Homework 1: Complex Numbers

MATH 3160
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Problem 1: Complex Number Reduction

Reduce each of these to a real number:

(a) $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$

$$
\begin{aligned}
\frac{1+2i}{3-4i} + \frac{2-i}{5i} &= \\
\frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} + \frac{(2-i)(-5i)}{(5i)(-5i)} &= \\
\frac{(1+2i)(3+4i)}{9-16i^2} + \frac{(2-i)(-5i)}{-25i^2} &= \\
\frac{(3+4i+6i+8\cancelto{-1}{i^2})}{9-16\cancelto{-1}{i^2}} + \frac{(-10i+5\cancelto{-1}{i^2})}{-25\cancelto{-1}{i^2}} &= \\
\frac{(3+4i+6i-8)}{25} + \frac{(-10i-5)}{25} &= \\
\frac{(3+4i+6i-8)}{25} + \frac{(-10i-5)}{25} &= \\
\frac{(-5+10i)}{25} + \frac{(-10i-5)}{25} &= \\
\frac{(-5-5+10i-10i)}{25} = \boxed{-\frac{2}{5}}
\end{aligned}
$$

(b) $\frac{5i}{(1-i)(2-i)(3-i)}$

$$\frac{5i}{(1-i)(2-i)(3-i)} = \frac{5i}{(2-i-2i+i^2)(3-i)} =$$

$$\frac{5i}{(2-i-2i+\cancelto{-1}{i^2})(3-i)} = \frac{5i}{(1-3i)(3-i)} =$$

$$\frac{5i}{3-i-9i+3i^2} = \frac{5i}{3-i-9i+3\cancelto{-1}{i^2}} =$$

$$\frac{5i}{3-i-9i-3} = \frac{5i}{-10i} = \boxed{-\frac{1}{2}}$$

(c) $(1-i)^4$

$$(1-i)^4 = \left(\frac{2}{\sqrt{2}}\right)^4 \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^4 =$$

$$\left(\frac{2}{\sqrt{2}}\right)^4 \left(e^{-\frac{i\pi}{4}}\right)^4 = \left(\frac{2}{\sqrt{2}}\right)^4 e^{-i\pi} = -\left(\sqrt{2}\right)^4 = -2^2 = \boxed{-4}$$

Problem 2: Vector Addition and Subtraction

Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially by drawing a graph when:

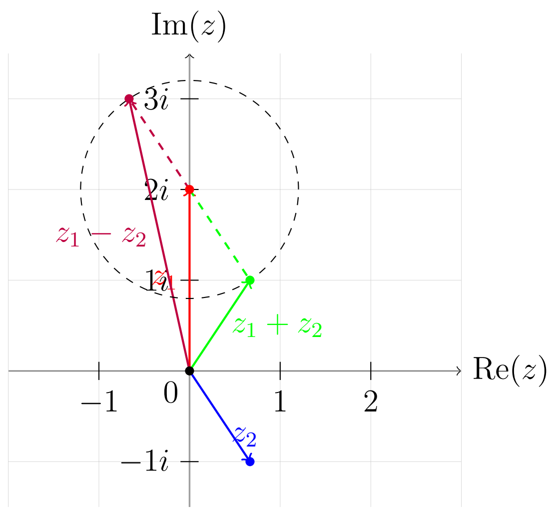
(a) $z_1 = 2i$, $z_2 = 2/3 - i$

(b) $z_1 = -\sqrt{3} + i$, $z_2 = \sqrt{3}$

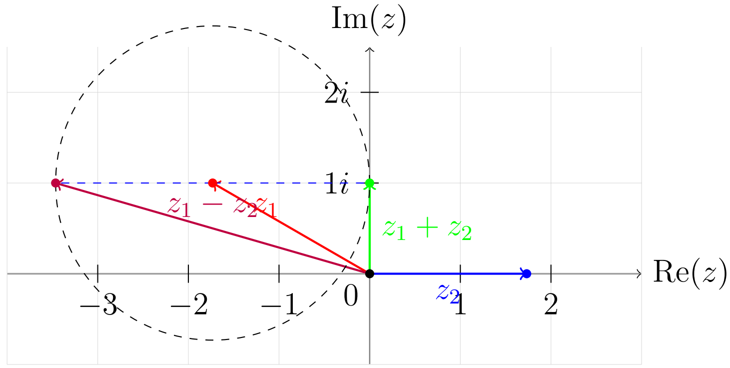
(c) $z_1 = (3, 1)$, $z_2 = (1, 4)$

(d) $z_1 = x_1 + iy_1$, $z_2 = x_1 - iy_1$

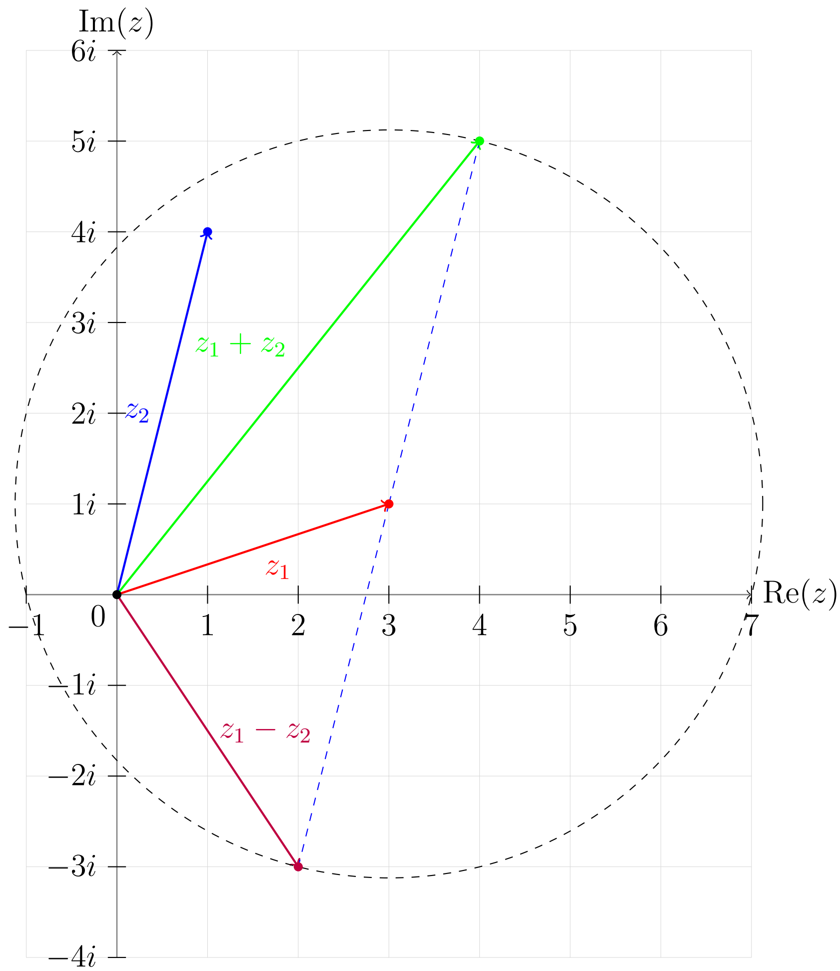
$z_1 = 2i$, $z_2 = 2/3 - i$



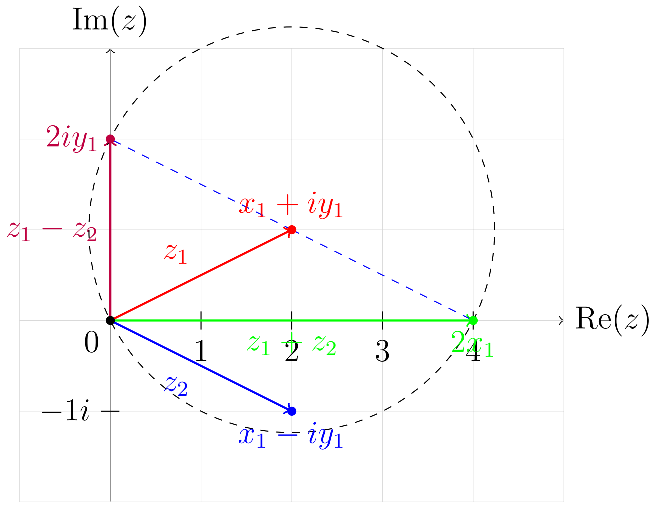
$z_1 = -\sqrt{3} + i$, $z_2 = \sqrt{3}$



$z_1 = (3, 1)$, $z_2 = (1, 4)$



$z_1 = x_1 + iy_1$, $z_2 = x_1 - iy_1$



Problem 3: Geometric Sets in the Complex Plane

In each case, sketch the set of points determined by the given condition:

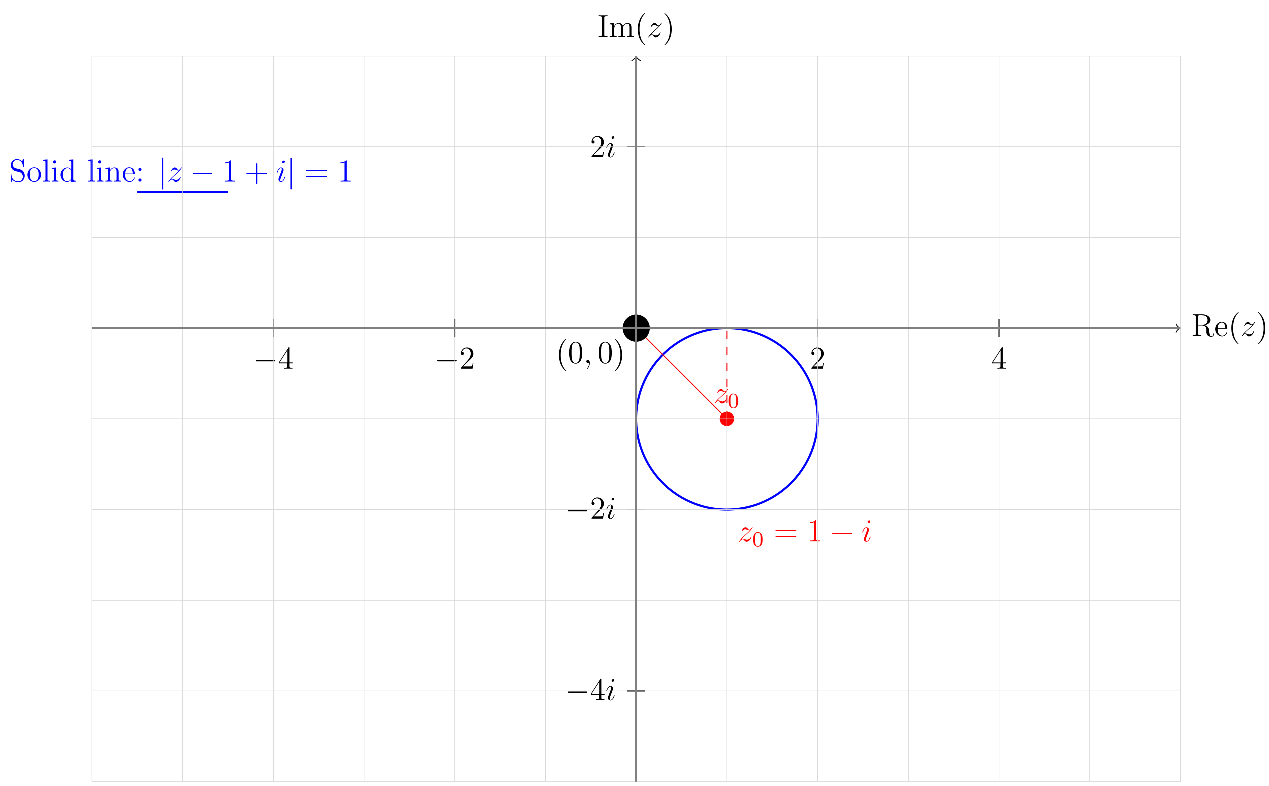
(a) $|z - 1 + i| = 1$

(b) $|z + i| \leq 3$

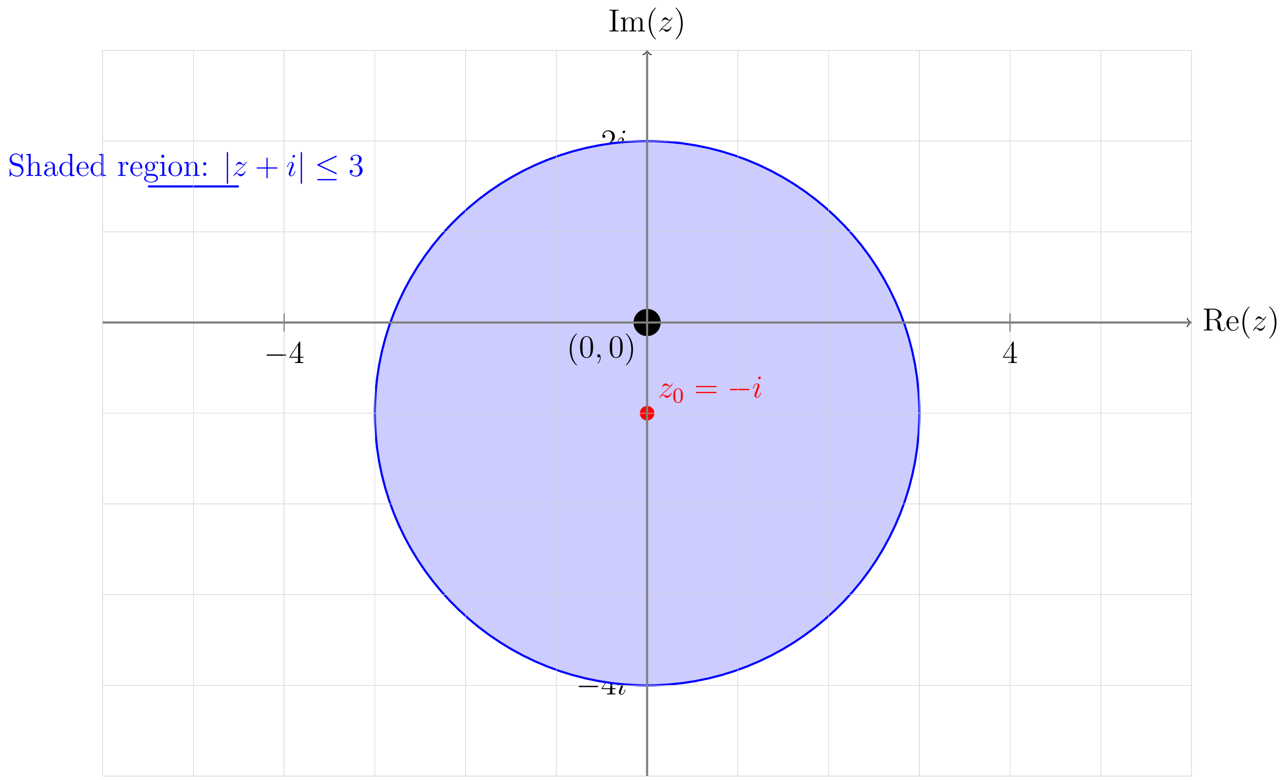
(c) $|z - 4i| \geq 4$

Hint: Note that for any two complex numbers z_1, z_2, the absolute value $|z_1 - z_2|$ is the distance between z_1 and z_2 in the complex plane.

