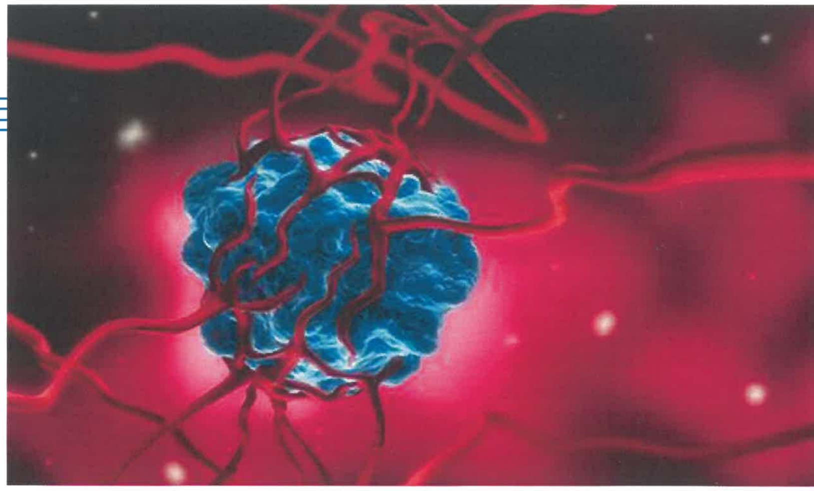
15 Multiple Integrals



15.6 Triple Integrals

Triple Integrals (1 of 1)

We have defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.

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Triple Integrals over Rectangular Boxes

Triple Integrals over Rectangular Boxes (1 of 6)

Let's first deal with the simplest case where f is defined on a rectangular box:

$$1 \quad B = \{(x, y, z) | a \le x \le b, c \le y \le d, r \le z \le s\}$$

$$B = [a, b] \times [c, d] \times [r, s]$$

The first step is to divide B into sub-boxes.

We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz.

Triple Integrals over Rectangular Boxes (2 of 6)

The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = \underbrace{[x_{i-1}, x_i]}_{\Delta x} \times \underbrace{[y_{j-1}, y_j]}_{\Delta y} \times [z_{k-1}, z_k]$$

which are shown in Figure 1. Each sub-box has volume $\Delta V = \Delta x \, \Delta y \, \Delta z$.

$$\Delta x = x_i - x_{i-1}$$
$$\Delta y = y_j - y_{j-1}$$
$$\Delta z = z_k - z_{k-1}$$

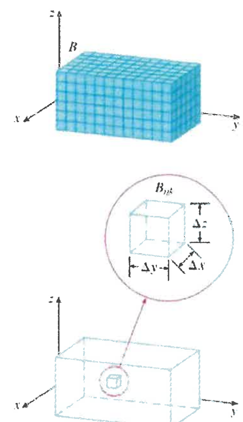

Figure 1

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Triple Integrals over Rectangular Boxes (3 of 6)

Then we form the **triple Riemann sum**

$$\mathbf{2} \quad \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}.

By analogy with the definition of a double integral, we define the triple integral as the limit of the triple Riemann sums in (2).

Triple Integrals over Rectangular Boxes (4 of 6)

3 Definition The **triple integral** of *f* over the box *B* is

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

if this limit exists.

Again, the triple integral always exists if *f* is continuous.

We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\,\Delta V$$

Triple Integrals over Rectangular Boxes (5 of 6)

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \underline{\int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz}$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z.

Triple Integrals over Rectangular Boxes (6 of 6)

There are five other possible orders in which we can integrate, all of which give the same value.

For instance, if we integrate with respect to y, then z, and then x, we have

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_r^s \int_c^d f(x,y,z)\,dy\,dz\,dx$$

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Example 1

Evaluate the triple integral $\iiint_B x\cos(y+z)\,dV,$ where B is the rectangular box given by

$$B = [0,1] \times [0,2] \times [0,3]$$

$$\iiint_B x\cos(y+z)\,dV = \int_0^3\int_0^2\int_0^1 x\cos(y+z)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^2 \left[\frac{x^2}{2}\cos(y+z)\right]_{x=0}^{1} dy\,dz = \int_0^3\int_0^2 \frac{1}{2}\cos(y+z)\,dy\,dz$$

$$= \int_0^3 \left[\frac{1}{2}\sin(y+z)\right]_{y=0}^{2} dz = \int_0^3 \left(\frac{1}{2}\sin(2+z) - \frac{1}{2}\sin(z)\right) dz$$

$$= \left[-\frac{1}{2}\cos(2+z) + \frac{1}{2}\cos(z)\right]_{z=0}^{3}$$

$$= -\frac{1}{2}\cos(5) + \frac{1}{2}\cos(3) + \frac{1}{2}\cos(2) - \frac{1}{2}\cos(0)$$

