Analytical solution of the Dirac equation near the event horizon of a charged black hole

Meysam Hassandoust
(Dated: August 22, 2023)

This study investigates the wave function of a 1/2 particle by the Dirac equation in curved space-time for the Reissner-Nordstrom metric. The research focuses on the behavior of particles in the vicinity of a charged black hole and reveals that near the event horizon, there is only one level of energy with $\frac{eQ}{r_+}$values, and the wave function curve goes to infinity. As we move away from the event horizon towards infinity, the wave function returns to a normal state, resembling a sine wave. The study provides a deeper understanding of the fundamental principles of physics and contributes to our knowledge of the behavior of particles in extreme conditions. The findings have important implications for the study of black holes and the nature of space-time. Overall, this research highlights the significance of the Dirac equation in understanding the behavior of particles in curved space-time and opens up new avenues for further research in this field. The results of this study have the potential to advance our understanding of the universe and the laws that govern it.

I. INTRODUCTION

In the realm of theoretical physics, the Dirac equation stands as a cornerstone of quantum mechanics, describing the behavior of fermions, such as electrons, in a relativistic framework. When combined with the effects of gravity, the Dirac equation offers a fascinating insight into the behavior of particles in the presence of a gravitational field. In this article, we delve into the intriguing realm of the Dirac equation in a gravitational field, specifically focusing on its application in the context of the Reissner-Nordström black hole.

The Reissner-Nordström black hole is a solution to Einstein's field equations that describes a charged, non-rotating black hole. It provides a unique setting to explore the interplay between gravity and quantum mechanics, as it incorporates both the effects of a gravitational field and an electromagnetic field. By examining the behavior of fermions within the vicinity of a Reissner-Nordström black hole, we can gain valuable insights into the nature of matter and the fundamental forces of the universe. When fermions, such as electrons, are subjected to a gravitational field, their behavior is governed by the Dirac equation modified to account for the effects of gravity. The Dirac equation in a gravitational field describes the wave function of a fermion and how it evolves in the presence of both gravity and electromagnetic fields. It takes into account the curvature of spacetime caused by the black hole's mass and charge, as well as the electromagnetic field surrounding it.

Solving the Dirac equation in a gravitational field for the Reissner-Nordström black hole is a complex task that requires advanced mathematical techniques. However, the resulting solutions provide valuable insights into the behavior of fermions near the event horizon and the influence of the black hole's charge on their properties. These solutions reveal intriguing phenomena, such as the existence of bound states near the black hole and the modification of the fermion's energy levels due to the gravitational and electromagnetic fields.

Studying the Dirac equation in a gravitational field within the context of the Reissner-Nordström black hole not only deepens our understanding of the fundamental nature of matter but also sheds light on the intricate interplay between gravity and quantum mechanics. Furthermore, it offers a glimpse into the behavior of particles in extreme astrophysical environments, where the effects of gravity and electromagnetic fields are intertwined. In conclusion, the Dirac equation in a gravitational field provides a powerful framework for exploring the behavior of fermions in the presence of a Reissner-Nordström black hole. By unraveling the intricate mathematics and analyzing the resulting solutions, scientists can uncover valuable insights into the fundamental nature of matter and the profound influence of gravity and electromagnetic fields on particle behavior.

In this paper we study and solve the Dirac equation in radial part near the event horizon and check the solution at infinity and we want find relation between solutions near the black hole and solution at infinity . We consider the radial part in this paper and assume we have quantum orbital in everywhere .

II. THE DIRAC EQUATION IN THE REISNNER-NORDSTRÖM METRIC

The Reisnner-Nordström metric for charged black hole is as follows

$$ds^2 = fdt^2 - f^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2 \tag{1}$$

where M is the mass of the black hole , Q is the charge of black hole and $f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$. we assume $c = G = \hbar = 1$. In this paper we consider the case , $|Q| < M$. In this case we have two horizon . Let $f(r) = 0$ then we have two roots , $r = r_\pm = M \pm \sqrt{M^2 - Q^2}$,the **Event Horizon** (r_+) and the **Cauchy Horizon** (r_-) . In this paper we just study about near the Event Horizon , because we dont have any information about below of the event horizon .

