Evaluating a Complex Limit with Geometric Algebra: A Comparative Approach

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Abstract

This document explores the evaluation of a fundamental limit in complex analysis, $\lim_{z\to i} z$, from two perspectives. First, we use the standard method of complex variables. Second, we employ the framework of Geometric Algebra (GA) to provide a more geometrically intuitive and generalizable solution. The goal is to demonstrate the direct translation between the two notations and highlight the conceptual advantages of the GA approach.

1 The Problem

We wish to evaluate the limit of the identity function $f(z) = z$ as z approaches the imaginary unit i. While trivial in standard analysis, this problem serves as an excellent illustration of foundational concepts.

- **In Complex Notation:** We evaluate $\lim_{z\to i} z$.
- **In Geometric Algebra Notation:** We evaluate $\lim_{Z\to I} Z$.

2 The Geometric Algebra Framework

In the geometric algebra of a 2D plane (G_2), a complex number $z = x + iy$ is represented as a **multivector** $Z = x + yI$.

- x is a scalar (grade-0 element).
- I is the unit **pseudoscalar** of the plane, representing a directed unit area. It is the geometric product of the two orthonormal basis vectors ($I = \mathbf{e}_1\mathbf{e}_2$).
- A key property, derived from the geometric product, is that $I^2 = -1$. Therefore, I is the direct geometric equivalent of the imaginary unit i.

3 The "All Paths at Once" Approach

To prove a limit exists in the complex plane, one must show that the same value is approached regardless of the path. We can parameterize all straight-line paths simultaneously using a polar displacement from the limit point.

3.1 Standard Complex Analysis

We represent a point z approaching i as:

$$z = i + re^{i\theta}$$

Here, $r \in \mathbb{R}^+$ is the radial distance from i, and θ is the angle of approach. The limit is found by letting $r \to 0$.

3.2 Geometric Algebra

Similarly, we represent a point Z approaching I as:

$$Z = I + \epsilon e^{I\theta}$$

Here, $\epsilon \in \mathbb{R}^+$ is the magnitude of the displacement, and $u = e^{I\theta} = \cos\theta + I\sin\theta$ is a unit multivector that defines the direction. In GA, $e^{I\theta}$ is known as a **rotor**, as it generates rotations. The limit is found by letting $\epsilon \to 0$.

4 Side-by-Side Evaluation

The following table provides a direct translation of the evaluation process.

Complex Notation	Geometric Algebra Notation
1. State the Problem	
$\lim_{z\to i} z$	$\lim_{Z\to I} Z$
2. Define a General Path of Approach	
Let $z = i + re^{i\theta}$. The limit is taken as the distance $r \to 0$. The term $e^{i\theta}$ represents the direction.	Let $Z = I + \epsilon e^{I\theta}$. The limit is taken as the magnitude $\epsilon \to 0$. The rotor $e^{I\theta}$ represents the direction.
3. Apply the Function	
The function is $f(z) = z$. Substituting gives: $f(i + re^{i\theta}) = i + re^{i\theta}$.	The function is $f(Z) = Z$. Substituting gives: $f(I + \epsilon e^{I\theta}) = I + \epsilon e^{I\theta}$.
4. Evaluate the Limit	
$\lim_{r\to 0}(i + re^{i\theta})$ The magnitude of the displacement term is $\lvert re^{i\theta}\rvert = r$, which vanishes as $r \to 0$. $\lim_{r\to 0}(i + re^{i\theta}) = i + 0 = i$.	$\lim_{\epsilon\to 0}(I + \epsilon e^{I\theta})$ The magnitude of the displacement term is $\lvert\epsilon e^{I\theta}\rvert = \epsilon$, which vanishes as $\epsilon \to 0$. $\lim_{\epsilon\to 0}(I + \epsilon e^{I\theta}) = I + 0 = I$.
5. Conclusion	
The result is **i**. It is independent of the angle of approach θ, confirming the limit.	The result is **I**. It is independent of the directional rotor $e^{I\theta}$, confirming the limit.

5 Conceptual Discussion: Why GA Matters

For this 2D problem, the notations are functionally identical. The expression $re^{i\theta}$ is a direct analog of $\epsilon e^{I\theta}$. However, the GA framework provides a deeper geometric insight and is far more general.

- **Geometric Foundation:** In complex analysis, i is defined axiomatically by $i^2 = -1$. In geometric algebra, the equivalent property $I^2 = -1$ is not an axiom but a *consequence* of the geometric definition of the algebra. I is a concrete geometric entity (an oriented plane segment), not an abstract "imaginary" number.

- **Generalizability:** The true power of GA becomes apparent in higher dimensions. Complex numbers are limited to 2D transformations. In 3D, a rotation occurs in a specific *plane*, which is represented by a bivector B. A rotation in any arbitrary 3D plane can be described by a rotor $R = e^{B\theta}$. This unified and powerful concept of rotors for describing rotations in any dimension has no simple parallel in standard complex number theory.

6 Conclusion

Evaluating $\lim_{z \to i} z$ using both methods yields the same result, i (or its GA equivalent, I). The parallel structure of the derivation shows how complex analysis can be viewed as a specialized subset of the more general geometric algebra of the plane. While complex analysis is a perfectly tailored tool for 2D problems, geometric algebra provides a more fundamental and scalable framework that keeps the underlying geometry at the forefront.