12 Vectors and the Geometry of Space

12.6 Cylinders and Quadric Surfaces

Cylinders and Quadric Surfaces

We have already looked at two special types of surfaces: planes and spheres.

Here we investigate two other types of surfaces: cylinders and quadric surfaces.

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes.

These curves are called **traces** (or cross-sections) of the surface.

Cylinders

Cylinders (1 of 2)

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

Example 1

Sketch the graph of the surface $z = x^2$.

Solution:

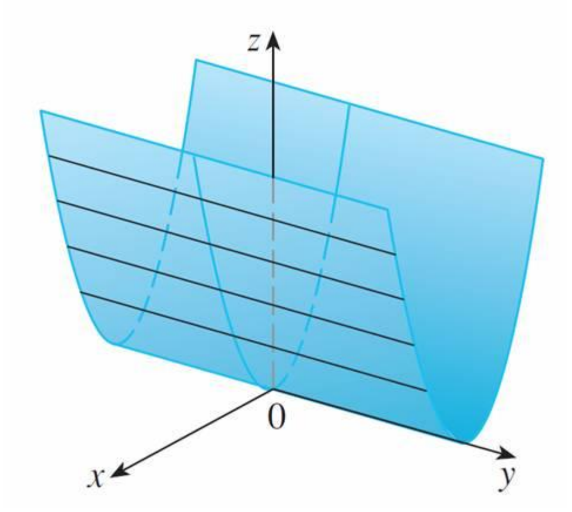
Notice that the equation of the graph, $z = x^2$, doesn't involve *y*.
This means that any vertical plane with equation *y* = *k*(parallel to the *xz*-plane) intersects the graph in a curve with equation $z = x^2$.

So these vertical traces are parabolas.

Example 1 – Solution

Figure 1 shows how the graph is formed by taking the parabola $z = x^2$ in the *xz*-plane and moving it in the direction of the *y*-axis.

The graph is a surface, called a **parabolic cylinder**, made up of infinitely many shifted copies of the same parabola. Here the rulings of the cylinder are parallel to the *y*-axis.



The surface $z = x^2$ is a parabolic cylinder

Figure 1

Cylinders (2 of 2)

In Example 1 the variable *y* is missing from the equation of the cylinder. This is typical of a surface whose rulings are parallel to one of the coordinate axes. If one of the variables *x*, *y* or *z* is missing from the equation of a surface, then the surface is a cylinder.

Note

When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the *xy*-plane is the circle with equations $x^2 + y^2 = 1, \quad z = 0.$

Quadric Surfaces

Quadric Surfaces (1 of 3)

A **quadric surface** is the graph of a second-degree equation in three variables *x*, *y*, and *z*. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where *A*,*B*,*C*,…, *J* are constants, but by translation and rotation it can be brought into one of the two *standard forms*

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0 \quad or \quad Ax^2 + Hy + Cz = 0$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane.

Example 3

Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

Solution:

By substituting $z = 0$, we find that the trace in the xy-plane is $x^2 + \frac{y^2}{9} = 1$,
which we recognize as an equation of an ellipse. In general, the horizontal trace in the plane $z = k$ is

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \quad z = k$$

which is an ellipse, provided that $k^2 < 4$, that is, $-2 < k < 2$.

Example 3 – Solution (1 of 2)

(If $|k| = 2,$ the trace consists of a single point, and the trace is empty for $|k| > 2.$)

Similarly, vertical traces parallel to the *yz*- and *xz*-planes are also ellipses:

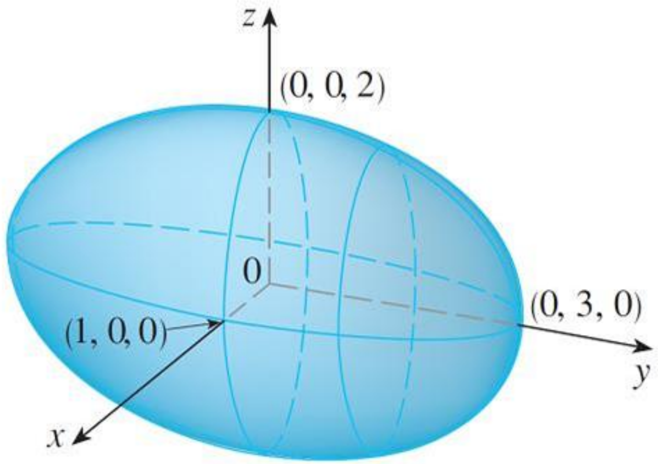
$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \quad x = k \quad (\text{if} -1 < k < 1)$$

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \quad y = k \quad (\text{if} -3 < k < 3)$$

Example 3 – Solution (2 of 2)

Figure 4 shows how drawing some traces indicates the shape of the surface. It’s called an **ellipsoid** because all of its traces are ellipses.

Notice that it is symmetric with respect to each coordinate plane; this is because its equation involves only even powers of *x*, *y*, and *z*.



The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

Figure 4

Example 4

Use traces to sketch the surface $z = 4x^2 + y^2$.

Solution:

If we put $x = 0$, we get $z = y^2$, so the yz-plane intersects the surface in a parabola. If we put $x = k$ (a constant), we get $z = y^2 + 4k^2$.

This means that if we slice the graph with any plane parallel to the yz-plane, we obtain a parabola that opens upward.

Similarly, if $y = k$, the trace is $z = 4x^2 + k^2$, which is again a parabola that opens upward.

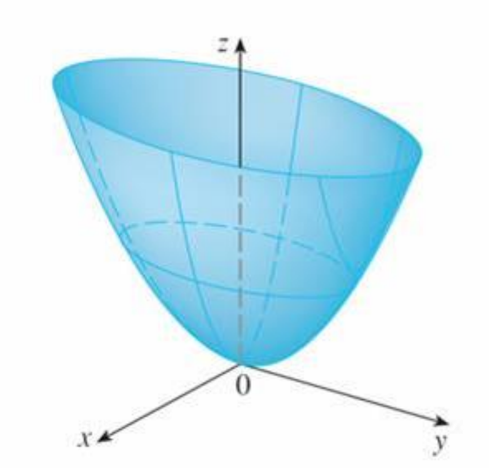
Example 4 – Solution

If we put $z = k$, we get the horizontal traces $4x^2 + y^2 = \mathrm{k}$, which we recognize as a family of ellipses ($k > 0$). Knowing the shapes of the traces, we can sketch the graph in Figure 5.

Because of the elliptical and parabolic traces, the quadric surface

$$z = 4x^2 + y^2,$$

is called an **elliptic paraboloid**.



The surface $z = 4x^2 + y^2$ is an elliptic paraboloid.
Horizontal traces are ellipses; vertical traces are parabolas.

Figure 5

Example 5

Sketch the surface $z = y^2 - x^2$.

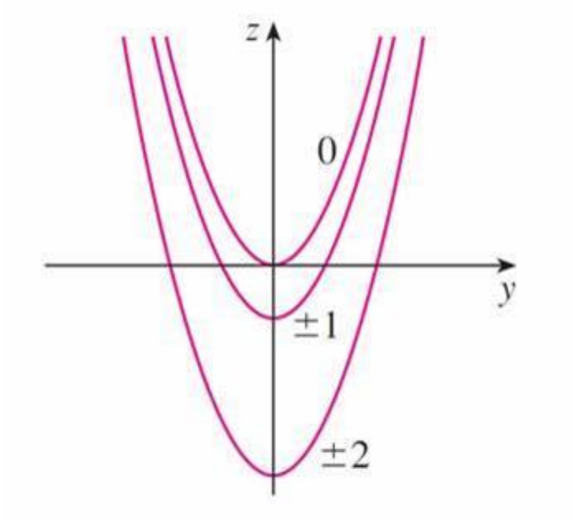
Solution:

The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward.

