Online Supplement to the *Estimation with Pairwise Observations* paper

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The Online Supplement presents additional summary tables and figures related to the full-pairwise Monte Carlo exercises. Namely, the results provide a more comprehensive overview about the coefficient estimates, the test statistics and the empirical distribution of the test statistics considering Normal and Uniform data generating processes, and Δx and $|\Delta x|$ weighting.

Appendix A: Additional Theoretical Results

1 Theoretical Results on the Differences in Bias between Estimators

Consider the following Data Generating Process

$$y_i = \beta_0 + \beta_1 x_i + u_i, \quad i = 1, \ldots, N \tag{A.1}$$

where (x_i, u_i) follows the the Eyraud-Farlie-Gumbel-Morgenstern (EFGM) Copulas distribution with the following joint density function

$$h(x_i, u_i) = [1 + \theta(1 - 2G(u_i))(1 - 2F(x_i))] \, f(x_i)g(u_i) \tag{A.2}$$

with $\theta \in [0, 1]$ controls the degree of dependence between x_i and u_i. It is well known in the literature that the marginal distribution of x_i and u_i in this case are $f(x_i)$ and $g(u_i)$, respectively. For more information on the properties of EFGM Copulas see Cambanis (1977) and Conway (2005). The following theoretical results will be useful in deriving the asymptotic properties of both OLS and EwPO with $w_{ij} = |\Delta x_{ij}|$.

Proposition 1. *Let x_i and u_i follow the joint distribution as defined in Equation (A.2) with $f(x_i)$ and $g(u_i)$ satisfy the properties of a continuous probability density function with support A_x and A_u, respectively. Furthermore, assume that $\int_{A_x} F(x)f(x)dx < \infty$ and $\int_{A_u} G(u)g(u)du < \infty$ then*

$$\mathbb{E}\left(u_i|x_i\right) = \mu_u + \theta\left(1 - 2F(x_i)\right)\mu_u - 2\theta(1 - 2F(x_i)) \int_{A_u} u_i G(u_i)g(u_i)du_i \tag{A.3}$$

where A_u denotes the support of u_i and $\mu_u = \mathbb{E}(u_i)$.

Proof.

$$\begin{aligned}
\mathbb{E}(u_i|x_i) &= \int_{A_u} u \frac{h(x_i, u)}{f(x_i)} du \\
&= \int_{A_u} u \left[1 + \theta(1 - 2F(x_i))(1 - 2G(u))\right] g(u)du \\
&= \int_{A_u} ug(u)du + \theta(1 - 2F(x_i)) \int_{A_u} ug(u)du \\
&\quad - 2\theta(1 - 2F(x_i)) \int_{A_u} uG(u)g(u)du
\end{aligned}$$

Since $\mathbb{E}(u) = \int u(g)du = \mu_u$, substitute μ_u into the expression above gives the result. □

Remark 1. In the present context, $\mu_u = 0$ and therefore

$$\mathbb{E}(u_i|x_i) = \theta(2F(x_i) - 1)\mu_{\tilde{u}} \tag{A.4}$$

where $\mu_{\tilde{u}} = \displaystyle\int_{A_u} uG(u)g(u)du$.

Remark 2. Integral such as

$$\int_{A_x|} F(x)f(x)dx \tag{A.5}$$

will always exists if $F(x)$ is a cumulative distribution function which density $F'(x) = f(x)$ is symmetric around 0. In fact, $2F(x)f(x)$ in such case is a class of skewed distributions as defined in Azzalini (1985).

Proposition 2. *Under the assumptions as Proposition (1) with* $\mu_u = 0$

$$\mathbb{E}(x_i u_i) = \theta \mu_{\tilde{u}} \left(\mu_{\tilde{x}} - \mu_x \right). \tag{A.6}$$

Proof. Following the result from Proposition 1 and under the assumption that $\mu_u = 0$, the Law of Iterated Expectation implies

$$\begin{aligned}\mathbb{E}(x_i u_i) =& \mathbb{E}\left[\mathbb{E}(x_i u_i | x_i)\right] \\ =& \int_{A_x} x\theta(2F(x) - 1)\mu_{\tilde{u}} f(x)dx \\ =& \theta\mu_{\tilde{u}} \left[2\int_{A_x} xF(x)f(x)dx - \int_{A_x} xf(x)dx\right]\end{aligned}$$

where A_x denotes the support of x_i. Substitute $\mu_{\tilde{x}} = 2\int_{A_x} xF(x)f(x)dx$ and $\mu_x = \int_{A_x} xf(x)dx$ gives the result. □

Proposition 3. *Under the assumptions of Proposition 1*

$$\mathbb{E}\left[sgn\left(\Delta x_{ij}\right)\Delta u_{ij}\right] = 2\mu_{\tilde{u}}\theta\left[-2\int_{A_x} \tilde{F}(x)f(x)dx + 2\tilde{F} - 1\right] \tag{A.7}$$

where $\tilde{F}(x) = 2\int_{L_x}^{x} F(x)f(x)dx$ *and* $\tilde{F} = \lim_{x\to U_x} \tilde{F}(x)$ *with* $A_x = (L_x, U_x)$.

