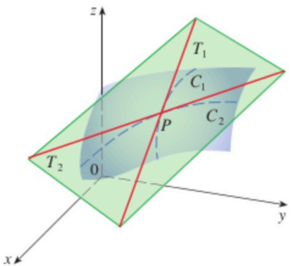
Section 14.4:

2 Equation of a Tangent Plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The tangent plane contains the tangent lines T_1 and T_2.



__Recall:__ $y = f(x)$ then at a point x_0, that f is differentiable at, $y = f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{Equation for the tangent line at } (x_0, f(x_0))}$ for x close to x_0.

Higher dimensional equivalent: • $z = f(x,y)$ is a surface in $\mathbb{R}^3$.

• fix a point $(x_0, y_0, \underbrace{f(x_0,y_0)}_{z_0})$ on the surface. Any plane that passes through the point is of the form $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

$\iff z - z_0 = \frac{-A}{C}(x-x_0) - \frac{B}{C}(y-y_0) \Rightarrow z = z_0 + a(x-x_0) + b(y-y_0)$ where $a = \frac{-A}{C}$, $b = \frac{-B}{C}$.

• By fixing $y = y_0$, the curve C_1 has tangent line T_1 with slope $f_x(x_0, y_0)$.

• By fixing $x = x_0$, the curve C_2 has tangent line T_2 with slope $f_y(x_0, y_0)$.

Def: The tangent plane at $(x_0, y_0, f(x_0, y_0))$ is the plane with lines T_1 and T_2.

Equation of the tangent plane $\longrightarrow$ $x = x_0 \Rightarrow z = z_0 + b(y - y_0) \Rightarrow z = z_0 + f_y(x_0, y_0)(y - y_0)$

$y = y_0 \Rightarrow z = z_0 + a(x - x_0) \Rightarrow z = z_0 + f_x(x_0, y_0)(x - x_0)$

$\Rightarrow$ Equation of tangent plane is

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 1

$f(1,1) = 2(1)^2 + 1^2 = 3$

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

$\hookrightarrow (x_0, y_0, f(x_0,y_0))$

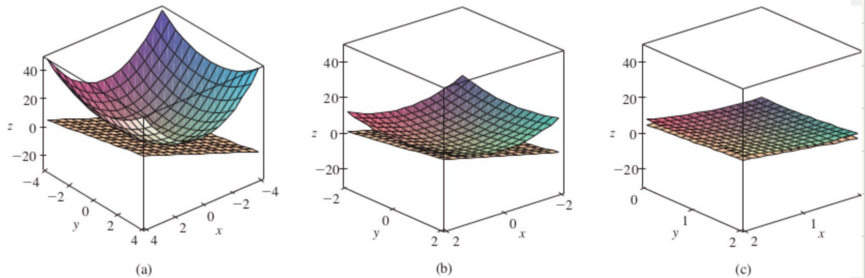
$$f(x,y) = 2x^2 + y^2 \Rightarrow f_x(x,y) = 4x, \quad f_y(x,y) = 2y$$

$$\Rightarrow f_x(1,1) = 4, \quad f_y(1,1) = 2$$

$$z = z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - x_0)$$

$$\Rightarrow z = 3 + 4(x-1) + 2(y-1) \Rightarrow \boxed{z = 4x + 2y - 3}$$

The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward $(1, 1, 3)$.



Def: • The linear function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is called the linearization of f at (a,b).

• The approximation

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

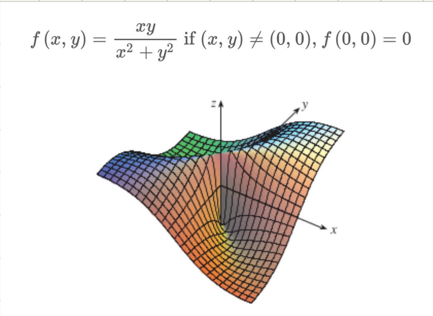
is called the linear approximation or the tangent plane approximation of f at (a,b).

Recall a consequence of differentiability from calc 1:

$y = f(x)$ is differentiable, then,

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

where $\Delta y = f(a+y) - f(y)$ and Δx describes a change from a to $a + \Delta x$.



