Exercises from Ch. 3: Modules, Lang

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Disclaimer: The proofs given are completed by myself. Thus, they are also reviewed by myself to ensure no logic flaws and incorrect arguments/deductions/conclusions. If you find any error in my proofs, please do not hesitate to contact me so we can discuss any potential errors and I can update the document as needed.

Exercises

Exercise 0.1. (Exercise 1) Let V be a vector space over a field K, and let U, W be subspaces. Show that

$$\dim U + \dim W = \dim(U + W) + \dim(U \cap W).$$

Proof. Define $\varphi \colon U \to (U + W)/W$ by $\varphi(u) = u + W$. Then $\ker \varphi = \{u \in U : u \in W\} = U \cap W$. By Theorem 5.3 (in-text),

$$\begin{aligned} \dim U &= \dim \ker \varphi + \dim \operatorname{Im} \varphi \\ &= \dim(U \cap W) + \dim \operatorname{Im} \varphi \end{aligned}$$

The image of φ is precisely

$$\operatorname{Im} \varphi = \{u + W : u \in U\} = (U + W)/W \quad \text{(since } \varphi \text{ is surjective)}$$

Thus,

$$\dim U = \dim(U \cap W) + \dim((U + W)/W)$$

Furthermore, since $W \subseteq U + W$, by Theorem 5.3, we have

$$\dim((U + W)/W) = \dim(U + W) - \dim W$$

Hence,

$$\dim U = \dim(U \cap W) + \dim(U + W) - \dim W$$

Therefore, rearranging the terms,

$$\dim U + \dim W = \dim(U + W) + \dim(U \cap W).$$

□

Exercise 0.2. (Exercise 3)
Let R be an entire ring containing a field k as a subring. Suppose that R is a finite dimensional vector space over k under the ring multiplication. Show that R is a field.

Proof. Define $\varphi_r \colon R \to R$ by $x \mapsto rx$. We first show that this is a k-linear map. Let $x, y \in R$. Then

$$\varphi_r(x+y) = \underbrace{r(x+y) = rx + ry}_{\because \text{ by distributivity of } R} = \varphi_r(x) + \varphi_r(y)$$

Let $s \in k$. Then

$$\varphi_r(sx) = \underbrace{rsx = srx}_{\because s \in k \subseteq R \text{ entire}} = s\varphi_r(x)$$

Thus, φ_r is a k-linear map. Suppose $r \neq 0$ and $\varphi_r(x) = 0$. Then $rx = 0$ which implies that $x = 0$ by the entirety of R. Thus, $\ker \varphi_r = \{0\}$. Since R is a finite dimensional vector space, injectivity implies surjectivity and thus the map is a bijection (Rank-Nullity Theorem). We know that $1 \in R$ and thus, by the surjectivity of φ_r, there exists some $x \in R$ such that $\varphi_r(x) = 1$. Hence, $rx = 1 \Rightarrow x = r^{-1}$. Thus, since $r \neq 0$ was arbitrary, R is a field.

□

Exercise 0.3. (Exercise 4) **Direct Sums**

a) Prove in detail that the conditions given in Proposition 3.2 for a sequence to split are equivalent. Show that a sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ splits if and only if there exists a submodule N of M such that M is equal to the direct sum $\operatorname{Im} f \oplus N$, and that if this is the case, then N is isomorphic to M''. Complete all the details of the proof of Proposition 3.2.

b) Let E and E_i $(i = 1, \ldots, m)$ be modules over a ring. Let $\varphi_i : E_i \to E$ and $\psi_i : E \to E_i$ be homomorphisms having the following properties:

$$\psi_i \circ \varphi_i = id., \quad \psi_i \circ \varphi_j = 0 \quad \text{if } i \neq j$$
$$\sum_{i=1}^{m} \varphi_i \circ \psi_i = id.$$

Show that the map $x \mapsto (\psi_1 x, \ldots, \psi_m x)$ is an isomorphism of E onto the direct product of the E_i $(i = 1, \ldots, m)$, and that the map

$$(x_1, \ldots, x_m) \mapsto \varphi_1 x_1 + \cdots + \varphi_m x_m$$

is an isomorphism of this direct product onto E.

Conversely, if E is equal to a direct product (or direct sum) of submodules E_i $(i = 1, \ldots, m)$, if we let φ_i be the inclusion of E_i in E, and ψ_i the projection of E on E_i, then these maps satisfy the above-mentioned properties.

Proof. a) We state **Proposition 3.2**: Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of modules. The following conditions are equivalent:

1 There exists a homomorphism $\varphi : M'' \to M$ such that $g \circ \varphi = id.$

2 There exists a homomorphism $\psi : M \to M'$ such that $\psi \circ f = id.$

If these conditions are satisfied, then we have isomorphisms:

$$M = \operatorname{Im} f \oplus \ker \psi, \quad M = \ker g \oplus \operatorname{Im} \varphi,$$
$$M \cong M' \oplus M''$$

We now proceed with the proof.

[Equivalence of 1 and 2]

$(1 \Rightarrow 2)$ Suppose there exists $\varphi : M'' \to M$ such that $g \circ \varphi = id$. Let $x \in M \Rightarrow g(x) \in M''$. Consider the difference

$$x - \varphi(g(x)) \in M.$$

Applying g yields,

$$\begin{aligned} g(x - \varphi(g(x)) &= g(x) - g(\varphi(g(x))) \\ &= g(x) - g(x) = 0 \end{aligned}$$

Therefore, $x - \varphi(g(x)) \in \ker g = \operatorname{Im} f$. Thus, $f(z) = x - \varphi(g(x))$ with $z \in M'$. Since f is injective, there is only one such z that maps to $x - \varphi(g(x))$ for some $x \in M$, i.e. z is uniquely determined by x. Thus, $\psi : M \to M'$ by $x \mapsto z$ is well-defined, where z satisfies $f(z) = x - \varphi(g(x))$. Let $m_1, m_2 \in M$. Then,

$$\begin{aligned} f(\psi(m_1 + m_2)) &= (m_1 + m_2) - \varphi(g(m_1 + m_2)) \\ &= m_1 + m_2 - \varphi(g(m_1) + g(m_2)) \\ &= m_1 + m_2 - \varphi(g(m_1)) - \varphi(g(m_2)) \\ &= m_1 - \varphi(g(m_1)) + m_2 - \varphi(g(m_2)) \\ &= f(\psi(m_1)) + f(\psi(m_2)) = f(\psi(m_1 + m_2)) \end{aligned}$$

Since f is injective, we get $\psi(m_1 + m_2) = \psi(m_1) + \psi(m_2)$. Now let $r \in A$ (the ring the module is over). Then,

$$\begin{aligned} f(\psi(rm)) &= rm - \varphi(g(rm)) \\ &= rm - r\varphi(g(m)) \qquad \text{since } \varphi, g \text{ are module-homomorphisms} \\ &= r(m - \varphi(g(m))) = f(r\psi(m)) \end{aligned}$$

Hence, by the injectivity of f, $\psi(rm) = r\psi(m)$. Therefore, ψ is a module homomorphism. Now, let $m' \in M'$. WTS $(\psi \circ f)(m') = m'$. Then,

$$\begin{aligned} f(\psi(f(m'))) &= f(m') - \varphi(g(f(m'))) \\ &= f(m') - \varphi(0) = f(m') \end{aligned}$$

Since f is injective, $\psi(f(m')) = m'$.

$(2 \Rightarrow 1)$ Assume that there exists a homomorphism $\psi : M \to M'$ such that $\psi \circ f = id$. Since $g : M \to M''$ is surjective, for all $x'' \in M''$, we many choose $m \in M$ such that $g(m) = x''$. Define $\varphi(x'') = m - f(\psi(m))$. We first check that this is well-defined. Suppose $m_1, m_2 \in\in M$ satisfy $g(m_1) = g(m_2)$. Then $g(m_1 - m_2) = 0 \Rightarrow m_1 - m_2 \in \ker g = \operatorname{Im} f$. Hence, $m_1 - m_2 = f(m')$ for some $m' \in M'$. Applying $\psi \circ f = id$, we have

$$m' = \psi(f(m')) = \psi(m_1 - m_2) = \psi(m_1) - \psi(m_2)$$

Thus,

$$\begin{aligned} (m_1 - f(\psi(m_1)) - (m_2 - f(\psi(m_2)) &= f(m') - f(\psi(m_1) - \psi(m_2)) \\ &= f(m' - (\psi(m_1) - \psi(m_2))) \\ &= f(0) = 0 \end{aligned}$$

Next, we show that φ is a homomorphism. Let $x_1'', x_2'' \in M''$ and let r be a scalar in the base ring. Choose $m_1, m_2 \in M$ such that $g(m_1) = x_1''$ and $g(m_2) = x_2''$. Then

$$
\begin{aligned}
\varphi(x_1'' + x_2'') &= (m_1 + m_2) - f(\psi(m_1 + m_2)) \\
&= m_1 + m_2 - f(\psi(m_1) + \psi(m_2)) \\
&= (m_1 - f(\psi(m_1))) + (m_2 - f(\psi(m_2))) \\
&= \varphi(x_1'') + \varphi(x_2'')
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\varphi(rx'') &= rm - f(\psi(rm)) \\
&= r(m - f(\psi(m))) = r\varphi(x''),
\end{aligned}
$$

so φ preservers both addition and scalar multiplication and is thus a module homomorphism. Finally, we check that $g \circ \varphi = id$. Then,

$$
\begin{aligned}
g(\varphi(x'')) &= g(m - f(\psi(m))) \\
&= g(m) - g(f(\psi(m))) \\
&= x'' - 0 = x''
\end{aligned}
$$

Hence, $g \circ \varphi = id$, and the sequence splits. Thus, equivalence of (1) and (2) in Proposition 3.2 is proved.

We now prove that a sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ splits if and only if there exists a submodule N of M such that $M = \operatorname{Im} f \oplus N$, and in that case, then N is isomorphic to M''.

($\Rightarrow$) Suppose $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ splits. Thus, let $h : M'' \to M$ be such that $g \circ h = \mathrm{id}.$. Let $N = \operatorname{Im} h \subseteq M$. To show that $\operatorname{Im} f \oplus N = M$, we first show that $N \cap \operatorname{Im} f = \{0\}$. Suppose

$$
\begin{aligned}
&x \in \operatorname{Im} f \cap N \\
\Rightarrow x = h(y) \quad &\text{and} \quad x \in \operatorname{Im} f = \ker g
\end{aligned}
$$

Applying g, we get $0 = g(x) = g(h(y)) = y$. Thus, since h is injective, by assumption of splitting and exactness, and $y = 0$, we have that $x = h(0) \Rightarrow x = 0$. Therefore, $N \cap \operatorname{Im} f = \{0\}$. Now we want to show for all $m \in M$, $m = a + b$ with $a \in \operatorname{Im} f$, $b \in N$. Consider

$$
m = \underbrace{m - h(g(m))}_{\substack{\in \operatorname{Im} f = \ker g \\ \because g(m - h(g(m))) = 0}} + \underbrace{h(g(m))}_{\in \operatorname{Im} h = N} \quad \text{(adding 0 to } m\text{)}
$$

Thus, $M = \operatorname{Im} f \oplus N$. Define

$$
g\Big|_N : N \to M''
$$

to be the restriction of g to N. (Surjective) Let $m'' \in M''$. Then $h(m'') \in N$. Applying the restriction of g,

$$
g\Big|_N (h(m'')) = m'', \quad (\because\ g \circ h = \mathrm{id.})
$$

(Injective) Suppose $g\Big|_N(x) = 0$ for some $x \in N = \operatorname{Im} h$. Thus, $x \in N$ and $x \in \ker g \Rightarrow x \in N \cap \ker g = N \cap \operatorname{Im} f = \{0\}$. Hence, $x = 0$ and thus $g\Big|_N$ is injective. Therefore, $N \cong M''$ with inverse map (isomorphism) h.

($\Leftarrow$) Assume there exists some $N \subseteq M$ such that $M = \operatorname{Im} f \oplus N$. Define

$$g\Big|_N : N \to M''$$

be the restriction of g to N. (Injective) Suppose $g\Big|_N(n) = 0$ for some $n \in N$. Then, $n \in \underbrace{N \cap \ker g = N \cap \operatorname{Im} f}_{\substack{\text{by defn of} \\ \operatorname{Im} f \oplus N}} = \{0\} \Rightarrow n = 0$. Therefore, $g\Big|_N$ is injective. (Surjective) Let $m'' \in M''$. Since g is surjective, there exists an element $m \in M$ such that $g(m) = m''$. Write $m = a + n$ with $a \in \operatorname{Im} f$ and $n \in N$. Then

$$g(n) = g(m - a) = g(m) - g(a) = m'' - 0 = m'', \quad \text{since } a \in \operatorname{Im} f = \ker g$$

so $g\Big|_N$ is surjective. Therefore, $g\Big|_N : N \to M''$ is an isomorphism, so $N \cong M''$. Now we prove that the sequence splits. Define

$$h : M'' \to M$$

by $h = \iota \circ \left(g\Big|_N\right)^{-1}$, where $\iota : N \hookrightarrow M$ is the inclusion and $\left(g\Big|_N\right)^{-1}$ exists since $g\Big|_N$ is an isomorphism. For any $m'' \in M''$

$$\begin{aligned} g(h(m'')) &= g\left(\left(\iota \circ \left(g\Big|_N\right)^{-1}(m'')\right)\right) \\ &= g\Big|_N\left(\left(g\Big|_N\right)^{-1}(m'')\right) = m'' \quad \left(\because g \circ \iota = g\Big|_N : N \to M''\right) \end{aligned}$$

Thus, $g \circ h = \text{id.}$, so the sequence splits.

b) Define $f \colon E \to \prod E_i$ by $f(x) = (\psi_1(x), \dots, \psi_m(x))$ and $g \colon \prod E_i \to E$ by $g((x_1, \dots, x_m)) = \sum_{i=1}^{m} \varphi_i(x_i)$. We show that f, g are isomorphisms by showing that $f \circ g = \text{id.}$ and $g \circ f = \text{id.}$. Let $(x_1, \dots, x_m) \in \prod E_i$. Then

$$g((x_1, \dots, x_m)) = \sum_{i=1}^{m} \varphi_i(x_i)$$

Applying f, we obtain

$$\begin{aligned} f\left(\sum_{i=1}^{m} \varphi_i(x_i)\right) &= \left(\psi_1\left(\sum_{i=1}^{m} \varphi_i(x_i)\right), \dots, \psi_m\left(\sum_{i=1}^{m} \varphi_i(x_i)\right)\right) \\ &= \left(\sum_{i=1}^{m} \psi_1(\varphi_i(x_i)), \dots, \sum_{i=1}^{m} \psi_m(\varphi_i(x_i))\right) \quad \text{(by the fact that } \psi_i \text{ are homomorphisms)} \end{aligned}$$

Thus, by th assumed properties of ψ_i, φ_i, namely $\psi_i \circ \varphi_i = id$ if $i = j$ and $\psi_i \circ \varphi_j = 0$ if $i \neq j$, we obtain

$$(f \circ g)(x_1, \dots, x_m) = (x_1, \dots, x_m)$$

and thus $f \circ g = \text{id}.$.

Now let $x \in E$. Then

$$
\begin{aligned}
(g \circ f)(x) &= g((\psi_1(x), \ldots, \psi_m(x))) \\
&= \sum_{i=1}^{m} \varphi_i(\psi_i(x)).
\end{aligned}
$$

By the assumed property that

$$\sum_{i=1}^{m} \varphi_i \circ \psi_i = \text{id}.,$$

it immediately follows that $g \circ f = \text{id}.$. Thus, $E \cong \prod E_i$. Hence, f, g are isomorphisms.

For the converse, assume that $E = \prod E_i$ with E_i submodules. Let $\varphi_i \colon E_i \hookrightarrow E$ by $\varphi_i(x) = (0, \ldots, 0, x, 0, \ldots, 0)$ with x in the i-th component and let $\psi_i \colon E \to E_i$ by $\psi_i((x_1, \ldots, x_m)) = x_i$. Suppose $i \neq j$. Consider $\psi_i \circ \varphi_j$. Then if $x \in E_j$,

$$
\begin{aligned}
(\psi_i \circ \varphi_j)(x) &= \psi_i((0, \ldots, 0, x, 0, \ldots, 0)) \\
&= 0
\end{aligned}
$$

Hence, $\psi \circ \varphi_j = 0$ if $i \neq j$. If $i = j$, we clearly have $\psi_i \circ \varphi_j = \text{id}.$. Finally, consider

$$\sum_{i=1}^{m} \varphi_i \circ \psi_i = \varphi_1 \circ \psi_1 + \cdots + \varphi_m \circ \psi_m$$

Let $x = (x_1, \ldots, x_m) \in E$. Then,

$$
\begin{aligned}
\left(\sum_{i=1}^{m} \varphi_i \circ \psi_i \right)(x) &= \sum_{i=1}^{m} \varphi_i(x_i) \\
&= (x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, x_m) \\
&= (x_1, \ldots, x_m) = x
\end{aligned}
$$

Therefore, $\sum_{i=1}^{m} = \text{id}.$. Hence, all properties are satisfied in this case.

□

Exercise 0.4. (Exercise 9)

a) Let A be a commutative ring and let M be an A-module. Let S be a multiplicative subset of A. Define $S^{-1}M$ in a manner analogous to the one we used to define $S^{-1}A$, show that $S^{-1}M$ is an $S^{-1}A$-module.

b) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, show that the sequence $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is exact.

Proof. a) We define $S^{-1}M = \{m/s : m \in M, s \in S\}$ with addition as $m_1/s_1 + m_2/s_2 = m_1 s_2 + m_2 s_1 / s_1 s_2$ and since $S \subseteq A$ is multiplicative and A is commutative, $m_i s = s m_i = (\underbrace{1 + 1 + \cdots + 1}_{s-many}) m_i$ and thus $s m_i \in M$. Hence, $m_1 s_2 + m_2 s_1 \in M$ and $s_1 s_2 \in S$ which implies that $S^{-1}M$ is closed. The identity element in $S^{-1}M$ is (e_M, s) for any $s \in S$ with e_M the identity in the A-module M. Each element $m/s \in S^{-1}M$ has inverse $-m/s$ since $-m \in M$ is the inverse of $m \in M$. Finally, take $m_1/s_1, m_2/s_2, m_3/s_3 \in S^{-1}M$. Then,

$$
\begin{aligned}
&(m_1/s_1) + [(m_2/s_2) + (m_3/s_3)] \\
&= (m_1/s_1) + (m_2s_3 + m_3s_2/s_2s_3) \\
&= (m_1s_2s_3 + s_1(m_2s_3 + m_3s_2)/s_1s_2s_3) \\
&= (m_1s_2s_3 + m_2s_1s_3 + m_3s_1s_2/s_1s_2s_3) \\
&= (m_1s_2 + m_2s_1/s_1s_2) + (m_3/s_3) \\
&= [(m_1/s_1) + (m_2/s_2)] + (m_3/s_3)
\end{aligned}
$$

Therefore, $S^{-1}M$ is abelian group with most of the group properties following from the structure of the abelian group M.

