Quiz: extended to 11:59 pm 6/16
Future quizzes will show both "due at" & "available until"

Multivariate Limits

Pre-lecture for 6/16

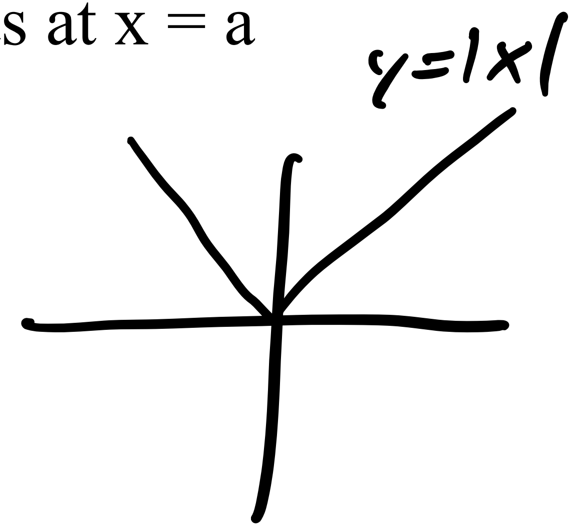
Homeworks: deadlines removed for future homeworks. will be available until 7/10. HOWEVER, do them past the advised "due date" at your own risk because you may suffer on quizzes &

exams from not having that practice

Directions for One Variable

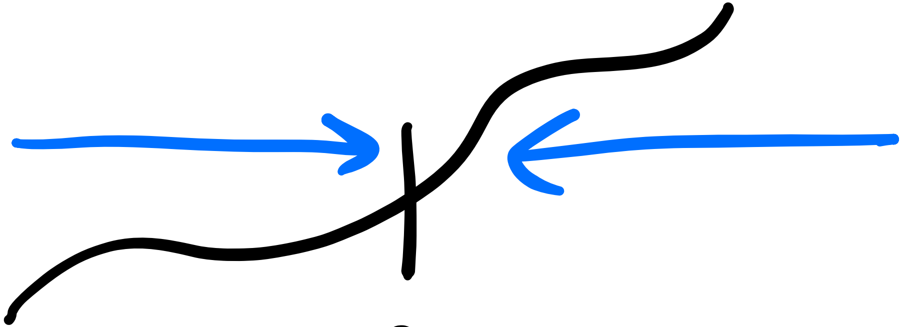
Materials: DW1,2 solutions up, DW3 ≈ 3pm today

- Recall $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x)$ then limit exists at $x = a$
- So we need left and right side directions equal



Also recall alternate forms of the limit:

- $\lim_{x\to a} f(x) = \lim_{x\to 0} f(x+a) = \lim_{|x|\to 0} f(x+a)$
- We will use the last alternate form



$x = a$

Directions for Multiple Variables

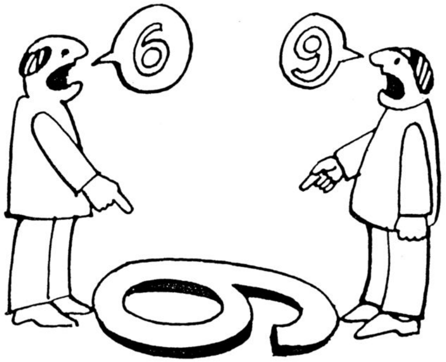
constant vector

Definition of multivariate limit:

- $\lim_{v \to c} f(v) = L$ if $f(v)$ approaches L regardless how v approaches c

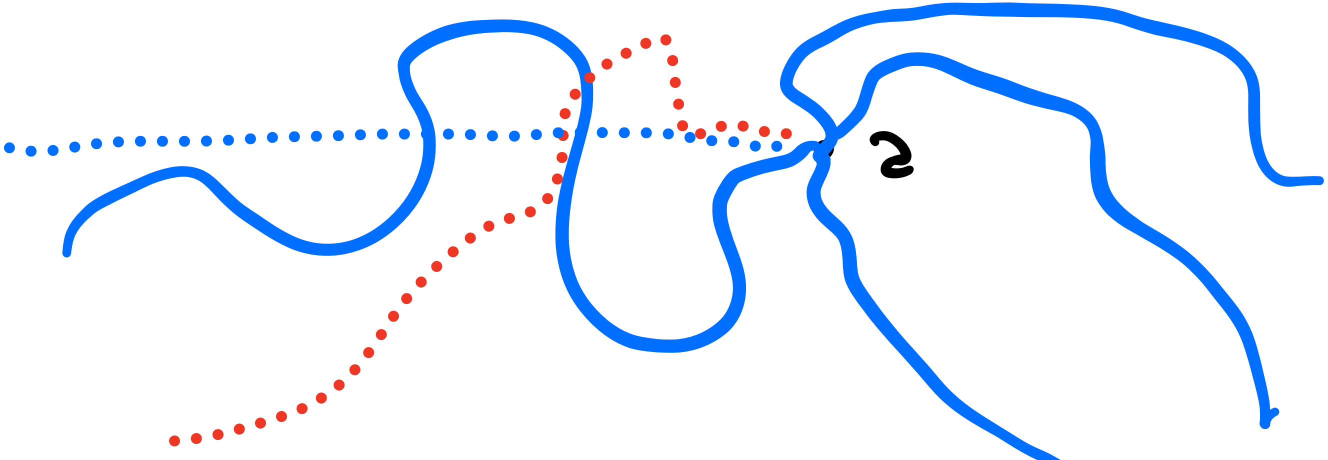
v vector variable

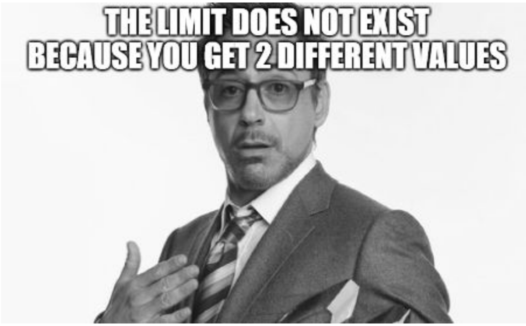
https://donmcminn.com/2018/02/yes-hand-value-dialectic-thinking/



Let's consider an alternate form:

- $\lim_{v \to c} f(v) = \lim_{\|v\| \to 0} f(v+c)$
- $\|v\|$ is a scalar, we can now use 1 variable tactics





Properties of Limits

- Sums: lim(f+g) = lim f + lim g
- Products: lim fg = (lim f)(lim g)
- Composition: lim f ∘ g = f ∘ (lim g) when f is continuous, 1-var func.
- Limits on right side must exist for rule to apply

Example with variables:

$$\lim_{v \to c}(f(v)+g(v)) = \lim_{v \to c} f(v) + \lim_{v \to c} g(v)$$

$$\lim_{v \to c} f(g(v)) = f\left(\lim_{v \to c} g(v)\right)$$

More Properties

These follow by using the previous slide

- lim cf = c lim f, lim (f-g) = lim f - lim g
- Limit of finite products or sums
- lim f/g = (lim f)/(lim g) provided lim g non-zero

1st prop: g=c in product property,
replace g with -g
$\lim(f-g) = \lim f + \lim(-g) = \lim f - \lim g$

2nd: $\lim(fgh) = \lim fg \lim h = \lim f \lim g \lim h$

3rd: $(\lim g)(\lim \frac{f}{g}) = \lim g \frac{f}{g} = \lim f$,

$\lim g \neq 0$ so divide

Continuity

Recall continuity for one variable

- If limit exists and $\lim_{x \to a} f(x) = f(a)$, f continuous at a
  - Function f cont. if it is continuous for every value in domain
- Same definition for multivariable functions
- We now omit writing the variable when it doesn't matter

$$\lim_{v \to c} f(v) = f(c)$$

omitted

Properties of Continuity

If f and g are continuous, these are too:

- $f \pm g$, fg, $f \circ g$ if composition makes sense
- f/g at any point where $g \neq 0$

Continuity is preserved when space gets upgraded:

- If $f(x)$ cont, then $g(a_1, a_2, a_3, \ldots) = f(a_i)$ continuous

Example: $f(x,y) = \dfrac{e^x + x + \cos(x)}{x^2+1}$

Checking Limits

- Limits will exist at almost all points (for math 243)
  - Use limit properties to work your way up
- For problem points, try cancelling factors $\frac{x(x-y)}{x-y} = x$
- Nothing to cancel, try directions
  - Get 2 different values and it doesn't exist
- All directions equal, try substitutions to prove existence
  - Polar: Put x = rcos(t), y = rsin(t) $\rightarrow$ $(x,y)\to(0,0) \Leftrightarrow r \to 0$
  - Linear: Put y = tx $\rightarrow$ $(x,y)\to(0,0) \Leftrightarrow (x,t)\to(0,0)$
- Also try squeeze theorem for existence

Checking for Continuity

Simple recipe:

- Find domain of function
- Check in domain where properties imply continuity
- Check the problem points by considering limits

try now or as part

of DW3

Practice Problems

Check the following functions are continuous:

- $f(x,y) = (x+y)/(x^2+y^2+1)$
- $f(x,y,z) = 3x^2z + xy\cos(x-y+z) + e^{\tan(x)}$

$g(x,y) = x$ cont.

$h(x,y) = y$ cont.

$g+h = x+y$ cont.

$g = x^2+1$ cont.

$h = y^2$ cont.

$g+h = x^2+y^2+1$ cont.

sum

Show the limit doesn't exist or find its value

- $(2x^2-xy-y^2)/(x^2-y^2)$ at $(1,1)$
- $(x^2+y^2)/(x^4+3y^4)$ at $(0,0)$
- $x^2\ln(x)/(x^2+y^2)$ at $(0,0)$

$3x^2z = \underbrace{(3x^2)}_{\text{cont}}\underbrace{(z)}_{\checkmark}$

$x^2+y^2+1 \geq 0+0+1$

$= 1 > 0$

prod 2 cont. is Scratchwork
cont.

$$\underbrace{x}_{ct}\,\underbrace{y}_{ct}\,\overbrace{\cos(\underbrace{x-y+z}_{ct})}^{\text{cont. by composition property}}$$

& also $\cos(t): \mathbb{R} \to \mathbb{R}$ is 1 var cont. function

$e^{\tan x}$ = cont. bcs 1-var cont. func.

$\Rightarrow \dfrac{x+y}{x^2+y^2+1}$ cont.

because $x+y$ cont., x^2+y^2+1 cont. & non-zero, and the $\frac{f}{g}$ property

can also be factored with quadratic formula

More Scratchwork

Show the limit doesn't exist or find

- $(2x^2-xy-y^2)/(x^2-y^2)$ at (1,1)
- $(x^2+y^2)/(x^4+3y^4)$ at (0,0)
- $x^2\ln(x)/(x^2+y^2)$ at (0,0)

$\ln(x+1)$

$$\frac{\overbrace{2x^2}^{a}\overbrace{-xy}^{b}\overbrace{-y^2}^{c}}{x^2-y^2} = \frac{(2x+y)(x-y)}{(x-y)(x+y)}$$

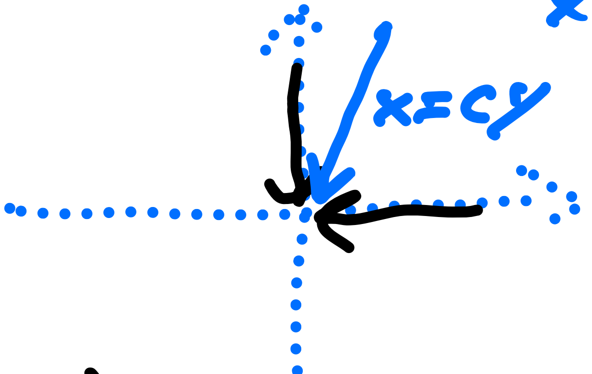
$x = y \pm \sqrt{\ldots}$

cont.

$$= \frac{2x+y}{x+y} = \frac{2+1}{1+1} = \frac{3}{2}$$

because $x-y=0$ isn't part of function domain

$$\frac{x^2+y^2}{x^4+3y^4} \overset{\substack{y\neq 0\\ x=0}}{=} \frac{y^2}{3y^4} = \frac{1}{3y^2} \to \infty$$



