MATH 243: Worksheet 3 **Name and Discussion Section:** ________________

Textbook Sections: 12.3, 12.4 and 12.5
Topics: The dot product, scalar and vector projections, and the cross product.
Instructions: Try each of the following problems, show the detail of your work. Clearly mark your choices in multiple choice items. Justify your answers. Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

1. Consider $\mathbf{a} = \langle -1, 4, 8\rangle$ and $\mathbf{b} = \langle 18, 2, 1\rangle$.
   (a) Find the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$.
   (b) Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.
2. Find the work (in J) done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$ that moves an object from the point $(0, 6, 4)$ to the point $(4, 14, 22)$ along a straight line. The distance is measured in meters and the force in newtons. (Hint: The work done by a constant force is the dot product of the force and displacement vectors).
3. Compute the dot product and cross product for the following pairs of vectors:
   (a) $\mathbf{u} = \langle -1, 1, 2\rangle$, $\mathbf{v} = \langle 4, 5, -2\rangle$
   (b) $\mathbf{u} = \langle 1, -1, 3\rangle$, $\mathbf{v} = \langle 2, -2, 6\rangle$
   (c) $\mathbf{u} = \langle 1, 2, 3\rangle$, $\mathbf{v} = \langle -3, 0, 1\rangle$
   (d) Based on your answer for parts (b) and (c), what conclusions can you make about the dot and cross products of two parallel vectors? Perpendicular vectors? Justify your answers.
4. Consider $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{j} + \mathbf{k}$.
   (a) Find the cross product $\mathbf{a} \times \mathbf{b}$.
   (b) Verify that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.
   (c) Find two unit vectors orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
5. Consider points $P(1, 2, 1), Q(2, 5, 4), R(6, 9, 12)$ and $S(5, 6, 9)$ in $\mathbb{R}^3$.
   (a) Find the area of the parallelogram with vertices $P(1, 2, 1), Q(2, 5, 4), R(6, 9, 12)$ and $S(5, 6, 9)$.
   (b) Find the area of the triangle PQS.
   (c) Show that the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are coplanar.
6. Find the following equations of the line through the point $P(2, 2, 4)$ and parallel to the vector $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k}$. Use the parameter t.
   (a) Vector equation
   (b) Parametric equations
   (c) Symmetric equations
7. (a) Find the symmetric equations for the line through $P(3, 4, 0)$ and perpendicular to both $2\mathbf{i} + 2\mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

(b) Find the point of intersection between the line you found in part (a) and the yz-plane.

8. Determine whether two lines L_1 and L_2 are parallel, intersecting or skew. If they intersect, find the angle between these two lines.

   (a) $L_1 : \mathbf{r}(t) = \langle -1+3t,\ 2+4t,\ 3-2t\rangle$, $\qquad L_2 : \dfrac{x-1}{2} = \dfrac{y}{-3} = \dfrac{z+1}{-3}$

   (b) $L_1 : x = 2t,\ y = -3+t,\ z = 5-t$

   $L_2 : x = 3-3s,\ y = 2 - \dfrac{3}{2}s,\ z = \dfrac{3}{2}s$

9. Find an equation for the plane through the points $A(0,1,2)$, $B(1,2,3)$, and $C(2,3,5)$.

Suggested Textbook Problems
Section 12.3: 1-13, 17, 19, 23, 25, 27, 28, 33, 40, 41, 43, 45, 47-52
Section 12.4: 1-20, 27-29, 31, 33-39, 41, 43-45, 53

DEFINITIONS AND FORMULAS

The Dot Product for vectors in $\mathbb{R}^3$

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The Cross Product (used for vectors in $\mathbb{R}^3$)

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The Scalar and Vector Projections of b onto a are respectively given by

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}, \qquad \text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right) = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}$$

Other useful relations

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta), \qquad \|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin(\theta)$$

$$\Big\{\text{Area of parallelogram formed by } \mathbf{a} \text{ and } \mathbf{b}\Big\} = \|\mathbf{a}\times\mathbf{b}\|$$

$$\Big\{\text{Volume of parallelepiped formed by } \mathbf{a},\ \mathbf{b}, \text{ and } \mathbf{c}\Big\} = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$