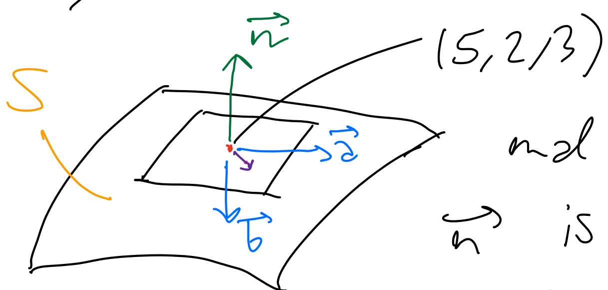
No logistical annoucements as of now

DW14 Q2: Find equation for tangent plane to S parametrized by $(x,y,z) = (u^2+1, v^3+1, u+v)$ at $(5,2,3)$.

First, let's see the values of u & v. $u^2+1=5 \Rightarrow u^2=4 \Rightarrow u=\pm 2$, $v^3+1=2 \Rightarrow v^3=1 \Rightarrow v=1$, $u+v=3 \Rightarrow u=3-v=2$. So $(u,v)=(2,1)$.



We just need to find a normal vector $\vec{n}$ to the plane. Recall $\vec{n}$ is perpendicular to any vector in the plane, so if we find 2 vectors $\vec{a}, \vec{b}$ in the plane, we can take $\vec{n} = \vec{a} \times \vec{b}$ by perp. property of cross product.

Consider $f: \mathbb{R}^2 \to \mathbb{R}^3$ as $f(u,v) = (u^2+1, v^3+1, u+v)$. Notice that at any point $f(u_0, v_0)$ on S, we can move slightly to $f(u_0+\varepsilon, v_0)$. Then $f(u_0+\varepsilon, v_0) - f(u_0, v_0)$ for ε sufficiently small resembles a tangent vector. For the sake of convenience, scale the vector, then $\dfrac{f(u_0+\varepsilon, v_0) - f(u_0,v_0)}{\varepsilon}$ resembles a tangent vector more & more as $\varepsilon \to 0$. But

$$\lim_{\varepsilon \to 0} \frac{f(u_0+\varepsilon, v_0) - f(u_0,v_0)}{\varepsilon} = f_u(u_0,v_0),$$ so we

can take f_u as one tangent vector. Similarly, f_v is another tangent vector.

$f = (u^2+1, v^3+1, u+v)$, $f_u = (2u, 0, 1)$, $f_v = (0, 3v^2, 1)$,

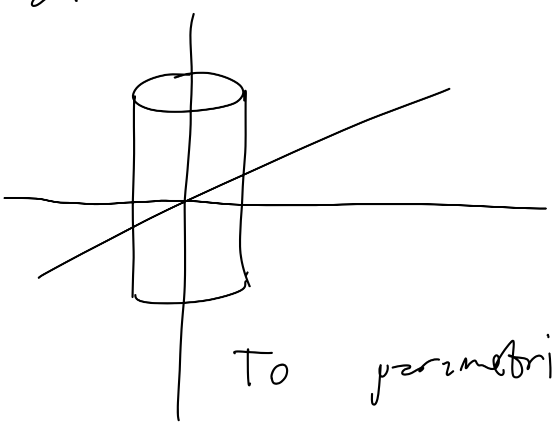
so $f_u \times f_v = (-3v^2, -2u, 6uv^2)$.

$$\left|\begin{matrix} i & j & k \\ 2u & 0 & 1 \\ 0 & 3v^2 & 1 \end{matrix}\right| \begin{matrix} i & j \\ 2u & 0 \\ 0 & 3v^2 \end{matrix}$$

Plug in $(u,v) = (2,1)$ to get $\vec{n} = (-3, -4, 12)$. Recall $(5,2,3)$ on plane, so one possible plane equation is

$$-3(x-5) - 4(y-2) + 12(z-3) = 0.$$

1b: Parametrize part of $x^2+z^2=9$ lying above xy-plane and between $y=-4$ & $y=4$



Above xy-plane $\Rightarrow$ above $z=0$ $\Rightarrow$ $z \geq 0$.

Also, $-4 \leq y \leq 4$.