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Example 1

Evaluate the triple integral $\iiint_B x\cos(y+z)\,dV,$ where B is the rectangular box given by

$$B = [0,1] \times [0,2] \times [0,3]$$

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Triple Integrals over General Regions

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Triple Integrals over General Regions (1 of 10)

Now we define the **triple integral over a general bounded region *E*** in three-dimensional space (a solid) by much the same procedure that we used for double integrals.

We enclose *E* in a box *B* of the type given by Equation 1. Then we define *F* so that it agrees with *f* on *E* but is 0 for points in *B* that are outside *E*.

By definition,

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

This integral exists if *f* is continuous and the boundary of *E* is “reasonably smooth.”

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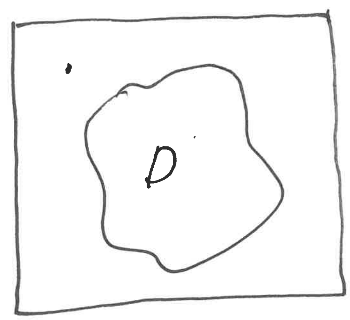
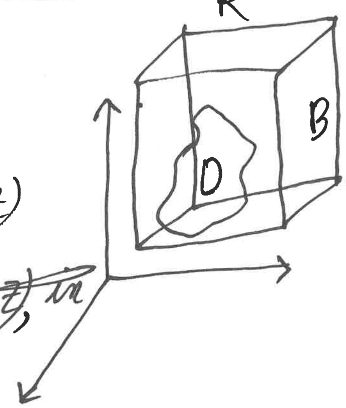
$f(x,y)$



R

$$F(x,y) = \begin{cases} f(x,y), & (x,y)\in D \\ 0, & \text{otherwise} \end{cases}$$



$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

$f(x,y,z)$

$$F(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z)\in D \\ 0, & \text{otherwise} \end{cases}$$

$\cancel{F(x,y,z) = \{ f(x,y,z), \text{in}}$

$$\iiint_B F(x,y,z)\,dV = \iiint_D f(x,y,z)\,dV$$

Triple Integrals over General Regions (4 of 10)

Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument, it can be shown that if E is a type 1 region given by Equation 5, then

$$6 \quad \iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

Example 5

Example: Find the volume of the solid, E in the first octant bounded by $x + y + z = 1$ and $x + y + 2z = 1$.

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Example 5

Example: Find the volume of the solid, E in the first octant bounded by $x + y + z = 1$ and $x + y + 2z = 1$.

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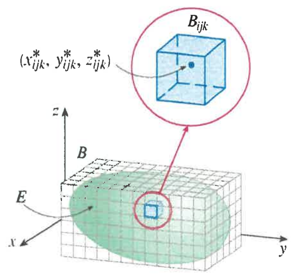
Applications of Triple Integrals (4 of 10)

All the applications of double integrals can be extended to triple integrals using analogous reasoning.

For example, suppose that a solid object occupying a region E has density $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z), in E.

Applications of Triple Integrals (5 of 10)

To find the total mass m of E we divide a rectangular box B containing E into sub-boxes B_{ijk} of the same size (as in Figure 18), and consider $\rho(x, y, z)$ to be 0 outside E.



The mass of each sub-box B_{ijk} is approximated by $\rho(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$

Figure 18

Example 2

Example: Let T be the tetrahedron with vertices $O(0,0,0), A(0,0,6), B(4,0,0)$ and $C(0,4,0)$.

(Note that the plane containing the points A, B and C has the equation $3x+3y+2z=12$)

(a) Express T as a solid region type 1.

(b) Express $\iiint_T f(x, y, z)\, dV$ as an iterated integral.

