

# Homework Assignment #1

ECE 6530: Digital Signal Processing  
September 2, 2023

Miguel Gomez U1318856

## Homework set #1

Due Date: Sep 1, 2023  
(100 points)

### 1

1.3 - Determine whether or not each of the following signals is periodic. In case a signal is periodic, specify its fundamental period.

a)  $x_a(t) = 3\cos(5t + \frac{\pi}{6})$

b)  $x(n) = 3\cos(5n + \frac{\pi}{6})$

c)  $x(n) = 2e^{j(\frac{n}{6} - \pi)}$

d)  $x(n) = \cos(\frac{n}{8})\cos(\frac{n\pi}{8})$

e)  $x(n) = \cos(\pi\frac{n}{2}) - \sin(\pi\frac{n}{8}) + 3\cos(\pi\frac{n}{4} + \frac{\pi}{3})$

---

Determining periodicity requires us to examine the following:

$$x_a(t) = x_a(t + T) \quad (1)$$

$$x(n) = x(n + N) \quad (2)$$

a) Consider the following expression to determine if  $x_a(t)$  is periodic:

We can expand  $x_a(t + T)$  as follows:

$$\begin{aligned} x_a(t + T) &= 3 \cos \left( 5(t + T) + \frac{\pi}{6} \right) \\ &= 3 \cos \left( 5t + 5T + \frac{\pi}{6} \right) \end{aligned}$$

We know that  $2\pi ft = 5t$ , so  $f = \frac{5}{2\pi}$  and  $T = \frac{2\pi}{5}$ .

$$\begin{aligned} x_a(t + T) &= 3 \cos \left( 5t + 5 \frac{2\pi}{5} + \frac{\pi}{6} \right) \\ &= 3 \cos \left( \cancel{2\pi} + \left( 5t + \frac{\pi}{6} \right) \right) = 3 \cos \left( 5t + \frac{\pi}{6} \right) \end{aligned}$$

The addition of  $2\pi$  is redundant since it represents a full rotation. Therefore,  $x_a(t + T) = x_a(t)$  and  $x_a(t)$  is periodic.

$$x(n) = 3\cos\left(5n + \frac{\pi}{6}\right)$$

b) Consider the expression to determine if  $x(n)$  is periodic:

We can expand  $x(n + N)$  as follows:

$$x(n + N) = 3\cos\left(5n + 5N + \frac{\pi}{6}\right)$$

For a discrete signal, the frequency is defined as  $f_0 = \frac{k}{N}$ :

$$\text{since } 2\pi f_0 n = 5n \quad f_0 = \frac{5}{2\pi} \quad T = \frac{2\pi}{5}$$

We can stop here since we know a discrete-time signal is only periodic if the frequency  $f_0$  can be expressed as a ratio of two integers. Since there is a factor of  $\pi$  in the expression, this is aperiodic.

c) Consider the following equations to determine if  $x(n)$  is periodic:

$$x(n) = 2e^{j\left(\frac{n}{6} - \pi\right)} = 2e^{-j\pi} e^{j\frac{n}{6}}$$

A phase shift not dependent on  $n$  does not change the periodicity:

$$= 2e^{j\frac{n}{6}} = 2\left(\cos\left(\frac{n}{6}\right) + j\sin\left(\frac{n}{6}\right)\right)$$

$$\text{since } 2\pi f_0 n = \frac{n}{6} \quad f_0 = \frac{1}{12\pi} \quad T = 12\pi$$

Again, we can stop here since we know a discrete-time signal is only periodic if the frequency  $f_0$  can be expressed as a ratio of two integers. Since there is a factor of  $\pi$  in the expression, this is aperiodic.

d) Consider the following equations to determine if  $x(n)$  is periodic:

$$x(n) = \cos\left(\frac{n}{8}\right) \cos\left(\frac{n\pi}{8}\right)$$

We have two frequencies to evaluate,  $f_{0_0} = \frac{n}{8}$  and  $f_{0_1} = \frac{\pi n}{8}$ :

$$\text{since } 2\pi f_{0_i} n = \left[\frac{n}{8}, \frac{n\pi}{8}\right] \quad f_{0_i} = \left[\frac{1}{16\pi}, \frac{1}{16}\right] \quad T_i = [16\pi, 16]$$

