

MILESTONE 1: THE BACKGROUND EVOLUTION OF THE UNIVERSE

AST5220 - COSMOLOGY 2

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[HTTPS://GITHUB.COM/METINSA/AST5220/TREE/MASTER/MILESTONE1](https://github.com/METINSA/AST5220/tree/master/milestone1)

ABSTRACT. We set up cosmological grids for the scale factor a , the redshift z and the parameter $x = \log a$. These are then applied to compute the time evolution of the background density and the Hubble parameter. Finally we compute the conformal time η . A small region, $x \in [-7.4, 0]$ of η is then interpolated between grid points through a spline resulting in a more continuous and smooth η estimate.

1. INTRODUCTION

The long-term goal of the numerical aspect of this course is to simulate the Cosmic Microwave Background (CMB); mainly through the calculation of the power spectrum. This will be a gradual process, with this report constituting the first milestone. The goals of milestone 1 will be to study the expansion history of the universe, as well as looking at how the uniform background densities of the various matter and energy components evolve. We will do so by exploring the Hubble parameter along with the conformal time.

The code used for the numerical calculations is written in FORTRAN 90 and is based on the provided skeleton code. The analysis of the data is done using Python. All source codes and tools used to produce the results and figures can be found on my Github by following the link below the author name on the front page.

The report will consist of a theory section where we introduce and motivate the assumptions and theory used to make our calculations. We then discuss the numerical implementation and methods use to solve the various equations in a implementation section which is followed by a results section where we present and discuss our results. We then conclude the report with a small section where we reflect back on the work done.

2. THEORY

We begin by introducing some cosmological concepts which are necessary in order to compute and study the expansion history of the Universe. We start by assuming a flat universe ($k = 0$) described by the Friedmann-Robertson-Walker metric in which the line element (in polar coordinates) is given by

$$ds^2 = -c^2 dt^2 + a^2(t) (dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)), \quad (1)$$

where $a(t)$ is the scale factor. The scale factor is connected to the redshift through the relation

$$1 + z = \frac{a(t_0)}{a(t)}, \quad (2)$$

where $a(t_0) = 1$ is the scale factor today. We will then define the parameter x

$$x \equiv \ln a. \quad (3)$$

This parameter represent the scale factor in a convenient way which makes it easier to work with the quantities we are about to compute; we mainly work with phenomena which vary over wide time ranges.

We proceed by considering a photon which propagates in this Universe. The line element for the photon is equal to zero, meaning that equation (1) can be written on the following form in Cartesian coordinates

$$\frac{dx}{dt} = \frac{c}{a(t)}. \quad (4)$$

We then define the conformal time which is given by integrating equation (4)

$$\eta(t) \equiv x = \int_0^t \frac{cdt}{a(t)}. \quad (5)$$

The conformal time acts as the particle horizon and tells us the maximum distance from which particles could have traveled to an observer in the age of the Universe. This is a useful quantity for our purposes as it relates the expansion of the Universe to a time. The line element can be expressed in terms of the conformal time

$$ds^2 = a^2(t) (-d^2\eta + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)). \quad (6)$$

In order to numerically compute the conformal time, we write it on the differential form

$$\frac{d\eta}{dt} = \frac{c}{a}. \quad (7)$$

We can then use the chain-rule to rewrite (7) to the following form

$$\frac{d\eta}{da} = \frac{c}{a^2 H} = \frac{c}{a \mathcal{H}}, \quad (8)$$

where the H is the Hubble parameter defined as $H \equiv \dot{a}/a$, and $\mathcal{H} \equiv aH$. This allows us to compute the conformal time as soon as we know how the scale factor evolves. The chain-rule can further be applied to rewrite (8) in terms of the variable x from definition (3) to simplify our calculations, giving us

$$\frac{d\eta}{dx} = \frac{c}{\mathcal{H}}. \quad (9)$$

The observable associated with the expansion of the Universe is the Hubble parameter which can be defined through the Friedmann equations as

$$H = H_0 \sqrt{(\Omega_b + \Omega_m)a^{-3} + (\Omega_r + \Omega_\nu)a^{-4} + \Omega_\Lambda}. \quad (10)$$

Here H_0 is the Hubble parameter today, and Ω_b , Ω_m , Ω_r , Ω_ν and Ω_Λ are the relative densities of the baryonic matter, dark matter, radiation, neutrinos and dark energy today, respectively. We will assume the neutrino density to be 0 through out all milestones.

