ECE 6310 - Advanced Electromagnetic Fields: Homework Set #1

Miguel Gomez

2024-01-22

1 Preliminaries

In this document, we use standard notation for electromagnetic theory. Key equations and concepts are summarized below:

Vector Notation

- $\boldsymbol{\mathscr{E}}$: Electric field intensity
- $\boldsymbol{\mathscr{H}}$: Magnetic field intensity
- $\boldsymbol{\mathscr{D}}$: Electric flux density
- $\boldsymbol{\mathscr{B}}$: Magnetic flux density
- $\boldsymbol{\mathscr{J}}$: Current density
- ρ_v: Volume charge density

Differential Operators

- $\nabla \cdot$: Divergence of a vector field
- $\nabla \times$: Curl of a vector field
- ∇ : Gradient of a scalar field
- ∂_i : Partial derivative with respect to the independent basis element i

Maxwell's Equations

In integral form, Maxwell's equations are given by:

$$\oint_{\partial V} \boldsymbol{\mathscr{E}} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_V \boldsymbol{\mathscr{B}} \cdot d\boldsymbol{\mathcal{S}} \qquad \text{(Faraday's Law of Induction)} \tag{1}$$

$$\oint_{\partial V} \boldsymbol{\mathscr{H}} \cdot d\boldsymbol{\ell} = \int_V \boldsymbol{\mathscr{J}} \cdot d\boldsymbol{\mathcal{S}} + \frac{d}{dt}\int_V \boldsymbol{\mathscr{D}} \cdot d\boldsymbol{\mathcal{S}} \qquad \text{(Ampère's Circuital Law)} \tag{2}$$

$$\oiint_{\partial V} \boldsymbol{\mathscr{D}} \cdot d\boldsymbol{\mathcal{S}} = \int_V \rho_v dV \qquad \text{(Gauss's Law for Electricity)} \tag{3}$$

$$\oiint_{\partial V} \boldsymbol{\mathscr{B}} \cdot d\boldsymbol{\mathcal{S}} = 0 \qquad \text{(Gauss's Law for Magnetism)} \tag{4}$$

Other Relevant Equations

- Continuity Equation: $\nabla \cdot \boldsymbol{\mathscr{J}} + \partial_t \rho_v = 0$
- Relationship between $\boldsymbol{\mathscr{E}}$, $\boldsymbol{\mathscr{D}}$: $\boldsymbol{\mathscr{D}} = \epsilon \boldsymbol{\mathscr{E}}$
- Relationship between $\boldsymbol{\mathscr{H}}$, $\boldsymbol{\mathscr{B}}$: $\boldsymbol{\mathscr{B}} = \mu \boldsymbol{\mathscr{H}}$

Boundary Conditions

Discuss the boundary conditions for $\boldsymbol{\mathscr{E}}$, $\boldsymbol{\mathscr{H}}$, $\boldsymbol{\mathscr{D}}$, and $\boldsymbol{\mathscr{B}}$ at interfaces between different media.

Problem 1.3

The electric flux density inside a cube is given by:

(a) $\vec{D} = \hat{a}_x(3 + x)$

(b) $\vec{D} = \hat{a}_y(4 + y^2)$

Find the total electric charge enclosed inside the cubical volume when the cube is in the first octant with three edges coincident with the x, y, z axes and one corner at the origin. Each side of the cube is 1 m long.
Ok, this one we can do by using the following expressions from the text [1]:

$$\oiint_s \mathscr{D} ds = \iiint_v q_{ev} dv = \mathscr{Q}_e \tag{5}$$

$$q_{ev} = \Delta \cdot \mathscr{D} \tag{6}$$

We apply expression 6 then plug into expression 5 to get the total electric charge.

