MAT4220: Partial Differential Equations

Tutorial 11 Slides[1]

Mou Minghao

CUHK(SZ)

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[1] All of the problems are taken from [Strauss, 2007].

Contents

Symmetry of Green's Function

Theorem (Symmetry of Green's Function)

For any region D we have a Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$. It is always symmetric

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = G(\boldsymbol{x}_0, \boldsymbol{x}).$$

Proof. We consider a pair of functions $u := G(\boldsymbol{x}, \boldsymbol{a})$ and $v := G(\boldsymbol{x}, \boldsymbol{b})$ defined on the domain D_ϵ, which denotes the domain D where two spheres centered at $\boldsymbol{a}$ and $\boldsymbol{b}$ with radius ϵ are excluded. Then the boundary of D_ϵ has three compnents

$$\partial D_\epsilon = \partial D \bigcup \{|\boldsymbol{x} - \boldsymbol{a}| = \epsilon\} \bigcup \{|\boldsymbol{x} - \boldsymbol{b}| = \epsilon\}.$$

Apply Green's function on D_ϵ to the pair u and v and split the integral over ∂D_ϵ into three parts, we immediately have

$$A_\epsilon + B_\epsilon = 0, \quad \text{(Why?)}$$

where $A_\epsilon = \iint_{|\boldsymbol{x}-\boldsymbol{a}|=\epsilon}(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}})dS$. One can show that

$$\lim_{\epsilon\to 0^+} A_\epsilon = v(\boldsymbol{a}); \quad \lim_{\epsilon\to 0^+} B_\epsilon = -u(\boldsymbol{b}).$$

Therefore,

$$0 = \lim_{\epsilon\to 0^+} A_\epsilon + B_\epsilon = v(\boldsymbol{a}) - u(\boldsymbol{b}).$$

Thus,

$$G(\boldsymbol{a}, \boldsymbol{b}) = G(\boldsymbol{b}, \boldsymbol{a}).$$

$\square$

The Half Space

The domain is

$$\Omega = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}.$$

We already know the function $1/(4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)$ satisfies two of the three conditions for the Green's function. We now want to modify it to satisfy all of the requirements. The corresponding Green's function should be of the following form

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|} + H(\boldsymbol{x}),$$

where $H(\boldsymbol{x})$ is harmonic and smooth on the domain Ω and $G(\boldsymbol{x}, \boldsymbol{x}_0) = 0$ on $\partial\Omega$. The idea is: we consider the reflected point of $\boldsymbol{x}_0$ across the z-axis, $\boldsymbol{x}_0^* = (x, y, -z)$, the function

$$\frac{1}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0^*|},$$

is harmonic and smooth in Ω. We therefore only need to check that $G(\boldsymbol{x}, \boldsymbol{x}_0)|_{\partial\Omega} = 0$. Note that $\partial\Omega = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. On $\partial\Omega$, it is obvious that

$$|\boldsymbol{x} - \boldsymbol{x}_0| = |\boldsymbol{x} - \boldsymbol{x}_0^*|.$$

Thus, the function G defined as

$$G(\boldsymbol{x}, \boldsymbol{x}_0) := -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} + \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0^*|},$$

is the desired Green's function in Ω of the point $\boldsymbol{x}_0$.

Now that we have the Green's function, we next derive the solution formula for the Dirichlet problem

$$\begin{cases} \Delta u = 0, \quad z > 0 \\ u(x, y, 0) = h(x, y). \end{cases}$$

The Half Space

Recall the theorem that *If $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function on the domain D of the point $\boldsymbol{x}_0$, then the solution formula of the Dirichlet problem is given by the formula*

$$u(\boldsymbol{x}_0) = \iint_{\partial D} u(\boldsymbol{x}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial \boldsymbol{n}} dS.$$

Note that

$$\frac{\partial}{\partial \boldsymbol{n}} = \boldsymbol{n} \cdot \nabla = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = -\frac{\partial}{\partial z}.$$

Therefore,

$$\begin{aligned} \frac{\partial G}{\partial \boldsymbol{n}} &= -\frac{\partial}{\partial z} G = -\frac{1}{4\pi} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|^2} \partial_z |\boldsymbol{x} - \boldsymbol{x}_0| + \frac{1}{4\pi} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0^*|^2} \partial_z |\boldsymbol{x} - \boldsymbol{x}_0^*| \\ &= -\frac{z - z_0}{4\pi |\boldsymbol{x} - \boldsymbol{x}_0|^3} + \frac{z - z_0^*}{4\pi |\boldsymbol{x} - \boldsymbol{x}_0^*|^3}. \end{aligned}$$

The Half Space

Since $z_0^* = -z_0$,

$$\frac{\partial G}{\partial \boldsymbol{n}} = \frac{1}{2\pi} \frac{z_0}{|\boldsymbol{x} - \boldsymbol{x}_0|^3}.$$

Therefore the solution is

$$u(\boldsymbol{x}_0) = \iint_{\partial D} \frac{h(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}_0|^3} dS.$$

This is the complete formula that solves the Dirichlet problem for the half-space.

The Sphere

The Green's function for the ball $\{|\boldsymbol{x}| < a\}$ of radius a can also be found by the reflection method. However, different from the case of half-space, the reflection is performed across the sphere $\{|\boldsymbol{x}| = a\}$. Consider the function G of the following form

$$G(\boldsymbol{x}, \boldsymbol{x}_0) := -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|} + \frac{C}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0^*|}.$$

- If $C = 1$, is it possible to find a $\boldsymbol{x}_0^* \notin B_a$ such that

$$|\boldsymbol{x} - \boldsymbol{x}_0| = |\boldsymbol{x} - \boldsymbol{x}_0^*|, \quad \forall x \in \partial B_a?$$

The answer is negative. (Why?)

The Sphere

- Is it possible to find a C such that

$$|\boldsymbol{x}-\boldsymbol{x}_0| = C|\boldsymbol{x}-\boldsymbol{x}_0^*|.$$

This is correct if $\boldsymbol{x}_0^* = \frac{a^2}{|\boldsymbol{x}_0|^2}\boldsymbol{x}_0$. Since $\frac{a^2}{|\boldsymbol{x}_0|^2} > 1$, the point $x_0^* \neq B_a$. Note that

$$|\boldsymbol{x}-\boldsymbol{x}_0^*| = \left|\boldsymbol{x} - \frac{a^2}{|\boldsymbol{x}_0|^2}\boldsymbol{x}_0\right| = \frac{a}{|\boldsymbol{x}_0|}\left|\frac{|\boldsymbol{x}_0|}{a}x - \frac{a}{|\boldsymbol{x}_0|}\boldsymbol{x}_0\right| = \frac{a}{|\boldsymbol{x}_0|}|\boldsymbol{x}-\boldsymbol{x}_0|.$$

The above equality implies

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|} = \frac{a}{|\boldsymbol{x}_0|}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0^*|}, \quad on\ \partial B_a.$$

Therefore, it would be wise to set $C = \frac{a}{|\boldsymbol{x}_0|}$. This means that

$$G(\boldsymbol{x},\boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}_0|} + \frac{a}{|\boldsymbol{x}_0|}\frac{1}{4\pi|x-x_0^*|}.$$

The Sphere

The above clearly only holds when $\boldsymbol{x}_0 \neq \mathbf{0}$. If $\boldsymbol{x}_0 = \mathbf{0}$, the Green's function for the domain can be shown to be

$$G(\boldsymbol{x}, \mathbf{0}) = -\frac{1}{4\pi|\boldsymbol{x}|} + \frac{1}{4\pi a}.$$

Again, we want to use the Green's function we find to solve the Dirichlet problem on the sphere.

$$\begin{cases} \Delta u = 0, & |\boldsymbol{x}| < a \\ u = h, & |\boldsymbol{x}| = a. \end{cases}$$

To this end, we need to find $\frac{\partial G}{\partial \boldsymbol{n}}$ on the on $|\boldsymbol{x}| = a$. Note that

$$\nabla G = \frac{\boldsymbol{x} - \boldsymbol{x}_0}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|^3} - \frac{a}{|\boldsymbol{x}_0|}\frac{\boldsymbol{x} - \boldsymbol{x}_0^*}{4\pi|\boldsymbol{x}_0 - \boldsymbol{x}_0^*|^3}.$$

The Sphere

Substitute $\boldsymbol{x}_0^* = (a/|\boldsymbol{x}_0|)^2\boldsymbol{x}_0$ and $|\boldsymbol{x}-\boldsymbol{x}_0^*| = (a/|\boldsymbol{x}_0|)|\boldsymbol{x}-\boldsymbol{x}_0|$, we get

$$\nabla G = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}_0|^3}\left[\boldsymbol{x} - (|\boldsymbol{x}_0|/a)^2\boldsymbol{x}\right].$$

Thus,

$$\frac{\partial G}{\partial \boldsymbol{n}} = \frac{\boldsymbol{x}}{a}\cdot\nabla G = \frac{a^2-|\boldsymbol{x}_0|^2}{4\pi a|\boldsymbol{x}-\boldsymbol{x}_0|^3}.$$

Therefore,

$$u(\boldsymbol{x}_0) = \frac{a^2-|\boldsymbol{x}_0|^2}{4\pi a}\iint_{|\boldsymbol{x}|=a}\frac{h(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x}_0|^3}dS.$$

This is the *three-dimensional* version of the *Poisson's formula*. In more classical notation, it would be written in the usual spherical coordinates as

$$u(r_0,\theta_0,\phi_0) = \frac{a(a^2-r_0^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{h(\theta,\phi)}{(a^2+r_0^2-2ar_0\cos\psi)^{3/2}}\sin\theta d\theta d\phi,$$

where $\psi := angle(\boldsymbol{x}_0,\boldsymbol{x})$ and $r_0 = |\boldsymbol{x}_0|$.

Distribution

The introduction of the theory of distributions, in our context, will provide a succinct and elegant interpretation of Green's functions. We have encountered the *approximate delta functions* in many places in this course. The delta function is defined (formally) to be infinite at $x = 0$ and zero at all $x \neq 0$. It should integrate to one: $\int_{-\infty}^{\infty} \delta(x)dx = 1$. Certainly it is NOT a function since a function defined in this way cannot satisfy the integrating to one requirement. It is indeed a more general object, which is called a *distribution*. Recall that a function is a rule which assigns numbers to numbers. However, a distribution is a rule that assigns numbers to functions.

Definition (Delta Function)

The delta function is the rule that assigns the number $\phi(0)$ to the function $\phi(x)$.

Distribution

We need to specify what function $\phi(x)$ is used here. A *test function* $\phi(x)$ is a real C^∞ function (a function all of whose derivatives exist) that vanishes outside a finite interval. Thus $\phi : \mathbb{R} \to \mathbb{R}$ is defined and differentiable for all $x \in \mathbb{R}$ and $\phi(x) \equiv 0$ for x large and for x small. Let $\mathcal{D}$ denote the collection of all test functions.

Definition (Distribution)

A distribution f is a functional (or a rule): $\mathcal{D} \to \mathbb{R}$ which is linear and continuous for the following sense. If $\phi \in \mathcal{D}$ is a test function, then we denote the corresponding real number as (f, ϕ). (A more intuitive way is to write $f(\phi)$)

Distribution

- The linearity of a functional f means

$$(f, a\phi + b\psi) = a(f, \phi) + b(f, \psi), \quad \forall a, b \in \mathbb{R}.$$

- The continuity means if $\{\phi_n\} \subset \mathcal{D}$ is a sequence of test functions that vanish outsider a compact set and converge uniformly to a test function ϕ, and **if all their derivatives do as well**, then

$$(f, \phi_n) \to (f, \phi), \quad n \to \infty.$$

Example

The delta function $\delta(x)$ is a distribution that assigns $\phi(x) \in \mathcal{D}$ to $\phi(0)$. We can specify the delta function as $\phi \mapsto \phi(0)$.

Example

The functional $\phi \mapsto \phi''(5)$ is a distribution. It is linear and a continuous functional, which follows from the assumption that $\phi_n \to \phi, \phi_n' \to \phi'$, and $\phi_n'' \to \phi''$ uniformly.

Distribution

Example

Let $f \in L^1$. It corresponds to the distribution

$$\phi \mapsto \int_{-\infty}^{\infty} f(x)\phi(x)dx.$$

Note that the RHS is a finite real number since $\phi \in \mathcal{D}$. It can be easily verified that the distribution defined above is linear and is also continuous. Note that the mathematical expectation is a particular case of the above distribution by picking $f(x) = x$.

$$\mathbb{E}_{x\sim p(x)}[X] := \int_{\Omega} xp(x)dx,$$

where Ω is the sample space. Expectation indeed maps a random variable $X : \Omega \to \mathbb{R}$, which is a measurable function, to a real number. Therefore, by definition it is a functional.

Distribution

Remark

Because of the above example, it is quite natural to use the integral notation for the delta function

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0).$$

and speak of the delta function as if it were a true function.

Convergence of Distributions

Definition

If f_n is a sequence of distributions and f is another distribution, we say that f_n converges weakly to f if

$$(f_n, \phi) \to (f, \phi), \quad n \to \infty.$$

Recall that several weeks ago we introduced the weak convergence in L^2, which says that a sequence of functions $f_n \in L^2(\Omega)$ is said to converge weakly to a function $f \in L^2(\Omega)$ if for any $g \in L^2(\Omega)$

$$\int_\Omega f_n \cdot g d\lambda \to \int_\Omega f \cdot g d\lambda, \quad n \to \infty.$$

Note that by Example 10.2 we know the integration above can be viewed as a distribution defined by f_n, we can equivalently rewrite it as

$$(f_n, g) \to (f, g), n \to \infty,$$

Convergence of Distributions

which conforms with the general definition of weak convergence. Indeed, in the L^p theory, we have the famous *Riesz Representation Theorem* (you might have also heard of this theorem in your advanced linear algebra course). The theorem says that any continuous linear functional defined on $L^p(\Omega), 1 \leq p \leq \infty$ can be represented as

$$\int_\Omega f \cdot g d\lambda = (g, f),$$

for some fixed $g \in L^q(\Omega)$, q is the conjugate of p (i.e. $1/p + 1/q = 1$).

Convergence of Distributions

Example

The fundamental solution for the diffusion equation on the whole real line is $\Phi(x,t) = 1/\sqrt{4\pi kt}e^{-x^2/4kt}$ *for* $t > 0$*. We have proved before that*

$$\int_{\mathbb{R}} \Phi(x,t)\phi(x)dx \to \phi(0), \quad N \to \infty.$$

Because for each t *we may consider the function* $\Phi(x,t)$ *at a distribution since it is of the form of an integral, this means that*

$$\Phi(x,t) \to \delta(x), \quad \text{weakly as } t \to \infty.$$

Convergence of Distributions

Example

Let $K_N(\theta)$ be the Dirichlet kernel.

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta = \frac{\sin[(N + \frac{1}{2}\theta)]}{\sin \frac{1}{2}\theta}.$$

We proved that

$$\int_{-\pi}^{\pi} K_N(\theta)\phi(\theta)d\theta \to 2\pi\phi(0), \quad N \to \infty.$$

Therefore,

$$K_N(\theta) \to 2\pi\delta(\theta), \quad \textit{weakly as } N \to \infty.$$

Derivative of a Distribution

The derivative of a distribution always exists and is another distribution. Consider the following motivating example. let $f(x)$ be any C^1 function and $\phi(x)$ be any test function. Integration by parts shows that

$$\int_{\mathbb{R}} f'(x)\phi(x)dx = -\int_{\mathbb{R}} f(x)\phi'(x)dx,$$

since $\phi(x) = 0$ for $|x|$ large.

Definition (Derivative)

For any distribution f, the derivative f' is defined by the formula

$$(f', \phi) := -(f, \phi'), \quad \phi \in \mathcal{D}.$$

Note that it is easy to check that f' is a distribution since it is linear and continuous. Moreover, if $f_n \to f$ weakly, then $f_n' \to f'$ weakly as well. The reason is that
$(f_n', \phi) = -(f_n, \phi') \to -(f, \phi') = (f', \phi), \forall \phi \in \mathcal{D}.$

Derivative of a Distribution

Example

The derivatives of the delta function are

$$(\delta', \phi) = -(\delta, \phi') = -\phi'(0).$$

$$(\delta'', \phi) = -(\delta', \phi') = (\delta, \phi'') = \phi''(0).$$

Example

The Heaviside function is defined by $H(x) = 1$ *for* $x > 0$ *and* $H(x) = 0$ *for* $x < 0$. *For any test function,* $(H', \phi) = -(H, \phi') = -\int_0^\infty \phi'(x)dx = \phi(0)$. *Thus,*

$$H' = \delta.$$

Thus plus function $p(x) = x^+$ *is defined as* $p(x) = x$ *for* $x \geq 0$, *and* $p(x) = 0$ *for* $x \leq 0$. *Then* $p' = H$ *and* $p'' = \delta$.

3D Distribution

A test function $\phi(\boldsymbol{x}) = \phi(x, y, z)$ is a real C^∞ function that vanishes outside some ball. $\mathcal{D}$ denotes the set of all test functions of $\boldsymbol{x}$. Then the definition of a distribution is identical to the one-dimensional case except we replace common intervals by common balls. The delta function δ is defined as the functional $\phi \mapsto \phi(\mathbf{0})$. Its partial derivative $\partial\delta/\partial z$ is defined as the functional $\phi \mapsto -(\partial\phi/\partial z)(\mathbf{0})$. If $f(\boldsymbol{x})$ is any ordinary intergrable function, it is considered to be the same as the distribution $\phi \mapsto \iiint_{\mathbb{R}^3} f(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x}$.

Revisit Green's Functions

Let $r = |\boldsymbol{x}|$ and $\phi(\boldsymbol{x}) \in \mathcal{D}$, then we know that

$$\phi(\mathbf{0}) = -\iiint \frac{1}{r}\Delta\phi(\boldsymbol{x})\frac{d\boldsymbol{x}}{4\pi}.$$

Note that if we interpret $\Delta(-\frac{1}{4\pi r})$ as a 3-d distribution, we know that

$$\iiint \Delta(-\frac{1}{4\pi r})\phi(\boldsymbol{x})d\boldsymbol{x} = -\iiint \frac{1}{r}\Delta\phi(\boldsymbol{x})\frac{d\boldsymbol{x}}{4\pi}.$$

By definition, we see that

$$\Delta(-\frac{1}{4\pi r}) = \delta(\boldsymbol{x}).$$

Because $\delta(\boldsymbol{x})$ vanishes except at the origin, the above formula explains why $1/r$ is a harmonic function away from the origin and it explains exactly how it differs from being harmonic at the origin.

Revisit Green's Functions

Consider now the Dirichlet problem for the Poisson's equation

$$\begin{cases} \Delta f = 0, & in\ D \\ u = 0, & on\ \partial D. \end{cases}$$

Its solution is

$$u(\boldsymbol{x}_0) = \iiint_D G(\boldsymbol{x}, \boldsymbol{x}_0) f(\boldsymbol{x}) d\boldsymbol{x},$$

where $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function. Now fix the point $\boldsymbol{x}_0 \in D$, the LHS can be written as

$$u(\boldsymbol{x}_0) = \iiint_D \delta(\boldsymbol{x} - \boldsymbol{x}_0) u(\boldsymbol{x}) d\boldsymbol{x}.$$

We assume that $u(\boldsymbol{x})$ is an arbitrary test function whose support is a bounded subset of D. The RHS is

Revisit Green's Functions

$$u(\boldsymbol{x}_0) = \iiint_D G(\boldsymbol{x}, \boldsymbol{x}_0)(\boldsymbol{x}) d\boldsymbol{x} = \iiint_D \Delta G(\boldsymbol{x}, \boldsymbol{x}_0) u(\boldsymbol{x}) d\boldsymbol{x},$$

where ΔG is understood in the sense of a distribution. Because $u(\boldsymbol{x})$ can be an arbitrary test function in D, we deduce that

$$\Delta G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \quad \textit{in } D.$$

As we know, $G(\boldsymbol{x}, \boldsymbol{x}_0) + (4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)^{-1}$ is harmonic in the whole domain D, including at $\boldsymbol{x}_0$. Thus,

$$\Delta G = -\Delta \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \quad \textit{in } D.$$

Indeed, $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the unique distribution that satisfies the following

$$\begin{cases} \Delta G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), & \textit{in } D, \\ G = 0, & \textit{on } \partial D. \end{cases}$$

Question 1

Problem (1)

Let $\phi \in C^2(\Omega)$ be defined on all of the three-dimensional space that vanishes outside some sphere. Show that

$$\phi(\mathbf{0}) = -\iiint \frac{1}{\|\boldsymbol{x}\|} \Delta \phi(\boldsymbol{x}) \frac{d\boldsymbol{x}}{4\pi}. \tag{1}$$

This integration is taken over the region where $\phi(\boldsymbol{x})$ is not zero.

Solution 1

Question 2

Problem (2)

Begin with the function $1/(2\pi)\log r$, show that the Green's function for the disk $\{|\boldsymbol{x}| < a\}$ is

$$G(\boldsymbol{x},\boldsymbol{x}_0) = \frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{x}_0| - \frac{1}{2\pi}\log\left(\frac{|\boldsymbol{x}_0|}{a}|\boldsymbol{x}_0-\boldsymbol{x}_0^*|\right),$$

where $\boldsymbol{x}_0^*$ is defined similarly as the 3d case. Moreover, show that the solution to the Dirichlet problem

$$\begin{cases}\Delta u = 0, & x^2+y^2<a^2\\ u = h, & x^2+y^2 = a^2.\end{cases}$$

is

$$u(\boldsymbol{x}_0) = \frac{a^2-|\boldsymbol{x}_0|^2}{2\pi a}\int_{|\boldsymbol{x}|=a}\frac{h(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x}_0|^2}ds.$$

Note that this is indeed the ***Poisson's formula***.

Solution 2

Question 3& 4

Problem (3)

Verify directly from the definition that $\phi \to \int_{-\infty}^{\infty} f(x)\phi(x)dx$ *is a distribution if* $f(x)$ *is any function that is integrable on each bounded set.*

Problem (4)

Let f *be any distribution. Verify that the functional* f' *defined by* $(f', \phi) := -(f, \phi)$ *satisfies the linearity and continuity properties and therefore is another distribution.*

Solution 3 & 4

References I

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