Esercizi sugli integrali tripli

Esercizio 1. Calcolare i seguenti integrali tripli:

1) $\displaystyle\int_\Omega xy^2z^3\,dx\,dy\,dz, \qquad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ 0\le x\le 1,\ \ 1\le y\le 3,\ \ 2\le z\le 3\right\}.$

2) $\displaystyle\int_\Omega 1\,dx\,dy\,dz, \qquad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ x+y+z\le 1,\ \ x\ge 0,\ \ y\ge 0,\ \ z\ge 0\right\}.$

3) $\displaystyle\int_\Omega z\,dx\,dy\,dz, \qquad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ x^2+y^2+z^2\le 2,\ \ z\ge\sqrt{x^2+y^2}\right\}.$

4) $\displaystyle\int_\Omega \frac{x}{x^2+z^2}\,dx\,dy\,dz, \qquad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ 1\le x^2+z^2\le 2x,\ \ 0\le y\le\sqrt{x^2+z^2}\right\}.$

5) $\displaystyle\int_\Omega 1\,dx\,dy\,dz, \quad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ x^2+y^2\le 9,\ \ 9x^2+y^2\ge 9,\ \ -x\le y\le 0,\ \ 0\le z\le 1+\frac{8x^2}{x^2+y^2}\right\}.$

6) $\displaystyle\int_\Omega \left(x^2+z^2\right)y\,dx\,dy\,dz, \qquad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ x^2+y^2+z^2\le 2,\ \ y\ge x^2+z^2\right\}.$

7) $\displaystyle\int_\Omega y\,dx\,dy\,dz, \qquad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ x^2+z^2\le y\le 1+x^2+z^2,\ \ y\le 2\right\}.$

8) $\displaystyle\int_\Omega \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\,dx\,dy\,dz, \qquad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ x^2+y^2+z^2\le 36,\ \ x\le -3\right\}.$

9) $\displaystyle\int_\Omega x\,dx\,dy\,dz, \qquad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ \sqrt{y^4+y^2+4}\le x\le\sqrt{y^4+2y^2+z^2},\ \ y^2+z^2\le 16\right\}.$

10) $\displaystyle\int_\Omega x\left(y^2+z^2\right)\,dx\,dy\,dz, \qquad \Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ x^2+y^2+z^2\ge 3,\ \ 0<x\le 3-y^2-z^2\right\}.$

SVOLGIMENTO

1) Consideriamo l'integrale $\int_\Omega xy^2z^3\,dx\,dy\,dz$, dove

$$\Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ 0\le x\le 1,\ \ 1\le y\le 3,\ \ 2\le z\le 3\right\}.$$

L'insieme Ω è un parallelepipedo con spigoli paralleli agli assi coordinati e la funzione integranda è prodotto di una funzione di x, di una di y e di una di z. Quindi l'integrale si può calcolare nel seguente modo:

$$\int_\Omega xy^2z^3\,dx\,dy\,dz = \left(\int_0^1 x\,dx\right)\left(\int_1^3 y^2\,dy\right)\left(\int_2^3 z^3\,dz\right) = \left[\frac{1}{2}x^2\right]_0^1\left[\frac{1}{3}y^3\right]_1^3\left[\frac{1}{4}z^4\right]_2^3 = \frac{845}{12}.$$

2) Consideriamo l'integrale $\int_\Omega 1\,dx\,dy\,dz$, dove

$$\Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ x+y+z\le 1,\ \ x\ge 0,\ \ y\ge 0,\ \ z\ge 0\right\}.$$

L'insieme Ω è un tetraedro. Integrando per fili paralleli all'asse z si ottiene

$$\int_\Omega 1\,dx\,dy\,dz = \int_D\left(\int_0^{1-x-y} 1\,dz\right)dx\,dy = \int_D\Big[z\Big]_0^{1-x-y}dx\,dy =$$

$$= \int_D (1-x-y)dx\,dy,$$

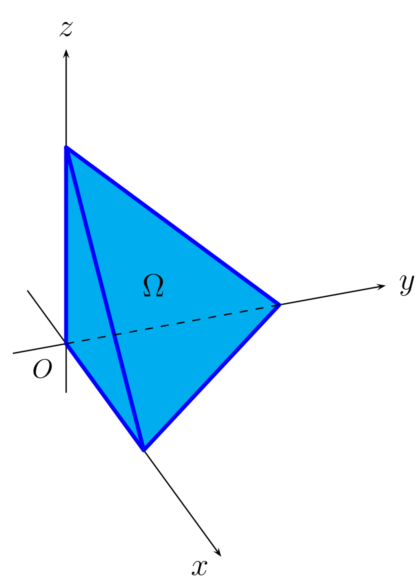
dove $D=\{(x,y)\in\mathbb{R}^2:\ \ 0\le x\le 1,\ 0\le y\le 1-x\}$.

