Homework # 3:

MATH 3160 – Complex Variables
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Problem 1:

(a) Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

(b) Show that the image of the points in the upper half plane ($y > 0$) that are exterior to the circle $|z| = 1$ are mapped under f to the entire upper half plane $v > 0$.

(a)

$$\begin{aligned}
f(z) &= z + \frac{1}{z} = (x+iy) + \frac{1}{(x+iy)} = \frac{(x+iy)(x^2+y^2)}{(x^2+y^2)} + \frac{x-iy}{(x^2+y^2)} \\
&= \frac{1}{x^2+y^2}((x+iy)(x^2+y^2) + x - iy) = \frac{1}{x^2+y^2}(x(x^2+y^2) + x + i(y(x^2+y^2) - y)) \\
&\therefore u(x,y) = \frac{1}{x^2+y^2}(x(x^2+y^2)+x) \quad \& \quad v(x,y) = \frac{1}{x^2+y^2}(y(x^2+y^2)-y) \\
&r^2 = x^2 + y^2 \\
&x = r\cos(\theta) \\
&y = r\sin(\theta)
\end{aligned}$$

$$\begin{aligned}
\therefore u(r,\theta) &= \frac{1}{r^2}(r^3\cos(\theta) + r\cos(\theta)) = r\cos(\theta) + \frac{1}{r}\cos(\theta) \\
&= \boxed{\left(r + \frac{1}{r}\right)\cos(\theta)} \\
v(r,\theta) &= \frac{1}{r^2}(r^3\sin(\theta) - r\sin(\theta)) = r\sin(\theta) - \frac{1}{r}\sin(\theta) \\
&= \boxed{\left(r - \frac{1}{r}\right)\sin(\theta)}
\end{aligned}$$

(b)

Since we have $r > 1$ for any point exterior to $|z| = 1$, then the condition for the y value:

$$\begin{aligned} v(r,\theta) &= \left(r - \frac{1}{r}\right)\sin(\theta) \\ &= \left(r - \frac{1}{r}\right) > 0 \quad \forall\, r > 1 \\ &= 0 \leq \sin(\theta) \leq 1 \quad \forall\, \theta \mid 0 \leq \theta \leq \pi \end{aligned}$$

Meaning all positive values for $y = r\sin(\theta)$ map to positive values for $v(r,\theta)$. Since a point inside the circle $|z| = 1$ would have $r \leq 1$, that would put the value of the factor on $\sin(\theta)$ less than 0. Taking the simplest case of $r = 1 + \epsilon$ where ϵ is a small increase, and the angle $\theta = 0$, we get a point v that is equal to 0. As we sweep the angle to $\frac{\pi}{2}$, we get a factor of 1 multiplied by something larger than 0.

$$\begin{aligned} v(r,\theta) &= \left(r - \frac{1}{r}\right)\sin(\theta) = \left((r+\epsilon) - \frac{1}{(r+\epsilon)}\right)\sin(\theta) \\ &= \left(\frac{(r+\epsilon)^2}{(r+\epsilon)} - \frac{1}{r+\epsilon}\right)\sin(\theta) = \left(\frac{(r+\epsilon)^2 - 1}{r+\epsilon}\right)\sin(\theta) \\ &= \left(\frac{((r+\epsilon)^2 - 1)(r-\epsilon)}{(r+\epsilon)(r-\epsilon)}\right)\sin(\theta) = \left(\frac{(r^2 + 2r\epsilon + \epsilon^2 - 1)(r-\epsilon)}{r^2 - \epsilon^2}\right)\sin(\theta) \\ &= \left(\frac{(r^2 + 2r\epsilon + \cancelto{0}{\epsilon^2} - 1)(r-\epsilon)}{r^2 - \cancelto{0}{\epsilon^2}}\right)\sin(\theta) = \left(\frac{(r^2 + 2r\epsilon - 1)(r-\epsilon)}{r^2}\right)\sin(\theta) \\ &= \left(\frac{(r^2 + 2r\epsilon - 1)(r-\epsilon)}{r^2}\right)\sin(\theta) = \left(\frac{(r^3 + 2r^2\epsilon - r - r^2\epsilon - 2r\epsilon^2 + \epsilon)}{r^2}\right)\sin(\theta) \\ &= \left(\frac{(r^3 + r^2\epsilon - r - \cancelto{0}{2r\epsilon^2} + \epsilon)}{r^2}\right)\sin(\theta) = \left(\frac{(r^3 + r^2\epsilon - r + \epsilon)}{r^2}\right)\sin(\theta) \\ &= \left(r + \epsilon - \frac{1}{r} + \frac{\epsilon}{r^2}\right)\sin(\theta) = \left(\left(r - \frac{1}{r}\right) + \left(\epsilon + \frac{\epsilon}{r^2}\right)\right)\sin(\theta) \end{aligned}$$

With $r = 1$ with the very small increase, we can plug 1 in for r and we see that the overall value is still positive.

$$\begin{aligned} &\left(\left(r - \frac{1}{r}\right) + \left(\epsilon + \frac{\epsilon}{r^2}\right)\right)\sin(\theta) = \left(\left(1 - \frac{1}{1}\right) + \left(\epsilon + \frac{\epsilon}{1}\right)\right)\sin(\theta) \\ &((1-1) + (2\epsilon))\sin(\theta) = 2\epsilon\sin(\theta) \end{aligned}$$

$\boxed{\therefore \text{ all values in the upper half plane outside the circle } |z| = 1 \text{ will map to the values such that } v > 0.}$

TODO:Showing the condition maps to the entire half-plane is not yet shown, but I think this is a decent argument so far. Come up with a method for showing this condition satisfies and covers the entire half-plane.

Problem 2:

Use the rectangular forms or exponential forms for the following functions to prove that

(a) $\lim_{z\to z_0} Re(z) = Re(z_0)$

$$\lim_{z\to z_0} Re(z) = \lim_{x\to x_0} x = x_0$$

We can stop here since this is sufficient.

(b) $\lim_{z\to z_0} \bar{z} = \bar{z}_0$

$$\begin{aligned}\lim_{z\to z_0} \bar{z} = \lim_{(x,y)\to(x_0,y_0)} \overline{(x+iy)} &= \lim_{(x,y)\to(x_0,y_0)} (x-iy) \\ &= \lim_{(x,y)\to(x_0,y_0)} = (x_0 - iy_0)\end{aligned}$$

We can stop here since this is sufficient.

(c) $\lim_{z\to 0} \frac{\bar{z}^2}{z} = 0$

$$\begin{aligned}\lim_{z\to 0} \frac{\bar{z}^2}{z} &= \lim_{r\to 0} \frac{\overline{re^{i\theta}}^2}{re^{i\theta}} \\ &= \lim_{z\to 0} \frac{(re^{-i\theta})^2}{re^{i\theta}} = \lim_{r\to 0} \frac{r^2 e^{-i2\theta}}{re^{i\theta}} \\ &= \lim_{r\to 0} r\frac{e^{-i2\theta}}{e^{i\theta}} = \lim_{r\to 0} re^{-i2\theta}e^{-i\theta} \\ &= \lim_{r\to 0} re^{-i3\theta}\end{aligned}$$

For this, no matter the angle used, we still have a dependence on r in the expression, so from any path, we will approach 0.

Problem 3:

Show that the limit of the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^3$$

as z tends to zero does not exist. Do so by examining several test paths going to zero.

$$z = re^{i\theta}$$
$$\bar{z} = re^{-i\theta}$$
$$\frac{z}{\bar{z}} = \frac{re^{i\theta}}{re^{-i\theta}} = re^{i\theta}r^{-1}e^{i\theta} = e^{i2\theta}$$
$$\left(\frac{z}{\bar{z}}\right)^3 = (e^{i2\theta})^3 = e^{i6\theta}$$

We can see that the dependence on r is now gone as the two canceled out. Evaluating two paths with differing value of θ will give two different magnitudes for the result. First with $\theta = 0$

$$\lim_{r \to 0} e^{i6\theta} = \lim_{r \to 0} e^0 = \lim_{r \to 0} = 1$$

as in it becomes 1 from the positive side. Now with $\theta = \frac{\pi}{2}$:

$$\lim_{r \to 0} e^{i6\theta} = \lim_{r \to 0} e^{i6\frac{\pi}{2}} = \lim_{r \to 0} e^{i3\pi} = -1$$

Here it becomes -1 by approaching from the negative side.

$\boxed{\therefore \text{ The limit DNE because different paths give different results.}}$

Problem 4:

Does $f(x+iy) = \dfrac{x+iy}{x+2iy}$ have a limit as $x+iy \to 0$?

No the function does not have a limit as it tends to 0 because the expression $x+iy$ can only become 0 if both x and y tend to 0:

$$\frac{x+iy}{x+2iy} = \frac{x+iy}{x+iy+iy} = \frac{x+iy}{(x+iy)+iy} \neq \frac{0}{iy}$$

This cannot equal 0 because if $x+iy$ implies that both x and y tend to 0, then that implies that the denominator is also going to become 0 which we know is an indeterminate form. Evaluating from two paths again as we have done previously. We can start with $y=0$ and evaluate the limit from the real axis.

$$\frac{x+iy}{x+2iy} = \frac{x+i(0)}{x+2i(0)} = \frac{x}{x} = 1$$

This loses all dependence on the variables and is 1 for any values approaching 0. Approaching now from the imaginary axis.

$$\frac{x+iy}{x+2iy} = \frac{(0)+iy}{(0)+2iy} = \frac{iy}{2iy} = \frac{1}{2}$$

This loses all dependence on the variables and is $\frac{1}{2}$ for any values approaching 0.

$\boxed{\therefore \text{ The limit DNE because different paths give different results.}}$