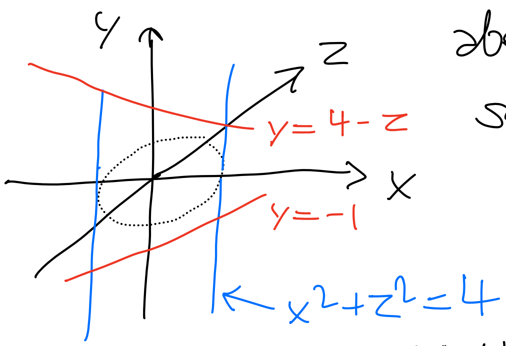
Midterm 2 graded this week, grades released next week. Standard disclaimer about makeups yada yada etc...

DW 11 #4: Volume $x^2+z^2=4$, $y=-1$, $y+z=4$ enclose.

Notice x & z both squared. Try cylindrical with y vertical.

So $x=r\cos\theta$, $z=r\sin\theta$, $y=y$. There is infinite space above $y=4-z$ & ∞ space below $y=-1$, so we seek $-1\leq y\leq 4-z=4-r\sin\theta$.



Need to be inside cylinder, so

$\leftarrow x^2+z^2=4$ $\quad 4\geq x^2+z^2=r^2 \Rightarrow 0\leq r\leq 2.$

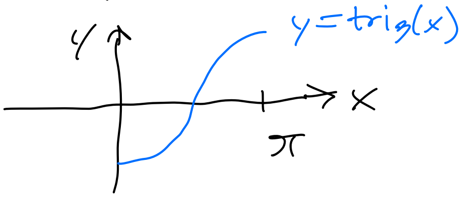
Finally, no restriction on θ found, so $0\leq\theta\leq 2\pi$.

$dV = r\,dy\,dr\,d\theta$ (do not accidentally put dz!), so

$$V=\int_{solid} dV=\int_0^{2\pi}\int_0^{2}\int_{-1}^{4-rs\theta} r\,dy\,dr\,d\theta=\int_0^{2\pi}\int_0^{2}(5-rs\theta)r\,dr\,d\theta$$

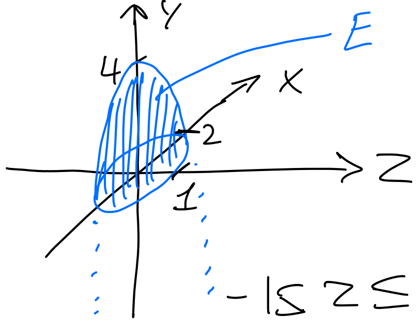
$$=\int_0^{2\pi}\left(\int_0^2\left(5r-r^2\sin\theta\right)dr\right)d\theta=\int_0^{2\pi}\left(10-\frac{8}{3}\sin\theta\right)d\theta=20\pi.$$

Note! you can get many trig integrals to be 0 by graphing and/or usin odd & even properties



From this graph with symmetry, $\int_0^{\pi}\text{trig}(x)\,dx=0$

Q5: Express $\iiint_E f\,dV$ in 6 ways, E bounded by $\begin{cases}y=4-x^2-4z^2\\ y=0\end{cases}$



There are $3! = 6$ ways to order $x, y \& z$. Let's write 3 of them for the sake of illustration. Note $0 \le y \le 4 - x^2 - 4z^2$, $-1 \le z \le 1$, $-2 \le x \le 2$, $x^2 + 4z^2 \le 4$ all bound E in some way. dy on inside $\Rightarrow$ blue & orange bound.

If dy inside, x outside: $4z^2 \le 4 - x^2 \Rightarrow z^2 \le 1 - \frac{x^2}{4} \Rightarrow$

$-\sqrt{1-\frac{x^2}{4}} \le z \le \sqrt{1-\frac{x^2}{4}}$, so

$$\iiint_E f\,dV = \int_{-2}^{2} \int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} \int_0^{4-x^2-4z^2} f\,dy\,dz\,dx$$

If dy inside, z outside: $x^2 \le 4 - 4z^2 = 4(1-z^2) \Rightarrow$

$-2\sqrt{1-z^2} \le x \le 2\sqrt{1-z^2}$, so

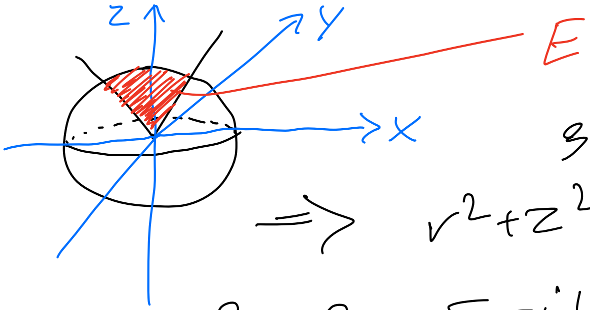
$$\iiint_E f\,dV = \int_{-1}^{1} \int_{-2\sqrt{1-z^2}}^{2\sqrt{1-z^2}} \int_0^{4-x^2-4z^2} f\,dy\,dx\,dz$$

