Modeling Airports Through Differential Equations

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Contents

1 Abstract

Airports experience frequent traffic everyday due to the demand of travel as well as the shortage of pilots to provide the travelling. Thus it is important to take a look at the number of planes at a given airport to predict the congestion at a given airport. As a result this should help people with their travel plans, pilots with their expected traffic levels upon landing or prior to departure, and also help the airports and airlines be able to increase the passengers and commuters experiences at the airport resulting in a more probable desire to travel via airfare.

2 Introduction

As the demand for air travel increases, air traffic controller (ATC), have experienced a growing amount of stress and lag during their jobs. Noting that air travel is in fact the safest mode of transportation, there still exists a danger of midair or on ground collisions. As well, a necessary thing to consider beyond commercial air and ground traffic, there is civilian and military aircraft too. One of the biggest things factors that air traffic congestion presents the encounter of unnecessary and excess fuel usage and associated air pollution as well as noise pollution namely close to major airports (FAA, 2023a).

If you imagine everyone commuted on the same road everyday to work without going on any diverging path simply for the sake of getting to the destination you would know what airport congestion is like. The way pilots develop a flight route, is via way points and SID's(Standard Instrument Departures) and STAR's(Standard Terminal Arrival). These are well defined routes for a pilots flight plan thus leading planes to the same destination via the same path resulting in congestion on the ground as the number of arrivals exceeds the amount of departing planes (FAA, 2023b).

Air travel, of course, depends on the atmospheric conditions such as wind direction, precipitation, and et al. Heavy winds such as cross winds or tail winds on landings and takeoffs, can delay airplanes arrivals and departures resulting in a dynamic and continuously changing inflow and outflow of airplanes. This results in various runway selections for landings which effects pre-planned airport routes (SID's/STAR's) causing landing delays and take off delays[4]. As an observer we hear and see a lot about winter storms or just storms in general, delaying planes for days at a time. This is a huge factor of course then to model airport occupancy.

Alongside the fact of weather delays there is layovers due to delays. For example, a plane can be flying from KLAX (Los Angeles, CA) to KJFK (New York, NY) but lets say there is heavy crosswinds delaying all departures and arrivals for the next three hours and this has been announced following the planes takeoff from KLAX and they are an hour away from New York. Before a plane takes off, the pilots will set the SID's/STAR's as mentioned but based off their STAR is what the secondary, "emergency" airport is defined as. So if our plane is doing this route they can have an "emergency" airport of KEWR (Newark, NJ), KHPN (Westchester County, NY), or KALB (Albany, NY). Thus it is not realistic to have a given plane always land at their desired airport but it is very important to note that planes will land at their destination even with heavy crosswinds due to the rigorous training of flight schools and also due to an airlines money. Thus, these diversions have a low chance of happening but still a notable thing to mention when discussing airport inflow and outflow.

Overall, the aim of this model is to help airport management understand potential amounts of aircraft at given times and eventually be expanded for holiday airfare congestion since the number of people using airfare will be heightened most of the time.

3 Schematic Diagrams

3.1 Airport Diagram Example (KLAS)

Every airport, perhaps majority of them, have an airport diagram like the one depicted above. In this case, we have Harry Reid International Airport (KLAS) located in Las Vegas, Nevada, USA, and there is some key things to note that will help in understanding the latter model. This diagram contains far more information than we need for this model as it is designed for pilots but some key notes are as such:

- For each runway, we see that technically there is two runways that are opposite directions, precisely by 180° thus depending on weather and wind direction is which runway is open. **The key part we will note is the fact that saying one runway implies two possible entries and exits onto the runway**.