Essendo D un insieme y-semplice, si ottiene

$$\int_\Omega 1\,dx\,dy\,dz = \int_0^1\left(\int_0^{1-x}(1-x-y)\,dy\right)dx =$$

$$= \int_0^1\left[(1-x)y-\frac{1}{2}y^2\right]_0^{1-x}dx = \frac{1}{2}\int_0^1(1-x)^2\,dx =$$

$$= \frac{1}{2}\left[-\frac{1}{3}(1-x)^3\right]_0^1 = \frac{1}{6}.$$

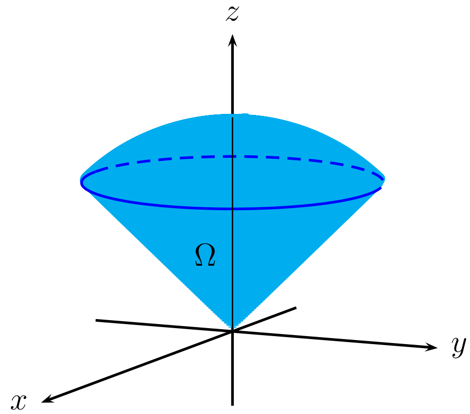


3) Consideriamo l'integrale $\int_\Omega z\,dx\,dy\,dz$, dove

$$\Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ x^2+y^2+z^2\le 2,\ \ z\ge\sqrt{x^2+y^2}\right\}.$$

Passiamo in coordinate polari nello spazio con la colatitudine misurata dall'asse z. Poniamo quindi

$$\Phi : \begin{cases} x = \rho \sin\vartheta \cos\varphi \\ y = \rho \sin\vartheta \sin\varphi, \\ z = \rho\cos\vartheta, \end{cases} \qquad \begin{array}{l} \rho \geq 0,\ 0 \leq \vartheta \leq \pi,\ 0 \leq \varphi \leq 2\pi, \\ \\ |\det J_\Phi(\rho,\vartheta,\varphi)| = \rho^2 \sin\vartheta. \end{array}$$



Allora

$$(x,y,z) \in \Omega \quad \Longleftrightarrow \quad \begin{cases} 0 \leq \rho \leq \sqrt{2} \\ 0 \leq \vartheta \leq \dfrac{\pi}{4} \\ 0 \leq \varphi \leq 2\pi. \end{cases}$$

Quindi si ha che $\Omega = \Phi(\Omega')$, dove

$$\Omega' = \left\{ (\rho,\vartheta,x) \in \mathbb{R}^3 : \ 0 \leq \rho \leq \sqrt{2},\ 0 \leq \vartheta \leq \frac{\pi}{4},\ 0 \leq \varphi \leq 2\pi \right\}.$$

Si ha che

$$\int_\Omega z\, dx\, dy\, dz = \int_{\Omega'} \rho^3 \sin\vartheta \cos\vartheta \, d\rho\, d\vartheta\, d\varphi =$$

essendo Ω' un parallelepipedo con spigoli paralleli agli assi coordinati e la funzione integranda è prodotto di una funzione di ρ, di una di ϑ e di una di φ, l'integrale si può calcolare come

$$= \left(\int_0^{\sqrt{2}} \rho^3\, d\rho \right) \left(\int_0^{\frac{\pi}{4}} \sin\vartheta\cos\vartheta\, d\vartheta \right) \left(\int_0^{2\pi} 1\, d\varphi \right) = 2\pi \left[\frac{1}{4}\rho^4 \right]_0^{\sqrt{2}} \left[\frac{1}{2}\sin^2\vartheta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{2}.$$

4) Consideriamo l'integrale $\displaystyle\int_\Omega \frac{x}{x^2+z^2}\, dx\, dy\, dz$, dove

