MAT4220: Partial Differential Equations

Tutorial 7 Slides[1]

Mou Minghao

CUHK(SZ)

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[1] All of the problems are taken from [Strauss, 2007].

Contents

Question 1

Problem (1 - Fourier Series)

Find the full Fourier series of the $f(x)$ on $(-l, l)$ in its real and complex forms.

(1). $f(x) = e^x$.

(2). $f(x) = \cosh x$.

(3). $f(x) = |x|$.

Question 2

Problem (2 - Gram-Schmidt)

If $X_1, X_2, \ldots$ is any sequence (finite or infinite) of linearly independent vectors in any inner product space, it can be replaced by a sequence of linear combinations that are mutually orthogonal. THe idea is that at each step one subtracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 = X_1/\|X_1\|$. Second, we define
$Y_2 = X_2 - \langle X_1, Z_1\rangle Z_1; \quad \text{and} \quad Z_2 = \frac{Y_2}{\|Y_2\|}$. Third, we define$Y_3 = X_3 - \langle X_3, Z_1\rangle Z_1 - \langle X_3, Z_2\rangle Z_2 \quad \text{and} \quad Z_3 = \frac{Y_3}{\|Y_3\|}$. and so on.

(1). Show that all the vectors $Z_1, Z_2, Z_3, \ldots$ are orthogonal to each other.

(2). Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ in the interval $(0, \pi)$ to get an orthogonal pair.

Question 3

Problem (3 - Nonnegative Eigenvalues)

(a). Prove the Green's First Identity: for every pair of functions $f(x)$ *and* $g(x)$ *on* (a, b),

$$\int_a^b f''(x)g(x)dx = -\int_a^b f'(x)g'(x) + f'g\big|_a^b.$$

(b). Show that the condition $ff'|_a^b \leq 0$ *is valid for any function* $f(x)$ *satisfies the Dirichlet, Neumann, or periodic boundary conditions.*
(c). Show that it is also valid for Robin BCs provided that the constants a_0 *and* a_l *are positive. (i.e. energy gets dissipated)*
(d). Prove that if for all f *satisfying the BCs, we have*

$$f(x)f'(x)\big|_{x=a}^{x=b} \leq 0,$$

then there is no negative eigenvalue.

Question 4

Problem (4 - Least Square Approximation)

Let $\{X_n\}$ *be any orthogonal set of functions. Let* $\|f\|_{L^2(\mathbb{C})} < \infty$*. Let* N *be a fixed positive integer. Among all the possible choices of* N *constants* $c_1, c_2, ..., c_N$*, the choice that minimizes*

$$\|f - \sum_{n=1}^{N} c_n X_n\|$$

is $c_1 = A_1, ..., c_n = A_n$*, where*

$$A_i := \frac{\langle f, X_i \rangle}{\langle X_i, X_i \rangle}$$

is the Fourier coefficient.

Exercises

Problem (5 - Parseval's Equality)

Prove the complex case of Theorem 6 in the textbook.

References I

Strauss, W. A. (2007).
Partial differential equations: An introduction.
John Wiley & Sons.