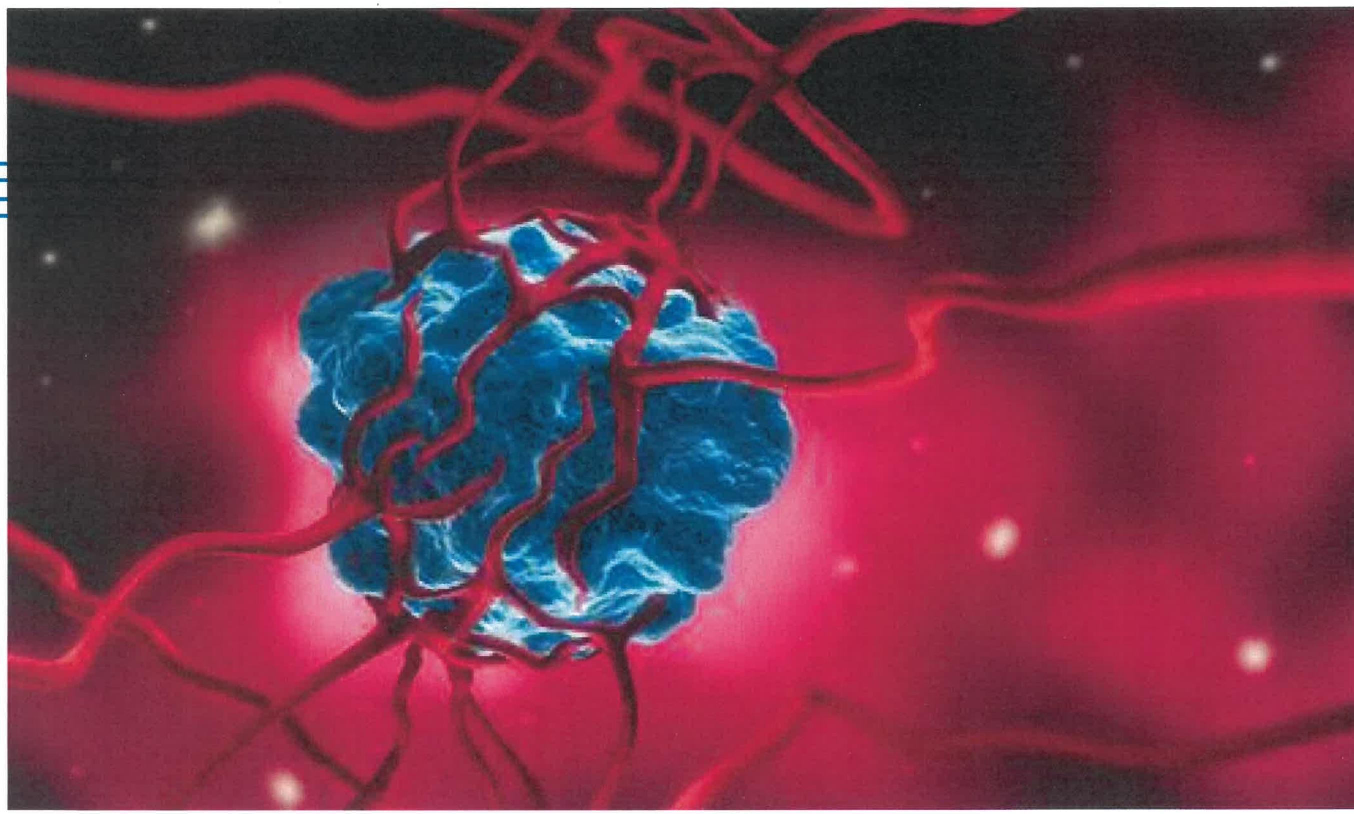
15 Multiple Integrals

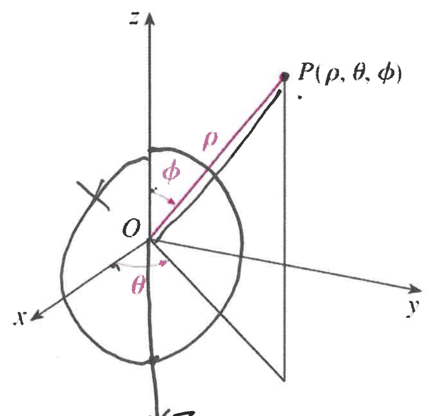


15.8 Triple Integrals in Spherical Coordinates

Spherical Coordinates

Spherical Coordinates (1 of 6)

The **spherical coordinates** (ρ, θ, φ) of a point *P* in space are shown in Figure 1 where $\rho = |OP|$ is the distance from the origin to *P*, θ is the same angle as in cylindrical coordinates, and φ is the angle between the positive *z*-axis and the line segment *OP*.



