Worksheet 2

MATH 3160
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Problem 1

Reduce each of these to a real number

(a) $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$

$$
\begin{aligned}
\frac{1+2i}{3-4i} + \frac{2-i}{5i} &= \\
\frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} + \frac{(2-i)(-5i)}{(5i)(-5i)} &= \\
\frac{(1+2i)(3+4i)}{9-16i^2} + \frac{(2-i)(-5i)}{-25i^2} &= \\
\frac{(3+4i+6i+8\cancelto{-1}{i^2})}{9-16\cancelto{-1}{i^2}} + \frac{(-10i+5\cancelto{-1}{i^2})}{-25\cancelto{-1}{i^2}} &= \\
\frac{(3+4i+6i-8)}{25} + \frac{(-10i-5)}{25} &= \\
\frac{(3+4i+6i-8)}{25} + \frac{(-10i-5)}{25} &= \\
\frac{(-5+10i)}{25} + \frac{(-10i-5)}{25} &= \\
\frac{(-5-5+10i-10i)}{25} &= \boxed{-\frac{2}{5}}
\end{aligned}
$$

(b) $\frac{5i}{(1-i)(2-i)(3-i)}$

$$
\begin{aligned}
\frac{5i}{(1-i)(2-i)(3-i)} &= \frac{5i}{(2-i-2i+i^2)(3-i)} = \\
\frac{5i}{(1-3i)(3-i)} &= \frac{5i}{(1-3i)(3-i)} = \\
\frac{5i}{-10i} &= \boxed{-\frac{1}{2}}
\end{aligned}
$$

Problem 2

Find the principal argument Arg z when..

preliminary necessary expressions:

$$\arg(z) = \text{Arg}(z) + 2 \cdot pi \cdot k \quad ; k \in \mathbb{Z}$$
$$e^{i\theta} = \cos\theta + i \sin\theta$$
$$-\pi < \theta \leq \pi$$

(a) $z = \frac{-2}{1+\sqrt{3}i}$

$$\frac{-2}{1+\sqrt{3}i} = \frac{-2}{1+\sqrt{3}i}\frac{1-\sqrt{3}i}{1-\sqrt{3}i} =$$
$$\frac{-2(1-\sqrt{3}i)}{1\cancelto{0}{-\sqrt{3}i+\sqrt{3}i} + \sqrt{3}^2\cancelto{-1}{i^2}} = \frac{-2+2\sqrt{3}i}{1-3}$$
$$2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2e^{i\frac{2\pi}{3}}$$
$$\therefore Arg z = \boxed{\frac{2\pi}{3}}$$

(b) $z = \frac{2i}{i-1}$

$$\frac{2i}{i-1} = \frac{2i(-i-1)}{(i-1)(-i-1)} = \frac{2i(-1-i)}{(-1+i)(-1-i)} =$$
$$\frac{(-2i-2i^2)}{1+i-i-i^2} = \frac{(-2i-2\cancelto{-1}{i^2})}{1\cancelto{0}{+i-i}-\cancelto{-1}{i^2}} = \frac{2-2i}{2} = 1-i = \sqrt{2}e^{-i\frac{\pi}{4}}$$
$$\therefore \text{Arg}(z) = \boxed{-\frac{\pi}{4}}$$

(c) $z = (\sqrt{3}-i)^6$

For this one, we need to first include a factor of 2^6 and divide by it as well to bring a half into the parentheses.

$$(\sqrt{3}-i)^6 = 2^6\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^6$$
$$2^6\left(e^{-i\frac{\pi}{6}}\right)^6 = 2^6 e^{-i\pi}$$
$$\therefore \text{Arg}(z) = \boxed{\pi}$$

Problem 3

For the next few questions write the individual factors on the left in exponential form, perform the needed operations on complex numbers, and finally change back to rectangular coordinates *Show that*:

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$

To convert into exponential form, we can add factors to get us the correct exponentials in normalized form:

$$i = e^{i\frac{\pi}{2}}$$
$$(1-\sqrt{3}i) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2e^{-i\frac{\pi}{3}}$$
$$(\sqrt{3}+i) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2e^{i\frac{\pi}{6}}$$

Note, $(\sqrt{3}+i)$ is $90°$ rotated from $(1-\sqrt{3}i)$. This is easily verified by adding $\pi/2$ to $\text{Arg}(1-\sqrt{3}i)$.

