Assignment 1

AST4320: Cosmology and Extragalatic Astronomy

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Exercise 1

(1)

The continuity equation for an unperturbed universe driven by hubble expansion is given as

$$\frac{d\bar{\rho}}{dt} + \bar{\rho}\nabla \cdot \mathbf{v}_0 = 0, \tag{1}$$

where $\bar{\rho}$ is the average density of the universe, and $\mathbf{v}_0$ comes from the Hubble law today which states that

$$\mathbf{v_0} = H_0\mathbf{r}. \tag{2}$$

Here H_0 is the Hubble parameter today, given as $H_0 = \dot{a}(t = t_0)/a(t = t_0)$, where a is the scale factor, and is defined so that $a(t = t_0) = 1$.

Inserting for (2) into the continuity equation we find

$$\frac{d\bar{\rho}}{dt} + \bar{\rho}\nabla \cdot \mathbf{r}\frac{da}{dt}\frac{1}{a} = 0,$$

where the del operator only works on $\mathbf{r}$ as its the scale factor is only a function of time. The operation results in $\nabla \cdot \mathbf{r} = 3$. We can then swap the right term over to the right hand side of the equation which leaves us with

$$\frac{d\bar{\rho}}{dt} = -3\bar{\rho}\frac{da}{dt}\frac{1}{a}.$$

Further, we can separate the equation

$$\frac{d\bar{\rho}}{\rho} = -3\frac{da}{a}.$$

This is then solved in the following way

$$\int_{\bar{\rho}(t=t_0)}^{\bar{\rho}(t)} \frac{d\bar{\rho}}{\rho} = \int_{a(t=t_0)}^{a(t)} -3\frac{da}{a},$$

$$\Rightarrow \ln\left(\frac{\bar{\rho}(t)}{\bar{\rho}(t = t_0)}\right) = \ln\left(\frac{a(t)}{a(t = t_0)}\right)^{-3}.$$

This can then be reduced to

$$\bar{\rho}(t) = \bar{\rho}(t = t_0)a(t)^{-3}, \tag{3}$$

where we have used that $a(t = t_0) = 1$.

(2)

The Poisson equation is given as

$$\nabla^2\phi = 4\pi G\rho. \tag{4}$$

We will now study how small perturbations in quantities of interest evolve in time. We introduce the definition $\psi = \psi_0 + \delta\psi$, where ψ is a variable of interest, ψ_0 is the unperturbed quantity and $\delta\psi$ is a little perturbation to that variable. The unperturbed Possion equation is then given as

$$\nabla^2 \phi_0 = 4\pi G \rho_0. \tag{5}$$

We can apply the perturbation definition to the gravitational potential ϕ and density ρ in Poisson's equation, to get

$$\nabla^2(\phi_0 + \delta\phi) = 4\pi G(\rho_0 + \delta\rho)$$

This can be written out on the form

$$\nabla^2 \phi_0 + \nabla^2 \delta\phi = 4\pi G \rho_0 + 4\pi G \delta\rho.$$

We spot that the two terms $\nabla^2 \phi_0$ and $4\pi G \rho_0$ equal the unperturbed Possion equation. These terms are then removed from the equation as they equal 0. This leaves us with the perturbed Poisson equation

$$\nabla^2 \delta\phi = 4\pi G \delta\rho, \tag{6}$$

Which describes how the perturbations evolve in time.

