Homework Assignment #3

ECE 6530: Digital Signal Processing
September 30, 2023

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Homework set #3

Due Date: Sep 29, 2023
(75 points)

1 Problem 3.2 parts a, b, d, f, and h

Determine the z-transform of the following signals and sketch the ROC of the following Note* I didn't catch it before, but the enumeration below is off. The letters above are the correct ones.

a) $x(n) = (1+n)u(n)$

b) $x(n) = (a^n + a^{-n})u(n)$ real a

c) $x(n) = (na^n \sin \omega_0 n)u(n)$

d) $x(n) = Ar^n \cos(\omega_0 n + \phi)u(n)$

e) $x(n) = \left[\frac{1}{2}\right]^n [u(n) - u(n-10)]$

a) Problem a can be split into two parts:

$$x(n) = (1+n)u(n) = u(n) + nu(n)$$

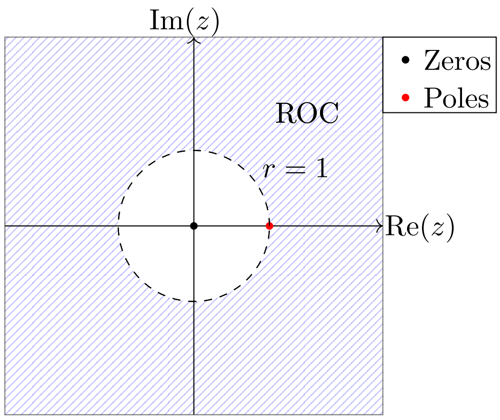
The first is a simple one that we can solve by geometric sum. But we have a table in the book that has these simple cases so we can skip ahead a bit:

$$\begin{aligned}
X_{tot}(z) &= X_1(z) + X_2(z) \\
X_{tot}(z) &= \left[\frac{1}{1-z^{-1}}\right] - z\frac{dX(z)}{dz} \\
X_{tot}(z) &= \left[\frac{1}{1-z^{-1}}\right] - z\left[\frac{-1}{(1-z^{-1})^2}\right](z^{-2}) \\
X_{tot}(z) &= \left[\frac{1}{1-z^{-1}}\right] + \left[\frac{z^{-1}}{(1-z^{-1})^2}\right] \\
X_{tot}(z) &= \left[\frac{1-z^{-1}}{(1-z^{-1})^2}\right] + \left[\frac{z^{-1}}{(1-z^{-1})^2}\right] \\
X_{tot}(z) &= \left[\frac{1}{(1-z^{-1})^2}\right]
\end{aligned}$$

The poles are clearly at 1 since a value of 1 for z would cause the denominator to go to 0. The zeros would need us to multiply top and bottom by z^2.

$$X_{tot}(z) = \left[\frac{z^2}{z^2(1-z^{-1})^2}\right] = \left[\frac{z^2}{(z-1)^2}\right]$$

This shows the zeros as well as the poles. both with multiplicity 2.



b)

$$\begin{aligned} x(n) &= (a^n + a^{-n})u(n) = a^n u(n) + a^{-n}u(n) \\ &= \sum_{n=-\infty}^{\infty} a^n u(n) + \sum_{n=-\infty}^{\infty} a^{-n}u(n) \end{aligned}$$

using the definition of the transform, we can now introduce z^{-1} and absorb the $u(n)$ into the sum:

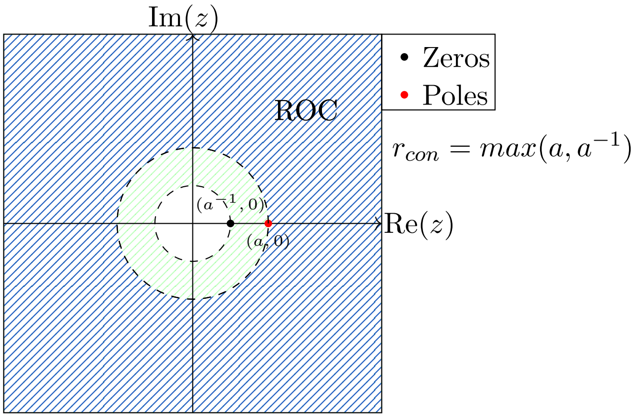
$$\begin{aligned} &= \sum_{n=0}^{\infty} a^n z^{-1} + \sum_{n=0}^{\infty} a^{-n} z^{-1} \\ X_1(z) &= \frac{1}{1-az^{-1}} \\ X_2(z) &= \frac{1}{1-(az)^{-1}} \end{aligned}$$

