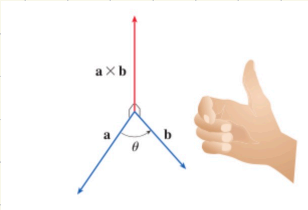
Recall:

Cross product: If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ then the cross product of a and b is the vector

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

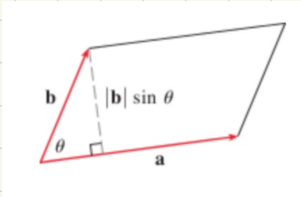


$$\boxed{\Rightarrow \quad a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - j \begin{vmatrix} a_1 & b_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}$$

$i = \langle 1, 0, 0 \rangle$
$j = \langle 0, 1, 0 \rangle$
$k = \langle 0, 0, 1 \rangle$

Facts:

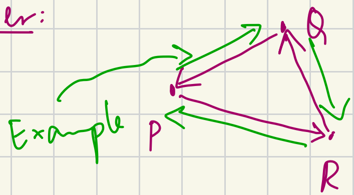
- $a \times b$ is orthogonal to a and b
- $\|a \times b\| = |a|\,|b| \sin\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$ (so $0 \le \theta \le \pi$)
- Two nonzero vectors are parallel $\Leftrightarrow$ $a \times b = 0$
- $\|a \times b\|$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$



i.e. Area = height × base = $\|b\| \sin\theta \times \|a\| = \|a \times b\|$

Example: Find a vector that is perpendicular to the plane that passes through the points $P(1,4,6)$, $Q(-2,5,-1)$, $R(1,-1,1)$

Answer:



- We need 2 vectors to determine the plane that P, Q, R are in.

  Eg: take $\vec{PQ}, \vec{QR}$ or $\vec{RP}, \vec{QP}, \ldots$

Eg:

$$\left.\begin{aligned} \vec{PQ} &= \langle -2-1,\ 5-4,\ -1-6 \rangle = \langle -3, 1, -7 \rangle \\ \vec{PR} &= \langle 1-1, -1-4, 1-6 \rangle = \langle 0, -5, -5 \rangle \end{aligned}\right\}$$

direction vectors for the plane with points P, Q, R

- If you find a vector perpendicular to $\vec{PQ}$ and $\vec{PR}$ it is perpendicular to the plane spanned by $\vec{PQ}$ and $\vec{PR}$.

- Take $\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} i - j \begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} + k \begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix}$

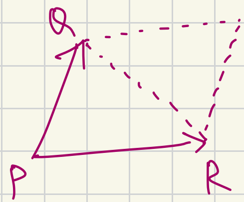
$$\left(\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \right)$$

$$= (-5-35)i - j(15+0) + k(15-0)$$

$$= -40i - 15j + 15k \qquad \blacksquare$$

Fact: $a\langle -40, -15, 15\rangle$ is $\perp$ to $\operatorname{span}(\overrightarrow{PQ}, \overrightarrow{PR})$ for any $a \in \mathbb{R} \setminus \{0\}$.

<u>Example</u>: find the area of the triangle with vertices P, Q, R.



$\Rightarrow$ Area of triangle PQR is half of the area of the parallelogram determined by $\overrightarrow{PQ}, \overrightarrow{PR}$.

$$\cdot \ \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \|\langle -40, -15, 15\rangle\| = \sqrt{(-40)^2 + (-15)^2 + (15)^2}$$

$$= 5\sqrt{82}$$

$$\Rightarrow \text{Area of } PQR = \boxed{\frac{5\sqrt{82}}{2}}.$$

Properties of the cross product:

Facts:

- $i \times j = k$
- $j \times i = -k$ pf: $j \times i = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = i \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}_{=0-0} - j \underbrace{\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}_{0-0} + k \underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{=0-1}$

$$= -k$$

- $a \times b = -(b \times a)$ pf: $a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$b \times a = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = i \begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix} - j \begin{vmatrix} b_1 & b_3 \\ a_1 & a_3 \end{vmatrix}$$

$$+ k \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} \Big) = -(a \times b)$$

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc) = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

11 Properties of the Cross Product

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

linearity

- $j \times k = i$
- $k \times j = -i$
- $k \times i = j$
- $i \times k = -j$

<u>eg</u>: $i \times (2j + 3k) = i \times (2j) + i \times (3k)$

$$= 2(i \times j) + 3(i \times k) = 2k - 3j$$

<u>Caution:</u> $\cdot\ a \times b \neq b \times a$

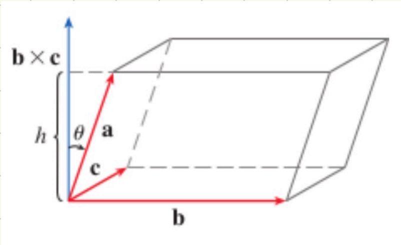
$\cdot\ (a \times b) \times c \neq a \times (b \times c)$