A similar derivation can be carried out for the Euler equation. The equation of motion, or the Euler equation is given as

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p - \nabla\phi. \tag{7}$$

In terms of the perturbed quantities, this becomes

$$\frac{d(\mathbf{v_0} + \delta\mathbf{v})}{dt} = -\frac{1}{(\rho_0 + \delta\rho)}\nabla(p_0 + \delta p) - \nabla(\phi_0 + \delta\phi),$$

where

$$\frac{d\mathbf{v_0}}{dt} = -\frac{1}{\rho_0}\nabla p_0 - \nabla\phi_0 \tag{8}$$

is the unperturbed equation. Since the perturbed quantities are very small, we will make the approximation that $1/(\rho_0 + \delta\rho) \approx 1/\rho_0$. In addition to implementing this approximation, we also split up the RHS terms

$$= -\frac{1}{\rho_0}\nabla p_0 - \frac{1}{\rho_0}\nabla\delta p - \nabla\phi_0 - \nabla\delta\phi.$$

We spot that the two terms containing the perturbed quantities are equal the RHS of the unperturbed Euler equation (8)

Exercise 2

In the lectures, we sketched how one could arrive at the second order differential equation

$$\frac{d^2\delta}{dt^2} + 2\frac{\dot{a}(t)}{a(t)}\frac{d\delta}{dt} = \delta(4\pi G\rho_0 - k^2 c_s^2), \tag{9}$$

which described the perturbation $\delta(t)$. Here G is the gravitational constant, k is the wavenumber of the perturbation given as $k = 2\pi/\lambda$, and c_s is the speed of sound in the medium.

(1)

The Friedman equations can be used to derive the following expression for the Hubble rate or the time evolution of the scale factor

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = H_0^2\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda\right], \tag{10}$$

where Ω_i is the fractional density parameter and $i = m, r, \Lambda$ represents the matter, radiation and dark energy contributions to the density. We will then specifically study the three scenarios where we have $(\Omega_m, \Omega_\Lambda) = (1.0, 0.0)$, $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$, and $(\Omega_m, \Omega_\Lambda) = (0.8, 0.2)$. We let $\Omega_r = 0$ for all scenarios as we want to look at the times close to the CMB.

Inserting these numbers into (7), we can find expressions for the $\dot{a}/a$ term for all three cases. Doing so gives us the following three expressions

$$\frac{\dot{a}}{a} = H_0 a^{-3/2} \tag{11}$$

$$\frac{\dot{a}}{a} = H_0\left[0.3a^{-3} + 0.7\right]^{1/2} \tag{12}$$

$$\frac{\dot{a}}{a} = H_0\left[0.8a^{-3} + 0.2\right]^{1/2} \tag{13}$$

The first expression (8) corresponds to the Einstein-de Sitter Universe.

Exercise 4

We will now consider the non-linear time-evolution of a spherical overdensity in the Einstein-de-Sitter Universe. We assume that this overdensity is confined into a sphere of radius $R(t) \equiv b(t)$, where $b(t)$ is the local scale factor. The acceleration of the radius of this sphere is then given by

$$\ddot{R} = -\frac{GM}{R^2}. \tag{14}$$

We will now show that the following parametrization

$$R = A(1 - \cos\theta) \tag{15}$$

$$t = B(\theta - \sin\theta) \tag{16}$$

$$A^3 = GMB^2 \tag{17}$$

satisfies the equation $(\ddot{R})$.

We start by inserting equation (t) solved for B into (A^3), which gives us

$$A^3 = \frac{GMt^2}{(\theta - \sin\theta)^2}.$$

We can further insert this new expression for A into equation (R) giving us the following equation for the radius of the sphere

$$R(t) = \frac{(GM)^{1/3}(1 - \cos\theta)}{(\theta - \sin\theta)^{2/3}} t^{2/3}. \tag{18}$$

We will then use the first order Taylor approximations $\cos\theta \approx 1 - \theta^2/2!$ and $\sin\theta \approx \theta - \theta^3/3!$. Inserting for these into equation (R) we see that

$$R(t) = \frac{(GM)^{1/3}\left(\frac{\theta^2}{2}\right)}{\left(\frac{\theta^3}{6}\right)^{2/3}} t^{2/3} = (GM)^{1/3}\left(\frac{6^{2/3}}{2}\right) t^{2/3}$$

the expression becomes independent of the angles. We will now differentiate the angleless expression for the radius twice in order to find the acceleration. Doing so leave us with

$$\ddot{R} = -(GM)^{1/3}\left(\frac{6^{2/3}}{2}\right)\left(\frac{2}{9}\right)\frac{1}{t^{4/3}}.$$

