AST4320 - Assignment 3

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Exercise 3

a)

In the lectures we derived the following second order differential equation for the density profile of an "isothermal" halo

$$-\frac{k_bT}{m_{\mathrm{DM}}r^2}\frac{d}{dr}r^2\frac{d}{dr}\ln\rho(r) = 4\pi G\rho(r). \tag{1}$$

We can show that

$$\rho(r) = \frac{A}{r^2}, \qquad A = \frac{k_bT}{2\pi G m_{\mathrm{DM}}}, \tag{2}$$

is a solution to equation (1) by substituting it in to the LHS. We start by rewriting the logarithm expression to the form

$$\frac{d\ln(\rho)}{dr} = \frac{1}{\rho}\frac{d\rho}{dr}.$$

Next up, we insert for ρ and compute the derivative

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{r^2}{A}\frac{d}{dr}\left(\frac{A}{r^2}\right) = -\frac{2}{r}.$$

Inserting this into equation (1) and computing the remaining derivative leaves us with

$$\frac{2k_bT}{m_{\mathrm{DM}}r^2} = 4\pi G\rho(r). \tag{3}$$

If we now substitute the constants on the LHS with A, we find

$$\frac{k_bT}{m_{\mathrm{DM}}} = 2\pi GA.$$

Finally we insert this into (3)

$$4\pi G\frac{A}{r^2} = 4\pi G\rho(r).$$

By reinserting for $\rho(r)$ from definition (2), we see that this is indeed the solution.

b)

For an isothermal gas, we have the following equation of state

$$p = \frac{k_b T}{m_\mathrm{p}}\rho. \tag{4}$$

For a gas in hydrostatic equilibrium we have

$$\frac{dp}{dr} = -\frac{GM(< r)\rho}{r^2}. \tag{5}$$

We will now show that an isothermal gas will end up in a similar state. We assume that the density profile is given similar to that of the isothermal halo, so that

$$\rho(r) = \frac{A_\mathrm{gas}}{r^2}, \qquad A_\mathrm{gas} = \frac{k_b T}{2\pi G m_\mathrm{p}}.$$

Rewriting the pressure in equation (4) in terms of A_gas results in

$$p = 2\pi G A_\mathrm{gas}\rho(r). \tag{6}$$

For a spherical symmetric gas, the mass which is smaller than some radius r is given as

$$M(< r) = 4\pi\int_0^r x^2 dx \rho(x) = \frac{4\pi}{3}r^3\rho(r). \tag{7}$$

We can now rewrite equation (6) in terms of M from equation (7). Doing so leaves us with the following equation of state

$$p = \frac{3}{2}\frac{GM(< r)\rho(r)}{r}. \tag{8}$$

Finally, we differentiate equation (8) with respect to r which leaves us with

$$\frac{dp}{dr} = \frac{-3GM(< r)\rho(r)}{r^2}. \tag{9}$$

We see that this is a very similar to the non-isothermal case seen in equation (5), the only difference being the factor 3. In both scenarios, the gas results in a state where

$$\frac{dp}{dr} \propto \frac{M(< r)\rho(r)}{r^2}. \tag{10}$$