Answer Key

University of Delaware

Department of Mathematical Sciences

MATH 243 Midterm Exam 1

Fall 2025

Monday 29th September, 2025

Instructions:

- The time allowed for completing this exam is **90** minutes in total.
- Check your examination booklet before you start. There should be **12** questions on **9** pages.
- Turn off your cell phone and put it away. Headsets, and any other electronic devices are prohibited.
- No calculators.
- Answer the questions in the space provided. If you need more space for an answer, continue your answer on the back of the page and/or the margins of the test pages. No extra paper. *Do not separate the pages from the exam booklet.*
- For full credit, sufficient work must be shown to justify your answer.
- Partial credit will not be given if appropriate work is not shown.
- Write legibly and clearly; indicate your final answer to every problem. Cross out any work that you do not want graded. If you produce multiple solutions for a problem, indicate clearly which one you want graded.
- **Any form of academic misconduct will result in a failing grade.**

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	5	7	8	2	8	8	8	8	17	13	6	10	100
Score:													

1. (5 points) Given two vectors $\mathbf{u} = \langle 2, -1, 2\rangle$ and $\mathbf{v} = \langle 1, 8, 3\rangle$. Find the angle formed by these two vectors.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\mathbf{u}\cdot\mathbf{v} = 2(1) - 1(8) + 2(3) = 0,$$

so $\cos\theta = 0$. The vectors are orthogonal, $\theta = \frac{\pi}{2}$.

2. Consider the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

   (a) (4 points) Compute the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$ ($\operatorname{comp}_{\mathbf{u}}\mathbf{v}$).

   First, we compute

$$\mathbf{u}\cdot\mathbf{v} = (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})\cdot(6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = -4$$
$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = 3$$

   Now, the scalar projection is

$$\operatorname{comp}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|} = -\frac{4}{3}.$$

   (b) (3 points) Compute the vector projection of $\mathbf{v}$ onto $\mathbf{u}$ ($\operatorname{proj}_{\mathbf{u}}\mathbf{v}$).

   The vector projection is

$$\operatorname{proj}_{\mathbf{u}}\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\right)\mathbf{u} = -\frac{4}{9}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.$$

3. Given three vectors, $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$ and $\mathbf{c} = \langle 0, -3, 6 \rangle$.

(a) (6 points) Find the volume of the parallelepiped determined by vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

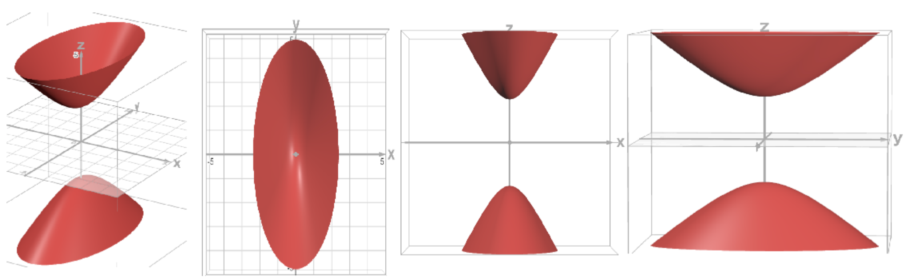
$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -3 & 6 \end{vmatrix} \\
&= 1 \begin{vmatrix} -1 & 4 \\ -3 & 6 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 6 \end{vmatrix} + (-7) \begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} \\
&= 1(6) - 4(12) - 7(-6) = 6 - 48 + 42 \\
&= 0.
\end{aligned}$$

So, the volume of the parallelepiped is 0.

(b) (2 points) Are these three vectors coplanar? Justify your answer.

Yes, they are coplanar because the volume of the parallelepiped determined by these 3 vectors is 0.

4. (2 points) Observe the following graphs:



Which equation below gives the surface shown above?

A. $\frac{z^2}{4} = x^2 + \frac{y^2}{4} + 1$

B. $z^2 + \frac{x^2}{4} + y^2 = 1$

C. $z = \frac{x^2}{4} - y^2$

D. $\frac{y^2}{4} + 1 = \frac{x^2}{4} + z^2$

ANSWER: A

5. Consider the following two points: $A(1,-5,1)$ and $B(3,2,-1)$.

(a) (2 points) Find the vector $\overrightarrow{AB}$.

