MAT4010: Functional Analysis
Tutorial 3: Compactness*

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1 Exercises

Problem 1.1 (Subsets of Separable Metric Spaces). *Show that every subset of a separable metric space is separable.*

Problem 1.2 (Bounded Linear Operator). *(a). A bounded linear operator from a normed space X to a normed space Y is bounded if and only if it maps bounded sets onto bounded sets. (b). Show that the inverse of a bounded linear operator need not be bounded.*

Problem 1.3 (Local Compactness). *(a). A metric space X is said to be* ***locally compact*** *if every point of X has a compact neighborhood. Show that $\mathbb{R}$ and $\mathbb{C}$ and, more generally, $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact. (b). Show that a compact metric space X is locally compact.*

Problem 1.4 (Compact Metric Space). *(a) If X is a compact metric space and $M \subset X$ is closed, show that M is compact. (b). Let X and Y be metric spaces, X is compact and $T : X \to Y$ bijective and continuous. Show that T is a homeomorphism.*

Problem 1.5 (Totally Bounded Sets). *Let A be a subset of a metric space. Show that A is totally bounded if and only if $\overline{A}$ is bounded.*

Problem 1.6 (The Sequence Space s). *Show that for an infinite subset M in the space s to be compact, it is necessary and sufficient that there are numbers $\gamma_1, \gamma_2, \cdots$ such that for all $x = (\xi_k(x)) \in M$ we have $|\xi_k(x)| \leq \gamma_k$.*

Problem 1.7 (Closed Subset). *If X is a compact metric space and $M \subset X$ is closed. Show that M is compact.*

Problem 1.8 (Continuous Image of Precompact Set). *If $A \subset X$ is precompact and $T : X \to Y$ is continuous, prove that $T(A) \subset Y$ is precompact.*

*The presentation of this note is based on [BS18] and [Kre91].

2 Compact Sets

Definition 2.1 (Boundedness)**.** *Let $(\mathcal{X}, d)$ be a metric space and $A \subset \mathcal{X}$. A is said to be* ***bounded*** *if $\exists x_0 \in \mathcal{X}$ and $r > 0$ such that $A \subset B(x_0, r)$, where*

$$B(x_0, r) := \{x \in \mathcal{X} : d(x, x_0) < r\}.$$

In finite-dimensional Euclidean space, finite bounded set must contain a convergent subsequence, which is the well-known ***Bolzano-Weierstrass Theorem****. However, this property fails to hold in general metric space.*

Theorem 2.1 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. This proof is constructive. Let $\{x_n\} \subset \mathbb{R}$ be a bounded sequence, say, $|x_n| \leq L$ for some $L > 0$. Then all the x_n's fall in the interval $[-L, L]$. First, we divide the interval into two halves. One of the two must contain infinitely many x_n's. Pick one of those x_n, denote it as x_{n_1}. Continue the dividing process and every time pick one x_{n_i} such that $n_i > n_{i-1}$. One should also notice that the endpoints of those intervals approaches 0. Then it is natural to choose that x_{n_i} converges to the limit of the endpoints, which exist by the monotone convergence theorem. □

Be careful that the Bolzano-Weierstrass Theorem fails in infinite-dimensional space. Here is an simple example.

Example 2.1. *In $C[0, 1]$, consider the sequence $f_n(t) := x^n \in C[0, 1]$. Obviously $\{f_n\} \subset B(\theta, 2)$, where θ denotes the constant zero function. However, $\{x_n\}$ does not contain a convergent subsequence. (closedness and boundedness does not imply sequential compactness)*

Proof. Notice that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwisely to the function

$$f := \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1, \end{cases}$$

which is not in $C[0, 1]$. For any subsequence of $\{f_n\}_{n\in\mathbb{N}}$, if it converges in $C[0, 1]$ to some function, the convergence is uniform and thus the limit must be f, which is impossible. □

