Worksheet # 4

MATH 3160 – Complex Variables
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Problem 1:

Write the following functions $f(z)$ in the form $f(z) = u(x,y) + iv(x,y)$

(a) $f(z) = z^3 + z + 1$

writing out each z as $x + iy$

$$
\begin{aligned}
f(z) &= z^3 + z + 1 = (x+iy)^3 + (x+iy) + 1 \\
(x+iy)^3 &= (x+iy)(x+iy)(x+iy) = (x^2 + 2ixy + i^2y^2)(x+iy) = \\
(x^2 + 2ixy - y^2)(x+iy) &= x^3 + ix^2y + 2ix^2y + 2i^2xy^2 - y^2x - iy^3 = \\
&= x^3 - 3xy^2 + i(3x^2y - y^3) \\
f(z) &= x^3 - 3xy^2 + i(3x^2y - y^3) + (x+iy) + 1 = \\
&= \boxed{(x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)}
\end{aligned}
$$

(b) $f(z) = \frac{\bar{z}^2}{z}$ for $z \neq 0$

$$
\begin{aligned}
f(z) = \frac{\bar{z}^2}{z} &= \frac{(\overline{x+iy})^2}{(x+iy)} = \frac{(x-iy)^2}{(x+iy)} = \\
&= \frac{(x^2 - 2ixy + i^2y^2)}{(x+iy)} = \frac{(x^2 - 2ixy - y^2)(x-iy)}{(x+iy)(x-iy)} = \\
&= \frac{(x^3 - x^2iy - 2ix^3y + 2i^2xy^2 - y^2x + iy^3)}{(x^2 + iyx - iyx - i^2y^2)} = \\
&= \frac{(x^3 - x^2iy - 2ix^3y - 2xy^2 - y^2x + iy^3)}{(x^2 + y^2)} = \\
&= \boxed{\frac{x^3 - 3xy^2}{x^2 + y^2} + i\frac{-3x^2y + y^3}{x^2 + y^2}}
\end{aligned}
$$

Problem 2:

Consider the mapping $z \to z^2$.

(a) What is the image of the line $z = x + i$?

$$z \to z^2 = (x + iy) \to (x + iy)^2$$
$$(x^2 + 2ixy - y^2) = (x^2 + 2ix(1) - (1)^2) = x^2 + 2xi - 1$$
$$= (x^2 - 1) + i(2x)$$

$\boxed{\therefore \text{ The image of the line } x + i \text{ turns out to be a parabola opening to the right in } \mathbb{R}.}$ Since we have x^2 term for $u(x, y)$ and is centered at -1. The parabola grows into complex plane.

(a) What is the image of the square bounded by the four lines $z = \pm 1 + iy$ and $z = x \pm i$?

case z has constant real components $z = \pm 1 + iy$:

$$z \to z^2 = (\pm 1 + iy) \to (\pm 1 + iy)^2$$
$$((\pm 1)^2 + 2i(\pm 1)y - y^2) = ((\pm 1)^2 + 2i(\pm 1)y - y^2) =$$
$$\text{positive branch}$$
$$((1)^2 + 2i(1)y - y^2) = ((1)^2 + 2i(1)y - y^2) = (-y^2 + 1) + i(2y)$$
$$\text{negative branch}$$
$$((-1)^2 + 2i(-1)y - y^2) = (1 - 2iy - y^2) = (-y^2 + 1) - i(2y)$$

These two lines appear to be the same parabola that opens to the left toward $-\mathbb{R}$. if y is negative, we get the same thing for $u(x, y)$ and the sign flips on $v(x, y)$. Same situation in the case y is positive. for the case of $x - i$:

$$z \to z^2 = (x - i) \to (x - i)^2$$
$$(x^2 - 2ix - 1) = (x^2 - 1) - i(2x)$$

Here, the parabola still opens to the right, and is the same parabola as we expected. with the imaginary components flipped.

$\boxed{\therefore \text{ The image of the square region is the region between the two parabolas.}}$

Problem 3:

Compute the following limits (or state that they do not exist)

(a) $\lim_{z\to i} \frac{iz^3-1}{z+i}$

$$\lim_{z\to i} \frac{iz^3-1}{z+i} = \frac{i(i)^3-1}{i+i} =$$
$$\frac{i^4-1}{2i} = \frac{1-1}{2} = 0$$

(b) $\lim_{z\to i} \left(z+\frac{1}{z}\right)$

$$\begin{aligned}\lim_{z\to i}\left(z+\frac{1}{z}\right) &= \left(\lim_{z\to i} z + \lim_{z\to i}\frac{1}{z}\right)\\ &= \lim_{z\to i} z = i\\ &= \lim_{z\to i}\frac{1}{z} = \frac{1}{i} = -i\\ \therefore \lim_{z\to i}\left(z+\frac{1}{z}\right) &= i - i = 0\end{aligned}$$

(c) $\lim_{z\to 0} \frac{1}{z^2}$

We can evaluate this by replacing z with $re^{i\theta}$ and then evaluating the limits in r and θ.

$$\lim_{z\to 0}\frac{1}{z^2} = \lim_{r\to 0}\frac{1}{(re^{i\theta})^2} = \lim_{r\to 0}\frac{1}{(r)^2}e^{-i2\theta}$$

From any direction, we will end up with a div by zero issue. meaning the limit is ∞. We could get ∞ if approaching from $\theta = 0$ or we could get $-\infty$ if we approach from $\theta = \frac{\pi}{2}$

Problem 4:

Does the following limit exist?

(a) $\lim_{z \to 0} \left(\frac{\bar{z}}{z}\right)^2$

$$\frac{\bar{z}}{z} = \frac{\overline{re^{i\theta}}}{re^{i\theta}} = re^{-i\theta}\frac{1}{r}e^{-i\theta} = e^{-i2\theta}$$

squaring this then doubles the angle theta and we see there is no more dependence on r

$$= \lim_{z \to 0} e^{-i4\theta}$$

Approaching from $\theta = 0$ we get 1, but approaching from $\theta = \frac{\pi}{4}$ we get -1.

$\therefore$ no, the limit DNE because different paths give different result.