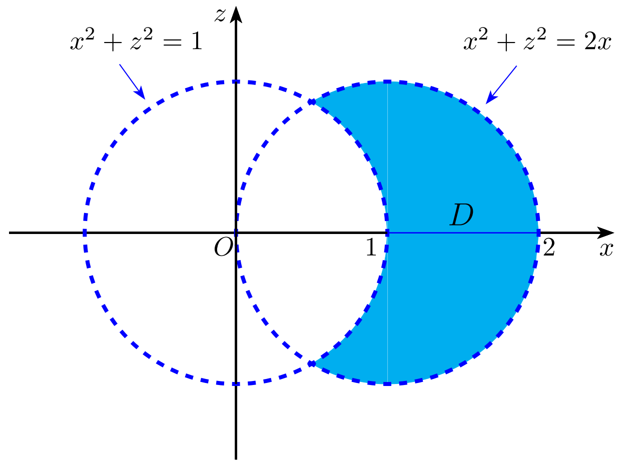
$$\Omega = \left\{ (x,y,z) \in \mathbb{R}^3 : \ 1 \leq x^2 + z^2 \leq 2x,\ 0 \leq y \leq \sqrt{x^2+z^2} \right\}.$$

Integrando per fili paralleli all'asse y si ha che

$$\int_\Omega \frac{x}{x^2+z^2}\, dx\, dy\, dz = \int_D \left(\int_0^{\sqrt{x^2+z^2}} \frac{x}{x^2+z^2}\, dy \right) dx\, dz = \int_D \frac{x}{\sqrt{x^2+z^2}}\, dx\, dz,$$

dove $D = \left\{ (x,z) \in \mathbb{R}^2 : \ 1 \leq x^2 + z^2 \leq 2x \right\}$.

Passiamo in coordinate polari nel piano xz centrate in $(0,0)$. Poniamo quindi



$$\Phi : \begin{cases} x = \rho\cos\vartheta \\ z = \rho\sin\vartheta, \end{cases} \qquad \begin{array}{l} \rho \ge 0,\ -\pi \le \vartheta \le \pi, \\ |\det J_\Phi(\rho,\vartheta)| = \rho. \end{array}$$

Allora

$$(x,z) \in D \quad \Longleftrightarrow \quad \begin{cases} 1 \le \rho \le 2\cos\vartheta \\ -\dfrac{\pi}{3} \le \vartheta \le \dfrac{\pi}{3}. \end{cases}$$

Quindi si ha che $D = \Phi(D')$, dove

$$D' = \left\{(\rho,\vartheta) \in \mathbb{R}^3 :\ -\frac{\pi}{3} \le \vartheta \le \frac{\pi}{3},\ 1 \le \rho \le 2\cos\vartheta\right\}.$$

Si ha che

$$\int_\Omega \frac{x}{x^2+z^2}\,dx\,dy\,dz = \int_D \frac{x}{\sqrt{x^2+z^2}}\,dx\,dz = \int_{D'} \rho\cos\vartheta\,d\rho\,d\vartheta =$$

essendo D' un insieme ρ-semplice, si ha

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\int_1^{2\cos\vartheta} \rho\cos\vartheta\,d\rho\right) d\vartheta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\vartheta \left[\frac{1}{2}\rho^2\right]_1^{2\cos\vartheta} d\vartheta = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\vartheta\left(4\cos^2\vartheta - 1\right) d\vartheta =$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(3\cos\vartheta - 4\cos\vartheta\sin^2\vartheta\right) d\vartheta = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[3\sin\vartheta - \frac{4}{3}\sin^3\vartheta\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \sqrt{3}.$$

5) Consideriamo l'integrale $\displaystyle\int_\Omega 1\,dx\,dy\,dz$, dove

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2 \le 9,\ 9x^2+y^2 \ge 9,\ -x \le y \le 0,\ 0 \le z \le 1 + \frac{8x^2}{x^2+y^2}\right\}.$$

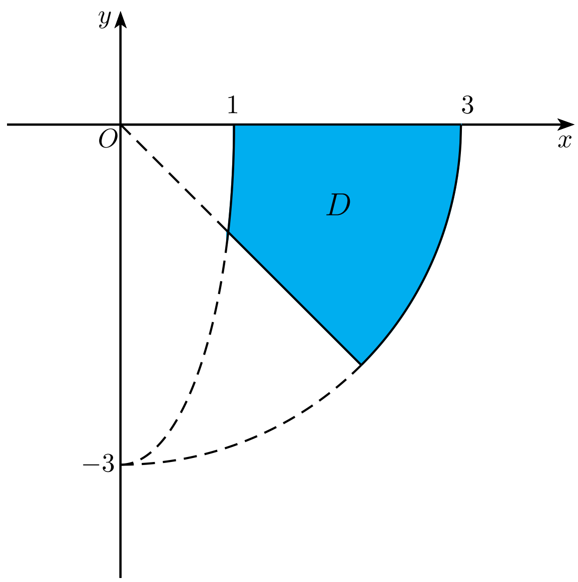
Integrando per fili paralleli all'asse z si ottiene

