Group Theory and Irreducible Representations of the Poincaré Group

Meysam Hassandoust

Physics Faculty

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Supervisor
Prof. S. S. Gousheh

Contents

1 Introduction

In this report, we examine the irreducible representations of the Poincaré group. For this, we start with the definition and structure of the theory of groups, and with the basic definitions, we try to define the groups and symmetries that we do in our daily life, and a little further, we try to examine them more deeply. After we encountered a simple definition and some simple and physical examples about groups in the first chapter, we try to define the structures of groups with a little focus. This will help us to have a closer connection with what we will see in the next chapters.

After these two chapters, we will discuss an important part of the theory of groups, which are actually very concrete examples of what we have dealt with so far. We will start with very simple groups like $GL_n(\mathbb{R})$ and gradually we will talk about groups that do not appear very commonly in physics like $SP(2,\mathbb{R})$ but are very important, and we will check all the information we got from the previous chapters for these groups.

In the following, we will examine how groups perform actions in reality and somehow establish a connection between mathematics or even physics with nature. After that, we will discuss one of the most important parts of the theory of groups called Lie group and Lie algebra, which contains the effects of all the previous chapters, and for this reason, we will try to define all the topics related to this part, such as Topology and Manifold, etc. in a very precise manner.

In the chapter on the theory of representations, which is the main part of this report, we have very briefly tried to define the basic definitions of representations in the theory of Lie groups and algebras, and we have stated and proved one of the most basic lemmas in this theory, which is strongly The need for our work is in the last chapters. In the last two chapters, we tried to summarize two of the most important groups by using the information obtained from the previous chapters, and then we obtained the irreducible representations for the Poincaré group using the previous chapters.

Finally, I must thank Mrs. Mohammad Sadeghi for her careful editing and proofreading of the text.

2 Group Theory

2.1 Introduction to the Group

In our daily life, we come across many symmetries. From the symmetry of a flower to a landscape or a building in terms of architecture or coloring. In the description of nature as we encounter it in physics, symmetry plays a more fundamental role and its effect is not limited to art and aesthetics. In physics and in general, we come across more and more symmetries. For example, three-dimensional space has translation and Rotation symmetry. In other words, it is the same as the laws of physics or nature regarding transfer and rotation in three-dimensional space. The four-dimensional space-time has more and deeper symmetries. The laws of nature are the same in the internal frame of reference that is moving in a straight line with respect to each other, and for this reason, we cannot know the movement of the internal frame of reference that I am riding on with any physical experiment.
At the microscopic level, there are other symmetries. These symmetries are not as obvious as the symmetries of a flower, but they have important physical effects. For example, the strong nuclear force is completely symmetric with respect to the exchange of protons and neutrons. This symmetry is a discrete symmetry that is described by the Z_2 group. We can say that the strong nuclear force acts on only one nucleon and it can be in one of two states, which we call proton and neutron, just like electron spin. For this reason, we call this symmetry **isospin symmetry**.
Now, on the other hand, in quantum mechanics, we know that we can expand the state of a particle in terms of the bases that make up that space, so in this example, we can define the state of a particle as

$$|\Psi\rangle = a\,|proton\rangle + b\,|neutron\rangle$$

Therefore the nucleon is described by the binary vector which has symmetry under the $U(2)$ group.
In the microscopic world, we have many symmetries like color symmetry and gauge symmetry[1][2][3][4]. Group theory is a branch of mathematics that describes symmetry precisely. In general, symmetry means that an entity or object does not change under the influence of a certain set of actions and leaves the feature or property of that object intact. For example, if we assume that the object is a vase that is symmetric about its z-axis, then it means that it is symmetric under a set of actions that rotate the object about its z-axis by an angle θ.
In other words, if we consider the object O and consider the property or characteristic we want about the object as $P(O)$ and action g, then g preserves

our property if we have

$$O^{'} = g(O) \to P(O^{'}) = P(g(O)) = P(O)$$

Now consider another action $g^{'}$ where

$$O^{''} = g^{'}(O^{'}) = g^{'}(g(O)) = (g^{'}g)(O)$$

Then $g^{'}g$ preservers P because

$$P(O^{''}) = P((g^{'}g)(O)) = P(g^{'}(g(O))) = P(g(O)) = P(O)$$

Therefore, we can say that the set of operations that preserves the property of the object is closed compared to the operation of multiplication. Also in this set, there is an action like e that does nothing on the object so $e(O) = O$ then we have

$$eg = ge = g$$

and if we have action, g^{-1} where

$$g^{-1}g = gg^{-1} = e$$

Then this set of actions with multiplication is **Group** [5].

2.2 Group

Now that we have explained symmetries and groups, we want to define groups more precisely and mathematically.

Definition 2.1 (Group)**.** A Group is a non-empty set G with a binary operation on G, here denoted " $*$ " that $* : G \times G \to G$ are satisfied

$$1.\ \forall a, b \in G \mid a * b \in G \tag{1}$$

$$2.\ \forall a, b, c \in G \mid (a * b) * c = a * (b * c) \tag{2}$$

$$3.\ \exists e \in G \mid e * a = a \tag{3}$$

$$4.\ \forall a \in G\ ,\ \exists a^{-1} \in G \mid a^{-1} * a = a * a^{-1} = e \tag{4}$$

if $a * b = b * a$ then G is **Abelian Group**.
Let us have some basic theorems of the group.

Theorem 2.1 (Uniqueness)**.** The identity element of the group is unique.

Proof. Assume, e and $e^{'}$ are identity elements of G. So

$$\forall g \in G | e * g = e^{'} * g = g \rightarrow e = e^{'}$$

□

Theorem 2.2 (Uniqueness)**.** The inverse element of the group is unique.

Proof. Assume, g_1^{-1} and g_2^{-1} are inverses elements for g. Then

$$g_1^{-1} * g = g_2^{-1} * g = e \rightarrow g_1^{-1} = g_2^{-1}$$

□

Theorem 2.3. For all $g \in G$ we have, $(g^{-1})^{-1} = g$

Proof. Assume, $g^{-1} \in G$. Then inverse element of g^{-1} is $(g^{-1})^{-1}$. So

$$(g^{-1})^{-1} * (g^{-1}) = e$$

If g is the right action then

$$(g^{-1})^{-1} = g$$

□

Theorem 2.4. For all $g_1, g_2 \in G$ we have, $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$

Proof. Assume, $(g_1 g_2)^{-1} \in G$. Then inverse element of $(g_1 g_2)^{-1}$ form Theorem 2.3 is $g_1 g_2$. So

$$(g_1 g_2)^{-1}(g_1 g_2) = e$$

If g_1 and g_2 right action then

$$(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$$

□

Definition 2.2 (Order)**.** The order of a group is the number of its elements and is donated by $|G|$. If a group is not finite, one says that its order is infinite.

There are different types of groups such as finite groups, infinite groups, and continuous groups....
First, we define finite groups.

2.3 Finite group

Definition 2.3 (Finite group)**.** In abstract algebra, a finite group is simply a group with a finite number of elements.

One of the most famous finite groups is the Cyclic group, which is defined as follows.

Definition 2.4. A group is called cyclic if it can be generated by a single element. In other words, there exists an element $a \in G$, all elements of G come from the set

$$< x >= \{\dots, x^{-1}, e, x^1, \dots\} \tag{5}$$

x is a **generator**.

The notation for the group generated by S is $< S >$.
Note for an infinite group, we need to consider inverses explicitly. For a finite group, inverses occur in the positive powers [6].

Example 2.1. The Cyclic Group Z_n define as below

$$Z_n := \{0, 1, ..., n-1\} \mod n$$

This Group is called a Cyclic group whose order is n.

Example 2.2. The Cyclic Group Z_p, where p is prime number and define as below

$$Z_p := \{0, 1, ..., p-1\} \mod p$$

This Group is called a Cyclic group whose order is p.

Example 2.3. Consider Figure 1. The following two actions can be performed on this form. An e action that does nothing and a π_y action that rotates the shape reflected around the y-axis. Therefore, we can say, the symmetry group of this object is $Z_2 = \{e, \pi_y\}$.

Example 2.4. Consider Figure 2. The following three actions can be performed on this form. An e action that does nothing a π_y action that rotates the shape reflected around the y-axis and a π_x action that rotates the shape reflected around the x-axis. Therefore, we can say, the symmetry group of this object is $Z_4 = \{e, \pi_x, \pi_y, \pi_x\pi_y\}$.

Figure 1: Fern - bilateral symmetry [7].

Figure 2: This photograph has bilateral symmetry from left to right and from top to bottom due to the reflection of the landscape in water [7].

Example 2.5 (Pauli group)**.** Two-dimensional matrices known as Pauli matrices are as follows

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

These matrices have this relation

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k$$

So we have a non-Abelian group

$$G_0 = \{\pm I, \pm\sigma_1, \pm\sigma_2, \pm i\sigma_3\} \tag{7}$$

and also

$$G_1 = \{\pm I, \pm\sigma_1, \pm\sigma_2, \pm\sigma_3, \pm iI, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3\} \tag{8}$$

Definition 2.5 (Symmetric group)**.** The symmetric group has some properties
1. The elements of the group are permutations on the given set (i.e., bijective maps from the set to itself).
2. The product of two elements is their composite as permutations, i.e., function composition.
3. The identity element of the group is the identity function from the set to itself.
4. The inverse of an element in the group is its inverse as a function [8].

Example 2.6 (Permutation group)**.** A famous example of this group is the Permutation group, which is represented by S_n, and a member of this group can be shown as follows

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ \alpha(1) & \alpha(2) & \dots & \alpha(n) \end{pmatrix} \tag{9}$$

Since this group itself is a type of the symmetric group, if $\alpha, \beta \in S_n$ then $\alpha\beta \in S_n$. Therefore S_n is closed under multiplication and for all $\alpha \in S_n$, there is inverse, $\alpha^{-1} \in S_n$.

Example 2.7. Consider S_2. The elements of S_2 are

$$e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad , \quad \alpha = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{10}$$

Then we have abelian group, $S_2 = \{e, \alpha\}$.

Example 2.8. Let's consider S_3. The elements of S_3 are

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad , \quad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \tag{11}$$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad , \quad \delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad , \quad \eta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \tag{12}$$

This group is non abelian and $S_3 = \{e, \alpha\,\beta, \gamma, \delta, \eta\}$.

2.4 Infinite group

Unlike finite groups, the order of an infinite group does not become a finite number. One of the most famous examples of infinite groups is the Permutation group.

Example 2.9 ($\mathcal{Z}$ group with integer addition)**.** Set of Integer numbers with integer addition also called $\mathcal{Z}$.

Example 2.10 ($\mathcal{Z} - \{0\}$ group with integer multiplication)**.** Set of Integer numbers except 0 with integer multiplication also called $\mathcal{Z} - \{0\}$.

Example 2.11 ($\mathcal{R}$ group with addition)**.** Set of Real numbers with addition also called $\mathcal{R}$.

Example 2.12. One of the most famous groups is **Braid group** [9]. This group is very important in the study of node topology.

Definition 2.6 (Braid group)**.** Suppose there are n points $p_1, \ldots, p_n$ in a plane and consider a plane parallel to this plane with these points. Now consider a set of curves that connect the bottom point to the top plane points, provided that these curves do not intersect each other, but can have any desired shape. Mathematically, we say that only the Homotopy class [10, 11] of curves is important to us, and not the curves themselves, and for this reason, every set of curves, or in other words, its homotopy class, is a member of the braid group [5]. In mathematically

$$\gamma : [0,1] \to \mathbb{R}^2 \times [0,1] \mid \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t)) \tag{13}$$

where

$$\gamma_1(0) = p_1, \ldots, \gamma_n(0) = p_n \tag{14}$$

and endpoints are permutations of initial points, in other words, $(\gamma_1(t), \ldots, \gamma_n(t))$ is permutations of initial points.
Each member of the Braid group is a curve set or homotopy class. We display the members of this group with letters $\alpha, \beta, \ldots$ and Braid group has called B_n
For example in Figure 3 one of element of B_3 which $\alpha = (\gamma_1(1), \gamma_2(1), \gamma_3(1)) = (1, 3, 2) \in B_3$.
And because homotopy class is important for us and the elements dependent on it, we have equivalence in some curves. For example Figure 4 we have equivalence in B_4.

Now we have to define an action, an identity element, and an inverse element for this group according to the definition. The operation of multiplying this group is to assume that α and β are two homotopy classes (set of curves) as follows

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad , \quad \beta = (\beta_1, \ldots, \beta_n) \tag{15}$$

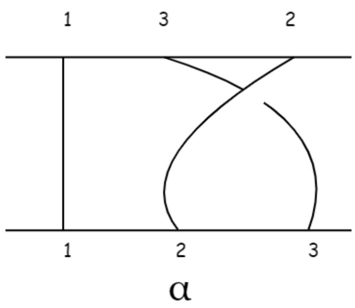

Figure 3: α is one of elements of B_3

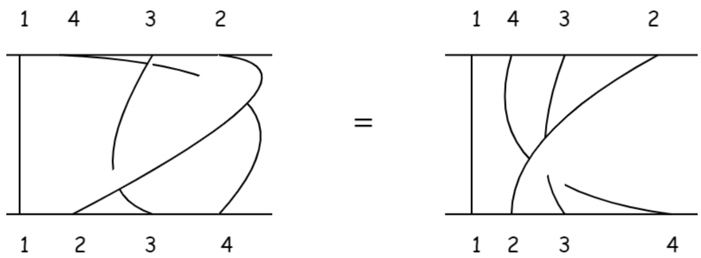

Figure 4: They are equivalent

that the indices define our initial points and the member's action and operation of multiplication is

$$\alpha : (1, \ldots, n) \to (\alpha_1, \ldots, \alpha_n) \tag{16}$$

$$\beta : (1, \ldots, n) \to (\beta_1, \ldots, \beta_n) \tag{17}$$

$$\beta\alpha : (\alpha_1, \ldots, \alpha_n) \to (\beta_{\alpha_1}, \ldots, \beta_{\alpha_n}) \tag{18}$$

So

$$\beta\alpha = (\beta_{\alpha_1}, \ldots, \beta_{\alpha_n})$$

For example, if in B_3 we have

$$\alpha = (2, 3, 1) \ , \ \ \beta = (1, 3, 2)$$
$$\beta\alpha = (1, 3, 2)$$

In Figure 5 we plot $\beta\alpha$ for B_3. So we define the multiplication operation for the Braid groups. Let's define the identity element in the Braid group. As we know the identity element, is an element that does not work on other elements, ie. so in the braid group the identity element is a homotopy class that does not work in the Braid group except that it connects the lower points to the upper points without any screws.

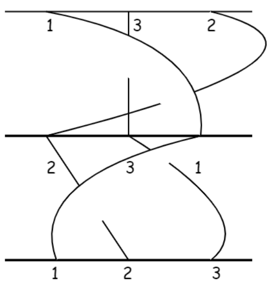

Figure 5: The multiplication operation in the Braid group, B_3 for $\alpha = (2, 3, 1)$ and $\beta = (1, 3, 2)$.

Now we can define the inverse element. Consider $\gamma : [0, 1] \to \mathbb{R}^3$ is curved. A curve that goes in the opposite direction is defined as follows

$$\gamma^{-1}(t) := \gamma(1 - t) \tag{19}$$

So the inverse element of the Braid group define

$$\gamma^{-1} = (\gamma_1^{-1}, \dots, \gamma_n^{-1}) \tag{20}$$

Therefore we define the Braid group by multiplication operator, identity element, and inverse element.

2.5 Continuous group

Definition 2.7 (Continuous group)**.** In a very general definition, it can be said that there are infinite groups whose elements are uncountable.

Example 2.13 (Invertible functions)**.** The set of invertible functions on $[0, 1] \in \mathbb{R}$, together with the multiplication operation, is a continuous group.

Example 2.14 (GL)**.** The set of invertible linear transformations on a vector space V that is represented by $GL(V)$ is called the group of general linear transformations on V. If V is a n-dimensional vector space on field $\mathbb{F}$ then $GL_n(\mathbb{F})$.

As we know, each of the inverse linear transformations in the vector space can be represented as a matrix.

Example 2.15 ($SL_n(\mathbb{R})$)**.** A subset of matrices $GL_n(\mathbb{R})$ that $det(A) = +1$ is the group of real n-dimensional matrices denoted by $SL_n(\mathbb{R})$.

Example 2.16 ($O_n(\mathbb{R})$)**.** A subset of matrices $GL_n(\mathbb{R})$ that are orthogonal, $A^{\dagger}A = I$, is the group of real n-dimensional matrices denoted by $O_n(\mathbb{R})$.

Example 2.17 ($SO_n(\mathbb{R})$)**.** A subset of matrices $O_n(\mathbb{R})$ that are orthogonal, $A^{\dagger}A = I$, and $det(A) = +1$ is the group of real n-dimensional matrices denoted by $SO_n(\mathbb{R})$.

Example 2.18 ($U(n)$)**.** A subset of matrices $GL_n(\mathbb{R})$ that is unitary is the group of real n-dimensional matrices denoted by $U(n)$.

We will discuss this section further.

2.6 Multiplication table

Now that we have defined the types of groups and given examples for them, we come to the point where we can define a multiplication table for each group. For example, consider the group S_3. From example 2.8, we know that the elements of this group are as follows

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad , \quad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \tag{21}$$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad , \quad \delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad , \quad \eta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \tag{22}$$

Now we can have a multiplication table for this group as follows

.	e	α	β	γ	δ	η
e	e	α	β	γ	δ	η
α	α	e	δ	η	β	γ
β	β	η	e	δ	γ	α
γ	γ	δ	η	e	α	β
δ	δ	γ	α	β	η	e
η	η	β	γ	α	e	δ

Example 2.19. Consider $G_1 = \{1, -1\}$ under multiplication. The Multiplication table of G_1 is

.	1	-1
1	1	-1
-1	-1	1

which acts like group $G = \{e, a\}$, $a^2 = e$.

Example 2.20. Consider $G_2 = \{0, 1\}$ under addition mod 2. The Multiplication table of G_2 is

+	0	1
0	0	1
1	1	0

which acts again like group $G = \{e, a\}$, $a^2 = e$.

Example 2.21. Consider $G_3 = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ under multiplication. The Multiplication table of G_3 is

.	e	α
e	e	α
α	α	e

where $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Which acts again like group $G = \{e, a\}$, $a^2 = e$.
As we have seen, these groups acted with a group of the equivalent or were in some way Isomorphism. In the next sections, we will discuss this similarity between groups.

3 Group Structure

Understanding group structures helps to understand groups and the relationships between them and their related issues.

3.1 Subgroup

Definition 3.1. A subset H of a group G is called a subgroup if it is not empty, closed under group operation, and has inverses. The notation $H \leq G$ denotes that H is a subgroup of G.

Every group G has two trivial subgroups, e and the group G itself. Let's look at a few examples

Example 3.1. $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.

Example 3.2. $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$ under addition.

Example 3.3. The set $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is a subgroup of $\mathbb{R}$ under addition.

Example 3.4. $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$ under addition.

