Supersymmetry Time Dependent In Quantum Mechanics

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Abstract

We established Supersymmetry in quantum mechanics for time-dependent systems and obtained a relation between eigenvectors as a time-independent state, and then we did the same for Pseudo-Supersymmetries and with this work, it is possible to solve time-dependent systems that are difficult to solve.

1 Introduction [1]

Supersymmetry (SUSY) arose as a response to attempts by physicists to obtain a unified description of all basic interactions of nature. SUSY relates bosonic and fermionic degrees of freedom combining them into superfields which provides a more elegant description of nature. The algebra involved in SUSY is a graded Lie algebra which closes under a combination of commutation and anti-commutation relations. It may be noted here that so far there has been no experimental evidence of SUSY being realized in nature.
Once people started studying various aspects of supersymmetric quantum mechanics (SUSY QM), it was soon clear that this field was interesting in its own right, not just asa model for testing field theory methods. It was realized that SUSY QM gives insight into the factorization method of Infeld and Hull which was the first attempt to categorize the analytically solvable potential problems. Gradually a whole technology was evolved based on SUSY to understand the solvable potential problems and even to discover new solvable potential problems. One purpose of this book is to introduce and elaborate on the use of these new ideas in unifying how one looks at solving bound state and continuum quantum mechanics problems.
Let us briefly mention some consequences of supersymmetry in quantum mechanics. It gives us insight into why certain one-dimensional potentials are analytically solvable and also suggests how one can discover new solvable potentials. For potentials which are not exactly solvable, supersymmetry allows us to develop an array of powerful new approximation methods. In this book, we review the theoretical formulation of SUSY QM and discuss how SUSY helps us find exact and approximate solutions

to many interesting quantum mechanics problems.

We will show that the reason certain potentials are exactly solvable can be understood in terms of a few basic ideas which include supersymmetric partner potentials and shape invariance. Familiar solvable potentials all have the property of shape invariance. We will also use ideas of SUSY to explore the deep connection between inverse scattering and isospectral potentials related by SUSY QM methods. Using these ideas we show how to construct multi-soliton solutions of the Korteweg-de Vries (KdV) equation. We then turn our attention to introducing approximation methods that work particularly well when modified to utilize concepts borrowed from SUSY. In particular we will show that a supersymmetry inspired WKB approximation is exact for a class of shape invariant potentials. Supersymmetry ideas also give particularly nice results for the tunneling rate in a double well potential and for improving large N expansions and variational methods.

In the continuation of this report, we will discuss the pseudo-hermits. Several years ago, D. Bessis conjectured on the basis of numerical studies that the spectrum of the Hamiltonian $H = p^2 + x^2 + ix^3$ is real and positive [2]. To date there is no rigorous proof of this conjecture. We claim that the reality of the spectrum of H is due to PT symmetry. Note that H is invariant neither under parity P, whose effect is to make spatial reflections, $p \to -p$ and $x \to -x$, nor under time reversal T , which replaces $p \to -p$, $x \to x$, and $i \to -i$. However, PT symmetry is crucial. For example, the Hamiltonian p^2+ix^3+ix has PT symmetry and our numerical studies indicate that its entire spectrum is positive definite; the Hamiltonian $p^2 + ix^3 + x$ not PT-symmetric, and the entire spectrum is complex. The connection between PT symmetry and positivity of spectra is simply illustrated by the harmonic oscillator $H = x^2+p^2$, whose energy levels are $E_n = 2n+1$. Adding ix to H does not break PTsymmetry, and the spectrum remains positive definite: $E_n = 2n + \frac{5}{4}$. Adding $-x$ also does not break PT symmetry if we define P as reflection about $x = \frac{1}{2}, x \to 1 - x$, and again the spectrum remains positive definite: $E_n = 2n + \frac{3}{4}$. By contrast, adding $ix - x$ does break PT symmetry, and the spectrum is now complex: $E_n = 2n + 1 + \frac{1}{2}i$.

The Hamiltonian studied by Bessis is just one example of a huge and remarkable class of non-Hermitian Hamiltonians whose energy levels are real and positive. The purpose of this Letter is to understand the fundamental properties of such a theory by examining the class of quantum-mechanical Hamiltonians

$$H = p^2 + m^2x^2 - (ix)^N \quad , \quad (N\ real).$$

As a function of N and mass m^2 we find various phases with transition points at which entirely real spectra begin to develop complex eigenvalues. There are many applications of non-Hermitian PT - invariant Hamiltonians in physics. Hamiltonians rendered non-Hermitian by an imaginary external field have been introduced recently to study delocalization transitions in condensed matter systems such as vortex fluxline depinning in type-II superconductors , or even to study population biology . Here, initially real eigenvalues bifurcate into the complex plane due to the increasing external field, indicating the unbinding of vortices or the growth of populations. We believe that one can also induce dynamic delocalization by tuning a physical parameter (here N) in a self-interacting theory. So by definition, a PT-symmetric Hamiltonian H satisfies