The spherical coordinates of a point

Figure 1

Spherical Coordinates (2 of 6)

Note that

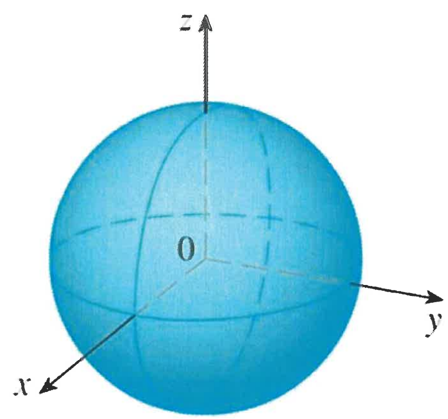
$$\rho \geq 0 \quad \underline{0 \leq \phi \leq \pi} \quad 0 \leq \theta \leq 2\pi$$

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

Spherical Coordinates (3 of 6)

For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure 2); this is the reason for the name "spherical" coordinates.

$x^2 + y^2 + z^2 = c^2$

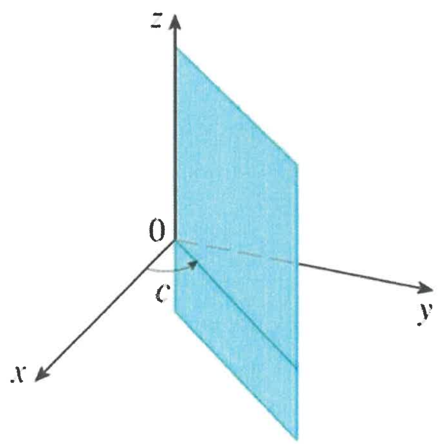


$\rho = c$, a sphere

Figure 2

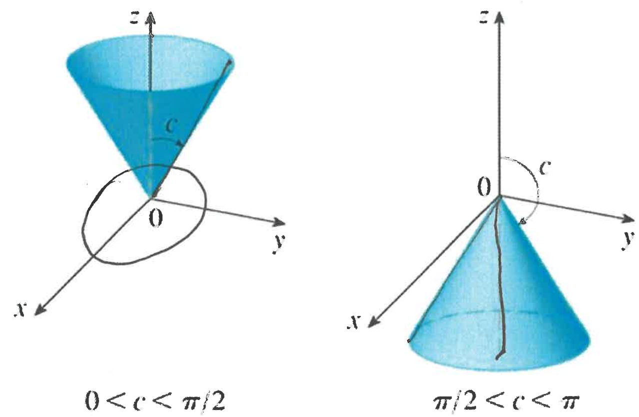
Spherical Coordinates (4 of 6)

The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 3), and the equation $\varphi = c$ represents a half-cone with the z-axis as its axis (see Figure 4).



$\theta = c$, a half-plane

Figure 3



$\varphi = c$, a half-cone

Figure 4

Spherical Coordinates (5 of 6)

The relationship between rectangular and spherical coordinates can be seen from Figure 5.

From triangles *OPQ* and OPP' we have

$$z = \rho\cos\phi \quad r = \rho\sin\phi$$

However,

$$x = r\cos\theta \quad y = r\sin\theta$$

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$$\frac{r}{z} = \frac{\sin\phi}{\cos\phi} = \tan\phi$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{r}{z}\right) = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right), \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

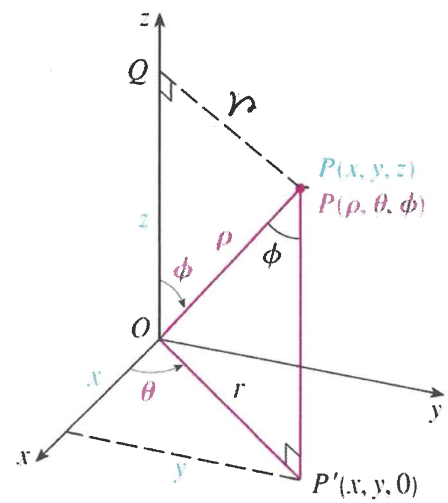


Figure 5

Spherical Coordinates (6 of 6)

But $x = r \cos \theta$ and $y = r \sin \theta$, so to convert from spherical to rectangular coordinates, we use the equations

1 $\quad x = \rho \sin \phi \cos\theta \quad y = \rho \sin \phi \sin\theta \quad z = \rho \cos \phi$

Also, the distance formula shows that

2 $\quad \rho^2 = x^2 + y^2 + z^2$

We use this equation in converting from rectangular to spherical coordinates.