Example 3.5. $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$ under addition.

Example 3.6. $SO_n(\mathbb{R})$ is a subgroup of $O_n(\mathbb{R})$ under addition.

Example 3.7. $\{e, \alpha\}$ is a subgroup of S_3.

Example 3.8. $\{e, \sigma_1\}$ is a subgroup of Pauli group.

Based on the definition and examples of subgroups, we express and prove theorems in subgroups.

Theorem 3.1. If H is a subset of G then H is a subgroup of G if and only if

1. $\forall a, b \in H \mid ab \in H$
2. $\forall a \in H \mid a^{-1} \in H$

Proof. If H is a subgroup of G then 1 and 2 are trivial. Let's assume 1 and 2 be established, we have

$$ab = aa^{-1} = e \in H \tag{1}$$

then $e \in H$ and H is a subgroup of G. □

Theorem 3.2. If G is a finite group and H is a subset of G then H is a subgroup of G if and only if

1. $\forall a, b \in H \ | \ ab \in H$

Proof. If H is a subgroup of G then 1 is trivial. Let's assume 1 is established. Consider $\{a, a^2, \ldots, a^n\} \in H$ and from 1 we have $aa^n = a^{n+1} \in H$. Then $\exists\, m < n+1$ where

$$a^{n+1} = a^m \in H \tag{2}$$

$$a^{n+1-m} = a^m a^{-m} = e \in H \tag{3}$$

then H has inverse element and $e \in H$ so H is a subgroup of G. □

3.2 Cosets

In the theory of numbers, two numbers a and b remain to the number n if their difference is a multiple of n, that is $a - b = kn$ then we call a and b are **equivalent**. For example if $a, b \in \mathbb{Z}$ and have $n\mathbb{Z}$ group then a and b are equivalent if $a - b$ in subgroup of $n\mathbb{Z}$. In other words

$$a \sim b \quad \rightarrow \quad a - b \in n\mathbb{Z} \tag{4}$$

Let's define equivalent in a subgroup.

Definition 3.2. If H is a subgroup of G then

$$a \sim_R b \ \ mod\, H \quad \rightarrow \quad ab^{-1} \in H \tag{5}$$

where $\sim_R$ is a right equivalent.

In other words, assume $h \in H$ then $ab^{-1} = h$ or $a = bh$. Therefore, if we want to find all the elements equivalent to a, we must multiply all the elements of $h \in H$ from the right side. In this way, a subgroup of G is obtained, which is as follows

$$aH := \{ah \ | \ h \in H\} \tag{6}$$

this set is called the Right coset of a.
In this set, all elements are equivalent to each other, and any element which is equivalent to a is in this set.

Definition 3.3. If H is a subgroup of G then

$$a \sim_L b \ \ mod\, H \quad \rightarrow \quad a^{-1}b \in H \tag{7}$$

where $\sim_L$ is a left equivalent.

As in the previous definition, we have

$$Ha := \{ha \mid h \in H\} \tag{8}$$

this set is called the Left coset of a.
Note that two equivalence relations 6 and 8 Partition a group into two different types and they are different from each other. For example, in Figures 1 and 2, we showed **Right cosets** and **Left cosets**. Now, if H is a finite

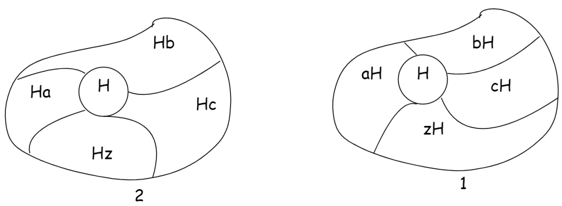

Figure 6: In 1 right cosets of $a, b, \ldots$ and in 2 left cosets of $a, b, \ldots$. As shown, these two types of equivalence are different.

subgroup, then all cosets have the same number of elements as the number of elements of H or the order of H, and at this point, we can define a new theorem.

Theorem 3.3 (Lagrange)**.** If H is a subgroup of a group G, then $|H| \mid |G|$.

Proof. Considering that the number of elements of each coset is equal to the order H, assume $\{a_1, \ldots, a_n\} \in G$ then

$$|a_1H| = |a_2H| = \ldots = |a_nH| = H \tag{9}$$

$$|G| = \sum_{i}^{n} |a_iH| + |H| = k|H| \quad \rightarrow \quad |H| \mid |G| \tag{10}$$

□

Definition 3.4. If $a \in G$ we call the smallest number m, which is $a^m = e$, the order of a, and denote it by $|a|$. If there is no exist m then the order of a is infinite.

Theorem 3.4. For all a in G we have $|a| \mid |G|$

Proof. As we know consider $< a >= \{e, a, a^2, \ldots\}$ and from proof of Theorem 2.6, $< a >$ is a subgroup of G. So from Theorem 2.7, we have $|a| \mid |G|$. □

Now, from this theorem, we can reach interesting theorems in group theory and number theory. We do not prove these theorems and can see their proof in [12][13][14].

3.3 Normal subgroup

As we have shown, the right coset and the left coset are not necessarily the same, but in a special case, these two can be the same. In other words, we can write bh as $h^{'}b$ for every $b \in G$ and $h \in H$ where $h^{'} \in H$.

Definition 3.5. H is a **normal subgroup** of G if

$$\forall\, g \in G \;,\; \forall\, h \in H \;\mid\; ghg^{-1} \in H \tag{11}$$

Note for normal subgroups the right and left cosets are equal. In Figure 7 this point is shown.

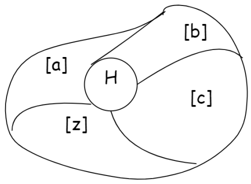

Figure 7: For a normal subgroup the right and left cosets are equal. That means $aH = Ha$ and it is shown as $[a], [b], \ldots$

Theorem 3.5. If H is a normal subgroup of G then for all $a \in G$ we have $aH = Ha$.

Proof. First, we prove that $aH \subset Ha$. So if $x \in aH$ then $x = ah$ and

$$x = ah = h^{'}a \in Ha \rightarrow aH \subset Ha \tag{12}$$

On the other hand to prove $Ha \subset aH$

$$x = ha = ah^{'} \in aH \rightarrow Ha \subset aH \tag{13}$$

So $aH = Ha$. □

Theorem 3.6. H is a normal subgroup of G if and only if

$$\forall\, g \in G \;\mid\; gHg^{-1} = H \tag{14}$$

Proof. First, assume H is a subgroup of G then from Definition 2.12 we have $gHg^{-1} \subset H$. Now consider $h \in H$ and then

$$h = g(g^{-1}hg)g^{-1} = gh^{'}g^{-1} \in gHg^{-1} \tag{15}$$

$$\rightarrow H \subset gHg^{-1} \tag{16}$$

So $gHg^{-1} = H$. □

Now, according to the definitions and theorems, we will give some examples of normal subgroups.

Example 3.9. $SL_n(\mathbb{R})$ is a normal subgroup of $SL_n(\mathbb{R})$.

Example 3.10. In $SL_n(\mathbb{R})$ the subgroups which have I and $-I$ matrices are normal subgroup.

Example 3.11. Every subgroup of an abelian group is the normal subgroup.

3.4 Factor group

According to the definition of a normal subgroup, now we want to make a new definition of all normal subgroups of a group.

Definition 3.6. Let H be a normal subgroup of G. Define the set G/H to be the set of all left(right) cosets of H in G and define a multiplication operation for all cosets which if $[a] = aH(Ha)$ then $[a][b] := [ab]$. So G/H is a group and called **Factor group**.

Let's take a few examples.

Example 3.12. Consider $G = \mathbb{Z}$ and $H = n\mathbb{Z}$. As mentioned in example 2.28, $H = n\mathbb{Z}$ is a normal subgroup. The subgroup H has the following cosets

$$[k] := k + H = k + n\mathbb{Z} = \{k + na \mid a \in \mathbb{Z}\} \quad , \quad k = 0, 1, \ldots \tag{17}$$

In other words, $[k]$ is the class of all numbers whose remainder when divided by n is equal to k. So

$$[k] + [l] = [k + l] \tag{18}$$

then $[k + n] = [k]$. Therefore we have

$$\mathbb{Z}/n\mathbb{Z} \equiv \mathbb{Z}_n \tag{19}$$

Example 3.13. Assume $G = U(2)$ and $H = SU(2)$ is a normal subgroup cause if $h \in H = SU(2)$ then $det(h) = det(ghg^{-1}) = 1$. Now we have

$$g = e^{i\phi/2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \in U(2) \quad , \quad [g] = gh = \{gh \in U(2) \mid det(g) = e^{i\phi}\} \tag{20}$$

and

$$[g][g^{'}] = [e^{i\phi}][e^{i\psi}] = [e^{i(\phi+\psi)}] = [gg^{'}] \tag{21}$$

So

$$U(2)/SU(2) \equiv U(1) \tag{22}$$

and more generally

$$U(n)/SU(n) \equiv U(n) \tag{23}$$

Example 3.14. Assume $G = U(2)$ and $H = U(1)$ is a normal subgroup cause if $h \in H = U(1)$ then $det(h) = det(ghg^{-1}) = 1$. Now

$$g = e^{i\phi/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U(2) \quad , \quad [g] = gh = \{gh \in U(2) \mid det(g) = 1\} \tag{24}$$

then

$$U(2)/U(1) \equiv SU(2) \tag{25}$$

3.5 Homomorphism and Isomorphism

As we said in the previous sections $G_1 = \{1, -1\}$ under multiplication, $G_2 = \{0, 1\}$ under addition mod 2 and $G_3 = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ under multiplication although they look completely different, they all act like $G = \{e, a\} \ , \ e = a^2$. All three are the equivalent of $G = \{e, a\}$. The equivalent of these three groups means that they can be mapped to each other so that their multiplication table does not change. These groups have the same structure and cannot be considered as three different groups. In this section, we discuss the equality or **Isomorphism** between groups.

Definition 3.7. Let $(G, .)$ and $(H, *)$ be groups and $f : G \to H$. Then f is a **homomorphism** if

$$\forall g_1, g_2 \in G \mid f(g_1.g_2) = f(g_1) * f(g_2) \tag{26}$$

Then G and H are homomorphic.

Definition 3.8. If a homomorphism is also a bijection, then it is called an **isomorphism**. Then G and H are isomorphic.

From the definitions above, two theorems can be reached.

Theorem 3.7. If e and $e^{'}$ are identity elements of G and H, respectively and G and H are isomorphic then

$$f(e) = e^{'} \tag{27}$$

Proof. From Definition 2.15, if $g_1 = e = g_2$ and $e' = f(g_1) = f(g_2)$ then

$$f(e) = e' \tag{28}$$

□

Theorem 3.8. If G and H are isomorphic then

$$f(g^{-1}) = f(g)^{-1} \tag{29}$$

Proof. From Definition 2.15, if $g_1 = g$ and $g^{-1} = g_2$ then we have

$$f(g.g^{-1}) = f(g) * f(g^{-1}) = f(e) = e' \to f(g^{-1}) = f(g)^{-1} \tag{30}$$

□

In the following examples, we discuss the homomorphism and isomorphism of the groups that we have met so far.

Example 3.15. A map $f : GL_n(\mathbb{R}) \to SL_n(\mathbb{R})$ where $f(g) = \frac{g}{det(g)}$, is a homomorphism.

Example 3.16. A map $f : GL_n(\mathbb{R}) \to \mathbb{R} - \{0\}$ where $f(g) = det(g)$, is a homomorphism.

Example 3.17. Consider $U(1) = \{e^{i\phi} \mid \phi \in [0, 2\pi]\}$ and
$SO(2) = \{\begin{pmatrix} cos\ \theta & sin\ \theta \\ -sin\ \theta & cos\ \theta \end{pmatrix}$, $\theta \in [0, 2\pi]\}$. Now define $f : SO(2) \to U(1)$. Then $U(1)$ and $SO(2)$ are isomorphic.(Proof [5] [15].)

Example 3.18. Consider $\mathbb{Z}$ and $n\mathbb{Z}$. Let's define $f : \mathbb{Z} \to n\mathbb{Z}$. So we have

$$f : \mathbb{Z} \to n\mathbb{Z},\ \ \forall z \in \mathbb{Z} \mid f(z) = nz\ \ ,\ \ f^{-1}(nz) = z \tag{31}$$

Thus

$$\forall z_1, z_2 \in \mathbb{Z} \mid f(z_1.z_2) = nz_1z_2 = f(z_1)f(z_2) \tag{32}$$

So $\mathbb{Z}$ and $n\mathbb{Z}$ are isomorphic.

Example 3.19. B_2 and $\mathbb{Z}$ are isomorphic.(Proof [16])

Now we represent theorems about homomorphism that are very practical and important.

Theorem 3.9. If $f : G \to H$ is a homomorphism, then $Ker(f)$ is a normal subgroup of G, and $Im(f)$ is a subgroup of H.

Proof. Assume $a, b \in Ker(f)$ then from definition of $ker(f)$ we have

$$f(a) = e^{'} = f(b) \;\;,\;\; f(ab) = f(a)f(b) = e^{'} \rightarrow ab \in Ker(f) \tag{33}$$
$$f(a) = e^{'} \;\;,\;\; (f(a))^{-1} = f(a^{-1}) = e^{'} \rightarrow a^{-1} \in Ker(f) \tag{34}$$

So from eqs.33 and 34, we obtain $Ker(f)$ is subgroup of G. Let's prove the $Ker(f)$ is a normal subgroup. So consider $a \in Ker(f)$ and $g \in G$ then

$$f(a) = f(gag^{-1}) = e^{'} \rightarrow gag^{-1} \in Ker(f) \tag{35}$$

So $Ker(f)$ is a normal subgroup of G.
Let's assume $a^{'}, b^{'} \in Im(f)$ then from definition of $Im(f)$ we have

$$a^{'} = f(a) \;\;,\;\; b^{'} = f(b) \rightarrow a^{'}b^{'} \in Im(f) \tag{36}$$
$$a^{'} = f(a) \;\;,\;\; a^{'-1} = f(a^{-1}) \rightarrow^{'-1} \in Im(f) \tag{37}$$

Therefore $Im(f)$ is a subgroup of H. □

Theorem 3.10. A map $f : G \rightarrow H$ is an isomorphism if and only if f is bijective and $Ker(f) = \{e\}$.

Proof. Consider $f(a) = f(b)$ then we have

$$f(a) = f(b) \rightarrow f(a)f^{-1}(b) = f(ab^{-1}) = e^{'} \tag{38}$$
$$f^{-1}(f(ab^{-1})) = ab^{-1} = e \rightarrow a = b \tag{39}$$

□

Theorem 3.11. If a map $f : G \rightarrow H$ is a homomorphism then $G/Ker(f) \sim Im(f)$.

Proof. As we know from Theorem 2.13, that $Ker(f)$ is a normal subgroup of G and so $G/Ker(f)$ is a closet. We define a map $f : G/Ker(f) \rightarrow Im(f)$ where

$$\forall [a] \in G/Ker(f) \;\;,\;\; f([a]) = x \in Im(f) \tag{40}$$

Now we prove the map is isomorphism. So

$$\forall [a], [b] \in G/Ker(f) \;\;,\;\; f([a])f([b]) = f([a][b]) \;\; : Homomorphism \tag{41}$$
$$if\ f([a]) = f([b]) \rightarrow f([a])f^{-1}([b]) = f(ab^{-1}) = e^{'} \rightarrow [a] = [b] \;\; : Bijective \tag{42}$$

So $G/Ker(f) \sim Im(f)$. □

Now let's examine an important example about isomorphic in physics.

Example 3.20 ($SU(2)/\mathbb{Z}_2 \sim SO(3)$)**.** As we know in Example 2.17, the rotation group in three-dimensional is $SO(3)$. Let's assume $A \in SO(3)$ is a rotation and $\overrightarrow{r}, \overrightarrow{r}' \in \mathbb{R}^3$ are vectors in euclidean space. Then

$$\overrightarrow{r}' = A\overrightarrow{r} \tag{43}$$

where $\overrightarrow{r}'$ is rotation of $\overrightarrow{r}$.
On the other hand, by using Pauli matrices, it can be said that every rotation can be represented by these matrices. In other words

$$\forall\overrightarrow{r} \in \mathbb{R}^3 \quad , \quad P := \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \equiv x_i\sigma_1 = \overrightarrow{r}.\overrightarrow{\sigma} \tag{44}$$

Now from $U \in SU(2)$ we have

$$P' = UPU^\dagger \tag{45}$$

and

$$\forall\overrightarrow{r}' \in \mathbb{R}^3 \quad , \quad P' := \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix} \equiv x_i'\sigma_1 = \overrightarrow{r}'.\overrightarrow{\sigma} \tag{46}$$

Therefore U rotate $\overrightarrow{r}$ to $\overrightarrow{r}'$ in three-dimensional and the interesting point is P and U are two-dimensional matrices! Therefore eqs.43 and 45 are equivalent. Now consider two roration

$$\overrightarrow{r}'' = A'\overrightarrow{r}' \quad , \quad \overrightarrow{r}' = A\overrightarrow{r} \tag{47}$$

$$P'' = U'P'U'^\dagger \quad , \quad P' = UPU^\dagger \tag{48}$$

so

$$\overrightarrow{r}'' = A'A\overrightarrow{r} \tag{49}$$

$$P'' = U'P'U'^\dagger = P'' = U'UPU^\dagger U'^\dagger = (U'U)P(U'U)^\dagger \tag{50}$$

Therefore $SU(2)$ and $SO(3)$ are homomorphic but not isomorphic because the map is not bijective and U and $-U$ generate rotation so we have

$$SU(2)/\mathbb{Z}_2 \sim SO(3) \tag{51}$$

4 Matrix groups

Whenever V is a real vector space, the set of invertible linear transformations from V to itself forms a group. When we choose a basis for the vector space, we can represent each transformation as a matrix. As a result, the node of linear transformations on a linear space is nothing but a group whose elements consist of invertible matrices, which are called matrix groups. Each group can be represented as a set of matrices, and in quantum mechanics, which is known as a fundamental framework for describing nature, it has a linear structure and physical states are represented in Hilbert space, and any type of transformation on these vectors are represented as a linear transformation with a matrix. In this section, we examine the most important matrix groups and in each case, we deal with infinitesimal transformations because these small transformations are used to obtain the generators of each group [5].

4.1 General linear group

A large and very important class of groups is defined as linear transformations on a vector space. The set of invertible linear transformations on a vector space V that is represented by $GL(V)$ is called the group of general linear transformations on V [5].
On the other hand, any transformation is an operator(map), and each mapping can be transformed into a matrix using the bases of the vector space on which the transformations were defined. Therefore if T is a map on vector space $\mathbb{V}$ with field $\mathbb{F}$, $\hat{T}$ is a matrix in which the elements of this matrix are in field $\mathbb{F}$. If the vector space has n dimensional then the group of general linear transformations on V with field $\mathbb{F}$ is called $GL_n(\mathbb{F})$.
Now consider $GL_2(\mathbb{R})$

$$GL_2(\mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \mid \ a, b, c, d \in \mathbb{R} \ , \ \ ad - bc \neq 0\} \tag{1}$$

This group is the simplest group of the general linear group. As we shown $GL_2(\mathbb{R})$ has 4 parameters so it is 4 dimensional group. On the other hand, this group is a continuous group or topological group and by changing the parameters, we can move from one point to another in this group.