$x\neq 0,\ y=0 \to \infty$

$$x = cy \Rightarrow \frac{(c^2+1)y^2}{(c^4+3)y^4} = \frac{c^2+1}{c^4+3} \cdot \frac{1}{y^2} \to \infty$$

$x = r\cos\theta$
$y = r\sin\theta$

$$\Rightarrow \frac{r^2}{\underbrace{r^4c^4 + 3r^4s^4}} = \frac{1}{r^2}\left(\frac{1}{\underbrace{c^4+3s^4}_{>0}}\right)$$

$r \to 0$

Partial Derivatives

Let m = minimum of $c^4+3s^4 > 0$

Lecture for 6/16

$$\geq \frac{1}{r^2} \cdot \frac{1}{m} = \frac{1}{mr^2} \to \infty$$ because $mr^2 \to 0$

$y=y,\ x=cy$ is a particular path we ... to approach ... $(0,0)$

can use to approach (c, s)

$y = \frac{1}{c}x$

Idea Behind Partial Derivatives

$x^2 = r^2c^2$

$y^2 = r^2s^2$

$x^2+y^2 = r^2(s^2+c^2)$

$= r^2$

- Hold all but 1 variable constant to get 1 variable function
  - Example: g(y) = f(2, y, 4) $\rightarrow g_1$
- We can now take usual derivative
- Chose (x, z) = (2, 4), can keep doing this for more choices
- Can we get a formula in terms of our choice?

$x^4 + 3y^4 = r^4c^4 + 3r^4s^4$

$g_2(y) = f(3, y, 1) \Rightarrow g_2'(y)$

$g_3(y) = f(2, y, 7) \Rightarrow g_3'(y)$

$(a, b) \longrightarrow$ derivative $g_{a,b}'(y)$

Definitions and Notations

almost like difference quotient in Calc I

- Define $f_x(x, y) = \lim_{h\to 0} [f(x+h, y) - f(x, y)]/h$
- Similarly, $f_y(x, y) = \lim_{h\to 0} [f(x, y+h) - f(x, y)]/h$
- If limits don't exist, partial derivatives don't exist
- Similarly if f has more variables

$\hookrightarrow f(x,y,z,w)$: define f_x, f_y, f_z, f_w using diff quot.

There are 3 notations:

- $(\partial/\partial x)\, f$, $\partial f/\partial x$, f_x

$\frac{\partial}{\partial x} f \qquad \frac{\partial f}{\partial x} \qquad f_x$

most common because
least work to write

Properties of Partials

Same properties from Calculus 1 apply

- Sums: $(f+g)_x = f_x + g_x$
- Products: $(fg)_x = f_x g + f g_x$
- Quotient: $(f/g)_x = (f_x g - f g_x)/g^2$
- Chain: $(f \circ g)_x = (f' \circ g) g_x$ if $f: R \to R$

$g: R^n \to R$

f has to be single variable

Variables besides the one differentiated act like constants:

- $(f(x, y)g(y))_x = f_x(x,y)g(y)$

Check your compositions make sense

Begin with $f(x,y) = 1$

$[\, g(y)_x = (1)_x\, g(y) = 0 \cdot g(y) = 0$

Notions of Increasing and Decreasing

Consequence of RGB: $f(x)_x = f'(x)$

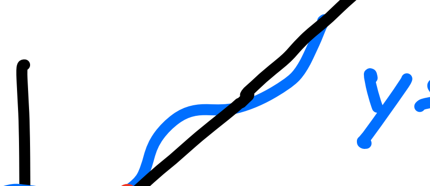
Consider the graph $z = f(x, y)$

- What happens to z if x fixed and y varies, or vice versa?
- Answer: consider f_x and f_y
- z is inc, dec. in x direction if $f_x \geq 0$, $f_x \leq 0$ respectively
- z is inc, dec in y direction if $f_y \geq 0$, $f_y \leq 0$

