Today:

- Three ways to represent the equation for a line $L \subseteq \mathbb{R}^3$.

- $\boxed{r(t) = r_0 + tv}$, $r_0 \in \mathbb{R}^3$, $v \in V_3$, each $r(t)$ is a position vector for a point on the line L.

Vector Equation of the line L

- For $v = \langle a, b, c \rangle$, $r_0 = \langle x_0, y_0, z_0 \rangle$, $r(t) = \langle x, y, z \rangle$

We have $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

$$\Leftrightarrow \boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

← Parametric Equations of the line L

- Solve for t: $\boxed{\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}}$ ← Symmetric Equations of L.

- Line segment $[r_0, r_1] = \{ (1-t)r_0 + t r_1 \mid 0 \leq t \leq 1 \}$

- Vector equation for a plane: $\boxed{n \cdot (r - r_0) = 0 \quad \text{OR} \quad n \cdot r = n \cdot r_0}$

where
- $\vec{n}$ is a normal vector to the plane
- r_0 is the position vector for the point on the plane.

- $n = \langle a, b, c \rangle$, $r = \langle x, y, z \rangle$, $r_0 = \langle x_0, y_0, z_0 \rangle \rightarrow \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$$\iff \boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

Scalar equation of the plane

- Collect terms: $\boxed{ax+by+cz+d=0}$ $\quad (d=-ax_0-by_0-cz_0)$

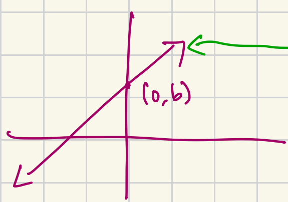
Linear Equation

Section 12.5: Equations of lines and planes.

Define: In $\mathbb{R}^2$ a line is a set of points $L=\{(x,y) : y=mx+b\}$ for some $m\in\mathbb{R}, b\in\mathbb{R}$.

slope (m), b ↳ y-intercept.

Note: $(x, mx+b) = (0,b) + x\langle 1, m\rangle$



a point in L + scalar multiple of $\langle 1, m\rangle$

Definition: Vector equation for a line in $\mathbb{R}^3$:

$$r(t) = r_0 + t\vec{v}$$

$\vec{v}$ → direction

$r(t)$: position vector of any

r_0: point on the line

other point on the line

- Let $r(t) = (x, y, z)$, $r_0 = (x_0, y_0, z_0)$, $\vec{v} = \langle a, b, c \rangle$.

So $r(t) = r_0 + t\vec{v} \iff (x, y, z) = (x_0, y_0, z_0) + t\langle a, b, c \rangle$

$\iff (x, y, z) = (x_0 + ta, y_0 + tb, z_0 + tc)$

$$\iff \left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}\right\}$$ Parametric Equations for the line L.

$$\iff \text{Solving for } t: \left.\begin{aligned} \frac{x - x_0}{a} &= t \\ \frac{y - y_0}{b} &= t \\ \frac{z - z_0}{c} &= t \end{aligned}\right\} \underbrace{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

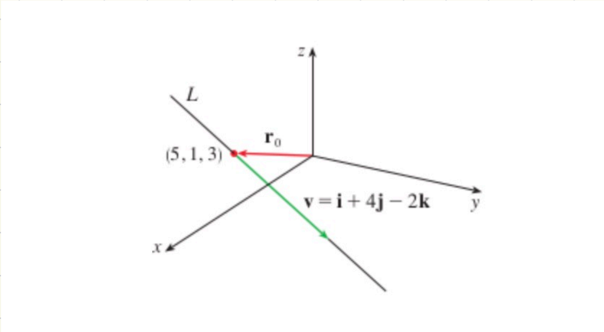
Symmetric Equation for L.

Example 1

(a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

(b) Find two other points on the line.

(a) $r(t) = (5,1,3) + t\langle 1,4,-2\rangle$



Alternatively:

$r = 5i + j + 3k + t(i + 4j - 2k)$

$= (5+t)i + (1+4t)j + (3-2t)k$

$\Rightarrow$ Parametric Equations are: $x = 5+t,\ y = 1+4t,\ z = 3-2t$

(b) $t = 1 \Rightarrow x = 6,\ y = 5,\ z = 1 \Rightarrow (6,5,1)$ is on the line.