$$PTH(PT)^{-1} = H \quad or \quad [PT, H] = 0, \tag{1}$$

where P and T are respectively the operators of parity and time-reversal transformations. These are defined according to

$$PxP = -x \ , \ PpP = TpT = -p \ , \ TiIT = -iI.$$

As we mentioned above, the only reason for relating the concept of PT-symmetry and nonHermitian Hamiltonians with a real spectrum is that most of the known examples of the latter satisfy Eq.1.
Certainly there are Hermitian Hamiltonians with a real spectrum that are not PT-symmetric and there are PT-symmetric Hamiltonians that do not have a real spectrum.[3] Therefore, PT-symmetry is neither a necessary nor a sufficient condition for a Hamiltonian to have a real spectrum. Therefore, PT symmetry includes a larger class of Hermitians, which is a real spectrum property . Now let's define the Pseudo-Hermitian.

1.1 Pseudo-Hermitian Hamiltonians [4]

We first give a few definitions. Throughout this paper we will assume that all the inner product spaces are complex. The generalization to real inner product spaces is straightforward.
$\boldsymbol{Definition 1:}$ Let V± be two inner product spaces endowed with Hermitian linear automorphisms $\eta_\pm$ (invertible operators mapping $V_\pm$ to itself and satisfying

$$\forall v_\pm, w_\pm \in V_\pm, \ \ (v_\pm \eta_\pm, w_\pm)_\pm = (v_\pm, \eta_\pm w_\pm)_\pm$$

where $(,)_\pm$ stands for the inner product of $V_\pm$ and $O : V_+ \to V_-$be a linear operator.Then the $\eta_\pm$-pseudo-Hermitian adjoint $O^\sharp : V_- \to V_+$ of O is defined by $O^\sharp := \eta_+^{-1} O^\dagger \eta_-$ .In particular, for $V_\pm = V$ and $\eta_\pm =$, the operator O is said to be η-pseudo-Hermitian if $O^\sharp = O$.

$\boldsymbol{Definition 2:}$ Let V be an inner product space. Then a linear operator $O : V \to V$ is said to be pseudo-Hermitian, if there is a Hermitian linear automorphism η such that O is η-pseudo-Hermitian.
So if Hamiltonian , H , is η-pseudo-Hermitian then we have

$$H^\dagger = \eta H \eta^{-1} \tag{2}$$

when $\eta = I$ then

$$H^\dagger = H$$

and this is normal hermitian we know . η is called a metric operator.
Now we have some proposition for pseudo-Hermitian .

***Proposition*1 :** The Hermitian indefinite inner product $\ll, \gg_\eta$ defined by η, i.e.,

$$\ll \psi_1|\psi_2 \gg_\eta := < \psi_1|\eta|\psi_2 > \ ; \ |\psi_1 >, |\psi_2 > \in H$$

is invariant under the time-translation generated by the Hamiltonian H if and only if H is η-pseudo-Hermitian.
***Proof* :** First note that the η-pseudo-Hermiticity of H is equivalent to the condition

$$H^\dagger = \eta H \eta^{-1}$$

Now, using the Schr¨odinger equation

$$i\frac{d}{dt}|\psi(t) >= H|\psi(t) >$$

and

$$\begin{aligned}
\frac{d}{dt} \ll \psi_1|\psi_2 \gg_\eta &= \frac{d}{dt} < \psi_1|\eta|\psi_2 > \\
&= [\frac{d}{dt} < \psi_1|]\eta|\psi_2 > + < \psi_1|\eta\frac{d}{dt}|\psi_2 > \\
&= i < \psi_1|H^\dagger\eta - \eta H|\psi_2 >= 0 \\
&\to H^\dagger\eta - \eta H = 0 \\
&\to H^\dagger = \eta H \eta^{-1}
\end{aligned}$$

Therefore, $\ll \psi_1|\psi_2 \gg_\eta$ is a constant if and only if $H^\dagger = \eta H \eta^{-1}$ holds.

***Proposition*2 :** An η-pseudo-Hermitian Hamiltonian has the following properties
a) The eigenvectors with a non-real eigenvalue have vanishing η-semi-norm, i.e.,

$$E_i \notin \Re \quad implies \quad ||E_i >|_\eta^2 := \ll E_i|E_i \gg_\eta = 0$$

b) Any two eigenvectors are η-orthogonal unless their eigenvalues are complex conjugates, i.e.,

$$E_i \neq E_j^* \quad implies \quad \ll E_i|E_j \gg_\eta = 0$$

In particular, the eigenvectors with distinct real eigenvalues are η-orthogonal. Now, in the continuation of the report, we will define pseudo-supersymmetries (section 3).