Proof.

$$\begin{aligned}\mathbb{E}\left[\text{sgn}\left(\Delta x_{ij}\right)\Delta u_{ij}\right] =& \mathbb{E}\left[\text{sgn}\left(\Delta x_{ij}\right)\Delta u_i\right] - \mathbb{E}\left[\text{sgn}\left(\Delta x_{ij}\right)\Delta u_j\right] \\ =& \mathbb{E}\left\{\mathbb{E}\left[\text{sgn}\left(\Delta x_{ij}\right)u_i | x_i, x_j\right]\right\} - \mathbb{E}\left\{\mathbb{E}\left[\text{sgn}\left(\Delta x_{ij}\right)u_j | x_i, x_j\right]\right\}.\end{aligned}$$

Consider first $\mathbb{E}\left[\text{sgn}\left(\Delta x_{ij}\right)u_i | x_i, x_j\right]$. Under the assumption that $\mu_u = 0$,

$$\begin{aligned}\mathbb{E}\left[\text{sgn}\left(\Delta x_{ij}\right)u_i | x_i, x_j\right] =& \text{sgn}\left(\Delta x_{ij}\right)\mathbb{E}(u_i | x_i) \\ =& \text{sgn}\left(\Delta x_{ij}\right)\theta\mu_{\tilde{u}}\left(2F(x_i) - 1\right).\end{aligned}$$

Conditional of x_j, the expectation of the expression above yields

$$
\begin{aligned}
&\mathbb{E}\left[\operatorname{sgn}\left(\Delta x_{ij}\right)\theta\mu_{\tilde{u}}\left(2F(x_i)-1\right)|x_j\right] \\
=&\theta\mu_{\tilde{u}}\left[-\int_{L_x}^{x_j}\left(2F(x_i)-1\right)f(x_i)dx_i+\int_{x_j}^{U_x}\left(2F(x_i)-1\right)f(x_i)dx_i\right] \\
=&\theta\mu_{\tilde{u}}\left[-2\tilde{F}(x_j)+\tilde{F}+2F(x_j)-1\right]
\end{aligned}
$$

Taking the final expectation of the expression above leads to

$$
\theta\mu_{\tilde{u}}\left[-2\int_{A_x}\tilde{F}(x)f(x)dx+2\tilde{F}-1\right].
$$

From symmetry, it is straightforward to deduce that

$$
\mathbb{E}\left\{\mathbb{E}\left[\operatorname{sgn}\left(\Delta x_{ij}\right)u_j|x_i,x_j\right]\right\}=-\theta\mu_{\tilde{u}}\left[-2\int_{A_x}\tilde{F}(x)f(x)dx+2\tilde{F}-1\right].
$$

Combine the last two expressions gives the result. This completes the proof. □

Remark 3. Note that when $f(x)$ is a continuous function that satisfies the properties of a probability density function then it can be shown that

$$
\int_{A_x}\tilde{F}(x)f(x)dx=\frac{1}{3}.
$$

This is due to the fact that $\tilde{F}(x)=F^2(x)$ using integration-by-parts and substitute this expression into the main integral and again, evaluate the integral using integration-by-parts, gives the result.

Proposition 4. *Under the assumption of Proposition 1,*

$$
\mathbb{E}\left(|\Delta x_{ij}|\right)=2\left(\mu_{\tilde{x}}-\mu_x\right). \tag{A.8}
$$

Proof.

$$
\begin{aligned}
\mathbb{E}\left(|\Delta x_{ij}|\right)=&\mathbb{E}\left[\mathbb{E}\left(|\Delta x_{ij}||x_j\right)\right] \\
\mathbb{E}\left(|\Delta x_{ij}||x_j\right)=&-\int_{L_x}^{x_j}(x-x_j)f(x)dx+\int_{x_j}^{U_x}(x-x_j)f(x)dx \\
=&-\int_{L_x}^{x_j}xf(x)dx+\int_{L_x}^{x_j}x_jf(x)dx+\int_{x_j}^{U_x}xf(x)dx-x_j\int_{x_j}^{U_x}f(x)dx \\
=&\mu_x-2\int_{L_x}^{x_j}xf(x)dx-x_j\left(1-2F(x_j)\right) \\
=&\mu_x-2\mu_x(x_j)-x_j\left(1-2F(x_j)\right).
\end{aligned}
$$

