

Homework #2

MATH 3160 – complex variables
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Problem 1

By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i)$

$$\begin{aligned} i &= e^{i\frac{\pi}{2}} \\ (1 - \sqrt{3}i) &= 2 \left(\frac{1 - \sqrt{3}i}{2} \right) = 2e^{-i\frac{\pi}{3}} \\ (\sqrt{3} + i) &= 2 \left(\frac{\sqrt{3} + i}{2} \right) = 2e^{i\frac{\pi}{6}} \\ e^{i\frac{\pi}{2}} 2e^{i\frac{\pi}{6}} 2e^{-i\frac{\pi}{3}} &= 4e^{i\pi(\frac{1}{2} + \frac{1}{6} - \frac{1}{3})} \\ &= 4e^{i\pi(\frac{2}{6} + \frac{1}{6} - \frac{2}{6})} \\ &= 4e^{i\pi(\frac{2}{6})} \\ &= 4e^{i(\frac{\pi}{3})} \\ &= 4 \left(\frac{1 + \sqrt{3}i}{2} \right) \\ &= 2(1 + \sqrt{3}i) \quad \square \end{aligned}$$

(b) $\frac{5i}{2+i} = 1 + 2i$

$$\begin{aligned} \frac{5i}{2+i} &= \frac{5i(2-i)}{(2+i)(2-i)} = \frac{(10i - 5i^2)}{4 - 2i + 2i - i^2} \\ &= \frac{(10i - 5i^2)}{4 - 2i + 2i - i^2} \stackrel{-1}{=} \frac{(10i + 5)}{5} \\ &\stackrel{0}{=} \frac{4 - 2i + 2i - i^2}{5} \stackrel{-1}{=} 1 + 2i \quad \square \end{aligned}$$

(c) $(-1 + i)^7 = -8(1 + i)$

$$\begin{aligned}
 -1 + i &= \sqrt{2} \left(\frac{-\sqrt{2} + \sqrt{2}i}{2} \right) = \sqrt{2} e^{i \frac{3\pi}{4}} \\
 (-1 + i)^7 &= (\sqrt{2} e^{i \frac{3\pi}{4}})^7 = \sqrt{2}^7 e^{i \frac{21\pi}{4}} \\
 &= \sqrt{2}^7 e^{i \frac{21\pi}{4}} = \sqrt{2}^7 e^{i(\frac{24\pi}{4} - \frac{3\pi}{4})} = \sqrt{2}^7 e^{i(6\pi - \frac{3\pi}{4})} \\
 &= \sqrt{2}^7 \cancel{e^{i6\pi}} e^{-i(\frac{3\pi}{4})} = \sqrt{2}^7 e^{-i(\frac{3\pi}{4})} \\
 &= \sqrt{2}^7 \frac{1}{\sqrt{2}} (-1 - i) = -2^3 (1 + i) \\
 &= -8(1 + i) \quad \square
 \end{aligned}$$

(d) $(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)$

$$\begin{aligned}
 (1 + \sqrt{3}i)^{-10} &= \left(2 \frac{1 + \sqrt{3}i}{2} \right)^{-10} = 2^{-10} \left(\frac{1 + \sqrt{3}i}{2} \right)^{-10} = 2^{-10} (e^{i \frac{\pi}{3}})^{-10} \\
 &= 2^{-10} e^{-i \frac{10\pi}{3}} = 2^{-10} e^{-i(\frac{6\pi}{3} + \frac{4\pi}{3})} = 2^{-10} e^{-i \frac{4\pi}{3}} \\
 &= 2^{-10} e^{i \frac{2\pi}{3}} = 2^{-10} \left(\frac{-1 + \sqrt{3}i}{2} \right) = 2^{-11}(-1 + \sqrt{3}i) \quad \square
 \end{aligned}$$

Problem 2

Find the square roots of (a) $2i$ and (b) $(1 - \sqrt{3}i)$ express them in rectangular coordinates

(a):

$$\begin{aligned}\sqrt{2i} &= \sqrt{2}\sqrt{i} = \sqrt{2}i^{\frac{1}{2}} = \sqrt{2}(e^{i\frac{\pi}{2}})^{\frac{1}{2}} = \sqrt{2}e^{i\frac{\pi}{4}} \\ \sqrt{2}e^{i\frac{\pi}{4}} &= \sqrt{2}(1 + i)\end{aligned}$$

