MAT4010: Functional Analysis
Tutorial 2: Banach Spaces*

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1 Exercises

Problem 1.1 extends the basic assertion in first year analysis that every absolutely convergent series of real numbers converges.

Problem 1.1 (Absolute Convergence). *Let $(X, \|\cdot\|)$ be a normed space. Prove that X is complete if and only if every series $\sum_{j=1}^{\infty} x_j$ in X satisfying $\sum_{j=1}^{\infty} \|x_j\|$ converges to a limit in X. Give an example of a space X and a series for which $\sum_{j=1}^{\infty} \|x_j\| < \infty$ but $\sum_{j=1}^{\infty} x_j$ does not converge in X.*

Problem 1.2 (Schauder Basis). *Show that if a normed space has a Schauder basis, it is separable. (Note that the converse is not true)*

Problem 1.3 (Separability). *Show that if there is a sequence $e_1, e_2, \ldots \in X$ such that the linear subspace of all (finite) linear combinations of the e_i is dense in X, then X is separable.*

Problem 1.4. *Let X be a normed vector space. Show that*
(a). *X^* is separable $\Rightarrow$ X is separable.*
(b). *X is reflexive and separable $\Rightarrow$ X^* is separable.*

Problem 1.5 (Equivalent Norms). *Prove the following:*

1. *Show that the Definition 2.1 is equivalent to the definition in the textbook.*
2. *Definition 2.1 yields an equivalence relation on the set of all norms on X.*
3. *Two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are equivalent if and only if the identity maps $id : (X, \|\cdot\|) \to (X, \|\cdot\|')$ and $id : (X, \|\cdot\|') \to (X, \|\cdot\|)$ are bounded linear operators.*
4. *Two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if and only if they induce the same topologies on X, i.e. $\mathscr{U}(X, \|\cdot\|) = \mathscr{U}(X, \|\cdot\|')$.*

*The presentation of this note is based on [BS18], [Kre91], [RF88], and [AL06].

5. *Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norms on X. Show that $(X, \|\cdot\|)$ is complete if and only if $(X, \|\cdot\|')$ is complete.*

Problem 1.6 (Nonequivalent Norms)**.** *Show that for every infinite-dimensional vector space X, we can always define two norms which are not equivalent.*

Problem 1.7 (Hamel Basis)**.** *Show that an infinite-dimensional Banach space cannot have a countable Hamel basis. (This problem is hard. You can skip it.)*

Problem 1.8 (Riesz Lemma)**.** *Show that if $\dim X < \infty$, the δ in Riesz lemma 2.7 can be chosen to be 0.*

2 Finite-dimensional Banach Spaces

Finite-dimensional normed vector spaces are complete, their linear subspaces are closed, linear functionals on them are continuous, and their closed unit balls are compact. However, these properties in general fail to hold for infinite-dimensional spaces.

Definition 2.1 (Equivalent Norms)**.** *Let X be a real vector space. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are called **equivalent** if there is a constant $c \geq 1$ such that*

$$\frac{1}{c}\|x\| \leq \|x\|' \leq c\|x\|, \quad \forall x \in X.$$

Theorem 2.1. *Let X be a finite-dimensional real vector space. Then any two norms on X are equivalent.*

Proof. Choose an ordered basis $e_1, ..., e_n \in X$ and define

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2}, \quad x = \sum_{i=1}^{n} x_i e_i, \quad x_i \in \mathbb{R}.$$

This is a norm on X. We prove in two steps that every norm on X is equivalent to $\|\cdot\|_2$ and by Problem 1.5 (4) the proof is done. Fix any norm function $X \to \mathbb{R} : x \mapsto \|x\|$.
Step 1. There is a constant $c > 0$ such that $\|x\| \leq c\|x\|_2$ for all $x \in X$.
Step 2. There is a constant $\delta > 0$ such that $\delta\|x\|_2 \leq \|x\|$ for all $x \in X$. □

Theorem 2.1 has several important consequences that are special to finite-dimensional normed vector spaces and do not carry over to infinite dimensions.

