Homework # 6

MATH 3160 – Complex Variables
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Problem 1

Find parameterized representations $z(t)$ of the following contours in the plane including t-ranges.

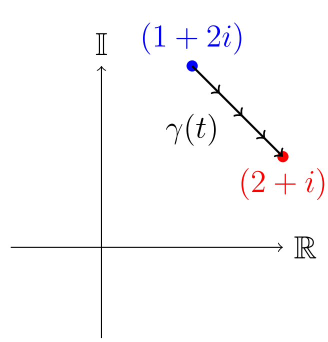
1. A straight line from point $(1 + 2i)$ to point $(i + 2)$
2. A line from $(0, 0)$ to point $(1 + \sqrt{3}i)$
3. A half-ellipse from point 2 to -2 passing through i centered at the origin. Recall that such an ellipse is defined by an equation of the form $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ in the xy-plane (for some real constants $a, b > 0$). *Hint: First find the suitable values of a and b defining the said ellipse. Then try parametrizing it similar to how $(\cos(t), \sin(t))$ parametrizes the unit cirle.*

(a)

A straight line from point $(1 + 2i)$ to point $(i + 2)$

This one will require the expression for a line between points: $P + t(Q - P)$ where P and Q are the points and t runs from $0 \leq t \leq 1$.

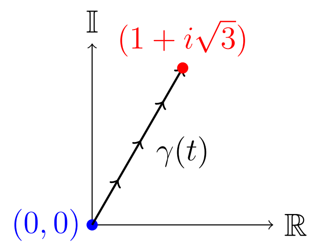
$$P = (1 + 2i)$$
$$Q = (i + 2)$$
$$Q - P = (i + 2) - (1 + 2i) = 1 - i$$
$$\gamma(t) = 1 + 2i + t(1 - i)$$



(b)

A line from $(0, 0)$ to point $(1 + \sqrt{3}i)$.

This one is quite simple as we only have to multiply the point by t as the first point P is the origin and that handles moving from the origin to the point $(1 + \sqrt{3}i)$ as it moves from $0 \leq t \leq 1$.



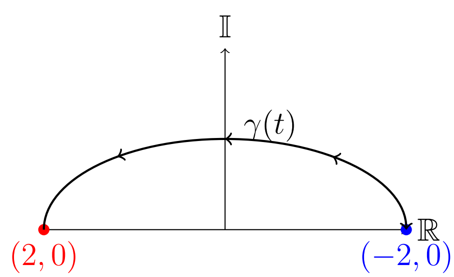
(c)

A half-ellipse from point 2 to -2 passing through i centered at the origin.

$$x = a\cos(t) = 2\cos(t)$$

$$y = b\sin(t) = \sin(t)$$

$$z(t) = x + iy = 2\cos(t) + \sin(t)$$



Problem 2

Evaluate the following integrals:

1. $\int_1^2 (\frac{1}{t} - i)^2 dt$
2. $\int_0^{\pi/6} e^{i2t} dt$
3. $\int_0^{\infty} e^{izt} dt$ where $Im(z) > 0$

(a)

$$\begin{aligned}
\int_1^2 (\frac{1}{t} - i)^2 dt &= \int_1^2 (\frac{1}{t^2} - 2i\frac{1}{t} - i^2) dt \\
&= \int_1^2 \frac{1}{t^2} dt - 2i \int_1^2 \frac{1}{t} dt + \int_1^2 dt \\
&= -\frac{1}{3} \int_1^2 -3t^{-2} dt - 2i \ln(t)|_1^2 + t|_1^2 \\
&= -\frac{1}{3} t^{-3}|_1^2 + -2i \ln(t)|_1^2 + t|_1^2 \\
&= \left(\frac{1}{3} - \frac{1}{3 * 2^3}\right) - 2i(\ln(2) - \ln(1)) + 1 \\
&= \left(\frac{1}{3} - \frac{1}{3 * 2^3} + 1\right) - 2i(\ln(2) - 0) \\
&= \left(\frac{4}{3} - \frac{1}{3 * 8}\right) - 2i \ln(2) \\
&= \left(\frac{32}{24} - \frac{1}{24}\right) - 2i \ln(2) \\
&= \left(\frac{31}{24}\right) - 2i \ln(2)
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{\pi/6} e^{i2t} dt &= \frac{1}{2i} \int_0^{\pi/6} 2ie^{i2t} dt \\
&= \frac{1}{2i} e^{i2t}|_0^{\pi/6} = -\frac{1}{2} i(e^{i\pi/3} - 1)
\end{aligned}$$

