Contents

1 - 6.19 **4**

2 - 6.20 **5**

3 - 6.25 **5**

4 - 6.26 **6**

5 - 6.38 **6**

6 - 7.3 **7**

7 - 7.15 **8**

8 - 7.37 **8**

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$$\Big[\hat{a}_{\mathsf{x}\,(\frac{\{}{}\partial 1_{z}\}\{\partial\, y\}\ -\ \{\frac{}{\partial}1_{y}\}\{\partial\, z\} - \hat{a}_{\mathsf{y}\,(\frac{\{}{}\partial 1_{z}\}\{\partial\, x\}\ -\ \{\frac{}{\partial}1_{x}\}\{\partial\, z\}\ + \hat{a}_{\mathsf{z}\,(\frac{\{}{}\partial 1_{y}\}\{\partial\, x\}\ -\ \{\frac{}{\partial}1_{x}\}\{\partial\, y\}\}}}}$$

ECE 6310 - Advanced Electromagnetic Fields: Homework Set #3

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February 21, 2024

Preliminaries

In this document, we use standard notation for electromagnetic theory. Key equations and concepts are summarized below:

Vector Notation

- $\mathcal{E}$: Electric field intensity
- $\mathcal{H}$: Magnetic field intensity
- $\mathcal{D}$: Electric flux density
- $\mathcal{B}$: Magnetic flux density
- $\mathcal{J}$: Current density
- ρ_v: Volume charge density

Differential Operators

- $\nabla\cdot$: Divergence of a vector field
- $\nabla\times$: Curl of a vector field
- ∇ : Gradient of a scalar field
- ∂_i : Partial derivative with respect to the independent basis element i

Maxwell's Equations

In integral form, Maxwell's equations are given by:

$$\oint_{\partial V} \mathcal{E} \cdot d\updownarrow = -\frac{d}{dt}\int_V \mathcal{B} \cdot d\mathcal{S} \qquad \text{(Faraday's Law of Induction)} \qquad (1)$$

$$\oint_{\partial V} \mathcal{H} \cdot d\updownarrow = \int_V \mathcal{J} \cdot d\mathcal{S} + \frac{d}{dt}\int_V \mathcal{D} \cdot d\mathcal{S} \qquad \text{(Ampère's Circuital Law)} \qquad (2)$$

$${}_{\partial V}\mathcal{D} \cdot d\mathcal{S} = \int_V \rho_v dV \qquad \text{(Gauss's Law for Electricity)} \qquad (3)$$

$${}_{\partial V}\mathcal{B} \cdot d\mathcal{S} = 0 \qquad \text{(Gauss's Law for Magnetism)} \qquad (4)$$

Other Relevant Equations

- Continuity Equation: $\nabla \cdot \mathcal{J} + \partial_t \rho_v = 0$
- Relationship between $\mathcal{E}$, $\mathcal{D}$: $\mathcal{D} = \epsilon\mathcal{E}$
- Relationship between $\mathcal{H}$, $\mathcal{B}$: $\mathcal{B} = \mu\mathcal{H}$

Boundary Conditions

Discuss the boundary conditions for $\mathcal{E}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{B}$ at interfaces between different media.

1 - 6.19

Show that for observations made at very large distances ($\beta r \gg 1$) the electric and magnetic fields of Example 6-3 reduce to the following:

$$
\begin{aligned}
E_\theta &= j\eta \frac{\beta I_e \ell e^{-j\beta r}}{4\pi r} \sin(\theta) \\
H_\phi &\approx \frac{E_{theta}}{\eta} \\
E_r &\approx 0 \\
E_\phi &= H_r = H_\theta = 0
\end{aligned}
$$

All of the above can be found bf considering a typical hertzian dipole. The example shows that we have an infinitesimal dipole with the length much smaller than a wavelength λ. Therefore, we can take the expressions as shown with all parts E and H getting a factor of $e^{-j\beta R}$ for the decay, and will have another dependent on the distance from the location of the dipole. We can ignore the factors present in all the values and look to only the factors containin r^{-1}.

$$
\begin{aligned}
E_r &= A \cdot \left(\frac{1}{r^2}\right)\left(1 + \frac{1}{j\beta r}\right) \\
&= \left(\frac{1}{r^2} + \frac{1}{j\beta r^3}\right)
\end{aligned}
$$

With r being much greater than 1, we can approximate that $E_r \to 0$ as the distance gets larger.

$$H_r = H_\theta = 0$$

These follow from the reduction of the vector potential equation after converting to spherical. With $A_{phi} = 0$, we do not end up with components in the H direction for these. Then, H_{phi} contains the same factors discussed above and we ignore the ones with higher order r's in the denominator. Leaving us with the same constant multiplied across all terms, but divided by η, which substituting E_{theta} for that factor gives the expected $H_{phi} \to \frac{E_{theta}}{\eta}$

2 - 6.20

For 6.19, show that

- the time average power density is:

$$\nabla\times\mathbf{A} = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{a}_\theta}{r}\left[\frac{\partial rA_r}{\partial\phi} - \frac{\partial A_\phi}{\partial r}\right]\begin{pmatrix} \\ \hat{a}_\phi\end{pmatrix}\times\begin{pmatrix} \\ \frac{1}{r}\frac{\partial A_\theta}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(rA_\phi)\end{pmatrix}$$

- Radiation Intensity

$$\mathbf{U} = r^2 S_{av} = \frac{\eta}{8}\left|\frac{\mathbf{I}_o\ell}{\lambda}\right|^2$$

- Radiated Power

$$\mathbf{P}_{rad} = \int_0^{2\pi}\int_0^{\pi}(U)(\theta,\phi)\times\sin(\theta)d\phi = \eta\left(\frac{\pi}{3}\right)\left|\frac{\mathbf{I}_o\ell}{\lambda}\right|^2$$