List of all equations needed for conversion

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

Example 1

Convert from Cartesian Coordinate $(x, y, z) = (-1, 1, \sqrt{\frac{2}{3}})$ to spherical coordinate (ρ, θ, φ).

$$\rho^2 = x^2 + y^2 + z^2.$$

$$\Rightarrow \rho^2 = (-1)^2 + 1^2 + \left(\sqrt{\frac{2}{3}}\right)^2 = 1 + 1 + \frac{2}{3} = \frac{8}{3}$$

$$\Rightarrow \rho = \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(-1) = \cancel{\frac{7\pi}{4}} - \tan^{-1}(1) = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{\frac{2}{3}}}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$\left(2\sqrt{\frac{2}{3}}, \frac{7\pi}{4}, \frac{\pi}{3}\right)_S$$

Exercise for class

Convert from spherical Coordinate $\left(2,\frac{\pi}{2},\frac{\pi}{4}\right)_{S}$ to Cartesian coordinate (x,y,z).

$x = \rho \sin\phi \cos\theta = 0$

$y = \rho \sin\phi \sin\theta = 2 \times \frac{1}{\sqrt{2}} \times 1 = \sqrt{2}$

$z = \rho \cos\phi = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$

$(0, \sqrt{2}, \sqrt{2})_R$

Triple Integrals in Spherical Coordinates

Triple Integrals in Spherical Coordinates (1 of 8)

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

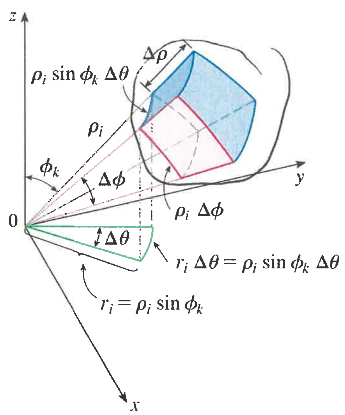
$$E = \{(\rho, \theta, \varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, c \le \varphi \le d\}$$

where $a \ge 0$ and $\beta - \alpha \le 2\pi$, and $d - c \le \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result.

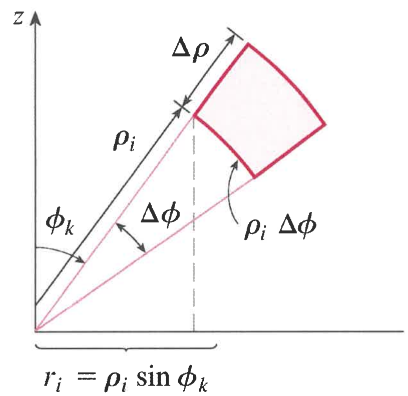
So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\varphi = \varphi_k$.

Triple Integrals in Spherical Coordinates (2 of 8)

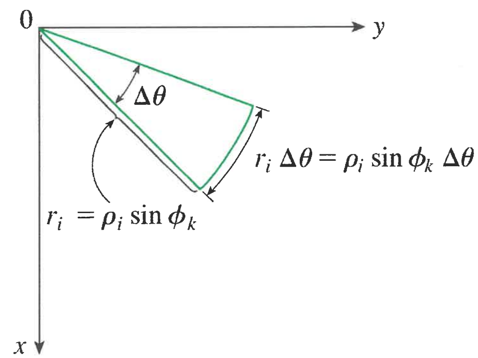
Figure 7 shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i \Delta\varphi$ (arc of a circle with radius ρ_i, angle $\Delta\varphi$), and $\rho_i \sin \varphi_k \Delta\theta$ (arc of a circle with radius $\rho_i \sin \varphi_k$, angle $\Delta\theta$).



(a) A spherical wedge



(b) Side view



(c) Top view

Figure 7

Triple Integrals in Spherical Coordinates (3 of 8)

So an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i \Delta\phi)(\rho_i \sin\phi_k \Delta\theta) = \rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi$$

In fact, it can be shown, with the aid of the Mean Value Theorem, that the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin \tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\varphi}_k)$ is some point in E_{ijk}.

