MAT4220: Partial Differential Equations

Tutorial 9 Slides[1]

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[1] All of the problems are taken from [Strauss, 2007].

Contents

Laplacian Δ

Under polar coordinates, the Laplacian Δ becomes

- **2D case.** $x = r\cos\theta; y = r\sin\theta$

$$\Delta_{\boldsymbol{x}} = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2.$$

- **3D case.** $x = \rho\cos\theta; y = \rho\sin\theta; \rho = r\cos\phi; z = r\sin\phi$

$$\Delta_{\boldsymbol{x}} = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left[\partial_\phi^2 - \tan\theta\partial_\phi + \frac{1}{\cos^2\phi}\partial_\theta^2\right].$$

Remark (Proof Idea)

Since $\Delta_{\boldsymbol{x}} = \partial_x^2 + \partial_y^2 + \partial_z^2$, apply the 2d results to $\partial_x^2 + \partial_y^2$ and then apply the 2d results to $\partial_\rho^2 + \partial_z^2$.

Special Harmonic Solutions

We look for some special harmonic functions in 2d and 3d that do not change under rotations, that is, which depend only on r.

- $$0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r,$$
  which is equivalent to
  $$(ru_r)_r = 0.$$
  Therefore,
  $$u = c_1 \log r + c_2, \quad c_1, c_2 \in \mathbb{R}.$$
  The term $\log r$ will play a central role later.
- $$0 = u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{2}{r}u_r \Leftrightarrow \partial_r\left(r^2 \partial_r u\right) = 0.$$
  Therefore,
  $$u = -C_1\frac{1}{r} + C_2, \quad C_1, C_2 \in \mathbb{R}.$$

Separation of Variables: Rectangle & Cube

Remark (General Procedure)

Special geometry can be solved by separation of variables. The general procedure is as follows

1. *Look for separated solutions of the PDE*
2. *Put in the homogeneous boundary conditions to get the eigenvalues*
3. *Sum the series*
4. *Put in the inhomogeneous boundary initial or boundary conditions*

Example (Cube)

$$\begin{cases} \Delta_2 u = u_{xx} + u_{yy} = 0, \quad \mathit{in}\ D := \{0 < x < a, 0 < y < b\}, \\ u_y(x,0) + u(x,0) = h(x); \quad u(x,b) = g(x) \\ u(0,y) = j(y); \quad u_x(a,y) = k(y). \end{cases}$$

Separation of Variables: Rectangle & Cube

- We consider a separated solution $u(x,t) = X(x)Y(y)$
- $$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$
- there is a constant λ such that $X'' + \lambda X = 0$ for $0 \leq x \leq a$ and $Y'' - \lambda Y = 0$ for $0 \leq y \leq b$
- Solve the eigenvalue problem with the boundary conditions $X(0) = X'(a) = 0 \Rightarrow \beta_n^2 = \lambda_n = (n + \frac{1}{2})^2 \frac{\pi^2}{a^2}, \quad n = 0, 1, 2, \ldots$
- eigenfunctions are $X_n(x) = \sin \frac{(n+\frac{1}{2})\pi x}{a}$
- $Y'' - \lambda Y = 0$ with the boundary conditions $Y'(0) + Y(0) = 0$
- $Y_n(y) = A_n \cosh \beta_n y + B_n \sinh \beta_n y$
- By the homogeneous boundary condition we know that $\beta_n B_n + A_n = 0$, without losing any information we may assume that $B_n = 1$, then $A_n = \beta_n$
- $u(x,y) = \sum_{n=1}^{\infty} A_n \sin \beta_n x(\beta_n \cosh \beta_n y - \sinh \beta_n y)$

Separation of Variables: Rectangle & Cube

- Finally we plug in the inhomogeneous boundary condition $u(x,b) = g(x)$
- $g(x) = \sum_{n=0}^{\infty} A_n(\beta_n \cosh \beta_n b - \sinh \beta_n b) \sin \beta_n x, \quad 0 < x < a$
- $$A_n = \frac{2}{a}(\beta_n \cosh \beta_n b - \sinh \beta_n b)^{-1} \int_0^a g(x) \sin \beta_n x dx.$$

$\square$

Example (Cube)

$$\begin{cases} \Delta_3 u = u_{xx} + u_{yy} + u_{zz} = 0, \quad in\ D \\ u(\pi, y, z) = g(y,z) \\ u(0,y,z) = u(x,0,z) = u(x,\pi,z) = u(x,y,0) = u(x,y,\pi) = 0. \end{cases}$$

