MAT4220: Partial Differential Equations

Tutorial 8 Slides[1]

Mou Minghao

CUHK(SZ)

November 8, 2022

[1] All of the problems are taken from [Strauss, 2007].

Contents

Review: Notions of Convergence

Definition (Pointwise Convergence)

$f_n(x)$ is said to converge to $f(x)$ in (a,b) pointwisely if for every $x \in (a,b)$ and for any $\epsilon > 0$, there exists a $M(x) \in \mathbb{N}$ such that for $n \geq M(x)$,

$$|f_n(x) - f(x)| < \epsilon.$$

Definition (Uniform Convergence)

$f_n(x)$ is said to converge to $f(x)$ in (a,b) uniformly if for any $\epsilon > 0$, there exists a M, which does not depend on the choice of x, such that for $\forall n \geq M$,

$$|f_n(x) - f(x)| < \epsilon, \quad \forall x \in (a,b).$$

Review: Notions of Convergence

Definition (L^2 Convergence / Mean-Square Convergence)

Suppose $f_n, f \in L^2$, f_n is said to converge to f in L^2 if

$$\lim_{n\to\infty} \|f_n - f\|_2 \to 0.$$

Remark

You can view f_n and f as two vectors in the vector space L^2. The L^2 convergence plainly says that the two vectors are getting closer and closer as $n \to \infty$.

Definition (Weak Convergence)

A sequence of functions $\{f_n\}_{n=1}^{\infty} \subset L^2(\Omega)$ is said to converge weakly to $f \in L^2(\Omega)$ if

$$\int_\Omega f_n \cdot g d\lambda = \int_\Omega f \cdot g d\lambda, \quad \forall g \in L^2(\Omega).$$

Review: Notions of Convergence

Let's play with those concepts by considering the following example

Example

$$f_n(x) := \sqrt{\frac{n}{1+n^2x^2}} - \sqrt{\frac{n-1}{1+(n-1)x^2}}, \quad x \in (0, l).$$

Then,

$$F_n(x) = \sum_{i=1}^{n} f_i(x) = \sqrt{\frac{1}{1+x^2}} - \sqrt{\frac{n}{1+n^2x^2}}, \quad (\textit{by telescoping})$$

(a). Does $F_n(x)$ converges pointwisely? If yes, find the limit. If no, explain.
(b). Does $F_n(x)$ converges uniformly? If yes, find the limit. If no, explain.
(c). Does $F_n(x)$ converges in L^2? If yes, find the limit. If no, explain.

Review: Notions of Convergence

(a). We claim that F_n converges to $F(x) := \sqrt{\frac{1}{1+x^2}}$. The reason is that

$$|F_n(x) - F(x)| = \sqrt{\frac{n}{1+n^2x^2}} \to 0,\ n \to \infty,\ \text{for each fixed } x.$$

(b). If F_n converges uniformly, the limit must be $F(x)$ (why? Explain). Note that

$$\sup_{x\in(0,l)} |F_n(x) - F(x)| = \sup_{x\in(0,l)} \sqrt{\frac{n^2}{1+n^2x^2}} = n \to \infty,\ as\ n \to \infty.$$

Therefore, no uniform convergence.
(c). We prove that F_n does not converge in L^2 to F

$$\|F_n - F\|_2^2 = \int_0^l \frac{N}{1+N^2x^2}dx = arc\tan(nx)\Big|_0^l \to \frac{\pi}{2} \neq 0.$$

Convergence Theorems

Theorem ($C^1 \Rightarrow$ Pointwise Convergence)

The classical Fourier series converges to $f(x)$ pointwisely on (a,b) provided that $f(x) \in C^1[a,b]$.

Theorem ($C^1 \Rightarrow$ Uniform Convergence)

Suppose $f \in C^1([a,b])$ s.t. B.C., then $S_N(f)$ converges uniformly to f on $\mathbb{T}$.

Theorem (L^2 Convergence)

The Fourier series converges to $f(x)$ in the mean-square sense in (a,b) provided only that $f(x)$ is any function for which

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 dx\right)^{1/2} < \infty.$$

Proof Ideas

Dirichlet Kernel $K_N(\theta)$

The Dirichlet Kernel $K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta$ occurs in the proof of pointwise convergence. It has the following properties:

- $\int_{-\pi}^{\pi} K_N(\theta)\frac{d\theta}{2\pi} = 1.$
- $K_N(\theta)$ can be summed. This is because

$$K_N(\theta) = \sum_{n=-N}^{N} e^{in\theta} = \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} = \frac{\sin[(N+1/2)\theta)}{\sin(1/2)\theta}.$$

Question 1

Problem (1 - Notions of Convergence)

Prove or disprove the followings

- *On a finite domain, uniform convergence implies* L^2 *convergence.*
- L^2 *convergence implies uniform convergence.*
- L^2 *convergence implies pointwise convergence.*

Solution 1

The first assertion is TRUE. Assume Ω is finite (i.e. it has finite measure, $\mu(\Omega) < \infty$). If $f_n \to f$ uniformly, consider

$$\|f_n - f\|_2^2 = \int_\Omega |f_n - f|^2 d\mu \leq \sup_\Omega |f_n - f|^2 \mu(\Omega) \to 0, \quad as\ n \to \infty.$$

