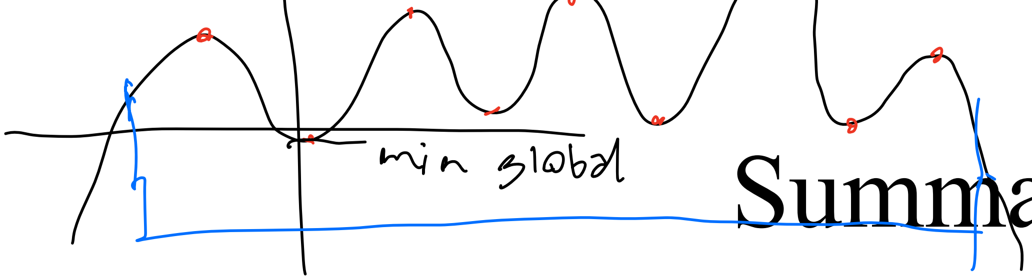
Global Extrema

Pre-lecture for 6/24

check all of $f' = 0$ values



Summary

$\nabla f = 0$, pass 2nd or Hess test $\to$ local min, local max, saddle

Suppose we want to find max or min for an entire region:

- Check that the max and min exist
- Any global extrema is either a local extrema or on boundary
- Solve $\nabla f = 0$, check points within the region
- Find the boundary
- Check value of f on the boundary

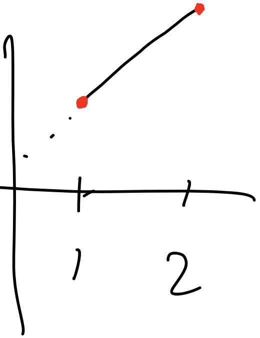
$y = \frac{1}{x}$

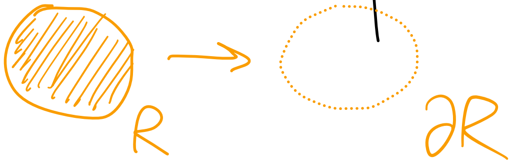
max & min on $(0,1)$

max DNE

min DNE

$f' \neq 0$ on $[1,2]$, but max @ 2, min @ 1



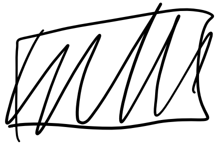
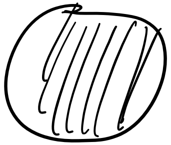


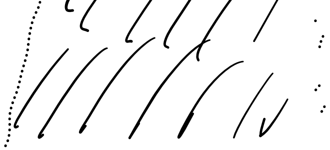
Extrema Existence

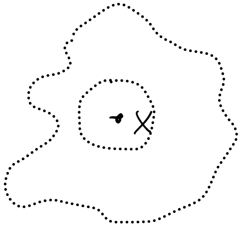
out of scope for this course

Topology Facts:

- Maximum of continuous function on compact set exists
- Subsets of R^n are compact iff they are closed and bounded
  - Lookup Heine-Borel Theorem for more info
- A set is closed iff it contains its boundary points

Bounded formal def: Set S in R^n is bounded if there exists M with $x \in S \Rightarrow \|x\| \leq M$,  i.e. entire set  contained inside some interval, circle,  sphere etc.





Boundaries of Sets

Open sets in R^n

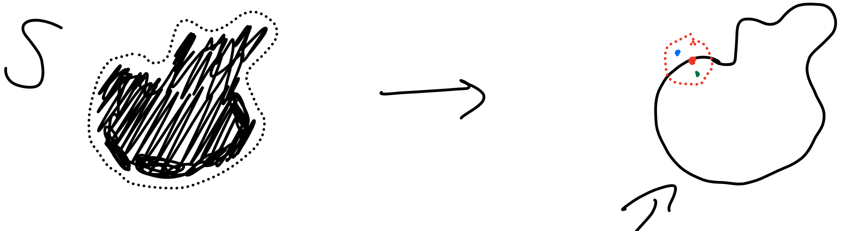
expand this out

- Let $B_r(x) = \{y : ||x-y|| < r\}$ be the open ball of radius r
- Any open neighborhood of x contains a ball, conversely any ball counts as a neighborhood

Example: $B_r(0)$ in
1D is $|x| < r$
2D is $\sqrt{x^2+y^2} < r$
3D is $\sqrt{x^2+y^2+z^2} < r$

Boundary Points

- A boundary point of a set S is a point x such that every neighborhood of x contains a point in S and a point outside of S



$\partial S = \partial R$, but R is closed & S isn't closed

R 

"Not closed" ≠ "open"

