Worksheet 2

MATH 3160
Miguel Gomez

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Problem 1

Reduce each of these to a real number

(a) $\frac{1+2i}{3-4i}+\frac{2-i}{5i}$

$$\frac{1+2i}{3-4i}+\frac{2-i}{5i}=$$
$$\frac{(1+2i)(3+4i)}{(3-4i)(3+4i)}+\frac{(2-i)(-5i)}{(5i)(-5i)}=$$
$$\frac{(1+2i)(3+4i)}{9-16i^2}+\frac{(2-i)(-5i)}{-25i^2}=$$
$$\frac{(3+4i+6i+8\cancelto{-1}{i^2})}{9-16\cancelto{-1}{i^2}}+\frac{(-10i+5\cancelto{-1}{i^2})}{-25\cancelto{-1}{i^2}}=$$
$$\frac{(3+4i+6i-8)}{25}+\frac{(-10i-5)}{25}=$$
$$\frac{(3+4i+6i-8)}{25}+\frac{(-10i-5)}{25}=$$
$$\frac{(-5+10i)}{25}+\frac{(-10i-5)}{25}=$$
$$\frac{(-5-5+10i-10i)}{25}=\boxed{-\frac{2}{5}}$$

(b) $\frac{5i}{(1-i)(2-i)(3-i)}$

$$\frac{5i}{(1-i)(2-i)(3-i)}=\frac{5i}{(2-i-2i+i^2)(3-i)}=$$
$$\frac{5i}{(1-3i)(3-i)}=\frac{5i}{(1-3i)(3-i)}=$$
$$\frac{5i}{-10i}=\boxed{-\frac{1}{2}}$$

Problem 2

Find the principal argument Arg z when..

preliminary necessary expressions:

$$\arg(z) = \text{Arg}(z) + 2 \cdot pi \cdot k \quad ; k \in \mathbb{Z}$$
$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$-\pi < \theta \leq \pi$$

(a) $z = \frac{-2}{1+\sqrt{3}i}$

$$\frac{-2}{1+\sqrt{3}i} = \frac{-2}{1+\sqrt{3}i}\frac{1-\sqrt{3}i}{1-\sqrt{3}i} =$$
$$\frac{-2(1-\sqrt{3}i)}{1\cancelto{0}{-\sqrt{3}i+\sqrt{3}i}+\sqrt{3}^2\cancelto{-1}{i^2}} = \frac{-2+2\sqrt{3}i}{1-3}$$
$$2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2e^{i\frac{2\pi}{3}}$$
$$\therefore Argz = \boxed{\frac{2\pi}{3}}$$

(b) $z = \frac{2i}{i-1}$

$$\frac{2i}{i-1} = \frac{2i(-i-1)}{(i-1)(-i-1)} = \frac{2i(-1-i)}{(-1+i)(-1-i)} =$$
$$\frac{(-2i-2i^2)}{1+i-i-i^2} = \frac{(-2i-2\cancelto{-1}{i^2})}{1\cancelto{0}{+i-i}-\cancelto{-1}{i^2}} = \frac{2-2i}{2} = 1-i = \sqrt{2}e^{-i\frac{\pi}{4}}$$
$$\therefore \text{Arg}(z) = \boxed{-\frac{\pi}{4}}$$

(c) $z = (\sqrt{3}-i)^6$

For this one, we need to first include a factor of 2^6 and divide by it as well to bring a half into the parentheses.

$$(\sqrt{3}-i)^6 = 2^6\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^6$$
$$2^6\left(e^{-i\frac{\pi}{6}}\right)^6 = 2^6 e^{-i\pi}$$
$$\therefore \text{Arg}(z) = \boxed{\pi}$$

Problem 3

For the next few questions write the individual factors on the left in exponential form, perform the needed operations on complex numbers, and finally change back to rectangular coordinates *Show that*:

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$

To convert into exponential form, we can add factors to get us the correct exponentials in normalized form:

$$i = e^{i\frac{\pi}{2}}$$
$$(1-\sqrt{3}i) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2e^{-i\frac{\pi}{3}}$$
$$(\sqrt{3}+i) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2e^{i\frac{\pi}{6}}$$

Note, $(\sqrt{3}+i)$ is 90° rotated from $(1-\sqrt{3}i)$. This is easily verified by adding $\pi/2$ to $\mathrm{Arg}(1-\sqrt{3}i)$.

$$\therefore i(1-\sqrt{3}i) = e^{i\frac{\pi}{2}}2e^{-i\frac{\pi}{3}} = 2e^{i\left(\frac{\pi}{2}-\frac{\pi}{3}\right)}$$
$$2e^{i\left(\frac{3\pi}{6}-\frac{2\pi}{6}\right)} = 2e^{\frac{\pi}{6}}$$

Including the next factor gives:

$$2e^{\frac{\pi}{6}}2e^{i\frac{\pi}{6}} = 4e^{i\frac{2\pi}{6}} = 4e^{i\frac{\pi}{3}} = \boxed{2(1+\sqrt{3}i)}$$

(b) $(\sqrt{3}+i)^6 = -64$

This is the same as problem 2c with the angle being $\pi/6$ instead of the negative angle. Same work shows the result:

$$(\sqrt{3}+i)^6 = 2^6\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^6$$
$$2^6\left(e^{i\frac{\pi}{6}}\right)^6 = 2^6e^{i\pi} = 2^6(-1) = \boxed{-64}$$

(c) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$

Here we can first rationalize the fraction

$$(1+\sqrt{3}i)^{-10} = \left(\frac{1}{1+\sqrt{3}i}\right)^{10} =$$
$$\left(\frac{1-\sqrt{3}i}{(1+\sqrt{3}i)(1-\sqrt{3}i)}\right)^{10} = \left(\frac{1-\sqrt{3}i}{1-\sqrt{3}i+\sqrt{3}i-\sqrt{3}^2i^2}\right)^{10} =$$
$$\left(\frac{1-\sqrt{3}i}{4}\right)^{10} = 2^{-11}\left(\frac{1-\sqrt{3}i}{2}\right)^{10} =$$
$$2^{-11}\left(e^{-i\frac{\pi}{3}}\right)^{10} = 2^{-11}e^{-i\frac{10\pi}{3}} = 2^{-11}e^{-i\left(2\pi+\frac{4\pi}{3}\right)}$$
$$2^{-11}e^{-i\left(\pi+\frac{1\pi}{3}\right)} = 2^{-11}e^{-i\left(\frac{-2\pi}{3}\right)} = \boxed{2^{-11}(-1+\sqrt{3}i)}$$

Problem 4

Use exponential form to find $(1-i)^5$

$$(1-i)^5 = \left(\frac{2}{\sqrt{2}}\right)^5 \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^5 = \left(\frac{2}{\sqrt{2}}\right)^5 \left(e^{-i\frac{\pi}{4}}\right)^5 =$$

$$\text{angle } \frac{-5\pi}{4} \text{ corresponds to angle } \frac{3\pi}{4}$$

$$\left(\frac{2}{\sqrt{2}}\right)^5 e^{-i\frac{5\pi}{4}} = \left(\frac{2}{\sqrt{2}}^5\right) e^{i\frac{3\pi}{4}} =$$

$$\left(\frac{2}{\sqrt{2}}\right)^5 \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \left(\frac{2}{\sqrt{2}}\right)^4 (-1+i)$$

$$\text{recall}\frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}^2} = \sqrt{2}$$

$$\therefore \left(\frac{2}{\sqrt{2}}\right)^4 = \sqrt{2}^4 = 2^2 = 4$$

$$\therefore (1-i)^5 = \boxed{4(-1+i)}$$