Special matrices (square)

Diagonal

$$\begin{bmatrix} a & & & \bar{\bar{0}} \\ & b & & \\ & & c & \\ \bar{\bar{0}} & & & \ddots \; z \end{bmatrix}$$

Triangular

lower/upper

$$\begin{bmatrix} 0 & & \bar{\bar{d}} \\ \vdots & \ddots & \\ 0 & \cdots & 0 \end{bmatrix}$$

Scalar

$$\begin{bmatrix} k & & & \bar{\bar{0}} \\ & k & & \\ & & \ddots & \\ \bar{\bar{0}} & & & k \end{bmatrix} = k \cdot \bar{\bar{I}}$$

Identity

$$\begin{bmatrix} 1 & & & \bar{\bar{0}} \\ & 1 & & \\ & & \ddots & \\ \bar{\bar{0}} & & & 1 \end{bmatrix} = \bar{\bar{I}}$$

Null (in various forms) $\bar{\bar{0}}$?

Transposée d'une matrice carrée

$$\begin{pmatrix} 1 & 5 \\ 6 & 8 \end{pmatrix}^T = \begin{pmatrix} 1 & 6 \\ 5 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 7 & 5 \\ 1 & 0 & 7 \\ 4 & 2 & 6 \end{pmatrix}^T = \begin{pmatrix} 9 & 1 & 4 \\ 7 & 0 & 2 \\ 5 & 7 & 6 \end{pmatrix}$$

Étapes :

- La 1ère ligne devient la 1ère colonne
- La 2 -ème ligne devient la 2 -ème colonne
- La 3 -ème ligne devient la 3 -ème colonne
- Ainsi de suite…

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^T$$

Special matrices (square)

Symmetric $\quad \bar{\bar{A}}^T = \bar{\bar{A}}$

e.g. $\begin{bmatrix} 5 & -1 & 0 \\ -1 & 9 & 5 \\ 0 & 5 & 7 \end{bmatrix}$

EXAM! side 1

Special matrices (square)

Symmetric $\bar{\bar{A}}^T = \bar{\bar{A}}$

e.g. $\begin{bmatrix} 5 & -1 & 0 \\ -1 & 9 & 5 \\ 0 & 5 & 7 \end{bmatrix}$

Skew-symmetric $\bar{\bar{A}}^T = -\bar{\bar{A}}$

e.g. $\begin{bmatrix} 0 & -6 & 2 \\ 6 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$

OBS what's particular about it?

Example [edit]

For example, the following matrix is skew-Hermitian

$$A = \begin{bmatrix} -i & +2+i \\ -2+i & 0 \end{bmatrix}$$

because

$$-A = \begin{bmatrix} i & -2-i \\ 2-i & 0 \end{bmatrix} = \begin{bmatrix} \overline{-i} & \overline{-2+i} \\ \overline{2+i} & \overline{0} \end{bmatrix} = \begin{bmatrix} \overline{-i} & \overline{2+i} \\ \overline{-2+i} & \overline{0} \end{bmatrix}^{\mathsf{T}} = A^{\mathsf{H}}$$

Basic remarks [edit]

A square matrix $\mathbf{A}$ with entries a_{ij} is called

- Hermitian or self-adjoint if $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$; i.e., $a_{ij} = \overline{a_{ji}}$.
- Skew Hermitian or antihermitian if $\mathbf{A} = -\mathbf{A}^{\mathrm{H}}$; i.e., $a_{ij} = -\overline{a_{ji}}$.
- Normal if $\mathbf{A}^{\mathrm{H}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{H}}$.
- Unitary if $\mathbf{A}^{\mathrm{H}} = \mathbf{A}^{-1}$, equivalently $\mathbf{A}\mathbf{A}^{\mathrm{H}} = \boldsymbol{I}$, equivalently $\mathbf{A}^{\mathrm{H}}\mathbf{A} = \boldsymbol{I}$.

Even if $\mathbf{A}$ is not square, the two matrices $\mathbf{A}^{\mathrm{H}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathrm{H}}$ are both Hermitian and in fact positive semi-definite matrices.

Matrix algebra (addition)

Addition

$\bar{\bar{A}} + \bar{\bar{B}} = \bar{\bar{B}} + \bar{\bar{A}}$ commutative

$(\bar{\bar{U}} + \bar{\bar{V}}) + \bar{\bar{W}} = \bar{\bar{U}} + (\bar{\bar{V}} + \bar{\bar{W}})$ associative

$\bar{\bar{A}} + \bar{\bar{0}} = \bar{\bar{A}}$ null element

$\bar{\bar{A}} + (-\bar{\bar{A}}) = \bar{\bar{0}}$ negative (anti) element

Multiplication

Matrix multiplication

$\bar{\bar{A}}\,\bar{\bar{B}} \neq \bar{\bar{B}}\,\bar{\bar{A}}$ can happen but in general not

$$\begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 7 & -3 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$-3\cdot 5 + 1\cdot 7 = -15+7 = -8$ $\quad -3\cdot 3 - 3\cdot 1 = -9-3 = -12$

$2\cdot 5 + 5\cdot 7 = 10+35 = 45$ $\quad 2\cdot 3 - 3\cdot 5 \quad 6-15 = -9$

Matrix algebra (mult. cont.)

$\bar{\bar{A}}(\bar{\bar{B}}\,\bar{\bar{C}}) = (\bar{\bar{A}}\,\bar{\bar{B}})\bar{\bar{C}}$ associative

$(\bar{\bar{A}} + \bar{\bar{B}})\bar{\bar{C}} = \bar{\bar{A}}\,\bar{\bar{C}} + \bar{\bar{B}}\,\bar{\bar{C}}$

$\bar{\bar{C}}(\bar{\bar{A}} + \bar{\bar{B}}) = \bar{\bar{C}}\,\bar{\bar{A}} + \bar{\bar{C}}\,\bar{\bar{B}}$ distributive (wrt. addition)

! observe order

We can say more about the matrix product when we know the matrix <u>rank</u> (later)

Matrix algebra (scalars)

$$c(\bar{\bar{A}}+\bar{\bar{B}}) = c\bar{\bar{A}} + c\bar{\bar{B}}$$
$$(c+k)\bar{\bar{A}} = c\bar{\bar{A}} + k\bar{\bar{A}}$$

distributive (wrt. add./mult.)

$$c(k\bar{\bar{A}}) = (ck)\bar{\bar{A}}$$

$$(k\bar{\bar{A}})\bar{\bar{B}}) = k(\bar{\bar{A}}\bar{\bar{B}}) = \bar{\bar{A}}(k\bar{\bar{B}})$$

associative

$$1\bar{\bar{A}} = \bar{\bar{A}}$$

identity

Matrix algebra (transpose)

$$(\bar{\bar{A}}+\bar{\bar{B}})^T = \bar{\bar{A}}^T + \bar{\bar{B}}^T$$
$$(c\bar{\bar{A}})^T = c\bar{\bar{A}}^T$$

distributive (wrt. matrix add. scalar mult.)

$$(\bar{\bar{A}}\bar{\bar{B}})^T = \bar{\bar{B}}^T\bar{\bar{A}}^T$$

! NOT distributive (wrt. matr. mult.)

$$(A^n)^T = (A^T)^n$$

→ we will see later that a similar reverse order applies to matrix inverse

Matrix determinant (Minor)

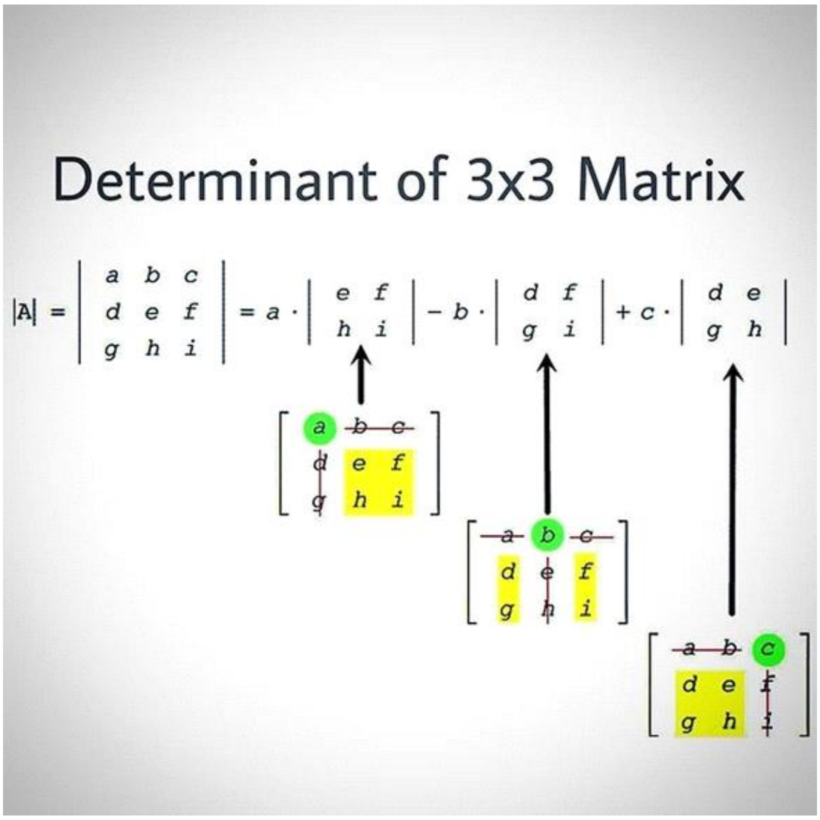
$$\begin{bmatrix} a_{11} & \cdots & & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ & & a_{jk} & & \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{bmatrix}$$ row j expan.

remove elements on red lines ...

... and calculate subdeterminant, i.e. minor M_{jk} of order $n-1$

$$\begin{bmatrix} a_{11} & \cdots & & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ & & a_{jk} & & \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{bmatrix}$$

column k expan.



Echelon form

A triangular matrix, such that:

- all 0's are below, left
- first $a'_{ij} \neq 0$ in each row is to the right of the $a' (\neq 0)$ in the row above AND has only 0's below it

∇ first a'_{ij}'s are not necessarily on one diagonal

∇ there can be rows with all 0's

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{mn} & \cdots & a_{mn} \end{bmatrix} \quad (= \bar{\bar{A}})$$

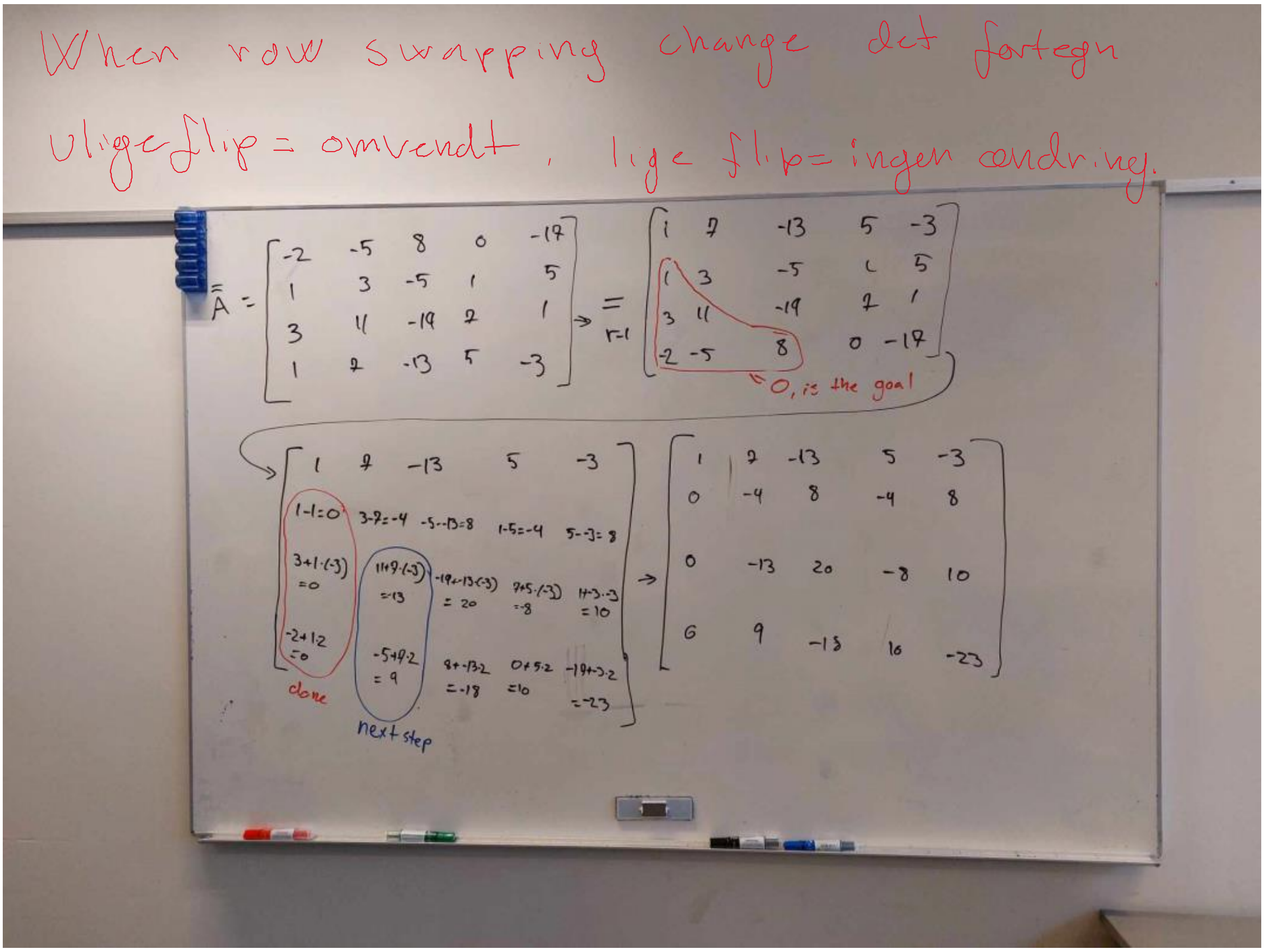
pivot - "the central element"

$$\begin{bmatrix} a'_{11} & \cdots & & a'_{1n} \\ 0 & & & \\ \vdots & 0 & a'_{ij} \cdots & a'_{in} \\ & & 0\,0 & a'_{pn} \\ 0 & \cdots & 0\,0 & 0 \\ 0 & \cdots & & 0 \end{bmatrix} \quad (= \bar{\bar{A}}')$$

row equivalent BUT different $\bar{\bar{A}}$

When row swapping change det fortegn

Uligeflip = omvendt, lige flip = ingen ændring.