To parametrize $x^2+z^2=9$, take $x = 3\cos\theta$ & $z = 3\sin\theta$ with $0 \leq \theta \leq 2\pi$. So

$$(x,y,z) = (3\cos\theta, y, 3\sin\theta), \quad 0 \leq \theta \leq 2\pi, \ -4 \leq y \leq 4$$

Textbook/instructors may write $(3\cos t, s, 3\sin t)$. - - - , note this is the same answer.

4a: Find $\iint_S x^2yz\,dS$ where S is part of plane $z = 1+2x+3y$ lying above $[0,3]\times[0,2]$.

Note that for any double/triple integral, we generally have 4 steps: convert differential, convert variables, change bounds, evaluate new integral.

Since S is described by $z=f(x,y)$ for $f=1+2x+3y$, we may apply the formula for surfaces parametrized by 1 function:

$\frac{dS}{dx\,dy} = \sqrt{1+f_x^2+f_y^2} = \sqrt{1+2^2+3^2} = \sqrt{14}$, so

$dS = \sqrt{14}\,dx\,dy$. Also, plug in for z:

$x^2yz\,dS = x^2y(1+2x+3y)\sqrt{14}\,dx\,dy$.

Rectangle $[0,3]\times[0,2]$ described by $0\le x\le 3,\ 0\le y\le 2$.

So $\iint_S \cdots dS = \int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{14}\,dy\,dx =$

$\sqrt{14}\int_0^3\int_0^2 (x^2y+2x^3y+3x^2y^2)\,dy\,dx =$

$\sqrt{14}\int_0^3 (2x^2+4x^3+8x^2) = \sqrt{14}\int_0^3 (10x^2+4x^3)\,dx$

$= \sqrt{14}\left(\frac{10}{3}x^3+x^4\right)\Big|_0^3 = \sqrt{14}(90+81) = 171\sqrt{14}$

$\int_0^3\int_0^{3-x}\int_0^{3-x-y} y^2\,dz\,dy\,dx = \int_0^3\int_0^{3-x} \underbrace{y^2(3-x-y)}_{y^2(3-x)-y^3}\,dy\,dx$

$= \int_0^3\left(\frac{(3-x)^3}{3}(3-x) - \frac{(3-x)^4}{4}\right)dx =$

$$\int_0^3 \frac{(3-x)^4}{3} - \frac{(3-x)^4}{4} = \int_0^3 \frac{(3-x)^4}{12} = \int_0^3 \frac{x^4}{12} =$$

$u = 3-x$
$dx = -du$
$0 \to 3 \Rightarrow 3 \to 0$
$\Downarrow$
$0 \to 3$

$$\frac{x^5}{60}\Big|_0^3 = \frac{243}{60} = \frac{81}{20}.$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$u = a+b-x$

Q5: Find $\iint_S F \cdot d\vec{S}$ for $F = \langle y, -x, 2z \rangle$, S hemisphere $x^2+y^2+z^2=4$, $z \geq 0$.

Note: There is a fast solution using divergence theorem from next week. For now, here is the solution using prior content.

If we can get one variable in terms of the others, then there is a nice formula for dS. In this case, note the "$z \geq 0$" suggests solving for z. Let's try:

$z^2 = 4 - x^2 - y^2 \Rightarrow z = \pm\sqrt{4-x^2-y^2}$.

But $z \geq 0$, so $z = +\sqrt{4-x^2-y^2}$.

Square root needs to be defined, so $4-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 4$. Let $f(x,y) = \sqrt{4-x^2-y^2}$, $R = \{(x,y): x^2+y^2 \leq 4\}$.

So our surface integral is $z = f(x,y)$ over the region R, and $\frac{dS}{dx\,dy} =$

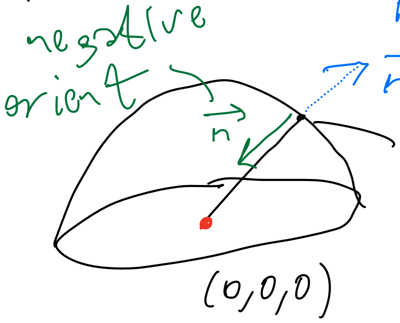
$$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+\left(\frac{-2y}{2\sqrt{\cdots}}\right)^2+\left(\frac{-2x}{2\sqrt{\cdots}}\right)^2}$$

$$= \sqrt{1+\frac{y^2}{4-x^2-y^2}+\frac{x^2}{4-x^2-y^2}} = \sqrt{\frac{4}{4-x^2-y^2}}$$

$$= \frac{2}{\sqrt{4-x^2-y^2}} \Rightarrow dS = \frac{2\,dx\,dy}{\sqrt{4-x^2-y^2}}.$$

However, we need to find $F \cdot d\vec{S}$.