These density fractions are given through each components individual energy density as $\Omega_i = \rho_i/\rho_c$, where $i = m, b, r, \nu, \Lambda$ denotes the different species, and ρ_c is

the critical density given as $\rho_c = 3H^2/8\pi G$. The Friedmann equations also provide a description of how each component evolves with time through the expanding Universe, which are

$$\rho_m = \rho_{m,0}a^{-3} \quad (11)$$

$$\rho_b = \rho_{b,0}a^{-3} \quad (12)$$

$$\rho_r = \rho_{r,0}a^{-4} \quad (13)$$

$$\rho_\nu = \rho_{\nu,0}a^{-4} \quad (14)$$

$$\rho_\Lambda = \rho_{\Lambda,0}, \quad (15)$$

where the subscript 0 indicate today's values.

3. IMPLEMENTATION

The implementation for this milestone mainly consist of completing the provided **time_mod.f90** module. We will now briefly discuss the structure of the code in addition to the completed methods and algorithms.

The code begins by initializing the different variables, parameters, and constants we are to work with. What follows is the allocation of arrays for the different quantities of interest. These include the scale factor a , the redshift z , the parameter x and the conformal time η , along with arrays associated to each of the density components ρ_i . The grids for a , x and z are then filled in a linear manner using the specified initial and end conditions ($z \in [0, 1630]$). We proceed by setting up a second, independent x -grid for the conformal time using the initial condition $a_{\text{init}} = 10^{-10}$, and the scale factor today, $a_0 = 1$ as the end condition.

With these grids in place, we are able to calculate the time evolution of the different density species in the Universe through equations (11 - 15). We can also compute the Hubble parameter through equation (10) using H_0 and Ω_i parameters found in the **params.f90** file. This is done in the lower section of the code as a separate function *get_H(x)* which returns the H for a given x . Similarly we have two additional functions *get_H_p(x)* and *get_dH_p(x)* which returns \mathcal{H} and its derivative $d\mathcal{H}$.

The next part of the code computes the conformal time η . In order to do so we need to solve equation (9). This is integrated with the use of the program **ode_solver.f90**. The calculation does however require an initial condition for η . We find this by considering equation (8). This equation can be solved for eta

$$\eta(a) = \int_0^a \frac{c da}{a^2 H(a)}. \quad (16)$$

We are only interested in the value for $a = a_{\text{init}}$, meaning that the expression is reduced to

$$\eta(a_{\text{init}}) = \frac{c}{a_{\text{init}} H(a_{\text{init}})}. \quad (17)$$

We know that the Universe was radiation dominated at early times meaning that the Hubble parameter can be written as $H = H_0 \sqrt{\Omega_r} a^{-4}$. Inserting this into (17), we find the follow initial condition for the conformal time

$$\eta(a_{\text{init}}) = \frac{c a_{\text{init}}}{H_0 \Omega_r^{1/2}}. \quad (18)$$

Once we have aquired the conformal time, we proceed by computing a spline through the resulting η data in the region $x \in [-7.4, 0]$ (the non-conformal time grids) using the provided program **spline_1D_mod.f90**. This helps us resolve the