$$\int_\Omega 1\,dx\,dy\,dz = \int_D \left(1 + \frac{8x^2}{x^2+y^2}\right) dx\,dy,$$

dove $D = \{(x,y) \in \mathbb{R}^2 :\ x^2+y^2 \le 9,\ 9x^2+y^2 \ge 9,\ -x \le y \le 0\}$.

Allora

$$(x,y) \in D \implies \begin{cases} x^2+y^2 \le 9 \\ 9x^2+y^2 \ge 9 \\ -x \le y \le 0 \end{cases} \implies \begin{cases} \rho \le 3 \\ \rho^2\left(9\cos^2\vartheta + \sin^2\vartheta\right) \ge 9 \\ -\cos\vartheta \le \sin\vartheta \le 0 \end{cases} \implies \begin{cases} \dfrac{3}{\sqrt{9\cos^2\vartheta + \sin^2\vartheta}} \le \rho \le 3 \\ -\frac{\pi}{4} \le \vartheta \le 0. \end{cases}$$



L'insieme D è l'insieme dei punti IV quadrante compresi fra l'ellisse di equazione $x^2 + \frac{y^2}{9} = 1$, la circonferenza $x^2 + y^2 = 9$, la retta $y = -x$ e l'asse x.

Passiamo in coordinate polari nel piano. Poniamo quindi

$$\Phi : \begin{cases} x = \rho \cos \vartheta \\ y = \rho \sin \vartheta, \end{cases} \qquad \rho \geq 0, \; -\pi \leq \vartheta \leq \pi, \qquad |\det J_\Phi(\rho, \vartheta)| = \rho.$$

Poiché $9\cos^2\vartheta + \sin^2\vartheta = 8\cos^2\vartheta + \cos^2\vartheta + \sin^2\vartheta = 8\cos^2\vartheta + 1$, si ha che

$$D' = \left\{(\rho, \vartheta) \in \mathbb{R}^2 : \; \frac{3}{\sqrt{8\cos^2\vartheta + 1}} \leq \rho \leq 3, \; -\frac{\pi}{4} \leq \vartheta \leq 0\right\}.$$

Essendo D' un insieme ρ semplice, si ha che

$$\int_D \left(1 + \frac{8x^2}{x^2+y^2}\right) dx\, dy = \int_{D'} \left(1 + 8\cos^2\vartheta\right) \rho\, d\rho\, d\vartheta = \int_{-\frac{\pi}{4}}^{0} \left(\int_{\frac{3}{\sqrt{8\cos^2\vartheta+1}}}^{3} \left(1 + 8\cos^2\vartheta\right) \rho\, d\rho\right) d\vartheta =$$

$$= \int_{-\frac{\pi}{4}}^{0} \left(1 + 8\cos^2\vartheta\right) \left[\frac{1}{2}\rho^2\right]_{\frac{3}{\sqrt{8\cos^2\vartheta+1}}}^{3} d\vartheta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{0} \left(1 + 8\cos^2\vartheta\right) \left(9 - \frac{9}{8\cos^2\vartheta + 1}\right) d\vartheta =$$

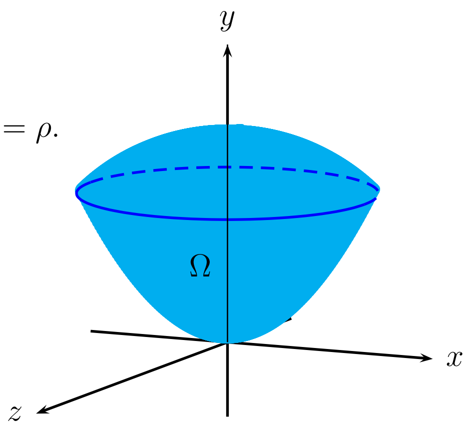
$$= 36 \int_{-\frac{\pi}{4}}^{0} \cos^2\vartheta\, d\vartheta = 36 \left[\frac{1}{2}(\vartheta + \sin\vartheta\cos\vartheta)\right]_{-\frac{\pi}{4}}^{0} = \frac{9}{2}\pi + 9.$$