Triple Integrals in Spherical Coordinates (4 of 8)

Let $\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)$ be the rectangular coordinates of this point.

Then

$$\iiint_F f(x, y, z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(\tilde{\rho}_i \sin\tilde{\varphi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\varphi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\varphi}_k\right)\tilde{\rho}_i^2 \sin\tilde{\varphi}_k\, \Delta\rho\, \Delta\theta\, \Delta\varphi$$

Triple Integrals in Spherical Coordinates (5 of 8)

But this sum is a Riemann sum for the function

$$F(\rho, \theta, \phi) = f(\rho \sin\phi \ \cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi$$

Consequently, we have arrived at the following **formula for triple integration in spherical coordinates**.

3 $$\iiint\limits_E f(x,y,z)\, dv = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \ \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

where *E* is a spherical wedge given by

$$E = \{(\rho, \theta, \varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, c \le \varphi \le d\}$$

Triple Integrals in Spherical Coordinates (6 of 8)

Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

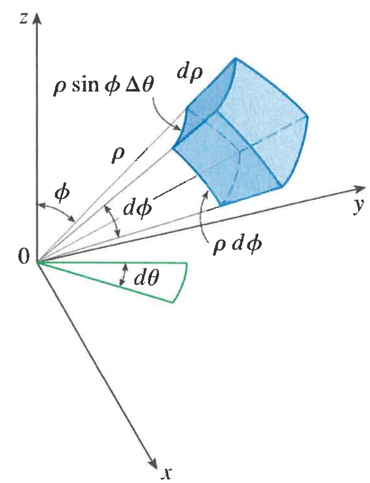
$$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$$

using the appropriate limits of integration, and replacing *dv* by $\rho^2 \sin\phi \ d\rho \, d\theta \, d\phi$.

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \quad \theta = \arctan(\frac{y}{x})$$

Triple Integrals in Spherical Coordinates (7 of 8)

This is illustrated in Figure 8.



Volume element in spherical coordinates: $dv = \rho^2 \sin\phi\ d\rho\, d\theta\, d\phi$.

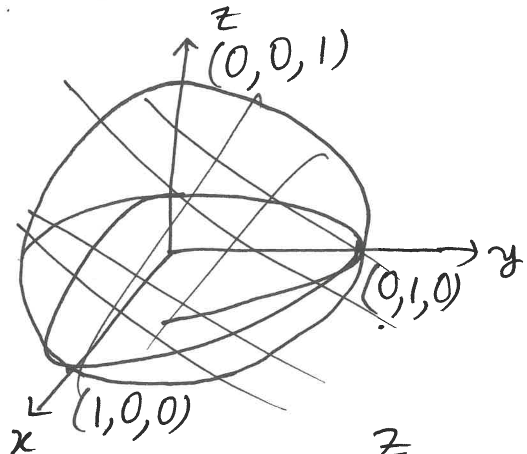
Figure 8

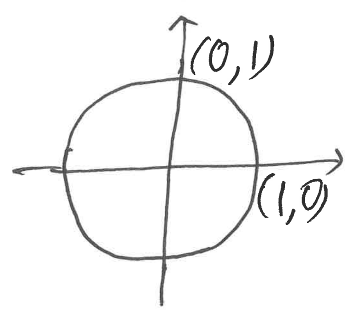
Example 2

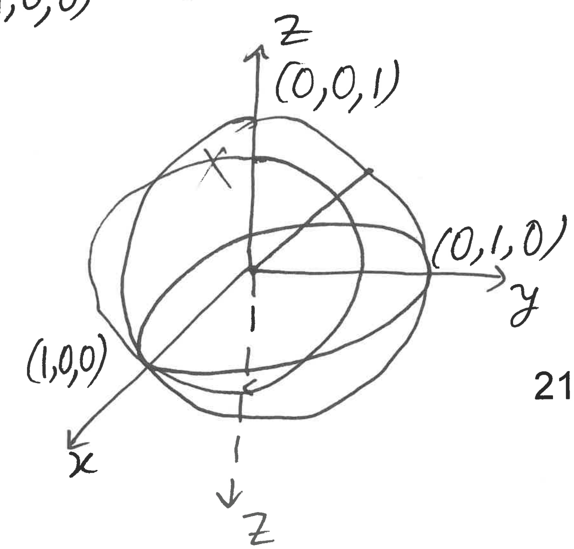
Evaluate $\int\int\int_B e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}dV$ where B is the ball $x^2+y^2+z^2 \le 1$.

