

Worksheet # 4

MATH 3160 – Complex Variables
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Problem 1:

Write the following functions $f(z)$ in the form $f(z) = u(x, y) + iv(x, y)$

(a) $f(z) = z^3 + z + 1$

writing out each z as $x + iy$

$$\begin{aligned} f(z) &= z^3 + z + 1 = (x + iy)^3 + (x + iy) + 1 \\ (x + iy)^3 &= (x + iy)(x + iy)(x + iy) = (x^2 + 2ixy + i^2y^2)(x + iy) = \\ (x^2 + 2ixy - y^2)(x + iy) &= x^3 + ix^2y + 2ix^2y + 2i^2xy^2 - y^2x - iy^3 = \\ &= x^3 - 3xy^2 + i(3x^2y - y^3) \\ f(z) &= x^3 - 3xy^2 + i(3x^2y - y^3) + (x + iy) + 1 = \\ &= \boxed{(x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)} \end{aligned}$$

(b) $f(z) = \frac{\bar{z}^2}{z}$ for $z \neq 0$

$$\begin{aligned} f(z) &= \frac{\bar{z}^2}{z} = \frac{(\overline{x + iy})^2}{(x + iy)} = \frac{(x - iy)^2}{(x + iy)} = \\ &= \frac{(x^2 - 2ixy + i^2y^2)}{(x + iy)} = \frac{(x^2 - 2ixy - y^2)(x - iy)}{(x + iy)(x - iy)} = \\ &= \frac{(x^3 - x^2iy - 2ix^3y + 2i^2xy^2 - y^2x + iy^3)}{(x^2 + iyx - iyx - i^2y^2)} = \\ &= \frac{(x^3 - x^2iy - 2ix^3y - 2xy^2 - y^2x + iy^3)}{(x^2 + y^2)} = \\ &= \boxed{\frac{x^3 - 3xy^2}{x^2 + y^2} + i\frac{-3x^2y + y^3}{x^2 + y^2}} \end{aligned}$$

Problem 2:

Consider the mapping $z \rightarrow z^2$.

- (a) What is the image of the line $z = x + i$?

$$\begin{aligned} z \rightarrow z^2 &= (x + iy) \rightarrow (x + iy)^2 \\ (x^2 + 2ixy - y^2) &= (x^2 + 2ix(1) - (1)^2) = x^2 + 2xi - 1 \\ &= (x^2 - 1) + i(2x) \end{aligned}$$

\therefore The image of the line $x + i$ turns out to be a parabola opening to the right in \mathbb{R} . Since we have x^2 term for $u(x, y)$ and is centered at -1 . The parabola grows into complex plane.

- (b) What is the image of the square bounded by the four lines $z = \pm 1 + iy$ and $z = x \pm i$?

case z has constant real components $z = \pm 1 + iy$:

$$\begin{aligned} z \rightarrow z^2 &= (\pm 1 + iy) \rightarrow (\pm 1 + iy)^2 \\ ((\pm 1)^2 + 2i(\pm 1)y - y^2) &= ((\pm 1)^2 + 2i(\pm 1)y - y^2) = \\ &\text{positive branch} \\ ((1)^2 + 2i(1)y - y^2) &= ((1)^2 + 2i(1)y - y^2) = (-y^2 + 1) + i(2y) \\ &\text{negative branch} \\ ((-1)^2 + 2i(-1)y - y^2) &= (1 - 2iy - y^2) = (-y^2 + 1) - i(2y) \end{aligned}$$

These two lines appear to be the same parabola that opens to the left toward $-\mathbb{R}$. if y is negative, we get the same thing for $u(x, y)$ and the sign flips on $v(x, y)$. Same situation in the case y is positive. for the case of $x - i$:

$$\begin{aligned} z \rightarrow z^2 &= (x - i) \rightarrow (x - i)^2 \\ (x^2 - 2ix - 1) &= (x^2 - 1) - i(2x) \end{aligned}$$

Here, the parabola still opens to the right, and is the same parabola as we expected. with the imaginary components flipped.

\therefore The image of the square region is the region between the two parabolas.

Problem 3:

Compute the following limits (or state that they do not exist)

(a) $\lim_{z \rightarrow i} \frac{iz^3 - 1}{z + i}$

$$\begin{aligned} \lim_{z \rightarrow i} \frac{iz^3 - 1}{z + i} &= \frac{i(i)^3 - 1}{i + i} = \\ &= \frac{i^4 - 1}{2i} = \frac{1 - 1}{2i} = 0 \end{aligned}$$

$$\boxed{\therefore \lim_{z \rightarrow i} \frac{iz^3 - 1}{z + i} = 0}$$

(b) $\lim_{z \rightarrow i} \left(z + \frac{1}{z} \right)$

$$\begin{aligned} \lim_{z \rightarrow i} \left(z + \frac{1}{z} \right) &= \left(\lim_{z \rightarrow i} z + \lim_{z \rightarrow i} \frac{1}{z} \right) \\ &= \lim_{z \rightarrow i} z = i \\ &= \lim_{z \rightarrow i} \frac{1}{z} = \frac{1}{i} = -i \end{aligned}$$

$$\boxed{\therefore \lim_{z \rightarrow i} \left(z + \frac{1}{z} \right) = i - i = 0}$$

(c) $\lim_{z \rightarrow 0} \frac{1}{z^2}$

We can evaluate this by replacing z with $re^{i\theta}$ and then evaluating the limits in r and θ .

$$\lim_{z \rightarrow 0} \frac{1}{z^2} = \lim_{r \rightarrow 0} \frac{1}{(re^{i\theta})^2} = \lim_{r \rightarrow 0} \frac{1}{(r)^2} e^{-i2\theta}$$

From any direction, we will end up with a div by zero issue. meaning the limit is ∞ . We could get ∞ if approaching from $\theta = 0$ or we could get $-\infty$ if we approach from $\theta = \frac{\pi}{2}$

\therefore no, the limit DNE because different paths give different result.

Problem 4:

Does the following limit exist?

(a) $\lim_{z \rightarrow 0} \left(\frac{\bar{z}}{z} \right)^2$

$$\frac{\bar{z}}{z} = \frac{\overline{re^{i\theta}}}{re^{i\theta}} = re^{-i\theta} \frac{1}{r} e^{-i\theta} = e^{-i2\theta}$$

squaring this then doubles the angle theta and we see there is no more dependence on r

$$= \lim_{z \rightarrow 0} e^{-i4\theta}$$

Approaching from $\theta = 0$ we get 1, but approaching from $\theta = \frac{\pi}{4}$ we get -1 .

\therefore no, the limit DNE because different paths give different result.