Challenge problems: 2b, 4b, 7, 9

Midterms graded, grades waiting to be released

For Shashank's section! whether discussion attendance will be graded still TBD, this week not graded

Unrelated: don't buy iPads. Tech support sucks

DW7 #1: 1st order partial derivatives of

$$f(x,y,z) = 4x^3y^2 - e^z y^4 + \frac{z^3}{x^2} + 4y - x^5$$

$$f_x = 12x^2y^2 - \cancel{0} - \frac{2z^3}{x^3} + \cancel{0} - 5x^4 = 12x^2y^2 - \frac{2z^3}{x^3} - 5x^4$$

$$f_y = 8x^3y - 4y^3e^z + 4$$

$$f_z = -e^z y^4 + \frac{3z^2}{x^2}$$

$$\frac{3}{c} \leftarrow \cancel{\left(\frac{z}{x}\right)^2 \cdot \frac{-}{c^2}}$$

3a: Tangent plane to $z = 4x^2 + y^2 - 9y$ at $(x,y) = (1,4)$.

Recall: formula for tangent plane to $z = f(x,y)$ at (x_0, y_0) is $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = z - f(x_0,y_0)$

Note! if you forget this, one way to remember this is the plane must pass through $(x_0, y_0, f(x_0,y_0))$, the slope in the x direction is f_x since slopes represent derivatives & you can imagine the plane $y = y_0$

as a 2D cross section where x varies & z varies as f_x. Similarly, slope in y-direction is f_y.

Now plug in using $f(x,y) = 4x^2 + y^2 - 9y$.

$$f_x\big|_{(1,4)} = 8x\big|_{(1,4)} = 8$$

$$f_y\big|_{(1,4)} = (2y-9)\big|_{(1,4)} = 2\cdot 4 - 9 = -1$$

$$f(1,4) = 4 + 16 - 36 = -16$$

Subsitute: $8(x-1) - (y-4) = z+16$
is the tangent plane equation.

#6: Find $\frac{dz}{dt}$, $z = \frac{x-y}{x+2y}$, $x = e^{\pi t}$, $y = e^{-\pi t}$.

2 steps to any chain rule question.

1st: figure out the relevant chain rule equation

2nd: plug in for the RHS of equation

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Note: to remember what goes in the denominator & numerator, you can imagine that you're cancelling out the fractions. Each one should get back to where you started.

For example, $\frac{\partial z}{\cancel{\partial x}} \frac{\cancel{\partial x}}{\partial t} = \frac{\partial z \cancel{\partial x}}{\partial t \cancel{\partial x}} = \frac{\partial z}{\partial t} = \frac{dz}{dt}$.

$x = e^{\pi t} \Rightarrow x_t = \pi e^{\pi t}$ $\qquad z = \frac{x-y}{x+2y}$

$y = e^{-\pi t} \Rightarrow y_t = -\pi e^{-\pi t}$

$z = 1 - \frac{3y}{x+2y}$, so $\frac{\partial z}{\partial x} = -3y \frac{\partial}{\partial x}(x+2y)^{-1}$

$= -3y \cdot (-1) \cdot (x+2y)^{-2} = \frac{3y}{(x+2y)^2}$.

$z = -\frac{1}{2} + \frac{1.5x}{x+2y}$, so $\frac{\partial z}{\partial y} = \frac{-1.5x}{(x+2y)^2}$.

So $\frac{\partial z}{\partial t} = \frac{3y}{(x+2y)^2} \pi e^{\pi t} - \frac{1.5x}{(x+2y)^2}(-\pi) e^{-\pi t} =$

$\frac{1.5\pi}{(x+2y)^2}\left(2y e^{\pi t} + x e^{-\pi t}\right) = \frac{1.5\pi}{(x+2y)^2}(2+1) =$

$\frac{4.5\pi}{(e^{\pi t} + 2e^{-\pi t})^2}$.

$\boxed{9:}$ $f(x,y) = f(g(t), h(t))$ where $g(2) = 4$, $g'(2) = -3$, $h(2) = 5$, $h'(2) = 6$, $f_x(4,5) = 2$, $f_y(4,5) = 8$.