$$\therefore i(1-\sqrt{3}i) = e^{i\frac{\pi}{2}}2e^{-i\frac{\pi}{3}} = 2e^{i\left(\frac{\pi}{2}-\frac{\pi}{3}\right)}$$
$$2e^{i\left(\frac{3\pi}{6}-\frac{2\pi}{6}\right)} = 2e^{\frac{\pi}{6}}$$

Including the next factor gives:

$$2e^{\frac{\pi}{6}}2e^{i\frac{\pi}{6}} = 4e^{i\frac{2\pi}{6}} = 4e^{i\frac{\pi}{3}} = \boxed{2(1+\sqrt{3}i)}$$

(b) $(\sqrt{3}+i)^6 = -64$

This is the same as problem 2c with the angle being $\pi/6$ instead of the negative angle. Same work shows the result:

$$(\sqrt{3}+i)^6 = 2^6\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^6$$
$$2^6\left(e^{i\frac{\pi}{6}}\right)^6 = 2^6e^{i\pi} = 2^6(-1) = \boxed{-64}$$

(c) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$

Here we can first rationalize the fraction

$$(1+\sqrt{3}i)^{-10} = \left(\frac{1}{1+\sqrt{3}i}\right)^{10} =$$
$$\left(\frac{1-\sqrt{3}i}{(1+\sqrt{3}i)(1-\sqrt{3}i)}\right)^{10} = \left(\frac{1-\sqrt{3}i}{1-\sqrt{3}i+\sqrt{3}i-\sqrt{3}^2i^2}\right)^{10} =$$
$$\left(\frac{1-\sqrt{3}i}{4}\right)^{10} = 2^{-11}\left(\frac{1-\sqrt{3}i}{2}\right)^{10} =$$
$$2^{-11}\left(e^{-i\frac{\pi}{3}}\right)^{10} = 2^{-11}e^{-i\frac{10\pi}{3}} = 2^{-11}e^{-i\left(2\pi+\frac{4\pi}{3}\right)}$$
$$2^{-11}e^{-i\left(\pi+\frac{1\pi}{3}\right)} = 2^{-11}e^{-i\left(\frac{-2\pi}{3}\right)} = \boxed{2^{-11}(-1+\sqrt{3}i)}$$

Problem 4

Use exponential form to find $(1-i)^5$

$$(1-i)^5 = \left(\frac{2}{\sqrt{2}}\right)^5 \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^5 = \left(\frac{2}{\sqrt{2}}\right)^5 \left(e^{-i\frac{\pi}{4}}\right)^5 =$$

$$\text{angle } \frac{-5\pi}{4} \text{ corresponds to angle } \frac{3\pi}{4}$$

$$\left(\frac{2}{\sqrt{2}}\right)^5 e^{-i\frac{5\pi}{4}} = \left(\frac{2}{\sqrt{2}}^{5}\right) e^{i\frac{3\pi}{4}} =$$

$$\left(\frac{2}{\sqrt{2}}\right)^5 \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \left(\frac{2}{\sqrt{2}}\right)^4 (-1+i)$$

$$\text{recall} \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}^2} = \sqrt{2}$$

$$\therefore \left(\frac{2}{\sqrt{2}}\right)^4 = \sqrt{2}^4 = 2^2 = 4$$

$$\therefore (1-i)^5 = \boxed{4(-1+i)}$$

Worksheet # 3

MATH 3160 – Complex Variables
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Completed: August 31, 2025

Problem 1

Show that $\text{Re}(z) = \frac{z+\bar{z}}{2}$ and $\text{Im}(z) \cdot i = \frac{z-\bar{z}}{2}$ for any complex number $z = a + bi$:

Expressing the two as complex numbers and reducing:

$$\text{Re}(z) = \frac{z+\bar{z}}{2} = \frac{a+bi+a-bi}{2} = \frac{2a}{2} = a = \text{Re}(z)$$
$$\text{Im}(z) \cdot i = \frac{z-\bar{z}}{2} = \frac{a+bi-a+bi}{2} = \frac{2bi}{2} = bi = \text{Im}(z) \cdot i$$

Problem 2

Find the fourth roots of $-8-8\sqrt{3}i$. express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out the principal root.

