# Homework # 6

MATH 3160 – Complex Variables
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## Problem 1

Find parameterized representations $z(t)$ of the following contours in the plane including $t$-ranges.

1. A straight line from point $(1 + 2i)$ to point $(i + 2)$
2. A line from $(0, 0)$ to point $(1 + \sqrt{3}i)$
3. A half-ellipse from point 2 to $-2$ passing through $i$ centered at the origin. Recall that such an ellipse is defined by an equation of the form $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ in the $xy$-plane (for some real constants $a, b > 0$). *Hint: First find the suitable values of $a$ and $b$ defining the said ellipse. Then try parametrizing it similar to how $(\cos(t), \sin(t))$ parametrizes the unit cirle.*

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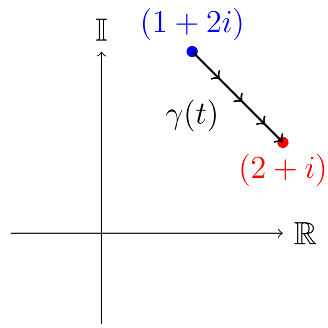
### (a)

A straight line from point $(1 + 2i)$ to point $(i + 2)$

This one will require the expression for a line between points: $P + t(Q - P)$ where $P$ and $Q$ are the points and $t$ runs from $0 \le t \le 1$.

$$P = (1 + 2i)$$
$$Q = (i + 2)$$
$$Q - P = (i + 2) - (1 + 2i) = 1 - i$$
$$\gamma(t) = 1 + 2i + t(1 - i)$$



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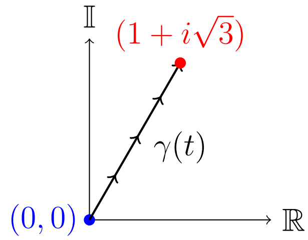
### (b)

A line from $(0,0)$ to point $(1+\sqrt{3}i)$.

This one is quite simple as we only have to multiply the point by $t$ as the first point $P$ is the origin and that handles moving from the origin to the point $(1+\sqrt{3}i)$ as it moves from $0 \le t \le 1$.

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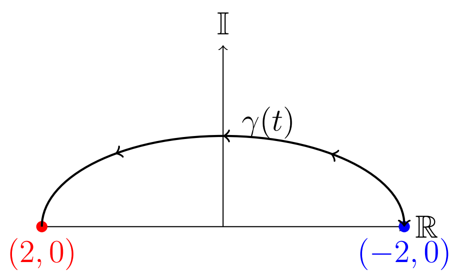
### (c)

A half-ellipse from point 2 to $-2$ passing through $i$ centered at the origin.

$$x = a\cos(t) = 2\cos(t)$$
$$y = b\sin(t) = \sin(t)$$
$$z(t) = x + iy = 2\cos(t) + \sin(t)$$



# Problem 2

Evaluate the following integrals:

1. $\int_1^2 (\frac{1}{t} - i)^2 dt$
2. $\int_0^{\pi/6} e^{i2t} dt$
3. $\int_0^{\infty} e^{izt} dt$ where $Im(z) > 0$

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## (a)

$$\begin{aligned}
\int_1^2 (\frac{1}{t} - i)^2 dt &= \int_1^2 (\frac{1}{t^2} - 2i\frac{1}{t} - i^2) dt \\
&= \int_1^2 \frac{1}{t^2} dt - 2i \int_1^2 \frac{1}{t} dt + \int_1^2 dt \\
&= -\frac{1}{3} \int_1^2 -3t^{-2} dt - 2i \ln(t)|_1^2 + t|_1^2 \\
&= -\frac{1}{3} t^{-3}|_1^2 + -2i \ln(t)|_1^2 + t|_1^2 \\
&= \left(\frac{1}{3} - \frac{1}{3 * 2^3}\right) - 2i(\ln(2) - \ln(1)) + 1 \\
&= \left(\frac{1}{3} - \frac{1}{3 * 2^3} + 1\right) - 2i(\ln(2) - \ln(1))
\end{aligned}$$

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## (b)

$$\begin{aligned}
\int_0^{\pi/6} e^{i2t} dt &= \frac{1}{2i} \int_0^{\pi/6} 2ie^{i2t} dt \\
&= \frac{1}{2i} e^{i2t}|_0^{\pi/6} = -\frac{1}{2} i (e^{i\pi/3} - 1)
\end{aligned}$$

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## (c)

## Problem 3

Sketch the oriented curve defined by the following four contours and compute $\int_C f(z)dz$ where $f(z) = z - 1$:

1. $C_1$: A semicircle $z = 2e^{i\theta}$ for $\theta \in [\pi, 2\pi]$.
2. $C_2$: A full circle $z = 2e^{i\theta}$ for $\theta \in [0, 2\pi]$.
3. $C_3$: A line on the real axis from 2 to $-2$.
4. $C_4 = C_1 + C_3$ where $+$ denotes concatenation.