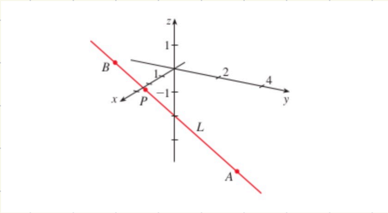
$t = -1 \Rightarrow x = 4,\ y = -3,\ z = 5 \Rightarrow (4,-3,5)$ is on the line.

Example 2

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.

(b) At what point does this line intersect the xy-plane?

(a) $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are direction vectors for the line that passes through A and B.



$\overrightarrow{AB} = \langle 3-2, -1-4, 1+3\rangle$

$= \langle 1, -5, 4 \rangle$

$\Rightarrow r(t) = 2i + 4j - 3k + t(i - 5j + 4k)$

$= (2+t)i + (4-5t)j + (-3+4t)k$

$\Rightarrow$ Parametric Equations are: $x = 2+t$, $y = 4-5t$, $z = -3+4t$

$\Rightarrow$ Symmetric equations: $\dfrac{x-2}{1} = \dfrac{y-4}{-5} = \dfrac{z+3}{4}$

(b) $xy\text{-plane} = \{(x, y, 0) : x, y \in \mathbb{R}\}$

Use the fact that $z = 0$: $\dfrac{x-2}{1} = \dfrac{y-4}{-5} = \dfrac{3}{4}$

$\Rightarrow x - 2 = \dfrac{3}{4} \Rightarrow x = \dfrac{11}{4}$, $y - 4 = -\dfrac{15}{4} \Rightarrow y = \dfrac{1}{4}$

$\Rightarrow$ the line intersects the xy-plane at $\left(\dfrac{11}{4}, \dfrac{1}{4}, 0\right)$

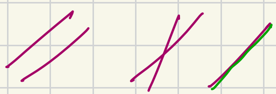
Example 3

Show that the lines L_1 and L_2 with parametric equations

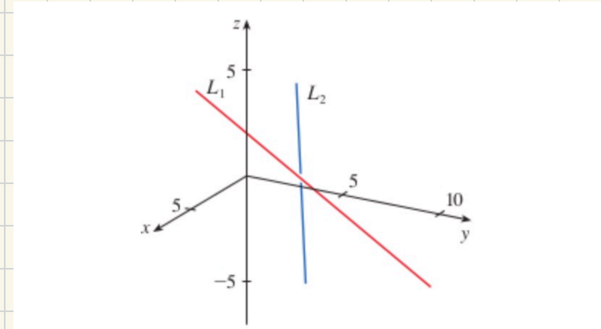
$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

In 2-D:



Recall: two vectors are parallel $\iff$ one is a scalar multiple of the other.



Vector equations for L_1 & L_2:

L_1: $r(t) = (1+t)i + (-2+3t)j + (4-t)k$

$= i - 2j + 4k + t(i + 3j - k) \Rightarrow L_1$ has direction vector $\vec{v_1} = \langle 1, 3, -1 \rangle$

L_2: $r(t) = (2s)i + (3+s)j + (-3+4s)k$

$= 3j - 3k + s(2i + j + 4k) \Rightarrow L_2$ has direction vector $\vec{v_2} = \langle 2, 1, 4 \rangle$

<u>Claim</u>: v_1 and v_2 are not parallel.

<u>Proof</u>: Suppose that they were. Then $v_1 = c v_2$ for some $c \in \mathbb{R}$.

$$\langle 1, 3, -1 \rangle = c \langle 2, 1, 4 \rangle \iff 1 = 2c, \underbrace{3 = c}_{\Downarrow}, -1 = 4c$$

$2c = 6 \neq 1$

$\Rightarrow$ you need two different c values for this to work $\Rightarrow$ not parallel.

Claim: L_1 and L_2 do not intersect.