$$-8-8\sqrt{3}i = -8(1+\sqrt{3}) = 16(-1)\left(\frac{1+\sqrt{3}}{2}\right) =$$

$$16e^{i\pi}e^{\frac{\pi}{3}} = 16e^{i\frac{3\pi}{3}}e^{\frac{\pi}{3}} = 16e^{i\frac{4\pi}{3}} = 16e^{i\frac{-2\pi}{3}}$$

$$16 = 2^4$$

$$16e^{i\frac{-2\pi}{3}} = 2^4e^{i\frac{-2\pi}{3}}$$

Starting from this point, we can take the 4^{th} root and then rotate that root by $\frac{2\pi}{4} = \frac{\pi}{2}$ to find the rest of the points.

$$(e^{i\frac{-2\pi}{3}})^{\frac{1}{4}} = e^{i\frac{-2\pi}{12}} = e^{i\frac{-\pi}{6}}$$

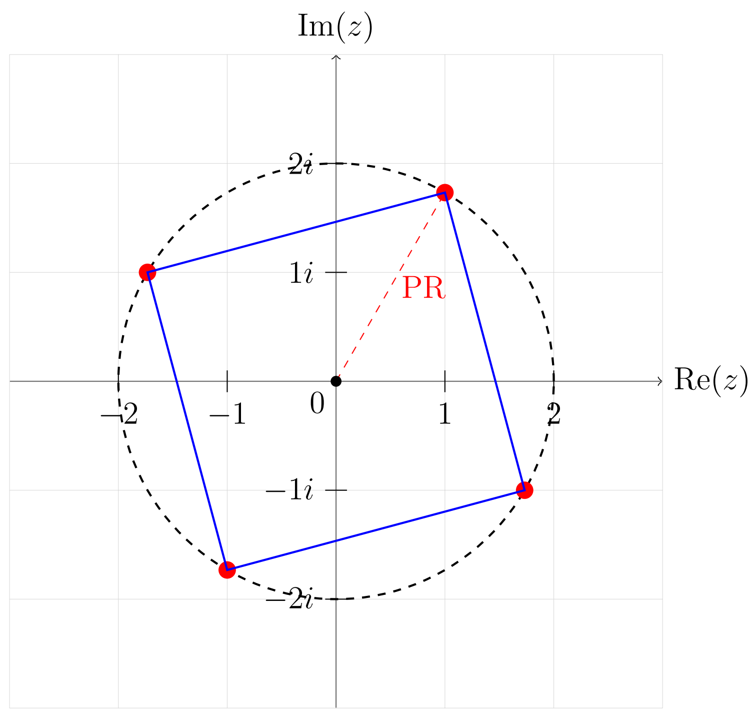
$$e^{i(\frac{-\pi}{6}+\frac{3\pi}{6})} = e^{i\frac{2\pi}{6}} = e^{i\frac{\pi}{3}}$$

$$e^{i\frac{2\pi+3\pi}{6}} = e^{i\frac{5\pi}{6}}$$

$$e^{i\frac{5+3\pi}{6}} = e^{i\left(\pi+\frac{2\pi}{6}\right)} = e^{i\left(\pi+\frac{\pi}{3}\right)}$$

So the principal root is the first root we get when moving counter-clockwise from 0, we get

$$= 2e^{i\frac{\pi}{3}}$$



Problem 3

Find the four zeros of the polynomial $z^4 + 4$, given that one of them is:

$$z_0 = \sqrt{2}e^{i\frac{\pi}{4}} = 1 + i$$

Use these zeros to factor $z^4 + 4$ into quadratic factors with real coefficients.

The zeros would be equally spaced because the polynomial can be factored into four roots.

$$z^4 + 4 = (z^2)^2 - -4 = (z^2)^2 - (2i)^2 =$$

$$(z^2 - 2i)(z^2 + 2i)$$

First root can also be put in terms of a difference of squares

$$(z^2 - 2i) = (z^2 - (\sqrt{2i})^2) = (z - \sqrt{2i})(z + \sqrt{2i})$$

Second root can also be put in terms of a difference of squares

$$(z^2 + 2i) = (z^2 - -2i) = (z^2 - (\sqrt{2i}i)^2) = (z - \sqrt{2i}i)(z + \sqrt{2i}i)$$

