# MAT4010: Functional Analysis
## Tutorial 4: The Dual Spaces*

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## 1 Exercises

**Problem 1.1** (Sequences Converging to Zero)**.** *Let*

$$c_0 := \left\{ x = (x_n)_{n\in\mathbb{N}} \middle| \lim_{n\to\infty} x_n = 0 \right\}.$$

*Prove that*
***(a).*** *$c_0$ is a closed linear subspace of $\ell^\infty$.*
***(b).*** *Let $e_i := (\delta_{ij})_{j\in\mathbb{N}} \in c_0$, prove that the linear subspace span$\{e_i | i \in \mathbb{N}\}$ is dense in $c_0$.*
***(c).*** *Show that $\ell^1 = c_0^*$.*

**Problem 1.2** (Bounded Linear Functional)**.** *Let $X$ be an infinite dimensional normed vector space and let $\Lambda : X \to \mathbb{R}$ be a nonzero linear functional. Show that the following are equivalent.*

1. *$\Lambda$ is bounded.*
2. *The kernel of $\Lambda$ is a closed linear subspace of $X$.*
3. *The kernel of $\Lambda$ is **not** dense in $X$.*

**Problem 1.3** ($X^* \neq X'$)**.** *If $X$ is an infinite-dimensional normed vector space. show that $X^* \neq X'$.*

**Problem 1.4** (Bounded Linear Operator)**.** ***(a).*** *If $T \neq 0$ is a bounded linear operator, show that for any $x \in D(T)$ such that $\|x\| < 1$ we have $\|Tx\| < \|T\|$.*
***(b).*** *Given $T$ a bounded linear operator, show that $\sup_{\|x\|=1} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|\leq 1} \frac{\|Tx\|}{\|x\|}$.*
***(c).*** *Show that the operator $T : \ell^\infty \to \ell^\infty$ defined by $y = Tx$, $y_j = x_j/j$, $x = (x_j)_{j\in\mathbb{N}}$ is linear and bounded.*
***(d).*** *Show that the range of $T$ of a bounded linear operator need not be closed.*

*The presentation of this note is based on [BS18], [Sal16], and [Kre91].

**Problem 1.5.** *If $X$ and $Y$ are normed spaces over $\mathbb{R}$ and $B(X,Y)$ is complete, show that $Y$ is complete.*

**Problem 1.6.** ***(a).*** *If $x$ and $y$ are different vectors in a finite dimensional vector space $X$, show that there is a linear functional $f$ on $X$ such that $f(x) \neq f(y)$.*
***(b).*** *If $f_1...., f_p$ are linear functionals on an $n$-dimensional vector space $X$, where $p < n$, show that there is a vector $x \neq 0$ in $X$ such that $f_1(x) = 0, ..., f_p(x) = 0$.*
***(c).*** *Let $Z$ be a proper subspace of an $n$-dimensional vector space $X$, and let $f$ be a linear functional on $Z$. Show that $f$ can be extended linearly to $X$, that is, there is a linear functional $\tilde{f}$ on $X$ such that $\tilde{f}|_Z = f$.*
***(d).*** *Let $Z$ be a proper subspace of an $n$-dimensional vector space $X$, and let $x_0 \in X \backslash Z$. Show that there is a linear functional $f$ on $X$ such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in Z$.*

**Problem 1.7** (Annihilator)**.** *Let $M \neq \emptyset$ be any subset of a normed space $X$. The **annihilator** of $M^\perp$ of $M$ is the set of bounded linear functionals on $X$ which vanish on $M$,*

$$M^\perp := \left\{ f \in X' \middle| f|_M = 0 \right\}.$$

*Show that*
***(a).*** *$M^\perp$ is a closed linear subspace.*
***(b).*** *$M$ is dense if and only if $M^\perp = \{0\}$.*

# 2 The Dual Spaces of $L^p(\mu)$

When $1 < p < \infty$ it turns out that the dual space of $L^p(\mu)$ is always isomorphic to $L^q(\mu)$ where $1/p+1/q = 1$. For $p = \infty$ the natural homomorphism $L^1(\mu) \to L^\infty(\mu)^*$ is an isometric embedding. However, in most cases, the dual space of $L^\infty(\mu)$ is much larger than $L^1(\mu)$. For $p = 1$ situation is more subtle. The natural homomorphism $L^\infty(\mu) \to L^1(\mu)^*$ need not be injective or surjective. However, it is bijective for a large class of measure spaces and one can characterize those measure spaces for which it is injective, respectively bijective. This requires the following definitions.

