3. Integrador de Adams-Moulton. Demuestre la formula de iteracion para tres y cuatro puntos

$$\beta_i = \int_{t_n}^{t_{n+1}} \mathcal{L}_i(x)\,dx$$

$$y_{n+1} = y_n + \sum_{i=0}^{k} \beta_i f_i$$

Para tres puntos

$$\mathcal{L}_{n-1}(x) = \prod_{j=n-1,\ j\neq n-1}^{n+1} \frac{x - x_j}{x_i - x_j} = \frac{x - x_n}{x_{n-1} - x_n} \cdot \frac{x - x_{n+1}}{x_{n-1} - x_{n+1}}$$

$$= \frac{x - (x_n - x_n)}{(x_n - h) - x_n} \cdot \frac{x - (x_n + h - x_n)}{(x_n - h) - (x_n + h)} = \frac{x \cdot (x - h)}{-h\,(-2h)}$$

$$= \frac{x\,(x-h)}{2h^2}.$$

$$\int_0^h \frac{x\,(x-h)}{2h^2}\,dx = -\frac{1}{12}h.$$

$$\boxed{\beta_{n-1} = -\frac{h}{12}}$$

$$\mathcal{L}_n(x) = \prod_{j=n-1\ j\neq n}^{n+1} \frac{x - x_j}{x_i - x_j} = \frac{x - x_{n-1}}{x_n - x_{n-1}} \cdot \frac{x - x_{n+1}}{x_n - x_{n+1}}$$

$$= \frac{x - ((x_n - h) - x_n)}{x_n - (x_n - h)} \cdot \frac{x - ((x_n + h) - x_n)}{x_n - (x_n + h)} = \frac{x+h}{h} \cdot \frac{x - h}{-h}$$

$$= \frac{(x+h)(x-h)}{-(h^2)}.$$

$$\int_0^h \frac{(x+h)(x-h)}{-(h^2)}\,dx = \frac{2}{3}h.$$

$$\boxed{\beta_n = \frac{2}{3}h}$$

$$\mathcal{L}_{n+1}(x) = \prod_{\substack{j=n-1 \\ j\neq n+1}}^{n+1} \frac{x - x_j}{x_i - x_j} = \left(\frac{x - x_{n-1}}{x_{n+1} - x_{n-1}}\right)\left(\frac{x - x_n}{x_{n+1} - x_n}\right)$$

$$= \left(\frac{x - ((x_n - h) - x_n)}{(x_n + h) - (x_n - h)}\right)\left(\frac{x - ((x_n - x_n))}{x_n + h - x_n}\right)$$

$$= \frac{x + h}{2h} \cdot \frac{x}{h}.$$

$$\int_0^h \frac{(x+h)x}{2h^2}\,dx = \frac{5}{12}h. \qquad \boxed{\beta_{n+1} = \frac{5}{12}h}$$

Finalmente

$$Y_{n+1} = Y_n + \sum_{i=n-1}^{n+1} \beta_i f_i.$$

$$Y_{n+1} = Y_n + \beta_{n+1} f_{n+1} + \beta_n f_n + \beta_{n-1} f_{n-1}$$

$$= Y_n + \frac{5}{12}h\, f_{n+1} + \frac{2}{3}h f_n - \frac{h}{12} f_{n-1}$$

$$= Y_n + \frac{h}{12}\left(5 f_{n+1} + 8 f_n - f_{n-1}\right).$$

Para 4 puntos

$$L_{n-2}(x) = \prod_{\substack{j=n-2 \\ j\neq n-2}}^{n+1} \frac{x - x_j}{x_{n-2} - x_j} = \left(\frac{x - x_{n-1}}{x_{n-2} - x_{n-1}}\right)\left(\frac{x - x_n}{x_{n-2} - x_n}\right)\left(\frac{x - x_{n+1}}{x_{n-2} - x_{n+1}}\right)$$

$$= \left(\frac{x-((x_n-h)-x_n)}{(x_n-2h)-(x_n-h)}\right)\left(\frac{x-((x_n-x_n))}{(x_n-2h)-x_n}\right)\left(\frac{x-((x_n+h)-x_n)}{(x_n-2h)-(x_n+h)}\right)$$

$$= \left(\frac{x+h}{-h}\right)\left(\frac{x}{-2h}\right)\left(\frac{x-h}{-3h}\right) = \frac{(x+h)\,x\,(x-h)}{-6h^3}$$

$$\int_0^h \frac{(x+h)\,x\,(x-h)}{-6h^3}\,dx = \frac{h}{24}. \qquad \boxed{\beta_{n-2} = \frac{h}{24}}$$

