MATH 243: Worksheet 5 **Discussion Section:** ____________

Textbook Sections: 13.3 and 13.4
Topics: Arc length and curvature; velocity and acceleration
Instructions: Try each of the following problems, show the detail of your work. Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

ARC LENGTH AND CURVATURE:

1. Find the length of the curve $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle$, $0 \le t \le 1$.

2. Consider the vector $\mathbf{r}(t) = \langle t, t^2, 4 \rangle$.
(a) Find the unit tangent vector $\mathbf{T}(t)$
(b) Find the unit normal vector $\mathbf{N}(t)$
(c) Find the curvature of the curve given by $\mathbf{r}(t)$ at any point t.

3. Consider the position function $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$. Find the vectors $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ at the point $(1, \frac{2}{3}, 1)$.

4. Find equations of the normal and osculating planes of the curve of intersection of the parabolic cylinders $x = y^2$ and $z = x^2$ at the point, $(1, 1, 1)$.

MOTION IN SPACE: VELOCITY and ACCELERATION

5. A particle moves with position function $\mathbf{r}(t) = \langle t \ln t, t, e^{-t} \rangle$. Find the velocity, speed, and acceleration of the particle.

6. Find the tangential and normal components of the acceleration vector for the position function $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + t^3\mathbf{j}$, $t \ge 0$.

7. If the acceleration of an object is given by $\mathbf{a}(t) = t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, find the object's position function given that the initial velocity is $\mathbf{v}(0) = \mathbf{k}$ and the initial position is $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

8. Consider a particle with the position function $\mathbf{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$. Find the velocity, acceleration, and speed of the particle at any time t.

Suggested Textbook Problems

Section 13.3	3-5, 7-9, 11, 13, 17, 19, 23, 24, 27, 29-33, 43, 47, 49, 50
Section 13.4	3, 5, 8, 9, 11, 15, 17-21, 23, 25, 28, 31, 32, 37, 39, 41

SOME USEFUL EQUATIONS:

The Arc Length Formula
The length L of a curve given by the vector function $\mathbf{r}(t)$, for $a \le t \le b$, can be computed using the formula

$$L = \int_a^b \|\mathbf{r}'(t)\| dt$$

Curvature

The *curvature* of the curve given by the vector function $\mathbf{r}$ is

$$\mathbf{k}(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

For a special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $r(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then

$$\mathbf{k}(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

Useful formula for curvature

$$\mathbf{k}(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Unit Tangent, Unit Normal and Unit Binormal Vectors

Unit tangent vector is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

Unit normal vector is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Unit binormal vector is given by

$$\mathbf{B}(t) = T(t) \times N(t)$$

The Tangential and Normal Components of Acceleration are denoted by $\mathbf{a_T}$ and $\mathbf{a_N}$ are respectively given by

$$a_T = v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(\mathbf{t})}{\|\mathbf{r}'(t)\|},$$

By using the curvature formula above, we have

$$a_N = \mathbf{k}v^2 = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|.\|\mathbf{r}'(t)\|^2}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$