

# Homework #5

MATH 3160 – Complex Variables  
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## Problem 1

Consider the analytic function  $f(z) = ze^{z^2}$ .

- (a) Show that the function  $u(x, y) = x e^{(x^2-y^2)} \cos(2xy) - y e^{(x^2-y^2)} \sin(2xy)$  is the real component of  $f(z)$ .
- (b) What is a harmonic conjugate for  $u(x, y)$ ?
- (c) Without computing the second partial derivatives of  $u(x, y)$ , explain why you know that  $u(x, y)$  is harmonic.

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Expanding  $f(z) = ze^{z^2}$  to see what the  $u$  and  $v$  turn out to be.

(a)

$$\begin{aligned} f(z) &= ze^{z^2} = (x + iy)e^{(x+iy)^2} = (x + iy)e^{x^2-y^2+2ixy} = (x + iy)e^{x^2-y^2}e^{2ixy} \\ &= e^{x^2-y^2}(x + iy)(\cos(2xy) + i\sin(2xy)) \\ &= e^{x^2-y^2}(x\cos(2xy) + ix\sin(2xy) + iy\cos(2xy) + i^2y\sin(2xy)) \\ &= e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy) + i(x\sin(2xy) + y\cos(2xy))) \\ \therefore u(x, y) &= e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy)) \end{aligned}$$

0.1 (b)

Utilizing the imaginary part of  $f$ ,  $v$  can serve as a conjugate up to an arbitrary constant, which we could set to 0 to recover  $f$ .

$$e^{x^2-y^2}(x\sin(2xy) + y\cos(2xy)) + C$$

**0.2 (c)**

Since we know that  $f$  is analytic as stated, and  $u$  is the real part of  $f$ , an analytic function, then  $u$  must be harmonic. This is due to the fact that the real and imaginary parts of any analytic function are harmonic functions.

**Problem 2**

Consider the function  $u(x, y) = x^3 - 3xy^2 - 3x^2y + y^3$ .

- (a) Show that  $u(x, y)$  is harmonic.
  - (b) Find a harmonic conjugate for  $u(x, y)$ .
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### Problem 3

Recall we learned of the following fact in class:

*Let  $u(x, y)$  be a harmonic function defined on a simply connected domain  $D$ .*

*Then  $u(x, y)$  has a harmonic conjugate on  $D$ .*

- (a) Show that  $u(x, y) = \ln(\sqrt{x^2 + y^2})$  is a harmonic function.
- (b) What is the domain of definition of  $u(x, y)$ ?
- (c) An aside: show that if  $f(z)$  and  $g(z)$  are two analytic functions on the same domain  $D$ , and we have  $\operatorname{Re}(f(z)) = \operatorname{Re}(g(z))$  for all  $z \in D$ , then  $f(z) = g(z) + c$  for some constant  $c \in \mathbb{C}$ .

*[Hint: show that the function  $h(z) = f(z) - g(z)$  has  $\operatorname{Re}(h(z)) = 0$ , and then use a result from class to conclude  $h(z)$  is a constant.]*

- (d) Explain why  $u(x, y)$  does *not* have a harmonic conjugate on its domain.

*[Hint: if such a conjugate existed, then  $u(x, y)$  would be the real component of some analytic function  $f(z)$ , but  $u(x, y)$  is already the real component of a familiar analytic function, which is discontinuous at its branch cut]*

1. Why does this not contradict the fact from class?
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## Problem 4

Find the following values, on the branches given:

(a)  $\log(3) \quad (-2\pi \leq \theta < 0)$

(b)  $\log(-1 + i) \quad (-\pi/2 < \theta \leq 3\pi/2)$

(c)  $\log(1 - i\sqrt{3}) \quad (\pi \leq \theta < 3\pi).$

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$$\log(z) = \log|z| + i\arg(z) = \log|z| + i(\operatorname{Arg}(z) + 2\pi k) \quad \forall k \in \mathbb{Z}$$

(a)

$$\log(3) \quad (-2\pi \leq \theta < 0)$$

$$\log(3) = \log|3| + i(\operatorname{Arg}(3) + 2\pi k) = \log(3) + i(0 + 2\pi k)$$

$$\log(3) + i(2\pi k) \quad k \in [-1, 0) = \log(3) - i2\pi$$

(b)

$$\log(-1 + i) \quad (-\pi/2 < \theta \leq 3\pi/2)$$

$$\log(-1 + i) = \log|-1 + i| + i(\operatorname{Arg}(-1 + i) + 2\pi k) = \log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right)$$

$$\log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right) \quad k \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$$= \log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right)$$

$$= \log(\sqrt{2}) + i\frac{3\pi}{4}$$

(c)

$$\log(1 - i\sqrt{3}) \quad (\pi \leq \theta < 3\pi)$$

**Problem 5**

Recall that power functions are defined by  $z^c = e^{c \log(z)}$ . In this exercise, we compute all power functions by using the branch  $(0 \leq \theta < 2\pi)$  for  $\log(z)$ .

- (a) For  $z = -i$  and  $c = i$ , compute the values of  $(z^c)^2$ ,  $(z^2)^c$ , and  $z^{(2c)}$ .
- (b) With the notation as in (a), which of these are true or false?

$$(z^c)^2 = (z^2)^c, \quad (z^c)^2 = z^{(2c)}, \quad (z^2)^c = z^{(2c)}.$$