AST 4320 - Assignment 2

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Exercise 1.

We will now consider the top-hat smoothing function in 1D, also known as the window function $W(x)$. The top-hat smoothing function is expressed as

$$W(x) = \begin{cases} 1, & \text{if } |x| < R \\ 0, & \text{otherwise} \end{cases}, \tag{1}$$

where R is the smoothing scale. We are interested in computing the Fourier conjugate $\tilde{W}$ of the smoothing function. The Fourier transformation of a function $f(x)$ is given as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{2}$$

Applying this transformation to $W(x)$ gives us the following expression to solve

$$\tilde{W}(k) = \int_{-\infty}^{\infty} W(x) e^{-ikx} dx.$$

If we insert for the definition of $W(x)$, we see that only the non-zero contributions occur in the limits R to $-R$. This reduces our expression to

$$\tilde{W}(k) = \int_{-R}^{R} e^{-ikx} dx.$$

This is a trivial integral to compute. Doing so results in

$$\tilde{W}(k) = -\frac{1}{ik}\left[e^{-ikx}\right]_{-R}^{R} = \frac{1}{ik}\left[e^{ikR} - e^{-ikR}\right].$$

From Euler's formula, we know that

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}.$$

We see that this can be substituted into the bracket on the right hand side, which leaves us with the final expression

$$\tilde{W}(k) = \frac{2}{k}\sin(kR). \tag{3}$$

We can now study this quantity. We start by plotting equation (3) for wavenumbers $k \in [-4\pi, 4\pi]$. The results are seen in figure 1. We are also interested in knowing what the full width at half the maximum is (FWHM). This quantity is given as $\text{FWHM} = 2\sqrt{2\ln 2}\sigma$, where σ is the standard deviation in the distribution. However, since we have a non-gaussian distribution with multiple amplitudes, we will have to find the value numerically. We do so by writing a short script in python which finds the the value of k at half the maximum.

We find that the FWHM of the Fourier conjugate of $W(x)$ with smoothing scale $R = 2$ is $\text{FWHM} = 1.899$.

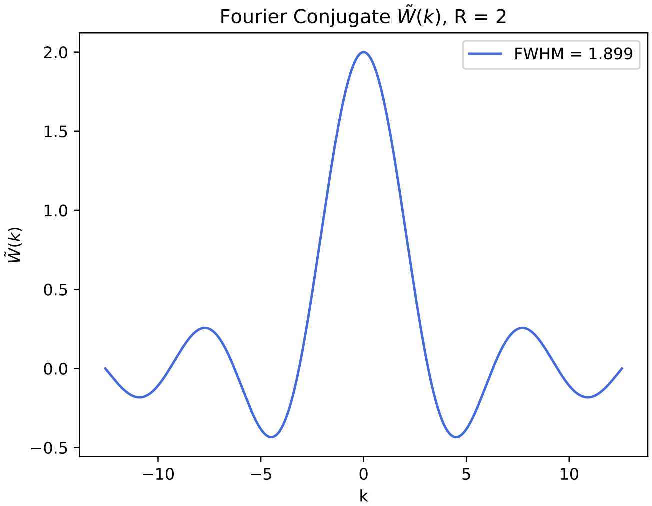

Figure 1: Top-hat smoothing function in 1D with a smoothing scale $R = 2$.

Exercise 2.

We will now look into the random walk process. We will consider a power spectrum on the form $P(k) = k$ and extract random numbers from Gaussian random distributions. We define the variance as a function of scale as

$$\sigma^2(S_c) = \frac{\pi}{S_c^4}, \tag{4}$$

where S_c, the smoothing scale is given as

$$S_c = \frac{2\pi}{k}. \tag{5}$$

The initial condition of k is found by requiring that we start at a radius so that $\sigma^2(S_c) = \sigma^2(S_1) < 10^{-4}$. We can find an expression for k which only depends on σ by solving equations for k. Doing so leaves us with

$$k = 2\pi \left(\frac{\pi}{\sigma^2}\right)^{-1/4}. \tag{6}$$

This can then be inserted back into equation (5), which leaves us with our initial scale

$$S_1 = \left(\frac{\pi}{\sigma^2}\right)^{1/4}. \tag{7}$$

By then inserting for a initial sigma $\sigma < 10^{-4}$, for instance 0.9×10^{-4}, we get our initial $S_1 = 13.66$.

