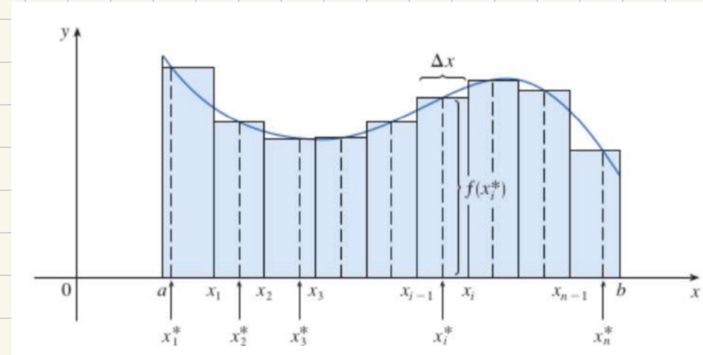
Recall:

Calc 1:

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

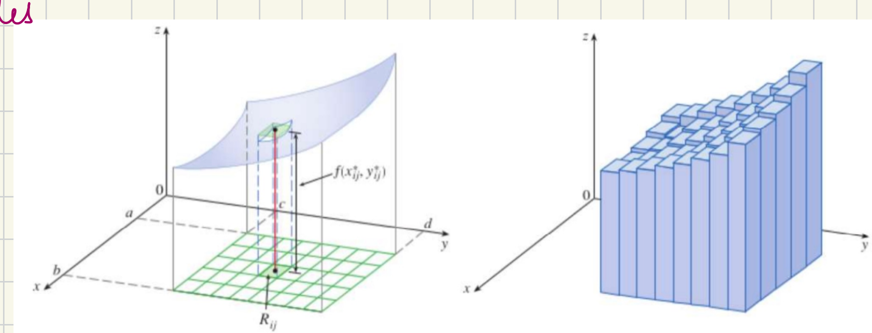


Section 15.1: Integrating over rectangles

Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=i}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

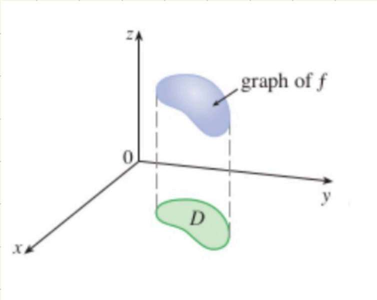
if this limit exists.



Fubini: If f is CTS on $R = [a,b] \times [c,d]$. Then

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy}$$

Section 15.2 : Integrating over more general regions.

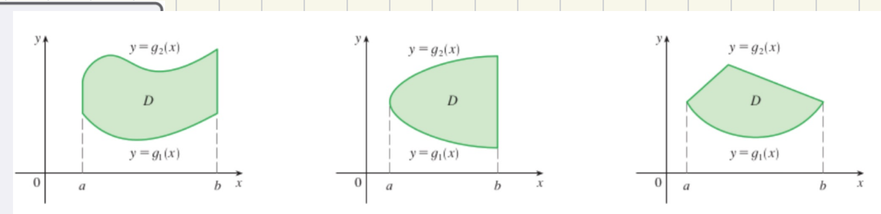


If f is continuous on a **type I region** D described by

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

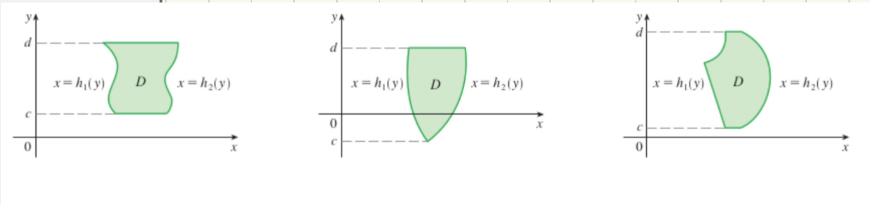


If f is continuous on a **type II region** D described by

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

then

$$\iint_D f(x, y)\, dA = \int_c^c \int_{h_1(x)}^{h_2(x)} f(x, y)\, dx\, dy$$



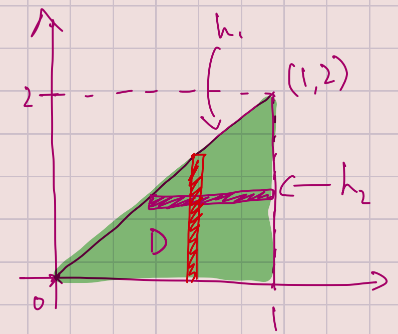
Changing order of integration: (Examples)