**Definition 2.1** ($\sigma$-finite / Semi-finite / Localizable)**.** *A measure space $(X, \mathcal{F}, \mu)$ is called **$\sigma$-finite** if there exists a partition $\{X_i\}_{i\in\mathbb{N}} \subset \mathcal{F}$ such that*

$$X = \bigcup_{n\in\mathbb{N}} X_i, \quad \mu(X_i) < \infty, \forall i \in \mathbb{N}.$$

*It is called **semi-finite** if for any measurable set $A \in \mathcal{F}$ satisfies*

$$\mu(A) > 0 \quad \Rightarrow \quad \exists E \in \mathcal{F} \text{ such that } E \subset A \text{ and } 0 < \mu(E) < \infty.$$

*It is called **localizable** if it is semi-finite and, for every collection of measurable sets $\Xi \subset \mathcal{F}$, there is a $H \in \mathcal{F}$ such that*
***(L1).*** *$\mu(E \backslash H) = 0, \forall E \in \Xi$.*
***(L2).*** *If $G \in \mathcal{F}$ such that $\mu(E \backslash G) = 0, \forall E \in \Xi$, then $\mu(H \backslash G) = 0$.*

**Lemma 2.1.** *Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1/p + 1/q = 1$. Let $g : X \to [0, \infty)$ be a measurable function and suppose that there exists a constant $c \geq 0$ such that*

$$f \in \mathcal{L}^p(\mu), \quad f \geq 0 \quad \Rightarrow \quad \int_X fgd\mu \leq c\|f\|_p. \tag{1}$$

*Then the following holds.*

1. *If $q = 1$ then $\|g\|_q \leq c$.*
2. *If $1 < q < \infty$ and $\|g\|_q < \infty$ then $\|g\|_q \leq c$.*
3. *If $1 < q < \infty$ and $(X, \mathcal{F}, \mu)$ is semi-finite then $\|g\|_q \leq c$.*
4. *If $q = \infty$ and $(X, \mathcal{F}, \mu)$ is semi-finite then $\|g\|_q \leq c$.*

*Proof.* **(i).** If $q = 1$, take $f = 1$, then we have $\|g\|_1 \leq c$.
**(ii).** Since $g \in L^q(\mu)$, we know that the set $A := \{x \in X | g(x) = \infty\}$ has measure zero. Define the function $h : X \to [0, \infty)$ by $h := g \cdot \mathbb{I}(X \backslash A)$. Then $h$ is measurable and

$$\|h\|_q = \|g\|_q < \infty, \quad \int_X fhd\mu = \int_X fgd\mu \leq c\|f\|_p.$$

for all $f \in \mathcal{L}^p(\mu)$ with $f \geq 0$. Define $f : X \to [0, \infty)$ by $f(x) := h(x)^{q-1}$ for all $x \in X$. Then $f^p = h^{p(q-1)} = h^q = fh$ and hence

$$\|f\|_p = \left(\int_X h^q d\mu\right)^{1-1/q} = \|h\|_q^{q-1}, \quad \int_X fhd\mu = \|h\|_q^q.$$

Thus, $f \in \mathcal{L}^p(\mu)$ and so $\|h\|_q^q = \int_X fhd\mu \leq c\|f\|_p = c\|h\|_q^{q-1}$. Since $\|h\|_q < \infty$ it follows that $\|g\|_q = \|h\|_q \leq c$.
**(iii).** Suppose for contradiction that $\|g\|_q > c$, we claim that there is a nonnegative function $h$ such that

$$0 \leq h \leq g, \quad c \leq \|h\|_q < \infty.$$

Therefore, $\int fhd\mu \leq \int ghd\mu \leq c\|f\|_p$. Since $\|h\|_q < \infty$, by (ii), we conclude that $\|h\|_q \leq c$, which yields a contradiction. Next, we prove the existence of such $h$. There exists a measurable step function $s : X \to [0, \infty)$ such that $0 \leq s \leq g$ and $\int_X s^q d\mu > c^q$ since $\|g\|_q > c$. If $\|s\|_q < \infty$, take $h = s$. Otherwise there exists a $A \in \mathcal{F}$ and $\delta > 0$ such that $\mu(A) = \infty$ and $\delta\mathbb{I}(A) \leq s \leq g$. Since the space is semi-finite, there exists a $E \subset A$, measurable such that $c^q < \delta^q \mu(E) < \infty$. Then the function $h := \delta\mathbb{I}(E)$ satisfies $0 \leq h \leq g$ and $\|h\|_q = \delta\mu(E)^{1/q} > c$.
**(iv).** Suppose by contradiction that $\|g\|_q > c$, then there exists a $\delta > 0$ such that the set $A := \{x \in X | g(x) \geq c + \delta\}$ has positive measure. There exists a $E \subset A$ such that $0 < \mu(E) < \infty$. Hence $f := \mathbb{I}(E) \in \mathcal{L}^1(\mu)$ and $\int_X fgd\mu \geq (c + \delta)\mu(E) > c\mu(E) = c\|f\|_1$, which is a contradiction to the assumption. □

**Theorem 2.2.** *Let $(X, \mathcal{F}, \mu)$ be a measure space and fix constants*

$$1 \le p \le \infty, \quad 1 \le q \le \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

*Then the following holds.*

1. *Let $g \in \mathcal{L}^q(\mu)$. Then the formula*

$$\Lambda_g([f]_\mu) := \int_X fg d\mu, \quad \forall f \in \mathcal{L}^p(\mu) \tag{2}$$

*defines a bounded linear functional $\Lambda_g : L^p(\mu) \to \mathbb{R}$ and*

$$\|\Lambda_g\| \le \|g\|_q.$$

2. *The map $g \mapsto \Lambda_g$ in (2) descends to a bounded linear operator*

$$L^q(\mu) \to L^p(\mu)^* : [g]_\mu \mapsto \Lambda_g. \tag{3}$$

3. *Assume that $1 < p \le \infty$, then $\|\Lambda_g\| = \|g\|_q$ for all $g \in L^q(\mu)$.*

4. *Assume $p = 1$. Then the map $L^\infty(\mu) \to L^1(\mu)^*$ is injective if and only if it is an isometric embedding if and only if $(X, \mathcal{F}, \mu)$ is semi-finite.*

*Proof.* **Step 1.** Given $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$, we have $fg \in \mathcal{L}^1$ by Holder's inequality. If $1 < p < \infty$, $\int_X |fg| d\mu \le \|f\|_p \|g\|_q$. If $p = 1$, $|fg| \le |f| \|g\|_\infty$ $\mu$-a.e., so $fg \in \mathcal{L}^1$ and $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$. If $p = \infty$, exchange the role of $f$ and $g$.

**Step 2.** We prove that (1) implies (2). By step 1 the map (2) is well-defined and it depends only on the equivalence class of $f$. The linearity is obvious and $\|\Lambda_g\| \le \|g\|_q$ follows from step 1. The bounded linear operator $\Lambda_g$ depends on $g$ only through its equivalence class. Therefore, the map (3) is well-defined and has norm less than or equal to 1.

**Step 3.** Let $1 < p \le \infty$ and $g \in \mathcal{L}^q$. If $f \in \mathcal{L}^p$ is nonnegative then the function $f \cdot sgn(g)$ is p-integrable and

$$\int_X f|g| d\mu = \Lambda_g(f \cdot sgn(g)) \le \|\Lambda_g\| \|f \cdot sgn(f)\|_p = \|\Lambda_g\| \|f\|_p.$$

Hence $\|g\|_q \le \|\Lambda_g\|$ by Lemma 2.1 and so $\|\Lambda_g\| = \|g\|_q$ by step 2.

**Step 4.** We prove that the map $L^\infty(\mu) \to L^1(\mu)^*$ is injective implies $(X, \mathcal{F}, \mu)$ is semi-finite. Let $A \in \mathcal{F}$ be such that $\mu(A) > 0$ and define $g := \mathbb{I}(A)$. Then $\Lambda_g : L^1(\mu) \to \mathbb{R}$ is nonzero by assumption. Hence there is a function $f \in \mathcal{L}^1$ such that

$$0 < \Lambda_g(f) = \int_A f d\mu.$$

For $i \in \mathbb{N}$ define $E_i = \{x \in A | f(x) > 2^{-i}\}$, then $E_i \subset A$ and $E_i \in \mathcal{F}$, we also have

$$\mu(E_i) \le 2^i \int_{E_i} f d\mu \le 2^i \|f\|_1 < \infty.$$

Moreover, $E := \bigcup_{i=1}^\infty E_i = \{x \in A | f(x) > 0\}$ has nonzero measure. Hence one of the sets $E_i$ must have nonzero measure. Therefore, $(X, \mathcal{F}, \mu)$ is semi-finite. □

**Example 2.1.** *Define* $(X, \mathcal{F}, \mu)$ *by*

$$X := \{a, b\}, \quad \mathcal{F} := 2^X, \quad \mu(\{a\}) = 1, \quad \mu(\{b\}) = \infty.$$

*This measure space is not semi-finite. Thus the linear map* $L^\infty(\mu) \to L^1(\mu)^*$ *is not injective. In fact,* $L^\infty(\mu)$ *has dimension two while* $L^1(\mu)$ *has dimension one.*

**Theorem 2.3** (The Dual Space of $L^p$)**.** *Let* $(X, \mathcal{F}, \mu)$ *be a measure space and fix constants*

$$1 \leq p < \infty, \quad 1 < q \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

*Then the following holds.*

1. *Assume* $1 < p < \infty$*. Then the map* $L^q(\mu) \to L^p(\mu)^* : [g]_\mu \mapsto \Lambda_g$ *defined by* (2) *is bijective and hence a Banach space isomorphism.*

2. *Assume* $p = 1$*. Then the map* $L^\infty(\mu) \to L^1(\mu)^* : [g]_\mu \mapsto \Lambda_g$ *defined by* (2) *is bijective if and only if* $(X, \mathcal{F}, \mu)$ *is localizable.*

*Proof.* The proof requires Radon-Nikodym theorem, for interested readers please refer to Chapter 5.2 of [Sal16]. □

**Example 2.2** (The Dual Space of $\ell^1$)**.** *Consider the measure space* $(\mathbb{N}, 2^{\mathbb{N}}, \#)$*. It is not difficult to see that* $\mathbb{N}$ *is localizable. (**Exercise:** prove this!) Therefore, according to theorem 2.3, the map* $L^\infty(\#) = \ell^\infty \to L^1(\#)^* := \ell^1 : x \to \Lambda_x$ *is a Banach space isomorphism.*

**Example 2.3** (The Dual Space of $\ell^p, 1 < p < \infty$)**.** *By theorem 2.3,* $l^{p*} = l^q$*.*

Problem 2.4 shows that, in general, theorem 2.3 does not extend to the case $p = \infty$. Note that by theorem 2.2 the Banach space $L^1(\mu)$ is equipped with an isometric inclusion $L^1(\mu) \to L^\infty(\mu)^*$, however, the dual space of $L^\infty(\mu)$ is typically much larger than $L^1(\mu)$.

**Problem 2.4** ($(\ell^\infty)^* \neq \ell^1$)**.** *Let* $\mu : 2^{\mathbb{N}} \to [0, \infty)$ *be the counting measure on the postive integers. Then* $\ell^\infty := L^\infty(\mu) = \mathcal{L}^\infty(\mu)$ *is the Banach space of bounded sequences* $x = (x_n)_{n\in\mathbb{N}}$ *of real numbers equipped with the supremum norm* $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$*. An interesting subspace is*

$$c := \left\{ x = (x_n)_{n\in\mathbb{N}} \middle| x \text{ is a Cauchy sequence} \right\}.$$

*It is equipped with a functional* $\Lambda_0 : c \to \mathbb{R}$*, defined by*

$$\Lambda_0(x) := \lim_{n\to\infty} x_n, \quad \text{for } x \in c. \tag{4}$$

***(a).*** *Show that* $\Lambda_0$ *is a bounded linear functional.*
***(b).*** *Use* (4) *to show that* $\ell^1 \neq (\ell^\infty)^*$*.*

**Remark 2.1.** *Part (b) of problem 2.4 shows that even though the underlying measure space has a very nice property (i.e. it is* $\sigma$*-finite, which also implies it is localizable), the map* $\ell^1 \to (\ell^\infty)^*$ *is not surjective.*

**Example 2.4** (Nonreflexive Spaces)**.** *By problem 1.1, we see*

$$\ell^1 = (c_0)^*, \quad c_0 \neq \ell^\infty = (\ell^1)^* = (c_0)^{**}, \quad \ell^1 \neq (\ell^\infty)^* = (\ell^1)^{**}.$$

*Therefore, the Banach spaces* $\ell^1$ *and* $c_0$ *are not reflexive.*

## References

[BS18] Theo Bühler and Dietmar A Salamon. *Functional analysis*, volume 191. American Mathematical Soc., 2018.

[Kre91] Erwin Kreyszig. *Introductory functional analysis with applications*, volume 17. John Wiley & Sons, 1991.

[Sal16] Dietmar Salamon. *Measure and integration*. Citeseer, 2016.