Lines:

- For any point $P(x,y,z)$ it's position vector is $\overrightarrow{OP} = \langle x, y, z\rangle$.
- A line L is defined by a point on the line $P_0(x_0, y_0, z_0)$ and direction vector $\vec{v} = \langle a, b, c\rangle$.

$$L = \{(x,y,z) \in \mathbb{R}^3 : \boxed{\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle} \text{ for some } t \in \mathbb{R}\}$$

$$= \{\vec{r}(t) \in \mathbb{R}^3 : \boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}} \text{ for some } t \in \mathbb{R}\}$$

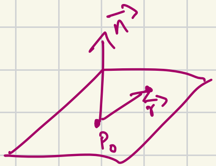
[where $\vec{r}(t)$ and $\vec{r}_0$ are position vectors of points on the line, t is a parameter, and $\vec{v}$ is a direction vector]

$$= \{(x,y,z) \in \mathbb{R}^3 : \boxed{x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct} \text{ for some } t \in \mathbb{R}\}.$$

$$= \{(x,y,z) \in \mathbb{R}^3 : \boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}\}$$

Planes:

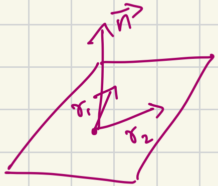
- A plane is defined by a normal vector $\vec{n} = \langle a, b, c\rangle$ and a point on the plane $P_0(x_0, y_0, z_0)$ with position vector $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$.



$$P = \{(x,y,z) \in \mathbb{R} : \langle a, b, c\rangle \cdot (\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle) = 0\}$$

$$= \{(x,y,z) \in \mathbb{R} : \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0\}$$

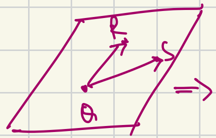
$$= \{(x,y,z) \in \mathbb{R} : a(x-x_0) + b(y-y_0) + c(z-z_0) = 0\}$$



$= \{ (x,y,z) \in \mathbb{R} : \quad ax + by + cz + d = 0 \}$ where $d = -(ax_0 + by_0 + cz_0)$.

$= \{ (x,y,z) \in \mathbb{R} : \quad \langle x,y,z \rangle = s\vec{r_1} + t\vec{r_2}$ for some parameters $s,t \in \mathbb{R} \}$.

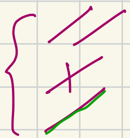
where $\vec{r_1}$ and $\vec{r_2}$ are direction vectors for the plane.



$\Longrightarrow \vec{n} = \overrightarrow{QR} \times \overrightarrow{QS}$

Intersections of Lines, and Planes.

- Intersections of just lines: In $\mathbb{R}^2$



parallel
intersect at a single pt
same line.

In $\mathbb{R}^3$

parallel
intersect
same line
skew lines.

- How to figure out which scenario?

fixed point on the line — direction vector

$L_1 : \quad X_1 = P_1 + a_1 t, \quad X_2 = P_2 + a_2 t, \quad X_3 = P_3 + a_3 t \qquad ((P_1, P_2, P_3), \langle a_1, a_2, a_3 \rangle)$

$L_2 : \quad Y_1 = Q_1 + b_1 t, \quad Y_2 = Q_2 + b_2 t, \quad Y_3 = Q_3 + b_3 t \qquad ((Q_1, Q_2, Q_3), \langle b_1, b_2, b_3 \rangle)$

Rewrite L_2 : $Y_1 = Q_1 + b_1 s$, $Y_2 = Q_2 + b_2 s$, $Y_3 = Q_3 + b_3 s$.

- Parallel: check if $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ are parallel.

- Intersecting: solve the system

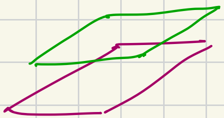
$$\left.\begin{aligned} P_1 + a_1 t &= Q_1 + b_1 s \\ P_2 + a_2 t &= Q_2 + b_2 s \\ P_3 + a_3 t &= Q_3 + b_3 s. \end{aligned}\right\}$$

That is find (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are the same.

- Prerequisite: Figure out how to solve a single equation and 2×2 systems.

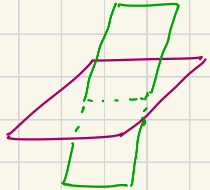
- Skew lines: prove that lines are not parallel, nor do they intersect.

- Intersections of planes:



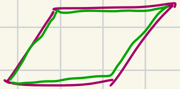
parallel

$\Leftrightarrow$ normal vectors are parallel



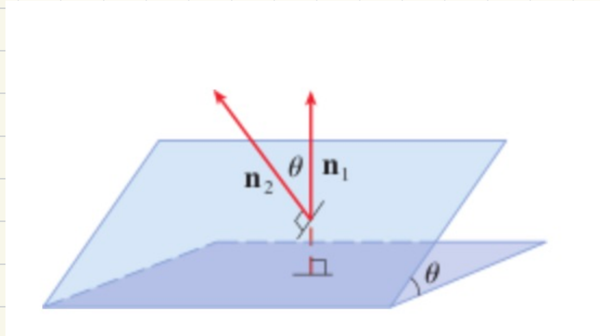
Intersecting planes

$\to$ Find line at the intersection



same plane

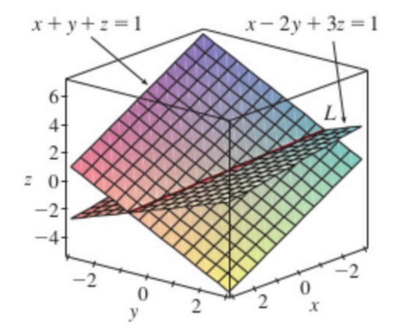
Definition: Angle between planes is the angle between the normal vectors.