On the other hand the Dirac equation in the curved space-time is

$$i\gamma^\mu(x)[\partial_\mu + ieA_\mu - \Gamma_\mu(x) + im]\Phi(x) = 0 \quad (2)$$

where A_μ is the electromagnetic vector potential which in this case the electromagnetic potential for a static black hole with charge Q is $A_\mu = (Q/r, 0, 0, 0)$ and $\Gamma_\mu(x)$ is the spin connection and $\gamma^\mu(x)$ are the space time dependent matrices. The spin connection relation is defined by

$$\Gamma_\mu(x) = \frac{1}{4} g_{\lambda\rho}(\partial_\mu e^\alpha_\nu e^\rho_\alpha - \Gamma^\rho_{\nu\mu})S^{\lambda\nu} \quad (3)$$

where $\Gamma^\rho_{\nu\mu}$ is the Christoffel symbol and $S^{\lambda\nu}$ has this relation

$$S^{\lambda\nu} = \frac{1}{2}[\gamma^\lambda(x), \gamma^\nu(x)]$$

Now the tetrad matrix has become

$$e^\mu_\alpha = \begin{pmatrix} f^{1/2} & 0 & 0 & 0 \\ 0 & f^{-1/2} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}$$

and

$$g_{\mu\nu} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

so the Dirac equation takes the form

$$\Bigg[\frac{\gamma^0}{f^{1/2}}(i\partial_t - \frac{eQ}{r}) + if^{1/2}\gamma^1[\partial_r + \frac{Q^2 - 3Mr + 2r^2}{2r^3}] + \frac{i}{r}[\gamma^2(\partial_r + \frac{\cot\theta}{2}) + \frac{\gamma^3}{\sin\theta}\partial_\phi]\Bigg]\Phi(t, r, \theta, \phi) = 0 \quad (4)$$

In [?] they redefine the wave function

$$\Phi = \frac{e^{-i\omega t}\Psi}{f^{1/4} r \sin^{1/2}\theta} \quad (5)$$

and single out the operator

$$\hat{\mathcal{L}} = \gamma^0\gamma^1(\gamma^2\frac{\partial}{\partial_\theta} + \gamma^3\frac{1}{\sin\theta}\frac{\partial}{\partial_\phi}) \quad (6)$$

with integer eigenvalues $\hat{\mathcal{L}}\Psi = k\Psi$, where $k = 0, \pm1, \pm2, ...$. In the future we obtain the angular part of this equation but now we just want to obtain radial part of wave function . If we writing Ψ as

$$\Psi = \mathcal{Z}(\theta, \phi)\begin{pmatrix} R_1(r)I_2 \\ iR_2(r)I_2 \end{pmatrix}$$

where I_2 is the column $(1, 1)^T$, we obtain a system of equations for the radial wave functions

$$[\partial_r - \frac{k}{rf^{1/2}}]R_1(r) = -[\frac{\omega - eQ/r}{f} + \frac{m}{f^{1/2}}]R_2(r) \quad (7)$$

$$[-\partial_r - \frac{k}{rf^{1/2}}]R_2(r) = +[\frac{\omega - eQ/r}{f} - \frac{m}{f^{1/2}}]R_1(r) \quad (8)$$

where $f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$.

III. SOLUTION OF NON-EXTERMAL BLACK HOLES WITH $|Q| < M$

In mathematics related to differential equations, we know that we need boundary conditions to obtain a unique solution of a differential equation. Here we are not able to analytically solve differential equations 7 and 8 because our boundary conditions are not defined on the surface of the black hole (event horizon) and at infinity we only know that our equation should give us the hydrogen atom model (Coulomb potential). to give Therefore, we can only talk about the local areas and we are excused from presenting an analytical solution using the method of separating variables.