Second root is at $\frac{2\pi}{2} = \pi$ from this one:

$$\sqrt{2}e^{-i\frac{3\pi}{4}}$$

(b):

$$\begin{aligned}\sqrt{(1 - \sqrt{3}i)} &= \sqrt{2\left(\frac{1 - \sqrt{3}i}{2}\right)} = \sqrt{2}\left(\frac{1 - \sqrt{3}i}{2}\right)^{\frac{1}{2}} = \sqrt{2}(e^{-i\frac{\pi}{3}})^{\frac{1}{2}} \\ &= \sqrt{2}e^{-i\frac{\pi}{6}}\end{aligned}$$

Again the other root is π away:

$$= \sqrt{2}e^{-i(\frac{\pi}{6} + \pi)} = \sqrt{2}e^{i\frac{5\pi}{6}}$$

Problem 3

Find all roots and indicate in rectangular coordinates

(a) $(-16)^{\frac{1}{4}}$

$$2^4 = 16$$

$$-1 = i^2$$

$$-16 = i^2 16 = i^2 4^2 = \sqrt{i^4} 2^4 = (2\sqrt{i})^4$$

roots are the four angles corresponding to

$$\left\{ \pm \frac{\pi}{4}, \pm \frac{3\pi}{4} \right\}$$

$$\{(1+i), (-1+i), (-1-i), (1-i)\}$$

(b) $(-8 - 8\sqrt{3}i)^{\frac{1}{4}}$ These make up a square with the first angle being $\frac{\pi}{3}$ with each separated by 90° .

Problem 4

Find the four zeros of $z^4 + 4$

Converting z into exponential form for $|z| = 1$

$$z^4 = (e^{i\theta})^4 = e^{i4\theta}$$

Four windings for this, so we should have 4 equally spaced roots that give us the zeros.

$$\begin{aligned} z^4 + 4 &= 0 \\ z^4 &= -4 \\ \sqrt[4]{z^4} &= \sqrt[4]{-4} \\ z^{\frac{1}{4}4} &= \sqrt[4]{-4} = e^{\frac{1}{4}i4\theta} \\ z &= \sqrt[4]{-4} = e^{i\theta} \\ &= \sqrt[4]{-2 \cdot 2} = \sqrt[4]{-\sqrt{2}^2 \cdot \sqrt{2}^2} = \sqrt[4]{-\sqrt{2}^4} \\ &= \sqrt{2} \cdot \sqrt[4]{-1} \\ \sqrt{z^4} &= \pm z^2 = \sqrt{-4} = \pm 2\sqrt{-1} \\ \sqrt{\pm z^2} &= \{\sqrt{z^2}, \sqrt{-z^2}\} = \{\pm z, \pm z\sqrt{-1}\} \\ &= \{\pm z, \pm zi\} \end{aligned}$$

This is also the following:

$$\begin{aligned} z^4 + 4 &= z^4 - (-4) = z^4 - (i^2 2^2) = (z^2)^2 - (2i)^2 \\ &= (z^2 - 2i)(z^2 + 2i) \end{aligned}$$

converting to exponential

$$\begin{aligned} i &= e^{i\frac{\pi}{2}} \\ \therefore \sqrt{i} &= (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}} \\ (z^2 - 2i) &= (z^2 - (\sqrt{2}\sqrt{i})^2) = (z - (\sqrt{2}\sqrt{i}))(z + (\sqrt{2}\sqrt{i})) \\ &= \boxed{(z - (\sqrt{2}\sqrt{i}))(z + (\sqrt{2}\sqrt{i}))} \end{aligned}$$

similarly, for the positive side

$$\begin{aligned} (z^2 + 2i) &= (z^2 - (-2i)) = (z^2 - i^2(\sqrt{2}\sqrt{i})^2) \\ &= \boxed{(z - (\sqrt{2}\sqrt{i})i)(z + (\sqrt{2}\sqrt{i})i)} \\ i\sqrt{i} &= e^{\frac{i\pi}{2}} e^{i\frac{\pi}{4}} = e^{i\pi(\frac{1+2}{4})} = e^{i\frac{3\pi}{4}} \\ \therefore z &= \sqrt{2}\{\pm e^{i\frac{\pi}{4}}, \pm e^{i\frac{3\pi}{4}}\} = \sqrt{2}\{(1+i), (-1+i), (-1-i), (1-i)\} \end{aligned}$$

