WaveFunction of the Universe in Anisotropic Minisuperspace Model

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August 11, 2023

Abstract

In this paper , we study the physical meaning of the wavefunction of the universe when we have scalar field . We will consider a model of gravity coupled to a scalar field ϕ moving in a potential $V(\phi)$ and obtain Wavefunction and density quantity , $\rho(a) = |\psi(a)|^2$ from Wheeler-DeWitt (WDW) equation in the anisotropic minisuperspace model for Slow Roll Inflation. After we investigated about order factor in boundary condition .

1 Introduction

The wavefunction of the universe is a concept that lies at the intersection of quantum mechanics and cosmology. It seeks to describe the entire universe as a single quantum mechanical system, providing insights into the fundamental nature of the universe and the laws that govern its behavior. In the anisotropic minisuperspace model, the universe is simplified to a three-dimensional space with a finite number of degrees of freedom, allowing for a more manageable analysis of the wavefunction. The Wheeler-DeWitt equation is a key tool used to describe the wavefunction of the universe in this model, which is a partial differential equation that describes the evolution of the wavefunction over time. Together, the wavefunction of the universe in the anisotropic minisuperspace model and the Wheeler-DeWitt equation provide a framework for understanding the universe at the smallest scales, where the principles of quantum mechanics and general relativity are both important. In this way, the wavefunction of the universe represents a fascinating and important area of research at the forefront of modern physics.

In Quantum Mechanics , we can completely described properties of particle by the wavefunction , $\Psi(x,t)$ and if we want to find particle at time Δt in interval Δx we can determined probability , $|\Psi(x,t)|^2 \Delta x$ and thus $\rho(x,t) = |\Psi(x,t)|^2$ is density of probability . In Standard Quantum Mechanics we have wavefunction and we can obtain it from Schrödinger equation and the wavefunction have many information about the system we work on it .

Solving the Wheeler-DeWitt equation in the anisotropic minisuperspace model is a challenging task, as it involves finding the wavefunction that satisfies the equation's constraints. This requires understanding the dynamics of the universe and the interplay between quantum mechanics and general relativity in this simplified setting.

By studying the Wheeler-DeWitt equation in the anisotropic minisuperspace model, researchers aim to gain insights into the early universe, the nature of cosmic inflation, and the fundamental laws that govern the evolution of the universe. While the equation's complete solution remains elusive, ongoing research and theoretical investigations continue to shed light on the behavior of the universe in this particular framework.

In Quantum Cosmology theory , the universe is described by a wavefunction $\psi(g_{\mu\nu}, \phi)$ determined by the quantum gravity equation , called the Wheeler-DeWitt (WDW) equation[1] , $\hat{H}\psi(g_{\mu\nu}, \phi) = 0$, where $g_{\mu\nu}$ is the metric and ϕ is a scalar field . In fact, like quantum mechanics, the wave function must contain all the information about the

universe for instance the probability density of the wavefunction of the universe should find a universe somewhere ,e.g.
Slow roll inflation is a theoretical concept in cosmology that describes a period of exponential expansion of the universe that occurred shortly after the Big Bang. During this period, the universe underwent a rapid expansion, which is believed to have smoothed out any irregularities in the distribution of matter and energy.Slow roll inflation is an important concept in cosmology because it helps to explain some of the observed properties of the universe, such as its large-scale homogeneity and isotropy. It also provides a framework for understanding the origin of the cosmic microwave background radiation, which is thought to be a remnant of the Big Bang.
In this paper we study the properties of the wavefunction of the uiverse in the anistropic minisuperspace model , in which the wavefunction of the universe can be determined by two parameters , scale factor (a) and scalar field (ϕ) .

2 Friedmann Equations

The Friedmann equation is a cornerstone of modern cosmology, describing the evolution of the universe as a function of time. It is a set of equations that describe the dynamics of the universe, taking into account the effects of gravity, matter, and energy. In the context of scalar fields, the Friedmann equation takes on a particularly important role, as scalar fields are a key component of many theories of fundamental physics, including string theory and inflationary cosmology. Scalar fields are fields that have a single value at each point in space, and they can be used to describe a wide range of physical phenomena, from the Higgs field that gives particles mass, to the inflaton field that is thought to have driven the rapid expansion of the universe in the earliest moments after the Big Bang. By incorporating scalar fields into the Friedmann equation, we can gain a deeper understanding of the behavior of the universe at the smallest scales, and explore the fundamental laws that govern the behavior of matter and energy. In this way, the Friedmann equation in scalar field represents a fascinating and important area of research at the forefront of modern cosmology and theoretical physics.