Taking expectation of the last expression above gives

$$\begin{aligned}
&\mu_x - 2\int_{A_x} \mu_x(x) f(x) dx - \mu_x + 2\int_{A_x} F(x) f(x) dx \\
=&2\left(\int_{A_x} xF(x)f(x)dx - \int_{A_x} \mu_x(x) f(x) dx\right).
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{A_x} \mu_x(x) f(x) dx =&\mu_(x) F(x)|_{A_x} - \int_{A_x} x f(x) F(x) dx \\
=&\mu_x - \frac{\mu_{\tilde{x}}}{2}.
\end{aligned}$$

This implies

$$\begin{aligned}
2\left(\int_{A_x} xF(x)f(x)dx - \int_{A_x} \mu_x(x) f(x) dx\right) =&\mu_{\tilde{x}} - 2\int_{A_x} \mu_x(x) f(x) dx \\
=&\mu_{\tilde{x}} - 2\left(\mu_x - \frac{\mu_{\tilde{x}}}{2}\right) \\
=&2\left(\mu_{\tilde{x}} - \mu_x\right).
\end{aligned}$$

This completes the proof. □

Now consider the case $f(x) = \exp(-x)$ and $g(u) = \phi(u)$, then direct calculation using Propositions 1 to 4 give

$$\begin{aligned}
\mathbb{E}\left(x_i u_i\right) =&\frac{\theta}{2\sqrt{\pi}} \\
\mathbb{E}\left[\left(x_i - \mu_x\right)^2\right] =&1 \\
\mathbb{E}\left(\operatorname{sgn}\left(\Delta x_{ij}\right)\Delta u_{ij}\right) =&\frac{\theta}{3\sqrt{\pi}} \\
\mathbb{E}\left(|\Delta x_{ij}|\right) =&1
\end{aligned}$$

and hence

$$\begin{aligned}
\hat{\beta}_1 - \beta_1 =&\frac{\theta}{4\sqrt{\pi}} \\
\tilde{\beta}_1 - \beta_1 =&\frac{\theta}{3\sqrt{\pi}} \\
\hat{\beta}_1 - \tilde{\beta}_1 =&-\frac{\theta}{6\sqrt{\pi}}.
\end{aligned}$$

Based on the same calculation, when $f(x) = \phi(x)$ and $g(u) = \phi(u)$, we have

$$\begin{aligned}
\mathbb{E}(x_i u_i) =& \frac{\theta}{\pi} \\
\mathbb{E}(x_i^2) =& 1 \\
\mathbb{E}\left[\operatorname{sgn}(\Delta x_{ij})\, \Delta u_{ij}\right] =& \frac{2\theta}{3\sqrt{\pi}} \\
\mathbb{E}\left[|\Delta x_{ij}|\right] =& \frac{2}{\sqrt{\pi}}
\end{aligned}$$

This means

$$\begin{aligned}
\hat{\beta}_1 - \beta_1 =& \frac{\theta}{\pi} \\
\tilde{\beta}_1 - \beta_1 =& \frac{\theta}{3} \\
\hat{\beta}_1 - \tilde{\beta}_1 =& \frac{\theta\,(3-\pi)}{3\pi} \approx -0.015\theta.
\end{aligned}$$

Tables 1 and 2 contain Monte Carlo simulation evidence for the theoretical results above. The date generating process follows

$$y_i = 1 + 0.5x_i + u_i$$

where x_i and u_i follows the probability distribution as defined in Equation (A.2) over different values of θ i.e., $\theta = 0, 0.2, 0.5$ and 0.9.

	θ			
n	0.0	0.2	0.5	0.9
50	-0.0005	-0.0179	-0.0359	-0.0694
100	0.0006	-0.0191	-0.0431	-0.0769
500	0.0004	-0.0187	-0.0465	-0.083
1000	0.0001	-0.019	-0.0466	-0.0841
5000	0.0	-0.0189	-0.0472	-0.0849

Table 1: $\hat{\beta}_1 - \tilde{\beta}_1$ with $f(x) = \exp(-x)$, $g(u) = \phi(u)$

	θ			
n	0.0	0.2	0.5	0.9
50	-0.0003	-0.0014	-0.0068	-0.0124
100	-0.0014	-0.0014	-0.0074	-0.0122
500	-0.0004	-0.0033	-0.0076	-0.0133
1000	-0.0003	-0.0031	-0.0077	-0.0133
5000	0.0	-0.0031	-0.0075	-0.0135

Table 2: $\hat{\beta}_1 - \tilde{\beta}_1$ with $f(x) = \phi(x)$, $g(u) = \phi(u)$

As shown in Tables 1 and 2, the difference between the two estimators approach the theoretical value in each case of θ as the sample size n increases.

