MATH 243 Midterm 2

1. Little John wants to find the volume of a hemisphere of radius R using spherical coordinates for his home extension project. He sets up the bounds $0 \leq \rho \leq R, 0 \leq \theta \leq \pi, 0 \leq \phi \leq \pi$ and begins to compute $V = \int_0^R \int_0^\pi \int_0^\pi \rho^2 \sin(\phi) d\phi\, d\theta d\rho$. Has he made a mistake?

A. Yes. Using spherical coordinates would be circular logic since the volume of a sphere is needed to derive the formulas for spherical coordinate conversions

B. Yes. He forgot to borrow screws from his aunt and these would change the volume once drilled in.

C. Yes. The bounds must be $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$ instead.

D. Yes. This integral only computes half the volume.

E. No. The bounds are correct.

2. Let B be the unit ball centered at the origin. Select all of the following equivalent to $\iiint_B f\, dV$

A. $8 \int_0^1 \int_0^{\pi/2} \int_0^{\sqrt{1-z^2}} r f(r\cos\theta, r\sin\theta, z)\, dr\, d\theta\, dz$

B. $\int_{-1}^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\, dx\, dy\, dz$

C. $\int_0^1 \int_0^\pi \int_0^{2\pi} f(\rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi)\rho^2 \sin(\phi) d\theta\, d\phi\, dr$

D. $2\iiint_{B_+} f\, dV$ where B_+ is the portion of B with $z \geq 0$

3. Suppose F is a continuously differentiable vector field on $\mathbb{R}^2$ which is not conservative. Select all of the following which must always be true

A. If $F = (A, B)$, then $A_y \neq B_x$

B. If D_R is the disk of radius R centered at the origin and $f(x,y)$ is a continuously differentiable function, then $\iint_{D_R} \|F - \nabla f\|\, dA > 0$ for some R

C. For any closed curve C of non-zero length, $\int_C F \cdot dr = 0$

D. If $P = (0,0), Q = (1,0)$, there exists paths C_1, C_2, both from P to Q, such that $\int_{C_1} F \cdot dr \neq \int_{C_2} F \cdot dr$

E. Given any $P \neq Q$, and any three distinct paths C_1, C_2, C_3 from P to Q, the integrals $\int_{C_i} F \cdot dr$ can't all have the same value

4. Let R be the annulus with inner radius 1, outer radius 2, and center at the origin. Let $K = \iint_R (|x| + y^2 + x^3)\, dA$. We can write $K = \frac{a}{b} + \frac{c}{d}\pi$ where a, b, c, d are integers and both fractions are in reduced form. Find $a + b + c + d$

5. Let $L = \int_C F \cdot dr$ where C is the line segment $(0,0,1) \to (1,1,1)$ and $F = (ze^{xz} + \frac{y}{z} + \sin x, \frac{x}{z} + \sin y, xe^{xz} - \frac{xy}{z^2} + \cos z)$. We can write $L = e + a - b\cos(c)$ where a, b, c are integers. Find $100a + 10b + c$

6. Let C be the curve obtained from the graph of $y = \tan x$ restricted to $-\pi/4 \leq x \leq \pi/4$. Let $M = \int_C (y^3 + 1)\, dx + (x^3 + 1)\, dy$. We can write $M = \frac{\pi}{a} + b$ for some integers a, b. Find $10a + b$

7. Let R be the same annulus as before. Find the surface area of the portion of the surface $z = xy + 1$ within R, then find the center of mass of R under the weight function $w(x, y) = |x - 2| + |y - 2|$.

8. Classify the critical points for $f(x, y, z) = x^2 + 2xy + y^2 + 2yz + z^2$

9. Let $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ be the same unit ball as before. Find the maximum of $x^2 + (x + y)^2 + (x + z)^2$ on B

10. Extra credit: Let C_x be the infinite cylinder with radius 1 and central axis the x-axis. Similarly, define C_y and C_z. Find the volume inside the intersection of C_x, C_y, C_z, and $x + y + z \geq 0$

11. Extra extra credit: Let $S = [0, 1]^3$ be the unit cube. Find the value of $\iiint_S \frac{z}{(1+x^2+y^2)^2}\, dx\, dy\, dz$