ECE 6310 - Advanced Electromagnetic Fields: Homework Set #1

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4 days - 12 hours - 39 min until deadline

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1 Preliminaries

In this document, we use standard notation for electromagnetic theory. Key equations and concepts are summarized below:

Vector Notation

- $\mathbf{E}$: Electric field intensity
- $\mathbf{H}$: Magnetic field intensity
- $\mathbf{D}$: Electric flux density
- $\mathbf{B}$: Magnetic flux density
- $\mathbf{J}$: Current density
- ρ_v: Volume charge density

Differential Operators

- $\nabla\cdot$: Divergence of a vector field
- $\nabla\times$: Curl of a vector field

- ∇ : Gradient of a scalar field
- ∂_i : Partial derivative with respect to the independent basis element i

Maxwell's Equations

In integral form, Maxwell's equations are given by:

$$\oint_{\partial V} \boldsymbol{\mathscr{E}} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_V \boldsymbol{\mathscr{B}} \cdot d\boldsymbol{\mathcal{S}} \qquad \text{(Faraday's Law of Induction)} \tag{1}$$

$$\oint_{\partial V} \boldsymbol{\mathscr{H}} \cdot d\boldsymbol{\ell} = \int_V \boldsymbol{\mathscr{J}} \cdot d\boldsymbol{\mathcal{S}} + \frac{d}{dt}\int_V \boldsymbol{\mathscr{D}} \cdot d\boldsymbol{\mathcal{S}} \qquad \text{(Ampère's Circuital Law)} \tag{2}$$

$$\oiint_{\partial V} \boldsymbol{\mathscr{D}} \cdot d\boldsymbol{\mathcal{S}} = \int_V \rho_v dV \qquad \text{(Gauss's Law for Electricity)} \tag{3}$$

$$\oiint_{\partial V} \boldsymbol{\mathscr{B}} \cdot d\boldsymbol{\mathcal{S}} = 0 \qquad \text{(Gauss's Law for Magnetism)} \tag{4}$$

Other Relevant Equations

- Continuity Equation: $\nabla \cdot \mathbf{J} + \partial_t \rho_v = 0$
- Relationship between $\mathbf{E}$, $\mathbf{D}$: $\mathbf{D} = \epsilon \mathbf{E}$
- Relationship between $\mathbf{H}$, $\mathbf{B}$: $\mathbf{B} = \mu \mathbf{H}$

Boundary Conditions

Discuss the boundary conditions for $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ at interfaces between different media.

Problem 1.3

The electric flux density inside a cube is given by:

(a) $\vec{D} = \hat{a}_x(3 + x)$

(b) $\vec{D} = \hat{a}_y(4 + y^2)$

Find the total electric charge enclosed inside the cubical volume when the cube is in the first octant with three edges coincident with the x, y, z axes and one corner at the origin. Each side of the cube is 1 m long.
Ok, this one we can do by using the following expressions from the text [1]:

$$\oiint_s \mathscr{D} ds = \iiint_v \mathscr{q}_{ev} dv = \mathscr{Q}_e \tag{5}$$

$$\mathscr{q}_{ev} = \Delta \cdot \mathscr{D} \tag{6}$$

We apply expression 6 then plug into expression 5 to get the total electric charge.

$$\iiint_V \mathscr{q}_{ev} = \iiint_V \Delta \cdot (\hat{a}_x(3 + x))dV = \iiint_V \partial_x(\hat{a}_x(3 + x))dV = \iiint_0^1 1dV = 1$$

$$\iiint_V \mathscr{q}_{ev} = \iiint_V \Delta \cdot (\hat{a}_y(4 + y^2))dV = \iiint_V \partial_y(\hat{a}_y(4 + y^2))dV = \iiint_0^1 2y\ dV$$

$$\iiint_0^1 2y\ dV = \iint_0^1 \int_0^1 2ydy\ ds = \iint_0^1 y^2|_0^1 ds = \iint_0^1 (1 - 0)\ ds = 1$$

Problem 1.4

An infinite planar interface between media, as shown in the figure, is formed by having air (medium #1) on the left of the interface and lossless polystyrene (medium #2) to the right of the interface. An electric surface charge density $\sigma_{es} = 0.2\,\mathrm{C/m}^2$ exists along the entire interface. The static electric flux density inside the polystyrene is given by $\vec{D}_2 = 6\hat{a}_x + 3\hat{a}_z\,\mathrm{C/m}^2$. Determine the corresponding vector:

(a) Electric field intensity inside the polystyrene.
we can find this using the following expression for the Electric field

intensity vector:

$$
\begin{aligned}
\epsilon_{poly} &= 2.56\epsilon_0 \\
\mathbf{D} &= \epsilon\mathbf{E} \\
\mathbf{E} &= \epsilon^{-1}\mathbf{D} = 0.390625\epsilon_0^{-1} \cdot \mathbf{D} = \epsilon_0^{-1}(2.34 \cdot a_x \ + \ 1.17 \cdot a_z)
\end{aligned}
$$

(b) Electric polarization vector inside the polystyrene.
we can find this using the following expression for the polarization vector:

$$
\begin{gathered}
\mathbf{P}_e = \epsilon_0\chi_e\mathbf{E} \\
\chi_e = \epsilon_r - 1 = 2.56 - 1 = 1.56 \\
\mathbf{P}_e = 1.56\cancelto{1}{\epsilon_0\epsilon_0^{-1}}(2.34 \cdot a_x \ + \ 1.17 \cdot a_z) = 3.6504 \cdot a_x \ + \ 1.8252 \cdot a_z
\end{gathered}
$$

