3 Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2 = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum. $\cup$

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum. $\cap$

(c) If $D < 0$, then $f(a, b)$ is a saddle point of f. (inflection point in calc 1)

(d) If $D = 0$ then the test is inconclusive.

Steps:

1. Find ∇f and points (a,b) such that $\nabla f(a,b) = \vec{0}$.

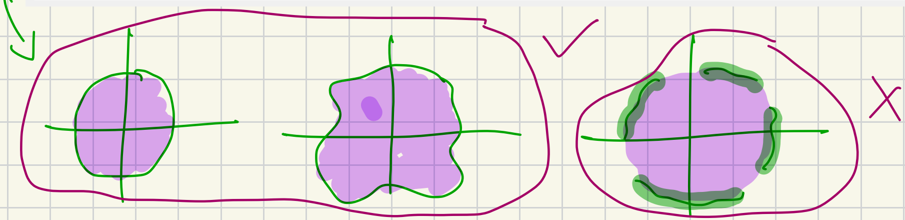
where $f_{xy} = f_{yx}$.

Remark: You can technically check the sign of f_{yy} as well.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

closed and bounded

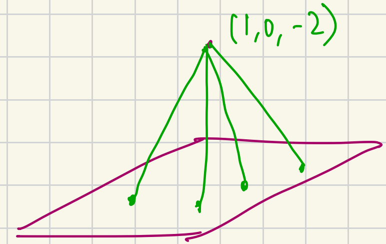


In calc 1: $f(x)$ is differentiable function (a,b) find absolute min / max on $[a,b]$.

Solution: Find c s.t $f'(c) = 0$ and check $f(c), f(a), f(b)$

Example 5

Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.



A:
- $d(x,y,z) = \sqrt{(x-1)^2 + (y-0)^2 + (z-(-2))^2}$
is the distance from any point in $\mathbb{R}^3$ and $(1,0,-2)$.

- Take $z = 4 - x - 2y$ and

$$f(x,y) = (x-1)^2 + y^2 + (\overbrace{4-x-2y+2}^{6-x-2y})^2$$

- Now use 2^{nd} derivative test:

- $f_x = 2(x-1) + 2(6-x-2y)(-1)$, $\quad f_y = 2y + 2(6-x-2y)(-2)$

$\quad = 2x - 2 - 12 + 2x + 4y \qquad\qquad = 2y - 24 + 4x + 8y$

$\quad = 4x + 4y - 14 \qquad\qquad = 4x + 10y - 24$

- $f_{xx} = 4$, $f_{yy} = 10$, $f_{xy} = 4 = f_{yx}$

Remark: minimizing $\sqrt{x}$ over $[a,b]$ is "the same" as minimizing x.

- Solve for (a,b) such that $f_x(a,b) = f_y(a,b) = 0$.

$$\left.\begin{aligned} 4x + 4y - 14 &= 0 \\ 4x + 10y - 24 &= 0 \end{aligned}\right\} \rightarrow 4y - 10y + 10 = 0 \rightarrow y = \frac{5}{3}$$

$$\Rightarrow 4x + \frac{20}{3} - 14 = 0 \Rightarrow 4x = \frac{22}{3} \Rightarrow x = \frac{11}{6}.$$

- So the critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$.
- $f_{xx} f_{yy} - (f_{xy})^2 = 40 - 16 = 24 > 0 \Rightarrow$ there exists either a minimum or maximum.
- $f_{xx} = 4 > 0 \Rightarrow$ local min.
- z at $\left(\frac{11}{6}, \frac{5}{3}\right)$ is

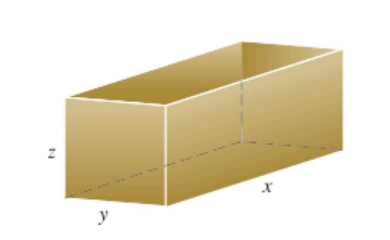
$$z = 4 - \frac{11}{6} - \frac{10}{3} = 4 - \frac{11 - 20}{6} = \frac{-7}{6}$$

$$\Rightarrow d(x,y,z) = \sqrt{\left(\frac{11}{6} - 1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{7}{6} + 2\right)^2}$$

$$= \boxed{\frac{5}{6}\sqrt{6}}$$

Example 6

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.