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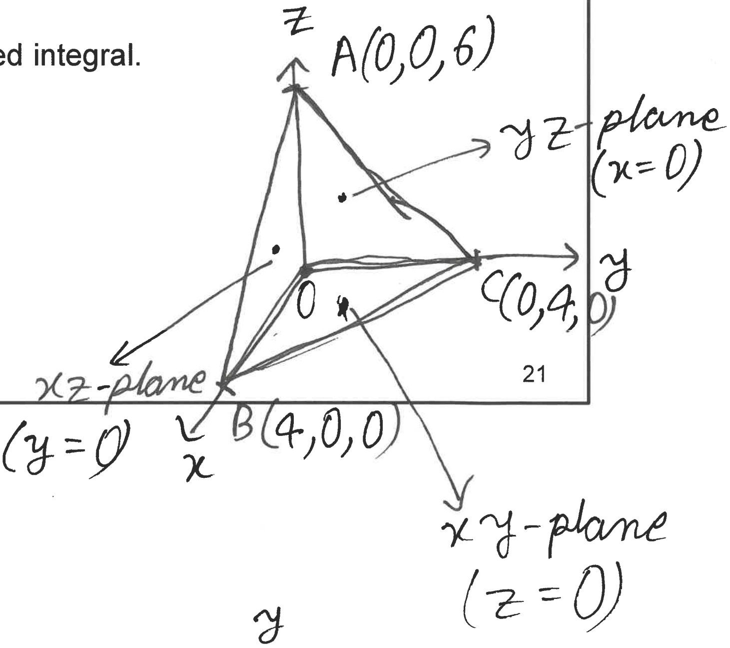


OAC: $x=0$

OBA: $y=0$

OBC: $z=0$

ABC: $3x+3y+2z=12$

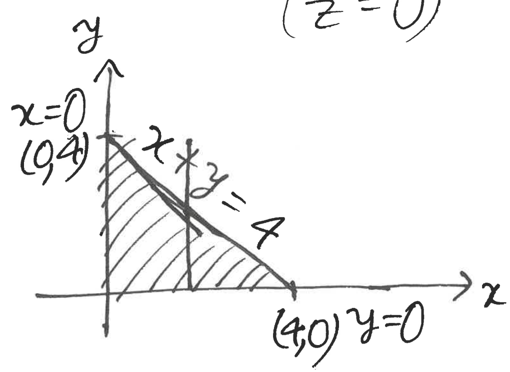
$\Rightarrow z = 6 - \frac{3}{2}x - \frac{3}{2}y$

Projection onto xy-plane:

$x=0$, $y=0$, $3x+3y=12$

$\Rightarrow x+y=4$

$D = \{(x,y) : 0 \leq x \leq 4, 0 \leq y \leq 4-x\}$



$0 \leq z \leq 6 - \frac{3}{2}x - \frac{3}{2}y$

$E = \{(x,y,z) : 0 \leq x \leq 4, 0 \leq y \leq 4-x, 0 \leq z \leq 6 - \frac{3}{2}x - \frac{3}{2}y\}$

Example 2

Example: Let T be the tetrahedron with vertices $O(0,0,0), A(0,0,6), B(4,0,0)$ and $C(0,4,0)$.

(Note that the plane containing the points A, B and C has the equation $3x + 3y + 2z = 12$)

(a) Express T as a solid region type 1.

(b) Express $\int\int_T\int f(x, y, z)\, dV$ as an iterated integral.

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Example 2

Example: Let T be the tetrahedron with vertices $O(0,0,0), A(0,0,6), B(4,0,0)$ and $C(0,4,0)$.

(Note that the plane containing the points A, B and C has the equation $3x + 3y + 2z = 12$)

(a) Express T as a solid region type 1.

(b) Express $\int\int_T\int f$(x, y, z) dV as an iterated integral.

Triple Integrals over General Regions (8 of 10)

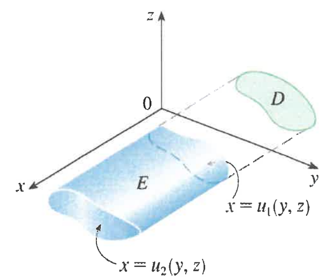
A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 8).

The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

10 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$



A type 2 region

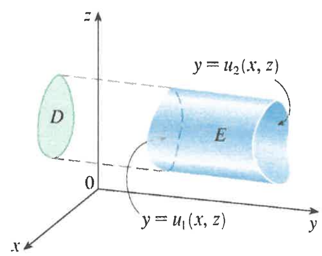
Figure 8

Triple Integrals over General Regions (9 of 10)

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 9).



A type 3 region

Figure 9

Triple Integrals over General Regions (10 of 10)

For this type of region we have

$$\textbf{11} \quad \iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether *D* is a type I or type II plane region (and corresponding to Equations 7 and 8).

