Jacobian and Vector Fields

Lecture for 7/1

Jacobian General Idea

- Suppose $x = f(u, v)$, $y = g(u, v)$
- Can we make the substitution and convert $dx\,dy$?
- Yes, provided the substitution is invertible
- Cylindrical, spherical, polar become special cases of Jacobian

Example:

$x = u + v$

$y = u - v$

$dx\,dy = du\,dv \cdot \underbrace{?}_{}$

what is this value?

Jacobian

Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in R^n and x_i depend on $t_1, \ldots, t_m$

- Define J to be n x m matrix with (i, j) entry $\partial x_i/\partial t_j$
- If n = m and det(J) ≠ 0, then $d\mathbf{x} = dx_1 \ldots dx_n = J(\mathbf{x})\, dt_1 \ldots dt_n$
    - Furthermore, $\int_A f(x_1, \ldots)\, d\mathbf{x} = \int_A f(x_1(t_1, \ldots), \ldots)\, \cancel{J}(\mathbf{x})\, d\mathbf{t}$ det J

$$\begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \cdots \\ \frac{\partial x_2}{\partial t_1} & \cdots & \\ \vdots & & \\ \frac{\partial x_n}{\partial t_1} & \cdots & \end{bmatrix}$$

Example in 2D:

$x = f(u,v)$

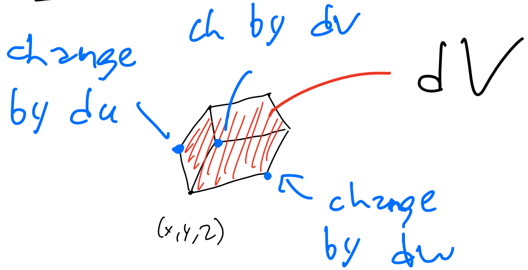
$y = g(u,v)$, then $J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

$= \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}$

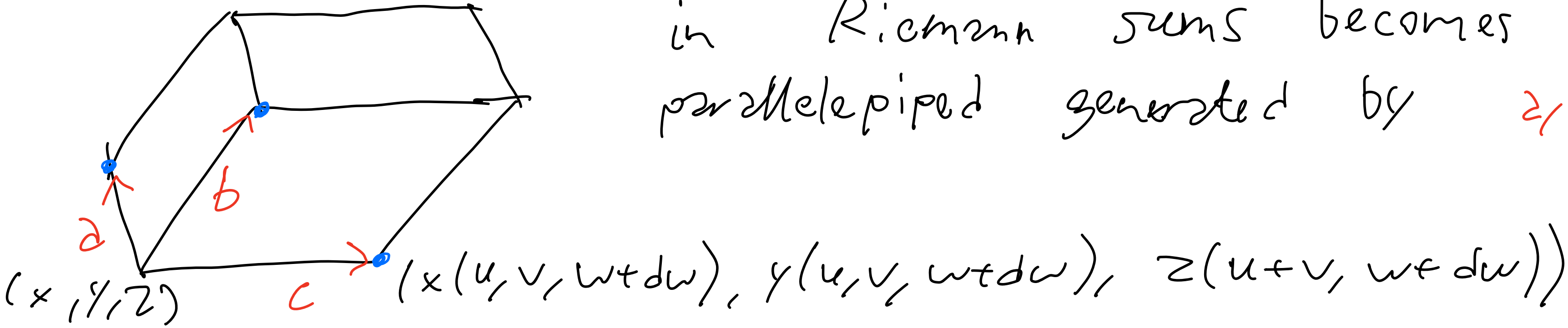
Jacobian Derivation

For sake of convenience, we derive Jacobian for 3 variables x, y, z with $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$. Proof for more or less variables is the same

Idea: imagine changing x, y, z by some small amounts du, dv, dw



Let a, b, c be vector pointing to new point when u, v, w respectively are slightly changed. Small prism in Riemann sums becomes small parallelepiped generated by a, b, c.



To continue, find better expressions for a, b, c.

Consider $C = (x(\ldots, w+dw), \ldots) - (x,y,z) = (\ldots, \ldots, \ldots)$.

1st component: $x(u,v,w+dw) - x(u,v,w) = \dfrac{x(u,v,w+dw) - x(u,v,w)}{dw}\, dw$

$= \dfrac{\partial x}{\partial w} dw$ by partial derivative def & fact that dw infitesm.

Similarly, other components are $\dfrac{\partial y}{\partial w} dw$, $\dfrac{\partial z}{\partial w} dw$

Plug in: $\vec{C} = \left(\dfrac{\partial x}{\partial w} dw, \dfrac{\partial y}{\partial w} dw, \dfrac{\partial z}{\partial w} dw \right) = (x_w, y_w, z_w)\, dw$

Similarly, $\vec{A} = (x_u, y_u, z_u)\, du$, $\vec{B} = (x_v, y_v, z_v)\, dv$.