Write down chain rule for f:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

$$\frac{\partial f}{\partial t}\Big|_{t=2} = f_x\Big|_{t=2} \frac{dx}{dt}\Big|_{t=2} + f_y\Big|_{t=2} \frac{dy}{dt}\Big|_{t=2}$$

When $t=2$, $x = g(2) = 4$, $y = h(2) = 5$, so

$(x,y) = (4,5)$, $f_x\big|_{t=2} = f_x(4,5) = 2$.

Similarly, we get $f_y\big|_{t=2} = f_y(4,5) = 8$.

$\frac{dx}{dt} = g'(2) = -3$, $\frac{dy}{dt} = h'(2) = 6$, so

$$\frac{\partial f}{\partial t} = 2\cdot(-3) + 8\cdot 6 = 48 - 6 = 42.$$

4: Find linear approximation of each function

(a) $f(-0.99, 1.01)$, $f = \frac{5\sqrt{y}}{x}$ (b) $f(2.01, 0.99)$, $f = \ln(x+y^7)$

Recall: $f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$

is the convenient form for linear approximation

$f(-1,1) = \frac{5\sqrt{1}}{-1} = -5$, $f_x = -\frac{5\sqrt{y}}{x^2} \Rightarrow f_x(-1,1) = -5$,

$f_y = \frac{5}{2x\sqrt{y}} \Rightarrow f_y(-1,1) = \frac{5}{-2} = -2.5$.

Choose $(x,y) = (-1,1) \Rightarrow (\Delta x, \Delta y) = (0.01, 0.01)$,

then $f(-.99, 1.01) \approx f(-1,1) + \frac{1}{100} f_x(-1,1) + 10^{-2} f_y(-1,1)$

$= -5 - 0.05 - .025 = -5.075$

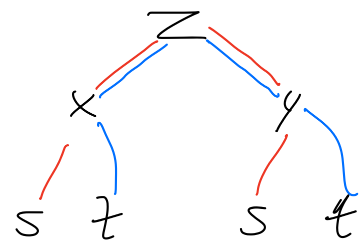
Now part b: $f(2,1) = \ln 3$, $f_x = \dfrac{1}{x+y^7} \Longrightarrow$

$f_x(2,1) = \dfrac{1}{3}$, $f_y = \dfrac{7y^6}{x+y^7} \Longrightarrow f_y = \dfrac{7}{3}$, so

$f(2.01, 0.99) \approx f(2,1) + .01 f_x(2,1) - .01 f_y(2,1) =$

$\ln 3 + .01\left(\frac{1}{3} - \frac{7}{3}\right) = \ln(3) - 0.02$

$\boxed{7:}$ Find $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}$, $z = \tan^{-1}(x^2+y^2)$, $x(s,t) = s\ln t$, $y = te^s$.

Try tree method: 

$\Longrightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial x} = \frac{2x}{1+(x^2+y^2)^2}, \quad \frac{\partial z}{\partial y} = \frac{2y}{1+(x^2+y^2)^2},$$

$$\frac{\partial x}{\partial s} = \ln t, \quad \frac{\partial y}{\partial s} = te^s, \quad \frac{\partial x}{\partial t} = \frac{s}{t}, \quad \frac{\partial y}{\partial t} = e^s,$$

so $$\frac{\partial z}{\partial s} = \frac{2x\ln t + 2yte^s}{1+(x^2+y^2)^2} = \frac{2s(\ln t)^2 + 2t^2(\ln s)^2}{1+(s^2\ln^2 t + t^2e^{2s})^2},$$

$$\frac{\partial z}{\partial t} = \frac{2xs/t + 2ye^s}{1+(x^2+y^2)^2} = \frac{2s^2\frac{\ln t}{t} + 2te^{2s}}{1+(s^2\ln^2 t + t^2e^{2s})^2}$$

$\boxed{8:}$ Implicit diff to find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ if $z^2 = yz + x\ln y$.

Let $F(x,y,z) = z^2 - yz - x\ln y \Rightarrow F = 0$.

Now implicitly differentiate both sides:

$$0 = \frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}.$$

We treat z as depending on x & y. Then $\frac{\partial x}{\partial x} = 1$ & $\frac{\partial y}{\partial x} = 0$ since x & y independent,

so $0 = F_x + F_z \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$.