combining the two into a single fraction:

$$\begin{aligned} &= \frac{1}{1-az^{-1}} + \frac{1}{1-(az)^{-1}} = \frac{1-(az)^{-1}}{(1-(az)^{-1})(1-az^{-1})} + \frac{1-az^{-1}}{(1-(az)^{-1})(1-az^{-1})} \\ &= \frac{1-az^{-1}+1-(az)^{-1}}{(1-(az)^{-1})(1-az^{-1})} = \frac{2-az^{-1}-(az)^{-1}}{(1-(az)^{-1})(1-az^{-1})} \end{aligned}$$

the zeros and the poles can be found by evaluating the top and bottom of the expression as we did in a) and we start with the poles:

$$
\begin{aligned}
(1-(az)^{-1})(1-az^{-1}) &= 0 \\
X_{poles} &= \{\, a, a^{-1} \} \\
{\color{red}z}(2-az^{-1}-a^{-1}z^{-1}) &= 0{\color{red}z} \\
2z-(a+a^{-1}) &= 0 \\
X_{zero} &= \left\{ \frac{a+a^{-1}}{2} \right\}
\end{aligned}
$$



ROC here is whichever of the two, a or a^{-1} is larger.

c)

$$x(n) = (na^n \sin(\omega_0 n))u(n)$$

The inclusion of the n in the expression means we need to do the derivative, and we can take the rest together as a whole:

$$
\begin{aligned}
&= n(a^n \sin(\omega_0 n))u(n) \\
nx(n) &= -z\frac{dX(z)}{dz} \\
&= -z\left(\frac{d}{dz}\cdot\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}\right) \\
\text{by } \frac{d}{dz}\frac{f(z)}{g(z)} &= \frac{f'g-fg'}{g^2}
\end{aligned}
$$

$$
\begin{aligned}
&= -z\left(\frac{(-1)az^{-2}\sin(\omega_0)\cdot(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)+ \\
&-z\left(\frac{az^{-1}\sin(\omega_0)\cdot((-1)(-2)az^{-2}\cos(\omega_0)+(-2)a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)
\end{aligned}
$$

$$X(z) = -z\left(\frac{-az^{-2}\sin(\omega_0)\cdot(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right) + \\ -z\left(\frac{az^{-1}\sin(\omega_0)\cdot(2az^{-2}\cos(\omega_0)-2a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

Combining into a single fraction:

$$= -z\left(\frac{-az^{-2}\sin(\omega_0)\cdot(1-2az^{-1}\cos(\omega_0)-a^2z^{-2})+az^{-1}\sin(\omega_0)\cdot(2az^{-2}\cos(\omega_0)+2a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$= -zaz^{-1}\sin(\omega_0)\cdot\left(\frac{z^{-1}(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})+(2az^{-2}\cos(\omega_0)-2a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$= -a\cancelto{1}{zz^{-1}}\sin(\omega_0)\cdot\left(\frac{(z^{-1}-2az^{-2}\cos(\omega_0)+a^2z^{-3})+(2az^{-2}\cos(\omega_0)-2a^2z^{-3})}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

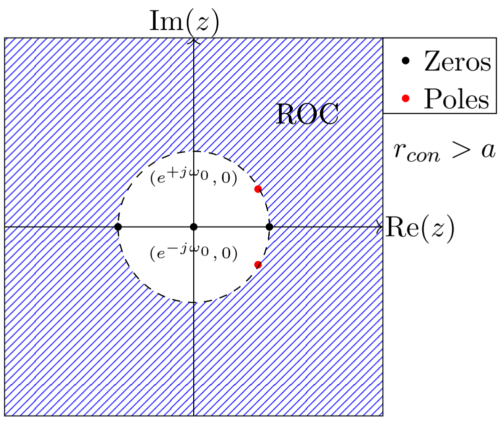
$$= -a\sin(\omega_0)\cdot\left(\frac{z^{-1}-2az^{-2}\cos(\omega_0)+a^2z^{-3}+2az^{-2}\cos(\omega_0)-2a^2z^{-3}}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$= -a\sin(\omega_0)\cdot\left(\frac{z^{-1}+\cancelto{0}{(2a-2a)}z^{-2}\cos(\omega_0)+\cancelto{0}{(2a^2-2a^2)}z^{-3}}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}\right)$$

$$= \frac{-az^{-1}\sin(\omega_0)}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}$$