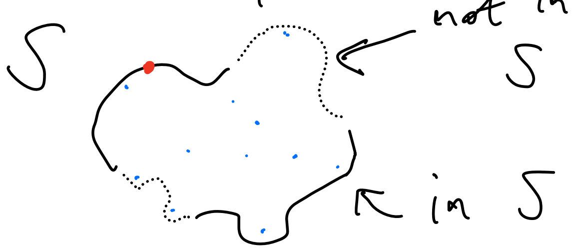
"Not open" ≠ "closed"

Sets in R^n are not doors, they can be neither closed or open

Tips to Remember

- If region described by g ≥ 0 or g>0 with g continuous, then g = 0 is the boundary
- When described by g ≥ 0, region is closed
- When described by g > 0, region is open
- R^n and empty set are both closed and open
- If the region isn't closed, try to show max and min don't exist
- Only use solutions to ∇f = 0 in the region

Example:



$\partial S \not\subseteq S$

because of parts not in S, so S not closed

Ex: $\{(x,y) : x^2+y^2 < 7\} = S$

$\partial S = \{(x,y) : x^2+y^2 = 7\}$

No neighborhood of red point is entirely in S, so S is not open

Exception:

$S = \{x^2+y^2<0\} = \emptyset$ Any of the blue points have a neighborhood contained inside S

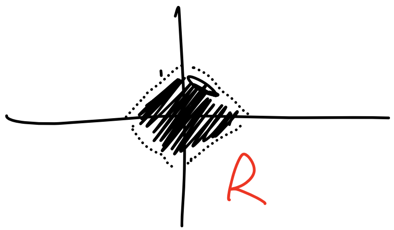
$\partial S = \emptyset$, not $\{x^2+y^2=0\} = \{(0,0)\}$

Practice Problems

Find the maximum and minimum, or show they don't exist:

- $f(x, y) = x^2+y^2-xy^2+1$ on $-1 \leq x, y \leq 1$
- $f(x, y, z) = |x|+|y|+|z|$ on $x^2+y^2+z^2 < 1$
- $f(x, y) = 2x^2-y^2+6y$ on $x^2+y^2 \leq 16$

Example for non-existence: $f(x,y) = x$ on $\underbrace{|x|+|y|<1}_{R}$



We have $-1<x<1$ on R, so range of f is $(-1,1)$.

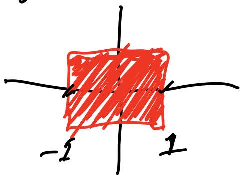
But $(-1,1)$ doesn't have a min or max because -1 & 1 are not included

Scratchwork

If instead, $f(x,y)=x^2$, then max DNE but min $= 0$

$f(x, y) = x^2+y^2-xy^2+1$ on $-1 \le x, y \le 1$

f is continuous, $R = \{(x,y): -1 \le x, y \le 1\}$ is closed & bounded because it's the solid square



, so min & max exist.

$$0 = \nabla f = (2x - y^2, 2y - 2xy) \Rightarrow \begin{array}{l} 2x = y^2 \\ y = xy \end{array}$$

$y = xy \Rightarrow y(1-x) = 0 \Rightarrow y = 0$ or $x = 1$

$y = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \Rightarrow (0,0)$

$x = 1 \Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2} \Rightarrow (1, \pm\sqrt{2})$

Check values: $f(0,0) = 1$, $f(1, \pm\sqrt{2}) = 2 + y^2 - y^2 = 2$

Check boundary: $x=1$, $x=-1$, $y=1$, or $y=-1$

$y=\pm 1$: $f(x,y) = x^2+1-x+1 = x^2-x+2 = \underbrace{\left(x-\tfrac{1}{2}\right)^2+\tfrac{7}{4}}_{\geq 7/4} \leq \left(-\tfrac{3}{2}\right)^2+\tfrac{7}{4} = \tfrac{9}{4}+\tfrac{7}{4} = 4$

$f(-1,1) = 4$

$x=1$: $f(x,y) = 1+y^2-y^2+1 = 2$

$x=-1$: $f(x,y) = 1+y^2+y^2+1 = 2+2y^2 \leq 2+2=4$

$\geq 2+0=2$

New values in pink

Minimum of all these values is 1

Maximum of these values is 4

min $f=1$, max $f=4$ on this region

Lagrange Multipliers

Lecture for 6/24

g cont $\Longrightarrow R = g^{-1}(\{0\})$ is closed. R also bounded, then max min exist

