Challenge Problems in RED

MATH 243: Worksheet 10 **Discussion Section:** ____________

Textbook Sections: 15.2 and 15.3
Topics: Double integrals over general domains, and double integrals in polar coordinates.
Instructions: Try each of the following problems, show the detail of your work.
Clearly mark your choices in multiple choice items. Justify your answers.
Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

Double Integrals over general Domains:

1. Given the region R bounded by the lines $y = 3x + 1$, $x = 1$ and $y = 1$, draw the region R and iterate the double integral (Do NOT evaluate).
$$\iint_R f(x, y)\, dA$$

2. Evaluate $\iint_D y^2 e^{xy}\, dA$, where D is the region bounded by $y = x,\ y = 4,\ x = 0$.

3. (a) Sketch the region of integration for $\int_0^1 \int_0^y f(x, y)\, dx\, dy$

   (b) Switch the order of integration for the integral in part a.

   (c) Calculate the integral from part b, for the function $f(x, y) = \sqrt{x} + 3y$.

4. Find the average value of $f(x, y) = xy$ over the triangle D with the vertices $(0, 0), (1, 0), (1, 3)$.

Double integrals in polar coordinates:

5. Use polar coordinates to find the volume of the solid $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leq x^2 + y^2 \leq 4$.

6. Calculate $\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy$ by changing to polar coordinates.

7. Use polar coordinates to combine the sum
$$\int_{\frac{1}{\sqrt{2}}}^{1} \int_{\sqrt{1-x^2}}^{x} xy\, dy\, dx + \int_1^{\sqrt{2}} \int_0^x xy\, dy\, dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} xy\, dy\, dx$$
into one double integral and evaluate the double integral.

Suggested Textbook Problems

Section 15.2	1-10, 13-32, 39-40, 45-57, 65, 68
Section 15.3	1-11, 13-16, 22-27, 29-35, 39-42, 49

SOME USEFUL DEFINITIONS, THEOREMS AND NOTATION:

The Average Formula

The average value of a function f of two variables defined on a region R with area $A(R)$, is given by

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

Integrals Over General Regions

If f is continuous on a type I region $D = \{\ (x, y)|\ a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$ where g_1 and g_2 are continuous functions, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

If f is continuous on a type II region $D = \{\ (x, y)|\ c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$, where h_1 and h_2 are continuous functions, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Properties of double integrals

Suppose the integrals of $f(x, y)$ and $g(x, y)$ exist, then

(1) $\displaystyle\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$

(2) $\displaystyle\iint_D cf(x, y)\, dA = c \iint_D f(x, y)\, dA$, where c is a constant

(3) If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then, $\displaystyle\iint_D f(x, y)\, dA \geq \iint_D g(x, y)\, dA$

(4) If $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except perhaps on their boundaries ,then,

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$

(5) If we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D, i. e. $\iint_D 1\, dA = Area(D)$

Change to Polar Coordinates in a Double Integral

(1) If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, rdr\, d\theta$$

(2) If f is continuous on a polar region of the form $D = \{\ (r, \theta)|\ \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$ then,

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, rdr\, d\theta$$