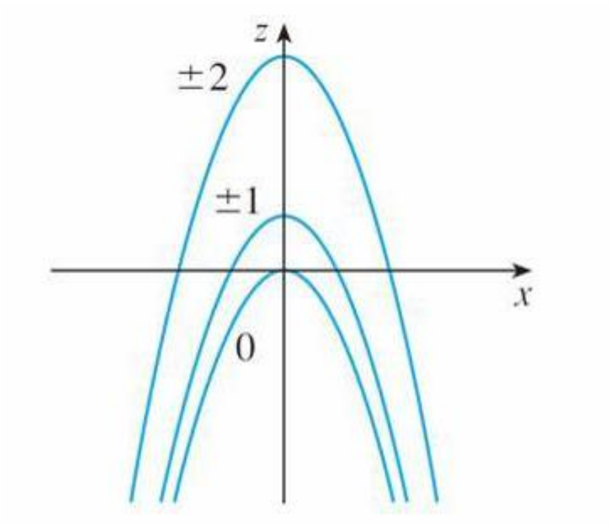
The horizontal traces are $y^2 - x^2 = k$, a family of hyperbolas.

Example 5 – Solution (1 of 3)

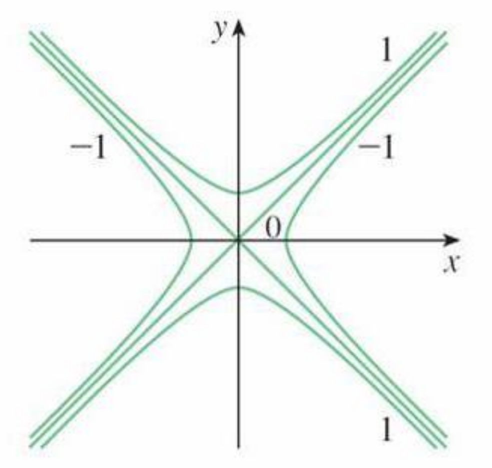
We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.



Traces in $x = k$ are $z = y^2 - k^2$.



Traces in $y = k$ are $z = -x^2 + k^2$.

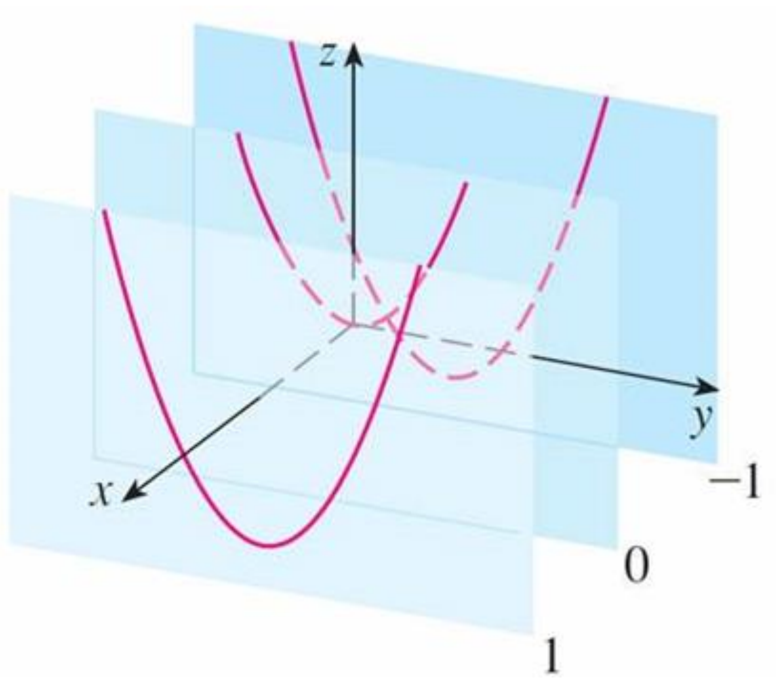


Trace in $z = k$ are $y^2 - x^2 = k$.