$$\iiint_V q_{ev} = \iiint_V \Delta \cdot (\hat{a}_x(3+x))dV = \iiint_V \partial_x(\hat{a}_x(3+x))dV = \iiint_0^1 1dV = 1$$

$$\iiint_V q_{ev} = \iiint_V \Delta \cdot (\hat{a}_y(4+y^2))dV = \iiint_V \partial_y(\hat{a}_y(4+y^2))dV = \iiint_0^1 2y\ dV$$

$$\iiint_0^1 2y\ dV = \iint_0^1 \int_0^1 2ydy\ ds = \iint_0^1 y^2|_0^1 ds = \iint_0^1 (1-0)\ ds = 1$$

Problem 1.4

An infinite planar interface between media, as shown in the figure, is formed by having air (medium #1) on the left of the interface and lossless polystyrene (medium #2) to the right of the interface. An electric surface charge density $\sigma_{es} = 0.2\,\mathrm{C/m^2}$ exists along the entire interface. The static electric flux density inside the polystyrene is given by $\vec{D}_2 = 6\hat{a}_x + 3\hat{a}_z\,\mathrm{C/m^2}$. Determine the corresponding vector:

(a) Electric field intensity inside the polystyrene.
we can find this using the following expression for the Electric field intensity vector:

$$\begin{aligned}\epsilon_{poly} &= 2.56\epsilon_0\\ \mathscr{D} &= \epsilon\mathscr{E}\\ \mathscr{E} &= \epsilon^{-1}\mathscr{D} = 0.390625\epsilon_0^{-1}\cdot\mathscr{D} = \epsilon_0^{-1}(2.34\cdot a_x\ +\ 1.17\cdot a_z)\end{aligned}$$

(b) Electric polarization vector inside the polystyrene.
we can find this using the following expression for the polarization vector:

$$\boldsymbol{\mathscr{P}}_e = \epsilon_0 \chi_e \boldsymbol{\mathscr{E}}$$
$$\chi_e = \epsilon_r - 1 = 2.56 - 1 = 1.56$$
$$\boldsymbol{\mathscr{P}}_e = 1.56\overset{1}{\cancel{\epsilon_0\epsilon_0^{-1}}}(2.34 \cdot a_x \ + \ 1.17 \cdot a_z) = 3.6504 \cdot a_x \ + \ 1.8252 \cdot a_z$$

(c) Electric flux density inside the air medium.
By the continuity of tangential components at the boundary:
$\hat{\mathbf{n}} \times (\boldsymbol{\mathscr{E}}_2 - \boldsymbol{\mathscr{E}}_1) = 0$
we apply the following.

$$\begin{aligned}
\boldsymbol{\mathscr{E}}_{air} &= \boldsymbol{\mathscr{E}}_{Poly} \\
\boldsymbol{\mathscr{D}}_{air} &= \epsilon_{air} \boldsymbol{\mathscr{E}}_{air} \\
\therefore \boldsymbol{\mathscr{D}}_{air} \epsilon_{air}^{-1} &= \boldsymbol{\mathscr{D}}_{Poly} \epsilon_{Poly}^{-1} \\
\boldsymbol{\mathscr{D}}_{air} &= \frac{\epsilon_{air}}{\epsilon_{Poly}} \boldsymbol{\mathscr{D}}_{Poly} = \frac{\cancel{\epsilon_0} \cdot 1}{\cancel{\epsilon_0} \cdot 2.56} \cdot \boldsymbol{\mathscr{D}}_{Poly} \\
&= \frac{6}{2.56} a_x = 2.34 \cdot a_x
\end{aligned}$$

The normal components at the boundary are not continuous and therefore we must do more.
The normal is $\hat{a}_z$
$\hat{\mathbf{n}} \cdot (\boldsymbol{\mathscr{D}}_2 - \boldsymbol{\mathscr{D}}_1) = q_{es}$

$$\hat{a}_z \cdot (\boldsymbol{\mathscr{D}}_2 - \boldsymbol{\mathscr{D}}_1) = 0.2$$
$$\boldsymbol{\mathscr{D}}_{1z} = \boldsymbol{\mathscr{D}}_{2z} - 0.2 = 3 - .2 = 2.8$$