<u>Triple products:</u>

$(a = \langle a_1, a_2, a_3 \rangle,\ b = \langle b_1, b_2, b_3 \rangle,\ c = \langle c_1, c_2, c_3 \rangle)$

<u>Def:</u> $$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

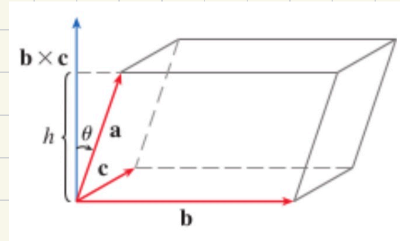
<u>Thm:</u>



Volume of parallelopiped determined by $\vec{a}, \vec{b}, \vec{c}$ is

$$|a \cdot (b \times c)|$$

Pf:



$\cos\theta = \frac{h}{\|a\|} \Rightarrow |h| = \big|\, \|a\| \cdot \cos\theta \,\big|$
$= \|a\| \cdot |\cos\theta|$

$\Rightarrow$ Volume $= \text{Area}(\vec{b}, \vec{c}) \times h$

$= \|\vec{b} \times \vec{c}\| \cdot \|a\| \cdot |\cos\theta|$

$= |a \cdot (\vec{b} \times \vec{c})|$

$$\boxed{\begin{array}{l} \text{Recall: } \vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta \\ \Rightarrow |\vec{u}\cdot\vec{v}| = \|\vec{u}\|\cdot\|\vec{v}\|\cdot|\cos\theta| \end{array}}$$

Example: Use the scaler triple product to show that $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$ and $\vec{c} = \langle 0, -9, 18 \rangle$ are coplanar.

For example: $\vec{a}, \vec{b}, \vec{c}$ are coplanar if $\vec{a}$ is in the plane spanned by $\vec{b}$ and $\vec{c}$

$\Rightarrow$ Volume $= \text{Area}(\vec{b}, \vec{c}) \times \underbrace{0}_{\text{height} = 0}$

So check if $a \cdot (b \times c) = 0$. If so then vectors are coplanar.

$$a\cdot(b\times c) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} + (-7)\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

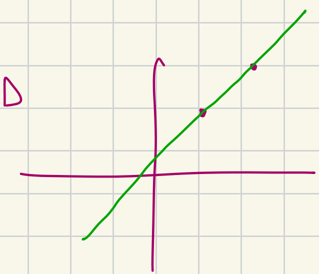
$$= 1(-18+36) - 4(36) - 7(-18)$$

$$= 1(18) - 4\times 2\times 18 + 7\times 18$$

$$= -7\times 18 + 7\times 18 = 0$$

12.5: Lines and Planes.

Eg: In 2-D

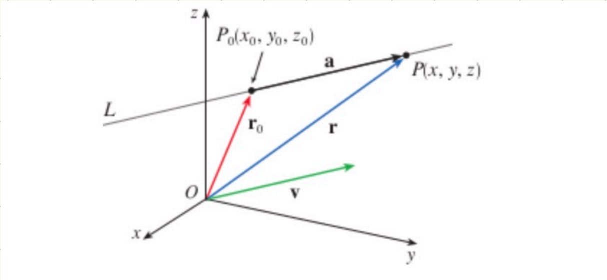


Given two points:

1. Find slope $= \dfrac{\text{rise}}{\text{run}}$
2. All other pts on the line are "in the direction of the slope".

In general, point-"slope" form of a line in $\mathbb{R}^n$ is:

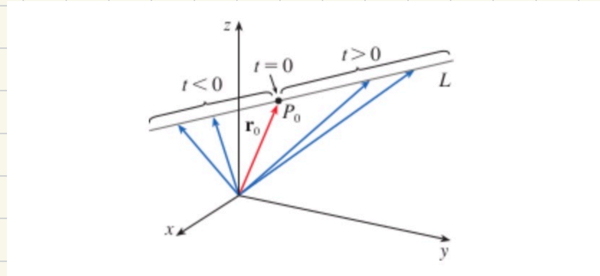
$$\underbrace{\vec{x} = x_0 + t\vec{v}}_{\text{vector form of the equation}} \quad \text{where } x_0 \in \mathbb{R}^n,\ \vec{v} \in V_n.$$



Given two points in $\mathbb{R}^3$, $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$

$\vec{v} = \overrightarrow{P_0P}$ is the vector in the direction of the line through P_0 and P.

So the vector form of the equation of the line through P_0 and P is $x = P_0 + t\,\overrightarrow{P_0P}$


z
t < 0
t = 0
t > 0
L
P_0
$\mathbf{r}_0$
x
y