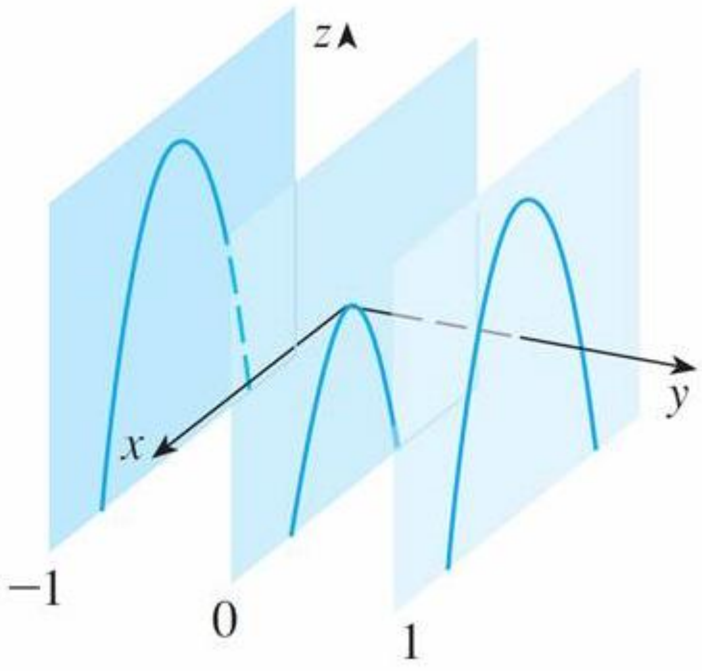
Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of *k*.

Figure 6

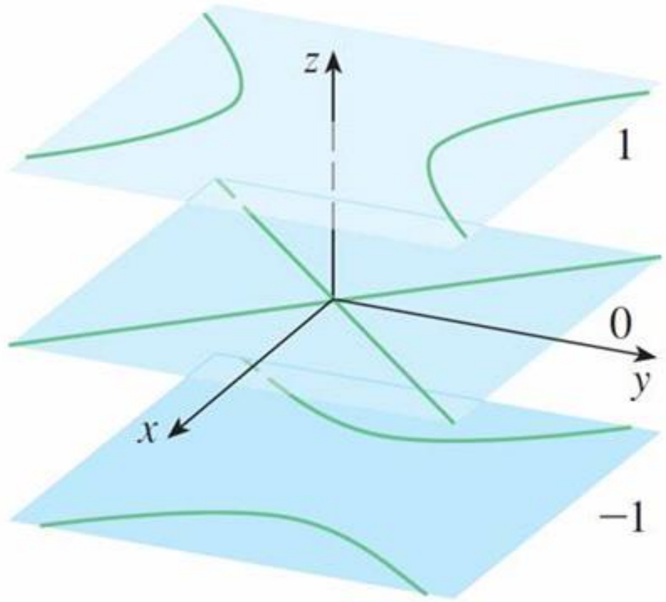
Example 5 – Solution (2 of 3)



