MILESTONE 1: THE BACKGROUND EVOLUTION OF THE UNIVERSE
FYS5220 - COSMOLOGY 2

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Abstract. We set up the cosmological grids for the scale factor a, the redshift z and the parameter $x = \log a$. These are then applied to compute the time evolution of the background density and the Hubble parameter. Finally we compute the conformal time η and spline it to interpolate between the grid points in order to achieve a better accuracy at .

1. Introduction

The long-term goal of the numerical aspect of this course is to simulate the Cosmic Microwave Background (CMB); mainly through the calculation of the power spectrum. This will be a gradual process, with this report constituting the first milestone. The goals of milestone 1 will be to study the expansion history of the universe, as well as looking at how the uniform background densities of the various matter and energy components evolve. We will do so by exploring the Hubble parameter along with the conformal time.

The code used for the numerical calculations is written in FORTRAN 90 and is based on the provided skeleton code. The analysis of the data is done using Python. All source codes and tools used to produce the results and figures can be found on my Github by following the link below the author name on the front page.

The report will consist of a theory section where we introduce and motivate the assumptions and theory used to make our calculations. What follows is the result section where we present and discuss our results. We then conclude the report with a small section where we reflect back on the work done.

2. Theory

We begin by introducing some cosmological concepts which are necessary in order to compute and study the expansion history of the Universe. We start by assuming a flat universe ($k = 0$) described by the Friedmann-Robertson-Walker metric in which the line element (in polar coordinates) is given by

$$ds^2 = -c^2dt^2 + a^2(t)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{1}$$

where $a(t)$ is the scale factor. The scale factor is connected to the redshift through the relation

$$1 + z = \frac{a(t_0)}{a(t)}, \tag{2}$$

where $a(t_0) = 1$ is the scale factor today. We will then define the parameter x

$$x \equiv \ln a. \tag{3}$$

This parameter makes it easier to work with the quantities we are about to compute as we will mainly work with phenomena that varies strongly over wide time ranges.

We proceed by considering a photon which propagates in this Universe. The line element for the photon is equal to zero, meaning that equation (1) can be written on the following form in Cartesian coordinates

$$\frac{dx}{dt} = \frac{c}{a(t)}. \tag{4}$$

We then define the conformal time which is given by integrating equation (4)

$$\eta(t) \equiv x = \int_0^t \frac{cdt}{a(t)}. \tag{5}$$

The conformal time acts as the particle horizon and tells us the maximum distance from which particles could have traveled to an observer in the age of the Universe. This is a useful quantity for our purposes as it relates the expansion of the Universe to a time. The line element can be expressed in terms of the conformal time

$$ds^2 = a^2(t)\left(-d^2\eta + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\right). \tag{6}$$

In order to numerically compute the conformal time, we write it on the differential form

$$\frac{d\eta}{dt} = \frac{c}{a}. \tag{7}$$

We can then use the chain-rule to rewrite (7) to the following form

$$\frac{d\eta}{da} = \frac{c}{a^2 H} = \frac{c}{a\mathcal{H}}, \tag{8}$$

where the H is the Hubble parameter defined as $H \equiv \dot{a}/a$, and $\mathcal{H} \equiv aH$. This allows us to compute the conformal time as soon as we know how the scale factor evolves. The chain-rule can further be applied to rewrite (8) in terms of the variable x from definition (3) to simplify our calculations, giving us

$$\frac{d\eta}{dx} = \frac{c}{\mathcal{H}}. \tag{9}$$

The observable associated with the expansion of the Universe is the Hubble parameter which can be defined through the Friedmann equations as

$$H = H_0\sqrt{(\Omega_b + \Omega_m)a^{-3} + (\Omega_r + \Omega_\nu)a^{-4} + \Omega_\Lambda}. \tag{10}$$

Here H_0 is the Hubble parameter today, and Ω_b, Ω_m, Ω_r, Ω_ν and Ω_Λ are the relative densities of the baryonic matter, dark matter, radiation, neutrinoes and dark energy today, respectively. We will assume the neutrino density to be 0 through out all milestones.

These density fractions are given through each components individual energy density as $\Omega_i = \rho_i/\rho_c$, where $i = m, b, r, \nu, \Lambda$ denotes the different species, and ρ_c is the critical density given as $\rho_c = 3H^2/8\pi G$. The Friedmann equations also provide a description of how each component evolves with time through the expanding

Universe, which are

$$\rho_m = \rho_{m,0} a^{-3} \tag{11}$$

$$\rho_b = \rho_{b,0} a^{-3} \tag{12}$$

$$\rho_r = \rho_{r,0} a^{-4} \tag{13}$$

$$\rho_\nu = \rho_{\nu,0} a^{-4} \tag{14}$$

$$\rho_\Lambda = \rho_{\Lambda,0}, \tag{15}$$

where the subscript 0 indicate today's values.

3. Implementation

The implementation for this milestone mainly consist of completing the provided **time_mod.f90** module. We will now briefly discuss the structure of the code in addition to the completed methods and algorithms.

