Homework # 4

MATH 3160 – Complex Variables
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Problem 1

Find $f'(z)$ using differentiation rules.

(a) $f(z) = 3z^2 - 2z + 4$

(b) $f(z) = (1 - 4z)^3$

(c) $f(z) = \frac{z-1}{2z+1}$, assume $z \neq -1/2$

(d) $f(z) = \frac{(z^2+1)^4}{z^2}$, assume $z \neq 0$

(e) $f(z) = z\, e^{z^2+3}$.

(a)

$$\begin{aligned} f(z) &= 3z^2 - 2z + 4 \\ f'(z) &= 6z - 2 \end{aligned}$$

(b)

$$\begin{aligned} f(z) &= a(z) = (1 - 4z)^3 = b^3 \mid b = 1 - 4z \\ f'(z) &= \frac{da}{db}\frac{db}{dz} = 3(1-4z)^2(-4) = -12(1-4z)^2 \end{aligned}$$

(c)

Assuming $z \neq -1/2$

$$\begin{aligned} f(z) &= \frac{z-1}{2z+1} = \frac{f}{g} \\ \frac{df}{dz} &= 1 \qquad \frac{dg}{dz} = 2 \\ f'(z) = \frac{f'g - fg'}{g^2} &= \frac{1(2z+1) - (z-1)2}{(2z+1)^2} \\ &= \frac{(2z-2z) + (1+2)}{(2z+1)^2} = \frac{3}{(2z+1)^2} \end{aligned}$$

(d)

Assuming $z \neq 0$

$$\begin{aligned}
f(z) &= \frac{(z^2+1)^4}{z^2} = \frac{a}{b} \\
\frac{da}{dz} &= 4(z^2+1)^3(2z) \qquad \frac{db}{dz} = 2z \\
f'(z) &= \frac{4(z^2+1)^3(2z)(z^2) - (z^2+1)^4(2z)}{z^4} \\
&= \frac{(2z)(z^2+1)^3[4(z^2) - (z^2+1)]}{z^4} \\
&= \frac{2(z^2+1)^3[3z^2-1]}{z^3}
\end{aligned}$$

(e)

$$\begin{aligned}
f(z) &= ze^{z^2+3} \\
f'(z) &= (1)e^{z^2+3} + (z)e^{z^2+3}(2z) = e^{z^2+3}(2z^2+1)
\end{aligned}$$

Problem 2

Show that $f'(z_0)$ does not exist at any point z_0 in two ways: using the limit definition and using the Cauchy-Riemann equations. Here, $z = x + iy$ and $x, y \in \mathbb{R}$.

(a) $f(z) = 2x + ixy^2$

(b) $f(z) = e^x e^{-iy}$

Via Cauchy-Riemann equations: The Cauchy-Riemann equations are the following:

$$\frac{du}{dx} = \frac{dv}{dy}$$
$$\frac{du}{dy} = -\frac{dv}{dx}$$

(a)

$$f(z) = 2x + ixy^2 \quad u(x,y) = 2x \quad v(x,y) = xy^2$$
$$\frac{du}{dx} = 2 \quad \frac{du}{dy} = 0 \quad \frac{dv}{dx} = y^2 \quad \frac{dv}{dy} = 2xy$$
$$\frac{du}{dx} \neq \frac{dv}{dy}$$
$$\therefore \text{not differentiable}$$

(b)

$$f(z) = e^x e^{-iy} = e^x(\cos(-y) + i\sin(-y))$$
$$u(x,y) = e^x \cos(-y) \quad v(x,y) = e^x \sin(-y)$$
$$\frac{du}{dx} = e^x \cos(y) \quad \frac{du}{dy} = -e^x \sin(y) \quad \frac{dv}{dx} = -e^x \sin(y) \quad \frac{dv}{dy} = -e^x \cos(y)$$
$$e^x \cos(y) = -e^x \cos(y) \to 2e^x \cos(y) = 0$$
$$\text{Only possible if } \cos(y) = 0$$
$$-e^x \sin(y) = e^x \sin(y) \to 2e^x \sin(y) = 0$$
$$\text{only possible if } \sin(y) \text{ is } 0$$
$$\text{Both sin and cos cannot be 0 simultaneously}$$
$$\therefore \text{not differentiable}$$

Now by using the limit definition:

Limit definition:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_o}$$
$$= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$
$$\Delta w = f(z_0 + \Delta z) - f(z_0)$$
$$\Delta z = z - z_0$$

(a)

$$\begin{aligned}
\Delta w &= 2(x_0 + \Delta x) + i(x_0 + \Delta x)(y_0 + \Delta y)^2 - (2x_0 + ix_0y_0^2) \\
&= 2x_0 + 2\Delta x - 2x_0 + i(x_0 + \Delta x)(y_0 + \Delta y)^2 - ix_0y_0^2 \\
&= 2\Delta x + i(x_0 + \Delta x)(y_0^2 + 2y_0\Delta y + \Delta y^2) - ix_0y_0^2 \\
&= 2\Delta x + i((x_0y_0^2 + 2x_0y_0\Delta y + x_0\Delta y^2) + (\Delta x y_0^2 + 2\Delta x y_0 \Delta y + \Delta x \Delta y^2)) - ix_0y_0^2 \\
&= 2\Delta x + i(((\cancelto{0}{x_0y_0^2 - x_0y_0^2}) + 2x_0y_0\Delta y + x_0\Delta y^2) + (\Delta x y_0^2 + 2\Delta x y_0 \Delta y + \Delta x \Delta y^2)) \\
&= 2\Delta x + i((2x_0y_0\Delta y + x_0\cancelto{0}{\Delta y^2}) + (\Delta x y_0^2 + \cancelto{0}{2\Delta x y_0 \Delta y} + \Delta x \cancelto{0}{\Delta y^2})) \\
&= 2\Delta x + i(2x_0y_0\Delta y + \Delta x y_0^2) \\
\frac{\Delta w}{\Delta z} &= \frac{2\Delta x + i(2x_0y_0\Delta y + \Delta x y_0^2)}{\Delta x + i\Delta y}
\end{aligned}$$

Approach with $\Delta y = 0$:

$$\begin{aligned}
\frac{\Delta w}{\Delta z} &= \frac{2\Delta x + i(2x_0y_0(0) + \Delta x y_0^2)}{\Delta x + i(0)} \\
&= \frac{2\Delta x + i(\Delta x y_0^2)}{\Delta x} = 2 + iy_0^2
\end{aligned}$$

Approach with $\Delta x = 0$:

$$\begin{aligned}
\frac{\Delta w}{\Delta z} &= \frac{2(0) + i(2x_0y_0\Delta y + (0)y_0^2)}{(0) + i\Delta y} \\
&= \frac{i2x_0y_0\Delta y}{i\Delta y} = 2x_0y_0
\end{aligned}$$

Different paths give different results. only true if $2x_0y_0 = 2 + iy_0$ and these cannot be true since x_0 and $y_0 \in \mathbb{R}$.

(b)

$$\begin{aligned}
f(z) &= e^x e^{-iy} \\
\Delta w &= f(z_0 + \Delta z) - f(z_0) \\
\Delta z &= z - z_0 \\
\Delta w &= e^{(x_0+\Delta x)} e^{-i(y_0+\Delta y)} - e^{x_0} e^{-iy_0} \\
&= e^{x_0} e^{\Delta x} e^{-iy_0} e^{-i\Delta y} - e^{x_0} e^{-iy_0} \\
&= e^{x_0} e^{-iy_0} e^{\Delta x} e^{-i\Delta y} - e^{x_0} e^{-iy_0} \\
&= e^{x_0} e^{-iy_0} (e^{\Delta x} e^{-i\Delta y} - 1) \\
&= e^{x_0} e^{-iy_0} (e^{\Delta x - i\Delta y} - 1) \\
\frac{\Delta w}{\Delta z} &= \frac{e^{x_0} e^{-iy_0} (e^{\Delta x - i\Delta y} - 1)}{\Delta x + i\Delta y}
\end{aligned}$$

Approach with $\Delta y = 0$:

$$\begin{aligned}
\frac{\Delta w}{\Delta z} &= \frac{e^{x_0} e^{-iy_0} (e^{\Delta x - i(0)} - 1)}{\Delta x + i(0)} \\
&= \frac{e^{x_0} e^{-iy_0} (e^{\Delta x} - 1)}{\Delta x} \\
&= e^{x_0} e^{-iy_0} \lim_{\Delta x \to 0} \frac{(e^{\Delta x} - 1)}{\Delta x} \\
&\text{limit definition of exponential resolves to 1} \\
&== e^{x_0} e^{-iy_0}
\end{aligned}$$

Approach with $\Delta x = 0$:

$$\begin{aligned}
\frac{\Delta w}{\Delta z} &= \frac{e^{x_0} e^{-iy_0} (e^{(0) - i\Delta y} - 1)}{(0) + i\Delta y} \\
&= \frac{e^{x_0} e^{-iy_0} (e^{-i\Delta y} - 1)}{i\Delta y} \\
&\text{limit definition of exponential resolves to -1} \\
&== -e^{x_0} e^{-iy_0}
\end{aligned}$$

Different paths give different results:

$$\therefore \text{Limit DNE}$$

Problem 3

Using the exponential function e^z, we can now define the complex cosine and sine function for any $z \in \mathbb{C}$ as follows:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Using these formulas,

(a) express $\cos(z)$ and $\sin(z)$ in rectangular coordinates $u(x, y) + iv(x, y)$ where $z = x + iy$.

(b) show that the complex cosine and sine functions are analytic over $\mathbb{C}$ and calculate their derivatives.

(a)

$$\begin{aligned}
\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\
&= \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2} \\
&= e^{ix} = (\cos(x) + i\sin(x)) \\
&= e^{-ix} = (\cos(x) - i\sin(x)) \\
&= \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2} = \frac{(\cos(x) + i\sin(x))e^{-y} + (\cos(x) - i\sin(x))e^{y}}{2} \\
&= \frac{\cos(x)(e^{-y} + e^{y}) + i\sin(x)(e^{-y} - e^{y})}{2} \\
&= \frac{\cos(x)(e^{-y} + e^{y})}{2} + i\frac{\sin(x)(e^{-y} - e^{y})}{2}
\end{aligned}$$

(b)

$$
\begin{aligned}
\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\
&= \frac{e^{ix-y} - e^{-ix+y}}{2i} = \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i} \\
&= e^{ix} = (\cos(x) + i\sin(x)) \\
&= e^{-ix} = (\cos(x) - i\sin(x)) \\
&= \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i} = \frac{(\cos(x) + i\sin(x))e^{-y} - (\cos(x) - i\sin(x))e^{y}}{2i} \\
&= \frac{\cos(x)(e^{-y} - e^{y}) + i\sin(x)(e^{-y} + e^{y})}{2i} \\
&= \frac{\cos(x)(e^{-y} - e^{y})}{2i} + \frac{\sin(x)(e^{-y} + e^{y})}{2} \\
&= \frac{\sin(x)(e^{-y} + e^{y})}{2} - i\frac{\cos(x)(e^{-y} - e^{y})}{2}
\end{aligned}
$$

These two in rectangular coordinates appear to be the same thing with their respective sin and cos terms swapped. Another place where we see the sin and cos terms swap is when we perform a rotation by 90°. I suspect that including i along with z is performing a rotation on z by 90° and that is shown in the work. Therefore, proving the case for cos also proves the case for sin because we know a. priori that sin and cos are derivatives of each other in a cycle.

$$
\begin{aligned}
\frac{df}{dz} &= \text{product rule here after splitting} \\
&\frac{\cos(x)(e^{-y} + e^{y})}{2} \\
&i\frac{\sin(x)(e^{-y} - e^{y})}{2}
\end{aligned}
$$