MAT4220: Partial Differential Equations

Tutorial 5 Slides

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Contents

Question 1

Kind Reminder: If you are not familiar with how to solve those problems, you are highly encouraged to review the lecture notes or the accompanying tutorial notes.

Problem (1)

Let $\phi(x)$ *be a continuous function such that* $|\phi(x)| \leq Ce^{ax^2}$. *Show that the solution of the diffusion equation*

$$\frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-s)^2}{4kt}} \phi(s) ds,$$

makes sense for $0 < t < 1/(4ak)$, *but not necessarily for large* t.

Solution 1

The fact that $|\phi(x)| \leq Ce^{ax^2}$ prompts us to consider the magnitude of $u(x,t)$.

$$|u(x,t)| = \frac{1}{\sqrt{4\pi kt}} \left| \int_{\mathbb{R}} e^{-\frac{(x-s)^2}{4kt}} \phi(s) ds \right|.$$

We can bring the absolute value sign to the integrand, provided we make this an inequality.

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} \left| e^{-\frac{(x-s)^2}{4kt}} \phi(s) \right| ds.$$

The exponential function is never negative, so we can remove the absolute values around it. It is here where we substitute $|\phi(x)| \leq Ce^{ax^2}$.

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} \left| e^{-\frac{(x-s)^2}{4kt}} \right| |\phi(s)| ds \leq \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-s)^2}{4kt}} Ce^{as^2} ds.$$

Solution 1

The problem statement says the integral won't necessarily make sense when $t = 1/(4ak)$, so let's make the substitution to see why.

$$t = \frac{1}{4ak} \to a = \frac{1}{4kt} \to \frac{a}{\pi} = \frac{1}{4\pi kt} \to \sqrt{\frac{a}{\pi}} = \frac{1}{\sqrt{4\pi kt}}.$$

The integral becomes

$$\left| u(x, \frac{1}{4ak}) \right| \leq C\sqrt{\frac{a}{\pi}} \int_{\mathbb{R}} e^{-a(x-s)^2} e^{as^2} \, ds.$$

Combine the exponential functions.

$$\left| u(x, \frac{1}{4ak}) \right| \leq C\sqrt{\frac{a}{\pi}} \int_{\mathbb{R}} e^{-ax^2+2axs} \, ds.$$

Solution 1

Pull the constant out in front of the integral and then proceed with the integration.

$$\left|u(x, \frac{1}{4ak})\right| \leq C\sqrt{\frac{a}{\pi}}e^{-ax^2}\int_{\mathbb{R}} e^{2axs}ds = C\sqrt{\frac{a}{\pi}}e^{-ax^2}\frac{e^{2axs}}{2ax}\Big|_{-\infty}^{\infty}$$
$$= C\sqrt{\frac{a}{\pi}}\frac{e^{-ax^2}}{2ax}\left(e^{\infty} - \frac{1}{e^{\infty}}\right).$$

Therefore,

$$\left|u(x, \frac{1}{4ak})\right| \leq \infty.$$

What this indicates is that the integral solution may or may not be bounded when $t = 1/4ak$. This is also the case for any later time. However, when t is any smaller than $1/4ak$, there is an e^{-s^2} term that makes the integral converge. Since the integral solution only holds for $t > 0$, the integral solution makes sense when $0 < t < 1/(4ak)$, but not necessarily for later times. $\square$

Question 2

Problem (2)

Consider the following problem with a Robin boundary condition:

$$\begin{aligned} &u_t = ku_{xx}, \quad 0 < x < \infty, 0 < t < \infty \\ &u(x,0) = x \\ &u_x(0,t) - 2u(0,t) = 0, \quad x = 0. \end{aligned} \tag{1}$$

Let $f(x) = x, x > 0$ *and* $f(x) = x + 1 - e^{2x}, x < 0$, *and let*

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y) dy.$$

Show that $v(x,t)$ *satisfies* (1) *for* $x > 0$. *Assuming uniqueness, deduce the solution of* (1) *is given by*