4.2 Special linear group

Definition 4.1. A subset of matrices $GL_n(\mathbb{R})$ that are $det(A) = 1$, is the group of real n-dimensional matrices denoted by $SL_n(\mathbb{R})$. In other words

$$SL_n(\mathbb{R}) = \{g \in GL_n(\mathbb{R}), \ \mid \ det(A) = 1\} \tag{2}$$

The simplest group of the special linear group is $SL_2(\mathbb{R})$ where define

$$SL_2(\mathbb{R}) = \{g \in GL_2(\mathbb{R}), \ \mid \ det(A) = 1\} \tag{3}$$

Let's obtain generators of this group. There are two methods to get the generators of a group, we use the first method to get the generators and we will present the second method in the further sections. We know that $SL_2(\mathbb{R})$ is a three-parameter group. On the other hand, when we are near the identity element of the group, the parameters of the group can be written as follows

$$g \in SL_2(\mathbb{R}) \ \ , \ \ g = I + \mathcal{L} \approx \begin{pmatrix} 1+\epsilon_3 & \epsilon_1 - \epsilon_2 \\ \epsilon_1 + \epsilon_2 & 1 - \epsilon_3 \end{pmatrix} \tag{4}$$

then

$$det(g) = det(I + \mathcal{L}) = 1 + tr(\mathcal{L}) = 1 \to tr(\mathcal{L}) = 0 \tag{5}$$

So

$$\begin{aligned} g &\approx I + \epsilon_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \epsilon_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \epsilon_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &(6)\\ &\approx I + \epsilon_1 T_1 + \epsilon_2 T_2 + \epsilon_3 T_3 &(7)\end{aligned}$$

Therefore

$$T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \ , \ \ T_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ \ , \ \ T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

are **generator** of $SL_2(\mathbb{R})$. Now, if we assume that $\epsilon_1, \epsilon_2, \epsilon_3$ is not small enough, we can assume θ_i that not very small and define N which is too large. Then we have

$$g \approx I + \frac{\theta_1}{N} T_1 + \frac{\theta_2}{N} T_2 + \frac{\theta_3}{N} T_3 \tag{9}$$

On the other hand from the property of the group, every group is closed so

$$g \approx (I + \frac{\theta_1}{N}T_1 + \frac{\theta_2}{N}T_2 + \frac{\theta_3}{N}T_3)^N \tag{10}$$

and in $N \to \infty$ we have

$$g = \lim_{N\to\infty}(I + \frac{\theta_1}{N}T_1 + \frac{\theta_2}{N}T_2 + \frac{\theta_3}{N}T_3)^N = e^{\theta_1 T_1 + \theta_2 T_2 + \theta_3 T_3} \tag{11}$$

Therefore if we change θ_i, an element of the group is generated. Note matrices in eq. 8, are pauli matrices

$$T_1 = \sigma_1 \ \ , \ \ T_2 = -i\sigma_2 \ \ , \ \ T_3 = \sigma_3 \tag{12}$$

so the generators and elements of $SL_2(\mathbb{R})$ are rotation in $\mathbb{R}^3$.

4.3 Orthogonal group

Definition 4.2. A subset of matrices $GL_n(\mathbb{R})$ that are orthogonal, $A^\dagger A = I$, is the group of real n-dimensional matrices denoted by $O_n(\mathbb{R})$. In other words

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}), \ \ | \ \ A^\dagger A = I\} \tag{13}$$

Example 4.1. The set of 2-dimensional and real orthogonal matrices is a group, which is denoted by $O_2(\mathbb{R})$.

Example 4.2. The set of 3-dimensional and real orthogonal matrices is a group, which is denoted by $O_3(\mathbb{R})$.

Let's consider $O_2(\mathbb{R})$. This group invariant the inner product in $\mathbb{R}^2$ space. According to the definition of the orthogonal group, it can be concluded from $A^\dagger A = I$, $det(A) = \pm 1$ so this group is not continuous or topological then we can not define a generator for this group. In Figure 8 we have shown $O_2(\mathbb{R})$ and its subgroup. A subset of these matrices whose determinant is equal to $+1$ is a group. This group name is **Special orthogonal group**.

4.4 Special orthogonal group

Definition 4.3. A subset of matrices $O_n(\mathbb{R})$ that are orthogonal, $A^\dagger A = I$, and $det(A) = +1$ is the group of real n-dimensional matrices denoted by $SO_n(\mathbb{R})$. In other words

$$SO_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}), \ \ | \ \ A^\dagger A = I \ , \ det(A) = +1\} \tag{14}$$

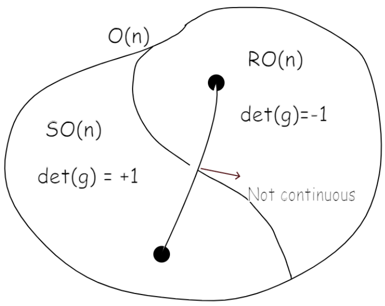

Figure 8: As we shown $O_n(\mathbb{R})$ is not continuous and not connected in topological. Cause we can not move between two points in $O(n)$.

In $SO_2(\mathbb{R})$, for all $A \in SO_2(\mathbb{R})$ we can show

$$A = \begin{pmatrix} cos\ \theta & sin\ \theta \\ -sin\ \theta & cos\ \theta \end{pmatrix} \quad , \quad \theta \in [0, 2\pi] \tag{15}$$

Given that this group is continuous, we can define and obtain the generator as before. So

$$A \in SO_2(\mathbb{R}) \quad , \quad A \approx I + \mathcal{L} \tag{16}$$

$$AA^\dagger = I + \mathcal{L} + \mathcal{L}^\dagger = I \rightarrow \mathcal{L} + \mathcal{L}^\dagger = 0 \tag{17}$$

$$det(A) = det(I + \mathcal{L}) = 1 + tr(\mathcal{L}) = 1 \rightarrow tr(\mathcal{L}) = 0 \tag{18}$$

Now consider

$$\mathcal{L} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{19}$$

from eqs.17 and 18, $\mathcal{L}$ has this properties, so

$$\mathcal{L} + \mathcal{L}^\dagger = \begin{pmatrix} \alpha + \alpha^* & \beta + \gamma^* \\ \gamma + \beta^* & \delta + \delta^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{20}$$

$$tr(\mathcal{L}) = \alpha + \delta = 0 \tag{21}$$

Therefore $\mathcal{L}$ can be

$$\begin{aligned} \mathcal{L} &= \begin{pmatrix} i\epsilon_3 & -\epsilon_2 + i\epsilon_1 \\ \epsilon_2 + i\epsilon_1 & -i\epsilon_3 \end{pmatrix} \\ &= \epsilon_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \epsilon_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + i\epsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{22}$$

Then

$$T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \ \ , \ \ T_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \ \ , \ \ T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_3 \quad (23)$$

So we obtain generators of $SO_2(\mathbb{R})$. Now all elements of this group are

$$g = e^{\epsilon_1 T_1 + \epsilon_2 T_2 + \epsilon_3 T_3} \quad (24)$$

by change ϵ_i.
In the next sections, we will discuss a subgroup of $O(1,3)$ that has very important effects in physics. This subgroup, represented by $SO^+(1,3)$, invariant the inner product $< \overrightarrow{x}, \overrightarrow{y} >= x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$, which we call **Lorentz group**.

4.5 Unitary group

Definition 4.4. The unitary group is a subgroup of the general linear group $GL(n, \mathbb{C})$ and is denoted by $U(n)$.
This group invariant inner product in $\mathbb{C}^n$ and because the determined of this group also is a pure phase, then this group is a continuous(Topological) group and we can define generators. Let's define the **Special unitary group**.

4.6 Special unitary group

Definition 4.5. A subset of matrices $U(n)$ that are orthogonal, $A^\dagger A = I$, and $det(A) = +1$ is the group of complex n-dimensional matrices denoted by $SU_n(\mathbb{R})$. In other words

$$SU_n(\mathbb{R}) = \{A \in U(n), \ \ | \ \ A^\dagger A = I \ , \ det(A) = +1\} \quad (25)$$

Let's consider $g \in SU_2(\mathbb{R}) = SU(2)$. So generators of this group are

$$\begin{aligned} \mathcal{L} &= \begin{pmatrix} i\epsilon_3 & -\epsilon_2 + i\epsilon_1 \\ \epsilon_2 + i\epsilon_1 & -i\epsilon_3 \end{pmatrix} \\ &= i\epsilon_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\epsilon_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + i\epsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (26)$$

Then

$$T_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_1 \ \ , \ \ T_2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_2 \ \ , \ \ T_3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_3 \quad (27)$$

Now we can assume $(\epsilon_1, \epsilon_2, \epsilon_3) = \theta(n_1, n_2, n_3)$ that $n = (n_1, n_2, n_3)$ is a unit vector, so we have

$$g \approx I + i\overrightarrow{n}.\overrightarrow{\sigma} = e^{i\overrightarrow{n}.\overrightarrow{\sigma}} \tag{28}$$

Therefore all elements of $SU(2)$ generate by $\overrightarrow{n}$ and $\overrightarrow{\sigma}$ from eq.28.

As mentioned, the group $U(n)$ can invariant the inner product in $\mathbb{C}^n$ and according to the defined metric, any subgroup of $U(p,q)$ can be defined that is true in the defined inner product. There is no need for these groups in this article, so we will not discuss them [17] [18] [19].

4.7 Sympletic group

This group is very important in classical mechanics. This group represents all canonical transformations on a phase space, which in classical mechanics is the phase space (x,p), and the vectors of this space, which act on a space $\mathbb{R}^2$, can be represented as $g = \begin{pmatrix} x \\ p \end{pmatrix}$. In this space, we define an inner product as follows

$$< g, f >:= x_g p_f - x_f p_g \tag{29}$$

where this inner product is from classical mechanics [11] [20] [21].On the other hand, we can show the inner product 29 in matrix notation

$$< g, f >:= g^T J f \tag{30}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is called **Sympletic matrix**. Let's define $SP(2,\mathbb{R})$ that invariant inner product

$$SP(2,\mathbb{R}) = \{S \in GL(2,\mathbb{R}) \ | \ S^T J S = J \ , \ det(S) = 1\} \tag{31}$$

So obtain the generator of this group.

$$S \approx I + \mathcal{L} \tag{32}$$

and

$$S^T J S = J \rightarrow \mathcal{L}^T J + J\mathcal{L} = 0 \tag{33}$$

$$det(S) = 1 \rightarrow tr(\mathcal{L}) = 0 \tag{34}$$

Therefore $\mathcal{L}$ is

$$\mathcal{L} = \begin{pmatrix} \epsilon_3 & \epsilon_1 + \epsilon_2 \\ \epsilon_1 - \epsilon_2 & -\epsilon_3 \end{pmatrix} = \epsilon_1 T_1 + \epsilon_2 T_2 + \epsilon_3 T_3 \tag{35}$$

where

$$T_1 = \sigma_1 \quad , \quad T_2 = -i\sigma_2 \quad , \quad T_3 = \sigma_3 \tag{36}$$

So all of the elements of $SP(2,\mathbb{R})$ generate by changing ϵ_i in $S = e^{\epsilon_1 T_1 + \epsilon_2 T_2 + \epsilon_3 T_3}$. As seen in special linear group $SL(2,\mathbb{R})$ the generators are same with the generators of $SP(2,\mathbb{R})$, so $SL(2,\mathbb{R})$ and $SP(2,\mathbb{R})$ are isomorphic, but just in 3-dimensional [5] [22] [23].

5 Group Action

After getting acquainted with matrix groups, in this section, we will discuss the action of groups on sets, and in general, we will examine the importance of the use of groups in nature. As it was said in the introduction of this chapter, we are looking for a property of the object or body that is under our investigation, and by performing actions on this object, we are looking for its new property. If the property of our object does not change before and after the action, we say that our object is symmetrical with respect to the action performed. For example, consider a jar. If we rotate this jar with respect to its z-axis and its geometric properties do not change, we say that this jar is symmetrical with respect to the jars around the z-axis. In this section, we consider the jar as a set and assign the rotations around the z-axis as a group as we discussed, and examine the action of this group with respect to the said set.

Definition 5.1. A left action of a group G on a set M is a map

$$\phi : G \times M \to M \tag{1}$$

such that:

1. $\forall m \in M \ \mid \ \phi(e, m) = m$
2. $\forall m \in M \ , \ \forall a, b \in G \ \mid \ \phi(a, \phi(b, m)) = \phi(ab, m)$

In Figure 9 we have shown the meaning of the action(left action) of a group on a set.

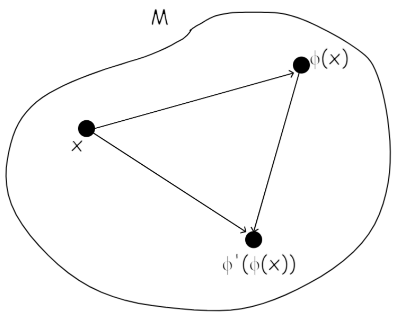

Figure 9: A left action of a group on a set means we can relate between points on a set by a map.

Let's give some examples.

Example 5.1. Consider $G = \mathbb{Z}_2 = \{I, -I\}$ and $M = \mathbb{R}$ and define left action $\phi : \mathbb{Z}_2 \times \mathbb{R} \to \mathbb{R}$. By definition we have

$$1.\forall m \in \mathbb{R} \ , \ \ \phi(I, m) = m \tag{2}$$

$$2.\forall m \in \mathbb{R} \ , \ \forall a, b \in \mathbb{Z}_2 \ | \ \phi(a, \phi(b, m)) = \phi(ab, m) \tag{3}$$

Example 5.2. Consider $G = \{\overrightarrow{a} \ | \ \overrightarrow{a} \in \mathbb{R}^3\}$ and $M = \mathbb{R}^3$. Let's define left action $\phi : G \times \mathbb{R}^3 \to \mathbb{R}^3$ which

$$\forall \overrightarrow{m} \in \mathbb{R}^3 \ , \ \forall \overrightarrow{a} \in G \ | \ \phi(\overrightarrow{a}, \overrightarrow{m}) = \overrightarrow{a + m} \tag{4}$$

Example 5.3. Assume $G = SO(2)$ and $M = S^1$. As we know $SO(2)$ is a rotation group in 2-dimensional and S^1 is a 2-dimensional sphere. So the left action $\phi : SO(2) \times S^1 \to S^1$ which

$$\phi(g, \overrightarrow{r}) = \overrightarrow{r}' \ \ , \ \ g \in SO(2) \ \ and \ \ \overrightarrow{r}, \overrightarrow{r}' \in S^1 \tag{5}$$

For more examples, you can refer to specialized sources of group theory [4] [5] [24] [25] [26]. Let's give some definitions and theorems which not proven.

Definition 5.2. If there is point like $m_0 \in M$ such that $\forall g \in G$, $\phi(g, m_0) = m_0$ then m_0 is a **fixed point** of action group on M.

For example in Example 2.24, $0 \in \mathbb{R}$ is a fixed point of action $\mathbb{Z}_2$.

Theorem 5.1. Assume $m \in M$ and G action on M. Then the elements of G which are fixed m are elements of a subgroup of G and it is called **fixed point subgroup**.

Definition 5.3. A action of a group G on a set M is called **transitive** when the set is nonempty and there is exactly one orbit [26].

Theorem 5.2. If a group G on a set M is transitive then the fixed point subgroups of all points are isomorphic.

Theorem 5.3. If an action of G on M is transitive and H is a fixed point subgroup then there is a bijection between G/H and M.

6 Lie group and Lie algebra

In this section, we will examine continuous groups or topological groups. These groups, as their name suggests, are groups that can be moved from one point to another point in the group by continuously changing one or more parameters, and then we will define the Lie group and the characteristics of this group. Therefore, at the beginning, we define topology and manifolds, and by using them, we get to know topology groups, Lie groups, and Lie algebras.

6.1 Topology

Definition 6.1. Let $\mathcal{X}$ be a set. A **topology** on $\mathcal{X}$ is a collection τ of subsets of $\mathcal{X}$ that satisfy the following three requirements:

1. $\emptyset \in \tau$ and $\mathcal{X} \in \tau$
2. Given $\mathcal{U} \subset \tau$, we have $\cup\{U \ : \ U \in \mathcal{U}\} \in \tau$ (Closure under arbitrary unions)
3. Given U_1 and $U_2 \in \tau$, we have $U_1 \cap U_2 \in \tau$(Closure under finite intersections)

Members of a topology are called open sets.

Example 6.1 (Discrete topological space)**.** Let $\mathcal{X}$ be an arbitrary set, $\tau = \mathcal{P}(\mathcal{X})$ where $\mathcal{P}(\mathcal{X})$ is the power set of $\mathcal{X}$: the set of all subset of $\mathcal{X}$. Then τ is called the **discrete topology**.

Example 6.2. When $\tau = \{\emptyset, \mathcal{X}\}$ then is called the **indiscrete topology**.

Definition 6.2. The elements of the **Euclidean topology** on R^n are unions of open balls in R^n. This topology is donated by $||.||_n$.

Definition 6.3. A set $\mathcal{X}$ together with a topology τ on $\mathcal{X}$ form a **topological space**. This is denoted by the pair $(\mathcal{X}, \tau)$.

We will not go further into the basic definitions of topology due to the large amount of material and the interestingness of the theorems, and it is suggested that if you are interested in these materials, refer to [10] [27] [28] [29] [30] [31] [32].

6.2 Manifold

Definition 6.4. M is an m-dimensional differentiable manifold if:

1. M is a topological space.

2. M is provided with a family of pairs $\{(U_i, \phi_i)\}$.

3. $\{U_i\}$ is a family of open sets which covers M, that is,$\cup_i U_i = M$. ϕ_i is a homeomorphism from U_i onto an open subset U_i' of $\mathcal{R}^m$.

4. Given U_i and U_j such that $U_i \cap U_j = \emptyset$, the map $\psi_{ij} = \phi_i \circ \phi_j^{-1}$ from $\phi_j(U_i \cap U_j)$ to $\phi_i(U_i \cap U_j)$ is infinitely differentiable.

In Figure 10 we have shown.

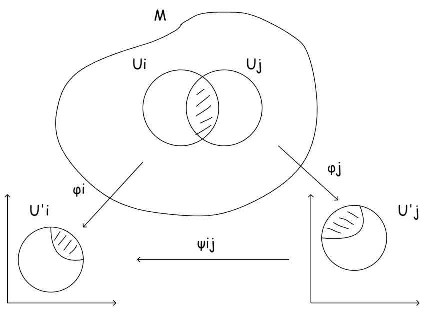

Figure 10: A homomorphism ϕ_i maps U_i onto an open subset $U_i' \subset \mathbb{R}^m$, providing coordinates to a point $p \in U_i$. If $U_i \cap U_j = \emptyset$, the transition from one coordinate system to another is smooth.

The pair (U_i, ϕ_i) is called a **chart** while the whole family $\{(U_i, \phi_i)\}$ is called, for obvious reasons, an **atlas**. The subset U_i is called the **coordinate neighborhood** while ϕ_i is the **coordinate**. On the other hand, the homeomorphism ϕ_i is represented by m functions $\{x^1(p), \ldots, x^m(p)\}$. The set $\{x^i(p)\}$ is also called the **coordinate**.
If the union of two atlases $\{(U_i, \phi_i)\}$ and $\{(V_j, \psi_j)\}$ is again an atlas, these two atlases are said to be **compatible**. The compatibility is an equivalence relation, the equivalence class of which is called the **differentiable structure**.

Example 6.3. The **Euclidean space** $\mathbb{R}^n$ is the most trivial example, where a single chart covers the whole space and ϕ may be the identity map.

Definition 6.5. Let $F : M \to N$ be a map from an m-dimensional manifold M to an n-dimensional manifold N. A point $p \in M$ is mapped to a point $F(p) \in N$ and take a chart (U, ϕ) on M and (V, ψ) on N, where $p \in U$ and $F(p) \in V$. Then F has the following coordinate presentation:

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \tag{1}$$

Now if we write $\phi(p) = \{x^i\}$ and $\psi(F(p)) = \{y^j\}$ then $y = \psi \circ F \circ \phi^{-1}(x)$. If $y = \psi \circ F \circ \phi^{-1}(x)$, or simply $y^j = F^j(x^i)$, is C^∞ with respect to each x^i, F is said to be **differentiable** at p.

Let's define the tangent vectors and then the tangent space to the manifold at a point. In the future, we will see that by using these definitions, we can generalize the concept of generators in Lie groups, and it will help us a lot in the theory of groups.

6.3 Tangent vectors and tangent fields

Definition 6.6. A **tangent vector** or **contravariant vector**, or simply a **vector** at $p_0 \in M$, where M is an n-dimensional manifold, call it $\mathbf{X}$, assigns to each chart (U, x) holding p_0, an n-tuple of real numbers

$$X_U^i = (X_U^1, \ldots, X_U^n) \tag{2}$$

such that if $p_0 \in U \cap V$, then

$$X_V^i = \sum_j \left[\frac{\partial x_V^i}{\partial x_U^j}(p_0) \right] X_U^i \tag{3}$$

and in matrix notation

$$X_V = c_{VU} X_U \tag{4}$$

where c_{VU} is called **transition function** that is the $n \times n$ Jacobian matrix at the point p_0.

Now we can define a new definition that considers vectors as differential operators.

Definition 6.7. In Euclidean space, an important role is played by the notation of differentiating a function f with respect to a vector at the point p

$$D_v(f) = \frac{d}{dt}[f(p + t\mathbf{v})]_{t=0} \tag{5}$$

and if (x) is any cartesian coordinate system we have

$$D_v(f) = \sum_j \left[\frac{\partial f}{\partial x^j}\right](p)v^j \tag{6}$$

This is the motivation for a similar operation on function on any manifold M.
If $\mathbf{X}$ is a vector at $p \in M^n$ we define the derivation of f with respect to the vector $\mathbf{X}$ by

$$\mathbf{X}_p(f) := D_{\mathbf{X}}(f) := \sum_j \left[\frac{\partial f}{\partial x^j}\right](p)X^j \tag{7}$$

So we can define

$$\mathbf{X}_p = \sum_j X^j \frac{\partial}{\partial x^j}\bigg|_p \tag{8}$$

Definition 6.8. The **tangent space** to M^n at the point $p \in M^n$, written M_p^n or T_pM^n, is the real vector space consisting of all tangent vectors to M^n at p. If (x) is a coordinate system holding p, then the n vectors

$$\frac{\partial}{\partial X^1}\bigg|_p, \ldots, \frac{\partial}{\partial X^n}\bigg|_p \tag{9}$$

form a basis of this n-dimensional vector space and this basis is called a **coordinate basis** or **coordinate frame**.

Definition 6.9. A **vector field** on an open set U will be the differentiable assignment of a vector $\mathbf{X}$ to each point of U; in terms of local coordinates

$$\mathbf{X} = \sum_j X^j(x)\frac{\partial}{\partial x^j} \tag{10}$$

where the components x^j are differentiable functions of (x). In particular, each $\frac{\partial}{\partial X^i}$ is a vector field in the coordinate patch. To make it easier to understand the definition, see Figure 11.

For more on manifolds and differential geometry, see [10] [11] [33] [34].

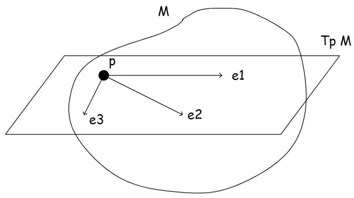

Figure 11: A coordinate basis, $e_1 = \frac{\partial}{\partial X^1}|_p$, $e_2 = \frac{\partial}{\partial X^2}|_p$, $e_2 = \frac{\partial}{\partial X^3}|_p$ in T_pM which are tangent to M.

6.4 Topological group

Definition 6.10. A **topological group** $(G, *, \tau)$ consists of a group $(G, *)$ and a topology τ on G and there is a continuous map which

$$\kappa : G \times G \to G \;\; , \;\; (g, h) \mapsto g^{-1}h \tag{11}$$

We then call τ a **group topology** on G.

We can define topological groups in another way, which we will need in the future

Definition 6.11. A **topological group** $(G, *, \tau)$ consists of a group $(G, *)$ and a topology τ on G for which the multiplication map

$$G \times G \to G \;\; , \;\; (g, h) \mapsto g * h = gh \tag{12}$$

and the inversion map

$$G \to G \;\; , \;\; g \mapsto g^{-1} \tag{13}$$

6.5 Lie group

After all the definitions of topology, manifold, and its details and topological group, we will define the Lie group.

Definition 6.12. A **Lie group** is a differentiable manifold G endowed with a product, that is, a map

$$G \times G \to G \;\; , \;\; (g, h) \mapsto gh \tag{14}$$

making G into a group. We demand that this map, as well as the inversion map

$$G \to G \;\; , \;\; g \mapsto g^{-1} \tag{15}$$

be differentiable.

Example 6.4. Consider $SL_2(\mathbb{R}) = SL(2,\mathbb{R})$. From eq.3 we know

$$g(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3) = \begin{pmatrix} \epsilon_0+\epsilon_3 & \epsilon_1-\epsilon_2 \\ \epsilon_1+\epsilon_2 & \epsilon_0-\epsilon_3 \end{pmatrix} \tag{16}$$

On the other hand, $det(g) = 1$, so

$$\epsilon_0^2 - \epsilon_1^2 - \epsilon_2^2 + \epsilon_3^2 = 1 \tag{17}$$

Therefore manifold $SL(2,\mathbb{R})$ is a hyperbole in 3-dimensional and

$$g(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)g(\theta_0,\theta_1,\theta_2,\theta_3) = g(\epsilon_0+\theta_0,\epsilon_1+\theta_1,\epsilon_2+\theta_2,\epsilon_3+\theta_3) \tag{18}$$

$$g(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3) = g(-\epsilon_0,-\epsilon_1,-\epsilon_2,-\epsilon_3) \tag{19}$$

So $SL(2,\mathbb{R})$ is a Lie group.

Example 6.5. Consider $SO_2(\mathbb{R}) = SO(2,\mathbb{R})$. From eq.23 we obtain the generators and every element is

$$g(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3) = \begin{pmatrix} \epsilon_0+i\epsilon_3 & i\epsilon_1-\epsilon_2 \\ i\epsilon_1+\epsilon_2 & \epsilon_0-i\epsilon_3 \end{pmatrix} \tag{20}$$

So the manifold of the Lie group $SO(2,\mathbb{R})$ is a sphere in 3-dimensional and we have

$$g(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)g(\theta_0,\theta_1,\theta_2,\theta_3) = g(\epsilon_0+\theta_0,\epsilon_1+\theta_1,\epsilon_2+\theta_2,\epsilon_3+\theta_3) \tag{21}$$

$$g(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3) = g(-\epsilon_0,-\epsilon_1,-\epsilon_2,-\epsilon_3) \tag{22}$$

So $SO(2,\mathbb{R})$ is a Lie group.

Example 6.6. Consider $G = U(1)$ and $g = e^{i\theta} \in U(1)$ so the manifold G is S^1 and the map and inversion are

$$g(\theta)g(\phi) = g(\theta+\phi) \tag{23}$$

$$g^{-1}(\theta) = g(-\theta) \tag{24}$$

So $U(1)$ is a Lie group.

Example 6.7. Consider $SU(2,\mathbb{R})$. From eq.27

$$g = \begin{pmatrix} \epsilon_0+i\epsilon_3 & -\epsilon_2+i\epsilon_1 \\ \epsilon_2+i\epsilon_1 & \epsilon_0-i\epsilon_3 \end{pmatrix}$$

So

$$\epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = 1 \tag{25}$$

Therefore manifold $SU(2,\mathbb{R})$ is a sphere in 3-dimensional and

$$g(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)g(\theta_0,\theta_1,\theta_2,\theta_3) = g(\epsilon_0+\theta_0,\epsilon_1+\theta_1,\epsilon_2+\theta_2,\epsilon_3+\theta_3) \tag{26}$$

$$g(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3) = g(-\epsilon_0,-\epsilon_1,-\epsilon_2,-\epsilon_3) \tag{27}$$

So $SU(2,\mathbb{R})$ is a Lie group.

6.6 Lie algebra

Definition 6.13. A **Lie algebra** is a vector space $\mathfrak{g}$ over a field F with an operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which we call a **Lie bracket** , such that

1. $\forall x, y \in \mathfrak{g}$, $[x, y] = -[y, x]$ **Skew symmetric**
2. $\forall x, y, z \in \mathfrak{g}$, $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ **Jacobi Identity**

Example 6.8. Consider $\mathbb{R}^3$ with the exterior product, $[\overrightarrow{a}, \overrightarrow{b}] = a \times b$ is Lie algebra.

Example 6.9. In classical mechanics, the function space in phase space by Lie bracket that is Poisson bracket, is a Lie algebra.

$$[f, g](p, q) = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right] \tag{28}$$

Example 6.10. Let $\mathfrak{A} = M^{n \times n}$ be a vector space, then by commutator bracket, $\mathfrak{A}$ is a Lie algebra.

Example 6.11. Consider $\mathfrak{U} = T_p M^n$ is a tangent space of manifold M^n. Then by commutator bracket, $\mathfrak{U}$ is a Lie algebra.

Example 6.12. If $B_n(\mathbb{R}) = \{a_{ij} \in GL_n(\mathbb{R}) \mid a_{ij} = 0 \; for \; i > j\}$ be a vector space then, $\mathfrak{b}_n(\mathbb{R})$ by commutator bracket is a Lie algebra.

Now that we have defined Lie algebra, we want to define the Lie algebra of the Lie group and use it to obtain the elements of the Lie group. As we know from Definition 6.12, a Lie group is a differentiable manifold with some properties, and from Definition 6.13, we got a Lie algebra is a vector space $\mathfrak{g}$ by Lie bracket. Now we want to know what is the relation between the Lie group and Lie algebra and if is there any relation or not.

Definition 6.14. A **Lie algebra homomorphism (isomorphism)** is a linear homomorphism (isomorphism) between Lie algebras $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that for all $x, y \in \mathfrak{g}_1$, $f([x, y]) = [f(x), f(y)]$.

Definition 6.15. Let $(G, *)$ be a Lie group and let $L_g : G \to G$ be the left multiplication diffeomorphism $h \mapsto g * h$. A vector field $\mathbf{X}$ is called **left-invariant** if $(L_g) * \mathbf{X} = \mathbf{X}$ for all $g \in G$.

Definition 6.16. Let $(G, *)$ be a Lie group and V is a vector field of manifold G, then **Lie(G)** is the set of all left-invariant vector fields on a Lie group G.

Theorem 6.1. The evaluation map $\pi : Lie(G) \to T_eG$ is a linear isomorphism.

So if G is a Lie group that is an n-dimensional real manifold. Then Lie(G) is an n-dimensional real vector space.

Theorem 6.2. Lie(G) is a **Lie algebra** under the bracket operation $[X, Y] = X \circ Y - Y \circ X$.

Definition 6.17. Let G be a Lie group. The **Lie algebra of** G, denoted $\mathfrak{g}$, is T_eG under the bracket operation $[X, Y] = X \circ Y - Y \circ X$.

Now suppose $\{T_1, \ldots, T_n\}$ are the basis of the Lie algebra $\mathfrak{g}$. So from Definition 6.13, because $\mathfrak{g}$ is a vector space we have

$$[T_i, T_j] = f_{ij}^k T_k \tag{29}$$

where f_{ij}^k is called the **Structure constants** of the Lie algebra and from the Jacobi Identity we have

$$f_{ij}^m f_{mk}^n + f_{jk}^m f_{mi}^n + f_{ki}^m f_{mj}^n = 0 \tag{30}$$

Theorem 6.3. Consider a map $exp : \mathfrak{g} \to G$ that is called **Expotential map**. Let's suppose $\{T_1, \ldots, T_n\}$ are the basis of $\mathfrak{g}$. Then $\{T_1, \ldots, T_n\}$ are the generators of the Lie group G.

So from Theorem 6.3, we can consider every element of the Lie group can obtain

$$g = e^{\theta^i T_i} \tag{31}$$

and the generators

$$T_i = \left.\frac{\partial g}{\partial \theta^i}\right|_{\theta^i = 0} \tag{32}$$

Theorem 6.4. For any matrix A, $det\ e^A = e^{tr\ A}$.

Proof. Consider A is a n-dimensional matrix with eigenproblem, $Av = \lambda v$, then from the power series for e^A we have $e^A v = e^\lambda v$. So the eigenvalues of e^A is $exp(\lambda_1), \ldots, exp(\lambda_n)$ then

$$det\ e^A = \Pi_i exp(\lambda_i) = e^{tr\ A} \tag{33}$$

□

Now we will examine Lie algebra for the groups we were familiar with.

Example 6.13. We know the generators of $SL(2,\mathbb{R})$ are

$$T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad T_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{34}$$

or for simply

$$S_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad S_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{35}$$

So the Lie algebra of $SL(2,\mathbb{R})$ is $\mathfrak{sl}(2,\mathbb{R})$ and the bases are 35 with commutation relations is

$$[S_1, S_2] = -S_3 \quad , \quad [S_2, S_3] = -S_1 \quad , \quad [S_3, S_1] = -S_2 \tag{36}$$

Therefore from Theorem 6.4

$$\mathfrak{sl}(2,\mathbb{R}) = \{X \in M_{2\times 2}(\mathbb{R}) \ \mid \ tr(X) = 0\} \tag{37}$$

Example 6.14. Consider $SO(2,\mathbb{R})$. The generators from eq.23 are

$$T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M_{23} \quad , \quad T_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_{31} \quad , \quad T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = M_{12} \tag{38}$$

So $\mathfrak{so}(2,\mathbb{R})$ is a Lie algebra and commutation relations is

$$[T_1, T_2] = -T_3 \quad , \quad [T_2, T_3] = -T_1 \quad , \quad [T_3, T_1] = -T_2 \tag{39}$$

Therefore

$$\mathfrak{so}(2,\mathbb{R}) = \{X \in M_{2\times 2}(\mathbb{R}) \ \mid \ X^T = -X^{-1} \ , \ tr(X) = 0\} \tag{40}$$

Example 6.15. Let's consider $SU(2,\mathbb{R})$. From eq.27 the generators are

$$T_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_1 \quad , \quad T_2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_2 \quad , \quad T_3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_3 \tag{41}$$

and the commutation relations are

$$[T_1, T_2] = -T_3 \quad , \quad [T_2, T_3] = -T_1 \quad , \quad [T_3, T_1] = -T_2 \tag{42}$$

So $\mathfrak{su}(2,\mathbb{R})$ is a Lie algebra of the Lie group $SO(2,\mathbb{R})$ and

$$\mathfrak{su}(2,\mathbb{R}) = \{X \in M_{2\times 2}(\mathbb{R}) \ \mid \ X^\dagger = -X^{-1} \ , \ tr(X) = 0\} \tag{43}$$

7 Representations theory

In this section, we define representation theory in groups, Lie groups, and Lie algebras. In simple words, the representation of a group is to assign to each element of the group, g, a matrix-like $D(g)$ that does not change the multiplication table of the group.
In order to better understand why we need to display groups, let me give an example. Suppose we have a group like $G = \{e, a\}$, and we have an object like a flower. Suppose the action of an on this flower causes the flower to rotate 180 degrees relative to the x-axis. We do not know exactly how this element of the group acts on a flower in space. For this to happen, we have to consider the flower as a quantum state $|\psi\rangle$ in the Hilbert space $\mathcal{H}$ and consider the action of the element on the flower as an operator that

$$A(a) : \mathcal{H} \to \mathcal{H} \tag{1}$$

so we have $A(a)\,|\psi\rangle$. On the other hand, we expect that the inner product of our quantum state does not change with itself, so $A(a)$ must be a unitary operator. Now, if this Hilbert space has bases, then $A(a)$ can be represented as a matrix in terms of those bases.

7.1 Representation of a group

Definition 7.1. Let G be a group. A representation D of G is a homomorphism from G to the endomorphism group of a vector space V over a field F ($End(V)$).

$$D : G \to End(V) \tag{2}$$

Note that for D to be a homomorphism, it needs to satisfy for all $g, h \in G$

- $D(g)D(h) = D(gh)$
- $D(e) = I$

Example 7.1. Consider C_4, the cyclic group of order 4

$$C_4 = \{e, a, a^2, a^3\} \tag{3}$$

Let's consider

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad D(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$D(a^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad D(a^3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $D(a^4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D(e)$. So this is one of the representations of C_4 in 2-dimensional. Let's represent C_4 in 1-dimensional. Consider $e^{in\pi}$ and then

$$D(e) = 1 \ \ , \ \ D(a) = e^{\frac{i\pi}{4}} \ \ , \ \ D(a^2) = e^{\frac{2i\pi}{4}} \ \ , \ \ D(a^3) = e^{\frac{3i\pi}{4}} \tag{4}$$

Example 7.2. Let's G be any group and define

$$D : G \to End(V) \tag{5}$$

by

$$\forall g \in G \ \mid \ D(g) = I \tag{6}$$

Then this is a **Trivial representation** of G.

Example 7.3. Let's consider Pauli's group that is defined in 8. We can represent this group in 2-dimensional

$$D(\sigma_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \ , \ \ D(\sigma_2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ \ , \ \ D(\sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

which for simply $D(\sigma_i) = \sigma_i$.

Example 7.4. Consider $G = S_n$. So the representation of S_n in 1-dimensional is

$$\begin{aligned} D(g) &= 1 \quad , \ \ g \ \textit{is even permutation} \\ D(g) &= -1 \ \ , \ \ g \ \textit{is odd permutation} \end{aligned}$$

Theorem 7.1. If $D : G \to End(V)$ is a representation and $S \in End(V)$ is a linear transformation then $D' = SDS^{-1}$ is a representation.