Traces in $x = k$



Traces in $y = k$



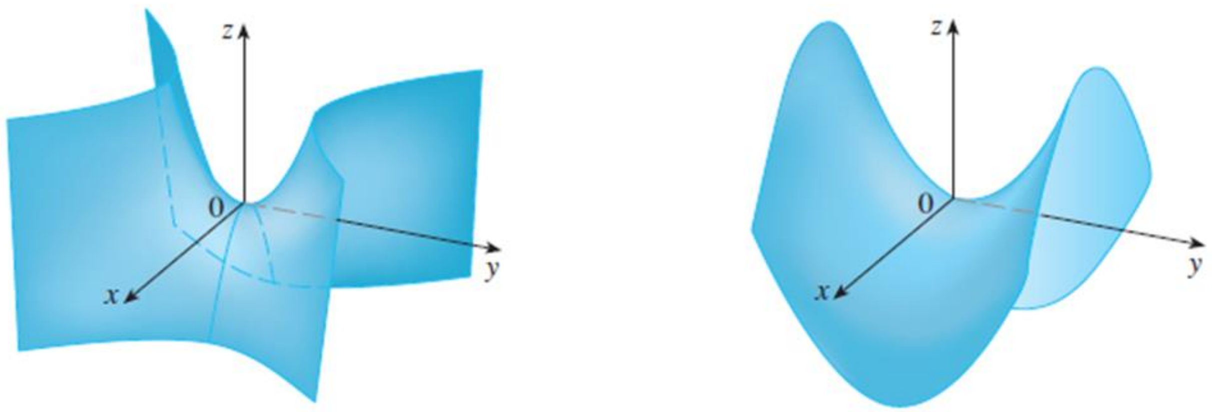
Traces in $z = k$

Traces moved to their correct planes

Figure 7

Example 5 – Solution (3 of 3)

In Figure 8 we fit together the traces from Figure 7 to form the surface $z = y^2 - x^2$, a **hyperbolic paraboloid**.



Two views of the surface $z = y^2 - x^2$, a hyperbolic paraboloid.

Figure 8

Notice that the shape of the surface near the origin resembles that of a saddle.

Example 6

Sketch the surface $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$.

Solution:

The trace in any horizontal plane z = *k* is the ellipse

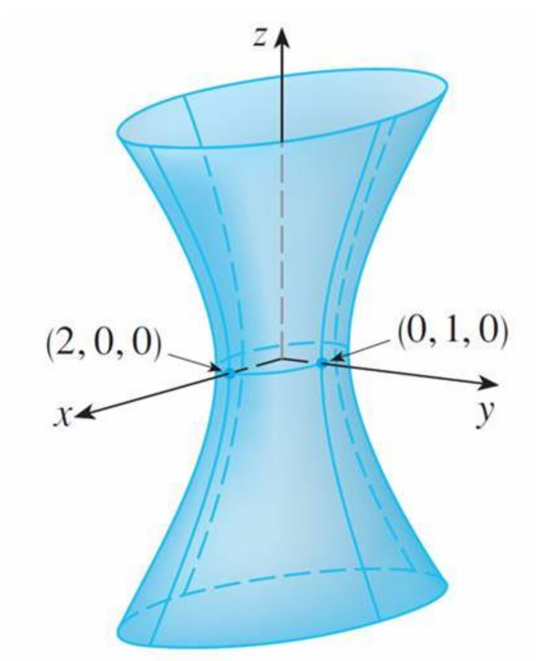
$$\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4} \quad z = k$$

but the traces in the *xz*- and *yz*-planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \quad y = 0 \text{ and } \quad y^2 - \frac{z^2}{4} = 1 \quad x = 0$$

Example 6 – Solution

This surface is called a **hyperboloid of one sheet** and is sketched in Figure 9.



The surface $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1,$

a hyperboloid of one sheet

Figure 9

Quadric Surfaces (2 of 3)

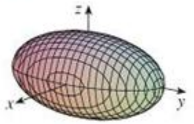
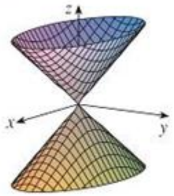
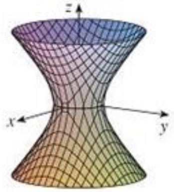
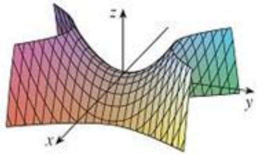
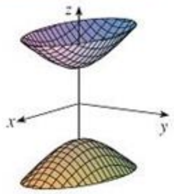
The idea of using traces to draw a surface is employed in three-dimensional graphing software.

In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k.

Quadric Surfaces (3 of 3)

Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form.

All surfaces are symmetric with respect to the z-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Graphs of Quadric Surfaces

Table 1

Applications of Quadric Surfaces

Applications of Quadric Surfaces (1 of 2)

Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles.

Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals.

Applications of Quadric Surfaces (2 of 2)

In a radio telescope, for instance, signals from distant stars that strike the bowl are all reflected to the receiver at the focus and are therefore amplified. The same principle applies to microphones and satellite dishes in the shape of paraboloids.

Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes.