Homework Assignment #3

ECE 6530: Digital Signal Processing
September 26, 2023

Miguel Gomez U1318856
Homework set #3

Due Date: Sep 29, 2023
(75 points)

1 Problem 3.2 parts a, b, d, f, and h

Determine the z-transform of the following signals and sketch the ROC of the following:

a) $x(n) = (1 + n)u(n)$

b) $x(n) = (a^n + a^{-n})u(n)$ real a

c) $x(n) = (na^n \sin \omega_0 n)u(n)$

d) $x(n) = Ar^n \cos(\omega_0 n + \phi)u(n)$

e) $x(n) = \left[\frac{1}{2}\right]^n [u(n) - u(n - 10)]$

a) Problem a can be split into two parts:

$$x(n) = (1 + n)u(n) = u(n) + nu(n)$$

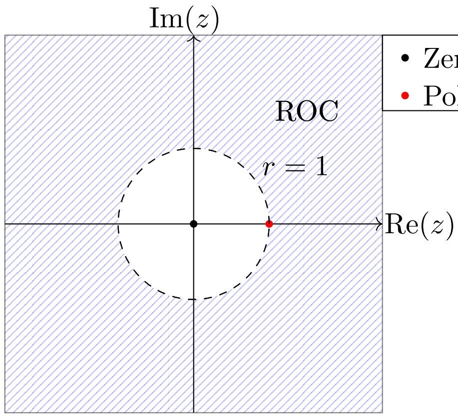
The first is a simple one that we can solve by geometric sum. But we have a table in the book that has these simple cases so we can skip ahead a bit:

$$\begin{aligned}
X_{tot}(z) &= X_1(z) + X_2(z) \\
X_{tot}(z) &= \left[\frac{1}{1 - z^{-1}}\right] - z\frac{dX(z)}{dz} \\
X_{tot}(z) &= \left[\frac{1}{1 - z^{-1}}\right] - z\left[\frac{-1}{(1 - z^{-1})^2}\right](z^{-2}) \\
X_{tot}(z) &= \left[\frac{1}{1 - z^{-1}}\right] + \left[\frac{z^{-1}}{(1 - z^{-1})^2}\right] \\
X_{tot}(z) &= \left[\frac{1 - z^{-1}}{(1 - z^{-1})^2}\right] + \left[\frac{z^{-1}}{(1 - z^{-1})^2}\right] \\
X_{tot}(z) &= \left[\frac{1}{(1 - z^{-1})^2}\right]
\end{aligned}$$

The poles are clearly at 1 since a value of 1 for z would cause the denominator to go to 0. The zeros would need us to multiply top and bottom by z^2.

$$X_{tot}(z) = \left[\frac{z^2}{z^2(1 - z^{-1})^2} \right] = \left[\frac{z^2}{(z-1)^2} \right]$$

This shows the zeros as well as the poles. both with multiplicity 2.



b)

$$\begin{aligned} x(n) &= (a^n + a^{-n})u(n) = a^n u(n) + a^{-n} u(n) \\ &= \sum_{n=-\infty}^{\infty} a^n u(n) + \sum_{n=-\infty}^{\infty} a^{-n} u(n) \end{aligned}$$

using the definition of the transform, we can now introduce z^{-1} and absorb the $u(n)$ into the sum:

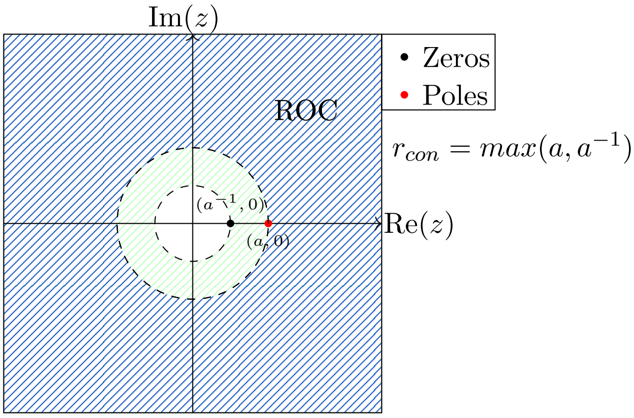
$$\begin{aligned} &= \sum_{n=0}^{\infty} a^n z^{-1} + \sum_{n=0}^{\infty} a^{-n} z^{-1} \\ X_1(z) &= \frac{1}{1 - az^{-1}} \\ X_2(z) &= \frac{1}{1 - (az)^{-1}} \end{aligned}$$