Proof. Let's assume $D' = SDS^{-1}$ is a representation so

$$\begin{aligned} D^{'}(g_1)D^{'}(g_2) = SD(g_1)S^{-1}SD(g_2)S^{-1} &= SD(g_1)D(g_2)S^{-1} \\ = SD(g_1g_2)S^{-1} &= D^{'}(g_1g_2) \end{aligned} \tag{8}$$

So $D' = SDS^{-1}$ is a representation of G. □

Definition 7.2. Let's $D : G \to End(V)$ and $D^{'} : G \to End(V)$ are two representations of G. Then if there is $S \in End(V)$ that $D' = SDS^{-1}$ the representations are **equivalent**.

Example 7.5. Consider $\mathbb{Z} = \{e, a\}$. Then we can obtain two representation

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

and

$$D^{'}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad D^{'}(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

These two representation of $\mathbb{Z}$ is equivalent cause there is $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ which $D'(g) = SD(g)S^{-1}$ for all $g \in G$.

Definition 7.3. Let's consider $D_1 : G \to End(V)$ and $D_2 : G \to End(W)$. Then there is a representation of G

$$(D_1 \oplus D_2)(g) : G \to End(V \oplus W) \tag{11}$$

and

$$(D_1 \oplus D_2)(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \tag{12}$$

Definition 7.4. A non-zero representation V of G or $\mathfrak{g}$ is called **irreducible** if it has no subrepresentations other than 0, V. Otherwise, V is called **reducible**.

This definition means that the representation that cannot be diagonalized in any basis is **irreducible representation**.

Example 7.6. Consider $\mathbb{Z} = \{e, a\}$. We know the 1-dimensional representations of $\mathbb{Z}$ are

$$D(e) = 1 \quad , \quad D(a) = 1 \tag{13}$$

and

$$D^{'}(e) = 1 \quad , \quad D^{'}(a) = -1 \tag{14}$$

and the 2-dimensional representations are

$$D^{''}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad D^{''}(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{15}$$

But from Example 7.5 we obtain S and the representation in 2-dimensional is diagonal so $D^{''}(g) = D(g) \oplus D^{'}(g)$.

Now we will state two simple theorems that will help us prove some concepts in the future.

Theorem 7.2. Any representation of a finite group is equivalent to **Unitary representation** of the finite group.

Proof. Let's consider $D : G \to End(V)$ and we have an inner product such that

$$< x, y >:= \sum_g (D(g)x, D(g)y) \tag{16}$$

Therefore

$$< x, y >=< D(g)x, D(g)y > \tag{17}$$

Now assume $\{e_i\}_{i=1}^N$ and $\{e_i^{'}\}_{i=1}^N$ are basis of $(,)$ and $<,>$, respectively and there is S which

$$Se_i = e_i^{'} \tag{18}$$

So

$$< Sx, Sy >=< S(x_i e_i), S(y_j e_j) >= x_i y_j < e_i^{'}, e_j^{'} >= (x, y) \tag{19}$$

Then

$$(D(g)x, D(g)y) = (S^{-1}D(g)Sx, S^{-1}D(g)Sy) = Sx, Sy = (x, y) \tag{20}$$

Therefore D is a Unitary representation. □

Theorem 7.3. If M commutes with unitary matrix U then hermitian matrices $M_\pm$ are commute with U where

$$M_+ = M + M^\dagger \ \ , \ \ M_- = i(M - M^\dagger) \tag{21}$$

Proof. If $[M, U] = 0$ then we have $[M^\dagger, U] = 0$ so

$$[M_+, U] = [M + M^\dagger, U] = [M, U] + [M^\dagger, U] = 0 \to [M_+, U] = 0 \tag{22}$$

$$[M_-, U] = [i(M - M^\dagger), U] = i[M, U] - i[M^\dagger, U] = 0 \to [M_-, U] = 0 \tag{23}$$

□

7.1.1 Schur's lemma

An important theorem, which is used to derive a lot of results on the irreducible representations of finite and compact groups is that known as **Schur's lemma** [35] [36] [37].

Lemma 1 (Schur's first lemma)**.** If D is an irreducible representation of finite group G and M is a matrix that commutes with $D(g)$ for all $g \in G$ then $M = \lambda I$ for some $\lambda \in \mathbb{C}$.

Proof. First from Theorem 7.2 and 7.3 we know D is a unitary representation and M_+ is commutes with D, so

$$\forall g \in G \;\mid\; [D(g), M_+] = 0 \tag{24}$$

On the other hand cause M_+ is a hermitian matrix so the eigenvectors are the basis of the vector space and we have

$$M_+ \left|m_n^i\right\rangle = \lambda_n \left|m_n^i\right\rangle \tag{25}$$

where i is the degenerate. Now from eq.24 we obtain

$$M_+(D(g) \left|m_n^i\right\rangle) = \lambda(D(g) \left|m_n^i\right\rangle) \tag{26}$$

So

$$D(g) = \begin{pmatrix} D_1(g) & 0 & \dots & 0 \\ 0 & D_2(g) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \end{pmatrix} \tag{27}$$

where $D_1(g)$ is a matrix with n_1-dimensional etc but because D is a irreducible representation so $M_+ = \alpha I$ and for M_- we have the same way so we obtain $M = \lambda I$ for some $\lambda \in \mathbb{C}$. □

Lemma 2. If D and $D^{'}$ are irreducible representations of finite group G with dimension d and $d^{'}$, respectively and there is $d \times d^{'}$ matrix M such satisfy

$$\forall g \in G \;\;,\;\; D(g)M = MD^{'}(g) \tag{28}$$

if d and $d^{'}$ are **not equal** then $M = 0$ and if d and $d^{'}$ are **equal** then M is invertible matrix.

Proof. Assume V is a vector space and the bases are $\{|m_i\rangle\}_{i=1}^{d}$. So the representation D is

$$D(g)\,|m_i\rangle = \sum_{j=1}^{d}(D(g))_{ji}\,|m_j\rangle \tag{29}$$

and every vector in V is shown by

$$|\phi_i\rangle = \sum_{j=1}^{d} M_{ji}\,|m_j\rangle \tag{30}$$

Now

$$D(g)\,|\phi_i\rangle = \sum_{j=1}^{d} M_{ji}D(g)\,|m_j\rangle = \sum_{j=1}^{d}\sum_{k=1}^{d} M_{ji}(D(g))_{kj}\,|m_k\rangle = \sum_{i=1}^{d} D^{'}(g)_{ik}\,|m_i\rangle$$

Therefore, the action $D(g)$ on each of the vectors is again a linear combination of the same vectors in our vector space but D is an irreducible representation, the vector space must be $\emptyset$ or V. If $\emptyset$ we have

$$\forall i \quad , \quad |m_i\rangle = 0 \rightarrow M = 0 \tag{31}$$

If V is a vector space we have two cases. If $|m_i\rangle$ are independent then $d = d^{'}$ and $det(M) \neq 0$ and if $|m_i\rangle$ are dependent then $d^{'} > d$. Now if we construct for the transpose of M then we conclude if $d \neq d^{'}$ then $M = 0$ and if $d^{'} = d$ then $det(M) \neq 0$. □

7.2 Representations of Lie algebras

Definition 7.5. A **representation of a Lie algebra** $\mathfrak{g}$ is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \rightarrow End(\mathfrak{g}) \tag{32}$$

Definition 7.6. If $A \in \mathfrak{g}$, we define **Adjoint action** of A on $\mathfrak{g}$ as the endomorphism $ad(A) : \mathfrak{g} \rightarrow \mathfrak{g}$ with $ad(A)(B) = [A, B]$ for all $B \in \mathfrak{g}$. The map $ad : \mathfrak{g} \rightarrow End(\mathfrak{g})$ is called the **Adjoint representation** of $\mathfrak{g}$.

Definition 7.7. Let's $\mathfrak{g}$ is a finite dimension the trace of the composition of two such endomorphisms defines a symmetric bilinear form

$$K(x, y) = tr(ad(x) \circ ad(y)) \tag{33}$$

is called **Killing form** of $\mathfrak{g}$.

Now, like the representation theory of groups, we can define equivalence definitions for the representation theory of Lie algebras

Definition 7.8. A subspace W of V is called **invariant** if

$$\rho(\mathfrak{g}) = \{\rho(x)w \mid x \in \mathfrak{g}\ ,\ w \in W\} \subseteq W \tag{34}$$

A representation ρ of $\mathfrak{g}$ on V is **reducible** if a proper nonvanishing invariant subspace W of V exists.
A representation ρ of $\mathfrak{g}$ on V is **irreducible** if no nontrivial invariant subspace of V exists [38].

7.2.1 Schur's lemma

Lemma 3. Let $\mathfrak{g}$ be a complex Lie algebra and ρ its representation on a finite-dimensional vector space V

- Let ρ be irreducible. Then any operator A on V which commutes with all $\rho(x)$,

$$\forall x \in \mathfrak{g} \mid [A, \rho(x)] = 0 \tag{35}$$

  has the form $A = \lambda I$ for some complex number λ.

- Let ρ be fully reducible and such that every operator A on V which commutes with all $\rho(x)$ has the form $A = \lambda I$ for some complex number λ. Then ρ is irreducible.

Therefore, according to Schur's lemmas for the representation theory of Lie algebras, if we can find an operator or operators that commute with the bases of our Lie algebras, then the representations of Lie algebra are **irreducible**.
In order to find these operators, it is necessary to know **Universal enveloping algebras**, which is discussed in detail in [39] [40] [41] [42], and we only examine the formation of these operators, which are called **Casimir operators**, in Lie algebras and apply them in the last chapters.
Let us consider a semisimple complex Lie algebra $\mathfrak{g}$ [43] with d-dimensional and its Killing form K. Consider $\{e_i\}_{i=1}^{d}$ and $\{e^{'i}\}_{i=1}^{d}$ are basis of $\mathfrak{g}$ and dual, respectively such that

$$K(e_i, e^{'j}) = \delta_i^j \tag{36}$$

and we have

$$[e_i, e_j] = f_{ij}^k e_k \tag{37}$$

where f_{ij}^k is structure constant. So from the property of the Killing form

$$K(e_i, [e_j, e'^k]) = -K([e_i, e_j], e'^k) = -f_{ij}^m K(e_m, e^k) = -f_{ij}^m \delta_m^k = -f_{ij}^k \quad (38)$$

$$= K(e_i, \sum_{m=1}^{d} f_{jm}^k e'^m) \quad (39)$$

So

$$[e_j, e'^k] = \sum_{m=1}^{d} f_{jm}^k e'^m \quad (40)$$

Let's construct an element of the universal enveloping algebra $\mathfrak{U}(g)$ of the form

$$C = \sum_{i=1}^{d} e_i \otimes e'^i = \sum_{i=1}^{d} e'^i \otimes e_i \quad (41)$$

Now commute C with any elements of $\mathfrak{g}$

$$\forall e_j \in \mathfrak{g} \ \mid \ [e_j, C] = [e_j, \sum_{i=1}^{d} e_i \otimes e'^i] = \sum_{i=1}^{d} [e_j, e_i \otimes e'^i] = 0 \quad (42)$$

So C is a **Casimir operator** of $\mathfrak{g}$. It is called the **quadratic Casimir operator** [38].

7.3 Irreducible Representation of $\mathfrak{su}(2)$

As before we obtain the generators of $SU(2)$ group then because the $SU(2)$ algebra is a Lie algebra, the generators are the basis of $\mathfrak{su}(2)$. Now we want to obtain the irreducible representation of this algebra and in the future, these representations help us to obtain the irreducible representations of the Poincaré group. Let's rewrite the generators or basis of $\mathfrak{su}(2)$

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_1 \ , \ T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\sigma_2 \ , \ T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_3$$

So the commutator relation is

$$[T_i, T_j] = i\epsilon_{ijk} T_k \quad (43)$$

Let's define the two operators are called **Raise and Lower operator**

$$T_\pm = T_1 \pm iT_2 \quad (44)$$

Notice that formally, these are not elements of the algebra $\mathfrak{su}(2)$ since we have taken a complex linear combination of the generators. These are elements of the complex algebra denoted by A_1 [38]. So

$$[T_\pm, T_3] = \pm T_\pm \tag{45}$$
$$[T_+, T_-] = 2T_3 \tag{46}$$

Now we can define **Casimir operator**

$$C = T_1^2 + T_2^2 + T_3^2 \tag{47}$$

so C commutes with all generators(basis) of $\mathfrak{su}(2)$. Therefore $[C, T_3] = 0$ and eigenstates of T_3 is

$$T_3 \left|j, m\right\rangle = m \left|j, m\right\rangle \tag{48}$$

The operators $T_\pm$ raise and lower the eigenvalue of T_3 since using

$$T_3 T_\pm \left|j, m\right\rangle = ([T_3, T_\pm] + T_\pm T_3) \left|j, m\right\rangle = (m \pm 1) T_\pm \left|j, m\right\rangle \tag{49}$$

We are interested in finite representations and therefore there can only exist a finite number of eigenvalues m in a given representation. Consequently, there must exist a state which possess the highest eigenvalue of T_3 which we denote j

$$T_+ \left|j, j\right\rangle = 0 \tag{50}$$

Again, since the representation is finite there must exist a positive integer l such that

$$(T_-)^{l+1} \left|j, j\right\rangle = 0 \tag{51}$$

and we can rewrite the Casimir operator by raising and lowering the operator

$$C = T_1^2 + T_2^2 + T_3^2 = T_3^2 + \frac{1}{2}(T_+ T_- + T_- T_+) \tag{52}$$

so

$$C \left|j, j\right\rangle = [T_3^2 + \frac{1}{2}(T_+ T_- + T_- T_+)] \left|j, j\right\rangle = j(j+1) \left|j, j\right\rangle \tag{53}$$

Then

$$C \left|j, m\right\rangle = j(j+1) \left|j, m\right\rangle \tag{54}$$

where $|j, m\rangle = (T_-)^n |j, j\rangle = |j, j - n\rangle$. From Schur's lemma, in an irreducible representation, the Casimir operator has to be proportional to the unity matrix and so

$$C = \mathbf{T}^2 = j(j+1)I \tag{55}$$

Now from eq.52 we have

$$\begin{aligned}
&T_+T_- = C - T_3^2 + T_3 \\
&T_+T_-(T_-)^{l+1} |j, j\rangle = [j(j+1) - (j-l)^2 + (j+l)] |j, j\rangle = 0
\end{aligned}$$

so

$$j(j+1) - (j-l)^2 + (j+l) = (2j - l)(l+1) = 0 \tag{56}$$

Since l is a positive integer, the only possible solution is $l = 2j$. Therefore, the spin satisfy is $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

8 Lorentz Group

As stated in the definition of the orthogonal group, a group represented by $O(p,q)$ is a group preserving the inner product $<\overrightarrow{x},\overrightarrow{y}>= x^{\dagger}\eta y$, where η is called the metric. A subgroup of this group is called the **Lorentz group** and is represented by x. In this section, we will review everything we have learned so far for the Lorentz group and in the next section for the Poincaré group. First, we review special relativity.

8.1 Special relativity

In 1905, Albert Einstein caused a great revolution in physics and mathematics by presenting his special theory of relativity [44] [45]. This theory is based on two principles:

- The velocity of light is the same in all internal systems.
- The fundamental laws of physics have the same form in all internal systems.

From the second principle, a point in Minkowski space the **contravariant** space-time four-vector

$$x^{\mu} = (x^0, x^1, x^2, x^3) = (ct, x^i) = (ct, \mathbf{x}) \tag{1}$$

and from the first principle follows for light in two different internal systems we have

$$(ct)^2 - \mathbf{x}^2 = (ct^{'})^2 - \mathbf{x}^{'2} \tag{2}$$

This condition defines the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3}$$

We can rewrite eq.2 as below

$$g_{\mu\nu}x^{\mu}x^{\nu} = g_{\mu\nu}x^{'\mu}x^{'\nu} \tag{4}$$

Now from the metric 3 we can define the **covaraint** space-time four-vector

$$x_{\mu} = g_{\mu\nu}x^{\nu} \tag{5}$$

where

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x^i) = (ct, -\mathbf{x}) \tag{6}$$

So the eq.4 is the inner product in Minkowski space-time and is invariant in different internal systems

$$x^\mu x_\mu = x^{'\mu} x^{'}_\mu \tag{7}$$

Furthermore, the obvious identity

$$g_{\mu\nu}\delta^\nu_\kappa = g_{\mu\kappa} \tag{8}$$

where

$$g^\nu_\mu = \delta^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Let's consider the **contravariant** energy-momentum four-vector. First from the second principle

$$E^2 = M^2c^4 + \mathbf{p}^2c^2 \quad , \quad E^{'2} = M^2c^4 + \mathbf{p}^{'2}c^2 \tag{10}$$

so

$$(\frac{E}{c})^2 - \mathbf{p}^2 = (\frac{E^{'}}{c})^2 - \mathbf{p}^{'2} \tag{11}$$

Therefore the contravariant energy-momentum four-vector is

$$p^\mu = (p^0, p^1, p^2, p^3) = (\frac{E}{c}, p^i) = (\frac{E}{c}, \mathbf{p}) \tag{12}$$

and we have

$$g_{\mu\nu}p^\mu p^\nu = g_{\mu\nu}p^{'\mu}p^{'\nu} \tag{13}$$

So the **covariant** energy-momentum four-vector is

$$p^\mu = (p_0, p_1, p_2, p_3) = (\frac{E}{c}, -p^i) = (\frac{E}{c}, -\mathbf{p}) \tag{14}$$

and the same eq.7 we have

$$p^\mu p_\mu = p^{'\mu}p^{'}_\mu \tag{15}$$

Now from eqs.10 and 11 we obtain

$$p^\mu p_\mu = M^2c^2 \tag{16}$$

8.2 Defining of Lorentz group

Now, as we said before, the orthogonal group was the set of elements that maintained our usual inner product. As you have seen, in special relativity our space is Minkowski space and its inner product depends on Minkowski metric 3. Now we want to check what elements preserve this inner product. We denote the set of all these elements by $O(1,3)$ where 1 represents time and 3 represents space coordinates.

Definition 8.1. A set of matrices that are $\Lambda^\dagger g \Lambda = g$ which g is a Minkowski metric is a **Lorentz group** and denoted by $O(1,3)$.