2 Supersymmetry In Quantum Mechanics

2.1 Supersymmetry time independent

We start with a simple generic description of supersymmetry in one-dimensional supersymmetric quantum mechanics [1] . In supersymmetric quantum mechanics, the supersymmetric partner Hamiltonians are given by $H_+ = A^\dagger A$ and $H_- = AA^\dagger$, where A and $A^\dagger$ are lowering and raising operators. We can write the Hamiltonian H in the form of a 2×2 matrix as

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}$$

where

$$A = A(0) = -\partial_x + w(x) \quad A^\dagger = A^\dagger(0) = \partial_x + w(x)$$

The operators A and $A^\dagger$ act on the n th state ψ_n in the following manner (n is an integer):

$$A\psi_n = c_n \psi_{n-1} \quad A^\dagger \psi_n = c^*_{n+1} \psi_{n+1}$$

and we have

$$AH_+ = H_- A$$

This allows us to identify

$$V_+(x) = w^2(x) + \partial_x w(x) \ , \ V_-(x) = w^2(x) - \partial_x w(x)$$

which is the well-known Riccati equation . The quantity $w(x)$ is generally referred to as the superpotential in SUSY QM literature . The solution for $w(x)$ in terms of the ground state wave function is

$$w(x) = -\frac{\partial_x \psi_0}{\psi_0} \tag{3}$$

This solution is obtained by recognizing that once we satisfy $A\psi_0 = 0$, we automatically have a solution of $H_+\psi_0 = A^\dagger A\psi_0 = 0$.
The supersymmetric partner Hamiltonian H_+ acts on a state ψ_n according to

$$H_+\phi_n = A^\dagger A\phi_n = E_n\phi_n \quad E_n = |c_n|^2 \tag{4}$$

When H_- acts on the partner state $\psi_n = A\phi_n$, it satisfies

$$H_-\psi_n = E_n\psi_n. \tag{5}$$

Equations 4 and 5 show that ϕ_n and ψ_n are the supersymmetric partner states with the same energy $E_n > 0$, as long as Aϕ_n does not vanish.However, the ground state ϕ_0 of the supersymmetric system is unique and satisfies

$$A\phi_0 = 0 \ , \ H_+\phi_0 = A^\dagger A\phi_0 = 0 \ , \ E_0 = 0.$$

Now if we have ϕ_0 , we can obtain $w(x)$ and then can construct H_- .
Let us Look at a well known potential , namely the infinite square well and determine its SUSY partner potential . Consider a particle of mass m in an infinite squarre well potential of width L :

$$\begin{aligned} V(x) &= 0 \ , \ 0 \le x \le L, \\ &= \infty \ , \ otherwise \end{aligned}$$

The normalized ground state wave function is known to be

$$\phi_0 = (\frac{2}{L})^{1/2} \sin(\frac{\pi x}{L})$$

and the ground state energy is

$$E_0 = \frac{\pi^2}{2mL^2}$$

Subtracting off the ground state energy so that the Hamiltonian can be factorized we have for $H^{'} = H_+ - E_0$ that the energy eigenvalues are

$$E_n^{'} = \frac{n(n+1)}{2mL^2}\pi^2 \ , \ n = 0, 1, 2, ...,$$

and the normalized eigenfunctions are

$$\phi_0^{'} = (\frac{2}{L})^{1/2} \sin(\frac{(n+1)\pi x}{L})$$

The superpotential for this problem is readily obtained using equation 3

$$w(x) = -\frac{1}{\sqrt{2m}} \frac{\pi}{L} \cot(\frac{\pi x}{L})$$

and hence the supersymmetric partner potential V_- is

$$V_-(x) = \frac{\pi^2}{2mL^2}[2cosec^2(\frac{\pi x}{L}) - 1]$$

The wave functions for H_- are obtained by applying the operator A to the wave functions of H_+ . In particular we find that the normalized ground and first excited state wave functions are

$$\psi_0 = -2\sqrt{\frac{2}{3L}} \sin^2(\frac{\pi x}{L})$$
$$\psi_1 = -\frac{2}{\sqrt{L}} \sin(\frac{\pi x}{L}) \sin(\frac{2\pi x}{L})$$

Now if we expand the above result , we have

$$V_{p+1}(x) = \frac{\pi^2}{2mL^2}[p(p+1)cosec^2(\frac{\pi x}{L}) - p^2]$$
$$\psi_0^{p+1} \propto \sin^{p+1}(\frac{\pi x}{L})$$

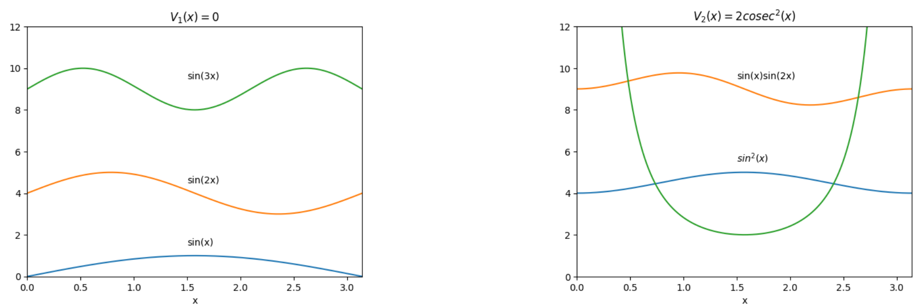

Figure 1: Wave functions at different potentials

2.2 Supersymmetry time dependent

In this report, we assumed that if the system has a time-dependent Hamiltonian, what are the eigenfunctions related to each other and can we define a relation between them or not . If the Hamiltonian is time-dependent, then according to the definition of operator A and the superpotential , $w(x)$, must be time-dependent because in this Hamiltonian, the potential is time-dependent. So we have a new define

