Egyptian Fractions

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Introduction

The notion of an Egyptian Fraction simply implies that every rational number can be expressed as the finite sum of distinct unit fractions. A unit fraction is a fraction that follows $\frac{1}{a}$ where $a \in \mathbb{Z}^+$, i.e. the numerator is always equivalent to 1. More succinctly, we can define an Egyptian Fraction for a rational number as,

$$\frac{a}{b} = \sum_{i=1}^{n} \frac{1}{x_i}$$

where $\frac{a}{b} \in \mathbb{Q}$ (for the rest of this paper, note that $\underline{\gcd(a, b) = 1}$) and $x_i \neq x_j$ for all $i \neq j$. A big factor in being able to expand a rational number into a sum of unit fractions is being able to factor the denominator but it is also case dependent. The following example is using small numbers and a simple algorithm.

The original application of this topic was in ancient Egypt where this concept was used to divide rations. If you had 7 loaves of bread and need to divide it amongst 8 people evenly, you can compute the fraction expansion of $\frac{7}{8} = \frac{1}{2} + \frac{1}{3} + \frac{1}{24}$. Hence, you would first cut the loaves in half and give everyone one half, then cut the remaining loaves into thirds and give everyone a third and then finally cut the remaining part into 24ths and give everyone a 24th.

<u>Ex.</u> Find the unit fractions that sum to $\frac{13}{19}$.
Solution:
The first step for this would be to figure out the largest unit fraction that is less than $\frac{13}{19}$. We see that $\frac{1}{2}$. Thus $\frac{13}{19} - \frac{1}{2} = \frac{7}{38}$. Then again, we can find that $\frac{1}{6}$ is the largest unit fraction less than $\frac{7}{38}$ and thus we again find the difference between them as $\frac{7}{38} - \frac{1}{6} = \frac{1}{57}$. Since the result is a unit fraction, we stop here and thus we can write $\frac{13}{19} = \frac{1}{2} + \frac{1}{6} + \frac{1}{57}$. The previous algorithm is known as the *greedy algorithm* where we subtract off the largest unit fraction less than the current fraction until we end up with a unit fraction. In more mathematical notation this algorithm will follow replacing $\frac{x}{y}$ by

$$\frac{x}{y} = \frac{1}{\lceil \frac{y}{x} \rceil} + \frac{(-y) \pmod{x}}{y \lceil \frac{y}{x} \rceil}$$

Proof. First, recall that calculating a negative number modulo another number is the same as adding the modulus to the negative number until you reach a positive number. Mathematically, this is equivalent to $-y \mod x = -b + ka$. However, $k = \lceil \frac{y}{x} \rceil$ and thus $-y \mod x = -y + \lceil \frac{y}{x} \rceil a$. Therefore assume, $x, y \in \mathbb{Z}^+$, then $\frac{x}{y} = \frac{1}{\lceil \frac{y}{x} \rceil} + \frac{-y \mod x}{y \lceil \frac{y}{x} \rceil}$. We want to show that the RHS indeed equals $\frac{x}{y}$. Thus substituting what $-y \mod x$ is equal to, we obtain $\frac{1}{\lceil \frac{y}{x} \rceil} + \frac{-y + \lceil \frac{y}{x} \rceil x}{y \lceil \frac{y}{x} \rceil}$. Adding these fractions we get $\frac{y - y + \lceil \frac{y}{x} \rceil x}{y \lceil \frac{y}{x} \rceil} = \frac{x}{y}$. Note that if you have an expansion of a number $z \in \mathbb{Q}^+$, then $-z$ is simply the subtraction of the unit fraction, i.e. if $z = \sum_{i=0}^{n} \frac{1}{x_i}$, then $-z = \sum_{i=0}^{n} \frac{-1}{x_i}$. Hence finding an Egyptian fraction expansion in $\mathbb{Q}^+$ means you have found an expansion for the same number in $\mathbb{Q}^-$.

$\square$

<u>**Ex Using Greedy Algorithm (Fibonacci)**:</u>
<u>Find the unit fractions that sum to $\frac{8}{17}$.</u>
Solution:
First we will calculate $\lceil \frac{y}{x} \rceil$, thus $\lceil \frac{19}{8} \rceil = 3$. Next we wish to find the least positive residue of $(-y) \pmod{x}$, hence $-19 \equiv 5 \pmod{8}$. Therefore in our first step of the algorithm we obtain $\frac{8}{19} = \frac{1}{3} + \frac{5}{57}$. Now we will use this same formula to make $\frac{5}{57}$ a sum of unit fractions. Thus we get $\frac{5}{57} = \frac{1}{12} + \frac{3}{684}$. But $\frac{3}{684} = \frac{1}{228}$. Thus, $\frac{8}{19} = \frac{1}{3} + \frac{1}{12} + \frac{1}{228}$.

Observe that this algorithm converged after three terms which is computationally feasible and also had small denominators. In some cases, this algorithm can be inefficient resulting in 10 plus terms with some terms having 500 digit denominators.

Sylvester's Sequence: Sylvester's sequence is an integer sequence that is the product of the previous terms plus 1. Thus we can define $s_n = 1 + \prod_{i=0}^{n-1} s_i$ where $s_0 = 2$. Therefore, $s_n = 2, 3, 7, 43, 1807, \ldots$. However, if you take the infinite sum of the reciprocal's of s_n, the sum will converge to 1. Hence, $\sum_{i=0}^{\infty} \frac{1}{s_i} \to 1$. Writing the partial sums of this up to some $j-1$ we get a closed formula of $\sum_{i=0}^{j-1} \frac{1}{s_i} = 1 - \frac{1}{s_j - 1}$. It is worth noting that this result comes from the fact that the original summation telescopes and we can write $\sum_{i=0}^{j-1} \frac{1}{s_i} = \sum_{i=0}^{j-1} (\frac{1}{s_i - 1} - \frac{1}{s_{i+1} - 1}) = \frac{1}{s_0 - 1} - \frac{1}{s_j - 1}$. Since this series converges and sums to 1, it forms an infinite Egyptian fraction representation of the number 1. We can truncate this process at any point depending on how accurate we want the approximation. For example, $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$ where we are only summing the first three reciprocals of s_n.