(d) Electric field intensity inside the air medium.
Again using the expression $\boldsymbol{\mathscr{D}} = \epsilon \boldsymbol{\mathscr{E}}$

$$\begin{aligned}
\boldsymbol{\mathscr{D}}_{air} &= \epsilon_{air} \boldsymbol{\mathscr{E}}_{air} \\
\boldsymbol{\mathscr{E}}_{air} &= \epsilon_{air}^{-1} \boldsymbol{\mathscr{D}}_{air} \\
&= \epsilon_0^{-1}(2.34 \cdot a_x + 2.8 \cdot a_z)
\end{aligned}$$

(e) Electric polarization vector inside the air medium.
Again using $\boldsymbol{\mathscr{P}}_e = \epsilon_0 \chi_e \boldsymbol{\mathscr{E}}$

$$
\begin{aligned}
\boldsymbol{\mathscr{P}}_{e_{air}} &= \epsilon_0 \chi_{e_{air}} \boldsymbol{\mathscr{E}}_{air} \\
&= \chi_{e_{air}} \cancelto{1}{\epsilon_0 \epsilon_0^{-1}} (2.34 \cdot a_x + 2.8 \cdot a_z) \\
\chi_{e_{air}} &= \epsilon_{r_{air}} - 1 = 1 - 1 = 0 \\
\therefore \boldsymbol{\mathscr{P}}_{e_{air}} &= \mathbf{0}
\end{aligned}
$$

Leave your answers in terms of ϵ_0, μ_0.

Problem 1.13

The instantaneous magnetic flux density in free space is given by:

$$\boldsymbol{\mathscr{B}} = \hat{a}_x B_x \cos(2y) \sin(\omega t - \pi z) + \hat{a}_y B_y \cos(2x) \cos(\omega t - \pi z)$$

where B_x and B_y are constants. Assuming there are no sources at the observation points x, y, determine the electric displacement current density.
We can obtain the electric displacement current $\mathscr{J}_d$ by using the following expressions:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_i + \boldsymbol{J}_c + \boldsymbol{J}_d \tag{7}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{8}$$

$$\boldsymbol{J}_d = \frac{\partial \boldsymbol{D}}{\partial t} \tag{9}$$

We take $\boldsymbol{J}_i + \boldsymbol{J}_c$ to be 0 since we have no sources at the observation points. Using expression 7 and solving expression 8 for $\boldsymbol{H}$:

$$
\begin{aligned}
\nabla \times \boldsymbol{H} &= \boldsymbol{J}_d \\
\boldsymbol{J}_d &= \nabla \times \mu^{-1}\boldsymbol{B} \\
\boldsymbol{J}_d &= \mu^{-1}\nabla \times \boldsymbol{B} \\
&= \mu^{-1}\left[\hat{a}_x\left(\frac{\partial \boldsymbol{B}_z}{\partial y} - \frac{\partial \boldsymbol{B}_y}{\partial z}\right) - \hat{a}_y\left(\frac{\partial \boldsymbol{B}_z}{\partial x} - \frac{\partial \boldsymbol{B}_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial \boldsymbol{B}_y}{\partial x} - \frac{\partial \boldsymbol{B}_x}{\partial y}\right)\right] \\
\boldsymbol{B}_z &= 0 \\
&= \mu^{-1}\left[\hat{a}_x\left(0 - \frac{\partial \boldsymbol{B}_y}{\partial z}\right) - \hat{a}_y\left(0 - \frac{\partial \boldsymbol{B}_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial \boldsymbol{B}_y}{\partial x} - \frac{\partial \boldsymbol{B}_x}{\partial y}\right)\right] \\
&= \mu^{-1}\left[-\hat{a}_x\frac{\partial \boldsymbol{B}_y}{\partial z} + \hat{a}_y\frac{\partial \boldsymbol{B}_x}{\partial z} + \hat{a}_z\left(\frac{\partial \boldsymbol{B}_y}{\partial x} - \frac{\partial \boldsymbol{B}_x}{\partial y}\right)\right] \\
\frac{\partial \boldsymbol{B}_x}{\partial z} &= -\pi B_x \cos(2y)\cos(\omega t - \pi z) \\
\frac{\partial \boldsymbol{B}_y}{\partial z} &= \pi B_y \cos(2x)\sin(\omega t - \pi z) \\
\frac{\partial \boldsymbol{B}_y}{\partial x} &= -2\sin(2x) cos(\omega t - \pi z) \\
\frac{\partial \boldsymbol{B}_x}{\partial y} &= -2\sin(2y) sin(\omega t - \pi z) \\
&= \mu^{-1}\left[\hat{a}_x(-\pi B_y \cos(2x)\sin(\omega t - \pi z))\right. \\
&+ \hat{a}_y(-\pi B_x \cos(2y)\cos(\omega t - \pi z)) \\
&\left.\hat{a}_z 2(\sin(2y) sin(\omega t - \pi z) - \sin(2x) cos(\omega t - \pi z))\right]
\end{aligned}
$$