Gauss elimination - by matrix ops.

Operation — rows : EA (pre)
— columns : AE (post)

Case of 3x3 for illustration

Multiply by a constant

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{this row}$$

this column — OR

Swap rows / columns

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{these rows}$$

these columns — OR

Multiply and add

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \leftarrow \text{this row}$$

this column — OR

 slide

Inverse Matrices

Example A: Find the inverse matrix A^{-1} if

a. $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Extend $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ to $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$

Apply row operations to tranform it so the identity matrix I is on the left side:

$$\left[\begin{array}{cc|cc} \overset{-1}{1} & \overset{-2}{2} & \overset{-1}{1} & \overset{0}{0} \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R1 \text{ Add to } R2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

(-2)R2 Add to R1

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

|| identity matrix

Hence $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. One checks easily that $AA^{-1} = A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1-1 & 3-2 & 0-1 & 1-0 \end{array}\right]$$

$$0 \quad 1 \quad -1 \quad 1$$

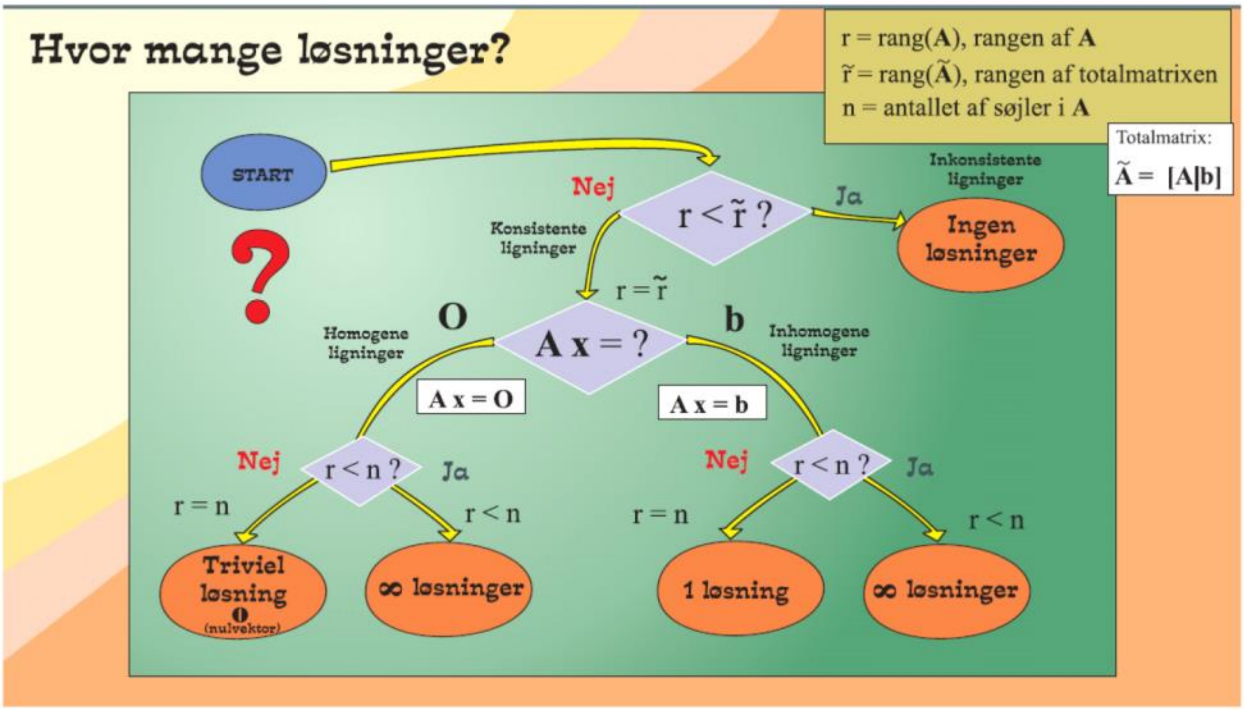
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1-0\cdot 2 & 2-2\cdot 1 & 1-2\cdot -1 & 0-2\cdot 1 \\ 1 & 0 & 1+2=3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right] \qquad A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

Lec 2 - More basis løsning af lignings systemer

Monday, 27 May 2024 10.26

Solutions to linear systems



Aalborg University, WCN – lineær algebra og dynamiske systemer slide

Solution approaches

Gauss elimination $(m \times n)$:

solves any case

Use of inverse (algebraic):

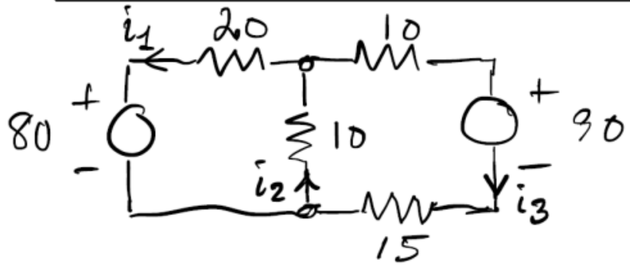
$$\bar{\bar{A}}\bar{x} = \bar{b} \Leftrightarrow \bar{x} = \bar{\bar{A}}^{-1}\bar{b} \quad \text{IF}$$

$\bar{\bar{A}}$ is square and non-singular

Use of Cramer's rule:

$$\bar{x}_k = \det(\bar{\bar{A}}_{(k)}) / \det(\bar{\bar{A}}) \quad \text{IF}$$

Ex.: Solution by Gauss elimination..



$$\left.\begin{aligned} i_1 - i_2 + i_3 &= 0 \\ -i_1 + i_2 - i_3 &= 0 \end{aligned}\right\} \text{KCL}$$

$$\left.\begin{aligned} 10 i_2 + 25 i_3 &= 90 \\ 20 i_1 + 10 i_2 \quad &= 80 \end{aligned}\right\} \text{KVL}$$

$\bar{x} = ?$

$$\tilde{\bar{A}} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \xrightarrow{\text{Gauss}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

using partial pivoting (swapping rows)

... and Backsubstitution

Backsubstitution

$$\tilde{\bar{\bar{A}}} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \bar{i}_1 = \bar{i}_2 - \bar{i}_3 = 2 \\ \bar{i}_2 = (90 - 25\bar{i}_3)/10 = 4 \\ \bar{i}_3 = -190/-95 = 2 \\ 0 = 0 \end{array}$$

- A unique solution (since $\text{rank}(\bar{\bar{A}}) = n$)
- An overdetermined system since $m > n$
- A consistent system since solutions exist

Row and column space

The column and row vectors of $\bar{\bar{A}}$ span a vector space of dimension $r = \text{rank}(\bar{\bar{A}})$ — as per def. of rank

Row space: a set of r linearly independent row vectors

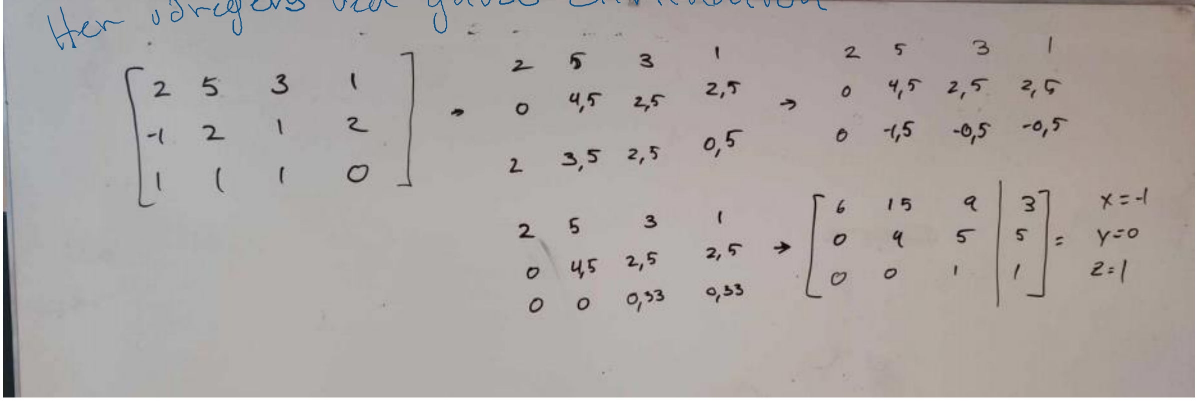
Column space: a set of r linearly independent column vectors
The essential vectors!

Så nu til hvordan man løser et lignings system:

$$\begin{array}{c} \\ x \\ y \\ z \end{array}
\begin{array}{c} \begin{array}{cccc} x & y & z & \text{ans} \end{array} \\ \left[\begin{array}{ccc|c} 2 & 5 & 3 & 1 \\ -1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

er vores matrice

Her udregnes ved gauss elimination



Og her ved hjælpen af cramners

$$\det\begin{bmatrix} 2 & 5 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 2\cdot(-1)^{1+1}\cdot(2\cdot 1 - 1\cdot 1) + 5\cdot(-1)^{1+2}\cdot(-1\cdot 1 - 1\cdot 1) + 3\cdot(-1)^{1+3}(-1\cdot 1 - 1\cdot 2) = 3$$

$$\det_1\begin{bmatrix} 1 & 5 & 3 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \; 1\cdot(-1)^{1+1}\cdot(2\cdot 1 - 1\cdot 1) + 5\cdot(-1)^{1+2}\cdot(2\cdot 1 - 0\cdot 1) + 3\cdot(-1)^{1+3}\cdot(2\cdot 1 - 0\cdot 2) = -8$$

$$\det_2\begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \; 2\cdot(-1)^{1+1}\cdot\underbrace{(2\cdot 1 - 0\cdot 1)}_{4} - 1\cdot(-1)^{2+1}\cdot\underbrace{(1\cdot 1 - 0\cdot 3)}_{1} + 1\cdot(-1)^{3+1}\cdot\underbrace{(1\cdot 1 - 2\cdot 3)}_{-5} = 0$$

$$\det_3\begin{bmatrix} 2 & 5 & 1 \\ -1 & 2 & 2 \\ 1 & 1 & 0 \end{bmatrix} \; 2\cdot(-1)^{1+1}\cdot(2\cdot 0 - 1\cdot 2) + 5\cdot(-1)^{1+2}\cdot(-1\cdot 0 - 1\cdot 2) + 1\cdot(-1)^{1+3}\cdot(-1\cdot 1 - 1\cdot 2) = 3$$

$$\frac{-3}{3} = -1 \qquad \frac{0}{3} = 0 \qquad \frac{3}{3} = 1 \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Linear transformations

A linear transformation is a mapping F from vector space X to space Y

$$\bar{x} \in X \qquad \bar{y} = F(\bar{x}) \qquad \bar{y} \in Y$$

"image"

$$\bar{\bar{A}}\,\bar{x} = \bar{y}$$

$\bar{x} = x_1\bar{e}_1 + \dots x_n\bar{e}_n$ $\qquad$ $F(\bar{x}) = x_1 F(\bar{e}_1) + \dots x_n F(\bar{e}_n)$

$\bar{e}_i$ $(n\times 1)$ basis $\qquad$ $\bar{\varepsilon}_i = F(\bar{e}_i)$ $(m\times 1)$ basis

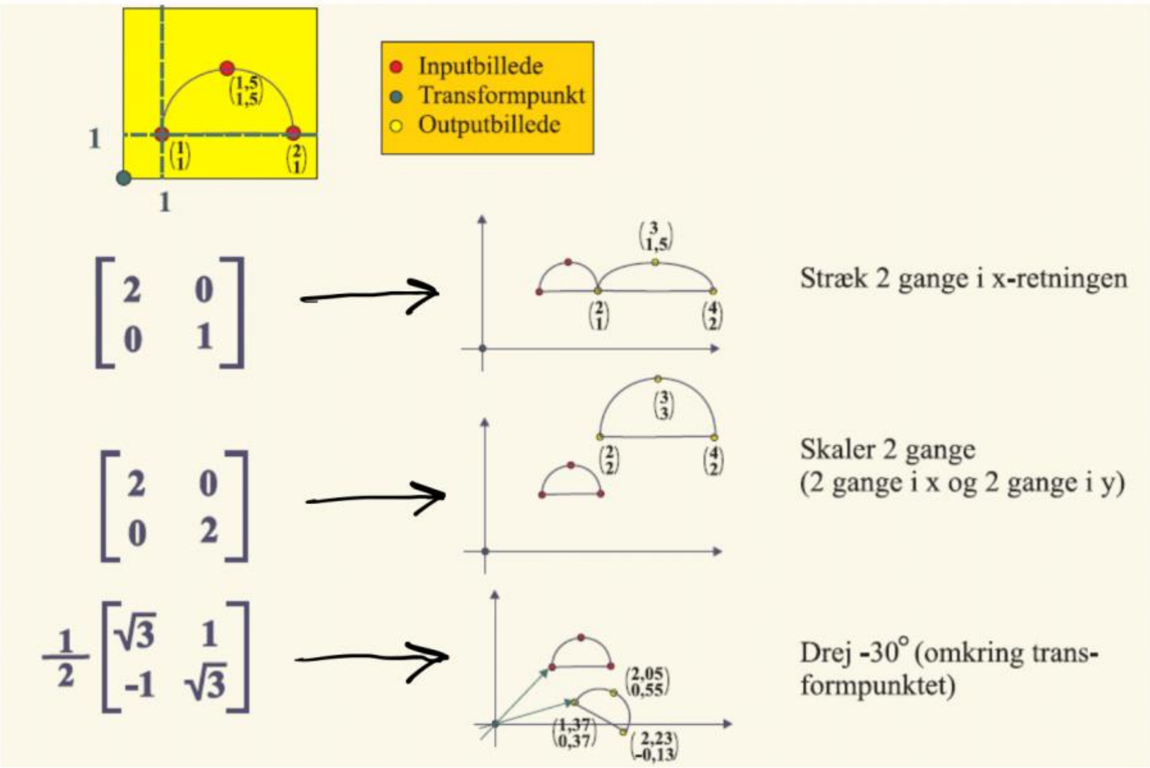
Linear transformations

If $\bar{\bar{A}}$ is square $(n \times n)$ and non-singular, $\bar{\bar{A}}^{-1}$ is defined, thus

$$\bar{x} = F^{-1}(\bar{y}) = \bar{\bar{A}}^{-1}\bar{y}$$

An orthogonal $(n \times n)$ matrix is one such, $\bar{\bar{A}}^{-1} = \bar{\bar{A}}^{T}$, i.e. a rotation in 2D and 3D Euclidean space. It preserves the inner product (length).

