MAT4010: Functional Analysis
Tutorial 1: Banach's Fixed Point Theorem*

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1 Exercises

Problem 1.1 (Homeomorphism)**.** *A **homeomorphism** is a continuous bijective mapping $T : X \to Y$ whose inverse is continuous; the metric spaces X and Y are then said to be **homeomorphic**.*
(a). Show that if X and Y are isometric, they are homeomorphic.
(b). Illustrate with an example that a complete and an incomplete metric space may be homeomorphic.

Problem 1.2 (Completion of a Metric Space)**.** *Let $(\mathcal{X}, d)$ be a metric space. A **completion** of $(\mathcal{X}, d)$ is a triple $(\overline{\mathcal{X}}, \overline{d}, \iota)$, consisting of a complete metric space $(\overline{\mathcal{X}}, d)$ and an isometric embedding $\iota : \mathcal{X} \to \overline{\mathcal{X}}$ with a dense image.*
*(a). Every completion $(\overline{\mathcal{X}}, \overline{d}, \iota)$ of $(\mathcal{X}, d)$ has the following **universality property**: if $(\mathcal{Y}, d_{\mathcal{Y}})$ is a complete metric space and $\phi : \mathcal{X} \to \mathcal{Y}$ is a **1-Lipschitz map** (i.e. a Lipschitz continuous map with Lipschitz constant one), then there exists a unique 1-Lipschitz map $\overline{\phi} : \overline{\mathcal{X}} \to \mathcal{Y}$ such that*

$$\phi = \overline{\phi} \circ \iota.$$

(b). If $(\overline{\mathcal{X}_1}, \overline{d_1}, \iota_1)$ and $(\overline{\mathcal{X}_2}, \overline{d_2}, \iota_2)$ are completions of $(\mathcal{X}, d)$, then there exists a unique isometry $\psi : \overline{\mathcal{X}_1} \to \overline{\mathcal{X}_2}$ such that $\psi \circ \iota_1 = \iota_2$.
(c). $(\mathcal{X}, d)$ admits a completion.
(d). Let $(\overline{\mathcal{X}}, \overline{d})$ be a complete metric space and let $\iota : \mathcal{X} \to \overline{\mathcal{X}}$ be a 1-Lipschitz map that satisfies the universality property in (a). Prove that $(\overline{\mathcal{X}}, \overline{d}, \iota)$ is a completion of $(\mathcal{X}, d)$.

Problem 1.3 (Completion of a Normed Vector Space)**.** *The completion of a normed vector space is a Banach space.*

*The presentation of this note is based on *Functional Analysis* by Gongqing Zhang and Yuanqu Lin, [Kre91], and [BS18].

Problem 1.4 (Continuously Differentiable Functions)**.** *Let $I := [0,1]$ be the unit interval and denote by $C^1(I)$ the space of continuously differentiable functions $f : I \to \mathbb{R}$ (with one-sided derivatives at $t = 0$ and $t = 1$). Define*

$$\|f\|_{C^1} := \sup_{0\le t\le 1} |f(t)| + \sup_{0\le t\le 1} |f'(t)|, \quad f \in C^1(I).$$

Prove that $(C^1(I), \|\cdot\|_{C^1})$ is a Banach space.

Problem 1.5 (Subset of ℓ^∞)**.** *Let X denote the set of all elements $(\xi_1, \xi_2, ...) \in \ell^\infty$ such that the sequence $(\xi_j)_{j\in\mathbb{N}}$ converges and let $X_0 \subset X$ be the set of all such elements for which $\xi_j \to 0$ as $j \to \infty$. Prove that X and X_0 are both complete metric space with the restricted ℓ^∞ metric.*

2 Banach's Fixed Point Theorem

Definition 2.1 (Cauchy Sequence & Completeness)**.** *A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space $(\mathcal{X}, d)$ is* ***Cauchy*** *if for $\forall\epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $m, n \ge N(\epsilon) \Rightarrow d(x_m, x_n) < \epsilon$. A metric space is* ***complete*** *if all the Cauchy sequences converge.*