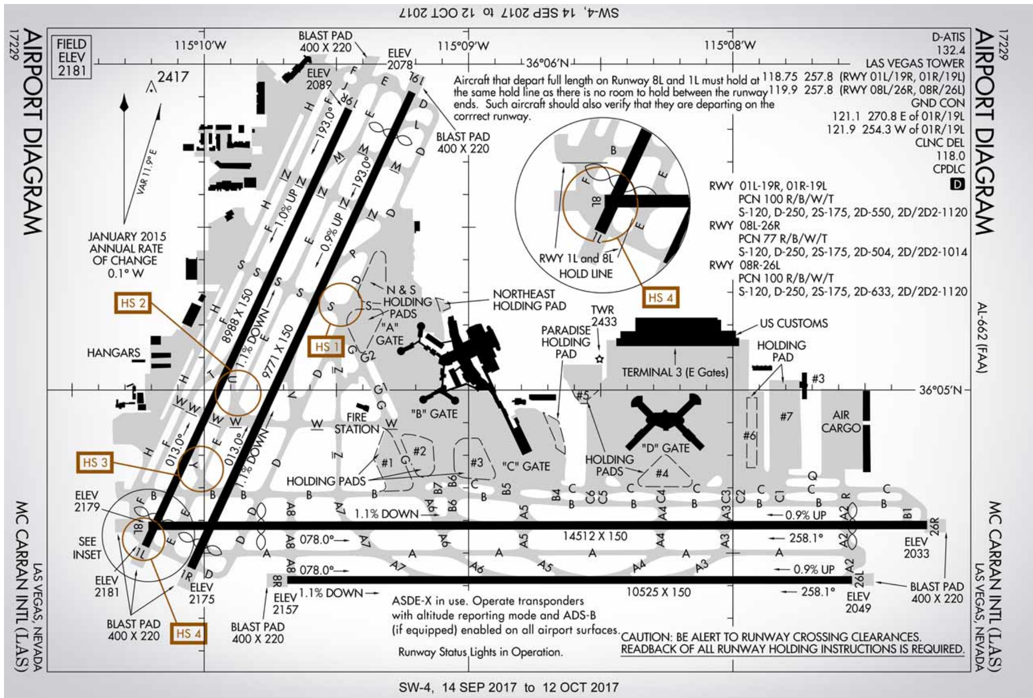
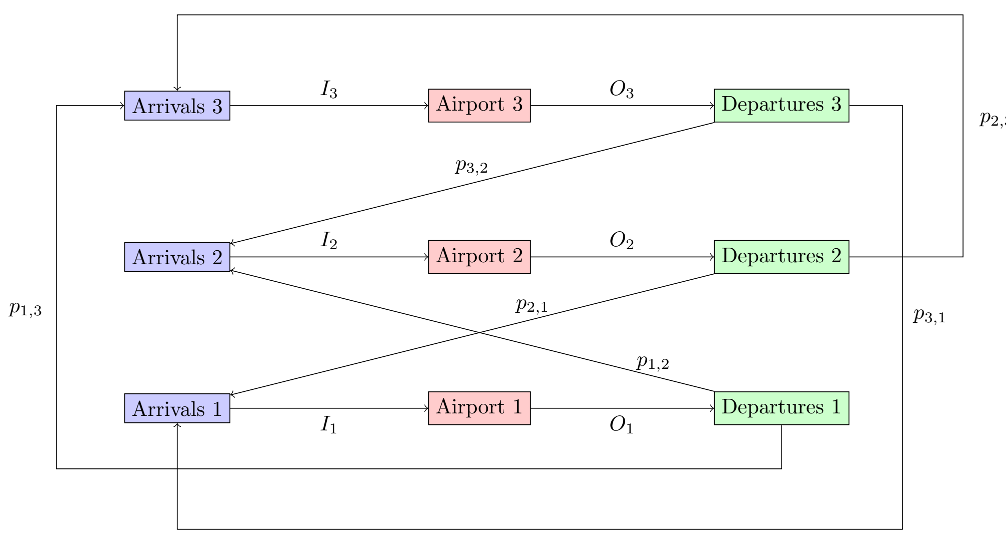


Figure 1: Visual representation of KLAS, showing the relationships between different components of a system (FlightAware, 2023).

3.2 Model Schematic Diagram



Note: can be expanded to more airports as needed

In our models schematic diagram, we can see that for each airport, there is a general inflow (arrivals) and outflow (departures) rates based off airports specified size which is proportional to how busy it is. A good way to think of them is as planes that might be coming from another airport outside of our model to one of these three modeled airports or departing our three airports going elsewhere. Then within our system of airports, a departure from one airport has a given probability of going to another airport in our system denoted by $P_{i,f}$, where i and f is the departing and arrival airport respectively. Also note that the sum of these probabilities does not have to add up to one since they are not part of a specific distribution.

4 Assumptions

Modeling an airports arrivals and departures is a very dynamic setting. Many variables are dependent on multiple factors such as weather at departing and arriving airports, time of day, size of the plane, and et al which would need to be modeled using partial differential equations or integer programming however we will focus and shape the model around ordinary differential equations. Our assumptions are as follows:

1) Probability of any plane globally landing at any given airport is equal for all airports. This is important to have for **6.3** if we want to allow the addition of airports outside the system into the model with ease.
2) Weather will be random and only cause delays of 15, 30, or 45 minutes. As no real weather data is involved currently but rather a vector of random numbers between 0 and 1 correlate to severity of delays.
3) Inflow and Outflow rates will follow that of a sine and cosine function, respectively. This is reasonable since if you think generally about inflow and outflow in a system there are moments when it is higher and some when its lower and these functions are wave-like functions which demonstrates that exactly.
4) Taxi to gate and runway times will **NOT** be considered and thus wont cause any model delays. This model doesn't involve time delays currently for the simplicity of modeling just airport inflow and outflow.
5) Planes can take off "immediately" after landing. This assumption is needed for the same reason as the above assumption.
6) There is only inflow from 12am-4am on a given day since departures normally don't start until 5am. In the real world, you don't have planes departing during these hours a majority of the time and usually only will have planes landing during these hours so for accuracy purposes it is important to include.
7) In section **6.3**, airport 1, airport 2, and airport 3, are defined as KJFK(John F. Kennedy Intl Airport, NY), KSAN(San Diego Intl Airport, CA), and KATL(Hartsfield-Jackson Intl Airport, GA), respectively.

Comparing varing sizes of international airports is important to see the dynamics of the system in these instances.

5 Equations/Parameter Table

5.1 Parameter Table

Parameter	Meaning	Value
ω	Frequency	$\frac{2\pi}{T}$
$I_{ai} \mid i = \{1,2,3\}$	Amplitude of Arrivals at Airport i	(See Appendix)
$O_{ai} \mid i = \{1,2,3\}$	Amplitude of Departures at Airport i	(See Appendix)
$I_{0i} \mid i = \{1,2,3\}$	Mean Inflow Rate Airport i	(See Appendix)
$O_{0i} \mid i = \{1,2,3\}$	Mean Outflow Rate Airport i	(See Appendix)
$K_i \mid i = \{1,2,3\}$	Carrying Capacity Airport i	(See Appendix)
G	Number of Global Flights Per Day	$151,435$
$P_i \mid i = \{1,2,3\}$	Probability of Plane Landing at Airport i Outside System	$\{0.02667, 0.007, 0.03\}$
$p_{i,j} \mid i = \{1,2,3\}, j = \{1,2,3\}$	Probability of Plane going from Airport i to Airport j	(See Matrix Below)
$m_i \mid i = \{1,2,3\}$	Population Dependent Departure Rate i	$\{0.02, 0.02, 0.02\}$

****Note: values are justified and explained in Appendix*

$$p_{general} = \begin{bmatrix} 0 & p_{1,2} & \cdots & p_{1,j} \\ p_{2,1} & 0 & p_{2,3} & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ p_{i,1} & p_{i,2} & \cdots & 0 \end{bmatrix}$$

The matrix above is how we define the probability of an airplane going from an airport to another airport in which the origin airport is the row index and the destination is the column index where if $i = j$ then $p_{i,j} = 0$. So for example, $p_{1,2}$ is the probability that an airplane flies from airport 1 to airport 2. Thus can be expanded to as many airports as you wish. If you have n airports then the matrix will be $n \times n$ with 0's on the diagonal, but in our case $n = 3$ will suffice. The numerical values for the last assumption in section **4** are defined in a matrix below:

$$p = \begin{bmatrix} 0 & 0.0075 & 0.0108 \\ 0.0254 & 0 & 0.019 \\ 0.0104 & 0.00444 & 0 \end{bmatrix}$$