The code begins by initializing the different variables, parameters, and constants we are to work with. What follows is the allocation of arrays for the different quantities of interest. These include the scale factor a, the redshift z, the parameter x and the conformal time η, along with arrays associated to each of the density components ρ_i. The grids for a, x and z are then filled in a linear manner using the specified initial and end conditions ($z \in [0, 1630]$). We proceed by setting up a second, independent x-grid for the conformal time using the initial condition $a_{\text{init}} = 10^{-10}$, and the scale factor today, $a_0 = 1$ as the end condition.

With these grids in place, we are able to calculate the time evolution of the different density species in the Universe through equations (11 - 15). We can also compute the Hubble parameter through equation (10) using H_0 and Ω_i parameters found in the **params.f90** file. This is done in the lower section of the code as a seperate function *get_H(x)* which returns the H for a given x. Similarely we have two additional functions *get_H_p(x)* and *get_dH_p(x)* which returns $\mathcal{H}$ and its derivative $d\mathcal{H}$.

The next part of the code computes the conformal time η. In order to do so we need to solve equation (9). This is integrated with the use of the program **ode_solver.f90**. The calculation does however require an initial condition for η. We find this by considering equation (8). This equation can be solved for eta

$$\eta(a) = \int_0^a \frac{c\,da}{a^2 H(a)}. \tag{16}$$

We are only interested in the value for $a = a_{\text{init}}$, meaning that the expression is reduced to

$$\eta(a_{\text{init}}) = \frac{c}{a_{\text{init}} H(a_{\text{init}})}. \tag{17}$$

We know that the Universe was radiation dominated at early times meaning that the Hubble parameter can be written as $H = H_0\sqrt{\Omega_r a^{-4}}$. Inserting this this into (17), we find the follow initial condition for the conformal time

$$\eta(a_{\text{init}}) = \frac{c a_{\text{init}}}{H_0 \Omega_r^{1/2}}. \tag{18}$$

We proceed by computing a spline through the resulting η data using the provided program **spline_1D_mod**. This helps us resolve the η values between the grid points of interest, and will come in handy in future calculations.

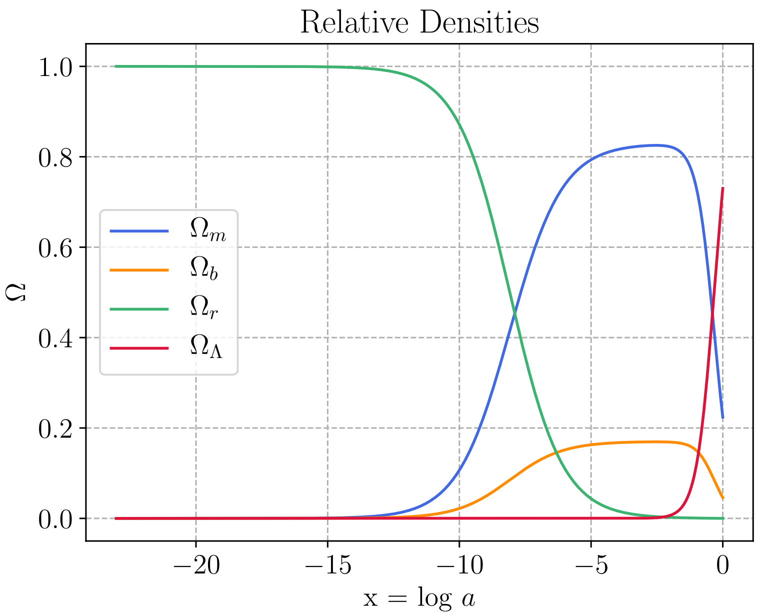

FIGURE 1. The evolution of the different fractional densities as a function of x.

4. Results

The results of the fractional density evolution calculation is seen in figure 1. As expected, we see that the universe starts of in an era dominated by Ω_r, the radiation. The dark matter, Ω_m overtakes the radiation at $x \approx 8$ and continues to dominate the universe until around $x \approx 1$ at which the dark energy Ω_Λ takes over. The baryons follow the same behavior as the dark matter as expected since they both scale with a^{-3} with a lower overall amplitude.

The results of the Hubble parameter calculation can be seen in figures 2 and 3 where we have plotted the results both as functions of the parameter x and the redshift z. Both curves follow the same behaviour. It should be noted that the the z-axis in figure 3 has been flipped in order to resemble figure 2.

The calculated conformal time can be seen in figure 4 which also includes the splined data. The splined curve only resolves the area of $x < 7$ as this seems to be the epoch where the conformal time changes behaviour. This is likely connected to the fact that the Universe goes from being radiation dominated to dark matter dominated.

5. Conclusion

We have calculated the expansion history of the universe in addition to the studying the evolution of the background densities. All quantities behave in an expected manner which suggest that our code is doing its job correctly. We are therefore ready to apply our methods and results to the coming milestone 2 where we are to study the recombination history of the Universe.