7 Definition

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy such that ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

8 Theorem

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 2

Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

$f_x(x,y) = e^{xy} + yxe^{xy}$, $f_y(x,y) = x^2 e^{xy}$

f_x, f_y are CTS since they linear combinations of terms like "polynomial x exponential"

we need, $f(1,0) = e^0 = 1$, $f_x(1,0) = e^0 + 0 = 1$, $f_y(1,0) = 1 \cdot e^0 = 1$

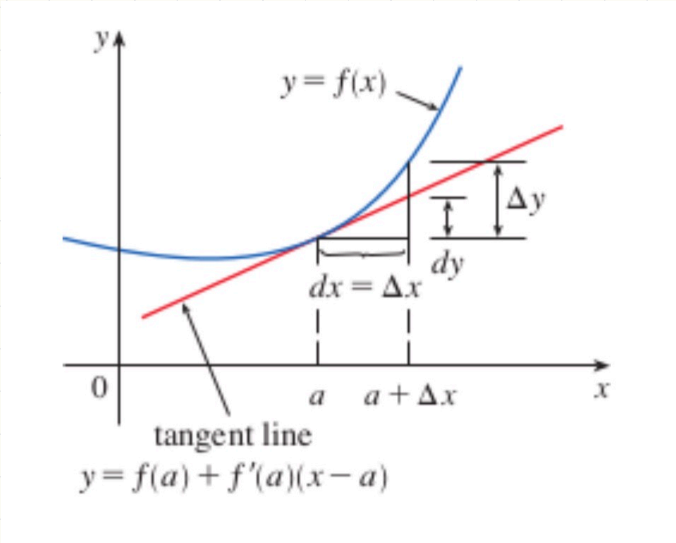
$\Rightarrow L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = 1 + 1(x-1) + 1(y-0) = x + y$.

$f(1.1, -0.1) \approx L(1.1, -0.1) = 1.1 - 0.1 = 1$

The actual value of $f(1.1, -0.1) = 1.1\, e^{-0.11} \approx 0.98542$.

Differentials.

- Define the differential dx to be an independent variable.
- Then for $y = f(x)$, $dy = f'(x)\,dx$
- When x changes by an amount $dx = \Delta x$, the function values change by Δy but the values along the tangent line change by $dy = f'(x)\,dx$
$$= f'(x)\,\Delta x$$

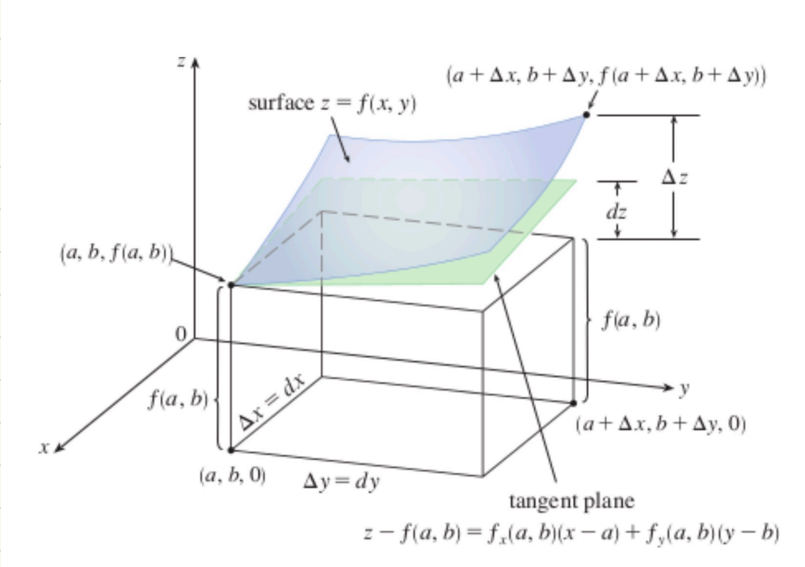


- So we can approximate changes along $f(x)$ by changes along the tangent line when Δx is "small". That is, $\Delta y \approx dy$.

Total differential:

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.}$$

where $f(x,y) \approx f(a,b) + dz$



Example 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

(a) $f_x = 2x + 3y \quad , \quad f_y = 3x - 2y$

$$\Rightarrow dz = (2x + 3y)\,dx + (3x - 2y)\,dy.$$

(b) $dx = \Delta x = 2.05 - 2 = 0.05$ and $dy = \Delta y = 2.96 - 3 = -0.04$

$$\Rightarrow dz = (2x + 3y)\Big|_{x=2,y=3} \Delta x + (3x - 2y)\Big|_{x=2,y=3} \Delta y$$

$$= (4 + 9)(0.05) + (6 - 6)(-0.04) = 0.65$$

and $\Delta z = f(2.05, 2.96) - f(2, 3) \approx 0.6449.$