Homework #5

MATH 3160 – Complex Variables
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Problem 1

Consider the analytic function $f(z) = ze^{z^2}$.

(a) Show that the function $u(x, y) = x\ e^{(x^2-y^2)} \cos(2xy) - y\ e^{(x^2-y^2)} \sin(2xy)$ is the real component of $f(z)$.

(b) What is a harmonic conjugate for $u(x, y)$?

(c) Without computing the second partial derivatives of $u(x, y)$, explain why you know that $u(x, y)$ is harmonic.

Expanding $f(z) = ze^{z^2}$ to see what the u and v turn out to be.

(a)

$$\begin{aligned} f(z) &= ze^{z^2} = (x+iy)e^{(x+iy)^2} = (x+iy)e^{x^2-y^2+2ixy} = (x+iy)e^{x^2-y^2}e^{2ixy} \\ &= e^{x^2-y^2}(x+iy)(\cos(2xy) + i\sin(2xy)) \\ &= e^{x^2-y^2}(x\cos(2xy) + ix\sin(2xy) + iy\cos(2xy) + i^2y\sin(2xy)) \\ &= e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy) + i(x\sin(2xy) + y\cos(2xy))) \\ &\therefore u(x,y) = e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy)) \end{aligned}$$

0.1 (b)

Utilizing the imaginary part of f, v can serve as a conjugate up to an arbitrary constant, which we could set to 0 to recover f.

$$v(x, y) = e^{x^2-y^2}(x\sin(2xy) + y\cos(2xy)) + C$$

0.2 (c)

Since we know that f is analytic as stated, and u is the real part of f, an analytic function, then u must be harmonic. This is due to the fact that the real and imaginary parts of any analytic function are harmonic functions.

Problem 2

Consider the function $u(x, y) = x^3 - 3xy^2 - 3x^2y + y^3$.

(a) Show that $u(x, y)$ is harmonic.

(b) Find a harmonic conjugate for $u(x, y)$.

(a)

For this, we can start by showing that the expression for u satisfies the Laplacian:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
\frac{\partial u}{\partial x} &= 3x^2 - 3y^2 - 6xy \\
\frac{\partial^2 u}{\partial x^2} &= 6x - 6y \\
\frac{\partial u}{\partial y} &= -6xy - 3x^2 + 3y^2 \\
\frac{\partial^2 u}{\partial y^2} &= -6x + 6y \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 6x - 6y + (-6x + 6y) = 0 \\
&\therefore u(x, y) \text{ is harmonic.}
\end{aligned}$$

(b)

For a conjugate, we can back solve to get the v expression that works.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$v(x,y) = \int 3x^2 - 3y^2 - 6xydy = 3x^2y - y^3 - 3xy^2 + G(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$v(x,y) = \int 6xy + 3x^2 - 3y^2dx = 3x^2y + x^3 - 3x^2y + G(y)$$

$$\therefore v(x,y)_1 = v(x,y)_2$$

$$G(x) = x^3 + C$$

$$G(y) = -y^3 + C$$

$$\therefore v(x,y) = \boxed{3x^2y + x^3 - 3x^2y + -y^3 + C}$$

Problem 3

Recall we learned of the following fact in class:

Let $u(x,y)$ be a harmonic function defined on a simply connected domain D.
Then $u(x,y)$ has a harmonic conjugate on D.

(a) Show that $u(x,y) = \ln(\sqrt{x^2+y^2})$ is a harmonic function.

(b) What is the domain of definition of $u(x,y)$?

(c) An aside: show that if $f(z)$ and $g(z)$ are two analytic functions on the same domain D, and we have $\mathrm{Re}(f(z)) = \mathrm{Re}(g(z))$ for all $z \in D$, then $f(z) = g(z) + c$ for some constant $c \in \mathbb{C}$.

[*Hint: show that the function $h(z) = f(z) - g(z)$ has Re$(h(z)) = 0$, and then use a result from class to conclude $h(z)$ is a constant.*]

(d) Explain why $u(x,y)$ does *not* have a harmonic conjugate on its domain.

[*Hint: if such a conjugate existed, then $u(x,y)$ would be the real component of some analytic function $f(z)$, but $u(x,y)$ is already the real component of a familiar analytic function, which is discontinuous at its branch cut*]

1. Why does this not contradict the fact from class?

(a)

$$u(T) = \ln(T), \quad T(S) = \sqrt{S}, \quad S = x^2 + y^2$$
$$\frac{du}{dT} = \frac{1}{T} \quad \frac{dT}{dS} = \frac{1}{2\sqrt{S}} \quad \frac{\partial S}{\partial x|y} = 2x|2y$$
$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\ln((x^2+y^2)^{\frac{1}{2}})) = ((x^2+y^2)^{-\frac{1}{2}})\left(\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\right)(2x)$$
$$= \left(\frac{2x}{2(x^2+y^2)}\right)$$
$$\frac{\partial u}{\partial x} = \left(\frac{x}{x^2+y^2}\right)$$
$$\frac{\partial u}{\partial y} = \left(\frac{y}{x^2+y^2}\right)$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} = \frac{f'g - fg'}{g^2} &= \left(\frac{(x)'(x^2+y^2) - (x)(x^2+y^2)'}{(x^2+y^2)^2}\right) \\
&= \left(\frac{(x^2+y^2) - (x)(2x)}{(x^2+y^2)^2}\right) \\
\frac{\partial^2 u}{\partial x^2} &= \left(\frac{(y^2-x^2)}{(x^2+y^2)^2}\right) \\
\frac{\partial^2 u}{\partial y^2} &= \left(\frac{(x^2-y^2)}{(x^2+y^2)^2}\right) \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2} = 0
\end{aligned}$$

(b)

$$\ln(\sqrt{x^2+y^2}) = \frac{1}{2}\ln(x^2+y^2) = \frac{1}{2}\ln(|z|)$$

Domain of definition is anywhere that the magnitude is not zero.

$$\mathbb{C} \setminus \{0\}$$

(c)

Defining a function $h = f - g$ as the hint suggests. If both f and g are analytic and their real parts are the same:

$$h(z) = f(z) - g(z) = g(z) + c - g(z) = i * (\text{Im}(f) + \text{Im}(g))$$

Since h is equal to c from this definition, because both f and g are analytic, then we know that the real part of h is 0, meaning we have $u(x,y) = 0$ for h. From the CR equations, we must have the partial derivative of h_{u_x} be the same as h_{u_y}. but we know that this should be 0, therefore all of h must be constant.

(d)

It does not because it is a function that has a behavior that exhibits periodicity due to the $\arg(z)$. Because of this, the answers repeat for integer values k. The inclusion of the branch cut means that it cannot be continuous across branch cuts.

Why no contradiction?

If I recall correctly, it is because we use the branch cut to define the domain and this kind of domain is not simply connected. Because of this, it does not contradict the definitions we discussed.

Problem 4

Find the following values, on the branches given:

(a) $\log(3)$ $(-2\pi \leq \theta < 0)$

(b) $\log(-1+i)$ $(-\pi/2 < \theta \leq 3\pi/2)$

(c) $\log(1-i\sqrt{3})$ $(\pi \leq \theta < 3\pi)$.

$$\log(z) = \log|z| + i arg(z) = \log|z| + i(Arg(z) + 2\pi k)\ \forall k \in \mathbb{Z}$$

(a)

$\log(3)$ $(-2\pi \leq \theta < 0)$

$$\begin{aligned}
&\log(3) = \log|3| + i(Arg(3) + 2\pi k) = \log(3) + i(0 + 2\pi k) \\
&\log(3) + i(2\pi k)\ k \in [-1, 0) \quad = -1 \\
&\qquad = \log(3) - i2\pi
\end{aligned}$$

(b)

$\log(-1+i)$ $(-\pi/2 < \theta \leq 3\pi/2)$

$$\begin{aligned}
&\log(-1+i) = \log|-1+i| + i(Arg(-1+i) + 2\pi k) = \log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right) \\
&\log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right)\ k \in \left(-\frac{1}{4}, \frac{3}{4}\right] \\
&\qquad = \log(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right)\ k = 0 \\
&\qquad = \log(\sqrt{2}) + i\frac{3\pi}{4}
\end{aligned}$$

(c)

$\log(1-i\sqrt{3})$ $(\pi \leq \theta < 3\pi)$

$$\begin{aligned}
\log(1-i\sqrt{3}) &= \log|1-i\sqrt{3}| + i(Arg(1-i\sqrt{3}) + 2\pi k) = \log(2) - i\left(\frac{\pi}{3} + 2\pi k\right) \\
&= \log(2) - i\left(\frac{\pi}{3} + 2\pi k\right)\ k \in \left[\frac{1}{2}, \frac{3}{2}\right) \quad k = 1 \\
&= \log(2) + i\frac{5\pi}{3}
\end{aligned}$$

Problem 5

Recall that power functions are defined by $z^c = e^{c\log(z)}$. In this exercise, we compute all power functions by using the branch $(0 \leq \theta < 2\pi)$ for $\log(z)$.

(a) For $z = -i$ and $c = i$, compute the values of $(z^c)^2$, $(z^2)^c$, and $z^{(2c)}$.

(b) With the notation as in (a), which of these are true or false?

$$(z^c)^2 = (z^2)^c, \qquad (z^c)^2 = z^{(2c)}, \qquad (z^2)^c = z^{(2c)}.$$

$$\begin{aligned}
\text{case: } (z^c)^2 &= (z^2)^c \\
((-i)^i)^2 &= (e^{i(\log(-i))})^2 \\
e^{i(\log(-i))} &= e^{i(\log(|-i|)+\arg(-i)+2\pi k)} \\
&= e^{i(0+\frac{3\pi}{2}+2\pi k)} \\
&= e^{i(\frac{3\pi}{2}+0)} = e^{i(\frac{3\pi}{2})} \\
(e^{i(\frac{3\pi}{2})})^2 &= e^{i3\pi} \\
(z^2)^c &= ((-i)^2)^i = ((-1)^2(i)^2)^i \\
&= (-1)^i = e^{i(\log(-1))} = e^{i(|-1|+\arg(-1)+2\pi k)} \\
&= e^{i(1+\pi+0)} = e^{i\pi} \\
\therefore (z^c)^2 &\neq (z^2)^c \text{ given the branch cut.}
\end{aligned}$$

$$\begin{aligned}
\text{case: } (z^c)^2 &= z^{(2c)} \\
(z^c)^2 &= e^{i3\pi} \\
z^{(2c)} &= (-i)^{(2i)} = e^{2i(\log(-i))} \\
&= e^{2i(\log|-i|+\arg(-i)+2\pi k)} \\
&= e^{2i(0+\frac{3\pi}{2}+0)} = e^{i3\pi} \\
\therefore (z^c)^2 &= z^{(2c)} \text{ given the branch cut.}
\end{aligned}$$

$$\begin{aligned}
\text{case: } (z^2)^c &= z^{(2c)} \\
(z^2)^c &= e^{i\pi} \\
z^{(2c)} &= e^{i3\pi} \\
\therefore (z^2)^c &\neq z^{(2c)} \text{ given the branch cut.}
\end{aligned}$$