Homework # 4

MATH 3160 – Complex Variables
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Problem 1

Find $f'(z)$ using differentiation rules.

(a) $f(z) = 3z^2 - 2z + 4$

(b) $f(z) = (1-4z)^3$

(c) $f(z) = \frac{z-1}{2z+1}$, assume $z \neq -1/2$

(d) $f(z) = \frac{(z^2+1)^4}{z^2}$, assume $z \neq 0$

(e) $f(z) = z\,e^{z^2+3}$.

(a)

$$\begin{aligned} f(z) &= 3z^2 - 2z + 4 \\ f'(z) &= 6z - 2 \end{aligned}$$

(b)

$$\begin{aligned} f(z) = a(z) &= (1-4z)^3 = b^3 \mid b = 1-4z \\ f'(z) &= \frac{da}{db}\frac{db}{dz} = 3(1-4z)^2(-4) = -12(1-4z)^2 \end{aligned}$$

(c)

Assuming $z \neq -1/2$

$$\begin{aligned} f(z) &= \frac{z-1}{2z+1} = \frac{f}{g} \\ \frac{df}{dz} &= 1 \qquad \frac{dg}{dz} = 2 \\ f'(z) = \frac{f'g - fg'}{g^2} &= \frac{1(2z+1) - (z-1)2}{(2z+1)^2} \\ &= \frac{(2z-2z)+(1+2)}{(2z+1)^2} = \frac{3}{(2z+1)^2} \end{aligned}$$

(d)

Assuming $z \neq 0$

$$
\begin{aligned}
f(z) &= \frac{(z^2+1)^4}{z^2} = \frac{a}{b} \\
\frac{da}{dz} &= 4(z^2+1)^3(2z) \qquad \frac{db}{dz} = 2z \\
f'(z) &= \frac{4(z^2+1)^3(2z)(z^2) - (z^2+1)^4(2z)}{z^4} \\
&= \frac{(2z)(z^2+1)^3[4(z^2) - (z^2+1)]}{z^4} \\
&= \frac{2(z^2+1)^3[3z^2-1]}{z^3}
\end{aligned}
$$

(e)

$$
\begin{aligned}
f(z) &= ze^{z^2+3} \\
f'(z) &= (1)e^{z^2+3} + (z)e^{z^2+3}(2z) = e^{z^2+3}(2z^2+1)
\end{aligned}
$$

Problem 2

Show that $f'(z_0)$ does not exist at any point z_0 in two ways: using the limit definition and using the Cauchy-Riemann equations. Here, $z = x + iy$ and $x, y \in \mathbb{R}$.

(a) $f(z) = 2x + ixy^2$

(b) $f(z) = e^x e^{-iy}$

Via Cauchy-Riemann equations: The Cauchy-Riemann equations are the following:

$$\frac{du}{dx} = \frac{dv}{dy}$$
$$\frac{du}{dy} = -\frac{dv}{dx}$$

(a)

$$f(z) = 2x + ixy^2 \quad u(x,y) = 2x \quad v(x,y) = xy^2$$
$$\frac{du}{dx} = 2 \quad \frac{du}{dy} = 0 \quad \frac{dv}{dx} = y^2 \quad \frac{dv}{dy} = 2xy$$
$$\frac{du}{dx} \neq \frac{dv}{dy}$$
$$\therefore \text{not differentiable}$$

(b)

$$f(z) = e^x e^{-iy} = e^x(\cos(-y) + i\sin(-y))$$
$$u(x,y) = e^x \cos(-y) \quad v(x,y) = e^x \sin(-y)$$
$$\frac{du}{dx} = e^x \cos(y) \quad \frac{du}{dy} = -e^x \sin(y) \quad \frac{dv}{dx} = -e^x \sin(y) \quad \frac{dv}{dy} = -e^x \cos(y)$$
$$e^x \cos(y) = -e^x \cos(y) \to 2e^x \cos(y) = 0$$

Only possible if $\cos(y) = 0$

$$-e^x \sin(y) = e^x \sin(y) \to 2e^x \sin(y) = 0$$

only possible if $\sin(y)$ is 0

Both sin and cos cannot be 0 simultaneously

$\therefore$ not differentiable

Now by using the limit definition:

Limit definition:

$$\begin{aligned}
&\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_o}\\
&\quad=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}\\
&\Delta w = f(z_0+\Delta z)-f(z_0)\\
&\Delta z = z-z_0
\end{aligned}$$

