

# Exercises from Ch. 3: Modules, Lang

Matthew Gergley

Last Updated: November 24, 2025

## Exercises

### Exercise 0.1. (Exercise 4) Direct Sums

a) Prove in detail that the conditions given in Proposition 3.2 for a sequence to split are equivalent.

Show that a sequence  $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$  splits if and only if there exists a submodule  $N$  of  $M$  such that  $M$  is equal to the direct sum  $\text{Im } f \oplus N$ , and that if this is the case, then  $N$  is isomorphic to  $M''$ . Complete all the details of the proof of Proposition 3.2.

b) Let  $E$  and  $E_i$  ( $i = 1, \dots, m$ ) be modules over a ring. Let  $\varphi_i : E_i \rightarrow E$  and  $\psi_i : E \rightarrow E_i$  be homomorphisms having the following properties:

$$\begin{aligned}\psi_i \circ \varphi_i &= \text{id.}, \quad \psi_i \circ \varphi_j = 0 \quad \text{if } i \neq j \\ \sum_{i=1}^m \varphi_i \circ \psi_i &= \text{id.}\end{aligned}$$

Show that the map  $x \mapsto (\psi_1 x, \dots, \psi_m x)$  is an isomorphism of  $E$  onto the direct product of the  $E_i$  ( $i = 1, \dots, m$ ), and that the map

$$(x_1, \dots, x_m) \mapsto \varphi_1 x_1 + \dots + \varphi_m x_m$$

is an isomorphism of this direct product onto  $E$ .

Conversely, if  $E$  is equal to a direct product (or direct sum) of submodules  $E_i$  ( $i = 1, \dots, m$ ), if we let  $\varphi_i$  be the inclusion of  $E_i$  in  $E$ , and  $\psi_i$  the projection of  $E$  on  $E_i$ , then these maps satisfy the above-mentioned properties.

*Proof.* a) We state **Proposition 3.2**: Let  $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$  be an exact sequence of modules. The following conditions are equivalent:

- 1 There exists a homomorphism  $\varphi : M'' \rightarrow M$  such that  $g \circ \varphi = \text{id.}$
- 2 There exists a homomorphism  $\psi : M \rightarrow M'$  such that  $\psi \circ f = \text{id.}$

If these conditions are satisfied, then we have isomorphisms:

$$\begin{aligned}M &= \text{Im } f \oplus \ker \psi, \quad M = \ker g \oplus \text{Im } \varphi, \\ M &\cong M' \oplus M''\end{aligned}$$

We now proceed with the proof.

[Equivalence of 1 and 2]

( $1 \Rightarrow 2$ ) Suppose there exists  $\varphi : M'' \rightarrow M$  such that  $g \circ \varphi = \text{id.}$  Let  $x \in M \Rightarrow g(x) \in M''$ . Consider the difference

$$x - \varphi(g(x)) \in M.$$

Applying  $g$  yields,

$$\begin{aligned} g(x - \varphi(g(x))) &= g(x) - g(\varphi(g(x))) \\ &= g(x) - g(x) = 0 \end{aligned}$$

Therefore,  $x - \varphi(g(x)) \in \ker g = \text{Im } f$ . Thus,  $f(z) = x - \varphi(g(x))$  with  $z \in M''$ . Since  $f$  is injective, there is only one such  $z$  that maps to  $x - \varphi(g(x))$  for some  $x \in M$ , i.e.  $z$  is uniquely determined by  $x$ . Thus,  $\psi : M \rightarrow M'$  by  $x \mapsto z$  is well-defined, where  $z$  satisfies  $f(z) = x - \varphi(g(x))$ . Let  $m_1, m_2 \in M$ . Then,

$$\begin{aligned} f(\psi(m_1 + m_2)) &= (m_1 + m_2) - \varphi(g(m_1 + m_2)) \\ &= m_1 + m_2 - \varphi(g(m_1) + g(m_2)) \\ &= m_1 + m_2 - \varphi(g(m_1)) - \varphi(g(m_2)) \\ &= m_1 - \varphi(g(m_1)) + m_2 - \varphi(g(m_2)) \\ &= f(\psi(m_1)) + f(\psi(m_2)) = f(\psi(m_1 + m_2)) \end{aligned}$$

Since  $f$  is injective, we get  $\psi(m_1 + m_2) = \psi(m_1) + \psi(m_2)$ . Now let  $r \in A$  (the ring the module is over). Then,

$$\begin{aligned} f(\psi(rm)) &= rm - \varphi(g(rm)) \\ &= rm - r\varphi(g(m)) \quad \text{since } \varphi, g \text{ are module-homomorphisms} \\ &= r(m - \varphi(g(m))) = f(r\psi(m)) \end{aligned}$$

Hence, by the injectivity of  $f$ ,  $\psi(rm) = r\psi(m)$ . Therefore,  $\psi$  is a module homomorphism. Now, let  $m' \in M'$ . WTS  $(\psi \circ f)(m') = m'$ . Then,

$$\begin{aligned} f(\psi(f(m'))) &= f(m') - \varphi(g(f(m'))) \\ &= f(m') - \varphi(0) = f(m') \end{aligned}$$

Since  $f$  is injective,  $\psi(f(m')) = m'$ .