2 Residuals Test

The residuals test is particularly useful when $\beta_0 = 0$. Consider $w_{ij} = |\Delta x_{ij}|$ which implies

$$\hat{\beta} = \beta + \delta_n \,, \tag{A.9}$$

where

$$\delta_n = \left(\sum_{i=2}^{n} \sum_{j=1}^{i-1} |\Delta x_{ij}| \right)^{-1} \left(\sum_{i=2}^{n} \sum_{j=1}^{i-1} \operatorname{sgn}(\Delta x_{ij}) \, \Delta u_{ij} \right) \tag{A.10}$$

and under $x_i \perp u_i$, $\delta_n = o_p(1)$. Now given the following specification

$$y_i = x_i \beta + u_i \tag{A.11}$$

the estimated residual from the EwPO estimator is

$$\begin{aligned} \hat{u}_i =& y_i - x_i \hat{\beta} \\ &= x_i(\beta - \hat{\beta}) + u_i \\ &= -x_i \delta_n + u_i . \end{aligned}$$

It is straightforward to show that

$$\begin{aligned} n^{-1} \sum_{i=1}^{n} \hat{u}_i =& - n^{-1} \sum_{i=1}^{n} x_i \delta_n + u_i \\ &= - \delta_n n^{-1} \sum_{i=1}^{n} x_i + n^{-1} \sum_{i=1}^{n} u_i \,. \end{aligned}$$

Now as $n \to \infty$ the last line above is

$$n^{-1} \sum_{i=1}^{n} \hat{u}_i = -\delta_n \mu_x + o_p(1) \tag{A.12}$$

under the assumption that $\mathbb{E}(u_i) = 0$. There are two cases when $n^{-1} \sum_{i=1}^{n} \hat{u}_i$ is $o_p(1)$ namely, $x_i \perp u_i$ or $\mu_x = 0$. Assuming $\mu_x \neq 0$, which can be easily verified in practice, it is possible to *directly* test $x_i \perp u_i$ by testing $H_0 : \mathbb{E}(u_i) = 0$. This provides the foundation for testing endogeneity by examining the mean of the estimated residuals using standard testing procedure, such as the t-test.[1]

The related codes were written by the authors and are available on GitHub.

[1]The argument here also applies to the usual Ordinary Least Squares (OLS) estimator. That is, when $\beta_0 = 0$, the estimated residuals do not have 0 mean. Thus, it also provides a test of endogeneity in this special case.

Appendix B: Monte Carlo Simulations Setups and Simulation Results for the EwPO Estimation

This Online Supplement presents additional Monte Carlo (MC) simulation results to assess the properties of the EwPO estimator with selected weights.

The data generating process for the MC simulations is based on the model

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

and the MC experiments consider two possible distributions for u_i, namely

1. $u_i \sim N(0,1)$,
2. $u_i \sim$ skewed normal distribution,

where the skewed normal distribution is generated as

$$u_i = \xi + \lambda |v_i| + z_i,$$

with $\xi = -\lambda\sqrt{\frac{2}{\pi}}$, $v_i \sim N(0,1)$ and $z_i \sim N(0,\sigma^2)$ such that v_i and w_i are independently distributed.

The MC experiments consider uniform distribution U(-10,10) for the regressor x_i.

The parameter vector, presented here (for purposes of robustness checking), is $(\beta_0, \beta_1) = (1, 1.5)$. The number of MC replications is 1000.

Sorted MC	Estimates and MC standard errors			
	Parameter	Estimate/S.e.	OLS	pairwise
n=50	$\hat{\beta}_0$	Estimate	0.9866	0.9866
		S.e.	0.1609	0.1609
	$\hat{\beta}_1$	Estimate	1.4999	1.4999
		S.e.	0.2928	0.2928
n = 500	$\hat{\beta}_0$	Estimate	1.0001	1.0001
		S.e.	0.0504	0.0504
	$\hat{\beta}_1$	Estimate	1.4971	1.4971
		S.e.	0.0948	0.0948
n = 5000	$\hat{\beta}_0$	Estimate	0.9992	0.9992
		S.e.	0.0169	0.0169
	$\hat{\beta}_1$	Estimate	1.4997	1.4997
		S.e.	0.0281	0.0281

