MILESTONE 2: THE RECOMBINATION HISTORY OF THE UNIVERSE
AST5220 - COSMOLOGY 2

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https://github.com/MetinSa/AST5220/tree/master/milestone2

ABSTRACT. We compute the free electron fraction, X_e, which describes the ionization history of the Universe. This is done through the use of the Saha equation for $X_e \geq 0.99$, while the Peebles equation is used for $X_e < 0.99$. We then apply X_e to acquire the electron number density n_e, which in turn is used to compute the optical depth τ and the visibility function g. The resulting τ and g indicate that recombination occurred around $z \approx 1100$.

1. Introduction

This is the second milestone on the path to simulating the Cosmic Microwave Background (CMB). The goal of this milestone will be to study the recombination history of the Universe. Our main objectives will be to calculate the optical depth and the visibility function of the Universe. These quantities are needed for the coming milestones where we are to integrate the Boltzmann-Einstein equations. We will acquire these by computing the ionization fraction of hydrogen in the early universe from which both the optical depth and visibility function can be derived.

Similarly to Milestone 1, the programs used for the numerical calculations are written in FORTRAN 90 and based on a provided skeleton code. The analysis of the results are done using Python. The source code and tools used to produce the following results and figures can be found on my Github by following the link below the author name on the front page.

The report will consist of a theory section where the relevant physics and assumptions are motivated and introduced. What follows is a section where we discuss the numerical implementation and methods used to solve the various equations. We continue by presenting the results of the study in a results section. The report is then concluded with a brief conclusion section where we reflect back on the work done.

2. Theory

We assume that the reader is familiar with Milestone 1 and will therefore not repeat the definitions and equations previously presented. The first quantity of interest in this milestone is the electron number density, n_e. We can find this through the free electron fraction, which is defined as

$$X_e \equiv \frac{n_e}{n_{\mathrm{H}}} = \frac{n_e}{n_b}. \tag{1}$$

Here n_{H} is the total number density of hydrogen which we approximate to be equal to the baryon number density, n_{b}, by ignoring helium. The hydrogen density is given as

$$n_{\mathrm{H}} = n_b \simeq \frac{\rho_b}{m_{\mathrm{H}}} = \frac{\Omega_b \rho_c}{m_{\mathrm{H}} a^3}, \tag{2}$$

where ρ_b and ρ_c are the baryon and the critical density today, respectively, and m_{H} is the mass of the hydrogen atom.

Prior to recombination, the electron fraction can be approximated by the Saha equation

$$\frac{X_e^2}{1 - X_e} \simeq \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b}. \tag{3}$$

Here, T_b is the baryon temperature, m_e the mass of the electron, and ϵ_0 the ionization energy of hydrogen. The baryon temperature, T_b is approximated to be equal to that of the photons as they were coupled during these early times

$$T_b \simeq T_r = \frac{T_0}{a}, \quad T_0 = 2.725\mathrm{K}. \tag{4}$$

The Saha equation is a good approximation as long as the system we are considering is in strong thermodynamic equilibrium (TE). However, during recombination, photons decouple from baryons resulting in the universe dropping out of TE. We will therefore have to use the more general form of the Saha equation, namely the Peebles equation, given as

$$\frac{dX_e}{dx} = \frac{C_r (T_b)}{H} \left[\beta (T_b) (1 - X_e) - n_{\mathrm{H}} \alpha^{(2)} (T_b) X_e^2 \right]. \tag{5}$$