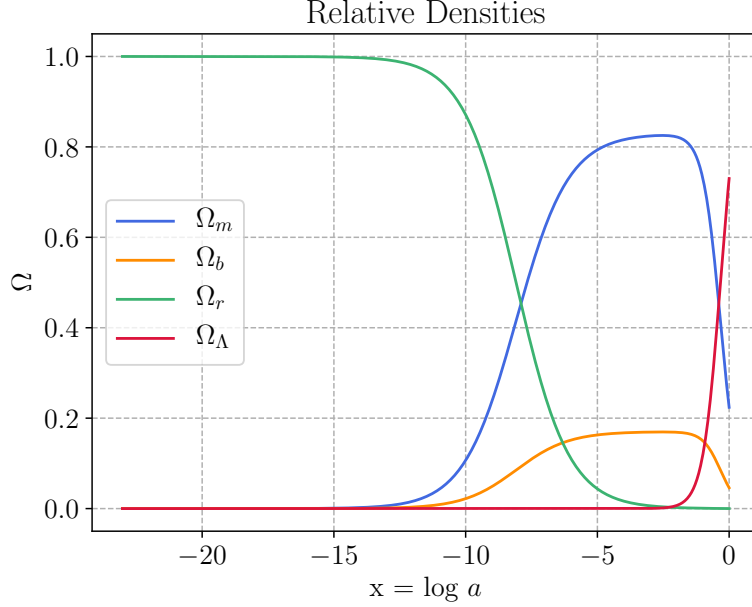


FIGURE 1. The evolution of the different fractional densities as a function of x .

η values between the grid points of interest, and will come in handy in future calculations.

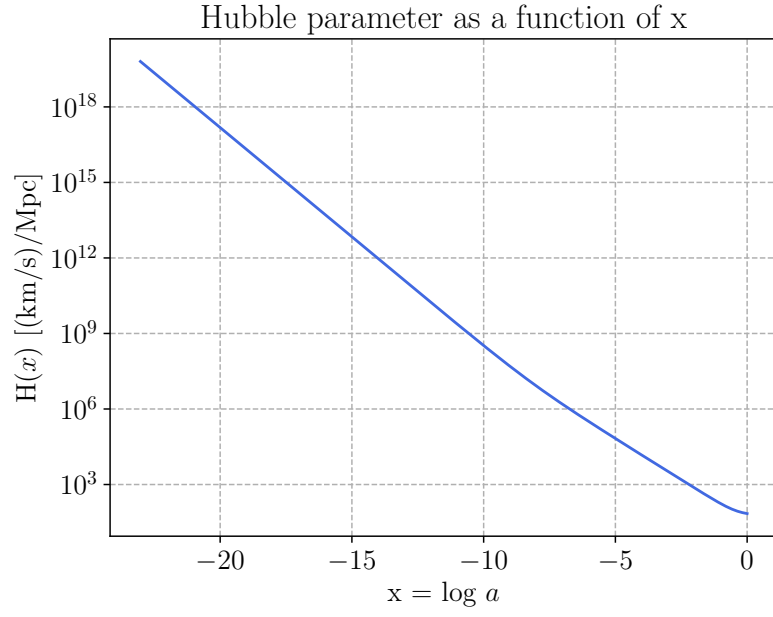
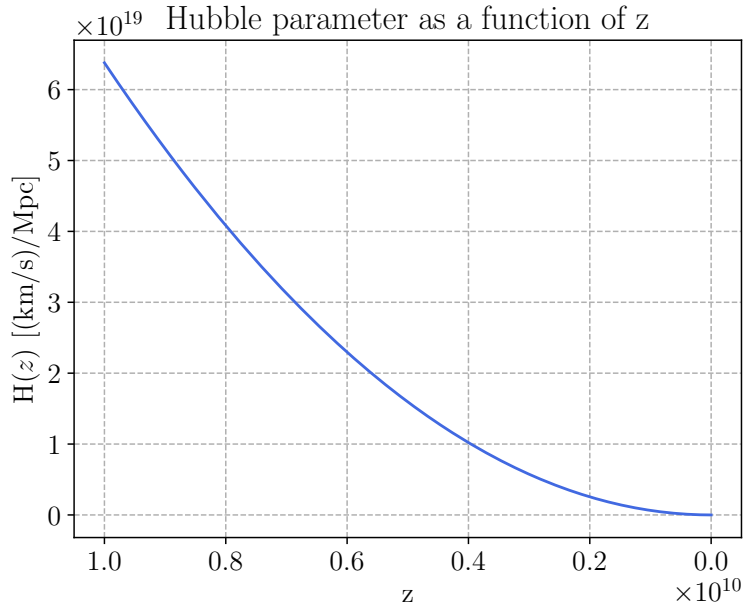
4. RESULTS