Example linear transformations



$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \longrightarrow$$

$$\frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix} \longrightarrow$$

Power series

A **power series** *in powers of* $z - z_0$ is a series of the form

(1)
$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

where z is a complex variable, $a_0, a_1, \cdots$ are complex (or real) constants, called the **coefficients** of the series, and z_0 is a complex (or real) constant, called the **center** of the series. This generalizes real power series of calculus.

If $z_0 = 0$, we obtain as a particular case a *power series in powers of* z:

(2)
$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots .$$

Power series - Convergence

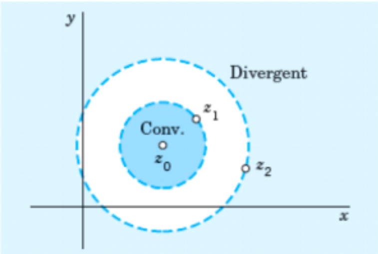
- If we fix z, all the concepts for series with constant terms apply.

Convergence of a Power Series

(a) *Every power series* (1) *converges at the center* z_0.

(b) *If* (1) *converges at a point* $z = z_1 \neq z_0$, *it converges absolutely for every* z *closer to* z_0 *than* z_1, *that is,* $|z - z_0| < |z_1 - z_0|$. See Fig. 365.

(c) *If* (1) *diverges at* $z = z_2$, *it diverges for every* z *farther away from* z_0 *than* z_2. See Fig. 365.



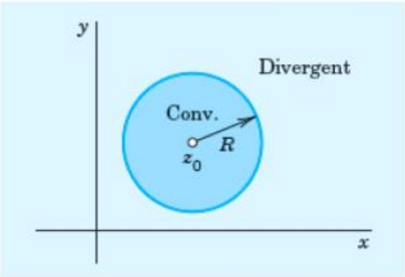
EXAM! side 18

Power series – Radius of convergence

- We consider the ***smallest*** circle with center that includes all the points at which a given power series converges. Let R denote its radius. The circle

$$|z - z_0| = R$$

is the circle of convergence and R is the radius of convergence.



Power series – Radius of convergence

Radius of Convergence R

Suppose that the sequence $|a_{n+1}/a_n|$, $n = 1, 2, \cdots$, *converges with limit* L . *If* $L = 0$, *then* $R = \infty$; *that is, the power series* (1) *converges for all* z. *If* $L \neq 0$ (*hence* $L > 0$), *then*

$$R = \frac{1}{L} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \qquad \textbf{(6)}$$

(Cauchy–Hadamard formula[1]**).**

If $|a_{n+1}/a_n| \to \infty$, *then* $R = 0$ (*convergence only at the center* z_0).

Functions given by power series

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$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R).$$

- f(z) is represented by the power series.
- A function f(z) cannot be represented by two different power series with the same center → uniqueness of a power series representation
- A derived series of a power series is obtained by termwise differentiation.

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots.$$

- The derived series of a power series has the same radius of convergence as the original series.
- The integrated series of a power series has the same radius of convergence as the original series.

7

Taylor and Maclaurin series

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- A Taylor series of a function f(z) is

a_n = det til R

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

- A Maclaurin series is a Taylor series with center $z_0=0$.

$$f(z) = f(z_0) + \frac{z - z_0}{1} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots$$
$$+ \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z).$$

remainder of the Taylor series

8

Taylor and Maclaurin series

- Geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

- Exponential function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \cdots.$$

- Trigonometric function

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{4} - + \cdots$$

- Logarithm

$$\mathrm{Ln}\,(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots$$

Exercises module 2

Basis formel

$$R = \frac{1}{L} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

A.

1) $\sum_{n=0}^{\infty} 4^n (z+1)^n \qquad z_0 = -1$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{4}$$

2) $\sum_{n=1}^{\infty} \frac{n^n}{n} (z - \pi i)^n \qquad z_0 = \pi i$

$$R = \lim_{n\to\infty} \left| \frac{\frac{n^n}{n}}{\frac{(n+1)^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n^n}{n} \cdot \frac{n+1}{(n+1)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n^n}{n} \cdot \frac{1}{(n+1)^n} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{1}{n} \cdot \left(\frac{n}{n+1} \right)^n \right| = \lim_{n\to\infty} \left| \frac{1}{n} \cdot \left(\frac{1}{1+\frac{1}{n}} \right)^n \right| =$$

$$= 0 \cdot \frac{1}{e} = 0$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

3) $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-i)^{2n}$ $\quad z_0 = i$

$$\sum_{n=0}^{\infty} a_n (z-z_0)^{2n} = \sum_{n=0}^{\infty} a_n \left[(z-z_0)^2\right]^n$$

$$(z-z_0)^2 < R$$

$$\Rightarrow |z-z_0| < \sqrt{R}$$

$$R = \lim_{n\to\infty} \left| \frac{\frac{n(n-1)}{3^n}}{\frac{(n+1)n}{3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{n-1}{n+1} \cdot 3 \right| = 3 \rightarrow \sqrt{R} = \sqrt{3}$$

4) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(z+3)^n$ $\quad z_0 = -3$

$$R = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n n}{4^n}}{\frac{(-1)^{n+1}(n+1)}{4^{n+1}}} \right| = \lim_{n\to\infty} \left| -4 \cdot \frac{n}{n+1} \right| = 4$$

6) $\sum_{n=0}^{\infty} n!\,(2z+1)^n = \sum_{n=0}^{\infty} 2^n \cdot n!\left(z+\frac{1}{2}\right)^n$

$$z_0 = -\frac{1}{2}$$

$$R = \lim_{n\to\infty} \left| \frac{2^n \cdot n!}{2^{n+1}(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{1}{2} \cdot \frac{n!}{(n+1)\cdot n!} \right| = 0$$

7) $\sum_{n=0}^{\infty} \frac{(z-2i)^n}{n^n}$ $\quad z_0 = 2i$

$$R = \lim_{n\to\infty} \left| \frac{\frac{1}{n^n}}{\frac{1}{(n+1)^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n} \right| = \lim_{n\to\infty} \left| (n+1) \cdot \left(\frac{n+1}{n}\right)^n \right| =$$

$$= \lim_{n\to\infty} \left| (n+1) \cdot \left(1+\frac{1}{n}\right)^n \right| = \infty$$

8) $\sum_{n=0}^{\infty} \left(\frac{1-i}{2+3i}\right)^n z^n$ $\quad z_0 = 0$

$$R = \lim_{n\to\infty} \left| \frac{\left(\frac{1-i}{2+3i}\right)^n}{\left(\frac{1-i}{2+3i}\right)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2+3i}{1-i} \right| = \sqrt{\frac{13}{2}}$$

The eigenvalue problem

$$\bar{\bar{A}}\bar{x} = \lambda \bar{x} \quad , \quad \underline{\bar{x} \neq \bar{0}} \quad !$$

unknown eigenvector (characteristic vector)

unknown eigenvalue (scalar) (characteristic value, latent value)

$\{\lambda_i\}$: spectrum of $\bar{\bar{A}}$

$\max|\lambda_i|$: spectral radius of $\bar{\bar{A}}$

$\{\bar{x}_i, \bar{0}\}_{\lambda_i}$: eigenspace of $\bar{\bar{A}}$ corresponding to λ_i

The eigenvalue problem (cont.)

We want to find solutions of

$$\bar{\bar{A}}\bar{x} = \lambda \bar{x} \iff (\bar{\bar{A}} - \lambda \bar{\bar{I}})\bar{x} = \bar{0}$$

a homogeneous linear system of equ.'s

For this to have solutions $(\bar{\bar{A}} - \lambda \bar{\bar{I}})$ needs to be singular, i.e., $\det(\bar{\bar{A}} - \lambda \bar{\bar{I}})$ needs to be zero:

$$D(\lambda) = \det(\bar{\bar{A}} - \lambda \bar{\bar{I}}) = 0$$

characteristic equation

characteristic determinant / polynomial (of degree n)

Recipe – eigenvalue problem

1. **Dan den karakteristiske matrix:**

$$\cancel{\mathbf{M}} = \mathbf{A} - \lambda \mathbf{I}$$

2. **Find den karakteriske determinant**

$$\Delta \cancel{\mathbf{M}} = |\mathbf{A} - \lambda \mathbf{I}|$$

polynomium / ligning

3. **Dette giver den karakteriske ligning:**

$$a\lambda^{n} + b\lambda^{n-1} + c\lambda^{n-2} + \cdots = 0$$

4. **Løsningerne til den karakteristiske ligning giver spektret:**

Spektral radius

$$S = \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$$

5. **Dette indeholder mellem 1 og n egenværdier**

Bemærk den algebraiske multiplicitet for λ_k (ordenen af roden).

6. **Indsæt hver λ_k i den karakteristiske matrix M og løs det homogene system (fx vha. Gaussisk elimination):**

$$(\mathbf{A} - \lambda \mathbf{I})\bar{x} = \bar{0}$$

Løsningerne er egenvektorerne.

7. **Bemærk den geometriske multiplicitet (antallet af lineært uafhængige egenvektorer for den givne λ_k)**

Eigenvalues (Square matrices)

Hermetian C^n $\quad \bar{A}^{*T} = \bar{A}$
(symmetric, R^n)

λ_i real (0)

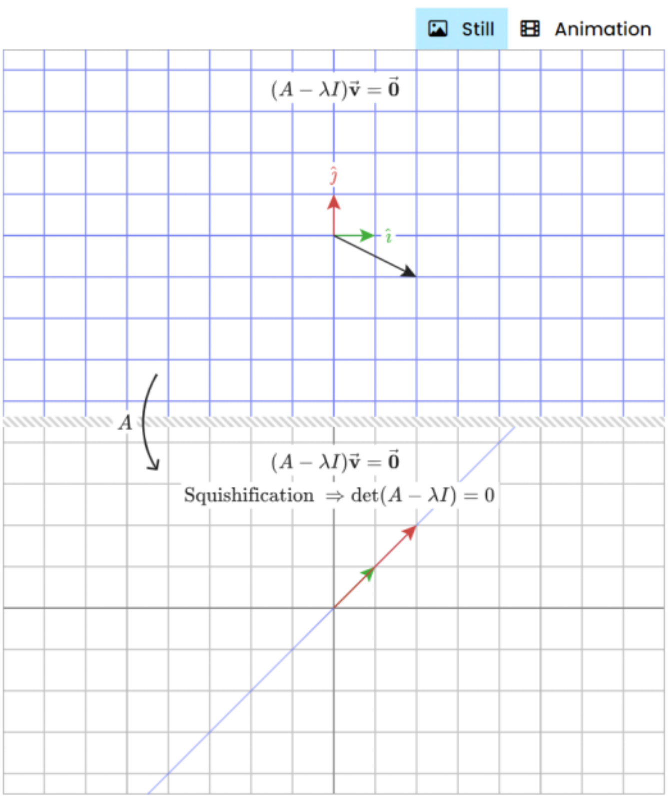
Skew-Hermetian C^n $\quad \bar{A}^{*T} = -\bar{A}$
(skew-symmetric, R^n)

λ_i imaginary (0)

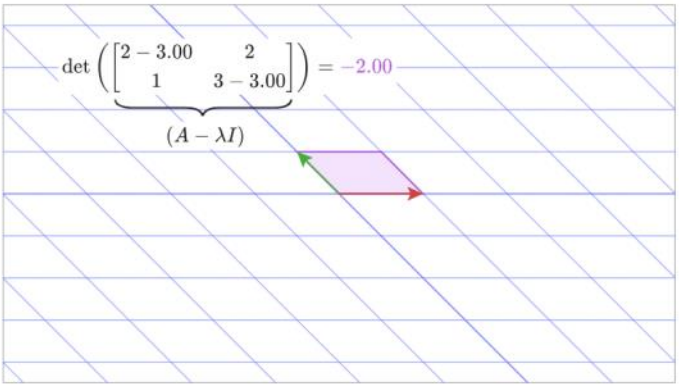
Unitary C^n $\quad \bar{A}^{*T} = \bar{A}^{-1}$
(orthogonal, R^n)

$|\lambda_i| = 1$

$*$ complex conjugate

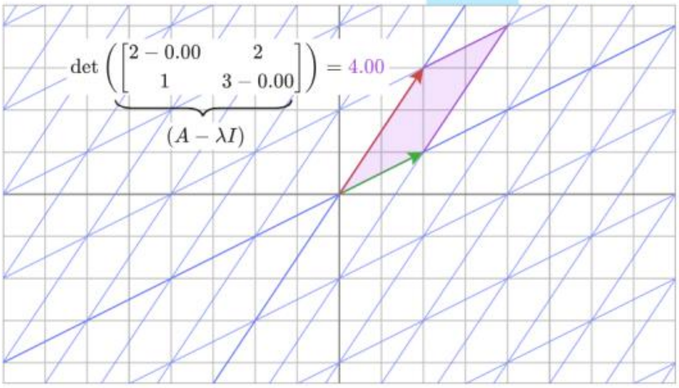


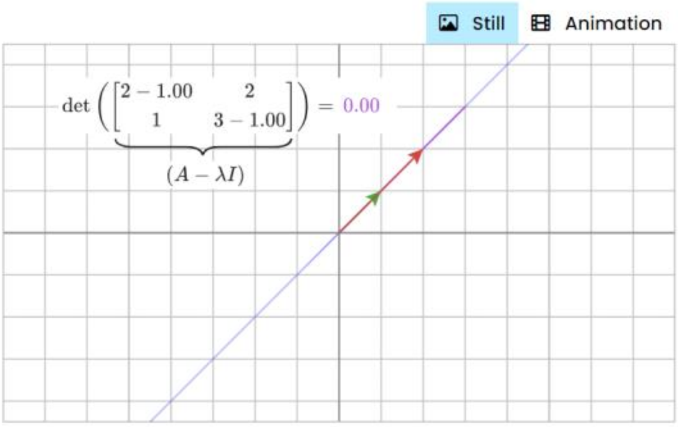
To be concrete, let's say your matrix is $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, and imagine subtracting off the variable amount lambda from each diagonal entry.



