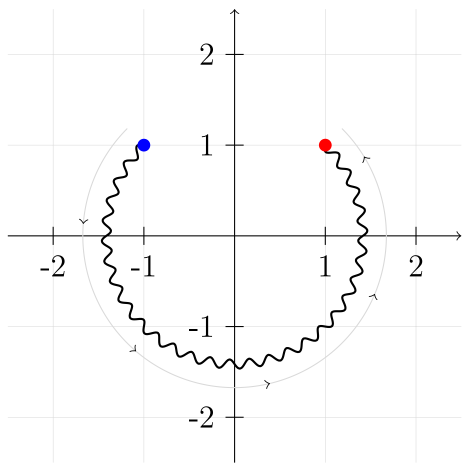
Worksheet # 9

MATH 3160 – Complex Variables
Miguel Gomez

Completed: October 25, 2025

Problem 1

let C be the contour shown below, traversed counter-clockwise from the blue point to the red. I have reconstructed this image from the worksheet and am confident this is similar to how it was done, but I must admit I am making an assumption about the structure.



find $\int_C \frac{1}{z} dz$ (Hint: Consider a new branch of the logarithm function by $\log\left(re^{i\theta}\right) = \ln(r) + i\theta$, where $-3\pi/2 < \theta \leq \pi/2$, and check that this is an anti-derivative of $1/z$.)

I get the hint, but I saw this from the start and wanted to work it out. In checking this path, we know that the result should have a value because it is not a closed path. Parametrizing this path works as follows given the diagram. placing a circle of radius $\sqrt{2}$ cuts through the sinusoid and it oscillates around it.

$$\begin{aligned}
z &= (\sqrt{2} + A\sin(\omega\theta))e^{i\theta} \\
z_0 &= \sqrt{2}e^{-i\frac{5\pi}{4}} \quad A\sin\left(-\omega\frac{5\pi}{4}\right) = 0 \\
z_f &= \sqrt{2}e^{i\frac{\pi}{4}} \quad A\sin\left(\omega\frac{\pi}{4}\right) = 0 \\
r(t) &= \sqrt{2} + A\sin(\omega\theta(t))
\end{aligned}$$

The path has a sinusoidal signal in superposition such that there is a change $A\sin(\omega\theta)$ in the magnitude of z. Instead of writing out so much, we can continue by treating this more generally:

$$\begin{aligned}
z(t) &= r(t)e^{i\theta(t)} \\
z'(t) &= r'(t)e^{i\theta(t)} + r(t)(i\theta'(t))e^{i\theta(t)} \\
\int_\gamma f(z)dz &= \int_{t_0}^{t_1} f(z(t))z'(t)dt \\
\int_\gamma f(z)dz &= \int_{t_0}^{t_1} \frac{1}{r(t)e^{i\theta(t)}}[r'(t)e^{i\theta(t)} + r(t)(i\theta'(t))e^{i\theta(t)}]dt \\
&= \int_{t_0}^{t_1} \frac{e^{i\theta(t)}}{r(t)e^{i\theta(t)}}[r'(t) + r(t)(i\theta'(t))]dt \\
&= \int_{t_0}^{t_1} \left[\frac{r'(t)}{r(t)} + i\theta'(t)\right] dt \\
&= \int_{t_0}^{t_1} \frac{r'(t)}{r(t)}dt + i\int_{t_0}^{t_1} \theta'(t)dt \\
&= \ln(r(t))|_{t_0}^{t_1} + i\theta(t)|_{t_0}^{t_1} \\
&= (\ln(r(t_1)) - \ln(r(t_0))) + i(\theta(t_1) - \theta(t_0))
\end{aligned}$$

Now, we can see that no matter the function $r(t)$, we get the final expressions by recognizing that the sinusoidal signal for the magnitude has the same value at t_0 and t_1, then $r(t_1) = r(t_0)$, and therefore $\ln(r(t_1)) = \ln(r(t_0))$. This then leaves us with the final expression:

$$\int_\gamma f(z)dz = i(\theta(t_1) - \theta(t_0))$$

This shows that the integral value only depends on the angle difference and would not change for any radius used. Therefore, since the angular difference is 3/4 of the unit circle, and we know the integral of $1/z$ is $2\pi i$, this integral is therefore $\frac{3\pi}{2}i$. Which is nice because it confirms the given hint and why it works as an antiderivative.

Problem 2

Show that $\int_C f(z)dz = 0$ for C the unit circle and :

(i) $f(z) = \frac{z^2}{z+3}$

(ii) $f(z) = \frac{1}{z^2+2z+2}$

(i)

Since we are evaluating with C within the unit circle, any point which lies outside of the unit circle does not matter for our evaluation as we only need the curve and its interior to be a simply connected domain D. in the denominator, we see that we have $z + 3$, meaning that it only becomes 0 if z is -3. So the point $z = -3$ in the complex plane will give a divide by zero issue. Since $|3| = 3 > 1$, the unit circle only contains points where the magnitude of z is less than or equal to 1, meaning it is analytic inside and on the unit circle.

$\therefore$ the C-G theorem holds and we have a path with a simply connected interior region with the same starting and ending point whose integral evaluates to 0.

(ii)

In this problem, we have a similar result as the denominator is 0 only where $z = -1 \pm i$ given the factoring of the denominator. Notice that the magnitude of z for these points will be $\sqrt{2}$, and therefore the points are also outside of the unit circle. With $\sqrt{2} > 1$ and the unit circle only contains points where $|z| \leq 1$, then we again have an integral that evaluates to 0.

Problem 3

Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, and $y = \pm i$, and let C_2 denote the positively oriented circle $|z| = 4$. Explain why:

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

when

(a) $f(z) = \frac{1}{2x^2+1}$

(b) $f(z) = \frac{z+2}{sin(z/2)}$

(c) $f(z) = \frac{z}{1-e^z}$