We will now multiply with $R(t)^2$ on both sides of the equation

$$R^2\ddot{R} = -(GM)^{1/3}\left(\frac{6^{2/3}}{2}\right)\left(\frac{2}{9}\right)\frac{1}{t^{4/3}} \cdot (GM)^{2/3}\left(\frac{6^{2/3}}{2}\right)^2 t^{4/3}.$$

We see that the constant numbers cancel each other, and we are simply left with

$$R^2\ddot{R} = -GM.$$

By dividing both sides with R^2, we end up with the desired results

$$\ddot{R} = -\frac{GM}{R^2},$$

which means that the parametrization is satisfied.

Exercise 5

We will now compute the infall velocity v from when the material in the overdensity first reaches the viral radius R_{vir}. We will do so by using the parametrization in exercise 4. The infall velocity is given as

$$v = \frac{dR}{dt},$$

where R is the radius from equation (15). By using the chain rule, we can express this as

$$\frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt}.$$

We start by computing $dR/d\theta$ by differentiating equation (15) with respect to θ

$$\frac{dR}{d\theta} = \frac{d}{d\theta}A(1-\cos\theta) = A\sin\theta.$$

Similarly, we compute $d\theta/dt$ by differentiating expression (16) with respect to θ

$$\frac{dt}{d\theta} = \frac{d}{d\theta}B(\theta - \sin\theta) = B(1-\cos\theta).$$

and then inverting it. Combining these terms gives us

$$v = \frac{A \sin\theta}{B(1 - \cos\theta)}.$$

If we now square each side of the equation, we get

$$v^2 = \frac{A^2 \sin^2\theta}{B^2(1 - \cos\theta)^2}.$$

Further we solve equation (18) for B^2 and substitute in

$$v^2 = \frac{GM \sin^2\theta}{A(1 - \cos\theta)^2}$$

and then insert for equation (15) solved for A, which gives us

$$v^2 = \frac{GM \sin^2\theta}{R(1 - \cos\theta)}.$$

We know that radius equals the virialization radius $R = R_{\text{vir}}$ when $\theta = 3\pi/2$. We plug in the virialization angle and find

$$v^2 = \frac{GM}{R_{\text{vir}}}.$$

Finally, by taking the square root on both sides, we find the following equation for the infall velocity

$$v = \sqrt{\frac{GM}{R_{\text{vir}}}}. \tag{19}$$

Exercise 6

We can derive the gravitational binding energy of a uniform sphere of radius R and mass M by assuming that the density within the sphere i constant. The density is then given as the total mass divided by the volume

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}.$$

We then imagine that the sphere is divided into shells with the mass of the outermost shell being

$$m_{\text{shell}} = 4\pi R^2 \rho \, dr.$$

The remaining mass within the outer shell is then

$$m_{\text{interior}} = \frac{4}{3}\pi R^3 \rho.$$

The energy required for the outer shell to escape the gravitational pull is then given as

$$dU = -G\frac{m_{\text{shell}} m_{\text{interior}}}{R}.$$

We find the total energy by inserting for the masses and integrating over all shells

$$
\begin{aligned}
U &= -G \int_0^R \frac{\left(4\pi R^2 \rho\right)\left(\frac{4}{3}\pi R^3 \rho\right)}{r} dr \\
&= -\frac{16}{3} G\pi^2 \rho^2 \int_0^R R^4 dr
\end{aligned}
$$

Solving the integral leaves us with

$$
U = -\frac{16}{15} G\pi^2 \rho^2 R^5.
$$

Finally by inserting for the density, we get

$$
U = -\frac{16}{15} G\pi^2 R^5 \left(\frac{M}{\frac{4}{3}\pi R^3}\right)^2 = -\frac{3}{5}\frac{GM^2}{R},
$$

which was to be shown.