Problem 1.20

The instantaneous electric field inside a conducting rectangular pipe (waveguide) of width a is given by:

$$
\boldsymbol{\mathscr{E}} = \hat{a}_y E_0 \sin\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta_z z)
$$

where β_z is the waveguide's phase constant. Assuming there are no sources within the free-space-filled pipe determine the:

(a) Corresponding instantaneous magnetic field components inside the conducting pipe. ($\boldsymbol{\mathscr{H}}$)

(b) Phase constant β_z.

The height of the waveguide is b. This can be found by relating $\boldsymbol{\mathscr{E}}$ to $\boldsymbol{\mathscr{H}}$ and we know how to get there using the instantaneous forms of these:

$$\boldsymbol{\mathscr{E}} = -j\omega\mu\nabla \times \boldsymbol{\mathscr{H}}$$
$$\boldsymbol{\mathscr{H}} = \frac{j}{\omega\mu}\nabla \times \boldsymbol{\mathscr{E}}$$

$$\frac{j}{\omega\mu}\nabla \times \boldsymbol{\mathscr{E}} = \frac{j}{\omega\mu}\left[\hat{a}_x\left(\frac{\partial \boldsymbol{E}_z}{\partial y} - \frac{\partial \boldsymbol{E}_y}{\partial z}\right) - \hat{a}_y\left(\frac{\partial \boldsymbol{E}_z}{\partial x} - \frac{\partial \boldsymbol{E}_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial \boldsymbol{E}_y}{\partial x} - \frac{\partial \boldsymbol{E}_x}{\partial y}\right)\right]$$

$$\frac{j}{\omega\mu}\nabla \times \boldsymbol{\mathscr{E}} = \frac{j}{\omega\mu}\left[\hat{a}_x\left(0 - \frac{\partial \boldsymbol{E}_y}{\partial z}\right) - \hat{a}_y\left(0 - 0\right) + \hat{a}_z\left(\frac{\partial \boldsymbol{E}_y}{\partial x} - 0\right)\right]$$
$$\frac{j}{\omega\mu}\nabla \times \boldsymbol{\mathscr{E}} = -\frac{j}{\omega\mu}\left[\hat{a}_x\frac{\partial \boldsymbol{E}_y}{\partial z} - \hat{a}_z\frac{\partial \boldsymbol{E}_y}{\partial x}\right]$$
$$\boldsymbol{\mathscr{E}} = \hat{a}_y E_0 \sin\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta_z z)$$
$$\frac{\partial \boldsymbol{E}_y}{\partial x} = \left(\frac{\pi}{a}\right) E_0 \cos\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta_z z)$$
$$\frac{\partial \boldsymbol{E}_y}{\partial z} = \left(\beta_z\right) E_0 \sin\left(\frac{\pi x}{a}\right)\sin(\omega t - \beta_z z)$$
$$\boldsymbol{\mathscr{H}} = -\frac{j}{\omega\mu}\left[\hat{a}_x\left(\beta_z\right) E_0 \sin\left(\frac{\pi x}{a}\right)\sin(\omega t - \beta_z z) - \hat{a}_z\left(\frac{\pi}{a}\right) E_0 \cos\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta_z z)\right]$$