A: $V = xyz$ Goal: Maximize V given that surface area of open box is $12\ m^2$.

$S = xy + 2zy + 2xz = 12$

- Solve for constraint in terms of z: $xy + 2z(y+x) = 12$

$$\Rightarrow z = \frac{12 - xy}{2(x+y)}$$

$$\Rightarrow V = xy\frac{(12-xy)}{2(x+y)} \Rightarrow \frac{\partial V}{\partial x} \overset{\text{DIY}}{=} \frac{y^2(12-2xy-x^2)}{2(x+y)^2}, \quad \frac{\partial V}{\partial y} \overset{\text{DIY}}{=} \frac{x^2(12-2xy-y^2)}{2(x+y)^2}$$

- Solve for points where $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$.

- Constraints from real life: ① $x, y, z \geq 0$.

② if $x = y = 0$ then $V = 0$

- So focus on numerators: we want $y^2(12 - 2xy - x^2) = 0$

and $x^2(12 - 2xy - y^2) = 0$

Since $x \neq 0$ and $y \neq 0$, we need to solve

$$12 - 2xy - x^2 = 0 \quad \text{and} \quad 12 - 2xy - y^2 = 0$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$x^2 = 12 - 2xy \qquad\qquad y^2 = 12 - 2xy$$

- $\Rightarrow x^2 = y^2$ so either $x = y$ OR $\cancel{x = -y}$.

$\Rightarrow x = y \Rightarrow x^2 = 12 - 2x^2 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \Rightarrow x = 2$ (reject the negative solution).

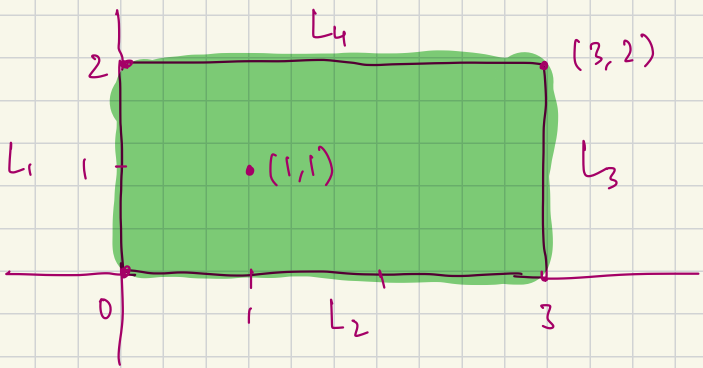
$\Rightarrow y = 2$. $z = \dfrac{12 - xy}{2(x+y)}$ so at $x = 2, y = 2$, $z = \dfrac{12 - 4}{2 \cdot 4} = 1$.

$\Rightarrow$ max volume at $(2,2,1)$ since minimum volume was 0.

(Don't need the 2^{nd} derivative test here)

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.



Step 1: Find critical points of f. $\boxed{\text{DIY}}$. Critical pt is $(1,1)$ $\quad (f(x,y) = x^2 - 2xy + 2y)$

$(f(1,1) = 1 - 2 + 2 = 1)$

Step 2: Find min/max on the lines L_1, L_2, L_3, L_4.

L_1: $x = 0$, $0 \leq y \leq 2$

$\Rightarrow f(0,y) = 2y \Rightarrow f(0) = 0$ is the abs min, $f(2) = 4$ is the abs max.

(since $f(0,y)$ is increasing on $[0,2]$).

L_2 : $y = 0$, $0 \leq x \leq 3$

$f(x,0) = x^2 \Rightarrow f(0) = 0$ is the abs min , $f(3) = 9$ is the abs max.
(since $f(x,0)$ is increasing on $[0,3]$).

L_3 : $x = 3$, $0 \leq y \leq 2$

$f(3,y) = 9 - 6y + 2y = 9 - 4y$

$\Rightarrow f(0) = 9$ is the abs max, $f(2) = 1$ is the abs min (since $f(3,y)$ is decreasing on $[0,2]$)

L_4 : $y = 2$, $0 \leq x \leq 3$

$f(x,2) = x^2 - 4x + 4 = g(x)$

Method 1: $g'(x) = 2x - 4 = 0 \Leftrightarrow x = 2$. $g(2) = 0$, $g(0) = 4$, $g(3) = 1$.

$\Rightarrow g(2) = 0$ is the abs min, $g(0) = 4$ is the abs max.