Imagine tweaking lambda, turning a knob to change its value. As that value of lambda changes, the matrix changes, and so the determinant of that matrix changes.

Still Animation





Remember that the lambda is tied to the matrix, so if we use another matrix, the lambda might not be 1. To unravel what that means, when $\lambda = 1$, the matrix $A - \lambda \cdot I$ squishes space onto a line. That means there's a nonzero vector $\mathbf{v}$ such that $(A - \lambda \cdot I) \cdot \mathbf{v} = 0$.

The reason we care about that is because it means $A \cdot \mathbf{v} = \lambda \cdot \mathbf{v}$, which you can read as saying the vector $\mathbf{v}$ is an eigenvector of A, staying on its own span during the transformation A. In this example, the corresponding eigenvalue is 1, so it actually just stays fixed in place.

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$A\mathbf{v} - \lambda I\mathbf{v} = 0$$

$$(A - \lambda I)\mathbf{v} = 0$$

$$\det(A - \lambda I) = 0$$

Man vil sku enlig bare gerne have noget der gør at ens det(A) = 0, når man ændre lambda som en værdi.
Det er der også en MATLAB kommando til: eig(A)

SÅDAN HER LØSER MAN!

What is the eigenvalue(s) of the matrix $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$?

- ○ 3
- ○ 5
- ◉ 5 and 1 ✓
- ○ 2

Correct! Reset

$$\det\left(\begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}\right) = 0$$

$$0 = (3-\lambda)(3-\lambda) - 4$$

$$0 = \lambda^2 - 6\lambda + 5$$

$$0 = (\lambda - 5)(\lambda - 1)$$

$$\lambda = 5$$

$$\lambda = 1$$

What is the eigenvalue(s) of the matrix $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$?

- (selected, correct) There are real eigenvalues.
- 2
- 3 and 1
- 1

Correct! Reset

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} 0 &= (2-\lambda)^2 + 1 \\ -1 &= (2-\lambda)^2 \\ \sqrt{-1} &= 2-\lambda \\ \lambda &= 2-\sqrt{-1} \\ \lambda &= 2+i \\ \lambda &= 2-i \end{aligned}$$

The fact that there are no real number solutions indicates that there are no eigenvectors.

What is the eigenvalue(s) of the matrix $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

- There are none.
- 2
- 3
- (selected, correct) -2

Correct! Reset

$$\det\left(\begin{bmatrix} -2-\lambda & 1 \\ 0 & -2-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} 0 &= (-2-\lambda)(-2-\lambda) \\ 0 &= (\lambda+2)(\lambda+2) \\ \lambda &= -2 \end{aligned}$$

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

Step 1: Find the Eigenvalues

The eigenvalues λ are found by solving the characteristic equation $\det(A - \lambda I) = 0$.

1. Compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix}$$

2. Compute the determinant $\det(A - \lambda I)$:

$$\det(A - \lambda I) = (4-\lambda)(3-\lambda) - (2 \cdot 1) = (4-\lambda)(3-\lambda) - 2$$

3. Expand and simplify the equation:

$$(4-\lambda)(3-\lambda) - 2 = 12 - 4\lambda - 3\lambda + \lambda^2 - 2 = \lambda^2 - 7\lambda + 10$$

4. Solve the characteristic equation $\lambda^2 - 7\lambda + 10 = 0$:

$$\lambda^2 - 7\lambda + 10 = 0$$

Husk maple kan dette her tjek sidebaren

Factoring the quadratic equation, we get:

$$(\lambda - 5)(\lambda - 2) = 0$$

So, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 2$.

Step 2: Find the Eigenvectors

Next, we find the eigenvectors for each eigenvalue by solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Eigenvector for $\lambda_1 = 5$:

1. Substitute $\lambda = 5$ into $A - \lambda I$:

$$A - 5I = \begin{pmatrix} 4-5 & 1 \\ 2 & 3-5 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$$

2. Solve $(A - 5I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

reglen for at gange matricen sammen

This gives the system of equations:

$$\begin{cases} -1v_1 + 1v_2 = 0 \\ 2v_1 - 2v_2 = 0 \end{cases}$$

Both equations simplify to $v_1 = v_2$. Hence, an eigenvector corresponding to $\lambda_1 = 5$ is:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Eigenvector for $\lambda_2 = 2$:

1. Substitute $\lambda = 2$ into $A - \lambda I$:

$$A - 2I = \begin{pmatrix} 4-2 & 1 \\ 2 & 3-2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

2. Solve $(A - 2I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the system of equations:

$$\begin{cases} 2v_1 + 1v_2 = 0 \\ 2v_1 + 1v_2 = 0 \end{cases}$$

Both equations simplify to $2v_1 + v_2 = 0$. Hence, an eigenvector corresponding to $\lambda_2 = 2$ is:

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Step 3: Form the Eigenbasis

The eigenbasis of matrix A is the set of eigenvectors corresponding to the eigenvalues. Thus, the eigenbasis for A is:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

These eigenvectors form a basis for the vector space in which the matrix A operates.

DIAGONALIZE YOUR MATIRX!

1. Find the Eigenvalues

- Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A. Here, I is the identity matrix of the same dimension as A, and $\det$ represents the determinant.

2. Find the Eigenvectors

- For each eigenvalue λ_i, find the corresponding eigenvector v_i by solving the equation $(A - \lambda_i I)v_i = 0$. This involves solving a system of linear equations to find the non-trivial solutions.

3. Form the Matrix P

- Construct the matrix P using the eigenvectors as its columns. That is, $P = [v_1\ v_2\ \ldots\ v_n]$.

4. Form the Diagonal Matrix D

- The diagonal matrix D will have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its diagonal. That is, $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

5. Verify

- Verify that $P^{-1}AP = D$. If the computation is correct, this should hold true.

Example

Let's diagonalize a 2x2 matrix A:

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

Step 1: Find the Eigenvalues

Solve $\det(A - \lambda I) = 0$:

$$\det \begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 - 2 = \lambda^2 - 7\lambda + 8 = 0$$

Solve the characteristic polynomial:

$$\lambda^2 - 7\lambda + 8 = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 6$.

Step 2: Find the Eigenvectors

For $\lambda_1 = 1$:

Solve $(A - I)v = 0$:

$$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the first row: $3v_1 + v_2 = 0 \Rightarrow v_2 = -3v_1$

Choose $v_1 = 1$, then $v_2 = -3$. So, $v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

For $\lambda_2 = 6$:

Solve $(A - 6I)v = 0$:

$$\begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the first row: $-2v_1 + v_2 = 0 \Rightarrow v_2 = 2v_1$

Choose $v_1 = 1$, then $v_2 = 2$. So, $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Step 3: Form the Matrix P

eig base 1

$$P = \begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$$

eig base 2

Step 4: Form the Diagonal Matrix D

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$ eig Val 1

eig Val 2

Step 5: Verify

d −b

Calculate P^{-1}:   −c a

$$P^{-1} = \frac{1}{(1)(2)-(1)(-3)} \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$$

Verify $P^{-1}AP = D$: det(A)

Step 5: Verify

Calculate P^{-1}:

$$P^{-1} = \frac{1}{(1)(2)-(1)(-3)} \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$$

Verify $P^{-1}AP = D$:

$$P^{-1}A = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 7 & 5 \\ 15 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 3 & 6 \end{pmatrix} = D$$

So, the matrix A is successfully diagonalized as D.

Basis of eigenvectors (cont.)

If an eigenbasis for $\bar{\bar{A}}$ exists $\{\bar{x}_i\}$:

$\bar{\bar{A}}\bar{x}_i = \lambda_i \bar{x}_i$

- Eigendecomposition

$$\bar{y} = \bar{\bar{A}}\bar{x} = \bar{\bar{A}}(c_1\bar{x}_1 + \cdots c_n\bar{x}_n)$$
$$= c_1\bar{\bar{A}}\bar{x}_1 + \cdots c_n\bar{\bar{A}}\bar{x}_n$$
$$= c_1\lambda_1\bar{x}_1 + \cdots c_n\lambda_n\bar{x}_n$$

- Diagonalization

$$\bar{\bar{D}} = \bar{\bar{X}}^{-1}\bar{\bar{A}}\,\bar{\bar{X}} \longrightarrow$$

Diagonalization

$$\bar{\bar{D}} = \bar{\bar{X}}^{-1}\,\bar{\bar{A}}\,\bar{\bar{X}}$$

$$\begin{bmatrix} \lambda_1 & & \bar{0} \\ & \ddots & \\ \bar{\bar{0}} & & \lambda_n \end{bmatrix}$$ (diagonal) matrix of eigenvalues

matrix of eigenvectors $$[\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n]$$

$\bar{\bar{X}}$ is said to diagonalize $\bar{\bar{A}}$

$\bar{\bar{D}}$ and $\bar{\bar{A}}$ are called similar* in that they have the same eigenvalues

(* generally, any non-singular matrix $\bar{\bar{X}}$ sufficise

Diagonalization (cont.)

To see this : $\bar{\bar{A}}\bar{x} = \lambda \bar{x}$, $\underline{\bar{x} \neq 0}$

$$\bar{\bar{X}}^{-1}(\bar{\bar{A}}\bar{x}) = \lambda \bar{\bar{X}}^{-1}\bar{x}$$

$$\Updownarrow$$

$$\bar{\bar{X}}^{-1}\bar{\bar{A}}(\bar{\bar{X}}\bar{\bar{X}}^{-1})\bar{x} = \lambda \bar{\bar{X}}^{-1}\bar{x}$$

$$\Updownarrow$$

$$(\bar{\bar{X}}^{-1}\bar{\bar{A}}\bar{\bar{X}})(\bar{\bar{X}}^{-1}\bar{x}) = \lambda(\bar{\bar{X}}^{-1}\bar{x})$$

So, λ is (also) an eigenvalue of $\bar{\bar{X}}^{-1}\bar{\bar{A}}\bar{\bar{X}} = \bar{\bar{D}}$,
BUT corresponding $\underline{\text{eigenvector } \hat{\bar{x}} = \bar{\bar{X}}^{-1}\bar{x}}$
($\underline{\bar{x}} = \bar{\bar{X}}\bar{\bar{X}}^{-1}\bar{x} = \bar{\bar{X}}\bar{0} = \underline{\bar{0}}$, so $(\bar{\bar{X}}^{-1}\bar{x}) \neq \bar{0}$)

DUM DUM STUFF

Tuesday, 28 May 2024 09.12

Husk der eksister en function der faktor dit tal I maple. BRUG DET TIL EIGEN VALUES.

FIND THE NULL SPACE

Tuesday, 28 May 2024 09.57

Solution.

We first obtain the reduced row echelon form matrix corresponding to the matrix A.

We reduce the matrix A as follows:

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$$

$$\xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & -1 \end{bmatrix} \xrightarrow[R_3+R_2]{R_1-2R_2} \begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last matrix is in reduced row echelon form. That is,

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (*$$

(a) Find a basis for the nullspace of A.

By the computation above, we see that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} x_1 &= -9x_3 - 2x_4 \\ x_2 &= 3x_3 - x_4, \end{aligned}$$

where x_3 and x_4 are free variables.

Thus, the vector form solution to $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 - 2x_4 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

It follows that the nullspace of the matrix A is given by

$$\mathcal{N}(A) = \left\{ \mathbf{x} \in \mathbb{R}^4 \;\middle|\; \mathbf{x} = x_3 \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \text{ for all } x_3, x_4 \in \mathbb{R}^4 \right\}$$

$$= \operatorname{Span} \left\{ \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus, the set

$$\left\{ \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a spanning set for the nullspace $\mathcal{N}(A)$.
It is straightforward to see that this set is linearly independent, and hence it is a basis for $\mathcal{N}(A)$.

(b) Find a basis for the row space of A.

Recall that the nonzero rows of $\mathrm{rref}(A)$ form a basis for the row space of A.
Thus,

fra reduced row echelon rref

$$\left\{ \begin{array}{c} r_1 \\ \begin{bmatrix} 1 \\ 0 \\ 9 \\ 2 \end{bmatrix} \end{array}, \begin{array}{c} r_2 \\ \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix} \end{array} \right\}$$

is a basis for the row space of A.