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y) dy.$$

Solution 2 - Step 1

We recognize that $v(x,t)$ as the convolution of the heat kernel $\Phi(x,t)$ and the initial condition

$$v(x,t) = \Phi(\cdot,t) * f = \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-s)^2}{4kt}} f(s) ds.$$

Therefore, $v(x,t)$ is the solution to the diffusion equation on the whole real line with initial data $f(x)$

$$v_t = v_{xx}, \quad v(x,0) = f(x) = \begin{cases} x, & x > 0 \\ x + 1 - 2e^{2x}, & x < 0 \end{cases}, \quad x \in \mathbb{R}.$$

Solution 2 - Step 2

Because any derivative of a solution to the diffusion equation is also a solution and any linear combination of solutions to the diffusion equation is also a solution, $w(x,t) := v_x(x,t) - 2v(x,t)$ is a solution to the diffusion equation. We can show this in a direct way as follows.

$$v_t = kv_{xx}.$$

Differentiate both sides w.r.t. x

$$(v_t)_x = k(v_{xx})_x.$$

Multiply both sides of the original equation by 2.

$$2v_t = 2kv_{xx}.$$

Now, we have

$$(v_t)_x - 2v_t = k(v_{xx})_x - 2kv_{xx}.$$

Solution 2 - Step 2

Change the order of differentiation in the first term on the left and the first term on the right.

$$(v_x)_t - 2v_t = k(v_x)_{xx} - 2kv_{xx}.$$

Factor the operator from both sides.

$$(v_x - 2v)_t = k(v_x - 2v)_{xx}.$$

Therefore, $w = v_x 2v$ satisfies the diffusion equation $w_t = kw_{xx}$. The initial condition for it is $w(x,0) = w_x(x,0) - 2w(x,0)$.

$$w(x,0) = f'(x) - 2f(x) = \begin{cases} 1 - 2x, & x > 0 \\ -1 - 2x & x < 0 \end{cases}.$$

Solution 2 - Step 3

Note that

$$
\begin{aligned}
w(-x,0) &= \begin{cases} 1-2(-x), & -x>0 \\ -1-2(-x), & -x<0 \end{cases} = \begin{cases} -1+2x, & x>0 \\ 1+2x, & x<0 \end{cases} \\
&= \begin{cases} -(1-2x), & x>0 \\ -(-1-2x), & x<0 \end{cases} = -w(x,0)
\end{aligned}
$$

Therefore, $w(x,0) = f'(x) - 2f(x)$ is an odd function w.r.t. the spatial variable x.

Solution 2 - Step 4

if the initial condition is an odd function of x, then the solution to the diffusion equation is also an odd function of x. Since in Step 3 we have proved that the initial condition $w(x,0)$ is odd, so $w(x,t)$ is odd in x as well, i.e. $w(x,t) = -w(-x,t)$.

Solution 2 - Step 5

Since $w(x,t)$ is odd, the boundary condition $w(0,t) = 0$ will be satisified automatically, and the corresponding problem on the half-line can be solved by taking the restriction $x > 0$. Therefore, we have

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-s)^2}{4kt}} [f'(s) - 2f(s)] ds.$$

Now that we know w, we can solve for v by using the original substitution $w = v_x - 2v$.

$$\begin{aligned}
v_x - 2v &= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-s)^2}{4kt}} [f'(s) - 2f(s)] ds \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-s)^2}{4kt}} f'(s) ds - 2\frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)}{4kt}}{}^2 f(s) ds \\
&:= v_x(x,t) - 2v(x,t).
\end{aligned}$$

Solution 2 - Step 5

Therefore,

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)}{4kt}^2} f(s) ds.$$

The restriction of $v(x,t)$ to $x > 0$ gives us the solution to the initial boundary value problem satisfied by $u(x,t)$. Because the solution to the problem is unique, this has to be the one and only one solution for $u(x,t)$. Therefore