Now, let $a/s, b/s' \in S^{-1}A$ and $(m_1/s_1)(m_2/s_2) \in S^{-1}M$. Then,

$$
\begin{aligned}
[(a/s) + (b/s')]\,(m_1/s_1) &= (as' + bs/ss')(m_1/s_1) \\
&= ((as' + bs)m_1/ss's_1) \\
&= (as'm_1 + bsm_1/ss's_1) \\
&= (am_1/ss_1) + (bm_1/s's_1) \\
&= (a/s)(m_1/s_1) + (b/s')(m_1/s_1)
\end{aligned}
$$

And,

$$
\begin{aligned}
(a/s)\,[(m_1/s_1) + (m_2/s_2)] &= (a/s)(m_1s_2 + m_2s_1/s_1s_2) \\
&= (a(m_1s_2 + m_2s_1)/ss_1s_2) \\
&= (am_1s_2 + am_2s_1/ss_1s_2) \\
&= (am_1/ss_1) + (am_2/ss_2) \\
&= (a/s)(m_1/s_1) + (a/s)(m_2/s_2)
\end{aligned}
$$

Therefore, $S^{-1}M$ is an $S^{-1}A$-module.

b) Suppose $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ is exact with α, β module homomorphisms. Hence, α is injective, $\operatorname{Im}\alpha = \ker\beta$, and β is surjective. Next, consider the localized sequence, $0 \to S^{-1}M' \xrightarrow{\delta} S^{-1}M \xrightarrow{\gamma} S^{-1}M'' \to 0$ where $\delta(m'/s) = \alpha(m')/s$ and $\gamma(m/s) = \beta(m)/s$. Thus, we must show three things: δ injective, $\operatorname{Im}\delta = \ker\gamma$, and γ surjective.

(δ injective) Suppose that $m'/s \in \ker\delta$. Hence, $\delta(m'/s) = \alpha(m')/s = 0/1$. Then, by localization, there exists some $t \in S$, such that $t\alpha(m') = 0$. Since, α is a module-homomorphism, we have $\alpha(tm') = 0 \Rightarrow tm' \in \ker\alpha = \{0\}$. Therefore,

$$m'/s = tm'/ts = 0/ts = 0 \in S^{-1}M'$$

Since we assumed that m'/s was an arbitrary element of the kernel of δ, we have shown that the kernel only contains 0 and is thus trivial proving that δ is injective.

($\operatorname{Im}\delta = \ker\gamma$) [$\operatorname{Im}\delta \subseteq \ker\gamma$] Let $m/s \in \operatorname{Im}\delta$. Hence, $m/s = \delta(m'/s)$. Applying γ, we obtain

$$\gamma(\delta(m'/s)) = \gamma(\alpha(m')/s) = \beta(\alpha(m'))/s$$

Since, $\operatorname{Im}\alpha = \ker\beta$, $\beta(\alpha(m')) = 0$. Hence, $\gamma(\delta(m'/s)) = 0/s$. Thus,

$$\delta(m'/s) \in \ker\gamma \Rightarrow \operatorname{Im}\delta \subseteq \ker\gamma.$$

[$\operatorname{Im}\delta \supseteq \ker\gamma$] Let $m/s \in S^{-1}M$ such that $\gamma(m/s) = 0 \in S^{-1}M''$, i.e. $m/s \in \ker\gamma$. Thus, $\gamma(m/s) = \beta(m)/s = 0/1$. By localization, there exists some $t \in S$, such that $t\beta(m) = 0$ and since

β is a module homomorphism, we get $\beta(tm) = 0$. Thus, $tm \in \ker\beta = \operatorname{Im}\alpha$. Hence, there exists some $m' \in M'$, such that $\alpha(m') = tm$. Thus,

$$m/s = tm/ts = \alpha(m')/ts = \delta(m'/ts)$$

Hence, $m/s \in \operatorname{Im}\delta \Rightarrow \ker\gamma \subseteq \operatorname{Im}\delta$.

Therefore, $\operatorname{Im}\delta = \ker\gamma$.

(γ surjective) Let $m''/s \in S^{-1}M''$. Since, β is srujective, choose some $m \in M$ such that $\beta(m) = m''$. Thus, $\gamma(m/s) = \beta(m)/s = m''/s$. Therefore, γ is surjective.

This proves that $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is exact.

□

Exercise 0.5. (Exercise 10)

a) If $\mathfrak{p}$ is a prime ideal, and $S = A - \mathfrak{p}$ is the complement of $\mathfrak{p}$ in the ring A, then $S^{-1}M$ is denoted by $M_\mathfrak{p}$. Show that the natural map
$$M \to \prod M_\mathfrak{p}$$
of a module M into the direct product of all localizations $M_\mathfrak{p}$ where $\mathfrak{p}$ ranges over all maximal ideals, is injective.

b) Show that a sequence $0 \to M' \to M \to M'' \to 0$ is exact if and only if the sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$ is exact for all primes $\mathfrak{p}$.

c) Let A be an entire ring and let M be a torsion-free module. For each prime $\mathfrak{p}$ of A show that the natural map $M \to M_\mathfrak{p}$ is injective. In particular $A \to A_\mathfrak{p}$ is injective, but you can see that directly from the imbedding of A in its quotient field K.

Proof. a) Let $\varphi : M \to \prod M_\mathfrak{p}$. Assume $m \in \ker\varphi$. Then for every maximal $\mathfrak{p}$, the image of m in $M_\mathfrak{p} = S^{-1}M$ ($S = A \setminus \mathfrak{p}$) is $0/1$. Thus, if $m/1 = 0/1$, by localization criterion, for each maximal $\mathfrak{p}$ there exists $t \notin \mathfrak{p}$ such that $tm = 0$. Set $I = \operatorname{Ann}(m) = \{a \in A : \ am = 0\}$. By this definition, for every maximal $\mathfrak{p}$, we have $I \cap (A \setminus \mathfrak{p}) \neq \varnothing$, hence I is not contained in any maximal ideal. But every proper ideal of A is contained in some maximal ideal; hence I cannot be proper. Therefore, $I = A \Rightarrow 1 \in I$, i.e. $1 \cdot m = m = 0$. Thus, $\ker\varphi = \{0\}$ (trivial), therefore, φ is injective.

b) ($\Rightarrow$) By **Exercise 9(b)**, exactness is preserved under localization at any multiplicative set. In particular, taking $S = A \setminus \mathfrak{p}$ for an arbitrary maximal (prime) $\mathfrak{p}$ gives exactness of the localized sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$.

($\Leftarrow$) Suppose $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$ is exact. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, "\pi_1"] & M \arrow[r, "\beta"] \arrow[d, "\pi_2"] & M'' \arrow[r] \arrow[d, "\pi_3"] & 0 \\
0 \arrow[r] & M'_\mathfrak{p} \arrow[r, "\delta"] & M_\mathfrak{p} \arrow[r, "\gamma"] & M''_\mathfrak{p} \arrow[r] & 0
\end{tikzcd}$$

with π_i being projections of to each localization. For example, $\pi_2(m) = m/1$ and likewise for π_1, π_3. By exactness of the localized sequence, we know δ is injective, $\operatorname{Im}\delta = \ker\gamma$, and γ is surjective. Thus, we need to show three things: α is injective, $\operatorname{Im}\alpha = \ker\beta$, and β surjective.