(a)

$$\begin{aligned}
\Delta w &= 2(x_0+\Delta x)+i(x_0+\Delta x)(y_0+\Delta y)^2-(2x_0+ix_0y_0^2)\\
&= 2x_0+2\Delta x-2x_0+i(x_0+\Delta x)(y_0+\Delta y)^2-ix_0y_0^2\\
&= 2\Delta x+i(x_0+\Delta x)(y_0^2+2y_0\Delta y+\Delta y^2)-ix_0y_0^2\\
&= 2\Delta x+i((x_0y_0^2+2x_0y_0\Delta y+x_0\Delta y^2)+(\Delta x y_0^2+2\Delta x y_0\Delta y+\Delta x\Delta y^2))-ix_0y_0^2\\
&= 2\Delta x+i(((\cancelto{0}{x_0y_0^2-x_0y_0^2})+2x_0y_0\Delta y+x_0\Delta y^2)+(\Delta x y_0^2+2\Delta x y_0\Delta y+\Delta x\Delta y^2))\\
&= 2\Delta x+i((2x_0y_0\Delta y+x_0\cancelto{0}{\Delta y^2})+(\Delta x y_0^2+\cancelto{0}{2\Delta x y_0\Delta y}+\Delta x\cancelto{0}{\Delta y^2}))\\
&= 2\Delta x+i(2x_0y_0\Delta y+\Delta x y_0^2)\\
\frac{\Delta w}{\Delta z} &= \frac{2\Delta x+i(2x_0y_0\Delta y+\Delta x y_0^2)}{\Delta x+i\Delta y}
\end{aligned}$$

Approach with $\Delta y = 0$:

$$\begin{aligned}
\frac{\Delta w}{\Delta z} &= \frac{2\Delta x+i(2x_0y_0(0)+\Delta x y_0^2)}{\Delta x+i(0)}\\
&= \frac{2\Delta x+i(\Delta x y_0^2)}{\Delta x} = 2+iy_0^2
\end{aligned}$$

Approach with $\Delta x = 0$:

$$\begin{aligned}
\frac{\Delta w}{\Delta z} &= \frac{2(0)+i(2x_0y_0\Delta y+(0)y_0^2)}{(0)+i\Delta y}\\
&= \frac{i2x_0y_0\Delta y}{i\Delta y} = 2x_0y_0
\end{aligned}$$

Different paths give different results. only true if $2x_0y_0 = 2+iy_0$ and these cannot be true since x_0 and $y_0 \in \mathbb{R}$.

(b)

$$
\begin{aligned}
f(z) &= e^x e^{-iy} \\
\Delta w &= f(z_0 + \Delta z) - f(z_0) \\
\Delta z &= z - z_0 \\
\Delta w &= e^{(x_0+\Delta x)} e^{-i(y_0+\Delta y)} - e^{x_0} e^{-iy_0} \\
&= e^{x_0} e^{\Delta x} e^{-iy_0} e^{-i\Delta y} - e^{x_0} e^{-iy_0} \\
&= e^{x_0} e^{-iy_0} e^{\Delta x} e^{-i\Delta y} - e^{x_0} e^{-iy_0} \\
&= e^{x_0} e^{-iy_0} (e^{\Delta x} e^{-i\Delta y} - 1) \\
&= e^{x_0} e^{-iy_0} (e^{\Delta x - i\Delta y} - 1) \\
\frac{\Delta w}{\Delta z} &= \frac{e^{x_0} e^{-iy_0} (e^{\Delta x - i\Delta y} - 1)}{\Delta x + i\Delta y}
\end{aligned}
$$

Approach with $\Delta y = 0$:

$$
\begin{aligned}
\frac{\Delta w}{\Delta z} &= \frac{e^{x_0} e^{-iy_0} (e^{\Delta x - i(0)} - 1)}{\Delta x + i(0)} \\
&= \frac{e^{x_0} e^{-iy_0} (e^{\Delta x} - 1)}{\Delta x} \\
&= e^{x_0} e^{-iy_0} \lim_{\Delta x \to 0} \frac{(e^{\Delta x} - 1)}{\Delta x} \\
&\text{limit definition of exponential resolves to 1} \\
&== e^{x_0} e^{-iy_0}
\end{aligned}
$$

Approach with $\Delta x = 0$:

$$
\begin{aligned}
\frac{\Delta w}{\Delta z} &= \frac{e^{x_0} e^{-iy_0} (e^{(0) - i\Delta y} - 1)}{(0) + i\Delta y} \\
&= \frac{e^{x_0} e^{-iy_0} (e^{-i\Delta y} - 1)}{i\Delta y} \\
&\text{limit definition of exponential resolves to -1} \\
&== -e^{x_0} e^{-iy_0}
\end{aligned}
$$

Different paths give different results:

$$\therefore \text{Limit DNE}$$

Problem 3

Using the exponential function e^z, we can now define the complex cosine and sine function for any $z \in \mathbb{C}$ as follows:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Using these formulas,

(a) express $\cos(z)$ and $\sin(z)$ in rectangular coordinates $u(x, y) + iv(x, y)$ where $z = x + iy$.

(b) show that the complex cosine and sine functions are analytic over $\mathbb{C}$ and calculate their derivatives.

(a)

$$\begin{aligned}\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\ &= \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2}\end{aligned}$$