****These probabilities, $p_{i,j}$, are discussed in the Appendix.*

5.2 Equations (Used in 6.3)

Note: I_i represents inflow at airport i, O_i represents outflow at airport i

$$I_1 = I_{01} + I_{a1} \cdot \sin(\omega \cdot t) \tag{1}$$

$$I_2 = I_{02} + I_{a2} \cdot \sin(\omega \cdot t) \tag{2}$$

$$I_3 = I_{03} + I_{a3} \cdot \sin(\omega \cdot t) \tag{3}$$

$$O_1 = O_{01} + O_{a1} \cdot \cos(\omega \cdot t) \tag{4}$$

$$O_2 = O_{02} + O_{a2} \cdot \cos(\omega \cdot t) \tag{5}$$

$$O_3 = O_{03} + O_{a3} \cdot \cos(\omega \cdot t) \tag{6}$$

$$\frac{dN_1}{dt} = (P_1 G + p_{2,1} O_2 + p_{3,1} O_3 - O_1) + (1 - \frac{N_1}{K_1}) - \frac{m_1 \cdot N_1}{(1 + \frac{N_1}{K_1})} \tag{7}$$

$$\frac{dN_2}{dt} = (P_2 G + p_{1,2} O_1 + p_{3,2} O_3 - O_2) + (1 - \frac{N_2}{K_2}) - \frac{m_2 \cdot N_2}{(1 + \frac{N_2}{K_2})} \tag{8}$$

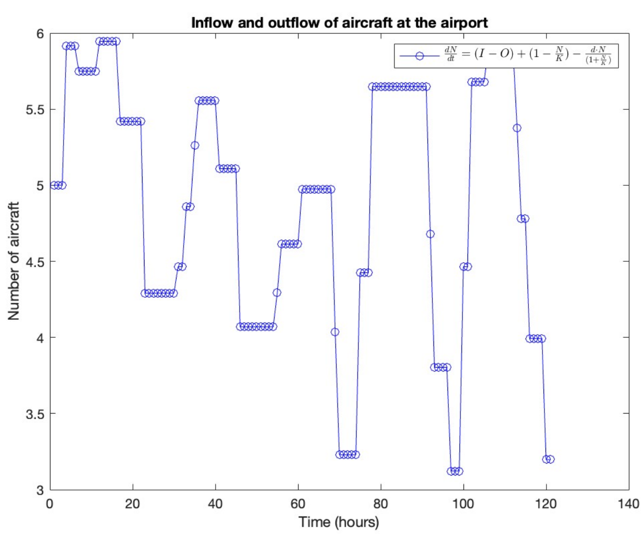
$$\frac{dN_3}{dt} = (P_3 G + p_{1,3} O_1 + p_{2,3} O_2 - O_3) + (1 - \frac{N_3}{K_3}) - \frac{m_3 \cdot N_3}{(1 + \frac{N_3}{K_3})} \tag{9}$$

6 Simulation and Results

Note: This simulation of one airport does not involve probability.

6.1 One Airport

The following simulation is of the equation $\frac{dN}{dt} = (I - O) + (1 - \frac{N}{K}) - \frac{m_1 \cdot N}{(1+\frac{N}{K})}$, where $I = I_0 + I_1 \cdot \sin(\omega \cdot t)$, $O = O_0 + O_1 \cdot \cos(\omega \cdot t)$ modeling the airport plane population at a given time t:



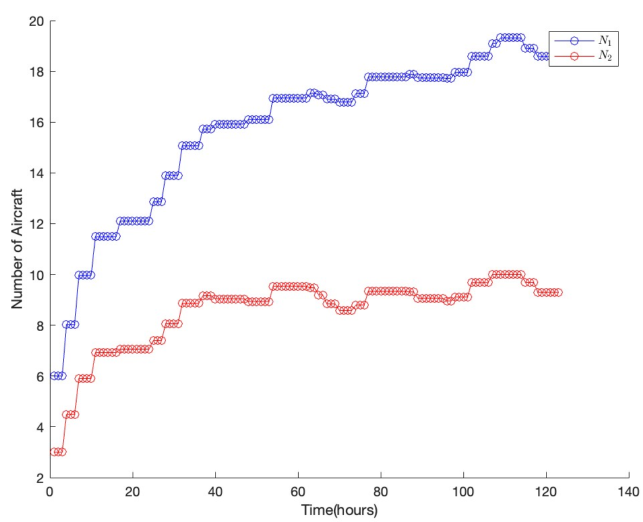
****Values for simulation are defined in Appendix.*

These results are consistent with what we expect over the 5 days the simulation was run for. As you can see in the graph, there are many instances of the population staying constant over time steps (hours) and this is due to the fact of random weather that would cause arrival and departures delays thus keeping the population constant. Another key aspect is the fact that in the first 4 hours of each day has no decrease in airport population since in our assumptions, there is no departures in the hours 12am-4am which leads to an expected observation of the population always increasing following those time frames as planes are only distinguished as arrivals in those time ranges. The reason why the population stays constant at those times instead of increasing all the time, since there is only arrivals in those hours, is due to the weather aspect which will override the arrivals if there is bad weather that would delay an arrival thus making sense in the graphic as well.