$$A(t) = -\partial_x + w(x,t) \quad A^\dagger(t) = \partial_x + w(x,t)$$

Then

$$\begin{aligned} H_+(t) = A^\dagger(t)A(t) = -\partial_x^2 + w^2(x,t) + \partial_x w(x,t) \\ H_-(t) = A(t)A^\dagger(t) = -\partial_x^2 + w^2(x,t) - \partial_x w(x,t) \end{aligned}$$

So we have

$$A(t)H_+(t) = H_-(t)A(t) \tag{6}$$

Now we will discuss whether operator $A(t)$ is still a relation between eigenfunctions or not . Assume $\psi_n^-(x,t)$ and $\phi_n^+(x,t)$ are eigenfunctions for $H_-(t)$ and $H_+(t)$, respectively . So if $\psi_n^-(x,t) = A(t)\phi_n^+(x,t)$

$$\begin{aligned} H_-(t)\psi_n^-(x,t) &= i\partial_t\psi_n^-(x,t) = i\partial_t[A(t)\phi_n^+(x,t)] = A(t)A^\dagger(t)\psi_n^-(x,t) \\ A(t)A^\dagger(t)A(t)\phi_n^+(x,t) &= i\partial_t[A(t)\phi_n^+(x,t)] \\ &= i\partial_t A(t)\phi_n^+(x,t) + iA(t)\partial_t\phi_n^+(x,t) \\ &= i\partial_t A(t)\phi_n^+(x,t) + A(t)H_+(t)\phi_n^+(x,t) \end{aligned}$$

So we have

$$\begin{aligned} &A(t)H_+(t) = i\partial_t A(t) + A(t)H_+(t) \\ &\partial_t A(t) = 0 \ \ , \ \ A(t) = A(0) = -\partial_x + w(x) \end{aligned}$$

Therefor if $A(t)$ is relation between eigenfunctions , then it must be time independent and that is incorrect .

Let us consider we have this relation between eigenfunctions , $\psi_n^-(x,t) = L(x,t)\phi_n^+(x,t)$, where $L(x,t)$ is a time dependent operator , now we have

$$\begin{aligned} H_-(t)\psi_n^-(x,t) &= i\partial_t\psi_n^-(x,t) = i\partial_t[L(x,t)\phi_n^+(x,t)] \\ H_-(t)L(x,t)\phi_n^+(x,t) &= i\partial_t L(x,t)\phi_n^+(x,t) + iL(x,t)\partial_t\phi_n^+(x,t) \\ &= i\partial_t L(x,t)\phi_n^+(x,t) + L(x,t)H_+(t)\phi_n^+(x,t) \end{aligned}$$

So

$$H_-(t)L(x,t) - L(x,t)H_+(t) = i\partial_t L(x,t) \tag{7}$$

Now from equations 6 and 7 we have

$$\left\{ \begin{array}{c} A(t)H_+(t) = H_-(t)A(t) \\ H_-(t)L(x,t) - L(x,t)H_+(t) = i\partial_t L(x,t) \end{array} \right.$$

Note , if $\lim t \to 0$ then $L(x,t) \to A(0)$, so $L(x,0) = A(0)$.
From equation 6

$$\begin{aligned} &A(t)H_+(t)\phi_n^+(x,t) = H_-(t)A(t)\phi_n^+(x,t) = A(t)i\partial_t\phi_n^+(x,t) \\ &\longrightarrow iA(t)\partial_t = H_-(t)A(t) \end{aligned}$$

So

$$iA(t)\partial_t L(x,t) = H_-(t)A(t)L(x,t)$$

From equation 7

$$\begin{aligned} &A(t)[H_-(t)L(x,t) - L(x,t)H_+(t)] = H_-(t)A(t)L(x,t) \\ &A(t)L(x,t)H_+(t) = [A(t), H_-(t)]L(x,t) \\ &L(x,t)H_+(t) = A^{-1}(t)[A(t), H_-(t)]L(x,t) = H_-(t)L(x,t) - i\partial_t L(x,t) \\ &i\partial_t L(x,t) = \Big[H_-(t) - A^{-1}(t)[A(t), H_-(t)]\Big]L(x,t) \\ &\qquad = \Big[H_-(t) - H_-(t) + A^{-1}(t)H_-(t)A(t)]\Big]L(x,t) \\ &\qquad = A^{-1}(t)H_-(t)A(t)L(x,t) = A^{-1}(t)A(t)H_+L(x,t) \\ &\qquad = H_+(t)L(x,t) \end{aligned}$$