(2  $\Rightarrow$  1) Assume that there exists a homomorphism  $\psi : M \rightarrow M'$  such that  $\psi \circ f = id$ . Since  $g : M \rightarrow M''$  is surjective, for all  $x'' \in M''$ , we may choose  $m \in M$  such that  $g(m) = x''$ . Define  $\varphi(x'') = m - f(\psi(m))$ . We first check that this is well-defined. Suppose  $m_1, m_2 \in M$  satisfy  $g(m_1) = g(m_2)$ . Then  $g(m_1 - m_2) = 0 \Rightarrow m_1 - m_2 \in \ker g = \text{Im } f$ . Hence,  $m_1 - m_2 = f(m')$  for some  $m' \in M'$ . Applying  $\psi \circ f = id$ , we have

$$m' = \psi(f(m')) = \psi(m_1 - m_2) = \psi(m_1) - \psi(m_2)$$

Thus,

$$\begin{aligned} (m_1 - f(\psi(m_1)) - (m_2 - f(\psi(m_2))) &= f(m') - f(\psi(m_1) - \psi(m_2)) \\ &= f(m' - (\psi(m_1) - \psi(m_2))) \\ &= f(0) = 0 \end{aligned}$$

Next, we show that  $\varphi$  is a homomorphism. Let  $x''_1, x''_2 \in M''$  and let  $r$  be a scalar in the base ring. Choose  $m_1, m_2 \in M$  such that  $g(m_1) = x''_1$  and  $g(m_2) = x''_2$ . Then

$$\begin{aligned}
\varphi(x''_1 + x''_2) &= (m_1 + m_2) - f(\psi(m_1 + m_2)) \\
&= m_1 + m_2 - f(\psi(m_1) + \psi(m_2)) \\
&= (m_1 - f(\psi(m_1))) + (m_2 - f(\psi(m_2))) \\
&= \varphi(x''_1) + \varphi(x''_2)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\varphi(rx'') &= rm - f(\psi(rm)) \\
&= r(m - f(\psi(m))) = r\varphi(x''),
\end{aligned}$$

so  $\varphi$  preserves both addition and scalar multiplication and is thus a module homomorphism. Finally, we check that  $g \circ \varphi = id$ . Then,

$$\begin{aligned}
g(\varphi(x'')) &= g(m - f(\psi(m))) \\
&= g(m) - g(f(\psi(m))) \\
&= x'' - 0 = x''
\end{aligned}$$

Hence,  $g \circ \varphi = id$ , and the sequence splits. Thus, equivalence of (1) and (2) in Proposition 3.2 is proved.

We now prove that a sequence  $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$  splits if and only if there exists a submodule  $N$  of  $M$  such that  $M = \text{Im } f \oplus N$ , and in that case, then  $N$  is isomorphic to  $M''$ .

( $\Rightarrow$ ) Suppose  $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$  splits. Thus, let  $h : M'' \rightarrow M$  be such that  $g \circ h = id$ . Let  $N = \text{Im } h \subseteq M$ . To show that  $\text{Im } f \oplus N = M$ , we first show that  $N \cap \text{Im } f = \{0\}$ . Suppose

$$\begin{aligned}
x &\in \text{Im } f \cap N \\
\Rightarrow x &= h(y) \quad \text{and} \quad x \in \text{Im } f = \ker g
\end{aligned}$$

Applying  $g$ , we get  $0 = g(x) = g(h(y)) = y$ . Thus, since  $h$  is injective, by assumption of splitting and exactness, and  $y = 0$ , we have that  $x = h(0) \Rightarrow x = 0$ . Therefore,  $N \cap \text{Im } f = \{0\}$ . Now we want to show for all  $m \in M$ ,  $m = a + b$  with  $a \in \text{Im } f$ ,  $b \in N$ . Consider

$$m = \underbrace{m - h(g(m))}_{\substack{\in \text{Im } f = \ker g \\ \because g(m - h(g(m))) = 0}} + \underbrace{h(g(m))}_{\substack{\in \text{Im } h = N}} \quad (\text{adding 0 to } m)$$

Thus,  $M = \text{Im } f \oplus N$ . Define

$$g|_N : N \rightarrow M''$$

to be the restriction of  $g$  to  $N$ . (Surjective) Let  $m'' \in M''$ . Then  $h(m'') \in N$ . Applying the restriction of  $g$ ,

$$g|_N(h(m'')) = m'', \quad (\because g \circ h = id)$$

(Injective) Suppose  $g|_N(x) = 0$  for some  $x \in N = \text{Im } h$ . Thus,  $x \in N$  and  $x \in \ker g \Rightarrow x \in N \cap \ker g = N \cap \text{Im } f = \{0\}$ . Hence,  $x = 0$  and thus  $g|_N$  is injective. Therefore,  $N \cong M''$  with inverse map (isomorphism)  $h$ .