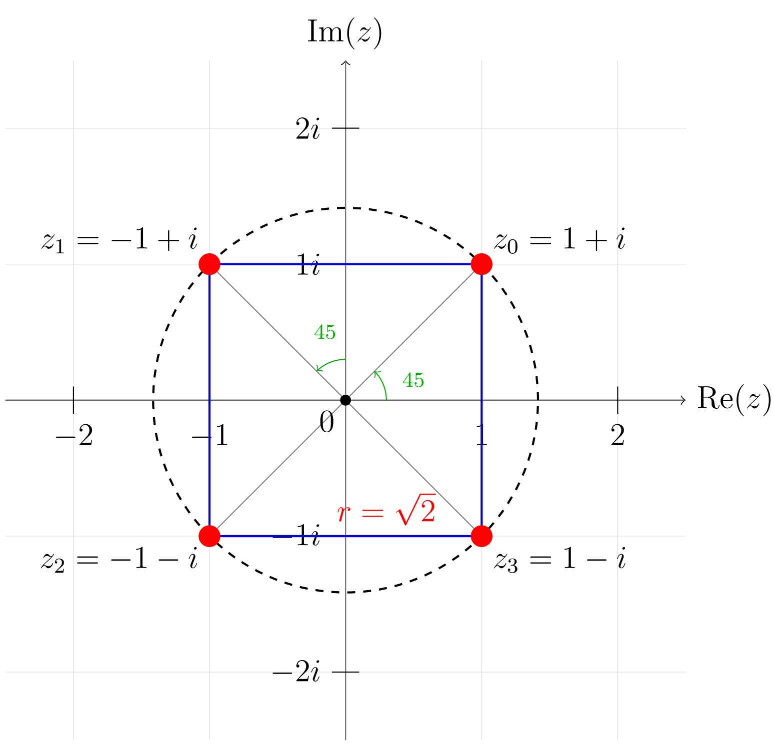
$$= (z - \sqrt{2i})(z + \sqrt{2i})(z - \sqrt{2i}i)(z + \sqrt{2i}i)$$

square root of i:

$$\sqrt{i} = i^{\frac{1}{2}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$$

magnitude of zero is $\sqrt{2}$

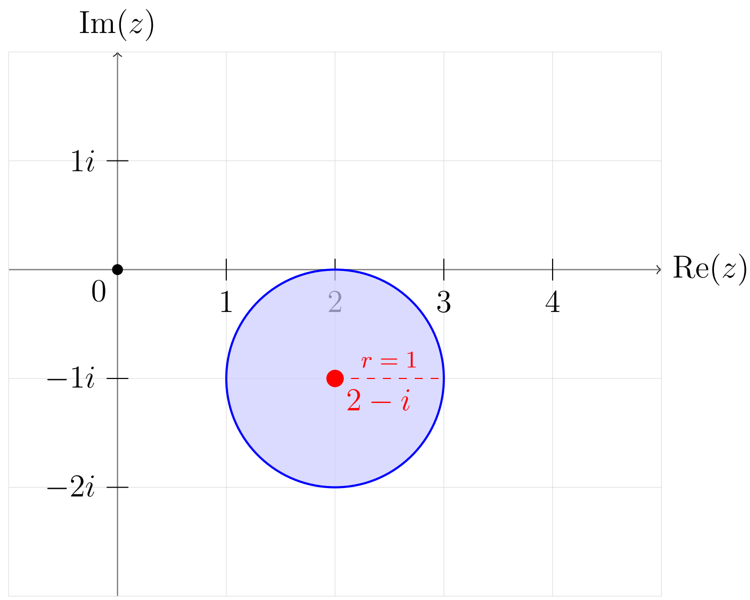
$\therefore$ The four zeros are the points at $\sqrt{2}$ from the center. These all have angles that are $\pm 45°$ from 0 and π.