While we came up with one expression for  $f_{0_i}$  that is rational, the other one is not.  $\therefore$  the overall signal will be aperiodic.

e) Consider the following equations to determine if  $x(n)$  is periodic:

$$x(n) = \cos\left(\pi\frac{n}{2}\right) - \sin\left(\pi\frac{n}{8}\right) + 3\cos\left(\pi\frac{n}{4} + \frac{\pi}{3}\right)$$

We have three frequencies to evaluate,  $f_{0_0} = \frac{\pi n}{2}$ ,  $f_{0_1} = \frac{\pi n}{8}$ , and  $f_{0_2} = \frac{\pi n}{4}$ :

$$\text{since } 2\pi f_{0_i} n = \left[\frac{\pi n}{2}, \frac{n\pi}{8}, \frac{n\pi}{4}\right] \quad f_{0_i} = \left[\frac{1}{4}, \frac{1}{16}, \frac{1}{8}\right] \quad T_i = [4, 16, 8]$$

Since the frequencies  $f_{0_i}$  are all rational. The signal is periodic with  $f_0 = \frac{1}{16}$ .

## 2

1.7 An analog signal contains frequencies up to 10 kHz.

- What range of sampling frequencies allows exact reconstruction of this signal from its samples?
- Suppose that we sample this signal with a sampling frequency  $F_s = 8$  kHz. Examine what happens to the frequency  $F_1 = 5$  kHz.
- Repeat part (b) for a frequency  $F_2 = 9$  kHz.

a) What range of sampling frequencies allows exact reconstruction of this signal from its samples?

By  $F_s > 2F_{max}$  we can find the range of frequencies:

$$\begin{aligned} -\frac{F_s}{2} &\leq F_{max} \leq \frac{F_s}{2} \\ -\frac{F_s}{2} &\leq 10kHz \leq \frac{F_s}{2} \\ -F_s &\leq 20kHz \leq F_s \end{aligned}$$

$\therefore$  the range is up to  $|20kHz|$

b and c) Suppose that we sample this signal with a sampling frequency  $F_s = 8$  kHz. Examine what happens to the frequency  $F_1 = 5$  kHz. Repeat part (b) for a frequency  $F_2 = 9$  kHz.

Sampling the signals if they are outside the range we found, at this frequency, would give us a mapping from that frequency to one within our range.

$$F_s = [5 \ 9] kHz$$

looking at what happens in the outputted frequencies will show:

$$\begin{aligned} F_i \bmod(F_N) &\rightarrow F_i - F_N - F_N \quad \text{i.f.f. } F_i > F_N \\ \text{else } F_i \bmod(F_N) &\rightarrow F_i \quad \text{if } F_i < F_N \end{aligned}$$

This will give the mapping of the frequencies and our  $F_N = \frac{F_s}{2} = 4kHz$ :

$$F_i \bmod(F_N) = [5 \ 9] kHz$$

Both 5 and 9 kHz are above the maximum frequency we can capture,  $\therefore F_i > F_N$

$$\begin{aligned} &= [5 \ 9] kHz - 2F_N \\ &= [-3 \ -1] kHz \end{aligned}$$

$\therefore$  both frequencies are mapped to others within the range.

### 3

1.9 - An analog signal  $x_a(t) = \sin(480\pi t) + 3\sin(720\pi t)$  is sampled 600 times per second.

- Determine the Nyquist sampling rate for  $x_a(t)$ .
- Determine the folding frequency.
- What are the frequencies, in radians, in the resulting discrete time signal  $x(n)$ ?
- If  $x(n)$  is passed through an ideal D/A converter, what is the reconstructed signal  $y_a(t)$ ?

a and b) Determine the Nyquist sampling rate for  $x_a(t)$ .

