T.D.E. 17/02/2023 - ESERCIZIO 2

$$\underline{y}'(t) = M\,\underline{y}(t) \qquad M = \begin{pmatrix} -3 & 10 \\ -2 & 6 \end{pmatrix}$$

(i) e^{tM}

(ii) integrale gen. sist. omog. e soluzione del pb. di Cauchy $\underline{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(iii) integrale gen. sist. non omog.

$$\underline{y}'(t) = M\,\underline{y}(t) + \begin{pmatrix} 2e^t \\ e^t \end{pmatrix}$$

e soluz. del pb. di Cauchy $\underline{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(i) Calcolo autovalori e autovettori:

$$\det\begin{pmatrix} -3-\lambda & 10 \\ -2 & 6-\lambda \end{pmatrix} = (-3-\lambda)(6-\lambda) + 20$$

$$= \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$$

autovalori $\lambda_1 = 1$, $\lambda_2 = 2$

Autospazio relativo a $\lambda_1 = 1$:

$$\begin{pmatrix} -4 & 10 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 2x = 5y$$

$$\left\{ \underline{v} = \begin{pmatrix} 5t \\ 2t \end{pmatrix}, t \in \mathbb{R} \right\} \qquad \underline{v}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Autospazio relativo a $\lambda_2 = 2$:

$$\begin{pmatrix} -5 & 10 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x = 2y$$

$$\left\{ \underline{v} = \begin{pmatrix} 2t \\ t \end{pmatrix},\ t \in \mathbb{R} \right\} \qquad \underline{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad e^{t\Lambda} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

$$S^{-1} = \frac{1}{5-4} \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$$

$$e^{tM} = S\, e^{t\Lambda} S^{-1} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^t & -2e^t \\ -2e^{2t} & 5e^{2t} \end{pmatrix} = \begin{pmatrix} 5e^t - 4e^{2t} & -10e^t + 10e^{2t} \\ 2e^t - 2e^{2t} & -4e^t + 5e^{2t} \end{pmatrix}$$

(ii) integrale gen. sist. omog.:

$$y_o(t) = e^{tM} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5e^t - 4e^{2t} & -10e^t + 10e^{2t} \\ 2e^t - 2e^{2t} & -4e^t + 5e^{2t} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} (5e^t - 4e^{2t})c_1 + (-10e^t + 10e^{2t})c_2 \\ (2e^t - 2e^{2t})c_1 + (-4e^t + 5e^{2t})c_2 \end{pmatrix}$$

soluz. pb. Cauchy tramite la formula:

$$y(t) = e^{(t-t_0)M} \cdot y_0 = e^{tM} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6e^{2t} - 5e^t \\ 3e^{2t} - 2e^t \end{pmatrix}$$

oppure tramite sostituzione:

$$\underline{y}_o(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \overset{\text{impongo}}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 1 \end{array}$$

quindi:

$$\underline{y}(t) = \begin{pmatrix} 6e^{2t} - 5e^{t} \\ 3e^{2t} - 2e^{t} \end{pmatrix}$$

che coincide con la sol. trovata prima

(iii) integrale gen. sist. completo: per il teor. di struttura

$$\underline{y}(t) = \underline{y}_o(t) + \underline{y}_p(t)$$

soluz. part. tramite la formula:

$$\underline{y}_p(t) = e^{tM} \int e^{-\tau M} \cdot \underline{b}(\tau)\, d\tau$$

$$e^{-\tau M} \cdot \underline{b}(\tau) = \begin{pmatrix} 5e^{-\tau} - 4e^{-2\tau} & -10e^{-\tau} + 10^{-2\tau} \\ 2e^{-\tau} - 2e^{-2\tau} & -4e^{-\tau} + 5e^{-2\tau} \end{pmatrix} \begin{pmatrix} 2e^{\tau} \\ e^{\tau} \end{pmatrix}$$

$$= \begin{pmatrix} 10 - 8e^{-\tau} - 10 + 10e^{-\tau} \\ 4 - 4e^{-\tau} - 4 + 5e^{-\tau} \end{pmatrix} = \begin{pmatrix} 2e^{-\tau} \\ e^{-\tau} \end{pmatrix}$$

$$\int e^{-\tau M} \cdot \underline{b}(\tau)\, d\tau = \begin{pmatrix} \int 2e^{-\tau}\, d\tau \\ \int e^{-\tau}\, d\tau \end{pmatrix} = \begin{pmatrix} -2e^{-t} \\ -e^{-t} \end{pmatrix}$$