(c) Electric flux density inside the air medium.
By the continuity of tangential components at the boundary:
$\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0$
we apply the following.

$$
\begin{aligned}
\mathbf{E}_{air} &= \mathbf{E}_{Poly} \\
\mathbf{D}_{air} &= \epsilon_{air}\mathbf{E}_{air} \\
\therefore \mathbf{D}_{air}\epsilon_{air}^{-1} &= \mathbf{D}_{Poly}\epsilon_{Poly}^{-1} \\
\mathbf{D}_{air} &= \frac{\epsilon_{air}}{\epsilon_{Poly}}\mathbf{D}_{Poly} = \frac{\cancel{\epsilon_0} \cdot 1}{\cancel{\epsilon_0} \cdot 2.56} \cdot \mathbf{D}_{Poly} \\
&= \frac{6}{2.56}a_x = 2.34 \cdot a_x
\end{aligned}
$$

The normal components at the boundary are not continuous and therefore we must do more.
The normal is $\hat{a}_z$
$\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = q_{es}$

$$
\begin{aligned}
\hat{a}_z \cdot (\mathbf{D}_2 - \mathbf{D}_1) &= 0.2 \\
\mathbf{D}_{1z} = \mathbf{D}_{2z} - 0.2 &= 3 - .2 = 2.8
\end{aligned}
$$

(d) Electric field intensity inside the air medium.
Again using the expression $\mathbf{D} = \epsilon \mathbf{E}$

$$\begin{aligned}
\mathbf{D}_{air} &= \epsilon_{air} \mathbf{E}_{air} \\
\mathbf{E}_{air} &= \epsilon_{air}^{-1} \mathbf{D}_{air} \\
&= \epsilon_0^{-1} (2.34 \cdot a_x + 2.8 \cdot a_z)
\end{aligned}$$

(e) Electric polarization vector inside the air medium.
Again using $\mathbf{P}_e = \epsilon_0 \chi_e \mathbf{E}$

$$\begin{aligned}
\mathbf{P}_{e_{air}} &= \epsilon_0 \chi_{e_{air}} \mathbf{E}_{air} \\
&= \chi_{e_{air}} \cancelto{1}{\epsilon_0 \epsilon_0^{-1}} (2.34 \cdot a_x + 2.8 \cdot a_z) \\
\chi_{e_{air}} &= \epsilon_{r_{air}} - 1 = 1 - 1 = 0 \\
\therefore \mathbf{P}_{e_{air}} &= \mathbf{0}
\end{aligned}$$

Leave your answers in terms of ϵ_0, μ_0.

Problem 1.13

The instantaneous magnetic flux density in free space is given by:

$$\boldsymbol{\mathscr{B}} = \hat{a}_x B_x \cos(2y) \sin(\omega t - \pi z) + \hat{a}_y B_y \cos(2x) \cos(\omega t - \pi z)$$

where B_x and B_y are constants. Assuming there are no sources at the observation points x, y, determine the electric displacement current density.
We can obtain the electric displacement current $\boldsymbol{J}$ by using the expression

Problem 1.20

The instantaneous electric field inside a conducting rectangular pipe (waveguide) of width a is given by:

$$\boldsymbol{\mathscr{E}} = \hat{a}_y E_0 \sin\left(\frac{\pi x}{a}\right) \cos(\omega t - \beta_z z)$$

where β_z is the waveguide's phase constant. Assuming there are no sources within the free-space-filled pipe determine the:

(a) Corresponding instantaneous magnetic field components inside the conducting pipe.

(b) Phase constant β_z.

(c) The height of the waveguide is b.

Problem 2.18

The time-varying electric field inside a lossless dielectric material of polystyrene, of infinite dimensions and with a relative permittivity (dielectric constant) of 2.56, is given by:

$$\vec{\mathscr{E}} = \hat{a}_x 10^{-3} \sin(2\pi \times 10^7 t)\,\mathrm{V/m}$$

Determine the corresponding:

(a) Electric susceptibility of the dielectric material.

(b) Time-harmonic electric flux density vector.

(c) Time-harmonic electric polarization vector.

(d) Time-harmonic displacement current density vector.

(e) Time-harmonic polarization current density vector defined as the partial derivative of the corresponding electric polarization vector.

Leave your answers in terms of ϵ_0, μ_0.

Problem 2.25

Aluminum has a static conductivity of about $\sigma = 3.96 \times 10^7\,\mathrm{S/m}$ and an electron mobility of $\mu_e = 2.2 \times 10^{-3}\,\mathrm{m^2/(V\text{-}s)}$. Assuming that an electric field of $\vec{E} = \hat{a}_x 2\,\mathrm{V/m}$ is applied perpendicularly to the square area of an aluminum wafer with cross-sectional area of about $10\,\mathrm{cm}^2$, find the:

(a) Electron charge density q_{ve}.

(b) Electron drift velocity v_e.

(c) Electric current density J.

(d) Electric current flowing through the square cross section of the wafer.

(e) Electron density N_e.

Leave your answers in terms of ϵ_0, μ_0.

References

[1] Constantine A. Balanis. *advanced engineering electromagnetics*, chapter 1, page 2–3. John Wiley & Sons, 2024.