What if y outside? $0 \le y \le 4$, $x^2 + 4z^2 \le 4 - y$. Let x be in the middle. $x^2 \le x^2 + 4z^2 \le 4 - y \Rightarrow -\sqrt{4-y} \le x \le \sqrt{4-y}$, then $z^2 \le \frac{1}{4}(4 - y - x^2) \Rightarrow -\frac{1}{2}\sqrt{\cdots} \le z \le \frac{1}{2}\sqrt{\cdots} \Rightarrow$

$$\int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-\frac{1}{2}\sqrt{4-y-x^2}}^{\frac{1}{2}\sqrt{4-y-x^2}} f\,dz\,dx\,dy.$$

Note: You may also express it as an iterated integral in $y, r \& \theta$ using cylindrical. This is the 7th way

Q8: Volume between $z = x^2 + y^2$ & $x^2 + y^2 + z^2 = 2$.



We seek the volume of E as graphed. $x = r\cos\theta$, $y = r\sin\theta$, $z = z$
$\Rightarrow r^2 + z^2 = 2$, $z = r^2$. Inside the sphere $\Rightarrow$ $r^2 + z^2 \le 2$. Inside paraboloid $\Rightarrow z \ge r^2$. We have $2 \ge r^4 + r^2 \Rightarrow r \le 1$, so $0 \le r \le 1$. Also, $z^2 \le 2 - r^2$
$\Rightarrow z \le \sqrt{2-r^2}$, so $r^2 \le z \le \sqrt{2-r^2}$. Lastly, no restriction on θ, so $0 \le \theta \le 2\pi$. $dV = r\,dr\,d\theta\,dz$.

So $V = \int_E dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta =$

$$2\pi \int_0^1 r\left(\sqrt{2-r^2} - r^2\right)dr = 2\pi \int_0^1 \left(r(2-r^2)^{1/2} - r^3\right)dr$$

$$= 2\pi\left(-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{3}\text{etc} + \frac{r^4}{4}\right)\Big|_1^0$$

$$= 2\pi\left(\frac{1}{3}2^{3/2} - \frac{1}{3}1^{3/2} - \frac{1}{4}\right) = 2\pi\,\frac{8\sqrt{2}-7}{12} = \frac{8\sqrt{2}-7}{6}\pi$$

Note: If you instead put z on outside, then $0 \le z \le \sqrt{2}$ and $r \ge 0$, but the upper bound for r will be a piecewise function & you'll need 2 separate integrals. So be careful choosing r & z orders

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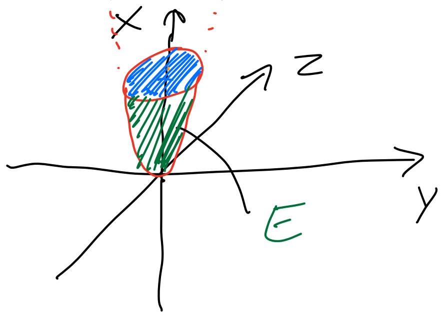
Q6: What surface does $z^2 + r^2 = 9$ describe in cylindrical?

It's hard to recognize equations in alternate coordinate systems like polar, cylindrical, spherical etc., so let's convert to Cartesian.

z is already there, now use $r^2 = x^2 + y^2$, so $9 = x^2 + y^2 + z^2$, equation of sphere with radius 3.

Q3: $\iiint_E x\,dV$ where E bounded by $\begin{cases} x = 4y^2 + 4z^2 \\ x = 4 \end{cases}$



Bounds from graphing:

Note: If you have 1 variable isolated and the other 2 dependent on it, make that variable your vertical axis & the other 2 a flat plane when graphing. Do not be afraid to relabel the axes so that x or y is up instead of z.

$x \leq 4$ & $4y^2+4z^2 \leq x$. Also from the graph, $x \geq 0$.

Use cylindrical with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$, then

$0 \leq x \leq 4$, $x \geq 4(y^2+z^2) = 4r^2 \Rightarrow 0 \leq r \leq \sqrt{\frac{x}{4}}$,

and no restriction on $\theta \Rightarrow 0 \leq \theta \leq 2\pi$.

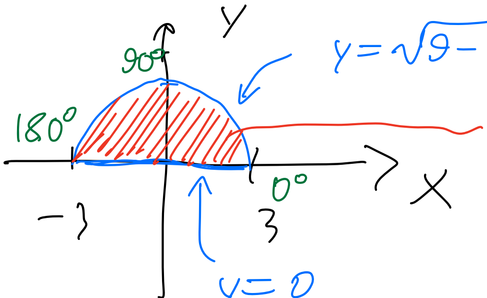
Convert $x\,dV = x\,r\,dr\,d\theta\,dx$, so

$$\iiint_E x\,dV = \int_0^4 \int_0^{2\pi} \int_0^{\sqrt{\frac{x}{4}}} rx\,dr\,d\theta\,dx = 2\pi \int_0^4 x \cdot \frac{x}{8}\,dx$$

Q10: Find $\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\,dz\,dy\,dx$ via cylindrical.

For cylindrical, use $z = z$, $x = r\cos\theta$, $y = r\sin\theta$ because of the prevalence of x^2+y^2 in the integral.

