Quiz 2 graded
Quiz 3 released, due 7/1 11:59

Midterm 2 may change format depending on
next ~~few~~ days & also poll, TBD

Cylindrical and Spherical Integrals

Lecture for 6/30

Derivation of coordinates themselves was on the prelec video for 6\13

Converting Integrals

$\iiint f(x,y,z)\,dx\,dy\,dz$

Cylindrical: (x, y, z) = (rcos(θ), rsin(θ), z)

- Bounds: $0 \le \theta < 2\pi$

match

$= \int_{??}^{??}\int_{?}^{?}\int_{?}^{?} f(\rho\cos\theta\sin\varphi, \ldots, \ldots)$ replace dxdydz

Spherical: (x, y, z) = ρ(cos(θ) sin(φ), sin(θ)sin(φ), cos(φ))

- Bounds: $0 \le \varphi \le \pi$, $0 \le \theta < 2\pi$

We can convert functions, but what about ~~dA~~ dV = dx dy dz?

- Cylindrical: d~~A~~V = r dr dθ dz
- Spherical: ~~dA~~ dV = $\rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$

don't forget the extra terms when converting

Last thing: converting bounds

Converting bounds on x, y, z to bounds on r, θ, z (for example) is a similar procedure to the one described last Thursday on converting bounds on x, y to bounds on r & θ

All of the comments & warnings on polar in the 6/26 lecture apply to cylindrical in this lecture as well. The general ideas, like the warning on including variable dependencies on top of just finding min & max for each variable, also apply to spherical.

Note: Some sources may swap θ & φ, letting $z = \rho\cos\theta$ and using φ to denote the same angle that is called θ in cylindrical. It doesn't matter which order you use as long as you keep it consistent with the π & 2π bounds on the angles, the cos & sin expressions for x, y, z.

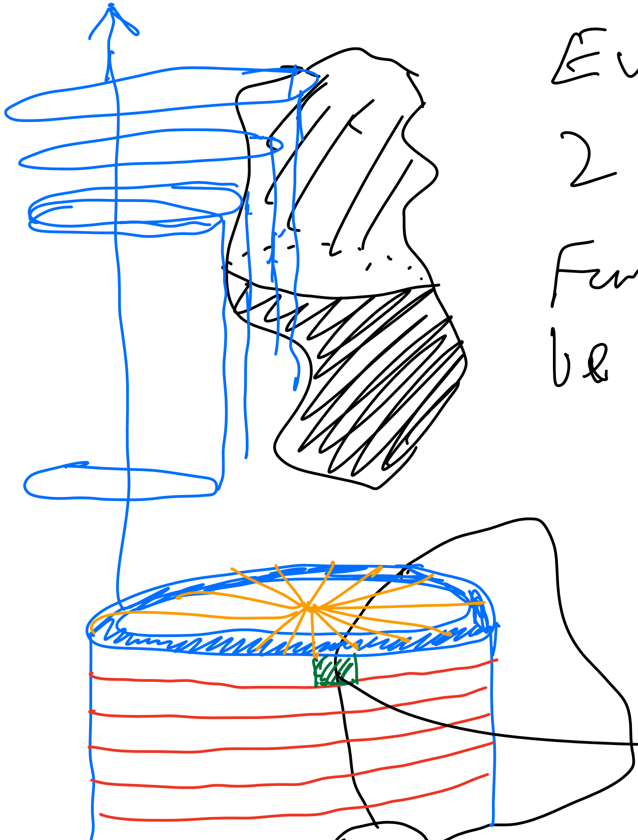
For example, if you write $z = \rho\cos\theta$ at some point, then θ will be forced to be in $[0, \pi]$ and you'll have φ in $[0, 2\pi)$.

Extra practice: If you struggle with cylindrical, go back to polar problems from last week (lecture problems &

worksheet, solve those 25 warmups, then come back to cylindrical.

Differential Derivation

Let's begin with cylindrical because it's simpler

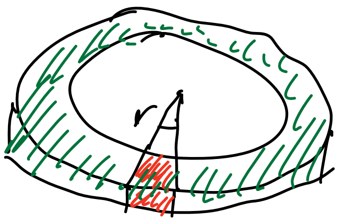


Every point in the shape is between 2 cylinders of almost the same size.

Furthermore, the gap between cylinders can be broken apart by height & angle.

After breaking the shape apart like this, each piece is almost a rectangular prism. This is what we want for dV since we justified triple integral with Riemann sums with small rectangular prisms on 6\27.