$\overrightarrow{AB} = \langle 3-1, 2-(-5), -1-1\rangle = \langle 2,7,-2\rangle$.

(b) (2 points) Find a vector equation for the line containing A and B using $\overrightarrow{AB}$ as a direction vector.

Easy choices for a point on the line: take A or B.

Equation with A : $\vec{r}(t) = \langle 1,-5,1\rangle + t\langle 2,7,-2\rangle$

Equation with B : $\vec{r}(t) = \langle 3,2,-1\rangle + t\langle 2,7,-2\rangle$

(c) (2 points) Express the vector equation of the line as parametric equations.

Equation with A : $x = 1+2t, \quad y = -5+7t, \quad z = 1-2t$

Equation with B : $x = 3+2t, \quad y = 2+7t, \quad z = -1-2t$

(d) (2 points) Express the parametric equations of the line as symmetric equations.

Equation with A : $\dfrac{x-1}{2} = \dfrac{y+5}{7} = \dfrac{z-1}{-2}$

Equation with B : $\dfrac{x-3}{2} = \dfrac{y-2}{7} = \dfrac{z+1}{-2}$

6. (8 points) Find a vector equation for the line of intersection between the planes

$$x + y + z = 2 \quad \text{and} \quad x + 2y - z = 1.$$

From the equations of the planes, we see that $\langle 1, 1, 1\rangle$ is normal to the first plane and $\langle 1, 2, -1\rangle$ is orthogonal to the second. Thus, the vector

$$\begin{aligned} \langle 1,1,1\rangle \times \langle 1,2,-1\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} \\ &= \mathbf{i}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ &= -3\mathbf{i} + 2\mathbf{j} + \mathbf{k} \end{aligned}$$

is orthogonal to both planes. Therefore, the vector $\langle -3, 2, 1\rangle$ points in the direction of the line. We now need a point on the line; for this, we observe that $(0, 1, 1)$ is a point on both planes and hence on the line. Thus, a vector equation for the line of intersection between the planes is

$$\mathbf{r}(t) = \langle 0, 1, 1\rangle + t\langle -3, 2, 1\rangle.$$

7. (8 points) Find a vector equation of the line tangent to the vector function

$$\mathbf{r}(t) = te^t\,\mathbf{i} + t^3\,\mathbf{j} + \ln(t)\,\mathbf{k}$$

at the point corresponding to $t = 1$.

First, we evaluate $\mathbf{r}(t)$ at $t = 1$,

$$\mathbf{r}(1) = 1(e)\,\mathbf{i} + 1^3\,\mathbf{j} + \ln(1)\,\mathbf{k} = e\,\mathbf{i} + \mathbf{j}.$$

Now, we take the derivative of $\mathbf{r}(t)$,

$$\mathbf{r}'(t) = (e^t + te^t)\,\mathbf{i} + 3t^2\,\mathbf{j} + \frac{1}{t}\,\mathbf{k}.$$

Evaluating the derivative at $t = 1$,

$$\mathbf{r}'(1) = (e + 1(e))\,\mathbf{i} + 3(1^2)\,\mathbf{j} + \tfrac{1}{1}\,\mathbf{k} = 2e\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}.$$

Now, the tangent line is given by

$$\mathbf{L}(t) = (2et + e)\,\mathbf{i} + (1 + 3t)\,\mathbf{j} + t\,\mathbf{k}.$$

8. (8 points) Find an equation of the plane passing through the point $(1, -1, 0)$ and containing the line defined by the parametric equations

$$x = 1 - t, \quad y = 3t - 1, \quad z = t + 2.$$

Let P be the point $(1, -1, 0)$ on the plane. The direction vector of the line: $\mathbf{v} = \langle -1, 3, 1 \rangle$. One point on the line: $Q(1, -1, 2)$. Now we can obtain another vector on the plane: $\overrightarrow{QP} = \langle 1-1, -1-(-1), 0-2 \rangle = \langle 0, 0, -2 \rangle$

To find the normal vector of the plane:

$$\begin{aligned}
\mathbf{n} = \mathbf{v} \times \overrightarrow{QP} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 0 & 0 & -2 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 3 \\ 0 & 0 \end{vmatrix} \\
&= -6\mathbf{i} - 2\mathbf{j} + 0\mathbf{k} \\
&= -6\mathbf{i} - 2\mathbf{j}
\end{aligned}$$