Problem 4

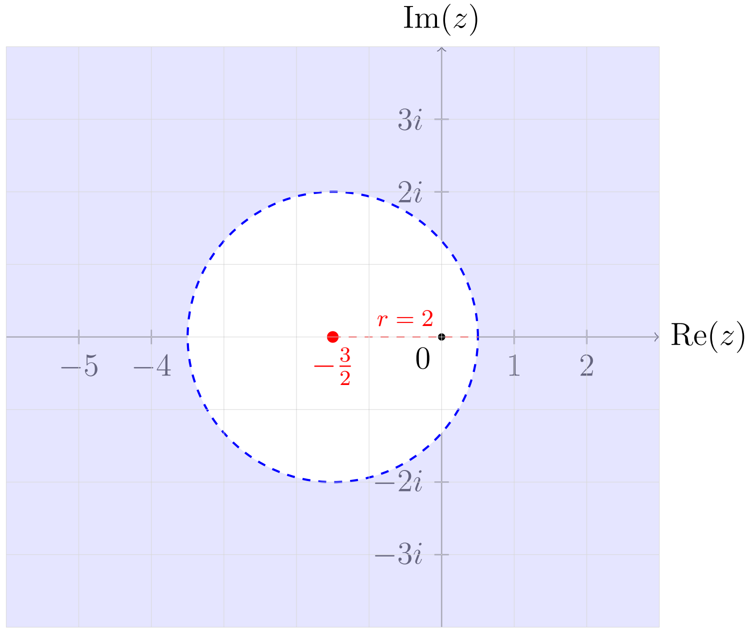
Sketch the following sets and state whether each set is open, connected, a domain, and whether it is bounded.

(a) $|z - 2 + i| \leq 1$



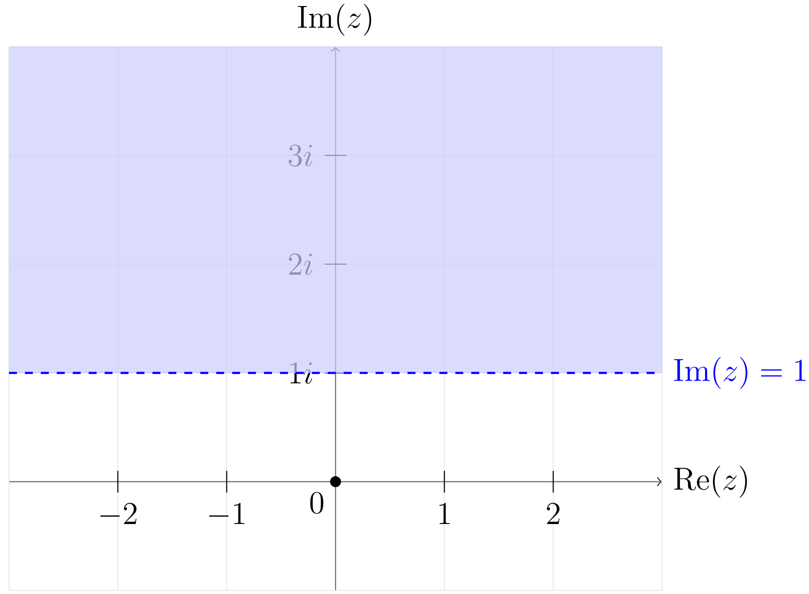
- **Open:** No, because the boundary is included ($\leq$ condition)
- **Connected:** Yes, it's a disk which is connected
- **Domain:** No, because it's not open
- **Bounded:** Yes, all points are within distance 1 from center $(2, -1)$

(b) $|2z + 3| > 4$



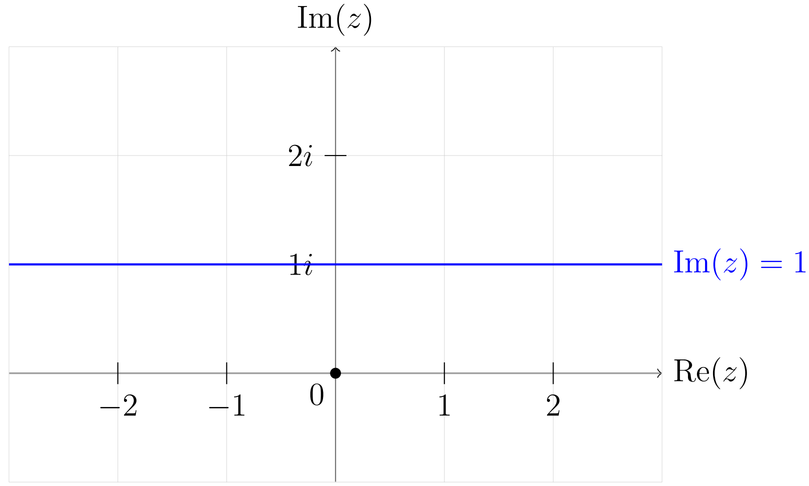
- **Open:** Yes, because boundary is not included
- **Connected:** Yes because the region outside the disk is connected
- **Domain:** Yes, open and connected are satisfied
- **Bounded:** No, the region extends to infinity