Separation of Variables: Rectangle & Cube

- $u = X(x)Y(y)Z(z)$
- $\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$
- boundary conditions $X(0) = Y(0) = Y(\pi) = Z(0) = Z(\pi) = 0$
- Each quotient $X''/X, Y''/Y, Z''/Z$ must be constant
- Solve two eigenvalue problems for Y and Z we have

$$\begin{aligned} Y(y) &= \sin my, \quad m = 1, 2, ... \\ Z(z) &= \sin nz, \quad n = 1, 2, ... \end{aligned}$$

- $X'' = (m^2 + n^2)X, \quad X(0) = 0 \Rightarrow X(x) = A\sinh(\sqrt{m^2+n^2}x)$
- $u(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sinh(\sqrt{m^2+n^2}x)\sin my \sin nz$
- plug in the inhomogeneous boundary condition $u(\pi, y, z) = g(y, z)$ and solve for the Fourier coefficients

Possion's Formula

Example (Circle)

Let's consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & x^2 + y^2 < a^2 \\ u = h(\theta), & x^2 + y^2 = a^2. \end{cases} \tag{1}$$

with radius a and any boundary data $h(\theta)$.

- $u(r, \theta) := R(r)\Theta(\theta)$
- Divide by $R\Theta$ and multiplying by r^2, we find that

$$\Theta'' + \lambda\Theta = 0$$
$$r^2 R'' + rR' - \lambda R = 0.$$

- **Solution:** $u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$

Possion's Formula

use the inhomogeneous boundary condition at $r = a$ to determine the coefficients. Set $r = a$

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta).$$

Thus,

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos n\phi d\phi.$$
$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin n\phi d\phi.$$

put the coefficients A_n and B_n into the expression of u

Possion's Formula

$$u(r,\theta) = \int_0^{2\pi} h(\phi) \left(1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right) \frac{d\phi}{2\pi}.$$

The term in red can be summed up explicitly and it is called the *Poisson's kernel.* To show this, note that the term in red can be written as

$$\begin{aligned}
&1 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} \\
&= 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}} \\
&= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2} \\
&= P(r,\theta).
\end{aligned}$$

Possion's Formula

Therefore,

Theorem (Poisson's Formula)

$$u(r,\theta) = (a^2 - r^2)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\frac{d\phi}{2\pi}.$$

The Poisson's formula can also be written in a more geometric way. Write $\boldsymbol{x} = (x, y)$ as a point with polar coordinate (r, θ). We could also think of $\boldsymbol{x}$ as the vector from the origin $\mathbf{0}$ to the point (x, y). Let $\boldsymbol{x}'$ be a point on the boundary.

Theorem (Poisson's Formula)

$$u(\boldsymbol{x}) = \frac{a - |\boldsymbol{x}|^2}{2\pi a}\int_{|\boldsymbol{x}'|=a} \frac{u(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^2} ds', \quad \boldsymbol{x} \in D.$$

Corollaries

Theorem (Mean Value Property)

Let u be a harmonic function in a disk D, continuous in its closure $\overline{D}$. Then the value of u at the center of D equals the average of u on its circumference.

Theorem (Strong Maximum Principle)

SMP states that the maximum of u cannot be attained in the interior of the connected domain D. (We have stated it before, please review)

Theorem (Differentiability)

Let u be a harmonic function in any open set D of the plane. Then $u(\boldsymbol{x}) = u(x, y)$ possesses all partial derivatives of all orders in D.

Proofs

Question 1: Prove the Proposition

Proposition (Properties of $P(r, \theta)$)

The Poisson kernel is defined to be

$$P(r, \theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = 1 + 2\sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^2 \cos n\theta.$$

It has the following properties:

- $P(r, \theta) > 0$ *for* $r < a$.
- $\int_0^{2\pi} P(r, \theta) \frac{d\theta}{2\pi} = 1.$
- $P(r, \theta)$ *is a harmonic function inside the circle.*

Solution 1

Question 2: Prove the Theorem

Theorem

Let $h(\phi) = u(\boldsymbol{x}')$ be any continuous function on the circle $C = \partial D$. Then the Poisson formula provides the only harmonic function in D for which

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} u(\boldsymbol{x}) = h(\boldsymbol{x}_0), \quad \forall \boldsymbol{x}_0 \in C.$$

Solution 2

References I

Strauss, W. A. (2007).
Partial differential equations: An introduction.
John Wiley & Sons.