The equation of the plane is given by:

$$\begin{aligned}
-6(x-1) - 2(y-(-1)) &= 0 \\
-6x + 6 - 2y - 2 &= 0 \\
-6x - 2y + 4 &= 0 \\
3x + y &= 2
\end{aligned}$$

9. Let $\mathbf{r}(t) = \langle 6\sin(2t),\, 3t,\, 6\cos(2t)\rangle$ and $P = \left(6, \frac{3\pi}{4}, 0\right)$.

(a) (2 points) Verify that P lies on the graph of $\mathbf{r}(t)$.

1) Find the parameter t corresponding to P. From the second component,

$$3t = \frac{3\pi}{4} \quad \Rightarrow \quad t = \frac{\pi}{4}.$$

Check with the other components:

$$x = 6\sin(2t) = 6\sin\left(\frac{\pi}{2}\right) = 6, \qquad z = 6\cos(2t) = 6\cos\left(\frac{\pi}{2}\right) = 0,$$

so indeed $\mathbf{r}\left(\frac{\pi}{4}\right) = P$.

(b) (6 points) Evaluate the unit tangent vector $\mathbf{T}$ at the point P .

2) Unit tangent $\mathbf{T}$ at P. Differentiate $\mathbf{r}$:

$$\mathbf{r}'(t) = \langle 12\cos(2t),\, 3,\, -12\sin(2t)\rangle.$$

The speed is

$$|\mathbf{r}'(t)| = \sqrt{(12\cos 2t)^2 + 3^2 + (-12\sin 2t)^2} = \sqrt{144(\cos^2 2t + \sin^2 2t) + 9} = 3\sqrt{17}.$$

Hence

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{3\sqrt{17}}\langle 12\cos(2t),\, 3,\, -12\sin(2t)\rangle.$$

At $t = \frac{\pi}{4}$ (so $\cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1$),

$$\boxed{\mathbf{T}(P) = \left\langle 0,\, \frac{1}{\sqrt{17}},\, -\frac{4}{\sqrt{17}} \right\rangle}.$$

(c) (6 points) Find the unit normal vector $\mathbf{N}$ at the point P.

3) Unit normal $\mathbf{N}$ at P. Differentiate $\mathbf{T}(t)$:

$$\mathbf{T}'(t) = \frac{1}{3\sqrt{17}}\langle -24\sin(2t),\, 0,\, -24\cos(2t)\rangle = \left\langle -\frac{8}{\sqrt{17}}\sin(2t),\, 0,\, -\frac{8}{\sqrt{17}}\cos(2t) \right\rangle.$$

Its magnitude is

$$|\mathbf{T}'(t)| = \frac{8}{\sqrt{17}}\sqrt{\sin^2(2t) + \cos^2(2t)} = \frac{8}{\sqrt{17}}.$$

Therefore

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin(2t),\, 0,\, -\cos(2t)\rangle.$$

At $t = \frac{\pi}{4}$,

$$\boxed{\mathbf{N}(P) = \langle -1,\, 0,\, 0\rangle}.$$

(d) (3 points) Determine the curvature κ at the point P.

4) Curvature κ at P. Using $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$,

$$\kappa(t) = \frac{8/\sqrt{17}}{3\sqrt{17}} = \frac{8}{51}.$$

$$\Rightarrow \kappa(\frac{\pi}{4}) = \boxed{\frac{8}{51}}$$

10. A particle moves in space with acceleration

$$\mathbf{a}(t) = \langle 6t,\ 4e^t,\ \frac{1}{t}\rangle.$$

You are given the following data:

$$\mathbf{v}(1) = \langle 5,\ 7,\ -3\rangle, \qquad \mathbf{r}(1) = \langle 1,\ 0,\ 2\rangle.$$

(a) (6 points) Find the velocity $\mathbf{v}(t)$.