We consider that the two internal systems are connected by a linear coordinate transformation Λ which is called **Lorentz transformation**

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{17}$$

Now, to show that the set of Λ with multiplication forms a group, we check the condition of the group

- Close

$$\begin{aligned}\forall \Lambda_1, \Lambda_2 \in O(1,3) \ \mid \ (\Lambda_1\Lambda_2)^\dagger g(\Lambda_1\Lambda_2) &= \Lambda_2^\dagger \Lambda_1^\dagger g \Lambda_1 \Lambda_2 \\ &= \Lambda_2^\dagger g \Lambda_2 \\ &= g\end{aligned}$$

- Associativity

$$\begin{aligned}\forall \Lambda_1, \Lambda_2, \Lambda_3 \in O(1,3) \ \mid \ [(\Lambda_1\Lambda_2)\Lambda_3]^\dagger g[(\Lambda_1\Lambda_2)\Lambda_3] &= \Lambda_3^\dagger[\Lambda_2^\dagger(\Lambda_1^\dagger g \Lambda_1)\Lambda_2]\Lambda_3 \\ &= \Lambda_3^\dagger(\Lambda_2^\dagger g \Lambda_2)\Lambda_3 \\ &= \Lambda_3^\dagger g \Lambda_3 \\ &= g\end{aligned}$$

- Identity element

$$\exists \Lambda_e, \forall \Lambda \in O(1,3) \ \mid \ (\Lambda_e\Lambda)^\dagger g(\Lambda_e\Lambda) = \Lambda^\dagger g \Lambda = g$$

- Inverse element

$$\forall \Lambda, \exists \Lambda^{-1} \in O(1,3) \ \mid \ \Lambda^{-1\dagger} g \Lambda^{-1} = g$$

So $O(1,3)$ is a group. Now we can be classified with respect to the following two properties:

1. From $O(1,3)$ we have

$$\Lambda^\dagger g\Lambda = g \quad , det(\Lambda^\dagger g\Lambda) = det(\Lambda^\dagger)det(g)det(\Lambda) = det(g)(det\Lambda)^2 = det(g)$$
$$\to (det\ \Lambda)^2 = 1$$
$$\to det(\Lambda) = \pm 1 \tag{18}$$

2. From condition of $O(1,3)$ we have

$$\Lambda^\sigma_\mu g_{\sigma\rho}\Lambda^\rho_\nu = g_{\mu\nu} \tag{19}$$

and for $\mu = \nu = 0$

$$\Lambda^\sigma_0 g_{\sigma\rho}\Lambda^\rho_0 = (\Lambda^0_0)^2 - (\Lambda^i_0)^2 = g_{00} = 1$$

So

$$(\Lambda^0_0)^2 = 1 + (\Lambda^i_0)^2 \geq 1 \tag{20}$$

A Lorentz transformation Λ with $\Lambda^0_0 \geq 1$ is called **orthochronous** and $\Lambda^0_0 \leq -1$ is called **non-orthochronous**.
So $O(1,3)$ has 4 districts such that

$$O(1,3)^+_\uparrow = \{\Lambda \in O(1,3)\ |\ det(\Lambda) = 1\ ,\ \Lambda^0_0 \geq 1\}$$
$$O(1,3)^+_\downarrow = \{\Lambda \in O(1,3)\ |\ det(\Lambda) = 1\ ,\ \Lambda^0_0 \leq -1\}$$
$$O(1,3)^-_\uparrow = \{\Lambda \in O(1,3)\ |\ det(\Lambda) = -1\ ,\ \Lambda^0_0 \geq 1\}$$
$$O(1,3)^-_\downarrow = \{\Lambda \in O(1,3)\ |\ det(\Lambda) = -1\ ,\ \Lambda^0_0 \leq -1\} \tag{21}$$

As it is known, $O(1,3)$ is not a continuous group (topological group) so we can not define the generators. From $O(1,3)$ and the districts we know $O(1,3)^-_\uparrow$ and $O(1,3)^-_\downarrow$ are not subgroup of $O(1,3)$ cause $det(\Lambda_1)det(\Lambda_2) = 1 \notin O(1,3)^-_\uparrow\ or\ O(1,3)^-_\downarrow$. In fact, these districts are about time reversal and parity inversion. The important district is $O(1,3)^+_\uparrow$ which is a subgroup and is called **Special Lorentz group**.

8.3 Generators

As we had in the previous chapters, we can obtain the generators of a group with infinitesimal transformations around the Identity element. So

$$\Lambda \approx I + \mathcal{L} \tag{22}$$

and from another condition

$$g\mathcal{L} + \mathcal{L}^{\dagger} g = 0 \tag{23}$$

so we obtain

$$\begin{aligned}\mathcal{L} &= \begin{pmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ \theta_1 & 0 & -\epsilon_3 & \epsilon_2 \\ \theta_2 & \epsilon_3 & 0 & -\epsilon_1 \\ \theta_3 & -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix} \\ &= \theta_1 M_1 + \theta_2 M_2 + \theta_3 M_3 + \epsilon_1 L_1 + \epsilon_2 L_2 + \epsilon_3 L_3 \end{aligned} \tag{24}$$

where

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;,\; M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;,\; M_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{25}$$

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \;,\; L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \;,\; L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

So all elements of $O(1,3)^{+}_{\uparrow}$ can generated by θ_i and ϵ_i

$$\Lambda = e^{\overrightarrow{\theta}.\overrightarrow{M} + \overrightarrow{\epsilon}.\overrightarrow{L}} \tag{27}$$

Now we want to prove that the M_i and L_i are generators of **Boost** and **Rotation**, respectively and $\overrightarrow{\theta}$ and $\overrightarrow{\epsilon}$ are the direction of boost and rotation.

- **Boost**.

  Consider $\overrightarrow{\theta} = (\theta_1, 0, 0)$ so

$$\Lambda = e^{\theta_1 M_1} = \sum_{n=0}^{\infty} \frac{(\theta_1 M_1)^n}{n!} = \sum_{even}^{\infty} \frac{(\theta_1 M_1)^{2n}}{2n!} + \sum_{odd}^{\infty} \frac{(\theta_1 M_1)^{2n+1}}{(2n+1)!} \tag{28}$$

  we know

$$M_1^{2n} = M_1^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;,\; M_1^{2n+1} = M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so

$$\Lambda = e^{\theta_1 M_1} = M_1^2 \sum_{even}^{\infty} \frac{\theta_1^{2n}}{2n!} + M_1 \sum_{odd}^{\infty} \frac{\theta_1^{2n+1}}{(2n+1)!}$$

$$= I + M_1^2(\cosh\theta_1 - 1) - M_1 \sinh\theta_1 = \begin{pmatrix} \cosh\theta_1 & -\sinh\theta_1 & 0 & 0 \\ -\sinh\theta_1 & \cosh\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{29}$$

On the other hand from Lorentz's transformation, we have

$$\begin{aligned} t^{'} &= \cosh\theta_1 t - \sinh\theta_1 x \\ x^{'} &= \cosh\theta_1 x - \sinh\theta_1 t \\ y^{'} &= y \\ z^{'} &= z \end{aligned}$$

so Lorentz's transformation matrix is

$$\begin{pmatrix} t^{'} \\ x^{'} \\ y^{'} \\ z^{'} \end{pmatrix} = \begin{pmatrix} \cosh\theta_1 & -\sinh\theta_1 & 0 & 0 \\ -\sinh\theta_1 & \cosh\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{30}$$

Therefore from eqs.29 and 30, we conclude M_1 is one of the generators of boost in direction x so the M_i are the generators of the **Boost**.

- **Rotation**

Consider $\overrightarrow{\epsilon} = (\epsilon_1, 0, 0)$

$$\Lambda = e^{\epsilon_1 L_1} = \sum_{n=0}^{\infty} \frac{(\epsilon_1 L_1)^n}{n!} = \sum_{even}^{\infty} \frac{(\epsilon_1 L_1)^{2n}}{2n!} + \sum_{odd}^{\infty} \frac{(\epsilon_1 L_1)^{2n+1}}{(2n+1)!} \tag{31}$$

and

$$L_1^{2n} = (-1)^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad L_1^{2n+1} = (-1)^n L_1 = (-1)^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So

$$\Lambda = e^{\epsilon_1 L_1} = L_1^2 \sum_{even}^{\infty} \frac{(-1)^n \epsilon_1^{2n}}{2n!} - L_1 \sum_{odd}^{\infty} \frac{(-1)^{n+1} \epsilon_1^{2n+1}}{(2n+1)!}$$

$$= I + L_1^2(1-\cos\epsilon_1) + L_1 \sin\epsilon_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\epsilon_1 & \sin\epsilon_1 \\ 0 & 0 & -\sin\epsilon_1 & \cos\epsilon_1 \end{pmatrix} \tag{32}$$

On the other hand, if we rotation coordinates along the x-axis we have

$$\begin{aligned} t^{'} &= t \\ x^{'} &= x \\ y^{'} &= \cos\epsilon_1 y + \sin\epsilon_1 z \\ z^{'} &= -\sin\epsilon_1 y + \cos\epsilon_1 z \end{aligned}$$

and

$$\begin{pmatrix} t^{'} \\ x^{'} \\ y^{'} \\ z^{'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\epsilon_1 & \sin\epsilon_1 \\ 0 & 0 & -\sin\epsilon_1 & \cos\epsilon_1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{33}$$

So from eqs.32 and 33, L_1 is the generator of rotation along the x-axis. Therefore the L_i are the generators of the **Rotation**.

8.4 Lorentz algebra

Now we obtain all the steps algebraically for the Lorentz group and show what structure the Lie algebra or Lorentz algebra has.
The Λ is described in total by 16 degrees of freedom, where the invariance 19 leads to 10 restrictions. Therefore the dimension of the Lorentz group is $16 - 10 = 6$. Now by infinitesimally from the unity element we have

$$\Lambda^{\mu}_{\nu} \approx I + \omega^{\mu}_{\nu} = g^{\mu}_{\nu} + \omega^{\mu}_{\nu} \tag{34}$$

Note ω^{μ}_{ν} is represent $\mathcal{L}$ in the previous section. Now from the condition of the Lorentz group we obtain

$$\begin{aligned} \Lambda^{\sigma}_{\mu}\Lambda^{\rho}_{\nu} g_{\sigma\rho} &= (g^{\sigma}_{\mu} + \omega^{\sigma}_{\mu})(g^{\rho}_{\nu} + \omega^{\rho}_{\nu}) g_{\sigma\rho} \\ &= g^{\sigma}_{\mu} g^{\rho}_{\nu} g_{\sigma\rho} + g^{\sigma}_{\mu}\omega^{\sigma}_{\mu} g_{\sigma\rho} + \omega^{\sigma}_{\mu} g^{\rho}_{\nu} g_{\sigma\rho} + \omega^{\sigma}_{\mu}\omega^{\rho}_{\nu} g_{\sigma\rho} \\ &\approx g_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} = g_{\mu\nu} \end{aligned} \tag{35}$$

So

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0 \tag{36}$$

The set of all anti-symmetric 4×4 matrices $\omega_{\mu\nu}$, are called **the Lorentz algebra of the Lorentz group**.
Now we can be represented the elements ω^{μ}_{ν} of the Lorentz algebra as

$$\omega^{\mu}_{\nu} = g^{\sigma\mu} g^{\kappa}_{\nu} \omega_{\sigma\kappa} = \frac{1}{2}(g^{\sigma\mu} g^{\kappa}_{\nu} - g^{\kappa\mu} g^{\sigma}_{\nu})\omega_{\sigma\kappa} \tag{37}$$

So we can define

$$(L^{\sigma\kappa})^{\mu}_{\nu} = g^{\sigma\mu} g^{\kappa}_{\nu} - g^{\kappa\mu} g^{\sigma}_{\nu} \tag{38}$$

By comparing 24 and 38, we find that these two equations are the same, and therefore $L^{\sigma\kappa}$ is proportional to the generators obtained in the previous section. So

$$L_i = \frac{1}{2}\epsilon_{ijk} L^{jk} \tag{39}$$
$$M_i = L^{0i} \tag{40}$$

Therefore from the Lie bracket in Lie algebra, $L^{\mu\nu}$ are the basis of Lorentz algebra and we can commute the generators so

$$\begin{aligned}
([L^{\mu\nu}, L^{\sigma\kappa}])^{\alpha}_{\beta} &= (L^{\mu\nu} L^{\sigma\kappa})^{\alpha}_{\beta} - (L^{\sigma\kappa} L^{\mu\nu})^{\alpha}_{\beta} = (L^{\mu\nu})^{\alpha}_{m} (L^{\sigma\kappa})^{m}_{\beta} - (L^{\sigma\kappa})^{\alpha}_{m} (L^{\mu\nu})^{m}_{\beta} \\
&= (g^{\mu\alpha} g^{\nu}_{m} - g^{\nu\alpha} g^{\mu}_{m})(g^{\sigma m} g^{\kappa}_{\beta} - g^{\kappa m} g^{\sigma}_{\beta}) \\
&- (g^{\sigma\alpha} g^{\kappa}_{m} - g^{\kappa\alpha} g^{\sigma}_{m})(g^{\mu m} g^{\nu}_{\beta} - g^{\nu m} g^{\mu}_{\beta}) \\
&= g^{\mu\alpha} g^{\nu}_{m} g^{\sigma m} g^{\kappa}_{\beta} - g^{\mu\alpha} g^{\nu}_{m} g^{\kappa m} g^{\sigma}_{\beta} - g^{\nu\alpha} g^{\mu}_{m} g^{\sigma m} g^{\kappa}_{\beta} + g^{\nu\alpha} g^{\mu}_{m} g^{\kappa m} g^{\sigma}_{\beta} \\
&- g^{\sigma\alpha} g^{\kappa}_{m} g^{\mu m} g^{\nu}_{\beta} + g^{\sigma\alpha} g^{\kappa}_{m} g^{\nu m} g^{\mu}_{\beta} + g^{\kappa\alpha} g^{\sigma}_{m} g^{\mu m} g^{\nu}_{\beta} - g^{\kappa\alpha} g^{\sigma}_{m} g^{\nu m} g^{\mu}_{\beta} \\
&= g^{\mu\alpha} g^{\nu\sigma} g^{\kappa}_{\beta} - g^{\mu\alpha} g^{\nu\kappa} g^{\sigma}_{\beta} - g^{\nu\alpha} g^{\mu\sigma} g^{\kappa}_{\beta} + g^{\nu\alpha} g^{\mu\kappa} g^{\sigma}_{\beta} - g^{\sigma\alpha} g^{\kappa\mu} g^{\nu}_{\beta} \\
&+ g^{\sigma\alpha} g^{\kappa\nu} g^{\mu}_{\beta} + g^{\kappa\alpha} g^{\sigma\mu} g^{\nu}_{\beta} - g^{\kappa\alpha} g^{\sigma\nu} g^{\mu}_{\beta} \\
&= g^{\mu\kappa}(g^{\nu\alpha} g^{\sigma}_{\beta} - g^{\sigma\alpha} g^{\nu}_{\beta}) + g^{\nu\sigma}(g^{\mu\alpha} g^{\kappa}_{\beta} - g^{\kappa\alpha} g^{\nu}_{\beta}) \\
&- g^{\mu\sigma}(g^{\nu\alpha} g^{\kappa}_{\beta} - g^{\kappa\alpha} g^{\nu}_{\beta}) - g^{\nu\kappa}(g^{\mu\alpha} g^{\sigma}_{\beta} - g^{\sigma\alpha} g^{\mu}_{\beta}) \\
&= (g^{\mu\kappa} L^{\nu\sigma} + g^{\nu\sigma} L^{\mu\kappa} - g^{\mu\sigma} L^{\nu\kappa} - g^{\nu\kappa} L^{\mu\sigma})^{\alpha}_{\beta}
\end{aligned} \tag{41}$$

So

$$[L^{\mu\nu}, L^{\sigma\kappa}] = g^{\mu\kappa} L^{\nu\sigma} + g^{\nu\sigma} L^{\mu\kappa} - g^{\mu\sigma} L^{\nu\kappa} - g^{\nu\kappa} L^{\mu\sigma} \tag{42}$$

and from Lie algebra

$$[L^{\mu\nu}, L^{\sigma\kappa}] = f^{\mu\nu\sigma\kappa}_{\epsilon\zeta} L^{\epsilon\zeta} \tag{43}$$

where the structure constants are given by

$$f^{\mu\nu\sigma\kappa}_{\epsilon\zeta} = g^{\mu\kappa} g^{\nu}_{\epsilon} g^{\sigma}_{\zeta} + g^{\nu\sigma} g^{\mu}_{\epsilon} g^{\kappa}_{\zeta} - g^{\mu\sigma} g^{\nu}_{\epsilon} g^{\kappa}_{\zeta} - g^{\nu\kappa} g^{\mu}_{\epsilon} g^{\sigma}_{\zeta} \tag{44}$$

Now from 39, 40, and 43 we can obtain the commutation relation with the boost and rotation generators

$$\begin{aligned}
[L_i, L_l] &= [\frac{1}{2}\epsilon_{i\mu\nu} L^{\mu\nu}, \frac{1}{2}\epsilon_{l\sigma\kappa} L^{\sigma\kappa}] = \frac{1}{4}\epsilon_{ijk}\epsilon_{lmn}[L^{\mu\nu}, L^{\sigma\kappa}] = \frac{1}{4}\epsilon_{i\mu\nu}\epsilon_{l\sigma\kappa} f^{\mu\nu\sigma\kappa}_{\epsilon\zeta} L^{\epsilon\zeta} \\
&= \frac{1}{4}\epsilon_{i\mu\nu}\epsilon_{l\sigma\kappa}(g^{\mu\kappa} g^{\nu}_{\epsilon} g^{\sigma}_{\zeta} + g^{\nu\sigma} g^{\mu}_{\epsilon} g^{\kappa}_{\zeta} - g^{\mu\sigma} g^{\nu}_{\epsilon} g^{\kappa}_{\zeta} - g^{\nu\kappa} g^{\mu}_{\epsilon} g^{\sigma}_{\zeta}) L^{\epsilon\zeta} \\
&= \frac{1}{4}\epsilon_{i\mu\nu}(g^{\mu\kappa} g^{\nu}_{\epsilon} \epsilon_{l\zeta\kappa} + g^{\nu\sigma} g^{\mu}_{\epsilon} \epsilon_{l\sigma\zeta} - g^{\mu\sigma} g^{\nu}_{\epsilon} \epsilon_{l\sigma\zeta} - g^{\nu\kappa} g^{\mu}_{\epsilon} \epsilon_{l\zeta\kappa}) L^{\epsilon\zeta} \\
&= \frac{1}{4}(g^{\mu\kappa}\epsilon_{i\mu\epsilon}\epsilon_{l\zeta\kappa} + g^{\nu\sigma}\epsilon_{i\epsilon\nu}\epsilon_{l\sigma\zeta} - g^{\mu\sigma}\epsilon_{i\mu\epsilon}\epsilon_{l\sigma\zeta} - g^{\nu\kappa}\epsilon_{i\epsilon\nu}\epsilon_{l\zeta\kappa}) L^{\epsilon\zeta} \\
&= \frac{1}{4}(\epsilon^{\kappa}_{i\epsilon}\epsilon_{l\zeta\kappa} + \epsilon^{\sigma}_{i\epsilon}\epsilon_{l\sigma\zeta} + \epsilon^{\sigma}_{i\epsilon}\epsilon_{l\sigma\zeta} - \epsilon^{\kappa}_{i\epsilon}\epsilon_{l\zeta\kappa}) L^{\epsilon\zeta} \\
&= \frac{1}{2}\epsilon^{\sigma}_{i\epsilon}\epsilon_{l\sigma\zeta} L^{\epsilon\zeta} = \epsilon^{\sigma}_{il} L_{\sigma} = \epsilon_{il\sigma} L_{\sigma}
\end{aligned}$$

and

$$\begin{aligned}
[M_i, M_j] &= [L^{0i}, L^{0j}] = g^{0j} L^{i0} + g^{i0} L^{0j} - g^{00} L^{ij} - g^{ij} L^{00} \\
&= -L^{ij} = -\epsilon_{ijk} L_k
\end{aligned}$$

and

$$\begin{aligned}
[L_i, M_j] &= [\frac{1}{2}\epsilon_{i\mu\nu} L^{\mu\nu}, L^{0j}] = \frac{1}{2}\epsilon_{i\mu\nu}[L^{\mu\nu}, L^{0j}] \\
&= \frac{1}{2}\epsilon_{i\mu\nu}(g^{\mu j} L^{\nu 0} + g^{\nu 0} L^{\mu j} - g^{\mu 0} L^{\nu j} - g^{\nu j} L^{\mu 0}) \\
&= \epsilon_{ijk} L^{0k} = \epsilon_{ijk} M_k
\end{aligned}$$