Let $(\mathcal{X}, d)$ be a metric space and let $K \subset \mathcal{X}$. Then the restriction of the distance function d to $K \times K$ is a distance function denoted by $d_K := d|_{K\times K} : K \times K \to \mathbb{R}$, so (K, d_K) is a metric space in its own right. The metric space $(\mathcal{X}, d)$ is called **sequentially compact** if every sequence in $\mathcal{X}$ has a convergent subsequence. The subset K is called **sequentially compact** if (K, d_K) is sequentially compact, i.e. if every sequence in K has a subsequence that converges to an element in K. It is called **precompact** if its closure is sequentially compact. Thus K is sequentially compact if and only if it is precompact and closed. The subset K is called **complete** if (K, d_K) is a complete metric space, i.e. if every Cauchy sequence in K converges to an element of K. It is called **totally bounded** if it is either empty or, for every $\epsilon > 0$, there exists finitely many elements $x_1, x_2, ..., x_m \in K$ such that

$$K \subset \bigcup_{i=1}^{m} B_\epsilon(x_i).$$

The next theorem characterizes the compact subsets of a metric space $(\mathcal{X}, d)$ in terms of open subsets of $\mathcal{X}$. It thus shows that compactness depends only on the topology $\mathscr{U}(\mathcal{X}, d)$ induced by the distance function d.

Example 2.2. *Show that a bounded metric space does not have to be totally bounded.*

Proof. Consider the discrete metric space $(\mathcal{X}, d)$ where $\mathcal{X}$ is an infinite set. □

The Heine-Borel Theorem fails in infinite-dimensional spaces and thus the characterization of compact sets is of special interests.

Theorem 2.2 (Characterization of Compact Sets)**.** *Let $(\mathcal{X}, d)$ be a metric space and let $K \subset \mathcal{X}$. Then the followings are equivalent.*

1. *K is sequentially compact.*
2. *K is complete and totally bounded.*
3. *Every open cover of K has a finite subcover.*

Lemma 2.3. *Let $(\mathcal{X}, d)$ be a metric space and let $K \subset \mathcal{X}$. Then the followings are equivalent.*

1. *Every sequence in K has a Cauchy subsequence.*
2. *K is totally bounded.*

Proof. □

Example 2.3. *Consider $c_0 \subset l^\infty$ the space of all sequences converging to zero. Fix a sequence $x \in c_0$ and let*

$$S_x := \{y \in c_0 | |y_n| \leq |x_n|\}.$$

Show that S_x is a compact subset of c_0.

It follows from Theorem 2.2 that every compact metric space is separable. Here are the relevant definitions.

Definition 2.2 (Separability)**.** *Let $\mathcal{X}$ be a topological space. A subset $S \subset \mathcal{X}$ is called **dense** in $\mathcal{X}$ if its closure is equal to $\mathcal{X}$, equivalently, every nonempty open subset of $\mathcal{X}$ contains an element of S. The space $\mathcal{X}$ is called **separable** if it admits a countable dense subset. (A set is called **countable** if it is either finite or countably infinite.)*

Corollary 2.4. *Every compact metric space is separable.*

Proof. Let $n \in \mathbb{N}$. Since $\mathcal{X}$ is totally bounded by Theorem 2.2, there exists a finite set $S_n \subset \mathcal{X}$ such that $\mathcal{X} = \bigcup_{\xi \in S_n} B_{1/n}(\xi)$. Hence $S := \bigcup_{n\in\mathbb{N}} S_n$ is a countable dense subset of $\mathcal{X}$ by the axiom of countable choice. $\square$

Remark 2.1. *Note that in the above proof we only use the fact that a compact metric space is totally bounded. Therefore, a totally bounded metric space is separable. Moreover, if every sequence has a Cauchy subsequence, the metric space is also separable.*

Corollary 2.5. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subset \mathcal{X}$. Then the followings are equivalent.*

1. *A is precompact.*
2. *Every sequence in A has a subsequence that converges in $\mathcal{X}$.*
3. *A is totally bounded and every Cauchy sequence in A converges in $\mathcal{X}$.*

Proof.

- (1) implies (2) directly follows from the definitions.
- We prove that (2) implies (3). By (2) every sequence in A has a Cauchy subsequence and so A is totally bounded by Lemma 2.3. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A, then by (2) there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ that converges in $\mathcal{X}$, and so the original sequence converges in $\mathcal{X}$ because a Cauchy sequence converges if and only if it has a convergent subsequence **(exercise: prove this!)**.
- We finally prove that (3) implies (1). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the closure $\overline{A}$ of A. Then, there exists a sequence $(a_n)_{n\in\mathbb{N}} \subset A$ such that $d(a_n, x_n) < 1/n$ for all $n \in \mathbb{N}$. Since A is totally bounded, it follows from Lemma 2.3 that the sequence $(a_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence $(a_{n_i})_{i\in\mathbb{N}}$. This subsequence converges in $\mathcal{X}$ by (3). Denote its limit by a. Then $a \in \overline{A}$ and $a = \lim_{i\to\infty} x_{n_i}$. Thus $\overline{A}$ is sequentially compact.

$\square$

3 The Arzela-Ascoli Theorem

It is a recurring theme in functional analysis to understand which subsets of a Banach space or topological vector space are compact. For the Euclidean space the answer is given by the **Heine-Borel Theorem**, which continues to hold for any finite-dimensional spaces (recall that the finite-dimensionality is characterized by the compactness of the closed unit ball). For infinite-dimensional spaces a necessary condition is that a compact set is bounded and closed but the reverse does not hold. For the Banach space of continuous mappings on a compact metric space, a characterization of compact sets is given by a theorem of Arzela and Ascoli.

Let (X, d_X) and (Y, d_Y) be metric spaces and assume that X is compact. Then the space

$$C(X,Y) := \{f : X \to Y | f \text{ is continuous}\}$$

of continuous maps from X to Y is a metric space with the distance function

$$d(f,g) := \sup_{x\in X} d_Y(f(x), g(x)), \quad f, g \in C(X,Y). \tag{1}$$

Problem 3.1. *Check that (1) is a well-defined distance function.*

Problem 3.2. *Check that when X is nonempty, the metric space $(C(X,Y), d(\cdot,\cdot))$ is complete if and only if Y is complete.*

Definition 3.1 (Equi-continuity/Pointwise Compactness)**.** *A subset $\mathscr{F} \subset C(X,Y)$ is called* ***equi-continuous*** *if, for every $\epsilon > 0$, there exists a constant $\delta > 0$ such that, for all $x, x' \in X$ and for all $f \in \mathscr{F}$,*

$$d_X(x,x') < \delta \quad \Rightarrow \quad d_Y(f(x), f(x')) < \epsilon.$$

It is called ***pointwise compact*** *if, for every element $x \in X$, the set*

$$\mathscr{F}(x) := \{f(x) | f \in \mathscr{F}\}$$

is a compact subset of Y. The ***pointwise precompactness*** *is defined similarly.*

Theorem 3.3 (Arzela-Ascoli)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces such that X is compact and let $\mathscr{F} \subset C(X,Y)$. Then the following are equivalent.*

1. *$\mathscr{F}$ is precompact.*
2. *$\mathscr{F}$ is pointwise precompact and equi-continuous.*

When the target space Y is the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$. The Arzela-Ascoli Theorem takes the following form.

Corollary 3.4. *Let (X,d) be a compact metric space and let $\mathscr{F} \subset C(X,\mathbb{R}^n)$. Then the following hold.*

1. *$\mathscr{F}$ is precompact if and only if it is bounded and equi-continuous.*
2. *$\mathscr{F}$ is compact if and only if it is closed, bounded, and equi-continuous.*

References

[BS18] Theo Bühler and Dietmar A Salamon. *Functional analysis*, volume 191. American Mathematical Soc., 2018.

[Kre91] Erwin Kreyszig. *Introductory functional analysis with applications*, volume 17. John Wiley & Sons, 1991.