Solving for β_z, we can use first expression for the $\nabla \times \boldsymbol{\mathscr{H}}$:

$$\nabla \times \boldsymbol{\mathscr{H}} = (-j\omega\mu)^{-1}\boldsymbol{\mathscr{E}}$$
$$\nabla \times \boldsymbol{\mathscr{H}} = \left[\hat{a}_x\left(\frac{\partial \boldsymbol{H}_z}{\partial y} - \frac{\partial \boldsymbol{H}_y}{\partial z}\right) - \hat{a}_y\left(\frac{\partial \boldsymbol{H}_z}{\partial x} - \frac{\partial \boldsymbol{H}_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial \boldsymbol{H}_y}{\partial x} - \frac{\partial \boldsymbol{H}_x}{\partial y}\right)\right]$$

The expression for $\boldsymbol{\mathscr{H}}$ we found does not have any dependence on y, and only has x and z components:

$$\begin{aligned}
\nabla \times \boldsymbol{\mathscr{H}} &= \left[\hat{a}_x (0-0) - \hat{a}_y \left(\frac{\partial \boldsymbol{H}_z}{\partial x} - \frac{\partial \boldsymbol{H}_x}{\partial z}\right) + \hat{a}_z (0-0)\right] \\
&= -\hat{a}_y \left(\frac{\partial \boldsymbol{H}_z}{\partial x} - \frac{\partial \boldsymbol{H}_x}{\partial z}\right) \\
&= -\frac{j}{\omega\mu}\hat{a}_y \left(\left(\frac{\pi}{a}\right)^2 E_0 \sin\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta_z z) - \beta_z^2 E_0 \sin\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta_z z)\right) \\
&= \hat{a}_y \frac{j}{\omega\mu}\left(\beta_z^2 - \left(\frac{\pi}{a}\right)^2\right) E_0 \sin\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta_z z) = \hat{a}_y \frac{j}{\omega\mu}\left(\beta_z^2 - \left(\frac{\pi}{a}\right)^2\right)\boldsymbol{\mathscr{E}} \\
\therefore \omega\mu &= \frac{1}{\omega\mu}\left(\beta_z^2 - \left(\frac{\pi}{a}\right)^2\right) \\
\beta_z^2 &= (\omega\mu)^2 + \left(\frac{\pi}{a}\right)^2 \\
\beta_z &= \pm\sqrt{(\omega\mu)^2 + \left(\frac{\pi}{a}\right)^2}
\end{aligned}$$

Problem 2.18

The time-varying electric field inside a lossless dielectric material of polystyrene, of infinite dimensions and with a relative permittivity (dielectric constant) of 2.56, is given by:

$$\vec{\mathscr{E}} = \hat{a}_z 10^{-3} \sin(2\pi \times 10^7 t)\ \mathrm{V/m}$$

Determine the corresponding:

(a) Electric susceptibility of the dielectric material.

$$\chi_e = \epsilon_r - 1 = 1.56$$

(b) Time-harmonic electric flux density vector.

$$\boldsymbol{D} = \epsilon \boldsymbol{E} = \hat{a}_z 2.56 \times 10^{-3}\epsilon_0 \sin(2\pi \times 10^7 t)$$

(c) Time-harmonic electric polarization vector.

$$\boldsymbol{P} = \chi_e \epsilon_0 \boldsymbol{E} = \hat{a}_z 1.56 \times 10^{-3}\epsilon_0 \sin(2\pi \times 10^7 t)$$

(d) Time-harmonic displacement current density vector.

$$\begin{aligned}\boldsymbol{J}_d = \frac{\partial \boldsymbol{D}}{\partial t} &= \hat{a}_z 2.56 \times 10^{-3} \times 2\pi \times 10^7 \epsilon_0 \cos(2\pi \times 10^7 t) \\ &= \hat{a}_z 1.65 \times 10^5 \epsilon_0 \cos(2\pi \times 10^7 t)\end{aligned}$$