6) Consideriamo l'integrale $\int_\Omega \left(x^2 + z^2\right) y\, dx\, dy\, dz$, dove

$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 + z^2 \leq 2, \; y \geq x^2 + z^2\right\}.$$

Passiamo in coordinate cilindriche con asse parallelo all'asse y. Poniamo quindi

$$\Phi : \begin{cases} x = \rho\cos\vartheta \\ y = y, \\ z = \rho\sin\vartheta, \end{cases} \qquad \rho \geq 0,\ 0 \leq \vartheta \leq 2\pi,\ y \in \mathbb{R} \qquad |\det J_\Phi(\rho,\vartheta,y)| = \rho.$$



Allora

$$(x,y,z) \in \Omega \quad \Longleftrightarrow \quad \begin{cases} \rho^2 \leq y \leq \sqrt{2-\rho^2} \\ 0 \leq \rho \leq 1 \\ 0 \leq \vartheta \leq 2\pi. \end{cases}$$

Quindi si ha che $\Omega = \Phi(\Omega')$, dove

$$\Omega' = \left\{(\rho,\vartheta,y) \in \mathbb{R}^3 :\ \ \rho^2 \leq y \leq \sqrt{2-\rho^2}, \quad 0 \leq \vartheta \leq 2\pi,\ \ 0 \leq \rho \leq 1\right\}.$$

Si ha che

$$\int_\Omega \left(x^2+z^2\right) y\,dx\,dy\,dz = \int_{\Omega'} y\rho^3\,d\rho\,d\vartheta\,dy = 2\pi\int_0^1 \rho^3\left(\int_{\rho^2}^{\sqrt{2-\rho^2}} y\,dy\right)d\rho = 2\pi\int_0^1 \rho^3\left[\frac{1}{2}y^2\right]_{\rho^2}^{\sqrt{2-\rho^2}} d\rho =$$

$$= \pi\int_0^1 \left(2\rho^3-\rho^5-\rho^7\right)d\rho = \pi\left[\frac{1}{2}\rho^4-\frac{1}{6}\rho^6-\frac{1}{8}\rho^8\right]_0^1 = \frac{5}{24}\pi.$$

7) Consideriamo l'integrale $\displaystyle\int_\Omega y\,dx\,dy\,dz$, dove

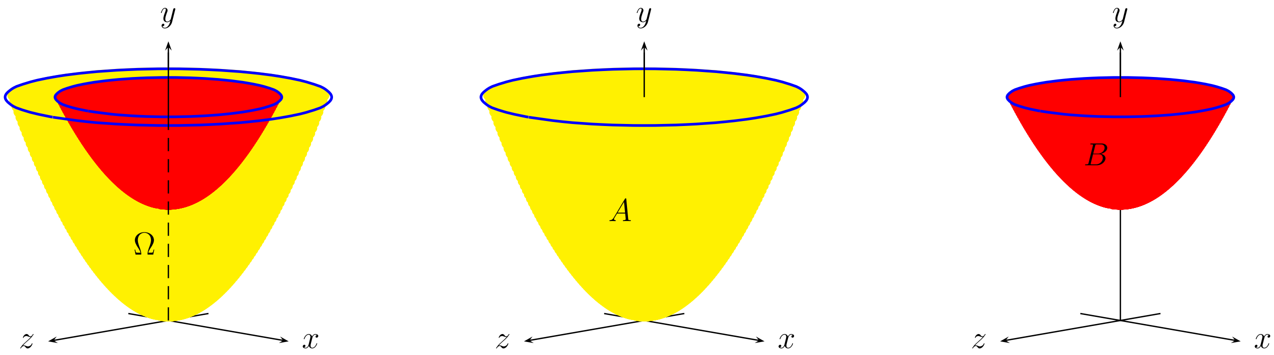
$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 :\ \ x^2+z^2 \leq y \leq 1+x^2+z^2,\ \ y \leq 2\right\}.$$

Osserviamo che $\Omega = A \setminus B$, dove

$$A = \left\{(x,y,z) \in \mathbb{R}^3 :\ \ x^2+z^2 \leq y \leq 2\right\}, \qquad B = \left\{(\rho,\vartheta,y) \in \mathbb{R}^3 :\ \ 1+x^2+z^2 < y < 2\right\}.$$