Definition 2.2 (Contraction Mapping)**.** *$T : (\mathcal{X}, d) \to (\mathcal{X}, d)$ is said to be a* ***contraction mapping*** *if there exists $0 < \alpha < 1$ such that $d(Tx, Ty) \le \alpha d(x, y), (\forall x, y \in \mathcal{X})$.*

Example 2.1. *Let $\mathcal{X} = [0,1]$, $T(x)$ is a differentiable function on $[0,1]$ that satisfies*

$$T(x) \in [0,1] \quad (\forall x \in [0,1]); \quad |T'(x)| \le \alpha < 1, \quad (\forall x \in [0,1]).$$

Then, T is a contraction mapping.

Suppose T is defined as in Example 2.1. Does T have any fixed points? If yes, is the fixed point unique? Consider pick an arbitrary $x_0 \in [0,1]$, define a sequence $(x_n)_{n\in\mathbb{N}} \subset [0,1]$ by $x_{n+1} := Tx_n (n = 0, 1, 2, ...)$, notice that

$$|x_{n+1} - x_n| = |Tx_n - Tx_{n-1}| \le \alpha|x_n - x_{n-1}| \le \cdots \le \alpha^n|x_1 - x_0|, \tag{1}$$

for any $p \in \mathbb{N}$,

$$\begin{aligned} |x_{n+p} - x_n| &\le \sum_{i=1}^{p} |x_{n+i} - x_{n+i-1}| \le \sum_{i=1}^{p} \alpha^{n+i-1}|x_1 - x_0| \\ &< \sum_{i=1}^{\infty} \alpha^{n+i-1}|x_1 - x_0| = \frac{\alpha^n}{1-\alpha}|x_1 - x_0| \to 0, \end{aligned} \tag{2}$$

as $n \to \infty$. Therefore, $(x_n)_{n\in\mathbb{N}}$ is Cauchy and thus converges (by the completeness of $\mathbb{R}$). If we send n to infinity on both sides of $x_{n+1} = Tx_n$ and by continuity of T,

$$x^* = Tx^*,$$

where $x^* := \lim_{n\to\infty} x_n$, which means that x^* is a fixed point of T. In fact, the fixed point for T is unique. Suppose for contradiction that there exists two distinct fixed points of T, x_1 and x_2, then

$$|x_1 - x_2| = |Tx_1 - Tx_2| \leq \alpha |x_1 - x_2|,$$

the inequality implies that $x_1 = x_2$. As you might have noticed, in the argument above, we only invoke the triangle inequality of the absolute value. Indeed, any distance function also has the triangle inequality property. Hence, if we replace all the absolute value $|\cdot|$ by an arbitrary distance function $d(\cdot,\cdot)$, the argument still applies. We would then have for $\forall p \in \mathbb{N}$

$$d(x_{n+p}, x_n) \leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0) \to 0, \quad \textit{as } n \to \infty,$$

which shows that $(x_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ is Cauchy. However, to ensure we can find such a $x^* \in \mathcal{X}$, we need $(\mathcal{X}, d)$ to be complete.

Theorem 2.1 (Contraction Mapping Theorem / Banach's Fixed Point Theorem)**.** *Let $(\mathcal{X}, d)$ be a* ***complete*** *metric space and $T : \mathcal{X} \to \mathcal{X}$ is a contraction, then T has a* ***unique*** *fixed point in $\mathcal{X}$.*

Banach's Fixed Point Theorem is very basic. It is one of the most common and widely-used existence theorems. Many of the existence theorems in mathematical analysis are its special cases. Let us consider some simple examples. First, consider the ODE initial value problem

$$\begin{cases} \frac{dx}{dt} = F(t,x), \\ x(0) = \xi \end{cases} \tag{3}$$

or its equivalent formulation, i.e., find a continuous function $x(t)$ that satisfies the integral equation

$$x(t) = \xi + \int_0^t F(\tau, x(\tau))d\tau, \tag{4}$$

The problem can also be viewed as a fixed point problem. To see that, we investigate the metric space $C[-h,h]$ and introduce the mapping

$$(Tx)(t) := \xi + \int_0^t F(\tau, x(\tau))d\tau, \tag{5}$$

then the IVP is equivalent to finding a point $x \in C[-h,h]$ such that $x = Tx$, which is same as finding the fixed point of T.

