Homework #5

MATH 3160 – Complex Variables
Miguel Gomez

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Problem 1

Consider the analytic function $f(z) = ze^{z^2}$.

(a) Show that the function $u(x, y) = x\ e^{(x^2-y^2)} \cos(2xy) - y\ e^{(x^2-y^2)} \sin(2xy)$ is the real component of $f(z)$.

(b) What is a harmonic conjugate for $u(x, y)$?

(c) Without computing the second partial derivatives of $u(x, y)$, explain why you know that $u(x, y)$ is harmonic.

Expanding $f(z) = ze^{z^2}$ to see what the u and v turn out to be.

(a)

$$\begin{aligned}
f(z) = ze^{z^2} &= (x+iy)e^{(x+iy)^2} = (x+iy)e^{x^2-y^2+2ixy} = (x+iy)e^{x^2-y^2}e^{2ixy} \\
&= e^{x^2-y^2}(x+iy)(\cos(2xy) + i\sin(2xy)) \\
&= e^{x^2-y^2}(x\cos(2xy) + ix\sin(2xy) + iy\cos(2xy) + i^2 y\sin(2xy)) \\
&= e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy) + i(x\sin(2xy) + y\cos(2xy))) \\
&\therefore u(x, y) = e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy))
\end{aligned}$$

0.1 (b)

Utilizing the imaginary part of f, v can serve as a conjugate up to an arbitrary constant, which we could set to 0 to recover f.

$$v(x, y) = e^{x^2-y^2}(x\sin(2xy) + y\cos(2xy)) + C$$

0.2 (c)

Since we know that f is analytic as stated, and u is the real part of f, an analytic function, then u must be harmonic. This is due to the fact that the real and imaginary parts of any analytic function are harmonic functions.

Problem 2

Consider the function $u(x, y) = x^3 - 3xy^2 - 3x^2y + y^3$.

(a) Show that $u(x, y)$ is harmonic.

(b) Find a harmonic conjugate for $u(x, y)$.

(a)

For this, we can start by showing that the expression for u satisfies the Laplacian:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x - 6y$$

$$\frac{\partial u}{\partial y} = -6xy - 3x^2 + 3y^2$$

$$\frac{\partial^2 u}{\partial y^2} = -6x + 6y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6y + (-6x + 6y) = 0$$

$$\therefore u(x, y) \text{ is harmonic.}$$

For a conjugate, we can back solve to get the v expression that works.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$v(x, y) = \int 3x^2 - 3y^2 - 6xydy = 3x^2y - y^3 - 3xy^2 + G(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$v(x, y) = \int 6xy + 3x^2 - 3y^2dx = 3x^2y + x^3 - 3x^2y + G(y)$$

$$\therefore v(x, y)_1 = v(x, y)_2$$

$$G(x) = x^3 + C$$

$$G(y) = -y^3 + C$$

$$\therefore v(x, y) = \boxed{3x^2y + x^3 - 3x^2y + -y^3 + C}$$

Problem 3

Recall we learned of the following fact in class:

Let $u(x, y)$ be a harmonic function defined on a simply connected domain D. Then $u(x, y)$ has a harmonic conjugate on D.

(a) Show that $u(x, y) = \ln(\sqrt{x^2 + y^2})$ is a harmonic function.

(b) What is the domain of definition of $u(x, y)$?

(c) An aside: show that if $f(z)$ and $g(z)$ are two analytic functions on the same domain D, and we have $\text{Re}(f(z)) = \text{Re}(g(z))$ for all $z \in D$, then $f(z) = g(z) + c$ for some constant $c \in \mathbb{C}$.

[*Hint: show that the function $h(z) = f(z) - g(z)$ has Re$(h(z)) = 0$, and then use a result from class to conclude $h(z)$ is a constant.*]

(d) Explain why $u(x, y)$ does *not* have a harmonic conjugate on its domain.

[*Hint: if such a conjugate existed, then $u(x, y)$ would be the real component of some analytic function $f(z)$, but $u(x, y)$ is already the real component of a familiar analytic function, which is discontinuous at its branch cut*]

1. Why does this not contradict the fact from class?

Problem 4

Find the following values, on the branches given:

(a) $\log(3)$ $(-2\pi \leq \theta < 0)$

(b) $\log(-1+i)$ $(-\pi/2 < \theta \leq 3\pi/2)$

(c) $\log(1-i\sqrt{3})$ $(\pi \leq \theta < 3\pi)$.

$$\log(z) = \log|z| + iarg(z) = \log|z| + i(Arg(z) + 2\pi k)\ \forall k \in \mathbb{Z}$$

(a)

$\log(3)$ $(-2\pi \leq \theta < 0)$

$$\log(3) = \log|3| + i(Arg(3) + 2\pi k) = \log(3) + i(0 + 2\pi k)$$
$$\log(3) + i(2\pi k)\ k \in [-1, 0) = \log(3) - i2\pi$$

(b)

$\log(-1+i)$ $(-\pi/2 < \theta \leq 3\pi/2)$

$$\log(-1+i) = \log|-1+i| + i(Arg(-1+i) + 2\pi k) = \log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right)$$
$$\begin{aligned} \log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right)\ k \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ = \log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right) \\ = \log(\sqrt{2}) + i\frac{3\pi}{4} \end{aligned}$$

(c)

$\log(1-i\sqrt{3})$ $(\pi \leq \theta < 3\pi)$

Problem 5

Recall that power functions are defined by $z^c = e^{c\log(z)}$. In this exercise, we compute all power functions by using the branch $(0 \leq \theta < 2\pi)$ for $\log(z)$.

(a) For $z = -i$ and $c = i$, compute the values of $(z^c)^2$, $(z^2)^c$, and $z^{(2c)}$.

(b) With the notation as in (a), which of these are true or false?

$$(z^c)^2 = (z^2)^c, \qquad (z^c)^2 = z^{(2c)}, \qquad (z^2)^c = z^{(2c)}.$$