- $D = \{ (x,y) : 0 \le y \le 2,\ \underbrace{\frac{y}{2}}_{} \le x \le \underbrace{1}_{} \}$

$h_2(y) = 1$

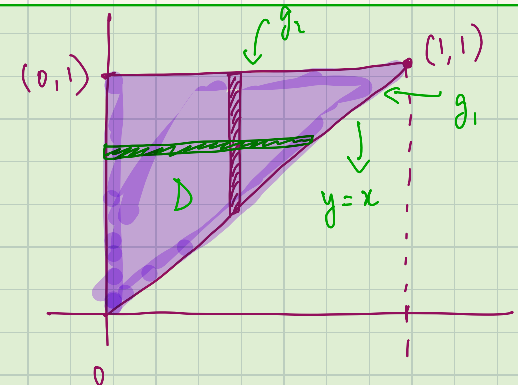
$h_1(y) = \frac{y}{2}$

i.e. $x = \frac{y}{2} \iff y = 2x$



- $D = \{ (x,y) : 0 \le x \le 1,\ 0 \le y \le 2x \}$

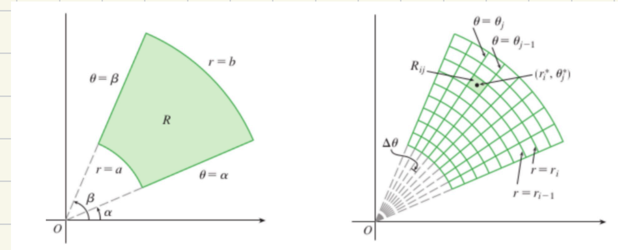
$D = \{ (x,y) : 0 \le x \le 1,\ \underbrace{x}_{\substack{\shortparallel \\ g_1(x) = x}} \le y \le \underbrace{1}_{\substack{\shortparallel \\ g_2(x) = 1}} \}$



$D = \{ (x,y) : 0 \le y \le 1,\ 0 \le x \le y \}$

Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x,y)\,\underset{\substack{\uparrow \\ dx\,dy}}{dA} = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$



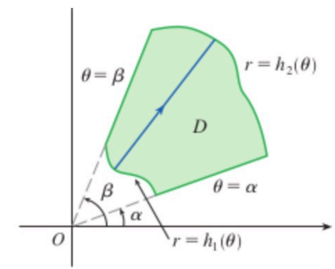
3 If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$D = \{(r,\theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$



Area of D, denoted by $A(D)$, is $A(D) = \displaystyle\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} 1\, r\,dr\,d\theta$

Eg: in 1-D, $\displaystyle\int_a^b 1\,dx = x\Big|_a^b = b-a$ = length of $[a,b]$.

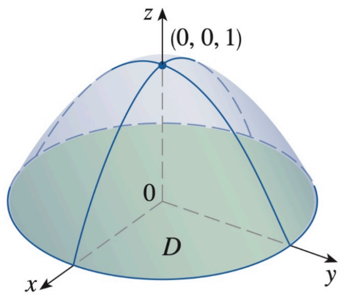
bounded above by the paraboloid

Question 4. Find the volume of the solid E ~~bounded below the paraboloid~~

$$z = \underbrace{1 - x^2 - y^2}_{= 1 - (x^2+y^2)} = f(x,y)$$

below by $z = 0$

and ~~above the plane $z = 0$.~~



$f(x,y) = 0 \iff x^2 + y^2 = 1$

So $D = \{(x,y) : \underbrace{x^2 + y^2}_{= r^2} \leq 1\}$

$= \{(r,\theta) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1\}$

$f(r\cos\theta, r\sin\theta)$

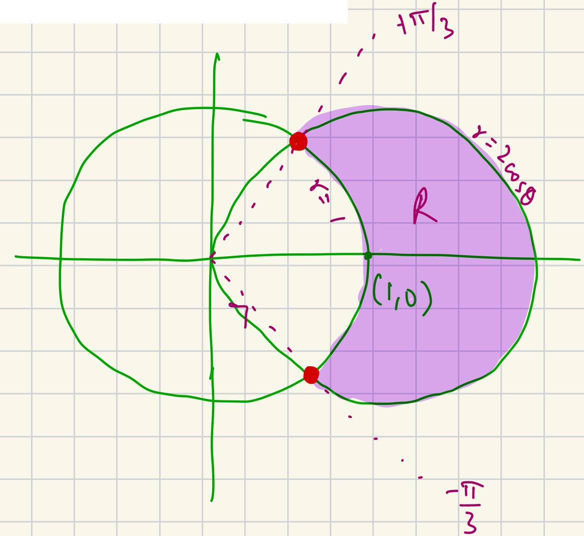
$= 1 - (r^2\cos^2\theta + r^2\sin^2\theta) = 1 - r^2$

$$V = \iint_D 1 - x^2 - y^2 \, dA = \int_0^{2\pi} \int_0^1 \underbrace{(1 - r^2)\, r}_{= r - r^3} \, dr \, d\theta \overset{\boxed{\text{DIY}}}{=} \pi/2$$

