Homework # 3:

MATH 3160 – Complex Variables
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Problem 1:

(a) Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r,\theta) + iv(r,\theta)$.

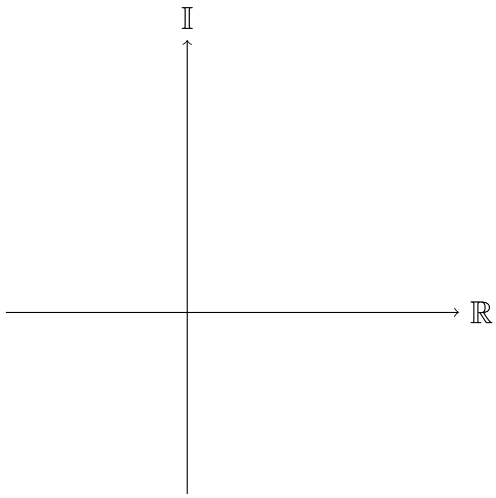
(b) Show that the image of the points in the upper half plane ($y > 0$) that are exterior to the circle $|z| = 1$ are mapped under f to the entire upper half plane $v > 0$.

(a)

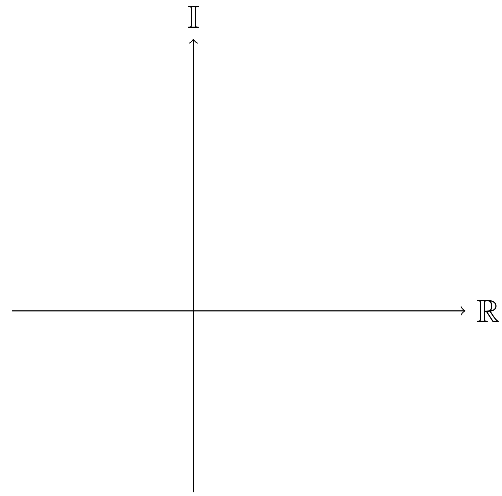
$$\begin{aligned}
f(z) &= z + \frac{1}{z} = (x+iy) + \frac{1}{(x+iy)} = \frac{(x+iy)(x^2+y^2)}{(x^2+y^2)} + \frac{x-iy}{(x^2+y^2)} \\
&= \frac{1}{x^2+y^2}((x+iy)(x^2+y^2) + x - iy) = \frac{1}{x^2+y^2}(x(x^2+y^2) + x + i(y(x^2+y^2) - y)) \\
&\therefore u(x,y) = \frac{1}{x^2+y^2}(x(x^2+y^2)+x) \quad \& \quad v(x,y) = \frac{1}{x^2+y^2}(y(x^2+y^2)-y) \\
&r^2 = x^2+y^2 \\
&x = r\cos(\theta) \\
&y = r\sin(\theta)
\end{aligned}$$

$$\begin{aligned}
\therefore u(r,\theta) = \frac{1}{r^2}(r^3\cos(\theta) + r\cos(\theta)) &= r\cos(\theta) + \frac{1}{r}\cos(\theta) \\
&= \boxed{\left(r + \frac{1}{r}\right)\cos(\theta)} \\
v(r,\theta) = \frac{1}{r^2}(r^3\sin(\theta) - r\sin(\theta)) &= r\sin(\theta) - \frac{1}{r}\sin(\theta) \\
&= \boxed{\left(r - \frac{1}{r}\right)\sin(\theta)}
\end{aligned}$$

(b)



z-plane



$f(z)$-plane

Problem 2:

Use the rectangular forms or exponential forms for the following functions to prove that

(a) $\lim_{z \to z_0} Re(z) = Re(z_0)$

(b) $\lim_{z \to z_0} \bar{z} = \bar{z}_0$

(c) $\lim_{z \to 0} \frac{\bar{z}^2}{z} = 0$

Problem 3:

Show that the limit of the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^3$$

as z tends to zero does not exist. Do so by examining several test paths going to zero.

Problem 4:

Does $f(x+iy) = \dfrac{x+iy}{x+2iy}$ have a limit as $x+iy \to 0$?