( $\Leftarrow$ ) Assume there exists some  $N \subseteq M$  such that  $M = \text{Im } f \oplus N$ . Define

$$g|_N : N \rightarrow M''$$

be the restriction of  $g$  to  $N$ . (Injective) Suppose  $g|_N(n) = 0$  for some  $n \in N$ . Then,  $n \in N \cap \ker g = N \cap \text{Im } f = \{0\} \Rightarrow n = 0$ . Therefore,  $g|_N$  is injective. (Surjective) Let  $m'' \in M''$ . Since  $g$  is surjective, there exists an element  $m \in M$  such that  $g(m) = m''$ . Write  $m = a + n$  with  $a \in \text{Im } f$  and  $n \in N$ . Then

$$g(n) = g(m - a) = g(m) - g(a) = m'' - 0 = m'', \quad \text{since } a \in \text{Im } f = \ker g$$

so  $g|_N$  is surjective. Therefore,  $g|_N : N \rightarrow M''$  is an isomorphism, so  $N \cong M''$ . Now we prove that the sequence splits. Define

$$h : M'' \rightarrow M$$

by  $h = \iota \circ (g|_N)^{-1}$ , where  $\iota : N \hookrightarrow M$  is the inclusion and  $(g|_N)^{-1}$  exists since  $g|_N$  is an isomorphism. For any  $m'' \in M''$

$$\begin{aligned} g(h(m'')) &= g\left(\left(\iota \circ (g|_N)^{-1}(m'')\right)\right) \\ &= g|_N\left((g|_N)^{-1}(m'')\right) = m'' \quad (\because g \circ \iota = g|_N : N \rightarrow M'') \end{aligned}$$

Thus,  $g \circ h = \text{id}$ , so the sequence splits.

### NEED TO DO THE LAST PART OF THIS PROOF, i.e. DETAILS OF PROP 3.2 PROOF

b) ??????????????????????????????????

□

### Exercise 0.2. (Exercise 9)

- a) Let  $A$  be a commutative ring and let  $M$  be an  $A$ -module. Let  $S$  be a multiplicative subset of  $A$ . Define  $S^{-1}M$  in a manner analogous to the one we used to define  $S^{-1}A$ , show that  $S^{-1}M$  is an  $S^{-1}A$ -module.
- b) If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is an exact sequence, show that the sequence  $0 \rightarrow S^{-1}M' \rightarrow S^{-1}M \rightarrow S^{-1}M'' \rightarrow 0$  is exact.

*Proof.* a) We define  $S^{-1}M = \{m/s : m \in M, s \in S\}$  with addition as  $m_1/s_1 + m_2/s_2 = m_1s_2 + m_2s_1/s_1s_2$  and since  $S \subseteq A$  is multiplicative and  $A$  is commutative,  $m_i s = s m_i = (\underbrace{1 + 1 + \dots + 1}_{s-\text{many}}) m_i$

and thus  $s m_i \in M$ . Hence,  $m_1 s_2 + m_2 s_1 \in M$  and  $s_1 s_2 \in S$  which implies that  $S^{-1}M$  is closed. The identity element in  $S^{-1}M$  is  $(e_M, s)$  for any  $s \in S$  with  $e_M$  the identity in the  $A$ -module  $M$ . Each element  $m/s \in S^{-1}M$  has inverse  $-m/s$  since  $-m \in M$  is the inverse of  $m \in M$ . Finally, take  $m_1/s_1, m_2/s_2, m_3/s_3 \in S^{-1}M$ . Then,

$$\begin{aligned}
& (m_1/s_1) + [(m_2/s_2) + (m_3/s_3)] \\
&= (m_1/s_1) + (m_2s_3 + m_3s_2/s_2s_3) \\
&= (m_1s_2s_3 + s_1(m_2s_3 + m_3s_2)/s_1s_2s_3) \\
&= (m_1s_2s_3 + m_2s_1s_3 + m_3s_1s_2/s_1s_2s_3) \\
&= (m_1s_2 + m_2s_1/s_1s_2) + (m_3/s_3) \\
&= [(m_1/s_1) + (m_2/s_2)] + (m_3/s_3)
\end{aligned}$$

Therefore,  $S^{-1}M$  is abelian group with most of the group properties following from the structure of the abelian group  $M$ .

Now, let  $a/s, b/s' \in S^{-1}A$  and  $(m_1/s_1)(m_2/s_2) \in S^{-1}M$ . Then,

$$\begin{aligned}
[(a/s) + (b/s')] (m_1/s_1) &= (as' + bs/ss')(m_1/s_1) \\
&= ((as' + bs)m_1/ss's_1) \\
&= (as'm_1 + bsm_1/ss's_1) \\
&= (am_1/ss_1) + (bm_1/s's_1) \\
&= (a/s)(m_1/s_1) + (b/s')(m_1/s_1)
\end{aligned}$$

And,

$$\begin{aligned}
(a/s) [(m_1/s_1) + (m_2/s_2)] &= (a/s)(m_1s_2 + m_2s_1/s_1s_2) \\
&= (a(m_1s_2 + m_2s_1)/ss_1s_2) \\
&= (am_1s_2 + am_2s_1/ss_1s_2) \\
&= (am_1/ss_1) + (am_2/ss_2) \\
&= (a/s)(m_1/s_1) + (a/s)(m_2/s_2)
\end{aligned}$$

Therefore,  $S^{-1}M$  is an  $S^{-1}A$ -module.

b) Suppose  $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$  is exact with  $\alpha, \beta$  module homomorphisms. Hence,  $\alpha$  is injective,  $\text{Im } \alpha = \ker \beta$ , and  $\beta$  is surjective. Next, consider the localized sequence,  $0 \rightarrow S^{-1}M' \xrightarrow{\delta} S^{-1}M \xrightarrow{\gamma} S^{-1}M'' \rightarrow 0$  where  $\delta(m'/s) = \alpha(m')/s$  and  $\gamma(m/s) = \beta(m)/s$ . Thus, we must show three things:  $\delta$  injective,  $\text{Im } \delta = \ker \gamma$ , and  $\gamma$  surjective.

