COMPLEX VARIABLES CHEATSHEET II

Harmonic Functions

Definition A real-valued function $u(x, y)$ is **harmonic** in a domain D if it has continuous second-order partials and satisfies **Laplace's equation**:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- If $f(z) = u + iv$ is analytic, then u and v are both harmonic.

Harmonic Conjugate $v(x, y)$ is a **harmonic conjugate** of $u(x, y)$ if $f(z) = u(x, y) + iv(x, y)$ is analytic. (This means they must satisfy the C-R equations).

Methods for Finding v from u

1. **Integrate C-R Equations**
   - **Step 1:** Use $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$. Integrate w.r.t. y:

     $$v(x, y) = \int \frac{\partial u}{\partial x} dy + h(x)$$

     (where $h(x)$ is an unknown function of x).
   - **Step 2:** Use $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Differentiate the result from Step 1 w.r.t. x:

     $$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\int \frac{\partial u}{\partial x} dy\right) + h'(x)$$
   - **Step 3:** Set equal and solve for $h'(x)$:

     $$h'(x) = -\frac{\partial u}{\partial y} - \frac{\partial}{\partial x}\left(\int \frac{\partial u}{\partial x} dy\right)$$
   - **Step 4:** Integrate $h'(x)$ to find $h(x)$ and add the final constant C.
2. **Total Differential (Line Integral)**
   - The total differential for v is $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$.
   - **Step 1:** Use C-R to write dv in terms of u:

     $$dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$$
   - **Step 2:** Integrate dv from a fixed point (x_0, y_0) to (x, y) along a simple path (e.g., $(x_0, y_0) \to (x, y_0) \to (x, y)$).

     $$v(x, y) = \int_{x_0}^{x} -\frac{\partial u}{\partial y}(t, y_0)dt + \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t)dt + C$$
3. **Inspection / Guess** $f(z)$
   - Works for simple polynomials.
   - Let $y = 0$, giving $u(x, 0)$.
   - Try to guess the analytic function $f(z)$ by replacing x with z.
   - **Ex:** $u(x, y) = x^2 - y^2$. Let $y = 0 \implies u(x, 0) = x^2$.
   - **Guess:** $f(z) = z^2$.
   - **Check:** $f(z) = (x + iy)^2 = (x^2 - y^2) + i(2xy)$.
   - This works! $u = x^2 - y^2$ and $v = 2xy$.

Contour Integration

Parametrization A contour C is a curve $z(t) = x(t) + iy(t)$ for $a \le t \le b$.

- $z'(t) = \frac{dz}{dt} = x'(t) + iy'(t)$.
- C is **simple** if it doesn't cross itself.
- C is **closed** if $z(a) = z(b)$.

Definition of the Contour Integral

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Common Parametrizations

- **Line Segment** from z_1 to z_2:
  - $z(t) = z_1 + t(z_2 - z_1)$, for $0 \le t \le 1$.
  - $z'(t) = z_2 - z_1$.
- **Circle** $|z - z_0| = R$:
  - **CCW (Positive):** $z(t) = z_0 + Re^{it}$
  - $z'(t) = iRe^{it}$, for $0 \le t \le 2\pi$.
  - **CW (Negative):** $z(t) = z_0 + Re^{-it}$
  - $z'(t) = -iRe^{-it}$, for $0 \le t \le 2\pi$.

Properties of Integrals

- **Linearity:** $\int_C(\alpha f + \beta g)dz = \alpha \int_C f dz + \beta \int_C g dz$.
- **Path Reversal:** $\int_{-C} f(z)dz = -\int_C f(z)dz$.
- **Additivity:** $\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$.

ML-Inequality (Estimation Bound) If $|f(z)| \le M$ for all z on C, and L is the arc length of C, then:

$$\left|\int_C f(z)dz\right| \le M \cdot L$$

Antiderivatives FTC

Definition: Antiderivative A function $F(z)$ is an **antiderivative** of $f(z)$ in a domain D if $F'(z) = f(z)$ for all $z \in D$.

Complex Fundamental Theorem of Calculus If $f(z)$ is continuous in a domain D and has an antiderivative $F(z)$ in D, then for any contour C in D from z_1 to z_2:

$$\int_C f(z)dz = F(z_2) - F(z_1)$$

Key Consequences:

- **Path Independence:** The integral's value depends only on the endpoints.
- **Closed Loop Theorem:** If C is closed ($z_1 = z_2$), then $\oint_C f(z)dz = 0$.

Existence of an Antiderivative An analytic function $f(z)$ has an antiderivative in a domain D **if and only if** D is **simply connected**.

- **Ex:** $f(z) = 1/z$ is analytic on $D = \mathbb{C} \setminus \{0\}$, which is **not** simply connected. $\oint_{|z|=1} \frac{1}{z}dz = 2\pi i \neq 0$. Thus $1/z$ has no single antiderivative (like Log(z)) on this domain.

Cauchy Theorems (Deformation)

Cauchy-Goursat Theorem If $f(z)$ is analytic at all points **inside and on** a simple closed contour C, then:

$$\oint_C f(z)dz = 0$$

(Goursat's contribution was proving this without assuming $f'(z)$ is continuous).

Principle of Deformation of Paths If C_1 and C_2 are two simple closed, positively oriented (CCW) contours, and $f(z)$ is analytic on both contours and in the region **between** them, then:

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

This allows "deforming" a contour around singularities.

Multiply Connected Domains Let C be an outer CCW contour and $C_1, \ldots, C_n$ be inner contours, all **clockwise (CW)**. If $f(z)$ is analytic in the region, then:

$$\oint_C f(z)dz + \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 0$$

If all contours are **CCW**:

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$$

(The integral around the outer boundary equals the sum of integrals around the "holes").

Keyhole Contours Used to make a multiply connected domain simply connected by using a "cut". This is common for branch cuts. The integral along the cut path L_1 and the return path L_2 cancel each other out.

Cauchy's Integral Formulas

(Review: Using the formulas to compute integrals) (Assumes $f(z)$ is analytic inside and on a simple closed CCW contour C)

The Integral Formula Gives the value of $f(z)$ at any point z_0 **inside** C.

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz$$

To compute $\oint_C \frac{g(z)}{z-z_0}dz$:

1. Identify z_0. If z_0 is **outside** C, the integral is 0 (by Cauchy-Goursat, if $g(z)$ is analytic).
2. If z_0 is **inside** C, let $f(z) = g(z)$.
3. The integral is $\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i \cdot f(z_0)$.

The Generalized Formula (for derivatives) Gives the n-th derivative of $f(z)$ at z_0.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}dz$$

To compute $\oint_C \frac{g(z)}{(z-z_0)^{n+1}}dz$:

1. Identify z_0 (must be inside C) and the power $n + 1$. This gives you n.
2. Let $f(z) = g(z)$ (the analytic numerator).
3. Find the n-th derivative, $f^{(n)}(z)$.
4. Evaluate $f^{(n)}(z_0)$.
5. The integral is $\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} \cdot f^{(n)}(z_0)$.

Example: $\oint_{|z|=2} \frac{e^z}{(z-1)^3}dz$

- $z_0 = 1$ (inside C).
- $n + 1 = 3 \implies n = 2$.
- $f(z) = e^z$.
- $f'(z) = e^z$, $f''(z) = e^z$.
- $f''(1) = e^1 = e$.
- Result: $\frac{2\pi i}{2!}f''(1) = \frac{2\pi i}{2}(e) = \pi ie$.