Bernoulli Numbers and Polynomials

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1 Brief History

The Bernoulli numbers are terms of a sequence of rational numbers discovered independently by Jakob Bernoulli and Seki Takakazu. These numbers were discovered in an effort to define a general formula for the sum of integer powers,

$$\sum_{k=1}^{n} k^m$$

for integers $m \geq 1$. We do not discuss here the procedure in which Bernoulli and Takakazu each uncovered these numbers; the curious reader can seek out derivations that were followed by both renown mathematicians.

2 Applications

Here are just a few of the applications of Bernoulli numbers:

- Riemann-Zeta function (Number Theory)
- Kervaire-Milnor formula (Topology and Geometry)
- Quantum Harmonic Oscillator (Quantum Physics)
- Modular Forms and Elliptic Curves
- p-adic Analysis and p-adic Zeta functions (Number Theory)
- Algebraic K-theory and Motivic Cohomology (Algebraic geometry and topology)

$$\vdots$$

3 Definition/Construction of Bernoulli Numbers

One of the ways that Bernoulli numbers are defined is the following

$$F(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

where the Bernoulli numbers are the coefficients of the power series, B_n. Hence, we derived a formula for these numbers.
We start with the exponential power series and turn it into a power series for $\frac{t}{e^t-1}$. Thus,

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots$$

Then,

$$\begin{aligned} e^t - 1 &= t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots \\ &= \sum_{n=1}^{\infty} b_n t^n \end{aligned}$$

with $b_1 = 1$, $b_2 = \frac{1}{2}$, $b_3 = \frac{1}{6}, \ldots, b_k = \frac{1}{k!}, \ldots$. We want the following power series,

$$\frac{t}{e^t - 1} = t \cdot \left(\frac{1}{\sum\limits_{n=1}^{\infty} b_n t^n} \right)$$

Observe in $e^t - 1 = t\left(1 + \frac{t}{2} + \frac{t^2}{6} + \cdots\right)$ that $1 + \frac{t}{2} + \frac{t^2}{6} + \cdots \to 1$ as $t \to 0$. Thus, near 0, $e^t - 1 \sim t$. Then

$$\frac{1}{e^t - 1} \sim \frac{1}{t}$$

Therefore we need a Laurent series since $\frac{t}{e^t-1}$ is not analytic at 0. The coefficient of t^{-1} is called the **residue** or **pole's leading coefficient**. In this case, the residue must be 1 due to the asymptotic behavior $\frac{1}{e^t-1} \sim \frac{1}{t}$. Let

$$\frac{1}{e^t - 1} = \sum_{n=-1}^{\infty} c_n t^n$$

since we expect a $\frac{1}{t}$ pole. The first term is $c_{-1}t^{-1}$ since the singularity is exactly a first-order pole, i.e. not $1/t^2$, $1/t^3$, etc., we only need one negative power term. Now consider

$$1 = (e^t - 1)\left(\frac{1}{e^t - 1}\right) = \left(\sum_{k=1}^{\infty} b_k t^k\right)\left(\sum_{n=-1}^{\infty} c_n t^n\right)$$

By multiplying these power series (series convolution), the coefficient of t^m in the product is

$$\sum_{k+n=m} b_k c_n$$

Since $\left(\sum\limits_{k=1}^{\infty} b_k t^k\right)\left(\sum\limits_{n=-1}^{\infty} c_n t^n\right) = 1$, the coefficient of t^0 must be 1, and the coefficient of every t^m for $m \geq 1$ must be 0. We know what the coefficients of b_j are thus we wish to solve for the c_i. We solve a few of these below and note that $k \geq 1$ and $n \geq -1$ by their respective series definitions:

<u>$(m = 0)$ constant term; coeff. of t^0</u>:

$$\sum_{k+n=m} b_k c_n = 1$$

Since $b_1 = 1 \Rightarrow c_{-1} = 1$.

<u>$(m = 1)$ coeff. of t^1</u>:

$$\sum_{k+n=m} b_k c_n = \sum_{k+n=1} b_k c_n = b_1 c_0 + b_2 c_{-1} = 0$$

Thus, since $b_1 = 1$, $b_2 = \frac{1}{2}$, $c_{-1} = 1$, we have $c_0 = -\frac{1}{2}$.