(α injective) Let $m' \in \ker\alpha$. Suppose $m' \neq 0$. Then, there exists a prime ideal, $\mathfrak{p} \supseteq \operatorname{Ann}(m') = \{a \in A : \ am' = 0\}$. Take $\pi_1(m') = m'/1 \in M'_\mathfrak{p}$. By construction, $m'/1 \neq 0 \in M'_\mathfrak{p}$ (Since if $m'/1 = 0/1 \Rightarrow \exists\, t \notin \mathfrak{p}$ such that $tm' = 0$ which is impossible since such t would be in $\operatorname{Ann}(m') \subseteq p$ which means $t \notin A \setminus \mathfrak{p}$). But $\delta(m'/1) = \alpha(m')/1 = 0/1 = 0 \in M_\mathfrak{p}$ which is a contradiction since $\ker\delta = \{0\}$ by assumption of exactness. Hence, $m' = 0$. Thus, $\ker\alpha = \{0\}$ (trivial), therefore α is injective.

($\operatorname{Im}\alpha = \ker\beta$) [$\ker\beta \subseteq \operatorname{Im}\alpha$] Let $m \in \ker\beta$ (i.e. $\beta(m) = 0$). Then, $\gamma(\pi_2(m)) = \gamma(m/1) = \beta(m)/1 = 0/1 \in M''_\mathfrak{p}$. Thus, $\pi_2(m) \in \ker\gamma = \operatorname{Im}\delta$. Therefore, there exists some $m'/1 \in M'_\mathfrak{p}$ such that $\pi_2(m) = \delta(m'/1) = \alpha(m')/1 = m/1 \in M_\mathfrak{p}$. By localization,

$m/1 = \alpha(m')/1$ means there exists $t \notin \mathfrak{p}$ such that $t(m - \alpha(m')) = 0 \Rightarrow tm = \alpha(tm') \in \operatorname{Im}\alpha$. Define

$$I = \{a \in A : \; am \in \operatorname{Im}\alpha\}.$$

For every prime $\mathfrak{p}$, we have $t \in I$ with $t \notin \mathfrak{p}$. Suppose $I \neq A$. Then there exists a maximal ideal, thus prime, $\mathfrak{q}$ of A such that $I \subseteq \mathfrak{q}$. By definition of I, there exists $t \in I$ such that $t \notin \mathfrak{q}$ which is clearly a contradiction of $I \subseteq \mathfrak{q}$. Therefore, $I = A \Rightarrow 1 \in I$, i.e. $1 \cdot m = m \in \operatorname{Im}\alpha$. Thus, $\ker\beta \subseteq \operatorname{Im}\alpha$.

[$\ker\beta \supseteq \operatorname{Im}\alpha$] Let $m = \alpha(m') \in \operatorname{Im}\alpha$. Consider $\pi_1(m') = m'/1 \in M'_{\mathfrak{p}}$ and $\pi_2(\alpha(m')) = \alpha(m')/1 \in M_{\mathfrak{p}}$. Thus, we have an equality of $\delta(\pi_1(m')) = \alpha(m')/1 = \pi_2(\alpha(m')) \in M_{\mathfrak{p}} \supseteq \ker\gamma = \operatorname{Im}\delta$. Thus, since the localization is exact, if we apply γ to this equality, we get $\pi_2(\alpha(m')) \in \ker\gamma$ because $\gamma \circ \delta = 0$. By definition, $\gamma(\pi_2(\alpha(m'))) = \gamma(\alpha(m')/1) = \beta(\alpha(m'))/1 \in M''_{\mathfrak{p}}$. Thus, $\beta(\alpha(m')) = 0/1$. By localization, there exists some $t \notin \mathfrak{p}$ such that $t\beta(\alpha(m')) = 0$. Set

$$J = \{a \in A : \; a\beta(\alpha(m')) = 0\}.$$

Thus, $t \in J$ for every prime ideal $\mathfrak{p}$ but, $t \notin \mathfrak{p}$. Suppose $J \neq A$, and thus there exists a maximal ideal (prime) $\mathfrak{q}$ containing J. But $t \in J$ and $t \notin \mathfrak{q}$ (since in the localization relation, t is taken to not be in the complement of the ideal in A, just like we did in the other inclusion). Therefore, $J = A \Rightarrow 1 \in J$, i.e. $1 \cdot \beta(\alpha(m')) = \beta(\alpha(m')) = 0$. Thus, $\alpha(m') \in \ker\beta$. Therefore, $\operatorname{Im}\alpha \subseteq \ker\beta$.

Thus, $\operatorname{Im}\alpha = \ker\beta$.

(β surjective) Let $m'' \in M''$. Consider $\pi_3(m'') = m''/1 \in M''_{\mathfrak{p}}$. By exactness of the localization, we know that γ i surjective, therefore, we can choose $m/1 \in M_{\mathfrak{p}}$ such that $\gamma(m/1) = m''/1$, i.e. $m/1 = \pi_2(m)$. Thus, $\gamma(m/1) = \beta(m)/1 = m''/1 \in M''_{\mathfrak{p}}$. Then, by localization, there exists $t \notin \mathfrak{p}$ such that $t(\beta(m) - m'') = 0$. Set

$$K = \{a \in A : \; a(\beta(m) - m'') = 0\}.$$

Hence, for every prime ideal $\mathfrak{p}$, there exists a $t \in K$ with $t \notin \mathfrak{p}$. Suppose $K \neq A$ which implies there exists a maximal ideal (prime) $\mathfrak{q}$ containing K. But again, $t \in K$, but $t \notin \mathfrak{q}$ (by defn of equivalence in the localization like both previous inclusions). Hence, $K = A \Rightarrow 1 \in K$, i.e. $1 \cdot (\beta(m) - m'') = 0 \Rightarrow \beta(m) - m'' = 0$ and thus, $\beta(m) = m''$. Whence, β is surjective.

Therefore, $0 \to M' \to M \to M'' \to 0$ is exact.

c) Define $\psi : \; M \to M_{\mathfrak{p}}$ by $m \mapsto m/1$. Suppose $m \in \ker\psi \Rightarrow \psi(m) = m/1 = 0/1 \in M_{\mathfrak{p}}$. By localization, there exists $t \notin \mathfrak{p}$ such that $tm = 0$. But since M is torsion-free, i.e. the only $m \in M$ for which we can find an $a \in A$ such that $am = 0$ is $m = 0$ ($M_{\text{tor}} = 0$), and $t \neq 0 \in A \setminus \mathfrak{p}$, we must have $m = 0$. Thus, $\ker\psi = \{0\}$, hence ψ is injective.