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)}{4kt}^2} f(s) ds, \quad x > 0.$$

$\square$

Question 3

Problem (3)

Solve the following wave equation on the half-line

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0, \quad t \in \mathbb{R}, x > 0 \\ u(x,0) = \phi(x), \quad \partial_t u(x,0) = \psi(x) \\ u_x(0,t) = e^t \end{cases} \tag{2}$$

Solution 3

Since the Neumann boundary condition is inhomogeneous, we cannot directly apply the reflection method. Therefore, we first need to reduce the inhomogeneous boundary condition to a homogeneous one. Set $v(x,t) := u(x,t) - xe^t$, we then have v satisfies the following system

$$\begin{cases} v_{tt} - c^2 v_{xx} = -xe^t, \quad x > 0, t \in \mathbb{R} \\ v(x,0) = \phi(x) - x; \quad v_t(x,0) = \psi(x) - x \\ v_x(0,t) = 0 \end{cases} \tag{3}$$

Note that now we have homogeneous Neumann boundary condition, so next we do even extension on all the data $f(x,t) := -xe^t$, $\phi(x) - x$, and $\psi(x) - x$ to get

$$-|x|e^t; \quad \phi(|x|) - |x|; \quad \psi(|x|) - |x|.$$

Solution 3

Consider the solution $\tilde{v}$ to the following system defined on the whole real line

$$\begin{cases} \tilde{v}_{tt} - c^2 \tilde{v}_{xx} = -|x|e^t, & x \in \mathbb{R}, t \in \mathbb{R} \\ \tilde{v}(x,0) = \phi(|x|) - |x|; & \tilde{v}_t(x,0) = \psi(|x|) - |x| \end{cases} \tag{4}$$

Since all the data are even, we immediately know that the solution $\tilde{v}(x,t)$ is also even. Hence, $\tilde{v}_x(0,t) = 0$. If we restrict the solution $\tilde{v}$ to the positive half-line, we then get a solution satisfying (3).

- When $x > c|t|$, we have

$$\tilde{v}(x,t) = -x + \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} (\psi(s) - s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} -ye^s dyds.$$

Solution 3

- when $x < c|t|$, we have

$$\begin{aligned}\tilde{v}(x,t) = &-ct + \frac{1}{2}[\phi(ct+x) + \phi(ct-x)] \\ &+ \frac{1}{2c}\int_0^{ct-x}(\psi(s)-s)ds + \frac{1}{2c}\int_0^{ct+x}(\psi(s)-s)ds \\ &+ \frac{1}{2c}\iint_D -ye^s dyds,\end{aligned}$$

  where D is the domain of dependence shown in Fig.1.

Since we now have $\tilde{v}$, we restrict it to the positive half-line to get a solution v to the system (3), i.e. $v(x,t) = \tilde{v}(x,t), x > 0$. Finally, we recover u by $u(x,t) = v(x,t) + xe^t$. (Please write down the solution formula for $u(x,t)$ by yourself...) □

Question 4

Problem (4)

Solve the following wave equation on the half-line with inhomogeneous Dirichlet boundary condition

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < \infty \\ u(0,t) = t^2, & \\ u(x,0) = x, & u_t(x,0) = 0. \end{cases} \tag{5}$$

Solution 4

Note that the Dirichlet boundary condition is inhomogeneous. Therefore, if we want to apply the reflection method, we need first consider to transform the inhomogeneous boundary condition to a homogeneous one. Let

$$v(x,t) := u(x,t) - t^2,$$

then v satisfies

$$\begin{cases} v_{tt} - c^2 v_{xx} = -2, \quad 0 < x < \infty \\ v(x,0) = x := \phi(x); \quad v_t(x,0) = 0 := \psi(x) \\ v(0,t) = 0 := f(x,t) \end{cases} \tag{6}$$

Now, since the *Dirichlet* boundary condition is homogeneous, we can apply the reflection method. First, we do odd extension on the data f, ϕ, and ψ w.r.t. the spatial variable x.

Solution 4

$$\tilde{f} := \begin{cases} -2, & x > 0 \\ 2, & x < 0 \end{cases} \quad \tilde{\phi} := \begin{cases} x, & x > 0 \\ x, & x < 0 \end{cases} \quad \tilde{\psi} := \begin{cases} 0, & x > 0 \\ 0, & x < 0 \end{cases}$$

Then, we know that the solution $\tilde{v}$ to the below system

$$\begin{cases} \tilde{v}_{tt} - c^2 \tilde{v}_{xx} = -2, \quad -\infty < x < \infty \\ \tilde{v}(x,0) = \tilde{\phi}(x); \quad \tilde{v}_t(x,0) = \tilde{\psi}(x) \\ \tilde{v}(0,t) = \tilde{f}(x,t) \end{cases} \tag{7}$$

is odd, so $\tilde{v}(0,t) = 0$ automatically, which also implies that $v(0,t) = 0$. Next, we only need to solve for $\tilde{v}$ and then restrict it to the positive half-line to recover v.

Solution 4

- when $x > c|t|$, the solution formula, by d'Alembert's fomula, is given by

$$\tilde{v}(x,t) = \frac{1}{2}[\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(s)ds \\ + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t+s)} \tilde{f}(y,s)dyds. \tag{8}$$

Thus, plug in the data $\tilde{f}$, $\tilde{\phi}$, and $\tilde{\psi}$, we directly get

$$\tilde{v}(x,t) = x + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} -2dyds = x - t^2, x > c|t|.$$

Solution 4

- When $x < ct$, the situation is a little bit more complicated. Note that now the domain of dependence is given by Fig.1

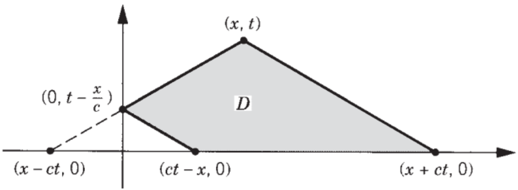

Figure 1: Domain of dependence D.

The area of D is $ct^2 - (ct - x)(t - x/c) = 2tx - x^2/c$, so the solution is

$$\tilde{v}(x,t) = \frac{1}{2}[ct + x - (ct - x)] + \frac{1}{2c}\iint_D -2dS = x - \frac{2tx}{c} + \frac{x^2}{c^2}, x < c|t|.$$

Solution 4

By restrcting $\tilde{v}$ to the positive half-line, we recover v. Since $v(x,t) = u(x,t) + t^2$, we finally conclude that

$$u(x,t) = \begin{cases} x, & x > c|t| \\ x + \left(t - \frac{x}{c}\right)^2, & 0 < x < c|t| \end{cases}$$

$\square$

Exercises

The below problems are left as exercises for you to enhance your problem-solving skills.

Problem (5)

Show by direct substitution that $u(x,t) = h(t - \frac{x}{c})$ *for* $x < ct$ *and* $u(x,t) = 0$ *for* $x \geq ct$ *solves the homogeneous wave equation on the half-line with zero initial data and boundary condition* $u(0,t) = h(t)$.

Problem (6)

Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{cases} u_t - ku_{xx} = f(x,t), \quad 0 < x < \infty, 0 < t < \infty, \\ u(0,t) = 0; \quad u(x,0) = \phi(x). \end{cases}$$

Using the method of reflection.

Exercises

Problem (7)

Solve the Neumann diffusion equation on the half-line

$$\begin{cases} w_t - kw_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ w_x(0,t) = h(t); & w(x,0) = \phi(x). \end{cases}$$

By the subtraction method.

Problem (8)

Derive the solution formula of the fully inhomogeneous wave equation on the half-line

$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x,t), & 0 < x < \infty \\ v(x,0) = \phi(x); & v_t(x,0) = \psi(x), \\ v(0,t) = h(t) \end{cases}$$