( $\delta$  injective) Suppose that  $m'/s \in \ker \delta$ . Hence,  $\delta(m'/s) = \alpha(m')/s = 0/1$ . Then, by localization, there exists some  $t \in S$ , such that  $t\alpha(m') = 0$ . Since,  $\alpha$  is a module-homomorphism, we have  $\alpha(tm') = 0 \Rightarrow tm' \in \ker \alpha = \{0\}$ . Therefore,

$$m'/s = tm'/ts = 0/ts = 0 \in S^{-1}M'$$

Since we assumed that  $m'/s$  was an arbitrary element of the kernel of  $\delta$ , we have shown that the kernel only contains 0 and is thus trivial proving that  $\delta$  is injective.

( $\text{Im } \delta = \ker \gamma$ ) [ $\text{Im } \delta \subseteq \ker \gamma$ ] Let  $m/s \in \text{Im } \delta$ . Hence,  $m/s = \delta(m'/s)$ . Applying  $\gamma$ , we obtain

$$\gamma(\delta(m'/s)) = \gamma(\alpha(m')/s) = \beta(\alpha(m'))/s$$

Since,  $\text{Im } \alpha = \ker \beta$ ,  $\beta(\alpha(m')) = 0$ . Hence,  $\gamma(\delta(m'/s)) = 0/s$ . Thus,

$$\delta(m'/s) \in \ker \gamma \Rightarrow \text{Im } \delta \subseteq \ker \gamma.$$

[ $\text{Im } \delta \supseteq \ker \gamma$ ] Let  $m/s \in S^{-1}M$  such that  $\gamma(m/s) = 0 \in S^{-1}M''$ , i.e.  $m/s \in \ker \gamma$ . Thus,  $\gamma(m/s) = \beta(m)/s = 0/1$ . By localization, there exists some  $t \in S$ , such that  $t\beta(m) = 0$  and since

$\beta$  is a module homomorphism, we get  $\beta(tm) = 0$ . Thus,  $tm \in \ker \beta = \text{Im } \alpha$ . Hence, there exists some  $m' \in M'$ , such that  $\alpha(m') = tm$ . Thus,

$$m/s = tm/ts = \alpha(m')/ts = \delta(m'/ts)$$

Hence,  $m/s \in \text{Im } \delta \Rightarrow \ker \gamma \subseteq \text{Im } \delta$ .

Therefore,  $\text{Im } \delta = \ker \gamma$ .

( $\gamma$  surjective) Let  $m''/s \in S^{-1}M''$ . Since,  $\beta$  is surjective, choose some  $m \in M$  such that  $\beta(m) = m''$ . Thus,  $\gamma(m/s) = \beta(m)/s = m''/s$ . Therefore,  $\gamma$  is surjective.

This proves that  $0 \rightarrow S^{-1}M' \rightarrow S^{-1}M \rightarrow S^{-1}M'' \rightarrow 0$  is exact.  $\square$

### Exercise 0.3. (Exercise 10)

- a) If  $\mathfrak{p}$  is a prime ideal, and  $S = A - \mathfrak{p}$  is the complement of  $\mathfrak{p}$  in the ring  $A$ , then  $S^{-1}M$  is denoted by  $M_{\mathfrak{p}}$ . Show that the natural map

$$M \rightarrow \prod M_{\mathfrak{p}}$$

of a module  $M$  into the direct product of all localizations  $M_{\mathfrak{p}}$  where  $\mathfrak{p}$  ranges over all maximal ideals, is injective.

- b) Show that a sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is exact if and only if the sequence  $0 \rightarrow M'_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \rightarrow M''_{\mathfrak{p}} \rightarrow 0$  is exact for all primes  $\mathfrak{p}$ .

- c) Let  $A$  be an entire ring and let  $M$  be a torsion-free module. For each prime  $\mathfrak{p}$  of  $A$  show that the natural map  $M \rightarrow M_{\mathfrak{p}}$  is injective. In particular  $A \rightarrow A_{\mathfrak{p}}$  is injective, but you can see that directly from the imbedding of  $A$  in its quotient field  $K$ .

*Proof.* a) Let  $\varphi : M \rightarrow \prod M_{\mathfrak{p}}$ . Assume  $m \in \ker \varphi$ . Then for every maximal  $\mathfrak{p}$ , the image of  $m$  in  $M_{\mathfrak{p}} = S^{-1}M$  ( $S = A \setminus \mathfrak{p}$ ) is  $0/1$ . Thus, if  $m/1 = 0/1$ , by localization criterion, for each maximal  $\mathfrak{p}$  there exists  $t \notin \mathfrak{p}$  such that  $tm = 0$ . Set  $I = \text{Ann}(m) = \{a \in A : am = 0\}$ . By this definition, for every maximal  $\mathfrak{p}$ , we have  $I \cap (A \setminus \mathfrak{p}) \neq \emptyset$ , hence  $I$  is not contained in any maximal ideal. But every proper ideal of  $A$  is contained in some maximal ideal; hence  $I$  cannot be proper. Therefore,  $I = A \Rightarrow 1 \in I$ , i.e.  $1 \cdot m = m = 0$ . Thus,  $\ker \varphi = \{0\}$  (trivial), therefore,  $\varphi$  is injective.

- b) ( $\Rightarrow$ ) By **Exercise 9(b)**, exactness is preserved under localization at any multiplicative set. In particular, taking  $S = A \setminus \mathfrak{p}$  for an arbitrary maximal (prime)  $\mathfrak{p}$  gives exactness of the localized sequence  $0 \rightarrow M'_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \rightarrow M''_{\mathfrak{p}} \rightarrow 0$ .