Example: Find the area of D where D is the region inside the circle $\underbrace{(x-1)^2+y^2=1}_{D_1}$ and outside the circle $\underbrace{x^2+y^2=1}_{D_2}$.

circle of radius 1 with center at (1,0)

We want $A(R) = \iint_R 1\,dA = \int_{-\frac{\pi}{3}}^{\pi/3}\int_{1}^{2\cos\theta} 1\; r\,dr\,d\theta$



$D_1 = \{(x,y) : (x-1)^2+y^2 \le 1\}$

$(x-1)^2+y^2=1 \iff (r\cos\theta-1)^2+r^2\sin^2\theta=1$

$\iff r^2\cos^2\theta+\cancel{1}-2r\cos\theta+r^2\sin^2\theta=\cancel{1}$

$\iff r^2=2r\cos\theta \underset{(r\neq 0)}{\iff} r=2\cos\theta$

Boundary of D_1 is $r=2\cos\theta$ where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$\Rightarrow D_1 = \{(r,\theta) : -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ r\le\cos\theta\}$

$\boxed{\text{Answer: } A(R) = \frac{3\sqrt{3}+2\pi}{6}}$

Boundary of D_2 is $r = 1$ so the boundaries intersect when $r = 2\cos\theta = 1$

$\iff \cos\theta = \frac{1}{2} \iff \theta = \pm\frac{\pi}{3}$.

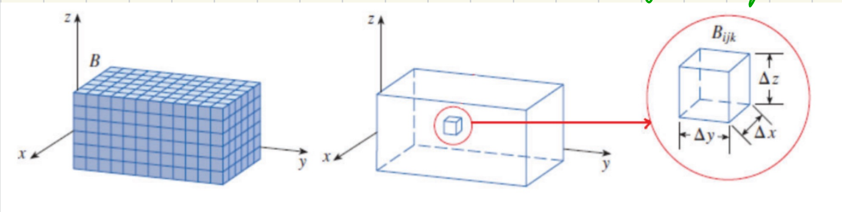
Section 15.6: Triple Integrals.

Volume of a box in 4-D: $\underbrace{f(x^*, y^*, z^*) \cdot \Delta x \cdot \Delta y \cdot \Delta z}_{\text{product of 4 lengths}}$

Definition The **triple integral** of f over the box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

if this limit exists.



Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Example: Evaluate $\underbrace{\iiint_B x\cos(y+z)\,dV}_{= I}$ where $B = \underset{x}{[0,1]} \times \underset{y}{[0,2]} \times \underset{z}{[0,3]}$.

$$I = \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} x\cos(y+z)\,dz\,dy\,dx = \int_0^1 \int_0^2 \left(x\sin(y+3) - x\sin y\right) dy\,dx$$

$$\text{Let } I_1 = \int_0^3 x\cos(y+z)\,dz = x\sin(y+z)\Big|_{z=0}^{3} = x\left(\sin(y+3)-\sin(y+0)\right)$$

$$\text{Let } I_2 = \int_0^2 \left(x\sin(y+3) - x\sin y\right)dy$$

$$= x\int_0^2 \sin(y+3)\,dy - x\int_0^2 \sin y\,dy = -x\cos(y+3)\Big|_0^2 + x\cos y\Big|_0^2$$

$$= -x\left(\cos(5)-\cos(3)\right) + x\left(\cos(2)-\cos(0)\right)$$

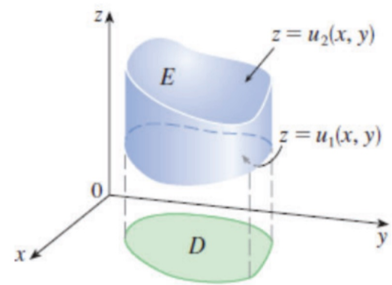
$$I = \int_0^1 -x\cos(5) + x\cos(3) + x\cos(2) - x\,dx = \left(-\cos(5)+\cos(3)+\cos(2)-1\right)\int_0^1 x\,dx$$

