To save you some time. Focus and remember to breath.

COMPLEX VARIABLES CHEATSHEET

Complex Numbers Algebra

Fundamental Representations
- Let $z \in \mathbb{C}$.
- **Cartesian Form:** $z = x + iy$
- $x = \text{Re}(z)$ is the real part.
- $y = \text{Im}(z)$ is the imaginary part.
- **Polar Form:** $z = r(\cos\theta + i\sin\theta)$
- $r = |z| = \sqrt{x^2 + y^2}$ is the modulus (magnitude).
- $\theta = \arg(z)$ is the argument (angle).
- **Exponential Form (Euler's Formula):**
- $e^{i\theta} = \cos\theta + i\sin\theta$
- $z = re^{i\theta}$

Complex Conjugate
- If $z = x + iy$, the conjugate is $\bar{z} = x - iy$.
- $\bar{z} = re^{-i\theta}$
- $z\bar{z} = |z|^2 = x^2 + y^2$
- $\text{Re}(z) = \frac{z+\bar{z}}{2}$, $\text{Im}(z) = \frac{z-\bar{z}}{2i}$

Multiplication & Division
- Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.
- **Multiplication:** $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$
- Magnitudes multiply, angles add.
- **Division:** $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$
- Magnitudes divide, angles subtract.

Functions of a Complex Variable

Mapping A complex function $f(z)$ maps a point z in the complex plane (the domain) to a point $w = f(z)$ in another complex plane (the codomain or image).
- $w = f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$.
- $u(x, y)$ is the real part of the output.
- $v(x, y)$ is the imaginary part of the output.

Limits
- $\lim_{z\to z_0} f(z) = L$ means $f(z)$ approaches L as z approaches z_0 **from any direction**.
- If the limit differs along two different paths to z_0, the limit does not exist.

Strategies for Evaluating Limits
1. **Direct Substitution:** If $f(z_0)$ is defined and the function is continuous, the limit is $f(z_0)$.
2. **Test Along Paths:** To show a limit DNE, approach z_0 along two paths and get different results.
   - Along the real axis: let $z = x + iy_0$, take $x \to x_0$.
   - Along the imaginary axis: let $z = x_0 + iy$, take $y \to y_0$.
   - Along a line: let $z = z_0 + re^{i\phi}$, take $r \to 0$ (for fixed ϕ).
3. **Squeeze Theorem:** If $|f(z)| \le g(z)$ and $\lim_{z\to z_0} g(z) = 0$, then $\lim_{z\to z_0} f(z) = 0$.

Continuity A function $f(z)$ is continuous at z_0 if:
1. $f(z_0)$ exists.
2. $\lim_{z\to z_0} f(z)$ exists.
3. $\lim_{z\to z_0} f(z) = f(z_0)$.

Derivatives Analyticity

The Complex Derivative The derivative of $f(z)$ at z_0 is:

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

- The limit must be the same regardless of how Δz approaches 0.

Cauchy-Riemann Equations A function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$ if and only if the partial derivatives of u and v exist and satisfy the Cauchy-Riemann (C-R) equations.

Cartesian Form:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If these hold and the partials are continuous, the derivative is:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Polar Form: For $z = re^{i\theta}$ and $f(z) = u(r, \theta) + iv(r, \theta)$.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

If these hold and the partials are continuous, the derivative is:

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Analyticity
- A function $f(z)$ is **analytic** at a point z_0 if it is differentiable at z_0 and in a small disk around z_0.
- A function is **analytic in a region** if it is analytic at every point in that region.
- An **entire function** is analytic on the entire complex plane $\mathbb{C}$. Examples: $e^z, \sin(z), \cos(z)$, polynomials.
- If C-R equations hold for a region, $f(z)$ is analytic there.

Elementary Transformations

Powers of z
- Let $z = re^{i\theta}$. The function $f(z) = z^n$ for integer n is:

$$w = z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

- **Geometric Effect:**
  - The magnitude is raised to the power n: $|w| = |z|^n$.
  - The angle is multiplied by n: $\arg(w) = n \cdot \arg(z)$.
- This means points are rotated by a factor of n and their distance from the origin is scaled by a power of n.
- A sector of angle α in the z-plane is mapped to a sector of angle $n\alpha$ in the w-plane.

Roots of Complex Numbers
- The n-th roots of a complex number $z_0 = r_0 e^{i\theta_0}$ are the solutions to $w^n = z_0$.
- There are exactly n distinct roots, given by:

$$w_k = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0 + 2\pi k}{n}\right)\right]$$

- for $k = 0, 1, 2, \ldots, n - 1$.

How to Calculate Roots:
1. Write the number z_0 in exponential form $r_0 e^{i\theta_0}$. Be sure to use the principal argument for θ_0.
2. The magnitude of all roots is the same: $\sqrt[n]{r_0}$.
3. Find the angle of the first root ($k = 0$): $\frac{\theta_0}{n}$.
4. The other roots are spaced evenly around a circle. Add increments of $\frac{2\pi}{n}$ to the angle for each subsequent root.

Example: Cube roots of $8i$
1. Polar form: $z = 8i = 8e^{i\pi/2}$. Here $r_0 = 8, \theta_0 = \pi/2, n = 3$.
2. Magnitude of roots: $\sqrt[3]{8} = 2$.
3. Angles: $\frac{\pi/2+2\pi k}{3}$ for $k = 0, 1, 2$.
   - $k = 0 : \frac{\pi/2}{3} = \frac{\pi}{6}$
   - $k = 1 : \frac{\pi/2+2\pi}{3} = \frac{5\pi}{6}$
   - $k = 2 : \frac{\pi/2+4\pi}{3} = \frac{9\pi}{6} = \frac{3\pi}{2}$
4. The roots are: $w_0 = 2e^{i\pi/6}$, $w_1 = 2e^{i5\pi/6}$, $w_2 = 2e^{i3\pi/2}$.

The Exponential Function
- $f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$.
- $|e^z| = e^x$ and $\arg(e^z) = y$.
- Periodic with period $2\pi i$: $e^{z+2\pi i} = e^z$.
- Maps horizontal lines ($y = c$) to rays from the origin.
- Maps vertical lines ($x = c$) to circles of radius e^c.

Logarithmic Function (Principal Value)
- The inverse of e^z, but multi-valued.
- Principal Value: $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$
- where $\text{Arg}(z)$ is the principal argument, $-\pi < \text{Arg}(z) \le \pi$.
- The "branch cut" is usually on the negative real axis.

Complex Trigonometric Functions

Definitions from Euler's Formula
- $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$
- $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$

These are entire functions. Unlike their real counterparts, complex sine and cosine are **unbounded**.

Hyperbolic Functions
- $\cosh(z) = \frac{e^z+e^{-z}}{2}$
- $\sinh(z) = \frac{e^z-e^{-z}}{2}$

Relations:
- $\cos(iy) = \cosh(y)$
- $\sin(iy) = i\sinh(y)$
- $\cosh(iz) = \cos(z)$
- $\sinh(iz) = i\sin(z)$

Rectangular Form of Sin/Cos
- $z = x + iy$
- $\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$
- $\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$

Important Definitions

- **Contour:** A continuous chain of a finite number of smooth curves.
- **Simple Contour:** A contour that does not cross itself.
- **Closed Contour:** A contour whose start and end points are the same.
- **Domain:** An open connected set of points.
- **Simply Connected Domain:** A domain with no "holes". Any simple closed contour in the domain encloses only points within the domain.
- **Singular Point (Singularity):** A point where a function is not analytic.
- **Harmonic Functions:** Real-valued functions $u(x, y)$ and $v(x, y)$ that satisfy Laplace's equation ($\nabla^2 u = u_{xx} + u_{yy} = 0$). The real and imaginary parts of an analytic function are harmonic conjugates.