(b) Find a basis for the row space of A.

Recall that the nonzero rows of $\mathrm{rref}(A)$ form a basis for the row space of A.
Thus,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix} \right\}$$

is a basis for the row space of A.

(c) Find a basis for the range of A that consists of column vectors of A.

Recall that by the leading 1 method, the columns of A corresponding to columns of $\mathrm{rref}(A)$ that contain leading 1 entries form a basis for the range $\mathcal{R}(A)$ of A.
From (*), we see that the first and the second columns contain the leading 1 entries. Thus,

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Tilbage fra vore orginale A for c_1 & c_2

is a basis for the range $\mathcal{R}(A)$ of A.

(d) For each column vector which is not a basis vector that you obtained in part (c), express it as a linear combination of the basis vectors for the range of A.

Let us write A_1, A_2, A_3, and A_4 for the column vectors of the matrix A.
In part (c), we showed that $\{A_1, A_2\}$ is a basis for the range $\mathcal{R}(A)$.
Thus, we need to express the vectors A_3 and A_4 as a linear combination of A_1 and A_2, respectively.

A shortcut is to note that the entries of third column vector of $\mathbf{rref}(A)$ give the coefficients of the linear combination for A_3. That is, we have

Husk dit span

$$A_3 = 9A_1 - 3A_2.$$

$0 = -9A_1 + 3A_2 + A_3 + 0A_4 \Rightarrow A_3 = 9A_1 - 3A_2$

Husk fra Span

Similarly, the entries of the fourth column of $\mathbf{rref}(A)$ yield

$0 = -2A_1 - 1A_2 + 0A_3 + A_4 \Rightarrow A_4 = 2A_1 + 1A_2$

$$A_4 = 2A_1 + A_2.$$

Differential equations (cont.)

The diff. equ. is linear what does that mean?

You probably remember:

$r(t) = 0$ homogeneous diff. equ.

"forcing function" $r(t) \neq 0$ nonhomogeneous – " –

Further, if we specify

$y(t_0) = K_0, \; y'(t_0) = K_1, \; \dots \; y^{(n-1)}(t_0) = K_{n-1}$

we get an initial value problem

Differential equations (cont.)

The general solution for $y(t)$ is

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is a solution of the homogenous equation, and $y_p(t)$ is any particular solution of the nonhomogeneous equation (integration constant 0).

Systems of equations

Stacking differentials

$$\bar{y}(t) = [y_1(t), y_2(t), \dots y_n(t)]^T$$

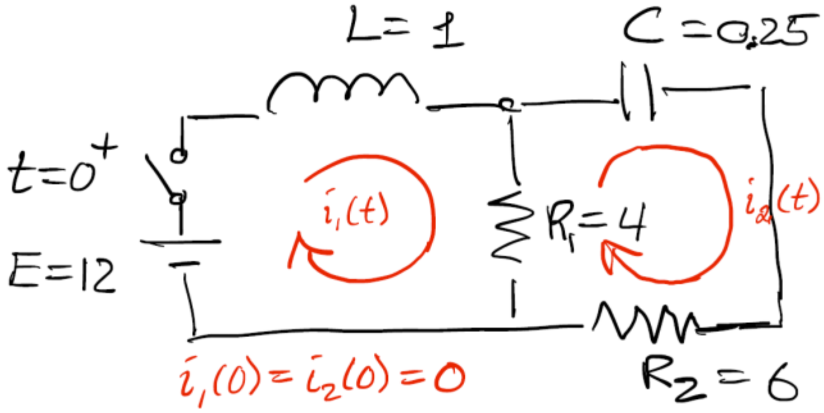
$$\bar{y}'(t) = [y_1'(t), y_2'(t), \dots y_n'(t)]^T$$

differentiate each entry

$$\bar{y}'(t) = \bar{\bar{A}}\,\bar{y}(t)$$ stacked differential equations

Example 2, EK4.1

SI units



$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

$$v_C(t) = \frac{1}{C} \cdot \int_{t_0}^{t} i_C(\tau)\,d\tau$$

KVL

1) $-E + L \cdot i_1'(t) + R(i_1(t) - i_2(t)) = 0$

2) $-R_1(i_1(t) - i_2(t)) + \frac{1}{C}\int_{t_0}^{t} i_2(\tau)\,d\tau + R_2 i_2(t) = 0$

Example (cont.)

1) $i_1'(t) = -4i_1(t) + 4i_2(t) + 12$

2) $10\,i_2'(t) = 4i_1'(t) - 4i_2(t) = -16i_1(t) + 16i_2(t) + 48 - 4i_2(t)$

$\Updownarrow$

$i_2'(t) = -1.6\,i_1(t) + 1.2\,i_2(t) + 4.8$

$$\begin{bmatrix} i_1'(t) \\ i_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & +4 \\ -1.6 & +1.2 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \begin{bmatrix} 12 \\ 4.8 \end{bmatrix}$$

koblet 1. ordens i $i_1(t)$ og $i_2(t)$ eller 2. ordens i $i_2(t)$

OR

$$\bar{J}'(t) = \bar{\bar{A}}\,\bar{J}(t) + \bar{g}$$

$\Updownarrow$

$$\bar{J}'(t) - \bar{\bar{A}}\,\bar{J}(t) = \bar{g}\quad,\ \bar{J}_h = \bar{x}e^{At},\ \bar{J}_p = \bar{a}$$

$\hookrightarrow$ a nonhomogeneous first order equ.

A first order system

$$\left\{\begin{aligned} y_1' &= f_1(t, y_1, y_2, \dots, y_n, r(t)) \\ y_2' &= f_2(t, y_1, y_2, \dots, y_n, r(t)) \\ &\ \vdots \\ y_n' &= f_n(t, y_1, y_2, \dots, y_n, r(t)) \end{aligned}\right\}\quad \bar{y}' = \bar{f}(t, \bar{y}, r(t))$$

autonomous (time invariant) $= f(\bar{y}, r(t))$

whose general solution $\tilde{\bar{y}}(t) = [\tilde{y}_1(t), \dots \tilde{y}_n(t)]^T$ satisfies $\tilde{\bar{y}}'(t) = \bar{f}(t, \tilde{\bar{y}}(t))$, $\alpha \le t \le \beta$ and unique solution further satisfies $\tilde{\bar{y}}(0) = [K_1, K_2, \dots K_n]^T$

A first order linear system

If the system is linear in $y_i(t)$ we may write

$$y_1' = f_1 = a_{11}(t)y_1 + a_{12}(t)y_2 + \dots a_{1n}(t)y_n + g_1(t)$$

$$\vdots \qquad \vdots$$

$$y_n' = f_n = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \dots a_{nn}(t)y_n + g_n(t)$$

$$\bar{y}' = \bar{\bar{A}}(t)\bar{y} + \bar{g}(t)$$

$g(t)=0$ Homogeneous

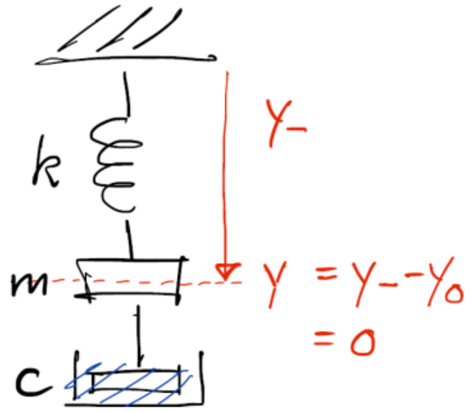
$g(t)\neq 0$ Nonhomogeneous

- Same form for coupled equ's

Homogeneous equ. (ex.)

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$$

$$y_1 = y,\ y_2 = y'$$



$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Constant coefficients

$$\Downarrow\quad \bar{y} = \bar{v} e^{\lambda t}$$

homogeneous?
autonomous?

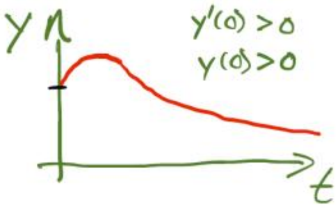
$$m Y_-'' = -kY_- + mg - cY_-'$$

$$\Updownarrow\quad = -ky - \underbrace{kY_0 + mg}\ - cY_-'$$

$$my'' = -ky - cy'$$

Homogeneous equ. (cont.)

$$\bar{y} = \bar{v} e^{\lambda t}$$



$$\bar{y}' = \lambda \bar{v} e^{\lambda t} = \bar{\bar{A}} \bar{y} = \bar{\bar{A}} \bar{v} e^{\lambda t}$$

$$\Updownarrow\quad \bar{\bar{A}} \bar{v} = \lambda \bar{v}$$

an eigenvalue problem

1) $\det(\bar{\bar{A}} - \lambda \bar{\bar{I}}) = 0$

autonomous

$m = 1,\ c = 2,\ k = 0.75$ ($c^2 > 4mk$, overdamped)

$\lambda^2 + 2\lambda + 0.75 \iff \lambda_1 = -0.5,\ \lambda_2 = -1.5$

2) $(\bar{\bar{A}} - \lambda \bar{\bar{I}})\bar{v} = 0 \implies \bar{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ \bar{v}_2 = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$

$$\bar{y} = c_1 \bar{v}_1 e^{-0.5t} + c_2 \bar{v}_2 e^{-1.5t}$$

$$\Rightarrow y = y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$$

Homogeneous (auton.) formalized

Generally, in the constant coefficient (autonomous) homogeneous equation

$$\bar{y}' = \bar{\bar{A}}\bar{y} \quad , \quad \bar{y} = \bar{v} e^{\lambda t}$$

$$\Updownarrow$$

$$\bar{\bar{A}}\bar{v} = \lambda \bar{v}$$ eigenvalue problem

If $\bar{\bar{A}}$ has a basis of eigenvectors $\{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n\}$ it follows that

$$W([\bar{v}_1 e^{\lambda_1 t}, \bar{v}_2 e^{\lambda_2 t}, \ldots \bar{v}_n e^{\lambda_n t}]) \neq 0$$

Homogeneous equ. – Stability

which means that

$$\bar{y} = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t} + \dots c_n \bar{v}_n e^{\lambda_n t}$$

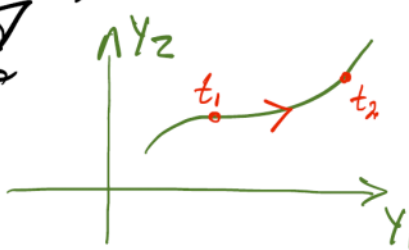
is a general solution (spans all solutions)

Since it generalises all solutions we can use it to analyse whether the (physical) system produces stable solutions, i.e.

"a small change changes the behaviour of the system only slightly at all future times"

Critical points – phase plane

Stability is related to critical points in the (so-called) phase-plane, illustrated here by "second-order" systems, and to the eigenvalues of $\bar{\bar{A}}$!

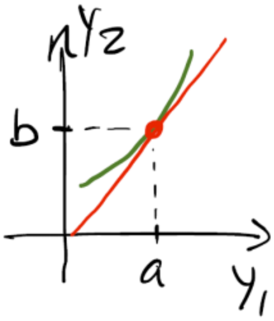


Phase plane $\bar{y}(t)$

parametric curves (trajectories) of
$y_2(t)$ versus $y_1(t)$ – a phase portrait

Phase plane tangent

At the point $P:(y_1,y_2)=(a,b)$ we can define the tangent:



$$\frac{dy_2}{dy_1}=\frac{y_2'}{y_1'}=\frac{a_{21}y_1+a_{22}y_2}{a_{11}y_1+a_{12}y_2}$$

Generally, if $\tilde{y}_1=y_1-a$, $\tilde{y}_2=y_2-b$, we may think of the point (a,b) at $(0,0)$ in $\tilde{y}_1,\tilde{y}_2$ coordinates. Equivalently, we can consider $P:(a,b)$ at $(0,0)$.

Critical points

The tangent is well defined except at $(y_1,y_2)=(0,0)$ where we have

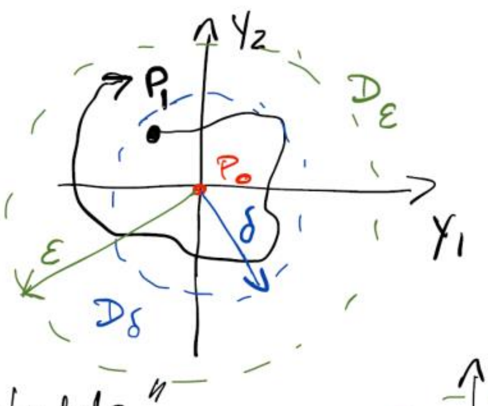
$$\frac{dy_2}{dy_1}=\frac{0}{0}=?$$ undetermined!

When this happens, we call P a critical point of the linear system $\bar{y}'=\bar{\bar{A}}\bar{y}$

where trajectories could "spin off"!