So we yield

$$[L_i, L_l] = \epsilon_{il\sigma} L_{\sigma} \tag{45}$$

$$[M_i, M_j] = -\epsilon_{ijk} L_k \tag{46}$$

$$[L_i, M_j] = \epsilon_{ijk} M_k \tag{47}$$

8.5 Representations of Lorentz group in physical field

Let's consider a scalar field $\phi(x^\mu)$. We know it is invariant with respect to any Lorentz transformation so from eq.17 we have

$$\phi^{'}(x^{'\mu}) = \phi(x^\mu) = \phi((\Lambda^{-1})^\mu_\nu x^\nu) \tag{48}$$

Now by infinitesimal Lorentz transformations in first order and eq.37 we obtain

$$\phi^{'}(x^\mu) = \phi(x^\mu + \frac{1}{2}\omega_{\sigma\kappa}(L^{\sigma\kappa})^\mu_\nu x^\nu) = (1 - \frac{1}{2}\omega_{\sigma\kappa}\hat{L}^{\sigma\kappa})\phi(x^\mu) \tag{49}$$

where

$$\hat{L}^{\sigma\kappa} = -(L^{\sigma\kappa})^\mu_\nu x^\nu \partial_\mu \tag{50}$$

From eq.38 we can obtain

$$\begin{aligned}\hat{L}^{\sigma\kappa} &= -(L^{\sigma\kappa})^\mu_\nu x^\nu \partial_\mu = -(g^{\sigma\mu} g^\kappa_\nu - g^{\kappa\mu} g^\sigma_\nu) x^\nu \partial_\mu \\ &= -(g^{\sigma\mu} g^\kappa_\nu x^\nu \partial_\mu - g^{\kappa\mu} g^\sigma_\nu x^\nu \partial_\mu) = x^\sigma \partial^\kappa - x^\kappa \partial^\sigma\end{aligned} \tag{51}$$

Note $\hat{L}^{\sigma\kappa}$ is **angular momentum operator**. Let's obtain the commutation relation

$$\begin{aligned}[\hat{L}^{\mu\nu}, \hat{L}^{\sigma\kappa}] &= [x^\mu\partial^\nu - x^\nu\partial^\mu, x^\sigma\partial^\kappa - x^\kappa\partial^\sigma] \\ &= [x^\mu\partial^\nu, x^\sigma\partial^\kappa] - [x^\mu\partial^\nu, x^\kappa\partial^\sigma] - [x^\nu\partial^\mu, x^\sigma\partial^\kappa] + [x^\nu\partial^\mu, x^\kappa\partial^\sigma] \\ &= (g^{\nu\sigma}x^\mu\partial^\kappa - g^{\mu\kappa}x^\sigma\partial^\nu) - (g^{\nu\kappa}x^\mu\partial^\sigma - g^{\mu\sigma}x^\mu\partial^\nu) \\ &- (g^{\mu\sigma}x^\nu\partial^\kappa - g^{\nu\kappa}x^\sigma\partial^\mu) + (g^{\mu\kappa}x^\nu\partial^\sigma - g^{\nu\sigma}x^\nu\partial^\mu) \\ &= g^{\nu\sigma}(x^\mu\partial^\kappa - x^\kappa\partial^\mu) + g^{\mu\kappa}(x^\nu\partial^\sigma - x^\sigma\partial^\nu) \\ &+ g^{\nu\kappa}(x^\sigma\partial^\mu - x^\mu\partial^\sigma) + g^{\mu\sigma}(x^\kappa\partial^\nu - x^\nu\partial^\kappa) \\ &= g^{\nu\sigma}\hat{L}^{\mu\kappa} + g^{\mu\kappa}\hat{L}^{\nu\sigma} - g^{\nu\kappa}\hat{L}^{\mu\sigma} - g^{\mu\sigma}\hat{L}^{\nu\kappa}\end{aligned}$$

So

$$[\hat{L}^{\mu\nu}, \hat{L}^{\sigma\kappa}] = g^{\nu\sigma}\hat{L}^{\mu\kappa} + g^{\mu\kappa}\hat{L}^{\nu\sigma} - g^{\nu\kappa}\hat{L}^{\mu\sigma} - g^{\mu\sigma}\hat{L}^{\nu\kappa} \tag{52}$$

Now we can obtain commute $\hat{L}^{\mu\nu}$ between $L^{\alpha\beta}$

$$\begin{aligned}[\hat{L}^{\mu\nu}, L^{\alpha\beta}] &= [-(L^{\mu\nu})^\sigma_\kappa x^\kappa\partial_\sigma, L^{\alpha\beta}] = -[L^{\mu\nu}x^\sigma\partial_\sigma, L^{\alpha\beta}] \\ &= -L^{\mu\nu}[x^\sigma\partial_\sigma, L^{\alpha\beta}] + [L^{\mu\nu}, L^{\alpha\beta}]x^\sigma\partial_\sigma \\ &= 0\end{aligned} \tag{53}$$

On the other hand, from the four-momentum operator in quantum mechanics

$$\hat{p}^{\mu} = i\hbar\partial^{\mu} \tag{54}$$

so

$$\hat{L}^{\sigma\kappa} = \frac{i}{\hbar}(x^{\sigma}\hat{p}^{\kappa} - x^{\kappa}\hat{p}^{\sigma}) \tag{55}$$

Now from eq.49, we can define the operator which infinitesimal transform [1]

$$\hat{R}(\omega) = \frac{1}{2}\omega_{\sigma\kappa}\hat{L}^{\sigma\kappa} \quad , \quad \delta x^{\mu} = \hat{R}(\omega)x^{\mu} = \omega^{\mu}_{\nu}x^{\nu} \tag{56}$$

Let's Consider a vector field $A^{\nu}(x^{\mu})$. The vector field is transformed by Lorentz transformation so

$$\begin{aligned}
A'^{\nu}(x^{\mu}) &= \Lambda^{\nu}_{\sigma}A^{\sigma}(x^{\mu}) \\
&= \Lambda^{\nu}_{\sigma}A^{\sigma}((\Lambda^{-1})^{\mu}_{\kappa}x^{\kappa}) \\
&= (g^{\nu}_{\sigma} + \omega^{\nu}_{\sigma})A^{\sigma}((g^{\mu}_{\kappa} - \omega^{\mu}_{\kappa})x^{\kappa}) = (g^{\nu}_{\sigma} + \omega^{\nu}_{\sigma})A^{\sigma}(x^{\mu} - \omega^{\mu}_{\kappa}x^{\kappa}) \\
&= (g^{\nu}_{\sigma} + \omega^{\nu}_{\sigma})(A^{\sigma}(x^{\mu}) - \omega^{\mu}_{\kappa}x^{\kappa}\partial_{\mu}A^{\sigma}(x)) \\
&= A^{\nu}(x^{\mu}) + \omega^{\nu}_{\sigma}A^{\sigma}(x^{\mu}) - \omega^{\mu}_{\kappa}x^{\kappa}\partial_{\mu}A^{\nu}(x) \\
&= (g^{\nu}_{\sigma} + \omega^{\nu}_{\sigma} - g^{\nu}_{\sigma}\omega^{\mu}_{\kappa}x^{\kappa}\partial_{\mu})A^{\sigma}(x^{\mu}) \\
&= \left[g^{\nu}_{\sigma} + \frac{1}{2}\omega^{\mu\kappa}(g_{\kappa\sigma}g^{\nu}_{\mu} - g_{\mu\sigma}g^{\nu}_{\kappa}) + \frac{1}{2}\omega^{\mu\kappa}g^{\nu}_{\sigma}(x_{\mu}\partial_{\kappa} - x_{\kappa}\partial_{\mu})\right]A^{\sigma}(x^{\mu}) \\
&= \left[I - \frac{1}{2}\omega^{\mu\kappa}\hat{M}_{\mu\kappa}\right]^{\nu}_{\sigma}A^{\sigma}(x^{\mu})
\end{aligned} \tag{57}$$

where

$$(\hat{M}_{\mu\kappa})^{\nu}_{\sigma} = (g_{\kappa\sigma}g^{\nu}_{\mu} - g_{\mu\sigma}g^{\nu}_{\kappa}) + g^{\nu}_{\sigma}(x_{\mu}\partial_{\kappa} - x_{\kappa}\partial_{\mu}) = (L_{\mu\kappa})^{\nu}_{\sigma} + g^{\nu}_{\sigma}\hat{L}_{\mu\kappa} \tag{58}$$

and the commutation relation is

$$\begin{aligned}
[\hat{M}_{\mu\nu}, \hat{M}_{\sigma\kappa}] &= [L_{\mu\nu} + \hat{L}_{\mu\nu}, L_{\sigma\kappa} + \hat{L}_{\sigma\kappa}] \\
&= [L_{\mu\nu}, L_{\sigma\kappa}] + [\hat{L}_{\mu\nu}, L_{\sigma\kappa}] + [L_{\mu\nu}, \hat{L}_{\sigma\kappa}] + [\hat{L}_{\mu\nu}, \hat{L}_{\sigma\kappa}] \\
&= [L_{\mu\nu}, L_{\sigma\kappa}] + [\hat{L}_{\mu\nu}, \hat{L}_{\sigma\kappa}] \\
&= g_{\mu\kappa}L_{\nu\sigma} + g_{\nu\sigma}L_{\mu\kappa} - g_{\mu\sigma}L_{\nu\kappa} - g_{\nu\kappa}L_{\mu\sigma} \\
&+ g_{\mu\kappa}\hat{L}_{\nu\sigma} + g_{\nu\sigma}\hat{L}_{\mu\kappa} - g_{\mu\sigma}\hat{L}_{\nu\kappa} - g_{\nu\kappa}\hat{L}_{\mu\sigma} \\
&= g_{\mu\kappa}\hat{M}_{\nu\sigma} + g_{\nu\sigma}\hat{M}_{\mu\kappa} - g_{\mu\sigma}\hat{M}_{\nu\kappa} - g_{\nu\kappa}\hat{M}_{\mu\sigma}
\end{aligned}$$

So

$$[\hat{M}_{\mu\nu}, \hat{M}_{\sigma\kappa}] = g_{\mu\kappa}\hat{M}_{\nu\sigma} + g_{\nu\sigma}\hat{M}_{\mu\kappa} - g_{\mu\sigma}\hat{M}_{\nu\kappa} - g_{\nu\kappa}\hat{M}_{\mu\sigma} \tag{59}$$

9 Poincaré Group

In this chapter, we will examine one of the most important groups in the Minkowski space called **Poincaré Group**, and like all groups, we will examine the structures of the group and its algebra and in the last part of this chapter, we will examine the irreducible representations of this group.

9.1 Defining of Poincaré group

Definition 9.1. Poincaré transformation in Minkowski space are put together from a Lorentz transformation Λ^μ_ν and a shift a^μ

$$x^{'\mu} = \Lambda^\mu_\nu x^\nu + a^\mu \tag{1}$$

Note the Poincaré transformation only leaves distances between four-vectors invariant:

$$g_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) = g_{\mu\nu}(x^{'\mu} - y^{'\mu})(x^{'\nu} - y^{'\nu}) \tag{2}$$

Therefore, Poincaré transformations are also called to be **inhomogeneous Lorentz transformations**.
Now we show a set of (Λ, a) forms a group $\mathcal{P}$ (Poincaré group).

- Close

$$\begin{aligned}\forall(\Lambda_1, a_1), (\Lambda_2, a_2) \in \mathcal{P} \;\mid\; x_2^\mu &= \Lambda^\mu_{2\nu} x_1^\nu + a_2^\mu \\ &= \Lambda^\mu_{2\nu}(\Lambda^\nu_{1\sigma} x_1^\sigma + a_1^\nu) + a_2^\mu \\ &= \Lambda^\mu_{2\nu}\Lambda^\nu_{1\sigma} x_1^\sigma + \Lambda^\mu_{2\nu} a_1^\nu + a_2^\mu \\ &= \Lambda^\mu_\sigma x_1^\sigma + a^\mu\end{aligned}$$

  So

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$$

- Associativity

$$\begin{aligned}&\forall(\Lambda_1, a_1), (\Lambda_2, a_2), (\Lambda_3, a_3) \in \mathcal{P} \\ &(\Lambda_3, a_3)[(\Lambda_2, a_2)(\Lambda_1, a_1)] = (\Lambda_3, a_3)(\Lambda_2\Lambda_1, \Lambda_2 a_2 + a_1) \\ &= (\Lambda_3\Lambda_2\Lambda_1, \Lambda_3\Lambda_2 a_2 + \Lambda_3 a_1 + a_3) = [(\Lambda_3, a_3)(\Lambda_2, a_2)](\Lambda_1, a_1)\end{aligned}$$

- Identity element

$$\exists(I, 0), \forall(\Lambda, a) \in \mathcal{P} \;\mid\; (I, 0)(\Lambda, a) = (\Lambda, a)$$

- Inverse element

$$\forall(\Lambda, a), \exists(\Lambda, a)^{-1} \in \mathcal{P} \;\mid\; (\Lambda, a)^{-1}(\Lambda, a) = (\Lambda^{-1}, -\Lambda^{-1}a)(\Lambda, a) = (I, 0)$$

So the set of (Λ, a) forms a Poincaré group $\mathcal{P}$.

9.2 Poincaré algebra

First, in order to investigate the Poincaré algebra, it is necessary to obtain the infinitesimally transforms for the **rotation** and **translation**, so

- **Lorentz transformations**
  As we obtained in Lorentz transformations eq.56, we know the operator of the infinitesimal rotation is

$$\hat{R}(\omega) = \frac{1}{2}\omega_{\sigma\kappa}\hat{L}^{\sigma\kappa} \quad , \quad \delta x^{\mu} = \hat{R}(\omega)x^{\mu} = \omega^{\mu}_{\nu}x^{\nu} \tag{3}$$

- **Translations**
  Let's consider the infinitesimal space-time translation of the form

$$\delta x^{\mu} = \epsilon^{\mu} \tag{4}$$

can be generated by the infinitesimal translation operator

$$\hat{R}(\epsilon) = \epsilon^{\mu}\hat{P}_{\mu} = \epsilon^{\mu}\partial_{\mu} \tag{5}$$

so that

$$\hat{R}(\epsilon)x^{\mu} = \epsilon^{\nu}\hat{P}_{\nu}x^{\mu} = \epsilon^{\nu}\partial_{\nu}x^{\mu} = \epsilon^{\mu} = \delta x^{\mu} \tag{6}$$

Therefore the generator of the translation is P_{μ} and Lie bracket is

$$[\hat{P}_{\mu}, \hat{P}_{\nu}] = [\partial_{\mu}, \partial_{\nu}] = 0 \tag{7}$$

Note $\hat{P}$ is proportional with $\hat{p}$.

Now if in addition to infinitesimal Lorentz transformations we consider the infinitesimal translations we have the infinitesimal Poincaré transformation

$$\delta x^{\mu} = \epsilon^{\mu} + \omega^{\mu}_{\nu}x^{\nu} = \hat{R}(\epsilon, \omega)x^{\mu} \tag{8}$$

where

$$\hat{R}(\epsilon, \omega) = \epsilon_{\mu}\hat{P}^{\mu} + \frac{1}{2}\omega_{\mu\nu}\hat{L}^{\mu\nu} \tag{9}$$

So the generators of the Poincaré group are $\hat{P}^{\mu}$ and $\hat{L}^{\mu\nu}$. Therefore the Lie algebra of the Poincaré group (Poincaré algebra) basis are the generators so we have a commutation relation by Lie bracket. As before we obtained 52 and 7 so

$$\begin{aligned}
[\hat{P}^\mu, \hat{L}^{\nu\sigma}] &= [\partial^\mu, x^\nu\partial^\sigma - x^\sigma\partial^\nu] = [\partial^\mu, x^\nu\partial^\sigma] - [\partial^\mu, x^\sigma\partial^\nu] \\
&= g^{\mu\nu}\partial^\sigma - x^\nu\partial^\sigma\partial^\mu - g^{\mu\sigma}\partial^\nu - x^\sigma\partial^\mu\partial^\nu = g^{\mu\nu}\partial^\sigma - g^{\mu\sigma}\partial^\nu \\
&= g^{\mu\nu}\hat{P}^\sigma - g^{\mu\sigma}\hat{P}^\nu \qquad (10)
\end{aligned}$$

So

$$[\hat{P}^\mu, \hat{P}^\nu] = 0 \qquad (11)$$

$$[\hat{L}^{\mu\nu}, \hat{L}^{\sigma\kappa}] = g^{\nu\sigma}\hat{L}^{\mu\kappa} + g^{\mu\kappa}\hat{L}^{\nu\sigma} - g^{\nu\kappa}\hat{L}^{\mu\sigma} - g^{\mu\sigma}\hat{L}^{\nu\kappa} \qquad (12)$$

$$[\hat{P}^\mu, \hat{L}^{\nu\sigma}] = g^{\mu\nu}\hat{P}^\sigma - g^{\mu\sigma}\hat{P}^\nu \qquad (13)$$

Therefore $\hat{P}_\mu$ and $\hat{L}^{\nu\sigma}$ are generators of Poincaré algebra and they have the commutation relation 11, 12 and 13.
Note that usually in books and articles, they write $\hat{L}^{\nu\sigma}$ as $\hat{M}^{\nu\sigma}$ and we continue in the same general way [1] [2] [46] [47].