combining the two into a single fraction:

$$\begin{aligned} &= \frac{1}{1 - az^{-1}} + \frac{1}{1 - (az)^{-1}} = \frac{1 - (az)^{-1}}{(1 - (az)^{-1})(1 - az^{-1})} + \frac{1 - az^{-1}}{(1 - (az)^{-1})(1 - az^{-1})} \\ &= \frac{1 - az^{-1} + 1 - (az)^{-1}}{(1 - (az)^{-1})(1 - az^{-1})} = \frac{2 - az^{-1} - (az)^{-1}}{(1 - (az)^{-1})(1 - az^{-1})} \end{aligned}$$

the zeros and the poles can be found by evaluating the top and bottom of the expression as we did in a) and we start with the poles:

$$\begin{aligned}
(1-(az)^{-1})(1-az^{-1}) &= 0 \\
X_{poles} &= \left\{ a, a^{-1} \right\} \\
{\color{red}z}(2-az^{-1}-a^{-1}z^{-1}) &= 0{\color{red}z} \\
2z-(a+a^{-1}) &= 0 \\
X_{zero} &= \left\{ \frac{a+a^{-1}}{2} \right\}
\end{aligned}$$



ROC here is whichever of the two, a or a^{-1} is larger.

c)

$$x(n) = (na^n \sin(\omega_0 n))u(n)$$

The inclusion of the n in the expression means we need to do the derivative, and we can take the rest together as a whole:

$$\begin{aligned}
&= n(a^n \sin(\omega_0 n))u(n) \\
nx(n) &= -z\frac{dX(z)}{dz} \\
&= -z\left(\frac{d}{dz}\cdot\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}\right) \\
\text{by } \frac{d}{dz}\frac{f(z)}{g(z)} &= \frac{f'g-fg'}{g^2}
\end{aligned}$$

$$= -z\left(\frac{(-1)az^{-2}\sin(\omega_0)\cdot(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)+$$

$$-z\left(\frac{az^{-1}\sin(\omega_0)\cdot((-1)(-2)az^{-2}\cos(\omega_0)+(-2)a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$X(z) = -z\left(\frac{-az^{-2}\sin(\omega_0)\cdot(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right) + \\ -z\left(\frac{az^{-1}\sin(\omega_0)\cdot(2az^{-2}\cos(\omega_0)-2a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

Combining into a single fraction:

$$= -z\left(\frac{-az^{-2}\sin(\omega_0)\cdot(1-2az^{-1}\cos(\omega_0)-a^2z^{-2})+az^{-1}\sin(\omega_0)\cdot(2az^{-2}\cos(\omega_0)+2a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$= -zaz^{-1}\sin(\omega_0)\cdot\left(\frac{z^{-1}(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})+(2az^{-2}\cos(\omega_0)-2a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$= -a\cancelto{1}{zz^{-1}}\sin(\omega_0)\cdot\left(\frac{(z^{-1}-2az^{-2}\cos(\omega_0)+a^2z^{-3})+(2az^{-2}\cos(\omega_0)-2a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$= -a\sin(\omega_0)\cdot\left(\frac{z^{-1}-2az^{-2}\cos(\omega_0)+a^2z^{-3}+2az^{-2}\cos(\omega_0)-2a^2z^{-3}}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$= -a\sin(\omega_0)\cdot\left(\frac{z^{-1}+\cancelto{0}{(2a-2a)}z^{-2}\cos(\omega_0)+\cancelto{0}{(2a^2-2a^2)}z^{-3}}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$= \frac{-az^{-1}\sin(\omega_0)}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}$$

unfortunately, I believe I dropped a negative somewhere and I am not able to see where. It should have two terms because the exponential form has two terms. Doing it using the exponential instead. We could

$$sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2j}$$
$$\sin(\omega_0) = \frac{e^{i\theta}-e^{-i\theta}}{2j}$$

2 Problem 3.3 a-d

a)

$$x_1(n) = \begin{cases} \left(\frac{1}{3}\right)^n & \text{if } n \geq 0 \\ \left(\frac{1}{2}\right)^{-n} & \text{if } n < 0 \end{cases}$$

b)

$$x_2(n) = \begin{cases} \left(\frac{1}{3}\right)^n - 2^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

c) $x_3(n) = x_1(n+4)$

d) $x_4(n) = x_1(-n)$

3 Problem 3.7