(e) Time-harmonic polarization current density vector defined as the partial derivative of the corresponding electric polarization vector.

$$\begin{aligned}\boldsymbol{J}_p = \frac{\partial \boldsymbol{P}}{\partial t} &= \hat{a}_z 1.56 \times 10^{-3} \times 2\pi \times 10^7 \epsilon_0 \cos(2\pi \times 10^7 t) \\ &= \hat{a}_z 9.80 \times 10^4 \epsilon_0 \cos(2\pi \times 10^7 t)\end{aligned}$$

Leave your answers in terms of ϵ_0, μ_0.

Problem 2.25

Aluminum has a static conductivity of about $\sigma = 3.96 \times 10^7\,\mathrm{S/m}$ and an electron mobility of $\mu_e = 2.2 \times 10^{-3}\,\mathrm{m^2/(V\text{-}s)}$. Assuming that an electric field of $\vec{E} = \hat{a}_x 2\,\mathrm{V/m}$ is applied perpendicularly to the square area of an aluminum wafer with cross-sectional area of about $10\,\mathrm{cm}^2$, find the:
Here are a few things that will be useful in this set of problems:

$$\begin{aligned}v_e &= -\mu_e \boldsymbol{E} \\ \boldsymbol{J} &= -q_{ve}\mu_e \boldsymbol{E} \\ \sigma_s &= -q_{ve}\mu_e \\ q_{ve} &= N_e q_e\end{aligned}$$

(a) Electron charge density q_{ve}.

$$\begin{aligned}\sigma_s &= -q_{ve}\mu_e \\ q_{ve} &= -\sigma_s \mu_e^{-1} = -3.96 \times 10^7 (2.2 \times 10^{-3})^{-1} = -\frac{3.96}{2.2} \times 10^{10} \left[\frac{C}{m^3}\right]\end{aligned}$$

(b) Electron drift velocity v_e.

$$\begin{aligned}v_e &= -\mu_e \boldsymbol{E} \\ v_e &= -2.2 \times 10^{-3} \hat{a}_x 2 = -\hat{a}_x 4.4 \times 10^{-3} \left[\frac{m}{s}\right]\end{aligned}$$

(c) Electric current density J.

$$\begin{aligned}
\boldsymbol{J} &= -q_{ve}\mu_e\boldsymbol{E} \\
&= --\frac{3.96}{2.2} \times 10^{10} \times 2.2 \times 10^{-3}\hat{a}_x \times 2 \\
&= \frac{3.96}{\cancelto{1}{2.2}} \times 10^{\cancelto{7}{10}} \times \cancelto{1}{2.2 \times 10^{-3}}\hat{a}_x \times 2 \\
&= 2 \times 3.96 \times 10^7\hat{a}_x = \hat{a}_x 7.92 \times 10^7 \left[\frac{A}{m^2}\right]
\end{aligned}$$

(d) Electric current flowing through the square cross section of the wafer.

$$\begin{aligned}
I &= \iint_0^A \boldsymbol{J} dA = \boldsymbol{J} \iint_0^A 1 dA = \boldsymbol{J} A = 10cm^2 \times 7.92 \times 10^7\hat{a}_x \\
&= 10^{-3} \times 7.92 \times 10^7\hat{a}_x = \hat{a}_x 7.92 \times 10^4 [A]
\end{aligned}$$

(e) Electron density N_e.

$$\begin{aligned}
q_{ve} &= N_e q_e \\
q_e &= 1.6 \times 10^{-19} \\
\therefore N_e &= \left|\frac{q_{ve}}{q_e}\right| = \frac{3.96}{2.2 \cdot 1.6} \times 10^{10} \times 10^{19} = 1.125 \times 10^{29} \left[\frac{e}{m^2}\right]
\end{aligned}$$

Leave your answers in terms of ϵ_0, μ_0.

References

[1] Constantine A. Balanis. *advanced engineering electromagnetics*, chapter 1, page 2–3. John Wiley & Sons, 2024.