At first Einstein-Hilbert action for the model can be written as

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \quad (1)$$

On the other hand the universe is homogeneous and isotropic so the metric of the universe in the minisuperspace model is given by

$$ds^2 = -dt^2 + a^2(t)d\Omega_3^2 \quad (2)$$

where $d\Omega_3^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ is the metric on sphere . From equation2 we can obtain scalar Ricci

$$R = 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}) \quad (3)$$

and

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^{-2} \end{pmatrix} \quad (4)$$

and assume scalar field depends only on time , so we have

$$\begin{aligned} S &= \int d^4x a^3\left[3(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}) - \frac{1}{2}\dot{\phi}^2 - V(\phi)\right] \\ &= 2\pi^2\int dt\left[3(a^2\ddot{a} + a\dot{a}^2 + ka) - a^3[\frac{1}{2}\dot{\phi}^2 + V(\phi)]\right] \\ &= 2\pi^2\int dt\left[3(-a\dot{a}^2 + ka) - a^3[\frac{1}{2}\dot{\phi}^2 + V(\phi)]\right] \end{aligned}$$

So

$$L = 3(-a\dot{a}^2 + ka) - a^3[\frac{1}{2}\dot{\phi}^2 + V(\phi)] \quad (5)$$

Now we can obtain Friedmann Equation for scalar field by Euler-Lagrange equation , so

$$\begin{aligned} &\frac{\partial L}{\partial a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{a}} = 0 \quad , \quad \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \\ &\frac{\partial L}{\partial a} = 3(-a\dot{a}^2 + k) - 3a^2[\frac{1}{2}\dot{\phi}^2 + V(\phi)] \\ &\frac{\partial L}{\partial \dot{a}} = -6a\dot{a} \ , \ \frac{d}{dt}\frac{\partial L}{\partial \dot{a}} = -6(2\dot{a + a}\ddot{a}) \\ &\frac{\partial L}{\partial \phi} = -a^3\frac{d}{d\phi}V(\phi) \\ &\frac{\partial L}{\partial \dot{\phi}} = -a^3\dot{\phi} \ , \ \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = -3a^2\dot{a}\dot{\phi} - a^3\ddot{\phi} \end{aligned}$$

then we have

$$2(\frac{\ddot{a}}{a})+(\frac{\dot{a}}{a})^2+\frac{k}{a^2}-[\frac{1}{2}\dot{\phi}^2-V(\phi)]=0 \tag{6}$$

$$\ddot{\phi}+3(\frac{\dot{a}}{a})\dot{\phi}-\frac{d}{d\phi}V(\phi)=0 \tag{7}$$

These equations are Friedmann equations for scalar field .

3 Wheeler-DeWitt equation in anisotropic minisuperspace model

The Wheeler-DeWitt equation in the anisotropic minisuperspace model takes into account the anisotropy of the universe, meaning that the spatial dimensions can have different expansion rates. It incorporates the effects of gravity and matter fields, providing a framework to study the behavior of the universe at the smallest scales.
Now we can found Hamiltonian by Lagrangian

$$\begin{aligned} H &= \sum \dot{q}\frac{\partial L}{\partial \dot{q}} - L \\ &= \dot{a}\frac{\partial L}{\partial \dot{a}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L \\ &= -3(a\dot{a}^2+ka) - a^3[\frac{1}{2}\dot{\phi}^2 - V(\phi)] \end{aligned}$$

so

$$H=-3(a\dot{a}^2+ka)-a^3[\frac{1}{2}\dot{\phi}^2-V(\phi)] \tag{8}$$

and

$$p_a=\frac{\partial L}{\partial \dot{a}}=-6a\dot{a} \quad , \quad p_\phi=-a^3\dot{\phi} \tag{9}$$

By placing in the Hamiltonian, we have

$$H=-3P_a^2-\frac{1}{2a^2}P_\phi^2+U(a,\phi,k) \tag{10}$$

where $U(a,\phi,k)=a^2[a^2V(\phi)-3k]$ and P_a is the canonically conjugate momentum , P_ϕ is the canonically conjugate angular momentum , $k=1,0,+1$is the curvature parameter . The scale factor, a, is such that $0\leq a<\infty$.
Now, in order to write down the most general form of the Wheeler-DeWitt equation, we need to do the quantization of the momentum, as follows

$$P_a^2=-a^{-p}\frac{\partial}{\partial a}(a^p\frac{\partial}{\partial a}) \quad , \quad P_\phi^2=-\frac{\partial^2}{\partial^2\phi}$$

So

$$H=-3a^{-p}\frac{\partial}{\partial a}(a^p\frac{\partial}{\partial a})+\frac{1}{2a^2}\frac{\partial^2}{\partial^2\phi}+U(a,\phi,k) \tag{11}$$

The parameter p represents the uncertainty in the ordering of the operator factors a and $\frac{\partial}{\partial a}$ [2].The Wheeler-Dewitt equation is as follows

$$\left[-3a^{-p}\frac{\partial}{\partial a}(a^p\frac{\partial}{\partial a})+\frac{1}{2a^2}\frac{\partial^2}{\partial^2\phi}+U(a,\phi,k)\right]\Phi(a,\phi)=0 \tag{12}$$