Table 3: Sorted – full-pairwise MC, $x_i \sim U(-10,10)$, $u_i \sim$ skewed normal, Δx weighted estimator

Sorted MC	Estimates and MC standard errors			
	Parameter	Estimate/S.e.	OLS	pairwise
n=50	$\hat{\beta}_0$	Estimate	0.9866	0.9866
		S.e.	0.1609	0.1609
	$\hat{\beta}_1$	Estimate	1.4999	1.4999
		S.e.	0.2928	0.2928
n = 500	$\hat{\beta}_0$	Estimate	1.0001	1.0001
		S.e.	0.0504	0.0504
	$\hat{\beta}_1$	Estimate	1.4971	1.4971
		S.e.	0.0948	0.0948
n = 5000	$\hat{\beta}_0$	Estimate	0.9992	0.9992
		S.e.	0.0169	0.0169
	$\hat{\beta}_1$	Estimate	1.4997	1.4997
		S.e.	0.0281	0.0281

Table 4: Non-sorted full-pairwise MC, $x_i \sim U(-10, 10)$, Δx weighted estimator

Note 1: It is no mistake, the sorted and non-sorted results are identical here.

Note 2: In general, the standard errors are much larger for β_0 than β_1, but when the distribution of the x_i-s is 'informative', they are in fact quite close to the OLS ones.

Non-sorted MC	Estimates and MC standard errors			
	Parameter	Estimate/S.e.	OLS	pairwise
n=50	$\hat{\beta}_0$	Estimate	1.0009	0.9967
		S.e.	0.1408	0.1997
	$\hat{\beta}_1$	Estimate	1.5007	1.5018
		S.e.	0.0251	0.0291
n = 500	$\hat{\beta}_0$	Estimate	0.9967	0.9966
		S.e.	0.0446	0.0640
	$\hat{\beta}_1$	Estimate	1.5001	1.4998
		S.e.	0.0081	0.0095
n = 5000	$\hat{\beta}_0$	Estimate	1.001	1.0004
		S.e.	0.0144	0.0192
	$\hat{\beta}_1$	Estimate	1.4999	1.4999
		S.e.	0.0025	0.0031

Table 5: Non-sorted adjacent MC, $x_i \sim U(-10, 10)$, $u_i \sim N(0, 1)$, $|\Delta x|$ weighted estimator

Full-pairwise MC	Estimates and MC standard errors			
	Parameter	Estimate/S.e.	OLS	pairwise
n=50	$\hat{\beta}_0$	Estimate	1.0009	1.0009
		S.e.	0.1408	0.1408
	$\hat{\beta}_1$	Estimate	1.5007	1.5007
		S.e.	0.0251	0.0251
n = 500	$\hat{\beta}_0$	Estimate	0.9967	0.9967
		S.e.	0.0446	0.0446
	$\hat{\beta}_1$	Estimate	1.5001	1.5001
		S.e.	0.0081	0.0081
n = 5000	$\hat{\beta}_0$	Estimate	1.0001	1.001
		S.e.	0.0144	0.0144
	$\hat{\beta}_1$	Estimate	1.4999	1.4999
		S.e.	0.0025	0.0025

Table 6: Sorted full-pairwise MC, $x_i \sim U(-10, 10)$, $u_i \sim N(0, 1)$, Δx weighted estimator

Appendix C: Monte Carlo Simulations Setups and Simulation Results for the Test

The MC setup considers sample size $n = 50$, 500, and 5000 with 1000 replications.

Step 1. Generate model the model with one explanatory variable namely,

$$y_i = \alpha + x_i\beta + u_i$$

with $\alpha = 1$, $\beta = 0.5$ to start with, and u_i is generated as $N(0, 1)$. Finally, x should be generated as $N(0, 1)$ and also $U(-5, 5)$.

The simulation of x_i and u_i is conducted under four different correlations namely $\rho = 0$ (benchmark ideal case), $\rho = 0.2$ (small), $\rho = 0.5$ (medium), and $\rho = 0.8$ (large).