( $\Leftarrow$ ) Suppose  $0 \rightarrow M'_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \rightarrow M''_{\mathfrak{p}} \rightarrow 0$  is exact. Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \longrightarrow 0 \\ & & \downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_3 \\ 0 & \longrightarrow & M'_{\mathfrak{p}} & \xrightarrow{\delta} & M_{\mathfrak{p}} & \xrightarrow{\gamma} & M''_{\mathfrak{p}} \longrightarrow 0 \end{array}$$

with  $\pi_i$  being projections of to each localization. For example,  $\pi_2(m) = m/1$  and likewise for  $\pi_1, \pi_3$ . By exactness of the localized sequence, we know  $\delta$  is injective,  $\text{Im } \delta = \ker \gamma$ , and  $\gamma$  is surjective. Thus, we need to show three things:  $\alpha$  is injective,  $\text{Im } \alpha = \ker \beta$ , and  $\beta$  surjective.

( $\alpha$  injective) Let  $m' \in \ker \alpha$ . Suppose  $m' \neq 0$ . Then, there exists a prime ideal,  $\mathfrak{p} \supseteq \text{Ann}(m') = \{a \in A : am' = 0\}$ . Take  $\pi_1(m') = m'/1 \in M'_{\mathfrak{p}}$ . By construction,  $m'/1 \neq 0 \in M'_{\mathfrak{p}}$  (Since if  $m'/1 = 0/1 \Rightarrow \exists t \notin \mathfrak{p}$  such that  $tm' = 0$  which is impossible since such  $t$  would be in  $\text{Ann}(m') \subseteq \mathfrak{p}$  which means  $t \notin A \setminus \mathfrak{p}$ ). But  $\delta(m'/1) = \alpha(m')/1 = 0/1 = 0 \in M_{\mathfrak{p}}$  which is a contradiction since  $\ker \delta = \{0\}$  by assumption of exactness. Hence,  $m' = 0$ . Thus,  $\ker \alpha = \{0\}$  (trivial), therefore  $\alpha$  is injective.

( $\text{Im } \alpha = \ker \beta$ ) [ $\ker \beta \subseteq \text{Im } \alpha$ ] Let  $m \in \ker \beta$  (i.e.  $\beta(m) = 0$ ). Then,  $\gamma(\pi_2(m)) = \gamma(m/1) = \beta(m)/1 = 0/1 \in M''_{\mathfrak{p}}$ . Thus,  $\pi_2(m) \in \ker \gamma = \text{Im } \delta$ . Therefore, there exists some  $m'/1 \in M'_{\mathfrak{p}}$  such

that  $\pi_2(m) = \delta(m'/1) = \alpha(m')/1 = m/1 \in M_{\mathfrak{p}}$ . By localization,  $m/1 = \alpha(m')/1$  means there exists  $t \notin \mathfrak{p}$  such that  $t(m - \alpha(m')) = 0 \Rightarrow tm = \alpha(tm') \in \text{Im } \alpha$ . Define

$$I = \{a \in A : am \in \text{Im } \alpha\}.$$

For every prime  $\mathfrak{p}$ , we have  $t \in I$  with  $t \notin \mathfrak{p}$ . Suppose  $I \neq A$ . Then there exists a maximal ideal, thus prime,  $\mathfrak{q}$  of  $A$  such that  $I \subseteq \mathfrak{q}$ . By definition of  $I$ , there exists  $t \in I$  such that  $t \notin \mathfrak{q}$  which is clearly a contradiction of  $I \subseteq \mathfrak{q}$ . Therefore,  $I = A \Rightarrow 1 \in I$ , i.e.  $1 \cdot m = m \in \text{Im } \alpha$ . Thus,  $\ker \beta \subseteq \text{Im } \alpha$ .

[ $\ker \beta \supseteq \text{Im } \alpha$ ] Let  $m = \alpha(m') \in \text{Im } \alpha$ . Consider  $\pi_1(m') = m'/1 \in M'_{\mathfrak{p}}$  and  $\pi_2(\alpha(m')) = \alpha(m')/1 \in M_{\mathfrak{p}}$ . Thus, we have an equality of  $\delta(\pi_1(m')) = \alpha(m')/1 = \pi_2(\alpha(m')) \in M_{\mathfrak{p}} \supseteq \ker \gamma = \text{Im } \delta$ . Thus, since the localization is exact, if we apply  $\gamma$  to this equality, we get  $\pi_2(\alpha(m')) \in \ker \gamma$  because  $\gamma \circ \delta = 0$ . By definition,  $\gamma(\pi_2(\alpha(m'))) = \gamma(\alpha(m')/1) = \beta(\alpha(m'))/1 \in M''_{\mathfrak{p}}$ . Thus,  $\beta(\alpha(m')) = 0/1$ . By localization, there exists some  $t \notin \mathfrak{p}$  such that  $t\beta(\alpha(m')) = 0$ . Set

$$J = \{a \in A : a\beta(\alpha(m')) = 0\}.$$

Thus,  $t \in J$  for every prime ideal  $\mathfrak{p}$  but,  $t \notin \mathfrak{p}$ . Suppose  $J \neq A$ , and thus there exists a maximal ideal (prime)  $\mathfrak{q}$  containing  $J$ . But  $t \in J$  and  $t \notin \mathfrak{q}$  (since in the localization relation,  $t$  is taken to not be in the complement of the ideal in  $A$ , just like we did in the other inclusion). Therefore,  $J = A \Rightarrow 1 \in J$ , i.e.  $1 \cdot \beta(\alpha(m')) = \beta(\alpha(m')) = 0$ . Thus,  $\alpha(m') \in \ker \beta$ . Therefore,  $\text{Im } \alpha \subseteq \ker \beta$ .