Ne segue che

$$\int_\Omega y\,dx\,dy\,dz \int_A y\,dx\,dy\,dz - \int_B y\,dx\,dy\,dz.$$



Passiamo in coordinate cilindriche con asse parallelo all'asse y. Poniamo quindi

$$\Phi : \begin{cases} x = \rho\cos\vartheta \\ y = y \\ z = \rho\sin\vartheta, \end{cases} \qquad \rho \geq 0,\ 0 \leq \vartheta \leq 2\pi,\ y \in \mathbb{R}, \quad |\det J_\Phi(\rho,\vartheta,y)| = \rho.$$

Quindi $A = \Phi(A')$ e $B = \Phi(B')$, dove

$$A' = \left\{(\rho,\vartheta,y) \in \mathbb{R}^3 :\ 0 \leq \rho \leq \sqrt{2},\ 0 \leq \vartheta \leq 2\pi,\ \ \rho^2 \leq y \leq 2\right\},$$

$$B' = \left\{(\rho,\vartheta,y) \in \mathbb{R}^3 :\ 0 \leq \rho \leq 1,\ 0 \leq \vartheta \leq 2\pi,\ \ 1+\rho^2 < y < 2\right\}.$$

Ne segue che

$$\int_\Omega y\,dx\,dy\,dz = \int_A y\,dx\,dy\,dz - \int_B y\,dx\,dy\,dz = \int_{A'} \rho y\,d\rho\,d\vartheta\,dy - \int_{B'} \rho y\,d\rho\,d\vartheta\,dy =$$

$$= 2\pi \int_0^{\sqrt{2}} \rho \left(\int_{\rho^2}^2 y\,dy\right) d\rho - 2\pi \int_0^1 \rho \left(\int_{1+\rho^2}^2 y\,dy\right) d\rho =$$

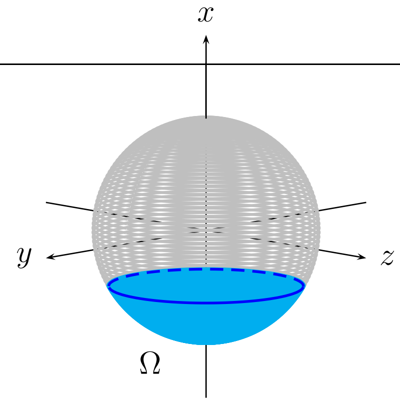
$$= 2\pi \left(\int_0^{\sqrt{2}} \rho \left[\frac{1}{2}y^2\right]_{\rho^2}^2 d\rho - \int_0^1 \rho \left[\frac{1}{2}y^2\right]_{1+\rho^2}^2 d\rho\right) =$$

$$= \pi \left(\int_0^{\sqrt{2}} \left(4\rho - \rho^5\right) d\rho - \int_0^1 \left(3\rho - 2\rho^3 - \rho^5\right) d\rho\right) =$$

$$= \pi \left(\left[2\rho^2 - \frac{1}{6}\rho^6\right]_0^{\sqrt{2}} - \left[\frac{3}{2}\rho^2 - \frac{1}{2}\rho^4 - \frac{1}{6}\rho^6\right]_0^1\right) = \frac{11}{6}\pi.$$

8) Consideriamo l'integrale $\displaystyle\int_\Omega \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz$, dove

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 \leq 36,\ \ x \leq -3\right\}.$$



Passiamo in coordinate polari centrate in $(0,0,0)$ con la colatitudie misurata dall'asse x. Poniamo

$$\Phi: \begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta\cos\varphi \\ z = \rho\sin\vartheta\sin\varphi \end{cases} \quad \rho \geq 0, \quad 0 \leq \vartheta \leq \pi, \quad 0 \leq \varphi \leq 2\pi, \qquad |\det J_\Phi(\rho,\vartheta,\varphi)| = \rho^2\sin\vartheta.$$

Ne segue che

$$\int_\Omega \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz = \int_{\Omega'} \cos\vartheta\sin\vartheta\,d\rho\,d\vartheta\,d\varphi,$$

dove $\Omega' \subseteq \mathbb{R}^3$ è tale che $\Phi(\Omega') = \Omega$. Si ha che

$$(x,y,z)\in\Omega \quad \Longrightarrow \quad \begin{cases} x^2+y^2+z^2 \leq 36 \\ x \leq -3 \end{cases} \quad \Longrightarrow \quad \begin{cases} \rho^2 \leq 36 \\ \rho\cos\vartheta \leq -3 \\ \rho \geq 0 \\ 0 \leq \vartheta \leq \pi \\ 0 \leq \varphi \leq 2\pi \end{cases} \quad \Longrightarrow \quad \begin{cases} -\dfrac{3}{\cos\vartheta} \leq \rho \leq 6 \\ \dfrac{2}{3}\pi \leq \vartheta \leq \pi \\ 0 \leq \varphi \leq 2\pi. \end{cases}$$