Step 2. Estimate the model with EwPO with $w_{ij} = \Delta x_{ij}$ and $w_{ij} = |\Delta x_{ij}|$. In each case, calculate the test statistics as defined in Equation (C.1).

$$S(\mathbf{w}) = n^{-2} \sum_{p=2}^{n} \sum_{q=1}^{p-1} \Delta x_{pq} \Delta \hat{u}_{pq}. \tag{C.1}$$

Full-pairwise MC	Average test statistics				
	Parameter	Pairwise	Standard deviation	Skewness	Kurtosis
n=50	Exogen	0.0204	0.6551	-0.0043	3.2285
	$\rho = 0.2$	-0.4338	0.7196	0.0537	2.7248
	$\rho = 0.5$	-0.6800	0.5816	0.0045	3.0386
	$\rho = 0.8$	-2.2359	0.4627	0.0591	2.9205
n = 500	Exogen	0.0014	0.2100	-0.0992	3.1302
	$\rho = 0.2$	-0.4109	0.2199	-0.0596	3.1559
	$\rho = 0.5$	-0.9357	0.1888	0.0415	2.9658
	$\rho = 0.8$	-1.4518	0.1335	0.1038	3.1749
n = 5000	Exogen	0.0001	0.0671	0.0696	2.8285
	$\rho = 0.2$	-0.3970	0.0680	-0.0949	2.8970
	$\rho = 0.5$	-1.0143	0.0594	-0.2178	3.1821
	$\rho = 0.8$	-1.5325	0.0407	-0.0089	3.0313

Table 7: Average test statistics, full-pairwise MC, $x_i \sim N(0,5)$, $|\Delta x|$ weighted estimator

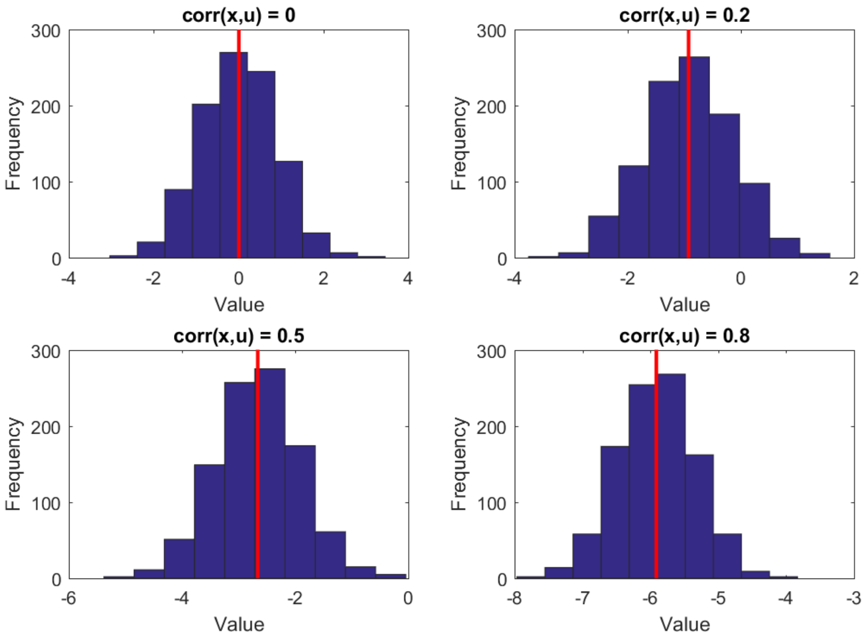

Figure 1: Distribution of the test-statistics, $x_i \sim$ Uniform(-5,5), Δx weighted full-pairwise estimator, $n = 50$

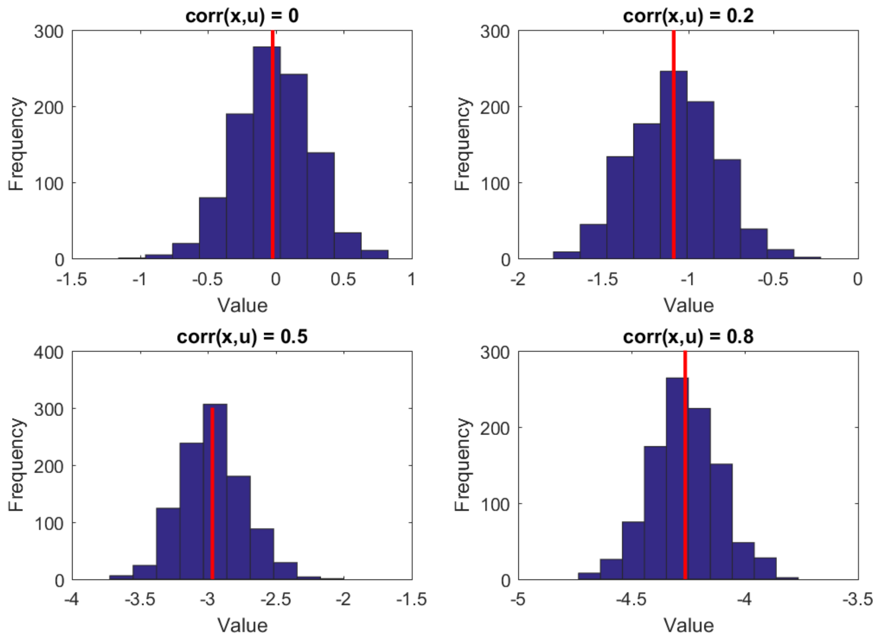

Figure 2: Distribution of the test-statistics, $x_i \sim$ Uniform(-5,5), Δx weighted full-pairwise estimator, $n = 500$