$$= \left(-\cos(5)+\cos(3)+\cos(2)-1\right)\frac{1}{2}$$

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

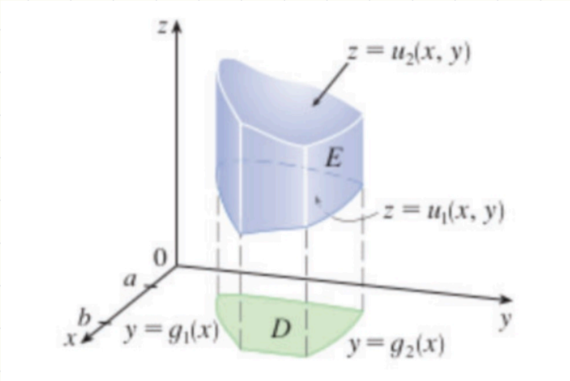
where D is the projection of E onto the xy-plane.

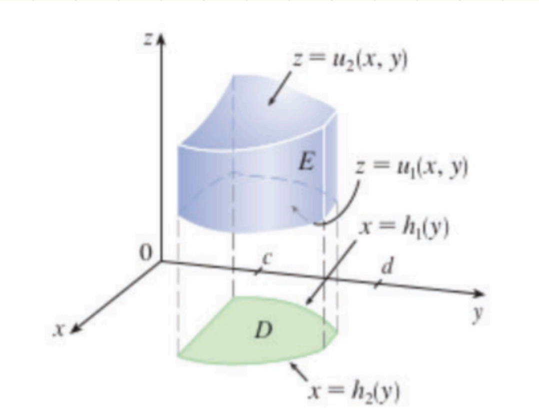


- Upper boundary of the solid E is the surface $z = u_2(x, y)$
- Lower boundary of the solid E is the surface $z = u_1(x, y)$.

If E is a type 1 region, then:

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$$



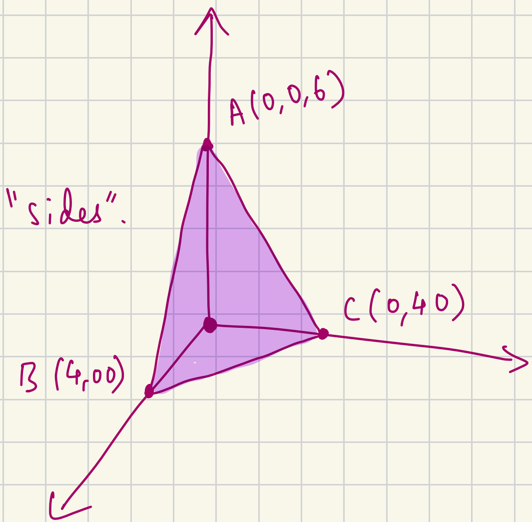


Example: Let T be the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$ and $C(0,4,0)$.
(Note that the plane containing the points A, B and C has the equation $3x+3y+2z=12$)

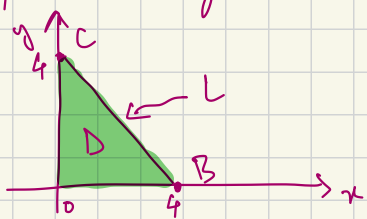
(a) Express T as a solid region type 1.

(b) Express $\iiint\limits_T f(x,y,z)\,dV$ as an iterated integral.

- Faces of the tetrahedron are parts of the following planes: $3x+3y+2z=12$ (in pink), xy plane (base), xz & yz planes are the "sides".



- Project the surface $3x+3y+2z=12$ onto the xy-plane to get the following region:



- D is the region in the xy plane bounded by the lines $x=0$, $y=0$, and L.
- L is the line between $\underbrace{B(4,0,0) \text{ and } C(0,4,0)}$

think of these as $B(4,0)$ and $C(0,4)$

$\Rightarrow$ L has slope $\dfrac{0-4}{4-0} = -1$ and y-intercept 4 $\Rightarrow$ $y = -x+4$

OR $x+y=4$

OR $x = -y+4$

- $E = \{(x,y,z) : 0 \le x \le 4,\ 0 \le y \le -x+4,\ 0 \le z \le 6 - \frac{3}{2}x - \frac{3}{2}y\}$

$3x + 3y + 2z = 12 \Rightarrow z = 6 - \dfrac{3}{2}x - \dfrac{3}{2}y$

$= \{(x,y,z) : 0 \le y \le 4,\ 0 \le x \le -y+4,\ 0 \le z \le 6 - \frac{3}{2}x - \frac{3}{2}y\}$

