Cooling of graphene resonators with superconducting circuit

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Contents

List of symbols

C_C Coupling capacitance of microwave cavity to feedline or input/output line

G_0 classical opto-mechanical coupling parameter $G_0 = g_0/x_{\mathrm{ZPF}}$ in Hz/m.

$P_{in}(\omega)$ Input power applied at source ($P_{in}(\omega) = 10$ dBm). Input power at sample: $P_{in}(\omega) \cdot loss$

$P_{out}(\omega)$ Output power measured in spectrum analyzer. Output power at sample: $P_{out}(\omega)/gain$

Z_0 Impedance of the input and output transmission lines (R_L in Hertzberg thesis)

$\Delta\omega$ detuning from cavity resonance frequency $\Delta\omega = \omega - \omega_c$ (for sideband pumping $\Delta\omega_p = \pm\omega_m$).

Γ_m decay rate of mechanical resonator ($\Gamma_m \approx 0.1 - 20$ kHz).

κ total superconducting cavity loss rate ($\kappa = \kappa_{ext} + \kappa_{int} \approx 1 - 10$ MHz).

κ_{ext} external loss rate of superconducting cavity ($\kappa_{ext} =\approx 1 - 10$ MHz).

κ_{int} internal loss rate of superconducting cavity ($\kappa_{int} =\approx 0.1 - 1$ MHz).

ω_c Resonance frequency of the cavity ($\omega_c \approx 2\pi \times 7.5$ GHz).

ω_m Resonance frequency of the mechanical resonator ($\omega_m \approx 2\pi \times 10 - 100$ MHz).

ω_p Pump frequency (for sideband pumping $\omega_p = \omega_c \pm \omega_m$).

g parametric coupling rate $g = g_0\alpha_p$, with α_p the amplitude of the pump field in the cavity.

g_0 optomechanical single-photon coupling strength. Shift in cavity resonance frequency induced by zero point displacement ($\kappa_{int} =\approx 0.1 - 1$ MHz).

$gain(\omega)$ Frequency dependent gain between the sample output and the spectrum analyzer (typical 40 dB).

$loss(\omega)$ Frequency dependent loss between source and sample due to cabling and attenuators (typical 65 dB).

x motional amplitude of the mechanical resonator ($x \approx 1$ nm).

x_{ZPF} amplitude of the zero point fluctuations ($x \approx 1$ nm).

I. QUARTER WAVELENGTH RESONATOR COUPLED TO TRANSMISSION LINE

A. parallel RLC

B. equivalent RLC for CPW cavities

equivalent $RLC_{||}$ for $\lambda/2$ and $\lambda/4$

$$C_{||}^{\lambda/2} = C_{||}^{\lambda/4} = \frac{\ell C_\ell}{2}$$

$$L_{||}^{\lambda/2} = \frac{1}{\omega_0^2 C_{||}} = L_\ell \cdot \frac{2}{\pi^2}\ell \qquad L_{||}^{\lambda/4} = \frac{1}{\omega_0^2 C_{||}} = L_\ell \cdot \frac{8}{\pi^2}\ell$$

$$R_{||} = \frac{Z_{0,\ell}}{\alpha\ell}$$

with C_ℓ and C_ℓ the capacitance and inductance per length ℓ of the cavity, α the damping of the electro-magnetic wave in the cavity and $\omega_0 = 1/\sqrt{C_\ell L_\ell}$ the resonance frequency of the cavity and $Z_{0,\ell} = \sqrt{\frac{L_\ell}{C_\ell}}$ the characeristic impedence of the cavity waveguide.

- Due to the shorter length of the $\lambda/4$ compared to $\lambda/2$ cavity, the capacitance is a factor of two smaller while the inductance is a factor of two larger. This is benefical for the coupling that depends on the change of the total capacitance per mechanical motion (see below). but as well for the reduced loss due to the shorter path of the resonating cavity photons. The total capacitance can be reduced as well by increasing $Z_{0,\ell}$.
- It is interesting to note, that the internal dissipation is not affected by the above modifications of $Z_{0,\ell}$ as it solely depends on the damping α

$$Q_{int} = \omega_0 R_{||} C_{tot} \approx \frac{\omega_0}{\alpha\ell}\sqrt{\frac{L_\ell}{C_\ell}} C_\ell \ell = \frac{1}{\alpha}.$$

C. Transmission spectra S_{21} in a $\lambda/4$ cavity: useful expressions

The transmission of a quarter wavelength resonator coupled to a feedline can be calculated using a norten equivalent circuit to transform the feedline and the capacitance (see [1]) or alternatively by calculating an effective impedance (see [2])

$$S_{21} = 1 - \frac{Q_{\mathrm{tot}}/Q_{\mathrm{ext}}}{1 + 2iQ_{\mathrm{tot}}\frac{\delta\omega}{\omega_0}}$$

On resonance ($\delta\omega = 0$) the maximum power is coupled into the resonator and lost due to internal dissipation

$$S_{21,\mathrm{min}} = \frac{Q_{\mathrm{ext}}}{Q_{\mathrm{ext}} + Q_{\mathrm{int}}} = \frac{Q_{\mathrm{tot}}}{Q_{\mathrm{int}}}$$

The transmission in power is given by

$$|S_{21}|^2 = 1 - \frac{1 - \left(\frac{Q_{\mathrm{tot}}}{Q_{\mathrm{int}}}\right)^2}{1 + 4Q_{\mathrm{tot}}^2\frac{\delta\omega^2}{\omega_0^2}} \qquad = 1 - \frac{1 - (S_{21,\mathrm{min}})^2}{1 + 4Q_{\mathrm{tot}}^2\frac{\delta\omega^2}{\omega_0^2}} \qquad = 1 - \frac{\kappa_{\mathrm{ext}}^2}{\kappa_{\mathrm{tot}}^2 + 4\delta\omega^2}$$

By measuring the width of the resonance $\Delta\omega_{-3\mathrm{dB}}$ -3 dB below the offresonant transmission, the quality factors can be calculated with $S_{21,\mathrm{min}}$

$$Q_{\mathrm{tot}} = \omega_0 \frac{\sqrt{1 - 2\cdot(S_{21,\mathrm{min}})^2}}{\Delta\omega_{-3\mathrm{dB}}}$$

$$Q_{\mathrm{int}} = \frac{Q_{\mathrm{tot}}}{S_{21,\mathrm{min}}}, \qquad Q_{\mathrm{ext}} = \frac{1}{1/Q_{\mathrm{tot}} - 1/Q_{\mathrm{int}}}$$

- **over-coupled case**: $Q_{\rm int} \gg Q_{\rm ext}$, $Q_{\rm tot} \approx Q_{\rm ext}$. Then the transmitted power is given by

$$|S_{21}|^2 \approx 1 - \frac{1}{1 + 4Q_{\rm tot}^2 \frac{\delta\omega^2}{\omega_0^2}}.$$

In this limit the on-resonance transmission $S_{21,\rm min} = 0$ and the cavity dissipation $2\delta\omega = \kappa$ is the full width half minimum of the peak and can be read off the -3 dB point below the unperturbed transmission ($\delta\omega = \kappa/2 = \frac{\omega_0}{2Q_{\rm tot}}$).

- **critical coupling**: $Q_{\rm ext} = Q_{\rm int}$ and $Q_{\rm tot} = Q_{\rm int}/2 = Q_{\rm ext}/2$

$$|S_{21}|^2 = 1 - \frac{3/4}{1 + 4Q_{\rm tot}^2 \frac{\delta\omega^2}{\omega_0^2}}.$$

with $S_{21,\rm min} = -6$ dB and κ is the full width of the dip at $|S_{21}|^2 = 5/8 = -2$ dB.

- **under coupling**: $Q_{\rm int} \ll Q_{\rm ext}$, $Q_{\rm tot} \approx Q_{\rm int}$ How much smaller can Q_{int} be with respect to Q_{ext} in order to still show a detectable resonance? From $S_{21,\rm min} = \frac{Q_{\rm ext}}{Q_{\rm int}+Q_{\rm ext}} = \frac{1}{Q_{\rm int}/Q_{\rm ext}+1}$ we get

$$\frac{Q_{\rm int}}{Q_{\rm ext}} = \frac{1}{S_{21,\rm min}} - 1$$

Compared with the transmission loss in a symmetrically coupled half wavelength cavity $loss_{\lambda/2} = Q_{\rm ext}^2/Q_{\rm tot}^2$. we get in the undercoupled case

$$loss_{\lambda/2} \approx \frac{Q_{\rm int}^2}{Q_{\rm ext}^2} = \left(\frac{1}{S_{21,\rm min}} - 1\right)^2.$$

An undercoupling ratio of $Q_{\rm ext} = 3 \cdot Q_{\rm int}$ ($10 \cdot Q_{\rm int}$, $100 \cdot Q_{\rm int}$) corresponds to a dip of $|S_{21,\rm min}|^2$ = -2.4 dB (-0.8 dB, -0.08 dB) and a loss in the half-wavelength cavity of $loss_{\lambda/2} = 10$ dB (20 dB, 40 dB). Hence we should still be able to resolve a transmission signal despite of significant loss.

II. WHAT DO WE NEED TO RESOLVE THERMAL- OR ZERO-POINT MOTION OF A GRAPHENE RESONATOR

A. Coupling

The coupling of the mechanical resonator and the superconducting cavity G (in [Hz/m]) is obtained from the shift of the cavity resonance frequency ω_c as a function of the resonator displacement x

$$G_0 = \frac{\partial\omega_c}{\partial x} = \frac{\omega_c}{2C_{\rm tot}}\frac{\partial C_g}{\partial x} \approx \frac{\omega_c}{2C_{\rm tot}}\frac{\epsilon_0 A}{d^2}$$

source: [3].

Quantum picture: The radiation pressure interaction between a (near resonant) cavity field and a mechanical resonator is

$$H_{\rm int} = \hbar g_0 \hat{a}^\dagger \hat{a}\left(\hat{b} + \hat{b}^\dagger\right).$$

Here $\hat{a}$ ($\hat{a}^\dagger$) are the photon annihilation (creation) operator and $\hat{b}(\hat{b}^\dagger)$ the phonon annihilation (creation) operator. The optomechanical coupling rate g_0 corresponds to the shift in cavity resonance frequency induced by the zero point motion.

$$g_0 = G_0 \cdot x_{ZPF}$$

When driving the system with a strong (therefore classical), red detuned pump beam α_p, the Hamiltonian can be linearized $\hat{a}_{tot} = \alpha_p + \hat{a}$ and reduced to an effective beamsplitter hamiltonian

$$H_{\rm int} = \hbar g\left(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger\right)$$

with g the parametric coupling strength given by $g = g_0\alpha_p = g_0\sqrt{n_p}$. [4]

The optomechanical coupling rate $\Omega_c = 2g_0\alpha_p = 2g_0\sqrt{n_p}$ is the analogon to the rabi frequency in EIT[5].

The cooperativity C is defined as

$$C = \frac{\Omega_c^2}{\Gamma_m \kappa} = \frac{4 \cdot n_p \cdot g_0^2}{\Gamma_m \kappa}$$

- Parameter estimation: Plate capacitor model with $d = 300$ nm and $A = 1\ \mu\text{m}^2$: $\frac{\partial C_g}{\partial x} = \frac{\epsilon_0 A}{d^2} \approx 10^{-10}$ F/m
  With $C_{tot} = 0.3$ pF and $\omega_c = 2\pi \times 7$ GHz $G_0 \approx 2\pi \times 1 \frac{\text{kHz}}{\text{nm}}$
  For sample ke2: $G_0 = 2\pi \times 6 \frac{\text{kHz}}{\text{nm}}$, $g_0 = 0.86$ Hz, $n_p = 4 \cdot 10^5$, $g = 1150$ Hz and $C = 3 \cdot 10^{-5}$ for $P_{in} = 20$ dBm, $loss = -42$ dB, $\omega_m = 2\pi \cdot 100$ MHz.
  For sample tm1: $G_0 = 2\pi \times 41 \frac{\text{kHz}}{\text{nm}}$, $g_0 = 8$ Hz, $n_p = 8 \cdot 10^6$, $g = 23$ kHz and $C = 0.3$ for $P_{in} = 20$ dBm, $loss = -62$ dB, $\omega_m = 2\pi \cdot 50$ MHz, .
  Delft estimated there parameters to be $g_0 = 1$ Hz , $n_p = 10^{10}$, $Q_{tot} = \omega_c/\kappa_{tot} = 30$k.

B. Cavity pumping and sideband output power

(see also Herzberg PhD thesis) *Derivation of average energy in resonator $E = 1/2CV^2$, with cavity voltage calculated from norton equivalent circuit. The difference between quarter and walfwavelength resonators are factors of 2 for the equivalent current source I_{No} and the equivalent resistor R_{No}. $R_{No,\lambda/2} = R_{L,eq}/2 = \frac{1}{2Z_0\omega^2 C_C^2}$, $R_{No,\lambda/4} = \frac{2}{Z_0\omega^2 C_C^2}$ $I_{No,\lambda/2} = 2V_{in} \cdot i\omega C_C$, $I_{No,\lambda/4} = V_{in} \cdot i\omega C_C$. The voltage at the input of the resonator is $2V_0$ in case of $\lambda/2$ because of the high impedance termination of the line which gives rise to $2 \times V_0$ as the incomming an reflected wave add up.*

The number of **pump photons in the cavity** n_p are related to the cavity energy by $E_{\text{SMR}}(\omega_p) = n_p \cdot \hbar\omega_p$ and are given by

$$n_p = \frac{1}{\hbar\omega} P_{p,in} \cdot loss \cdot \frac{2}{\kappa_{ext}} \cdot \frac{\kappa_{ext}^2}{\kappa^2 + 4\Delta\omega_p^2}.$$

For $\lambda/4$ measured in transmission, the line is matched and the voltage at the resonator is proportional to the applied input voltage. For the quarter-wavelength resonator coupled to a feedline the same amount of pump photons are coupled into the cavity (coupling rate $\kappa_{ext}/2$) (see also Ref. [6] eq. (S26)). See Fig. 1 for a measure/estimate of the limit for $P_{p,in}$ before heating. *Calculate limit given on n_p from superconductivity of Nb.*

If the pump frequency is on a sideband a finite photon population n_c will be created at the resonance frequency of the microwave cavity by up- or downconversion of the pump. This mode leaks out at both sides of the cavity with rate $\kappa_{ext}/2$. The resuling **output power** $P_{out}(\omega_c)$ is then given by

$$P_{out} = n_c\hbar\omega_c \cdot \frac{\kappa_{ext}}{2} \cdot gain(\omega_c).$$

or

$$P_{out}(\omega_c) = P_{p,in} \cdot loss(\omega_p) \cdot \frac{\kappa_{ext}^2}{\kappa^2 + 4\Delta\omega_p^2} \cdot \left(\frac{1}{\kappa}\frac{\partial\omega_c}{\partial x}\right)^2 \cdot 2\left\langle x^2 \right\rangle \cdot gain(\omega_c)$$

here $\kappa = \omega_c/Q = \kappa_{ext} + \kappa_{int}$ is the (angular) frequency line-width of the microwave cavity.

With small internal losses we have $\kappa \approx \kappa_{ext}$. In the resolved sidband regime $\Delta\omega_p = \omega_m \gg \kappa$ and the ouput power simplifies to

$$P_{out}(\omega_c) = P_{p,in} \cdot loss(\omega_p) \cdot \left(\frac{\partial\omega_c}{\partial x}\right)^2 \cdot 2\left\langle x^2 \right\rangle \cdot \frac{1}{4\Delta\omega_p^2} \cdot gain(\omega_c)$$

Note that the output power does not depend on the cavity linewidth. On the one hand a larger (external) coupling increases the output signal, on the other hand the photons stay longer in the cavity with smaller coupling and increase the effective coupling.

In the case of critical coupling $\kappa_{int} = \kappa_{ext} = \kappa/2$ the output power gets reduced by a factor of 4 ($P_{out,c}$ scales with κ_{ext}^2/κ^2).

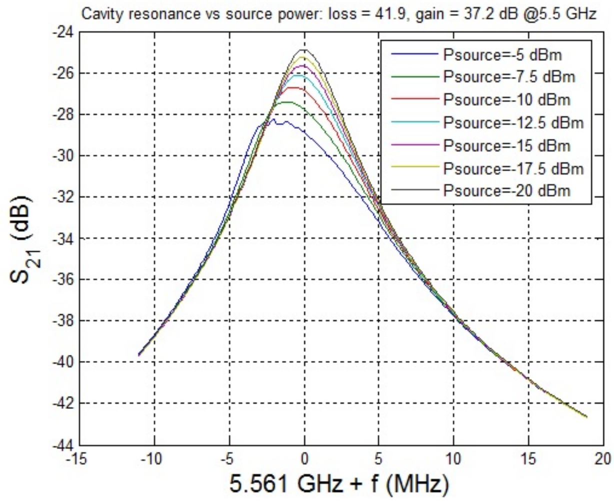

FIG. 1: Strong pumping of cavity. 130216 device kd2: strong drive induces nonlinearity of cavity starting above -20 dB at the source. At -7.5 dB source power the loss has increased by 3 dB (analyzer input: -38 dBm) here $n_p = 5e9$ (with input line loss = 42 dB and output gain = 37.2 dB at 5.5 GHz and $Q_{tot} = 2300$ ($\kappa = 2.4$ MHz)) From S21 measurements while warming up, a peak transmission of $S_{21} = -27$ dB roughly corresponds to a temperature $T_{MC} = 150$ mK. This peak transmission is reached for the $P_{in} = -10$ dB where $n_p = 4e8$. For $P_{in} = -20$ dB no heating/saturation effect is observed and $n_p = 6e7$ (on resonance). ToDo: Double check that loss is due to heating and not due to saturation of spectrum analyzer.

	PT1	PT2	Still	MC	Sample		
drive (dBm)	10 dB att	20 dB att	6 dB att	20 dB att			
20	20	10	-10	-16	-36	dissipation in dB	
	100	10	0.1	0.02512	0.00025	dissipation in mW	
25	25	15	-5	-11	-31	dissipation in dB	
	316.228	31.6228	0.31623	0.07943	0.00079	dissipation in mW	
30	30	20	0	-6	-26	dissipation in dB	
	1000	100	1	0.25119	0.00251	dissipation in mW	
35	35	25	5	-1	-21	dissipation in dB	
	3162.28	316.228	3.16228	0.79433	0.00794	dissipation in mW	
40	40	30	10	4	-16	dissipation in dB	
	10000	1000	10	2.51189	0.02512	dissipation in mW	
45	45	35	15	9	-11	dissipation in dB	
	31622.8	3162.28	31.6228	7.94328	0.07943	dissipation in mW	
50	50	40	20	14	-6	dissipation in dB	
	100000	10000	100	25.1189	0.25119	dissipation in mW	

FIG. 2: Calculation of dissipation in Triton at different stages.

1. *Displacement measurement*

Description with noise power spectral density $S(f)$ (at output of cavity) and displacement spectral density $S_x(f)$. The averaged displacement is $\langle x \rangle$ is related to the spectral density by

$$\langle x^2 \rangle = \int_0^\infty S_x \frac{d\omega}{2\pi}$$

with

$$P_{out}(\omega_c) = P_{p,in} \cdot loss(\omega_p) \cdot \left(\frac{1}{\kappa} \frac{\partial \omega_c}{\partial x} \right)^2 \cdot 2 \left\langle x^2 \right\rangle \cdot \frac{\kappa_{ext}^2}{\kappa^2 + 4\Delta\omega_p^2} \cdot gain(\omega_c)$$

$$S(\omega) = P_{p,in} \cdot loss(\omega_p) \cdot \left(\frac{\kappa_{ext}}{\kappa} \right)^2 G_0^2 \cdot 2 \cdot S_x(\omega) \cdot \frac{1}{\kappa^2 + 4\Delta\omega_p^2} \cdot gain(\omega) + S_N(\omega)$$

with $S_N(\omega)$ the noise power spectral density of the background (LNF amplifier noise). Hence

$$S_x(\omega) = \frac{1}{2}(\kappa^2 + 4\Delta\omega_p^2)\left(\frac{\kappa}{\kappa_{ext}G_0}\right)^2 \frac{S(\omega) - S_N(\omega)}{P_{p,in} \cdot loss(\omega_p) \cdot gain(\omega)}$$

2. Optical spring effect and resonator damping

Using the optical spring effect the coupling strength can be calibrated independently from the mechanical amplitude. While pumping on the red sideband the mechanical resonance frequency gets shifted due to the so called optical spring effect. Using a fixed pumping power the shifted mechanical frequency Ω_m' is given by (see Ref. [7])

$$\Omega_m' \approx \Omega_m + \frac{4g^2\delta}{\kappa^2 + 4\delta^2}.$$

Here $\delta = (\omega_p + \Omega_m) - \omega_c$ is the detuning of the pump frequency from the red side band condidtion ($\omega_p + \Omega_m = \omega_c$). If the mechnical resonator is driven we need for every spectrum around ω_c while sweeping δ, as well to step the drive frequency ω_d (3D measurement with ω_{probe}, δ (ω_p), ω_d).

The damping of the mechanical resonator as a function of pump detuning δ is given by

$$\Gamma_m' \approx \Gamma_m + \frac{4g^2\kappa}{\kappa^2 + 4\delta^2}$$

The damping might be visible even if we don't step the drive frequency.

C. Mechanical motion

1. Zero point motion

$$x_{\mathrm{ZPF}} = \sqrt{\frac{\hbar}{2m\omega_{\mathrm{m}}}} = \sqrt{\frac{\hbar}{2\rho_{\mathrm{2D}}A\omega_{\mathrm{m}}}}$$

For graphene: $\rho_{\mathrm{2D}} = 7.6 \times 10^{-7}$ kg/m^2 $= 7.6 \times 10^{-19}$ kg/μm^2. For a $A = 1\mu$m^2 graphene resonator with $f = 10$ MHz $x_{ZPF} = 1$ pm, for a bilayer resonator with $\omega_m = 2\pi \cdot 50$ MHz we get $x_{ZPF} = 300$ fm.

2. Thermal motion

Without driving the resonator, the vibrational energy is given over the thermal energy $k_B T = \frac{1}{2}m\omega_m^2 x_{th}^2$. And we get the thermal displacement as

$$x_{th} = \sqrt{\frac{2k_B \cdot T_{bath}}{m\omega_m^2}}.$$

We get $x_{th} = 30$ pm for a single layer graphene sheet ($A = 1\mu$m^2) $m = m_0$ at $T_{bath} = 100$ mK with $\omega_m = 2\pi \cdot 10$ MHz. For bilayer with $\omega_m = 2\pi \cdot 50$ MHz we get $x_{th} = 8.5$ pm.

3. Driven motion

As long as we don't see the thermal motion we can drive our resonator by applying an AC gate voltage V_g^{AC} at $\omega = \omega_m$.

$$V_g^{AC} = \sqrt{2 \cdot loss \cdot P_{d,out} \cdot 50\ \Omega};$$

This is the voltage amplitude $V_g^{AC}(t) = V_g^{AC}\cos(\omega t)$ with $P_{sample} = 1/2 * Vsdac^2/R$. If the ac voltage is applied at the gate then the the amplitude is douple due to the reflection part ($V_g^{AC2} = 2 \cdot \sqrt{2 \cdot loss \cdot P_{d,out} \cdot 50\ \Omega}$). If the voltage is applied at either source or drain, then the the effective voltage amplitude at the center of the flake is roughly half and the first expression is valid. This voltage induces a capacitive force on the graphene resonator which is proportional to the product of the DC and AC gate voltage

$$\begin{aligned} F_{cap}(\omega) &= \frac{1}{2}\frac{dC_g}{dx}V_g(\omega)^2 \\ F_{cap}(\omega_m) &= \frac{dC_g}{dx}V_g^{DC}V_g^{AC} \end{aligned}$$

The force induced displacement δx_{ind} is estimated by modeling the vibration by a harmonic oscillator with the equation of motion

$$\delta x_{ind} + \gamma_m \delta \dot{x}_{ind} + \omega_m^2 \delta \ddot{x}_{ind} = \frac{F_{cap}}{m}\cos(\omega_d t)$$

with the damping rate $\gamma_m = \omega_m/Q$ and quality factor Q. The stationary solution is

$$\delta x_{ind}(t) = x_{ind}(\omega_d)\cos(\omega_d t - \phi), \quad x_{ind}(\omega_d) = \frac{F_{cap}/m}{\sqrt{(\omega_m^2 - \omega_d^2)^2 + \gamma_m^2\omega_m^2}}, \quad \phi = \arctan\left(\frac{\gamma_m\omega_d}{\omega_m^2 - \omega_d^2}\right).$$

On resonance $\omega_d = \omega_m$ we have

$$x_{ind} = \frac{\partial C_g}{\partial x}\frac{V_g^{DC}V_g^{AC}}{m\gamma_m\omega_m}.$$

Using the measured values from the sample ke2 ($V_g^{DC} = 10$ V, $\omega_m = 150$ MHz, $L = 1.55\mu$m, $w = 1.2\mu$m, $m = 2{\cdot}m_0$): at a drive of $P_{d,out} = -50$ dBm we get $x_{ind} = 140$ pm. By assuming a linear dependence we expect to be able to resolve the 100 mK thermal motion. Hence we need to increase the readout sensitivity $P_{s,out}$ approximately by 40 dB to detect the thermal motion $x_{th} = 1.4$ pm and by 60 dB in order to detect the zero point motion $x_{zp} = 0.14$ pm. This could be achieved by:

- Reduce 35 dB loss on cavity resonance
- gain 20 dB in coupling by reducing the graphene - gate distance d from 130 to 40 nm
- gain 10 dB in coupling by reducing total capacitance with a meander and a $\lambda/4$ cavity design
- gain up to 15 dB (30 x S_x) by using a JPA (Teufel et.al. 2011 ground-state paper)

4. *Gate voltage dependence*

Static displacement:

$$F_{el}(z) = \frac{1}{2}\frac{\partial C_g}{\partial z}V_g^2 \approx \frac{1}{2}\frac{\epsilon A_g}{(d-z)^2}V_g^2$$

by modeling the gate capacitor C_g as a parallal plate capacitor with plates of area A_g separated by d.

$$F_{mech}(z) = -m\omega_m^2 z - \alpha z^3$$

Static displacement $z_s(V_g)$ can be calculated from the equilibrium at $|F_{el}| = |F_{mech}|$

$$V_g = \sqrt{(m\omega_m^2 z_s + \alpha z_s^3)\frac{2(d-z_s)^2}{\epsilon A_g}}$$

The change in the resonance frequency due to the gate voltage and the nonlinearity can be calculated by linearizing the equation of motion around the static displacement with $z_{tot}(t) = z_s + z(t)$.

$$F_{el}(z_{tot}(t)) \approx F_{el}(z_s) + \frac{\partial F_{el}(z_s)}{\partial z}z(t)$$

$$\alpha z_{tot}^3 \approx \alpha z_s^3 + 3\alpha z_s^2 z(t)$$

The effective resonance frequency $\omega_{\mathrm{m,eff}}$ is then given by the the terms proportional to $z(t)$

$$\omega_{\mathrm{m,eff}}^2 = \omega_{\mathrm{m,0}}^2 - \frac{1}{m}\frac{\partial F_{el}}{\partial z} + 3\frac{\alpha}{m}z_s^2$$

with the plate capacitance model

$$\omega_{\mathrm{m,eff}}^2 = \omega_{\mathrm{m,0}}^2 - \frac{1}{m}\frac{\epsilon A_g}{(d-z_s)^3}V_g^2 + 3\frac{\alpha}{m}z_s^2$$

$$\alpha_{eff} \approx -\frac{1}{12}C^{IV}V_g^2 - \frac{10}{9k_0}\frac{1}{16}(C^{III})^2V_g^4 + \frac{1}{2}C^{III}V_g^2\frac{10}{9k_0}(\beta + 3\alpha z_s)$$

5. *Nonlinear lineshape*

$$F_{el}(t) = C'(z_s)\cdot V_g^{ac}\cdot V_g^{dc}\cos(\omega t) = G\cos(\omega t)$$

with $z(t) = z_0\cos(\omega t)$

$$z_0^2 = \frac{\left(\frac{G}{2m\omega_0^2}\right)^2}{\left(\frac{\omega-\omega_0}{\omega_0} - \frac{3}{8}\frac{\alpha}{m\omega_0^2}z_0^2\right)^2 + \left(\frac{1}{2Q} + \frac{1}{8}\frac{\eta}{m\omega_0}z_0^2\right)^2}$$

The frequency with maximum amplitude is given by

$$\omega_{max} = \omega_0 + \frac{3}{8}\frac{\alpha}{m\omega_0}z_0^2$$

with the amplitude

$$z_{0,max} = \frac{C'(z_s)\cdot V_g^{ac}\cdot V_g^{dc}2\omega 0/\Gamma_m}{2m\omega_0^2} = \frac{C'(z_s)\cdot V_g^{ac}\cdot V_g^{dc}}{m\omega_0\Gamma_m}$$

D. Josephson parametric amplifier

1. Parametric processes in optics

(from Scully Quantum optics p. 471ff., EichlerDiss13 p. 78) The term *parametric* in quantum optics describe processes where the refractive index is the parameter that is periodically changed over the dependence of the polarization of a nonlinear material by a pump field. In a medium with a quadratic non-linearity ($\chi^{(2)}$) the energy of a pump photon is converted into a signal and idler photon $\omega_p = \omega_i + \omega_s$. This *three wave mixing* process is used to create entangled pairs of photons if signal and idler are in the vacuum state (spontaneous parametric down conversion).

In parametric processes that are mediated by a cubic non-linearity $\chi^{(3)}$, all the frequencies can be similar. In this so called *four wave mixing* process two pump photons are converted into signal and idler $2\omega_p = \omega_i + \omega_s$. [If signal and idler are in the vacuum state, the process is called spontaneous four wave mixing. First generation of squeezed state was achieved like this.] In an optical setup the mixing process consists of two counter-propagating pump waves E_2 and E_2' (to satisfy momentum conservation), that interact with a probe wave to produce a fourth wave which is the spacial complex conjugate of the probe wave.

To see the influence of the polarization it is instructive to look at the wave equation

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

with $E(r,t) = E_s + E_p + E_p' + E_i$ and the nonlinear polarization $P = \chi^{(3)}E^3$. From the many coupling terms only few are relevant if the pump is much larger than the signal power. If in addition the fields are slowly varying $|d^2E_n/dz^2| \ll |k_n dE_n/dz|$, the evolution of the field amplitudes $E_s, i(z)$ can be described by

$$\begin{aligned}\frac{dE_s(z)}{dz} &= i\kappa_1 E_s + i\kappa E_i^* \\ \frac{dE_i(z)}{dz} &= -i\kappa_1 E_i - i\kappa E_s^*\end{aligned}$$

with the coupling parameters given over

$$\begin{aligned}\kappa_1 &\propto \xi^{(3)}(|E_2|^2 + |E_{2'}|^2) \\ \kappa &\propto \xi^{(3)}E_2E_{2'}\end{aligned}$$

The equations indicate that the signal and idler modes can be mixed or amplified by the pump fields. See Scully1997 p. 474 for the evaluation of the position dependent solution from these equations.

2. Parametric processes in the microwave domain

(see also Eichler Diss 2013) Analog to the role of the refractive index in optics the signal propagation at microwave frequencies is described by the effective impedance. Parametric processes can be induced by changing the capacitance or inductance over time. A time varying inductance can be achieved by passing a small current trough a Josephson junction

$$L \approx L_J\left(1 + \frac{1}{6}\left(\frac{I(t)}{I_C}\right)^2\right).$$

The I^2 dependence of the inductance leads to a four wave mixing. This can be seen from the propagation of an microwave in a transmission line (telegrapher equations, e.g. Pozar eqn. 2.4b), where

$$\begin{aligned}\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) &= 0 \\ \frac{d^2I(z)}{dz^2} - \gamma^2 I(z) &= 0\end{aligned}$$

with $\gamma = \alpha + i\beta$. In the loss-less case $\gamma = i\omega\sqrt{L'C'}$ with L', C' the inductance and capacitance per unit length of the line. Hence we get a $I^3(z)$ dependence in analogy to the general wave equation.

Squids instead of single junctions to be able to tune the resonance frequency over flux. E_J per squid is 0.5-5 THz (or 2-20 meV, 250 GHz $\equiv$ 1 meV) which is small compared to E_J in transmon qubits.

Weak non-linearity (effective Kerr non-linearity K) is beneficial to have high dynamic range as the critical value of the system (before bifurcation) depends on the product of K and the input pump power ($K|\alpha_{in}|^2$). Typically $K/\omega_0 \approx 10^{-6}$.

Cavity to select band of amplification (parameters on page 99 $Q_{int} = 2500, Q_{ext} = 430$ ($\kappa_{ext} = 15$ MHz), $\omega_0 = 6.267$ GHz).

Why $\lambda/4$ resonator? Drawback compared to asymmetric $\lambda/2$: pump signal has to be canceled.

Kerr effect: change of refractive index with electric field. *DC (electro-optical) Kerr effect*; $\Delta n = \lambda K E^2$ with K the Kerr constant. *AC (optical) Kerr effect*; For the Kerr effect the $\xi^{(3)}$ term of the electric susceptibility is significant. For $E = E_0 + E_\omega \cos(\omega t)$ For the DC Kerr effect the electric polarization reduces to

$$P \approx \epsilon_0(\xi^{(1)} + 3\xi^{(3)}|E_0|^2)E_\omega \cos(\omega t)$$

Input-Output Formalism

–¿ Heisenberg equation of motion –¿

3. *Parametric amplification within a cavity*

The Hamiltonian that describes the parametric amplification process in a cavity is given as

$$H_{JPA} = \hbar\widetilde{\omega}_0 A^\dagger A + \hbar\frac{K}{2}(A^\dagger)^2 A^2$$

with A the annihilation operator of the intra-resonator field and $\widetilde{\omega}_0/2\pi$ the resonance frequency. The tilde indicates that the resonance frequency of the cavity has shifted due to the coupling with the environment

$$\widetilde{\omega}_j^2 = \frac{\omega_j^2}{1 + C_\kappa/C_j} = \frac{1}{(C_j + C_\kappa)L_j}.$$

The index $j = 0, 1, 2, 3, \ldots$ indicates the different normal modes of the $\lambda/4$ resonator with $k_j \ell_{\lambda/4} = \pi/2(1 + 2j)$. The decay of the modes is given by

$$\kappa_j = \frac{\widetilde{\omega}_j^2 C_\kappa^2 R_L}{C_j + C_\kappa}$$

with $R_L = 50\Omega$ the internal impedance of the source (see goeppl08 with one coupling capacitance).

linear regime of the para-amp: Without internal loss the reflection is unity and the response is completely characterized by the phase ϕ of the reflected field. As we have a nonlinear system, ϕ does not only depend on the frequency but also on the drive power. For large pump power we get into a bistable regime, where the system can no longer be used as an amplifier. Below a critical value however, the reflected phase depends sensitively on the power which can be used to amplify a signal.

Use input-output formalism to solve *classical nonlinear system response* (with operator as a complex number $A = \alpha$). For the mean number of pump photons n in the resonator we get

$$1 = (\delta^2 + \frac{1}{4})n - 2\delta\xi n^2 + \xi^2 n^3$$

with the scale invariant quantities

$$n \equiv \frac{|\alpha^2|}{|\tilde{\alpha}_{in}|^2}, \qquad \tilde{\alpha}_{in} \equiv \frac{\sqrt{\kappa\alpha_{in}}}{\kappa + \gamma}, \qquad \delta = \frac{\omega_p - \tilde{\omega}_0}{\kappa + \gamma}, \qquad \xi \equiv \frac{|\tilde{\alpha}_{in}|^2 K}{|\kappa + \gamma|}$$

where $\tilde{\alpha}_{in}$ is the dimensionless drive amplitude, δ the detuning between pump and resonance frequency normalized by the total resonator line-width and ξ the *cavity non-linearity parameter* with the critical value $\xi_{crit} = -1/\sqrt{27}$ above which the system gets bistable. The critical detuning is given by $\delta_{crit} = -\sqrt{3}/2$ below which the system can get bistable as well.

experimentally the properties are measured over the complex reflection coefficient $\Gamma \equiv \alpha_{in}/\alpha_{out}$

$$S_{11} \equiv \Gamma = \frac{\kappa}{\kappa+\gamma}\frac{1}{\frac{1}{2}-i\delta+i\xi n}-1.$$

See fig. 4.3 inDissEichler.

Response with additional weak quantum signal If the photon flux of the pump $|\alpha_{in}|^2$ is much larger than the signal $\left\langle a_{in}^\dagger a_{in}\right\rangle$, the equation of motion for $A(t)$

$$\dot{A} = -i\omega_0 A - iKA^\dagger AA - \frac{\kappa+\gamma}{2}A + \sqrt{\kappa}A_{in}(t) + \sqrt{\gamma}b_{in}(t)$$

can be linearized using $A(t) = (a(t)+\alpha(t))\exp(-i\omega_p t)$. Depending on the signal pump detuning Δ we get for small losses $\gamma/\kappa \to 0$

$$a_{\mathrm{out},\Delta} = g_{S,\Delta}a_{\mathrm{in},\Delta} + g_{I,\Delta}a^\dagger_{\mathrm{in},-\Delta} \tag{2.1}$$

with the gain of the signal and idler given by

$$\begin{aligned} g_{S,\Delta} &= -1 + \frac{\kappa}{\kappa+\gamma}\frac{i(\delta-2\xi n-\Delta)+\frac{1}{2}}{(i\Delta-\lambda_-)(i\Delta-\lambda_+)} \\ g_{I,\Delta} &= \frac{\kappa}{\kappa+\gamma}\frac{-i\xi n e^{2i\phi}}{(i\Delta-\lambda_-)(i\Delta-\lambda_+)} \end{aligned}$$

where the ϕ denotes the phase of the intra resonator pump field $\alpha = |\alpha|e^{i\phi}$ and

$$\lambda_\pm = \frac{1}{2} \pm \sqrt{(\xi n)^2 - (\delta - 2\xi n)^2}.$$

For zero signal detuning Δ and $\gamma = \kappa$ the gain $G_\Delta = |g_{S,\Delta}|^2$ gets maximal for $\xi \approx \xi_{\mathrm{crit}}$ and pump tone detuning $\delta \approx -0.85$.

4. phase sensitive and insensitive mode of operation

Because tones with opposite detuning from the pump frequency gets mixed, the amplification of a signal with frequencies around the pump frequency is *phase sensitive*. One quadrature is amplified while the other is deamplified. With a proper choice of the pump phase and neglecting a global phase factor the phase sensitive parametric amplification can be described by the operator

$$a_{\mathrm{amp}} \approx \sqrt{G_0}a + \sqrt{G_0-1}a^\dagger$$

For the quadrature $X = (a+a^\dagger)/2$ we get amplification $X_{\mathrm{amp}} = (\sqrt{G_0}+\sqrt{G_0-1})X$ while the quadrature $P = i(a^\dagger - a)/2$ is deamplified $P_{\mathrm{amp}} = (\sqrt{G_0}-\sqrt{G_0-1})P$. The amplified quadrature can be selected over the pump phase $X_\phi = (ae^{-i\phi}+a^\dagger e^{i\phi})/2$. The phase sensitive amplification results from the interference of the phase correlated positive and negative frequency components in the parametric process. The amplification is noiseless for a single quadrature signal since there are no other modes mixed in.

This is different if the signal occupies a band that is detuned from the pump frequency. Here the amplification is *phase insensitive* but there is noise added from the mixing with the oppositely detuned frequency band

$$a_{\mathrm{amp}} = g_{S,\Omega}a + g_{I,\Omega}h^\dagger.$$

Here Ω is the detuning of the center of the detection band from the pump frequency and $h = \int d\Delta f^*_{-\Delta}a^\dagger_{out,\Delta}$ a noise mode which is independent from the signal if Ω is large ($[a,h^\dagger] \approx 0$). This transformation is called a two mode squeezing transformation.

continue with chapter 4.4 on page 98

5. Implementation in our setup

Frequency tuning over magnetic flux through SQUIDS from 4.5- 6.5 GHz in [8] and 4- 8 GHz over many flux periods in [9]. A coil underneath the sample is used to apply a DC flux and the AC flux, needed e.g. for Qubit operations, is created with an on chip flux line. In order to use the amplifier we need as well to cancel the cavity pump signal with a phase shifter and variable attenuator.

How much gain do we need? Castellanos calculates at $f = 5$ GHz for a $T_A = 5$ K noise temperature of the HEMT amplifier a needed gain of 16 dB. The noise quanta added by the amplifier are

$$N_A = \frac{k_B T_A}{hf} = 20.8.$$

The JPA as a quantum limited amplifier adds 1/2 quanta of noise. The total noise contribution at the sample output is then given by

$$N_{tot} = \frac{1}{2} + \frac{N_A}{G_{JPA}}$$

which are equal for $G_{JPA} = 40 = 16$ dB. The effective noise temperature of the amplifier depends as well on the attenuation from the sample output to the amplifier input. Eichler et al report a HEMT noise contribution of 50 quanta which corresponds to a noise temperature of 15 K at 6 GHz (5 dB loss with $T_A = 4.5$K or 9 dB with $T_A = 2$K).

Bandwidth: The measured gain in [8] is 10 dB at a bandwidth of 3.6 MHz. Castellanos et al.[9] have 16 dB at 2 MHz. Gain-bandwidth product is constant and $\sqrt{G_0}B = 11.4$ or 12.6 in the two cases.

DC flux line from Eichler Diss 2013 page 33: In addition to the flux control lines we use small coils mounted below the sample to apply a constant magnetic flux bias to the SQUIDs (Figure 2.9(c)). Using coils instead of on-chip flux lines for DC bias avoids a steady current flow onto the PCB, which can lead to significant heating. The coils are fed through a twisted pair of DC wires powered by a voltage source (14SIM928 from Stanford Research Systems) providing a maximum voltage of ±20 V and a resolution of 1mV. Above the 4K temperature stage the wires are made of copper, while below the 4K stage superconducting NbTi wires are used to avoid heating. The wires are thermalized at each temperature stage by twisting them around a copper heat sink. Current noise through the coil is reduced by adding a high impedance low pass filter at room temperature and shielding the cables outside the cryostat.

E. Loss in sH cavities

1. Loss due to small coupling capacitance

Can the significant loss of the sH cavities in the 4K system be explained by a larger external coupling capacitance? The external quality factor Q_{ext} in dependence of the coupling capacitance C_κ is given as

$$Q_{ext} = \frac{C_{||}}{2R_L\omega_c C_\kappa^2}$$

with the $C_{||} = C_\ell \ell/2$ the effective parallel LC resonator capacitance [see Goeppl08 eq. 21].

The external capacitance C_κ is obtained by simulations with ANSYS Maxwell. Using the above expression Q_{ext}^{cal} is calculated with the line resistance $R_L = 50\ \Omega$ and the $C_{||} = 300$ fF.

From the measurements in the 4K system we get a roughly device independent total Q up to 9000. This suggests that the total quality factor is dominated by the internal losses (under-coupled regime) and $Q_{int}^{4K} \approx 9000$. Over the insertion loss the external Q can then be estimated as

$$Q_{ext} = \frac{Q_{int}}{10^{IL(dB)/20}}.$$

For the sH40k cavity we get from the measurement at 4K $Q_{ext}^{cal} = 130k$ and $Q_{ext}^{IL} = 142k$. Hence part of the loss in the sH cavity seems to come from a too small coupling capacitance. In the sH12.5k device that we tested (15.6.2012) we measured a misleading Q of 15k. This Q was probably not limited by the external coupling but rather indicates the internal Q of the sH cavities at 1.4 K.

From this estimate we would need $C_\kappa = 3.5\ fF$ for $Q_{ext} = 5k$ and $C_\kappa = 5\ fF$ for $Q_{ext} = 2.5k$.

From static simulations we get 3.5 fF for a gap of 25 μm (2.5 fF for 50 μm) and 5 fF for a gap of 5 μm (4.6 fF for 10 μm).

device	capacitance gap width	simulated capacitance	calculated Q_{ext}	measured Q_{tot}	insertion loss	Q_{ext}
sH 5k	gap $= 100\ \mu$m	$C_\kappa = 1.4$ fF	$Q_{ext}^{cal} = 32$k			
sH 10k	gap $= 125\ \mu$m	$C_\kappa = 1.4$ fF	$Q_{ext}^{cal} = 32$k			
sH 12k	gap $= 140\ \mu$m	$C_\kappa = 0.9$ fF	$Q_{ext}^{cal} = 80$k	$Q_{4K} = 9000$	$IL = -15.5$ dB	$Q_{ext}^{IL} = 50$k
sH 40k	gap $= 175\ \mu$m	$C_\kappa = 0.7$ fF	$Q_{ext}^{cal} = 130$k	$Q_{4K} = 8500$	$IL = -24$ dB	$Q_{ext}^{IL} = 142$k

2. Loss due to parasitic gate capacitance

In the ke2 configuration the capacitance of the gate finger is estimated as $C_{g,para} = 2$ fF using the following parameters: along source and drain gate length of 12 um, gate width 0.85 um and a gap of 175 nm and an additional 10 um connection to the center conductor. In general the capacitance is roughly 55-200 aF/um gate length (0.5 um-0.1 um gap, 1 um width).

The loss on resonance due to the parasitic capacitance is then given as

$$loss = \frac{1}{(1 + \frac{C_{g,para}}{2C_\kappa})^2}.$$

For $ke2$ with $C_{g,para} = 2$ fF and $C_\kappa = 1.4$ fF the loss is roughly -5 dB. For the future with larger coupling capacitance between 3 and 5 fF we should not get over 2 fF even if keep the same geometry.

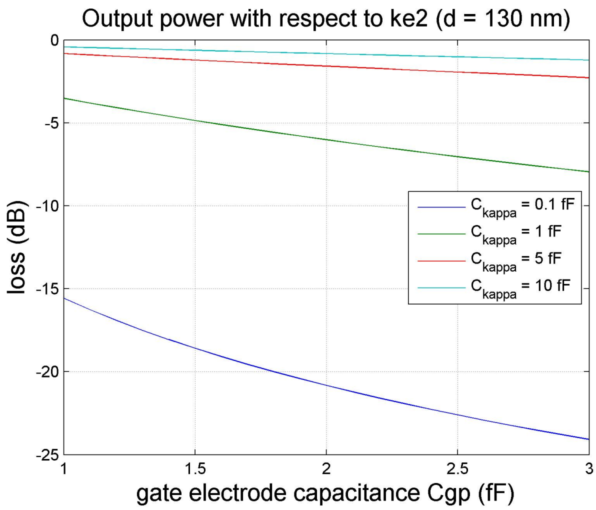

FIG. 3: See derivation 18.4.2013 and SCC - matlab - ke2_readout130424.m

III. $Q \cdot f$ PRODUCT

The number of coherent operations in the presence of a thermal equilibrium bath is proportional to the $Q \cdot f$ product.

$$N_{osc} = f \times \frac{1}{\Gamma \cdot n_{bath}} = Q \times f \times \frac{h}{k_B T}$$

with Γ the decay rate of the quantum system, the thermal occupation of the bath $n_{bath} \approx \frac{k_B T}{hf}$. At room temperature the constant is given by $h/k_B T = 1.6 \times 10^{-13}$ implying a minimum Qf product of 6e12 Hz for a single quantum operation.

With superconducting circuits a Qf product of $5e15$ Hz has been achieved by Schoelkopf. With quartz resonators a Qf product of $> 10^{16}$ Hz has been measured. Typically more on the order of 10^{13} Hz.

see e.g. http://podcastfichiers.college-de-france.fr/devoret-20120515.pdf

Quartz resonator with QF product ¿ $10^{1}6$ http://scitation.aip.org/content/aip/journal/apl/100/24/10.1063/1.4729292

Qf in silicon MEMS resonators:

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