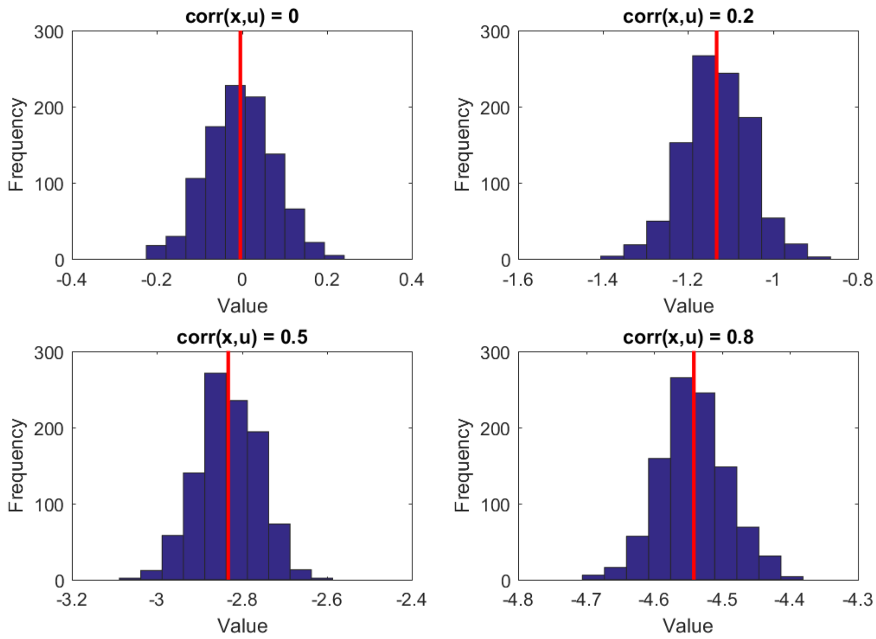

Figure 3: Distribution of the test-statistics, $x_i \sim$ Uniform(-5,5), Δx weighted full-pairwise estimator, $n = 5000$

Full-Pairwise MC	Estimates and MC standard errors			
	Parameter	Estimate/S.e.	OLS	pairwise
n=50	Exogen	Estimate	0.4997	0.4999
		S.e.	0.0602	0.0638
	$\rho = 0.2$	Estimate	0.5378	0.5408
		S.e.	0.0573	0.0607
	$\rho = 0.5$	Estimate	0.5852	0.5781
		S.e.	0.1609	0.1801
	$\rho = 0.8$	Estimate	0.6277	0.6348
		S.e.	0.0395	0.0408
n=500	Exogen	Estimate	0.5007	0.5006
		S.e.	0.0202	0.0207
	$\rho = 0.2$	Estimate	0.5426	0.5430
		S.e.	0.0190	0.0196
	$\rho = 0.5$	Estimate	0.6027	0.6000
		S.e.	0.0162	0.0165
	$\rho = 0.8$	Estimate	0.6586	0.6525
		S.e.	0.0113	0.0116
n=5000	Exogen	Estimate	0.5003	0.5003
		S.e.	0.0062	0.0063
	$\rho = 0.2$	Estimate	0.5394	0.5393
		S.e.	0.0060	0.0062
	$\rho = 0.5$	Estimate	0.5962	0.5963
		S.e.	0.0055	0.0056
	$\rho = 0.8$	Estimate	0.6571	0.6563
		S.e.	0.0038	0.0039

Table 8: β_1 coefficient estimates - Δx weighted full-pairwise MC, $x_i \sim$ N(0,5)