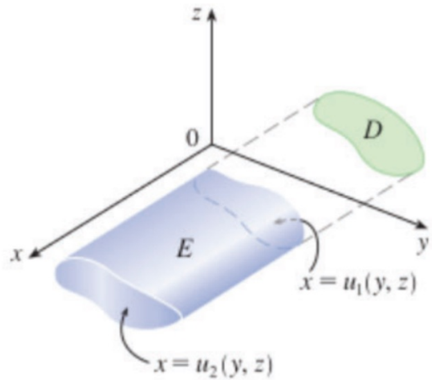
(b) $$\iiint_E f(x,y,z)\,dV = \int_0^4 \int_0^{-x+4} \int_0^{6-\frac{3}{2}x-\frac{3}{2}y} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_0^4 \int_0^{-y+4} \int_0^{6-\frac{3}{2}x-\frac{3}{2}y} f(x,y,z)\,dz\,dx\,dy$$

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\}$$

where D is the projection of E onto the yz-plane.



- The back surface is $x = u_1(y, z)$.
- The front surface is $x = u_2(y, z)$

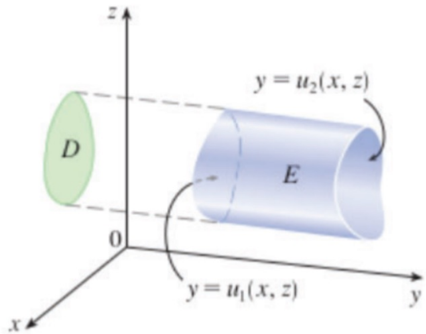
Then,

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$

A solid region E is of **type 3** if it is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

where D is the projection of E onto the xz-plane.



- The left surface is $y = u_1(x, z)$.
- The right surface is $y = u_2(x, z)$

Then,

$$\boxed{\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA}$$

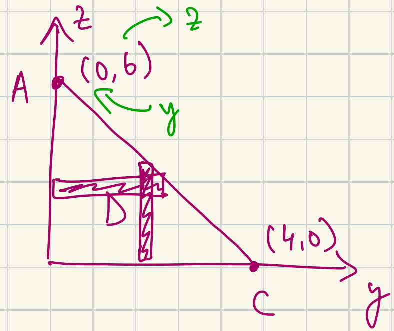
Note: In each of these equations, there may be two possible expressions for the integral depending on whether D is a type I or a type II plane region.

Example: Let T be the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$ and $C(0,4,0)$.
(Note that the plane containing the points A, B and C has the equation $3x+3y+2z = 12$)

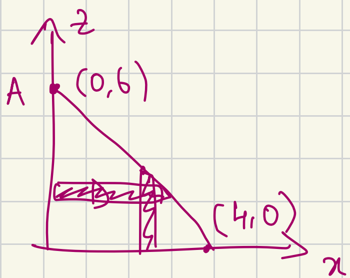
(a) Express T as a solid region type 2 and type 3.

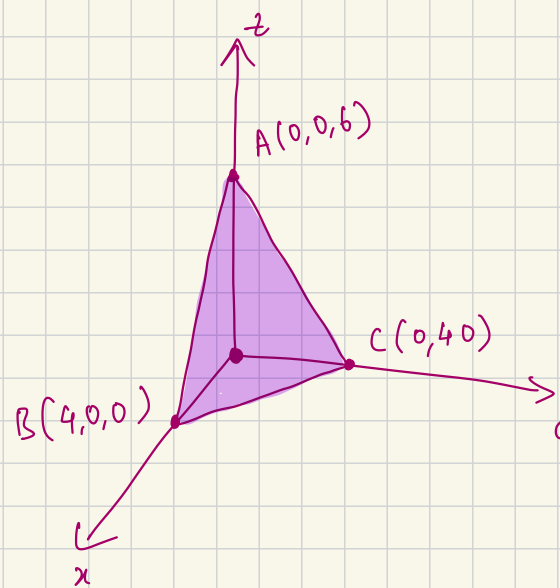
(b) Express $\displaystyle\iiint_T f(x,y,z)\,dV$ as an iterated integral.

(a) Type 2:



Type 3:





Example: Evaluate $\displaystyle\iiint_T e^z \, dV$ where T is the tetrahedron with vertices $O(0,0,0)$, $A(0,0,6)$, $B(4,0,0)$ and $C(0,4,0)$.