In our case, $\frac{\partial z}{\partial x} = -\frac{-\ln y}{2z-y} = \frac{\ln y}{2z-y}$.

Similarly, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-z-x/y}{2z-y} = \frac{z+\frac{x}{y}}{2z-y}$.

2: For $g = \frac{x\sin y}{z^2}$, calc. (a) 1st (b) 2nd order partial derivatives of g.

(a): $g_x = \frac{\sin y}{z^2}$, $g_y = \frac{x\cos y}{z^2}$, $g_z = -\frac{2x\sin y}{z^3}$

(b):

1st var \ 2nd var	x	y	z
x	$g_{xx} = 0$	$g_{xy} = \frac{\cos y}{z^2}$	$g_{xz} = -\frac{2\sin y}{z^3}$
y	$g_{yx} = g_{xy}$	$g_{yy} = -\frac{x\sin y}{z^2}$	$g_{yz} = -\frac{2x\cos y}{z^3}$
z	$g_{zx} = g_{xz}$	$g_{zy} = g_{yz}$	$g_{zz} = \frac{6x\sin y}{z^4}$

Note how we use Clairaut's Theorem to save on calculation time

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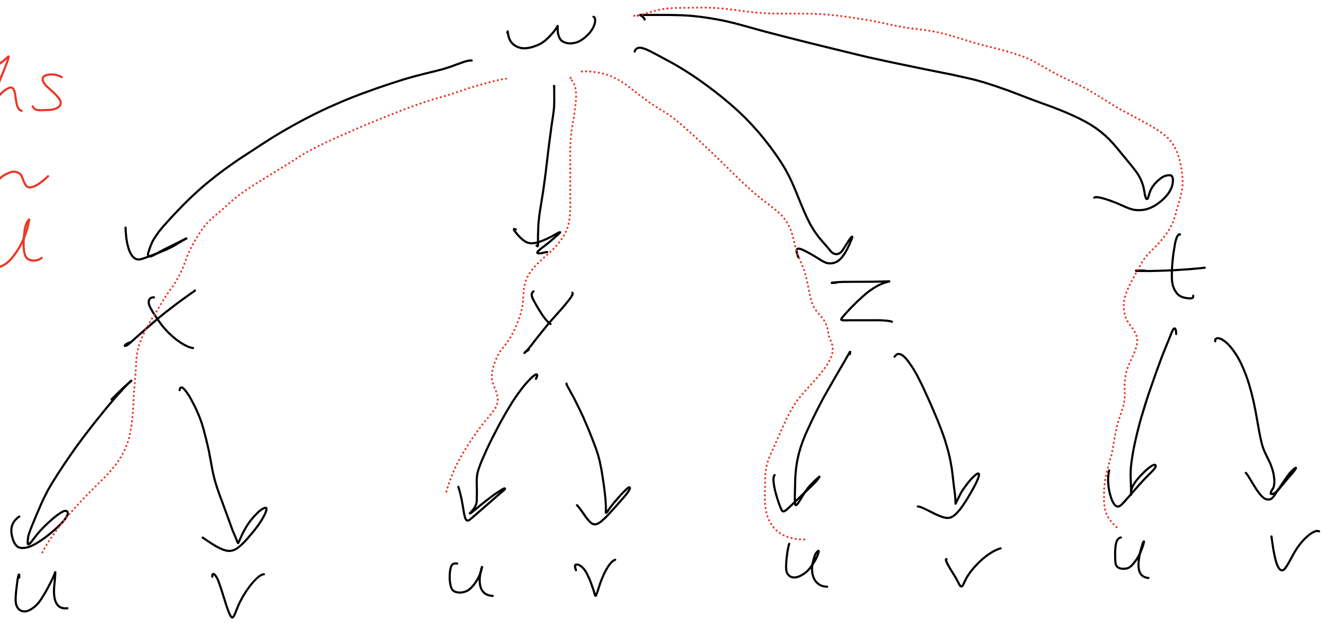
#5: $w = f(x, y, z, t)$, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $t = t(u, v)$.

4 paths from $w \to u$



To find $\frac{\partial w}{\partial u}$, follow all paths that start at w and end at u, and add up all the paths. So

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}.$$