Luckily, $F \cdot d\vec{S} = (F \cdot \vec{n})\,dS$ where $\vec{n}$ is the unit normal to the surface S.



From the diagram, a normal to (x_0,y_0,z_0) is $\langle x_0,y_0,z_0\rangle$, so a unit normal is $\frac{\langle x_0,y_0,z_0\rangle}{\sqrt{x_0^2+y_0^2+z_0^2}} = \frac{1}{2}\langle x_0,y_0,z_0\rangle$

since $x_0^2+y_0^2+z_0^2=4$ on S. So

$F\cdot\vec{n} = \langle y,-x,2z\rangle \cdot \frac{1}{2}\langle x,y,z\rangle =$

$\frac{1}{2}xy - \frac{1}{2}xy + z^2 = z^2 = 4-x^2-y^2.$

So $(F\cdot\vec{n})\,dS = 2\sqrt{4-(x^2+y^2)}\,dx\,dy.$

Integrating square roots is tricky, and we see a x^2+y^2, so use polar.

$x=r\cos\theta$, $y=r\sin\theta$, $4\geq x^2+y^2=r^2$

$\Rightarrow 0\leq r\leq 2$. No restrictions on θ

$\Rightarrow 0\leq\theta\leq 2\pi$. Integrand becomes

$2\sqrt{4-r^2}\,r\,dr\,d\theta$. So at last,

$$\iint_S F\cdot d\vec{S} = \iint_R 2\sqrt{4-(x^2+y^2)}\,dx\,dy =$$

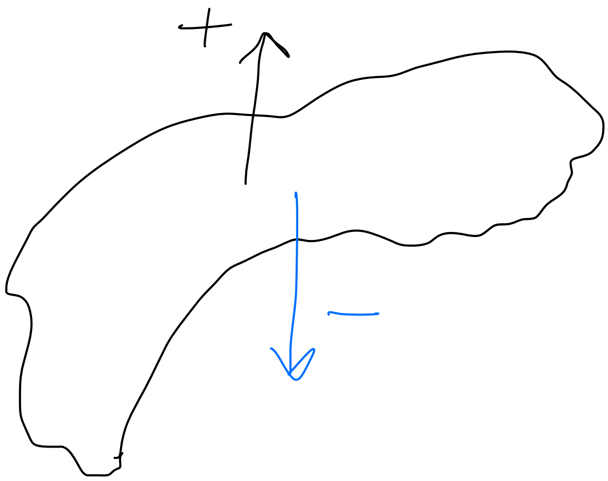
$$\int_0^2\int_0^{2\pi} 2r\sqrt{4-r^2}\,d\theta\,dr = 4\pi\int_0^2 r(4-r^2)^{1/2}\,dr$$

$$= 4\pi\left(-\tfrac{1}{3}\right)(4-r^2)^{3/2}\Big]_0^2 = -\frac{4\pi}{3}\cdot -4^{3/2}$$

$$= \frac{4\pi}{3}4\sqrt{4} = \frac{32\pi}{3}.$$

$\frac{3}{2}\cdot(-2r) = -3r$, so take $-\frac{1}{3}$ to balance out

However, this accidentally assumed $\vec{n}$ had positive orientation. So the answer is actually $-\dfrac{32\pi}{3}$.



outside surface = +
inside = −

16.7 #11: Verify divergence theorem for $F = \langle z, y, x \rangle$ on $E = \{x^2+y^2+z^2 \le 49\}$, that is, show $\iint_S F \cdot d\vec{S} = \iiint_E \operatorname{div} F \, dV$.

$$\operatorname{div} F = \frac{\partial}{\partial x} z + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} x = 0 + 1 + 0 = 1, \text{ so}$$

$$\iiint_E \operatorname{div} F \, dV = \iiint_E dV = \operatorname{Vol}(E) = \text{volume of ball of radius } 7 = \frac{4}{3}\pi \cdot 7^3 = \frac{343 \cdot 4\pi}{3}$$

$$\begin{array}{r} 343 \\ \cdot\ 4 \\ \hline 1372 \end{array}$$

$\dfrac{1372\pi}{3}$. Now let's find $F \cdot d\vec{S}$.

