Homework #2

MATH 3160 – complex variables
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Problem 1

By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i)$

$$\begin{aligned}
i &= e^{i\frac{\pi}{2}} \\
(1 - \sqrt{3}i) &= 2\left(\frac{1 - \sqrt{3}i}{2}\right) = 2e^{-i\frac{\pi}{3}} \\
(\sqrt{3} + i) &= 2\left(\frac{\sqrt{3} + i}{2}\right) = 2e^{i\frac{\pi}{6}} \\
e^{i\frac{\pi}{2}} 2e^{i\frac{\pi}{6}} 2e^{-i\frac{\pi}{3}} &= 4e^{i\pi\left(\frac{1}{2} + \frac{1}{6} - \frac{1}{3}\right)} \\
&= 4e^{i\pi\left(\frac{3}{6} + \frac{1}{6} - \frac{2}{6}\right)} \\
&= 4e^{i\pi\left(\frac{2}{6}\right)} \\
&= 4e^{i\left(\frac{\pi}{3}\right)} \\
&= 4\left(\frac{1 + \sqrt{3}i}{2}\right) \\
&= 2(1 + \sqrt{3}i) \quad \square
\end{aligned}$$

(b) $\frac{5i}{2+i} = 1 + 2i$

$$\begin{aligned}
\frac{5i}{2+i} &= \frac{5i(2-i)}{(2+i)(2-i)} = \frac{(10i - 5i^2)}{4 - 2i + 2i - i^2} \\
&= \frac{(10i - 5\cancelto{-1}{i^2})}{4\cancelto{0}{-2i + 2i} - \cancelto{-1}{i^2}} = \frac{(10i + 5)}{5} \\
&= 1 + 2i \quad \square
\end{aligned}$$

(c) $(-1+i)^7 = -8(1+i)$

$$-1+i = \sqrt{2}\left(\frac{-\sqrt{2}+\sqrt{2}i}{2}\right) = \sqrt{2}e^{i\frac{3\pi}{4}}$$

$$\begin{aligned}(-1+i)^7 &= (\sqrt{2}e^{i\frac{3\pi}{4}})^7 = \sqrt{2}^7 e^{i\frac{21\pi}{4}} \\ &= \sqrt{2}^7 e^{i\frac{21\pi}{4}} = \sqrt{2}^7 e^{i\left(\frac{24\pi}{4}-\frac{3\pi}{4}\right)} = \sqrt{2}^7 e^{i\left(6\pi-\frac{3\pi}{4}\right)} \\ &= \sqrt{2}^7 \overset{1}{\cancel{e^{i6\pi}}} e^{-i\left(\frac{3\pi}{4}\right)} = \sqrt{2}^7 e^{-i\left(\frac{3\pi}{4}\right)} \\ &= \sqrt{2}^7 \frac{1}{\sqrt{2}}(-1-i) = -2^3(1+i) \\ &= -8(1+i) \quad \square\end{aligned}$$

(d) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$

$$\begin{aligned}(1+\sqrt{3}i)^{-10} &= \left(2\frac{1+\sqrt{3}i}{2}\right)^{-10} = 2^{-10}\left(\frac{1+\sqrt{3}i}{2}\right)^{-10} = 2^{-10}\left(e^{i\frac{\pi}{3}}\right)^{-10} \\ &= 2^{-10}e^{-i\frac{10\pi}{3}} = 2^{-10}e^{-i\frac{(6\pi+4\pi)}{3}} = 2^{-10}e^{-i\frac{(4\pi)}{3}} \\ &= 2^{-10}e^{i\frac{2\pi}{3}} = 2^{-10}\left(\frac{-1+\sqrt{3}i}{2}\right) = 2^{-11}(-1+\sqrt{3}i) \quad \square\end{aligned}$$

Problem 2

Find the square roots of (a) $2i$ and (b) $(1-\sqrt{3}i)$ express them in rectangular coordinates

(a):

$$\sqrt{2i} = \sqrt{2}\sqrt{i} = \sqrt{2}i^{\frac{1}{2}} = \sqrt{2}(e^{i\frac{\pi}{2}})^{\frac{1}{2}} = \sqrt{2}e^{i\frac{\pi}{4}}$$
$$\sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}(1+1i)$$