□

Exercise 0.6. (Exercise 14)
Consider a commutative diagram of R-modules and homomorphisms such that each row is exact:

$$\begin{tikzcd} & M' \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' & \end{tikzcd}$$

Prove:

a) If f, h are monomorphisms, then g is a monomorphism.

b) If f, h are surjective, then g is surjective.

c) Assume in addition that $0 \to M' \to M$ is exact and $N \to N'' \to 0$ is exact. Prove that if any two of f, g, h are isomorphisms, then so is the third. [*Hint*: Use the snake lemma.]

Proof. a) We rewrite the diagram with additional labeling:

$$\begin{tikzcd} & M' \arrow[r, "\alpha"] \arrow[d, "f"] & M \arrow[r, "\beta"] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "\gamma"] & N \arrow[r, "\delta"] & N'' & \end{tikzcd}$$

with $\operatorname{Im}\alpha = \operatorname{Ker}\beta$ and $\operatorname{Im}\gamma = \operatorname{Ker}\delta$, since the rows are exact. We will show that g is a monomorphism by proving the kernel of g is trivial, i.e. $\operatorname{Ker} g = \{0\}$. Thus, let $m \in \operatorname{Ker} g \Rightarrow g(m) = 0$. From the commutativity of

$$\begin{tikzcd} M \arrow[r, "\beta"] \arrow[d, "g"] & M'' \arrow[d, "h"] \\ N \arrow[r, "\delta"] & N'' \end{tikzcd}$$

we have $\delta \circ g = h \circ \beta$. Hence,

$$\begin{aligned} h(\beta(m)) &= \delta(g(m)) \\ &= \delta(0) \\ &= 0 \end{aligned}$$

Thus, since h is injective and $h(\beta(m)) = 0$, we have $\beta(m) = 0 \Rightarrow m \in \operatorname{Ker}\beta$. By exactness, $\operatorname{Im}\alpha = \operatorname{Ker}\beta$, and thus $m = \alpha(m')$ for some $m' \in M'$. Now, by the commutativity of

$$\begin{tikzcd} M' \arrow[r, "\alpha"] \arrow[d, "f"] & M \arrow[d, "g"] \\ N' \arrow[r, "\gamma"] & N \end{tikzcd}$$

we have $g \circ \alpha = \gamma \circ f$. Thus,

$$\begin{aligned} 0 = g(m) &= g(\alpha(m)) \\ &= (g \circ \alpha)(m') \\ &= (\gamma \circ f)(m') \\ &= \gamma(f(m')) \end{aligned}$$

Hence, $\gamma(f(m')) = 0$ and since γ is injective (by the exactness of $0 \to N' \xrightarrow{\gamma} N$, i.e. the zero map with image 0 is equal to the kernel of γ, i.e. $\operatorname{Ker}\gamma = \{0\} = \operatorname{Im}(0 \to N')$), we get $f(m') = 0$ and thus $m' \in \operatorname{Ker} f$. But since f is injective, $\operatorname{Ker} f = \{0\}$ and thus $m' = 0$. Then, $m = \alpha(m') = \alpha(0)$. Because α is a module homomorphism, $\alpha(0) = 0$ and thus $m = 0$. Therefore, $\operatorname{Ker} g = \{0\}$ and hence g is injective.

b) Let $n \in N$. Then, $\delta(n) \in N''$ and since h is surjective, there exists an element $m'' \in M''$ such that $h(m'') = \delta(n)$. By the exactness of $M' \to M \to M'' \to 0$, $m'' = \beta(m)$ since β is surjective from $\operatorname{Im}\beta = \operatorname{Ker}(M'' \to 0) = M''$. Thus, $h(m'') = h(\beta(m)) = \delta(n)$. Furthermore, from the commutativity of the diagram, $h \circ \beta = \delta \circ g \Rightarrow h(\beta(m)) = \delta(g(m)) = \delta(n)$. Thus, $\delta(n) - \delta(g(m)) = \delta(n - g(m)) = 0$. Hence, $n - g(m) \in \operatorname{Ker}\delta$. Thus, by the exactness of $0 \to N' \to N \to N''$, there exists an element $n' \in N'$ such that $\gamma(n') = n - g(m)$. Since f is surjective, $n' = f(m')$ for some $m' \in M'$. Therefore, by commutativity of $g \circ \alpha = \gamma \circ f$,

$$\begin{aligned} g(\alpha(m')) &= \gamma(f(m')) \\ &= \gamma(n') \\ &= n - g(m) \end{aligned}$$

which implies that $n = g(m) + g(\alpha(m')) = g(m + \alpha(m'))$. Hence, g is surjective.

c) We rewrite the commutative diagram with the additional assumptions,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \\
 & & \Big\downarrow f & & \Big\downarrow g & & \Big\downarrow h & & \\
0 & \longrightarrow & N' & \xrightarrow{\gamma} & N & \xrightarrow{\delta} & N'' & \longrightarrow & 0
\end{array}
$$

These additional assumptions now add that α is injective and δ is surjective. We proceed through the three cases; albeit repetitive and potentially inefficient.

<u>(f, g isomorphisms)</u>: Let $m'' \in M''$ such that $h(m'') = 0$. Since β is surjective, there exists some $m \in M$ such that $\beta(m) = m''$. From the commutative subdiagram

$$
\begin{array}{ccc}
M & \xrightarrow{\beta} & M'' \\
\Big\downarrow g & & \Big\downarrow h \\
N & \xrightarrow{\delta} & N''
\end{array}
$$

we have

$$
\begin{aligned}
\delta(g(m)) &= h(\beta(m)) \\
&= h(m'') \\
&= 0
\end{aligned}
$$

Thus, $g(m) \in \ker\delta = \operatorname{Im}\gamma$. Hence, there exists some $n' \in N'$ such that $\gamma(n') = g(m)$. Since f is an isomorphism (surjection), there exists an $m' \in M'$ such that $f(m') = n'$. Then from

$$
\begin{array}{ccc}
M' & \xrightarrow{\alpha} & M \\
\Big\downarrow f & & \Big\downarrow g \\
N' & \xrightarrow{\gamma} & N
\end{array}
$$

we have

$$
\begin{aligned}
\gamma(f(m')) &= g(\alpha(m')) \\
\gamma(n') &= g(\alpha(m')) \\
g(m) &= g(\alpha(m'))
\end{aligned}
$$

Since g is an isomorphism (injective), $m = \alpha(m')$. Applying β to both sides, we obtain $\beta(m) = \beta(\alpha(m')) = 0$ since $\operatorname{Im}\alpha = \ker\beta$. Thus, from the above, $m'' = \beta(m) = 0$. Therefore, h is injective.

