2. Running fine structure constant of QED $\alpha(s)$.

The running fine structure constant of QED $\alpha(s)$ can be expressed as

$$\alpha(s) = \frac{\alpha(0)}{1-\Delta\alpha(s)}; \qquad \alpha(0) = 1/137{,}036$$

whereby $\Delta\alpha(s)$ is governed by the renormalized vacuum polarization function $\Pi_\gamma(s)$ to be defined by the Fourier transform of the time-ordered product of the EM currents $j^\mu_{em}(s)$ in the vacuum

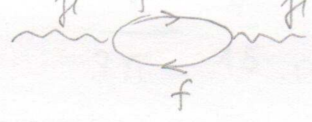
$$(q^\mu q^\nu - q^2 g^{\mu\nu})\,\Pi_\gamma(q^2) = i\int d^4x\, e^{iqx}\langle 0|T[j^\mu_{em}(x)\,j^\nu_{em}(0)]|0\rangle$$

as

$$\Delta\alpha(s) = -4\pi\alpha\,\mathrm{Re}\left[\Pi_\gamma(s) - \Pi_\gamma(0)\right].$$

The $\Delta\alpha(s)$, for instance at $s = M_Z^2$ is large, due to the large change in scale going from $s\to 0$ (Thomson limit) to the mass of Z-resonance

In perturbation theory, the leading order contribution represented by the free fermion loops

$$\Delta\alpha(s) = \sum_f$$ 

gives

$$\Delta\alpha(s) = \frac{\alpha(0)}{3\pi}\sum_f{}' Q^2 N_{cf}\left(\ln\frac{s}{m_f^2} - \frac{5}{3}\right)$$

Q ... the fermion charge

N_{cf} ... the color factor — 1 for leptons

— 3 for quarks

One distinguishes contributions in $\Delta\alpha(s)$

- from leptons
- from the 5 light quarks u, d, s, c, b
- from "top"-quark.

Then

$$\Delta\alpha(s) = \Delta\alpha_\ell(s) + \Delta\alpha^{(5)}_{had}(s) + \Delta\alpha_{top}(s).$$

The leptonic contributions are calculable in perturbation theory, where at leading order the free lepton loops yield

$$\Delta\alpha_\ell(s) = \sum_{\ell = e, \mu, \tau}' \frac{\alpha(0)}{3\pi}\left[\ln(s/m_\ell^2) - \frac{5}{3}\right]$$

$$\Rightarrow \quad \boxed{\Delta\alpha_\ell(M_Z^2) \simeq 0.031498}.$$

Since the t-quark is heavy ($m_t \gg M_Z$), one cannot use the light fermion approximation for it and it decouples like

$$\boxed{\Delta\alpha_{top}(s) \simeq -\frac{\alpha(0)}{3\pi}\frac{4}{15}\frac{M_Z^2}{m_t} \rightarrow 0.}$$

A serious problem is the five light qarks, u, d, s, c, b, contribution $\Delta\alpha^{(5)}_{had}(s)$, which cannot be calculated by using "perturbative" QCD.

Fortunately, one can evaluate it from $e^+e^- \to$ hadrons data, by using dispersion relation

$$\mathrm{Re}\,\Pi_\gamma(s) - \Pi_\gamma(0) = \frac{s}{\pi}\,\mathrm{Re}\int_{s_0}^{\infty} ds' \frac{\mathrm{Im}\,\Pi_\gamma(s')}{s'(s'-s-i\varepsilon)}\,ds'$$

and the optical theorem

$$\mathrm{Im}\,\Pi_\gamma(s) = \frac{s}{e^2}\,\sigma_{tot}(e^+e^- \to \text{hadrons})(s).$$

In terms of the total cross-section ratio

$$R(s) = \frac{\sigma_{tot}(e^+e^- \to \text{had})}{\sigma_{tot}(e^+e^- \to \mu^+\mu^-)},$$

where

$$\sigma_{tot}(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha(0)^2}{3s},$$

one obtaines

$$\Delta\alpha^{(5)}_{had}(M_Z^2) = -\frac{\alpha(0)\,M_Z^2}{3\pi}\,\mathrm{Re}\,P\!\!\int_{4m_\pi^2}^{\infty} \frac{R(s')}{s'(s'-M_Z^2-i\varepsilon)}\,ds'.$$