We start by creating an algorithm which first calculates the variance, σ^2 in equation (4) using the initial scale S_1. It then subtracts a small value ϵ from the scale S_c. It proceeds by computing a new variance using the new S_c value. It then finds the difference in variance between these two variances $\sigma_{12}^2 = \sigma^2(S_2) - \sigma^2(s_1)$. It then uses numpy's *random.normal* function which extracts a random normal distributed number δ with the above variance σ_{12}^2. This is then continued until the scale S_c drops to $S_c = 1$ at which point we extract the final δ value corresponding to $S_c = 1$.

This whole sequence is computed 10^5 times. The distribution of the extracted δ are then plotted in a histogram which is seen in figure 2.

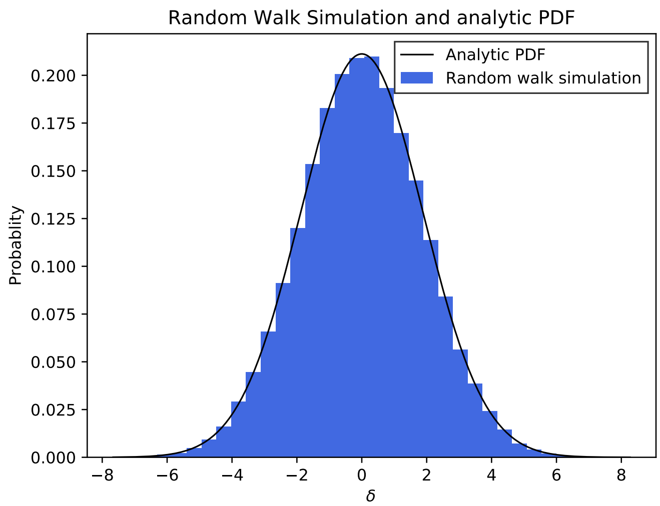

Figure 2: Distribution of 10^5 random walk realizations in addition to the Gaussian Probability distribution function (PDF).

We have also overplotted the analytic probability distribution for a Gaussian random field with overdensity δ, smoothed by S_c with a variance σ^2, which is given as

$$P(\delta \mid M) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\delta^2}{2\sigma^2}\right]. \tag{8}$$

We see that the analytic probability distribution perfectly represents the random walk simulation.

We can further choose to only study the chains which never crosses the threshold $\delta_{\text{crit}} = 1$. Analytically, the probability of this is given by

$$P_{\text{nc}}(\delta \mid M)\frac{1}{\sqrt{2\pi}\sigma}\left(\exp\left[-\frac{\delta^2}{2\sigma^2}\right] - \exp\left[-\frac{[2\delta_{\text{crit}} - \delta]^2}{2\sigma^2}\right]\right). \tag{9}$$

By editing our program to extract only $\delta <= 1$ We get the following distribution, with the analytic expression overplotted seen in figure 3.

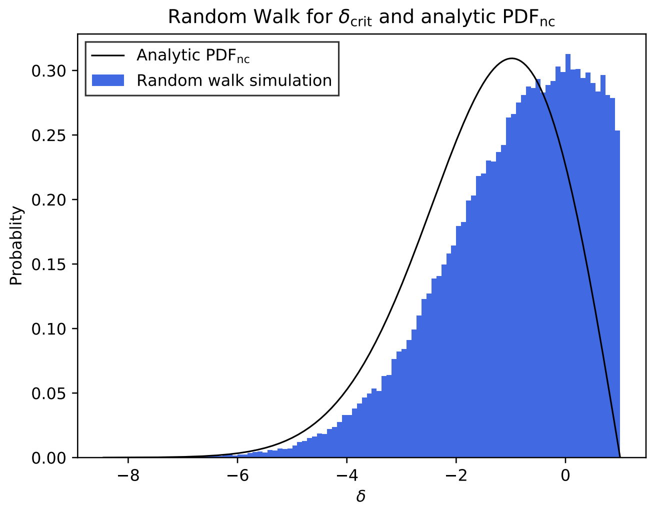

Figure 3: Distribution of 10^5 random walk realizations where only the chains up to $\delta \leq 1$ have been included. This is then compared to the analytic probability distribution PDF_{nc}.

