MAT4220: Partial Differential Equations

Tutorial 6 Slides[1]

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[1] All of the problems are taken from [Strauss, 2007].

Contents

Question 1

Problem (1)

Consider waves in a resistant medium that satisfy the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} - r u_t, \quad 0 < x < l \\ u(0,t) = 0; \quad u(l,t) = 0 \\ u(x,0) = \phi(x); \quad u_t(x,0) = \psi(x) \end{cases}$$

where r is a constant, $0 < r < 2\pi c/l$.
(a). Write down the series expansion of the solution.
(b). (exercise) Do the same for the case $2\pi c/l < r < 4\pi c/l$.

Solution 1 - Formulating Eigenvalue Problem

The PDE and its boundary conditions are linear and homogeneous, so the method of *separation of variables* can be applied to solve it. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and plug it into the PDE

$$u_{tt} = c^2 u_{xx} - ru_t \to XT'' = c^2 X'' T - rXT'.$$

and the boundary conditions.

$$u(0,t) = X(0)T(t) \to X(0) = 0$$
$$u(l,t) = X(l)T(t) \to X(l) = 0.$$

Separate variables now.

$$XT'' + rXT' = c^2 X'' T \to \frac{T'' + rT'}{c^2 T} = \frac{X''}{X}.$$

Solution 1 - Formulating Eigenvalue Problem

Note that c^2 is a constant and can go on either side. The final answer will be the same regardless. We have a function of t on the left side and a function of x on the right side. The only way both functions can be equal is if they are equal to a constant.

$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X} \equiv k \in \mathbb{R}.$$

(Note that in the lecture notes, the k here is replaced by $-\lambda$, but the derivation is the same.) Values of k for which $X(0) = 0$ and $X(l) = 0$ are satisfied are called the *eigenvalues*, and the nontrivial functions $X(x)$ associated with them are called *eigenfunctions*.

Solution 1 - Solving Eigenvalue Problem

- Assuming k is positive, set $k = \mu^2$, then the differential equation for X becomes

$$X'' = \mu^2 X$$

the general solution can be written in terms of hyperbolic sine and hyperbolic cosine functions

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x.$$

Now we use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0. \end{aligned}$$

We can see that $C_1 = 0$ and $C_2 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering positive values for k, and there are no positive eigenvalues.

Solution 1 - Solving Eigenvalue Problem

- Assuming that $k = 0$, the differential equation now becomes

  $$X'' = 0$$

  The general solution is just a linear function

  $$X(x) = C_3 x + C_4.$$

  Now use the boundary conditions to determine the costants.

  $$X(0) = C_3 = 0$$
  $$X(l) = C_3 l + C_4 = 0.$$

  We see that $C_3 = 0$ and $C_4 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering $k = 0$, and thus zero is not an eigenvalue.

Solution 1 - Solving Eigenvalue Problem

- Assuming k is negative, the differential equation for X becomes
$$\frac{X''}{X} = -\lambda^2.$$
The general solution can be written as
$$X(x) = C_5 \cos \lambda x + C_6 \sin \lambda x$$
Now use the boundary conditions to determine the constants.
$$\begin{aligned} X(0) &= C_5 = 0 \\ X(l) &= C_5 \cos \lambda l + C_6 \sin \lambda l = 0 \end{aligned}$$
The second equation simplifies to $C_6 \sin \lambda l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Doing so yields an equation for the eigenvalues.
$$\sin \lambda l = 0.$$

Solution 1 - Solving Eigenvalue Problem

Solve for λl.

$$\lambda l = n\pi, \quad n = 1, 2, \ldots$$

So then

$$\lambda = \lambda_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots$$

The eigenfunctions associated with these eigenvalues are

$$X_n(x) = \sin \frac{n\pi x}{l}, \quad n = 1, 2, \ldots$$

Now solve the differential equation for $T(t)$

$$\frac{T'' + rT'}{c^2 T} = -\lambda^2 \rightarrow T'' + rT' + c^2\lambda^2 T = 0.$$

This is an ODE with constant coefficients, so its solution is of the form

$$T = e^{st}.$$

Solution 1 - Solving Eigenvalue Problem

Substitute this into the ODE to determine s.

$$s^2 + rs + c^2\lambda^2 = 0, \quad \text{(characteristic polynomial.)}$$

This is a quadratic equation for s, so use the quadratic formula to solve for it.

$$s = \frac{-r \pm \sqrt{r^2 - 4c^2\lambda^2}}{2} = \frac{-r \pm \sqrt{r^2 - n^2(2\pi c/l)^2}}{2}.$$

Since $0 < r < 2\pi c/l$, the quantity under the square root is negative for every value that n takes. Factor out -1 and bring it out of the square root as i.