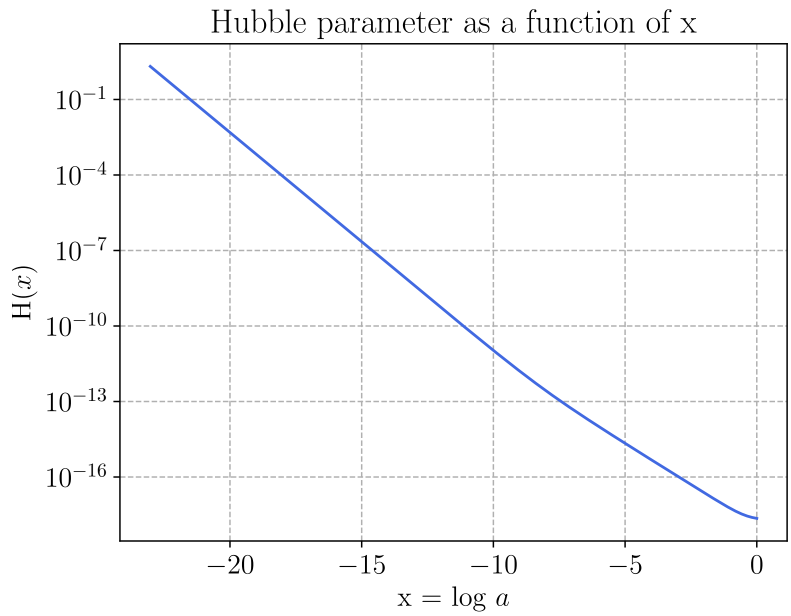

FIGURE 2. The evolution of the Hubble parameter as a function of x.

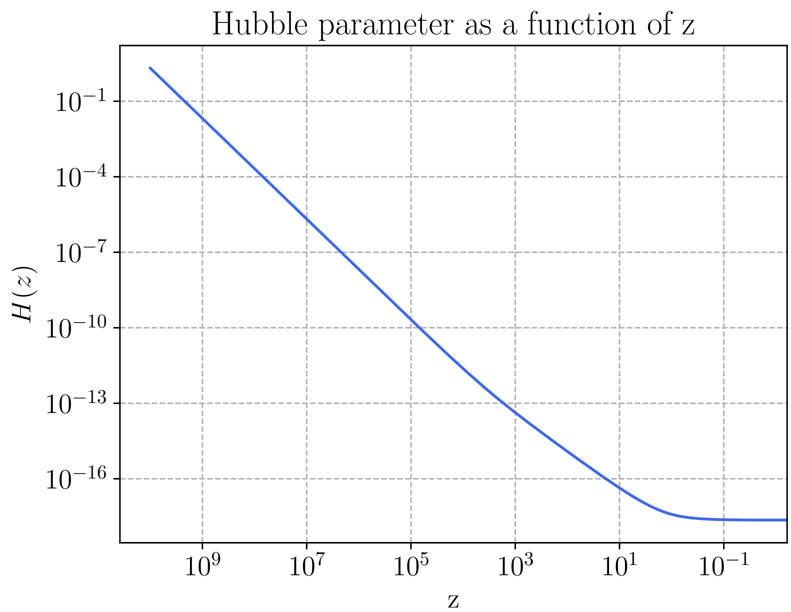

FIGURE 3. The evolution of the Hubble parameter as a function of z.

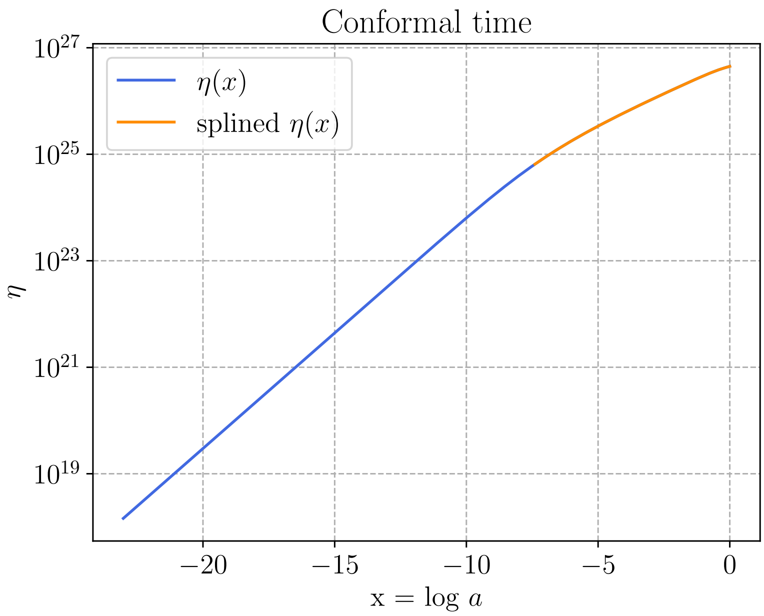

FIGURE 4. The evolution of the conformal time, plotted together with the splined conformal time.