(c) $\text{Im}(z) > 1$



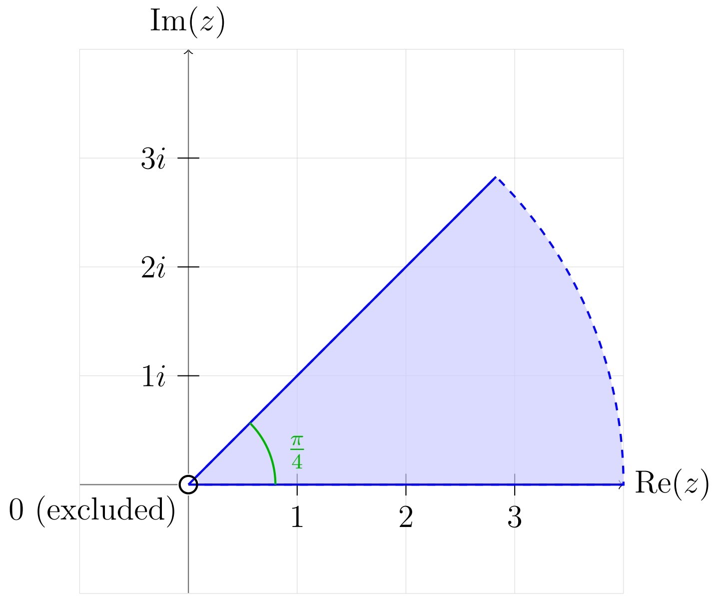
- **Open:** Yes, because boundary is not included
- **Connected:** Yes because the region above the line $\text{Im}(z) = 1$ is connected
- **Domain:** Yes, open and connected are satisfied
- **Bounded:** No, the region extends to infinity

(d) $\text{Im}(z) = 1$



- **Open:** No, because line is included
- **Connected:** No because no region of radius r can be formed on a line
- **Domain:** No, open and connected aren't satisfied
- **Bounded:** No, the line extends to infinity and there is no region

(e) $0 \leq \arg(z) \leq \frac{\pi}{4}$, where $z \neq 0$

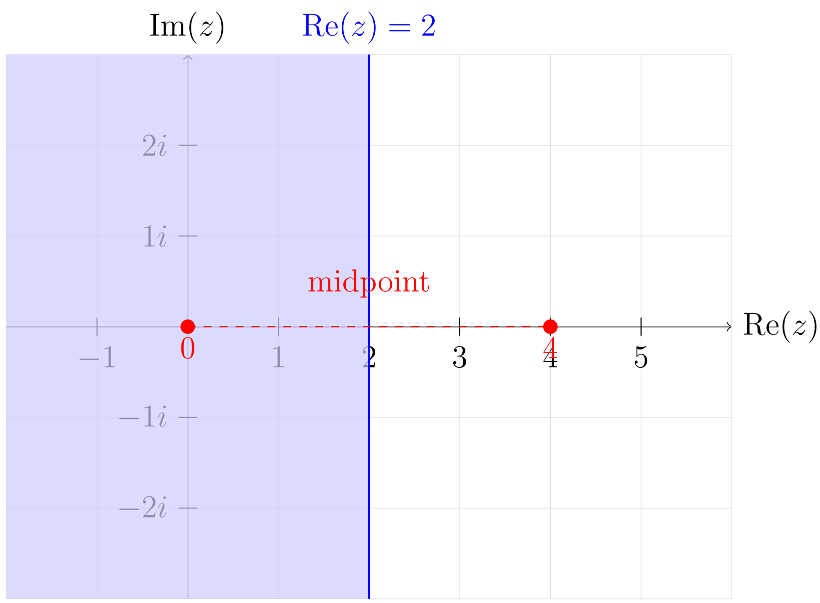


- **Open:** No, because boundary is included
- **Connected:** Yes because the region inside the arc is connected
- **Domain:** No, because both open and connected aren't satisfied
- **Bounded:** No, the region within the arc extends to infinity

(f) $|z-4| \geq |z|$

Equal at $z = 2$. If $z = 0$, then the expression holds. $4 \geq 0$. If $z > 2$, then we get a false condition. say it were 3:

$$|3-4| \geq |3| \rightarrow |-1| \ngeq |3|$$



- **Open:** No because the boundary is included.
- **Connected:** Yes because the region less than midpoint is connected
- **Domain:** No, because both open and connected aren't satisfied
- **Bounded:** No, the region less than midpoint extends to infinity