Now that we have converted the ODE initial value problem to a fixed point problem, we would like to investigate what conditions should $F(t,x)$ satisfy so that the mapping T becomes a contraction. Note that

$$d(Tx,Ty) = \max_{|t|\leq h} \left| \int_0^t F(\tau, x(\tau))d\tau - \int_0^t F(\tau, y(\tau))d\tau \right| \leq h \max_{|t|\leq h} |F(t,x(t)) - F(t,y(t))|,$$

for example, if $F(t,x)$ satisfies the local Lipschitz condition for the variable x, uniformly for the variable t (i.e., $\exists \delta > 0, L > 0$ such that when $|t| \leq h, |x_1 - \xi| \leq \delta, |x_2 - \xi| \leq \delta$)

$$|F(t,x_1) - F(t,x_2)| \leq L|x_1 - x_2|,$$

then we have

$$d(Tx,Ty) \leq Lhd(x,y), \quad \forall x,y \in \overline{B}_\delta(\xi),$$

where $\overline{B}_\delta(\xi) := \{x(t) \in C[-h,h] | \max_{|t|\leq h} |x(t) - \xi| \leq \delta\}$. However here we cannot take $C[-h,h]$ as the metric space in theorem 2.1. This is because when $Lh < 1$, T is a contraction only on the subset $\overline{B}_\delta(\xi) \subset C[-h,h]$. (Here we view ξ as a constant function on $[-h,h]$ which equals to $\xi \in \mathbb{R}$). Let $\mathcal{X} = \overline{B}_\delta(\xi)$, in order to make $T : \mathcal{X} \to \mathcal{X}$, set

$$M := \max_{(t,x)\in[-h,h]\times[\xi-\delta,\xi+\delta]} \{|F(t,x)|\}.$$

Take h to be sufficiently small such that

$$\max_{|t|\leq h} |(Tx)(t) - \xi| = \max_{|t|\leq h} \left| \int_0^t F(\tau, x(\tau))d\tau \right| \leq Mh \leq \delta.$$

Since $(C[-h,h],d)$ is a complete metric space and $\mathcal{X}$ is a closed subset of $C[-h,h]$, we conclude that $(\mathcal{X}, d_{\mathcal{X}\times\mathcal{X}})$ is complete. This proves the following corollary. $\square$

Corollary 2.2 (Local Existence and Uniqueness of IVP / Picard Lindelof Theorem)**.** *Let the function $F(t,x)$ be defined on $[-h,h] \times [\xi - \delta, \xi + \delta]$, continuous, and satisfies the **Lipschitz** condition, i.e., $\exists \delta > 0, L > 0$, such that whenever $|t| \leq h, |x_1 - \xi| \leq \delta, |x_2 - \xi| \leq \delta$,*

$$|F(t,x_1) - F(t,x_2)| \leq L|x_1 - x_2|,$$

then if $h < \min\{\delta/M, 1/L\}$, the initial value problem (3) has a unique solution on $[-h,h]$. Where $M := \max\{|F(t,x)| : (x,t) \in [-h,h] \times [\xi - \delta, \xi + \delta]\}$.

Corollary 2.3 (Implicit Function Theorem)**.** *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$ is a neiborhood of $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$. If f and $\partial f/\partial y$ are continuous in $U \times V$ and*

$$f(x_0,y_0) = 0; \quad \left[det\left(\frac{\partial f}{\partial y}\right)\right](x_0,y_0) \neq 0,$$