Quindi

$$\Omega' = \left\{(\rho,\vartheta,\varphi)\in\mathbb{R}^3: \ -\frac{3}{\cos\vartheta} \leq \rho \leq 6, \ \frac{2}{3}\pi \leq \vartheta \leq \pi, \ 0 \leq \varphi \leq 2\pi\right\}.$$

Ne segue che

$$\int_\Omega \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz = \int_{\Omega'} \cos\vartheta\sin\vartheta\,d\rho\,d\vartheta\,d\varphi = 2\pi\int_{\frac{2}{3}\pi}^{\pi}\left(\int_{-\frac{3}{\cos\vartheta}}^{6} \cos\vartheta\sin\vartheta\,d\rho\right)d\vartheta =$$

$$= 2\pi\int_{\frac{2}{3}\pi}^{\pi} (6\cos\vartheta\sin\vartheta + 3\sin\vartheta)\,d\vartheta = 2\pi\Big[3\sin^2\vartheta - 3\cos\vartheta\Big]_{\frac{2}{3}\pi}^{\pi} = -\frac{3}{2}\pi.$$

<u>Alternativa per il calcolo dell'integrale</u>: passare in coordinate cilindriche con asse parallelo all'asse x.

Poniamo

$$\Phi: \begin{cases} x = x \\ y = \rho\cos\vartheta \\ z = \rho\sin\vartheta \end{cases} \quad \rho \geq 0, \quad 0 \leq \vartheta \leq 2\pi, \quad x \in \mathbb{R}, \qquad |\det J_\Phi(\rho,\vartheta,x)| = \rho.$$

Ne segue che

$$\int_\Omega \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz = \int_{\Omega'} \frac{x}{(\rho^2+x^2)^{3/2}}\rho\,d\rho\,d\vartheta\,dx,$$

dove $\Omega' \subseteq \mathbb{R}^3$ è tale che $\Phi(\Omega') = \Omega$. Si ha che

$$(x,y,z)\in\Omega \quad\Longrightarrow\quad \begin{cases} x^2+y^2+z^2 \le 36 \\ x \le -3 \end{cases} \quad\Longrightarrow\quad \begin{cases} \rho^2+x^2 \le 36 \\ x\le -3 \\ \rho \ge 0 \\ 0\le\vartheta\le 2\pi \end{cases} \quad\Longrightarrow\quad \begin{cases} 0\le\rho\le\sqrt{36-x^2} \\ -6\le x\le -3 \\ 0\le\vartheta\le 2\pi. \end{cases}$$

Quindi

$$\Omega' = \left\{(\rho,\vartheta,x)\in\mathbb{R}^3:\ \ 0\le\rho\le\sqrt{36-x^2},\ -6\le x\le -3,\ 0\le\vartheta\le 2\pi\right\}.$$

Ne segue che

$$\int_\Omega \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz = \int_{\Omega'} \frac{x}{(\rho^2+x^2)^{3/2}}\rho\,d\rho\,d\vartheta\,dx = 2\pi\int_{-6}^{-3} x\left(\int_0^{\sqrt{36-x^2}} \rho\left(\rho^2+x^2\right)^{-3/2}\,d\rho\right)dx =$$

$$= 2\pi\int_{-6}^{-3} x\left[-\left(\rho^2+x^2\right)^{-1/2}\right]_0^{\sqrt{36-x^2}}\,dx = -2\pi\int_{-6}^{-3} x\left(\frac{1}{6}+\frac{1}{x}\right)dx = -2\pi\int_{-6}^{-3}\left(\frac{1}{6}x+1\right)dx =$$

$$= -2\pi\left[\frac{1}{12}x^2+x\right]_{-6}^{-3} = -\frac{3}{2}\pi.$$

9) Consideriamo l'integrale $\displaystyle\int_\Omega x\,dx\,dy\,dz$, dove

$$\Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ \sqrt{y^4+y^2+4}\le x\le\sqrt{y^4+2y^2+z^2},\ \ y^2+z^2\le 16\right\}.$$

Osserviamo che

$$\sqrt{y^4+y^2+4}\le\sqrt{y^4+2y^2+z^2} \quad\Longleftrightarrow\quad y^2+z^2\ge 4.$$