Thus,  $\text{Im } \alpha = \ker \beta$ .

( $\beta$  surjective) Let  $m'' \in M''$ . Consider  $\pi_3(m'') = m''/1 \in M''_{\mathfrak{p}}$ . By exactness of the localization, we know that  $\gamma$  is surjective, therefore, we can choose  $m/1 \in M_{\mathfrak{p}}$  such that  $\gamma(m/1) = m''/1$ , i.e.  $m/1 = \pi_2(m)$ . Thus,  $\gamma(m/1) = \beta(m)/1 = m''/1 \in M''_{\mathfrak{p}}$ . Then, by localization, there exists  $t \notin \mathfrak{p}$  such that  $t(\beta(m) - m'') = 0$ . Set

$$K = \{a \in A : a(\beta(m) - m'') = 0\}.$$

Hence, for every prime ideal  $\mathfrak{p}$ , there exists a  $t \in K$  with  $t \notin \mathfrak{p}$ . Suppose  $K \neq A$  which implies there exists a maximal ideal (prime)  $\mathfrak{q}$  containing  $K$ . But again,  $t \in K$ , but  $t \notin \mathfrak{q}$  (by defn of equivalence in the localization like both previous inclusions). Hence,  $K = A \Rightarrow 1 \in K$ , i.e.  $1 \cdot (\beta(m) - m'') = 0 \Rightarrow \beta(m) - m'' = 0$  and thus,  $\beta(m) = m''$ . Whence,  $\beta$  is surjective.

Therefore,  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is exact.

c) Define  $\psi : M \rightarrow M_{\mathfrak{p}}$  by  $m \mapsto m/1$ . Suppose  $m \in \ker \psi \Rightarrow \psi(m) = m/1 = 0/1 \in M_{\mathfrak{p}}$ . By localization, there exists  $t \notin \mathfrak{p}$  such that  $tm = 0$ . But since  $M$  is torsion-free, i.e. the only  $m \in M$  for which we can find an  $a \in A$  such that  $am = 0$  is  $m = 0$  ( $M_{\text{tor}} = 0$ ), and  $t \neq 0 \in A \setminus \mathfrak{p}$ , we must have  $m = 0$ . Thus,  $\ker \psi = \{0\}$ , hence  $\psi$  is injective.

□

#### Exercise 0.4. (Exercise 14)

Consider a commutative diagram of  $R$ -modules and homomorphisms such that each row is exact:

$$\begin{array}{ccccccc} M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' \end{array}$$

Prove:

- a) If  $f, h$  are monomorphisms, then  $g$  is a monomorphism.
- b) If  $f, h$  are surjective, then  $g$  is surjective.
- c) Assume in addition that  $0 \rightarrow M' \rightarrow M$  is exact and  $N \rightarrow N'' \rightarrow 0$  is exact. Prove that if any two of  $f, g, h$  are isomorphisms, then so is the third. [Hint: Use the snake lemma.]

*Proof.* a) We rewrite the diagram with additional labeling:

$$\begin{array}{ccccccc} M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N' & \xrightarrow{\gamma} & N & \xrightarrow{\delta} & N'' \end{array}$$

with  $\text{Im } \alpha = \text{Ker } \beta$  and  $\text{Im } \gamma = \text{Ker } \delta$ , since the rows are exact. We will show that  $g$  is a monomorphism by proving the kernel of  $g$  is trivial, i.e.  $\text{Ker } g = \{0\}$ . Thus, let  $m \in \text{Ker } g \Rightarrow g(m) = 0$ . From the commutativity of

$$\begin{array}{ccc} M & \xrightarrow{\beta} & M'' \\ \downarrow g & & \downarrow h \\ N & \xrightarrow{\delta} & N'' \end{array}$$

we have  $\delta \circ g = h \circ \beta$ . Hence,

$$\begin{aligned} h(\beta(m)) &= \delta(g(m)) \\ &= \delta(0) \\ &= 0 \end{aligned}$$

Thus, since  $h$  is injective and  $h(\beta(m)) = 0$ , we have  $\beta(m) = 0 \Rightarrow m \in \text{Ker } \beta$ . By exactness,  $\text{Im } \alpha = \text{Ker } \beta$ , and thus  $m = \alpha(m')$  for some  $m' \in M'$ . Now, by the commutativity of

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \downarrow f & & \downarrow g \\ N' & \xrightarrow{\gamma} & N \end{array}$$

we have  $g \circ \alpha = \gamma \circ f$ . Thus,

$$\begin{aligned} 0 &= g(m) = g(\alpha(m)) \\ &= (g \circ \alpha)(m') \\ &= (\gamma \circ f)(m') \\ &= \gamma(f(m')) \end{aligned}$$