So

$$\begin{aligned} &i\partial_t L(x,t) = H_+(t)L(x,t) \\ &i\partial_t L(x,t)L^{-1}(x,t) = H_+(t) \\ &dL(x,t)L^{-1}(x,t) = -iH_+(t)dt \ , \ \int_0^t dL(x,t^{'})L^{-1}(x,t^{'}) = -i\int_0^t H_+(t^{'})dt^{'} \\ &\ln[\frac{L(x,t)}{A(0)}] = -i\int_0^t H_+(t^{'})dt^{'} \ \ ; \ \ L(x,t) = A(0)\exp[-i\int_0^t H_+(t^{'})dt^{'}] \end{aligned}$$

we obtain relation between eigenfunctions , $L(x,t)$. Note that the $L(x,t)$ satisfy condition in $\lim t \to 0$.

$$\psi_n^-(x,t) = L(x,t)\phi_n^+(x,t) \quad , \quad L(x,t) = U_+(t,0)A(0) \tag{8}$$

$$U(t,0) = T\exp[-i\int_0^t H_+(t^{'})dt^{'}] \tag{9}$$

3 Pseudo Supersymmetric Quantum Mechanics

The application of the ideas of supersymmetric quantum mechanics [28] in constructing nonHermitian PT-symmetric Hamiltonians has been considered in Refs. [3, 7, 13, 17, 19] and a formulation of PT-symmetric supersymmetry has been outlined in Refs. [14, 20].In this section, we develop a straightforward generalization of supersymmetric quantum mechanics that applies for pseudo-Hermitian Hamiltonians.

3.1 Pseudo Supersymetry time independent

Definition3 : Consider a Z_2-graded quantum system [29] with the Hilbert space $H_+ \bigoplus H_-$ and the involution or grading operator τ satisfying

$$\tau = \tau^\dagger = \tau^{-1} \quad and \quad \forall|\psi_\pm> \in H_\pm, \quad \tau|\psi_\pm>= \pm|\psi_\pm>$$

Let η be an even Hermitian linear automorphism (i.e., $[\eta,\tau] = 0$) and suppose that the Hamiltonian H of the system is η-pseudo-Hermitian. Then H (alternatively the system) is said to have a pseudo-supersymmetry generated by an odd linear operator Q (i.e., $Q,\tau = 0$) if H and Q satisfy the pseudo-superalgebra

$$Q^2 = Q^{\sharp 2} = 0, \quad Q,Q^\sharp = 2H \tag{10}$$

A simple realization of pseudo-supersymmetry is obtained using the two-component representation of the Hilbert space where the state vectors $|\psi>$ are identified by the column vector $\begin{pmatrix}|\psi_+>\\|\psi_->\end{pmatrix}$ of their components $|\psi_\pm>$ belonging to $H_\pm$. In this representation, one can satisfy the η-pseudo-Hermiticity of the Hamiltonian H and the pseudo-superalgebra10 by setting

$$\tau = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \quad , \ \eta = \begin{pmatrix}\eta_+ & 0\\0 & \eta_-\end{pmatrix}$$

$$Q = \begin{pmatrix}0 & 0\\D & 0\end{pmatrix} \quad , \ H = \begin{pmatrix}H_+ & 0\\0 & H_-\end{pmatrix}$$

where $\eta_\pm$ is a Hermitian linear automorphism of $H_\pm$,$D : H_+ \to H_-$ is a linear operator, and

$$H_+ := D^\sharp D \quad , \quad H_+ := DD^\sharp$$

Note that, by definition, $Q^\sharp = \eta^{-1} Q^\dagger \eta$,

$$D^\sharp = \eta_+^{-1} D^\dagger \eta_-$$

and

$$DH_+ = H_- D \quad , \quad D^\sharp H_- = H_+ D^\sharp$$

As a consequence, H_+ and H_- are isospectral, D maps the eigenvectors of H_+ to those of H_- and $D^\sharp$ does the converse, except for those eigenvectors that are eliminated by these operators.
An interesting situation arises when one of the automorphisms $\eta_\pm$ is trivial, e.g., $\eta_+ = 1$. In this case, H_+ is a Hermitian Hamiltonian with a real spectrum, and pseudo-supersymmetry implies that the pseudo-Hermitian Hamiltonian H_- — which is generally non-Hermitian — must have a real spectrum as well.
Now we use this property and we construct a Pseudo-Hermitian time-independent Hamiltonian by a Hermitian time-independent Hamiltonian, and by using supersymmetry, we obtain the Pseudo-Hermitian Hamiltonian solutions through the Hermitian Hamiltonian solutions. So we define

$$\eta_+ : H_+ \to H_+ \quad , \quad \eta_- : H_- \to H_-$$
$$B : H_+ \to H_- \quad , \quad B^\sharp : H_- \to H_+$$

where

$$B^\sharp = \eta_+^{-1} B^\dagger \eta_-$$

then

$$H_+ = B^\sharp B = \eta_+^{-1} B^\dagger \eta_- B$$
$$H_- = B B^\sharp = B \eta_+^{-1} B^\dagger \eta_-$$