(c)

$$\int_0^\infty e^{izt}dt = \frac{1}{iz}e^{izt}|_0^\infty = \frac{1}{iz}(e^{iz\infty} - 1)$$
$$= \frac{1}{iz}(e^{i(x+iy)\infty} - 1) = \frac{1}{iz}(e^{(ix-y)\infty} - 1) = \frac{1}{iz}(e^{ix\infty}e^{-y\infty} - 1)$$

$e^{ix\infty}$ oscillates forever and always has a magnitude equal to 1 no matter the x value. $e^{-y\infty}$ reveals why we need to have the imaginary part grater than 0. If the imaginary part is greater than 0, we then get exponential decay for the expression involving the negative sign. With this, this eventually dissipates as t approaches infinity.

$$\therefore \int_0^\infty e^{izt}dt = -\frac{1}{iz}$$

Problem 3

Sketch the oriented curve defined by the following four contours and compute $\int_C f(z)dz$ where $f(z) = z - 1$:

1. C_1: A semicircle $z = 2e^{i\theta}$ for $\theta \in [\pi, 2\pi]$.
2. C_2: A full circle $z = 2e^{i\theta}$ for $\theta \in [0, 2\pi]$.
3. C_3: A line on the real axis from 2 to -2.
4. $C_4 = C_1 + C_3$ where $+$ denotes concatenation.

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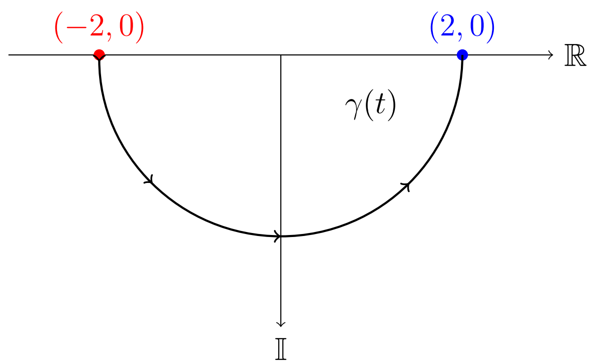
(1)



$$
\begin{aligned}
f(z) &= z - 1 \\
\gamma(t) &= 2e^{i\pi t} \\
1 \le\ & t \le 2 \\
t &= 1 \to e^{i\pi} \\
t &= 2 \to e^{i2\pi}
\end{aligned}
$$

$$
\begin{aligned}
\int_1^2 f(\gamma(t))\gamma(t)'dt &= \int_1^2 (\gamma(t) - 1)\gamma(t)'dt \\
= \int_1^2 (2e^{i\pi t} - 1)(2i\pi e^{i\pi t})dt &= 2\int_1^2 (2i\pi e^{i2\pi t})dt - 2\int_1^2 i\pi e^{i\pi t}dt \\
&= 2\int_1^2 (2i\pi e^{i2\pi t})dt - 2\int_1^2 i\pi e^{i\pi t}dt \\
&= 2e^{i2\pi t}|_1^2 - 2e^{i\pi t}|_1^2 = 2[(e^{i4\pi} - e^{i2\pi}) - (e^{i2\pi} - e^{i\pi})] \\
&= 2[(\cancelto{1}{e^{i4\pi}} - \cancelto{1}{e^{i2\pi}}) - (\cancelto{1}{e^{i2\pi}} - \cancelto{-1}{e^{i\pi}})] = 2[(1-1) - (1 - (-1))] \\
&= 2(-2) = -4
\end{aligned}
$$

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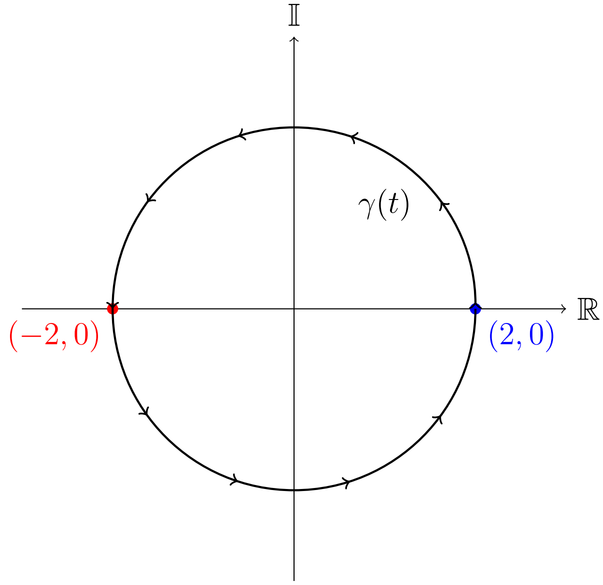
(2)

