Homework Assignment #1

ECE 6530: Digital Signal Processing
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Homework set #1

Due Date: Sep 1, 2023
(100 points)

1

1.3 - Determine whether or not each of the following signals is periodic. In case a signal is periodic, specify its fundamental period.

a) $x_a(t) = 3cos(5t + \frac{\pi}{6})$

b) $x(n) = 3cos(5n + \frac{\pi}{6})$

c) $x(n) = 2e^{j(\frac{n}{6} - \pi)}$

d) $x(n) = cos(\frac{n}{8})cos(\frac{n\pi}{8})$

e) $x(n) = cos(\pi\frac{n}{2}) - sin(\pi\frac{n}{8}) + 3cos(\pi\frac{n}{4} + \frac{\pi}{3})$

Determining periodicity requires us to examine the following:

$$x_a(t) = x_a(t + T) \tag{1}$$

$$x(n) = x(n + N) \tag{2}$$

a) Consider the following expression to determine if $x_a(t)$ is periodic:

We can expand $x_a(t + T)$ as follows:

$$\begin{aligned} x_a(t+T) &= 3\cos\left(5(t+T) + \frac{\pi}{6}\right) \\ &= 3\cos\left(5t + 5T + \frac{\pi}{6}\right) \end{aligned}$$

We know that $2\pi ft = 5t$, so $f = \frac{5}{2\pi}$ and $T = \frac{2\pi}{5}$.

$$\begin{aligned} x_a(t+T) &= 3\cos\left(5t + 5\cancelto{2\pi}{\frac{2\pi}{5}} + \frac{\pi}{6}\right) \\ &= 3\cos\left(\cancel{2\pi} + \left(5t + \frac{\pi}{6}\right)\right) = 3cos\left(5t + \frac{\pi}{6}\right) \end{aligned}$$

The addition of 2π is redundant since it represents a full rotation. Therefore, $x_a(t+T) = x_a(t)$ and $x_a(t)$ is periodic.

$$x(n) = 3cos\left(5n + \frac{\pi}{6}\right)$$

b) Consider the expression to determine if $x(n)$ is periodic:

We can expand $x(n+N)$ as follows:

$$x(n+N) = 3\cos\left(5n + 5N + \frac{\pi}{6}\right)$$

For a discrete signal, the frequency is defined as $f_0 = \frac{k}{N}$:

since $2\pi f_0 n = 5n \quad f_0 = \frac{5}{2\pi} \quad T = \frac{2\pi}{5}$

We can stop here since we know a discrete-time signal is only periodic if the frequency f_0 can be expressed as a ratio of two integers. Since there is a factor of π in the expression, this is aperiodic.

c) Consider the following equations to determine if $x(n)$ is periodic:

$$x(n) = 2e^{j\left(\frac{n}{6}-\pi\right)} = 2e^{-j\pi}e^{\frac{n}{6}}$$

A phase shift not dependent on n does not change the periodicity:

$$= 2e^{\frac{n}{6}} = 2\left(cos\left(\frac{n}{6}\right) + jsin\left(\frac{n}{6}\right)\right)$$

since $2\pi f_0 n = \frac{n}{6} \quad f_0 = \frac{1}{12\pi} \quad T = 12\pi$

Again, we can stop here since we know a discrete-time signal is only periodic if the frequency f_0 can be expressed as a ratio of two integers. Since there is a factor of π in the expression, this is aperiodic.

d) Consider the following equations to determine if $x(n)$ is periodic:

$$x(n) = cos\left(\frac{n}{8}\right)cos\left(\frac{n\pi}{8}\right)$$

We have two frequencies to evaluate, $f_{0_0} = \frac{n}{8}$ and $f_{0_1} = \frac{\pi n}{8}$:

since $2\pi f_{0_i} n = \left[\frac{n}{8}, \frac{n\pi}{8}\right] \quad f_{0_i} = \left[\frac{1}{16\pi}, \frac{1}{16}\right] \quad T_i = [16\pi, 16]$

While we came up with one expression for f_{0_i} that is rational, the other one is not. $\therefore$ the overall signal will be aperiodic.

e) Consider the following equations to determine if $x(n)$ is periodic:

$$x(n) = cos\left(\pi\frac{n}{2}\right) - sin\left(\pi\frac{n}{8}\right) + 3cos\left(\pi\frac{n}{4} + \frac{\pi}{3}\right)$$

We have three frequencies to evaluate, $f_{0_0} = \frac{\pi n}{2}$, $f_{0_1} = \frac{\pi n}{8}$, and $f_{0_2} = \frac{\pi n}{4}$:

since $2\pi f_{0_i} n = \left[\frac{\pi n}{2}, \frac{n\pi}{8}, \frac{n\pi}{4}\right]$ $\quad f_{0_i} = \left[\frac{1}{4}, \frac{1}{16}, \frac{1}{8}\right]$ $\quad T_i = [4, 16, 8]$

Since the frequencies f_{0_i} are all rational. The signal is periodic with $f_0 = \frac{1}{16}$.

2

1.7 An analog signal contains frequencies up to 10 kHz.

a) What range of sampling frequencies allows exact reconstruction of this signal from its samples?

b) Suppose that we sample this signal with a sampling frequency Fs = 8 kHz. Examine what happens to the frequency F1 = 5 kHz.

c) Repeat part (b) for a frequency F2 = 9 kHz.

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a) What range of sampling frequencies allows exact reconstruction of this signal from its samples?

By $F_s > 2F_{max}$ we can find the range of frequencies:

$$-\frac{F_s}{2} \leq F_{max} \leq \frac{F_s}{2}$$
$$-\frac{F_s}{2} \leq 10kHz \leq \frac{F_s}{2}$$
$$-F_s \leq 20kHz \leq F_s$$

$\therefore$ the range is up to |20kHz|

b and c) Suppose that we sample this signal with a sampling frequency Fs = 8 kHz. Examine what happens to the frequency F1 = 5 kHz. Repeat part (b) for a frequency F2 = 9 kHz.

Sampling the signals if they are outside the range we found, at this frequency, would give us a mapping from that frequency to one within our range.

$$F_s = [5\ 9]\, kHz$$

looking at what happens in the outputted frequencies will show:

$$F_i\ mod(F_N) \rightarrow F_i - F_N - F_N \quad \text{i.f.f.} \quad F_i > F_N$$
$$else\ F_i\ mod(F_N) \rightarrow F_i \quad \text{if} \quad F_i < F_N$$

This will give the mapping of the frequencies and our $F_N = \frac{F_s}{2} = 4kHz$:

$$F_i\ mod(F_N) = [5\ 9]\, kHz$$

Both 5 and 9 kHz are above the maximum frequency we can capture, $\therefore F_i > F_N$

$$= [5\ 9]\, kHz - 2F_N$$
$$= [-3 \quad -1]\, kHz$$

$\therefore$ both frequencies are mapped to others within the range.

3

1.9 - An analog signal $x_a(t) = sin(480\pi t) + 3sin(720\pi t)$ is sampled 600 times per second.

a) Determine the Nyquist sampling rate for $x_a(t)$.

b) Determine the folding frequency.

c) What are the frequencies, in radians, in the resulting discrete time signal $x(n)$?

d) If $x(n)$ is passed through an ideal D/A converter, what is the reconstructed signal $y_a(t)$?

a and b) Determine the Nyquist sampling rate for $x_a(t)$.

F_{max} of $360Hz$ gives a Nyquist rate of $2F_{max} = 720Hz$

sampling at $600Hz$ gives a Folding Freq of $\frac{F_s}{2} = 300Hz$

What are the frequencies, in radians, in the resulting discrete time signal $x(n)$?

$$F_i = [240\ 360]$$
$$F_1 = 240 \quad F_1 < F_N \rightarrow F_1 = 240$$
$$F_2 = 360 \quad F_2 > F_N \rightarrow F_2 = 60 - 300 = -240$$

$\therefore$ 360 maps to -240

$$F_i = \pm(2\pi 240) = \pm(480\pi)$$

If $x(n)$ is passed through an ideal D/A converter, what is the reconstructed signal $y_a(t)$?

$$\begin{aligned} x_a(t) &= sin(480\pi t) + 3sin(720\pi t) \\ y_a(t) &= sin(480\pi t) + 3sin(-480\pi t) \\ y_a(t) &= sin(480\pi t) - 3sin(480\pi t) \\ y_a(t) &= -2sin(480\pi t) \end{aligned}$$

4

1.10 - A digital communication link carries binary-coded words representing samples of an input signal:

$$x_a(t) = 3cos(600\pi t) + 2cos(1800\pi t)$$

The link is operated at 10,000 bits/s, and each input sample is quantized into 1024 different voltage levels.

a) What are the sampling frequency and the folding frequency?

b) What is the Nyquist rate for the signal $x_a(t)$?

c) What are the frequencies in the resulting discrete-time signal $x(n)$?

d) What is the resolution Δ?

What are the sampling frequency and the folding frequency?

To find this, we need to use the 1024 levels as a clue. With bits of data, we use the set $B = \{0\ 1\}$, meaning we need to use the log_2 function to find the number of bits we need to represent the data:

$$log_2(1024) = 10$$

With this number of bits, we can divide the operating rate by this to find the frequency of captures.

$$\begin{aligned} BPS &= rate * bits \\ rate = F_s &= \frac{BPS}{bits} = \frac{1\cancel{0}k\cancel{B}}{s\cancel{10B}} = \frac{1k}{s} = 1kHz \\ F_f &= \frac{F_s}{2} = 500Hz \end{aligned}$$

What is the Nyquist rate for the signal $x_a(t)$?

The signal has a max ω_i of 1800π:

$$\begin{aligned} 1800\pi &= 2\pi 900 \\ \therefore \text{Nyquist rate} &= 2F_{max} = 1800Hz \end{aligned}$$

What are the frequencies in the resulting discrete-time signal $x(n)$?

Frequencies in the signal would be:

$$\begin{aligned} f_i &= [300 \quad 900]\, Hz \\ f_{i_n} &= \frac{f_i n}{f_f} = \begin{bmatrix} \frac{300n}{500} & \frac{900n}{500} \end{bmatrix} Hz \\ &= \begin{bmatrix} \frac{3n}{5} & \frac{-1n}{5} \end{bmatrix} Hz \\ \therefore \omega_i &= \begin{bmatrix} -\frac{1\pi n}{5} & \frac{3\pi n}{5} \end{bmatrix} \end{aligned}$$

What is the resolution Δ?

Δ represents the quantization of the signal using the number of levels we have available to us. Additionally, we need to find the min and max of the signal we have.

$$\begin{aligned} f_i &= [300 \quad 900]\, Hz \\ 2\pi f_i &= [2\pi 300 \quad 2\pi 900] = [2\pi 300 \quad 3(2\pi 300)] \end{aligned}$$

Note: the second frequency here is an integer multiple of the first. $\therefore$ the max will happen at the same time every other rotation. by adding a phaseshift of π to both would make the min line up in the signal as well.

$$\begin{aligned} \therefore x_{max} &= 3 + 2 = 5 \\ x_{min} &= -3 + -2 = -5 \\ \Delta &= \frac{x_{max} - x_{min}}{L - 1} = \frac{5 - (-5)}{1024 - 1} \approx 9.8e^{-3}V \\ \therefore \Delta &= 9.8mV \end{aligned}$$