According to the mentioned feature, if $\eta_+ = I$, then H_+ is hermitian and has real spectrum , then H_- should has real spectrum because H_+ and H_- are supersymmetry partner of each other. So

$$if\ \eta_+ = I : H_+ = B^\sharp B = B^\dagger \eta_- B \quad , \quad H_- = B B^\sharp = B B^\dagger \eta_-$$

As we know, the H_+ is Hermitian because

$$H_+^\dagger = (B^\dagger \eta_- B)^\dagger = B^\dagger \eta_- B = H_+$$

Now we define an operator that works like ladder operators in supersymmetries, so we define:

$$A := \eta_-^{1/2} B = \rho B = -\partial_x + w(x) \quad , \quad \rho := \eta_-^{1/2}$$

So

$$H_+ = B^\dagger \eta_- B = A^\dagger A$$
$$H_- = B B^\dagger \eta_- = \rho^{-1} A A^\dagger \rho$$

The interesting point is that Hamiltonian 1 has changed like supersymmetry and Hamiltonian 2 has kept its supersymmetrical part and is influenced by the metric operator only on both sides. ***In*** fact, the physical Hamiltonian is placed between two operators that turn it into a non-physical Hamiltonian.
Now we want to find the relation between eigenfunctions , so

$$\begin{aligned}
&H_+\phi_n^+ = A^\dagger A\phi_n^+ = E_n^+\phi_n^+\\
&H_-\psi_n^- = \rho^{-1}AA^\dagger\rho\psi_n^- = E_n^-\psi_n^-\\
&\rho H_-\psi_n^- = AA^\dagger\rho\psi_n^- = \rho E_n^-\psi_n^- = E_n^-\rho\psi_n^- = H_-[\rho\psi_n^-]\\
&A^\dagger H_-[\rho\psi_n^-] = A^\dagger AA^\dagger\rho\psi_n^- = H_+A^\dagger[\rho\psi_n^-]\\
&\to A^\dagger H_- = H_+A^\dagger
\end{aligned}$$

As it was in Hermitian supersymmetry. So the relation is

$$\psi_n^- = A\phi_n^+$$

For simplicity, we define

$$h_- := AA^\dagger$$

Example

3.2 Pseudo Supersymmetry time dependent

Now consider the Hamiltonians are time dependent . First, we obtain a relation between physical and non-physical Hamiltonian eigenfunctions in the time-independent mode .For simplicity, we define

$$h_- := AA^\dagger$$

for physical Hamiltonian . Then we assume that the metric operator is dependent on time, and in this case we consider the general case for pseudo-Hermitians. If the metric operator is time-dependent, then the form of the pseudo-Hermitians changes as follows

$$\begin{aligned}
\frac{d}{dt} \ll \psi_1|\psi_2 \gg_\eta = &\frac{d}{dt} < \psi_1|\eta(t)|\psi_2 >\\
&= [\frac{d}{dt} < \psi_1|]\eta(t)|\psi_2 > + < \psi_1|\frac{d}{dt}\eta(t)|\psi_2 > + < \psi_1|\eta(t)\frac{d}{dt}|\psi_2 >\\
&= i < \psi_1|H^\dagger\eta(t) - \eta(t)H - i\dot{\eta(t)}|\psi_2 >= 0\\
&\to H^\dagger\eta - \eta H - i\dot{\eta(t)} = 0\\
&\to H^\dagger = \eta(t)H\eta^{-1}(t) + i\dot{\eta(t)}\eta^{-1}(t)
\end{aligned}$$

If the metric operator is time constant, then it becomes equation 2 .
Now we know in time independent the relation between physical and Non-physical eigenfunctions is

$$|\Psi_n^- >= \rho|\psi_n^- >$$

where

$$H_-|\psi_n^- >= E_n^-|\psi_n^- > \quad , \quad h_-|\Psi_n^- >= E_n^-|\Psi_n^- >$$

Now we consider the most general non-Hermitian time-dependent Hamiltonian $H(t)$ and its associated time-dependent metric operator $\eta(t)$.
The main assumption to be made is that the two time-dependent Schrodinger equations still holds

$$H(t)|\psi^-(t) >= i\partial_t|\psi^-(t) > \tag{11}$$

$$h(t)|\Psi^-(t) >= i\partial_t|\Psi^-(t) > \tag{12}$$

both Hamiltonians involved are explicitly time-dependent , with $H(t)$ being to be non-Hermitian whereas $h(t)$ is taken Hermitian, i.e. Next , we assume that the two solutions $|\psi^-(t) >$ and $|\Psi^-(t) >$ of eqs. 11 ,12 are related by a time -dependent invertible operator $\rho(t)$ as

$$|\Psi_n^-(t) >= \rho|\psi_n^-(t) > \tag{13}$$

it then follows immediately, by direct substitution of 13 into eqs. 11 and 12, that the two Hamiltonians are allied to each other as