*then there exists a neiborhood of (x_0,y_0), $U_0 \times V_0 \subset U \times V$ and a **unique** continuous function $\psi : U_0 \to V_0$ such that*

$$\begin{cases} f(x,\psi(x)) = 0, & x \in U_0, \\ \psi(x_0) = y_0. \end{cases}$$

Proof. Define a mapping $T : \psi \mapsto T\psi$ by

$$(T\psi)(x) := \psi(x) - \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^{-1} f(x,\psi(x)),$$

where $\psi \in C(\overline{B}_r(x_0), \mathbb{R}^m), r > 0$, which denotes the collection of vector-valued continuous functions defined on $\overline{B}_r(x_0)$ with values in $\mathbb{R}^m$. The metric is definded to be

$$d(\phi, \psi) := \max_{x \in \overline{B}_r(x_0), 1 \le i \le m} |\phi_i(x) - \psi_i(x)|,$$

where $\phi = (\phi_1, \phi_2, ..., \phi_m)$ and $\psi = (\psi_1, \psi_2, ..., \psi_m)$. For $x \in \mathbb{R}^n$ and $y_i \in \mathbb{R}^m (i = 1, 2, ..., m)$, denote

$$D_y f(x, y_1, y_2, ..., y_m) := \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(x, y_1) & \cdots & \frac{\partial f_1}{\partial y_m}(x, y_1) \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial y_1}(x, y_m) & \cdots & \frac{\partial f_m}{\partial y_m}(x, y_m) \end{pmatrix}.$$

Since $\partial f / \partial y$ is continuous on $U \times V$, there exists a $\delta > 0$ such that

$$\left| \delta_{ij} - \left[\left(\frac{\partial f}{\partial y}(x_0, y_0) \right)^{-1} D_y f(x, y_1, y_2, ..., y_m) \right]_{ij} \right| < \frac{1}{2m},$$
$$i, j = 1, 2, ..., m, x \in \overline{B}_\delta(x_0), y_1, ..., y_m \in \overline{B}_\delta(y_0),$$

where $[\cdot]_{ij}$ denotes the (i, j)-th entry of a matrix and δ_{ij} is the Kronecker delta notation. Denote $d_i(x) := \phi_i(x) - \psi_i(x)$. We have

$$\begin{aligned} d(T\phi, T\psi) &= \max_{x \in \overline{B}_r(x_0), 1 \le i \le m} |d_i(x) - \sum_{j=1}^{m} \left[\left(\frac{\partial f}{\partial y}(x_0, y_0) \right)^{-1} D_y f(x, \hat{y}_1, ..., \hat{y}_m) \right]_{ij} d_j(x)| \\ &< \frac{1}{2} \max_{x \in \overline{B}_r(x_0), 1 \le i \le m} |d_i(x)| = \frac{1}{2} d(\phi, \psi), \end{aligned}$$

where $r < \delta$ such that $\phi(x), \psi(x) \in \overline{B}_\delta(y_0)$. $\hat{y}_i(x) = \theta_i \phi(x) + (1 - \theta_i)\psi(x), 0 < \theta_i(x) < 1, i = 1, 2, ..., m$.

Let $\mathcal{X} := \{\psi \in C(\overline{B}_r(x_0), \mathbb{R}^m) | \psi(x_0) = y_0, \psi(x) \in \overline{B}_\delta(y_0)\}$, then it is easy to show that $\mathcal{X}$ is a closed subset of $C(\overline{B}_r(x_0), \mathbb{R}^m)$ and thus a complete metric space with the restricted distance function. We have shown that T is a contraction on $\mathcal{X}$. To apply theorem 2.1, we only need to show that $T : \mathcal{X} \to \mathcal{X}$. Note that

$$d(T\psi, y_0) \le d(T\psi, Ty_0) + d(Ty_0, y_0) \le \frac{1}{2} d(\psi, y_0) + \max_{x \in \overline{B}_r(x_0), 1 \le i \le m} \left| \left[\left(\frac{\partial f}{\partial y}(x_0, y_0) \right)^{-1} f(x, y_0) \right]_i \right|.$$