Problem 5

Show that if c is an n^{th} root of 1 other than 1 itself, then:

$$1 + c + c^2 + \dots + c^{n-1} = 0$$

Hint: multiply above by $(c - 1)$

multiplying the above by $(c - 1)$ gives the following

$$\begin{aligned} (c - 1) \cdot (1 + c + c^2 + \dots + c^{n-1}) &= (c - 1) \cdot 0 \\ c + c^2 + c^3 + \dots + c^{n-1} + c^n &+ \quad \text{expanding } c \\ -1 - c - c^2 - \dots - c^{n-1} &\quad \text{expanding } -1 \\ -1 + (c - c) + (c^2 - c^2) + (c^3 - c^3) + \dots &+ (c^{n-1} - c^{n-1}) + c^n = 0 \\ -1 + \cancel{(c - c)} + \cancel{(c^2 - c^2)} + \cancel{(c^3 - c^3)} + \dots &+ \cancel{(c^{n-1} - c^{n-1})} + c^n = 0 \\ -1 + c^n &= 0 \\ c^n &= 1 \\ \sqrt[n]{c^n} &= \sqrt[n]{1} \\ c &= 1 \end{aligned}$$

Now using something other than 1, i.e. $c \neq 1$, then the sum cannot be 0.

$$1 + c + c^2 + \dots + c^{n-1} = S$$

same steps as before

$$c^n - 1 = S(c - 1)$$

since $c^n = 1$; and $c \neq 1$ we can divide

$$\frac{c^n - 1}{c - 1} = S$$

since $c \neq 1$; $(c - 1)$ is not 0 but $c^n = 1$

$$\therefore \frac{\cancel{c^n}^1 - 1}{c - 1} = \frac{0}{c - 1} = S = 0 \quad \square$$

Problem 6

For each of the below, indicate the domain of definition.

(a) $f(z) = \frac{1}{z^2+1}$

We need $z^2 + 1 \neq 0$, which means $z^2 \neq -1$.

Solving $z^2 = -1$:

$$z^2 = -1 = i^2$$

$$z = \pm i$$

$$(\pm i)^2 + 1 = i^2 + 1 = -1 + 1 = 0$$

undefined only at $z = i$ and $z = -i$.

For all other complex numbers z , we have $z^2 + 1 \neq 0$, so the function is well-defined.

\therefore Domain of Definition: $\mathbb{C} \setminus \{\pm i\}$

(b) $f(z) = \text{Arg}\left(\frac{1}{z}\right)$

Here we need $z \neq 0$ Since the division is not defined when $z = 0$.

So for z :

Domain of definition: $\mathbb{C} \setminus \{0\}$

This results in the $\text{Arg}(z)$ being defined in the typical range:

$$-\pi < \text{Arg}(z) \leq \pi$$

(c) $f(z) = \frac{z}{z-\bar{z}}$

Recall $z - \bar{z} = 2i\text{Im}(z)$

$$\begin{aligned} \therefore \frac{z}{z-\bar{z}} &= \frac{z}{2i\text{Im}(z)} = \frac{z(-2i\text{Im}(z))}{-(2i\text{Im}(z))^2} \\ &= \frac{-2z\text{Im}(z)i}{-(4i^2\text{Im}(z)^2)} = \frac{-2z\text{Im}(z)i}{(4\text{Im}(z)^2)} \end{aligned}$$

Condition: $\text{Im}(z) \neq 0$

Domain of Definition: $\mathbb{C} \setminus \{z = a + bi \mid b = 0; a \in \mathbb{R}\}$

Note*: I mistakenly did this for minus in the denominator, it should be plus after looking over HW sheet again.

(c) corrected: $f(z) = \frac{z}{z+\bar{z}}$

Recall $z + \bar{z} = 2\operatorname{Re}(z)$

$$\therefore \frac{z}{z+\bar{z}} = \frac{z}{2\operatorname{Re}(z)}$$

Condition: $\operatorname{Re}(z) \neq 0$

Domain of Definition: $\mathbb{C} \setminus \{z = a + bi \mid a = 0; b \in \mathbb{R}\}$

(d) $f(z) = \frac{1}{(1-|z|^2)}$ Since $1 - |z|^2$ is in the denominator, our condition here is as follows:

$$|z|^2 \neq 1; \quad |z| = \sqrt{a^2 + b^2}$$

$$|z|^2 = a^2 + b^2$$

$$a^2 + b^2 \neq 1$$

This corresponds to all points on the unit circle.