Phase plane – stability

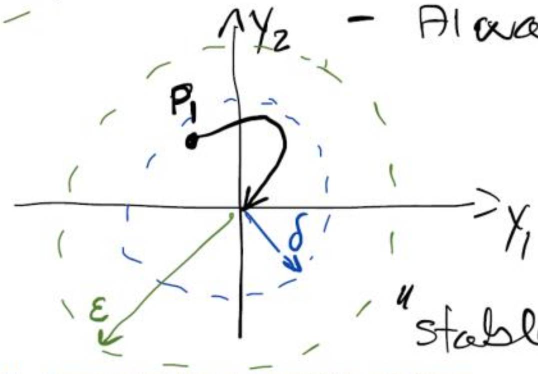
Lyapunov stability



"stable"

Trajectory such that:

- P_1 in D_δ at t_1
- Always in D_ε for $t \geq t_1$



"stable and attractive"

Types of critical points

Node(s)

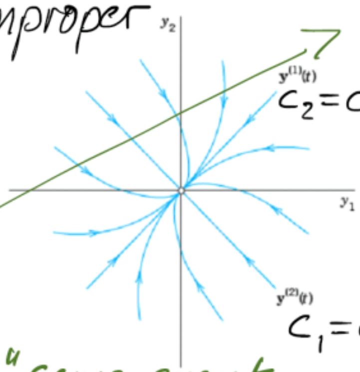
$\lambda_1 = -2, \lambda_2 = -4$

$M_\lambda = 1, m_\lambda = 1$

$\lambda_1 = 1, \lambda_2 = 1$

$M_\lambda = 2, m_\lambda = 2$!

improper

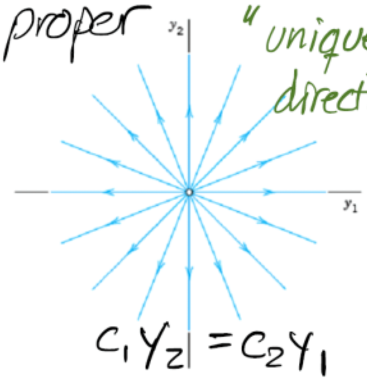


$c_2 = 0$

$c_1 = 0$

"same, except two"

proper



"unique directions"

$c_1 y_2 = c_2 y_1$

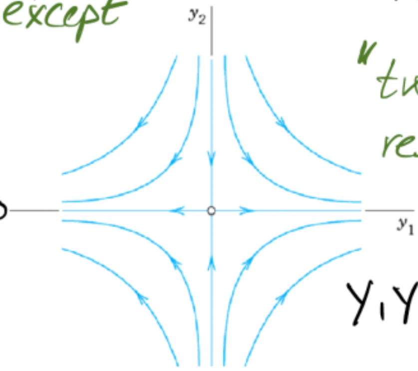
Saddle

$\lambda_1 = 1, \lambda_2 = -1$

$m_\lambda = 1, M_\lambda = 1$

[Examples refer to Kreyszig]

$c_2 = 0$



$c_1 = 0$

"two in, two out rest pass by"

$y_1 y_2 = c_1 c_2 = K$

Types of critical points

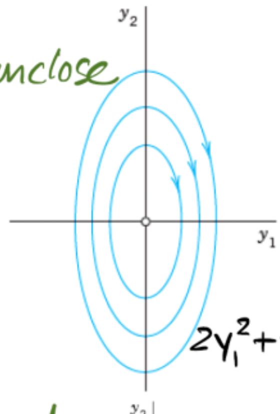
Center

$\lambda_1 = 2i, \lambda_2 = -2i$

$M_\lambda = 1, m_\lambda = 1$

Imaginary

"all enclose P_0"



$2y_1^2 + \frac{1}{2}y_2^2 = K$

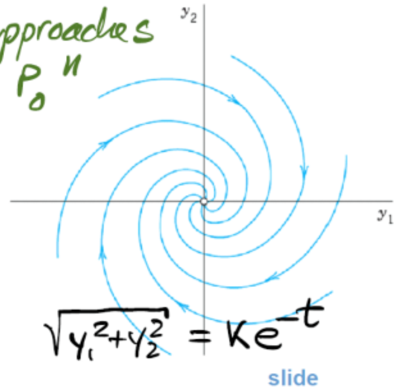
Spiral

$\lambda_1 = -1+i, \lambda_2 = -1-i$

$M_\lambda = 1, m_\lambda = 1$

Complex conjugate

"approaches P_0"



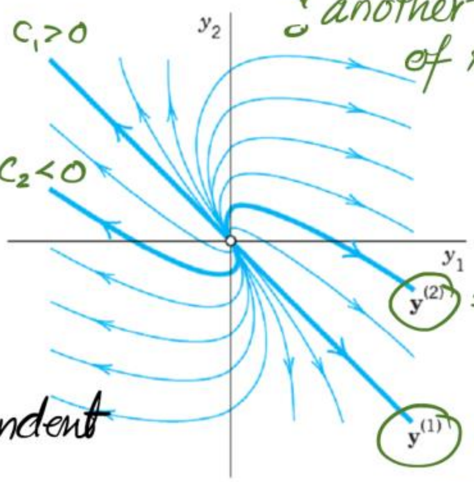
$\sqrt{y_1^2 + y_2^2} = Ke^{-t}$

Degenerate node

No eigenbasis!

The previous types all had an eigenbasis, but this one not

$\lambda = 3$: $M_\lambda = 2$, $m_\lambda = 1$! (defect)



However, it is possible to construct two linearly independent vectors:

$$\bar{Y}_{(1)} = \bar{v} e^{\lambda t} \quad \text{(from eigenvalue problem)}$$

$$\bar{Y}_{(2)} = \bar{v} t e^{\lambda t} + \bar{u} e^{\lambda t}$$

$$\hookrightarrow \quad (\bar{\bar{A}} - \lambda \bar{\bar{I}}) \bar{u} = \bar{v} \quad , \det(\bar{\bar{A}} - \lambda \bar{\bar{I}}) = 0 \,!$$

Stability and eigenvalues

$$\det(\bar{\bar{A}} - \lambda \bar{\bar{I}}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + \det(\bar{\bar{A}}) = 0$$

From LA we know:

$$\mathrm{trace}(\bar{A}) = \sum_i a_{ii} = a_{11} + a_{22} = \sum_i \lambda_i \equiv p$$

$$\det(\bar{\bar{A}}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \prod_i \lambda_i \equiv q$$

$$\det(\bar{\bar{A}} - \lambda \bar{\bar{I}}) = \lambda^2 - p\lambda + q \quad , \quad D = p^2 - 4q$$

Nodes and eigenvalues

$\underline{D \geq 0}$ (real eigenvalues)

$$q = \prod_i \lambda_i > 0 \overset{(p \neq 0)}{\Longrightarrow} \text{node}$$

$$q = \prod_i \lambda_i < 0 \Longrightarrow \text{saddle}$$

except $q \leq 0$

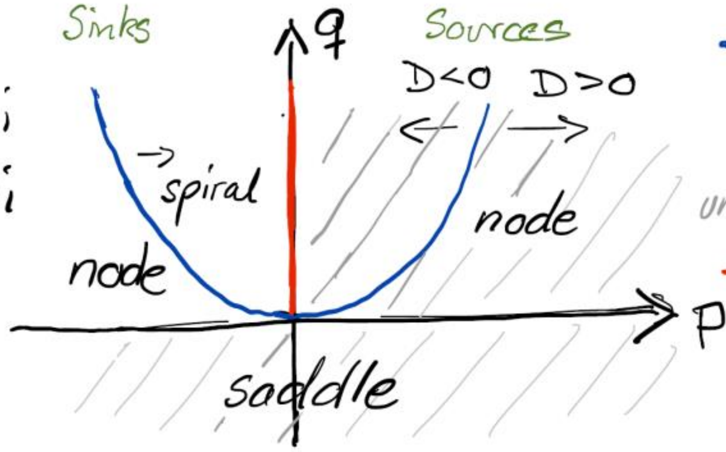
except $p = 0$

$$p = \sum_i \lambda_i = 0 \overset{(q > 0)}{\Longrightarrow} \text{center}$$

$$p = \sum_i \lambda_i \neq 0 \Longrightarrow \text{spiral point}$$

$\underline{D < 0}$ (complex eigenvalues)

Phase plane - stability



$p = \sum \lambda_i$

$q = \prod_i \lambda_i$

— degenerate (+ proper) $D=0$

— center ($p=0$)

- unstable if $q<0$ or $p>0$
- stable if $q>0$ and $p \leq 0$
- stable if $q>0$ and $p<0$ (attractive or asymptotically stable)

holds also for $n>2$

Phase plane stability (cont.)

We can rewrite this for $n \geq 2$:

If all eigenvalues (real or complex conjugate pairs) have negative real parts, then P is a stable attracting point.

10.4.2.1: Using Eigenvalues to Solve a System

A linear system will be solve by hand and using Eigenvalues[ ] expression in Mathematica simultaneously. Note that, in the Mathematica inputs below, "In[]:=" is not literally typed into the program, only what is after it. The syntax needed to be typed is the line following "In[]=" . The term is used here to more accurately demonstrate coding in Mathematica. To find a general solution of the linear system of ordinary differential equation:

$$\frac{dx}{dt} = 4x + 8y$$

$$\frac{dy}{dt} = 10x + 2y$$

We first put the system in matrix form:

$$A = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Where we can see that

$$A = \begin{bmatrix} 4 & 8 \\ 10 & 2 \end{bmatrix}$$

In mathematica, we can use the following code to represent A:

```
In[1]:= MatrixForm [{{4,8},{10,2}}]
```

Out[1]:= $\begin{bmatrix} 4 & 8 \\ 10 & 2 \end{bmatrix}$

The eigenvalues λ_1 and λ_2, are found using the characteristic equation of the matrix A, det(A- λI)=0.

$$det(A - \lambda I) = 0$$

The eigenvalues λ_1 and λ_2, are found using the characteristic equation of the matrix A, det(A- λI)=0.

$$det(A-\lambda I)=0$$

$$det(\begin{bmatrix}4 & 8\\ 10 & 2\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix})=0$$

$$det\begin{bmatrix}4-\lambda & 8\\ 10 & 2-\lambda\end{bmatrix}=0$$

$$(4-\lambda)(2-\lambda)-80=0$$

$$(\lambda-12)(\lambda+6)=0$$

Therefore, $\lambda_1 = 12$ and $\lambda_2 = -6$

We can use Mathematica to find the eigenvalues using the following code:

```
In[2]:= Eigenvalues[{{4,8},{10,2}}]

Out[2]:={12,-6}
```

Now, for each eigenvalue (λ_1=12 and λ_2=-6), an eigenvector associated with it can be found using $(A-\lambda I)\vec{v}=0$, where $\vec{v}$ is an eigenvector such that $A\vec{v}=\lambda\vec{v}$

i) For λ_1=12

$$(A-\lambda_1 I)\vec{v}=0$$

$$\begin{bmatrix}4-12 & 8-0\\ 10-0 & 2-12\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0$$

$$\begin{bmatrix}-8 & 8\\ 10 & -10\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0$$

This will lead to the equations (1) &(2):

$-8x+8y=0$**(1)**

$+10x-10y=0$**(2)**

ii) For λ_2=-6,

$$(A-\lambda_2 I)\vec{v}=0$$

$$\begin{bmatrix}4+6 & 8-0\\ 10-0 & 2+6\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0$$

$$\begin{bmatrix}10 & 8\\ 10 & 8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0$$

This will lead to the equations (3) & (4):

$10x+8y=0$**(3)**

$10x+8y=0$**(4)**

Equations (3) & (4) lead to the solution $y = -\frac{5}{4}x$.

Recall that the direction of a vector such as $\begin{bmatrix}1\\2\end{bmatrix}$ is the same as the vector $\begin{bmatrix}4\\8\end{bmatrix}$ or any other scalar multiple. Therefore, to get the eigenvector, we are free to choose for either the value x or y.

i) For $\lambda_1 = 12$

We have arrived at $y = x$. As mentioned earlier, we have a degree of freedom to choose for either x or y. Let's assume that x=1. Then, y=1 and the eigenvector $\vec{v_1}$ associated with the eigenvalue λ_1 is $\vec{v_1} = \begin{bmatrix}1\\1\end{bmatrix}$

ii) For $\lambda_2 = -6$

We have arrived at $y = -\frac{5}{4}x$. Let's assume that x = 4. Then, y = -5 and the eigenvector associated with the eigenvalue λ_2 is $\vec{v_2} = \begin{bmatrix}4\\-5\end{bmatrix}$.

These two eigenvalues and associated eigenvectors yield the solution:

$$\begin{bmatrix}x(t)\\y(t)\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{12t} + c_2\begin{bmatrix}4\\-5\end{bmatrix}e^{-6t}$$

Hence a general solution of the linear system in scalar form is:

$$lx(t) = c_1e^{12t} + c_2 4e^{-6t}$$
$$y(t) = c_1e^{12t} - c_2 5e^{-6t}$$

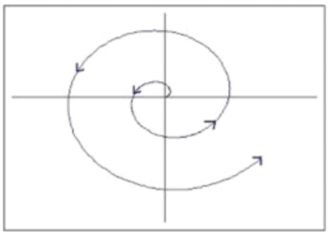
10.4.3.1: Imaginary (or Complex) Eigenvalues

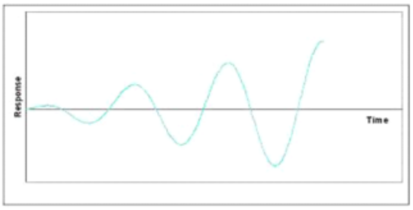
When eigenvalues are of the form $a + bi$, where $a$and b are real scalars and i is the imaginary number $\sqrt{-1}$, there are three important cases. These three cases are when the real part a is positive, negative, and zero. In all cases, when the complex part of an eigenvalue is non-zero, the system will be oscillatory.

10.4.3.1.1: Positive Real Part

When the real part is positive, the system is unstable and behaves as an unstable oscillator. This can be visualized as a vector tracing a spiral away from the fixed point. The plot of response with time of this situation would look sinusoidal with ever-increasing amplitude, as shown below.

This situation is usually undesirable when attempting to control a process or unit. If there is a change in the process, arising from the process itself or from an external disturbance, the system itself will not go back to steady state.

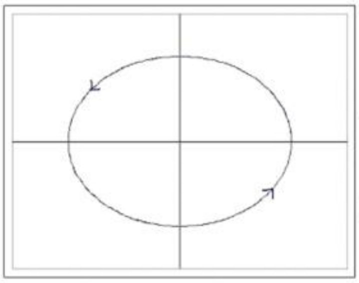


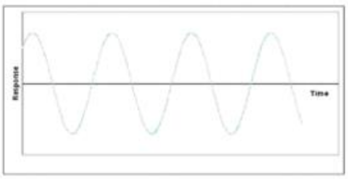


10.4.3.1.2: Zero Real Part

When the real part is zero, the system behaves as an undamped oscillator. This can be visualized in two dimensions as a vector tracing a circle around a point. The plot of response with time would look sinusoidal. The figures below should help in understanding.

