Worksheet 1: Complex Numbers

MATH 3160
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Problem 1: Simplify

Simplify the following complex expressions.

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i)$

$$\begin{aligned}
(\sqrt{2} - i) &- i(1 - \sqrt{2}i) = \\
(\sqrt{2} - i) &- i - \sqrt{2}i^2 = \\
(\sqrt{2} - i) &- i - \sqrt{2}\cancelto{-1}{i^2} = \\
\sqrt{2} - i &- i + \sqrt{2} = \\
\sqrt{2} + \sqrt{2} &- i - i = \\
2\sqrt{2} &- 2i = 2(\sqrt{2} - i)
\end{aligned}$$

(b) $(2 - 3i)(-2 + i)$

$$\begin{aligned}
(2 - 3i)(-2 + i) &= \\
(-2(2 - 3i) &+ i(2 - 3i)) = \\
(-4 + 6i) &+ (2i - 3i^2) = \\
(-4 + 6i) &+ (2i - 3\cancelto{-1}{i^2}) = \\
(-4 + 6i) &+ (2i + 3) = \\
(-4 + 3) &+ (2i + 6i) = (-1 + 8i)
\end{aligned}$$

(c) $(3 + i)(3 - i)\left(\frac{1}{5} + \frac{1}{10}i\right)$

$$\begin{aligned}
(3 + i)(3 - i)\left(\frac{1}{5} + \frac{1}{10}i\right) &= \\
(3(3 + i) - i(3 + i))\left(\frac{1}{5} + \frac{1}{10}i\right) &= \\
((9 + 3i) - 3i - i^2))\left(\frac{1}{5} + \frac{1}{10}i\right) &= \\
((9 + 3i) - 3i - \cancelto{-1}{i^2}))\left(\frac{1}{5} + \frac{1}{10}i\right) &= \\
(9 + \cancelto{0}{3i - 3i} + 1))\left(\frac{1}{5} + \frac{1}{10}i\right) &= \\
10\left(\frac{1}{5} + \frac{1}{10}i\right) &= (2 + i)
\end{aligned}$$

Problem 2: Verification

Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

First root: $1 + i$

$$
\begin{aligned}
z^2 - 2z + 2 &= 0 \\
(1+i)^2 - 2(1+i) + 2 &= 0 \\
(1+i)^2 - 2(1+i) + 2 &= 0 \\
(1 + 2i + i^2) - 2 - 2i + 2 &= 0 \\
(1 + \cancelto{-1}{i^2}) + (2i - 2i) + (2 - 2) &= 0 \\
\cancelto{0}{(1-1)} + \cancelto{0}{(2i - 2i)} + \cancelto{0}{(2-2)} &= 0 \\
0 &= 0 \quad \square
\end{aligned}
$$

Second root: $1 - i$

$$
\begin{aligned}
z^2 - 2z + 2 &= 0 \\
(1-i)^2 - 2(1-i) + 2 &= 0 \\
(1-i)^2 - 2(1-i) + 2 &= 0 \\
(1 - 2i + i^2) - 21 + 2i + 2 &= 0 \\
(1 - 2i + \cancelto{-1}{i^2}) - 2 + 2i + 2 &= 0 \\
\cancelto{0}{(1-1)} + \cancelto{0}{(2i - 2i)} + \cancelto{0}{(2-2)} &= 0 \\
0 &= 0 \quad \square
\end{aligned}
$$

Problem 3: Solving Equations

Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing $(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$ and then solving a pair of simultaneous equations in x and y.

Using expressions (3) and (4) from the textbook shown below:

$$(3)\ (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(4)\ (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2)$$

We can see that $x_1 = x_2$ and $y_1 = y_2$ for our problem. Solving for the expression in the form above:

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$
$$(x^2 - y^2 + x + 1, 2xy + y + 0) = (0, 0)$$

Case $x = 0$:

$$(x^2 - y^2 + x + 1, 2xy + y + 0) = (0, 0)$$
$$(\cancelto{0}{x^2} - y^2 + \cancelto{0}{x} + 1, \cancelto{0}{2xy} + y + 0) = (0, 0)$$
$$(-y^2 + 1, y + 0) = (0, 0)$$

We can see that we simultaneously have expressions $y = 0$ and $y = \pm 1$. This is a contradiction.

Case $y = 0$:

$$(x^2 - y^2 + x + 1, 2xy + y + 0) = (0, 0)$$
$$(x^2 - \cancelto{0}{y^2} + x + 1, \cancelto{0}{2xy} + \cancelto{0}{y} + 0) = (0, 0)$$
$$(x^2 + x + 1, 0) = (0, 0)$$

This is not factorable in $\mathbb{R}$. We can show this with the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$
$$x = \frac{-1 \pm \sqrt{-3}}{2} \quad ; i = \sqrt{-1}$$
$$x = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\therefore$ if x is complex, that implies that y is non-zero. Then we can plug $x = -\frac{1}{2}$ and $y = \pm\frac{\sqrt{3}}{2}$ back into the starting expression $(x^2 - y^2 + x + 1, 2xy + y + 0)$ to verify the answers come out to $(0, 0)$.

$$(x^2-y^2+x+1\ ,2xy+y+0)=(0,\ 0)$$

$$\left(\left(-\frac{1}{2}\right)^2-\left(\pm\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)+1,\ 2\left(-\frac{1}{2}\right)\left(\pm\frac{\sqrt{3}}{2}\right)+\left(\pm\frac{\sqrt{3}}{2}\right)+0\right)=(0,0)$$

$$\left(\frac{1}{4}-\frac{3}{4}-\frac{1}{2}+1,\ \left(\cancelto{-1}{-\frac{2}{2}}\right)\left(\pm\frac{\sqrt{3}}{2}\right)+\left(\pm\frac{\sqrt{3}}{2}\right)\right)=(0,0)$$

$$\left(\cancelto{-\frac{1}{2}}{\frac{1}{4}-\frac{3}{4}}-\frac{1}{2}+1,\ \left(\cancelto{\mp}{-1\pm}\frac{\sqrt{3}}{2}\right)+\left(\pm\frac{\sqrt{3}}{2}\right)\right)=(0,0)$$

$$\left(\cancelto{-1}{-\frac{1}{2}-\frac{1}{2}}+1,\ \left(\mp\frac{\sqrt{3}}{2}\right)+\left(\pm\frac{\sqrt{3}}{2}\right)\right)=(0,0)$$

$$\left(\cancelto{0}{1-1,}\ \cancelto{0}{\left(\mp\frac{\sqrt{3}}{2}\right)+\left(\pm\frac{\sqrt{3}}{2}\right)}\right)=(0,0)$$

$$(0,0)=(0,0)\quad\square$$