Unfortunately, I believe I dropped a negative somewhere and I am not able to see where. It should have two terms because the exponential form has two terms. Doing it using the exponential instead. We could use these, but I am calling it here and stating the results we obtained in our study session group together:

$$= \frac{\left(az^{-1}-a^{-3}z^{-3}\right)\sin(\omega_0)}{(1-2az^{-1}\cos(\omega_0)+a^2z^{-2})^2}$$



ROC here is greater than a, double poles at $e^{\pm j\omega_0}$, zeros at $\pm a$ and 0. The choice to place poles where they are in the plot is arbitrary.

d)

$$x(n) = Ar^n \cos{(\omega_0 n + \phi)} u(n)$$

Note, we can use the definition of cos as the exponential to extract the phase term from the expression.

$$\begin{aligned}
\cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\cos{(\omega_0 n + \phi)} &= \frac{e^{i\omega_0 n + \phi} + e^{-i\omega_0 n - \phi}}{2} \\
&= \frac{e^{i\omega_0 n} e^{i\phi} + e^{-i\omega_0 n} e^{-i\phi}}{2} \\
x(n) &= Ar^n \left(\frac{e^{i\omega_0 n} e^{i\phi} + e^{-i\omega_0 n} e^{-i\phi}}{2} \right) u(n) \\
x(n) &= Ar^n \left(\frac{e^{i\omega_0 n} e^{i\phi}}{2} + \frac{e^{-i\omega_0 n} e^{-i\phi}}{2} \right) u(n) \\
X(z) &= \frac{A}{2} \left[\frac{e^{i\phi}}{1 - re^{i\omega_0} z^{-1}} + \frac{e^{-i\phi}}{1 - re^{-i\omega_0} z^{-1}} \right]
\end{aligned}$$

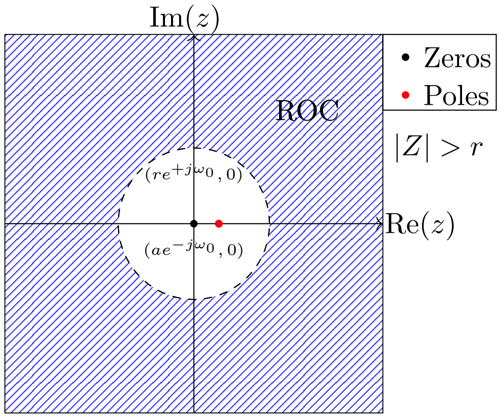
Combining the fractions:

$$\begin{aligned}
&= \frac{A}{2} \left[\frac{e^{i\phi}(1 - re^{-i\omega_0} z^{-1})}{(1 - re^{-i\omega_0} z^{-1})(1 - re^{i\omega_0} z^{-1})} + \frac{e^{-i\phi}(1 - re^{i\omega_0} z^{-1})}{(1 - re^{i\omega_0} z^{-1})(1 - re^{-i\omega_0} z^{-1})} \right] \\
&= \frac{A}{2} \left[\frac{e^{i\phi} + e^{-i\phi} - rz^{-1}(e^{i\omega_0} e^{-i\phi} + e^{-i\omega_0} e^{i\phi})}{(1 - r(e^{i\omega_0} + e^{-i\omega_0}) + r^2 z^{-2})} \right] \\
&= \frac{A}{2} \left[\frac{e^{i\phi} + e^{-i\phi} - rz^{-1}(e^{i\omega_0} e^{-i\phi} + e^{-i\omega_0} e^{i\phi})}{(1 - rz^{-1}(e^{i\omega_0} + e^{-i\omega_0}) + r^2 z^{-2})} \right]
\end{aligned}$$

