

Homework Assignment #3

ECE 6530: Digital Signal Processing
September 28, 2023

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Homework set #3

Due Date: Sep 29, 2023
(75 points)

1 Problem 3.2 parts a, b, d, f, and h

Determine the z-transform of the following signals and sketch the ROC of the following **Note*** I didn't catch it before, but the enumeration below is off. The letters above are the correct ones.

- a) $x(n) = (1 + n)u(n)$
- b) $x(n) = (a^n + a^{-n})u(n)$ real a
- c) $x(n) = (na^n \sin \omega_0 n)u(n)$
- d) $x(n) = Ar^n \cos(\omega_0 n + \phi)u(n)$
- e) $x(n) = \left[\frac{1}{2}\right]^n [u(n) - u(n - 10)]$

a) Problem a can be split into two parts:

$$x(n) = (1 + n)u(n) = u(n) + nu(n)$$

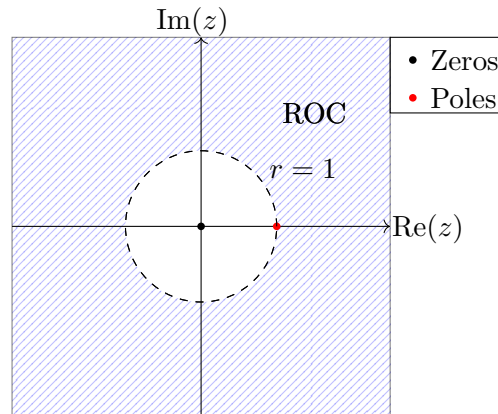
The first is a simple one that we can solve by geometric sum. But we have a table in the book that has these simple cases so we can skip ahead a bit:

$$\begin{aligned} X_{tot}(z) &= X_1(z) + X_2(z) \\ X_{tot}(z) &= \left[\frac{1}{1 - z^{-1}} \right] - z \frac{dX(z)}{dz} \\ X_{tot}(z) &= \left[\frac{1}{1 - z^{-1}} \right] - z \left[\frac{-1}{(1 - z^{-1})^2} \right] (z^{-2}) \\ X_{tot}(z) &= \left[\frac{1}{1 - z^{-1}} \right] + \left[\frac{z^{-1}}{(1 - z^{-1})^2} \right] \\ X_{tot}(z) &= \left[\frac{1 - z^{-1}}{(1 - z^{-1})^2} \right] + \left[\frac{z^{-1}}{(1 - z^{-1})^2} \right] \\ X_{tot}(z) &= \left[\frac{1}{(1 - z^{-1})^2} \right] \end{aligned}$$

The poles are clearly at 1 since a value of 1 for z would cause the denominator to go to 0. The zeros would need us to multiply top and bottom by z^2 .

$$X_{tot}(z) = \left[\frac{z^2}{z^2(1 - z^{-1})^2} \right] = \left[\frac{z^2}{(z - 1)^2} \right]$$

This shows the zeros as well as the poles. both with multiplicity 2.



b)

$$\begin{aligned} x(n) &= (a^n + a^{-n})u(n) = a^n u(n) + a^{-n} u(n) \\ &= \sum_{n=-\infty}^{\infty} a^n u(n) + \sum_{n=-\infty}^{\infty} a^{-n} u(n) \end{aligned}$$

using the definition of the transform, we can now introduce z^{-1} and absorb the $u(n)$ into the sum:

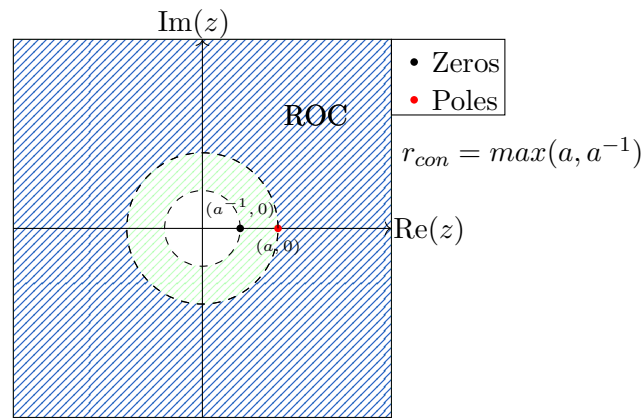
$$\begin{aligned} &= \sum_{n=0}^{\infty} a^n z^{-1} + \sum_{n=0}^{\infty} a^{-n} z^{-1} \\ X_1(z) &= \frac{1}{1 - az^{-1}} \\ X_2(z) &= \frac{1}{1 - (az)^{-1}} \end{aligned}$$