Let dV be the volume of this piece, then we will calculate dV & show it to be as expected.



Suppose the red piece represents some small change in height dz, small change in angle $d\theta$, small change in radius dr.

Volume of entire green Shell = (volume outer cylinder) − (volume of outer cylinder) =

$\pi\,(\text{out radius})^2 \cdot h_{out} - \pi\,(\text{inn radius})^2 \cdot h_{in} =$

$\pi\,(r+dr)^2 \cdot dz - \pi r^2 dz =$

$\pi \cdot dz\left[(r+dr)^2 - r^2\right] = \pi dz\left[2r\,dr + (dr)^2\right].$

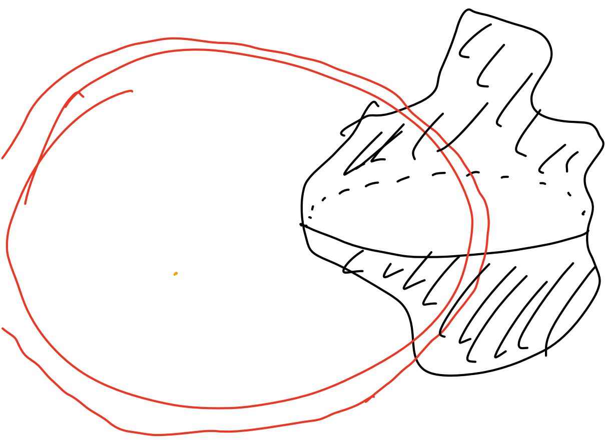
This corresponds to an angle of $360^\circ = 2\pi$. In order to get the portion for an angle of $d\theta$, we need to scale:

$$dV = \frac{d\theta}{2\pi}\left[\text{vol green shell}\right] = \frac{d\theta}{2} dz\left[2r\,dr + (dr)^2\right]$$

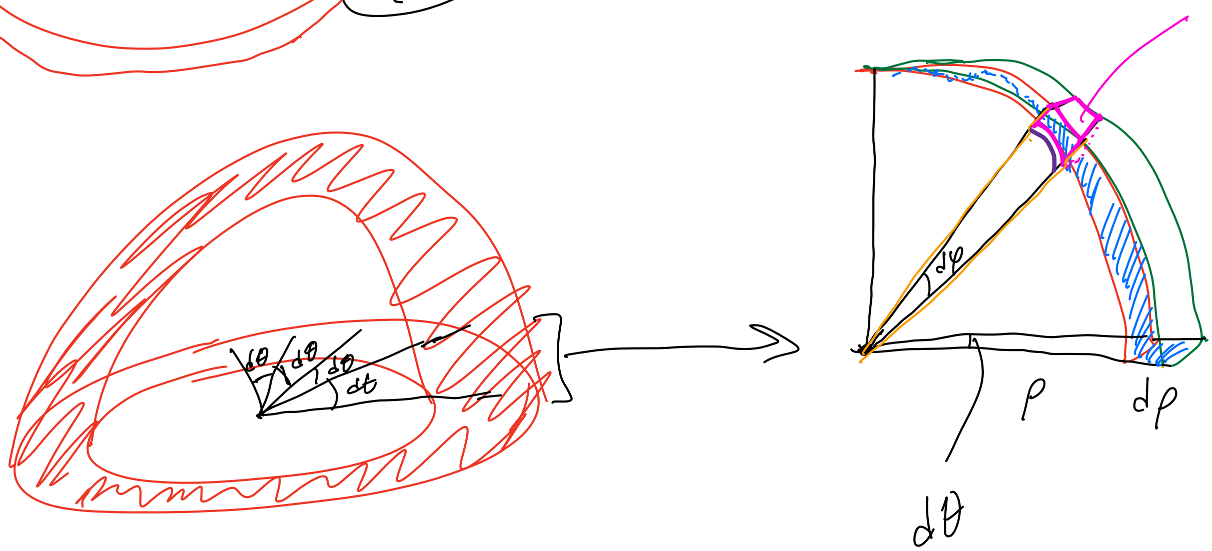
$$= r\,dr\,d\theta\,dz + \frac{1}{2}d\theta\,dz\,(dr)^2.$$

2nd term is 0 since as explained on 6/26 lecture, $(du)^2 = 0$ for any variable u. Same comment on "use Δ's and take $\Delta \to 0$ to prove things properly" applies.