Corollary 2.2. *Every finite-dimensional normed vector space is complete.*

Proof. It is easy to show that X is complete w.r.t. the 2-norm. □

Corollary 2.3. *Let X be a normed vector space. Then every finite-dimensional linear subspace of X is a closed subset of X.*

Corollary 2.4. *Let X be a finite-dimensional normed vector space and let $K \subset X$. Then K is compact if and only if K is a closed and bounded.*

Proof. It is easy to establish a homeomorphism between $(X, \|\cdot\|_2)$ and $(\mathbb{R}^n, \|\cdot\|_2)$. Therefore, the Heine-Borel theorem holds for $(X, \|\cdot\|_2)$ and thus holds for X with any norm. $\square$

Corollary 2.5. *Let X and Y be normed vector spaces and suppose $dim X < \infty$. Then every linear operator $A : X \to Y$ is bounded.*

Proof. Define the function $X \to \mathbb{R} : x \to \|x\|_A$ by

$$\|x\|_A := \|x\|_X + \|Ax\|_Y, \quad \forall x \in X.$$

This is a norm on X (**exercise**: show this). Hence, by theorem 2.1, there exists a constant $c \geq 1$ such that $\|x\|_A \leq c\|x\|_X$ for all $x \in X$. Hence A is bounded. $\square$

The above four corollaries spell out some of the standard facts in finite-dimensional linear algebra. The following four examples show that in none of these four corollaries can the hypothesis of finite-dimensionality be dropped. Thus in functional analysis one must dispense with some of the familiar features of linear algebra. In particular, linear subspaces need no longer be closed subsets and linear maps need no longer be continuous.

Example 2.1. ***(i: Norms may not be equivalent).*** *Consider the space $X := C([0,1])$ of continuous real valued functions on the closed unit interval $[0,1]$. Then the formulas*

$$\|f\|_\infty := \sup_{0 \leq t \leq 1} |f(t)|, \quad \|f\|_2 := \left(\int_0^1 |f(t)|^2 \right)^{1/2}$$

*for $f \in C([0,1])$ define norms on X. The space $C([0,1])$ is complete with $\|\cdot\|_\infty$ but not with $\|\cdot\|_2$. Thus the two norms are not equivalent. (**Exercise:** Find a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ that is Cauchy with respect to the L^2-norm and has no convergent subsequence).*
(ii: Linear subspace may not be closed). *The space $Y := C^1([0,1])$ of continuously differentiable real valued functions on the closed unit interval is a dense linear subspace (by Stone-Weierstrass) of $C([0,1])$ with the supremum norm and so is not a closed subset of $(C([0,1]), \|\cdot\|_\infty)$.*
(iii: Heine-Borel fails). *Consider the closed unit ball*

$$B := \{f \in C([0,1]) | \|f\|_\infty \leq 1\}$$

*in the Banach space $C([0,1])$ with the supremum norm. The set is closed and bounded, but not compact. (**Exercise:** Prove it. **Hint:** Prove it is not sequentially compact). Therefore, by Arzala-Ascoli, B is not equi-continuous.*
(iv: Unbounded linear operator). *Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space and choose an unordered basis $E \subset X$ such that $\|e\| = 1$ for all $e \in E$. By assumption E is an infinite set. (The existence of an unordered Hamel basis is guaranteed by the Zorn's Lemma, or equivalently, the axiom of choice. Indeed, E cannot be countable if it is complete, see Problem 1.7). Choose any unbounded function $\lambda : E \to \mathbb{R}$ and define the linear map $\Phi_\lambda : X \to \mathbb{R}$ by $\Phi_\lambda(\sum_{i=1}^l x_i e_i) := \sum_{i=1}^l \lambda(e_i) x_i$ for all $l \in \mathbb{N}$, all pairwise distinct l-tuples of basis vectors $e_1, ..., e_l \in E$, and all $x_1, ..., x_l \in \mathbb{R}$. Then Φ_λ is an unbounded linear functional.*

Theorem 2.6. *Let $(X, \|\cdot\|)$ be a normed vector space and denote the closed unit ball and the closed unit sphere in X by*

$$B := \{x \in X | \|x\| \leq 1\}, \quad S := \{x \in X | \|x\| = 1\}.$$

Then the following are equivalent

1. *$dim X < \infty$.*
2. *B is compact.*
3. *S is compact.*

Proof. (1) implies (2) follows from the Heine-Borel theorem. (2) implies (3) follows from the fact that a closed subset of a compact set in a topological space is compact. (3) implies (1) requires the Riesz Lemma. □