Graph bounds from the 2 outer integrals to find bounds on r & θ. $-3 \leq x \leq 3$, $0 \leq y \leq \sqrt{9-x^2}$.



The region is a semicircle of radius 3 going from 0° to 180°, so

$0 \leq r \leq 3$ and $0 \leq \theta \leq 180° = \pi$. Now let's substitute for all the terms in the integral.

Note: you should never have a negative bound for r. As θ varies, it will cover all the coordinates with negative x or y values.

$\sqrt{x^2+y^2}\,dz\,dy\,dx = r\,dV = r^2\,dr\,d\theta\,dz$.

$z \leq 9-x^2-y^2 = 9-r^2$.

Substitute orange terms to get

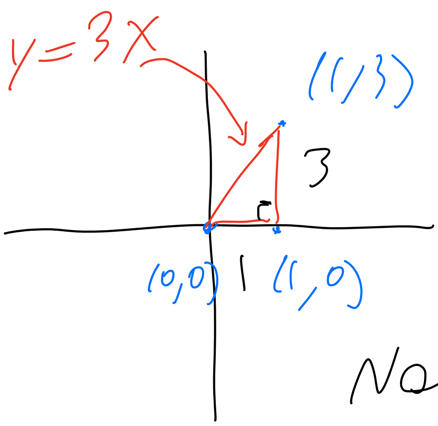
$$\int_0^{\pi}\int_0^3\int_0^{9-r^2} r^2\,dz\,dr\,d\theta = \pi\int_0^3 r^2(9-r^2)\,dr$$

$$= \pi\int_0^3 (9r^2-r^4)\,dr = \pi\left(3r^3-\frac{r^5}{5}\right)\Big|_0^3 = \pi\left(81-\frac{243}{5}\right)$$

$$= \frac{162\pi}{5}.$$

DW 10 #4: Average of $f=xy$ on triangle D with vertices $(0,0), (1,0), (1,3)$.

Recall formula: $\text{avg}_D f = \frac{1}{\text{area}(D)}\iint_D f\,dA$



D is right triangle with legs 1 & 3, so area $= \frac{1}{2}\cdot 1\cdot 3 = \frac{3}{2}$.

Now find bounds on x & y for D. $0 \leq x \leq 1$, $0 \leq y \leq 3x$.

$$\iint_D f\,dA = \int_0^1\int_0^{3x} xy\,dy\,dx = \int_0^1 x\,\frac{9x^2}{2}\,dx =$$

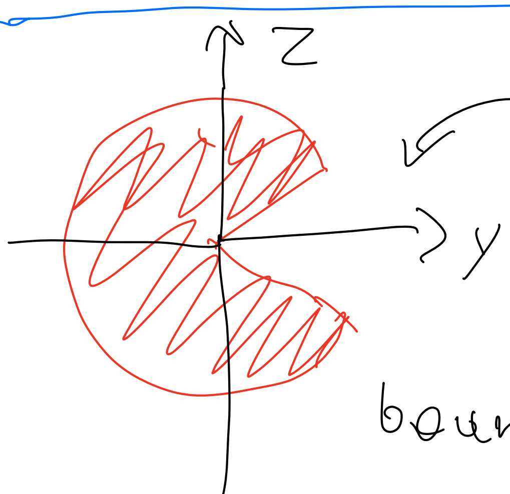
$\frac{9}{2}\int_0^1 x^3\,dx = \frac{9}{2}\cdot\frac{1}{4} = \frac{9}{8}$, so

$$\text{average} = \frac{1}{3/2}\,\frac{9}{8} = \frac{2}{3}\cdot\frac{9}{8} = \frac{3}{4},$$

If we estimate average x & average y in triangle, then $x \approx \frac{1}{2}$ & $y \approx \frac{3}{2}$ on average since $0 \le x \le 1$ & $0 \le y \le 3$.

Then $f = xy \approx \frac{1}{2}\cdot\frac{3}{2} = 3/4$.

Note: It's a coincidence we get $\frac{3}{4}$.
This is a good way to check your work, but the estimate may not always be exact. You still have to actually solve the problem & find the integral.



For the 2 ways to assign $\{y, z\} = \{r\cos\theta, r\sin\theta\}$, you will get different bounds on θ. However, you can never make a mistake choosing the wrong order of $\cos\theta$ & $\sin\theta$.

Every problem can be solved with either order. This is because if you choose $\cos\theta$ for y & $\sin\theta$ for z, winding up with $a \le \theta \le b$, observe that the transformation $\theta \to \frac{\pi}{2} - \theta$ sends $[a,b]$ to $[\frac{\pi}{2} - b, \frac{\pi}{2} - a]$ and $\{(\cos\theta, y), (\sin\theta, z)\}$ to $\{(\cos(\frac{\pi}{2} - \theta), y), (\sin(\frac{\pi}{2} - \theta), z)\}$ $= \{(\sin\theta, y), (\cos\theta, z)\}$, so we will have the same exact value for the integral with sin & cos chosen in the other order, just that the new bounds on θ will be $\frac{\pi}{2} - b \le \theta \le \frac{\pi}{2} - a$.