Method of Lagrange Multipliers

Consider the region R described by g = 0 for continuous g

- What if R is complicated but we want to maximize f on R?
- Solve $\nabla f = \lambda \nabla g$
- Plug in all the values
  - Smallest will be min, largest will be max

If you have a c, note that $g = c \Longleftrightarrow g - c = 0$

What if g is 1 dimensional curve drew in R^2? Then $\partial R = R$, so "$\nabla f = 0$" provides no info and we are forced to check ∂R manually.

which means checking all of R manually, so we have not accomplished anything

Why This Works

because $0=0$ is always true

Recall fact on graphs:

- If G_c is the graph of $f(x, y) = c$, then ∇f and G_c normal

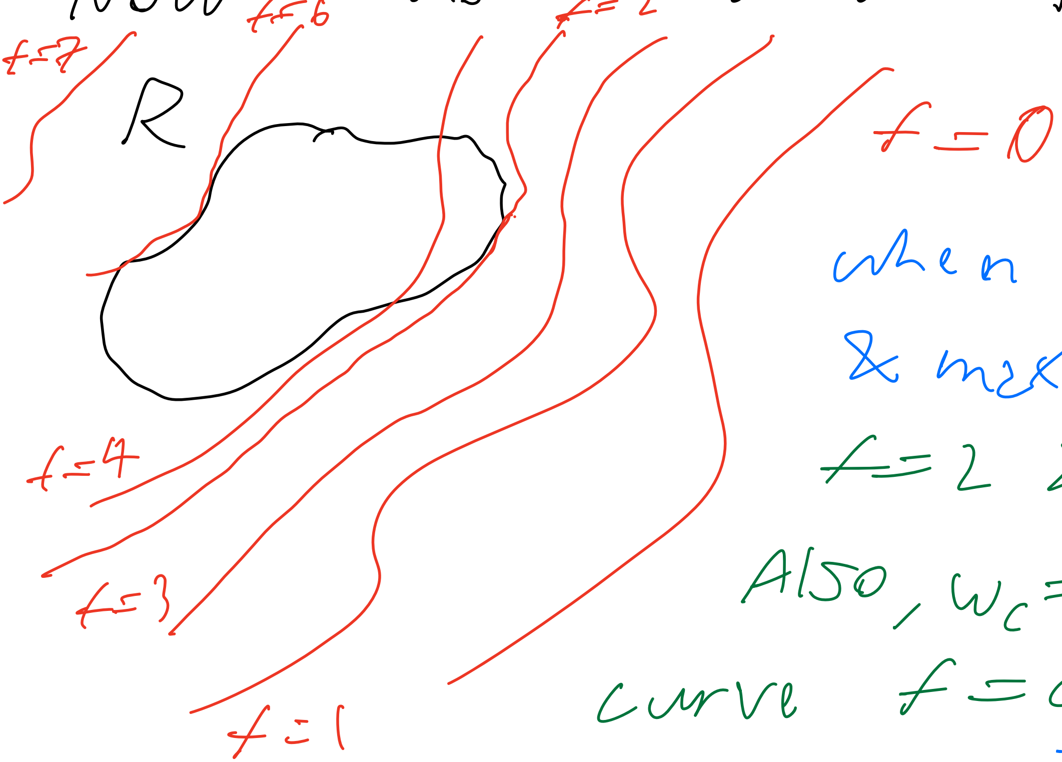
Why it might work: if $g=0$, then R is actually all of R^n, so we need to solve $\nabla f=0$. And so we have to solve $\nabla f = 0 = \lambda \cdot 0 = \lambda \nabla g$, which is exactly a special case of Lagrange multipliers.

Let G_c be graph of $f=c$ and suppose $f(v_c)=c$.

Then $u_c=(\nabla g)(v_c)$ is normal to the graph of

$g=0$, i.e. u_c is normal to R.

Now consider what happens when c varies:

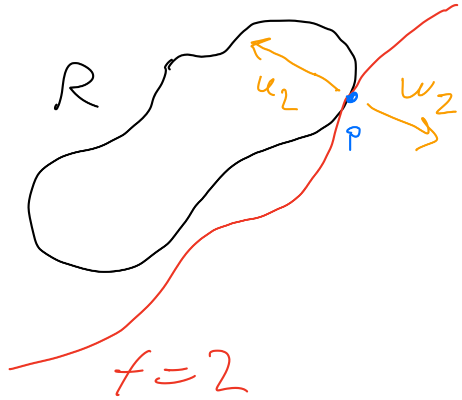


In this diagram, the curves $f=c$ intersect R when $2 \le c \le 6$. So $\min = 2$ & $\max = 6$.