Lemma 2.7 (Riesz)**.** *Let $(X, \|\cdot\|)$ be a normed vector space and let $Y \subset X$ be a closed linear subspace that is not equal to X. Fix a constant $\delta \in (0, 1)$. Then there exists a vector $x \in X$ such that*

$$\|x\| = 1, \quad \inf_{y \in Y} \|x - y\| \geq 1 - \delta.$$

3 L^p Space

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Recall that $\mathcal{L}^p(\Omega, \mathcal{F}, \mu), 0 < p \leq \infty$ (for notational simplicity we will write $\mathcal{L}^p(\Omega)$), is the collection of all measurable functions f on Ω such that $\|f\|_p < \infty$, where for $0 < p < \infty$,

$$\|f\|_p := \left(\int |f|^p d\mu \right)^{\min\{\frac{1}{p}, 1\}}$$

and for $p = \infty$,

$$\|f\|_\infty := \sup\{k : \mu(|f| > k) > 0\}$$

One should recall that $\mathcal{L}^p(\Omega)$ is a vector space, i.e. $af + bg \in \mathcal{L}^p(\Omega), \forall f, g \in \mathcal{L}^p(\Omega), a, b \in \mathbb{R}$. The proof is divided into three cases: $0 < p \leq 1$, use the inequality

$$\left(\frac{x}{x+y} \right)^p + \left(\frac{y}{x+y} \right)^p \geq \frac{x}{x+y} + \frac{y}{x+y} = 1, \quad \forall x, y \geq 0.$$

When $1 < p < \infty$, use $|af + bg|^p \leq 2^{p-1}(|af|^p + |bg|^q)$. When $p = \infty$, you need to play with the definition.

Consider an equivalence relation on $\mathcal{L}^p(\Omega)$: $f \sim g$ if $f = g, \mu$-a.e. Then $\mathcal{L}^p(\Omega)$ is partitioned into disjoint equivalence classes. We denote the new space by $L^p(\Omega)$, equipped with the norm $\|[f]\|_p := (\int |f|^p d\mu)^{1/p}$, where $f \in [f] \in L^p(\Omega)$. For $0 < p < 1$, if one defines $d_p(f, g) := \int |f - g|^p d\mu$, then $(L^p(\Omega), d_p)$ is a metric space.

3.1 Completeness

Theorem 3.1 (Riesz-Fischer). *For $0 < p < \infty$, $(L^p(\Omega), \|\cdot\|)$ is a Banach space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be Cauchy in $L^p(\Omega)$.
Step 1. Let $\{\epsilon_k\}_{k\geq 1}$ and $\{\delta_k\}_{k\geq 1}$ be sequences of positive numbers decreasing to zero. Since $\{f_n\}$ is Cauchy, for each $k \geq 1$, there exists an integer n_k such that

$$\int |f_n - f_m|^p d\mu \leq \epsilon_k, \quad \forall n, m \geq n_k.$$

WLOG, let $n_{k+1} > n_K$ for each $k \geq 1$. Then, by Markov's inequality

$$\mu(\{|f_{n_{k+1}} - f_{n_k}| \geq \delta_k\}) \leq \delta_k^{-p} \int |f_{n_{k+1}} - f_{n_k}|^p d\mu \leq \delta_k^{-p}\epsilon_k.$$

Let $A_k := \{|f_{n_{k+1}} - f_{n_k}| \geq \delta_k\}$ and $A := \limsup_{n\to\infty} A_n = \bigcap_{j=1}^{\infty} \bigcup_{k\geq j} A_k$. If δ_k and ϵ_k satisfy

$$\sum_{k=1}^{\infty} \delta_k^{-p}\epsilon_k < \infty,$$

then $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ and by Borel-Cantelli lemma $\mu(A) = 0$. Note that for $\omega \in A^c$, $|f_{n_{k+1}}(\omega) - f_{n_k}(\omega)| < \delta_k$ for all k large. Thus, if $\sum_{k=1}^{\infty} \delta_k < \infty$, then for $\omega \in A^c$, $\{f_{n_k}(\omega)\}$ is a Cauchy sequence in $\mathbb{R}$ and hence, it converges to some $f(\omega) \in \mathbb{R}$. Setting $f(\omega) = 0$ for $\omega \in A$, one gets