Second root is at $\frac{2\pi}{2} = \pi$ from this one:

$$\sqrt{2}e^{-i\frac{3\pi}{4}}$$

(b):

$$\sqrt{(1-\sqrt{3}i)} = \sqrt{2\left(\frac{1-\sqrt{3}i}{2}\right)} = \sqrt{2}\left(\frac{1-\sqrt{3}i}{2}\right)^{\frac{1}{2}} = \sqrt{2}(e^{-i\frac{\pi}{3}})^{\frac{1}{2}}$$
$$= \sqrt{2}e^{-1\frac{\pi}{6}}$$

Again the other root is π away:

$$= \sqrt{2}e^{-i(\frac{\pi}{6}+\pi)} = \sqrt{2}e^{i\frac{5\pi}{6}}$$

Problem 3

Find all roots and indicate in rectangular coordinates

(a) $(-16)^{\frac{1}{4}}$

$$2^4 = 16$$
$$-1 = i^2$$
$$-16 = i^2 16 = i^2 4^2 = \sqrt{i}^4 2^4 = (2\sqrt{i})^4$$

roots are the four angles corresponding to

$$\{\pm\frac{\pi}{4},\ \pm\frac{3\pi}{4}\}$$
$$2 \cdot \{(1+i),\ (-1+i),\ (-1-i),\ (1-i)\}$$

(b) $(-8-8\sqrt{3}i)^{\frac{1}{4}}$ These make up a square with the first angle being $\frac{\pi}{3}$ with each separated by 90°.

$$(-8-8\sqrt{3}i)^{\frac{1}{4}} = \left(-16\left(\frac{1+\sqrt{3}i}{2}\right)\right)^{\frac{1}{4}} =$$
$$(-16e^{i\frac{\pi}{3}})^{\frac{1}{4}} = (-16)^{\frac{1}{4}}(e^{i\frac{\pi}{3}})^{\frac{1}{4}}$$

This is the same as the roots of (a) rotated by $\dfrac{\pi}{12}$

$$(-16)^{\frac{1}{4}}(e^{i\frac{\pi}{12}}) =$$

roots are the four angles corresponding to

$$2 \cdot \{\pm\frac{\pi}{4} + \frac{\pi}{12},\ \pm\frac{3\pi}{4} + \frac{\pi}{12}\}$$
$$2 \cdot \{\pm\frac{3\pi}{12} + \frac{\pi}{12},\ \pm\frac{9\pi}{12} + \frac{\pi}{12}\}$$
$$2 \cdot \{\frac{4\pi}{12},\ -\frac{2\pi}{12},\ \frac{10\pi}{12}, -\frac{8\pi}{12}\}$$
$$2 \cdot \{\frac{\pi}{3},\ -\frac{\pi}{6},\ \frac{5\pi}{6}, -\frac{2\pi}{3}\}$$
$$2 \cdot \{\frac{(1+\sqrt{3}i)}{2},\ \frac{(\sqrt{3}-i)}{2},\ \frac{-(1+\sqrt{3}i)}{2},\ \frac{(-\sqrt{3}+i)}{2}\}$$
$$\therefore \text{roots are:}\quad \{(1+\sqrt{3}i),\ (\sqrt{3}-i),\ -(1+\sqrt{3}i),\ (-\sqrt{3}+i)\}$$

Problem 4

Find the four zeros of $z^4 + 4$

Converting z into exponential form for $|z| = 1$

$$z^4 = (e^{i\theta})^4 = e^{i4\theta}$$

Four windings for this, so we should have 4 equally spaced roots that give us the zeros.