- Directivity

$$\mathbf{D}_o = \frac{4\pi\mathbf{U}_{max}(\theta,\phi)}{\mathbf{P}_{rad}}$$

- Radiation Resistance

$$\mathbf{R}_{rad} = \frac{2\mathbf{P}_{rad}}{|\mathbf{I}_0|^2} = 80\pi^2\left(\frac{\ell}{\lambda}\right)^2$$

3 - 6.25

The current distribution on a very thin wire dipole antenna of overall length ℓ is given by:

$$I_e = \begin{cases} \hat{a}zI_0\sin[\beta\left(\frac{\ell}{2} - z'\right)] & \text{if } 0 \le z' \le \frac{\ell}{2} \\ \hat{a}zI_0\sin[\beta\left(\frac{\ell}{2} + z'\right)] & \text{if } -\frac{\ell}{2} \le z' \le 0\end{cases}$$

where I0 is a constant. Representing the distance R of (6-112) by the far-field approximations of (6-112a) through (6-112b), derive the far-zone electric and magnetic fields radiated by the dipole using (6-97a) and the far-field formulations of Section 6.7

4 - 6.26

Show that the radiated far-zone electric and magnetic fields derived in Problem 6.25 reduce for a half-wavelength dipole ($\ell = \frac{\lambda}{2}$) to:

$$E_\theta \approx j\eta \frac{I_0 e^{-j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right]$$

$$H_\phi \approx \frac{E_\theta}{\eta}$$

$$E_r \approx E_\phi \approx H_r \approx H_\theta \approx 0$$

*Note: Verify your solutions for the far field E_{theta} for a half-wavelength dipole (6.26) in HFSS; design the antenna to be resonant at $1GHz$.

5 - 6.38

A coaxial line of inner and outer radii a and b, respectively, is mounted on an infinite conducting ground plane. Assuming that the electric field over the aperture of the coax is:

$$E_a = -\hat{a}_\rho \frac{V}{\varepsilon \ln(\frac{b}{a})\rho\,'}$$

where $a \leq \rho\,' \leq b$

where V is the applied voltage and is the permittivity of medium in the coax, find the far-zone spherical electric and magnetic field components radiated by the aperture.

[scale=2,tdplot$_{m}ain_{c}oords$]
[rotate=0] [inner color=gray!80,outer color=gray!30,opacity=0.75] (0,0) ellipse (1 and 0.5*1); [white] (0,0) ellipse (0.5 and 0.3*0.5); [gray!30,opacity=0.5] (-1.5,-1.5) rectangle (1.75,1.75); [-Latex[length=3mm]] (0,0) – (2.5,0) node[anchor=south east] y; [-Latex[length=3mm]] (0,0) – (0,2.5) node[anchor=north west] z; [-Latex[length=3mm]] (0,0) – (-1.5,-1.5) node[anchor=north] x; [rotate=0] (0,0) -> node[midway,above] a (-0.5/2,.15); (0,0) -> node[midway,above] b (-1/2,.45); [anchor=south] at (0,-.5) ε; at (1.3,-1.3) $\sigma = \infty$;
[below] at (current bounding box.south) Figure P6-38;

6 - 7.3

An infinitesimal vertical magnetic dipole of length l and constant current Im is placed symmetrically about the origin and it is directed along the z axis, as shown in Figure 6-2 a. Derive expressions valid every- where, near and far field, for the:

- Electric vector potential components $(F_r, F_{theta}, F_{phi})$.
- Electric field components $(E_r, E_{theta}, E_{phi})$.
- Magnetic field components $(H_r, H_{theta}, H_{phi})$.
- Time-average power density, defined as

$$\mathbf{S} = \frac{1}{2}\text{Re}[\mathbf{E} \times \mathbf{H}^*]$$

- Radiation intensity, defined in the far field $U \approx r^2 S_{av}$
- Power Radiated, defined as

$$\mathbf{P}_{rad} = \int_0^{2\pi} \int_0^{\pi} U(\theta, \phi) \sin(\theta) d\theta d\phi$$

- Maximum directivity, defined as:

$$\mathbf{D}_0 = \frac{4\pi U(\theta, \phi)}{\mathbf{P}_{rad}}$$

- Radiation resistance, defined as:

$$\mathbf{R}_r = \frac{2\mathbf{P}_{rad}}{|I_m|^2}$$

7 - 7.15

An infinitesimal electric dipole is placed at an angle of 30 at a height h above a perfectly conducting electric ground plane. Determine the location and orientation of its image. Do this by sketching the image.

[gray!50] (-2,-2) rectangle (2,0); (-2,0) – (2,0);
at (1,-0.5) $\sigma = \infty$;
[thick, dotted] (0,1.25) – (1.5,1.25); [->,thick] (0,1) – +(30:1) node[midway, above right, xshift=.5em, yshift=.5em] $30°$; (.425,1.25) circle (1pt); [<->] (-0.5,0) – node[midway, fill=white] h (-0.5,1.25);
[below] at (0,-2.5) Figure P7-15;

8 - 7.37

For the aperture shown in Figure 6-4c and assuming it is mounted on an infinite PEC ground plane:

a) Form the most practical, exact or approximate (when necessary to solve the problem), equivalent currents $\mathbf{J}_s$ and $\mathbf{M}_s$.

b) Find the far-zone electric and magnetic fields. The electric field distribution at the aperture is given by (Eo is a constant):

$$\mathbf{E}_a = \hat{a}_y E_0$$
$$-\frac{a}{2} \leq x\,' \leq \frac{a}{2} \quad ; \quad -\frac{b}{2} \leq y\,' \leq \frac{b}{2}$$

References