Exercise 3.

(1) From the analysis in exercise 2, we know that the probability distribution function for $\delta \leq \delta_{\text{crit}}$ at some scale $M' > M$ is given by equation (9). We know that the probability of finding mass larger than zero is given as

$$P(> 0) = P(> M) + P(< M) = 1.$$

We can rewrite this to

$$P(> M) = 1 - P(< M). \tag{10}$$

Mass smaller than M corresponds to the density being smaller than critical overdensity δ_{crit}. From the randomwalk analysis, we know that this has the probability of not crossing P_{nc} seen in equation (9). By inserting this into the equation (11) we get the following expression

$$P(> M) = 1 - \int_{-\infty}^{\delta_{\text{crit}}} P_{\text{nc}}(\delta \mid M)d\delta, \tag{11}$$

where we have integrated $P_{\rm nc}$ from $-\infty$ to $\delta_{\rm crit}$.

(2) We will now show how the factor 2 in the Press-Schechter formalism naturally occurs. We do so by first combingin equations (9) and (11). This leaves us with

$$P(>M) = 1 - \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\delta_{\rm crit}} \left(\exp\left[-\frac{\delta^2}{2\sigma^2}\right] - \exp\left[-\frac{[2\delta_{\rm crit}-\delta]^2}{2\sigma^2}\right]\right) \tag{12}$$

We split the integral into two integrals, I and II

$$\mathrm{I}: \quad \int_{-\infty}^{\delta_{\rm crit}} e^{-\delta^2/2\sigma^2} d\delta,$$

$$\mathrm{II}: \quad \int_{-\infty}^{\delta_{\rm crit}} e^{-(2\delta_{\rm crit}-\delta)^2/2\sigma^2} d\delta.$$

We start with computing I

$$\mathrm{I}: \quad \int_{-\infty}^{\delta_{\rm crit}} e^{-\delta^2/2\sigma^2} d\delta = \frac{1}{2}\sqrt{2\pi}\sigma\left[\mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)+1\right],$$

where we have substituted $\nu = \delta_{\rm crit}/\sigma$, and erf is the error function given as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

Similarely, we compute II

$$\mathrm{II}: \quad \int_{-\infty}^{\delta_{\rm crit}} e^{-(2\delta_{\rm crit}-\delta)^2/2\sigma^2} d\delta = \frac{1}{2}\sqrt{2\pi}\sigma\ \mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right),$$

where erfc is given as

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x).$$

We now solve the integral expression in equation (12) by computing I - II

$$\begin{aligned}
\mathrm{I}-\mathrm{II} &= \frac{1}{2}\sqrt{2\pi}\sigma\left[\mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)+1\right] - \frac{1}{2}\sqrt{2\pi}\sigma\ \mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \\
&= \frac{1}{2}\sqrt{2\pi}\sigma\left[\mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)+1-\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\right] \\
&= \frac{1}{2}\sqrt{2\pi}\sigma\left[\mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)+1-1+\mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right] \\
&= \sqrt{2\pi}\sigma\ \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)
\end{aligned}$$

We can then insert this expression into the integral in equation (12), which leaves us with

$$P(> M) = 1 - \frac{1}{\sqrt{2\pi}\sigma} \sqrt{2\pi}\sigma \; \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right).$$

We see that the factor $\sqrt{2\pi}\sigma$ disappears, and we are left with the Press-Schelter expression

$$P(> M) = 1 - \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right), \tag{13}$$

where the factor 2 now naturally comes in.