$$s = -\frac{r}{2} \pm i\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}.$$

Solution 1 - Determine Coefficients

Thus, the general solution to the ODE for $T(t)$ is

$$T(t) = C_7 e^{-\frac{r}{2}t} \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right) + C_8 e^{-\frac{r}{2}t} \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right).$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $X(x)T(t)$ over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n e^{-\frac{r}{2}t}\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t + B_n e^{-\frac{r}{2}t}\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right]\sin\frac{n\pi x}{l}.$$

Solution 1 - Determine Coefficients

The final task is to use Fourier's method to express the coefficients, A_n and B_n, in terms of the provided initial data, $\phi(x)$ and $\psi(x)$.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = \phi(x).$$

Thus, we have

$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx.$$

In order to use the second initial condition, differentiate the boxed solution for u with respect to t and plug $t = 0$

$$\begin{aligned} u_t(x,0) &= \sum_{n=1}^{\infty}\left(-\frac{r}{2}\right)A_n \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}\sin\frac{n\pi x}{l} \\ &= \psi(x). \end{aligned}$$

Solution 1 - Determine Coefficients

The previous expression is equivalent to

$$\sum_{n=1}^{\infty} B_n \frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l} \sin\frac{n\pi x}{l} = \frac{r}{2}\phi(x) + \psi(x).$$

Therefore, finally we have

$$B_n = \frac{4}{\sqrt{4n^2\pi^2c^2 - r^2l^2}} \int_0^l \left[\frac{r}{2}\phi(x) + \psi(x)\right] \sin\frac{n\pi x}{l} dx.$$

$\square$

Question 2

Problem (2)

Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0,t) = 0$ and $u(l,t) = 0$ (Neumann at the left, Dirichlet at the right).

(a). Show that the eigenfunctions are

$$\cos\left(\frac{n+\frac{1}{2}}{l}\pi x\right).$$

(b). Write the series expansion for a solution $u(x,t)$.

Solution 2

Since the wave equation and its boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and plug it into the PDE

$$u_{tt} = c^2 u_{xx} \to XT'' = c^2 X'' T$$

and the boundary conditions

$$\begin{aligned} u_x(0,t) = X'(0)T(t) = 0 &\to X'(0) = 0 \\ u(l,t) = X(l)T(t) = 0 &\to X(l) = 0. \end{aligned}$$

Separate variables now.

$$\frac{T''}{c^2 T} = \frac{X''}{X}.$$

Solution 2

Note that c^2 is a constant and can go on either side. The final answer will be the same regardless. We have a function of t on the left side and a function of x on the right side. The only way both functions can be equal is if they are equal to a constant.

$$\frac{T''}{c^2 T} = \frac{X''}{X} = p \in \mathbb{R}.$$

(Note that in the lecture notes, the p here is replaced by $-\lambda$, but the derivation is the same.) Values of p for which $X'(0) = 0$ and $X(l) = 0$ are satisfied are called the *eigenvalues*, , and the nontrivial functions $X(x)$ associated with them are called the *eigenfunctions*.

Solution 2

- Assume $p = \mu^2$ is positive, then

$$X'' = \mu^2 X.$$

The general solution can be written as

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x.$$

Now use the boundary conditions to determine C_1 and C_2

$$\begin{aligned} X'(0) &= C_2 \mu = 0 \\ X(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0. \end{aligned}$$

We see that $C_1 = C_2 = 0$. Hence, only the trivial function $X(x) \equiv 0$ results from considering positive values for p, and there are no positive eigenvalues.

Solution 2

- Assume $p = 0$, then
$$X'' = 0 \to X = C_3 x + C_4.$$
Now use the boundary conditions to determine C_3 and C_4.
$$\begin{aligned} X'(0) &= C_3 = 0 \\ X(l) &= C_3 l + C_4 = 0. \end{aligned}$$
We see that $C_3 = C_4 = 0$. Thus, zero is NOT an eigenvalue.

Solution 2

- Assume $p = -\lambda^2$ is negative, then

$$X'' = -\lambda^2 X.$$

The general solution is

$$X(x) = C_5 \cos \lambda x + C_6 \sin \lambda x.$$

Now use the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X'(0) &= C_6 \lambda = 0 \\ X(l) &= C_5 \cos \lambda l + C_6 \sin \lambda l = 0. \end{aligned}$$

The second equation simplifies to

$$C_5 \cos \lambda l = 0.$$

Solution 2

To avoid getting the trivial solution, we insist that $C_5 \neq 0$. Doing so yields an equation for the eigenvalues.

