Assignment 1

AST4320: Cosmology and Extragalatic Astronomy

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Exercise 1

(1)

The continuity equation for an unperturbed universe driven by hubble expansion is given as

$$\frac{d\bar{\rho}}{dt} + \bar{\rho}\nabla \cdot \mathbf{v}_0 = 0, \tag{1}$$

where $\bar{\rho}$ is the average density of the universe, and $\mathbf{v}_0$ comes from the Hubble law today which states that

$$\mathbf{v_0} = H_0\mathbf{r}. \tag{2}$$

Here H_0 is the Hubble parameter today, given as $H_0 = \dot{a}(t = t_0)/a(t = t_0)$, where a is the scale factor, and is defined so that $a(t = t_0) = 1$.

Inserting for (2) into the continuity equation we find

$$\frac{d\bar{\rho}}{dt} + \bar{\rho}\nabla \cdot \mathbf{r}\frac{da}{dt}\frac{1}{a} = 0,$$

where the del operator only works on $\mathbf{r}$ as its the scale factor is only a function of time. The operation results in $\nabla \cdot \mathbf{r} = 3$. We can then swap the right term over to the right hand side of the equation which leaves us with

$$\frac{d\bar{\rho}}{dt} = -3\bar{\rho}\frac{da}{dt}\frac{1}{a}.$$

Further, we can separate the equation

$$\frac{d\bar{\rho}}{\rho} = -3\frac{da}{a}.$$

This is then solved in the following way

$$\int_{\bar{\rho}(t=t_0)}^{\bar{\rho}(t)} \frac{d\bar{\rho}}{\rho} = \int_{a(t=t_0)}^{a(t)} -3\frac{da}{a},$$

$$\Rightarrow \ln\left(\frac{\bar{\rho}(t)}{\bar{\rho}(t = t_0)}\right) = \ln\left(\frac{a(t)}{a(t = t_0)}\right)^{-3}.$$

This can then be reduced to

$$\bar{\rho}(t) = \bar{\rho}(t = t_0)a(t)^{-3}, \tag{3}$$

where we have used that $a(t = t_0) = 1$.

(2)

The unperturbed Poisson equation is given as

$$\nabla^2\phi_0 = 4\pi G\rho_0. \tag{4}$$

By rewriting in terms of the perturbation terms, we get the following equation

$$\nabla^2(\phi + \delta\phi) = 4\pi G(\rho + \delta\rho)$$

We can then write this out to the form

$$\nabla^2\phi + \nabla^2\delta\phi = 4\pi G\rho + 4\pi G\delta\rho.$$

Since the two terms $\nabla^2\phi$ and $4\pi G\rho$ simply makes up the Possion equation which is equal to 0, we are left with the perturbed Possion equation

$$\nabla^2\delta\phi = 4\pi G\delta\rho. \tag{5}$$

Exercise 2

In the lectures, we sketched how one could arrive at the second order differential equation

$$\frac{d^2\delta}{dt^2} + 2\frac{\dot{a}(t)}{a(t)}\frac{d\delta}{dt} = \delta(4\pi G\rho_0 - k^2c_s^2), \tag{6}$$

which described the perturbation $\delta(t)$. Here G is the gravitational constant, k is the wavenumber of the perturbation given as $k = 2\pi/\lambda$, and c_s is the speed of sound in the medium.

(1)

The Friedman equations can be used to derive the following expression for the Hubble rate or the time evolution of the scale factor

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = H_0^2\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda\right], \tag{7}$$

where Ω_i is the fractional density parameter and $i = m, r, \Lambda$ represents the matter, radiation and dark energy contributions to the density. We will then specifically study the three scenarios where we have $(\Omega_m, \Omega_\Lambda) = (1.0, 0.0)$, $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$, and $(\Omega_m, \Omega_\Lambda) = (0.8, 0.2)$. We let $\Omega_r = 0$ for all scenarios as we want to look at the times close to the CMB.

Inserting these numbers into (7), we can find expressions for the $\dot{a}/a$ term for all three cases. Doing so gives us the following three expressions

$$\frac{\dot{a}}{a} = H_0 a^{-3/2} \tag{8}$$

$$\frac{\dot{a}}{a} = H_0\left[0.3a^{-3} + 0.7\right]^{1/2} \tag{9}$$

$$\frac{\dot{a}}{a} = H_0\left[0.8a^{-3} + 0.2\right]^{1/2} \tag{10}$$

The first expression (8) corresponds to the Einstein-de Sitter Universe.