Fact 1: If $x \in \mathbb{Q}$, then by definition there exists $c, d \in \mathbb{Z}$ s.t. $x = \frac{c}{d}$. Then if we can write $c = k + l$ s.t. $d = kl - 1$ for some integers $k, l \in \mathbb{Z}$, then we can write,

$$\frac{c}{d} = \frac{1}{k} + \frac{1}{k(lk-1)} + \frac{1}{l} + \frac{1}{l(kl-1)}$$

Fact 2: If $x \in \mathbb{Q}$ s.t. $x = \frac{1}{d}$ for some integer d, then we can write,

$$\frac{1}{d} = \frac{1}{d+1} + \frac{1}{d(d+1)}$$

Using algebra of fractions, we can easily prove these expansions.

Proof. (1) Let $c = k + l$ and $d = kl - 1$, then $\frac{k+l}{kl-1} = \frac{1}{k} + \frac{1}{k(kl-1)} + \frac{1}{l} + \frac{1}{l(kl-1)}$. We can sum the RHS to get $\frac{l(kl-1)+l+k(kl-1)+k}{kl(kl-1)} = \frac{kl^2-l+l+k^2l-k+k}{kl(kl-1)} = \frac{kl^2+k^2l}{kl(kl-1)} = \frac{kl(k+l)}{kl(kl-1)} = \frac{k+l}{kl-1}$ as desired.

(2) Let $d \in \mathbb{Z}$. Then $\frac{1}{d} = \frac{1}{d+1} + \frac{1}{d(d+1)}$. Again, summing the RHS we get $\frac{d+1}{d(d+1)} = \frac{1}{d}$ as desired.

$\square$

We can consider the following lemma.

Lemma 1: If $a, b \in \mathbb{Z}$ s.t. $\gcd(a, b) = 1$ and $\frac{a}{b} = \sum_{i=1}^{n} \frac{1}{x_i}$ where $x_i \in \mathbb{Z}$ for all $1 \leq i \leq n$, then b is a divisor of $\prod_{i=1}^{n} x_i$.

Proof. Summing the RHS of $\frac{a}{b} = \sum_{i=1}^{n} \frac{1}{x_i}$ we obtain

$$\frac{\prod_{i=2}^{n} x_i + x_1 \prod_{i=3}^{n} x_i + \prod_{i=1}^{2} x_i \prod_{i=4}^{n} x_i + \cdots + \prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n} x_i}$$

Let $c = \prod_{i=2}^{n} x_i + x_1 \prod_{i=3}^{n} x_i + \prod_{i=1}^{2} x_i \prod_{i=4}^{n} x_i + \cdots + \prod_{i=1}^{n-1} x_i$ and $d = \prod_{i=1}^{n} x_i$. Then $\frac{a}{b} = \frac{c}{d}$. Thus $ad = bc$. Then ad is a multiple of b. But $\gcd(a, b) = 1$ so $b \nmid a$ so then $b \mid d$. Hence $b \mid \prod_{i=1}^{n} x_i$.

$\square$

Applications

Egyptian fractions have applications in electrical engineering and circuitry. When you have resistors in parallel, they the reciprocals add to form an equivalent resistance. More formally,

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \text{(Resistors in parallel)}$$

where R_i are all in parallel in a circuit and R_{eq} is the equivalent resistance. It is also the case that capacitors in series add this way to get an equivalent capacitance and inductors in parallel add this way to get an equivalent inductance. Looking at this in reverse, if we know an equivalent circuit element value, we can find an Egyptian fraction expansion of it to find each individual circuit elements value.

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \text{(Capacitors in series)}$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots \text{(Inductors in parallel)}$$

Modern Number Theory's Connection to Egyptian Fractions

Currently there are a few open problems in Number Theory that relate to Egyptian Fractions as well as applications in Cryptography.

- **Erdös-Straus Conjecture:** For every integer $n \geq 2$, there exists positive integers x, y, z such that $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.

  Since we are only looking for integer solutions, this is very much so an unsolved problem in the realm of finding solutions to Diophantine Equations. In fact though, this has shown to be true for $n \leq 10^{17}$.

- **Stream Cipher:** The concept of splitting rationals numbers into a sum of distinct unit fractions is similar to that of the process involved in the Stream Cipher. The stream cipher is an algorithm for encryption/decryption of data by encrypting bits using pseudo-random stream of bits called the key stream. The decryption then relies on the ability to split the stream back into bits. Hence, we are taking a larger object, the stream, and breaking it into smaller pieces that make it up, bits. In comparison to Egyptian fractions, the stream is the initial rational number you start with and the unit fractions are like the bits you get after decrypting the stream.

- **Znam's Problem:** An unsolved problem in number theory related to Egyptian fractions is if every integer can be represented as a sum of unit fractions. So far it has only been proven for 1. Formally, for what $y \in \mathbb{Z}$ does

$$y = \sum \frac{1}{x_i} + \prod \frac{1}{x_i}$$

where $x_i \in \mathbb{Z}$. As mentioned, this is proven true for the case of $y = 1$.

This problem and solutions to such a Diophantine equation have been found to have applications in topology and the classification of singularities on surfaces.

Works Cited

References

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