$$\cos \lambda l = 0.$$

Solve for λl,

$$\lambda l = \frac{1}{2}(2n+1)\pi, n = 0, 1, 2, ...$$

So then

$$\lambda = \lambda_n = \frac{(n+\frac{1}{2})}{l}\pi, n = 0, 1, 2, ...$$

Therefore,

$$X_n(x) = \cos \frac{(n+\frac{1}{2})}{l}\pi x, n = 0, 1, 2, ...$$

Solution 2

Now solve the differential equation for $T(t)$.

$$T'' = -\lambda^2 c^2 T.$$

The general solution is

$$T(t) = C_7 \cos c\lambda t + C_8 \sin c\lambda t.$$

According to the principle of linear superposition, the solution to the PDE for $u(x, t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$u(x, t) = \sum_{n=0}^{\infty} (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) \cos \lambda_n x.$$

If two initial conditions $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$ are provided, one can determine the coefficients A_n and B_n. $\square$

Question 3

Problem (3)

For the Robin BCs, show that

$$E_R := \frac{1}{2}\int_0^l \left(c^{-2}u_t^2 + u_x^2\right)dx + \frac{1}{2}a_l u^2(l,t) + \frac{1}{2}a_0 u^2(0,t).$$

is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is 'lost' to the boundary if a_0 and a_l are positive and 'gained' from the boundary if a_0 and a_l are negative.

Solution 3

Set

$$E := \frac{1}{2}\int_0^l c^{-2}u_t^2 + u_x^2 dx.$$

Then we have

$$\begin{aligned}
\frac{dE}{dt} &= u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t) \quad \text{(WHY?)} \\
&= u_t(l,t)[-a_l u(l,t)] - u_t(0,t)[a_0 u(0,t)] \\
&= -a_l u_t(l,t)u(l,t) - a_0 u(0,t)u_t(0,t) \\
&= -a_l \left[\frac{1}{2}\frac{d}{dt}[u(l,t)^2]\right] - a_0\left[\frac{1}{2}\frac{d}{dt}[u(0,t)]^2\right] \\
&= -\frac{1}{2}a_l\frac{d}{dt}[u(l,t)]^2 - \frac{1}{2}a_0\frac{d}{dt}[u(0,t)]^2.
\end{aligned}$$

Bring all terms to the left side.

$$\frac{dE}{dt} + \frac{1}{2}a_l\frac{d}{dt}[u(l,t)]^2 + \frac{1}{2}a_0\frac{d}{dt}[u(0,t)]^2 = 0.$$

Solution 3

The sum of the derivatives is the derivative of the sum.

$$\frac{d}{dt}\left(\frac{dE}{dt} + \frac{1}{2}a_l\frac{d}{dt}[u(l,t)]^2 + \frac{1}{2}a_0\frac{d}{dt}[u(0,t)]^2\right) = 0.$$

Use the definition of E here.

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)dx + \frac{1}{2}a_l\frac{d}{dt}[u(l,t)]^2 + \frac{1}{2}a_0\frac{d}{dt}[u(0,t)]^2\right) = 0.$$

Therefore, E_R is conserved when there are Robin boundary conditions. $\square$

Question 4

Problem (4)

Let $\phi(x) = x^2$ *for* $0 \le x \le 1 := l$*.*

(a). Calculate its Fourier sine series.

(b). Calculate its Fourier cosine series.

Solution 4 - (a)

The Fourier sine series of our function is defined as

$$x^2 = \sum_{n=1}^{\infty} B_n \sin n\pi x.$$

Note that the Fourier coefficients is given by the formula

$$\begin{aligned} B_n &= \frac{2}{l} \int_0^l x^2 \sin n\pi x dx \\ &= 2 \int_0^1 x^2 \sin n\pi x dx \\ &= 2 \frac{-2 + (-1)^n (2 - n^2\pi^2)}{n^3\pi^3} \end{aligned}$$

$\square$

Solution 4 - (b)

The Fourier cosine series of our function is defined as

$$x^2 = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x.$$

To solve for A_0, simply integrate both sides from 0 to 1

$$\int_0^1 x^2 dx = \int_0^1 \frac{1}{2}A_0 dx + 0 \rightarrow A_0 = \frac{2}{3}.$$

Note that the Fourier coefficients A_n is given by the formula

$$A_n = 2\int_0^1 x^2 \cos n\pi x dx = \frac{4(-1)^n}{n^2\pi^2}.$$

$\square$

Exercises

Problem (5)

Finish the part (b) of Prob 1.1.

Problem (6)

Find the Fourier cosine series of the function $|\sin x|$ *in the interval* $(-\pi, \pi)$. *Use it to find the sums*

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}; \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

References I

Strauss, W. A. (2007).
Partial differential equations: An introduction.
John Wiley & Sons.