$f=2$ & $f=6$ are tangent to R.

Also, $w_c = (\nabla f)(v_c)$ is normal to the curve $f=c$.

Let p be a intersection of $f=2$ & R, i.e. the blue tangency point below.



Since $f(p)=2$, we may take $v_2 = p$. Plug in this in:

$w_2 = (\nabla f)(p)$ normal to $f=2$

$u_2 = (\nabla g)(p)$ normal to R

u_2 & w_2 are parallel because they are normal vectors to the same curve at the same point. (because R & $f=2$ are tangent to p, w_2 is also normal to R)

Since u_2 & w_2 are parallel, $w_2 = c\,u_2$ for some c.

$\Longrightarrow \quad (\nabla f)(p) = c\,(\nabla g)(p)$.

Use λ instead of c: $\quad \nabla f = \lambda \nabla g$

Practice Problems

Find the max and min

- $f(x, y, z) = xyz$ subject to $x+y+z = 1$ and $x, y, z \geq 0$
- $f(x, y) = 4x^2+10y^2$ subject to $x^2+y^2 \leq 4$
- $f(x, y, z) = xyz$ subject to $x^2+y^2+z^2 = 1$

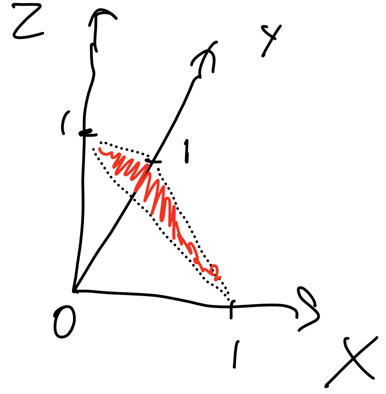
$g(x,y,z) = x+y+z-1$

We have 3 steps:

1. Show min & max exist
2. Solve $\nabla f = \lambda \nabla g$
3. Plug in solutions, see which is largest & smallest

Scratchwork

Step 1: $g = x+y+z-1$ is continuous, $R = \{(x,y,z) : x+y+z=1$ & $x,y,z \geq 0\}$. R is closed because R is intersection of closed sets $\{x+y+z=1\}$, $\{x\geq 0\}$, $\{y \geq 0\}$, $\{z \geq 0\}$. R is also bounded because it's the red triangle drawn to the left.



Alternatively, we can try to bound R:

If $(x,y,z) \in R$, $\|(x,y,z)\|^2 = x^2+y^2+z^2$

$\leq \underline{x^2+y^2+z^2+2(xy+yz+zx)} = \underline{(x+y+z)^2}$

$= 1^2 = 1 \Rightarrow \|(x,y,z)\| \leq 1 \Rightarrow R$ bounded by 1.

Step 2: $\nabla f = (yz, xz, xy)$, $\nabla g = (1,1,1)$.

So solve $(yz, xz, xy) = (\lambda, \lambda, \lambda)$, so

$$\left\{\begin{array}{l} yz=\lambda \\ xz=\lambda \\ xy=\lambda \end{array}\right\} \Rightarrow \begin{array}{l} \lambda^3=(xyz)^2 \geq 0 \Rightarrow \lambda \geq 0. \\ xyz = \lambda^{3/2} \end{array}$$

$$x = \frac{xyz}{yz} = \frac{\lambda^{3/2}}{\lambda} = \sqrt{\lambda} \text{ if } \lambda \neq 0, \text{ similarly,}$$

$y = \sqrt{\lambda}$, $z = \sqrt{\lambda}$. Then $0 = g = x+y+z-1$

$= 3\sqrt{\lambda} - 1 \Rightarrow \sqrt{\lambda} = \frac{1}{3} \Rightarrow \lambda = \frac{1}{9} \Rightarrow$

$(x,y,z) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \Rightarrow f(x,y,z) = 1/27$.

If $\lambda = 0$, then $yz = 0 \Rightarrow y=0$ or $z=0$.

If $y=0 \Rightarrow xz=0 \Rightarrow x=0$ or $z=0$.

So ≥ 2 among x, y, z must be 0.

Suppose $x=y=0$. Then $0=g=x+y+z-1=z-1$

$\Rightarrow z=1$. Then $f(0,0,1)=0\cdot 0\cdot 1=0$

Similarly, $x=z=0 \Rightarrow y=1 \Rightarrow f(0,1,0)=0$

Similarly, $y=z=0 \Rightarrow x=1 \Rightarrow f(1,0,0)=0$.

In orange are all the candidates. So,

min $f=0$, max $f=\frac{1}{27}$.

Try remaining problems before the discussion