Now let $n'' \in N''$. Since δ is surjective, there exists some $n \in N$ such that $\delta(n) = n''$. Furthermore, by the fact that g is assumed to be an isomorphism (surjection), there exists an element $m \in M$ such that $g(m) = n$. Then from the subdiagram

$$
\begin{array}{ccc}
M & \xrightarrow{\beta} & M'' \\
\Big\downarrow g & & \Big\downarrow h \\
N & \xrightarrow{\delta} & N''
\end{array}
$$

we get

$$
\begin{aligned}
h(\beta(m)) &= \delta(g(m)) \\
&= \delta(n) \\
&= n''
\end{aligned}
$$

Since β is surjective, $\beta(M) = M''$ and thus, h is surjective. Hence, h is an isomorphism.

<u>$(g, h$ isomorphisms$)$</u> : Let $m' \in M'$ such that $f(m') = 0$. Then from

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ N' & \xrightarrow{\gamma} & N \end{array}$$

we obtain

$$\begin{aligned} g(\alpha(m')) &= \gamma(f(m')) \\ &= \gamma(0) \\ &= 0 \end{aligned}$$

Since g is an isomorphism (injective), we have $\alpha(m') = 0$ and by the injectivity of α, it follows that $m' = 0$. Hence, f is injective.

Now let $n' \in N'$. Then $\gamma(n') \in N$. Since g is an isomorphism (surjective), there exists some $m \in M$ such that $g(m) = \gamma(n')$. Applying δ, we obtain $\delta(g(m)) = \delta(\gamma(n')) = 0$. Since $\operatorname{Im}\gamma = \ker\delta$. From

$$\begin{array}{ccc} M & \xrightarrow{\beta} & M'' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} \\ N & \xrightarrow{\delta} & N'' \end{array}$$

we have

$$\begin{aligned} h(\beta(m)) &= \delta(g(m)) \\ &= 0 \end{aligned}$$

Since h is an isomorphism, it follows that $\beta(m) = 0$. By exactness of the first row, there exists an element $m' \in M'$ such that $\alpha(m') = m$. Then from the subdiagram

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ N' & \xrightarrow{\gamma} & N \end{array}$$

we get

$$\begin{aligned} \gamma(f(m')) &= g(\alpha(m')) \\ &= g(m) \\ &= \gamma(n') \end{aligned}$$

Since γ is injective, $f(m') = n'$. Hence, f is surjective and is thus an isomorphism.

<u>$(f, h$ isomorphisms$)$</u> : Let $m \in M$ such that $g(m) = 0$. From

$$\begin{array}{ccc} M & \xrightarrow{\beta} & M'' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} \\ N & \xrightarrow{\delta} & N' \end{array}$$

we obtain

$$
\begin{aligned}
h(\beta(m)) &= \delta(g(m)) \\
&= \delta(0) \\
&= 0 \quad (\because \delta \text{ is a homomorphism})
\end{aligned}
$$

By the fact that h is an isomorphism, $\beta(m) = 0$. By the exactness of the top row, there exists some $m' \in M'$ such that $\alpha(m') = m$. Then by

$$
\begin{array}{ccc}
M' & \xrightarrow{\alpha} & M \\
\downarrow f & & \downarrow g \\
N' & \xrightarrow{\gamma} & N
\end{array}
$$

we obtain

$$
\begin{aligned}
\gamma(f(m')) &= g(\alpha(m')) \\
&= g(m) \\
&= 0
\end{aligned}
$$

Therefore by the injectivity of γ and the fact that f is an isomorphism, we have $m' = 0$. Thus, by the above, $m = \alpha(m') = \alpha(0) = 0$ (since α is a homomorphism). Therefore, g is injective.

Now let $n \in N$. Thus, $\delta(n) \in N''$. Since h is an isomorphism, there exists some $m'' \in M''$ such that $h(m'') = \delta(n)$. By the surjectivity of β, there exists some $m \in M$ such that $\beta(m) = m''$. Thus, from

$$
\begin{array}{ccc}
M & \xrightarrow{\beta} & M'' \\
\downarrow g & & \downarrow h \\
N & \xrightarrow{\delta} & N''
\end{array}
$$

we get

$$
\begin{aligned}
\delta(g(m)) &= h(\beta(m)) \\
&= h(m'') \\
&= \delta(n)
\end{aligned}
$$

Thus, $\delta(n) - \delta(g(m)) = 0 \Rightarrow \delta(n - g(m)) = 0$ (since δ is a homomorphism). Furthermore, $n - g(m) \in \ker \delta = \operatorname{Im} \gamma$ which implies that there exists some $n' \in N'$ such that $\gamma(n') = n - g(m)$. Since f is an isomorphism (surjective), there exists some $m' \in M'$ such that $f(m') = n'$. Thus, from the commutative subdiagram

$$
\begin{array}{ccc}
M' & \xrightarrow{\alpha} & M \\
\downarrow f & & \downarrow g \\
N' & \xrightarrow{\gamma} & N
\end{array}
$$

we have

$$
\begin{aligned}
g(\alpha(m')) &= \gamma(f(m')) \\
&= \gamma(n') \\
&= n - g(m)
\end{aligned}
$$

Hence, $n = g(\alpha(m')) + g(m) = g(\alpha(m') + m)$. Thus, g is surjective and therefore an isomorphism. $\square$

Exercise 0.7. (Exercise 15)
The five lemma. Consider a commutative diagram of R-modules and homomorphisms such that each row is exact:

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & M_4 \arrow[r] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5
\end{tikzcd}$$

Prove:

a) If f_1 is surjective and f_2, f_4 are monomorphisms, then f_3 is a monomorphism.

b) If f_5 is a monomorphism and f_2, f_4 are surjective, then f_3 is surjective. [*Hint:* Use the snake lemma.]