$$\lim_{k\to\infty} f_{n_k} = f \quad \mu\text{-}a.e.$$

f is μ-measurable as an a.e. limit of μ-measurable functions. A choice of ϵ_k and δ_k is given by $\epsilon_k = (2)^{-(p+1)k}$ and $\delta_k = 2^{-k}$.
Step 2. By Fatou's lemma,

$$\epsilon_k \geq \liminf_{j\to\infty} \int |f_{n_k} - f_{n_{k+j}}|^p d\mu \geq \int |f_{n_k} - f|^p d\mu.$$

Since $f_{n_k} \in L^p(\Omega)$, this shows that $f \in L^p(\Omega)$ by triangle inequality. Now, letting $k \to \infty$, we have

$$\lim_{k\to\infty} \int |f_{n_k} - f|^p d\mu = 0.$$

Step 3. By triangle inequality, for any $k \geq 1$ fixed,

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\|.$$

By step 2, for $n \geq n_k$, the right hand side of the above is $\leq 2\tilde{\epsilon}_k$, where $\tilde{\epsilon}_k = \epsilon_k$ if $0 < p \leq 1$ and $\tilde{\epsilon}_k = \epsilon_k^{1/p}$ if $1 < p < \infty$. Now letting $k \to \infty$, we have $\|f_n - f\| \to 0$. $\square$

Theorem 3.2 (Riesz-Fischer). *$(L^\infty(\Omega), \|\cdot\|_\infty)$ is Banach.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence. For any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\|f_m - f_n\| < \epsilon, \quad \forall m, n \geq N.$$

For $m, n \in \mathbb{N}$, set

$$F_{m,n} := \{x \in X | |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\}$$

Clearly, $\mu(F_{m,n}) = 0$ for all $m, n \in \mathbb{N}$. Set $F = \bigcup_{m,n\in\mathbb{N}} F_{m,n}$ and $E := F^c$. Note that $\mu(E^c) = \mu(F) = 0$. Moreover,

$$\begin{aligned} E &= \bigcap_{m,n\in\mathbb{N}} \{x \in X | |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty\} \\ &= \{x \in X | |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty, \forall m, n \in \mathbb{N}\}. \end{aligned}$$

For any $x \in E$, we have when $m, n \geq N$

$$|f_m(x) - f_n(x)| \leq \|f_n - f_m\|_\infty < \epsilon.$$

This implies that $\{f_n(x)\}_{n\geq 1}$ is Cauchy in $\mathbb{R}$ for any $x \in E$. Define $f(x) := \lim_{n\to\infty} f_n(x), \forall x \in E$ and $f(x) = 0, \forall x \in E$. Now, we have for $m \geq N$,

$$|f_m(x) - f(x)| \leq \epsilon, \forall x \in E.$$

Then,

$$\|f_m - f\|_\infty \leq \epsilon.$$

Therefore $f \in L^\infty(\Omega)$ since $\|f\|_\infty \leq \|f_m\|_\infty + \|f_m - f\|_\infty \leq \|f_m\|_\infty + \epsilon < \infty$. Moreover,

$$\|f_m - f\|_\infty \to 0, \quad m \to \infty.$$

□

Example 3.1. *By taking $\Omega = \mathbb{N}$ and μ to be the counting measure, we can show that l^p and l^∞ are complete.*

3.2 Separability

In this subsection, we only consider the $\mathbb{R}$ case. Suppose $E \subset \mathbb{R}$ be a Lebesgue measurable set.