$$\begin{aligned}
z^4 + 4 &= 0 \\
z^4 &= -4 \\
\sqrt[4]{z^4} &= \sqrt[4]{-4} \\
z^{\frac{1}{4}4} &= \sqrt[4]{-4} = e^{\frac{1}{4}i4\theta} \\
z &= \sqrt[4]{-4} = e^{i\theta} \\
&= \sqrt[4]{-2 \cdot 2} = \sqrt[4]{-\sqrt{2}^2 \cdot \sqrt{2}^2} = \sqrt[4]{-\sqrt{2}^4} \\
&= \sqrt{2} \cdot \sqrt[4]{-1}
\end{aligned}$$

$$\begin{aligned}
\sqrt{z^4} &= \pm z^2 = \sqrt{-4} = \pm 2\sqrt{-1} \\
\sqrt{\pm z^2} &= \{\sqrt{z^2}, \sqrt{-z^2}\} = \{\pm z, \pm z\sqrt{-1}\} \\
&= \{\pm z, \pm zi\}
\end{aligned}$$

This is also the following:

$$\begin{aligned}
z^4 + 4 &= z^4 - (-4) = z^4 - (i^2 2^2) = (z^2)^2 - (2i)^2 \\
&= (z^2 - 2i)(z^2 + 2i)
\end{aligned}$$

converting to exponential

$$\begin{aligned}
i &= e^{i\frac{\pi}{2}} \\
\therefore \sqrt{i} &= (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}} \\
(z^2 - 2i) &= (z^2 - (\sqrt{2}\sqrt{i})^2) = (z - (\sqrt{2}\sqrt{i}))(z + (\sqrt{2}\sqrt{i})) \\
&= \boxed{(z - (\sqrt{2}\sqrt{i}))(z + (\sqrt{2}\sqrt{i}))}
\end{aligned}$$

similarly, for the positive side

$$\begin{aligned}
(z^2 + 2i) &= (z^2 - (-2i)) = (z^2 - i^2(\sqrt{2}\sqrt{i})^2) \\
&= \boxed{(z - (\sqrt{2}\sqrt{i})i)(z + (\sqrt{2}\sqrt{i})i)} \\
i\sqrt{i} &= e^{\frac{i\pi}{2}} e^{i\frac{\pi}{4}} = e^{i\pi(\frac{1+2}{4})} = e^{i\frac{3\pi}{4}} \\
\therefore z &= \sqrt{2}\{\pm e^{i\frac{\pi}{4}}, \pm e^{i\frac{3\pi}{4}}\} = \sqrt{2}\{(1+i), (-1+i), (-1-i), (1-i)\}
\end{aligned}$$

Problem 5

Show that if c is an n^{th} root of 1 other than 1 itself, then:

$$1 + c + c^2 + ... + c^{n-1} = 0$$

Hint: multiply above by $(c - 1)$

multiplying the above by $(c - 1)$ gives the following

$$(c-1) \cdot (1 + c + c^2 + ... + c^{n-1}) = (c-1) \cdot 0$$

$$c + c^2 + c^3 + ... + c^{n-1} + c^n + \quad \text{expanding } c$$

$$-1 - c - c^2 - ... - c^{n-1} \quad \text{expanding } -1$$

$$-1 + (c - c) + (c^2 - c^2) + (c^3 - c^3) + ... + (c^{n-1} - c^{n-1}) + c^n = 0$$

$$-1 + \cancelto{0}{(c - c)} + \cancelto{0}{(c^2 - c^2)} + \cancelto{0}{(c^3 - c^3)} + ... + \cancelto{0}{(c^{n-1} - c^{n-1})} + c^n = 0$$

$$-1 + c^n = 0$$

$$c^n = 1$$

$$\sqrt[n]{c^n} = \sqrt[n]{1}$$

$$c = 1$$

Now using something other than 1, i.e. $c \neq 1$, then the sum cannot be 0.

$$1 + c + c^2 + ... + c^{n-1} = S$$

same steps as before

$$c^n - 1 = S(c-1)$$

since $c^n = 1$; and $c \neq 1$ we can divide

$$\frac{c^n - 1}{c - 1} = S$$

since $c \neq 1$; $(c - 1)$ is not 0 but $c^n = 1$

$$\therefore \frac{\cancelto{1}{c^n} - 1}{c - 1} = \frac{0}{c - 1} = S = 0 \quad \square$$

Problem 6

For each of the below, indicate the domain of definition.

(a) $f(z) = \frac{1}{z^2+1}$

We need $z^2 + 1 \neq 0$, which means $z^2 \neq -1$.