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$$L_{n-1}(x) = \prod_{\substack{j=n-2 \\ j\neq n-1}}^{n+1} \frac{x - x_j}{x_{n-1} - x_j} = \left(\frac{x - x_{n-2}}{x_{n-1} - x_{n-2}}\right)\left(\frac{x - x_n}{x_{n-1} - x_n}\right)\left(\frac{x - x_{n+1}}{x_{n-1} - x_{n+1}}\right)$$

$$= \frac{x-((x_n-2h)-x_n)}{(x_n-h)-(x_n-2h)} \cdot \frac{x-(x_n-x_n)}{(x_n-h)-x_n} \cdot \frac{x-((x_n+h)-x_n)}{(x_n-h)-(x_n+h)}$$

$$= \frac{x+2h}{h} \cdot \frac{x}{-h} \cdot \frac{x-h}{-2h} = \frac{(x+2h)\,x\,(x-h)}{2h^3}$$

$$\int_0^h \frac{(x+2h)\,x\,(x-h)}{2h^3}\,dx = -\frac{5}{24}h. \qquad \boxed{\beta_{n-1} = -\frac{5}{24}h}$$

$$L_n(X) = \prod_{\substack{j=n-2 \\ j\neq n}}^{n+1} \frac{X - X_j}{X_n - X_j} = \left(\frac{X - X_{n-2}}{X_n - X_{n-2}}\right)\left(\frac{X - X_{n-1}}{X_n - X_{n-1}}\right)\left(\frac{X - X_{n+1}}{X_n - X_{n+1}}\right)$$

$$= \left(\frac{X-((X_n-2h)-X_n)}{2h}\right)\left(\frac{X-((X_n-h)-X_n)}{h}\right)\left(\frac{X-((X_n+h)-X_n)}{-h}\right) = \left(\frac{(X+2h)(X+h)(X-h)}{-2h^3}\right)$$

$$\int_0^h \frac{(X+2h)(X+h)(X-h)}{-2h^3}\,dX = \frac{19}{24}h. \quad \boxed{B_n = \frac{19}{24}h}$$

$$L_{n+1}(X) = \prod_{\substack{j=n-2 \\ j\neq n+1}}^{n+1} \frac{X - X_j}{X_{n+1} - X_j} = \left(\frac{X - X_{n-2}}{X_{n+1} - X_{n-2}}\right)\left(\frac{X - X_{n-1}}{X_{n+1} - X_{n-1}}\right)\left(\frac{X - X_n}{X_{n+1} - X_n}\right)$$

$$= \left(\frac{X-(X_n-2h-X_n)}{X_n+h-(X_n-2h)}\right)\left(\frac{X-(X_n-h-X_n)}{X_n+h-(X_n-h)}\right)\left(\frac{X-(X_n-X_n)}{X_n+h-X_n}\right)$$

$$= \frac{X+2h}{3h}\,\frac{X+h}{2h}\,\frac{X}{h} = \frac{(X+2h)(X+h)X}{6h^3}.$$

$$\int_0^h \frac{(X+2h)(X+h)X}{6h^3}\,dX = \frac{3h}{8}. \quad \boxed{B_{n+1} = \frac{3h}{8}}$$

Entonces

$$y_{n+1} = y_n + \sum_{i=n-2}^{n+1} \beta_i f_i$$

$$y_{n+1} = y_n + \beta_{n+1} f_{n+1} + \beta_n f_n + \beta_{n-1} f_{n-1} + \beta_{n-2} f_{n-2}$$

$$y_{n+1} = y_n + \frac{3h}{8} f_{n+1} + \frac{19}{24} h f_n - \frac{5}{24} h f_{n-1} + \frac{h}{24} f_{n-2}$$

$$y_{n+1} = y_n + \frac{h}{24}\left( 9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2} \right) \quad \square$$