Proof. a) We rewrite the diagram with the mappings labeled:

$$\begin{tikzcd}
M_1 \arrow[r, "\alpha"] \arrow[d, "f_1"] & M_2 \arrow[r, "\beta"] \arrow[d, "f_2"] & M_3 \arrow[r, "\gamma"] \arrow[d, "f_3"] & M_4 \arrow[r, "\delta"] \arrow[d, "f_5"] & M_5 \arrow[d, "f_5"] \\
N_1 \arrow[r, "\varepsilon"] & N_2 \arrow[r, "\eta"] & N_3 \arrow[r, "\varphi"] & N_4 \arrow[r, "\psi"] & N_5
\end{tikzcd}$$

We show that f_3 is a monomorphism by showing that $\operatorname{Ker} f_3 = \{0\}$. Let $m_3 \in \operatorname{Ker} f_3 \Rightarrow f_3(m_3) = 0$. Consider the commutative subdiagram

$$\begin{tikzcd}
M_3 \arrow[r, "\gamma"] \arrow[d, "f_3"] & M_4 \arrow[d, "f_4"] \\
N_3 \arrow[r, "\varphi"] & N_4
\end{tikzcd}$$

with $f_4 \circ \gamma = \varphi \circ f_3$. Thus, $f_4(\gamma(m_3)) = \varphi(f_3(m_3)) = \varphi(0) = 0$ (since φ is a module homomorphism). Hence, $f_4(\gamma(m_3)) = 0 \Rightarrow \gamma(m_3) = 0$ (since f_4 is injective). Hence, $m_3 \in \operatorname{Ker} \gamma$. By exactness, $\operatorname{Im} \beta = \operatorname{Ker} \gamma$, we have $m_3 = \beta(m_2)$ with $m_2 \in M_2$. Now consider the commutative subdiagram

$$\begin{tikzcd}
M_2 \arrow[r, "\beta"] \arrow[d, "f_2"] & M_3 \arrow[d, "f_3"] \\
N_2 \arrow[r, "\eta"] & N_3
\end{tikzcd}$$

with $\eta \circ f_2 = f_3 \circ \beta$. Then,

$$\begin{aligned}
\eta(f_2(m_2)) &= f_3(\beta(m_2)) \\
&= f_3(m_3) = 0
\end{aligned}$$

Thus, $\eta(f_2(m_2)) = 0 \Rightarrow f_2(m_2) \in \operatorname{Ker} \eta$. By the exactness, $\operatorname{Im} \varepsilon = \operatorname{Ker} \eta$. Hence, $\varepsilon(n_1) = f_2(m_2)$ with $n_1 \in N_1$. Therefore, since f_1 is surjective, $n_1 = f(m_1)$ with $m_1 \in M_1$. Then, $\varepsilon(f_1(m_1)) = f_2(m_2)$. Consider the commutative subdiagram

$$\begin{tikzcd}
M_1 \arrow[r, "\alpha"] \arrow[d, "f_1"] & M_2 \arrow[d, "f_2"] \\
N_1 \arrow[r, "\varepsilon"] & N_2
\end{tikzcd}$$

with $\varepsilon \circ f_1 = f_2 \circ \alpha$. Hence,

$$\begin{aligned}
&\varepsilon(f_1(m_1)) = f_2(\alpha(m_1)) \\
&\Rightarrow \varepsilon(n_1) = f_2(\alpha(m_1)) \\
&\Rightarrow f_2(m_2) = f_2(\alpha(m_1))
\end{aligned}$$

Since f_2 is injective, $m_2 = \alpha(m_1)$. By exactness, $\operatorname{Im} \alpha = \operatorname{Ker} \beta$. Hence, since $m_2 \in \operatorname{Im} \alpha$, $\beta(m_2) = m_3 \in \operatorname{Ker} \beta$. Thus,

$$\begin{aligned}
&\beta(m_2) = m_3 \\
&\Rightarrow \beta(\alpha(m_1)) = m_3
\end{aligned}$$

But again by $\operatorname{Im}\alpha = \operatorname{Ker}\beta$, everything in the image of α gets sent to 0 in M_3. Thus, $\beta(\alpha(m_1)) = 0 \Rightarrow m_3 = 0$. Therefore, $\operatorname{Ker} f_3 = \{0\}$ and hence f_3 is a monomorphism.

b) Let $n_3 \in N_3$. Then, $\varphi(n_3) \in N_4$. Since f_4 is surjective, there exists an element $m_4 \in M_4$ such that $f_4(m_4) = \varphi(n_3)$. Consider the commutative subdiagram

$$\begin{array}{ccc} M_4 & \xrightarrow{\delta} & M_5 \\ \big\downarrow{\scriptstyle f_4} & & \big\downarrow{\scriptstyle f_5} \\ N_4 & \xrightarrow{\psi} & N_5 \end{array}$$

with $f_5 \circ \delta = \psi \circ f_4$. Thus,

$$\begin{aligned} f_5(\delta(m_4)) &= \psi(f_4(m_4)) \\ &= \psi(\varphi(n_3)) \\ &= 0 \quad (\text{since } \operatorname{Im}\varphi = \operatorname{Im}\psi) \end{aligned}$$

Then, $f_5(\delta(m_4)) = 0$ and since f_5 is injective, $\delta(m_4) = 0 \Rightarrow m_4 \in \operatorname{Ker}\delta = \operatorname{Im}\psi$. Hence, $\psi(m_3) = m_4$ with $m_3 \in M_3$. Moving left in the diagram, consider the commutative subdiagram

$$\begin{array}{ccc} M_3 & \xrightarrow{\psi} & M_4 \\ \big\downarrow{\scriptstyle f_3} & & \big\downarrow{\scriptstyle f_4} \\ N_3 & \xrightarrow{\varphi} & N_4 \end{array}$$

with $f_4 \circ \gamma = \varphi \circ f_3$. Then,

$$\begin{aligned} & f_4(\gamma(m_3)) = \varphi(f_3(m_3)) \\ & \Rightarrow f_4(m_4) = \varphi(f_3(m_3)) \\ & \Rightarrow \varphi(n_3) = \varphi(f_3(m_3)) \\ & \Rightarrow \varphi(n_3 - f_3(m_3)) = 0 \\ & \Rightarrow n_3 - f_3(m_3) \in \operatorname{Ker}\varphi = \operatorname{Im}\eta \\ & \Rightarrow \eta(n_2) = n_3 - f_3(m_3) \end{aligned}$$

Since f_2 is surjective, there exists an element $m_2 \in M_2$ such that $f_2(m_2) = n_2$. Finally, consider the commutative subdiagram

$$\begin{array}{ccc} M_2 & \xrightarrow{\beta} & M_3 \\ \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f_3} \\ N_2 & \xrightarrow{\eta} & N_3 \end{array}$$

with $f_3 \circ \beta = \eta \circ f_2$. Then,

$$\begin{aligned} f_3(\beta(m_2)) &= \eta(f_2(m_2)) \\ &= \eta(n_2) \\ &= n_3 - f_3(m_3) \\ &\Rightarrow n_3 = f_3(m_3 + \beta(m_2)) \end{aligned}$$

Therefore, f_3 is surjective.

$\square$