6.2 Two Airports

Note: This simulation of two generic airports does not involve probability.

The following simulation is of the equations $\frac{dN_1}{dt} = (I_1+O_2-O_1)+(1-\frac{N_1}{K_1})-\frac{m_1 \cdot N_1}{(1+\frac{N_1}{K_1})}$ and $\frac{dN_2}{dt} = (I_2+O_1-O_2)+(1-\frac{N_2}{K_2})-\frac{m_2 \cdot N_2}{(1+\frac{N_2}{K_2})}$, where $I_i = I_{0i}+I \cdot \sin(\omega \cdot t)$, $O = O_{0i} + O \cdot \cos(\omega \cdot t)$ modeling the two plane populations at each airport at a given time t:

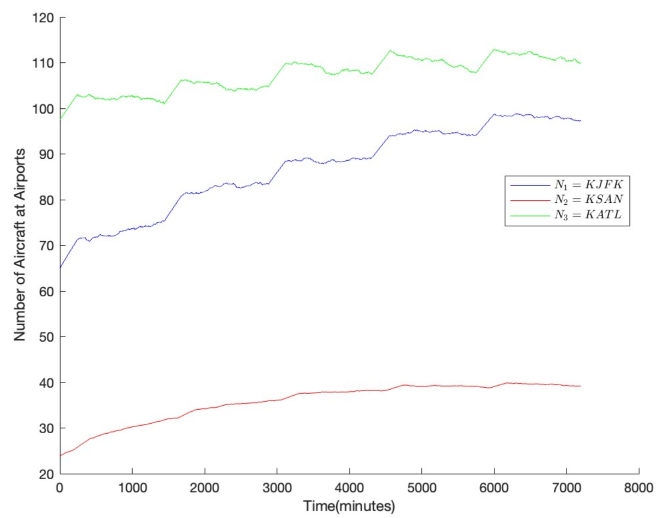


****Values for simulation are defined in Appendix.*

From the results above, recalling the parameter table, Airport 1 (blue) is the larger airport with a higher carrying capacity and a greater mean inflow rate. Thus from the simulation we see Airport 1 grows substantially faster than Airport 2 (red). Again, we see that the time delays for weather are depicted as well as the fact of there only being arrivals to each airport in the first 4 hours of each day as is the case in the real world in our assumptions.

6.3 Three Airports With Probability

The following is a simulation of three airports, using the equations in **5.2** namely KJFK (John F. Kennedy International Airport, NY), KSAN (San Diego International Airport, CA), and KATL (Hartsfield Jackson International Airport, GA) respectively:



****Values for simulation are defined in Appendix.*

This result above is expected which is always good since it reflects that our model is demonstrating this scenario well. Note that $K_3 > K_1 > K_2$ so it makes sense that we don't see drastic sinusoidal patterns in N_2 and also, due to the size of the airport, there is a lower probability of planes going to that airport compared to the others. This also includes weather and all time delays which can be seen through the rough nature and note this simulation is iterated per minute as supposed to every hour for 5 days. And you see immediately all the airports increase from the initial 50% of their capacity due to the fact that most airports operate in generally at 60% of their capacity which can be seen in this simulation. Also we see the graphs leveling off and even decreasing slightly to hover around that mark since as an airport approaches it capacity, it starts to have more departures which is why the m_i term in the equations is very important.

7 Future Works

7.1 Limitations & Improvements

As with any mathematical model, it is impossible to model a scenario perfectly due to time constraints, model constraints, availability of data, randomness of the real world, and et al. Hence we need to go over what this model is limited to and what needs to be done better in future iterations of the model.