Undamped oscillation is common in many control schemes arising out of competing controllers and other factors. Even so, this is usually undesirable and is considered an unstable process since the system will not go back to steady state following a disturbance.



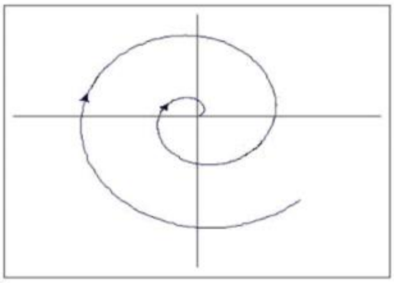


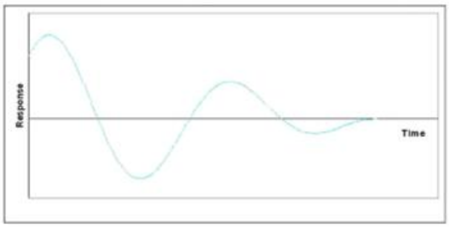
10.4.3.1.3: Negative Real Part

When the real part is negative, then the system is stable and behaves as a damped oscillator. This can be visualized as a vector tracing a spiral toward the fixed point. The plot of response with time of this situation would look sinusoidal with ever-decreasing amplitude, as shown below.

This situation is what is generally desired when attempting to control a process or unit. This system is stable since steady state will be reached even after a disturbance to the system. The oscillation will quickly bring the system back to the setpoint, but will over shoot, so if overshooting is a large concern, increased damping would be needed.

While discussing complex eigenvalues with negative real parts, it is important to point out that having all negative real parts of eigenvalues is a necessary and sufficient condition of a stable system.





10.4.3.2: Real Eigenvalues

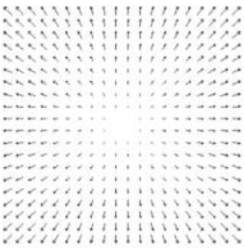
We've seen how to analyze eigenvalues that are complex in form, now we will look at eigenvalues with only real parts.

10.4.3.2.1: Zero Eigenvalues

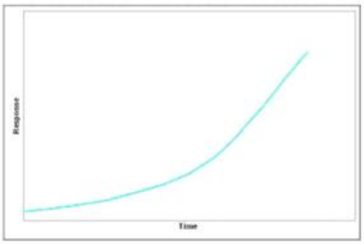
If an eigenvalue has no imaginary part and is equal to zero, the system will be unstable, since, as mentioned earlier, a system will not be stable if its eigenvalues have any non-negative real parts. This is just a trivial case of the complex eigenvalue that has a zero part.

10.4.3.2.2: Positive Eigenvalues

When all eigenvalues are real, positive, and distinct, the system is unstable. On a gradient field, a spot on the field with multiple vectors circularly surrounding and pointing out of the same spot (a node) signifies all positive eigenvalues. This is called a source node.

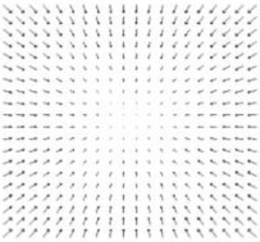


Graphically, real and positive eigenvalues will show a typical exponential plot when graphed against time.

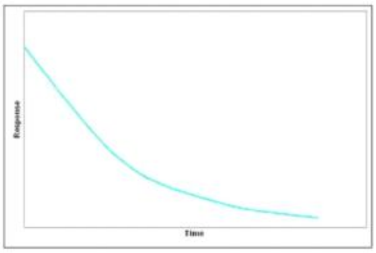


10.4.3.2.3: Negative Eigenvalues

When all eigenvalues are real, negative, and distinct, the system is unstable. Graphically on a gradient field, there will be a node with vectors pointing toward the fixed point. This is called a sink node.

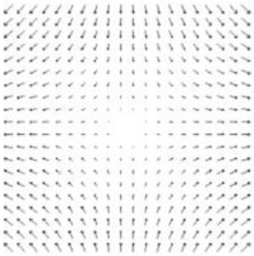


Graphically, real and negative eigenvalues will output an inverse exponential plot.



10.4.3.2.4: Positive and Negative Eigenvalues

If the set of eigenvalues for the system has both positive and negative eigenvalues, the fixed point is an unstable saddle point. A saddle point is a point where a series of minimum and maximum points converge at one area in a gradient field, without hitting the point. It is called a saddle point because in 3 dimensional surface plot the function looks like a saddle.

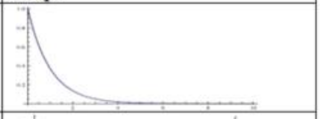
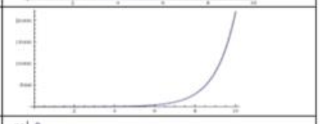
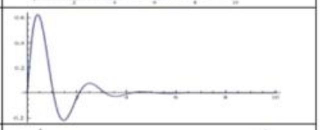
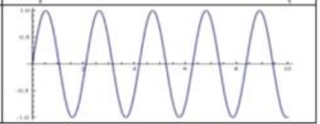


10.4.3.3: Repeated Eigenvalues

If the set of eigenvalues for the system has repeated real eigenvalues, then the stability of the critical point depends on whether the eigenvectors associated with the eigenvalues are linearly independent, or orthogonal. This is the case of degeneracy, where more than one eigenvector is associated with an eigenvalue. In general, the determination of the system's behavior requires further analysis. For the case of a fixed point having only *two* eigenvalues, however, we can provide the following two possible cases. If the two repeated eigenvalues are positive, then the fixed point is an unstable source. If the two repeated eigenvalues are negative, then the fixed point is a stable sink.

10.4.3.4: Summary of Eigenvalue Graphs

Below is a table summarizing the visual representations of stability that the eigenvalues represent.

Eigenvalues	Graph
All real and negative	
All real and one or more are positive	
All real eigenvalues are negative and there are imaginary parts	
One or more eigenvlaues have a positive real part and there are imaginary parts	
Real parts of the eigenvlaues are zero and there are imaginary parts	

Note that the graphs from Peter Woolf's lecture from Fall'08 titled Dynamic Systems Analysis II: Evaluation Stability, Eigenvalues were used in this table.

Non linear systems

Consider a non-linear homogeneous system (1st order, $n=2$)

$$\bar{y}' = f(t, \bar{y}) = \bar{f}(\bar{y}) = \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix}$$

! autonomous system

where f is non linear in y_1, y_2.

We are interested in the stability of the system at a critical point P_0

Linearization procedure

non-linear system (model)

$$y^{(n)}(t) = f(y^{(n-1)}(t), \ldots, y(t), r(t))$$

stationary point (often $y^{(i)}(t) = 0$)

or generally, critical point

linearization (Taylor of nonlinear f)

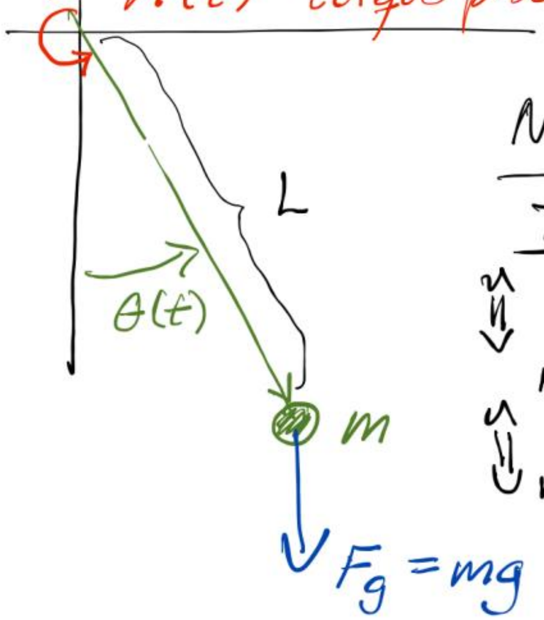
(new delta-variables)

linear model

(Laplace and transfer function)

Example (pendulum)

O $M(t)$ torque produced by motor at Origo



Newton's 2nd law (rotating)

$$I \bar{\theta}''(t) = \sum \bar{\tau}_{ext}(t) \quad (\bar{\tau} = \bar{r} \times \bar{F})$$

$$\Updownarrow$$

$$mL^2 \bar{\theta}'' = \bar{M} + \bar{L} \times \bar{F}_g$$

$$\Updownarrow$$

$$mL^2 \theta'' = M - mgL\sin\theta$$

Example pendulum (cont.)

Non linear system $\theta'' = \frac{1}{mL^2} M - \frac{g}{L} \sin\theta$

stationary point $\theta'' = \frac{1}{mL^2} M_0 - \frac{g}{L} \sin\theta_0 = 0$

thus $\theta_0 = \sin^{-1}\left(\frac{M_0}{mgL}\right)$

Linearization (around stationary point)

$$\theta(t) = \theta_0 + \Delta\theta(t)$$
$$\theta''(t) = 0 + \Delta\theta''(t)$$
$$M(t) = M_0 + \Delta M(t)$$

Example pendulum (cont.)

The nonlinear function: $\theta'' = f(\theta', \theta, t)$

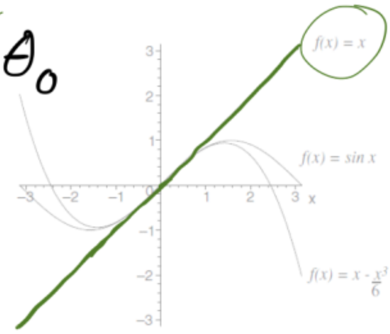
$$f(\theta) \approx f(\theta_0) + \left.\frac{\partial f}{\partial \theta}\right|_{\theta=\theta_0} \underbrace{(\theta - \theta_0)}_{\Delta\theta}$$

there's only one dependency here

$$\Delta\theta'' \approx \frac{1}{mL^2}(M_0 + \Delta M) - \frac{g}{L}\sin\theta_0 - \frac{g}{L}\cos\theta_0 \, \Delta\theta$$

@ stationary point

$$\Delta\theta'' = \frac{1}{mL^2}\Delta M - \frac{g}{L}\cos\theta_0 \, \Delta\theta$$



Example pendulum (cont.)

Compare to Kreyszig ($y_1 = \Delta\theta$, $y_2 = \Delta\theta'$)

$$y_1' = f_1(y_1, y_2) = y_2 \qquad \underline{\text{CASE } M=0}$$

$$y_2' = f_2(y_1, y_2) = -\frac{g}{L}\cos\theta_0 \cdot y_1$$

Critical points in nonlinear function occur for $y_1' = y_2 = \theta' = 0$ and $y_2' = \theta'' = -\frac{g}{L}\sin\theta_0 = 0 \iff \theta_0 = n\pi,\ n \in \mathbb{Z}$.

Example pendulum (cont.)

When we consider n even:

$$\bar{Y}' = \bar{\bar{A}}\bar{Y} = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix} \bar{Y}$$

n even ∇_0
$\cos\theta_0 = 1$

We get

$$p = \text{trace}(\bar{\bar{A}}) = 0$$
$$q = \det(\bar{\bar{A}}) = g/L > 0$$
$$D = p^2 - 4q = -4g/L < 0$$

Stable center

Example pendulum (cont.)

Looking at the points when n is odd we shift ($n=1$): $\tilde{\theta} = \theta - \theta_0$, $\theta_0 = \pi$

$\Delta\tilde{\theta}' = \Delta\theta' = y_2$ still $\Delta\theta$ but around $\theta_0 = \pi$

$\theta_0 = \pi$ gives $\cos\theta_0 = -1$, thus

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix} \bar{y}$$

n odd !

unstable

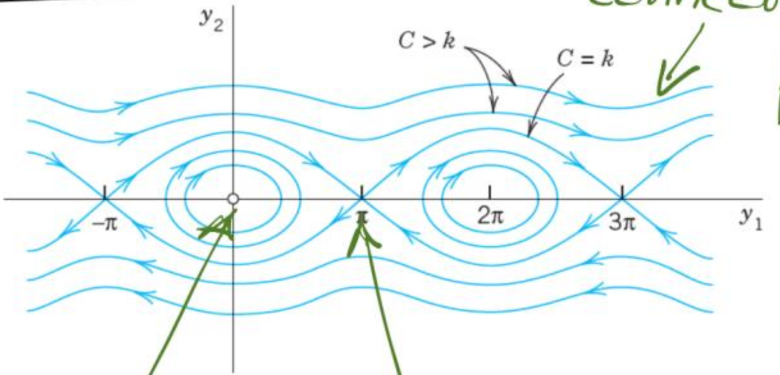
$p = 0$, $q = -g/L < 0$ saddle point

Example pendulum (cont.)

Undamped case

$C \propto$ energy

$k = g/L$



"rotations" — connects critical points (we can fall in any of them)

center with stable osc.

$(C = -k)$

saddle with unstable osc.

$(C = k)$

Example pendulum (cont.)