So we investigate the eqs.7 , 8 in $r \to \infty$ and $r \to r_+$. In $r \to \infty$ we have $f \to 1$ and the equations are as follows

$$[\partial_r - \frac{k}{r}]R_1(r) = -(\omega + m)R_2(r) \quad (9)$$

$$[-\partial_r - \frac{k}{r}]R_2(r) = +(\omega - m)R_1(r) \quad (10)$$

from these equations we have

$$[-\frac{d^2}{dr^2} - \frac{k - k^2}{r^2} + \omega^2 - m^2]R(r) = 0 \quad (11)$$

and the answer is

$$R(r) = \sqrt{r}\Bigg[C_1 J_{-\frac{1}{2}+k}(-i\sqrt{m^2 - \omega^2}r) + C_2 Y_{-\frac{1}{2}+k}(-i\sqrt{m^2 - \omega^2}r)\Bigg] \ , \ (r \to \infty) \quad (12)$$

Now we obtain the results in $r \to r_+$, so from eqs.7 , 8 we know in the event horizon $f(r) = 0$ and thus in near of the event horizon $f(r) \to 0$ so we can use Taylor approximation for obtain $f(r)$ in $r \to r_+$ so

$$
\begin{aligned}
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} &\simeq f(r_+) + \frac{d}{dr}f(r)|_{r_+}(r - r_+) \\
&\simeq 0 + \frac{d}{dr}[1 - \frac{2M}{r} + \frac{Q^2}{r^2}]_{r_+}(r - r_+) \\
&= 2(\frac{M}{r_+} - \frac{Q^2}{r_+^2})(\frac{r}{r_+} - 1)
\end{aligned}
$$

Let define

$$y := \sqrt{\frac{r}{r_+} - 1} \quad , \quad F^{-2} := 2(\frac{M}{r_+} - \frac{Q^2}{r_+^2}) \tag{13}$$

so

$$f(r) \simeq F^{-2}y^2 \tag{14}$$

This approximation is only true near the event horizon and in other places it is not the correct answer because $f(r)$ has a dependence on the radius . Now by inserting in equations 7 and 8 and consider the new definition 13 we have

$$[y\frac{d}{dy} - 2kFy]R_1(y) = -F[F(\omega - \frac{eQ}{r_+}) + my]R_2(y) \tag{15}$$

$$[-y\frac{d}{dy} - 2kFy]R_2(y) = F[F(\omega - \frac{eQ}{r_+}) - my]R_1(y) \tag{16}$$

Now, as in [?], the allowed energy levels can be obtained using the series of writing functions , let consider

$$R_1(y) = y^\mu \sum_{n=0}^{\infty} \alpha_n y^n \quad , \quad R_2(y) = y^\nu \sum_{n=0}^{\infty} \beta_n y^n \tag{17}$$

By inserting in equations 15 and 16 and taking into account that the coefficients of equal degrees of y are equal to each other, we find that $\mu = \nu$ and the first coefficient is equal to

$$\mu\alpha_0 = -F^2(\omega - \frac{eQ}{r_+})\beta_0 \quad , \quad -\mu\beta_0 = +F^2(\omega - \frac{eQ}{r_+})\alpha_0$$

therefore we must have

$$\omega = \frac{eQ}{r_+} \tag{18}$$

and this means the energy level of the particle near the event horizon must be one and the value is $\frac{eQ}{r_+}$.
Now considering this result, eqs.15 , 16 can be solved more easily , so

$$[y\frac{d}{dy} - 2kFy]R_1(y) = -FmyR_2(y) \tag{19}$$

$$[-y\frac{d}{dy} - 2kFy]R_2(y) = -FmyR_1(y) \tag{20}$$

and we have

$$[y^2\frac{d^2}{dy^2} + F^2(m^2 - 4k^2)y^2]R(y) = 0 \tag{21}$$

and the answer of this equation is

$$
\begin{aligned}
R(y) &= D_1 exp[F\sqrt{4k^2 - m^2}y] \\
&+ D_2 exp[-F\sqrt{4k^2 - m^2}y] \ , \ (r \to r_+)
\end{aligned} \tag{22}
$$

