Textbook Sections: 12.5, 13.1 and 13.2

Topics: Lines and Planes; Vector Functions and Space Curves; Derivatives and Integrals of Vector Functions

Instructions: Try each of the following problems, show the detail of your work. Cellphones, graphing calculators, computers and any other electronic devices are not to be used during the solving of these problems. Discussions and questions are strongly encouraged.

1. Find an equation for the plane that passes through the points $(0, -2, 5)$ and $(-1, 3, 1)$ and is perpendicular to the plane $2z = 5x + 4y$.

2. What is the angle between the vectors $x\vec{i} - \vec{j} + \vec{k}$ and $x\vec{i} + 2\vec{j} + 3\vec{k}$? Justify your answer.

   A. 0 degrees

   B. less than 90 degrees.

   C. greater than 90 degrees

   D. It can be any of the above depending on the value of x.

3. Find the parametric equations for the line of intersection of the planes $2x + 3y + 5z = 7$ and $x - y + 2z = 3$.

4. Find an equation of the plane through the $P(7, -2, -4)$ and parallel to the plane $z = 4x - 5y$.

5. Find an equation of the plane that passes through the point $P(10, -1, 5)$ and contains the line with symmetric equations $\dfrac{x}{4} = y + 6 = \dfrac{z}{5}$.

6. Determine if the line L given by $\mathbf{r}(t) = \langle -2t, 2 + 7t, -1 - 4t \rangle$, intersects the plane given by $4x + 9y - 2z + 8 = 0$.

   VECTOR FUNCTIONS and SPACE CURVES:

7. Find the domain of the vector function $\mathbf{r}(t) = \left\langle \ln(t+1), \dfrac{t}{4 - t^2}, \sqrt{4 - t} \right\rangle$

8. Evaluate the limit:

$$\lim_{t \to 1} \mathbf{r}(t) = \lim_{t \to 1} \left\langle \frac{t^2 - 1}{t^2 - 3t + 2}, \frac{t - 1}{\sqrt{t + 3} - 2}, \frac{\sin(t - 1)}{t - 1} \right\rangle$$

9. Find a vector function that represents the curve of intersection of the surfaces $x^2 + y^2 = 1$ and $z = y + 2$. Specify the domain of the vector function so that the curve is covered exactly once. Sketch the curve.

DERIVATIVES and INTEGRALS of VECTOR FUNCTIONS:

10. Evaluate the derivative of the vector function $\mathbf{r}(t) = \left\langle e^{t^2+2t}, \ln(\cos(t)), t\arctan(t) \right\rangle$.

11. Evaluate the integrals:

$$\int \left(\frac{1}{t+1}\mathbf{i} + \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k} \right) dt$$

and

$$\int_0^1 \left(\frac{1}{t+1}\mathbf{i} + \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k} \right) dt$$

12. For the vector function $\mathbf{r}(t) = \langle 3\sin(t), 2\cos(t) \rangle$, find the unit tangent vector $\mathbf{T}(\pi/3)$.

Suggested Textbook Problems

Section 12.5	1-13,15,19-41, 45,48-51,53,57-59,61,63,65-69, 71,73,76-79
Section 13.1	1-6, 9, 11-13, 17, 19, 21-32, 41-46, 49-50
Section 13.2	3, 4, 8, 9, 13, 15, 18, 19, 21, 22, 24-28, 33-42

SOME USEFUL FACTS and EQUATIONS:

Equations for Lines

The **vector equation** of a line is: $\mathbf{r}(t) = \mathbf{r}_0(t) + t\mathbf{v}$
where $\mathbf{r}(t) = \langle x, y, z \rangle$ is the position vector of an arbitrary point on the line, $\mathbf{r}_0$ is the position vector of a specific point on the line, and $\mathbf{v}$ is the direction vector. Assuming that $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$, the component equations of the vector equation give the **parametric equations** for the line:

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct, \qquad -\infty < t < \infty$$

Manipulating each of the component equations for the line to solve for the parameter t (when a, b, c are non-zero), and equating the results give the **symmetric equations** for the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Equations for Planes

Say that $\mathbf{r} = \langle x, y, z \rangle$ is the position vector of an arbitrary point in the plane, $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ is the position vector of some specific point on the plane, and $\mathbf{n} = \langle a, b, c \rangle$, the **normal vector**, is orthogonal to the plane. Then a **vector equation** of the plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Plugging in the components of $\mathbf{n}$, $\mathbf{r}$, and $\mathbf{r}_0$ gives a **scalar equation of the plane**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Multiplying out and collecting terms in the scalar equation of the plane gives a **linear equation** in x, y, and z:

$$ax + by + cz + d = 0$$

Domain of a Vector Function

The domain of a vector function is the intersection of the domains of all of its component functions.

Limit and Continuity of Vector Functions

Limits of vector functions are evaluated component-wise: If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t),\, \lim_{t\to a} g(t),\, \lim_{t\to a} h(t) \right\rangle$$

A vector function is continuous at a point a if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$. A vector function is continuous on some domain if and only if all of its component functions are continuous on that domain.

Derivative of a Vector Function

The derivative of a vector function is defined as: $\dfrac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \dfrac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$

Since limits of vector functions are evaluated component-wise, derivatives are evaluated component-wise as well.

Tangent Vector, Unit Tangent Vector, Tangent Line

Say that the point P corresponds to $\mathbf{r}(a)$ on the curve $\mathcal{C}$ given by $\mathbf{r}(t)$. Then the vector $\mathbf{r}'(a)$ is tangent to $\mathcal{C}$ at the point P, and $\mathbf{r}'(a)$ is the **tangent vector** to $\mathcal{C}$ at P. The **tangent line** to the curve $\mathcal{C}$ at the point P passes through the point P given by $\mathbf{r}(a)$ and is parallel to the tangent vector $\mathbf{r}'(a)$. The **unit tangent vector** to the curve $\mathcal{C}$ at point P is given by

$\mathbf{T}(a) = \frac{\mathbf{r}'(a)}{|\mathbf{r}'(a)|}$, as long as $|\mathbf{r}'(a)| \neq 0$.

Differentiation Rules

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then:

1. $\dfrac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$
2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$
4. $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$
6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Integral of a Vector Function

Integrals of vector functions are evaluated component-wise: If $\mathbf{r}(t)=\langle f(t), g(t), h(t)\rangle$, then we have the definite integral:

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle = \left(\int_a^b f(t)\,dt\right)\mathbf{i}+\left(\int_a^b g(t)\,dt\right)\mathbf{j}+\left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

and the indefinite integral:

$$\int \mathbf{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle = \left(\int f(t)\,dt\right)\mathbf{i}+\left(\int g(t)\,dt\right)\mathbf{j}+\left(\int h(t)\,dt\right)\mathbf{k}$$

There is a scalar constant of integration associated to each scalar indefinite integral, so the indefinite integral of a vector function requires a vector constant of integration C. If $\mathbf{r}(t)$ has the particular antiderivative $\mathbf{R}(t)$, then the most general antiderivative of $\mathbf{r}(t)$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t)+\mathbf{C}$$