Let's use the same idea for spherical coordinates:

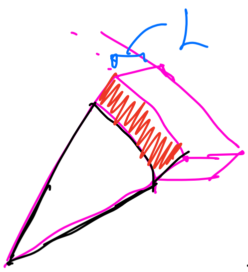


Draw spherical shells intersecting object, split each shell by θ & then by φ.



dV = volume of this piece will represent small change in ρ, θ & φ.

$$dV = (\text{volume larger wedge}) - (\text{volume smaller wedge}).$$

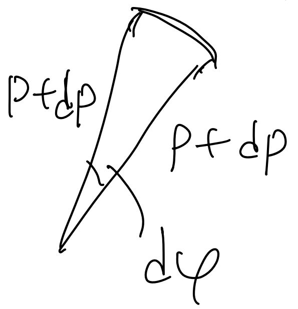


$$= (\text{area larger base})\cdot\text{height} - (\text{area smaller base})\cdot\text{height}$$

$$= \left(\text{area}(\text{sector})\cdot L\right) - \left(\text{area}(\triangle)\cdot L\right)$$

$$= L\left[\frac{d\varphi}{2}(\rho+d\rho)^2 - \frac{d\varphi}{2}\rho^2\right]$$

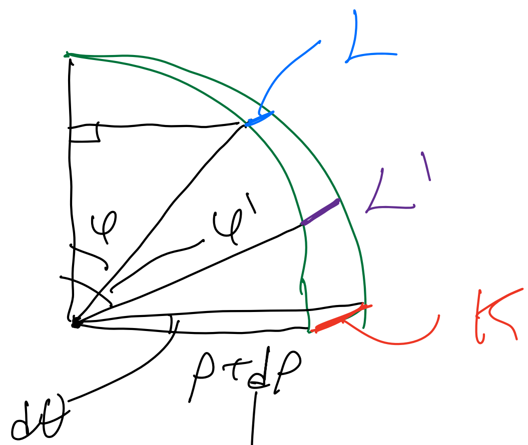
(from area of circular sector formula)

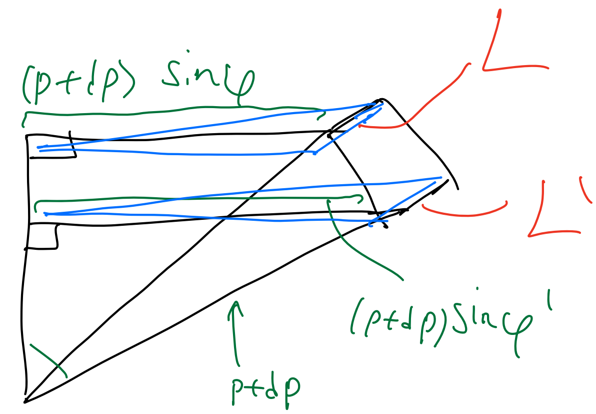


$$= L\,\frac{d\varphi}{2}\left[2\rho\, d\rho + (d\rho)^2\right].$$

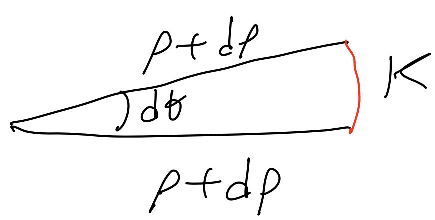
For the same reason as earlier derivations with squared differentials, $(d\rho)^2 = 0$, so this is

$$= L\,\rho\, d\rho\, d\varphi.$$





$$\frac{L'}{L} = \frac{(\rho+d\rho)\sin\varphi'}{(\rho+d\rho)\sin\varphi} = \frac{\sin\varphi'}{\sin\varphi}$$



Entire circumference is $C = 2\pi(\rho+d\rho)$, but we only have angle of $d\theta$, so

$$K = \frac{d\theta}{2\pi} C = d\theta(\rho+d\rho)$$

So setting $L' = K$, we have $\varphi' = 90^\circ \Rightarrow$

$$L = L'\,\frac{\sin\varphi}{\sin\varphi'} = K\sin\varphi = d\theta(\rho+d\rho)\sin\varphi$$

So finally, $dV = \angle\, \rho\, d\rho\, d\varphi =$

$$d\theta(\rho + d\rho)\sin\varphi\, \rho\, d\rho\, d\varphi =$$

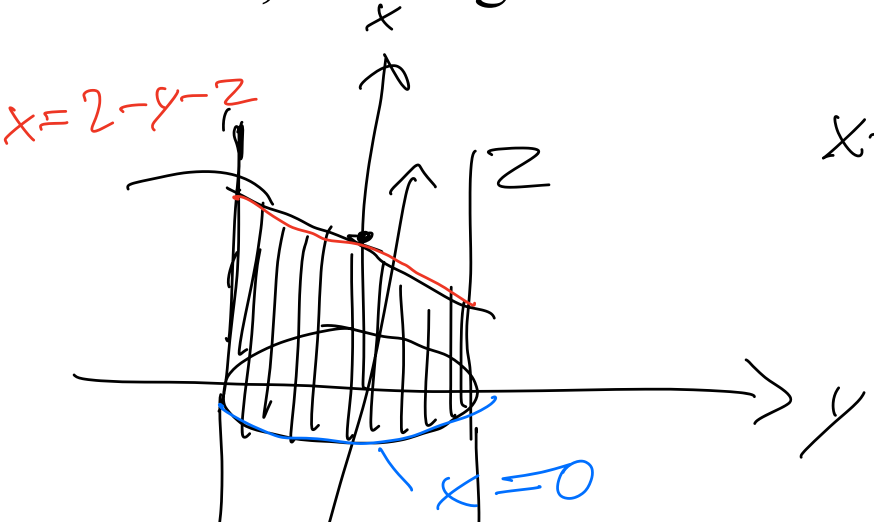
$$\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta + \overbrace{\rho \sin\varphi\, d\theta\, (d\rho)^2 d\varphi}^{(d\rho)^2 = 0} =$$

$$\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Practice Problems

Evaluate $\iiint_E f\,dV$ for the following functions and regions:

- $f = z$, E is region inside $y^2+z^2 = 1$ and between $x+y+z = 2$, $x = 0$
- $f = x^2+y^2$, E is portion of $x^2+y^2+z^2 = 4$ with $y \geq 0$
- $f = x^2$, E is region inside $x^2+y^2+z^2 = 36$ and $z = -(3x^2+3y^2)^{1/2}$



$$x+y+z=2 \Longrightarrow x = 2-y-z$$

Scratchwork

Bounds: $0 \leq y, z \leq 1$, $0 \leq x \leq 2 - y - z$.

$$\left.\begin{aligned} y &= r\cos\theta \\ z &= r\sin\theta \\ x &= x \end{aligned}\right] \Rightarrow dA = r\,dr\,d\theta\,\underbrace{dx}_{\text{dx now, not dz}}$$

$y^2 + z^2 \leq 1 \implies 0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

$0 \leq x \leq 2 - r\cos\theta - r\sin\theta$.

$$f\,dV = 2r\,dr\,d\theta\,dx = r^2\sin\theta\,dr\,d\theta\,dx$$

$$\int_E f\,dV = \int_0^{2\pi}\int_0^{1}\int_0^{2-r\cos\theta-r\sin\theta} r^2\sin\theta\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} r^2\sin\theta\,(2-r\cos\theta-r\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1}\left[2r^2 s - r^3 sc - r^3 s^2\right]dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{2}{3}\sin\theta - \underbrace{\frac{1}{4}\sin\theta\cos\theta}_{=\frac{1}{8}\sin 2\theta} - \frac{1}{4}\sin^2\theta\right]d\theta$$

$$= \underbrace{\left[-\frac{2}{3}\cos\theta + \frac{1}{16}\cos 2\theta\right]\Bigg|_0^{2\pi}}_{0} - \frac{1}{4}\int_0^{2\pi}\sin^2\theta\,d\theta$$

$$= -\frac{1}{4}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{1}{8}\int_0^{2\pi}(\cos 2\theta - 1)\,d\theta$$

$$c2\theta = 1 - 2s^2\theta$$

$$\Rightarrow s^2\theta = \frac{1-c2\theta}{2}$$

$$= \frac{1}{8}\left[\frac{1}{2}\sin 2\theta - \theta\right]\Bigg|_0^{2\pi}$$

$$= \frac{1}{8}\cdot -2\pi = -\frac{\pi}{4}.$$

Remark: remember your double angle formulas & how to get from $\cos^2$, $\sin^2$ into double angles.

$\int \cos^2$, $\int \sin^2$, $\int \sin^4$, . . . & other even powers which can't be done by u-sub appear now in cylindrical & spherical integral problems.