Quindi

$$\Omega = \left\{(x,y,z)\in\mathbb{R}^3:\ \ \sqrt{y^4+y^2+4}\le x\le\sqrt{y^4+2y^2+z^2},\ \ 4\le y^2+z^2\le 16\right\}.$$

Integrando per fili paralleli all'asse x si ottiene

$$\int_\Omega x\,dx\,dy\,dz = \int_D\left(\int_{\sqrt{y^4+y^2+4}}^{\sqrt{y^4+2y^2+z^2}} x\,dx\right)dy\,dz = \int_D\left[\frac{1}{2}x^2\right]_{\sqrt{y^4+y^2+4}}^{\sqrt{y^4+2y^2+z^2}}\,dy\,dz =$$

$$= \frac{1}{2}\int_D\left(y^2+z^2-4\right)dx\,dy,$$

dove $D = \{(y,z) \in \mathbb{R}^2 : \ 4 \le y^2 + z^2 \le 16\}$.

Passando in coordinate polari nel piano yz si ottiene

$$\int_\Omega x \, dx \, dy \, dz = \frac{1}{2} \int_{D'} \rho \left(\rho^2 - 4\right) \, d\rho \, d\vartheta =$$

con $D' = [2,4] \times [0, 2\pi]$

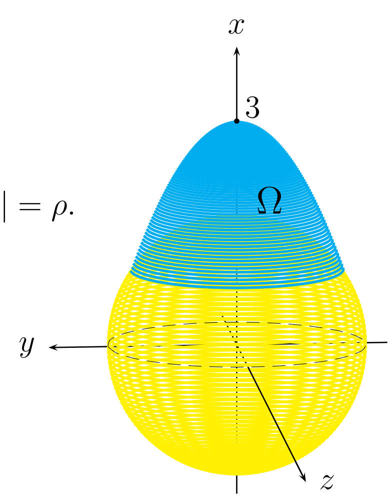
$$= \pi \int_2^4 \rho \left(\rho^2 - 4\right) \, d\rho = \pi \left[\frac{1}{4} \left(\rho^2 - 4\right)^2 \right]_2^4 = 36\pi.$$

10) Consideriamo l'integrale $\displaystyle\int_\Omega x \left(y^2 + z^2\right) \, dx \, dy \, dz$, dove

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 \ge 3, \ \ 0 < x \le 3 - y^2 - z^2 \right\}.$$

Passiamo in coordinate cilindriche con asse parallelo all'asse x. Poniamo quindi

$$\Phi : \begin{cases} x = x \\ y = \rho \cos\vartheta, \\ z = \rho \sin\vartheta, \end{cases} \quad \rho \ge 0, \ 0 \le \vartheta \le 2\pi, \ x \in \mathbb{R} \quad |\det J_\Phi(\rho, \vartheta, x)| = \rho.$$



Allora

$$(x,y,z) \in \Omega \quad \Longleftrightarrow \quad \begin{cases} \sqrt{3 - \rho^2} \le x \le 3 - \rho^2 \\ 0 \le \rho \le \sqrt{2} \\ 0 \le \vartheta \le 2\pi. \end{cases}$$

Quindi si ha che $\Omega = \Phi(\Omega')$, dove

$$\Omega' = \left\{(\rho, \vartheta, x) \in \mathbb{R}^3 : \ \sqrt{3 - \rho^2} \le x \le 3 - \rho^2, \ \ 0 \le \vartheta \le 2\pi, \ 0 \le \rho \le \sqrt{2} \right\}.$$

Si ha che

$$\int_\Omega x \left(y^2 + z^2\right) \, dx \, dy \, dz = \int_{\Omega'} x \rho^3 \, d\rho \, d\vartheta \, dx = 2\pi \int_0^{\sqrt{2}} \rho^3 \left(\int_{\sqrt{3-\rho^2}}^{3-\rho^2} x \, dx \right) d\rho =$$

$$= \pi \int_0^{\sqrt{2}} \rho^3 \left(6 - 5\rho^2 + \rho^4\right) \, d\rho = \pi \int_0^{\sqrt{2}} \left(6\rho^3 - 5\rho^5 + \rho^7\right) \, d\rho = \pi \left[\frac{3}{2}\rho^4 - \frac{5}{6}\rho^6 + \frac{1}{8}\rho^8 \right]_0^{\sqrt{2}} = \frac{4}{3}\pi.$$