Worksheet # 3

MATH 3160 – Complex Variables
Miguel Gomez

Completed: August 31, 2025

Problem 1

Show that $\text{Re}(z) = \frac{z+\bar{z}}{2}$ and $\text{Im}(z) \cdot i = \frac{z-\bar{z}}{2}$ for any complex number $z = a + bi$:

Expressing the two as complex numbers and reducing:

$$\text{Re}(z) = \frac{z+\bar{z}}{2} = \frac{a+bi+a-bi}{2} = \frac{2a}{2} = a = \text{Re}(z)$$
$$\text{Im}(z) \cdot i = \frac{z-\bar{z}}{2} = \frac{a+bi-a+bi}{2} = \frac{2bi}{2} = bi = \text{Im}(z) \cdot i$$

Problem 2

Find the fourth roots of $-8 - 8\sqrt{3}i$. express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out the principal root.

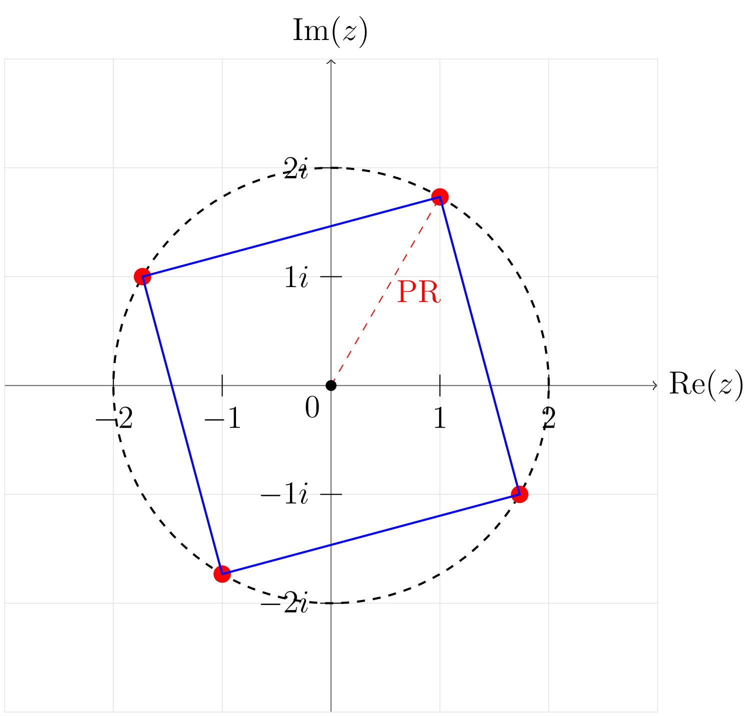
$$-8 - 8\sqrt{3}i = -8(1 + \sqrt{3}) = 16(-1)\left(\frac{1+\sqrt{3}}{2}\right) =$$
$$16e^{i\pi}e^{\frac{\pi}{3}} = 16e^{i\frac{3\pi}{3}}e^{\frac{\pi}{3}} = 16e^{i\frac{4\pi}{3}} = 16e^{i\frac{-2\pi}{3}}$$
$$16 = 2^4$$
$$16e^{i\frac{-2\pi}{3}} = 2^4 e^{i\frac{-2\pi}{3}}$$

Starting from this point, we can take the 4^{th} root and then rotate that root by $\frac{2\pi}{4} = \frac{\pi}{2}$ to find the rest of the points.

$$(e^{i\frac{-2\pi}{3}})^{\frac{1}{4}} = e^{i\frac{-2\pi}{12}} = e^{i\frac{-\pi}{6}}$$
$$e^{i(\frac{-\pi}{6}+\frac{3\pi}{6})} = e^{i\frac{2\pi}{6}} = e^{i\frac{\pi}{3}}$$
$$e^{i\frac{2\pi+3\pi}{6}} = e^{i\frac{5\pi}{6}}$$
$$e^{i\frac{5+3\pi}{6}} = e^{i\left(\pi+\frac{2\pi}{6}\right)} = e^{i\left(\pi+\frac{\pi}{3}\right)}$$

So the principal root is the first root we get when moving counter-clockwise from 0, we get

$$= 2e^{i\frac{\pi}{3}}$$



Problem 3

Find the four zeros of the polynomial $z^4 + 4$, given that one of them is:

$$z_0 = \sqrt{2}e^{i\frac{\pi}{4}} = 1 + i$$

Use these zeros to factor $z^4 + 4$ into quadratic factors with real coefficients.

The zeros would be equally spaced because the polynomial can be factored into four roots.

$$z^4 + 4 = (z^2)^2 - -4 = (z^2)^2 - (2i)^2 =$$

$$(z^2 - 2i)(z^2 + 2i)$$

First root can also be put in terms of a difference of squares

$$(z^2 - 2i) = (z^2 - (\sqrt{2i})^2) = (z - \sqrt{2i})(z + \sqrt{2i})$$

Second root can also be put in terms of a difference of squares

$$(z^2 + 2i) = (z^2 - -2i) = (z^2 - (\sqrt{2i}i)^2) = (z - \sqrt{2i}i)(z + \sqrt{2i}i)$$

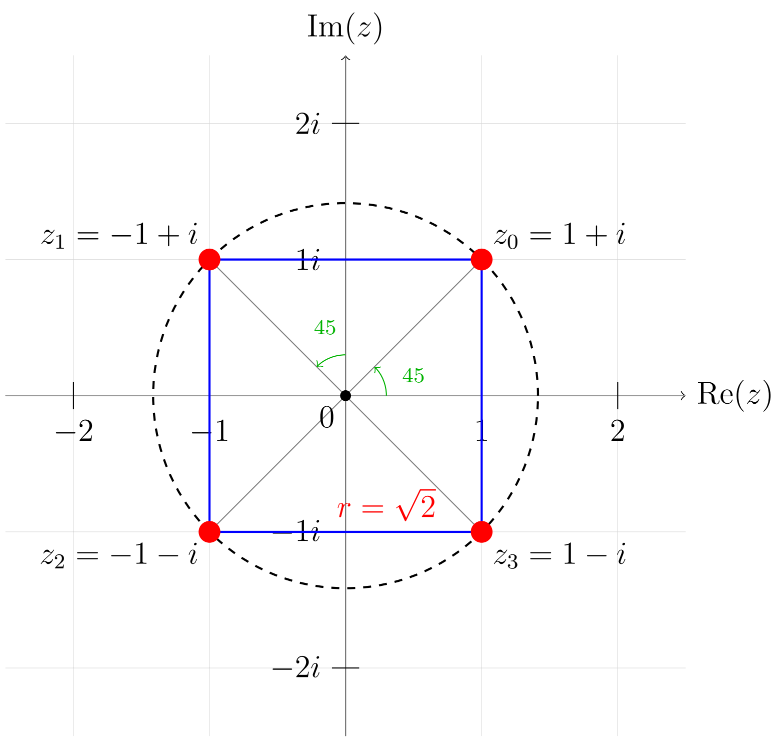
$$= (z - \sqrt{2i})(z + \sqrt{2i})(z - \sqrt{2i}i)(z + \sqrt{2i}i)$$

square root of i:

$$\sqrt{i} = i^{\frac{1}{2}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$$

magnitude of zero is $\sqrt{2}$

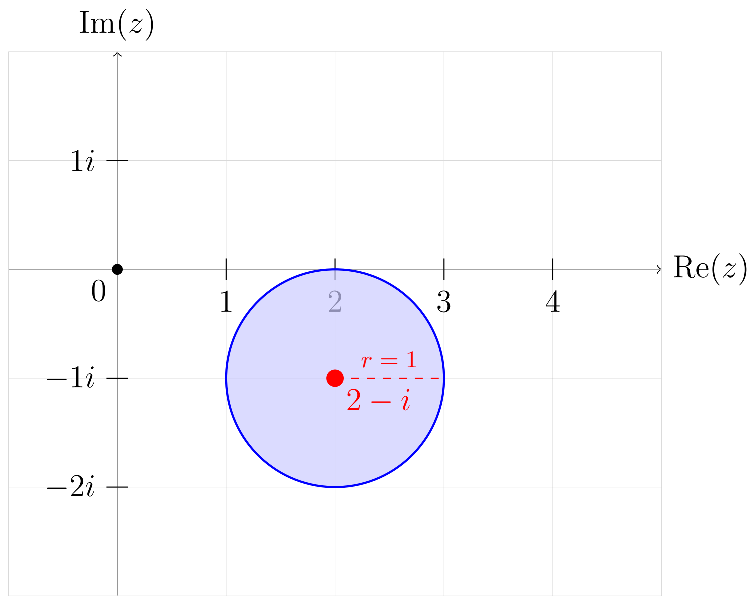
$\therefore$ The four zeros are the points at $\sqrt{2}$ from the center. These all have angles that are $\pm 45°$ from 0 and π.



Problem 4

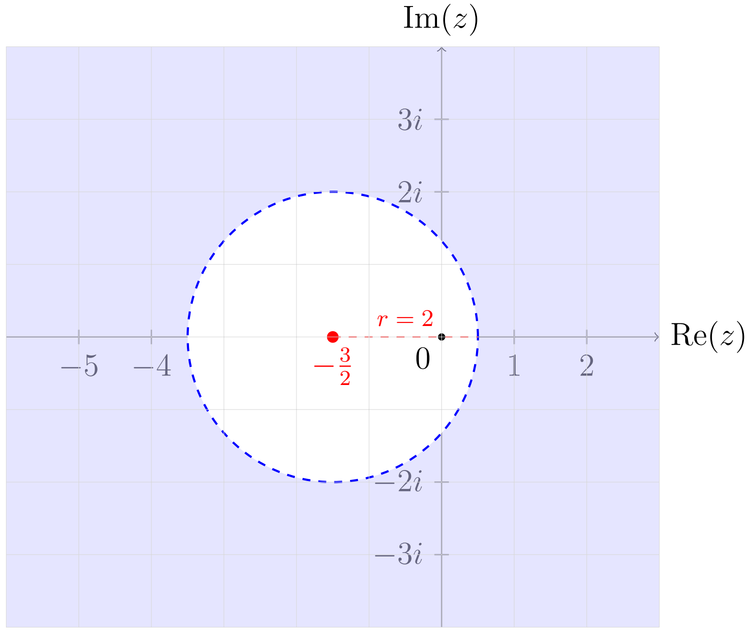
Sketch the following sets and state whether each set is open, connected, a domain, and whether it is bounded.

(a) $|z - 2 + i| \leq 1$



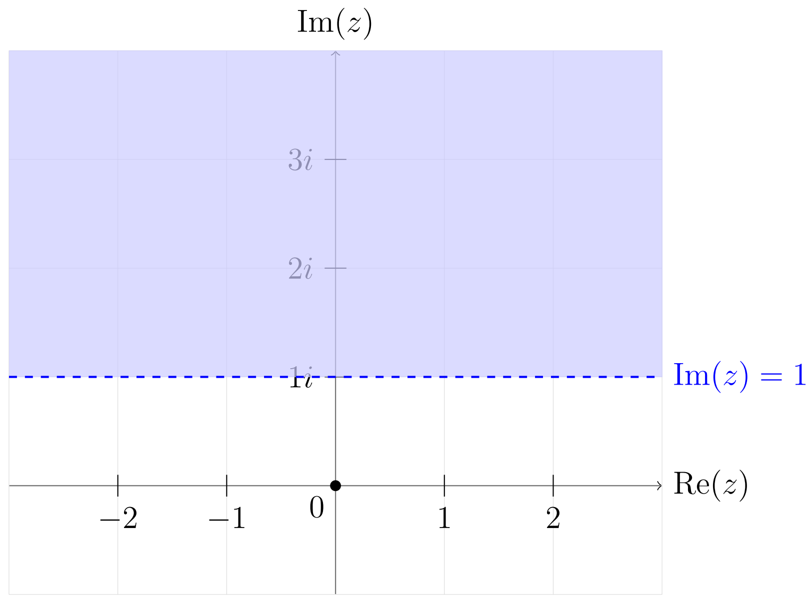
- **Open:** No, because the boundary is included ($\leq$ condition)
- **Connected:** Yes, it's a disk which is connected
- **Domain:** No, because it's not open
- **Bounded:** Yes, all points are within distance 1 from center $(2, -1)$

(b) $|2z + 3| > 4$



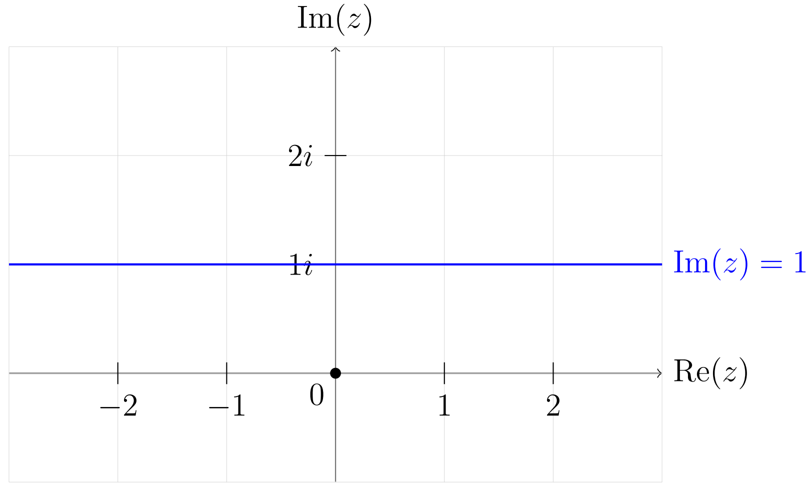
- **Open:** Yes, because boundary is not included
- **Connected:** Yes because the region outside the disk is connected
- **Domain:** Yes, open and connected are satisfied
- **Bounded:** No, the region extends to infinity

(c) $\mathrm{Im}(z) > 1$



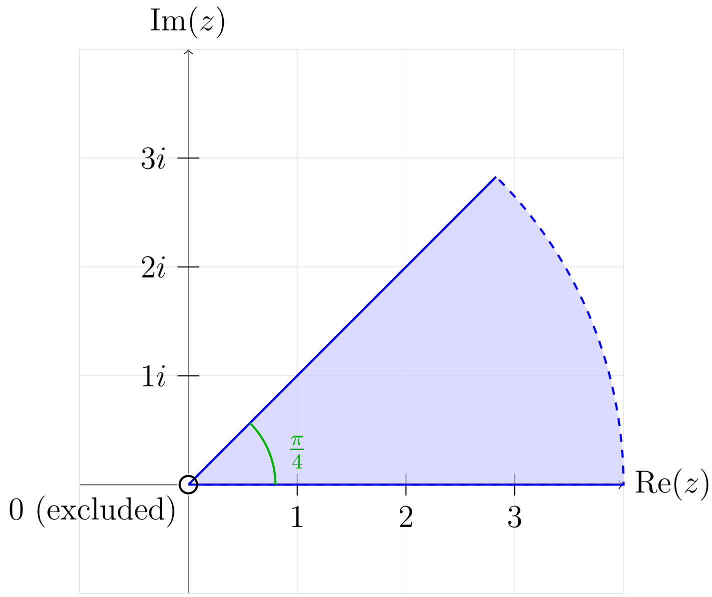
- **Open:** Yes, because boundary is not included
- **Connected:** Yes because the region above the line $\mathrm{Im}(z) = 1$ is connected
- **Domain:** Yes, open and connected are satisfied
- **Bounded:** No, the region extends to infinity

(d) $\text{Im}(z) = 1$



- **Open:** No, because line is included
- **Connected:** No because no region of radius r can be formed on a line
- **Domain:** No, open and connected aren't satisfied
- **Bounded:** No, the line extends to infinity and there is no region

(e) $0 \leq \arg(z) \leq \frac{\pi}{4}$, where $z \neq 0$

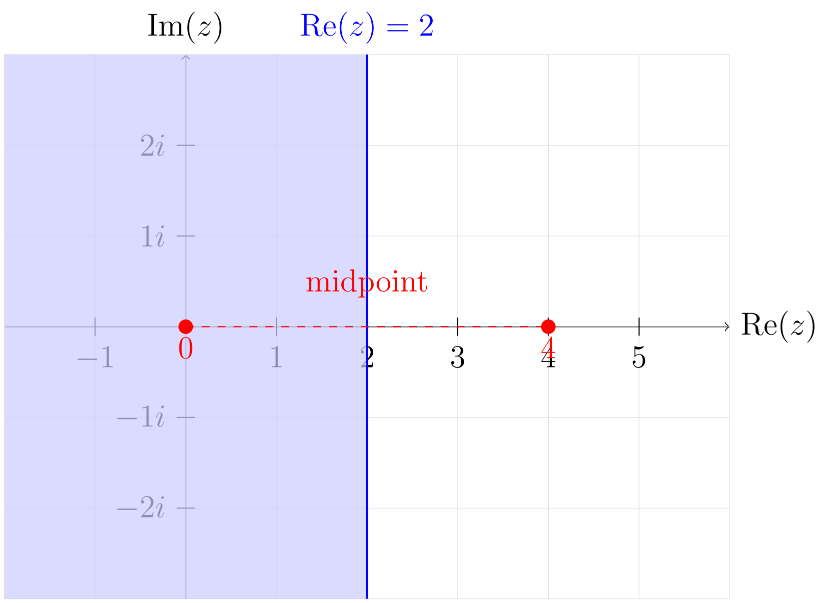


- **Open:** No, because boundary is included
- **Connected:** Yes because the region inside the arc is connected
- **Domain:** No, because both open and connected aren't satisfied
- **Bounded:** No, the region within the arc extends to infinity

(f) $|z-4| \geq |z|$

Equal at $z = 2$. If $z = 0$, then the expression holds. $4 \geq 0$. If $z > 2$, then we get a false condition. say it were 3:

$$|3-4| \geq |3| \rightarrow |-1| \ngeq |3|$$



- **Open:** No because the boundary is included.
- **Connected:** Yes because the region less than midpoint is connected
- **Domain:** No, because both open and connected aren't satisfied
- **Bounded:** No, the region less than midpoint extends to infinity