AST4320 - Assignment 3

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Exercise 1

a)

We assume that the intergalactic medium (IGM) contains only hydrogen and helium with respective mass fractions $X = 0.76$ and $Y = 0.24$ respectively. In general, the mean molecular weight μ is given as

$$\mu = \sum_i \left(\frac{X_i}{A_i}\right)^{-1}, \tag{1}$$

where X_i are the atomic species of consideration, which in our case is X and Y for hydrogen and helium, and A_i is the atomic weight of the species divided by the ionization number. We will assume that the entire IGM is completely ionized so that $A_X = 1/2$ because the atomic weight of hydrogen is 1 and the ionization number is 2 as one electron and 1 proton is produced. Similarly, $A_Y = 4/3$ as the atomic weight of helium is 4, and 3 particles are produced (1 helium core and 2 electrons). Using these numbers we can calculate the mean molecular weight of the IGM

$$\mu = \frac{1}{2X + (3/4)Y} = 0.5882. \tag{2}$$

b)

The Jeans mass is given as

$$M_J = \frac{\pi}{6}\rho\lambda_J^3, \tag{3}$$

where ρ is the density and λ_J is the Jeans length given as

$$\lambda_J = c_s\left(\frac{\pi}{G\rho}\right)^{1/2}. \tag{4}$$

Here, c_s is the speed of sound given as

$$c_s = \left(\frac{k_B T}{\mu m_p}\right)^{1/2}. \tag{5}$$

Since we are considering the IGM, the density is given as

$$\rho = \rho_m(1+z)^3 = \Omega_m \rho_c (1+z)^3, \tag{6}$$

where we have used that $\Omega_m = \rho_m/\rho_c = 0.308$ where $\rho_c \approx 10^{-26}$kg m^{-3} is the critical density. If we now use the definitions in equations (5), (6), we can rewrite the expressions of the Jeans length and mass in terms of redshift. The Jeans mass is then given as

$$M_J = \frac{\pi^{5/2}}{6}\left(\frac{k_B T}{G\mu m_p}\right)^{3/2} (\Omega_m \rho_c)^{1/2}(1+z)^{3/2}, \tag{7}$$

and similarely the Jeans length is given as

$$\lambda_J = \left(\frac{\pi k_B T}{G\mu m_p \Omega_m \rho_c}\right)^{1/2} (1+z)^{-3/2}. \tag{8}$$

If we use that the IGM has a temperature of $T = 10^4$K, then the Jeans Mass at a redshift $z = 4$ with the corresponding wave number k is equal to

$$M_J(z=4) = 5.515 \times 10^{15}\text{kg}, \qquad k = \frac{2\pi}{\lambda_J(z=4)} = 1.516 \times 10^{-21}\text{m}^{-1}. \tag{9}$$

Exercise 2

In the lectures we derived that the total optical depth of the ionized IGM due to electron scattering is given by

$$\tau_\text{e}(z) = c\int_0^z \frac{n_e(z)\sigma_\text{T} dz}{(1+z)H(z)}, \tag{10}$$

where $\sigma_\text{T} = 6.65 \times 10^{-25}$ is the cross-section for electron scattering, $\overline{n}_\text{H}(z) \sim 1.9\times10^{-7}(1+z)^3\text{cm}^{-3}$ is the average number density of hydrogen. We have the following cosmological parameters $\Omega_\Lambda = 0.692, \Omega_m = 0.308$ and $\Omega_r = 0$. $H(z)$ is the Hubble parameter and is given as

$$H^2(z) = H_0^2\left(\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda\right), \tag{11}$$

where $H_0 \approx 2.19\times10^{18}\text{s}^{-1}$ is the Hubble parameter today. If we assume that the IGM is completely ionized and consists only of hydrogen, then $\overline{n}_\text{H}(z) \approx n_e(z)$. We are then interested in calculating and plotting the optical depth $\tau_e(z)$ as a function of z in the redshift range $z \in [0, 10]$. We do this by creating a `python` program which solves equation (10) for the above values. The program is attached together with the delivery but can also be found on my GitHub[1].

The results of the calculation can be seen in figure 1. We see that the optical depth today is $\tau_e(z=0) \approx 0$. We also find $\tau_e(z=6) = 0.0332$ and $\tau_e(z=7) = 0.0699$.