Full-Pairwise MC	Estimates and MC standard errors			
	Parameter	Estimate/S.e.	OLS	pairwise
n=50	Exogen	Estimate	0.5014	0.5014
		S.e.	0.0156	0.0156
	$\rho = 0.2$	Estimate	0.5662	0.5663
		S.e.	0.0150	0.0151
	$\rho = 0.5$	Estimate	0.6677	0.6684
		S.e.	0.0131	0.0131
	$\rho = 0.8$	Estimate	0.7720	0.7707
		S.e.	0.0094	0.0094
n=500	Exogen	Estimate	0.4996	0.4997
		S.e.	0.0216	0.0221
	$\rho = 0.2$	Estimate	0.5406	0.5399
		S.e.	0.0191	0.0196
	$\rho = 0.5$	Estimate	0.6047	0.6038
		S.e.	0.0184	0.0192
	$\rho = 0.8$	Estimate	0.6456	0.6474
		S.e.	0.0122	0.0126
n=5000	Exogen	Estimate	0.5001	0.5000
		S.e.	0.0061	0.0062
	$\rho = 0.2$	Estimate	0.5404	0.5401
		S.e.	0.0061	0.0063
	$\rho = 0.5$	Estimate	0.6017	0.6016
		S.e.	0.0055	0.0056
	$\rho = 0.8$	Estimate	0.6539	0.6546
		S.e.	0.0038	0.0039

Table 9: β_1 coefficient estimates - $|\Delta x|$ weighted full-pairwise MC, $x_i \sim$ N(0,5)

Full-Pairwise MC	Estimates and MC standard errors			
	Parameter	Estimate/S.e.	OLS	pairwise
n=50	Exogen	Estimate	0.4999	0.4998
		S.e.	0.0464	0.0467
	$\rho = 0.2$	Estimate	0.5625	0.5629
		S.e.	0.0526	0.0529
	$\rho = 0.5$	Estimate	0.6613	0.6622
		S.e.	0.0445	0.0448
	$\rho = 0.8$	Estimate	0.7841	0.7875
		S.e.	0.0277	0.0278
n=500	Exogen	Estimate	0.5014	0.5014
		S.e.	0.0156	0.0156
	$\rho = 0.2$	Estimate	0.5662	0.5663
		S.e.	0.0150	0.0151
	$\rho = 0.5$	Estimate	0.6677	0.6684
		S.e.	0.0131	0.0131
	$\rho = 0.8$	Estimate	0.7720	0.7707
		S.e.	0.0094	0.0094
n=5000	Exogen	Estimate	0.5002	0.5002
		S.e.	0.0048	0.0048
	$\rho = 0.2$	Estimate	0.5681	0.5681
		S.e.	0.0047	0.0047
	$\rho = 0.5$	Estimate	0.6701	0.6700
		S.e.	0.0043	0.0043
	$\rho = 0.8$	Estimate	0.7717	0.7717
		S.e.	0.0030	0.0030

Table 10: β_1 coefficient estimates - Δx weighted full-pairwise MC, $x_i \sim$ U(-5,5)

Full-Pairwise MC	Estimates and MC standard errors			
	Parameter	Estimate/S.e.	OLS	pairwise
n=50	Exogen	Estimate	0.4986	0.4988
		S.e.	0.0492	0.0494
	$\rho = 0.2$	Estimate	0.5664	0.5673
		S.e.	0.0440	0.0442
	$\rho = 0.5$	Estimate	0.6738	0.6726
		S.e.	0.0437	0.0440
	$\rho = 0.8$	Estimate	0.7910	0.7970
		S.e.	0.0262	0.0265
n=500	Exogen	Estimate	0.5000	0.5000
		S.e.	0.0157	0.0157
	$\rho = 0.2$	Estimate	0.5690	0.5692
		S.e.	0.0145	0.0145
	$\rho = 0.5$	Estimate	0.6708	0.6705
		S.e.	0.0136	0.0136
	$\rho = 0.8$	Estimate	0.7726	0.7715
		S.e.	0.0092	0.0092
n=5000	Exogen	Estimate	0.5000	0.5000
		S.e.	0.0047	0.0047
	$\rho = 0.2$	Estimate	0.5679	0.5678
		S.e.	0.0048	0.0048
	$\rho = 0.5$	Estimate	0.6691	0.6691
		S.e.	0.0042	0.0042
	$\rho = 0.8$	Estimate	0.7688	0.7690
		S.e.	0.0029	0.0029

Table 11: β_1 coefficient estimates - $|\Delta x|$ weighted full-pairwise MC, $x_i \sim$ U(-5,5)

REFERENCES

Azzalini, A. (1985). A Class of Distributions Which Includes the Normal Ones. *12*(2), 171–178.

Cambanis, S. (1977, December). Some properties and generalizations of multivariate Eyraud-Gumbel-Morgenstern distributions. *Journal of Multivariate Analysis 7*(4), 551–559.

Conway, D. A. (2005, December). Farlie–Gumbel–Morgenstern Distributions. In S. Kotz, C. B. Read, N. Balakrishnan, and B. Vidakovic (Eds.), *Encyclopedia of Statistical Sciences* (2 ed.). Wiley.