Recall: volume formed by 3 vectors is the determinant formed by these 3 vectors as rows of the matrix.

This is also true in n dimensions & n vectors for any n, which is why Jacobian works in general

So $dV = (\text{vol formed by } A, B, C) = \det\begin{pmatrix} x_u\,du & y_u\,du & z_u\,du \\ x_v\,dv & y_v\,dv & z_v\,dv \\ x_w\,dw & y_w\,dw & z_w\,dw \end{pmatrix}$

$$= \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} du\,dv\,dw = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} du\,dv\,dw$$

factoring constants out of determinants

$\det(M) = \det(M^T)$

Note: $\begin{vmatrix} 2c & c & c \\ c & c & c \\ c & 2c & c \end{vmatrix} = c^3 \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$,

not just multiplying by c.

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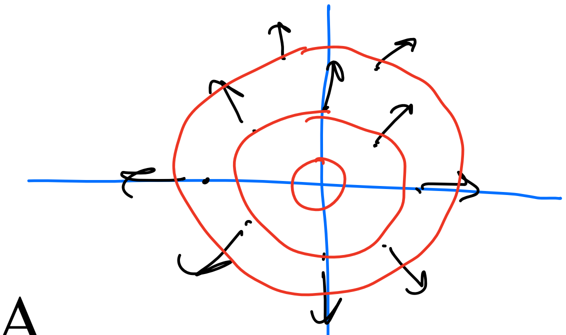
Vector Fields



A vector field for A assigns a vector to each point in A

- Any f: $R^n \to R^n$ can be considered a vector field for R^n
- We've already seen the field $\nabla f = (f_x, f_y, f_z)$
- Recall the field ∇f is perpendicular to graph cross sections $f = c$

$f = x^2 + y^2$

$\nabla f = (2x, 2y)$

$f = c$ is a circle of radius $\sqrt{c}$

Is there anything vector field the gradient can't do?

- Call F conservative if $F = \nabla f$ for some function f

Line Integrals

We've integrated over intervals, rectangles, prisms, and general solids.
What if we stretch an interval inside higher dimensions?

- Let $\mathbf{r}(t)$ with $a \leq t \leq b$ parametrize a curve C
- Can define $\int_C f(\mathbf{x})\, ds$ to be integral of f along C
- The previous integral expands to $\int_a^b f(\mathbf{r}(t))\, \|\mathbf{r}'(t)\|\, dt$
- Note: Value of integral depends on orientation of C

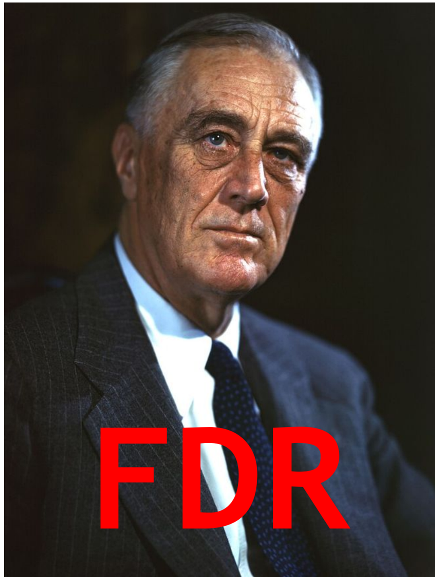
More Line Integrals

What if we only care about x or y when traveling along the curve?

- Let $r(t) = (x(t), y(t))$, $a \leq t \leq b$ parametrize C
- $\int_C f(x, y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$
- Similarly for y, similarly for more variables

What if we want to mix line integrals and vector fields?

- Consider $\int_C (F \cdot dr) = \int_C F(r(t))\,r'(t)\,dt$
- We have $\int_C (F \cdot dr) = \int_C (F \cdot T)\,ds$



Practice Problems

Evaluate $\int_C f\, ds$ for the following functions and curves:

- $f(x, y) = 3x^2-2y$, C is line segment from (3, 6) to (1,-1)
- $f(x, y) = 6x$, C is portion of $y = x^2$ from $x = -1$ to $x = 2$
- $f(x, y) = 16y^5$, C is $x = y^4$ from $y = 0$ to $y = 1$, followed by segment from (1,1) to (1, -2), followed by segment from (1, -2) to (2,0)

Evaluate $\int_C (x^2\, dy - yz\, dz)$ where C is segment from (4, -1, 2) to (1, 7, -1)

Scratchwork