Since f is continuous, we have

$$\begin{aligned} &\max_{x \in \overline{B}_r(x_0), 1 \le i \le m} \left| \left[\left(\frac{\partial f}{\partial y}(x_0, y_0) \right)^{-1} f(x, y_0) \right]_i \right| \\ &= \max_{x \in \overline{B}_r(x_0), 1 \le i \le m} \left| \left[\left(\frac{\partial f}{\partial y}(x_0, y_0) \right)^{-1} \{f(x, y_0) - f(x_0, y_0)\} \right]_i \right| < \frac{\delta}{2}, \quad \textit{given } r < \eta. \end{aligned}$$

Therefore, as long as $0 < r < \min\{\delta, \eta\}$, $d(T\psi, y_0) < \delta$. Moreover,

$$(T\psi)(x_0) = \psi(x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0) \right)^{-1} f(x_0, \psi(x_0)) = y_0.$$

Hence, $T : \mathcal{X} \to \mathcal{X}$. □

Banach's Fixed Point Theorem has important applications to iteration methods for solving systems of linear algebraic equations and yields sufficient conditions for convergence and error bounds. Set $\mathcal{X} = \mathbb{R}^n$, define on $\mathcal{X}$ a metric d by $d(x, z) := \max_{1 \le i \le n} |x_i - z_i|$, where $x = (x_1, x_2, ..., x_n)^T$ and $z = (z_1, z_2, ..., z_n)^T$. Note that $(\mathcal{X}, d)$ is complete. The proof is left as a simple exercise. On $\mathcal{X}$ we define a linear map $T : \mathcal{X} \to \mathcal{X}$ by

$$y = Tx = Cx + b, \tag{6}$$

where $C = (c_{jk}) \in \mathbb{R}^{n \times n}$ is a fixed real $n \times n$ matrix and $b = (b_j) \in \mathcal{X}$ is a fixed vector. Under what conditions will T become a contraction? First, write (6) in components, we have

$$y_j = \sum_{k=1}^{n} c_{jk} x_k + b_j, \quad j = 1, 2, ..., n.$$

Setting $w = (w_j) = Tz$, we thus have

$$\begin{aligned} d(y, w) = d(Tx, Tz) = \max_j |y_j - w_j| &= \max_j \left| \sum_{k=1}^{n} c_{jk}(x_k - z_k) \right| \\ &\le \max_i |x_i - z_i| \max_j \sum_{k=1}^{n} |c_{jk}| = d(x, z) \max_j \sum_{k=1}^{n} |c_{jk}|. \end{aligned}$$

We see that a sufficient condition for T to be a contraction is $\max_j \sum_{k=1}^{n} |c_{jk}| < 1$. Banach's Fixed Point Theorem 2.1 then yields

Theorem 2.4 (Linear Equation)**.** *If a system*

$$x = Cx + b, \quad C = (c_{jk}) \in \mathbb{R}^{n \times n}, b = (b_j) \in \mathbb{R}^n \tag{7}$$

of n linear equations in n unknowns $x_1, x_2, ..., x_n$ satisfies

$$\sum_{k=1}^{n} |c_{jk}| < 1, \quad j = 1, 2, ..., n \Leftrightarrow \|C\|_\infty < 1.$$

it has precisely one solution x. This solution can be obtained as the limit of the iterative sequence $(x^{(0)}, x^{(1)}, ...)$, where $x^{(0)}$ is arbitrary and

$$x^{(m+1)} = Cx^{(m)} + b, \quad m = 0, 1, 2, ...$$

How is theorem 2.4 related to methods used in practice? A system of n linear equations in n unknowns is usually written as

$$Ax = c, \quad A \in \mathbb{R}^{n \times n}, c \in \mathbb{R}^n.$$

We assume that $det(A) \neq 0$ and $A = B - G$ for some nonsingular B. Then,

$$Bx = Gx + c \quad \Leftrightarrow \quad x = B^{-1}(Gx + c).$$

Example 2.2 (Jacobi Method). *The Jacobi method is defined by*

$$x_j^{(m+1)} = \frac{1}{a_{jj}}(c_j - \sum_{k=1,k\neq j}^{n} a_{jk}x_k^{(m)}), \quad j = 1, 2, ..., n,$$