Absolute min at $(0,0)$, $(2,2)$ where value is 0

Absolute max at $(3,0)$ where value is 0.

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

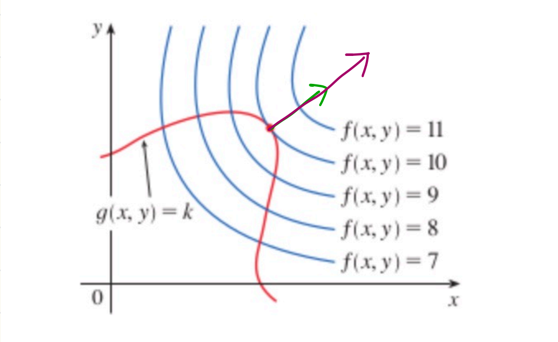
1. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.



Main idea: at a point where f is min/max, ∇f and ∇g are parallel.

i.e: $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$.

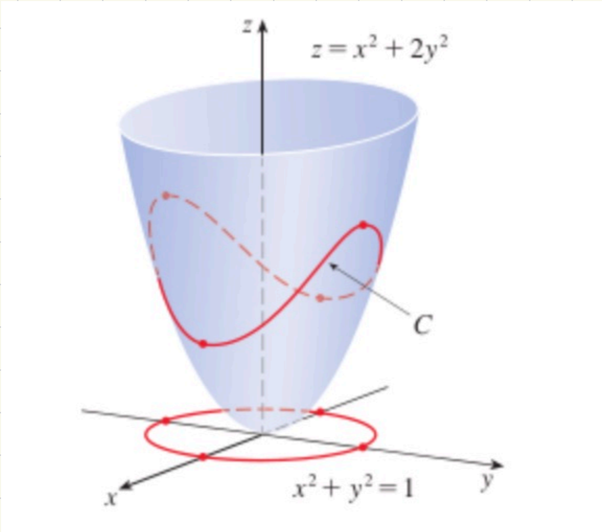
In two variables: $\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$

$\Leftrightarrow f_x = \lambda g_x, \ f_y = \lambda g_y$

and $g = k$

Example 1

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.



<u>A:</u> $g(x,y) = x^2 + y^2 \Rightarrow g_x = 2x, \; g_y = 2y$

$f_x = 2x \; , \; f_y = 4y$

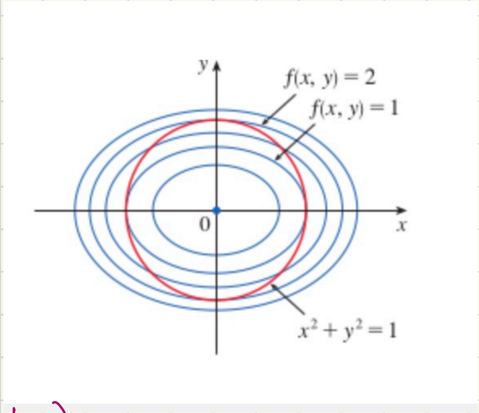
So $\nabla f(x,y) = \lambda \nabla g(x,y)$

$\Leftrightarrow \langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle$

We need to solve: $2x = 2\lambda x \Leftrightarrow 2x(\lambda - 1) = 0 \Leftrightarrow x = 0, \lambda = 1$

$4y = 2\lambda y$

$x^2 + y^2 = 1$



$x = 0 \Rightarrow 0^2 + y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow$ Test $(0, 1)$ and $(0, -1)$

$\lambda = 1 \Rightarrow 4y = 2y \Rightarrow y = 0 \Rightarrow x^2 + 0^2 = 1 \Rightarrow$ Test $(1, 0)$ and $(-1, 0)$.

