2. Integrador de Adams-Bashforth. Demuestre la formula de iteración para tres y cuatro puntos

$$\mathcal{L}_i(x) = \prod_{j=0,\, j\neq i}^{k} \frac{x - x_j}{x_i - x_j}$$

$$y_{n+1} = y_n + \sum_{i=0}^{k} f_i \int_{t_n}^{t_{n+1}} \mathcal{L}_i(x)\, dx$$

Para tres puntos:

$K = 2$

$$\mathcal{L}_0(x) = \prod_{j=0,\, j\neq 0}^{2} \frac{x - x_j}{x_i - x_j} = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2}.$$

$$\frac{x - x_1}{x_0 - x_1} = \frac{x - (x_1 - x_2)}{x_0 - (x_0 + h)} = \frac{x - (x_0 + h - (x_0 + 2h))}{-h} = \frac{x + h}{-h}.$$

$$\frac{x - x_2}{x_0 - x_2} = \frac{x - (x_2 - x_2)}{x_0 - (x_0 + 2h)} = \frac{x}{-2h}.$$

$$\mathcal{L}_0(x) = \frac{x + h}{-h} \cdot \frac{x}{-2h} = \frac{x^2 + xh}{2h^2}$$

$$\int_{t_n}^{t_{n+1}} \frac{x^2 + xh}{2h^2}\, dx = \int_0^h \frac{x^2 + xh}{2h^2}\, dx = \frac{5h}{12}.$$

$$\mathcal{L}_1(X) = \prod_{j=0,\ j\neq 1}^{2} \frac{x - x_j}{x_i - x_j} = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2}$$

$$= \frac{x-(x_0-x_2)}{h}\,\frac{x}{-h} = \frac{(x_0-(x_0-x_0-2h))\cancel{x}}{-(h^2)} = \frac{(x+2h)\,x}{-(h^2)}$$

$$\int_0^h \frac{x(x+2h)}{-(h^2)}\,dx = -\frac{4}{3}h.$$

$$\mathcal{L}_2(X) = \prod_{j=0,\ j\neq 2}^{2} \frac{x - x_j}{x_i - x_j} = \frac{x-(x_0-x_2)}{x_2-x_0}\cdot\frac{x-(x_1-x_2)}{x_2-x_1}$$

$$= \frac{x-(x_0-(x_0-2h))}{2h}\cdot\frac{x-(x_0+h-(x_0+2h))}{h}$$

$$= \frac{x+2h}{2h}\cdot\frac{x-(-h)}{h} = \frac{x+2h}{2h}\cdot\frac{x+h}{h} = \frac{(x+2h)(x+h)}{2h^2}.$$

$$\int_0^h \frac{(x+2h)(x+h)}{2h^2}\,dx = \frac{23}{12}h.$$

finalmente

$$Y_{n+1} = Y_n + \sum_{i=0}^{2} f_i \int_0^h \mathcal{L}_i(X)\,dx = Y_n + \frac{23}{12}h f_2 - \frac{4}{3}h f_1 + \frac{5}{12}h f_0$$

Para cuatro puntos

$$L_{n-3}(x) = \prod_{j=n-3,\ j\neq n-3}^{n} \frac{x-x_j}{x_i-x_j} = \frac{x-x_{n-2}}{x_{n-3}-x_{n-2}} \cdot \frac{x-x_{n-1}}{x_{n-3}-x_{n-1}} \cdot \frac{x-x_n}{x_{n-3}-x_n}$$

$$= \frac{x-(x_{n-2}-x_n)}{-h} \cdot \frac{x-(x_{n-1}-x_n)}{-2h} \cdot \frac{x-(x_n-x_n)}{-3h} = \frac{x+2h}{-h} \cdot \frac{x+h}{-2h} \cdot \frac{x-0}{-3h}$$

$$= \frac{(x+2h)(x+h)x}{-6h^3}$$

$$\int_0^h \frac{(x+2h)(x+h)x}{-6h^3}\,dx = -\frac{3}{8}h$$

$$L_{n-2}(x) = \prod_{j=n-3,\ j\neq n-2}^{n} \frac{x-x_j}{x_i-x_j} = \frac{x-x_{n-3}}{x_{n-2}-x_{n-3}} \cdot \frac{x-x_{n-1}}{x_{n-2}-x_{n-1}} \cdot \frac{x-x_n}{x_{n-2}-x_n}$$

$$= \frac{x+3h}{h} \cdot \frac{x+h}{-h} \cdot \frac{x}{-2h} = \frac{(x+3h)(x+h)x}{2h^3}$$

$$\int_0^h \frac{(x+3h)(x+h)x}{2h^3}\,dx = \frac{37}{24}h$$

$$L_{n-1}(x) = \prod_{j=n-3,\ j\neq n-1}^{n} \frac{x-x_j}{x_i-x_j} = \frac{x-x_{n-3}}{x_{n-1}-x_{n-3}} \cdot \frac{x-x_{n-2}}{x_{n-1}-x_{n-2}} \cdot \frac{x-x_n}{x_{n-1}-x_n}$$

$$= \frac{x+3h}{2h} \cdot \frac{x+2h}{h} \cdot \frac{x}{-h} = -\frac{(x+3h)(x+2h)x}{2h^3}$$

$$\int_0^h -\frac{(x+3h)(x+2h)x}{2h^3}\,dx = -\frac{59}{24}h$$

$$\mathcal{L}_n(x) = \prod_{\substack{j=n-3 \\ j\neq n}}^{n} \frac{x-x_j}{x_i-x_j} = \frac{x-x_{n-3}}{x_n-x_{n-3}} \cdot \frac{x-x_{n-2}}{x_n-x_{n-2}} \cdot \frac{x-x_{n-1}}{x_n-x_{n-1}}$$

$$= \frac{x-(-3h)}{3h} \cdot \frac{x-(-2h)}{2h} \cdot \frac{x-(-h)}{h}$$

$$= \frac{x+3h}{3h} \cdot \frac{x+2h}{2h} \cdot \frac{x+h}{h} = \frac{(x+h)(x+2h)(x+3h)}{6h^3}.$$

$$\int_0^h \frac{(x+h)(x+2h)(x+3h)}{6h^3}\,dx = \frac{55}{24}h.$$

finalmente

$$y_{n+1} = y_n + \sum_{i=n-3}^{n} f_i \int_0^h \mathcal{L}_i(x)\,dx$$

$$y_{n+1} = y_n + \frac{55}{24}h f_n - \frac{59}{24}h f_{n-1} + \frac{37}{24}h f_{n-2} - \frac{3}{8}h f_{n-3}$$

$$y_{n+1} = y_n + \frac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right). \quad \square$$