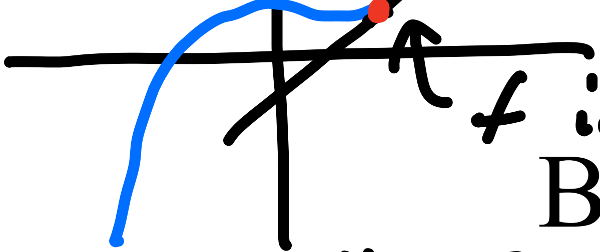
apply to $f(x, y, z) = w$

- Same principle applies to higher dimensions
- Will later see how to check inc/dec around the whole point

all partial derivatives come from using this + b + g



$y = f(x)$



f inc around • because $f' \geq 0$

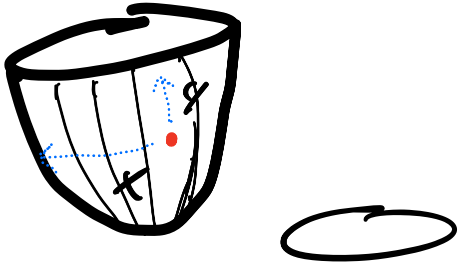
Back to Tangent Plane

actually, strict inc bec. $f' > 0$

$z = x^2 + y^2$

We left off not knowing f_x, f_y for tangent plane video

- Rewatch any parts of the video not understood
- Try the practice problem with $z = x^2 + y^2$



Formula for plane tangent at $(x,y) = (x_0,y_0)$ from video:

- $f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) = z-f(x_0,y_0)$

$f(x,y) = x^2+y^2$

$f_x = 2x+0$

$= 2x > 0$

if $x > 0$

$f_y = 2y > 0$

what if $(x,y) = (0,0)$

$z = x^2+y^2 > 0$

anywhere else

if $y>0$

Practice Problems

Try these from 12:47 - 1:00

Find all of the 1st order partial derivatives

- $x^y+y^x+e^{xy}$
- $x^4\sin(3y) - x/y + \cos(x/y)$
- $xyz/(x+y+z)$

Let c be a constant and $g(x) = f(x, c)$. Show that $f_x(x, c) = g'(x)$

Note: This vindicates our idea that taking the partial derivative produces the same value as plugging in constants and taking an ordinary derivative

Scratchwork

$\ln(x+1)$

$x^2 \ln(x)/(x^2+y^2)$ at $(0,0)$

$$\frac{x^2\ln(x+1)}{x^2+y^2}$$

try path $y=0$: $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{x^2\ln(x+1)}{x^2+y^2} =$

$$\lim_{x\to 0} \frac{x^2\ln(x+1)}{x^2} = \lim_{x\to 0} \ln(x+1) = \ln(0+1) = 0$$

try $y=x$: $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} \left(\frac{\cdots}{\cdots}\right) = \lim\limits_{x\to 0} \frac{x^2\ln(x+1)}{x^2+x^2} =$

$\lim \frac{1}{2}\ln(x+1) = 0$. Try when $x \neq 0$:

Scratchwork

$x \to 0$

$y = cx$

$$\frac{x^2\ln(x+1)}{x^2+y^2} = \frac{\ln(x+1)}{1+(y/x)^2} = \frac{\ln(x+1)}{1+c^2} \to \frac{0}{1+c^2} = 0$$

So we can guess limit is 0, but why?

Limits will exist at → N/A
- Use limit prope

For problem points → yes, problem point

Nothing to cancel, t
- Get 2 different → no luck on 2 diff directions

All directions equal
- Polar: Put x = r
- Linear: Put y = → just tried that

Also try squeeze th

yes, all dir. equal. So

let's see what's left: Polar & squeeze

Polar: $x = r\cos\theta$, $y = r\sin\theta$ $\Longrightarrow$ $x^2+y^2 = r^2$, $x^2\ln(x+1) = r^2\cos^2\theta\ln(r\cos\theta+1)$

so $$\frac{x^2\ln(x+1)}{x^2+y^2} = \cos^2\theta\ln(r\cos\theta+1)$$

Now what? We're guessing 0 from before, so

use squeeze: $|\cos^2\theta\ln(r\cos\theta+1)| =$

$$|\cos\theta|^2|\ln(r\cos\theta+1)| \leq |\ln(r\cos\theta+1)| \longrightarrow$$

$$|\ln(0+1)| = 0 \quad \text{because} \quad \underbrace{\lim_{r\to 0} r\cos\theta = 0}$$

Note: