Worksheet # 6

MATH 3160 – Complex Variables
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Problem 1

Show that $u(x, y)$ is harmonic and find the harmonic conjugate $v(x, y)$ when:

(a) $u(x, y) = 2x(1 - y)$

(b) $u(x, y) = \cos(x) \cosh(y)$ where $\cosh(y) = \frac{e^y + e^{-y}}{2}$

(c) $u(x, y) = \frac{y}{x^2 + y^2}$

(d) $u(x, y) = \cos(x)\, e^y$

A function $u(x, y) : D \to \mathbb{R}$ is harmonic if:

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0$$

(a)

For $u(x, y) = 2x(1 - y)$, we must first find the partials of u and then apply these to the usual Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = 2(1 - y)$$

$$\text{By Cauchy-Riemann, this must equal} \quad \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = 2(1 - y)$$

Given this expression, we must integrate so that we have something that would give the result when evaluated as $\frac{\partial v}{\partial y}$.

$$\frac{\partial C}{\partial y} = \frac{\partial}{\partial y}\left(\int 2(1 - y)dy\right) = 2(1 - y)$$

$$\therefore \int 2(1 - y)dy = C = 2\left(y - \frac{1}{2}y^2 + c\right)$$

Additionally, we need to have the derivative of v wrt x to be the negative of u wrt y

$$\frac{\partial u}{\partial y} = -2x$$
$$\frac{\partial v}{\partial x} = 2x$$
$$\therefore v(x,y) = x^2 - y^2 + 2y + c$$

$$u(x,y) = 2x(1-y)$$
$$v(x,y) = x^2 - y^2 + 2y + c$$
$$\frac{\partial u}{\partial x} = 2(1-y) \qquad \frac{\partial u}{\partial y} = -2x$$
$$\frac{\partial v}{\partial y} = 2(1-y) \qquad \frac{\partial v}{\partial x} = 2x$$

Using these expressions for u and v now satisfy the CR equations. In order for this to be harmonic, the second derivatives must cancel to 0.

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$
$$\frac{\partial^2 u(x,y)}{\partial x^2} = 0$$
$$\frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$\therefore u(x,y)$ is harmonic and

$v(x,y) = x^2 - y^2 + 2y + c$ is the harmonic conjugate.

(b)

Once again, we will follow the same steps as above but I will only show the work now for the rest to save space. $u(x,y) = \cos(x)\cosh(y)$.

$$\frac{\partial u}{\partial x} = -\sin(x)\cosh(y) \qquad \frac{\partial u}{\partial y} = \cos(x)\sinh(y)$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -\sin(x)\cosh(y)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\cos(x)\sinh(y)$$

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$$-\cos(x)\cosh(y) + \cos(x)\cosh(y) = 0$$

$$\therefore v(x,y) \text{ is harmonic and}$$

$$\therefore v(x,y) = -\sin(x)\sinh(y) + c \quad \text{ is the HC}$$

(c)

$$u(x,y) = \frac{y}{x^2+y^2} = y(x^2+y^2)^{-1}$$

$$\frac{\partial u}{\partial x} = (-1)y(x^2+y^2)^{-2}(2x)$$

$$\frac{\partial u}{\partial y} = \frac{\partial (fg)}{\partial y} = f'g + g'f$$

$$f = y \qquad f' = 1$$

$$g = (x^2+y^2)^{-1} \qquad g' = (-1)(x^2+y^2)^{-2}(2y)$$

$$f'g + g'f = (1)(x^2+y^2)^{-1} + y(-1)(x^2+y^2)^{-2}(2y)$$

$$= (x^2+y^2)^{-1} - 2y^2(x^2+y^2)^{-2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -2xy(x^2+y^2)^{-2}$$

This appears to be a form that looks to be related to a complex number divided by its magnitude. We can try to guess the function from here:

$$\frac{z}{|z|^2} = \frac{x+iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

Rotating it by i:

$$\begin{aligned}
\frac{-iz}{|z|^2} &= -i\frac{x+iy}{x^2+y^2} = -i\frac{x}{x^2+y^2} - i^2\frac{y}{x^2+y^2} \\
&= \frac{y}{x^2+y^2} - i\frac{x}{x^2+y^2} \\
|z|^2 &= z\bar{z} \\
\therefore f = \frac{-iz}{|z|^2} &= \frac{-iz}{z\bar{z}} = -\frac{i}{\bar{z}} \\
\text{HC} &= \text{Im}\left(-\frac{i}{\bar{z}}\right)
\end{aligned}$$

(d)

$$u(x,y) = \cos(x)\, e^y$$

This one is the real part of a complex number defined as $f = e^w$ where $w = y + ix$:

$$\begin{aligned}
f &= e^y(\cos(x) + i\sin(x)) \\
\frac{\partial u}{\partial x} &= -e^y \sin(x) \quad \frac{\partial u}{\partial y} = e^y\cos(x) \\
\frac{\partial v}{\partial x} &= -e^y \cos(x) \quad \frac{\partial v}{\partial y} = e^y\sin(x) \\
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}
\end{aligned}$$

$\frac{\partial v}{\partial y}$ needs a minus sign to be present implying that we need to have v be negative.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$-e^y\cos(x) + e^y\cos(x) = 0$$

using this we can infer that the HC is the following:

$$v(x,y) = -\sin(x)\, e^y + c$$

Problem 2

Suppose that v is a harmonic conjugate of u and u is a harmonic conjugate of v on some domain D. Show that u, v must then be constant on D. (Hint: show that all partial derivatives of u, v vanish on D) if v were a harmonic conjugate of u, then the following expressions should hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (1) \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (2)$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad (3) \qquad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad (4)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (5)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = 0 \quad (6)$$

Notice that if this were the case for both u and v, there would be an issue with fulfilling the first Cauchy equation for both x and y. If v were a harmonic conjugate of u. then (1) must hold. But if this were the case, and u is also a harmonic conjugate of v, then the same would have to hold in the case of (4).

This means that $\frac{\partial u}{\partial x}$ must be positive and negative, but the only case where this could be possible is in the case that it is 0. We have a similar problem in the case of expressions (3) and (2). Meaning that $\frac{\partial v}{\partial x}$ must be positive and negative. Once again, this is only possible in the case that it is 0.