Limitations	**Improvements**
- The model assumes that a plane outside the scope of the model has an equal probability of landing at any given international airport in the world which is not practical.	- Adding further weighted probabilities for a given airport and expanding to more airports in the model namely major airports.
- Weather is randomly generated each iteration of the model while in the real world specific airports have climate related trends and also the weather will at most cause delays of 45minutes.	- Import real weather data and use predictive modeling to forecast future airport populations using the forecasted weather and implement more delay levels.
- Using cosine and sine functions for inflow and outflow rates might not be the most optimal choice.	- Expand inflow and outflow functions to have time delays for ground taxiing and et al.
- This model doesn't take into effect any ground related delays such as taxiing, airport traffic, and boarding times.	- Explore time dependent probabilities in relations to time of year which can play a role in peoples desired destinations.

8 Conclusion

This model is great for seeing a general picture of what an airports can look like over the course of 5 days. The strength of it is that the system of differential equations is very easily expandable to as many airports as you desire. The weather delays are somewhat accurate since weather is random in nature and can change drastically in some instances which was an important thing that was considered when adapting a one minute time step. Also this model is functioning well since the populations reach an equilibrium around 60% of its carrying capacity which is the capacity most airports prefer to operate at. This model will continue to be developed to be an optimal model for forecasting congestion at airports.

9 Appendix

The following justifies the given values in the parameter table:

1) $\omega = \frac{2\pi}{T}$ is an input into the inflow and outflow function where T is the period so it can be adapted to a variety of different time periods and time steps.

2) Values for **6.1**: $I_0 = 1$, $O_0 = 1$, $I_1 = 0.5$, $O_1 = 0.25$, $\omega = \frac{2\pi}{24}$, $K = 10$, $m = 0.25$, $N(1) = 5$. The mean inflow and outflow rates are both 1 since you can assume one plane lands and one plane takes off normally to keep balance, and the amplitudes are 0.5 since in the case where this is one airport with one runway we can say hypothetically we can have a plane take off and land at the same time which would result in that "half" value. The given density-dependent departure rate is trial and error to balance the simulation and $K = 10$ is chosen using an estimation of a smaller airports carrying capacity.

3) Values for **6.2**: $I_{01} = 2$, $I_{02} = 1$, $O_{01} = 1$, $O_{02} = 1$, $I = 0.5$, $O = 0.5$, $\omega = \frac{2\pi}{24}$, $K_1 = 20$, $K_1 = 10$, $m_1 = 0.2$, $m_2 = 0.2$, $N_1(1) = 6$, $N_2(1) = 3$. The values are justified the same as the previous bullet point minus the fact that there are now two airports with one runway each but airport 1 is bigger and busier which is why it has a higher average inflow rate, carrying capacity, and starting population.

4) The values for **6.3** use real world data of the airports described in the last assumption as well as the simulation in 6.3 as well. In general, mean inflow rates for each airport were calculated by find how many planes on average land at the airport and then dividing it by $24 \cdot 60$ to get how many per minute and this value would be the same as the average outflow rate which appears to be a valid assumption. The amplitudes of inflow and outflow for each airport is just simply how many runways it has. All the probabilities are calculated using proportions of flights going to a given airports for example, there are $45,000$ FAA flights per day and $1,200$ planes land at KJFK each day so $p_1 = \frac{1,200}{45,000}$. A rundown of the values, minus the $p_{i,j}$ since they are defined in the matrix p in section 5; $G = 73.913$, $I_{01} = 0.833 = O_{01}$, $I_{02} = 0.21875 = O_{02}$, $I_{03} = 0.9375 = O_{03}$, $I_{a1} = 4 = O_{a1}$, $I_{a2} = 1 = O_{a2}$, $I_{a3} = 5 = O_{a3}$, $p_1 = 0.02667$, $p_2 = 0.007$, $p_3 = 0.03$, $K_1 = 130$, $K_2 = 48$, $K_3 = 195$, $m_1 = m_2 = m_3 = 0.02$, $\omega = \frac{2\pi}{7200}$(for minute time steps), and all starting populations for each airport is 50% of their carrying capacity.

Sources that were used throughout the paper for various details and parameters: Wikipedia, 2023, Union of International Associations, 2021, Federal Aviation Administration, 2019, Federal Aviation Administration, 2021, FinancesOnline, 2023, NASA, 2015, New York Art Life, 2013, World Data, 2023, FlightAware, 2023, FAA, 2023a, FAA, 2023b

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