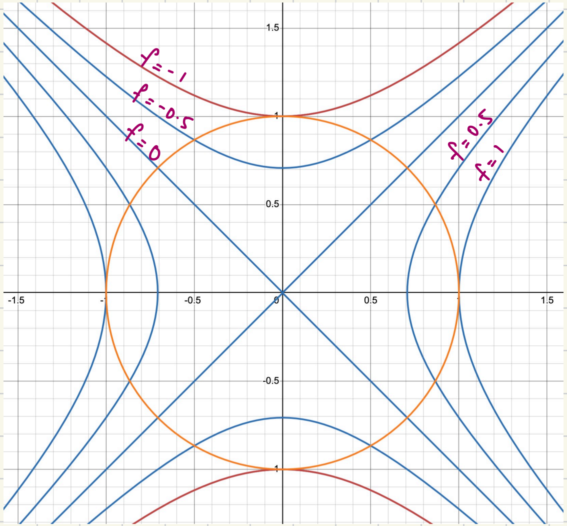
	$(0,1)$	$(0,-1)$	$(1,0)$	$(-1,0)$
$f(x,y)=x^2+2y^2$	2	2	1	1

max at $(0,\pm 1)$, min at $(\pm 1, 0)$

Example 5. *Find the extreme values of the function* $f(x,y) = x^2 - y^2$ *on the circle* $\underbrace{x^2+y^2}_{g(x,y)} = 1$.

- We want x, y, λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$.

- $\nabla f = \langle 2x, -2y \rangle$, $\nabla g = \langle 2x, 2y \rangle$



So $\nabla f = \lambda \nabla g \iff \langle 2x, -2y \rangle = \langle 2\lambda x, 2\lambda y \rangle$

$$\iff \begin{cases} 2x = 2\lambda x \Rightarrow 2x(\lambda - 1) = 0 \iff x = 0 \text{ OR } \lambda = 1 \\ -2y = 2\lambda y \end{cases}$$

3^{rd} equation $\rightsquigarrow$ $x^2 + y^2 = 1$

$x = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow$ Test $(0,1)$, $(0,-1)$

$\lambda = 1 \Rightarrow -2y = 2y \iff y = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

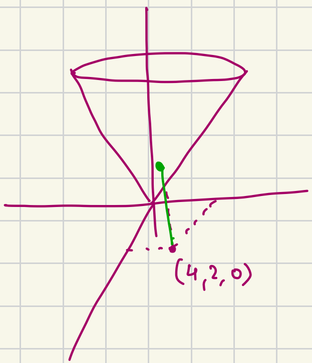
$\Rightarrow$ Test $(1,0)$, $(-1,0)$.

	$(0,1)$	$(0,-1)$	$(1,0)$	$(-1,0)$
x^2-y^2	-1	-1	1	1
	MIN		MAX	

Example 7. *Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.*

distance between any point (x,y,z) and $(4,2,0)$ is

$$\sqrt{(x-4)^2 + (y-2)^2 + z^2}$$



So we want to minimize distance subject to $z^2 = x^2 + y^2$

For ease: we will instead minimize

$$f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$$

subject to $\underbrace{x^2 + y^2 - z^2}$ $= 0$

$\Rightarrow g = x^2 + y^2 - z^2$

$$\nabla f = \langle 2(x-4), 2(y-2), 2z \rangle = \lambda \nabla g = \lambda \langle 2x, 2y, -2z \rangle$$

Equations: $2(x-4) = 2\lambda x \Rightarrow \frac{(x-4)}{x} = \lambda \Rightarrow \frac{x-4}{x} = \frac{y-2}{y}$

$2(y-2) = 2\lambda y \Rightarrow \frac{(y-2)}{y} = \lambda \Rightarrow \cancel{yx} - 4y = \cancel{xy} - 2x$

$2z = -2\lambda z \Rightarrow x = 2y$

$\Downarrow$

$2z(1+\lambda) = 0 \Rightarrow z = 0, \lambda = -1$

$x^2 + y^2 - z^2 = 0$

$z = 0 \Rightarrow x^2 + y^2 = 0 \Rightarrow 4y^2 + y^2 = 0 \Rightarrow y = 0 \Rightarrow x = 0$

$\lambda = -1 \Rightarrow 2(x-4) = -2x \Rightarrow 4x = 8 \Rightarrow x = 2$

$2(y-2) = -2y \Rightarrow 4y = 4 \Rightarrow y = 1$

$\Rightarrow z^2 = 2^2 + 1^2 = 5 \Rightarrow z = \pm\sqrt{5}$

	$(0,0,0)$	$(2,1,\sqrt{5})$	$(2,1,-\sqrt{5})$
f	20	10	10

$(x-4)^2 + (y-2)^2 + z^2$

Example 8. *Find the extreme values of the function* $f(x, y) = x^2 - y^2$ *on the closed disk* $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.