4 Solution for Slow Roll inflation

The term "slow roll" refers to the gradual decrease in the rate of expansion during this period. This gradual decrease in the rate of expansion is thought to be due to the presence of a scalar field, which is a hypothetical field that permeates the universe and is responsible for the inflationary expansion.
Two methods can be used to solve the WDW equation most of the time. The first method is **separate the variable** and the second method is **Conserved Current** and use Noether's theorem and constants.
In the first method, due to the fact that parameters a and ϕ are somehow coupled, it becomes difficult to solve analytically, and usually approximate solutions can be obtained for boundary conditions, such as $a<<1$.
In the second method, by using the stability and shape of the wavefunction in terms of action, accurate analytical solutions can be obtained, which have difficult mathematics to calculate.
In this paper we use the second method to solve .
For this we have first consider wavefuction is

$$\Phi(a,\phi)=R(a,\phi)e^{iS(a,\phi)}$$

where $R(a,\phi)$ and $S(a,\phi)$ are real function . Since the currents for WDW equation can be obtained as

$$\begin{aligned} j^a &= ia^p(\Phi^*\partial_a\Phi - \Phi\partial_a\Phi^*) \\ &= -a^pR^2(a,\phi)\partial_a S(a,\phi) \end{aligned}$$

and

$$\begin{aligned} j^\phi &= -ia^{p-2}(\Phi^*\partial_\phi\Phi - \Phi\partial_\phi\Phi^*) \\ &= a^{p-2}R^2(a,\phi)\partial_\phi S(a,\phi) \end{aligned}$$

and we know

$$\partial_\mu j^\mu = 0$$

so

$$\partial_a j^a + \partial_\phi j^\phi = 0 \tag{13}$$

$$\partial_a\left[a^pR^2(a,\phi)\partial_a S(a,\phi)\right] - \tag{14}$$

$$\partial_\phi\left[a^{p-2}R^2(a,\phi)\partial_\phi S(a,\phi)\right] = 0 \tag{15}$$

thus

$$\begin{aligned} \partial_a(a^pR^2\partial_a S) - \partial_a(a^{p-2}R^2\partial_\phi S)\frac{\partial_a}{\partial_\phi} = 0 \\ \partial_a(a^pR^2\partial_a S) - \partial_a(a^{p-2}R^2\partial_\phi S)\frac{a^2\dot{\phi}}{6\dot{a}} = 0 \end{aligned}$$

and

$$\begin{aligned} \partial_a\left[(a^{p-2}R^2\partial_\phi S)\frac{a^2\dot{\phi}}{6\dot{a}}\right] &= \partial_a(a^{p-2}R^2\partial_\phi S)\frac{a^2\dot{\phi}}{6\dot{a}} \\ &+ (a^{p-2}R^2\partial_\phi S)\partial_a\left[\frac{a^2\dot{\phi}}{6\dot{a}}\right] \end{aligned}$$

Now if we consider slow roll inflation we have $\dot{\phi} \approx 0$ $\partial_a\left[\frac{a^2\dot{\phi}}{6\dot{a}}\right] \to 0$ so

$$\begin{aligned} \partial_a(a^pR^2\partial_a S) &= \partial_a\left[(a^{p-2}R^2\partial_\phi S)\frac{a^2\dot{\phi}}{6\dot{a}}\right] \\ a^pR^2\partial_a S &= a^{p-2}R^2\partial_\phi S\frac{a^2\dot{\phi}}{6\dot{a}} + c_1 \end{aligned}$$

and

$$R = \left[\frac{c_1}{a^p\partial_a S + a^{p-2}\frac{a^2\dot{\phi}}{6\dot{a}}\partial_\phi S}\right]^{1/2} \tag{16}$$

and we have from Lagrangian

$$\begin{aligned} \partial_a S &= \frac{\partial L}{\partial \dot{a}} = -6a\dot{a} \\ \partial_\phi S &= \frac{\partial L}{\partial \dot{\phi}} = -a^3\dot{\phi} \end{aligned}$$

so

$$R = \left[\frac{-c_1\dot{a}}{36\dot{a}a^{p+1} + a^{p+1}\dot{\phi}^2}\right]^{1/2} \tag{17}$$

$$= a^{-\frac{1+p}{2}}\left[\frac{-c_1\dot{a}}{36\dot{a} + \dot{\phi}^2}\right]^{1/2} \tag{18}$$

and

$$\rho = R^2 = a^{-1-p}\left[\frac{-c_1\dot{a}}{36\dot{a} + \dot{\phi}^2}\right] \tag{19}$$

if we consider $\phi \approx 0$ again we have

$$\rho = R^2 = a^{-1-p} \tag{20}$$

and by boundary condition in $a \to 0$

$$p \leq -1$$

and in $a \to \infty$

$$p \geq -1$$

so we can say in Slow Roll inflation the order factor have $p = -1$.

5 Conclusion

In summary , we write Einstein-Hilbert action for anisotropic minisuperspace model and obtain Friedmann equations in scalar field and next section we obtain Hamiltonian for Wheeler-DeWitt equation and present two mathematical method for solve the equation and after by consider the slow roll inflation we obtain the wavefunction of the universe and probability density for the model and investigate the order factor p in this model by boundary condition must be $p = -1$ and possible the order factor in boundary condition be not exactly equal to -2 .

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