Worksheet # 10

MATH 3160 – Complex Variables
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Problem 1

Use the Taylor series of $\sin(z)$, $\cos(z)$, and e^z at $z = 0$ to prove Euler's formula:

$$e^{iz} = \cos(z) + i\sin(z)$$

Using the known expression for the expansion of the exponential shown above:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$
$$= 1 + \frac{(iz)^1}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + ...$$
$$= 1 + i(z)^1 + i^2\frac{(z)^2}{2!} + i^3\frac{(z)^3}{3!} + i^4\frac{(z)^4}{4!} + i^5\frac{(z)^5}{5!} + i^6\frac{(z)^6}{6!} + ...$$

The resulting pattern shows that we have i in many of the expressions, but we can reduce all the products of i with itself to the following.

$$i^0 = i^4 = i^{4n} = 1$$
$$i^1 = i^5 = i^{4n+1} = i$$
$$i^2 = i^6 = i^{4n+2} = -1$$
$$i^3 = i^7 = i^{4n+3} = -i$$

This shows that the expansion can be reduced to either have a factor of i or not. Therefore, we can reduce the expansion further:

$$= 1 + i(z)^1 + (-1)\frac{(z)^2}{2!} - i\frac{(z)^3}{3!} + (1)\frac{(z)^4}{4!} + i\frac{(z)^5}{5!} - (1)\frac{(z)^6}{6!} + ...$$
$$= 1 + i(z)^1 - \frac{(z)^2}{2!} - i\frac{(z)^3}{3!} + \frac{(z)^4}{4!} + i\frac{(z)^5}{5!} - (1)\frac{(z)^6}{6!} + ...$$
$$= 1 - \frac{(z)^2}{2!} + \frac{(z)^4}{4!} - (1)\frac{(z)^6}{6!} + i(z)^1 - i\frac{(z)^3}{3!} + i\frac{(z)^5}{5!}...$$
$$= \left(1 - \frac{(z)^2}{2!} + \frac{(z)^4}{4!} - \frac{(z)^6}{6!} + ...\right) + i\left((z)^1 - \frac{(z)^3}{3!} + \frac{(z)^5}{5!}...\right)$$

The expansion of $\sin(z)$ and $\cos(z)$:

$$\sin(z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{2n+1!}$$
$$= (z)^1 - \frac{(z)^3}{3!} + \frac{(z)^5}{5!} - \frac{(z)^7}{7!} + ...$$
$$\cos(z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{2n!}$$
$$= 1 - \frac{(z)^2}{2!} + \frac{(z)^4}{4!} - \frac{(z)^6}{6!} + ...$$

Notice how the $\sin(z)$ and $\cos(z)$ expansions are exactly what we see in the expression above when expanding out and grouping the expression of e^{iz}

$$\therefore e^{iz} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{2n!} + i\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{2n+1!}$$
$$= \cos(z) + i\sin(z)$$

Problem 2

Find $f^{(10)}(3)$ for $f(z)$ as given below. You can leave the answer in terms of powers, factorials, etc.

$$f(z) = \sum_{n=0}^{\infty} n^2(z-3)^n$$

To do this, we can employ the following expressions:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$
$$f^{(n)}(z) = n!a_n$$
$$a_n = \frac{f^n(z_0)}{n!}$$

So any term that would be present at the 10th derivative summation greater than n would disappear from being a constant or having a factor of $(z-3)$. Meaning we only get the final term being:

$$f^{(10)}(3) = 10!a_n$$
$$a_n = n^2 = 10^2 = 100$$
$$\therefore f^{(10)}(3) = 100 \cdot 10!$$

Problem 3

List the first four nonzero terms of the Taylor series for $f(z) = \cos(3z+2)$ centered at $z = -\frac{2}{3}$.

Here, we need to change the expression to look like $a_n(z - z_0)$:

$$\cos(3z+2) = \cos\left(3\left(z + \frac{2}{3}\right)\right)$$

expanding this out with the expansion we showed before:

$$\cos(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{2n!}$$
$$= 1 - \frac{(z)^2}{2!} + \frac{(z)^4}{4!} - \frac{(z)^6}{6!} + ...$$

This is already the first four terms, we only need to replace the z with our expression:

$$1 - \frac{(3(z+\frac{2}{3}))^2}{2!} + \frac{(3(z+\frac{2}{3}))^4}{4!} - \frac{(3(z+\frac{2}{3}))^6}{6!}$$