MAT4220: Partial Differential Equations*

Tutorial Notes

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Abstract

This is the tutorial notes for MAT4220: Partial Differential Equations. There are in total 10 sections. This notes mainly cover first-order linear PDEs and solution methods, second-order linear PDEs and their classifications, the wave equation, the heat equation, the reflection method and inhomogeneous wave and heat, separation of variables, Fourier series, convergence theorems, completeness, the Laplace's equation, Green's functions, and the distribution theory. If you find any intolerable mistakes or typos, please contact me through [email me].

Contents

*Much of the content is selected from [Str07]. There are also materials adopted from [Eva10] and Prof. Wang Xuefeng's notes.

1 Tutorial 1

1.1 Some Concepts

Definition 1.1 (PDE)**.** *An expression of the form*

$$F(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x) = 0, \ \ x \in U \tag{1}$$

is called a k-th order partial differential equation, where

$$F : \mathbb{R}^{n^k} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

is given, and

$$u : U \to \mathbb{R}$$

is the unknown.

The general PDE can be written in the operator form

$$\mathcal{L}[u](x) = f(x), \tag{2}$$

where $x = (x_1, ..., x_n)$ are the independent variables, $u = u(x_1, ..., x_n)$ the unknown function, $f(x) = f(x_1, ..., x_n)$ the given function. The highest order of the partial derivative of the unknown function $u(x)$ that appears in a PDE is called the *order* of the PDE. If $f = 0$, then the PDE is said to be *homogeneous*. The PDE is said to be *linear*, if the differential operator $\mathcal{L}$ satisfies the following condition: (for nonlinear PDEs, we do not talk about homogeneity since there is no benefit to do that...)

$$\mathcal{L}[\alpha u_1 + \beta u_2] = \alpha\mathcal{L}[u_1] + \beta\mathcal{L}[u_2]. \tag{3}$$

Why do we care about linearity? Because we have *superposition principle*.

Example 1.1. *The Possion equation $-\Delta u = f$ is linear since the differential operator $\Delta := \sum_{i=1}^n \partial_{x_i x_i}$ is a linear operator.*

Problem 1.1. *Check the order, linearity, homogeneity of the following PDEs:*

1. $u_t + V u_x = 0$
2. $u_{tt} - c^2 u_{xx} = A \sin t$
3. $u_{xx} + u_{yy} = e^{x+y}$
4. $u_t + u u_x = k u_{xx}$
5. $u_t + u_{xxxx} + (x^2 + 1)(1 + u)^{2/3} = x^{99} + 10^{100}$
6. $i u_t - u_{xx} + x^2 = 0$
7. $f(x, y, z)u_{xyz} + g(x, y)u_{xy} + h(z)u_z = a(x, z)$

1.2 Method of Characteristics

Let us start with the simplest transport/traffic problem[1]

$$\begin{cases} \partial_t u + \boldsymbol{V}\partial_x u = 0 \\ u(x,t)|_{t=0} = \phi(x) \end{cases} \tag{4}$$

where $\boldsymbol{V}(x, t)$ is a constant field. We would like to show by this simple example the basic idea of **the method of characteristics**.

The idea used to solve this problem is to reduce the PDE to an ODE, by restricting the solution $u(x, t)$ to a curve on the xt-plane so that u now is a function of one variable and satisfies an ODE. The curve

[1] This example is taken from Prof. Wang Xuefeng's notes.

is called the **characteristic curve** of the PDE. Consider a curve $x = x(t)$; after restricting u on the curve it becomes a function of t only. Then apply the chain rule, we have

$$\frac{du}{dt} = u_t + u_x \frac{dx}{dt}. \tag{5}$$

We wish to relate the righthand side to the PDE, so we demand that

$$\frac{dx}{dt} = \boldsymbol{V}. \tag{6}$$

Thus

$$x = x(t) = \boldsymbol{V}t + \boldsymbol{C} \tag{7}$$

where C is an arbitrary constant. (Note that all these characteristics are parallel lines and fill the entire xt plane.) Now by our choice of the curve, we have

$$\frac{du}{dt} = u_t + \boldsymbol{V}u_x = 0, \tag{8}$$

hence $u = M$ where the constant M depends on the characteristic curve and hence on the constant $\boldsymbol{C}$, i.e, u is a function of $\boldsymbol{C}$, which is just $x - \boldsymbol{V}t$ according to (7). Thus, we write

$$u(x,t) = f(x - \boldsymbol{V}t). \tag{9}$$

for some arbitrary function f. As you could verify, provided f is smooth enough (e.g. $f \in C^2(\Omega)$), the solution given by (9) is always a solution to the PDE (4) w.o. considering the initial condition. (we would do that later...)
We invoke the initial condition and it is easy to check that $u(x,t) = \phi(x - \boldsymbol{V}t)$.

Example 1.2. *solve the first-order equation* $2u_t + 3u_x = 0$ *with the auxiliary condition* $u(x, 0) = \sin x$.

Solution. Set $\frac{dx}{dt} = \frac{3}{2}$, then $x(t) = \frac{3}{2}t + C, C \in \mathbb{R}$. Note that $\frac{d}{dx}u(x(t),t) = u_x\frac{dx}{dt} + u_t = 0$. Therefore, $u(x(t),t) = u(x(0),0) = u(C,0) = u(x - \frac{3}{2}t, 0) := f(x - \frac{3}{2}t)$. As $f(x) = \sin x$, we conclude $u(x,t) = \sin(x - \frac{3}{2}t)$. □

Before we discuss the method of characteristics further, let us get some feel of this solution. If we take the snapshot of the graph of u as a function of x at time t, it is just the graph of the initial value $\phi(x)$ shifted to the right by V (if $V \in \mathbb{R}$; to the left if $V < 0$). Otherwise, it is shifted along the direction $\boldsymbol{V}$. The graph is travelling with velocity V without changing its shape. For this reason, u is called a **travelling wave solution**. This solution can be interpreted as modelling a lump of (non-diffusive) pollutant in a river with water moving with constant speed $\boldsymbol{V}$.

After understanding the mechanism of characteristic method, let's consider a (more) complicated example.

Example 1.3 (Linear Inhomogeneous Equation)**.**

$$\begin{cases} \partial_x u + y\partial_y u = u + y \\ u(x,y)|_{y=1} = \phi(x) \end{cases} \tag{10}$$

Solution. the characteristics are given by

$$\frac{dy}{dx} = y,$$

thus

$$y = Ce^x.$$

so u, when restricted on a fixed characteristic curve (so the constant C is fixed), satisfies

$$\frac{du}{dx} = u_x + yu_y = u + Ce^x. \tag{11}$$

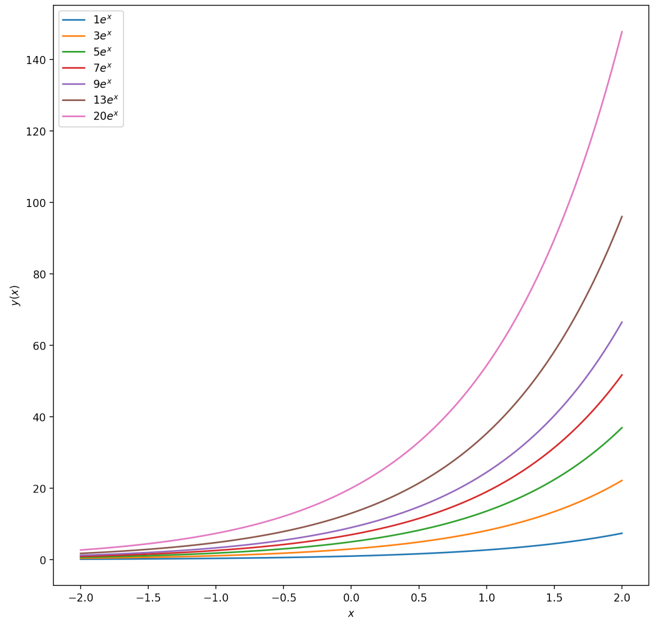


Figure 1: Characteristic Lines $y = Ce^x$.

solving the ODE, we obtain

$$u(x, y(x)) = Cxe^x + f(ye^{-x})e^x, \tag{12}$$

Thus, we obtain the general solution to the PDE:

$$u(x, y) = xy + f(ye^{-x})e^x. \tag{13}$$

Because of the initial condition $\phi(x, y)|_{y=1} = \phi(x)$, we have

$$f(x) = x(\phi(-\ln x) + \ln x)),$$

Think: Something important is missing here!!!

therefore,

$$u(x, y) = y \ln y + y\phi(x - \ln y). \tag{14}$$

In this example, the initial value is not given on the x-axis, because the x-axis is a characteristic curve on which the only choice for u is $Me^x, M \in \mathbb{R}$. And even such an initial value is prescribed, it does not affect the value of u off the x-axis. that is, such an initial value does not determine a unique solution. Let check the case $u(x, y)|_{y=0} = e^x$. Then we get

$$f(0) = 1,$$

which almost gives nothing about f. Therefore, we cannot *uniquely* determine $u(x, y)$ when $y \neq 0$. The moral of this story is that if an initial condition is imposed on a characteristic curve, then there may not exist a solution; if there exists a solution, the solution is not unique. (you can solve one of the problems in your assignment 1 by using the idea aforementioned!)

1.3 Coordinate Method

The basic idea of *coordinate method* is still to convert a first-order linear ODE with *constant coefficients* to an ODE, which is easier to solve. Consider the following example:

Example 1.4. *solve the equation $au_x + bu_y = 0$.*

note that

$$au_x + bu_y = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{pmatrix} u_x \\ u_y \end{pmatrix} \tag{15}$$

which is equivalent to

$$\boldsymbol{v}^T \nabla u = 0, \quad \boldsymbol{v} = \frac{(a,b)^T}{\|(a,b)^T\|}.$$

That is, the directional derivative of u along $\boldsymbol{v}$ is zero. We now consider to apply a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ to change from an orthogonal coordinate system to another orthogonal coordinate system. Consider, in specific, the linear transformation represented by the matrix:

$$A := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

that is, $x' = ax + by, y' = -bx + ay$. Then, we may check (easily):

$$\partial_x = a\partial_{x'} - b\partial_{y'}; \quad \partial_y = b\partial_{x'} + a\partial_{y'} \tag{16}$$

Hence,

$$(a\partial_x + b\partial_y) = (a^2 + b^2)\partial_{x'} \tag{17}$$

Therefore, the orginal PDE is equivalent to the ODE:

$$\partial_{x'}\tilde{u} = 0, \tilde{u} := u(x'(x,y), y'(x,y)), \tag{18}$$

□

Remark 1.1. *You can also check that if u_1, u_2 are orthogonal in $\mathbb{R}^2$, then Au_1, Au_2 are also orthogonal in $\mathbb{R}^2$.*

Remark 1.2. *Indeed, it might be good to use the matrix*

$$A = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{19}$$

which consists of two orthogonal normalized vectors, it can be easily checked that $A \in O(2)$ and in this case $a\partial_x + b\partial_y = \partial_{x'}$.

The coordinate method can also be applied to solve the below PDE

Example 1.5. $au_x + bu_y + cu = 0$.

use the A defined above, set

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{20}$$

then the orginal PDE is equivalent to

$$(a^2 + b^2)\tilde{u}_{x'} + c\tilde{u} = 0. \tag{21}$$

□

Problem 1.2 (p10,Q13)**.** *Use the coordinate method to solve the equation*

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2. \tag{22}$$

Solution. set $x' = x+2y; y' = 2x-y$, then $\partial_x + 2\partial_y = 5\partial_{x'}$. The orginal PDE becomes $5\tilde{u}_{x'} + y'\tilde{u} = x'y'$. This equation is ODE-solvable. Consider multiplying on both sides an integration factor $e^{\frac{x'y'}{5}}$. Then we have $(e^{x'y'/5}\tilde{u})_{x'} = \frac{1}{5}e^{x'y'/5}x'y'$. Solving it yields $\tilde{u}(x', y') = x' - \frac{5}{y'} + f(y')e^{-x'y'/5}$, where f is arbitrary. Transform it back to xy-plane. We finally have $u(x,y) = f(2x-y)e^{-(x+2y)(2x-y)/5} - \frac{5}{2x-y} + x + 2y$.□

Equation	Method of Characteristics	Coordinate Method
$au_x + bu_y = 0$	✓	✓
$au_x + bu_y = f(x, y)$	✓	✓
$au_x + bu_y + cu = 0$	✓	✓
$a(x, y)u_x + b(x, y)u_y = 0$	✓	×

Table 1: Camparison of method of characteristics and coordinate method

1.4 Review of Analysis

(optinal) If time allows.

In this section, we would briefly review some of the important facts covered in the introductory analysis courses. Most of the proofs will be omitted. You could check the proofs in the lecture notes of MAT2006/MAT2007 or in any standard analysis textbooks. This section is NOT required. If you, however, want to understand more rigorously, it is recommended to review those analysis facts.

Theorem 1.3 (Continuity)**.** *Given $f(x, y)$ is continuous in the rectangle $R = [\alpha, \beta] \times [a, b]$, the integral*

$$F(x) := \int_a^b f(x, s)ds$$

is a continuous function of x on $[\alpha, \beta]$.

Theorem 1.4. *Suppose $f(x, y) \in C^1(R := [\alpha, \beta] \times [a, b])$, we may change the order of differentiation and integration:*

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^b f(x, s)ds = \int_a^b f_x(x, s)ds.$$

Proof Sketch. Note that

$$\frac{F(x+h) - F(x)}{h} = \int_a^b \frac{f(x+h, s) - f(x, s)}{h}ds = \int_a^b f_x(x + \gamma h, s)ds, \gamma \in (0, 1).$$

then

$$\left|\frac{F(x+h) - F(x)}{h} - \int_a^b f_x(x, s)ds\right| \le \int_a^b |f_x(x + \gamma h, s) - f_x(x, s)|\, ds < \int_a^b \frac{\epsilon}{b+a}ds = \epsilon$$

Given h sufficiently small. □

Theorem 1.5. *Assume that $\psi_1(x), \psi_2(x) \in C^1([\alpha, \beta])$, where $a < \psi_1(x) < \psi_2(x) < b, \forall \alpha \le x \le \beta$ and $f(x, y), f_x(x, y)$ are continuous in $[\alpha, \beta] \times [a, b]$, then*

$$\frac{d}{dx}\int_{\psi_1(x)}^{\psi_2(x)} f(x, s)ds = \int_{\psi_1(x)}^{\psi_2(x)} f_x(x, s)ds + f(x, \psi_2(x))\psi_2'(x) - f(x, \psi_1(x))\psi_1'(x).$$

Hint. Set $v = \psi_1(x)$, $u = \psi_2(x)$ and $\phi(u, v, x) := \int_v^u f(x, s)ds$. Notice that $\frac{d}{dx}\phi(u, v, x) = \phi_u \frac{du}{dx} + \phi_v \frac{dv}{dx} + \phi_x$. □

Definition 1.2 (Uniform Convergence)**.** *We say $\int_a^\infty f(s, y)ds$ converges uniformly for $y \in [c, d]$, if for any $\epsilon > 0$, there exists a constant $M(\epsilon) \ge a$ such that whenever $A \ge M$,*

$$\left|\int_A^\infty f(s, y)ds\right| < \epsilon,$$

Theorem 1.6 (Weierstrass' M-test)**.** *If $|f(x, y)| < F(x)$ for sufficiently large x and for all $c \le y \le d$, and $\int_a^\infty F(s)ds$ converges, then $\int_a^\infty f(s, y)ds$ converges uniformly for $c \le y \le d$.*

Theorem 1.7. *If (i) f, f_y are continuous in $[a, \infty) \times [c, d]$;(ii) $\int_a^\infty f_y(s, y)ds$ converges uniformly for $y \in [c, d]$; (iii) $\psi(y) = \int_a^\infty f(s, y)ds$ exists in $[c, d]$, then $\psi'(y) = \int_a^\infty f_y(s, y)ds$.*

1.5 Exercises

You may apply the techniques covered in this tutorial to solve the following simple exercises. (some of them might have appeared in the homework assignments.)

Problem 1.8 (p10,Q10). *solve* $u_x + u_y + u = e^{x+2y}$ *with* $u(x,0) = 0$.

Problem 1.9. *Find the general solution of*

$$u_{xy} + u_x = 0. \tag{23}$$

Problem 1.10. *solve the equation*

$$u_x + 2xy^2 u_y = 0 \tag{24}$$

Problem 1.11. *solve the equation* $xu_x + yu_y = 0$.

2 Tutorial 2

2.1 Classification of Second-Order Linear PDEs

The general form of a second-order linear PDE is given as

$$(\nabla_x^T A \nabla_x + b^T \nabla_x + c)u = 0, \tag{25}$$

where $A = (\alpha_{ij}) \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n, c \in \mathbb{R}$.

We first present some simple insights on why there are three different types of second-order linear PDEs. Consider the Poisson equation $-(u_{xx} + u_{yy}) = f$. We perform Fourier transformation on both sides w.r.t. to x with f being a δ-function, and then dropping the Fourier transform of u. Then we obtain $\xi^2 + \eta^2 = 1$, which corresponds to an ellipse. In the same manner, the wave equation $u_{tt} - u_{xx} = f$ corresponds to a hyperbola $\xi^2 - \eta^2 = 1$. Hence we say that the wave equation is *hyperbolic*. Moreover, the heat equation $u_t - u_{xx} = f$ corresponds to $\xi = -\eta^2$, a parabola. That is why we say the heat equation is *parabolic*.

Lemma 2.1 (Spectral Theorem (MAT2040/MAT3040))**.** *Every real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is orthogonally diagonalizable. i.e. $\exists U \in O(n)$ such that*

$$U^T A U = \Lambda$$

That is, A is similar to a diagonal matrix.

Theorem 2.2 (Classfication)**.** *For a PDE given as (25), denote the eigenvalues of A as $\lambda_1, ..., \lambda_n$*

1. *if all λ_i are of the same sign, then the PDE is elliptic.*
2. *if none of λ_i vanishes but exactly one has the opposite sign as others, then the PDE is hyperbolic.*
3. *if exactly one of λ_i vanishes and all others have the same sign, the PDE is called parabolic.*

Proof Sketch. The proof is very simple, you might need to use the spectral theorem. □

Example 2.1. *Show that among all the equations of the form (25), the only ones when $n = 2$ that are unchanged under all rotations (rotationally invariant) have the form*

$$(\nabla_x^T \Lambda \nabla_x + c)u = 0, \tag{26}$$

where $\Lambda \in \mathbb{R}^{2\times 2}$ is diagonal.

Example 2.2. *Consider the equation*

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0,$$

1. *what is the type of the equation?*
2. *reduce the equation by a change of variable $u = ve^{\alpha x+\beta y}$ (α, β are parameters) and then do a change of scale $y' = \gamma y$.*

2.2 More on Method of Characteristics

Example 2.3. *use the method of characteristics to solve the following inhomogeneous PDE*

$$a(x, y)u_x + b(x, y)u_y = c(x, y),$$

where $a(x, y), b(x, y)$, and $c(x, y)$ are smooth functions.

In this section we will discuss **characteristic curves for general first-order PDEs**. After seeing so many examples of solving PDE by MC, it is now obvious that the characteristic curves of the general quasilinear PDE

$$a(x, t, u)u_t + b(x, t, u)u_x = f(x, t, u), \tag{27}$$

are defined by the differential equation

$$a(x,t,u)dx - b(x,t,u)dt = 0. \tag{28}$$

(We use the differentials because we have the advantage of re-writing the equation in the form of either $dx/dt = \cdots$ or $dt/dx = \cdots$). In the linear case (a and b independent of u), (28) is an ODE that may be solved as in the previous examples. But in the nonlinear case, (28) appears to be useless because it involves the unknown solution u. This is not so as we can see from the following.

Example 2.4. *Consider the following quasilinear traffic equation*

$$u_t + c(u)u_x = 0; \quad u(x,0) = \phi(x) \tag{29}$$

The ODE for characteristic curves is

$$\frac{dx}{dt} = c(u(x,t)). \tag{30}$$

On a fixed characteristic curve, we have

$$\frac{du}{dt} = u_t + u_x \frac{dx}{dt} = u_t + c(u)u_x = 0.$$

Thus on each characteristic curve, $u = M \in \mathbb{R}$. Now going back to (30), we obtain the equation for the characteristic curves

$$x = c(M)t + x_0.$$

which is a straight line that intersects the x-axis at x_0. So

$$u(x,t) = u(x_0,0) = \phi(x_0) = \phi(x - tc(u(x,t))). \tag{31}$$

This formula for u is not given in explicit form; this is a nonlinear phenomenon. □

Example 2.5 (Discontinuous Initial Data)**.** *Find the solution of the following Burger's equation with discontinuous initial condition:*

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases} \end{cases} \tag{32}$$

We study the following problem:

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) = \begin{cases} -1, & x < -\epsilon \\ x/\epsilon, & -\epsilon \le x \le \epsilon \\ 1, & x > \epsilon \end{cases} \end{cases} \tag{33}$$

By (31), we have

$$u(x,t) = \phi(x - u(x,t)t) = \begin{cases} -1, & x - ut < -\epsilon \\ (x-ut)/\epsilon, & -(t+\epsilon) \le x - ut \le \epsilon \\ 1, & x - ut > \epsilon \end{cases}$$

i.e.

$$u(x,t) = \begin{cases} -1, & x < -(t+\epsilon) \\ x/(t+\epsilon) & -(t+\epsilon) \le x \le (t+\epsilon) \\ 1, & x > t+\epsilon \end{cases}$$

Sending $\epsilon \to 0$, we get the solution of the original problem

$$u(x,t) = \begin{cases} -1, & x < -t \\ x/t & -t \le x \le t \\ 1, & x > t \end{cases}$$

□

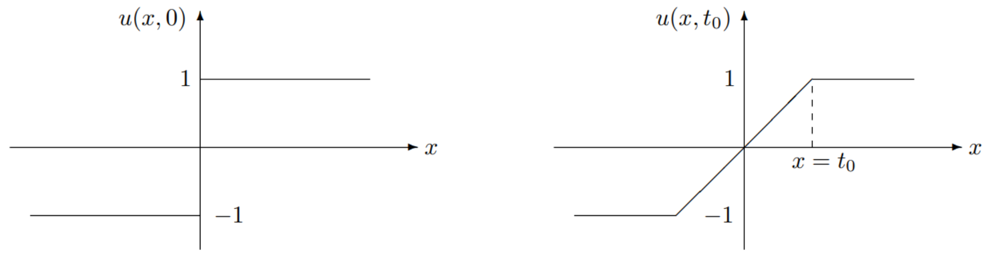


Figure 2: Burger's equation with discontinuous initial data

2.3 Boundary Problem

I do not see what is the relation of the below problems with PDE course, but they are covered in last year's tutorial notes.

Example 2.6. *Consider the problem*

$$\begin{aligned} u''(x) + u'(x) =& f(x) \\ u'(0) = u(0) =& \frac{1}{2}[u'(l) + u(l)] \end{aligned} \tag{34}$$

with $f(x)$ given. Is the solution unique? Does the solution necessarily exist? Is there any condition f must satisfy to ensure the existence of a solution?

Proof. NOT unique, consider $u_0 + K(e^{-x} - 2)$, where u_0 is a solution. f needs to satisfy $\int_0^l f(x)dx = 0$.

Example 2.7.

$$\begin{aligned} u'' =& 0 \quad 0 < x < 1 \\ u'(0) + ku(0) =& 0 \\ u'(1) \pm ku(1) =& 0 \end{aligned} \tag{35}$$

Solve the problem with $+, -$ resp. What would happen if $k = 2$?

Proof. for '+':

$$u(x) = \begin{cases} 0, & k \neq 0 \\ b, & k = 0 \end{cases}$$

For '$-$':

$$u(x) = \begin{cases} 0 & k \neq 0, 2 \\ b & k = 0 \\ -2bx + b & k = 2 \end{cases}$$

2.4 Review of Infinite Series

To every series $\sum_{n=1}^{\infty} a_n$, we associate the positive series $\sum_{n=1}^{\infty} |a_n|$. If $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} a_n$. In this way $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent.* If $\sum_{n=1}^{\infty} |a_n|$ diverges while $\sum_{n=1}^{\infty} a_n < \infty$, it is said to be *conditionally convergent.*

Theorem 2.3 (Comparison Test)**.** *if $|a_n| \leq b_n, \forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\sum_{n=1}^{\infty} |a_n|$ converges. The contrapositive necessarily follows.*

Next, we consider series of functions $\sum_{n=1}^{\infty} f_n(x)$. We say that $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ pointwisely in (a, b) if for each $x \in (a, b)$ we have:

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0, \quad \text{as } N \to \infty.$$

However, note that the choice of N here depends on x, which makes this mode of convergence not that strong to make general deductions. We say that the series *converges uniformly* to $f(x)$ in $[a, b]$ if

$$\sup_{a \le x \le b} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0, \quad \text{as } N \to \infty.$$

That is to say, the N chosen here is general, which does not depend on the specific choice of x.

Theorem 2.4. *If $|f_n(x)| \le c_n \in \mathbb{R}, \forall n \in \mathbb{N}$ and for all $x \in [a, b]$, and if $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly in the interval $[a, b]$, as well as absolutely.*

Theorem 2.5 (Term-by-term Integration)**.** *If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to $f(x)$ in $[a, b]$ and if all the functions $f_n(x)$ are continuous on $[a, b]$, then the sum $f(x)$ is also continuous in $[a, b]$ and*

$$\sum_{n=1}^{\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

That is, we can interchange taking limit and integration.

Remark 2.1. *Of course, if you have learned MAT3006: Real Analysis or other advanced analysis course, you might see the above is merely an application of monotone convergence theorem ($\{f_n\}$ measurable and nonnegative) or dominated convergence theorem.*

Proof Sketch. Let $x_m \to x$,

$$|f(x_m) - f(x)| \le |f(x_m) - \sum_{n=1}^{N} f_n(x_m)| + |f(x) - \sum_{n=1}^{N} f_n(x)| + |\sum_{n=1}^{N}(f_n(x_m) - f_n(x))| < 3\frac{\epsilon}{3} = \epsilon.$$

□

Theorem 2.6 (Term-by-term Differentiation)**.** *If $f_n(x)$ are differentiable for all n in $[a, b]$ and if the series $\sum_{n=1}^{\infty} f_n(c)$ converges at some point c, and if $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to $g(x)$, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to some $f(x)$ and $g(x) = f'(x)$, namely,*

$$\sum_{n=1}^{\infty} f_n'(x) = f'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} f_n(x).$$

2.5 Exercises

Problem 2.7. *Consider the equation*

$$x u_{xx} + y u_{yy} - u_{xy} + 2011 u_x + 210 u_y + u = x^2 + y, \tag{36}$$

In what region on xy-plane is the PDE elliptic, hyperbolic or parabolic type?

Problem 2.8. *Classify the followings PDEs:*

1. $(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + x u_x + y u_y = 0$
2. $u_{xx} + y u_{yy} = 0$
3. $e^{2x} u_{xx} + 2e^{x+2y} u_{yy} + e^{2y} u_{xy} = 0$

Problem 2.9. *Find the general solution of $u_x + x u_y = u$.*

Problem 2.10. *Solve the initial value problem: $u_t + u_x = x, u(x, 0) = 1/(1 + x^2)$.*

3 Tutorial 3

3.1 Wave Equation

Why do we care about wave equation defined on the whole real line?

1. Physically speaking, sitting far away from the boundary takes a substantial amount of time before the boundary effect takes place.

2. Mathematically, ignoring the boundary provides a significant simplification so that we can find properties of PDEs.

Theorem 3.1 (d'Alembert's Formula)**.**

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{37}$$

Example 3.1. *Solve the Cauchy Problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R}, t \in \mathbb{R} \\ u(x,0) = e^{-x^2}, \quad u_t(x,0) = \sin(x), & x \in \mathbb{R}. \end{cases} \tag{38}$$

Remark 3.1. *Since the derivation of d'Alembert's formula is constructive, the uniqueness of the solution of Cauchy problem is also proved. (Therefore, we have two methods to prove the uniqueness of wave equation.)*

Travelling Wave. We recall that the general solution to the wave equation $u_t - c^2 u_{xx} = 0$ is given by

$$u(x,t) = f(x-ct) + g(x+ct), \tag{39}$$

we consider $f(\cdot)$ first. The function $f(x-ct)$ is a **shift** of $f(x)$ to the right with a distance ct as illustrated below. Obviously the **travelling speed** is c. Thus $f(x-ct)$ is called a **right-travelling**

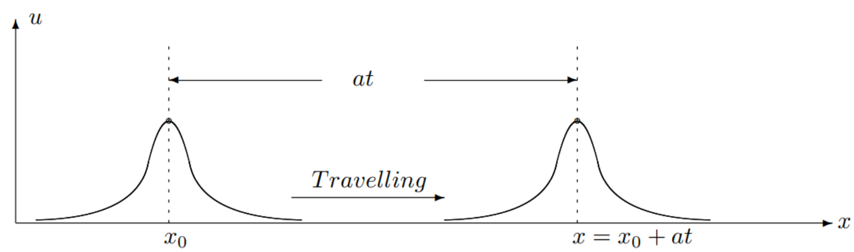


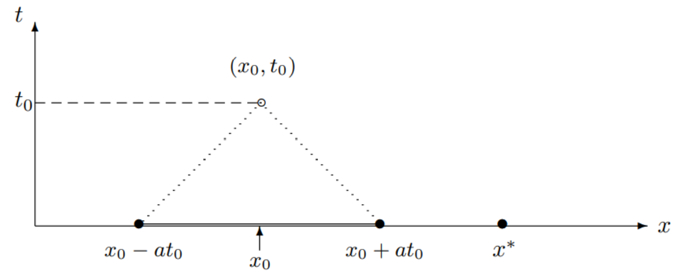
Figure 3: Travelling Wave

wave with speed c. (Thus, the propagation speed of wave is **finite**.) Similarly $g(x+ct)$ is called a **left-travelling wave** with speed c. Notice that $c = \sqrt{\frac{T}{\rho}}$. Therefore the wave propagation is faster, if the tension T is larger or the density ρ is smaller.

Domain of Dependence. since

$$u(x_0,t_0) = \frac{1}{2}[\phi(x_0+ct_0) + \phi(x_0-ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(s)ds,$$

it is clear that the solution at (x_0,t_0) is determined by the initial information in interval $[x_0-ct_0, x_0+ct_0]$. In daily life, the phenomenon of *sound propagation* is the simplest example. A person who is standing at position x_0 can only hear at time $t_0 > 0$ the sound produced at the distance ct_0 at time $t = 0$, where c is the speed of sound propagation; the solution value at (x_0,t_0) has no relation to the initial "information" at position x^* ($x^* > x_0 + ct_0$ or $x^* < x_0 - ct_0$), since the initial information is too far away to reach the position x_0.



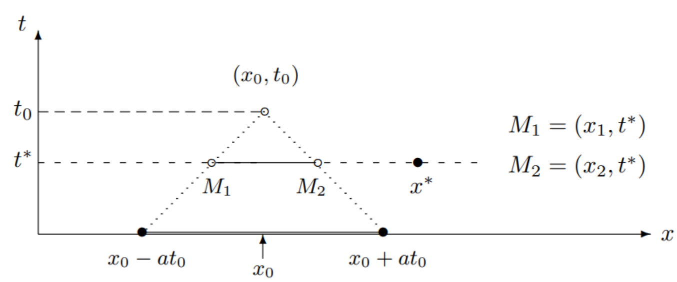
The solution at any point on $\overline{M_1M_2}$, $u(x, t^*)$ can be determined by the initial conditions in $(x_0 - ct_0, x_0 + ct_0)$ and is well-defined. We denote

$$\phi^*(x) = u(x, t^*), \quad \psi^*(x) = u_t(x, t^*), \quad \forall x_1 < x < x_2.$$

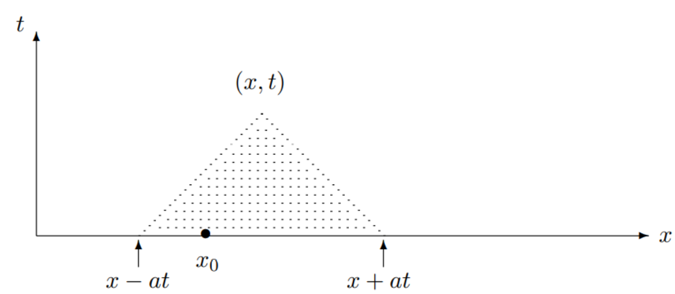
With respect to $t_0 > t^*$, t^* may also be considered as the 'initial time'; then the elapsed time from t^* to t_0 is $t_0 - t^* := \tau_0$. So by d'Alembert's formula the solution value $u(x_0, t_0)$ is given by

$$u(x_0, t_0) = \frac{1}{2}[\phi^*(x_0 + c\tau_0) + \phi^*(x_0 - c\tau_0)] + \frac{1}{2c}\int_{x_0 - c\tau_0}^{x_0 + c\tau_0} \psi^*(s)ds.$$

the solution $u(x_0, t_0)$ can be determined by the information on any line segment in the triangle. Meanwhile the information in the exterior of the triangle has no effect to the solution value $u(x_0, t_0)$. Thus the triangle is called the **domain of dependence** of point (x_0, t_0).



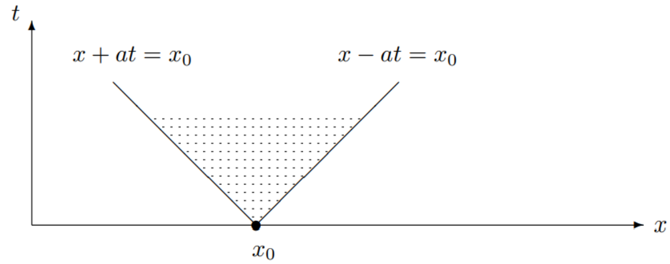
Domain of Influence. Now let us consider the influence of the initial conditions at point x_0. Pick a point (x, t) in upper-half of the xt-plane and draw its domain of dependence as illustrated below. It is clear that the initial conditions at x_0 will affect the solution at (x, t), if and only if $(x_0, 0)$ is an



interior point of the bottom of the domain of dependence, i.e. $x_0 > x - at$ and $x_0 < x + at$. In other words, the initial conditions at x_0 have influence on the domain

$$\{(x, t) : x < x_0 + ct \wedge x > x_0 - ct \wedge t > 0\}.$$

which is illustrated as below



3.2 Energy Method

Let us consider an infinite string with constant linear density ρ and tension magnitude T. The wave equation describing the vibrations of the string is then[2]

$$\rho u_{tt} = T u_{xx}, \quad -\infty < x < \infty. \tag{40}$$

Since this equation describes the mechanical motion of a vibrating string, we can compute the kinetic energy associated with the motion of the string. Recall that the kinetic energy is $\frac{1}{2}mv^2$. In this case the string is infinite, and the speed differs for different points on the string. However, we can still compute the energy of small pieces of the string, add them together, and pass to a limit in which the lengths of the pieces go to zero. This will result in the following integral

$$KE = \frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2 dx.$$

We will assume that the initial data vanishes outside of a large interval $|x| \le R$, so that the above integral is convergent due to the *finite speed of propagation*. We would like to see if the kinetic energy KE is conserved in time. For this, we differentiate the above integral with respect to time to see whether it is zero, as is expected for a constant function, or whether it is different from zero.

$$\frac{d}{dt}KE = \frac{1}{2}\rho\int_{-\infty}^{\infty} 2u_t u_{tt} dx = \int_{-\infty}^{\infty} \rho u_t u_{tt} dx.$$

Using the wave equation (40), we can replace ρu_{tt} by $T u_{xx}$, obtaining

$$\frac{d}{dt}KE = T\int_{-\infty}^{\infty} u_t u_{xx} dx.$$

The last quantity does not seem to be zero in general, thus the next best thing we can hope for, is to convert the last integral into a full derivative in time. In that case the difference of the kinetic energy and some other quantity will be conserved. To see this, we perform an integration by parts in the last integral

$$\frac{d}{dt}KE = T u_t u_x|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T u_{xt} u_x dx.$$

Due to the finite speed of propagation, the endpoint terms vanish. The last integral is a full derivative, thus we have

$$\frac{d}{dt}KE = -\int_{-\infty}^{\infty} T u_{xt} u_x dx = -\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} T u_x^2 dx\right).$$

Defining

$$PE := \frac{1}{2}T\int_{-\infty}^{\infty} u_x^2 dx,$$

we see that

$$\frac{d}{dt}(KE + PE) = 0.$$

[2] https://web.math.ucsb.edu/ grigoryan/124A/lecs/lec7.pdf

The quantity $E = KE + PE$ is then conserved, which is the total energy of the string undergoing vibrations. Notice that PE plays the role of the potential energy of a stretched string, and the conservation of energy implies conversion of the kinetic energy into the potential energy and back without a loss.
Another way to see that the energy

$$E = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2) dx \tag{41}$$

is conserved, is to multiply equation (40) by u_t and integrate with respect to x over the real line.

$$0 = \int_{-\infty}^{\infty} \rho u_{tt} u_t dx = \int_{-\infty}^{\infty} T u_{xx} u_t dx.$$

The first integral above is a full derivative in time. Integrating by parts in the second term, and realizing that the subsequent integral is a full derivative as well, while the boundary terms vanish, we obtain the identity

$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2 + T u_x^2 dx\right) = 0,$$

which is exactly the conservation of total energy.
The conservation of energy provides a straightforward way of showing that the solution to an IVP associated with the linear equation is unique. We demonstrate this for the wave equation next, while a similar procedure will be applied to establish uniqueness of solutions for the heat IVP in the next section.

Example 3.2. *Show that the initial value problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \quad x \in \mathbb{R} \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x). \end{cases} \tag{42}$$

has a unique solution.

The procedure used in the last example, called the energy method, is quite general, and works for other linear evolution equations possessing a conserved (or decaying) positive definite energy. We will discuss energy method for *heat equation* later.

Example 3.3. *Consider the wave equation on the whole real line with the usual initial data ϕ and ψ. If ϕ, ψ are both odd (even) functions, show that the solution $u(x,t)$ is also an odd (even) function of x.*

3.3 High-dimensional Wave Equation

We prove the energy conservation of the high-dimensional homogeneous wave equation with homogeneous Dirichlet / Neumann boundary.

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, \quad x \in \Omega \subset \mathbb{R}^n, \\ u|_{\partial\Omega} = 0 / \frac{\partial u}{\partial \boldsymbol{n}} = 0, \quad x \in \partial\Omega. \end{cases} \tag{43}$$

Note that the energy is defined to be

$$E(t) := \frac{1}{2}\int_\Omega |u_t|^2 + c^2 \|\nabla u\|^2 dx.$$

we differentiate the energy with respect to t

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_\Omega u_t u_{tt} + c^2 \nabla u \cdot \partial_t \nabla u dx, \quad \text{(why?)} \\ &= c^2 \int_\Omega u_t \Delta u dx + c^2 \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{n}} \partial_t u dS - c^2 \int_\Omega \partial_t u \Delta u dx, \quad \text{(why?)} \\ &= c^2 \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{n}} \partial_t u dS = 0. \end{aligned}$$

□

You might need to review some materials regarding Guass-Green, Integration by parts, and Green's formula. Please refer to the appendix of [Eva10]. For more materials on high-dimensional wave, please refer to Chapter 9 of [Str07].

3.4 Exercises

Problem 3.2. *If $u(x,t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity*

$$u(x+h,t+k) + u(x-h,t-k) = u(x+k,t+h) + u(x-k,t-h),$$

for all x, t, h and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.

Problem 3.3. *For the damped string $u_{tt} - c^2 u_{xx} + r u_t = 0$, show that the energy decreases.*

3.5 Review of Integration

Theorem 3.4 (Vanishing Theorem)**.** *Let $f(x)$ be a continuous function in a finite closed interval $[a,b]$. Assume that $f(x) \geq 0$ in the interval and that $\int_a^b f(x)dx = 0$. Then $f(x)$ is identically zero.*

proof sketch. Suppose not. WLOG, assume there exists $c \in [a,b]$ such that $f(c) > 0$. By continuity, near c the integral of f is larger than 0. □

Theorem 3.5 (First Vanishing Theorem)**.** *Let $f(x)$ be a continuous function in $\overline{D}$ where D is a bounded domain. Assume that $f(x) \geq 0$ in $\overline{D}$ and that $\iiint_D f(x)dx = 0$. Then $f(x)$ is identically zero.*

Theorem 3.6 (Second Vanishing Theorem)**.** *Let $f(x)$ be a continuous function in D_0 such that $\iiint_D f(x)dx = 0$ for all subdomains $D \subset D_0$. Then $f(x)$ is identically zero.*

Theorem 3.7 (Green's Theorem[3])**.** *Let C be a smooth Jordan curve and D is the bounded region enclosed by C. If $P(x,y)$ and $Q(x,y)$ are both in $C^1(\overline{D})$, then*

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy,$$

where C takes the positive orientation.

Remark 3.2. *When D is multiply-connected with finite number of holes, Green's theorem has a generalization.*

A completely equivalent formulation of Greens's theorem is obtained by substituing $p = -g$ and $q = +f$. If $\boldsymbol{f} = (f,g)$ is any C^1 vector field in $\overline{D}$, then $\iint_D (f_x + g_y)dxdy = \int_C (-gdx + fdy)$. If $\boldsymbol{n}$ is the unit outward normal vector of C, then $\boldsymbol{n} = (dy/ds, -dx/ds)$. Hence Green's theorem takes the form

$$\iint_D \nabla \cdot \boldsymbol{f} dxdy = \int_C \boldsymbol{f} \cdot \boldsymbol{n} ds. \tag{44}$$

In three dimensions we have the divergence theorem, also known as *Gauss's theorem*, which is the natural generalization of (44).

Theorem 3.8 (Divergence Theorem)**.** *Let D be a bounded spatial domain with a piecewise C^1 boundary surface S. Let $\boldsymbol{n}$ be the unit outward normal vector on S. Let $\boldsymbol{f}(\boldsymbol{x})$ be any C^1 vector field on $\overline{D} = D \cup S$. Then*

$$\iiint_D \nabla \cdot \boldsymbol{f} d\boldsymbol{x} = \iint_S \boldsymbol{f} \cdot \boldsymbol{n} dS. \tag{45}$$

[3] You can also check you MAT2007 notes to review line integral, double integral, triple integral and surface integral.

4 Tutorial 4

4.1 Heat Equation

Maximum Principle. Consider a refrigerator occupying region Ω; and think about the maximum of the temperature function $u(x,t)$ inside the refrigerator during time interval $[0,T]$. The maximum of $u(x,t)$ must be achieved either at a boundary point at some time between $t=0$ and $t=T$, or inside Ω at time $t=0$ (as in the case of a refrigerator which is turned on at time $t=0$). Thus

$$\max_{\overline{\Omega_T}} u = \max_{\partial_T \Omega} u, \tag{46}$$

where $\Omega_T := \Omega \times (0,T]$ and $\partial_T\Omega := (\partial\Omega \times (0,T]) \bigcup (\overline{\Omega} \times \{0\})$. Ω_T is called the **parabolic interior**, and $\partial_T\Omega$ is called the **parabolic boundary** of the cylinder $\overline{\Omega} \times [0,T]$.

Weak Maximum Principle. Let Ω be a bounded region in $\mathbb{R}^n$ and $T>0$. Suppose on Ω_T, $u(x,t)$ satisfies

$$u_t - k\Delta u \leq 0. \tag{47}$$

then (46) holds.

Proof. We first claim that if the strict inequality in (47) holds, then the maximum of u on $\overline{\Omega}_T$ cannot be achieved at a point (x_0,t_0) in the parabolic interior Ω_T. Suppose otherwise, Then $u_t(x_0,t_0) \geq 0$ and $\nabla^2 u$ is NSD, so the trace is non-positive. But the trace is just the Laplacian of u and hence

$$\Delta u(x_0,t_0) \leq 0.$$

This would arise a contradiction.

Now we suppose we do not have the strict inequality. Define a new function

$$v(x,t) := u(x,t) + \epsilon e^{-t},$$

where $\epsilon > 0$ is a constant. The new function satisfies the strict inequality in (47) and so it satisfies

$$\max_{\overline{\Omega_T}} v = \max_{\partial_T\Omega} v,$$

Sending $\epsilon \to 0^+$ (be careful with this, can you directly send ϵ to 0 on both sides simultaneously?), we conclude the proof of the weak maximum principle. $\square$

Strong Maximum Principle. Assume that $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ solves the heat equation in Ω_T. If Ω is connected and there exists a point $(x_0,t_0) \in \Omega_T$ such that

$$u(x_0,t_0) = \max_{\overline{\Omega_T}} u,$$

then u is constant in $\overline{\Omega_{t_0}} := \overline{\Omega} \times [0,t_0]$.

Remark 4.1.
- *by replacing u with $-u$, one can also prove the (strong) minimum principle.*
- ***Interpretation.*** *if u attains its maximum (or minimum) at an interior point, then u is constant at all earlier times. This accords with our strong intuitive understanding of the variable t as denoting time: the solution will be constant on the time interval $[0,t_0]$ provided the initial and boundary conditions are constant. However, the solution may change at times $t > t_0$ provided the boundary conditions alter after t_0. The solution will not respond to changes in the boundary conditions until these changes happen.*

NOTE: We will not give a proof at this level because it is somewhat a little bit technical. A quick proof requires the mean-value property of heat equation. For those who are interested, please read p.55-56 of [Eva10].

Example 4.1. *Consider the diffusion equation $u_t = u_{xx}$ in $\{0<x<1, 0<t<\infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$. Show that $0<u(x,t)<1$ for all $t>0$ and $0<x<1$. Also show that the energy is strictly decreasing.*

Proof. In class. $\square$

Comparison Principle. Let u and v satisfy

$$\begin{cases} u_t - k\Delta u \geq v_t - k\Delta v, & x \in \Omega, 0 < t \leq T \\ u(x,t) \geq v(x,t), & x \in \partial\Omega, 0 < t \leq T \\ u(x,0) \geq v(x,0), & x \in \Omega \end{cases} \tag{48}$$

Then

$$u(x,t) \geq v(x,t), \quad x \in \overline{\Omega} \times [0,T].$$

Moreover, if there exists $(x_0, t_0) \in \Omega \times (0,T]$ where u and v touch each other, then u and v are identical, at least before time t_0.

Proof. consider $w := v - u$ and apply the weak maximum priciple. If u and v touch at a point in the parabolic interior, apply the strong maximum principle. □

Corollary 1. Uniqueness of Heat Equation. The initial-Dirichlet boundary value problem for heat equation

$$\begin{cases} u_t - k\Delta u = f(x,t), & x \in \Omega, T \geq t > 0 \\ u(x,t) = g(x,t), & x \in \partial\Omega, T \geq t > 0 \\ u(x,0) = \phi(x), & x \in \Omega \end{cases} \tag{49}$$

has at most one solution.

Proof. Suppose there is another solution v, apply comparison principle twice and we can conclude that $u = v$. This can also be proved by the energy method, which will be discussed later. □

Corollary 2. Stability. For each of $i = 1, 2$, let u_i be the solution of

$$\begin{cases} u_{it} - k\Delta u_i = f_i(x,t) \\ u_i(x,t) = g_i(x,t), & x \in \partial\Omega, T \geq t > 0 \\ u_i(x,0) = \phi(x), & x \in \Omega \end{cases}$$

Then

$$\max_{\overline{\Omega_T}} |u_1 - u_2| \leq \max_{\overline{\Omega}} |\phi_1 - \phi_2| + \max_{\partial\Omega \times [0,T]} |g_1 - g_2| + T \max_{\overline{\Omega_T}} |f_1 - f_2|. \tag{50}$$

Proof. In class. **Hint:** use comparsion principle. □

Remark 4.2. *In applications, the structure components, i.e. the source term f, the boundary value g and the initial value ϕ are measured experimentally and hence are not given precisely. Equation (50) says that small errors in measuring these data result in a small error in the solution. Thus we have **stability**.*

Example 4.2 (Good Problem)**.** *Let Ω be a physical domain in $\mathbb{R}^n$, $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \Omega$. Suppose $u = u(\boldsymbol{x}, t)$ is a smooth solution of the following system*

$$\begin{cases} \partial_t u - a(\boldsymbol{x},t)\Delta u = b(\boldsymbol{x},t)|\nabla u|^2 - e^t, & (\boldsymbol{x},t) \in \Omega_T := \Omega \times (0,T]; \\ u(\boldsymbol{x},0) = \phi(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \\ u(\boldsymbol{x},t) = \psi(\boldsymbol{x},t), & \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{51}$$

(a). Suppose $a(\boldsymbol{x},t) > 0$ and $b(\boldsymbol{x},t) \equiv 0$ for $(\boldsymbol{x},t) \in \Omega_T$. Prove the weak maximum principle of the system. That is, prove

$$\max_{\Omega_T} u = \max_{\partial_T \Omega} u.$$

(b). If $-e^t$ is replaced by a general function $f(\boldsymbol{x},t)$, what condition should f satisfy in order that the proof in (a) still works?
(c). Does the result in (a) holds for the case $b \neq 0$? Explain.
(d). For the case $a(\boldsymbol{x},t) > 0, b \equiv 0$, prove that the solution to the system is unique.
(e). Suppose $a(x,t) \equiv a \geq 0$, $b(\boldsymbol{x},t) \leq a$, and $\psi \equiv 0$. Prove that the energy of u, the solution to the system, is non-increasing.

Weak maximum principle for Cauchy problem. Let $u(x,t) \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ and satisfies

$$\begin{cases} u_t - k\Delta u \le 0, & x \in \mathbb{R}^n, 0 < t < T \\ u(x,0) = \phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{52}$$

and satisfies

$$u(x,t) \le Ae^{a|x|^2} \tag{53}$$

for some constants $A, a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} \phi. \tag{54}$$

Proof. In class. □

From this maximum principle, it follows immediately **the uniqueness of solutions of the Cauchy problem with exponential growth rate**.

$$\begin{cases} u_t - k\Delta u = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$$

We mention that without the bounded growth rate condition, there is no uniqueness: the Cauchy problem has infinitely many solutions that grow extremely fast and have the 0 initial value.

Nonphysical Solutions. There are in fact infinitely many solutions of

$$\begin{cases} u_t - \Delta u = 0, & \mathbb{R}^n \times (0,T) \\ u = 0, & \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{55}$$

You can find instances in [Eva10]. Each of these solutions besides $u \equiv 0$ grows very rapidly as $|x| \to \infty$. There is an interesting point here: although $u \equiv 0$ is certainly the *physically correct* solution of (55), this initial-value problem in fact admits other nonphysical solutions. However, the theorem regarding the uniqueness of Cauchy problem helps eliminate some wrong solutions.

Theorem 4.1 (Smoothness)**.** *Suppose $u \in C_1^2(\Omega_T)$ solves the heat equation in Ω_T. Then*

$$u \in C^\infty(\Omega_T).$$

This regularity assertion is valid even if u attains nonsmooth boundary values on $\partial_T\Omega$.

Proof. This proof is not easy and thus we omit it here... □

Example 4.3. *Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2+y^2}$ is the cylinderical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$.*

Example 4.4. *Solve the above problem in a ball except that the temperature depends only on the spherical coordinate $\sqrt{x^2+y^2+z^2}$. Derive the equation $u_t = k(u_{rr} + 2u_r/r)$.*

Example 4.5. *Solve the following problems:*

1. *Show that the temperature of a metal rod, insulated at the end $x = 0$, satisfies the boundary condition $\partial u/\partial x = 0$.*
2. *Do the same for the diffusion of gas along a tube that is closed off at the end $x = 0$.*
3. *Show that the three-dimensional version of (a) and (b) leads to the boundary condition $\partial u/\partial n = 0$. (hint: $\boldsymbol{F} = -k\nabla u$)*

Example 4.6. *A homogeneous body occupying the solid region D is completely insulated. Its initial temperature is $f(x)$. Find the steady-state temperature that it reaches after a long time. (Hint: No heat is gained or lost)*

Solution. Use the formula you learned in high school $\Delta Q = cm\Delta t$, where c is the specific heat. At steady state $u_t = 0$ and thus by the diffusion equation $\Delta u = 0$. Since D is fully insulated, $\frac{\partial u}{\partial n} = 0$ on the boundary ∂D. Use the uniqueness that the Neumann problem up to constant, we know that u must be a constant.

4.2 Fundamental Solution

Definition 4.1 (Fundamental Solution of Heat Equation)**.**

$$\Phi(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right) & t > 0 \\ 0 & t \le 0 \end{cases} \tag{56}$$

We consider more generally a function defined as

$$G(x,t;\xi) := \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-\xi)^2}{4kt}\right), \quad t > 0. \tag{57}$$

where ξ is a parameter. Note that when setting $\xi = 0$, we get what we are familiar with. It can be directly verified that $G \in C^\infty$ w.r.t. x and t, and satisfies the homogeneous heat conduction equation for any ξ.

Properties of $G(x,t;\xi)$**.**

- $\frac{\partial^m G}{\partial x^m} \to 0, \quad x \to \infty, \quad t > 0, \xi \in (-\infty, \infty).$
- $\int_{-\infty}^{\infty} G(x,t;\xi)dx = 1, \quad t > 0, \xi \in (-\infty,\infty).$
- $\lim_{t\to 0^+} G(x,t;\xi) = 0 (x \neq \xi) / \infty (x = \xi).$
- $\lim_{t\to 0^+} \int_{\mathbb{R}} G(x,t;\xi) f(\xi) d\xi = f(x)$, where f is bounded on $\mathbb{R}$ and is continuous at x.

Proof. We will only prove the last property. Note that what we want to prove is

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} (f(\xi) - f(x)) G(x,t;\xi) d\xi = 0. \tag{58}$$

Introducing a new variable

$$\eta = \frac{\xi - x}{\sqrt{4kt}},$$

we rewrite the integral on the left hand side of (58) as

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} (f(x + \sqrt{4kt}\eta) - f(x)) d\eta. \tag{59}$$

Now it is clear that (58) directly follows from the *Lebesgue's Dominated Convergence Theorem* by considering $f_n(y) := \frac{1}{\sqrt{\pi}} e^{-y^2} (f(x + \sqrt{4kt_n}y) - f(x))$, where $\{t_n\} \to 0^+$ is arbitrary.

If the reader has not learned the Lebesgue theorem, then we offer a more elementary proof. Since f is bounded, we can find a constant M such that $|f(y)| \le M$ for any y. Since $e^{-\eta^2}$ is integrable over $\mathbb{R}$, for any $\epsilon > 0$, there exists a large L such that

$$\frac{1}{\sqrt{\pi}} \int_{|\eta| \ge L} e^{-\eta^2} d\eta \le \frac{\epsilon}{2M}. \tag{60}$$

Then

$$\frac{1}{\sqrt{\pi}} \left| \int_{|\eta|\ge L} e^{-\eta^2} (f(x + \sqrt{4kt}\eta) - f(x)) d\eta \right| \le 2M \frac{\epsilon}{2M} = \epsilon.$$

Since f is continuous at x, there exists a small $\tau > 0$ such that

$$|f(x + \sqrt{4kt}\eta) - f(x)| \le \epsilon$$

for any $0 < t < \tau$, $|\eta| \le L$. Hence,

$$\frac{1}{\sqrt{\pi}} \left| \int_{|\eta|\le L} e^{-\eta^2} (f(x + \sqrt{4kt}\eta) - f(x)) d\eta \right| \le \epsilon.$$

□

Infinite Propogation Speed. Consider the IVP

$$\begin{cases} u_t - k\Delta u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u = g, & (x,t) \in \mathbb{R}^n \times \{t=0\}. \end{cases}$$

If g is bounded, continuous, $g \geq 0$, and g is not 0, then

$$u(x,t) = \frac{1}{(4\pi kt)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{\|x-y\|^2}{4kt}} g(y)dy,$$

is in fact positive for *all* points and times $t > 0$. We interpret this observations by saying that the heat equation forces *infinite propagation speed* for disturbances. If the initial temperature is nonnegative and is positive somewhere, the temperature at any later time (no matter how small) is everywhere positive.

Below are three interesting questions about diffusion equation. After finishing Example 4.7 and 4.8, think about what is the connection to ODE.

Example 4.7 (Diffusion with Constant Dissipation)**.** *Solve the diffusion equation with constant dissipation*

$$u_t - ku_{xx} + bu = 0, \quad -\infty < x < \infty; \quad u(x,0) = \phi(x),$$

where $b > 0$ is a constant. (Hint: make the change of variables $u(x,t) = e^{-bt}v(x,t)$)

Example 4.8 (Diffusion with Variable Dissipation)**.**

$$u_t - ku_{xx} + bt^2u = 0, \quad -\infty < x < \infty; \quad u(x,0) = \phi(x),$$

where $b > 0$ is a constant. (Hint: make the change of variable $u(x,t) = e^{-bt^3/3}v(x,t)$)

Example 4.9 (Diffusion with Convection)**.** *Solve the diffusion equation with convection:*

$$u_t - ku_{xx} + Vu_x = 0, \quad -\infty < x < \infty; \quad u(x,0) = \phi(x),$$

where V is a constant. (Hint: consider $v(x,t) := u(x+Vt,t)$).

4.3 Energy Method

Consider the heat equation with initial condition and one of the three boundary conditions. Let Ω be a bounded region in $\mathbb{R}^n$ with piecewise C^1-smooth boundary curve $\partial\Omega$ (so the divergence theorem) applies

$$\begin{cases} u_t = k\Delta u + f(x,t), & x \in \Omega, t > 0 \\ u(x,0) = \phi(x), & x \in \Omega \\ \text{Boundray conditions on } \partial\Omega. \end{cases} \tag{61}$$

Now we define the *energy* of u to be

$$E[u](t) := \frac{1}{2}\int_\Omega |u(x,t)|^2 dx. \quad \text{(Dirichlet Energy)} \tag{62}$$

which is always positive, and decreasing, if u solves the heat equation. Indeed, differentiating the energy with respect to time, and using the heat equation we get

$$\frac{d}{dt}E[u] = \int_\Omega uu_t dx = k\int_\Omega u\Delta u dx = k\left[\int_{\partial\Omega} u\frac{\partial u}{\partial \boldsymbol{n}}dS - \int_\Omega \|\nabla u\|^2 dx\right].$$

Example 4.10. *Show that the solution to (61) is unique.*

Solution. Just as in the case of the wave equation, we argue from the inverse by assuming that there are two functions, u, and v, that both solve the inhomogeneous heat equation and satisfy the initial and boundary conditions of (61). Then their difference, $w = u - v$, satisfies the homogeneous heat equation with zero initial-boundary conditions. Note that

$$\frac{d}{dt}E[w] = k\left[\int_{\partial\Omega} w\frac{\partial w}{\partial \boldsymbol{n}}dS - \int_{\Omega}\|\nabla w\|^2 dx\right] = -k\int_{\Omega}\|\nabla w\|^2 dx \leq 0$$

Note also

$$E[w](0) = 0,$$

by the initial condition. Therefore $0 \leq E[w](t) \leq E[w](0) = 0$. Thus, (some steps are skipped) $w = u - v = 0$. □

Theorem 4.2 (Backward Uniqueness*). *Suppose $u, v \in C^2(\overline{\Omega_T})$ both satisfy*

$$\begin{cases} u_t - k\Delta u = 0, & \text{in } \Omega_T \\ u = g, & \text{on } \partial\Omega \times [0, T] \end{cases} \tag{63}$$

for some function g. If

$$u(x, T) = v(x, T), x \in \Omega$$

then $u \equiv v$ within Ω_T.

Remark 4.3. *The theorem implies that if two temperature distributions on Ω agree at some time $T > 0$ and have had the same boundary values for times $0 \leq t \leq T$, then these temperatures must have been identically equal within Ω at all earlier times. This is not at all obvious.*

Proof. In class. (Why we cannot directly apply the strong MP?) □

4.4 Exercises

Problem 4.3. *Prove the comparision principle for the diffusion equation: if u and v are two solutions, and if $u \leq v$ for $t = 0, x = 0$, and $x = l$, then $u \leq v$ for $0 \leq t < \infty, 0 \leq x \leq l$.*

Problem 4.4. *More generally, if $u_t - ku_{xx} = f$, $v_t - kv_{xx} = g$, $f \leq g$, and $u \leq v$ at $x = 0, x = l$, and $t = 0$, prove that $u \leq v$ for $0 \leq x \leq l, 0 \leq t < \infty$. (Hint: the proof of Maximum Principle is still valid if $u_t - ku_{xx} \leq 0$.)*

Problem 4.5. *If $v_t - v_{xx} \geq \sin x$ for $0 \leq x \leq \pi, 0 < t < \infty$, and if $v(0, t) \geq 0$, $v(\pi, t) \geq 0$, and $v(x, 0) \geq \sin x$, use the above problem to prove that $v(x, t) \geq (1 - e^{-t})\sin x$.*

Problem 4.6. *Solve the diffusion equation with the initial condition*

$$\phi(x) = 1, \quad |x| < l; \qquad \phi(x) = 0, \quad |x| > l.$$

4.5 Probabilistic Perspective: Heat, Random Walk and CLT*

Central Limit Theorem. [4]Let $X_1, X_2, \ldots$ be independently and identically distributed random variables with mean μ and variance σ^2. Let

$$Z_n := \frac{\frac{1}{n}\sum_{i=1}^n X_i - \mu}{\sigma/\sqrt{n}}$$

As $n \to \infty$, the probability distribution of Z_n increasingly resembles a standard normal distribution $\mathcal{N}(0, 1)$.

Diffusion is the continuum limit of a bunch of random walkers. Instead of describing the location of a single walker with a probability or the positions of a distribution of walkers, we consider the number density of walkers as a function of position and time $\rho(x, t)$ and see how that evolves.

Flux is numbers of walkers going through a surface per unit area on the surface per unit time. The

[4] http://astro.pas.rochester.edu/ aquillen/phy256/lectures/Diffusion_walks.pdf

flux of walkers depends on the gradient of random walkers. If all walkers start at the same spot, their density distribution spreads out. If there are more walkers in one spot than there are in a neighboring spot, more walkers will move from the denser area than will move from sparser area to the denser area. The flux of walkers depends on the gradient of walkers. The rate that the density of walkers changes

$$\frac{\partial \rho}{\partial t}$$

The gradient of walkers

$$\nabla \rho$$

The flux of walkers we assume depends on the gradient of the density of walkers

$$\boldsymbol{F} = -k\nabla \rho$$

The walkers should move from regions of high density to low density, and this gives the minus sign. The coefficient k determine how large a step the walkers take each time step. The change in density depends on the gradient of the flux

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{F}$$

If the coefficient k is independent of position then we can write this as

$$\frac{\partial \rho}{\partial t} = k\Delta \rho \tag{64}$$

This gives a partial differential equation for the evolution of the density of walkers $\rho(x,t)$. Equation (64) is a continuum equation, describing diffusion. It is sometimes called the *heat equation* as heat transfer is often described with the same equation.

Using variables that have moments, the central limit theorem states that the distribution of their sum will resemble a Gaussian distribution and the standard deviation of the Gaussian will be proportional to $\sqrt{N}$ where N is the number of variables used in the sum. A random walk is the sum of random variables. A ensemble of walkers that starts all in the same spot expands with standard deviation proportional to $\sqrt{N}$ where N is the number of steps. This resembles the *fundamental solution* for the heat or diffusion equation. A distribution of walkers will obey the heat/diffusion equation.

5 Tutorial 5

5.1 Reflection Methods

Diffusion on The Half-Line. Consider the domain $\Omega = (0, \infty)$ and take the *Dirichlet boundary condition* at the single point $x = 0$. So the problem is

$$\begin{cases} v_t - kv_{xx} = 0, & 0 < x < \infty, 0 < t < \infty \\ v(x,0) = \phi(x) \\ v(0,t) = 0 \end{cases} \tag{65}$$

We want to use the representation formula we have derived in class. The idea is to perform **odd extension** on $\phi(x)$. That is,

$$\phi_{odd}(x) \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \\ 0 & x = 0 \end{cases} \tag{66}$$

Then we can solve the diffusion equation on the whole line with initial data ϕ_{odd}. The solution automatically satisfies the boundary condition because the solution is *odd* (see Example 5.1). If we restrict the solution to the positive half-line, we get the solution.

$$u(x,t) = \int_0^\infty [\Phi(x-y,t) - \Phi(x+y,t)]\phi(y)dy.$$

For the Neumann problem, we just do even extension and the solution is given as

$$u(x,t) = \int_0^\infty [\Phi(x-y,t) + \Phi(x+y,t)]\phi(y)dy.$$

The derivation is as follows. Note that by d'Alemberts' formula

$$\begin{aligned} u(x,t) = \int_{\mathbb{R}} \Phi(x-y,s)\phi_{even}(y)dy &= \int_0^\infty \Phi(x-y,t)\phi(y)dy + \int_{-\infty}^0 \Phi(x-y,t)\phi(-y)dy \\ &= \int_0^\infty [\Phi(x-y,t) + \Phi(x+y,t)]\phi(y)dy. \end{aligned}$$

Example 5.1. *Consider the diffusion equation on the whole line with the usual initial condition $u(x,0) = \phi(x)$. If $\phi(x)$ is an odd (even resp.) function, show that the solution $u(x,t)$ is also an odd (even) function of x.*

Proof Sketch. I only prove the odd case. Note that $-u(-x,t)$ is also a solution to the IVP. Since the solution to the IVP is unique (why?). QED! □

Example 5.2. *Consider the following problem with a Robin boundary condition:*

$$\begin{aligned} &u_t = ku_{xx}, \quad 0 < x < \infty, 0 < t < \infty \\ &u(x,0) = x \\ &u_x(0,t) - 2u(0,t) = 0, \quad x = 0. \end{aligned} \tag{67}$$

Let $f(x) = x, x > 0$ and $f(x) = x + 1 - e^{2x}, x < 0$, and let

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^\infty e^{-(x-y)^2/4kt} f(y)dy.$$

Show that $v(x,t)$ satisfies (67) for $x > 0$. Assuming uniqueness, deduce the solution of (67) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^\infty e^{-(x-y)^2/4kt} f(y)dy.$$

Proof Sketch. First show that $v(x,t)$ satisfies the heat equation with initial condition f; show that $w := v_x - 2v$ satisfies the heat equation with initial condition $f' - 2f$; show that $f' - 2f$ is odd; show that w is an odd function; show that v satisfies the original boundary condition. See the tutorial 5 slides for a detailed solution. □

Remark 5.1. *You are encouraged to generalize the idea discussed here to solve the problem 5 on page 61 of [Str07].*

Wave on The Half-Line. Consider the *Dirichlet problem* on the half-line:

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, \quad 0 < x < \infty, -\infty < t < \infty \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x) \\ v(0,t) = 0 \end{cases} \tag{68}$$

The *reflection method* is carried out similarly. We do odd extension on both ϕ and ψ. Again, by Example 3.3, we know that the solution to the wave equation on the whole line with initial data ϕ_{odd} and ψ_{odd} is itself odd. Then the boundary condition is automatically satisfied. Thus, the solution formula is

$$v(x,t) = \begin{cases} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds, \quad x > c|t| \\ \frac{1}{2}[\phi(ct+x) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(s)ds, \quad 0 < x < c|t| \end{cases}$$

For Neumann problem, we just apply even extension to ϕ and ψ, the solution formula is

$$v(x,t) = \begin{cases} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds, \quad x > c|t| \\ \frac{1}{2}[\phi(ct+x) + \phi(ct-x)] + \frac{1}{2c}\left[\int_0^{ct+x} \psi(s)ds + \int_0^{ct-x} \psi(s)ds\right], \quad 0 < x < c|t| \end{cases}$$

The derivation for the case $x < c|t|$ is as follows, by d'Alembert's formula, we have

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\phi_{even}(x-ct) + \phi_{even}(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(s)ds \\ &= \frac{1}{2}[\phi(ct+x) + \phi(ct-x)] + \int_{x-ct}^{0} \psi(-s)ds + \frac{1}{2c}\int_0^{x+ct} \psi(s)ds \\ &= \frac{1}{2}[\phi(ct+x) + \phi(ct-x)] + \frac{1}{2c}\left[\int_0^{ct-x} + \int_0^{ct+x} \psi(s)ds\right]. \end{aligned}$$

Example 5.3. *Solve the following wave equation on the half-line*

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0, \quad t \in \mathbb{R}, x > 0 \\ u(x,0) = \phi(x), \quad \partial_t u(x,0) = \psi(x) \\ u(0,t) = e^t \end{cases} \tag{69}$$

You might need the knowledge from the next section...

Solution. Set $v(x,t) := u(x,t) - e^t$, then v satisfies the following system

$$\begin{cases} \partial_t^2 v - c^2 \partial_x^2 v = -e^t \\ v(x,0) = \phi(x) - 1; \quad v_t(x,0) = \psi(x) - 1 \\ v(0,t) = 0 \end{cases}$$

Do odd extension on the inhomogeneous term and the initial data and apply the reflection method we immediately have the solution

$$u(x,t) = e^t + \frac{1}{2}[\phi(ct+x) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} (\psi(s)+1)ds + \iint_D -e^s dyds, \quad 0 < x < c|t|.$$

Please write down the solution formula for the case $x > c|t|$ by yourself. □

THINK: What is the solution formula to the above problem if the initial condition is replaced by $v(x, t_0) = \phi(x) - 1, v_t(x, t_0) = \psi(x) - 1$.

5.2 Inhomogeneous Wave/Heat Equations

Diffusion with a Source. In this section we consider the *inhomogeneous* diffusion equation on the whole line,

$$\begin{cases} u_t - k u_{xx} = f(x,t), \quad -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi(x) \end{cases} \tag{70}$$

where $f(x,t)$ and $\phi(x)$ are given functions. We will show that the solution formula is

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\phi(y)dy + \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)f(y,s)dyds. \tag{71}$$

Let us give the motivation. Consider a first order linear ODE, the unknown function $u : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function.

$$\frac{du}{dt} + Au(t) = f(t), \quad u(0) = \boldsymbol{\phi}. \tag{72}$$

We can solve the equation by integrating factor. The solution is given by

$$u(t) = e^{-tA}\boldsymbol{\phi} + \int_0^t e^{(s-t)A} f(s)ds. \tag{73}$$

Set $S(t) := e^{-tA}$, then the above becomes

$$u(t) = S(t)\boldsymbol{\phi} + \int_0^t S(t-s)f(s)ds. \tag{74}$$

The equation (74) gives us some insight on how to solve inhomogeneous equation given we can solve the corresponding homogeneous equation. It is called the *Duhamel's Principle*. Since the solution formula to the homogeneous diffusion equation is

$$\int_{\mathbb{R}} \Phi(x-y,t)\phi(y)dy := (\mathcal{S}(t)\phi)(x),$$

here, $\mathcal{S}(t)$ is called the *source operator* which sends ϕ to $\int_{\mathbb{R}} \Phi(x-y,t)\phi(y)dy$. Then, according to (74), we *guess* the solution to diffusion equation with a source is

$$u(x,t) = \mathcal{S}(t)[\phi] + \int_0^t \mathcal{S}(t-s)[f(s,t)]ds. \tag{75}$$

It can be checked by direct computation that the $u(x,t)$ given above indeed solve the inhomogeneous diffusion equation. Since we know that the solution is unique (why?). We are able to safely conclude that $u(x,t)$ is the *unique* solution to the inhomogeneous diffusion equation. For readers who are interested, please refer to p.49 of [Eva10].

Here we only check that (75) satisfies the initial conditition. For checking that it satisfies the PDE, please read P.69 of [Str07]. Note that

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \int_{\mathbb{R}} \Phi(x-y,t)\phi(y)dy.$$

By **Weak Convergence** to the dirac-δ of the fundamental solution (as we have shown in 4.2), we know that $\lim_{t\to 0^+} u(x,t) = \phi(x)$. $\square$

Wave with a Source. Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \phi(x); \quad u_t(x,0) = \psi(x) \end{cases} \tag{76}$$

where f, ϕ, ψ are all given. $f(x,t)$ can be interpreted as an *external force* acting on an infinitely long vibrating string. We can without loss of generality assume that both $\phi = 0$ and $\psi = 0$ since it is easy to see that we can split the system into two, one is homogeneous with initial data while the other one is inhomogeneous with homogeneous initial data. We can solve the former one explicitly so we only need to take care of the latter one. We claim that the solution is given by

$$\frac{1}{2c}\iint_{\Delta} f = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds. \tag{77}$$

Section 3.4 in [Str07] gives three different derivations to the solution formula (I think the one using *Green's Theorem* is interesting). Here, we only discuss the Duhamel's principle. Note that $\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$ solves the PDE

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,s) = 0; \quad u_t(x,s) = f(x,s). \end{cases}$$

[Please think about how to solve the above IVP...]. Then

$$\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{1}{2c} f(y,s)dyds$$

solves

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = 0; \quad u_t(x,0) = 0 \end{cases}$$

If one can check that $\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{1}{2c} f(y,s)dyds$ indeed satisfies the PDE and initial data, then we are done because the solution is unique. $\square$

Diffusion with a Source on a Half-Line. For inhomogeneous diffusion on the half-line we can also apply the *reflection method* discussed before.

Consider the Dirichlet problem on the half-line

$$\begin{cases} v_t - kv_{xx} = f(x,t), \quad 0 < x < \infty, 0 < t < \infty \\ v(0,t) = h(t), \\ v(x,0) = \phi(x) \end{cases} \tag{78}$$

By setting $V(x,t) := v(x,t) - h(t)$, the boundary condition becomes homogeneous. Under this circumstance, we could apply reflection method to solve for V and then recover $v(x,t) = V(x,t) + h(t)$.

For the inhomogeneous *Neumann* problem on the half-line,

$$\begin{cases} w_t - kw_{xx} = f(x,t), \quad 0 < x < \infty, 0 < t < \infty \\ w_x(0,t) = h(t), \\ w(x,0) = \phi(x) \end{cases} \tag{79}$$

By setting $W(x,t) = w(x,t) - xh(t)$, the problem can be solved. $\square$

Wave with a Source on a Half-Line. Consider the general inhomogeneous problem on a half-line

$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x,t), \quad 0 < x < \infty \\ v(x,0) = \phi(x); \quad v_t(x,0) = \psi(x) \\ v(0,t) = h(t) \end{cases} \tag{80}$$

we can WLOG assume that $f = \phi = \psi = 0$ since we have already understood them.

Example 5.4. *Show by direct substitution that $u(x,t) = h(t - \frac{x}{c})$ for $x < ct$ and $u(x,t) = 0$ for $x \geq ct$ solves the homogeneous wave equation on the half-line with zero initial data and boundary condition $u(0,t) = h(t)$.*

Let's derive the boundary term $h(x - \frac{x}{c})$ for $x < ct$. Since the general solution to the wave equation is $v(x,t) = j(x+ct) + g(x-ct)$. From the initial condition $\phi = \psi = 0$, we have $j' = g' = 0$ and $j = -g$. Therefore, $j(s) = -g(s) = C \in \mathbb{R}$. Since $v(0,t) = h(t)$, we thus have $j(ct) + g(-ct) = h(t)$. Then, $h(t/c) = j(t) + g(-t)$. Finally, we have $v(x,t) = j(x+ct) + g(x-ct) = j(ct-x) + g(x-ct) = h((ct-x)/c) = h(t - x/c)$. $\square$

Example 5.5. *Solve the completely inhomogeneous diffusion equation on the half-line*

$$
\begin{aligned}
&v_t - kv_{xx} = f(x,t), \quad 0 < x < \infty, 0 < t < \infty \\
&v(0,t) = h(t), \quad v(x,0) = \phi(x),
\end{aligned}
\tag{81}
$$

Remark 5.2. *You are encouraged to solve problem 3 on page 71 of [Str07].*

Example 5.6. *Derive the solution of the fully inhomogeneous wave equation on the half-line*

$$
\begin{aligned}
&v_{tt} - c^2 v_{xx} = f(x,t), \quad 0 < x < \infty, \\
&v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), \\
&v(0,t) = h(t).
\end{aligned}
\tag{82}
$$

5.3 Exercises

Problem 5.1. *Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:*

$$
\begin{cases}
u_t - ku_{xx} = f(x,t), \quad 0 < x < \infty, 0 < t < \infty, \\
u(0,t) = 0; \quad u(x,0) = \phi(x).
\end{cases}
$$

Using the method of reflection.

Problem 5.2. *Solve the Neumann diffusion equation on the half-line*

$$
\begin{cases}
w_t - kw_{xx} = 0, \quad 0 < x < \infty, 0 < t < \infty, \\
w_x(0,t) = h(t); \quad w(x,0) = \phi(x).
\end{cases}
$$

By the subtraction method.

5.4 Duhamel's Principle: Intuitive Explanation

The core idea of Duhamel's Principle is that *if one can solve the homogeneous equation, then one can also solve the inhomogeneous equation by aggregating the initial information*[5].

Let me do the ODE version first: Let $X : \mathbb{R} \to \mathbb{R}^n$ be a vector-valued function of one (time) variable. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation that is *independent* of the time variable t. Let $\mathcal{R}_t$ denote the solution operator, which is a mapping that maps a vector $X_0 := X(0)$ to a function $X(t) = \mathcal{R}_t X_0$ that solves the ODE. That is,

$$\frac{d}{dt} X(t) = AX.$$

with the initial condition $X(0) = X_0$. Now let $Y(t)$ be another vector valued function, consider the following expression

$$X(t) = \int_0^t \mathcal{R}_{t-s} Y(s) ds.$$

Computing it explicitly we have that

$$\left(\frac{d}{dt} - A\right) X(t) = \int_0^t \left(\frac{d}{dt} - A\right) [\mathcal{R}_{t-s} Y(s)] ds + \mathcal{R}_{t-s} Y(s)\Big|_{s=t}.$$

The second term comes from the fundamental theorem of calculus when the $\frac{d}{dt}$ derivative hits the integral sign.

By definition the term under the integral sign evaluates to 0, since $Z(t) := \mathcal{R}_{t-s} Y(s)$ solves the homogeneous equation with initial equation with initial data $Z(s) = Y(s)$. So we are left with

$$\left(\frac{d}{ds} - A\right) X(t) = \mathcal{R}_0 Y(t) = Y(t).$$

[5] For more details, please refer to MathStackExchange

That is, $X(t) = \int_0^t \mathcal{R}_{t-s} X(s) ds$ solves the inhomogeneous equation.

For the PDE version, you just replace $\mathbb{R}^n$ with a Banach or Hilbert space, and the computation formally carries through in exactly the same way.

imagine you have a family of initial data $\{Y(s)\}_{s\in[a,b]}$. And you write down the expression

$$X(t) = \int_a^b \mathcal{R}_{t-s} Y(s) ds.$$

which would be what we do if we were to just add (integrate) the contributions from all of the linear waves coming from the "inhomogeneity", by linearity it is clear that $X(t)$ will still solve the homogeneous equation, since it is a fixed (as in the limits of the integral) sum of many solutions.

The magic of Duhamel's principle is in that the upper-limit of the integral is time! That is, we defined

$$x(t) = \int_0^t \mathcal{R}_{t-s} Y(s) ds.$$

As you see from the derivation above, it is this upper-limit which, when acted on using the fundamental theorem of calculus, give you the inhomogeneous term. So what is the physical interpretation then? That the upper-limits also changes represents the fact that the solution at $X(t + \Delta t)$ consists of the forward time evolution of the solution at $X(t)$ plus a new contribution from the data in $(t, t + \Delta t)$ which was not included in the computation in $X(t)$. This "adding a new contribution" is precisely what we imagine the inhomogeneous term as, that is, a source term!

Example 5.7. *Try to use the framework discussed above to understand the solution formula for the heat and wave equations presented in the lectures.*

6 Tutorial 6

6.1 Separation of Variables

In the preceding lectures and tutorials, methods about how to solve heat and wave equations on the *whole real line / half-line* were covered. What about the equations are restricted to a *finite domain*?

The method of *separation of variables* is to consider a *separated solution* of the form $u(x,t) := X(x)T(t)$. Substitute the separated solution into the PDE then together with boundary conditions we can construct *eigenvalue problems*. Write down the solution in an infinite series form, we finally determine the coefficients by results in Fourier series.

The Dirichlet Condition. For the homogeneous wave equation, after substituting the separated solution, we have

$$\begin{cases} X'' + \beta^2 X = 0; \quad X(0) = 0 = X(l) \\ T'' + c^2\beta^2 T = 0. \end{cases} \tag{83}$$

Solve the above two equations, we have

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2; \quad X_n(x) = \sin\frac{n\pi}{l}; \quad T_n(t) = A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}. \tag{84}$$

Since any finite sum of $u_n(x,t) := (A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l})\sin\frac{n\pi x}{l}$ solves the BVP. It is natural to consider the infinite sum of those u_n's. (If the series converges and can be differentiated) It still solves the PDE. In order to let it satisfy the B.C.'s. We need to require

$$\begin{aligned} \phi(x) &= \sum_n A_n \sin\frac{n\pi x}{l}. \\ \psi(x) &= \sum_n \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l}. \end{aligned} \tag{85}$$

Do those series converge? How to determine these coefficients A_n and B_n. We would solve the questions later.
What about if we only use the *finite* sum? Then a lot of ϕ cannot be represented. For example, if $\phi(x) \equiv 1$ and we only consider $A_1\sin(\frac{\pi}{l}x)$, $\phi \neq A_1\sin(\frac{\pi}{l}x)$, no matter what A_1 you choose.

For homogeneous diffusion equation, the strategy is almost the same except we have instead

$$T' = -\beta^2 kT.$$

Thus,

$$T_n(t) = A_n\exp(-(n\pi/l)^2kt).$$

Therefore, the solution to the BVP is

$$u(x,t) = \sum_n A_n\exp(-(n\pi/l)^2kt)\sin\frac{n\pi x}{l}. \tag{86}$$

To satisfy the boundary condition, we need also

$$\phi(x) = \sum_n A_n\sin\frac{n\pi x}{l}. \tag{87}$$

THINK: What is the relation between the solutions given by separation of variables with the solutions given by d'Alemberts' formula and the fundamental solution?

The Neumannn Condition. The same method works for both the Neumann and Robin (a little bit more complicated, please refer to section 4.3 of [Str07]) boundary conditions. Now, the boundary conditions are replaced by $u_x(0,t) = u_x(l,t) = 0$. Then, the eigenvalue problem becomes

$$\begin{cases} X'' = -\lambda X \\ X'(0) = X'(l) = 0. \end{cases} \tag{88}$$

We claim that the eigenvalues are always *nonnegative* and set $\lambda = \beta^2 \geq 0$ (see Prob. 6.13). This fact will be proved later. We first search for eigenfunctions when $\lambda > 0$. Since $X(x) = C\cos\beta x + D\sin\beta x$, by boundary conditions we directly have $D = 0$ and $\beta = \frac{n\pi}{l}, n \in \mathbb{N}$. Next, suppose $\lambda = 0$, then $X(x) = Cx + D$, again by boundary conditions we have $D = 0$ and C can be any constant. Therefore, $\lambda = 0$ is an eigenvalue and any constant function except 0 is its eigenfunction. To sum up, the eigenvalues are $(\frac{n\pi}{l})^2, n \in \mathbb{N} \cup \{0\}$ and eigenfunctions are $X_0(x) = C$ and $X_n(x) = \cos\frac{n\pi x}{l}, (n \geq 1)$. Based on the discussion above, we can immediately write down the solution formulas to the homogeneous heat and wave equations with Neumann boundary conditions

$$\begin{aligned} u(x,t) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \exp\left((-n\pi/l)^2 kt\right)\cos\frac{n\pi x}{l}. \\ u(x,t) &= \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\cos\frac{n\pi x}{l}. \end{aligned} \tag{89}$$

Note that if initial conditions are given, we can determine the coefficients A_n and B_n by integration.

Robin Boundnary Condition. Please read section 4.3 of [Str07].

Mixed Boundary Condition. A mixed boundary condition would be Dirichlet at one end and Neumann at the other. For instance, if $u(0,t) = u_x(l,t) = 0$. Then the eigenvalues are $(n+\frac{1}{2})^2\pi^2/l^2, n \in \mathbb{N}$ and corresponding eigenfunctions are $\sin((n+\frac{1}{2})\pi x/l)$.

Example 6.1. *Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0,t) = u(l,t) = 0$. Show that the eigenfunctions are $\cos((n+\frac{1}{2})\pi x/l)$ and write down the series expansion of the solution $u(x,t)$.*

Solution. Note that the eigenvalue problem is

$$\begin{cases} X'' = -\lambda X \\ X(l) = 0; \quad X'(0) = 0 \end{cases}$$

Solve the eigenvalue problem then the problem is done. □

Periodic Boundary Condition. Consider the periodic boundary condition

$$\begin{cases} u(-l,t) = u(l,t) \\ u_x(-l,t) = u_x(l,t) \end{cases}$$

Then the eigenvalue problem becomes

$$\begin{cases} X''(x) = -\lambda X(x) \\ X(-l) = X(l) \\ X'(-l) = X'(l) \end{cases}$$

It can still be proven that $\lambda \geq 0$ (see Prob. 6.13). Note that $X_n(x) = A_n\cos(\frac{n\pi}{l}x) + B_n\sin(\frac{n\pi}{l}x)$ and $T_n(t) = \exp(-k(\frac{n\pi}{l})^2 t)$ (we illustrate by using the heat equation here). Thus, the final solution is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n\cos(\frac{n\pi}{l}x) + B_n\sin(\frac{n\pi}{l}x))\exp(-k(\frac{n\pi}{l})^2 t).$$

Note that the part in red is a full Fourier series.

Example 6.2. *Please derive the above solution formula for periodic boundary condition by yourself and finish prob.4 in p.92 of [Str07].*

Example 6.3. *Let $K > 0$ be a positive constant. Consider the following initial boundary value problem for the heat equation*

$$\begin{cases} \partial_t u = K\partial_x^2 u, \quad x \in (0,2), t > 0 \\ u(x,0) = f(x) \\ u(0,t) = u(2,t), u_x(0,t) = u_x(2,t). \end{cases} \tag{90}$$

(a). Use separation of variables to derive the corresponding eigenvalue problem.
(b). Show that the eigenvalues are sign-determined and then solve the eigenvalue problem.
(c). Find the series solution to the system if $f(x) = x^2$.

Hint:
(a). Formulate the eigenvalue problem. Note that the boundary condition is a periodic one.
(b). see Prob. 6.13.
(c). Use Fourier theory. □

Summary. The following is a summary of the separation of variables:

1. Let $u(x,t) = X(x)T(t)$. Derive the eigenvalue problem for $X(x)$ from PDE and BCs
2. Find the eigenvalues λ_n and eigenfunctions $X_n(x)$ from the eigenvalue problem
3. Find $T_n(t)$ for each λ_n from the equation derived in step 1.
4. Set
$$u(x,t) = \sum_{n=1}^{\infty} A_n T_n(t) X_n(x).$$
Find the coefficients A_n by the initial condition(s).

6.2 Eigenvalue problems: Sturm-Liouville theory and eigen-expansion

Theorem 6.1 (Orthogonality)**.** *Suppose that λ_m and λ_n ($\lambda_m \neq \lambda_n$) are two eigenvalues of the following general eigenvalue problem.*

$$\begin{cases} X''(x) + \lambda X(x) = 0, \quad a < x < b \\ c_1 X'(a) + c_2 X(a) = 0, \\ d_1 X'(b) + d_2 X(b) = 0. \end{cases} \tag{91}$$

where c_1, c_2 and d_1, d_2 are not all zeros. Suppose X_m and X_n are eigenfunctions corresponding to λ_m and λ_n resp. Then the eigenfunctions are orthogonal, i.e.

$$\int_a^b X_m X_n dx = 0, \text{ or simply, } \langle X_m, X_n \rangle = 0.$$

Proof Hint. Use $X'' = -\lambda X$ and integration by parts. □

Remark 6.1. *Note that Dirichlet, Neumann, and Robin B.C.'s are all of the form of the boundary conditions in (91). Indeed, they can be generally said to be **symmetric boundary conditions**. (please refer to p.119 of [Str07])*

Theorem 6.2 (Sturm-Liouville)**.**

1. *The eigenvalues of (91) are real and form an increasing and diverging sequence*
$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty.$$
2. *for each n, the eigenspace V_n corresponding to the eigenvalue λ_n is one-dimensional.*

We emphasize that only when the boundary conditions are given in the form in (91) will the eigenspace V_n has dimension one. For instance, if the boundary conditions are *periodic*, then all except the first eigenspaces are two-dimensional. Sturm-Liouville theorem also holds when X'' is replaced more generally by $(a(x)X')' + c(x)X$.

Suppose we have obtained all eigenvalues $\{\lambda_n\}$ and their corresponding eigenfunctions $\{X_n\}$. To solve the heat equation $u_t = ku_{xx}$ with the B.C. in (91), we form

$$u_n(x,t) = \phi_n \exp(-\lambda_n kt) X_n(x).$$

To satisfy the initial condition, we construct

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n \exp(-\lambda_n kt) X_n(x).$$

Formally, to satisfy the initial condition, we choose ϕ_n such that

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n X_n(x). \tag{92}$$

Then by Theorem 6.1, we deduce

$$\int_a^b \phi(x) X_m(x) dx = \phi_m \int_a^b (X_m(x))^2 dx, \ \forall m.$$

Thus the coefficients are given by

$$\phi_n = \frac{\langle \phi, X_n \rangle}{\langle X_n, X_n \rangle}.$$

The right hand side of (92) with ϕ_n given above is called the **generalized Fourier series** or **the eigenexpansion** of ϕ. There is a subtle point in the above discussion: what we have actually shown is that if the right hand side of (92) converges to ϕ, then ϕ_n is of the abovementioned form. Therefore, we need to answer the question, under what conditions will the generalized Fourier Series of ϕ converges? The problem will be addressed later in this course. If you are interested in those theoretic guarantees, you are encouraged to read Section 5.4 and 5.5 in the textbook [Str07].

6.3 Eigenvalue Problems in Linear Algebra*

This section is optional. Just for those who want to connect knowledge learned.

In this section, we will assume all the vector spaces are finite-dimensional. V is an inner product space.

Definition 6.1 (Adjoint Operator). *Let $T : V \to V$ be a linear operator, the adjoint operator $T^* : V \to V$ is defined as for $\forall v, w \in V$*

$$\langle T^* v, w \rangle = \langle v, Tw \rangle.$$

Definition 6.2 (Self-adjoint Operator). *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, we say that $T : V \to V$ is self-adjoint if $T^* = T$.*

Example 6.4. *$V = \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the dot product. The linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ defined to be $Tx = Ax$ is self-adjoint iff $A = A^T$. If $V = \mathbb{C}^n$, then T is self-adjoint iff $A = \bar{A}^T$.*

As you can see, the abstract definition of self-adjoint operator, when restricted to the simple Euclidean space, shows us that real symmetric matrices indeed are special examples.

Proposition 6.3. *Let $T : V \to V$ be self-adjoint, then all the eigenvalues of T are real.*

Proof. Suppose $Tv = \lambda v, \lambda \in \mathbb{C}$ (note that we are not showing the existence of eigenvalue, what we want to show is if there is one eigenvalue, it must be real.)

$$\bar{\lambda} \langle v, v \rangle = \langle \lambda v, v \rangle = \langle v, T^* v \rangle = \langle v, Tv \rangle = \lambda \langle v, v \rangle.$$

Therefore, $\bar{\lambda} = \lambda$. □

Proposition 6.4. *Let $T : V \to V$ be self-adjoint and $Tv = \lambda v$, $Tw = \mu w$; $(\lambda \neq \mu)$, then*

$$\langle v, w\rangle = 0.$$

Proof. note that

$$\begin{aligned}\lambda\langle v, w\rangle &= \bar{\lambda}\langle v, w\rangle = \langle Tv, w\rangle\\ \mu\langle v, w\rangle &= \bar{\mu}\langle v, w\rangle = \langle v, Tw\rangle = \langle Tv, w\rangle.\\ &\Rightarrow (\lambda - \mu)\langle v, w\rangle = 0. \Rightarrow \langle v, w\rangle = 0.\end{aligned}$$

Remark. Do you see the similarity of the proposition to theorem 6.1?

Proposition 6.5. *All self-adjoint operators have an eigenvalue.*

Proof Sketch. For $\mathbb{C}$-space, we even do not need self-adjointness since the characteristic polynomial always has a root in the $\mathbb{C}$. However, for real space, we can do complexfication thus at least one eigenvalue is guaranteed. Since the operator is self-adjoint. The eigenvalue must be real. ☐

Theorem 6.6. *Let V be a finite-dimensional inner product space (Hilbert space), and $T : V \to V$ is self-adjoint, then T has an orthonormal basis consisting of eigenvectors. (i.e., T is diagonalizable)*

Proof. To prove this theorem, some other pre-knowledge is needed. If you are interested, please refer to [RAG05]. ☐

6.4 Eigenvalue Problems in Functional Analysis*

THIS SECTION IS OPTIONAL. JUST FOR THOSE WHO WANT TO CONNECT KNOWLEDGE LEARNED.

In this section, we will not merely restrict ourselves to *finite-dimensional* spaces. Instead, we also care about infinite-dimensional spaces, which is the main topic of functional analysis. We would show (not in detail) the spectral theory for *compact* and *self-adjoint* operators in a *separable* Hilbert space. The content of this section is mainly taken from [Kre91].

Definition 6.3 (Hilbert Space)**.** *A complex vector space equipped with an inner-product is called an inner product space. A complete (with respect to the induced norm) inner product space is called a Hilbert space.*

Example 6.5. *Consider the space of all continuous (real-valued) functions on the interval $[a, b]$ equipped with the inner product*

$$\langle f, g\rangle := \int_a^b f \cdot g.$$

You can check that this indeed satisfies the axioms of an inner product. However, it is NOT a Hilbert space since it is NOT complete. (The completion is $L^2[a, b]$, if you know measure theory...)

Theorem 6.7 (Riesz' Representation Theorem)**.** *Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely*

$$f(x) = \langle x, z\rangle$$

where z depends on f and has the norm $\|z\| = \|f\|$.

Our next goal is a slight generalization of the Riesz representation theorem which will allow us to define the notion of the *Hilbert adjoint.*

Definition 6.4 (Sesquilinear Form)**.** *Let X and Y be vector spaces over $\mathbb{K}$. Then a sesquilinear form h on $X \times Y$ is a mapping*

$$h : X \times Y \to \mathbb{K}$$

such that for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ and all scalars $\alpha, \beta \in \mathbb{K}$ we have h is linear in the first argument and conjugate linear in the second argument.

When X and Y are normed we can talk about bounded sesquilinear forms: h is said to be *bounded* if there exists a $c \in \mathbb{R}$ such that

$$|h(x,y)| \leq c\|x\|\|y\|.$$

For a bounded sesquilinear form h we define the norm of h to be

$$\|h\| := \sup_{0 \neq x \in X, 0 \neq y \in Y} \frac{|h(x,y)|}{\|x\|\|y\|} = \sup_{\|x\|=1, \|y\|=1} |h(x,y)|.$$

Example 6.6. *Show that the usual inner product is a bounded sesquilinear form.*

The next theorem shows that every sesquilinear form can be represented as an inner product.

Theorem 6.8. *Let H_1 and H_2 be Hilbert spaces and*

$$h : H_1 \times H_2 \to \mathbb{K}$$

be a bounded sesquilinear form. Then h has a representation

$$h(x,y) = \langle Sx, y \rangle$$

where $S : H_1 \to H_2$ is a bounded linear operator. S is uniquely determined by h and $\|h\| = \|S\|$.

Proof. Consider the map $y \mapsto \overline{h(x,y)}$. This is a bounded linear functional on y, so by RRT

$$\overline{h(x,y)} = \langle y, z \rangle,$$

for a unique $z \in H_2$, which of course depends on the choice of $x \in H_1$. This gives rise to a map $S : H_1 \to H_2, x \mapsto z$. What remains is to check that S is linear, bounded and unique. □

The previous theorem will give us existence of the Hilbert adjoint.

Definition 6.5 (Hilbert Adjoint)**.** *Let H_1 and H_2 be Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator. The Hilbert adjoint of T is the operator $T^* : H_2 \to H_1$ such that for all $x \in H_1$ and $y \in H_2$ we have*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle.$$

Theorem 6.9 (Existence and Uniqueness)**.** *The Hilbert adjoint operator T^* of T exists, is unique and satisfies $\|T^*\| = \|T\|$.*

Proof. Define $h(y,x) = \langle y, Tx \rangle$. This is indeed a sesquilinear form defined on $H_2 \times H_1$. It can be shown easily that $\|h\| = \|T\|$. By the previous theorem we know there exists a bounded linear operator, say T^* such that $h(y,x) = \langle T^* y, x \rangle$. Moreover, $\|T^*\| = \|h\| = \|T\|$. □

Spectral Theory: Compact Operators. One of the goals that was outlined in the motivation was to link functional analysis to the theory of ODEs and PDEs. Consider the following *Sturm-Liouville* eigenvalue problem

$$L\phi = \lambda\phi, \quad L = \frac{d^2}{dx^2} - q(x),$$

which comes together with suitable boundary conditions on the solution ϕ, say $\phi(a) = \phi(b) = 0$. The task is to determine those λ's for which one can solve $L\phi = \lambda\phi$ with the given boundary conditions, i.e. in other words to determine the eigenvalues (spectrum) of L. Note that this is an infinite dimensional version of the familiar finite-dimensional eigenvalue problem from linear algebra.

Definition 6.6 (Compact Operator)**.** *Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. Then T is compact if and only if it maps every bounded sequence $\{x_n\}$ to a sequence $\{Tx_n\}$ which has a convergent subsequence.*

It can be easily checked that any linear operator in finite-dimensional spaces are automatically compact operators. Now that we have some intuition for compact operators we can prove the spectral theorem. In the following, H is a separable infinite-dimensional Hilbert space. In the cases of finite dimensional Hilbert space, the theorem reduces to the familiar theorem in linear algebra about diagonalizing symmetric matrices.

Theorem 6.10 (Spectral Theorem). *Suppose T is a compact self-adjoint operator $T : H \to H$ for H a (separable) infinite dimensional Hilbert space. Then there exists an orthonormal basis $\{\psi_k\}_{k=1}^{\infty}$ of H consisting of eigenvectors of T.*

$$T\psi_k = \lambda_k \psi_k.$$

Moreover, $\lambda_k \in \mathbb{R}$ and $\lambda_k \to 0$ as $k \to \infty$.

6.5 Exercises

Problem 6.11. *Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary condition $u(0,t) = u_x(l,t) = 0$.*

Problem 6.12. *Solve the Schrodinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < l$ with the boundary condition $u_x(0,t) = 0, u(l,t) = 0$.*

Problem 6.13. *Show that the eigenvalues $\{\lambda_n\}$ for Dirichlet, Neumann, periodic boundary problem are nonnegative. (this does not hold for Robin B.C.)*

Proof Sketch. Consider

$$\int_{-l}^{l} X'' X dx = -\int_{-l}^{l} \lambda_n X X dx = -\lambda_n \int_{-l}^{l} |X|^2 dx \leq 0.$$

Apply intergration by parts on the LHS and use the boundary conditions. □

Problem 6.14. *Consider waves in a resistant medium that satisfy the problem*

$$\begin{cases} u_{tt} = c^2 u_{xx} - r u_t, \quad 0 < x < l \\ u = 0, \quad \text{at both ends} \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \end{cases}$$

Solution. See the tutorial 6 slides. □

7 Tutorial 7

7.1 Fourier Series

In the lectures you might have seen Fourier series of various types. For example, the Dirichlet boundary condition corresponds to Fourier sine series while the Neumann boundary condition corresponds to Fourier cosine series. Let's temporarily forget about the convergence issues, which will be solved later.

Sine Series. Let us consider a Forier sine series

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

How should we determine the coefficient A_n? Fortunately, we have a surprising mathematical fact, which can help us determine the coefficient.

Proposition 7.1.

$$\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = 0, \quad m \neq n.$$

Proof Sketch. Use the fact $\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$. □
Then, let consider

$$\begin{aligned}\int_0^l \phi(x) \sin \frac{m\pi x}{l} dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx &= \sum_{n=1}^{\infty} A_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx \quad \text{(why?)} \\ &= \frac{l}{2} A_m.\end{aligned}$$

Therefore, we have

$$A_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx.$$

Cosine Series. Let us consider the consine series

$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.$$

Again we can verify

Proposition 7.2.

$$\int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} dx = 0, \quad m \neq n,$$

where m and n are nonnegative integers.

Therefore,

$$\int_0^l \phi(x) \cos \frac{m\pi x}{l} dx = A_m \int_0^l \cos^2 \frac{m\pi x}{l} dx = \frac{l}{2} A_m, \quad m \neq 0.$$

If $m = 0$, we have

$$\int_0^l \phi(x) \cdot 1 dx = \frac{1}{2} A_0 \int_0^l 1^2 dx = \frac{1}{2} l A_0.$$

To sum up, we have the following coefficient formula

$$A_n = \frac{2}{l} \int_0^l \phi(x) \cos \frac{n\pi x}{l} dx.$$

Full Series. The full Fourier series, or simply the Fourier series, of $\phi(x)$ on the interval $(-l, l)$, is defined as

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}).$$

Again, we have the coincidence that

$$\int_{-l}^{l} \cos\frac{n\pi x}{l} \sin\frac{m\pi x}{l} dx = 0, \quad \forall n, m$$
$$\int_{-l}^{l} \cos\frac{n\pi x}{l} \cos\frac{m\pi x}{l} dx = 0, \quad m \neq n$$
$$\int_{-l}^{l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} dx = 0, \quad m \neq n$$
$$\int_{-l}^{l} 1 \cdot \cos\frac{n\pi x}{l} dx = 0 = \int_{-l}^{l} 1 \cdot \sin\frac{m\pi x}{l} dx.$$

Therefore, following the same procedure before, we obtain the coefficient formula

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x) \cos\frac{n\pi x}{l} dx, \quad (n = 0, 1, 2, ...)$$
$$B_n = \frac{1}{l}\int_{-l}^{l} \phi(x) \sin\frac{n\pi x}{l} dx, \quad (n = 1, 2, ...)$$

□

Example 7.1. *Let $\phi(x) \equiv 1$ in the interval $[0, l]$. Find its Fourier sine and cosine series respectively.*

Example 7.2. *Let $\phi(x) = x$ in the interval $[0, l]$. Find its Fourier sine and cosine series respectively.*

Example 7.3. *Let $\phi(x) = x$ in the interval $[-l, l]$. Find its full series.*

Question: What is the relation between the Fourier sine and cosine series to Fourier full series? Here we only illustrate the cosine case and leave the sine case as an exercise for you. Consider

$$\begin{cases} \partial_t u = k\partial_x^2 u, & 0 < x < l \\ \partial_x u(0,t) = \partial_x u(l,t) = 0 \\ u(x,0) = \phi(x) \end{cases} \tag{93}$$

We do even extension on u to $[-l, l]$, then

$$\begin{cases} \partial_t u = k\partial_x^2 u, & -l < x < l \\ \partial_x u(-l,t) = \partial_x u(l,t) = 0 \\ u(x,0) = \tilde{\phi}(x) \end{cases} \tag{94}$$

We claim the system below

$$\begin{cases} \partial_t u = k\partial_x^2 u, & -l < x < l \\ u(-l,t) = u(l,t) \\ u_x(-l,t) = u_x(l,t) \\ u(x,0) = \tilde{\phi}(x). \end{cases} \tag{95}$$

has the same solution as (93).

- **PDE Perspective.** For (95), the solution is

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi}{l}x + B_n \sin\frac{n\pi}{l}x)e^{k(n\pi/l)^2 t}.$$

Since $\tilde{\phi}$ is even, we know that $B_n = 0$ and $A_n = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi}{l}x dx$. The solution with these constants is exactly the solution to (93). Note that this is the only solution to (95) since its solution is *unique*. (Why?)

- **Another Idea.** Suppose $u(x,t)$ is a solution to (95), then $u(x,t) = u(-x,t)$ (Why?). We thus have
$$\begin{cases} u_x(-l,t) = -u_x(l,t) \\ u_x(-l,t) = u_x(l,t) \end{cases}$$
This implies that $u_x(l,t) = u_x(-l,t) = 0$. Note that this is exactly the boundary condition of (94). Conversely, suppose $u(x,t)$ is a solution to (93). Do even extension of u to $[-l,l]$. We have $u(-l,t) = u(l,t)$ and $u_x(-l,t) = u_x(l,t)$ (Why?). This implies that if you have a solution to (93), then by even extension, a solution to (95) is obtained.

Example 7.4. *Show that if $\phi(x)$ is odd, then its full Fourier series on $(-l,l)$ has only cosine terms. Moreover, show that if $\phi(x)$ is even, then its full Fourier series on $(-l,l)$ has only the cosine terms.*

Therefore, you can conclude from example 7.4 that if ϕ is odd, then the sine series equals the full series and if ϕ is even, then the cosine series equals the full series.

Complex Form. By De'Moivre's formula, we directly have
$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta,$$
thus we have
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}; \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$
Based on these facts, a solution of the form
$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l},$$
can be rewritten as
$$\sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}, \quad C_n := C_n(A_n, B_n).$$
Note also we have the fact
$$\int_{-l}^{l} e^{\frac{in\pi x}{l}} e^{-\frac{in\pi x}{l}} dx = \begin{cases} 2l, & n = m \\ 0, & n \neq m. \end{cases}$$
The coefficient can be computed by
$$C_n = \frac{1}{2l}\int_{-l}^{l} \phi(x) e^{-\frac{in\pi x}{l}} dx.$$

7.2 General Fourier Series

Consider the following *eigenvalue problem* with more general boundary conditions
$$\begin{cases} X'' = -\lambda X \\ \alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0 \\ \alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0 \end{cases} \tag{96}$$
All of the coefficients in the B.C.'s are real constants. Such a set of boundary conditions is called *symmetric* if
$$f'g - fg'\big|_{x=a}^{x=b} = 0,$$
for any pair of functions $f(x)$ and $g(x)$ satisfying the boundary conditions. It is easy to see that all of the four common types of boundary conditions we have encountered in this course: Dirichlet, Neumann, Robin, and periodic are symmetric boundary conditions. Symmetric boundary conditions guarantee orthogonality of eigenfunctions.

- **Orthogonality of eigenfunctions corresponding to distinct eigenvalues**. It is proved in Theorem 6.1. Some slight modifications need to be adopted since here the boundary conditions are more general. The implication of this property is that if ϕ can be represented as its general Fourier series $\phi = \sum_{n=1}^{\infty} A_n X_n$, then the coefficient is determined

$$A_n = \frac{\langle \phi, X_n \rangle}{\langle X_n, X_n \rangle}.$$

(to rigorously prove this, you need to invoke the *continuity of inner product.*)

More generally, we do not need to restrict ourselves to the real case. If we allow f and g to be complex-valued, then the inner product is defined to be

$$\langle f, g \rangle := \int_a^b f\bar{g}dx.$$

Similarly, we say that the boundary conditions in (96) are *symmetric* if

$$f'\bar{g} - f\bar{g}'\big|_{x=a}^{x=b} = 0,$$

for all f and g satisfying the B.C.'s. Under the complex setting, we can still prove the orthogonality of eigenfunctions corresponding to distinct eigenvalues. Moreover, we have a much nicer property, which is similar to the behaviour of *self-adjoint* operator in linear algebra

- **All the eigenvalues from the eigenvalue problem are real numbers**. To prove this, notice that if X is an eigenfunction, then $-\bar{X}'' = \bar{\lambda}\bar{X}$, which means that $\bar{\lambda}$ is also an eigenvalue. Consider

$$\int_a^b -X''\bar{X} + X\bar{X}''dx = (-X'\bar{X} + X\bar{X}')\big|_{x=a}^{x=b} = 0. \quad \text{(Why?)}$$

The LHS is just

$$(\lambda - \bar{\lambda})\int_a^b X\bar{X}dx.$$

Therefore, $\lambda = \bar{\lambda}$.

Note that given the symmetric boundary conditions, we can show that $D := -\frac{d^2}{dx^2}$ is indeed a self-adjoint operator defined on $C^2[a,b]$. This is due to the *Green's second identity* (plainly, integration by parts):

$$\langle Df, g \rangle = \int_a^b f''gdx = \int_a^b fg''dx = \langle f, Dg \rangle,$$

for all $f, g \in C^2[a,b]$ satisfying the boundary conditions.

Finally, as we have seen before, most of the eigenvalues are *nonnegative*. (This is at least true for Dirichlet, Neumann, and periodic boundary conditions; see Prob. 6.13). We provide a sufficient condition under which the all of the eigenvalues are nonnegative.

Theorem 7.3 (Nonnegativity)**.** *If*

$$ff'\big|_{x=a}^{x=b} \leq 0,$$

for all (real-valued) funcitions f satisfying the B.C.'s, then there is no negative eigenvalues.

Proof. Let f be an eigenfunction corresponding to the eigenvalue λ. Then

$$\int_a^b f''fdx = f'f\big|_a^b - \int_a^b |f'|^2 dx \leq 0.$$

Therefore, $\lambda \geq 0$. (Why? Please fill in the missing details.) □

As has been shown before, eigenfunctions in different eigenspaces are orthogonal. However, if there are two eigenfunctions corresponding to the same eigenvalue, they do not have to be orthogonal. But if they are linearly independent, we can always make them orthogonal by the following algorithm.

The Gram-Schmidt Orthogonalization Procedure. If $X_1, X_2, \dots$ is any sequence (finite or infinite) of linearly independent vectors in any inner product space, it can be replaced by a sequence of linear combinations that are mutually orthogonal. The idea is that at each step one subtracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 = X_1/\|X_1\|$. Second, we define

$$Y_2 = X_2 - \langle X_1, Z_1\rangle Z_1; \quad \text{and} \quad Z_2 = \frac{Y_2}{\|Y_2\|}.$$

Third, we define

$$Y_3 = X_3 - \langle X_3, Z_1\rangle Z_1 - \langle X_3, Z_2\rangle Z_2 \quad \text{and} \quad Z_3 = \frac{Y_3}{\|Y_3\|}.$$

and so on.

Example 7.5. *(a). Show that all the vectors $Z_1, Z_2, \dots$ are orthogonal to each other. (b). Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ in the interval $(0, \pi)$ to get an orthogonal pair.*

We would have already noticed the close analogy of our analysis with linear algebra. Not only are functions acting as if they were vectors, but the operator $-\frac{d^2}{dx^2}$ is acting like a matrix. In fact, it is a *linear transformation*. *Orthogonality* of eigenfunctions corresponding to distinct eigenvalues and *all eigenvalues are real* are like the properties of real symmetric matrices in linear algebra. For instance, if $A \in \mathbb{R}^{n\times n}$ is symmetric and $x, y \in \mathbb{R}^n$ are vectors, then we have

$$\langle Ax, y\rangle = x^T A^T y = x^T Ay = \langle x, Ay\rangle.$$

In our present case, A is replaced by a differential operator with symmetric boundary conditions and vectors x, y are replaced by functions. The same identity $\langle Ax, y\rangle = \langle x, Ay\rangle$ still holds in this setting. The two main differences are

- Our vector space now is a space of *functions*, which is infinite-dimensional. Many nice properties that hold in finite-dimensional vector spaces break down in infinite-dimensional spaces. (For instance, if the vector space is equipped with a *topology*, i.e. topological vector space. According to your real analysis course, we know that a bounded and closed set is *compact*, which is entitled to the famous *Heine-Borel Theorem*. However, in infinite-dimensional spaces, this is generally NOT true.)
- The *boundary conditions* must comprise part of the defnition of the differential operator. (Why? I've explained it before, think by yourself...)

7.3 Convergence Theorems and Completeness

WARNING: This section is quite lengthy and contains a lot of analysis. For those who do not care mathematical rigor, you can simply skip this section[6].

7.3.1 Background Information

$L^2(\mathbb{T})$**.** Although we could represent a 2π-periodic function by a function f defined on a closed interval $[a, a+2\pi]$ (here the choice of a is arbitrary) such that $f(a) = f(a+2\pi)$, we do not adopt this setting. Instead, we will take the modern setting that indentifying a 2π-periodic function on $\mathbb{R}$ by a function on the circle, or the one-dimensional torus, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, which we define by identifying points in $\mathbb{R}$ that differ by $2n\pi$ for some $n \in \mathbb{Z}$. To be precise, we define a binary relation $\sim$ on $\mathbb{R}$. $x \sim y$ if $x - y = 2n\pi$, for some $n \in \mathbb{Z}$. It is easy to show that $\sim$ is indeed an equivalence relation on $\mathbb{R}$. Then, $\mathbb{T}$ denotes the collection of all equivalence classes of $\sim$.

[6] This part is mainly taken from my project report, you can find the full report at github if you are interested

Construction. Since we can identify $\mathbb{T}$ as any interval with length 2π in $\mathbb{R}$, it enables us to equip $\mathbb{T}$ with a σ-algebra $M(\mathbb{T})$ (the collection of all Lebesgue measurable sets, defined by the *Caratheodory's Criterion*) and the Lebesgue measure λ (The restriction of outer measure on $M(\mathbb{T})$). This is justified by the Measure Extension Theorem. Then, In this case, $\|f\| \triangleq \left(\int_{\mathbb{T}} |f|^2 d\mu\right)^{1/2}$. If we want to specify $M(\mathbb{T})$ and μ, we use the notation $L^2(\mathbb{T}, M(\mathbb{T}), \mu)$. In short, $L^2(\mathbb{T}, M(\mathbb{T}), \mu)$ denotes the collection of all equivalence classes of Lebesgue measurable, square integrable functions from $\mathbb{T}$ to $\mathbb{C}$ with respect to the equivalence relation μ almost-everywhere.

Suppose we are given the collection of functions $\left\{e_n = \frac{1}{\sqrt{2\pi}} e^{inx}\right\} \subset L^2(\mathbb{T})$ and an arbitrary $f \in L^2(\mathbb{T})$. Let's take the inner product of f with e_n, this gives a collection of complex numbers $\{\langle f, e_n\rangle\}$.

Definition 7.1 (Parseval's Identity)**.**

$$\sum_{n=1}^{\infty} \langle f, e_n\rangle^2 = \|f\|^2.$$

Then, we can define what is called the completeness of a collection of functions.

Definition 7.2 (Completeness in L^2)**.** An infinite collection of functions $\{x_n\}$ is said to be *complete* in L^2 if for any $f \in L^2$, the Parseval's identity holds.

Hence, to show that $\{e_n\}$ is complete in L^2, we only need to prove the collection of functions $\{e_n(x)\}_{n=1}^{\infty}$, $e_n = \frac{1}{\sqrt{2\pi}} e^{inx}$ forms an orthonormal basis of $L^2(\mathbb{T})$.

Definition 7.3 (Orthonormal)**.** A sequence $\{x_n\}$ in an inner product space is said to be *orthonormal* if they are orthogonal to each other (i.e. $x_i \perp x_j, \forall i \neq j$) and $\|x_n\| = 1, \forall n$.

Definition 7.4 (Fourier Coefficients)**.** The *Fourier Coefficients* of a vector $x \in V$ with respect to an orthonormal collection B is the set $\{\langle x, b\rangle : b \in B\}$.

Proposition 7.4. $\{e_n\}_{n=1}^{\infty}$ is orthonormal.

Proof.

$$\langle e_n, e_m\rangle = \int_{\mathbb{T}} \overline{\frac{1}{\sqrt{2\pi}} e^{imx}} \frac{1}{\sqrt{2\pi}} e^{inx} d\mu = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x} dx = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

□

We are half done for our objective. The remaining is to show that $\{e_n\}_{n=1}^{\infty}$ is complete. That is, for any $f \in L^2(\mathbb{T})$, we have the following identity

$$\|f\|^2 = \sum_{n=1}^{\infty} |A_n|^2.$$

Where $A_n = \langle f, e_n\rangle$.

Not surprisingly, if the given collection of vectors is $\{e_n\}$, then the coefficients that give the best approximation are the *Fourier Coefficients*.

Theorem 7.5 (Least Square Approximation)**.** Let $(V, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let $\{x_n\} \subset V$ be a collection of orthogonal vectors. $N \in \mathbb{N}$ is fixed. If $\|f\| < \infty$, then the Fourier coefficients minimize

$$\|f - S_N(f)\|.$$

Proof. The error term is given by

$$E_N^2 = \|f - \sum_{n=1}^{N} c_n x_n\|^2 = \int_{\mathbb{T}} |f - \sum_{n=1}^{N} c_n x_n|^2 d\mu.$$

Expanding the square, we have

$$E_N^2 = \|f\|^2 - \sum_{n=1}^{N} c_n \langle f, x_n\rangle - \sum_{n=1}^{N} \overline{c_n} \langle x_n, f\rangle + \sum_{n=1}^{N} \sum_{m=1}^{N} c_n \overline{c_m} \langle x_n, x_m\rangle.$$

By the orthogonality,

$$\sum_{n=1}^{N}\sum_{m=1}^{N} c_n\overline{c_m}\langle x_n, x_m\rangle = \sum_{n=1}^{N} c_n^2\|x_n\|^2.$$

By completing the square, we have

$$E_N^2 = \sum_{n=1}^{N} \|x_n\|^2 \left[c_n - \frac{\langle f, x_n\rangle}{\|x_n\|^2}\right]^2 + \|f\|^2 - \sum_{n=1}^{N} \frac{\langle f, x_n\rangle}{\|x_n\|^2}.$$

It is obvious that the above is minimized if and only if $c_n = \frac{\langle f, x_n\rangle}{\|x_n\|^2}$, which is indeed the Fourier coefficient if $\{x_n\}$ is chosen to be orthonormal. □

Directly, we derive a useful corollary from the above theorem.

Corollary 7.6 (Bessel's Inequality)**.** Let $\{x_n\}$ be orthonormal in an inner product space $(V, \langle\cdot,\cdot\rangle)$. Then, for any $x \in V$,

$$\sum_{n=1}^{\infty}\langle x, x_n\rangle^2 \leq \|x\|^2. \tag{97}$$

Proof. In the proof of least square approximation, if we substitute c_n to be Fourier coefficients, then

$$0 \leq E_N^2 = \|f\|^2 - \sum_{n=1}^{N}\langle f, x_n\rangle,\ \forall N \in \mathbb{N}.$$

This is equivalent to

$$\|f\|^2 \geq \sum_{n=1}^{N}\langle f, x_n\rangle,\ \forall N \in \mathbb{N}.$$

By sending $N \to \infty$,

$$\|f\|^2 \geq \sum_{n=1}^{\infty}\langle f, x_n\rangle.$$

□

7.3.2 Theorems

Theorem 7.7 ($C^1 \Rightarrow$ Pointwise Convergence)**.** The classical Fourier series converges to $f(x)$ pointwisely on (a, b) provided that $f(x) \in C^1[a, b]$.

Proof. We assume that $l = \pi$, which can easily be arranged through a change of scale. Thus the Fourier series is

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n\cos nx + B_n\sin nx).$$

The Nth partial sum is

$$S_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N}(A_n\cos nx + B_n\sin nx).$$

Plug the coefficients A_n, B_n into the partial sum, we have

$$S_N(x) = \int_{-\pi}^{\pi} K_N(x-y)f(y)\frac{dy}{2\pi},$$

where $K_N(\theta) = 1 + 2\sum_{n=1}^{N}\cos n\theta$. Note that

$$K_N(\theta) = \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}.$$

This can be proved by complexification. Hence,

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} K_N(\theta)[f(x+\theta) - f(x)]\frac{d\theta}{2\pi},$$

or

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} g(\theta) sin[(N + \frac{1}{2})\theta]\frac{d\theta}{2\pi},$$

where

$$g(\theta) = \frac{f(x+\theta) - f(x)}{\sin\frac{1}{2}\theta}.$$

Notice that the functions

$$\phi_N(\theta) = \sin[(N + \frac{1}{2})\theta],\ N \in \mathbb{N}$$

forms an orthogonal set. Therefore, by Bessel's inequality,

$$\sum_{N=1}^{\infty} \frac{|\langle g, \phi_N\rangle|^2}{\|\phi_N\|^2} \leq \|g\|^2.$$

Since $\|\phi_N\|^2 = \pi$ and $\|g\| < \infty$, we have $\langle g, \phi_N\rangle \to 0$ and the proof is complete. □

Theorem 7.8 ($C^1 \Rightarrow$ Uniform Convergence)**.** Suppose $f \in C^1(\mathbb{T})$ s.t. B.C., then $S_N(f)$ converges uniformly to f on $\mathbb{T}$.

Proof. We only prove this when the collection of orthonormal vectors is given by $\{e_n\}$.
The Fourier coefficient is given by

$$\begin{aligned} c_n &= \int_{\mathbb{T}} \frac{1}{\sqrt{2\pi}} f e^{inx} d\mu \\ &= \frac{1}{in\sqrt{2\pi}} \int_0^{2\pi} f e^{inx} dx \\ &= \frac{1}{in} \int_{\mathbb{T}} \frac{1}{\sqrt{2\pi}} f' e^{inx} d\mu = \frac{1}{in} c_n', \end{aligned}$$

where c_n' stands for the Fourier coefficient of f'. Note that since $f \in C^1(\mathbb{T})$, we know that $S_N(f) \to f$ pointwisely. Also, since $f \in C^1(\mathbb{T})$, we have $\|f\| = \left(\int_{\mathbb{T}} |f|^2 d\mu\right)^2 < \infty$.

$$\begin{aligned} \sup|f - S_N(f)| &\leq \sup \sum_{n=N+1}^{\infty} |c_n e_n| \\ &= \frac{1}{n\sqrt{2\pi}} \sum_{n=N+1}^{\infty} |c_n'| \\ &\leq \left(\sum_{n=N+1}^{\infty} \frac{1}{2n^2\pi}\right)^{1/2} \left(\sum_{n=N+1}^{\infty} |c_n'|^2\right)^{1/2} \to 0,\ as\ N \to \infty. \end{aligned}$$

This is because $\sum_{n=1}^{\infty} \frac{1}{2n^2\pi} < \infty$ and by Bessel's inequality 7.6, $\sum_{n=1}^{\infty} |c_n'|^2 < \|f\|^2 < \infty$. This shows that $S_N(f) \to f$ uniformly. □

Remark. The requirement can be weaken to f is continuous but f' is with countably many discontinuities, since a function with countably many discontinuities is still Riemann-integrable.

Lemma 7.9 (Approximation)**.** $C_0^{\infty}(\mathbb{T})$ is dense in $L^2(\mathbb{T})$.

Proof. The proof is not easy, please see the appendix. □

Remark. Suppose $U \subset V$ is a subspace of V and V is equipped with a metric d. By denseness we mean that for any $v \in V$ and $\forall\epsilon > 0$, there exists a $u \in U$ such that $d(u, v) < \epsilon$. A concrete example is $\mathbb{Q}$ *is dense in* $\mathbb{R}$*, where* $\mathbb{R}$ *is equipped with the usual Euclidean norm* $|\cdot|$.

Theorem 7.10 (**Main Theorem**)**.** $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{T})$.

Proof. Fix a $\epsilon > 0$. Note that by *lemma 7.9*, we can find a $g \in C_0^\infty(\mathbb{T})$ such that

$$\|f - g\| < \frac{\epsilon}{3}.$$

The distance between f and $S_N(f)$ can be decomposed into three parts, we try to control each part respectively.

$$\|f - S_N(f)\| = \|f - g\| + \|g - S_N(g)\| + \|S_N(g) - S_N(f)\|.$$

Since g is infinitely differentiable, by invoking *theorem 7.8*, we know that $S_N(g) \to g$ uniformly, thus, there exists a $K \in \mathbb{N}$ such that for $\forall N \geq K$,

$$\begin{aligned}
\|g - S_N(g)\| &= \left(\int_{\mathbb{T}} |g - S_N(g)|^2 d\mu\right)^{1/2} \\
&\leq \left(\sup_{[0,2\pi]} |S_N(g) - g|^2\right)^{1/2} \left(\int_0^{2\pi} 1dx\right)^{1/2} \\
&= \sqrt{2\pi}\left(\sup_{[0,2\pi]} |S_N(g) - g|^2\right)^{1/2} < \frac{\epsilon}{3}.
\end{aligned}$$

For the third term, since $S_N(f-g) \perp [(f-g) - S_N(f-g)]$, by *theorem 7.7*, we have

$$\|f - g\| = \|S_N(f-g)\| + \|(f-g) - S_N(f-g)\|$$

This implies

$$\|S_N(f) - S_N(g)\| = \|S_N(f-g)\| \leq \|f - g\| < \frac{\epsilon}{3}.$$

Thus, for any $\epsilon > 0$, there exists a $K \in \mathbb{N}$ such that for all $N \geq K$,

$$\|f - S_N(f)\| < \epsilon.$$

This is equivalent to

$$\lim_{N\to\infty} \|f - S_N(f)\| = 0.$$

□

As we have stated before, an infinite collection of orthogonal functions is said to be complete in L^2 if the Parseval's identity holds. By the main theorem, we know that $\{e_n\}$ forms an orthonormal basis for $L^2(\mathbb{T})$. If we are given $f \in L^2(\mathbb{T})$, then $f = \sum_{n=1}^{\infty} c_n e_n$, where c_n's are some coefficients to be determined. Finally, we prove the Parseval's identity as a corollary of the previous theorem.

Corollary 7.11 (Parseval's Identity)**.** If $f \in L^2(\mathbb{T})$ and $\{x_n\}$ is an orthonormal basis, then the following equation holds

$$\sum_{n=1}^{\infty} \langle f, x_n\rangle^2 = \|f\|^2$$

Proof. By the previous theorem, we know that if $f \in L^2(\mathbb{T})$, then

$$\lim_{N\to\infty} E_N = \lim_{N\to\infty} \|f - S_N(f)\| = 0.$$

Thus,

$$\sum_{n=1}^{\infty} \langle f, x_n\rangle^2 = \|f\|^2.$$

□

Remark. The Parseval's identity equates the mean-square norm of the function with a corresponding norm of its Fourier coefficients.

Finally, we give two interesting results based on *Bessel's inequality* and *Parseval's identity*. Since the terms of a converging series tend to 0, we have the the following results.

Corollary 7.12 (Riemann-Lebesgue Lemma)**.** If $f \in L^2(\mathbb{T})$, then $\hat{f}(n) \to 0$ as $|n| \to \infty$.

An equivalent reformulation of this proposition is if f is integrable on $[0, 2\pi]$, then

$$\int_0^{2\pi} f(x)\sin(Nx)dx \to 0, \quad as\ N \to \infty.$$

and

$$\int_0^{2\pi} f(x)\cos(Nx)dx \to 0, \quad as\ N \to \infty.$$

The last lemma is a more general version of the Parseval's identity.

Lemma 7.13. Suppose F and G are both in $L^2(\mathbb{T})$ with

$$F \sim \sum a_n e^{inx} \quad and \quad G \sim \sum b_n e^{inx}.$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} F(x)\overline{G(x)}dx = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}.$$

Proof. The proof follows from Parseval's identity and the fact that

$$(F, G) = \frac{1}{4}[\|F + G\|^2 - \|F - G\|^2 + i(\|F + iG\|^2 - \|F - iG\|^2)],$$

which holds in every Hermitian inner product space. □

7.4 Gibbs Phenomenon

The Gibbs Phenomenon is what happens to Fourier series at jump discontinuities. The key is that it happens when the convergence is only pointwise, not uniform. Gibbs showed that near the jump discontinuity, the partial sum $S_N(f)$ always differs from f by an overshoot of about 9 percent. Though the width of the overshoot goes to 0 as N goes to ∞, the extra height remains at 9 percent of the jump. That is to say

$$\lim_{N \to \infty} |S_N(f)(x) - f(x)| \neq 0,$$

when x is near the jump discontinuity. However, for x that does not jump, $S_N(f)(x) \to f(x)$. Consider a concrete example given in the textbook [Str07].

Example 7.6.

$$\begin{cases} \frac{1}{2}, & 0 < x < \pi \\ -\frac{1}{2}, & -\pi < x < 0 \end{cases}$$

f has the Fourier series

$$\sum_{n=1, n odd} \frac{2}{n\pi}\sin(nx).$$

For the complete mathematical derivation, please read the textbook, I only provide the graphical illustration (see Fig 4) for different choices of $N \in \{5, 10, 25, 50, 100, 200\}$.

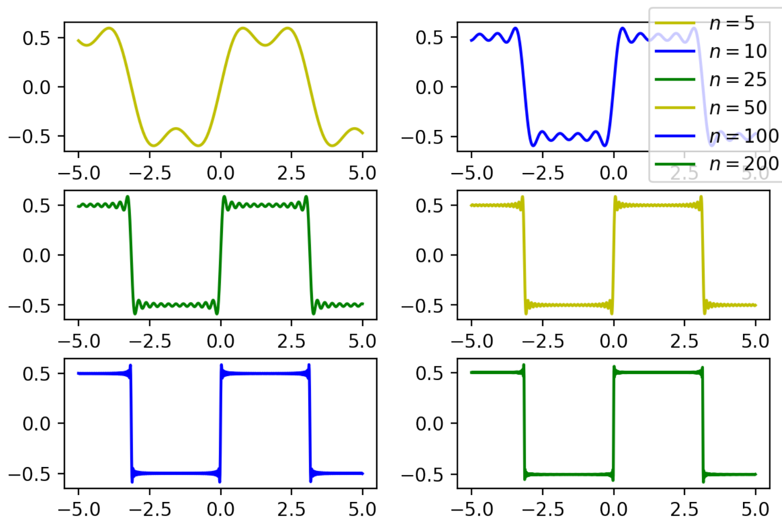


Figure 4: Gibbs phenomenon

7.5 Appendix: Approximation*

To prove this approximation theorem, we first define the *Schwarz Space*.

Definition 7.5 (Schwarz Space)**.** The class $\mathbb{S}$ consists of functions f on $\mathbb{R}^d$ that are smooth (infinitely differentiable) and such that for each multi-index α and β, the function $x^\alpha(\frac{\partial}{\partial x})^\beta$ is bounded on $\mathbb{R}^d$.[7]

Theorem 7.14 (Denseness of $\mathbb{S}$ in L^2)**.** The space $\mathbb{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. In other words, given any $f \in L^2(\mathbb{R}^d)$, there exists a sequence $\{f_n\} \subset \mathbb{S}(\mathbb{R}^d)$ such that

$$\|f - f_n\|_{L^2(\mathbb{R}^d)} \to 0, \ as \ n \to \infty$$

Proof. For the proof of the lemma, we fix $f \in L^2(\mathbb{R}^d)$ and $\epsilon > 0$. Then, for each $M > 0$, we define

$$g_M(x) = \begin{cases} f(x) & |x| \le M \ and \ |f(x)| \le M, \\ 0 & Otherwise. \end{cases}$$

Then, $|f(x) - g_M(x)| \le 2|f(x)|$, hence $|f(x) - g_M(x)|^2 \le 4|f(x)|^2$, and since $g_M(x) \to f(x)$ as $M \to \infty$ for almost every x, then dominated convergence theorem guarantees that for some M, we have

$$\|f - g_M\|_{L^2(\mathbb{R}^d)} < \epsilon.$$

We write $g = g_M$, note that this function is bounded and supported on a bounded set, and observe that it now suffices to approximate g by functions in the Schwartz space. TO achieve this goal, we use a method called **regularization**, which consists of smoothing g by convolving it with an approximation of the identity. Consider a function $\psi(x)$ on $\mathbb{R}^d$ with the following properties:

- ψ is smooth.
- ψ is supported in the unit ball.
- $\psi \ge 0$.
- $\int_{\mathbb{R}^d} \psi(x)dx = 1$.

For instance, one can take

$$\psi(x) = \begin{cases} ce^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & |x| \ge 1. \end{cases}$$

[7]Recall that $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ and $(\frac{\partial}{\partial x})^\beta = (\frac{\partial}{\partial x_1})^{\beta_1} (\frac{\partial}{\partial x_2})^{\beta_2} \cdots (\frac{\partial}{\partial x_d})^{\beta_d}$.

where the constanct c is chosed so that the forth property is satisfied.
Next, we consider the approximation to the identity defined by

$$K_\delta(x) = \delta^{-d}\psi(x/\delta).$$

The key observation is that $g * K_\delta$ belongs to $\mathbb{S}(\mathbb{R}^d)$, with this convolution in fact bounded and supported on a fixed bounded set, uniformly in δ (assuming for example that $\delta \leq 1$). Indeed, we may write

$$(g * K_\delta)(x) = \int g(y)K_\delta(x-y)dy = \int g(x-y)K_\delta(y)dy.$$

We note that since g is supported on some bounded set and K_δ vanishes outside the ball of radius δ, the function $g * K_\delta$ is supported in some fixed bounded set independent of δ. Also, the functionn g is bounded by construction, hence

$$|(g * K_\delta(x))| \leq \int |g(x-y)||K_\delta(y)|dy \leq \sup_{z\in\mathbb{R}^d} |g(z)|.$$

which shows that $g * K_\delta$ is also uniformly bounded in δ. Moreover, from the first integral expression for $g * K_\delta$ above, one may differentiate under the integral sign to see that $g * K_\delta$ is smooth and all of its derivatives have support in some fixed bounded set.
The proof of the lemma will be complete if we can show that $g * K_\delta$ converges to g in $L^2(\mathbb{R}^d)$. Now Theorem 2.1 in Chapter 3[8] guarantees that for almost every x, the quantity $|(g * K_\delta)(x) - g(x)|^2$ converges to 0 as δ tends to 0. An application of the bounded convergence theorem yields

$$\|(g * K_\delta) - g\|^2_{L^2(\mathbb{R}^d)} \to 0,\ as\ \delta \to 0.$$

In particular, $\|(g * K_\delta) - g\|_{L^2(\mathbb{R}^d)} < \epsilon$ for an appropriate δ and hence $\|f - g * K_\delta\|_{L^2(\mathbb{R}^d)} < 2\epsilon$, and choosing a sequence of ϵ tending to zero gives the construction of the desired sequence $\{f_n\}$. □ □

Remark. In the proof of the main theorem in this text, we pick a function in $C_0^\infty(\mathbb{T})$. Indeed, we could choose one from $\mathbb{S}(\mathbb{T})$, which can also achieve the same effect.

7.6 Exercises

Problem 7.15. *Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums*

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1}; \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

Problem 7.16. *Given the Fourier sine series of $\phi(x) = x$ on $(0, l)$. Assume that the series can be integrated term by term (under what conditions is this valid?)*
(a). Find the Fourier cosine series of the function $x^2/2$. Find the constant of integration that will be the first term in the cosine series.
(b). Then by setting $x = 0$ in your result, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Problem 7.17. *Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi$, with the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and the initial conditions $u(x,0) = 0$, $u_t(x,0) = \cos^2 x$.*

[8] Since the proof requires a lot of prerequisite knowledge and might be quite lengthy, please refer to *Real analysis* by Stein.

8 Tutorial 8

8.1 Laplace's Equation

In this section, we will briefly discuss the so-called Laplace equations. We will introduce the maximum principle and the invariance properties of Δ.

Laplace's Equation. If a diffusion or a wave is *stationary*, which means the process does not depend on time, then $u_t = 0$, the equations reduce to the *Laplace equation*.

$$\Delta u = 0. \tag{98}$$

A solution to the Laplace's equation is called a *harmonic function*. In one dimension case, $0 = \Delta u = u_{xx}$, thus the only harmonic functions are of the form $u(x) = Ax + B$. The *inhomogeneous* version of Laplace's equation

$$-\Delta u = f, \tag{99}$$

where f is a given function, is called *Poisson's equation*.

Maximum Principle. Let D be a connected, bounded, and open set. Let either $u(\boldsymbol{x})$ be a harmonic function in D that is continuous on $\overline{D}$. Then the maximum and the minimum values of u are attained on ∂D and nowhere inside (unless $u \equiv$ const.)

Proof. Suppose for contradiction that u achieves its maximum inside the domain. Then by second derivative test we will have $\nabla^2 u$ is NSD. This implies $\Delta u \leq 0$. The *idea* is, if we have $\Delta u < 0$, we arrive at a contradiction. Therefore, we instead consider an auxiliary function $v := u + \epsilon|x|^2, \epsilon > 0$. Then

$$\Delta v = \Delta u + \epsilon\Delta(x^2 + y^2) > 0.$$

If v achieves it maximum point inside the domain, we immediately have a contradiction due to the *second-order optimality condition*. Thus, v has *no* interior maximum point. (Similarly, you can prove v does not have interior minimum point.) Since v is continuous on a compact set, it must achieve its maximum at some point, say $\boldsymbol{x}_0$. Then

$$u(\boldsymbol{x}) \leq v(\boldsymbol{x}) \leq v(\boldsymbol{x}_0) = u(\boldsymbol{x}_0) + \epsilon\|\boldsymbol{x}_0\|^2 \leq \max_{\partial D} u + \epsilon l^2,$$

where $l := \sup_{\boldsymbol{x}\in\partial D} \|\boldsymbol{x}\| < \infty$. Since this is true for any $\epsilon > 0$, we send $\epsilon \to 0^+$. Hence,

$$u(\boldsymbol{x}) \leq \max_{\partial D} u, \quad \forall x \in D.$$

Therefore,

$$\max_{D} u \leq \max_{\partial D} u.$$

The reverse direction holds trivially. □

Problem 8.1. *Prove the uniqueness of the Dirichelet Problem*

$$\begin{cases} \Delta u = f \\ u = h \end{cases}$$

Example 8.1. *Prove the uniqueness of the Dirichlet problem $\Delta u = f$ in D, $u = g$ on ∂D by the energy method.*

Proof. Suppose u_1 and u_2 are both solutions, set $w := u_1 - u_2$. Consider

$$0 = \int_D w\Delta w = \int_{\partial D} w\frac{\partial w}{\partial \boldsymbol{n}} - \int_D \|\nabla w\|^2$$

□

Example 8.2. *A function $u(\boldsymbol{x})$ is subharmonic in D if $\Delta u \geq 0$ in D. Prove that its maximum value is attained on ∂D. [This is NOT true for the minimum value. Why? Any counterexample?]*

Proof. The proof follows exactly the same idea as the proof of the maximum princple, consider $v(x) := u(x) + \epsilon|x|^2$, show that $\Delta v > 0$, if v attains its maximum at an interior point, a contradiction arises. Try to fill in the details by yourself. □

Strong Maximum Principle. Let $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$, Ω is connected, then the maximum value M of u can only be taken on $\partial\Omega$. Otherwise, if there exists $\boldsymbol{x}_0 \in \Omega$ such that $u(\boldsymbol{x}_0) = M$, then

$$u(\boldsymbol{x}) \equiv M, \quad \forall \boldsymbol{x} \in \Omega$$

Proof. We will come back to prove this after introducing *Mean Value Property.* □

Invariance of Δ. We will briefly discuss the invariance properties of Δ in n-d case.

- **Translation Invariance.** Consider a translation in $\mathbb{R}^n$ given by

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{c},$$

where $\boldsymbol{c} \in \mathbb{R}^n$ is a constant vector. For any fixed $i \in \{1, 2, ..., n\}$, we have

$$\partial_{x_i} = \partial_{y_i} \frac{\partial y_i}{\partial x_i} = \partial_{y_i}.$$

Therefore, $\partial^2_{x_i} = \partial^2_{y_i}$, which implies

$$\Delta_{\boldsymbol{y}} = \sum_{i=1}^n \partial^2_{y_i} = \sum_{i=1}^n \partial^2_{x_i} = \Delta_{\boldsymbol{x}}.$$

- **Rotational Invariance.** Consider the linear transfomration

$$\boldsymbol{y} = \boldsymbol{U}\boldsymbol{x}, \quad \boldsymbol{U} \in O(n),$$

in particular, we have

$$y_j = \sum_{i=1}^n u_{ji} x_i.$$

$$\partial_{x_i} = \sum_{j=1}^n \partial_{y_j} \frac{\partial y_j}{\partial x_i} = \sum_{j=1}^n u_{ji} \partial_{y_j}. \tag{100}$$

Therefore, we have

$$\partial^2_{x_i} = \left(\sum_{j=1}^n u_{ji} \partial_{y_i} \right)^2 = \sum_{j,k} u_{ki} u_{ji} \partial_{y_k} \partial_{y_j}.$$

Thus

$$\sum_{i=1}^n \partial^2_{x_i} = \sum_{i=1}^n \left(\sum_{j,k} u_{ki} u_{ji} \partial_{y_k} \partial_{y_j} \right) = \sum_{j,k} \left(\sum_{i=1}^n u_{ki} u_{ji} \right) \partial_{y_k} \partial_{y_j} = \sum_{j,k} \delta_{kj} \partial_{y_k} \partial_{y_j} = \sum_{k=1}^n \partial^2_{y_k}.$$

There is one thing interesting, notice that from (100), we have

$$\nabla_{\boldsymbol{x}} = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n u_{j1} \partial_{y_j} \\ \sum_{j=1}^n u_{j2} \partial_{y_j} \\ \vdots \\ \sum_{j=1}^n u_{jn} \partial_{y_j} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \\ \vdots \\ \partial_{y_n} \end{pmatrix} = \boldsymbol{U}^T \nabla_{\boldsymbol{y}}.$$

The rotational invariance suggests that the two-dimensional laplacian should take a particularly simple form in *polar coordinates.* The transformation is

$$x = r\cos\theta, \quad y = r\sin\theta$$

Note that

$$\partial_r = \cos\theta\partial_x + \sin\theta\partial_y$$
$$\partial_\theta = -r\sin\theta\partial_x + r\cos\theta\partial_y,$$

which is equivalent to

$$\begin{pmatrix}\partial_r\\ \partial_\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}.$$

Thus, we have

$$\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}^{-1}\begin{pmatrix}\partial_r\\ \partial_\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & -\frac{\sin\theta}{r}\\ \sin\theta & \frac{\cos\theta}{r}\end{pmatrix}\begin{pmatrix}\partial_r\\ \partial_\theta\end{pmatrix}.$$

Therefore we have

$$\partial_x = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta$$
$$\partial_y = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta.$$

Based on the above results, we can compute the Laplacian in polar coordinates

$$\begin{aligned}
&\Delta_{\boldsymbol{x}} = (\cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta)^2 + (\sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta)^2\\
&= \left(\cos^2\theta\partial_r^2 + \frac{1}{r^2}\sin\theta\cos\theta\partial_\theta + \frac{1}{r^2}\sin^2\theta\partial_\theta^2 - \frac{1}{r}\sin\theta\cos\theta\partial_r\partial_\theta + \frac{1}{r}\sin^2\theta\partial_r - \frac{1}{r}\sin\theta\cos\theta\partial_r\partial_\theta + \frac{1}{r^2}\sin\theta\cos\theta\partial_\theta\right)\\
&+ \left(\sin^2\theta\partial_r^2 - \frac{1}{r^2}\sin\theta\cos\theta\partial_\theta + \frac{1}{r}\sin\theta\cos\theta\partial_r\partial_\theta + \frac{1}{r}\cos^2\theta\partial_r + \frac{\sin\theta\cos\theta}{r}\partial_\theta\partial_r - \frac{1}{r^2}\sin\theta\cos\theta\partial_\theta + \frac{1}{r^2}\cos^2\theta\partial_\theta^2\right)\\
&= \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2.
\end{aligned}$$

It is natural to look for special harmonic functions that are rotationally invariant. In two dimensions this means that we use polar coordinate (r, θ) and look for solutions depending only on r.

$$0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r,$$

which is equivalent to

$$(ru_r)_r = 0.$$

Therefore,

$$u = c_1\log r + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

The term $\log r$ will play a central role later.

8.2 Separation of Variables: Rectangle & Cube

Special geometry can be solved by separation of variables. The general procedure is as follows

1. Look for separated solutions of the PDE
2. Put in the homogeneous boundary conditions to get the eigenvalues
3. Sum the series
4. Put in the inhomogeneous boundary initial or boundary conditions

Rectangle. Consider the following problem

$$\begin{cases} \Delta_2 u = u_{xx} + u_{yy} = 0, \quad in\ D := \{0 < x < a, 0 < y < b\}, \\ u_y(x,0) + u(x,0) = h(x); \quad u(x,b) = g(x) \\ u(0,y) = j(y); \quad u_x(a,y) = k(y). \end{cases}$$

Note that the solution to the above system is actually a linear combination of u_1, u_2, u_3, u_4 such that each u_i solves the system with only one inhomogeneous boundary condition while all others conditions are set to be homogeneous. For simplicity, let assume that $h = 0, j = 0$, and $k = 0$. Thus, we are only left with one inhomogeneous boundary condition $u(x,b) = g(x)$. We consider a separated solution $u(x,t) = X(x)Y(y)$, then we have

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Hence there is a constant λ such that $X'' + \lambda X = 0$ for $0 \le x \le a$ and $Y'' - \lambda Y = 0$ for $0 \le y \le b$. Solve the eigenvalue problem with the boundary conditions $X(0) = X'(a) = 0$ we immediately have

$$\beta_n^2 = \lambda_n = (n + \frac{1}{2})^2 \frac{\pi^2}{a^2}, \quad n = 0, 1, 2, ...$$

The eigenfunctions are

$$X_n(x) = \sin \frac{(n + \frac{1}{2})\pi x}{a}.$$

Next we look at the y variable. We have

$$Y'' - \lambda Y = 0$$

with the boundary conditions $Y'(0) + Y(0) = 0$. From the ODE, we know that

$$Y_n(y) = A_n \cosh \beta_n y + B_n \sinh \beta_n y.$$

By the homogeneous boundary condition we know that $\beta_n B_n + A_n = 0$, without losing any information we may assume that $B_n = 1$, then $A_n = \beta_n$. Therefore,

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \beta_n x(\beta_n \cosh \beta_n y - \sinh \beta_n y).$$

Finally we plug in the inhomogeneous boundary condition $u(x,b) = g(x)$, it requires that

$$g(x) = \sum_{n=0}^{\infty} A_n(\beta_n \cosh \beta_n b - \sinh \beta_n b) \sin \beta_n x, \quad 0 < x < a.$$

The Fourier coefficients are given by

$$A_n = \frac{2}{a}(\beta_n \cosh \beta_n b - \sinh \beta_n b)^{-1} \int_0^a g(x) \sin \beta_n x dx.$$

Cube. The same method works for a three-dimensional box

$$D := \{0 < x < \pi, 0 < y < \pi, 0 < z < \pi\}.$$

Consider the problem

$$\begin{cases} \Delta_3 u = u_{xx} + u_{yy} + u_{zz} = 0, \quad in\ D \\ u(\pi, y, z) = g(y,z) \\ u(0,y,z) = u(x,0,z) = u(x,\pi,z) = u(x,y,0) = u(x,y,\pi) = 0. \end{cases}$$

To solve the problem, we again consider a separated solution

$$u = X(x)Y(y)Z(z),$$

then we have
$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$
with the boundary conditions $X(0) = Y(0) = Y(\pi) = Z(0) = Z(\pi) = 0$. Each quotient $X''/X, Y''/Y, Z''/Z$ must be constant (why?). Solve two eigenvalue problems for Y and Z we have

$$\begin{aligned} Y(y) &= \sin my, \quad m = 1, 2, ... \\ Z(z) &= \sin nz, \quad n = 1, 2, ... \end{aligned}$$

so that
$$X'' = (m^2 + n^2)X, \quad X(0) = 0.$$
Therefore, $X(x) = A \sinh(\sqrt{m^2 + n^2}x)$. Summing them up, the complete solution is

$$u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sinh(\sqrt{m^2 + n^2}x) \sin my \sin nz.$$

The remaining step is to plug in the inhomogeneous boundary condition $u(\pi, y, z) = g(y, z)$ and solve for the Fourier coefficients. Note that here we have a double Fourier series in the variables y and z, but the theory is similar to that of the single Fourier series. Please see the below problem. For more details, please read the p.164 of [Str07].

Problem 8.2. *Prove that the eigenfunctions* $\{\sin my \sin nz\}$ *are orthogonal on the square* $\{0 < y < \pi, 0 < z < \pi\}$.

8.3 Separation of Variables: Circle; Possion's Formula

The *rotational invariance* of the Laplacian operator Δ immediately provides a hint that the circle is a natural shape for harmonic functions. Let's consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & x^2 + y^2 < a^2 \\ u = h(\theta), & x^2 + y^2 = a^2. \end{cases} \tag{101}$$

with radius a and any boundary data $h(\theta)$.

Our method to solve the system is still *separation of variables.* However, instead of separating the solution w.r.t. the spatial coordinates, we separate it w.r.t. the polar coordinates. Set

$$u(r, \theta) := R(r)\Theta(\theta).$$

We then can write

$$0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''.$$

Divide by $R\Theta$ and multiplying by r^2, we find that

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \\ r^2 R'' + rR' - \lambda R &= 0. \end{aligned}$$

What kind of boundary conditions should we assign? It is quite natural to consider the periodic boundary condition for θ.
$$\Theta(\theta + 2\pi) = \Theta(\theta), \quad -\infty < \theta < \infty.$$
Thus, solving it we have

$$\lambda = n^2, \quad \Theta(\theta) = A \cos n\theta + B \sin n\theta, \quad n = 1, 2, 3, ...$$

There is also the solution $\lambda = 0$ with $\Theta(\theta) = A$.

The equation for R is also easy to solve since it is of the *Euler* type. Consider the solution of the form $R(r) = r^\alpha$. Since $\lambda = n^2$ it reduces to

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0. \tag{102}$$

Therefore, $\alpha = \pm n$. Thus $R(r) = Cr^n + Dr^{-n}$ and we have the separated solutions

$$u = \left(Cr^n + Dr^{-n}\right)(A\cos n\theta + B\sin n\theta), \quad n = 1, 2, 3, \ldots$$

In case $n = 0$, one should be more careful. Now we have $rR'' + rR' = 0$, so the solution is $R(r) = C + D\log r$. Notice that we have NOT used any boundary conditions for r yet. We should require that the solution u to be finite in the domain. Specifically, at $r = 0$, the solution should be *finite*. Therefore, it would be sensible to drop r^{-n} and $\log r$ terms since they are infinite at $r = 0$. Finally, the solution is given by

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta). \tag{103}$$

We use the inhomogeneous boundary condition at $r = a$ to determine the coefficients. Set $r = a$, we require that

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n(A_n\cos n\theta + B_n\sin n\theta).$$

Thus,

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos n\phi d\phi.$$
$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin n\phi d\phi.$$

Next, we derive the famous *Poisson's formula*. Let us first put the coefficients A_n and B_n into the expression of u, then we have

$$u(r,\theta) = \int_0^{2\pi} h(\phi)\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right)\frac{d\phi}{2\pi}.$$

The term in red can be summed up explicitly and it is called the *Poisson's kernel*. To show this, note that the term in red can be written as

$$1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} = 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}} = \frac{a^2-r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}$$
$$= P(r,\theta).$$

Example 8.3. *Show that $P(r,\theta)$ is a harmonic function in D by using polar coordinates.*

Therefore,

$$u(r,\theta) = (a^2 - r^2)\int_0^{2\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\frac{d\phi}{2\pi}. \quad \text{(Poisson's formula)}$$

The Poisson's formula can also be written in a more geometric way. Write $\boldsymbol{x} = (x, y)$ as a point with polar coordinate (r,θ). We could also think of $\boldsymbol{x}$ as the vector from the origin $\mathbf{0}$ to the point (x, y). Let $\boldsymbol{x}'$ be a point on the boundary.

$$\boldsymbol{x} : \text{polar coordinates } (r,\theta)$$
$$\boldsymbol{x}' : \text{polar coordinates } (a,\phi).$$

By the law of cosines

$$|\boldsymbol{x} - \boldsymbol{x}'| = a^2 + r^2 - 2ar\cos(\theta-\phi).$$

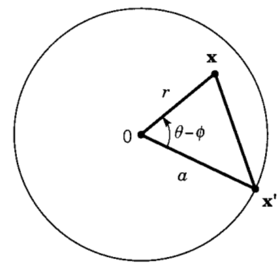


Figure 5: Poisson's formula: geometry version

The arc length element on the circumference is $ds' = ad\phi$. Therefore, Poisson's formula takes the alternative form

$$u(\boldsymbol{x}) = \frac{a - |\boldsymbol{x}|^2}{2\pi a} \int_{|\boldsymbol{x}'|=a} \frac{u(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^2} ds', \quad \boldsymbol{x} \in D.$$

Based on the *Poisson's formula*, we can prove some interesting results

Theorem 8.3. *Let $h(\phi) = u(\boldsymbol{x}')$ be any continuous function on the circle $C = \partial D$. Then the Poisson formula provides the only harmonic function in D for which*

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} u(\boldsymbol{x}) = h(\boldsymbol{x}_0), \quad \forall \boldsymbol{x}_0 \in C.$$

Before we prove theorem. 8.3, we introduce some properties of the *Poisson kernel* $P(r, \theta)$, which will be used in the proof.

Proposition 8.4 (Properties of $P(r, \theta)$)**.** *The Poisson kernel is defined to be*

$$P(r, \theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = 1 + 2\sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^2 \cos n\theta.$$

It has the following properties:

- $P(r, \theta) > 0$ *for* $r < a$.
- $\int_0^{2\pi} P(r, \theta) \frac{d\theta}{2\pi} = 1$.
- $P(r, \theta)$ *is a harmonic function inside the circle.*

Proof. The first property follows from the observation that $a^2 - 2ar\cos\theta + r^2 \geq a^2 - 2ar + r^2 = (a - r)^2 > 0$. The second property follows from the fact that $\int_0^{2\pi} \cos n\theta d\theta = 0$, for $n = 1, 2, ...$ and $P(r, \theta) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^2 \cos n\theta$. The third property follows from the fact that each term $(r/a)^n \cos n\theta$ in the series is harmonic and therefore so is the sum. □

Proof of theorem. 8.3. We differentiate under the integral sign (why?) to get

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \int_0^{2\pi} \left(P_{rr} + \frac{1}{r}P_r + \frac{1}{r^2}P_{\theta\theta}\right)(r, \theta - \phi)h(\phi)\frac{d\phi}{2\pi} = 0, \quad r < a, \quad \text{(By property 3)}$$

So u is harmonic in D. To prove the limit holds, let us first fix an angle θ_0 and consider a radius r near a. Then we will estimate the difference.

$$u(r, \theta_0) - h(\theta_0) = \int_0^{2\pi} P(r, \theta_0 - \phi)[h(\phi) - h(\theta_0)]\frac{d\phi}{2\pi}.$$

By property 2, $P(r, \theta)$ is centered around $\theta = 0$. This is true in the precise sense that, for $\delta \leq \theta \leq 2\pi - \delta$,

$$|P(r, \theta)| = \frac{a^2 - r^2}{(a - r)^2 + 4ar\sin^2(\theta/2)} < \epsilon,$$

for r sufficiently close to a. Now, by property 1, we have

$$|u(r,\theta_0)-h(\theta_0)| \le \int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta_0-\phi)\epsilon\frac{d\phi}{2\pi} + \epsilon\int_{|\phi-\theta_0|>\delta} |h(\phi)-h(\theta_0)|\frac{d\phi}{2\pi} \le \epsilon + 2H\epsilon.$$

where $|h| \le H$ for some constant H. Note that

$$|u(r,\theta)-h(\theta_0)| \le |u(r,\theta_1)-u(r,\theta_0)| + |u(r,\theta_0)-h(\theta_0)|.$$

The first term on the RHS can be made small by continuity and the second can be made small by our previous argument. So in fact we do not need to fix θ_0. □

Theorem 8.5 (Mean Value Property)**.** *Let u be a harmonic function in a disk D, continuous in its closure $\overline{D}$. Then the value of u at the center of D equals the average of u on its circumference.*

Proof. Choose coordinates with the origin $\mathbf{0}$ at the center of the circle. Put $\boldsymbol{x} = 0$ in Poisson's formula, then

$$u(\mathbf{0}) = \frac{a^2}{2\pi a}\int_{|\boldsymbol{x}'|=a} \frac{u(\boldsymbol{x}')}{a^2} ds'.$$

This is indeed the average of u on the circumference $|\boldsymbol{x}'| = a$. □

Theorem 8.6 (Strong Maximum Principle)**.** *SMP states that the maximum of u cannot be attained in the interior of the connected domain D. (We have stated it before, please review)*

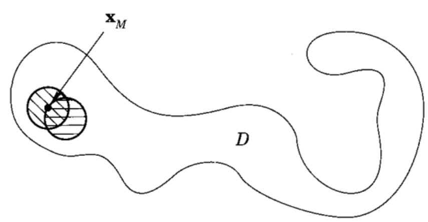


Figure 6: Strong Maximum Principle

Proof Sketch. We give a proof for the strong version based on the *Mean Value Property*. Suppose for contradiction that the maximum is attained at some $\boldsymbol{x}_0 \in D$, then by MVP, we know that any points lie in a circle centered at $\boldsymbol{x}_0$ also have the same value as $\boldsymbol{x}_0$. By connectedness, for any other point $\boldsymbol{x}' \in D$ that cannot be covered by a circle centered at $\boldsymbol{x}_0$, we can connect it with $\boldsymbol{x}_0$ through a curve. We can prove by contradiction that all the points on the curve must achieves the maximum value. (I think the argument is quite common in complex analysis, please formulate a detailed argument by yourself) □

Theorem 8.7 (Differentiability)**.** *Let u be a harmonic function in any open set D of the plane. Then $u(\boldsymbol{x}) = u(x,y)$ possesses all partial derivatives of all orders in D.*

Proof. If D is a disk, according to the Poisson's formula we are done. If D is a general domain, consider a point $\boldsymbol{x}_0 \in D$ and a disk centered at $\boldsymbol{x}_0$, which completely lies in D. By Poisson's formula, we know u is infinitely differentiable in the disk, in particular, differentiable at $\boldsymbol{x}_0$. Since $\boldsymbol{x}_0 \in D$ is arbitrary, we are done. □

8.4 Separation of Variables: Wedge, Annuli, and Exterior of Circles

This section briefly presents more examples of the application of separation of variables to special domains: wedge $\{0 < \theta < \theta_0, 0 < r < a\}$, annulus $\{0 < a < r < b\}$, and the exterior of a circle $\{a < r < \infty\}$. For more detailed descriptions, please refer to section 6.4 of the textbook [Str07].

The Wedge. Let us consider the Laplace's equation on the wedge with three sides $\theta = 0, \theta = \beta, r = a$ and solve the Laplace's equation with the homogeneous Dirichlet boundary condition on the straight sides and the inhomogeneous Neumann boundary conditon on the curved side. The BCs are

$$u(r,0) = 0 = u(r,\beta); \quad \frac{\partial u}{\partial r}(a,\theta) = h(\theta).$$

The *polar-coordinate* version of the separation of variables works just as for the circle

$$\Theta'' + \lambda\Theta = 0; \quad r^2 R'' + rR' - \lambda R = 0.$$

The corresponding eigenvalue problem is

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(0) = \Theta(\beta) = 0. \end{cases}$$

The solutions are

$$\lambda_n = \left(\frac{n\pi}{\beta}\right)^2; \quad \Theta(\theta) = \sin\frac{n\pi\theta}{\beta}.$$

As shown before, the solution to the below ODE

$$r^2 R'' + rR' - \lambda R = 0,$$

is $R_n(r) = r^{\pm n\pi/\beta}$. The negative exponent is rejected again because we are looking for a solution that is continuous in the wedge as well as its boundary. Thus we end up with the series

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta}.$$

Finally, we substitute the inhomogeneous boundary condition

$$h(\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} a^{-1+n\pi/\beta} \sin\frac{n\pi\theta}{\beta}.$$

By Fourier's theory

$$\frac{n\pi}{\beta} a^{-1+n\pi/\beta} A_n = \frac{2}{\beta}\int_0^{\beta} h(\theta)\sin\frac{n\pi\theta}{\beta}\,d\theta.$$

The Annulus. The Dirichlet problem for an annulus is

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad in\ 0 < a^2 < x^2 + y^2 < b^2 \\ u = g(\theta), \quad x^2 + y^2 = a^2 \\ u = h(\theta), \quad x^2 + y^2 = b^2. \end{cases}$$

The separated solutions are just the same as for a circle except that we do not throw out the functions r^{-n} and $\log r$, as these functions are perfectly finite within the annulus. So the solution is

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0\log r) + \sum_{n=1}^{\infty}\left[(C_n r^n + D_n r^{-n})\cos n\theta + (A_n r^n + B_n r^{-n})\sin n\theta\right].$$

These coefficients are determined by setting $r = a$ and $r = b$ and use the inhomogeneous boundary conditions.

The Exterior of a Circle. The Dirichlet problem for the exterior of a circle is

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad x^2 + y^2 > a^2 \\ u = h(\theta), \quad x^2 + y^2 = a^2. \\ u \text{ is bounded as } x^2 + y^2 \to \infty. \end{cases}$$

The reasoning is almost the same as the interior case. But here, instead of requiring finiteness at the *origin*, we have imposed boundedness at *infinity*. Therefore, r^n is excluded while r^{-n} is preserved. So we have

$$u(r, \theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta).$$

The boundary condition means

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$

So that

$$A_n = \frac{a^n}{n} \int_{-\pi}^{\pi} h(\theta) \cos n\theta d\theta$$
$$B_n = \frac{a^n}{n} \int_{-\pi}^{\pi} h(\theta) \sin n\theta d\theta$$

Similarly, substitute A_n, B_n into $u(r, \theta)$, the solution can be explicitely summed. Therefore, we arrive at a modified *Poisson's formula*

$$u(r, \theta) = (r^2 - a^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{d\phi}{2\pi}, \quad r > a.$$

8.5 Exercise

Problem 8.8. *Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell $0 < a < r < b$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$, where A and B are constants.*

Problem 8.9. *Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with boundary condition*

$$u = 1 + 3\sin\theta, \quad \text{on } r = a.$$

9 Tutorial 9

9.1 Green's First Identity

Green's First Identity. Suppose both u and v are univariate. We have the product rule

$$(vu_x)_x = v_x u_x + v u_{xx}.$$

Now, suppose $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}$ have n independent variables. We know that

$$\nabla \cdot (v \nabla u) = \sum_{i=1}^{n} (v u_{x_i})_{x_i} = \sum_{i=1}^{n} v_{x_i} u_{x_i} + v u_{x_i x_i} = \nabla v \cdot \nabla u + v \Delta u.$$

Then we integrate and use the divergence theorem on both sides to get

$$\iint_{\partial D} v \frac{\partial u}{\partial \boldsymbol{n}} dS = \iiint_D \nabla v \cdot \nabla u d\boldsymbol{x} + \iiint_D v \Delta u d\boldsymbol{x}.$$

Note that this is just the high-dimensional version of *integration by parts.*

Example 9.1 (Energy Method)**.** *Prove the uniqueness of the Dirichlet problem by energy method*

$$\begin{cases} \Delta u = f \\ u|_{\partial D} = 0. \end{cases}$$

Proof. Suppose we have two solutions u_1 and u_2. Set $u := u_1 - u_2$, then $\Delta u = 0$ and $u|_{\partial D} = 0$. Use the Green's first identity we have

$$0 = \iint_{\partial D} u \frac{\partial u}{\partial n} dS = \iiint_D |\nabla u|^2 d\boldsymbol{x} \Rightarrow |\nabla u| = 0.$$

Therefore, $u \equiv const$. Since u vanishes on the boundary, we know that u is constantly zero on D. □

Problem 9.1. *Prove the uniqueness of the Neumann Problem up to a constant by energy method.*

Mean Value Property in $\mathbb{R}^n$. In $\mathbb{R}^n$ the mean value property states that the average value of any harmonic function over any hypersphere equals its value at the center, i.e.

$$u(\boldsymbol{x}_0) = \frac{1}{|\partial B_r(\boldsymbol{x}_0)|} \int_{\partial B_r(\boldsymbol{x}_0)} u dS_r,$$

where $B_r(\boldsymbol{x}_0) \subset \mathbb{R}^n$ is a ball centered at $\boldsymbol{x}_0$ with radius r and S_r stands for the surface measure.

Proof. WLOG assume $\boldsymbol{x}_0 = 0$, otherwise, we just move the coordinate frame. Set $w_n := |\partial B_1|$, which is the surface area for the sphere of radius 1. Consider

$$f(r) := \frac{1}{|\partial B_r|} \int_{\partial B_r} u dS_r.$$

Recall that $|\partial B_r| = r^{n-1} |\partial B_1| = r^{n-1} w_n$, then

$$f(r) = \frac{1}{r^{n-1} w_n} \int_{\partial B_r} u dS_r.$$

Do a change of variable $r\boldsymbol{y} = \boldsymbol{x}$, we have

$$f(r) = \frac{1}{r^{n-1} w_n} \int_{\partial B_1} u(r\boldsymbol{y}) r^{n-1} dS_1 = \frac{1}{w_n} \int_{\partial B_1} u(r\boldsymbol{y}) dS_1.$$

Remember that

$$\begin{cases} x \in \partial B_r, \\ y \in \partial B_1, \\ dS_r = r^{n-1} dS_1. \end{cases}$$

Differentiate $f(r)$ we have

$$f'(r) = \frac{1}{w_n} \int_{\partial B_1} \boldsymbol{y} \cdot \nabla_x u(r\boldsymbol{y}) dS_1.$$

Note that

$$0 = \int_{B_r} \Delta u d\boldsymbol{x} = \int_{\partial B_r} \frac{\partial u}{\partial \boldsymbol{n}} dS_r = \int_{\partial B_r} \nabla_x u \cdot \boldsymbol{n} dS_r.$$

Since $\boldsymbol{n} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$,

$$\int_{\partial B_r} \nabla_x u \cdot \boldsymbol{n} dS_r = \int_{\partial B_r} \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \cdot \nabla_x u dS_r = \ (\text{set } r\boldsymbol{y} = \boldsymbol{x}) \ \int_{\partial B_1} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \cdot \nabla_x u(r\boldsymbol{y}) r^{n-1} dS_1$$

The above implies that

$$f'(r) = 0.$$

Therefore, f is independent of r. Since

$$\lim_{r \to 0^+} \frac{1}{|\partial B_r|} \int_{\partial B_r} u(\boldsymbol{x}) dS_r = u(\boldsymbol{0}), \quad \text{(Why?)}$$

We conclude that

$$u(\boldsymbol{x}_0) = \frac{1}{|\partial B_r(\boldsymbol{x}_0)|} \int_{\partial B_r(\boldsymbol{x}_0)} u dS_r.$$

□

Problem 9.2. *Prove the Maximum Principle for harmonic functions by the mean value property.*

Dirichlet Principle. The *Dirichlet Principle* states that among all the functions $w(\boldsymbol{x})$ in D that satisfies the Dirichlet boundary condition

$$w = h(\boldsymbol{x}), \quad \boldsymbol{x} \in D,$$

the lowest energy occurs for the harmonic function.

Proof. Let $u(\boldsymbol{x})$ be the unique (Why?) harmonic function in D that satisfies the boundary condition. Let $w(\boldsymbol{x})$ be any function in D that satisfies the boundary condition, we want to show that

$$E[w] \geq E[u].$$

We let $v = u - w$ and expand the square in the integral

$$E[w] = \frac{1}{2} \iint_D |\nabla(u - v)|^2 d\boldsymbol{x} = E[u] - \iiint_D \nabla u \cdot \nabla v d\boldsymbol{x} + E[v].$$

Apply Green's first identity to the middle term we then know that it vanishes since $v = 0$ on ∂D and u is a harmonic function on D. Therefore,

$$E[w] = E[u] + E[v] \quad \Leftrightarrow \quad E[w] \geq E[u].$$

□

Remark 9.1. *There is an alternate approach to prove the Dirichlet principle in p.183 of the textbook [Str07]. It requires some function approximation skills.*

9.2 Green's Second Identity

Recall that the Green's first identity is

$$\iint_{\partial D} v\frac{\partial u}{\partial \boldsymbol{n}}dS = \iiint_D \nabla v\cdot\nabla u d\boldsymbol{x} + \iiint_D v\Delta u d\boldsymbol{x}.$$

If we exchange the role of u and v and subtract the two equalities, we would obtain

$$\iiint_D (u\Delta v - v\Delta u)d\boldsymbol{x} = \iint_{\partial D}\left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}\right)dS,$$

which is the so-called *Green's second identity.* We can extend the definition of *symmetric boundary conditions.* We say that a boundary condition is *symmetric* for the operator Δ if the right hand side of Green's second identity vanishes. You should easily check that each of the three classical boundary conditions is symmetric.

Representation formula. The reason why we introduce Green's second identity is that it leads to a representation formula for any *harmonic* function as an integral over the boundary.

Theorem 9.3 (Representation Formula)**.** *If $\Delta u = 0$ in D, then*

$$u(\boldsymbol{x}_0) = \iint_{\partial D}\left(-u(\boldsymbol{x})\frac{\partial}{\partial \boldsymbol{n}}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|}\right) + \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|}\frac{\partial u}{\partial \boldsymbol{n}}\right)\frac{dS}{4\pi}.$$

Proof. The idea is to apply the Green's second identity with $v(\boldsymbol{x}) = (-4\pi|\boldsymbol{x}-\boldsymbol{x}_0|)^{-1}$. However, we cannot naively apply it over the whole domain D since $v(\boldsymbol{x})$ is unbounded at $\boldsymbol{x}_0$. Therefore, instead we consider a domain where a ball centered at $\boldsymbol{x}_0$ is excluded. Let us denote the domain by D_ϵ. WLOG assume that $\boldsymbol{x}_0$ is the origin (Why WLOG?), then $v(\boldsymbol{x}) = -1/(4\pi r)$, where $r = |\boldsymbol{x}|$. Writing down the Green's second identity for this choice of v, then we have

$$-\iint_{\partial D_\epsilon}\left(u\cdot\frac{\partial}{\partial \boldsymbol{n}}\frac{1}{r} - \frac{\partial u}{\partial \boldsymbol{n}}\cdot\frac{1}{r}\right)dS = 0.$$

Note that

$$\partial D_\epsilon = \partial D\bigcup\{r=\epsilon\}.$$

Therefore we have (note that $\partial/\partial\boldsymbol{n} = \partial/\partial r$ on $\{r=\epsilon\}$).

$$-\iint_{\partial D}\left(u\cdot\frac{\partial}{\partial \boldsymbol{n}}\frac{1}{r} - \frac{\partial u}{\partial \boldsymbol{n}}\cdot\frac{1}{r}\right)dS = -\iint_{r=\epsilon}\left(u\cdot\frac{\partial}{\partial r}\frac{1}{r} - \frac{\partial u}{\partial r}\cdot\frac{1}{r}\right)dS.$$

The above identity is valid for any choice of $\epsilon > 0$. To prove the theorem, we only need to show the RHS approaches $4\pi u(\mathbf{0})$ as $\epsilon \to 0^+$. Note that the RHS is just

$$\frac{1}{\epsilon^2}\iint_{r=\epsilon} u dS + \frac{1}{\epsilon}\iint_{r=\epsilon}\frac{\partial u}{\partial r}dS = 4\pi\bar{u} + 4\pi\epsilon\overline{\frac{\partial u}{\partial r}},$$

where the overline stands for the average value. By continuity of u and boundedness of $\partial u/\partial r$, we conclude that

$$4\pi\bar{u} + 4\pi\epsilon\overline{\frac{\partial u}{\partial r}} \to 4\pi u(\mathbf{0}) + 0, \quad \text{as } \epsilon\to 0^+.$$

□

Problem 9.4. *Show that the corresponding representation formula for two-dimensional case is*

$$u(\boldsymbol{x}_0) = \frac{1}{2\pi}\int_{\partial D}\left[u(\boldsymbol{x})\frac{\partial}{\partial \boldsymbol{n}}\log|\boldsymbol{x}-\boldsymbol{x}_0| - \frac{\partial u}{\partial \boldsymbol{n}}\log|\boldsymbol{x}-\boldsymbol{x}_0|\right]ds.$$

Hint. Choose $v(\boldsymbol{x}) = \log|\boldsymbol{x}-\boldsymbol{x}_0|$ and apply the same method.

9.3 Green's Function

The motivation of introducting *Green's function* is that we want to remove one of the terms in the representation formula. The modified function is called the *Green's function* for the domain D.

Definition 9.1 (Green's function)**.** *The Green's function $G(\boldsymbol{x})$ for the operator $-\Delta$ and the domain D at the point $\boldsymbol{x}_0 \in D$ is a function defined for $\boldsymbol{x} \in D$ such that*

1. *$G(\boldsymbol{x}) \in C^2(D)$ and $\Delta G = 0$ in D except at the point $\boldsymbol{x}_0$.*
2. *$G(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial D$.*
3. *The function $G(\boldsymbol{x}) + 1/(4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)$ is finite at $\boldsymbol{x}_0$ and has continuous second derivatives everywhere and is harmonic at $\boldsymbol{x}_0$.*

It can be shown that Green's function exists and is unique.

Problem 9.5. *Show that the Green's function is unique.*

Theorem 9.6. *If $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function, then the solution of the Dirichlet problem is given by the formula*

$$u(\boldsymbol{x}_0) = \iint_{\partial D} u(\boldsymbol{x}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial \boldsymbol{n}} dS.$$

Proof Sketch. Set $v(\boldsymbol{x}) = -(4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)^{-1}$. Write $G(\boldsymbol{x}, \boldsymbol{x}_0) = v(\boldsymbol{x}) + H(\boldsymbol{x})$. By definition of Green's function we know that H is a harmonic function throughout the domain D. We apply Green's second identity to the pair of harmonic functions $u(\boldsymbol{x})$ and $H(\boldsymbol{x})$. □

Remark 9.2. *Although the theorem provides a general way to represent the solution to the homogeneous Dirichlet problem, it is usually not easy to find the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$ especially when the domain D is NOT regular.*

Problem 9.7. *Prove the solution formula of the problem*

$$\Delta u = f, \text{ in } D; \quad u = h, \text{ on } \partial D, \tag{104}$$

is given by

$$u(\boldsymbol{x}_0) = \iint_{\partial D} h(\boldsymbol{x}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial \boldsymbol{n}} dS + \iiint_D f(\boldsymbol{x}) G(\boldsymbol{x}, \boldsymbol{x}_0) d\boldsymbol{x}. \tag{105}$$

Lemma 9.8 (Representation Formula for Poisson Equation)**.** *If u solves the Poisson equation $\Delta u = f$ on Ω, then*

$$u(\boldsymbol{x}_0) = \iint_{\partial\Omega} u \frac{\partial v}{\partial \boldsymbol{n}} - \frac{\partial u}{\partial \boldsymbol{n}} v dS + \iiint_\Omega v f d\boldsymbol{x}. \tag{106}$$

where $v = -\frac{1}{4\pi} \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_0\|}$.

Problem 9.9. *Let $\phi \in C^2(\Omega)$ be defined on all of the three-dimensional space that vanishes outside some sphere. Show that*

$$\phi(\mathbf{0}) = -\iiint \frac{1}{\|\boldsymbol{x}\|} \Delta\phi(\boldsymbol{x}) \frac{d\boldsymbol{x}}{4\pi}. \tag{107}$$

This integration is taken over the region where $\phi(\boldsymbol{x})$ is not zero.

Symmetry of the Green's Function. For any region D we have a Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$. It is always symmetric

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = G(\boldsymbol{x}_0, \boldsymbol{x}).$$

Proof. We consider a pair of functions $u := G(\boldsymbol{x}, \boldsymbol{a})$ and $v := G(\boldsymbol{x}, \boldsymbol{b})$ defined on the domain D_ϵ, which denotes the domain D where two spheres centered at $\boldsymbol{a}$ and $\boldsymbol{b}$ with radius ϵ are excluded. Then the boundary of D_ϵ has three compnents

$$\partial D_\epsilon = \partial D \bigcup \{|\boldsymbol{x} - \boldsymbol{a}| = \epsilon\} \bigcup \{|\boldsymbol{x} - \boldsymbol{b}| = \epsilon\}.$$

Apply Green's function on D_ϵ to the pair u and v and split the integral over ∂D_ϵ into three parts, we immediately have

$$A_\epsilon + B_\epsilon = 0, \quad \text{(Why?)}$$

where $A_\epsilon = \iint_{|\boldsymbol{x}-\boldsymbol{a}|=\epsilon}(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}})dS$. One can show that

$$\lim_{\epsilon\to 0^+} A_\epsilon = v(\boldsymbol{a}); \quad \lim_{\epsilon\to 0^+} B_\epsilon = -u(\boldsymbol{b}).$$

Therefore,

$$0 = \lim_{\epsilon\to 0^+} A_\epsilon + B_\epsilon = v(\boldsymbol{a}) - u(\boldsymbol{b}).$$

Thus,

$$G(\boldsymbol{a}, \boldsymbol{b}) = G(\boldsymbol{b}, \boldsymbol{a}).$$

□

9.4 Half-Space and Sphere

We solve for the harmonic functions in a half-space and a sphere by combining the Green's function with the method of reflection.

The Half-Space. The domain is

$$\Omega = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}.$$

We already know the function $1/(4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)$ satisfies two of the three conditions for the Green's function. We now want to modify it to satisfy all of the requirements. The corresponding Green's function should be of the following form

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|} + H(\boldsymbol{x}),$$

where $H(\boldsymbol{x})$ is harmonic and smooth on the domain Ω and $G(\boldsymbol{x}, \boldsymbol{x}_0) = 0$ on $\partial\Omega$. The idea is: we consider the reflected point of $\boldsymbol{x}_0$ across the z-axis, $\boldsymbol{x}_0^* = (x, y, -z)$, the function

$$\frac{1}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0^*|},$$

is harmonic and smooth in Ω. We therefore only need to check that $G(\boldsymbol{x}, \boldsymbol{x}_0)|_{\partial\Omega} = 0$. Note that $\partial\Omega = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. On $\partial\Omega$, it is obvious that

$$|\boldsymbol{x} - \boldsymbol{x}_0| = |\boldsymbol{x} - \boldsymbol{x}_0^*|.$$

Thus, the function G defined as

$$G(\boldsymbol{x}, \boldsymbol{x}_0) := -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} + \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0^*|},$$

is the desired Green's function in Ω of the point $\boldsymbol{x}_0$.

Now that we have the Green's function, we next derive the solution formula for the Dirichlet problem

$$\begin{cases} \Delta u = 0, \quad z > 0 \\ u(x, y, 0) = h(x, y). \end{cases}$$

Recall the theorem that *If $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function on the domain D of the point $\boldsymbol{x}_0$, then the solution formula of the Dirichlet problem is given by the formula*

$$u(\boldsymbol{x}_0) = \iint_{\partial D} u(\boldsymbol{x})\frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial \boldsymbol{n}}dS.$$

Note that

$$\frac{\partial}{\partial \boldsymbol{n}} = \boldsymbol{n}\cdot\nabla = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = -\frac{\partial}{\partial z}.$$

Therefore,

$$\frac{\partial G}{\partial \boldsymbol{n}} = -\frac{\partial}{\partial z}G = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|^2}\partial_z|\boldsymbol{x}-\boldsymbol{x}_0| + \frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0^*|^2}\partial_z|\boldsymbol{x}-\boldsymbol{x}_0^*| = -\frac{z-z_0}{4\pi|\boldsymbol{x}-\boldsymbol{x}_0|^3} + \frac{z-z_0^*}{4\pi|\boldsymbol{x}-\boldsymbol{x}_0^*|^3}.$$

Since $z_0^* = -z_0$,

$$\frac{\partial G}{\partial \boldsymbol{n}} = \frac{1}{2\pi}\frac{z_0}{|\boldsymbol{x}-\boldsymbol{x}_0|^3}.$$

Therefore the solution is

$$u(\boldsymbol{x}_0) = \iint_{\partial D} \frac{h(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x}_0|^3}dS.$$

This is the complete formula that solves the Dirichlet problem for the half-space.

The Sphere. The Green's function for the ball $\{|\boldsymbol{x}| < a\}$ of radius a can also be found by the reflection method. However, different from the case of half-space, the reflection is performed across the sphere $\{|\boldsymbol{x}| = a\}$. Consider the function G of the following form

$$G(\boldsymbol{x},\boldsymbol{x}_0) := -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|} + \frac{C}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0^*|}.$$

- If $C = 1$, is it possible to find a $\boldsymbol{x}_0^* \notin B_a$ such that

  $$|\boldsymbol{x}-\boldsymbol{x}_0| = |\boldsymbol{x}-\boldsymbol{x}_0^*|, \quad \forall x \in \partial B_a?$$

  The answer is negative. (Why?)

- Is it possible to find a C such that

  $$|\boldsymbol{x}-\boldsymbol{x}_0| = C|\boldsymbol{x}-\boldsymbol{x}_0^*|.$$

  This is correct if $\boldsymbol{x}_0^* = \frac{a^2}{|\boldsymbol{x}_0|^2}\boldsymbol{x}_0$. Since $\frac{a^2}{|\boldsymbol{x}_0|^2} > 1$, the point $x_0^* \neq B_a$. Note that

  $$|\boldsymbol{x}-\boldsymbol{x}_0^*| = \left|\boldsymbol{x} - \frac{a^2}{|\boldsymbol{x}_0|^2}\boldsymbol{x}_0\right| = \frac{a}{|\boldsymbol{x}_0|}\left|\frac{|\boldsymbol{x}_0|}{a}x - \frac{a}{|\boldsymbol{x}_0|}\boldsymbol{x}_0\right| = \frac{a}{|\boldsymbol{x}_0|}|\boldsymbol{x}-\boldsymbol{x}_0|. \quad \text{(Consider the geometry)}$$

  The above equality implies

  $$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|} = \frac{a}{|\boldsymbol{x}_0|}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0^*|}, \quad \textit{on } \partial B_a.$$

  Therefore, it would be wise to set $C = \frac{a}{|\boldsymbol{x}_0|}$. This means that

  $$G(\boldsymbol{x},\boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}_0|} + \frac{a}{|\boldsymbol{x}_0|}\frac{1}{4\pi|x-x_0^*|}.$$

  The above clearly only holds when $\boldsymbol{x}_0 \neq \boldsymbol{0}$. If $\boldsymbol{x}_0 = \boldsymbol{0}$, the Green's function for the domain can be shown to be

  $$G(\boldsymbol{x},\boldsymbol{0}) = -\frac{1}{4\pi|\boldsymbol{x}|} + \frac{1}{4\pi a}.$$

Problem 9.10. *Show that for the sphere case, if $\boldsymbol{x}_0 = \boldsymbol{0}$, the Green's function is*

$$G(\boldsymbol{x},\boldsymbol{0}) = -\frac{1}{4\pi|\boldsymbol{x}|} + \frac{1}{4\pi a}.$$

Again, we want to use the Green's function we find to solve the Dirichlet problem on the sphere.

$$\begin{cases} \Delta u = 0, & |\boldsymbol{x}| < a \\ u = h, & |\boldsymbol{x}| = a. \end{cases}$$

To this end, we need to find $\frac{\partial G}{\partial \boldsymbol{n}}$ on the on $|\boldsymbol{x}| = a$. Note that

$$\nabla G = \frac{\boldsymbol{x} - \boldsymbol{x}_0}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|^3} - \frac{a}{|\boldsymbol{x}_0|} \frac{\boldsymbol{x} - \boldsymbol{x}_0^*}{4\pi|\boldsymbol{x}_0 - \boldsymbol{x}_0^*|^3}.$$

Substitute $\boldsymbol{x}_0^* = (a/|\boldsymbol{x}_0|)^2\boldsymbol{x}_0$ and $|\boldsymbol{x} - \boldsymbol{x}_0^*| = (a/|\boldsymbol{x}_0|)|\boldsymbol{x} - \boldsymbol{x}_0|$, we get

$$\nabla G = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|^3} \left[\boldsymbol{x} - (|\boldsymbol{x}_0|/a)^2\boldsymbol{x}\right].$$

Thus,

$$\frac{\partial G}{\partial \boldsymbol{n}} = \frac{\boldsymbol{x}}{a} \cdot \nabla G = \frac{a^2 - |\boldsymbol{x}_0|^2}{4\pi a|\boldsymbol{x} - \boldsymbol{x}_0|^3}.$$

Therefore,

$$u(\boldsymbol{x}_0) = \frac{a^2 - |\boldsymbol{x}_0|^2}{4\pi a} \iint_{|\boldsymbol{x}|=a} \frac{h(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}_0|^3} dS.$$

This is the *three-dimensional* version of the *Poisson's formula*. In more classical notation, it would be written in the usual spherical coordinates as

$$u(r_0, \theta_0, \phi_0) = \frac{a(a^2 - r_0^2)}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{h(\theta, \phi)}{(a^2 + r_0^2 - 2ar_0 \cos\psi)^{3/2}} \sin\theta d\theta d\phi,$$

where $\psi := angle(\boldsymbol{x}_0, \boldsymbol{x})$ and $r_0 = |\boldsymbol{x}_0|$.

Problem 9.11. *Begin with the function* $1/(2\pi)\log r$, *show that the Green's function for the disk* $\{|\boldsymbol{x}| < a\}$ *is*

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{2\pi} \log|\boldsymbol{x} - \boldsymbol{x}_0| - \frac{1}{2\pi} \log(\frac{|\boldsymbol{x}_0|}{a}|\boldsymbol{x}_0 - \boldsymbol{x}_0^*|),$$

where $\boldsymbol{x}_0^*$ *is defined similarly as the 3d case. Moreover, show that the solution to the Dirichlet problem*

$$\begin{cases} \Delta u = 0, & x^2 + y^2 < a^2 \\ u = h, & x^2 + y^2 = a^2. \end{cases}$$

is

$$u(\boldsymbol{x}_0) = \frac{a^2 - |\boldsymbol{x}_0|^2}{2\pi a} \int_{|\boldsymbol{x}|=a} \frac{h(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}_0|^2} ds.$$

Note that this is indeed the ***Poisson's formula***.

10 Tutorial 10

10.1 Distributions

Distribution. The introduction of the theory of distributions, in our context, will provide a succinct and elegant interpretation of Green's functions. We have encountered the *approximate delta functions* in many places in this course. The delta function is defined (formally) to be infinite at $x = 0$ and zero at all $x \neq 0$. It should integrate to one: $\int_{-\infty}^{\infty} \delta(x)dx = 1$. Certainly it is NOT a function since a function defined in this way cannot satisfy the integrating to one requirement. It is indeed a more general object, which is called a *distribution.* Recall that a function is a rule which assigns numbers to numbers. However, a distribution is a rule that assigns numbers to functions.

Definition 10.1 (Delta Function)**.** *The delta function is the rule that assigns the number $\phi(0)$ to the function $\phi(x)$.*

We need to specify what function $\phi(x)$ is used here. A *test function* $\phi(x)$ is a real C^∞ function (a function all of whose derivatives exist) that vanishes outside a finite interval. Thus $\phi : \mathbb{R} \to \mathbb{R}$ is defined and differentiable for all $x \in \mathbb{R}$ and $\phi(x) \equiv 0$ for x large and for x small. Let $\mathcal{D}$ denote the collection of all test functions.

Definition 10.2 (Distribution)**.** *A distribution f is a functional (or a rule): $\mathcal{D} \to \mathbb{R}$ which is linear and continuous for the following sense. If $\phi \in \mathcal{D}$ is a test function, then we denote the corresponding real number as (f, ϕ). (A more intuitive way is to write $f(\phi)$)*

- The linearity of a functional f means
$$(f, a\phi + b\psi) = a(f, \phi) + b(f, \psi), \quad \forall a, b \in \mathbb{R}.$$
- The continuity means if $\{\phi_n\} \subset \mathcal{D}$ is a sequence of test functions that vanish outsider a compact set and converge uniformly to a test function ϕ, and **if all their derivatives do as well**, then
$$(f, \phi_n) \to (f, \phi), \quad n \to \infty.$$

Example 10.1. *The delta function $\delta(x)$ is a distribution that assigns $\phi(x) \in \mathcal{D}$ to $\phi(0)$. We can specify the delta function as $\phi \mapsto \phi(0)$.*

Example 10.2. *The functional $\phi \mapsto \phi''(5)$ is a distribution. It is linear and a continuous functional, which follows from the assumption that $\phi_n \to \phi, \phi_n' \to \phi'$, and $\phi_n'' \to \phi''$ uniformly.*

Example 10.3. *Let $f \in L^1$. It corresponds to the distribution*
$$\phi \mapsto \int_{-\infty}^{\infty} f(x)\phi(x)dx.$$
Note that the RHS is a finite real number since $\phi \in \mathcal{D}$. It can be easily verified that the distribution defined above is linear and is also continuous. Note that the mathematical expectation is a particular case of the above distribution by picking $f(x) = x$.
$$\mathbb{E}_{x\sim p(x)}[X] := \int_\Omega xp(x)dx,$$
where Ω is the sample space. Expectation indeed maps a random variable $X : \Omega \to \mathbb{R}$, which is a measurable function, to a real number. Therefore, by definition it is a functional.

Remark 10.1. *Because of the above example, it is quite natural to use the integral notation for the delta function*
$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0).$$
and speak of the delta function as if it were a true function.

Problem 10.1. *Verify directly from the definition that $\phi \to \int_{-\infty}^{\infty} f(x)\phi(x)dx$ is a distribution if $f(x)$ is any function that is integrable on each bounded set.*

Problem 10.2. *Let f be any distribution. Verify that the functional f' defined by $(f', \phi) := -(f, \phi)$ satisfies the linearity and continuity properties and therefore is another distribution.*

Convergence of Distributions. We start by introducing the concept of weak convergence.

Definition 10.3. *If f_n is a sequence of distributions and f is another distribution, we say that f_n converges weakly to f if*

$$(f_n, \phi) \to (f, \phi), \quad n \to \infty.$$

Recall that several weeks ago we introduced the weak convergence in L^2, which says that a sequence of functions $f_n \in L^2(\Omega)$ is said to converge weakly to a function $f \in L^2(\Omega)$ if for any $g \in L^2(\Omega)$

$$\int_\Omega f_n \cdot g d\lambda \to \int_\Omega f \cdot g d\lambda, \quad n \to \infty.$$

Note that by Example 10.2 we know the integration above can be viewed as a distribution defined by f_n, we can equivalently rewrite it as

$$(f_n, g) \to (f, g), n \to \infty,$$

which conforms with the general definition of weak convergence. Indeed, in the L^p theory, we have the famous *Riesz Representation Theorem* (you might have also heard of this theorem in your advanced linear algebra course). The theorem says that any continuous linear functional defined on $L^p(\Omega), 1 \leq p \leq \infty$ can be represented as

$$\int_\Omega f \cdot g d\lambda = (g, f),$$

for some fixed $g \in L^q(\Omega)$, q is the conjugate of p (i.e. $1/p + 1/q = 1$).

Example 10.4. *The fundamental solution for the diffusion equation on the whole real line is $\Phi(x,t) = 1/\sqrt{4\pi kt} e^{-x^2/4kt}$ for $t > 0$. We have proved before that*

$$\int_\mathbb{R} \Phi(x,t)\phi(x)dx \to \phi(0), \quad N \to \infty.$$

Because for each t we may consider the function $\Phi(x,t)$ at a distribution since it is of the form of an integral, this means that

$$\Phi(x,t) \to \delta(x), \quad \textit{weakly as } t \to \infty.$$

Example 10.5. *Let $K_N(\theta)$ be the Dirichlet kernel.*

$$K_N(\theta) = 1 + 2\sum_{n=1}^N \cos n\theta = \frac{\sin[(N + \frac{1}{2}\theta)]}{\sin \frac{1}{2}\theta}.$$

We proved that

$$\int_{-\pi}^{\pi} K_N(\theta)\phi(\theta)d\theta \to 2\pi\phi(0), \quad N \to \infty.$$

Therefore,

$$K_N(\theta) \to 2\pi\delta(\theta), \quad \textit{weakly as } N \to \infty.$$

Derivative of a Distribution. The derivative of a distribution always exists and is another distribution. Consider the following motivating example. let $f(x)$ be any C^1 function and $\phi(x)$ be any test function. Integration by parts shows that

$$\int_\mathbb{R} f'(x)\phi(x)dx = -\int_\mathbb{R} f(x)\phi'(x)dx,$$

since $\phi(x) = 0$ for $|x|$ large.

Definition 10.4 (Derivative)**.** *For any distribution f, the derivative f' is defined by the formula*

$$(f', \phi) := -(f, \phi'), \quad \phi \in \mathcal{D}.$$

Note that it is easy to check that f' is a distribution since it is linear and continuous. Moreover, if $f_n \to f$ weakly, then $f_n' \to f'$ weakly as well. The reason is that $(f_n', \phi) = -(f_n, \phi') \to -(f, \phi') = (f', \phi), \forall \phi \in \mathcal{D}$.

Example 10.6. *The derivatives of the delta function are*

$$(\delta', \phi) = -(\delta, \phi') = -\phi'(0).$$

$$(\delta'', \phi) = -(\delta', \phi') = (\delta, \phi'') = \phi''(0).$$

Example 10.7. *The Heaviside function is defined by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. For any test function, $(H', \phi) = -(H, \phi') = -\int_0^\infty \phi'(x)dx = \phi(0)$. Thus,*

$$H' = \delta.$$

Thus plus function $p(x) = x^+$ is defined as $p(x) = x$ for $x \geq 0$, and $p(x) = 0$ for $x \leq 0$. Then $p' = H$ and $p'' = \delta$.

Distribution in Three Dimensions. A test function $\phi(\boldsymbol{x}) = \phi(x, y, z)$ is a real C^∞ function that vanishes outside some ball. $\mathcal{D}$ denotes the set of all test functions of $\boldsymbol{x}$. Then the definition of a distribution is identical to the one-dimensional case except we replace common intervals by common balls. The delta function δ is defined as the functional $\phi \mapsto \phi(\mathbf{0})$. Its partial derivative $\partial\delta/\partial z$ is defined as the functional $\phi \mapsto -(\partial\phi/\partial z)(\mathbf{0})$. If $f(\boldsymbol{x})$ is any ordinary intergrable function, it is considered to be the same as the distribution $\phi \mapsto \iiint_{\mathbb{R}^3} f(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x}$.

10.2 Green' Functions Revisited

Let $r = |\boldsymbol{x}|$ and $\phi(\boldsymbol{x}) \in \mathcal{D}$, then we know that

$$\phi(\mathbf{0}) = -\iiint \frac{1}{r}\Delta\phi(\boldsymbol{x})\frac{d\boldsymbol{x}}{4\pi}.$$

Note that if we interpret $\Delta(-\frac{1}{4\pi r})$ as a 3-d distribution, we know that

$$\iiint \Delta(-\frac{1}{4\pi r})\phi(\boldsymbol{x})d\boldsymbol{x} = -\iiint \frac{1}{r}\Delta\phi(\boldsymbol{x})\frac{d\boldsymbol{x}}{4\pi}.$$

By definition, we see that

$$\Delta(-\frac{1}{4\pi r}) = \delta(\boldsymbol{x}).$$

Because $\delta(\boldsymbol{x})$ vanishes except at the origin, the above formula explains why $1/r$ is a harmonic function away from the origin and it explains exactly how it differs from being harmonic at the origin.

Consider now the Dirichlet problem for the Poisson's equation

$$\begin{cases} \Delta f = 0, & in\ D \\ u = 0, & on\ \partial D. \end{cases}$$

Its solution is

$$u(\boldsymbol{x}_0) = \iiint_D G(\boldsymbol{x}, \boldsymbol{x}_0)f(\boldsymbol{x})d\boldsymbol{x},$$

where $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function. Now fix the point $\boldsymbol{x}_0 \in D$, the LHS can be written as

$$u(\boldsymbol{x}_0) = \iiint_D \delta(\boldsymbol{x} - \boldsymbol{x}_0)u(\boldsymbol{x})d\boldsymbol{x}.$$

We assume that $u(\boldsymbol{x})$ is an arbitrary test function whose support is a bounded subset of D. The RHS is

$$u(\boldsymbol{x}_0) = \iiint_D G(\boldsymbol{x}, \boldsymbol{x}_0)(\boldsymbol{x})d\boldsymbol{x} = \iiint_D \Delta G(\boldsymbol{x}, \boldsymbol{x}_0)u(\boldsymbol{x})d\boldsymbol{x},$$

where ΔG is understood in the sense of a distribution. Because $u(\boldsymbol{x})$ can be an arbitrary test function in D, we deduce that

$$\Delta G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \quad \textit{in } D.$$

As we know, $G(\boldsymbol{x}, \boldsymbol{x}_0) + (4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)^{-1}$ is harmonic in the whole domain D, including at $\boldsymbol{x}_0$. Thus,

$$\Delta G = -\Delta \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \quad \textit{in } D.$$

Indeed, $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the unique distribution that satisfies the following

$$\begin{cases} \Delta G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \quad \textit{in } D, \\ G = 0, \quad \textit{on } \partial D. \end{cases}$$

END

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