$$h(t) = \rho(t)H(t)\rho^{-1}(t) + i\dot{\rho(t)}\rho^{-1}(t)$$

Proof if $\rho = \rho(t)$

$$h(t)|\Psi^-(t) >= i\partial_t|\Psi^-(t) > \ : \ \ h(t)\rho(t)|\psi^-(t) >= i\partial_t[\rho(t)|\psi^-(t) >]$$

$$\begin{aligned} h(t)\rho(t)|\psi^-(t) > &= i[\dot{\rho(t)}|\psi^-(t) > +\rho(t)\partial_t|\psi^-(t) >] \\ &= i\dot{\rho(t)}|\psi^-(t) > +\rho(t)[i\partial_t|\psi^-(t) >] \\ &= i\dot{\rho(t)}|\psi^-(t) > +\rho(t)H(t)|\psi^-(t) > \end{aligned}$$

so

$$\begin{aligned} & h(t)\rho(t) = i\dot{\rho(t)} + \rho(t)H(t) \\ & \Rightarrow h(t) = \rho(t)H(t)\rho^{-1}(t) + i\dot{\rho(t)}\rho^{-1}(t) \end{aligned}$$

Now we want to solve Schrodinger equation eq.11 . From previous section we known what is relation between Supersymmetry time dependent 15 , so from eqs. 13 and 15 we have

$$\begin{aligned} & \Psi_n^-(t) = \rho(t)\psi_n^-(t) = L(x,t)\phi_n^+(t) \\ & \Rightarrow \psi_n^-(t) = \rho^{-1}(t)L(x,t)\phi_n^+(t) \end{aligned}$$

so

$$\boxed{\begin{aligned} & \psi_n^-(x,t) = \rho^{-1}(t)L(x,t)\phi_n^+(x,t)\ ,\ L(x,t) = \rho^{-1}(t)U_+(t,0)A(0) && (14) \\ & U_+(t,0) = T\exp[-i\int_0^t H_+(t^{'})dt^{'}] && (15) \end{aligned}}$$

So we obtained time-dependent eigenfunctions for non-Hermitian systems by Pseudo Supersymmetry .

4 EXPLICITLY TIME-DEPENDENT INVARIANTS [10]

Now we have to solve the time-dependent Schrödinger equation analytically, because we need to solve the primary system in order to obtain the special functions of the secondary system.
We consider a system whose Hamiltonian operator $H(t)$ is an explicit function of time, and we assume the existence of another explicitly time-dependent nontrivial Hermitian operator $I(t)$, which is an invariant. That is, $I(t)$ satisfies the conditions

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i}[I, H]$$

and

$$I^{\dagger} = I$$

Now we have

$$H(t)\phi_n(x,t) = i\partial_t\phi_n(x,t)$$

if $\frac{dI}{dt} = 0$(invariant)

$$\frac{\partial I}{\partial t} = i[I, H]$$

then

$$\begin{aligned}\frac{\partial I}{\partial t}\phi_n(x,t) &= i[I,H]\phi_n(x,t) = i(IH - HI)\phi_n(x,t) \\ &= iI[H\phi_n(x,t)] - iH[I\phi_n(x,t)] = iI[i\partial_t\phi_n(x,t)] - iH[I\phi_n(x,t)] \\ &= -I[\partial_t\phi_n(x,t)] - iH[I\phi_n(x,t)]\end{aligned}$$

so

$$\begin{aligned}&\frac{\partial I}{\partial t}\phi_n(x,t) + I\partial_t\phi_n(x,t) = \frac{\partial}{\partial t}[I\phi_n(x,t)] = -iH[I\phi_n(x,t)] \\ &\Rightarrow H[I\phi_n(x,t)] = i\partial_t[I\phi_n(x,t)]\end{aligned}$$

Now we have in bra-ket

$$\begin{aligned}&I(t)|\phi_n(t)>= \lambda_n|\phi_n(t)> \\ &\partial_t I(t)|\phi_n(t)> + I(t)\partial_t|\phi_n(t)>= \lambda_n\partial_t|\phi_n(t)> \\ &\partial_t I(t)|\phi_n(t)> + I(t)[\frac{1}{i}H(t)|\phi_n(t)>] = \lambda_n[\frac{1}{i}H(t)|\phi_n(t)>] \\ &i\partial_t I(t)|\phi_n(t)>= [\lambda_n - I(t)]H(t)|\phi_n(t)>\end{aligned}$$

Now

$$\begin{aligned}&i<\phi_{n'}(t)|\partial_t I(t)|\phi_n(t)>=<\phi_{n'}(t)|[\lambda_n - I(t)]H(t)|\phi_n(t)> \\ &i<\phi_{n'}(t)|\partial_t I(t)|\phi_n(t)>=<\phi_{n'}(t)|[\lambda_n - \lambda_{n'}]H(t)|\phi_n(t)>\end{aligned}$$

so

$$i < \phi_{n'}(t)|\partial_t I(t)|\phi_n(t) >=< \phi_{n'}(t)|[\lambda_n - \lambda_{n'}]H(t)|\phi_n(t) > \qquad (16)$$