where we assume that $a_{jj} \neq 0$ *for all* $j = 1, 2, ..., n$. *This method is suggested by solving the* j*-th equation of* $Ax = c$ *for* x_j *with* $x_k^{(m)}, k \neq j$ *all given. It is easy to see that in this case*

$$C = -D^{-1}(A - D), \quad b = D^{-1}c,$$

where $D := diag(A)$. *The condition* $\|C\|_\infty < 1$ *is sufficient for the convergence of the Jacobi iteration. Since the form of* C *is simple, we can obtain the condition in closed form. The result is the* ***row sum criterion*** *for the Jacobi iteration*

$$\sum_{k=1,k\neq j} |a_{jk}| < |a_{jj}|, \quad j = 1, 2, ..., n,$$

which is equivalent to requiring that A *is* ***strictly diagonally dominant***. □

One drawback of Jacobi method is that when computing $x_j^{(m+1)}$, $x_k^{(m+1)}, k < j$ are available but are not used in computation. **Gauss-Seidel Method** makes up the drawback.

Example 2.3 (Gauss-Seidel Method). *The method is defined by*

$$x_j^{(m+1)} := \frac{1}{a_{jj}}\left(c_j - \sum_{k=1}^{j-1} a_{jk}x_k^{(m+1)} - \sum_{k=j+1}^{n} a_{jk}x_k^{(m)}\right),$$

where $j = 1, 2, ..., n$ *and we again assume* $a_{jj} \neq 0$ *for all* j. *We decompose* A *as* $A = -L + D - U$, *where* $-L, -U$ *are the lower triangular part and the upper triangular part of* A *respectively and* $D := diag(A)$. *Therefore, the update formula can be written as*

$$Dx^{(m+1)} = c + Lx^{(m+1)} + Ux^{(m)},$$

which further implies

$$C = (D - L)^{-1}U, \quad b = (D - L)^{-1}c.$$

Note that $\|C\|_\infty < 1$ *is sufficent for the convergence of Gauss-Seidel iteration. However, the form of* C *is complicated. The remaining problem is to get simpler sufficient conditions.* □

The final note is that in the above discussion our metric space is $(\mathcal{X}, d)$, where $d(x, z) := \max_j |x_j - z_j|$. We can use other metrics like $d_1(x, z) := \sum_{j=1}^n |x_j - z_j|$ and $d_2(x, z) := \left(\sum_{j=1}^n (x_j - z_j)^2\right)^{1/2}$. (it can be proved that $(\mathcal{X}, d_1 \text{ } or \text{ } d_2)$ are still complete). In these cases, the sufficient conditions become $\|C\|_1 < 1$ and $\|C\|_F < 1$ respectively. The proof is left as an exercise.

3 More on Fixed Point Theorems[1]

Theorem 3.1 (Brouwer Fixed Point Theorem[2])**.** *Let $B \subset \mathbb{R}^n$ be the closed unit ball, let $T : B \to B$ be continuous, then T has a fixed point $x \in B$.*

Corollary 3.2. *Let $C \subset \mathbb{R}^n$ be a compact convex subset and $T : C \to C$ is continuous, then T has a fixed point on C.*

Proof. Since C is homeomorphic to a unit ball in $\mathbb{R}^m$ $(m \leq n)$[3], denote this homeomorphism as $\psi : B^m(\theta, 1) \to C$, consider the mapping

$$T_\psi := \psi^{-1} \circ T \circ \psi.$$

Obviously, $T_\psi : B^m(\theta, 1) \to B^m(\theta, 1)$. Apply the Brouwer Fixed Point Theorem to T_ψ, there exists a $x \in B^m(\theta, 1)$ such that $T_\psi x = x$, therefore $T(\psi(y)) = \psi(y)$. □

The generalization of the fixed point theorem in finite-dimensional spaces to infinite-dimensional spaces is

Theorem 3.3 (Schauder Fixed Point Theorem)**.** *Let $C \subset (X, \|\cdot\|)$ be a closed convex subset, $T : C \to C$ continuous, and $T(C)$ is sequentially compact. Then T has a fixed point on C.*

