AST4320 - Assignment 3

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Exercise 2

In the lectures we derived that the total optical depth of the ionized IGM due to electron scattering is given by

$$\tau_{\mathrm{e}}(z) = c\int_0^z \frac{n_e(z)\sigma_{\mathrm{T}}dz}{(1+z)H(z)}, \tag{1}$$

where $\sigma_{\mathrm{T}} = 6.65\times10^{-25}$ is the cross-section for electron scattering, $\overline{n}_{\mathrm{H}}(z) \sim 1.9\times10^{-7}(1+z)^3\mathrm{cm}^{-3}$ is the average number density of hydrogen. We have the following cosmological parameters $\Omega_\Lambda = 0.692, \Omega_m = 0.308$ and $\Omega_r = 0$. $H(z)$ is the Hubble parameter and is given as

$$H^2(z) = H_0^2\left(\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda\right), \tag{2}$$

where $H_0 \approx 2.19\times10^{18}\mathrm{s}^{-1}$ is the Hubble parameter today. If we assume that the IGM is completely ionized and consists only of hydrogen, then $\overline{n}_{\mathrm{H}}(z) \approx n_e(z)$. We are then interested in calculating and plotting the optical depth $\tau_e(z)$ as a function of z in the redshift range $z\in[0,10]$. We do this by creating a `python` program which solves equation (1) for the above values. The program is attached together with the delivery but can also be found on my GitHub[1].

The results of the calculation can be seen in figure 1. We see that the optical depth today is $\tau_e(z=0)\approx 0$. We also find $\tau_e(z=6) = 0.0332$ and $\tau_e(z=7) = 0.0699$.

Exercise 3

a)

In the lectures we derived the following second order differential equation for the density profile of an "isothermal" halo

$$-\frac{k_bT}{m_{\mathrm{DM}}r^2}\frac{d}{dr}r^2\frac{d}{dr}\ln\rho(r) = 4\pi G\rho(r). \tag{3}$$

We can show that

$$\rho(r) = \frac{A}{r^2}, \qquad A = \frac{k_bT}{2\pi G m_{\mathrm{DM}}}, \tag{4}$$

[1] `github.com/metinsa/AST4320/assignment3/exercise2.py`

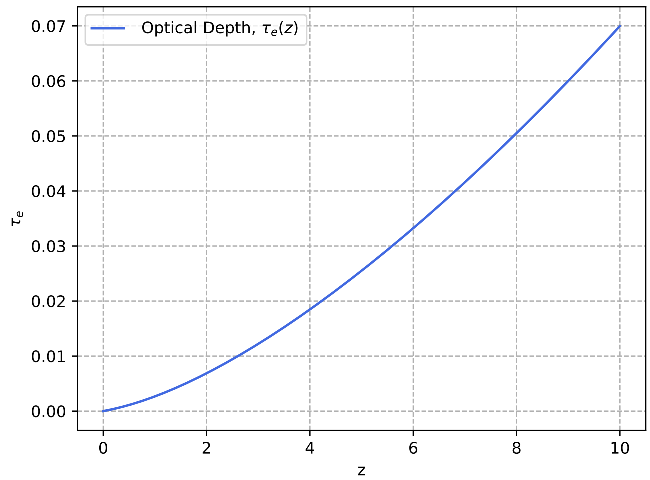

Figure 1: The optical depth $\tau_e(z)$ of the IGM for redshift $z \in [0, 10]$

is a solution to equation (3) by substituting it in to the LHS. We start by rewriting the logarithm expression to the form

$$\frac{d\ln(\rho)}{dr} = \frac{1}{\rho}\frac{d\rho}{dr}.$$

Next up, we insert for ρ and compute the derivative

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{r^2}{A}\frac{d}{dr}\left(\frac{A}{r^2}\right) = -\frac{2}{r}.$$

Inserting this into equation (3) and computing the remaining derivative leaves us with

$$\frac{2k_bT}{m_{\mathrm{DM}}r^2} = 4\pi G\rho(r). \tag{5}$$

If we now substitute the constants on the LHS with A, we find

$$\frac{k_bT}{m_{\mathrm{DM}}} = 2\pi GA.$$

Finally we insert this into (5)

$$4\pi G\frac{A}{r^2} = 4\pi G\rho(r).$$

By reinserting for $\rho(r)$ from definition (4), we see that this is indeed the solution.

b)

For an isothermal gas, we have the following equation of state

$$p = \frac{k_b T}{m_{\mathrm{p}}} \rho. \tag{6}$$

For a gas in hydrostatic equilibrium we have

$$\frac{dp}{dr} = -\frac{GM(< r)\rho}{r^2}. \tag{7}$$

We will now show that an isothermal gas will end up in a similar state. We assume that the density profile is given similar to that of the isothermal halo, so that

$$\rho(r) = \frac{A_{\mathrm{gas}}}{r^2}, \qquad A_{\mathrm{gas}} = \frac{k_b T}{2\pi G m_{\mathrm{p}}}.$$

Rewriting the pressure in equation (6) in terms of A_{gas} results in

$$p = 2\pi G A_{\mathrm{gas}} \rho(r). \tag{8}$$

For a spherical symmetric gas, the mass which is smaller than some radius r is given as

$$M(< r) = 4\pi \int_0^r x^2 dx \rho(x) = \frac{4\pi}{3} r^3 \rho(r). \tag{9}$$

We can now rewrite equation (8) in terms of M from equation (9). Doing so leaves us with the following equation of state

$$p = \frac{3}{2} \frac{GM(< r)\rho(r)}{r}. \tag{10}$$

Finally, we differentiate equation (10) with respect to r which leaves us with

$$\frac{dp}{dr} = \frac{-3GM(< r)\rho(r)}{r^2}. \tag{11}$$

We see that this is a very similar to the non-isothermal case seen in equation (7), the only difference being the factor 3. In both scenarios, the gas results in a state where

$$\frac{dp}{dr} \propto \frac{M(< r)\rho(r)}{r^2}. \tag{12}$$