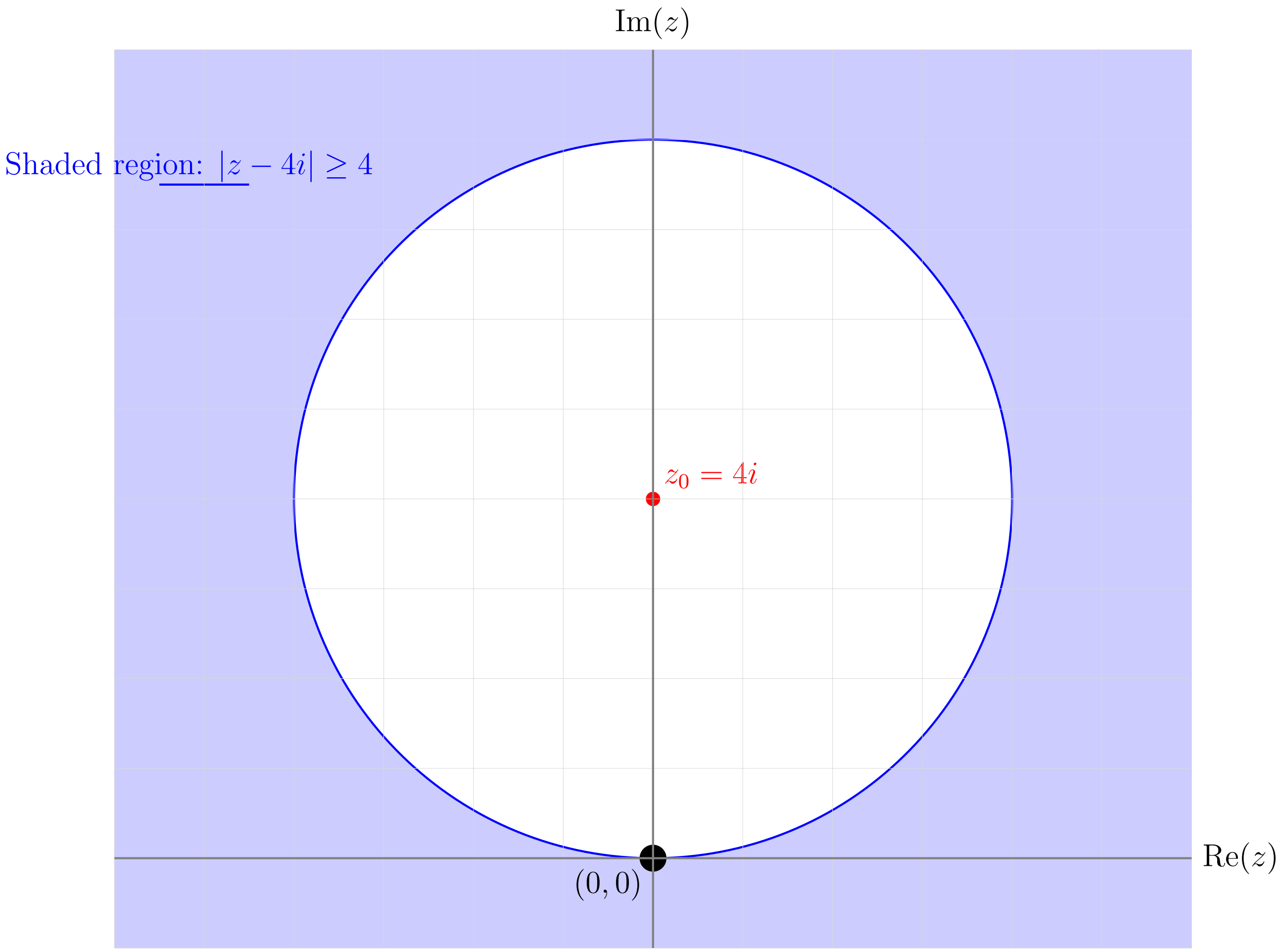
$|z - 1 + i| = 1$



$|z+i| \leq 3$



$|z - 4i| \geq 4$



Problem 4: Principal Arguments

preliminary necessary expressions:

$$\text{Arg}(z) = \theta + 2 \cdot pi \cdot k \quad ; k \in \mathbb{Z}$$
$$e^{i\theta} = \cos\theta + i \sin\theta$$

Find the principal argument $\text{Arg}(z)$ when:

(a) $z = \frac{i}{-2-2i}$

$$\frac{i}{-2-2i} \cdot \frac{-2+2i}{-2+2i} = \frac{i(-2+2i)}{(-2-2i)(-2+2i)} =$$
$$\frac{(-2i+2i^2)}{(4-4i+4i-4i^2)} = \frac{(-2i+2\cancelto{-1}{i^2})}{(4\cancelto{0}{-4i+4i}-4\cancelto{-1}{i^2})} =$$
$$-\frac{2i+2}{8} = -\frac{i+1}{4} = -\frac{i+1}{2 \cdot 2} = \frac{1}{2} \cdot -\frac{1+i}{2}$$

Introduce a factor of $\sqrt{2}$ to top and bottom of fraction to convert into normalized vector

$$\frac{1}{2\sqrt{2}} \cdot -\frac{\sqrt{2}(1+i)}{2} = \frac{1}{2\sqrt{2}} \cdot e^{-\frac{3\pi i}{4}}$$
$$\therefore \text{Arg}(z) = \boxed{-\frac{3\pi}{4}}$$

(b) $z = (\sqrt{3} - i)^6$

For this one, we are already very close. We can just introduce a factor of 2 over 2 six times to be able to bring the half into the parentheses:

$$(\sqrt{3}-i)^6 = \left(\frac{2}{2}\right)^6 \cdot (\sqrt{3}-i)^6 =$$
$$(2)^6 \cdot \left(\frac{\sqrt{3}-i}{2}\right)^6 = (2)^6 \cdot (e^{\frac{-i\pi}{6}})^6 = 2^6 \cdot e^{-i\pi} = 2^6 \cdot e^{i\pi}$$
$$\therefore \text{Arg}(z) = \boxed{\pi}$$

Problem 5: Argument Properties

For any two non-zero complex numbers z_1, z_2, show that any angle θ in the set $\arg(z_1 z_2)$ can be written as

$$\theta = \theta_1 + \theta_2$$

where $\theta_1 \in \arg(z_1)$ and $\theta_2 \in \arg(z_2)$. Also find an example where the principal argument $\operatorname{Arg}(z_1 z_2)$ is not equal to $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$.