$$\underline{y}_p(t) = \begin{pmatrix} 5e^{t} - 4e^{2t} & -10e^{t} + 10e^{2t} \\ 2e^{t} - 2e^{2t} & -4e^{t} + 5e^{2t} \end{pmatrix} \cdot \begin{pmatrix} -2e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -2e^{t} \\ -e^{t} \end{pmatrix}$$

oppure soluz. part. tramite metodo di somiglianza:

$$z(t)=\begin{pmatrix}Ae^t\\Be^t\end{pmatrix} \qquad z'(t)=\begin{pmatrix}Ae^t\\Be^t\end{pmatrix} \qquad z'=Mz+\begin{pmatrix}2e^t\\e^t\end{pmatrix}$$

$$\begin{pmatrix}Ae^t\\Be^t\end{pmatrix}=\begin{pmatrix}-3 & 10\\-2 & 6\end{pmatrix}\begin{pmatrix}Ae^t\\Be^t\end{pmatrix}+\begin{pmatrix}2e^t\\e^t\end{pmatrix}$$

$$=\begin{pmatrix}-3Ae^t+10Be^t\\-2Ae^t+6Be^t\end{pmatrix}+\begin{pmatrix}2e^t\\e^t\end{pmatrix}=\begin{pmatrix}(-3A+10B+2)e^t\\(-2A+6B+1)e^t\end{pmatrix}$$

$$\begin{cases}A=-3A+10B+2\\B=-2A+6B+1\end{cases} \iff 2A=5B+1$$

ad esempio: $B=1$, $A=3$

soluz. part. $y_p(t)=\begin{pmatrix}3e^t\\e^t\end{pmatrix}$

Prendendo invece $A=-2$, $B=-1$ si ritrova la soluzione determinata con il metodo precedente.

Soluzione del problema di Cauchy tramite la formula:

$$y(t)=e^{tM}\int_{t_0}^{t}e^{-\tau M}\cdot \underline{b}(\tau)\,d\tau+e^{(t-t_0)M}y_0$$

$$=e^{tM}\int_0^t\begin{pmatrix}2e^{-\tau}\\e^{-\tau}\end{pmatrix}d\tau+e^{tM}y_0$$

$$=e^{tM}\cdot\begin{pmatrix}2(1-e^{-t})\\1-e^{-t}\end{pmatrix}+e^{tM}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \begin{pmatrix} 5e^t - 4e^{2t} & -10e^t + 10e^{2t} \\ 2e^t - 2e^{2t} & -4e^t + 5e^{2t} \end{pmatrix} \cdot \begin{pmatrix} 2(1-e^{-t}) \\ 1 - e^{-t} \end{pmatrix} +$$

$$+ \begin{pmatrix} 5e^t - 4e^{2t} & -10e^t + 10e^{2t} \\ 2e^t - 2e^{2t} & -4e^t + 5e^{2t} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2e^t + 2e^{2t} \\ -e^t + e^{2t} \end{pmatrix} + \begin{pmatrix} 6e^{2t} - 5e^t \\ 3e^{2t} - 2e^t \end{pmatrix} = \begin{pmatrix} 8e^{2t} - 7e^t \\ 4e^{2t} - 3e^t \end{pmatrix}$$

Soluzione del problema di Cauchy tramite sostituzione: abbiamo determinato l'int. gen. del sist. completo

$$\begin{pmatrix} (5e^t - 4e^{2t})c_1 + (-10e^t + 10e^{2t})c_2 \\ (2e^t - 2e^{2t})c_1 + (-4e^t + 5e^{2t})c_2 \end{pmatrix} + \begin{pmatrix} -2e^t \\ -e^t \end{pmatrix}$$

sostituiamo $t=0$ e imponiamo le cond. iniz.

$$\begin{pmatrix} c_1 - 2 \\ c_2 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{matrix} c_1 = 3 \\ c_2 = 2 \end{matrix}$$

quindi:

$$\begin{pmatrix} 3(5e^t - 4e^{2t}) + 2(-10e^t + 10e^{2t}) - 2e^t \\ 3(2e^t - 2e^{2t}) + 2(-4e^t + 5e^{2t}) - e^t \end{pmatrix} =$$

$$\begin{pmatrix} 8e^{2t} - 7e^t \\ 4e^{2t} - 3e^t \end{pmatrix}$$

che coincide con la sol. trovata prima