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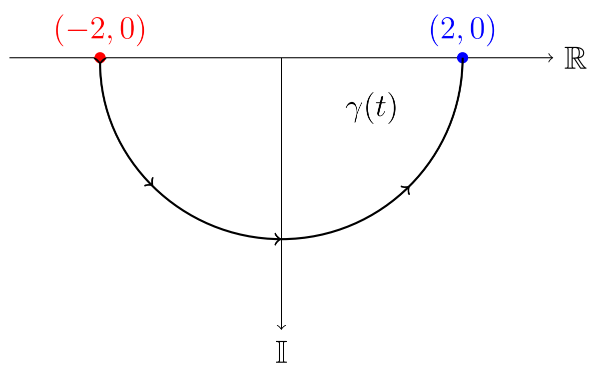
**(1)**

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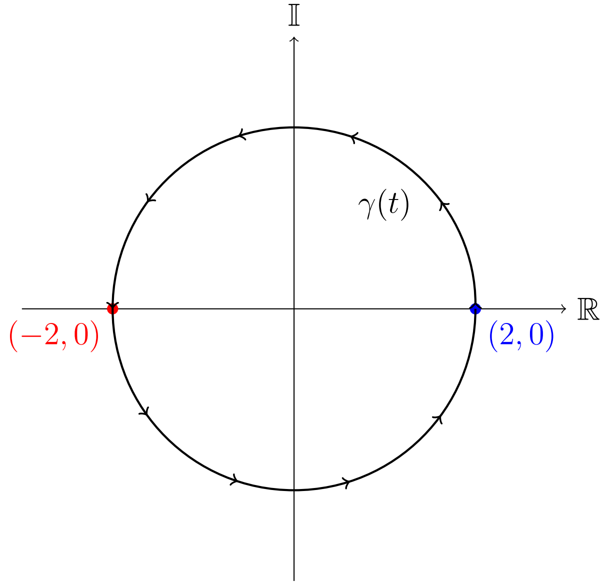
**(2)**

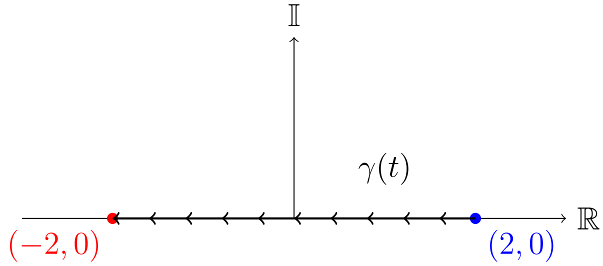
The oriented curve is defined as follows:



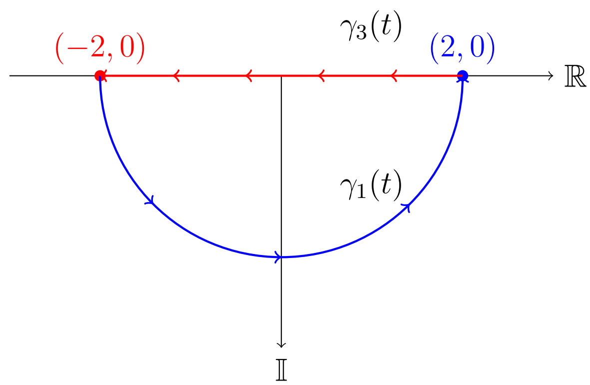

Since we know that the curve here starts and ends at the same point, we know that the overall expression here should evaluate to 0 by the Fundamental Theorem of Calculus. the path $\gamma$ here has the same starting and ending point, and the function $f(z)$ is continuous everywhere with no discontinuities or issues with branch cuts. The result is then:

$$\begin{aligned}
f(z) &= z - 1 \\
\gamma(t) &= 2e^{i\pi t} \\
\int_0^2 f(\gamma(t))\gamma(t)'dt &= \int_0^2 (\gamma(t) - 1)\gamma(t)'dt \\
&= \int_0^2 (2e^{i\pi t} - 1)(2i\pi e^{i\pi t})dt \\
&= \int_0^2 (2e^{i\pi t}e^{i\pi t} - e^{i\pi t})(2i\pi)dt \\
&= (2i\pi)\int_0^2 (2e^{i2\pi t} - e^{i\pi t})dt \\
&= (2i\pi)\left(\frac{2}{i2\pi}e^{i2\pi t} - \frac{1}{i\pi}e^{i\pi t}\right)\Big|_0^2 \\
&= \left(2e^{i2\pi t} - 2e^{i\pi t}\right)\Big|_0^2 \\
&= \left(2e^{i4\pi} - 2e^{i2\pi}\right) - \left(2e^0 - 2e^0\right) \\
&= (2\cdot 1 - 2\cdot 1) - (2 - 2) \\
&= 0
\end{aligned}$$

**(3)**


$\mathbb{I}$
$\gamma(t)$
$\mathbb{R}$
$(-2, 0)$
$(2, 0)$

**(4)**


$\gamma_3(t)$
$(-2, 0)$
$(2, 0)$
$\mathbb{R}$
$\gamma_1(t)$
$\mathbb{I}$