The results of the fractional density evolution calculation is seen in figure 1. As expected, we see that the universe starts of in an era dominated by Ω_r , the radiation as it scales with a^{-4} . The dark matter, Ω_m overtakes the radiation at $x \approx 8$ and continues to dominate the universe until around $x \approx 1$ at which the dark energy Ω_Λ takes over. The baryons follow the same behavior as the dark matter as expected since they both scale with a^{-3} with a lower overall amplitude. The sum of the densities are at all time unity as expected. A curiosity is that the matter contribution ($\Omega_m + \Omega_b$) roughly equals that of the dark energy contribution today.

The results of the Hubble parameter calculation can be seen in figures 2 and 3 where we have plotted the results both as functions of the parameter x and the redshift z . $H(x)$ has been plotted in a log-y plot to better resolve the behaviour. The plots are of units $[(\text{km/s})/\text{Mpc}]$ which are the standard units used when describing the Hubble parameter. A quick comparison shows that our computed H_0 is $H(x = 0) = 70.007 (\text{km/s})/\text{Mpc}$. Estimates of the Hubble parameter today range from $65 - 75 (\text{km/s})/\text{Mpc}$, with the Planck estimate¹ being $67.74 \pm 0.46 (\text{km/s})/\text{Mpc}$. This means that our model (even without accounting for neutrinos) does a good job of reproducing the Hubble parameter.

The calculated conformal time can be seen in figure 4 which also includes the splined data. The splined curve only resolves the area of $x \in [-7.4, 0]$ as this seems to be the epoch where the conformal time changes behaviour. This might be connected

¹<http://sci.esa.int/planck/60504-measurements-of-the-hubble-constant/>

FIGURE 2. The evolution of the Hubble parameter as a function of x .FIGURE 3. The evolution of the Hubble parameter as a function of z .

to the fact that the Universe goes from being radiation dominated to dark matter dominated around this time.

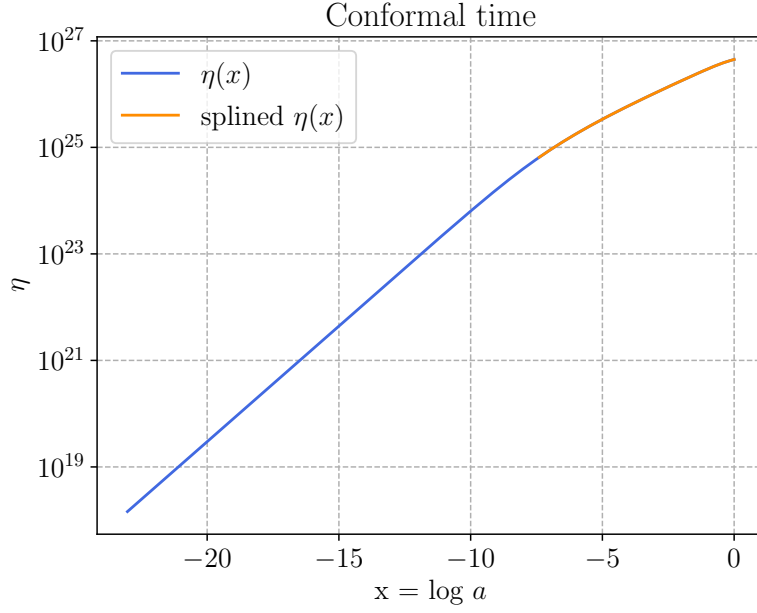


FIGURE 4. The evolution of the conformal time, plotted together with the splined conformal time.

5. CONCLUSION

We have calculated the expansion history of the universe in addition to the studying the evolution of the background densities. All quantities behave in an expected manner which suggest that our code is doing its job correctly. We are therefore ready to apply our methods and results to the coming milestone 2 where we are to study the recombination history of the Universe.