combining the two into a single fraction:

$$\begin{aligned} &= \frac{1}{1 - az^{-1}} + \frac{1}{1 - (az)^{-1}} = \frac{1 - (az)^{-1}}{(1 - (az)^{-1})(1 - az^{-1})} + \frac{1 - az^{-1}}{(1 - (az)^{-1})(1 - az^{-1})} \\ &= \frac{1 - az^{-1} + 1 - (az)^{-1}}{(1 - (az)^{-1})(1 - az^{-1})} = \frac{2 - az^{-1} - (az)^{-1}}{(1 - (az)^{-1})(1 - az^{-1})} \end{aligned}$$

the zeros and the poles can be found by evaluating the top and bottom of the expression as we did in a) and we start with the poles:

$$\begin{aligned}
 (1 - (az)^{-1})(1 - az^{-1}) &= 0 \\
 X_{poles} &= \{a, a^{-1}\} \\
 \cancel{z}(2 - az^{-1} - a^{-1}z^{-1}) &= 0 \cancel{z} \\
 2z - (a + a^{-1}) &= 0 \\
 X_{zero} &= \left\{ \frac{a + a^{-1}}{2} \right\}
 \end{aligned}$$



ROC here is whichever of the two, a or a^{-1} is larger.

c)

$$x(n) = (na^n \sin(\omega_0 n))u(n)$$

The inclusion of the n in the expression means we need to do the derivative, and we can take the rest together as a whole:

$$\begin{aligned}
 &= n(a^n \sin(\omega_0 n))u(n) \\
 nx(n) &= -z \frac{dX(z)}{dz} \\
 &= -z \left(\frac{d}{dz} \cdot \frac{az^{-1} \sin(\omega_0)}{1 - 2az^{-1} \cos(\omega_0) + a^2 z^{-2}} \right) \\
 &\quad \text{by } \frac{d}{dz} \frac{f(z)}{g(z)} = \frac{f'g - fg'}{g^2} \\
 &= -z \left(\frac{(-1)az^{-2} \sin(\omega_0) \cdot (1 - 2az^{-1} \cos(\omega_0) + a^2 z^{-2})}{(1 - 2az^{-1} \cos(\omega_0) + a^2 z^{-2})^2} \right) + \\
 &\quad -z \left(\frac{az^{-1} \sin(\omega_0) \cdot ((-1)(-2)az^{-2} \cos(\omega_0) + (-2)a^2 z^{-3})}{(1 - 2az^{-1} \cos(\omega_0) + a^2 z^{-2})^2} \right)
 \end{aligned}$$

$$X(z) = -z \left(\frac{-az^{-2} \sin(\omega_0) \cdot (1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})}{(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})^2} \right) +$$

$$-z \left(\frac{az^{-1} \sin(\omega_0) \cdot (2az^{-2} \cos(\omega_0) - 2a^2z^{-3})}{(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})^2} \right)$$

Combining into a single fraction:

$$= -z \left(\frac{-az^{-2} \sin(\omega_0) \cdot (1 - 2az^{-1} \cos(\omega_0) - a^2z^{-2}) + az^{-1} \sin(\omega_0) \cdot (2az^{-2} \cos(\omega_0) + 2a^2z^{-3})}{(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})^2} \right)$$

$$= -zaz^{-1} \sin(\omega_0) \cdot \left(\frac{z^{-1}(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2}) + (2az^{-2} \cos(\omega_0) - 2a^2z^{-3})}{(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})^2} \right)$$

$$= -\cancel{az} \cancel{z}^1 \sin(\omega_0) \cdot \left(\frac{(z^{-1} - 2az^{-2} \cos(\omega_0) + a^2z^{-3}) + (2az^{-2} \cos(\omega_0) - 2a^2z^{-3})}{(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})^2} \right)$$