Recall $F \cdot d\vec{S} = (F \cdot \vec{n})\, dS$

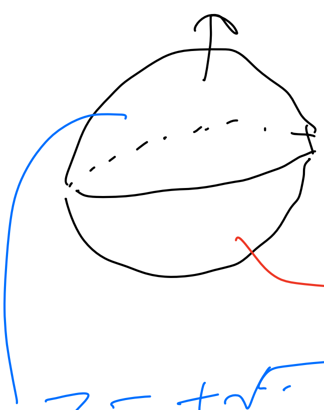
where $\vec{n}$ is positive unit normal to S. We have (scroll to the drawing of a sphere from before & same explanation) that $\vec{n} = \frac{\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}} = \frac{1}{7}\langle x,y,z\rangle$ for top $\frac{1}{2}$ of sphere, $\vec{n} = -\frac{1}{7}\langle x,y,z\rangle$ else.

If we have 1 variable in terms of the others there is a simple formula for dS. When $z = f(x,y)$, then $dS = \sqrt{1+f_x^2+f_y^2}$.

Note: $S = \partial E = \{x^2+y^2+z^2=49\}$, the surface of the ball.

$z^2 = 49 - x^2 - y^2 \Rightarrow z = \pm\sqrt{49-x^2-y^2}$. On S^+, $z = +\sqrt{\cdots}$. On S^-, $z = -\sqrt{\cdots}$.

Either way, $f(x,y) = +$ or $-\sqrt{49-x^2-y^2}$, so



$\} z \geq 0 := S^+$

$\} z \leq 0 := S^-$

$z = -\sqrt{\cdots}$

$z = +\sqrt{\cdots}$

$f_x = \left(\frac{-2x}{2\sqrt{\cdots}}\right)^2 = \frac{x^2}{49-x^2-y^2}$, $f_y = \frac{y^2}{49-x^2-y^2}$

$\Rightarrow 1+f_x^2+f_y^2 = \frac{49}{\cdots} \Rightarrow dS = \frac{7\,dx\,dy}{\sqrt{49-x^2-y^2}}$

$F \cdot \vec{n} = \langle z, y, x \rangle \cdot \pm \frac{1}{7} \langle x, y, z \rangle =$

$\pm \frac{1}{7}(2xz + y^2)$, so $(F \cdot \vec{n})\, dS =$

$$\pm \frac{(2xz+y^2)\,dx\,dy}{\sqrt{49-(x^2+y^2)}} = \frac{(2xz+y^2)\,dx\,dy}{\pm\sqrt{49-(x^2+y^2)}}$$

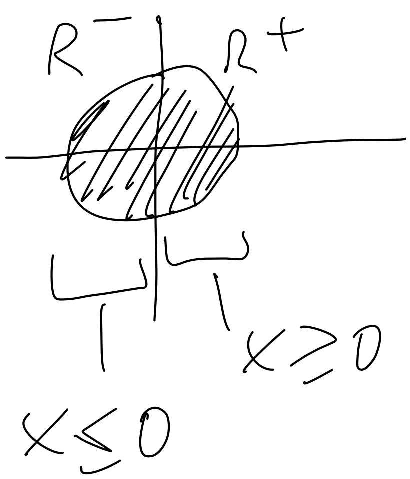
$$= \frac{(2xz+y^2)\,dx\,dy}{z} = \left(2x + \frac{y^2}{z}\right) dx\,dy.$$

For $z = \pm\sqrt{49-x^2-y^2}$ to exist, $49-x^2-y^2 \geq 0$

$\Rightarrow x^2+y^2 \leq 49$, so let $R = \{x^2+y^2 \leq 49\}$.

Then $\iint_S F \cdot d\vec{S} = \iint_S (F \cdot \vec{n})\, dS = \iint_R \cdots dx\,dy$

$$= \iint_R \left(2x + \frac{y^2}{z}\right) dx\,dy =$$



$$\iint_R 2x = \iint_{R^+} 2x - \left(\iint_{R^-} (-2x)\right)$$

$$= \iint_{R^+} 2x - \iint_{R^+} 2x = 0$$