Now, according to equation 5, the wave function is

$$\psi(r) = \begin{cases} \frac{D_1 e^{F\sqrt{4k^2-m^2}\sqrt{\frac{r}{r_+}-1}} + D_2 e^{-F\sqrt{4k^2-m^2}\sqrt{\frac{r}{r_+}-1}}}{r} \ , \ r \to r_+ \\ \frac{C_1 J_{-\frac{1}{2}+k}(-i\sqrt{m^2-\omega^2}r) + C_2 Y_{-\frac{1}{2}+k}(-i\sqrt{m^2-\omega^2}r)}{\sqrt{r}} \ , \ r \to \infty \end{cases} \tag{23}$$

Therefore, the wave function in the two regions is as above . In figure 1 we plot the wave function of r . As we expected, the wave function at infinity behaves normally in a sinusoidal manner, and on the other hand, another result that can be obtained is that the wave function tends to infinity around the event horizon. Of course, it should also be taken into account that the particle under our investigation has an energy level level, and using this amount of energy, we obtained that the wave function of a particle with spin 1/2 tends to infinity near the event horizon but it dependent what value have the coefficients , that means if the coefficients can create zero wave function near the event horizon. Slow down and pay attention that we can only consider this result for the proximity of the event horizon and we cannot talk about the surface of the black hole, the event horizon.
Note that we only obtained the solution for the limits of the event horizon and infinity, and for the general wave function for the quantum particle we have to solve equations 7 and 8, which is analytically very difficult.

IV. CONCLUSION

In conclusion, the study of the wave function of a 1/2 particle by the Dirac equation in curved space-time for the Reisnner-Nordström metric has revealed some fascinating insights. The research has shown that near the event horizon, there is only one level of energy with $\frac{eQ}{r_+}$ values, and the wave function curve goes to infinity. This finding is significant because it sheds light on the behavior of particles in the vicinity of a charged black hole. Furthermore, the study has demonstrated that as we move away from the event horizon towards infinity, the wave function returns to a normal state, resembling a sine wave. These findings have important implications for our understanding of the behavior of particles in the extreme conditions of space-time near a charged black hole. Overall, this research has contributed significantly to our knowledge of the fundamental principles of physics and has opened up new avenues for further exploration in this field.

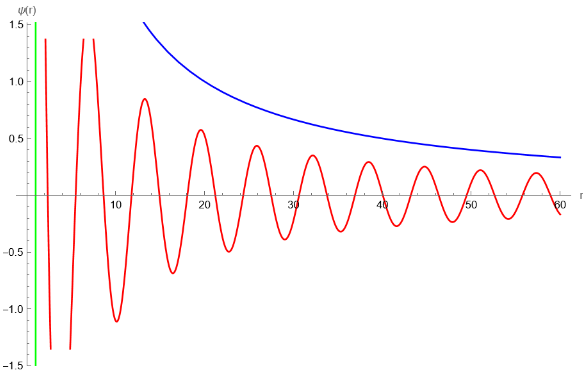

FIG. 1. In this image, the values were selected as $\omega = 1, C_1 = C_2 = D_1 = D_2 = 10, m = 0, F = 1, r_+ = 1$. As we expected, at infinite distance, the wave function has its normal state (red shape) and it alternates sinusoidally, and on the other hand, at the limits close to the event horizon (blue shape), the wave function tends to infinity. The green shape is radius of event horizon .

[1] V.I. Dokuchaev, Yu.N. Eroshenko *Stationary Solutions of the Dirac Equation in the Gravitational Field of a Charged Black Hole*.

[2] Vojtěch Witzany, Gabriel Andres Piovano *Analytic solutions for the motion of spinning particles near spherically symmetric black holes and exotic compact objects*.

[3] S. Weinberg *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (Wiley, New York, 1972; Platon, Volgograd, 2000)..*

[4] V.I. Dokuchaev *Class. Quant. Grav. 28, 235015 (2011)..*

[5] Ozlem Yesiltas *Dirac Equation In The Curved Spacetime and Generalized Uncertainty Principle: A fundamental quantum mechanical approach with energy dependent potentials*.

[6] Ahmed Farag Ali, Mohammed M. Khalil, Elias C. Vagenas *Minimal Length in quantum gravity and gravitational measurements*.