Homework #2

MATH 3160 – complex variables
Your Name Here

Completed: August 28, 2025

Problem 1

By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$

(b) $\frac{5i}{2+i} = 1+2i$

(c) $(-1+i)^7 = -8(1+i)$

(d) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$

Problem 2

Find the square roots of (a) $2i$ and (b) $(1 - \sqrt{3}i)$ express them in rectangular coordinates

Problem 3

Find all roots and indicate in rectangular coordinates

(a) $(-16)^{\frac{1}{4}}$

(b) $(-8-8\sqrt{3}i)^{\frac{1}{4}}$

Problem 4

Find the four zeros of $z^4 + 4$

Converting z into exponential form for $|z| = 1$

$$z^4 = (e^{i\theta})^4 = e^{i4\theta}$$

Four windings for this, so we should have 4 equally spaced roots that give us the zeros.

$$\begin{aligned}
z^4 + 4 &= 0 \\
z^4 &= -4 \\
\sqrt[4]{z^4} &= \sqrt[4]{-4} \\
z^{\frac{1}{4}4} &= \sqrt[4]{-4} = e^{\frac{1}{4}i4\theta} \\
z &= \sqrt[4]{-4} = e^{i\theta} \\
&= \sqrt[4]{-2 \cdot 2} = \sqrt[4]{-\sqrt{2}^2 \cdot \sqrt{2}^2} = \sqrt[4]{-\sqrt{2}^4} \\
&= \sqrt{2} \cdot \sqrt[4]{-1}
\end{aligned}$$

$$\begin{aligned}
\sqrt{z^4} &= \pm z^2 = \sqrt{-4} = \pm 2\sqrt{-1} \\
\sqrt{\pm z^2} &= \{\sqrt{z^2}, \sqrt{-z^2}\} = \{\pm z, \pm z\sqrt{-1}\} \\
&= \{\pm z, \pm zi\}
\end{aligned}$$

This is also the following:

$$\begin{aligned}
z^4 + 4 = z^4 - (-4) &= z^4 - (i^2 2^2) = (z^2)^2 - (2i)^2 \\
&= (z^2 - 2i)(z^2 + 2i)
\end{aligned}$$

converting to exponential

$$\begin{aligned}
i &= e^{i\frac{\pi}{2}} \\
\therefore \sqrt{i} &= (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}} \\
(z^2 - 2i) &= (z^2 - (\sqrt{2}\sqrt{i})^2) = (z - (\sqrt{2}\sqrt{i}))(z + (\sqrt{2}\sqrt{i})) \\
&= \boxed{(z - (\sqrt{2}\sqrt{i}))(z + (\sqrt{2}\sqrt{i}))}
\end{aligned}$$

similarly, for the positive side

$$\begin{aligned}
(z^2 + 2i) &= (z^2 - (-2i)) = (z^2 - i^2(\sqrt{2}\sqrt{i})^2) \\
&= \boxed{(z - (\sqrt{2}\sqrt{i})i)(z + (\sqrt{2}\sqrt{i})i)} \\
i\sqrt{i} &= e^{\frac{i\pi}{2}} e^{i\frac{\pi}{4}} = e^{i\pi(\frac{1+2}{4})} = e^{i\frac{3\pi}{4}} \\
\therefore z &= \sqrt{2}\{\pm e^{i\frac{\pi}{4}}, \pm e^{i\frac{3\pi}{4}}\} = \sqrt{2}\{(1+i), (-1+i), (-1-i), (1-i)\}
\end{aligned}$$

Problem 5

Show that if c is an n^{th} root of 1 other than 1 itself, then:

$$1 + c + c^2 + ... + c^{n-1} = 0$$

Hint: multiply above by $(c - 1)$

multiplying the above by $(c - 1)$ gives the following

$$(c-1) \cdot (1 + c + c^2 + ... + c^{n-1}) = (c-1) \cdot 0$$

$$c + c^2 + c^3 + ... + c^{n-1} + c^n + \quad \text{expanding } c$$

$$-1 - c - c^2 - ... - c^{n-1} \quad \text{expanding } -1$$

$$-1 + (c - c) + (c^2 - c^2) + (c^3 - c^3) + ... + (c^{n-1} - c^{n-1}) + c^n = 0$$

$$-1 + \cancelto{0}{(c - c)} + \cancelto{0}{(c^2 - c^2)} + \cancelto{0}{(c^3 - c^3)} + ... + \cancelto{0}{(c^{n-1} - c^{n-1})} + c^n = 0$$

$$-1 + c^n = 0$$

$$c^n = 1$$

$$\sqrt[n]{c^n} = \sqrt[n]{1}$$

$$c = 1$$

Now using something other than 1, i.e. $c \neq 1$, then the sum cannot be 0.

$$1 + c + c^2 + ... + c^{n-1} = S$$

same steps as before

$$c^n - 1 = S(c - 1)$$

since $c^n = 1$; and $c \neq 1$ we can divide

$$\frac{c^n - 1}{c - 1} = S$$

since $c \neq 1$; $(c - 1)$ is not 0 but $c^n = 1$

$$\therefore \frac{\cancelto{1}{c^n} - 1}{c - 1} = \frac{0}{c - 1} = S = 0 \quad \square$$

Problem 6

For each of the below, indicate the domain of definition.

(a) $f(z) = \frac{1}{z^2+1}$

We need $z^2 + 1 \neq 0$, which means $z^2 \neq -1$.

Solving $z^2 = -1$:

$$z^2 = -1 = i^2$$
$$z = \pm i$$
$$(\pm i)^2 + 1 = i^2 + 1 = -1 + 1 = 0$$

undefined only at $z = i$ and $z = -i$.

For all other complex numbers z, we have $z^2 + 1 \neq 0$, so the function is well-defined.

$\therefore$ Domain of Definition: $\mathbb{C} \setminus \{\pm i\}$

(b) $f(z) = \text{Arg}\left(\frac{1}{z}\right)$

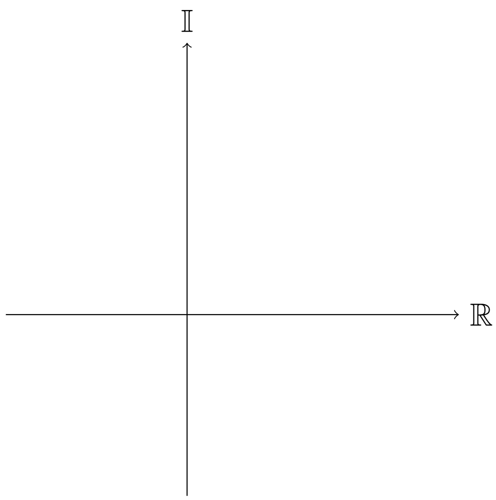
(c) $f(z) = \frac{z}{z+\bar{z}}$

(d) $f(z) = \frac{1}{(1-|z|^2)}$

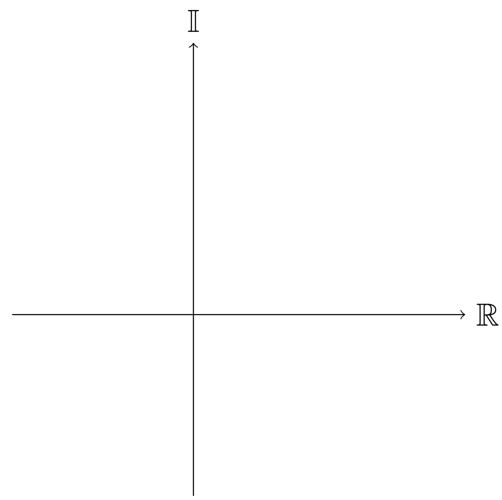
1 Problem 7

Sketch the region onto which the sector $r \leq 1;\ 0 \leq \theta \leq \frac{\pi}{4}$ in the z-plane is mapped to the $w = f(z)$-plane by the transformations

(a) $w = z^2$

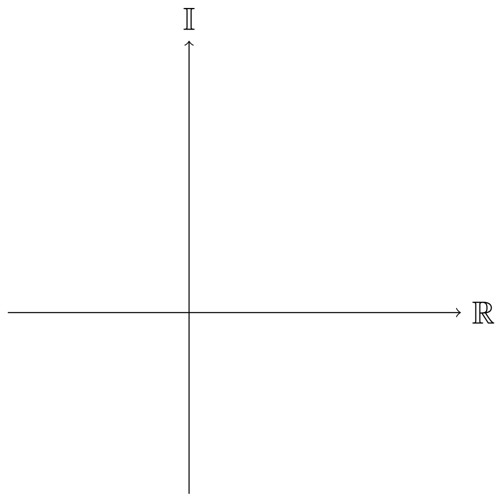


z-plane



w-plane

(b) $w = z^3$



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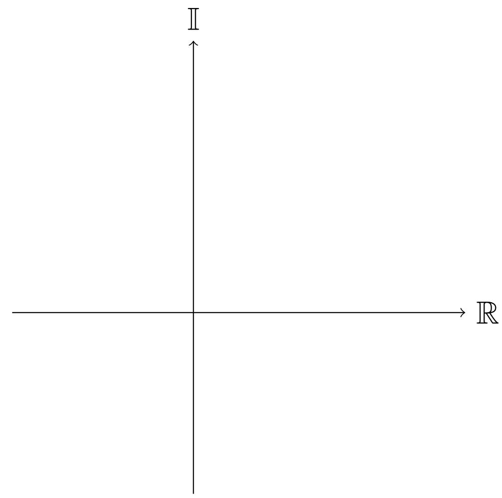
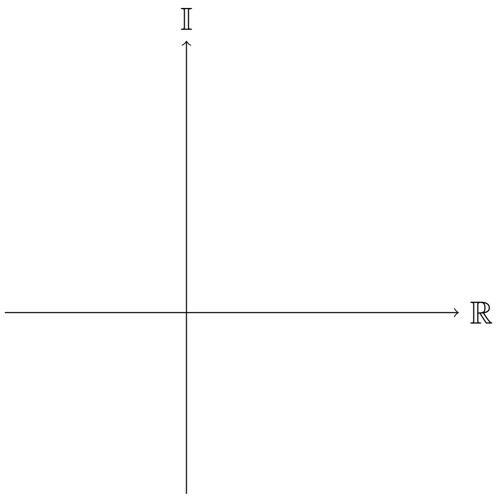


w-plane

(c) $w = z^4$



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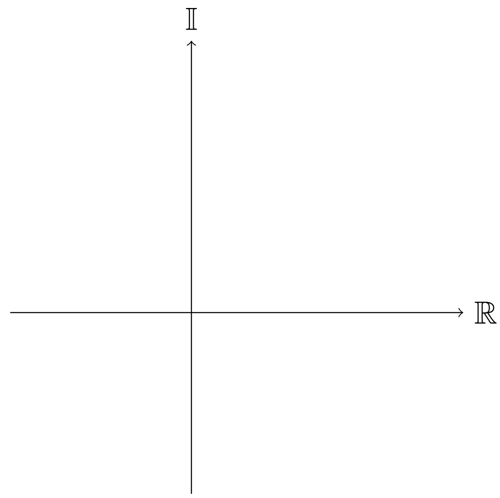

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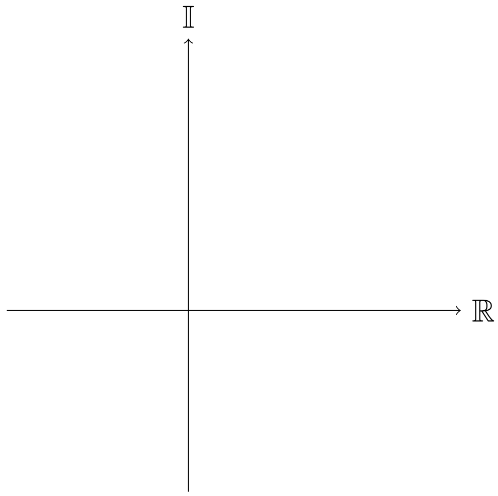
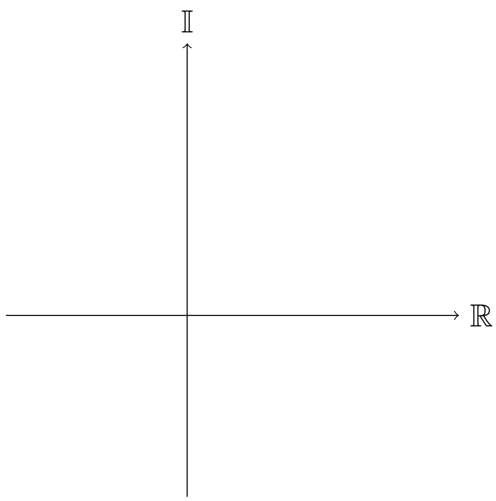
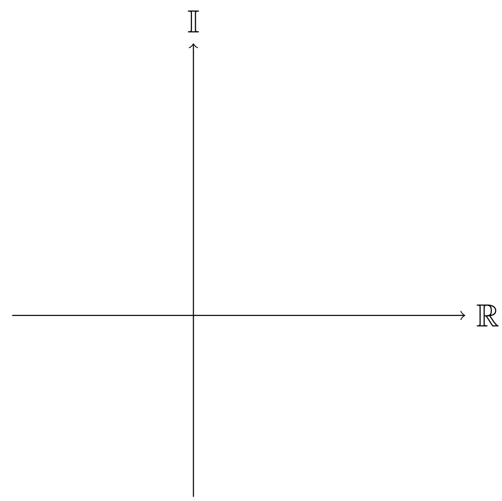




z-plane



w-plane