The Peebles equation account for important aspects of particle physics which result in a more accurate calculation of X_e. Here C_r, $\beta(T_b)$, and $\alpha^{(2)}(T_b)$ are substitutions defined as

$$C_r (T_b) = \frac{\Lambda_{2s \to 1s} + \Lambda_\alpha}{\Lambda_{2s \to 1s} + \Lambda_\alpha + \beta^{(2)} (T_b)}, \tag{6}$$

$$\Lambda_{2s \to 1s} = 8.227\mathrm{s}^{-1}, \quad \Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}}, \tag{7}$$

$$n_{1s} = (1 - X_e) n_H, \tag{8}$$

$$\beta^{(2)} (T_b) = \beta (T_b) e^{3\epsilon_0 / 4T_b}, \tag{9}$$

$$\beta (T_b) = \alpha^{(2)} (T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b}, \tag{10}$$

$$\alpha^{(2)} (T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2 (T_b), \tag{11}$$

$$\phi_2 (T_b) \simeq 0.448 \ln (\epsilon_0 / T_b). \tag{12}$$

It should be noted that the above equations are written in terms of natural units; one should perform a unit analysis and include the missing units before implementation these into code.

The electron density is of importance to us as it allows us to calculate the optical depth of the Universe

$$\tau(\eta) = \int_\eta^{\eta_0} n_e \sigma_T a d\eta', \tag{13}$$

where σ_T is the Thompson cross section, η is the conformal time, and the prime denotes $' = d/dx$. The optical depth can be interpreted as the probability that a photon scatters of an electron. We can also express the optical depth in a differential form

$$\tau' = \frac{d\tau}{dx} = -\frac{n_e \sigma_T a}{\mathcal{H}}, \tag{14}$$

which makes it easier to compute τ numerically.

Once acquired, the optical depth allows us the study the final item of interest during Milestone 2 which is the visibility function

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x), \tag{15}$$

$$\tilde{g}(x) \equiv -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}(x)}. \tag{16}$$

Here $\tilde{g}$ is the scaled visibility function, and $\dot{\tau} = d\tau/dt$. The visibility function is normalized as

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1, \tag{17}$$

meaning that it gives us the probability for a given observed photon to have scattered at conformal time η. We will use $\tilde{g}$ during our calculations as we are interested in working with functions of x, meaning that when we reference to the visibility function or g, we will imply equation (16).

3. Implementation

The implementation for Milestone 2 consists of completing the **rec_mod.f90** module which considers quantities and calculations related to recombination.

Similar to milestone 1, the code starts by initializing the relevant parameters and arrays that are to be used during the calculations. These include recombination grids for the scale factor a_{rec}, the redshift z_{rec}, and the parameter x_{rec}. In addition, arrays are allocated for the electron density, n_e, the electron fraction X_e, the optical depth τ, and the visibility function, g. We proceed by filling the grids in a linear manner using initial conditions corresponding to $a_{\text{start}} = 1$ and $a_{\text{stop}} = 10^{-10}$.

With the grids initialized, we move on to calculating the free electron fraction X_e. This is done in a loop which runs over all values of a and x. The loop starts by computing the electron number density seen in equation (2). What follows is a logic statement which checks the value of X_e. A value of $X_e \geq 0.99$ corresponds to an early Universe in TE, meaning that we make use of the Saha equation seen in (3). The Saha equation is solved for X_e which results in the following quadratic equation

$$X_{e,i} = \frac{-\gamma_i \pm \sqrt{\gamma_i^2 + 4\gamma_i}}{2}, \qquad \gamma_i = \frac{1}{n_{b,i}} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b} \tag{18}$$

where the quadratic constants are $a = 1$, $b = c = \gamma$, and $n_{b,i}$ is the baryon density at a given $a_i \in [a_{\text{start}}, a_{\text{stop}}]$. We have chosen to implement an alternate formulation of the quadratic solution

$$X_{e,i} = \frac{2}{1 \pm \sqrt{1 + 4/\gamma_i}}. \tag{19}$$

This expression provides the same roots in addition to being a more stable solution in the case where $\gamma_i \approx 4$.

However, if $X_e < 0.99$, we assume that the Universe is out of TE and we use Peebles' equation (5) instead. This is a first order differential equation, meaning that we solve it using a similar ODE routine as in Milestone 1. For this purpose, a subroutine is implemented which computes the RHS of (5). This is then fed into the ODE routine to return X_e.

