A Galois-Theoretic Proof of the Irrationality of $\sqrt{2}$

$\sqrt{2}$ is irrational.

Proof. Observe that

$$|\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})| = 2 = [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] \quad (\because\ \mathbb{Q}(\sqrt{2}) \text{ is normal and separable over } \mathbb{Q}).$$

Hence, $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \cong \mathbb{Z}_2$, with automorphisms

$$\phi_1 \colon a + b\sqrt{2} \mapsto a + b\sqrt{2}, \quad \text{and} \quad \phi_2 \colon a + b\sqrt{2} \mapsto a - b\sqrt{2},$$

for all $a, b \in \mathbb{Q}$.

Any automorphism in $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ must fix every element of $\mathbb{Q}$. That is, $\phi(q) = q$ for all $q \in \mathbb{Q}$. However, we have

$$\phi_2(\sqrt{2}) = -\sqrt{2}.$$

Clearly, $\sqrt{2} \neq -\sqrt{2}$. Hence, $\sqrt{2} \notin \mathbb{Q}$, and so $\sqrt{2}$ is irrational. $\square$