MAT4220: Partial Differential Equations

Tutorial 9 Slides[1]

Mou Minghao

CUHK(SZ)

November 12, 2022

[1] All of the problems are taken from [Strauss, 2007].

Contents

Green's First Identity

Suppose both u and v are univariate. We have the product rule

$$(vu_x)_x = v_x u_x + vu_{xx}.$$

Now, suppose $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}$ have n independent variables. We know that

$$\nabla \cdot (v\nabla u) = \sum_{i=1}^{n} (vu_{x_i})_{x_i} = \sum_{i=1}^{n} v_{x_i} u_{x_i} + vu_{x_i x_i} = \nabla v \cdot \nabla u + v\Delta u.$$

Then we integrate and use the divergence theorem on both sides to get

$$\iint_{\partial D} v \frac{\partial u}{\partial \boldsymbol{n}} dS = \iiint_D \nabla v \cdot \nabla u d\boldsymbol{x} + \iiint_D v\Delta u d\boldsymbol{x}.$$

Note that this is just the high-dimensional version of *integration by parts*.

Mean Value Property in $\mathbb{R}^n$

In $\mathbb{R}^n$ the mean value property states that the average value of any harmonic function over any hypersphere equals its value at the center, i.e.

$$u(\boldsymbol{x}_0) = \frac{1}{|\partial B_r(\boldsymbol{x}_0)|} \int_{\partial B_r(\boldsymbol{x}_0)} u dS_r,$$

where $B_r(\boldsymbol{x}_0) \subset \mathbb{R}^n$ is a ball centered at $\boldsymbol{x}_0$ with radius r and S_r stands for the surface measure.

Proof. WLOG assume $\boldsymbol{x}_0 = 0$, otherwise, we just move the coordinate frame. Set $w_n := |\partial B_1|$, which is the surface area for the sphere of radius 1. Consider

$$f(r) := \frac{1}{|\partial B_r|} \int_{\partial B_r} u dS_r.$$

Recall that $|\partial B_r| = r^{n-1} |\partial B_1| = r^{n-1} w_n$, then

$$f(r) = \frac{1}{r^{n-1} w_n} \int_{\partial B_r} u dS_r.$$

Mean Value Property in $\mathbb{R}^n$

Do a change of variable $r\boldsymbol{y} = \boldsymbol{x}$, we have

$$f(r) = \frac{1}{r^{n-1}w_n}\int_{\partial B_1} u(r\boldsymbol{y})r^{n-1}dS_1 = \frac{1}{w_n}\int_{\partial B_1} u(r\boldsymbol{y})dS_1.$$

Remember that

$$\begin{cases} x \in \partial B_r, \\ y \in \partial B_1, \\ dS_r = r^{n-1}dS_1. \end{cases}$$

Differentiate $f(r)$ we have

$$f'(r) = \frac{1}{w_n}\int_{\partial B_1} \boldsymbol{y}\cdot\nabla_x u(r\boldsymbol{y})dS_1.$$

Note that

$$0 = \int_{B_r} \Delta u d\boldsymbol{x} = \int_{\partial B_r} \frac{\partial u}{\partial \boldsymbol{n}} dS_r = \int_{\partial B_r} \nabla_x u \cdot \boldsymbol{n} dS_r.$$

Mean Value Property in $\mathbb{R}^n$

Since $\boldsymbol{n} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$,

$$\begin{aligned}\int_{\partial B_r} \nabla_x u \cdot \boldsymbol{n} dS_r &= \int_{\partial B_r} \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \cdot \nabla_x u dS_r \text{ (set } r\boldsymbol{y} = \boldsymbol{x}) \\ &= \int_{\partial B_1} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \cdot \nabla_x u(r\boldsymbol{y}) r^{n-1} dS_1\end{aligned}$$

The above implies that

$$f'(r) = 0.$$

Therefore, f is independent of r. Since

$$\lim_{r \to 0^+} \frac{1}{|\partial B_r|} \int_{\partial B_r} u(\boldsymbol{x}) dS_r = u(\boldsymbol{0}), \quad \text{(Why?)}$$

We conclude that

$$u(\boldsymbol{x}_0) = \frac{1}{|\partial B_r(\boldsymbol{x}_0)|} \int_{\partial B_r(\boldsymbol{x}_0)} u dS_r.$$

$\square$

Dirichlet Principle

The *Dirichlet Principle* states that among all the functions $w(\boldsymbol{x})$ in D that satisfies the Dirichlet boundary condition

$$w = h(\boldsymbol{x}), \quad \boldsymbol{x} \in D,$$

the lowest energy occurs for the harmonic function.
Proof. Let $u(\boldsymbol{x})$ be the unique (Why?) harmonic function in D that satisfies the boundary condition. Let $w(\boldsymbol{x})$ be any function in D that satisfies the boundary condition, we want to show that

$$E[w] \geq E[u].$$

We let $v = u - w$ and expand the square in the integral

$$E[w] = \frac{1}{2} \iint_D |\nabla(u-v)|^2 d\boldsymbol{x} = E[u] - \iiint_D \nabla u \cdot \nabla v d\boldsymbol{x} + E[v].$$

Apply Green's first identity to the middle term we then know that it vanishes since $v = 0$ on ∂D and u is a harmonic function on D. Therefore,

$$E[w] = E[u] + E[v] \quad \Leftrightarrow \quad E[w] \geq E[u]. \quad \square$$

Question 1

Problem (1 - Green's Function)

(a). What is the definition of Green's function on a domain $\Omega \subset \mathbb{R}^2$*?*
(b). Construct a Green's function if the domain Ω *is the upper plane.*
(c). Use it to represent the solution with boundary condition $u(x,0) = h(x)$ *for a given function* h*.*

Solution 1 - (a)

- $G(x, x_0)$ is smooth and harmonic on Ω except at the point x_0.
- $G(x, x_0) - \frac{1}{2\pi} \log |x - x_0|$ is smooth and harmonic on the whole domain.
- $G(x, x_0) = 0$ on $\partial\Omega$.

Solution 1 - (b)

Define $\boldsymbol{x}_0^* := (x_0, y_0)$,
Claim.

$$G(\boldsymbol{x}, \boldsymbol{x}_0) := \frac{1}{2\pi} \log |\boldsymbol{x} - \boldsymbol{x}_0| - \frac{1}{2\pi} \log |\boldsymbol{x} - \boldsymbol{x}_0^*|$$

is the Green's function on the upper plane.

Let's check that G indeed satisfies the requirements of the Green's function.

- It is smooth and harmonic except at the point $\boldsymbol{x}_0$.
- $G(\boldsymbol{x}, \boldsymbol{x}_0) - \frac{1}{2\pi} \log |\boldsymbol{x} - \boldsymbol{x}_0|$ is smooth and harmonic on the whole domain Ω.
- $G(\boldsymbol{x}, \boldsymbol{x}_0) = 0$ on $\partial\Omega$.

Solution 1 - (c)

Recall the theorem about the representation of the solution to Laplace's equation by Green's function

Theorem (Theorem 1 in p.188 of [Strauss, 2007])

If $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function, then the solution of the Dirichlet problem is given by the formula

$$u(\boldsymbol{x}_0) = \iint_{\partial D} u(\boldsymbol{x}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial \boldsymbol{n}} dS.$$

Back to this problem, note that

$$\frac{\partial G}{\partial \boldsymbol{n}} = \boldsymbol{n} \cdot \nabla G = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \nabla G = -\partial_y G.$$

Note also that

$$-\partial_y G = -\frac{1}{2\pi} \left[\frac{y - y_0}{|\boldsymbol{x} - \boldsymbol{x}_0|^2} \right] = \frac{2y_0}{2\pi} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|^2} = \frac{y_0}{2\pi} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|^2}.$$

Solution 1 - (c)

Therefore, the solution is

$$u(x_0, y_0) = \frac{y_0}{\pi} \int_{\mathbb{R}} \frac{h(x)}{(x - x_0)^2 + y_0^2} dx.$$

$\square$

Question 2

Problem (2 - Representation Formula)

Show that the corresponding representation formula for two-dimensional case is

$$u(\boldsymbol{x}_0) = \frac{1}{2\pi} \int_{\partial D} \left[u(\boldsymbol{x}) \frac{\partial}{\partial \boldsymbol{n}} \log |\boldsymbol{x} - \boldsymbol{x}_0| - \frac{\partial u}{\partial \boldsymbol{n}} \log |\boldsymbol{x} - \boldsymbol{x}_0| \right] ds.$$

Hint. Choose $v(\boldsymbol{x}) = \log |\boldsymbol{x} - \boldsymbol{x}_0|$ and apply the same method.

Solution 2

Question 3

Problem (3 - Representation Formula (Inhomogeneous Dirichlet))

Prove the solution formula of the problem

$$\Delta u = f, \text{ in } D; \quad u = h, \text{ on } \partial D, \tag{1}$$

is given by

$$u(\boldsymbol{x}_0) = \iint_{\partial D} h(\boldsymbol{x}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial \boldsymbol{n}} dS + \iiint_D f(\boldsymbol{x}) G(\boldsymbol{x}, \boldsymbol{x}_0) d\boldsymbol{x}. \tag{2}$$

Lemma (Representation Formula for Poisson Equation)

If u solves the Poisson equation $\Delta u = f$ on Ω, then

$$u(\boldsymbol{x}_0) = \iint_{\partial\Omega} u \frac{\partial v}{\partial \boldsymbol{n}} - \frac{\partial u}{\partial \boldsymbol{n}} v dS + \iiint_\Omega v f d\boldsymbol{x}. \tag{3}$$

where $v = -\frac{1}{4\pi} \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_0\|}$.

Solution 3

Question 4

Problem (4)

Let $\phi \in C^2(\Omega)$ be defined on all of the three-dimensional space that vanishes outside some sphere. Show that

$$\phi(\mathbf{0}) = -\iiint \frac{1}{\|\boldsymbol{x}\|} \Delta\phi(\boldsymbol{x}) \frac{d\boldsymbol{x}}{4\pi}. \tag{4}$$

This integration is taken over the region where $\phi(\boldsymbol{x})$ is not zero.

Solution 4

References I

Strauss, W. A. (2007).
Partial differential equations: An introduction.
John Wiley & Sons.