(a) Find $\mathbf{v}(t)$ using a vector constant of integration. Integrate $\mathbf{a}(t)$ and add a constant vector $\mathbf{C} = \langle C_1, C_2, C_3\rangle$:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \langle 3t^2,\ 4e^t,\ \ln(t)\rangle + \mathbf{C}.$$

Use $\mathbf{v}(1) = \langle 5, 7, -3\rangle$:

$$\langle 3,\ 4e,\ \ln(1)\rangle + \mathbf{C} = \langle 5, 7, -3\rangle \ \Rightarrow\ \mathbf{C} = \big\langle 2,\ 7-4e,\ -3\big\rangle.$$

Hence

$$\boxed{\mathbf{v}(t) = \big\langle 3t^2,\ 4e^t,\ \ln(t)\big\rangle + \big\langle 2,\ 7-4e,\ -3\big\rangle.}$$

(b) (7 points) Find the position vector $\mathbf{r}(t)$.

(b) Find $\mathbf{r}(t)$ using a vector constant of integration. Integrate $\mathbf{v}(t)$ and add a constant vector $\mathbf{D} = \langle D_1, D_2, D_3\rangle$:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \left(\langle 3t^2, 4e^t, \ln(t)\rangle + \langle 2, 7-4e, -3\rangle\right) dt$$

$$= \big\langle t^3,\ 4e^t,\ t(\ln(t)-1)\big\rangle + \langle 2, 7-4e, -3\rangle t + \mathbf{D}.$$

Use $\mathbf{r}(1) = \langle 1, 0, 2\rangle$:

$$\langle 3, 7, -4\rangle + \mathbf{D} = \langle 1, 0, 2\rangle \ \Rightarrow\ \mathbf{D} = \langle 2, -7, 6\rangle.$$

Therefore

$$\boxed{\mathbf{r}(t) = \big\langle t^3,\ 4e^t,\ t(\ln(t)-1)\big\rangle + \big\langle 2,\ 7-4e,\ -3\big\rangle\, t + \big\langle 2,\ -7,\ 6\big\rangle.}$$

11. (6 points) Find the arc length of the curve given by

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle, \quad 0 \le t \le \frac{\pi}{2}.$$

Differentiating, we find that

$$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 1\rangle,$$

so

$$|\mathbf{r}'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1^2} = \sqrt{4(\sin^2 t + \cos^2 t) + 1} = \sqrt{5}.$$

Now, by the arc length formula,

$$L = \int_0^{\pi/2} |\mathbf{r}'(t)|\,dt = \int_0^{\pi/2} \sqrt{5}\,dt = \frac{\pi}{2}\sqrt{5}.$$

12. For the following statements, please clearly mark **True** or **False**.

(a) (2 points) The curvature of the circle of radius $r = \frac{1}{3}$ is $\kappa = \frac{1}{9}$.

Check one: ◯ **True** $\surd$ **False**

False

$$\kappa = \frac{1}{r} = \frac{1}{\frac{1}{3}} = 3$$

(b) (2 points) If the magnitude of the vector function $\mathbf{r}(t)$ is constant, then the vector $\mathbf{r}'(t)$ is orthogonal to the vector $\mathbf{r}(t)$ for all t.

Check one: $\surd$ **True** ◯ **False**

True

$$\|\mathbf{r}(t)\| = c \Rightarrow \|\mathbf{r}(t)\|^2 = c^2 \Rightarrow \mathbf{r}\cdot\mathbf{r} = c^2$$

Taking the derivative using product rule

$$\mathbf{r}'\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{r}' = 0 \Rightarrow 2\mathbf{r}\cdot\mathbf{r}' = 0 \Rightarrow \mathbf{r} \perp \mathbf{r}'$$

(c) (2 points) If $\mathbf{T}(t)$ is a unit tangent vector and $\mathbf{N}(t)$ is a unit normal vector to a curve at the same point P, then $\mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector.

Check one: $\surd$ **True** ◯ **False**

True

$$\|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}\|\|\mathbf{N}\| \sin 90^\circ = 1(1)(1) = 1$$

So, $\mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector

(d) (2 points) For all vectors $\mathbf{u}$ and $\mathbf{v}$ in three dimensional space, $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

Check one: $\surd$ **True** ◯ **False**

True
The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. In particular, since the vectors $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ are orthogonal, their dot product is zero.

(e) (2 points) Any two lines in three dimensional space that are not parallel must intersect.

Check one: ◯ **True** $\surd$ **False**

False
The lines can be skew.