Homework #5

MATH 3160 – Complex Variables
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Problem 1

Consider the analytic function $f(z) = ze^{z^2}$.

(a) Show that the function $u(x, y) = x\ e^{(x^2-y^2)} \cos(2xy) - y\ e^{(x^2-y^2)} \sin(2xy)$ is the real component of $f(z)$.

(b) What is a harmonic conjugate for $u(x, y)$?

(c) Without computing the second partial derivatives of $u(x, y)$, explain why you know that $u(x, y)$ is harmonic.

Problem 2

Consider the function $u(x, y) = x^3 - 3xy^2 - 3x^2y + y^3$.

(a) Show that $u(x, y)$ is harmonic.

(b) Find a harmonic conjugate for $u(x, y)$.

Problem 3

Recall we learned of the following fact in class:

Let $u(x, y)$ be a harmonic function defined on a simply connected domain D.
Then $u(x, y)$ has a harmonic conjugate on D.

(a) Show that $u(x, y) = \ln(\sqrt{x^2 + y^2})$ is a harmonic function.

(b) What is the domain of definition of $u(x, y)$?

(c) An aside: show that if $f(z)$ and $g(z)$ are two analytic functions on the same domain D, and we have $\mathrm{Re}(f(z)) = \mathrm{Re}(g(z))$ for all $z \in D$, then $f(z) = g(z) + c$ for some constant $c \in \mathbb{C}$.

[*Hint: show that the function $h(z) = f(z) - g(z)$ has Re$(h(z)) = 0$, and then use a result from class to conclude $h(z)$ is a constant.*]

(d) Explain why $u(x, y)$ does *not* have a harmonic conjugate on its domain.

[*Hint: if such a conjugate existed, then $u(x, y)$ would be the real component of some analytic function $f(z)$, but $u(x, y)$ is already the real component of a familiar analytic function, which is discontinuous at its branch cut*]

1. Why does this not contradict the fact from class?

Problem 4

Find the following values, on the branches given:

(a) $\log(3)\ \ (-2\pi \leq \theta < 0)$

(b) $\log(-1+i)\ \ (-\pi/2 < \theta \leq 3\pi/2)$

(c) $\log(1-i\sqrt{3})\ \ (\pi \leq \theta < 3\pi)$.

Problem 5

Recall that power functions are defined by $z^c = e^{c\log(z)}$. In this exercise, we compute all power functions by using the branch $(0 \le \theta < 2\pi)$ for $\log(z)$.

(a) For $z = -i$ and $c = i$, compute the values of $(z^c)^2$, $(z^2)^c$, and $z^{(2c)}$.

(b) With the notation as in (a), which of these are true or false?

$$(z^c)^2 = (z^2)^c, \qquad (z^c)^2 = z^{(2c)}, \qquad (z^2)^c = z^{(2c)}.$$