Example 7

(a) Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$.

(b) Find symmetric equations for the line of intersection L of these two planes.



(a) $M_1: x+y+z=1$

$M_2: x-2y+3z=1$

Let $\vec{n}_1$ be the normal vector for M_1 and $\vec{n}_2$ be that for M_2. $\Rightarrow \cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \cdot \|\vec{n}_2\|} \xrightarrow{\text{DIY}} \theta = \cos^{-1}\left(\dfrac{2}{\sqrt{42}}\right)$

(b) Method 1: Solve for x, y, z in $\begin{matrix} x+y+z=1 \\ x-2y+3z=1 \end{matrix}$. For example via row reduction. } If you know some linear algebra.

Method 2: Find two points on the line.

Eg: · plug in $z=0$ in the system $\rightarrow$ $\left.\begin{array}{l} x+y=1 \\ x-2y=1 \end{array}\right\} \Rightarrow \underbrace{(x,y,z)=(1,0,0)}$

Call this point P.

· plug in $z=1$ in the system $\rightarrow$ $x+y=0 \rightarrow y=-x$

$x-2y=-2 \rightarrow x+2x=-2$

$\rightarrow x=\frac{-2}{3}$

$\rightarrow y=\frac{2}{3}$

$\Rightarrow$ Call the second point $Q\left(\frac{-2}{3},\frac{2}{3},1\right)$

$\Rightarrow$ Take the direction vector as $\overrightarrow{PQ}=\left\langle\frac{-5}{3},\frac{2}{3},1\right\rangle$

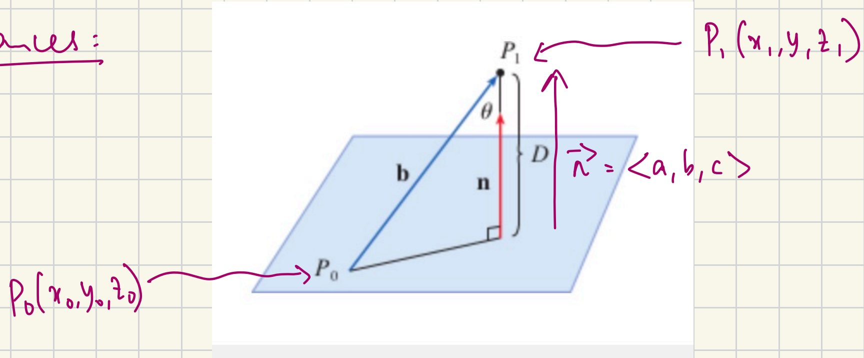
$\Rightarrow \quad \vec{r}(t)=\underbrace{\overrightarrow{OP}}+\overrightarrow{PQ} \Rightarrow \left.\begin{array}{l} x=1-\frac{5}{3}t \\ y=0+\frac{2}{3}t \\ z=0+t \end{array}\right\} \Rightarrow \frac{x-1}{-5/3}=\frac{y}{2/3}=\frac{z}{1}$

$P=(x,y,z)$

$\Rightarrow \overrightarrow{OP}=\langle x-0,y-0,z-0\rangle$

Method 3: Direction vector for L can be taken as $\vec{n}_1 \times \vec{n}_2$.

Distances:



We want the shortest distance of a point to the plane.

Let $\vec{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

Then $D = \left| \text{comp}_n b \right| = \left| \dfrac{n \cdot b}{|n|} \right| = \left| \dfrac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}} \right|$

$$= \left| \frac{ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)}{\sqrt{a^2 + b^2 + c^2}} \right| = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

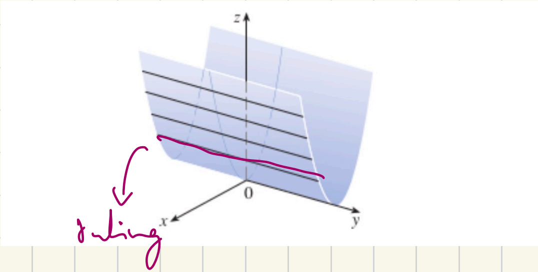
P_0 is a point on the plane $ax + by + cz + d = 0$

Cylinders:

Definition: Cylinder is a surface that consists of all lines that are parallel to a given line through a plane curve.

Example 1

Sketch the graph of the surface $z = x^2$.



Eg: any plane curve that has rulings parallel to an axis

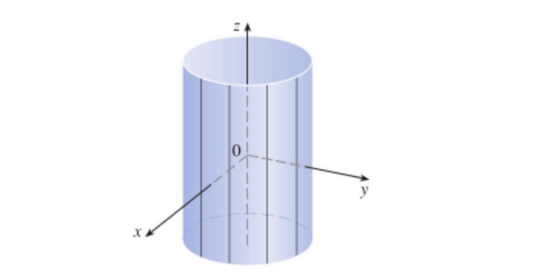
Example 2

Identify and sketch the surfaces.

(a) $x^2 + y^2 = 1$

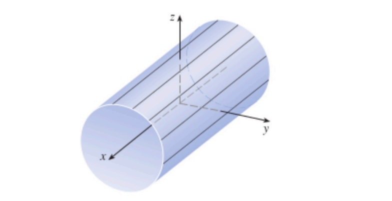
(b) $y^2 + z^2 = 1$

(a)



note: z is missing in $x^2+y^2=1$

(b)



Pop Quiz 1:

1. Find the cross product $a \times b$

$$a = 2j - 4k$$

$$b = -i + 3j + k$$

2. Find the $\underbrace{\text{scalar projection}}_{\text{comp}_a b = \frac{b \cdot a}{|a|}}$ of b onto a

$$a = \langle 4, 7, -4 \rangle$$

$$b = \langle 3, -1, 1 \rangle$$