The second assertion is FALSE. Set
$f_n(x) = (1-x)x^{n-1}, x \in (0,1)$, we know that its partial sum is $S_N(f) = 1 - x^N$. Obviously, $S_N(f) \to 1$ as $N \to \infty$. Note that

$$\|S_N(f) - 1\|^2 = \int_\Omega |x^N|^2 d\mu = \frac{1}{2N+1} \to 0, as\ N \to \infty.$$

However, it does not converge uniformly to 1. The reason is: fix $\epsilon = \frac{1}{2}$, for each $N \in \mathbb{N}$, choose $x \in [\frac{1}{2}^{1/N}, 1)$, then we have

$$\sup_{x\in(0,1)} |S_N(f) - 1| = \sup_{x\in(0,1)} |x^N| \geq \frac{1}{2} = \epsilon.$$

Solution 1

The third assertion is FALSE. Consider $f_n(x) = n^{\frac{1}{2}}\mathbb{I}((0, \frac{1}{n}])$, where $\mathbb{I}(\cdot)$ is a characteristic function. The sequence converges pointwisely to $[0, 1]$ to the function that is identically equal to zero but does not converge to this funciton with respect to the $L^2[0, 1]$ norm. This is because

$$\|f_n - 0\|_2^2 = \int_\Omega n\mathbb{I}((0, \frac{1}{n}])d\mu = 1.$$

$\square$

Remark

If the domain Ω is NOT of finite measure, then the first assertion is in general FALSE. Let $\Omega = \mathbb{R}$, obviously the domain is not finite. Consider the function $f_n = \frac{1}{\sqrt{n}} \cdot \mathbb{I}((0, n))$.

More Remarks

- Although in general if $f_n \to f$ in L^2, we cannot say that $f_n \to f$ pointwisely, however, we have the following theorem, which says that if f_n converges in L^2 to f, then there exists a subsequence of f_n converges pointwisely.

Remark (Riesz-Fischer Theorem)

If $f_n \to f$ in $L^2(\Omega)$, there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ pointwise almost everywhere on Ω.

- Also, if $f_n \to f$ pointwisely, we cannot conclude that $f_n \to f$ uniformly. However, pointwise convergence almost implies uniform convergence...

Remark (Egoroff's Theorem)

Suppose f_n converges almost everywhere on Ω of f. Then for any $\epsilon > 0$ there exists a subset A of Ω with $\mu(A) < \epsilon$ such that f_n converges uniformly to f on Ω/A.

More Remarks

- An interesting result feature of $L^2(\Omega)$ is that the **Heine-Borel Theorem** does NOT hold. That is, not every **bounded and closed** set is **compact**.

Example (Bounded + Closed $\neq$ Compact in $L^2(\Omega)$)

Consider the closed unit ball $B = \{f \in L^2([0,1]) : \|f\|_{L^2} \leq 1\}$ *in* $L^2([0,1])$. *Define* $f_n = (\sqrt{2})^n \cdot \mathbb{I}((0, 2^{-n})$. *Then* $\|f_n\| = 1$ *so* $f_n \in B$ *for each* n. *But if* $m > n$, *then*

$$\|f_n - f_m\|_{L^2} \geq 1.$$

Question 2

Problem (2)

Prove the validity of the Fourier series solution of the diffusion equation on $(0, l)$ *with* $u_x(x, 0) = u_x(x, l) = 0, u(x, 0) = \phi(x)$*, where* $\phi(x)$ *is continuous with a piecewise continuous derivative. That is, prove that the series truly converges to the solution. (**Hint:** Read p.144-145)*

Solution 2

Question 3

Problem (3)

*Show that if $f(x)$ is a C^1 function in $[-\pi, \pi]$ that satisfies the periodic BC and if $\int_{-\pi}^{\pi} f(x)dx = 0$, then $\int_{-\pi}^{\pi} |f|^2 dx \leq \int_{-\pi}^{\pi} |f'|^2 dx$. (**Hint:** Parseval's equality)*

Solution 3

Solution 3

- $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$
- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0.$
- By parseval's equality,

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=1}^{\infty} a_n^2 \left(\int_{-\pi}^{\pi} \cos^2 nx dx\right)^2 + b_n^2 \left(\int_{-\pi}^{\pi} \sin^2 nx dx\right)^2$$
$$= \sum_{n=1}^{\infty} \pi(a_n^2 + b_n^2).$$

- $f'(x) = \frac{1}{2}a_0' + \sum_{n=1}^{\infty} a_n' \cos nx + b_n' \sin nx;\ a_0' = \int_{-\pi}^{\pi} f'(x)dx = 0.$
- Note that $a_n' = nb_n$ and $b_n' = -na_n$.
- By parseval's equality,

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx = \pi \sum_{n=1}^{\infty} ((nb_n)^2 + (-na_n)^2) = \pi \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2)$$
$$\geq \pi \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx.$$

References I

Strauss, W. A. (2007).
Partial differential equations: An introduction.
John Wiley & Sons.