Since we have two non-zero complex numbers, we can express them in exponential form:

$$z_1 = |z_1|e^{i\theta_1}, \quad z_2 = |z_2|e^{i\theta_2}$$

where $\theta_1 = \operatorname{Arg}(z_1)$ and $\theta_2 = \operatorname{Arg}(z_2)$.

Converting multiplication into addition in the exponent:

$$\begin{aligned} z_1 z_2 &= |z_1||z_2|e^{i\theta_1} \cdot e^{i\theta_2} \\ &= |z_1||z_2|e^{i(\theta_1+\theta_2)} \end{aligned}$$

Since for any complex number z, we have $\arg(z) = \operatorname{Arg}(z) + 2\pi k$ where $k \in \mathbb{Z}$:

$$\begin{aligned} \theta_1 \in \arg(z_1) &\implies \theta_1 = \theta_1 + 2\pi k_1, \quad k_1 \in \mathbb{Z} \\ \theta_2 \in \arg(z_2) &\implies \theta_2 = \theta_2 + 2\pi k_2, \quad k_2 \in \mathbb{Z} \\ \theta \in \arg(z_1 z_2) &\implies \theta = (\theta_1 + \theta_2) + 2\pi k, \quad k \in \mathbb{Z} \end{aligned}$$

Therefore, any angle $\theta \in \arg(z_1 z_2)$ can be written as:

$$\begin{aligned} \theta &= (\theta_1 + 2\pi k_1) + (\theta_2 + 2\pi k_2) \\ &= (\theta_1 + \theta_2) + 2\pi k \\ &= \theta_1 + \theta_2 + 2\pi k \end{aligned}$$

where we choose $k_1, k_2 \in \mathbb{Z}$ such that $k = k_1 + k_2$.

Example where $\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$**:**

$$\begin{aligned} z_1 = e^{-i\pi/2}, &\quad z_2 = e^{-i\pi/2} \\ \mathrm{Arg}(z_1) &= -\frac{\pi}{2} \\ \mathrm{Arg}(z_2) &= -\frac{\pi}{2} \\ \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) &= -\frac{\pi}{2} + \left(-\frac{\pi}{2}\right) = -\pi \\ \text{Since } -\pi &\notin (-\pi, \pi] : \\ \mathrm{Arg}(z_1 z_2) &= -\pi + 2\pi = \pi \\ \therefore \mathrm{Arg}(z_1 z_2) &\neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \end{aligned}$$

Note: The principal arguments are equal, $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$, if and only if $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \in (-\pi, \pi]$. Otherwise, the sum must be adjusted by adding or subtracting 2π to bring it into the principal range.

Problem 6: Principal Argument Addition

Show that if $\text{Re}(z_1) > 0$ and $\text{Re}(z_2) > 0$, then $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$. Use polar form of z_1 and z_2 to do so.

Let $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$ where $\theta_1 = \text{Arg}(z_1)$ and $\theta_2 = \text{Arg}(z_2)$.

Since $\text{Re}(z_1) = |z_1|\cos(\theta_1) > 0$ and $|z_1| > 0$, we have $\cos(\theta_1) > 0$. Similarly, since $\text{Re}(z_2) = |z_2|\cos(\theta_2) > 0$ and $|z_2| > 0$, we have $\cos(\theta_2) > 0$.

For $\cos(\theta) > 0$ with $\theta \in (-\pi, \pi]$, we must have $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Therefore:

$$\begin{aligned} \theta_1 &\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \theta_2 &\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \implies \theta_1 + \theta_2 &\in (-\pi, \pi) \end{aligned}$$

Since $\theta_1 + \theta_2 \in (-\pi, \pi) \subset (-\pi, \pi]$, no adjustment is needed, and: $\text{Arg}(z_1 z_2) = \theta_1 + \theta_2 = \text{Arg}(z_1) + \text{Arg}(z_2)$