Theorem 3.3 (Separability)**.** *$(L^p(E), \|\cdot\|_p), 1 \leq p < \infty$ is separable.*

Proof. **Step 1.** $L^p(\mathbb{R})$ is separable. Let $S[a, b]$ be the collection of step functions on $[a, b]$ and $S'[a, b]$ be the collection of step functions ψ on $[a, b]$ that take rational values and for which there is a partition $\mathscr{P} = \{x_0, ..., x_n\}$ of $[a, b]$ with ψ constant on (x_{k-1}, x_k), for $1 \leq k \leq n$, and x_k rational for $1 \leq k \leq n - 1$. It is easy to show that $S'[a, b]$ is dense in $S[a, b]$ by the density of $\mathbb{Q}$ in $\mathbb{R}$. Moreover, $S'[a, b]$ is countable (**exercise:** prove this!). Since $S[a, b]$ is dense in $L^p([a, b])$, we conclude that $S'[a, b]$ is dense in $L^p([a, b])$.
Step 2. For each $n \in \mathbb{N}$, define $\mathcal{F}_n$ to be the functions on $\mathbb{R}$ that vanish outside $[-n, n]$

and whose restrictions to $[-n,n]$ belong to $S'[-n,n]$. Define $\mathcal{F} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. Then $\mathcal{F}$ is countable. By monotone convergence theorem,

$$\lim_{n\to\infty} \int |f|^p \cdot \mathbb{I}([-n,n]) = \int_{\mathbb{R}} |f|^p, \quad \forall f \in L^p(\mathbb{R}).$$

Therefore, $\mathcal{F}$ is a countable dense subset in $L^p(\mathbb{R})$.
Step 3. The collection of restrictions to E of functions in $\mathcal{F}$ is a countable dense subset of $L^p(E)$ and thus $L^p(E)$ is separable. □

Theorem 3.4. *$L^\infty([a,b])$ is not separable given that $[a,b]$ is non-degenerate.*

Proof. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a countable dense subset in $L^p([a,b])$. For each $x \in [a,b]$, select a natural number $\eta(x)$ such that

$$\|\mathbb{I}([a,x]) - f_{\eta(x)}\|_\infty < 1/2.$$

Observe that

$$\|\mathbb{I}([a,x_1]) - \mathbb{I}([a,x_2])\| = 1, \quad a \le x_1 < x_2 \le b.$$

Therefore, $\eta(\cdot)$ is a one-to-one mapping of $[a,b]$ onto $\mathbb{N}$, which yields a contradiction. □

In general, $L^\infty(\Omega,\mathcal{F},\mu)$ is separable if and only if $|\Omega| < \infty$ (the proof is hard, you can find a proof in the Wikipedia). $L^p(\Omega,\mathcal{F},\mu)$ is separable if and only if $(\Omega,\mathcal{F},\mu)$ is separable. For example, when $\Omega = \mathbb{N}, \mathcal{F} = 2^{\mathbb{N}}$, and $\mu = \#$, since Ω is separable. $L^p(\Omega,\mathcal{F},\mu) = l^p$ is separable. Since $|\Omega| = \infty$, l^∞ is not separable. Indeed, as long as we are dealing with Euclidean spaces with Lebesgue measure, $L^p, 1 \le p < \infty$ is always separable due to the first principle of Littlewood (i.e. every Lebesgue measure set is nearly a finite union of intervals), therefore we can still prove that the step functions is dense in $L^p(\Omega)$ and the proof follows.

Example 3.2 (A Separable Case)**.** *Let $(\Omega,\mathcal{F},\mu) = (M = \{1,2,...,n\}, 2^M, \#)$, then $L^\infty(\Omega,\mathcal{F},\mu) = (\mathbb{R}^n, \|\cdot\|_\infty)$. Since $\mathbb{Q} \subset \mathbb{R}$ is dense, we conclude that $L^\infty(\Omega,\mathcal{F},\mu)$ is separable.*

3.3 One More Thing

After we investigated the completeness and separability of $L^p(\mu)$, one might be curious about the relation between those spaces. Let $(\Omega,\mathcal{F},\mu)$ be a measure space.

Theorem 3.5. *Let $E \subset \mathcal{F}$ be such that $\mu(E) < \infty$ and $1 \le p_1 < p_2 \le \infty$, then $L^{p_2}(E) \subset L^{p_1}(E)$. Furthermore,*

$$\|f\|_{p_1} \le c\|f\|_{p_2}, \quad \forall f \in L^{p_2}(E).$$

for some $c > 0$.

