Worksheet # 6

MATH 3160 – Complex Variables
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Completed: September 24, 2025

Problem 1

Show that $u(x, y)$ is harmonic and find the harmonic conjugate $v(x, y)$ when:

(a) $u(x, y) = 2x(1 - y)$

(b) $u(x, y) = \cos(x) \cosh(y)$ where $\cosh(y) = \frac{e^y + e^{-y}}{2}$

(c) $u(x, y) = \frac{y}{x^2 + y^2}$

(d) $u(x, y) = \cos(x)\, e^y$

A function $u(x, y) : D \to \mathbb{R}$ is harmonic if:

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0$$

(a)

For $u(x, y) = 2x(1 - y)$, we must first find the partials of u and then apply these to the usual Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = 2(1 - y)$$

$$\text{By Cauchy-Riemann, this must equal} \quad \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = 2(1 - y)$$

Given this expression, we must integrate so that we have something that would give the result when evaluated as $\frac{\partial v}{\partial y}$.

$$\frac{\partial C}{\partial y} = \frac{\partial}{\partial y}\left(\int 2(1 - y)dy\right) = 2(1 - y)$$

$$\therefore \int 2(1 - y)dy = C = 2\left(y - \frac{1}{2}y^2 + c\right)$$

Additionally, we need to have the derivative of v wrt x to be the negative of u wrt y

$$\frac{\partial u}{\partial y} = -2x$$
$$\frac{\partial v}{\partial x} = 2x$$
$$\therefore v(x,y) = x^2 - y^2 + 2y + c$$

$$u(x,y) = 2x(1-y)$$
$$v(x,y) = x^2 - y^2 + 2y + c$$
$$\frac{\partial u}{\partial x} = 2(1-y) \qquad \frac{\partial u}{\partial y} = -2x$$
$$\frac{\partial v}{\partial y} = 2(1-y) \qquad \frac{\partial v}{\partial x} = 2x$$

Using these expressions for u and v now satisfy the CR equations. In order for this to be harmonic, the second derivatives must cancel to 0.

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$
$$\frac{\partial^2 u(x,y)}{\partial x^2} = 0$$
$$\frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$\therefore u(x,y)$ is harmonic and

$v(x,y) = x^2 - y^2 + 2y + c$ is the harmonic conjugate.

(b)

(c)

(d)

Problem 2

Suppose that v is a harmonic conjugate of u and u is a harmonic conjugate of v on some domain D. Show that u, v must then be constant on D. (Hint: show that all partial derivatives of u, v vanish on D)