Note: we can distribute the $\frac{1}{2}$ into the top and apply it to the expontentials. additionally, we can modify the bottom by multiplying the middle factor by $\frac{2}{2}$

$$\begin{aligned}
&= A \left[\frac{\frac{e^{i\phi} + e^{-i\phi}}{2} - rz^{-1} \frac{(e^{i\omega_0 - i\phi} + e^{-i\omega_0 + i\phi})}{2}}{(1 - r2z^{-1} \frac{e^{i\omega_0} + e^{-i\omega_0}}{2} + r^2 z^{-2})} \right] \\
\therefore X(z) &= A \left[\frac{\cos{(\phi)} - rz^{-1} \cos{(\omega_0 - \phi)}}{(1 - r2z^{-1} \cos{(\omega_0)} + r^2 z^{-2})} \right]
\end{aligned}$$

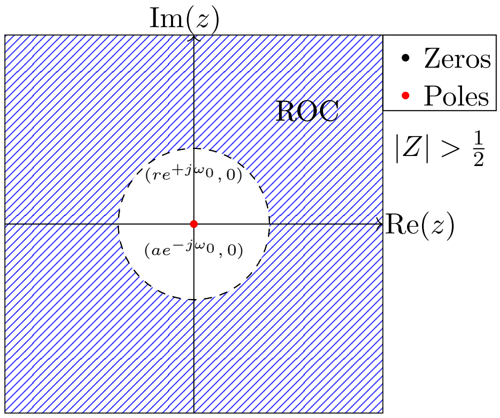
Poles at $z = re^{j\omega_0}$ and $z = ae^{-j\omega_0}$ and zeros at $z = 0$, and $z = r\frac{\cos{(\omega_0 - \phi)}}{\cos{(\phi)}}$. Triple pole at $z = \frac{1}{3}$ and zeros at $z = 0$ and $z = \frac{1}{3}$, so there is a pole-zero cancellation.



h)

$$x(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-10)] = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{n-10} u(n-10)$$
$$x(n) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{\left(\frac{1}{2}\right)^{10} z^{-10}}{1-\frac{1}{2}z^{-1}}$$
$$x(n) = \frac{1-\left(\frac{1}{2}\right)^{10} z^{-10}}{(1-\frac{1}{2}z^{-1})^2}$$

Here, we can see that we will have zeros at $z = \frac{1}{2}e^{\frac{j2\pi n}{M}}$ with multiplicity 10, and poles at $z = \frac{1}{2}$ with multiplicity 2.



Yeah, this one would be a pain to program in, so I will add the poles and zeros by hand.

2 Problem 3.3 a-d

a)

$$x_1(n) = \begin{cases} \left(\frac{1}{3}\right)^n & \text{if } n \geq 0 \\ \left(\frac{1}{2}\right)^{-n} & \text{if } n < 0 \end{cases}$$

b)

$$x_2(n) = \begin{cases} \left(\frac{1}{3}\right)^n - 2^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

c) $x_3(n) = x_1(n+4)$

d) $x_4(n) = x_1(-n)$

a) can be seen as the combination of the two systems, one for greater than or equal to 0 and the other for less than. Since the sum cannot start at 0, we must first remove that from the factor on the left and continue.

$$\begin{aligned}
x(n) &= \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^{-n} u(-n) \\
X(z) &= \frac{1}{1-\frac{1}{3}z^{-1}} + X(-z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z} - 1 \\
&= \frac{1-\frac{1}{2}z}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} + \frac{1-\frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} - \frac{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \\
&= \frac{1-\frac{1}{2}z+1-\frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} - \frac{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \\
&= \frac{1-\frac{1}{2}z+1-\frac{1}{3}z^{-1}-(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \\
&= \frac{1-\frac{1}{2}z+(1-\frac{1}{3}z^{-1})(1-(1-\frac{1}{2}z))}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \\
&= \frac{1-\frac{1}{2}z+(1-\frac{1}{3}z^{-1})(\frac{1}{2}z)}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \\
&= \frac{1\cancelto{0}{-\frac{1}{2}z+\frac{1}{2}z}-\frac{1}{3}\frac{1}{2}\cancelto{1}{z^{-1}z}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \\
&= \frac{1-\frac{1}{6}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \\
&= \frac{\frac{5}{6}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)}
\end{aligned}$$

The expression has poles at $z = \frac{1}{3}$ and at $z = 2$. $\quad \therefore$ ROC $\frac{1}{3} < |z| < 2$