MATH 243 Worksheet 1 Solutions

0: Solutions to the prerequisite quiz are in a separate document.

1: Recall the equation of a sphere with center (a, b, c) and radius r:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Now given the equation $2x^2 + 2y^2 + 2z^2 = 4x - 24z + 1$, we aim to write it in the form of the equation of a sphere. We proceed by completing the squares, that is

$$\begin{aligned} 2x^2 + 2y^2 + 2z^2 &= 4x - 24z + 1 \\ \Leftrightarrow 2(x^2 - 2x) + 2y^2 + 2(z^2 + 12z) &= 1 \\ \Leftrightarrow 2(x^2 - 2x + 1) - 2 + 2y^2 + 2(z^2 + 2 \times 6 \times z + 36) - 72 &= 1 \\ \Leftrightarrow 2(x-1)^2 + 2y^2 + 2(z+6)^2 &= 75 \\ \Leftrightarrow (x-1)^2 + y^2 + (z+6)^2 &= \left(\sqrt{\frac{75}{2}}\right)^2 \end{aligned}$$

Therefore, the sphere has a center located at $(a, b, c) = (1, 0, -6)$ and has radius $r = \sqrt{\frac{75}{2}} = \frac{5\sqrt{3}}{\sqrt{2}} = \frac{5\sqrt{6}}{2}$

2: The equation $x^2 + y^2 = 4$ represents a cylinder in $\mathbf{R^3}$ with radius $r = 2$ centered on the $z-$axis. It continues indefinitely in the positive and negative directions of the $z-$axis.
The equation $x = -1$ represents the set of all points in $\mathbb{R}^3$ whose x-coordinate is -1. The shape is a plane parallel to the coordinate plane $x = 0$, but shifted by one unit.

3: First, we find the sphere's diameter by computing the distance between the two given points, $(1, 2, 4)$ and $(4, 3, 10)$. We have

$$d = \sqrt{(1-4)^2 + (2-3)^2 + (4-10)^2} = \sqrt{46}$$

Since the radius r of the sphere is half the diameter, we get $r = \dfrac{d}{2} = \dfrac{\sqrt{46}}{2}$.
The center of the sphere, C, is located at the midpoint of its diameter. To find the coordinates of the midpoint of a segment, we take the average of the corresponding coordinates of the end points of the segment:

$$\left(\frac{4+1}{2}, \frac{2+3}{2}, \frac{4+10}{2}\right)$$

. Hence, the center is located at

$$C = \left(\frac{5}{2}, \frac{5}{2}, 7\right)$$

.
Therefore our sphere has radius $r = \dfrac{\sqrt{46}}{2}$ with a center at $\left(\dfrac{5}{2}, \dfrac{5}{2}, 7\right)$. Therefore its equation can be written as

$$\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 + (z-7)^2 = \frac{23}{2}.$$

4: The tangent line has slope: $6 = \dfrac{d}{dx}(x^2)|_{x=3}$, so any vector parallel to this tangent line has the form of $c\langle 1,6\rangle$, where c is a scalar. The only vector on the list that has this form is $\langle 1,6\rangle$ and answer is $\boxed{C}$.

5: Two nonzero vectors are parallel if they are scalar multiples of one another, Note that A can be written as $\frac{1}{6}\langle 1,-2,3\rangle$, similarly B can be written as $\frac{1}{\sqrt{14}}\langle 1,-2,3\rangle$, C can be written as $\frac{1}{14}\langle 1,-2,3\rangle$ D can be written as $-1\langle 1,-2,3\rangle$ and answer is $\boxed{F}$.

6: We are looking for a vector parallel to $\langle 2,1,2\rangle$ which has length 5. In other words, for a positive constant c, the solution satisfies the equation $||c\langle 2,1,2\rangle|| = c||\langle 2,1,2\rangle|| = c\sqrt{2^2+1^2+2^2} = 5$. Then, $c = \frac{5}{3}$, so answer is $\boxed{A}$.

7: $\mathbf{a}+\mathbf{b} = \langle 13,-1,-3\rangle$, $4\mathbf{a}+2\mathbf{b} = \langle 42,0,-14\rangle$, $||\mathbf{a}|| = 9$, $||\mathbf{a}-\mathbf{b}|| = \sqrt{43}$

8: $\mathbf{u}\times\mathbf{v} = (0,6-2t,0)$, so $\mathbf{0} = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (0,6-2t,0)\cdot(0,2,5) = 12-4t$ and $t=3$.

9: The 2nd vector is twice the 1st vector, so the vectors are parallel and the answer is $\mathbf{0}$.

10: a. Let $v = (a,b,c)$. Then $v\cdot v = a^2+b^2+c^2 = (\sqrt{a^2+b^2+c^2})^2 = \|v\|^2$
b. Let $\mathbf{w} = \mathbf{u}\times\mathbf{v}$. We need to check $\mathbf{w}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{v} = \mathbf{0}$. We show $\mathbf{w}\cdot\mathbf{u} = \mathbf{0}$ as the other equality is similar. The 1st component of $\mathbf{w}$ is $u_2v_3 - u_3v_2$, so the 1st summand for the dot product is $u_1u_2v_3 - u_1v_2u_3$. You may find the 2nd and 3rd summand, then add them up and see the result is 0.
Alternatively: The numbers cycle around, so we may express the sum as

$$\sum_{cyc}(u_iu_{i+1}v_{i+2} - u_iv_{i+1}u_{i+2}) = \sum_{cyc}u_{i+1}v_{i+2}u_i - \sum_{cyc}u_iv_{i+1}u_{i+2}$$

where indices are taken $\mod 3$, i.e. $4=1, 5=2$. Using $u_i = u_{i+3}$, rewrite the sum as

$$\sum_{cyc}u_{i+1}v_{i+2}u_{i+3} - \sum_{cyc}u_iv_{i+1}u_{i+2} = \sum_{cyc}u_iv_{i+1}u_{i+2} - \sum_{cyc}u_iv_{i+1}u_{i+2} = 0$$

c. Answers may vary, check 6/11 discussion recording for further explanation on how to find examples. Here's one example for each.
Dot product statement: $u = (1,0,0), v = (0,1,0), w = (0,0,0)$.
Cross product statement: $u = v = (1,0,0), w = (0,0,0)$.
Associativity statement: $u = (1,0,0), v = w = (0,1,0)$.