Hence,  $\gamma(f(m')) = 0$  and since  $\gamma$  is injective (by the exactness of  $0 \rightarrow N' \xrightarrow{\gamma} N$ , i.e. the zero map with image 0 is equal to the kernel of  $\gamma$ , i.e.  $\text{Ker } \gamma = \{0\} = \text{Im}(0 \rightarrow N')$ ), we get  $f(m') = 0$  and thus  $m' \in \text{Ker } f$ . But since  $f$  is injective,  $\text{Ker } f = \{0\}$  and thus  $m' = 0$ . Then,  $m = \alpha(m') = \alpha(0)$ . Because  $\alpha$  is a module homomorphism,  $\alpha(0) = 0$  and thus  $m = 0$ . Therefore,  $\text{Ker } g = \{0\}$  and hence  $g$  is injective.

b) Let  $n \in N$ . Then,  $\delta(n) \in N''$  and since  $h$  is surjective, there exists an element  $m'' \in M''$  such that  $h(m'') = \delta(n)$ . By the exactness of  $M' \rightarrow M \rightarrow M'' \rightarrow 0$ ,  $m'' = \beta(m)$  since  $\beta$  is surjective from  $\text{Im } \beta = \text{Ker}(M'' \rightarrow 0) = M''$ . Thus,  $h(m'') = h(\beta(m)) = \delta(n)$ . Furthermore, from the commutativity of the diagram,  $h \circ \beta = \delta \circ g \Rightarrow h(\beta(m)) = \delta(g(m)) = \delta(n)$ . Thus,  $\delta(n) - \delta(g(m)) = \delta(n - g(m)) = 0$ . Hence,  $n - g(m) \in \text{Ker } \delta$ . Thus, by the exactness of  $0 \rightarrow N' \rightarrow N \rightarrow N''$ , there exists an element  $n' \in N'$  such that  $\gamma(n') = n - g(m)$ . Since  $f$  is surjective,  $n' = f(m')$  for some  $m' \in M'$ . Therefore, by commutativity of  $g \circ \alpha = \gamma \circ f$ ,

$$\begin{aligned} g(\alpha(m')) &= \gamma(f(m')) \\ &= \gamma(n') \\ &= n - g(m) \end{aligned}$$

which implies that  $n = g(m) + g(\alpha(m')) = g(m + \alpha(m'))$ . Hence,  $g$  is surjective.

c) ??

□

**Exercise 0.5.** (Exercise 15)

**The five lemma.** Consider a commutative diagram of  $R$ -modules and homomorphisms such that each row is exact:

$$\begin{array}{ccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 \longrightarrow M_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 \longrightarrow N_5 \end{array}$$

Prove:

- a) If  $f_1$  is surjective and  $f_2, f_4$  are monomorphisms, then  $f_3$  is a monomorphism.
- b) If  $f_5$  is a monomorphism and  $f_2, f_4$  are surjective, then  $f_3$  is surjective. [Hint: Use the snake lemma.]

*Proof.* a) We rewrite the diagram with the mappings labeled:

$$\begin{array}{ccccccc} M_1 & \xrightarrow{\alpha} & M_2 & \xrightarrow{\beta} & M_3 & \xrightarrow{\gamma} & M_4 \xrightarrow{\delta} M_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_5 \\ N_1 & \xrightarrow{\varepsilon} & N_2 & \xrightarrow{\eta} & N_3 & \xrightarrow{\varphi} & N_4 \xrightarrow{\psi} N_5 \end{array}$$

We show that  $f_3$  is a monomorphism by showing that  $\text{Ker } f_3 = \{0\}$ . Let  $m_3 \in \text{Ker } f_3 \Rightarrow f_3(m_3) = 0$ . Consider the commutative subdiagram

$$\begin{array}{ccc} M_3 & \xrightarrow{\gamma} & M_4 \\ \downarrow f_3 & & \downarrow f_4 \\ N_3 & \xrightarrow{\varphi} & N_4 \end{array}$$

with  $f_4 \circ \gamma = \varphi \circ f_3$ . Thus,  $f_4(\gamma(m_3)) = \varphi(f_3(m_3)) = \varphi(0) = 0$  (since  $\varphi$  is a module homomorphism). Hence,  $f_4(\gamma(m_3)) = 0 \Rightarrow \gamma(m_3) = 0$  (since  $f_4$  is injective). Hence,  $m_3 \in \text{Ker } \gamma$ . By exactness,  $\text{Im } \beta = \text{Ker } \gamma$ , we have  $m_3 = \beta(m_2)$  with  $m_2 \in M_2$ . Now consider the commutative subdiagram

$$\begin{array}{ccc} M_2 & \xrightarrow{\beta} & M_3 \\ \downarrow f_2 & & \downarrow f_3 \\ N_2 & \xrightarrow{\eta} & N_3 \end{array}$$

with  $\eta \circ f_2 = f_3 \circ \beta$ . Then,

$$\begin{aligned} \eta(f_2(m_2)) &= f_3(\beta(m_2)) \\ &= f_3(m_3) = 0 \end{aligned}$$

Thus,  $\eta(f_2(m_2)) = 0 \Rightarrow f_2(m_2) \in \text{Ker } \eta$ . By the exactness,  $\text{Im } \varepsilon = \text{Ker } \eta$ . Hence,  $\varepsilon(n_1) = f_2(m_2)$  with  $n_1 \in N_1$ . Therefore, since  $f_1$  is surjective,  $n_1 = f_1(m_1)$  with  $m_1 \in M_1$ . Then,  $\varepsilon(f_1(m_1)) = f_2(m_2)$ . Consider the commutative subdiagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\alpha} & M_2 \\ \downarrow f_1 & & \downarrow f_2 \\ N_1 & \xrightarrow{\varepsilon} & N_2 \end{array}$$

with  $\varepsilon \circ f_1 = f_2 \circ \alpha$ . Hence,

$$\begin{aligned} \varepsilon(f_1(m_1)) &= f_2(\alpha(m_1)) \\ \Rightarrow \varepsilon(n_1) &= f_2(\alpha(m_1)) \\ \Rightarrow f_2(m_2) &= f_2(\alpha(m_1)) \end{aligned}$$