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{-(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{-\rho^3} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$







Example 2

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$$\int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{-\rho^3}\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$u = \rho^3$

$\Rightarrow du = 3\rho^2 d\rho$

$\Rightarrow \rho^2 d\rho = \frac{1}{3}du$

ρ	0	1
u	0	1

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{-u} \sin\phi \, \frac{1}{3} du \, d\theta \, d\phi$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{2\pi}\left[-e^{-u}\right]_{u=0}^{1} \sin\phi \, d\theta \, d\phi$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{2\pi}\left(-\frac{1}{e}+1\right) \sin\phi \, d\theta \, d\phi$$

Triple Integrals in Spherical Coordinates (8 of 8)

This formula can be extended to include more general spherical regions such as

$$\mathrm{E} = \{(\rho, \theta, \phi) \mid \alpha \le \theta \le \beta,\ \mathrm{c} \le \phi \le d, g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi)\}$$

In this case the formula is the same as in (3) except that the limits of integration for ρ are $g_1(\theta, \varphi)$ and $g_2(\theta, \varphi)$.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi,\quad \phi=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$=\frac{\pi}{3}$

Example 3

Find out the volume of the domain that lies inside the sphere $x^2+y^2+z^2=4$ and above the plane $z=1$.

$x^2+y^2+1^2=4$

$\Rightarrow x^2+y^2=4-1=3$

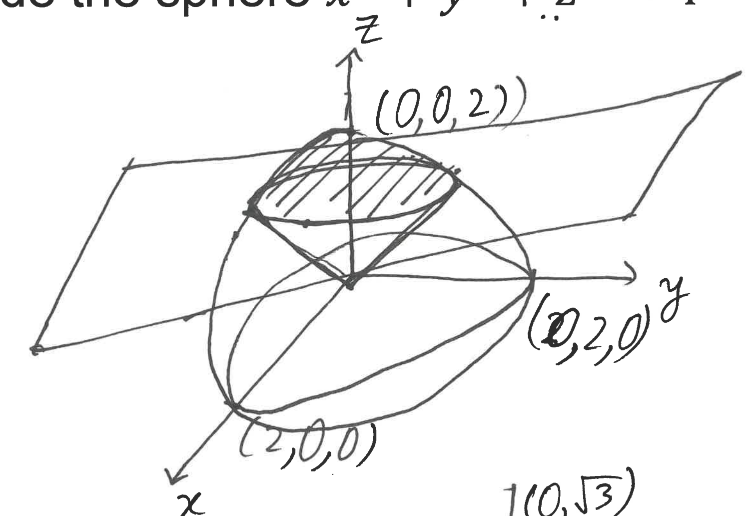
$\cancel{z=\rho\cos\phi}$

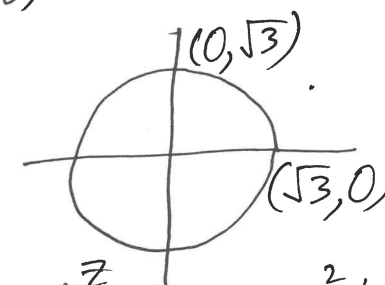
$\Rightarrow \rho\cos\phi=1$

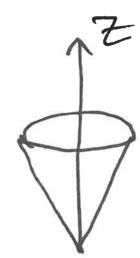
$\Rightarrow \rho=\frac{1}{\cos\phi}=\sec\phi$

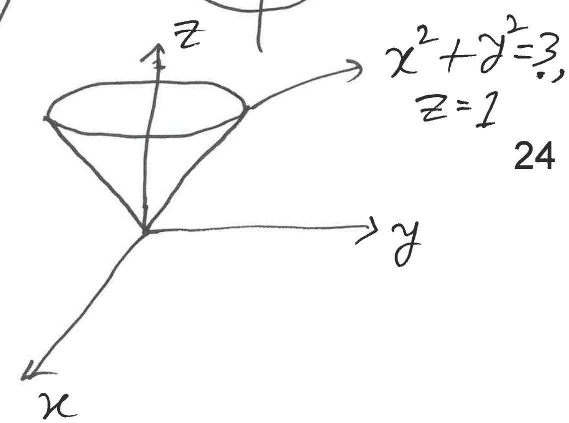
$$\int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_{\sec\phi}^{2} 1\ \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$=\int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_{\sec\phi}^{2} \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$









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