MAT4010: Functional Analysis

Tutorial 5: The Baire's Category Theorem*

Mou, Minghao
The Chinese University of Hong Kong, Shenzhen

March 1, 2023

1 Exercises

Problem 1.1 (Cantor's Intersection Theorem)**.** *The **diameter** of a nonempty subset A of a metric space (X, d) is defined by*

$$diam(A) := \sup_{x,y \in A} d(x, y).$$

(a). *Prove that a metric space (X, d) is complete if and only if every nested sequence $A_1 \supset A_2 \supset A_3 \supset \cdots$ of nonempty closed subsets $A_n \subset X$ satisfying $\lim_{n\to\infty} diam(A_n) = 0$ has a nonempty intersection (consisting of a single point).*
(b). *Find an example of a complete metric space and a nested sequence of nonempty closed bounded sets whose intersection is empty.*

Problem 1.2. *Consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous real-valued function on a complete metric space $(\mathcal{X}, d)$ and let*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exist for every $x \in \mathcal{X}$.
(a). *Give an example of a space X and a sequence of continous functions whose pointwise limit is not continuous.*
(b). *Show that the set of points where f is continuous is a residual set.*
(c). *Show that the Dirichlet function $f = \mathbb{I}_{\mathbb{Q}}$ is not the pointwise limit of any sequence of continuous function on the real line.*

Problem 1.3. *Show that an infinite-dimensional Banach space cannot have a countable Hamel basis.*

Problem 1.4 (Convergence Along Arithmetic Sequences)**.** *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that*

$$\lim_{n\to\infty} f(nt) = 0; \quad \forall t \geq 0.$$

*The presentation of this note is based on [BS18] and [Kre91].

Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

Problem 1.5 (Nowhere Differentiable Continuous Functions)**.** *Prove that the set*

$$\mathcal{R} := \left\{ f : [0,1] \to \mathbb{R} \middle| f \text{ is continuous and nowhere differentiable} \right\}$$

is residual in the Banach space $C([0,1])$ and hence is dense. (This result is due to Stefan Banach and was proved in 1931. You can find a proof that alternatively shows the set of continuous differentiable functions is meager here[1]*.)*

Remark. The problem shows that if one considers meager sets as small sets, then most of the continuous functions are indeed non-differentiable.

Problem 1.6 (Baire Category and Dependent Choice)**.** *Let $\boldsymbol{X}$ be a nonempty set and let $\boldsymbol{A} : \boldsymbol{X} \to 2^{\boldsymbol{X}}$ be a map which assigns to each $\boldsymbol{x} \in \boldsymbol{X}$ a nonempty subset $\boldsymbol{A}(\boldsymbol{x}) \subset \boldsymbol{X}$. Prove that there is a sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ in $\boldsymbol{X}$ such that $\boldsymbol{x}_{n+1} \in \boldsymbol{A}(\boldsymbol{x}_n)$ for all $n \in \mathbb{N}$.*

2 The Baire's Category Theorem

The Baire Category Theorem is a powerful tool in functional analysis. It provides conditions under which a subset of a complete metric space is dense. In fact, it describes a class of dense subsets such that every countable intersection of sets in this class belongs again to this class and hence is still a dense subset. Here are relevant definitions.

Definition 2.1 (Baire Category)**.** *Let $(\mathcal{X}, d)$ be a metric space.*

1. *A subset $A \subset \mathcal{X}$ is called **nowhere dense** if its closure $\overline{A}$ has an empty interior (i.e. $int(\overline{A}) = \emptyset$).*

2. *A subset $A \subset \mathcal{X}$ is said to be **meager** if it is a countable union of nowhere dense subsets of $\mathcal{X}$.*

3. *A subset $A \subset \mathcal{X}$ is said to be **nonmeager** if it is not meager.*

4. *A subset $A \subset \mathcal{X}$ is called **residual** if its complement is meager.*

Lemma 2.1. *Let $(\mathcal{X}, d)$ be a metric space. Then the followings hold.*

1. *A subset $A \subset \mathcal{X}$ is nowhere dense if and only if its complement A^c contains a dense open subset of $\mathcal{X}$.*

2. *If $B \subset \mathcal{X}$ is meager and $A \subset B$, then A is meager.*

3. *If $A \subset \mathcal{X}$ is nonmeager and $A \subset B \subset \mathcal{X}$, then B is nonmeager.*

4. *Every coutable union of meager sets is again meager.*

[1] http://homepages.math.uic.edu/ marker/math414/fs.pdf

5. *Every countable intersection of residual sets is again residual.*

6. *A subset of $\mathcal{X}$ is residual if and only if it contains a countable intersection of dense open subsets of $\mathcal{X}$.*

Proof. It is not hard to see that

$$X \backslash int(\overline{A}) = \overline{X \backslash \overline{A}} = \overline{int(X \backslash A)}.$$

Therefore, $A \subset \mathcal{X}$ is nowhere dense if and only if $int(X \backslash A)$ is dense in $\mathcal{X}$. This proves (1). Parts (2), (3), (4), and (5) follow directly from the definitions. We prove (6). Let $R \subset \mathcal{X}$ be a residual set and define $A := \mathcal{X} \backslash R$. Then there is a sequence of nowhere dense subsets $A_i \subset \mathcal{X}$ such that $A = \cup_{i=1}^{\infty} A_i$. Define $U_i := \mathcal{X} \backslash \overline{A_i}$. Then U_i is a dense open set by (1) and

$$\bigcap_{i=1}^{\infty} U_i = \mathcal{X} \backslash \bigcup_{i=1}^{\infty} \overline{A_i} \subset \mathcal{X} \backslash \bigcup_{i=1}^{\infty} A_i = \mathcal{X} \backslash A = R.$$

Conversely, suppose there is a sequence of dense open subsets $U_i \subset \mathcal{X}$ such that $\cup_{i=1}^{\infty} U_i \subset R$. Define $A_i := \mathcal{X} \backslash U_i$ and $A := \cup_{i=1}^{\infty} A_i$. Then A_i is nowhere dense by (1) and hence A is meager by definition. Moreover,

$$\mathcal{X} \backslash R \subset \mathcal{X} \backslash \bigcap_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} (\mathcal{X} \backslash U_i) = \bigcup_{i=1}^{\infty} A_i = A.$$

Hence, $\mathcal{X} \backslash R$ is meager by (2) and this proves the lemma. □

Lemma 2.2. *Let $(\mathcal{X}, d)$ be a metric space. The followings are equivalent.*

1. *Every residual subset of $\mathcal{X}$ is dense.*

2. *If $U \subset \mathcal{X}$ is a nonempty open set, then U is nonmeager.*

3. *If $A_i \subset \mathcal{X}$ is a sequence of closed sets with empty interior, then their union has empty interior.*

4. *If $U_i \subset \mathcal{X}$ is a sequence of open dense subsets, then their intersection is dense in $\mathcal{X}$.*

Proof. In class. □

Theorem 2.3 (Baire Category Theorem)**.** *Let $(\mathcal{X}, d)$ be a nonempty complete metric space. Then the followings hold.*

1. *Every residual subset of $\mathcal{X}$ is dense.*

2. *If $U \subset \mathcal{X}$ is a nonempty open set, then U is nonmeager.*

3. *If $A_i \subset \mathcal{X}$ is a sequence of closed sets with empty interior, then their union has empty interior.*

4. *If $U_i \subset \mathcal{X}$ is a sequence of open dense sets, then their intersection is dense in $\mathcal{X}$.*

5. *Every residual subset of $\mathcal{X}$ is nonmeager.*

Proof. In class. $\square$

We emphasize that, while the assumption of Baire Category Theorem (completeness) depends on the distance function in a crucial way, the conclusion (every countable intersection of open dense subsets is dense) is purely topological. Thus the Baire Category Theorem extends to many metric spaces that are not complete. All that is required is the existence of a complete distance function that induces the same topology as the original distance function.

Example 2.1. *Let (M, d) be a complete metric space and let $X \subset M$ be a nonempty open set. Then the conclusions of the Baire Category Theorem hold for the metric space (X, d_X) with $d_X := d|_{X\times X} : X \times X \to [0, \infty)$, even though (X, d_X) may not be complete. To see this, let $U_i \subset X$ be a sequence of dense open subsets of X, choose $x_0 \in X$ and $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x_0) \subset X$, and repeat the argument in the proof of the Baire Category Theorem to show that $B_{\epsilon_0}(x_0) \cap \bigcap_{i=1}^{\infty} U_i \neq \emptyset$. All that is needed is the fact that the closure $\overline{B_{\epsilon_1}(x_1)}$ that contains the sequence x_k is complete with respect to the induced metric.*

Example 2.2. *The conclusions of the Baire's Category Theorem do not hold for the metric space $\mathcal{X} = \mathbb{Q}$ of rational numbers with the standard distance function given by $d(x, y) := |x-y|$ for $x, y \in \mathbb{Q}$. Every one element subset of $\mathcal{X}$ is nowhere dense and every subset of $\mathcal{X}$ is both meager and residual.*

Example 2.3. *A residual subset of $\mathbb{R}^n$ may have Lebesgue measure zero. Namely, choose a bijection $\mathbb{N} \to \mathbb{Q}^n : k \mapsto x_k$, and, for $\epsilon > 0$, define*

$$U_\epsilon := \bigcup_{k=1}^{\infty} B_{2^{-k}\epsilon}(x_k).$$

This is a dense open subset of $\mathbb{R}^n$ since $\mathbb{R}^n = \overline{\mathbb{Q}^n} \subset \overline{U_\epsilon}$ and its Lebesgue measure is less than $(2\epsilon)^n$. Hence $R := \cap_{i=1}^{\infty} U_{1/i}$ is a residual set of Lebesgue measure zero and its complement

$$A := \mathbb{R}^n \backslash R = \bigcup_{i=1}^{\infty} (\mathbb{R}^n \backslash U_{1/i})$$

is a meager set of full Lebesgue measure.

References

[BS18] Theo Bühler and Dietmar A Salamon. *Functional analysis*, volume 191. American Mathematical Soc., 2018.

[Kre91] Erwin Kreyszig. *Introductory functional analysis with applications*, volume 17. John Wiley & Sons, 1991.