---

$F_{max}$  of  $360Hz$  gives a Nyquist rate of  $2F_{max} = 720Hz$

sampling at  $600Hz$  gives a Folding Freq of  $\frac{F_s}{2} = 300Hz$

What are the frequencies, in radians, in the resulting discrete time signal  $x(n)$ ?

$$\begin{aligned} F_i &= [240 \ 360] \\ F_1 = 240 \quad F_1 < F_N &\rightarrow F_1 = 240 \\ F_2 = 360 \quad F_2 > F_N &\rightarrow F_2 = 60 - 300 = -240 \end{aligned}$$

$\therefore$  360 maps to -240

$$F_i = \pm(2\pi 240) = \pm(480\pi)$$

If  $x(n)$  is passed through an ideal D/A converter, what is the reconstructed signal  $y_a(t)$ ?

$$\begin{aligned}x_a(t) &= \sin(480\pi t) + 3\sin(720\pi t) \\y_a(t) &= \sin(480\pi t) + 3\sin(-480\pi t) \\y_a(t) &= \sin(480\pi t) - 3\sin(480\pi t) \\y_a(t) &= -2\sin(480\pi t)\end{aligned}$$


---

## 4

1.10 - A digital communication link carries binary-coded words representing samples of an input signal:

$$x_a(t) = 3\cos(600\pi t) + 2\cos(1800\pi t)$$

The link is operated at 10,000 bits/s, and each input sample is quantized into 1024 different voltage levels.

- What are the sampling frequency and the folding frequency?
  - What is the Nyquist rate for the signal  $x_a(t)$ ?
  - What are the frequencies in the resulting discrete-time signal  $x(n)$ ?
  - What is the resolution  $\Delta$ ?
- 

What are the sampling frequency and the folding frequency?

To find this, we need to use the 1024 levels as a clue. With bits of data, we use the set  $B = \{0, 1\}$ , meaning we need to use the  $\log_2$  function to find the number of bits we need to represent the data:

$$\log_2(1024) = 10$$

With this number of bits, we can divide the operating rate by this to find the frequency of captures.

$$\begin{aligned}BPS &= rate * bits \\rate &= F_s = \frac{BPS}{bits} = \frac{10k\cancel{B}}{s10\cancel{B}} = \frac{1k}{s} = 1kHz \\F_f &= \frac{F_s}{2} = 500Hz\end{aligned}$$

What is the Nyquist rate for the signal  $x_a(t)$ ?

The signal has a max  $\omega_i$  of  $1800\pi$ :

$$\begin{aligned} 1800\pi &= 2\pi 900 \\ \therefore \text{Nyquist rate} &= 2F_{max} = 1800Hz \end{aligned}$$

What are the frequencies in the resulting discrete-time signal  $x(n)$ ?

Frequencies in the signal would be:

$$\begin{aligned} f_i &= [300 \ 900] Hz \\ f_{i_n} &= \frac{f_i n}{f_f} = \left[ \frac{300n}{500} \ \frac{900n}{500} \right] Hz \\ &= \left[ \frac{3n}{5} \ \frac{-1n}{5} \right] Hz \\ \therefore \omega_i &= \left[ -\frac{1\pi n}{5} \ \frac{3\pi n}{5} \right] \end{aligned}$$

What is the resolution  $\Delta$ ?

$\Delta$  represents the quantization of the signal using the number of levels we have available to us. Additionally, we need to find the min and max of the signal we have.

$$\begin{aligned} f_i &= [300 \ 900] Hz \\ 2\pi f_i &= [2\pi 300 \ 2\pi 900] = [2\pi 300 \ 3(2\pi 300)] \end{aligned}$$

Note: the second frequency here is an integer multiple of the first.  $\therefore$  the max will happen at the same time every other rotation. by adding a phaseshift of  $\pi$  to both would make the min line up in the signal as well.

$$\begin{aligned} \therefore x_{max} &= 3 + 2 = 5 \\ x_{min} &= -3 + -2 = -5 \\ \Delta &= \frac{x_{max} - x_{min}}{L - 1} = \frac{5 - (-5)}{1024 - 1} \approx 9.8e^{-3}V \\ \therefore \Delta &= 9.8mV \end{aligned}$$