Changing the Order of Integration

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Changing the Order of Integration (1 of 1)

Fubini's Theorem for Triple Integrals allows us to express a triple integral as an iterated integral, and there are six different orders of integration in which we can do this.

Given an iterated integral, it may be advantageous to change the order of integration because evaluating an iterated integral in one order may be simpler than in another.

In the next example we investigate equivalent iterated integrals using different orders of integration.

$$\left.\begin{array}{l}\iiint f\,dz\,dy\,dx \\ \iiint f\,dz\,dx\,dy\end{array}\right\} \text{Type I}$$

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$$\left.\begin{array}{l}\iiint f\,dy\,dz\,dx \\ \iiint f\,dy\,dx\,dz\end{array}\right\} \text{Type II}$$

$$\left.\begin{array}{l}\iiint f\,dx\,dy\,dz \\ \iiint f\,dx\,dz\,dy\end{array}\right\} \text{Type III}$$

Example 4

Rewrite the iterated integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$ as an iterated integral in the following orders.

(a) Integrate first with respect to x, then z, and then y.

(b) Integrate first with respect to y, then x, and then z.

(b) $0 \leq x$, $x \leq 1$, $\sqrt{x} \leq y \Leftrightarrow x \leq y^2$, $y \leq 1$

$0 \leq z$, $z \leq 1-y \Leftrightarrow y \leq 1-z$

Bound for y:

$\sqrt{x} \leq y$, $\cancel{y \leq 1}$, $y \leq 1-z$

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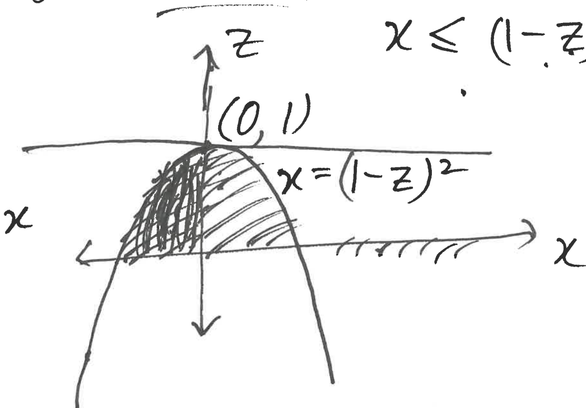
$\sqrt{x} \leq y \leq 1-z \implies \sqrt{x} \leq 1-z$

$0 \leq x$, $x \leq 1$, $\cancel{x \leq y^2}$ $0 \leq z$, $\sqrt{x} \leq 1-z \Leftrightarrow x \leq (1-z)^2$



$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f\,dy\,dx\,dz$$

Example 4

Rewrite the iterated integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$ as an iterated integral in the following orders.

(a) Integrate first with respect to x, then z, and then y.

(b) Integrate first with respect to y, then x, and then z.

(a) $dx\,dz\,dy$

$0 \leq x$, $x \leq 1$, $\sqrt{x} \leq y \Leftrightarrow x \leq y^2$, $y \leq 1$, $0 \leq z$

, $z \leq 1-y \Leftrightarrow y \leq 1-z$

Bounds for x :-

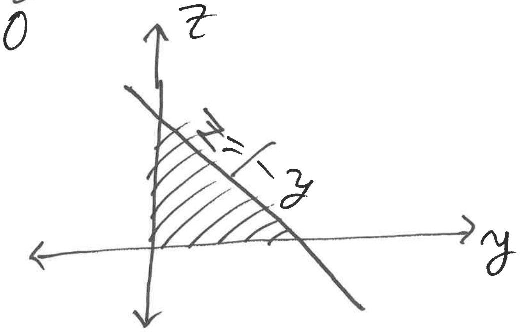
$0 \leq x$, $\cancel{x \leq 1}$, $x \leq y^2$

$0 \leq x \leq y^2$

Bounds for y and z :

$0 \leq y$, $y \leq 1$, $0 \leq z$, $z \leq 1-y$

$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f\, dx\, dz\, dy$$



Example 4

Rewrite the iterated integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$ as an iterated integral in the following orders.

(a) Integrate first with respect to x, then z, and then y.

(b) Integrate first with respect to y, then x, and then z.

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Applications of Triple Integrals (2 of 10)

Nonetheless, the triple integral $\iiint_E f(x, y, z)\, dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

$$\textbf{12} \quad V(E) = \iiint_E dV$$