<u>$(m = 2)$ coeff. of t^2</u>:

$$\sum_{k+n=m} b_k c_n = \sum_{k+n=2} b_k c_n = b_1 c_1 + b_2 c_0 + b_3 c_{-1} = 0$$

And thus, since $b_1 = 1$, $b_2 = \frac{1}{2}$, $b_3 = \frac{1}{6}$, $c_0 = -\frac{1}{2}$, $c_{-1} = 1$, we have $c_1 = \frac{1}{12}$.

Now we return back to the Laurent expansion of $\frac{1}{e^t-1}$ about $t = 0$,

$$\sum_{n=-1}^{\infty} c_n t^n$$

and multiply by t to obtain the Taylor expansion for $\frac{t}{e^t-1}$:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} d_n t^n = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

Therefore, $d_n = c_{n-1}$. Hence,

$$\begin{aligned}\frac{t}{e^t-1} &= c_{-1} + c_0 t + c_1 t^2 + \cdots \\ &= 1 - \frac{t}{2} + \frac{t^2}{12} + \cdots\end{aligned}$$

And by the series equivalence, we have $d_n = \frac{B_n}{n!} \Rightarrow B_n = d_n n! = c_{n-1} n!$. Thus, for example, $B_0 = c_{-1} = 1$, $B_1 = c_0 = -\frac{1}{2}$, and $B_2 = 2c_1 = \frac{1}{6}$.
Therefore, we have derived an explicit way to compute Bernoulli numbers through a construction starting from the exponential series.

4 Definition/Construction of Bernoulli Polynomials

Define the Bernoulli polynomials, $B_k(X)$, by the power series expansion

$$F_f(t, X) = \frac{te^{tX}}{e^t-1} = \sum_{k=0}^{\infty} B_k(X) \frac{t^k}{k!}$$

The explicit formula for $B_n(X)$ is

$$B_n(X) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} X^k \tag{1}$$

We derive this explicit formula below via the Taylor expansion, $\frac{t}{e^t-1}$, that we calculated in the previous section:

$$e^{tX} = \sum_{m=0}^{\infty} \frac{(tX)^m}{m!}$$

Thus,

$$\begin{aligned}\frac{te^{tX}}{e^t-1} = \left(\frac{t}{e^t-1}\right)\left(e^{tX}\right) &= \left(\sum_{k=0}^{\infty} d_k t^k\right)\left(\sum_{m=0}^{\infty} \frac{(tX)^m}{m!}\right) \\ &= \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} \frac{(tX)^m}{m!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k+m=n} B_k \frac{X^m}{k!m!}\right) t^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k \frac{X^{n-k}}{k!(n-k)!}\right) t^n \quad \because m = n-k\end{aligned}$$

Our goal is to get something of the form $\sum(\text{something})\frac{t^n}{n!}$. Hence, multiply $\frac{1}{k!(n-k)!}$ by $\frac{n!}{n!} \Rightarrow \frac{1}{n!} \cdot \binom{n}{k}$.
Thus we have

$$\sum_{k=0}^{n} B_k \frac{X^{n-k}}{k!(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} B_k X^{n-k}$$

Substituting this back into the series,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k \frac{X^{n-k}}{k!(n-k)!} \right) t^n \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} B_k X^{n-k} \right) t^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} B_k X^{n-k} \right) \frac{t^n}{n!}
\end{aligned}$$

Thus,

$$\frac{te^{tX}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(X) t^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} B_k X^{n-k} \right) \frac{t^n}{n!}$$

Therefore, by series equivalence,

$$B_n(X) = \sum_{k=0}^{n} \binom{n}{k} B_k X^{n-k} \tag{2}$$

as was to be shown (Note: Equations (1) and (2) differ only by a re-indexing of the summation; their equivalence follows by a imple change of variables). Below we calculate a few examples that demonstrate that the constant term of the *nth* Bernoulli polynomial is the *nth* Bernoulli number:

- $B_0(X) = \binom{0}{0} B_0 x^0 = B_0 = 1$
- $B_1(X) = \binom{1}{0} B_0 X^1 + \binom{1}{1} B_1 X^0 = B_0 X + B_1 = X - \frac{1}{2}$
- $B_2(X) = \binom{2}{0} B_0 X^2 + \binom{2}{1} B_1 X^1 + \binom{2}{2} B_2 X^0 = B_0 X^2 + 2B_1 X + B_2 = X^2 - X + \frac{1}{6}$