Proof. This proof of the theorem is left as an exercise. (problem 3.6) □

Problem 3.6. *Prove theorem 3.5.*

Remark 3.1. ***(a).*** *by theorem 3.5, we know that the natural inclusion $L^{p_2}(E) \to L^{p_1}(E) : g \mapsto g$ is bounded.*
(b). *The easiest way to see this by using the **Open Mapping Theorem**: the inclusion $L^2[0,1] \to L^1[0,1]$ is continuous but not onto. More explicitly, the set $B_n = \{f \in L^1 : \int |f|^2 \le n\}$ is easily seen to be closed and have empty interior and $L^2[0,1] = \bigcup_{n=1}^\infty B_n$. Similarly, if $1 \le p < q \le \infty$ then $L^q[0,1] \subset L^p[0,1]$ is meager.*

4 Signed Measures

Let μ_1 and μ_2 be two finite measures on a measurable space $(\Omega, \mathcal{F})$. Let

$$\nu(A) := \mu_1(A) - \mu_2(A), \quad \forall A \in \mathcal{F}.$$

Then $\nu : \mathcal{F} \to \mathbb{R}$ satisfies the following:

1. $\nu(\emptyset) = 0$.
2. For any mutually disjoint $\{A_n\}_{n\geq 1} \in \mathcal{F}$ such that $\bigcup_{n\geq 1} A_n = A$ with $\sum_{j=1}^{\infty} |\nu(A_j)| < \infty$,
$$\nu(A) = \sum_{i=1}^{\infty} \nu(A_i).$$
3. Let
$$\|\nu\| := \sup \left\{ \sum_{i=1}^{\infty} |\nu(A_i)| : A_n \in \mathcal{F}, A_i \cap A_j = \emptyset, i \neq j, \bigcup_{n\geq 1} A_n = \Omega \right\}.$$
Then, $\|\nu\|$ is finite.

Note that (3) holds since $\|\nu\| \leq \mu_1(\Omega) + \mu_2(\Omega) < \infty$.

Definition 4.1 (Signed Measure)**.** *A set function $\nu : \mathcal{F} \to \mathbb{R}$ satisfying (1), (2), and (3) above is called a **finite signed measure**.*

Proposition 4.1. *Let ν be a finite signed measure on $(\Omega, \mathcal{F})$. Let*

$$|\nu|(A) := \sup \left\{ \sum_{n=1}^{\infty} |\nu(A_n)| : A_n \in \mathcal{F}, A_i \cap A_j = \emptyset, i \neq j, \bigcup_{n\geq 1} A_n = A \right\}.$$

*Then $|\nu|(\cdot)$ is a finite measure on $(\Omega, \mathcal{F})$. It is called the **total variation measure**.*

Proof. In class. □

Let $\mathbb{S} := \{\nu : \nu \text{ is a finite signed measure on } (\Omega, \mathcal{F})\}$. For any $\alpha, \beta \in \mathbb{R}$ and $\nu_1, \nu_2 \in \mathbb{S}$,

$$\begin{aligned}
\alpha\nu_1 + \beta\nu_2 &= (\alpha_1^+ - \alpha_1^-)(\nu_1^+ - \nu_1^-) + (\alpha_2^+ - \alpha_2^-)(\nu_2^+ - \nu_2^-) \\
&= (\alpha_1^+\nu_1^+ + \alpha_1^-\nu_1^- + \alpha_2^+\nu_2^+ + \alpha_2^-\nu_2^-) - (\alpha_1^+\nu_1^- + \alpha_1^-\nu_1^+ + \alpha_2^+\nu_2^- + \alpha_2^-\nu_2^+) \\
&:= \lambda_1 - \lambda_2,
\end{aligned}$$

where both λ_1 and λ_2 are finite signed measures. It shows that $\mathbb{S}$ is a vector space over $\mathbb{R}$.

Definition 4.2 (Total Variation Norm)**.** *For a finite signed measure ν on a measurable space $(\Omega, \mathcal{F})$, the **total variation norm** ν is defined by $\|\nu\| := |\nu|(\Omega)$.*

Problem 4.2. *Show that the function $\|\cdot\|$ defined above is a norm on $\mathbb{S}$.*

Theorem 4.3 (Completeness)**.** *$(\mathbb{S}, \|\cdot\|)$ is Banach.*

Proof. In class □

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