Solving $z^2 = -1$:

$$z^2 = -1 = i^2$$
$$z = \pm i$$
$$(\pm i)^2 + 1 = i^2 + 1 = -1 + 1 = 0$$

undefined only at $z = i$ and $z = -i$.

For all other complex numbers z, we have $z^2 + 1 \neq 0$, so the function is well-defined.

$\therefore$ Domain of Definition: $\mathbb{C} \setminus \{\pm i\}$

(b) $f(z) = \text{Arg}\left(\frac{1}{z}\right)$

Here we need $z \neq 0$ Since the division is not defined when $z = 0$.

So for z:

$$\text{Domain of definition: } \mathbb{C} \setminus \{\mathbf{0}\}$$

This results in the $\text{Arg}(z)$ being defined in the typical range:

$$-\pi < \text{Arg}(z) \leq \pi$$

(c) $f(z) = \frac{z}{z-\bar{z}}$

$$\text{Recall } z - \bar{z} = 2i\text{Im}(z)$$
$$\therefore \frac{z}{z-\bar{z}} = \frac{z}{2i\text{Im}(z)} = \frac{z(-2i\text{Im}(z))}{-(2i\text{Im}(z))^2}$$
$$= \frac{-2z\text{Im}(z)i}{-(4i^2\text{Im}(z)^2)} = \frac{-2z\text{Im}(z)i}{(4\text{Im}(z)^2)}$$
$$\text{Condition: } \text{Im}(z) \neq 0$$
$$\text{Domain of Definition:} \mathbb{C} \setminus \{z = a + bi \mid b = 0; a \in \mathbb{R}\}$$

Note*: I mistakenly did this for minus in the denominator, it should be plus after looking over HW sheet again.

(c) corrected: $f(z) = \frac{z}{z+\bar{z}}$

$$\text{Recall } z + \bar{z} = 2\text{Re}(z)$$
$$\therefore \frac{z}{z+\bar{z}} = \frac{z}{2\text{Re}(z)}$$
$$\text{Condition: } \text{Re}(z) \neq 0$$
$$\text{Domain of Definition:} \mathbb{C} \setminus \{z = a + bi \mid a = 0; b \in \mathbb{R}\}$$

(d) $f(z) = \frac{1}{(1-|z|^2)}$ Since $1 - |z|^2$ is in the denominator, our condition here is as follows:

$$|z|^2 \neq 1; \quad |z| = \sqrt{a^2+b^2}$$
$$|z|^2 = a^2 + b^2$$
$$a^2 + b^2 \neq 1$$

This corresponds to all points on the unit circle.

$$\therefore \text{Domain of Definition: } \mathbb{C} \setminus \{|z| = a^2 + b^2 = 1\}$$

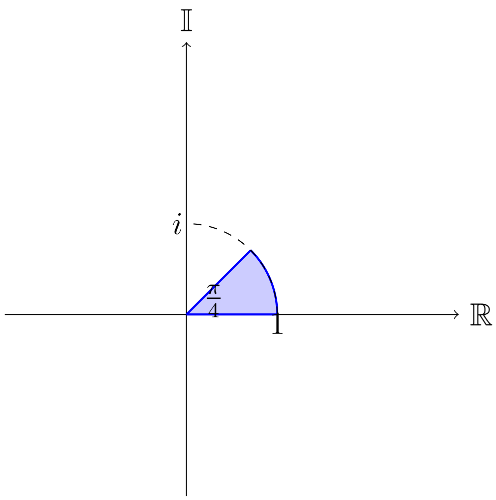
1 Problem 7

Sketch the region onto which the sector $r \leq 1;\ 0 \leq \theta \leq \frac{\pi}{4}$ in the z-plane is mapped to the $w = f(z)$-plane by the transformations

