

Worksheet # 6

MATH 3160 – Complex Variables
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Problem 1

Show that $u(x, y)$ is harmonic and find the harmonic conjugate $v(x, y)$ when:

(a) $u(x, y) = 2x(1 - y)$

(b) $u(x, y) = \cos(x) \cosh(y)$ where $\cosh(y) = \frac{e^y + e^{-y}}{2}$

(c) $u(x, y) = \frac{y}{x^2 + y^2}$

(d) $u(x, y) = \cos(x) e^y$

A function $u(x, y) : D \rightarrow \mathbb{R}$ is harmonic if:

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0$$

(a)

For $u(x, y) = 2x(1 - y)$, we must first find the partials of u and then apply these to the usual Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = 2(1 - y)$$

By Cauchy-Riemann, this must equal $\frac{\partial v}{\partial y}$

$$\frac{\partial v}{\partial y} = 2(1 - y)$$

Given this expression, we must integrate so that we have something that would give the result when evaluated as $\frac{\partial v}{\partial y}$.

$$\begin{aligned} \frac{\partial C}{\partial y} &= \frac{\partial}{\partial y} \left(\int 2(1 - y) dy \right) = 2(1 - y) \\ \therefore \int 2(1 - y) dy &= C = 2 \left(y - \frac{1}{2} y^2 + c \right) \end{aligned}$$

Additionally, we need to have the derivative of v wrt x to be the negative of u wrt y

$$\begin{aligned}\frac{\partial u}{\partial y} &= -2x \\ \frac{\partial v}{\partial x} &= 2x \\ \therefore v(x, y) &= x^2 - y^2 + 2y + c\end{aligned}$$

$$\begin{aligned}u(x, y) &= 2x(1 - y) \\ v(x, y) &= x^2 - y^2 + 2y + c \\ \frac{\partial u}{\partial x} &= 2(1 - y) & \frac{\partial u}{\partial y} &= -2x \\ \frac{\partial v}{\partial y} &= 2(1 - y) & \frac{\partial v}{\partial x} &= 2x\end{aligned}$$

Using these expressions for u and v now satisfy the CR equations. In order for this to be harmonic, the second derivatives must cancel to 0.

$$\begin{aligned}\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} &= 0 \\ \frac{\partial^2 u(x, y)}{\partial x^2} &= 0 \\ \frac{\partial^2 u(x, y)}{\partial y^2} &= 0\end{aligned}$$

$\therefore u(x, y)$ is harmonic and

$v(x, y) = x^2 - y^2 + 2y + c$ is the harmonic conjugate.

(b)

Once again, we will follow the same steps as above but I will only show the work now for the rest to save space. $u(x, y) = \cos(x) \cosh(y)$.

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\sin(x) \cosh(y) & \frac{\partial u}{\partial y} &= \cos(x) \sinh(y) \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = -\sin(x) \cosh(y) \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = -\cos(x) \sinh(y) \\ \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} &= 0 \\ -\cos(x) \cosh(y) + \cos(x) \cosh(y) &= 0 \\ \therefore v(x, y) &\text{ is harmonic and} \\ \therefore v(x, y) &= -\sin(x) \sinh(y) + c \quad \text{is the HC}\end{aligned}$$

(c)

$$\begin{aligned}u(x, y) &= \frac{y}{x^2 + y^2} = y(x^2 + y^2)^{-1} \\ \frac{\partial u}{\partial x} &= (-1)y(x^2 + y^2)^{-2}(2x) \\ \frac{\partial u}{\partial y} &= \frac{\partial(fg)}{\partial y} = f'g + g'f \\ f &= y & f' &= 1 \\ g &= (x^2 + y^2)^{-1} & g' &= (-1)(x^2 + y^2)^{-2}(2y) \\ f'g + g'f &= (1)(x^2 + y^2)^{-1} + y(-1)(x^2 + y^2)^{-2}(2y) \\ &= (x^2 + y^2)^{-1} - 2y^2(x^2 + y^2)^{-2} \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = -2xy(x^2 + y^2)^{-2}\end{aligned}$$

This appears to be a form that looks to be related to a complex number divided by its magnitude. We can try to guess the function from here:

$$\frac{z}{|z|^2} = \frac{x + iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i \frac{y}{x^2 + y^2}$$

Rotating it by i :

$$\begin{aligned}\frac{-iz}{|z|^2} &= -i \frac{x+iy}{x^2+y^2} = -i \frac{x}{x^2+y^2} - i^2 \frac{y}{x^2+y^2} \\ &= \frac{y}{x^2+y^2} - i \frac{x}{x^2+y^2} \\ |z|^2 &= z\bar{z} \\ \therefore f &= \frac{-iz}{|z|^2} = \frac{-iz}{z\bar{z}} = -\frac{i}{\bar{z}} \\ \text{HC} &= \text{Im} \left(-\frac{i}{\bar{z}} \right)\end{aligned}$$

(d)

$$u(x, y) = \cos(x) e^y$$

This one is the real part of a complex number defined as $f = e^w$ where $w = y + ix$:

$$\begin{aligned}f &= e^y(\cos(x) + i \sin(x)) \\ \frac{\partial u}{\partial x} &= -e^y \sin(x) & \frac{\partial u}{\partial y} &= e^y \cos(x) \\ \frac{\partial v}{\partial x} &= -e^y \cos(x) & \frac{\partial v}{\partial y} &= e^y \sin(x) \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\end{aligned}$$

$\frac{\partial v}{\partial y}$ needs a minus sign to be present implying that we need to have v be negative.

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ -e^y \cos(x) + e^y \cos(x) &= 0\end{aligned}$$

using this we can infer that the HC is the following:

$$v(x, y) = -\sin(x) e^y + c$$

Problem 2

Suppose that v is a harmonic conjugate of u and u is a harmonic conjugate of v on some domain D . Show that u, v must then be constant on D . (Hint: show that all partial derivatives of u, v vanish on D)