Proof. **Step 1:** Since $T(C)$ is sequentially compact, for any $n \in \mathbb{N}$, there exists a finite $1/n$-net, say $N_n = \{y_1, y_2, ..., y_{r_n}\} \subset T(C)$, i.e.,

$$T(C) \subset \bigcup_{i=1}^{r_n} B(y_i, 1/n).$$

Denote $E_n := Span(\{N_n\})$.
Step 2: Define $I_n : T(C) \to conv(N_n)$ by

$$I_n(y) := \sum_{i=1}^{r_n} y_i \lambda_i(y), \quad \forall y \in T(C),$$

where

$$\lambda_i(y) = \frac{m_i(y)}{\sum_{i=1}^{r_n} m_i(y)}, \quad m_i(y) = \begin{cases} 1 - n\|y - y_i\|, & y \in B(y_i, 1/n) \\ 0, & y \notin B(y_i, 1/n) \end{cases}$$

It can be shown that

$$\|I_n y - y\| \leq \frac{1}{n}, \quad \forall y \in T(C).$$

Step 3: Set $T_n := I_n \circ T$, then $T_n|_{conv(N_n)} : conv(N_n) \to conv(N_n)$. Since $conv(N_n) \subset E_n$ is bounded closed convex subset, there exists $x_n \in conv(N_n) \subset C$ such that

$$T_n x_n = x_n.$$

[1] Optional

[2] There are a number of proofs for this theorem, for example, one can refer to *J. Milno:, Analytic proofs of the 'hairy ball theorem'* for one elementary analytic proof.

[3] If $C \subset \mathbb{R}^n$ is compact and convex, there exists $m \leq n$ such that C is homeomorphic to $B^m(\theta, 1) \subset \mathbb{R}^m$.

Since $T(C)$ is sequentially compact and C is closed, there exists (n_k) and $x \in C$ such that

$$Tx_{n_k} \to x, \quad k \to \infty.$$

Therefore,

$$\begin{aligned}\|x_n - x\| &= \|T_n x_n - x\| = \|I_n T x_n - T x_n + T x_n - x\| \\ &\leq \|I_n T x_n - T x_n\| + \|T x_n - x\| < \frac{1}{n} + \|T x_n - x\|.\end{aligned}$$

which implies that $x_{n_k} \to x$. By continuity of T, $Tx = x$. $\square$

Corollary 3.4. *Let $C \subset (X, \|\cdot\|)$ is a bounded closed convex subset, $T : C \to C$ is continuous and compact, then T has a fixed point on C.*

Corollary 3.5 (Caratheodory Local Existence and Uniqueness Theorem for IVP)**.** *Let $f(t,x) \in C([-h,h] \times [\xi - b, \xi + b])$ and $|f(t,x)| \leq M$, then when $h < b/M$, the IVP*

$$\begin{cases} \frac{dx}{dt}(t) = f(t, x(t)), \\ x(0) = \xi. \end{cases}$$

has a unique solution $x(t)$ on $[-h,h]$.

Proof. Consider the mapping defined on $\overline{B}_b(\xi) \subset C[-h,h]$

$$(Tx)(t) := \xi + \int_0^t f(\tau, x(\tau))d\tau.$$

Notice that

$$\|Tx - \xi\|_{C[-h,h]} \leq Mh, \quad \forall x \in \overline{B}_b(\xi),$$

when $h \leq b/M$, $T : \overline{B}_b(\xi) \to \overline{B}_b(\xi)$. Because

$$|(Tx)(t) - (Tx)(t')| = \left| \int_{t'}^t f(\tau, x(\tau))d\tau \right| \leq M|t - t'|, \quad \forall t, t' \in [-h,h],$$

and

$$|(Tx)(t)| \leq |\xi| + Mh, \quad \forall t \in [-h,h],$$

we have T is continuous and $\overline{B}_b(\xi)$ is uniformly bounded and equicontinuous. By Arzela-Ascoli, we know that $T(\overline{B}_b(\xi)) \subset \overline{B}_b(\xi)$ is sequentially compact, Apply Schauder, there exists $x \in C[-h,h]$ such that $Tx = x$. $\square$

References

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