Homework # 4

MATH 3160 – Complex Variables
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Problem 1

Find $f'(z)$ using differentiation rules.

(a) $f(z) = 3z^2 - 2z + 4$

(b) $f(z) = (1-4z)^3$

(c) $f(z) = \frac{z-1}{2z+1}$, assume $z \neq -1/2$

(d) $f(z) = \frac{(z^2+1)^4}{z^2}$, assume $z \neq 0$

(e) $f(z) = z\,e^{z^2+3}$.

(a)

$$\begin{aligned} f(z) &= 3z^2 - 2z + 4 \\ f'(z) &= 6z - 2 \end{aligned}$$

(b)

$$\begin{aligned} f(z) &= a(z) = (1-4z)^3 = b^3 \mid b = 1-4z \\ f'(z) &= \frac{da}{db}\frac{db}{dz} = 3(1-4z)^2(-4) = -12(1-4z)^2 \end{aligned}$$

(c)

Assuming $z \neq -1/2$

$$\begin{aligned} f(z) &= \frac{z-1}{2z+1} = \frac{f}{g} \\ \frac{df}{dz} &= 1 \qquad \frac{dg}{dz} = 2 \\ f'(z) = \frac{f'g - fg'}{g^2} &= \frac{1(2z+1)-(z-1)2}{(2z+1)^2} \\ &= \frac{(2z-2z)+(1+2)}{(2z+1)^2} = \frac{3}{(2z+1)^2} \end{aligned}$$

(d)

Assuming $z \neq 0$

$$
\begin{aligned}
f(z) &= \frac{(z^2+1)^4}{z^2} = \frac{a}{b} \\
\frac{da}{dz} &= 4(z^2+1)^3(2z) \qquad \frac{db}{dz} = 2z \\
f'(z) &= \frac{4(z^2+1)^3(2z)(z^2) - (z^2+1)^4(2z)}{z^4} \\
&= \frac{(2z)(z^2+1)^3[4(z^2) - (z^2+1)]}{z^4} \\
&= \frac{2(z^2+1)^3[3z^2-1]}{z^3}
\end{aligned}
$$

(e)

$$
\begin{aligned}
f(z) &= ze^{z^2+3} \\
f'(z) &= (1)e^{z^2+3} + (z)e^{z^2+3}(2z) = e^{z^2+3}(2z^2+1)
\end{aligned}
$$

Problem 2

Show that $f'(z_0)$ does not exist at any point z_0 in two ways: using the limit definition and using the Cauchy-Riemann equations. Here, $z = x + iy$ and $x, y \in \mathbb{R}$.

(a) $f(z) = 2x + ixy^2$

(b) $f(z) = e^x e^{-iy}$

Problem 3

Using the exponential function e^z, we can now define the complex cosine and sine function for any $z \in \mathbb{C}$ as follows:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Using these formulas,

(a) express $\cos(z)$ and $\sin(z)$ in rectangular coordinates $u(x, y) + iv(x, y)$ where $z = x + iy$.

(b) show that the complex cosine and sine functions are analytic over $\mathbb{C}$ and calculate their derivatives.