Once we have acquired X_e we can easily compute the electron number density by solving equation (1) for n_e. We proceed by splining log n_e, such that it can be evaluated for arbitrary values of x when necessary. We spline the logarithm because n_e varies over many orders of magnitude. This means that we have to remember to exponentiate the results after interpolating.

With n_e in our possession, we can now compute the optical depth τ. This is done by solving equation (14). We observe that this is another ordinary differential equation, meaning that we solve it in a similar manner to Peebles' equation with the use of the ODE solver routine, and a subroutine which computes the RHS of (14). It follows from the definition of the optical depth that $\tau = 0$ today which we use as an initial condition. We proceed by splining log τ so that it too can be interpolated at arbitrary x. We also implement the calculations of the first and second derivatives of the optical depth τ' and τ'' with the use of the *spline* and *splint_deriv*, as these are required to compute the visibility function g, and will be of use to us in the coming milestones.

The final part of the implementation considers the visibility function. Unlike the other functions, g is directly solvable using τ and τ', resulting in a easy implementation of equation (16). Similarly to the case with the optical depth, we will again use the same methods as before to compute g' and g''. These are then properly splined to allow for arbitrary interpolation.

4. Results

The results of the fractional electron density is shown as a function of redshift in figure 1. This proves as a cosmic ionization history. We observe that the free electron fraction is 1 prior to some hundred redshift before recombination (z = 1100). Once recombination occurs, the free electron fraction rapidly drops. We observe a change in the rate of decrease around $z = 800$.

Figure 2 shows the optical depth of the universe as a function of $x = \log a$ along with its two derivatives. We observe that the optical depth closely follows the electron density, as expected from its definition (13). All three functions are continuous, and closely follow each other. At $x \approx -7$, corresponding to a redshift of $z \approx 1100$, we observe $\tau = 1$ which is defined as the boundary between a opaque and optical thick medium, meaning that the universe became opaque around this redshift. This coincides well with recombination. The second derivative of the optical depth experiences a slight increase around $x \approx -7$, likely as a result of recombination occurring.

The results of the visibility function calculations are seen in figure 3. We have chosen to focus in on the results centered around $x \approx 7$. Prior to this, the universe was too dense for scattering to occur. We observe a peak in the visibility function at $x \approx 7$. As mentioned in the theory section, the visibility function can be interpreted as a probability distribution of when a CMB photon observed today was last scattered. These results indicate that this did indeed occur around $z \approx 1100$ at the time of recombination.

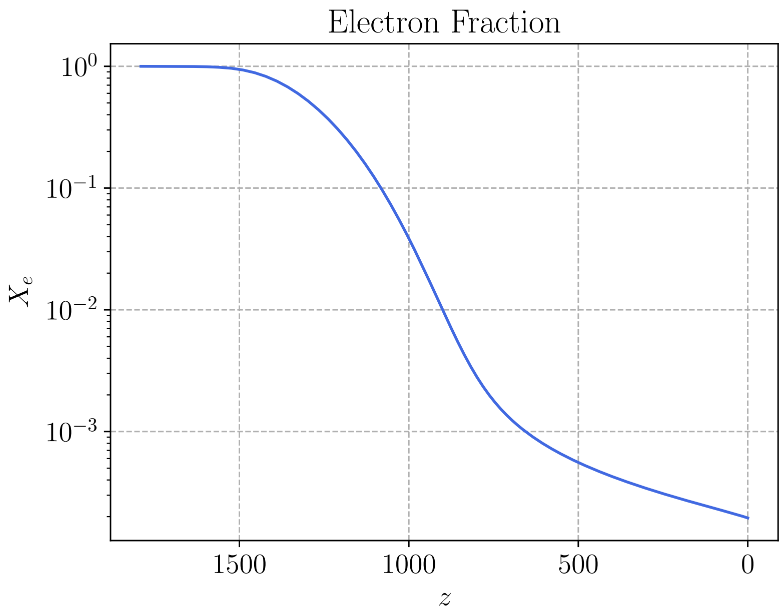

FIGURE 1. The free electron fraction X_e as a function of redshift z. The Saha equation (3) is used to obtain the values prior to $X_e \leq 0.99$ while the Peebles equation (5) is integrated to obtain the remaining values