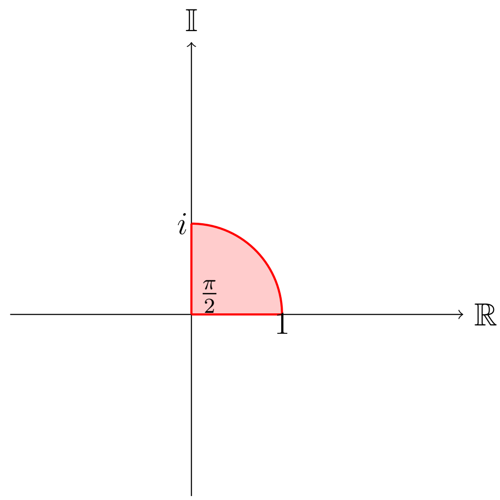
(a) $w = z^2$

$$z = e^{i\theta}$$
$$z^2 = (e^{i\theta})^2 = e^{i2\theta}$$

Corresponds to two windings of the angle θ, essentially moving by 2θ instead:

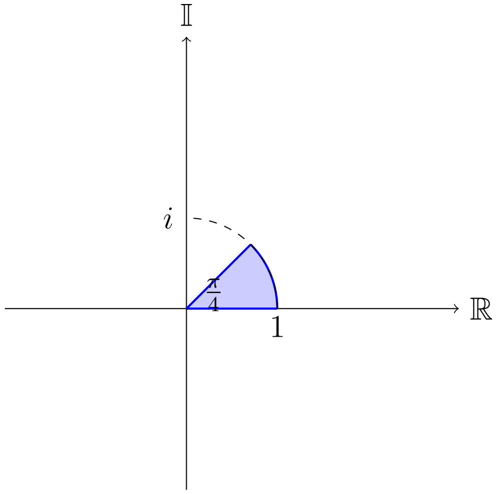


z-plane

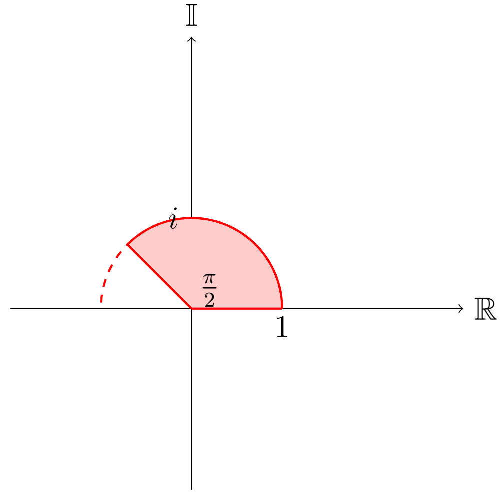


w-plane

(b) $w = z^3$ Corresponds to three windings of the angle like before but with 3θ:



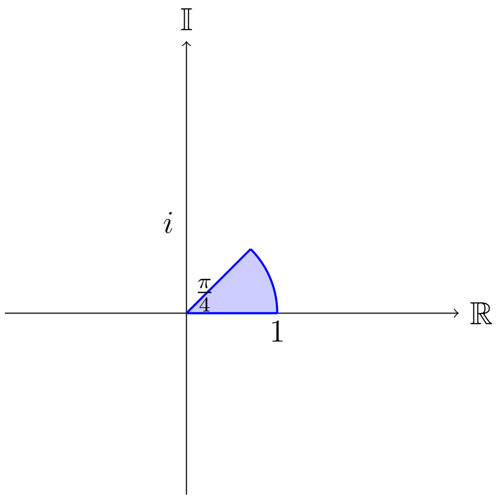
z-plane



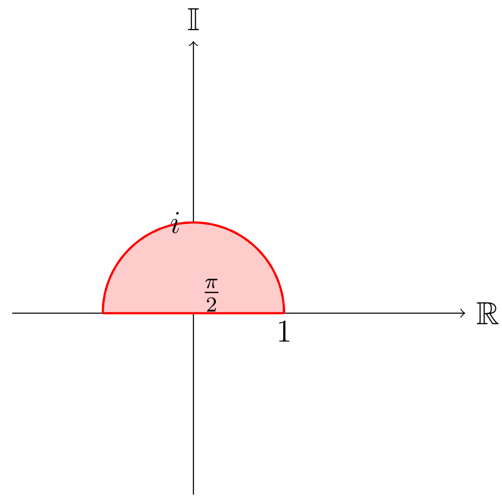
w-plane

(c) $w = z^4$

Corresponds to four windings of the angle like before but with 4θ:



z-plane



w-plane