Let consider in position space

$$H[I\phi_n(x,t)] = i\partial_t[I\phi_n(x,t)] \;\; ; \;\; I\phi_n(x,t) = \lambda_n\phi_n(x,t) = \psi_n(x,t)$$

where

$$H\psi_n(x,t) = i\partial_t\psi_n(x,t)$$

from equation 16

$$\begin{aligned}
&i\partial_t I(t)\phi_n(x,t) = [\lambda_n - I(t)]H(t)\phi_n(x,t)\\
&i\int_0^t dt\phi_m^*(x,t^{'})I(t^{'})\phi_n(x,t^{'}) = \int_0^t dt\phi_m^*(x,t^{'})[\lambda_n - I(t^{'})]H(t^{'})\phi_n(x,t^{'})\\
&i\int_0^t dt\phi_m^*(x,t^{'})I(t^{'})\phi_n(x,t^{'}) = \int_0^t dt\phi_m^*(x,t^{'})(\lambda_n - \lambda_m)H(t^{'})\phi_n(x,t^{'})
\end{aligned}$$

Now if $\lambda_n = \lambda_m$, then

$$i\int_0^t dt\phi_m^*(x,t^{'})I(t^{'})\phi_n(x,t^{'}) = 0$$

and then

$$\frac{\partial}{\partial t}\lambda_n = 0$$

so

$$i\int_0^t dt\phi_m^*(x,t^{'})\partial_{t'}I(t^{'})\phi_n(x,t^{'}) = (\lambda_n - \lambda_m)\int_0^t dt\phi_m^*(x,t^{'})H(t^{'})\phi_n(x,t^{'}) \qquad (17)$$

on the other hand

$$\begin{aligned}
&I(t)\phi_n(x,t) = \lambda_n\phi_n(x,t)\\
&\partial_t I(t)\phi_n(x,t) + I(t)\partial_t\phi_n(x,t) = \lambda_n\partial_t\phi_n(x,t)\\
&\partial_t I(t)\phi_n(x,t) = [\lambda_n - I(t)]\partial_t\phi_n(x,t)
\end{aligned}$$

so

$$i\int_0^t dt\phi_m^*(x,t^{'})\partial_{t'}I(t^{'})\phi_n(x,t^{'}) = (\lambda_n - \lambda_m)\int_0^t dt\phi_m^*(x,t^{'})\partial_{t'}\phi_n(x,t^{'}) \qquad (18)$$

from eqs.17 , 18

$$(\lambda_n - \lambda_m)\int_0^t dt\phi_m^*(x,t^{'})[H(t^{'}) - \partial_{t'}]\phi_n(x,t^{'}) = 0$$

if $\lambda_n \neq \lambda_m$

$$\int_0^t dt\phi_m^*(x,t^{'})[H(t^{'}) - \partial_{t^{'}}]\phi_n(x,t^{'}) = 0 \tag{19}$$

we can define a new set of eigenfunctions of $I(t)$ related to our initial set by a time-dependent gauge transformation

$$\Phi_n(x,t) = e^{i\alpha_n(t)}\phi_n(x,t)$$

now from equation 19

$$\int_0^t dte^{-i\alpha_m(t^{'})}\phi_m^*(x,t^{'})[H(t^{'}) - \partial_{t^{'}}]e^{i\alpha_n(t^{'})}\phi_n(x,t^{'}) = 0$$

we obtain

$$\int_0^t dt\phi_m^*(x,t^{'})[H(t^{'}) - \partial_{t^{'}}]\phi_n(x,t^{'}) = -\frac{d}{dt}\alpha_n(t)$$

so we obtain analytic solution of eigenfunctions of $H(t)$

$$H(t)\Phi_n(x,t) = i\partial_t\Phi_n(x,t)$$

where

$$\Phi_n(x,t) = e^{i\alpha_n(t)}\phi_n(x,t) \;\; ; \;\; I(t)\phi_n(x,t) = \lambda_n\phi_n(x,t) \;\; ; \;\; -i\partial_t I(t) = [I(t), H(t)]$$

$$\frac{d}{dt}\alpha_n(t) = -\int_0^t dt\phi_m^*(x,t^{'})[H(t^{'}) - \partial_{t^{'}}]\phi_n(x,t^{'})$$

5 Conclusions

In this report, we came to a new approach for solving the time-dependent Hamiltonian, which can be used to solve one time-dependent system by solving another system through supersymmetries, and we were also able to do this for systems that have a time-dependent non-Hermitian Hamiltonian. Apply the approach.

Now, the issue that is interesting to us is whether it is possible to create supersymmetry for many body systems and whether it is possible to describe their behavior in time, and also if the particles are entangled in a system, is there a system of There is a way of supersymmetries in which they are not entangled, and by solving this system, we can solve entangled systems or not, and pseudo-Hermitian can also be used in these systems. In the next step, we will look for answers to these questions.

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