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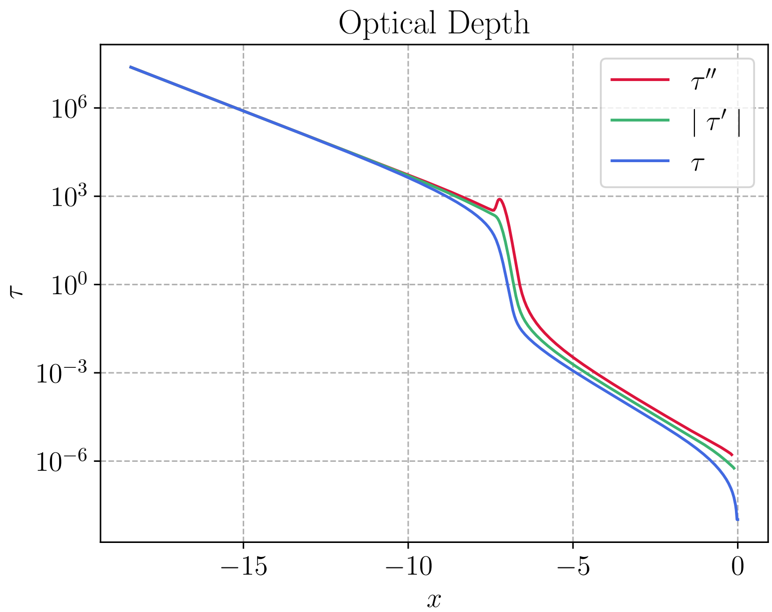

FIGURE 2. The optical depth τ along with the first and second derivatives $|\tau'|$ and τ''.

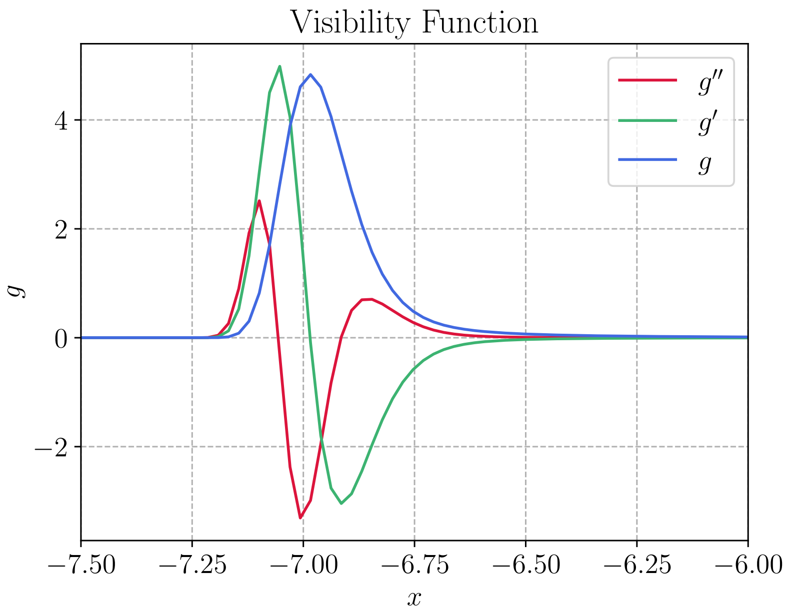

FIGURE 3. The visibility function g along with the first and second derivatives $g'/10$ and $g''/300$. The scaling of g' and g'' are chosen to make the curves fit into the same figure.

5. CONCLUSION

The implementation of Milestone 2 has seemingly been a success. We have studied the recombination history of the Universe by looking at the ionization of the Universe through calculating the free electron density. We have then used these results to study the optical depth of the Universe and the visibility function. We found that the CMB photons did indeed scatter around $z \approx 1100$. We are therefore ready to apply these results further to Milestone 3 where we start considering the evolution of structures in the Universe.