Since  $f_2$  is injective,  $m_2 = \alpha(m_1)$ . By exactness,  $\text{Im } \alpha = \text{Ker } \beta$ . Hence, since  $m_2 \in \text{Im } \alpha$ ,  $\beta(m_2) = m_3 \in \text{Ker } \beta$ . Thus,

$$\begin{aligned} \beta(m_2) &= m_3 \\ \Rightarrow \beta(\alpha(m_1)) &= m_3 \end{aligned}$$

But again by  $\text{Im } \alpha = \text{Ker } \beta$ , everything in the image of  $\alpha$  gets sent to 0 in  $M_3$ . Thus,  $\beta(\alpha(m_1)) = 0 \Rightarrow m_3 = 0$ . Therefore,  $\text{Ker } f_3 = \{0\}$  and hence  $f_3$  is a monomorphism.

b) Let  $n_3 \in N_3$ . Then,  $\varphi(n_3) \in N_4$ . Since  $f_4$  is surjective, there exists an element  $m_4 \in M_4$  such that  $f_4(m_4) = \varphi(n_3)$ . Consider the commutative subdiagram

$$\begin{array}{ccc} M_4 & \xrightarrow{\delta} & M_5 \\ \downarrow f_4 & & \downarrow f_5 \\ N_4 & \xrightarrow{\psi} & N_5 \end{array}$$

with  $f_5 \circ \delta = \psi \circ f_4$ . Thus,

$$\begin{aligned} f_5(\delta(m_4)) &= \psi(f_4(m_4)) \\ &= \psi(\varphi(n_3)) \\ &= 0 \quad (\text{since } \text{Im } \varphi = \text{Im } \psi) \end{aligned}$$

Then,  $f_5(\delta(m_4)) = 0$  and since  $f_5$  is injective,  $\delta(m_4) = 0 \Rightarrow m_4 \in \text{Ker } \delta = \text{Im } \psi$ . Hence,  $\psi(m_3) = m_4$  with  $m_3 \in M_3$ . Moving left in the diagram, consider the commutative subdiagram

$$\begin{array}{ccc} M_3 & \xrightarrow{\psi} & M_4 \\ \downarrow f_3 & & \downarrow f_4 \\ N_3 & \xrightarrow{\varphi} & N_4 \end{array}$$

with  $f_4 \circ \gamma = \varphi \circ f_3$ . Then,

$$\begin{aligned} f_4(\gamma(m_3)) &= \varphi(f_3(m_3)) \\ \Rightarrow f_4(m_4) &= \varphi(f_3(m_3)) \\ \Rightarrow \varphi(n_3) &= \varphi(f_3(m_3)) \\ \Rightarrow \varphi(n_3 - f_3(m_3)) &= 0 \\ \Rightarrow n_3 - f_3(m_3) &\in \text{Ker } \varphi = \text{Im } \eta \\ \Rightarrow \eta(n_2) &= n_3 - f_3(m_3) \end{aligned}$$

Since  $f_2$  is surjective, there exists an element  $m_2 \in M_2$  such that  $f_2(m_2) = n_2$ . Finally, consider the commutative subdiagram

$$\begin{array}{ccc} M_2 & \xrightarrow{\beta} & M_3 \\ \downarrow f_2 & & \downarrow f_3 \\ N_2 & \xrightarrow{\eta} & N_3 \end{array}$$

with  $f_3 \circ \beta = \eta \circ f_2$ . Then,

$$\begin{aligned} f_3(\beta(m_2)) &= \eta(f_2(m_2)) \\ &= \eta(n_2) \\ &= n_3 - f_3(m_3) \\ \Rightarrow n_3 &= f_3(m_3 + \beta(m_2)) \end{aligned}$$

Therefore,  $f_3$  is surjective.

**CONSIDER TRYING TO PROVE PART B WITH THE SNAKE LEMMA!!!!!!!!!!!!!!**

□

### Exercise 0.6. (Exercise 17)

a) Let  $n$  range over the positive integers and let  $p$  be a prime number. Show that the abelian groups  $A_n = \mathbb{Z}/p^n\mathbb{Z}$  form an inverse system under the canonical homomorphism if  $n \geq m$ . Let  $\mathbb{Z}_p$  be its inverse limit. Show that  $\mathbb{Z}_p$  maps surjectively on each  $\mathbb{Z}/p^n\mathbb{Z}$ ; that  $\mathbb{Z}_p$  has no divisors of 0, and has a unique maximal ideal generated by  $p$ . Show that  $\mathbb{Z}_p$  is factorial, with only one prime, namely  $p$  itself.

b) Next consider all non zero ideals of  $\mathbb{Z}$  as forming a directed system, by divisibility. Prove that

$$\varprojlim_{(a)} \mathbb{Z}/(a) = \prod_p \mathbb{Z}_p$$

where the limit is taken over all non zero ideals  $(a)$ , and the product is taken over all prime  $p$ .

*Proof.* a) ??????????????????????????????

b) ??????????????????????????????

□