\therefore Domain of Definition: $\mathbb{C} \setminus \{|z| = a^2 + b^2 = 1\}$

1 Problem 7

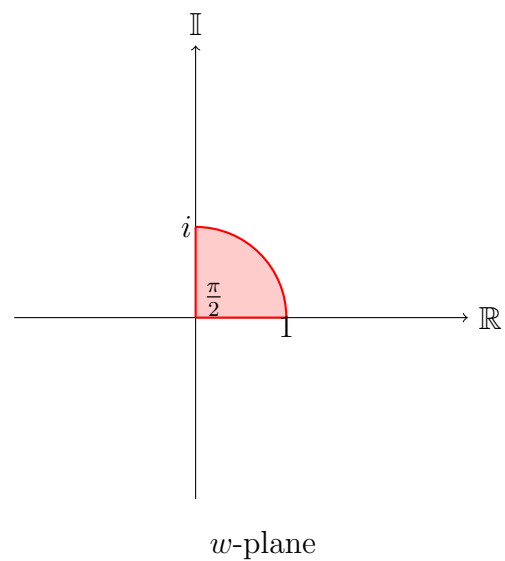
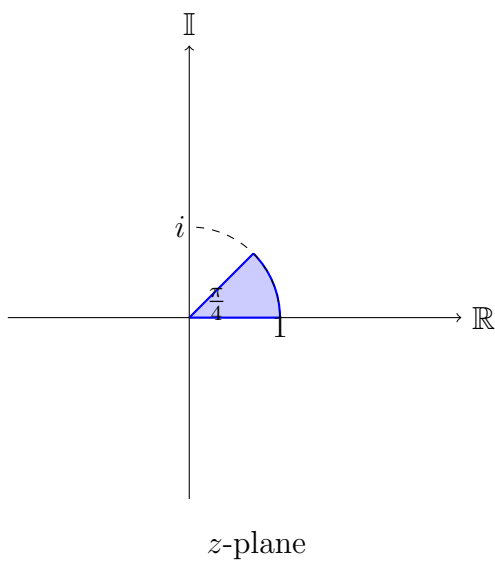
Sketch the region onto which the sector $r \leq 1$; $0 \leq \theta \leq \frac{\pi}{4}$ in the z -plane is mapped to the $w = f(z)$ -plane by the transformations

(a) $w = z^2$

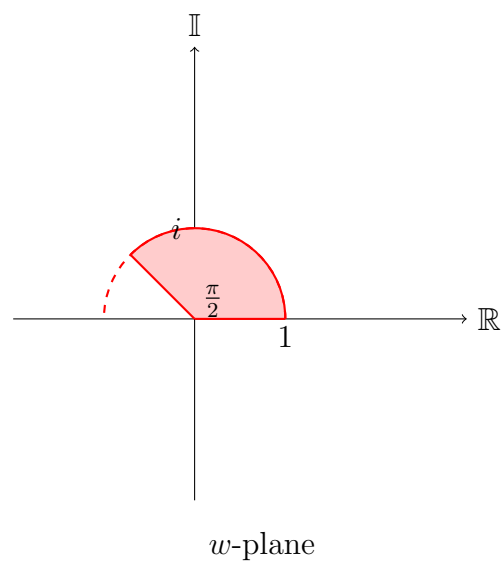
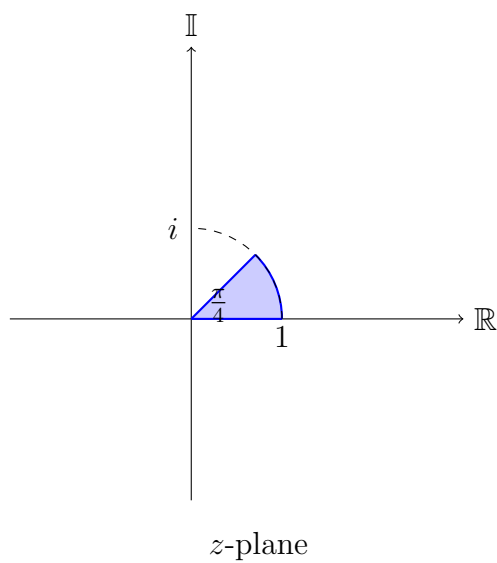
$$z = e^{i\theta}$$

$$z^2 = (e^{i\theta})^2 = e^{i2\theta}$$

Corresponds to two windings of the angle θ , essentially moving by 2θ instead:



(b) $w = z^3$ Corresponds to three windings of the angle like before but with 3θ :



(c) $w = z^4$

Corresponds to three windings of the angle like before but with 4θ :