9.3 Casimir operators of Poincaré algebra

As we found out in the chapter on Lie algebra representation Theory, according to Schur's lemma, if ρ is an irreducible representation and if an operator like $\rho(c)$ is found that commutes with all generators of the algebra, then that operator is proportional to the unit operator and called **Casimir operators**. Now in Poincaré algebra, the first Casimir operator is given by the scalar product of the momentum operator with itself

$$\hat{P}^2 = g_{\mu\nu}P^\mu P^\nu = P^\mu P_\mu \qquad (14)$$

Let's commute all generators of Poincaré algebra

$$\begin{aligned}
[\hat{P}^2, \hat{P}^\nu] &= [\hat{P}^\mu\hat{P}_\mu, \hat{P}^\nu] = \hat{P}^\mu[\hat{P}_\mu, \hat{P}^\nu] + [\hat{P}^\mu, \hat{P}^\nu]\hat{P}_\mu \\
&= 0 \qquad (15) \\
[\hat{P}^2, \hat{M}^{\mu\nu}] &= [\hat{P}^\sigma\hat{P}_\sigma, \hat{M}^{\mu\nu}] = \hat{P}^\sigma[\hat{P}_\sigma, \hat{M}^{\mu\nu}] + [\hat{P}^\sigma, \hat{M}^{\mu\nu}]\hat{P}_\sigma \\
&= \hat{P}^\sigma(g^\mu_\sigma\hat{P}^\nu - g^\nu_\sigma\hat{P}^\mu) + (g^{\mu\sigma}\hat{P}^\nu - g^{\nu\sigma}\hat{P}^\mu)\hat{P}_\sigma \\
&= \hat{P}^\mu\hat{P}^\nu - \hat{P}^\nu\hat{P}^\mu + \hat{P}^\mu\hat{P}^\nu - \hat{P}^\nu\hat{P}^\mu \\
&= 0 \qquad (16)
\end{aligned}$$

So $\hat{P}^2$ is the first Casimir operator of Poincaré algebra.
Let's define a new vector operator, known as the **Pauli-Lubanski operator**

[1] [48] [49] [50] [51], from the generators of the Poincaré algebra as

$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\hat{P}_\nu \hat{M}_{\lambda\rho} \tag{17}$$

Now commute the new operator with the generators of Poincaré algebra

$$\begin{aligned}
[\hat{W}^\mu, \hat{P}^\sigma] &= \hat{W}^\mu \hat{P}^\sigma - \hat{P}^\sigma \hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}(\hat{P}_\nu \hat{M}_{\lambda\rho}\hat{P}^\sigma - \hat{P}^\sigma \hat{P}_\nu \hat{M}_{\lambda\rho}) \\
&= \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}(\hat{P}_\nu \hat{M}_{\lambda\rho}\hat{P}^\sigma - \hat{P}_\nu \hat{P}^\sigma \hat{M}_{\lambda\rho} + \hat{P}_\nu \hat{P}^\sigma \hat{M}_{\lambda\rho} - \hat{P}^\sigma \hat{P}_\nu \hat{M}_{\lambda\rho}) \\
&= \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}(\hat{P}_\nu [\hat{M}_{\lambda\rho}, \hat{P}^\sigma] + [\hat{P}_\nu, \hat{P}^\sigma]\hat{M}_{\lambda\rho}) = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\hat{P}_\nu [\hat{M}_{\lambda\rho}, \hat{P}^\sigma] \\
&= \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\hat{P}_\nu (g^\sigma_\rho \hat{P}_\lambda - g^\sigma_\lambda \hat{P}_\rho) = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} g^\sigma_\rho \hat{P}_\nu \hat{P}_\lambda - \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} g^\sigma_\lambda \hat{P}_\nu \hat{P}_\rho \\
&= 0
\end{aligned} \tag{18}$$

and for simply we define [1]

$$\hat{\widetilde{M}}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\hat{M}_{\lambda\rho} \tag{19}$$

we can write Pauli-Lubanski operator also as

$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\hat{P}_\nu \hat{M}_{\lambda\rho} = \hat{P}_\nu \hat{\widetilde{M}}^{\mu\nu} \tag{20}$$

Now we obtain commutator with $\hat{M}_{\sigma\tau}$

$$\begin{aligned}
[\hat{\widetilde{M}}_{\mu\nu}, \hat{M}_{\sigma\tau}] &= [\frac{1}{2}\epsilon^{\lambda\rho}_{\mu\nu}\hat{M}_{\lambda\rho}, \hat{M}_{\sigma\tau}] = \frac{1}{2}\epsilon^{\lambda\rho}_{\mu\nu}[\hat{M}_{\lambda\rho}, \hat{M}_{\sigma\tau}] \\
&= \frac{1}{2}\epsilon^{\lambda\rho}_{\mu\nu}(g_{\rho\sigma}\hat{M}_{\lambda\tau} + g_{\lambda\tau}\hat{M}_{\rho\sigma} - g_{\rho\tau}\hat{M}_{\lambda\sigma} - g_{\lambda\sigma}\hat{M}_{\rho\tau}) \\
&= -\frac{1}{2}\epsilon^{\rho}_{\mu\nu\sigma}\hat{M}_{\rho\tau} + \frac{1}{2}\epsilon^{\rho}_{\mu\nu\tau}\hat{M}_{\rho\sigma} + \frac{1}{2}\epsilon^{\rho}_{\mu\nu\tau}\hat{M}_{\rho\sigma} - \frac{1}{2}\epsilon^{\rho}_{\mu\nu\sigma}\hat{M}_{\rho\tau} \\
&= -\epsilon^{\rho}_{\mu\nu\sigma}\hat{M}_{\rho\tau} + \epsilon^{\rho}_{\mu\nu\tau}\hat{M}_{\rho\sigma} = -\epsilon^{\rho}_{\mu\nu\sigma}(-\frac{1}{2}\epsilon^{\delta\zeta}_{\rho\tau}\hat{\widetilde{M}}_{\delta\zeta}) + \epsilon^{\rho}_{\mu\nu\tau}(-\frac{1}{2}\epsilon^{\delta\zeta}_{\rho\sigma}\hat{\widetilde{M}}_{\delta\zeta}) \\
&= -\frac{1}{2}\epsilon^{\rho}_{\mu\nu\tau}\epsilon^{\delta\zeta}_{\rho\sigma}\hat{\widetilde{M}}_{\delta\zeta} + \frac{1}{2}\epsilon^{\rho}_{\mu\nu\sigma}\epsilon^{\delta\zeta}_{\rho\tau}\hat{\widetilde{M}}_{\delta\zeta} \\
&= \frac{1}{2}[g_{\mu\tau}(g^\delta_\nu g^\zeta_\sigma - g^\delta_\sigma g^\zeta_\nu) + g^\delta_\mu(g^\zeta_\nu g_{\tau\sigma} - g^\zeta_\sigma g_{\tau\nu}) \\
&+ g^\zeta_\mu(g^\delta_\sigma g_{\nu\tau} - g^\delta_\nu g_{\sigma\tau}) - \sigma \leftrightarrow \tau]\hat{\widetilde{M}}_{\delta\zeta} \\
&= \frac{1}{2}[g_{\mu\tau}(\hat{\widetilde{M}}_{\nu\sigma} - \hat{\widetilde{M}}_{\sigma\nu}) + \hat{\widetilde{M}}_{\mu\nu}g_{\tau\sigma} - \hat{\widetilde{M}}_{\mu\sigma}g_{\tau\nu} - \hat{\widetilde{M}}_{\sigma\mu}g_{\nu\tau} - \hat{\widetilde{M}}_{\sigma\mu}g_{\nu\tau} \\
&- \hat{\widetilde{M}}_{\nu\mu}g_{\sigma\tau} - \sigma \leftrightarrow \tau] \\
&= \hat{\widetilde{M}}_{\mu\tau}g_{\nu\sigma} + \hat{\widetilde{M}}_{\nu\sigma}g_{\mu\tau} - \hat{\widetilde{M}}_{\mu\sigma}g_{\nu\tau} - \hat{\widetilde{M}}_{\nu\tau}g_{\mu\sigma}
\end{aligned}$$

So

$$[\hat{\widetilde{M}}_{\mu\nu}, \hat{M}_{\sigma\tau}] = \hat{\widetilde{M}}_{\mu\tau} g_{\nu\sigma} + \hat{\widetilde{M}}_{\nu\sigma} g_{\mu\tau} - \hat{\widetilde{M}}_{\mu\sigma} g_{\nu\tau} - \hat{\widetilde{M}}_{\nu\tau} g_{\mu\sigma} \tag{21}$$

Let's commute $\hat{W}^\mu$ with $\hat{M}_{\sigma\tau}$ With the help of equation 21

$$\begin{aligned}
[\hat{W}^\mu, \hat{M}_{\sigma\tau}] &= [\hat{P}_\nu \hat{\widetilde{M}}^{\mu\nu}, \hat{M}_{\sigma\tau}] = [\hat{P}_\nu, \hat{M}_{\sigma\tau}] \hat{\widetilde{M}}^{\mu\nu} + \hat{P}_\nu [\hat{\widetilde{M}}^{\mu\nu}, \hat{M}_{\sigma\tau}] \\
&= (g_{\nu\sigma} \hat{P}_\tau - g_{\nu\tau} \hat{P}_\sigma) \hat{\widetilde{M}}^{\mu\nu} + \hat{P}_\nu (\hat{\widetilde{M}}^{\mu}{}_{\tau} g^\nu_\sigma + \hat{\widetilde{M}}^{\nu}{}_{\sigma} g^\mu_\tau - \hat{\widetilde{M}}^{\mu}{}_{\sigma} g^\nu_\tau - \hat{\widetilde{M}}^{\nu}{}_{\tau} g^\mu_\sigma) \\
&= g^\mu_\sigma \hat{P}_\nu \hat{\widetilde{M}}^{\nu}{}_{\tau} - g^\mu_\tau \hat{P}_\nu \hat{\widetilde{M}}^{\nu}{}_{\sigma} = g^\mu_\sigma \hat{W}_\tau - g^\mu_\tau \hat{W}_\sigma
\end{aligned} \tag{22}$$

So $\hat{W}^\mu$ not commutes with the generator of Poincaré algebra. Let's define $\hat{W}^2 = \hat{W}^\mu \hat{W}_\mu$ and investigate can commute with all generators or not. So

$$[\hat{W}^2, \hat{P}^\nu] = [\hat{W}^\mu \hat{W}_\mu, \hat{P}^\nu] = \hat{W}^\mu [\hat{W}_\mu, \hat{P}^\nu] + [\hat{W}^\mu, \hat{P}^\nu] \hat{W}_\mu = 0 \tag{23}$$

and

$$\begin{aligned}
[\hat{W}^2, \hat{M}_{\sigma\tau}] &= [\hat{W}^\mu \hat{W}_\mu, \hat{M}_{\sigma\tau}] = \hat{W}^\mu [\hat{W}_\mu, \hat{M}_{\sigma\tau}] + [\hat{W}^\mu, \hat{M}_{\sigma\tau}] \hat{W}_\mu \\
&= \hat{W}^\mu (g_{\mu\sigma} \hat{W}_\tau - g_{\mu\tau} \hat{W}_\sigma) + (g_{\mu\sigma} \hat{W}_\tau - g_{\mu\tau} \hat{W}_\sigma) \hat{W}_\mu \\
&= \hat{W}_\sigma \hat{W}_\tau - \hat{W}_\sigma \hat{W}_\tau + \hat{W}_\tau \hat{W}_\sigma - \hat{W}_\tau \hat{W}_\sigma \\
&= 0
\end{aligned} \tag{24}$$

So the second Casimir operator of Poincaré algebra is $\hat{W}^2 = \hat{W}^\mu \hat{W}_\mu$. Therefore $\hat{P}^2$ and $\hat{W}^2$ represent the only Casimir operators of the algebra and consequently, the representations can be labeled by the **eigenvalues** of these operators.

9.4 Irreducible representations of the Poincaré Group

The irreducible representations of the Poincaré Group can be classified into two distinct categories.

9.4.1 Massive Representation

To find unitary irreducible representations of the Poincaré algebra, we choose the basis vectors of the representation to be eigenstates of the momentum operators. The eigenstates of the momentum operators $|p\rangle$ are, labeled by the momentum eigenvalues, p_μ, satisfying

$$\hat{P}_\mu |p\rangle = p_\mu |p\rangle \tag{25}$$

and in this basis, the eigenvalues of the operator $\hat{P}^2 = \hat{P}^\mu \hat{P}_\mu$ are

$$\hat{P}^2 \left|p\right\rangle = p^\mu p_\mu \left|p\right\rangle \tag{26}$$

On the other hand, we know in special relativity

$$p^2 = p^\mu p_\mu = m^2 \tag{27}$$

where m is the rest mass of the single particle state and we assume the rest mass to be non-zero. Now we want the operator $\hat{W}^2$ acting on a state so before we obtain $\hat{W}^2$ in new form

$$\begin{aligned}
\hat{W}^2 = \hat{W}^\mu \hat{W}_\mu &= \frac{1}{4}\epsilon^{\mu\nu\lambda\rho}\epsilon_\mu^{\sigma\tau\zeta}\hat{M}_{\lambda\rho}\hat{P}_\nu \hat{M}_{\tau\zeta}\hat{P}_\sigma \\
&= \frac{1}{4}[-g^{\nu\sigma}(g^{\lambda\tau}g^{\rho\zeta} - g^{\lambda\zeta}g^{\rho\tau}) - g^{\nu\tau}(g^{\lambda\zeta}g^{\rho\sigma} - g^{\lambda\sigma}g^{\rho\zeta}) - g^{\nu\zeta}(g^{\lambda\sigma}g^{\rho\tau} \\
&- g^{\lambda\tau}g^{\rho\sigma})]\hat{M}_{\lambda\rho}\hat{P}_\nu \hat{M}_{\tau\zeta}\hat{P}_\sigma \\
&= -\frac{1}{2}\hat{M}_{\lambda\rho}\hat{P}_\nu \hat{M}^{\lambda\rho}\hat{P}^\nu - \hat{M}_{\lambda\rho}\hat{P}_\nu \hat{M}^{\nu\lambda}\hat{P}^\rho \\
&= -\frac{1}{2}\hat{M}_{\lambda\rho}(\hat{M}^{\lambda\rho}\hat{P}^2 + g_\nu^\lambda \hat{P}^\rho P^\nu - g_\nu^\rho \hat{P}^\lambda P^\nu) - \hat{M}_{\lambda\rho}(\hat{M}^{\nu\lambda}\hat{P}_\nu \hat{P}^\rho + 4\hat{P}^\lambda \hat{P}^\rho \\
&- g_\nu^\lambda \hat{P}^\nu \hat{P}^\rho) \\
&= -\frac{1}{2}\hat{M}^{\lambda\rho}M_{\lambda\rho}\hat{P}^2 - \hat{M}_{\lambda\rho}\hat{M}^{\nu\lambda}\hat{P}_\nu \hat{P}^\rho
\end{aligned} \tag{28}$$

Let's go to the rest frame of the **massive particle**. In this frame we have

$$p_\mu = (m, 0, 0, 0) \;\;,\;\; p^2 = p^\mu p_\mu = m^2 \tag{29}$$

so the operator $\hat{W}^2$ acting on a state, takes the form

$$\begin{aligned}
\hat{W}^2 \left|p\right\rangle &= (-\frac{1}{2}\hat{M}^{\lambda\rho}M_{\lambda\rho}\hat{P}^2 - \hat{M}_{\lambda\rho}\hat{M}^{\nu\lambda}\hat{P}_\nu \hat{P}^\rho)\left|p\right\rangle \\
&= -\frac{1}{2}\hat{M}^{\lambda\rho}M_{\lambda\rho}\hat{P}^2\left|p\right\rangle - \hat{M}_{\lambda\rho}\hat{M}^{\nu\lambda}\hat{P}_\nu \hat{P}^\rho \left|p\right\rangle \\
&= -\frac{1}{2}m^2\hat{M}^{\lambda\rho}M_{\lambda\rho}\left|p\right\rangle - m^2\hat{M}_{\lambda 0}\hat{M}^{0\lambda}\left|p\right\rangle \\
&= -\frac{1}{2}m^2(2\hat{M}^{0\lambda}\hat{M}_{0\lambda} + \hat{M}^{ij}\hat{M}_{ij})\left|p\right\rangle + m^2\hat{M}^{0\lambda}\hat{M}_{0\lambda}\left|p\right\rangle \\
&= -\frac{1}{2}m^2\hat{M}^{ij}\hat{M}_{ij}\left|p\right\rangle
\end{aligned} \tag{30}$$

Now from eq.51, $\hat{M}^{ij} = \epsilon_{ij}^k \hat{J}_k$

$$\begin{aligned}
\hat{W}^2\left|p\right\rangle &= -\frac{1}{2}m^2\hat{M}^{ij}\hat{M}_{ij}\left|p\right\rangle = m^2\hat{J}^k\hat{J}_k\left|p\right\rangle = -m^2\mathbf{J}^2\left|p\right\rangle \\
&= -m^2 s(s+1)\left|p\right\rangle
\end{aligned} \tag{31}$$

We took help from eq.56 for the last line.
So $\hat{p}_\mu = (0, 0, 0, m)$ and $\hat{W}_\mu = (0, -m\mathbf{J})$ and the spin satisfy for **Massive particles** is $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. As is clear the eigenspace of the momentum operator is, as we have seen, a representation of $SU(2)$. So **Massive fields are classified by irreducible representations of $SU(2)$ that determines spin**.

9.4.2 Massless Representation

In contrast to the massive representations of the Poincaré group, the representations for a massless particle are slightly more involved. The basic reason behind this is that the **little group** of a lightcone vector is not so obvious [1] [52]. In this case, we have

$$p^\mu p_\mu = 0 \ \ , \ \ p_\mu = (p, 0, 0, p) \ \ , \ \ p \neq 0 \tag{32}$$

So

$$\hat{P}^2 \left|p\right\rangle = 0 \tag{33}$$

and from eq.18 we have

$$\hat{P}_\mu \hat{W}^\mu \left|p\right\rangle = 0 \tag{34}$$

There now appear two distinct possibilities for the action of the second Casimir $\hat{W}^2$ on the states of the representation, namely

$$\hat{W}^2 = \hat{W}^\mu \hat{W}_\mu \neq 0 \ \ or \ \ \hat{W}^2 = \hat{W}^\mu \hat{W}_\mu = 0 \tag{35}$$

In massless particles if $\hat{W}^2 \neq 0$ then we have infinite representations of spin values [1]. So we consider $\hat{W}^2 = 0$ then we can define $\hat{W}^\mu$ by the momentum operator so

$$\hat{W}^\mu = -p^\mu \hat{h} \tag{36}$$

where $\hat{h}$ is called **proportionality factor operator** [1]. Let's determine $\hat{h}$

$$\hat{W}^\mu \left|p\right\rangle = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\hat{P}_\nu \hat{M}_{\lambda\rho}\left|p\right\rangle \tag{37}$$

then from eq.32

$$W^0 \left|p\right\rangle = \frac{1}{2}\epsilon^{0ijk}p_i \hat{M}_{jk}\left|p\right\rangle = \frac{1}{2}\epsilon_{ijk}p_i(\epsilon^l_{jk}J_l)\left|p\right\rangle = -\mathbf{p}\cdot\mathbf{J}\left|p\right\rangle \tag{38}$$

Comparing with eq.36 we conclude that

$$\hat{h} = \frac{\mathbf{p}\cdot\mathbf{J}}{p} \tag{39}$$

Therefore we obtain **helicity** for **Massless particles**.

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