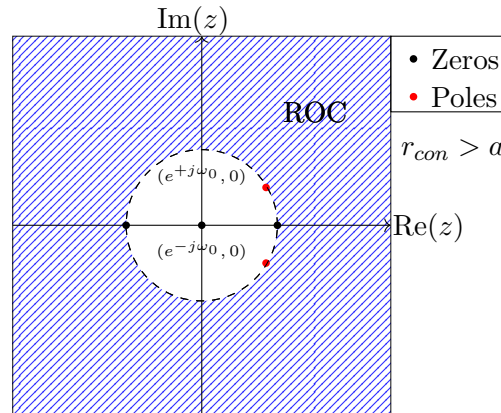
$$= -a \sin(\omega_0) \cdot \left(\frac{z^{-1} - 2az^{-2} \cos(\omega_0) + a^2z^{-3} + 2az^{-2} \cos(\omega_0) - 2a^2z^{-3}}{(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})^2} \right)$$

$$= -a \sin(\omega_0) \cdot \left(\frac{z^{-1} + (\cancel{2a} \cancel{z}^0 - \cancel{2a} \cancel{z}^0) \cos(\omega_0) + (\cancel{2a^2} \cancel{z}^0 - \cancel{2a^2} \cancel{z}^0)}{(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})^2} \right)$$

$$= \frac{-az^{-1} \sin(\omega_0)}{(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})^2}$$

Unfortunately, I believe I dropped a negative somewhere and I am not able to see where. It should have two terms because the exponential form has two terms. Doing it using the exponential instead. We could use these, but I am calling it here and stating the results we obtained in our study session group together:

$$= \frac{(az^{-1} - a^{-3}z^{-3}) \sin(\omega_0)}{(1 - 2az^{-1} \cos(\omega_0) + a^2z^{-2})^2}$$



ROC here is greater than a , double poles at $e^{\pm j\omega_0}$, zeros at $\pm a$ and 0. The choice to place poles where they are in the plot is arbitrary.

d)

$$x(n) = Ar^n \cos(\omega_0 n + \phi) u(n)$$

Note, we can use the definition of cos as the exponential to extract the phase term from the expression.

$$\begin{aligned} \cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \cos(\omega_0 n + \phi) &= \frac{e^{i\omega_0 n + \phi} + e^{-i\omega_0 n - \phi}}{2} \\ &= \frac{e^{i\omega_0 n} e^{i\phi} + e^{-i\omega_0 n} e^{-i\phi}}{2} \\ x(n) &= Ar^n \left(\frac{e^{i\omega_0 n} e^{i\phi} + e^{-i\omega_0 n} e^{-i\phi}}{2} \right) u(n) \\ x(n) &= Ar^n \left(\frac{e^{i\omega_0 n} e^{i\phi}}{2} + \frac{e^{-i\omega_0 n} e^{-i\phi}}{2} \right) u(n) \\ X(z) &= \frac{A}{2} \left[\frac{e^{i\phi}}{1 - re^{i\omega_0} z^{-1}} + \frac{e^{-i\phi}}{1 - re^{-i\omega_0} z^{-1}} \right] \end{aligned}$$

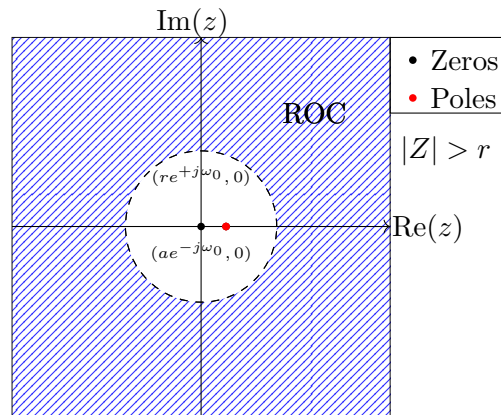
Combining the fractions:

$$\begin{aligned} &= \frac{A}{2} \left[\frac{e^{i\phi}(1 - re^{-i\omega_0} z^{-1})}{(1 - re^{i\omega_0} z^{-1})(1 - re^{i\omega_0} z^{-1})} + \frac{e^{-i\phi}(1 - re^{i\omega_0} z^{-1})}{(1 - re^{i\omega_0} z^{-1})(1 - re^{-i\omega_0} z^{-1})} \right] \\ &= \frac{A}{2} \left[\frac{e^{i\phi} + e^{-i\phi} - rz^{-1}(e^{i\omega_0} e^{-i\phi} + e^{-i\omega_0} e^{i\phi})}{(1 - r(e^{i\omega_0} + e^{-i\omega_0}) + r^2 z^{-2})} \right] \\ &= \frac{A}{2} \left[\frac{e^{i\phi} + e^{-i\phi} - rz^{-1}(e^{i\omega_0} e^{-i\phi} + e^{-i\omega_0} e^{i\phi})}{(1 - rz^{-1}(e^{i\omega_0} + e^{-i\omega_0}) + r^2 z^{-2})} \right] \end{aligned}$$

Note: we can distribute the $\frac{1}{2}$ into the top and apply it to the exponentials. additionally, we can modify the bottom by multiplying the middle factor by $\frac{2}{2}$

$$\begin{aligned} &= A \left[\frac{\frac{e^{i\phi} + e^{-i\phi}}{2} - rz^{-1} \frac{(e^{i\omega_0 - i\phi} + e^{-i\omega_0 + i\phi})}{2}}{(1 - r2z^{-1} \frac{e^{i\omega_0} + e^{-i\omega_0}}{2} + r^2 z^{-2})} \right] \\ \therefore X(z) &= A \left[\frac{\cos(\phi) - rz^{-1} \cos(\omega_0 - \phi)}{(1 - r2z^{-1} \cos(\omega_0) + r^2 z^{-2})} \right] \end{aligned}$$

Poles at $z = re^{j\omega_0}$ and $z = ae^{-j\omega_0}$ and zeros at $z = 0$, and $z = r \frac{\cos(\omega_0 - \phi)}{\cos(\phi)}$. Triple pole at $z = \frac{1}{3}$ and zeros at $z = 0$ and $z = \frac{1}{3}$, so there is a pole-zero cancellation.



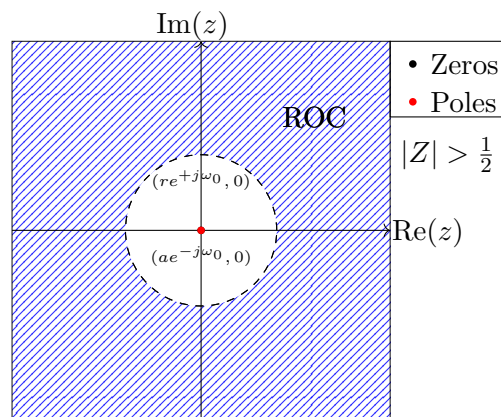
h)

$$x(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-10)] = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{n-10} u(n-10)$$

$$x(n) = \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{\left(\frac{1}{2}\right)^{10} z^{-10}}{1 - \frac{1}{2}z^{-1}}$$

$$x(n) = \frac{1 - \left(\frac{1}{2}\right)^{10} z^{-10}}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

Here, we can see that we will have zeros at $z = \frac{1}{2}e^{\frac{j2\pi n}{M}}$ with multiplicity 10, and poles at $z = \frac{1}{2}$ with multiplicity 2.



Yeah, this one would be a pain to program in, so I will add the poles and zeros by hand.

2 Problem 3.3 a-d

a)

$$x_1(n) = \begin{cases} \left(\frac{1}{3}\right)^n & \text{if } n \geq 0 \\ \left(\frac{1}{2}\right)^{-n} & \text{if } n < 0 \end{cases}$$

b)

$$x_2(n) = \begin{cases} \left(\frac{1}{3}\right)^n - 2^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

c) $x_3(n) = x_1(n + 4)$

d) $x_4(n) = x_1(-n)$

3 Problem 3.7