L_1: $x = 1+t$, $y = -2+3t$, $z = 4-t$ $\longleftarrow$ $r_1(t) = X + t v_1$

L_2: $x = 2s$, $y = 3+s$, $z = -3+4s$ $\longleftarrow$ $r_2(s) = Y + s v_2$

Proof: Suppose they did intersect. Then there are s and t values such that $r_1(t) = r_2(s)$

$$\Leftrightarrow \quad \boxed{\begin{matrix} 1+t = 2s \\ -2+3t = 3+s \end{matrix}} \rightarrow \begin{matrix} t - 2s = -1 \\ 3t - s = 5 \rightarrow s = 3t-5 \end{matrix}$$

$$4 - t = -3+4s$$

$\Leftrightarrow \quad t - 2(3t-5) = -1$

$\Leftrightarrow \quad t - 6t + 10 = -1$

$\Leftrightarrow \quad -5t = -11 \Leftrightarrow t = 11/5$

$$\Rightarrow s = \frac{33}{5} - 5 = \frac{8}{5}$$

$$\Leftrightarrow 4 - \left(\frac{11}{5}\right) = -3 + 4\left(\frac{8}{5}\right)$$

$$\Leftrightarrow \frac{20-11}{5} = \frac{-15+32}{5} \Leftrightarrow \frac{9}{5} = \frac{17}{5} \Rightarrow \text{contradiction!!}$$

$\Rightarrow$ L_1 and L_2 are not parallel nor do they intersect.

(they are also not the same line).

Planes:

- Lines require one direction vector, what set do you get with 2 direction vectors?

Eg: xy plane has two direction vectors, namely, $\hat{i}, \hat{j}$

Also, any point on the xy-plane is perpendicular to $\hat{k}$.

Proof: $k \cdot \langle x, y, 0 \rangle = \langle 0, 0, 1 \rangle \cdot \langle x, y, 0 \rangle$
$= 0.$

Definition: Vector equation of the plane with point $P(x_0, y_0, z_0)$ and normal vector $\vec{n}$ is

$$\vec{n} \cdot (r - r_0) = 0$$

where $r = (x, y, z)$ is a point on the plane.

- Let $\vec{n} = \langle a, b, c \rangle$, $r_0 = \langle x_0, y_0, z_0 \rangle$. Then

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\Leftrightarrow \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \leftarrow \text{scalar equation for the plane}$$

$$\Leftrightarrow \quad ax + by + cz + d = 0 \quad \leftarrow \text{linear equation for the plane.}$$

where $d = -ax_0 - by_0 - cz_0$

Aside: Lines require one parameter, planes require two.

Example 5

Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$.

A: Find two direction vectors:

$$\vec{PQ} = \langle 3-1, -1-3, 6-2 \rangle = \langle 2, -4, 4 \rangle$$

$$\vec{PR} = \langle 5-1, 2-3, 0-2 \rangle = \langle 4, -1, -2 \rangle$$

Aside: plane is the set $\{ P + s\vec{PQ} + t\vec{PR} : s, t \in \mathbb{R} \}$

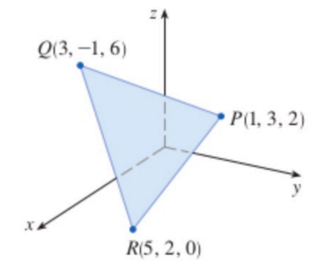
In order to find the equation for this plane we need a vector orthogonal to $\vec{PQ}$ and $\vec{PR}$: take $\vec{PQ} \times \vec{PR}$.

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} \underset{\text{DIY}}{=} 12i + 20j + 14k.$$

$\Rightarrow$ Equation is $12(x-1) + 20(y-3) + 14(z-2) = 0$.

$(P(1,3,2))$

$\overset{\text{(Simplify)}}{\Leftrightarrow}$ $6x + 10y + 7z = 50$



Example 6

Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

A: $x = 2 + 3t, \quad y = -4t, \quad z = 5 + t$

$$4(2+3t) + 5(-4t) - 2(5+t) = 18$$

$$\Leftrightarrow 8 + 12t - 20t - 10 - 2t = 18$$

$$\Leftrightarrow -10t = 20 \Leftrightarrow t = -2$$

$\Rightarrow$ point of intersection is $(-4, 8, 3)$