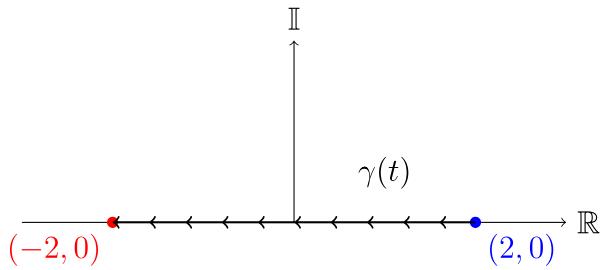
The oriented curve is defined as follows:



Since we know that the curve here starts and ends at the same point, we know that the overall expression here should evaluate to 0 by the Fundamental Theorem of Calculus. the path γ here has the same starting and ending point, and the function $f(z)$ is continuous everywhere with no discontinuities or issues with branch cuts. The result is then:

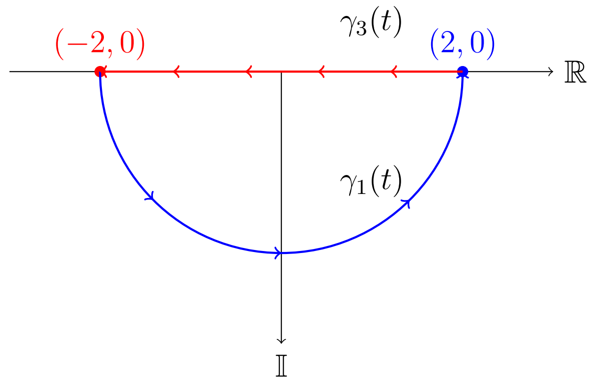
$$
\begin{aligned}
f(z) &= z - 1 \\
\gamma(t) &= 2e^{i\pi t} \\
\int_0^2 f(\gamma(t))\gamma(t)'dt &= \int_0^2 (\gamma(t) - 1)\gamma(t)'dt \\
&= \int_0^2 (2e^{i\pi t} - 1)(2i\pi e^{i\pi t})dt \\
&= \int_0^2 (2e^{i\pi t}e^{i\pi t} - e^{i\pi t})(2i\pi)dt \\
&= (2i\pi)\int_0^2 (2e^{i2\pi t} - e^{i\pi t})dt \\
&= (2i\pi)\left(\frac{2}{i2\pi}e^{i2\pi t} - \frac{1}{i\pi}e^{i\pi t}\right)\Big|_0^2 \\
&= \left(2e^{i2\pi t} - 2e^{i\pi t}\right)\Big|_0^2 \\
&= \left(2e^{i4\pi} - 2e^{i2\pi}\right) - \left(2e^0 - 2e^0\right) \\
&= (2\cdot 1 - 2\cdot 1) - (2 - 2) \\
&= 0
\end{aligned}
$$

(3)



$$
\begin{aligned}
f(z) &= z-1 \\
\gamma(t) &= z(t) = x(t) + iy(t) \\
y(t) &= 0 \\
(t) = P + t(Q-P) &= (2,0) + t[(-2,0) - (2,0)] = 2 - 4t \\
\gamma(t) &= x(t) = 2 - 4t \\
0 &\leq t \leq 1 \\
\int_0^1 f(\gamma(t))\gamma(t)'dt &= \int_0^1 (\gamma(t) - 1)\gamma(t)'dt \\
&= \int_0^1 (2 - 4t - 1)(-4)dt = (-4)\int_0^1 (1-4t)dt \\
&= -4(t - 2t^2)|_0^1 = -4[(1-2) - (0)] = -4 * (-1) = 4
\end{aligned}
$$

(4)



Similar to the case in two, we have a path that that is closed between two points. Since we have evaluated the path previously, we can conclude that this path will also resolve to 0. Taking the answers for 1 and 3 and taking their sum shows this resolves to 0.

$$\int C_1 + \int C_3 = -4 + 4 = 0$$