[1] `github.com/metinsa/AST4320/assignment3/exercise2.py`

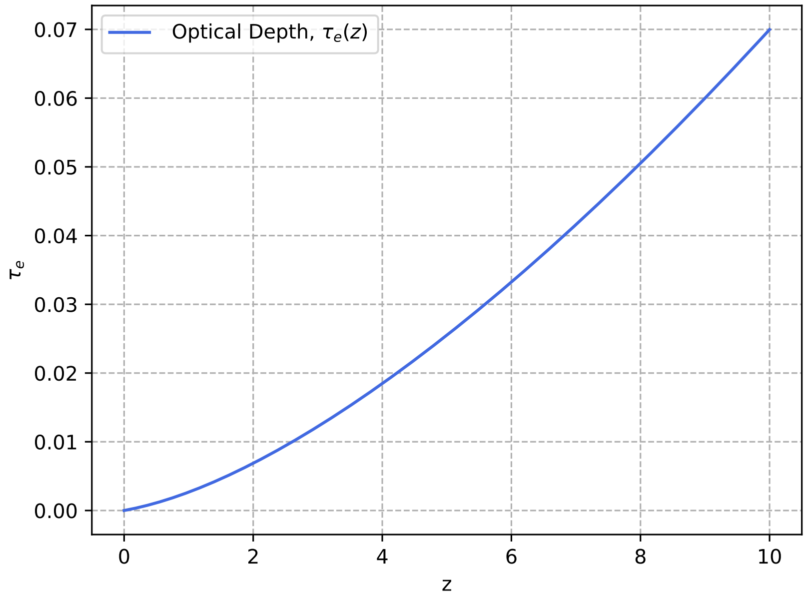

Figure 1: The optical depth $\tau_e(z)$ of the IGM for redshift $z \in [0, 10]$.

Exercise 3

a)

In the lectures we derived the following second order differential equation for the density profile of an "isothermal" halo

$$-\frac{k_b T}{m_{\mathrm{DM}} r^2} \frac{d}{dr} r^2 \frac{d}{dr} \ln \rho(r) = 4\pi G \rho(r). \tag{12}$$

We can show that

$$\rho(r) = \frac{A}{r^2}, \qquad A = \frac{k_b T}{2\pi G m_{\mathrm{DM}}}, \tag{13}$$

is a solution to equation (12) by substituting it in to the LHS. We start by rewriting the logarithm expression to the form

$$\frac{d \ln(\rho)}{dr} = \frac{1}{\rho} \frac{d\rho}{dr}.$$

Next up, we insert for ρ and compute the derivative

$$\frac{1}{\rho} \frac{d\rho}{dr} = \frac{r^2}{A} \frac{d}{dr} \left(\frac{A}{r^2} \right) = -\frac{2}{r}.$$

Inserting this into equation (12) and computing the remaining derivative leaves us with

$$\frac{2k_bT}{m_{\rm DM}r^2} = 4\pi G\rho(r). \tag{14}$$

If we now substitute the constants on the LHS with A, we find

$$\frac{k_bT}{m_{\rm DM}} = 2\pi GA.$$

Finally we insert this into (14)

$$4\pi G\frac{A}{r^2} = 4\pi G\rho(r).$$

By reinserting for $\rho(r)$ from definition (13), we see that this is indeed the solution.

b)

For an isothermal gas, we have the following equation of state

$$p = \frac{k_bT}{m_{\rm p}}\rho. \tag{15}$$

For a gas in hydrostatic equilibrium we have

$$\frac{dp}{dr} = -\frac{GM(<r)\rho}{r^2}. \tag{16}$$

We will now show that an isothermal gas will end up in a similar state. We assume that the density profile is given similar to that of the isothermal halo, so that

$$\rho(r) = \frac{A_{\rm gas}}{r^2}, \qquad A_{\rm gas} = \frac{k_bT}{2\pi Gm_{\rm p}}.$$

Rewriting the pressure in equation (15) in terms of $A_{\rm gas}$ results in

$$p = 2\pi GA_{\rm gas}\rho(r). \tag{17}$$

For a spherical symmetric gas, the mass which is smaller than some radius r is given as

$$M(<r) = 4\pi\int_0^r x^2dx\rho(x) = \frac{4\pi}{3}r^3\rho(r). \tag{18}$$

We can now rewrite equation (17) in terms of M from equation (18). Doing so leaves us with the following equation of state

$$p = \frac{3}{2}\frac{GM(<r)\rho(r)}{r}. \tag{19}$$

Finally, we differentiate equation (19) with respect to r which leaves us with

$$\frac{dp}{dr} = \frac{-3GM(<r)\rho(r)}{r^2}. \tag{20}$$

We see that this is a very similar to the non-isothermal case seen in equation (16), the only difference being the factor 3. In both scenarios, the gas results in a state where

$$\frac{dp}{dr} \propto \frac{M(< r)\rho(r)}{r^2}. \tag{21}$$