With damping we get (c damping coefficient)
$c > 0$

$\Delta\theta' = y_1' = f_1(y_1, y_2) = y_2$

$\Delta\theta'' = y_2' = f_2(y_1, y_2) = -g/L \cdot \cos\theta_0 \cdot y_1 - c y_2$

Still, critical points at $(y_1', y_2') = (0, 0)$

<u>n even</u> $\cos\theta_0 = 1$

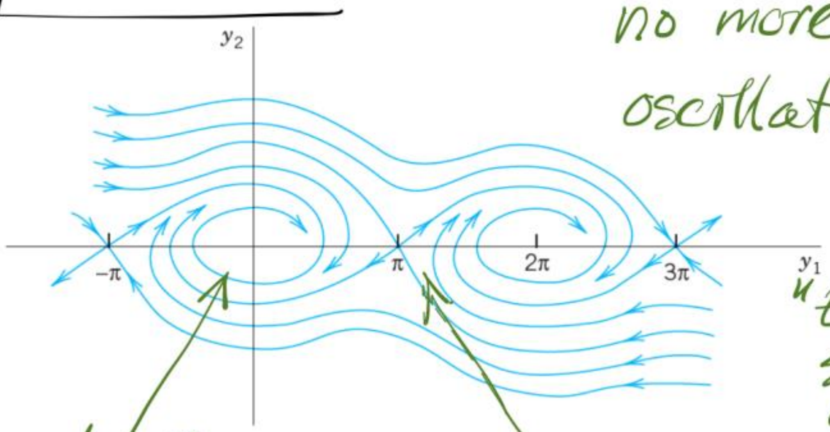
$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ -g/L & -c \end{bmatrix} \bar{y}$$

<u>n odd</u> $\cos\theta_0 = -1$

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ g/L & -c \end{bmatrix} \bar{y}$$

Example pendulum (cont.)

Damped case



no more steady oscillations

"turns into stable decr. osc."

spiral with damped (decreasing) oscillation

$p<0,\ q>0,\ D=c^2-4g/L$

saddle with unstable oscillations, still, but

$p<0,\ q<0$

Nonhomogeneous systems

Systems with a forcing function

$$\bar{y}'(t) = \bar{\bar{A}}(t)\,\bar{y}(t) + \bar{g}(t)$$

the general solution of which is

$$\bar{y}(t) = \bar{y}_h(t) + \bar{y}_p(t)$$

on interval $I: \alpha < t < \beta$

Nonhomogeneous systems (cont.)

If $\bar{\bar{A}}(t) = \bar{\bar{A}}$ (constant coeff.)

$\hookrightarrow$ method of undetermined coef.
(special forcing functions)

If $\bar{\bar{A}}(t)$ and general $\bar{g}(t)$

$\hookrightarrow$ method of variation of parameters

Undetermined coeff. (y_p)

We check $\bar{g}(t)$ for standard form and apply exponential, polynomial, trigonometric function, combinations, ...

In case $\bar{g}$ contains $e^{\lambda t}$, λ being an eigenvalue of $\bar{\bar{A}}$ we try

$$\bar{y}_p = \bar{u}\, t\, e^{\lambda t} + \bar{v}\, e^{\lambda t}$$

standard modification

extra term!

Variation of param. (y_p)

We need to know a (general) solution of the homogeneous system, $\bar{y}_h$

fundamental matrix

$$\bar{y}_h = c_1 \bar{y}_1 + \dots c_n \bar{y}_n$$

$$= [\bar{y}_1, \bar{y}_2, \dots \bar{y}_n] \bar{c} = \bar{\bar{Y}}(t) \bar{c}$$

We replace $\bar{c}$ (constant) by $\bar{u}(t)$

$$\underline{\bar{y}_p = \bar{\bar{Y}}(t) \bar{u}(t)}$$

$W(\bar{\bar{Y}}(t)) \neq 0$ on I
(Wronskian)

Variation of param. (cont.)

Substitution yields ($\bar{y}'(t) = \bar{\bar{A}}(t)\bar{y}(t) + \bar{g}(t)$)

$$\bar{y}_p' = \bar{\bar{Y}}'\bar{u} + \bar{\bar{Y}}\bar{u}' = \bar{\bar{A}}\bar{\bar{Y}}\bar{u} + \bar{g}$$

But $\bar{\bar{Y}}' = \bar{\bar{A}}\bar{\bar{Y}}$, thus

$$\bar{\bar{Y}}\bar{u}' = \bar{g}$$

Since $\det(\bar{\bar{Y}}) \neq 0$ (Wronskian),

$\bar{\bar{Y}}^{-1}$ exists, therefore

$$\bar{\bar{Y}}^{-1}\bar{\bar{Y}}\bar{u}' = \bar{u}' = \bar{\bar{Y}}^{-1}\bar{g}$$

Variation of param. (cont.)

$$\bar{u}' = \bar{\bar{Y}}^{-1}\bar{g} \quad , \quad [t_0 \ldots t] \in I$$

$$\Updownarrow$$

$$\bar{u}(t) = \int_{t_0}^{t} \bar{\bar{Y}}^{-1}(\tau)\bar{g}(\tau)\,d\tau$$

$$\bar{y}(t) = \bar{y}_h(t) + \bar{y}_p(t)$$

$$= \bar{\bar{Y}}(t)\left(\bar{c} + \int_{t_0}^{t} \bar{\bar{Y}}^{-1}(\tau)\bar{g}(\tau)\,d\tau\right)$$

integrate component - wise

Diagonalization

$$\bar{y}'(t) = \bar{\bar{A}}(t)\,\bar{y}(t) + \bar{g}(t)$$

e.g. normal matrix

If $\bar{\bar{A}}$ has an eigenbasis, then $\bar{\bar{D}} = \bar{\bar{X}}^{-1}\bar{\bar{A}}\,\bar{\bar{X}}$ where $\bar{\bar{D}} = \mathrm{diag}(\lambda_1, \lambda_2, \dots \lambda_n)$ and $\bar{\bar{X}}$ is a constant with eigenvectors of $\bar{\bar{A}}$, then defining $\bar{z} = \bar{\bar{X}}^{-1}\bar{y} \Leftrightarrow \bar{y} = \bar{\bar{X}}\,\bar{z}$ we get

$$\bar{y}' = \bar{\bar{X}} z' = \bar{\bar{A}}\,\bar{\bar{X}}\,\bar{z} + \bar{g}$$

$\bar{\bar{X}}$ is a constant

$$\Updownarrow \quad \bar{\bar{X}}^{-1}\bar{\bar{X}}\,\bar{z}' = \bar{\bar{X}}^{-1}\bar{\bar{A}}\,\bar{\bar{X}}\,\bar{z} + \bar{\bar{X}}^{-1}\bar{g}$$

$$\Updownarrow \quad \bar{z}' = \bar{\bar{D}}\,\bar{z} + \bar{X}^{-1}\bar{g}$$

Diagonalization (cont.)

or, with $\bar{h} = \bar{\bar{X}}^{-1}\bar{g}$,

$$\bar{z}'(t) = \bar{\bar{D}}\,\bar{z}(t) + \bar{h}(t)$$

$$z_j'(t) = \lambda_j z_j(t) + h_j(t), \quad 1 \leq j \leq n$$

first order linear ("Panserformel")

$(\bar{y} = \bar{\bar{X}}\bar{z})$

$$m_j(t) = \int p(t)\,dt = \int -\lambda_j\,dt = -\lambda_j t$$

$$z_j(t) = e^{\lambda_j t}\int e^{-\lambda_j t} h_j(t)\,dt + c_j e^{\lambda_j t}$$

Linear state space models

$$\bar{x}'(t) = \bar{\bar{A}}\,\bar{x}(t) + \bar{\bar{B}}\,u(t)$$

state   input

$$y(t) = \bar{C}\,\bar{x}(t) + D\,u(t)$$

output   (feedthrough)

Linear time-invariant (autonomous) first order vector differential equ. with initial condition $\bar{x}(t) = \bar{x}_0$.

Matrix exp function

$$\exp(\bar{\bar{A}}) = e^{\bar{\bar{A}}} = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{\bar{A}}^n \quad \left(\frac{1}{0!} \bar{\bar{A}}^0 = \bar{\bar{I}} \right)_0^?$$

If $\bar{\bar{X}}$ is invertible (eigenbasis of $\bar{\bar{A}}$)

$$\bar{\bar{A}} = \bar{\bar{X}}^{-1} \bar{\bar{D}} \bar{\bar{X}}, \quad \bar{\bar{D}} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

$$\bar{\bar{A}}^n = \bar{\bar{X}}^{-1} \bar{\bar{D}}^n \bar{\bar{X}}$$

$$\Downarrow$$

$$e^{\bar{\bar{A}}} = \bar{\bar{X}}^{-1} e^{\bar{\bar{D}}} \bar{\bar{X}}$$

Matrix exponential (cont.)

$$\bar{\bar{X}}(t) = \bar{\bar{A}}t \quad \Rightarrow$$

$$e^{\bar{\bar{X}}(t)} = e^{\bar{\bar{A}}t} = \sum_{n=0}^{\infty} \frac{\bar{\bar{A}}^n}{n!} t^n$$

$$\Downarrow$$

$$\frac{d e^{\bar{\bar{A}}t}}{dt} = \bar{\bar{A}} e^{\bar{\bar{A}}t} = e^{\bar{\bar{A}}t} \bar{\bar{A}} \quad \text{as usual}$$

Also, $e^{-\bar{\bar{X}}} e^{\bar{\bar{X}}} = \bar{\bar{X}}^0 = \bar{\bar{I}}$

Solution for state vector

$$\bar{x}'(t) = \bar{\bar{A}}\,\bar{x}(t) + \bar{\bar{B}}\,u(t) \;;\; \bar{y}(t) = \bar{\bar{C}}\,\bar{x}(t) + \bar{\bar{D}}u(t)$$

$e^{-\bar{\bar{A}}t}\cdot$ $\Updownarrow$

$$\underbrace{e^{-\bar{\bar{A}}t}\,\bar{x}'(t) - e^{-\bar{\bar{A}}t}\,\bar{\bar{A}}\,\bar{x}(t)}_{\text{chain rule}} = e^{-\bar{\bar{A}}t}\,\bar{\bar{B}}\,u(t)$$

$\Updownarrow$

$$\frac{d}{dt}\left(e^{-\bar{\bar{A}}t}\,\bar{x}(t)\right) = e^{-\bar{\bar{A}}t}\,\bar{\bar{B}}\,u(t)$$

$\Updownarrow$

$$\int_0^t d\left(e^{-\bar{\bar{A}}\tau}\,\bar{x}(\tau)\right) = \int_0^t e^{-\bar{\bar{A}}\tau}\,\bar{\bar{B}}\,u(\tau)\,d\tau$$

Solution for state vector (cont.)

$$\int_0^t d\left(e^{-\bar{\bar{A}}\tau}\bar{x}(\tau)\right) = e^{-\bar{\bar{A}}t}\bar{x}(t) - \bar{\bar{I}}\bar{x}_0$$

$$= \int_0^t e^{-\bar{\bar{A}}\tau}\bar{\bar{B}}u(\tau)d\tau$$

$$\Updownarrow$$

$$\bar{x}(t) = e^{\bar{\bar{A}}t}\left(\bar{x}_0 + \int_0^t e^{-\bar{\bar{A}}\tau}\bar{\bar{B}}u(\tau)d\tau\right)$$

$$= e^{\bar{\bar{A}}t}\bar{x}_0 + \int_0^t e^{\bar{\bar{A}}(t-\tau)}\bar{\bar{B}}u(\tau)d\tau$$

Impulse response

$$y(t) = \bar{c}\left(e^{\bar{\bar{A}}t}\bar{x}_0 + \int_0^t e^{\bar{\bar{A}}(t-\tau)}\bar{\bar{B}}\,u(\tau)\,d\tau + Du(t)\right)$$

The system's response is generally

$$y(t) = h(t) * u(t) = \int_{-\infty}^{t} h(t-\tau)u(\tau)\,d\tau$$

where $h(t)$ is the system's impulse response (to input $\delta(t)$): "by inspection"

$$h(t) = \bar{c}\,e^{\bar{\bar{A}}t}\bar{\bar{B}} + D\,\delta(t),\ t \geq 0$$

Numerically, we may approximate $e^{\bar{\bar{A}}t}$ and integrate (convolve)

Discrete time signals - Sequences

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- Discrete signals can be represented as a sequence of numbers

$$x = \{x[n]\} \qquad -\infty < n < \infty$$

where n is an integer.

- In case such sequences arise from periodic sampling of an analog signal:

$$x[n] = x_a[nT_s] \qquad -\infty < n < \infty$$

where Ts is the *sampling interval* and $f_s=1/T_s$ is the *sampling frequency*.

6

Discrete time signals - Sequences

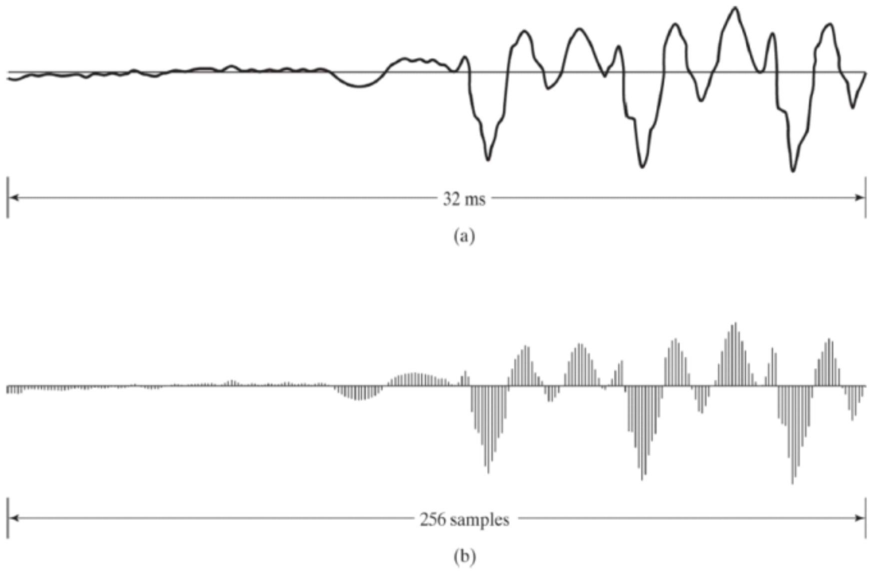


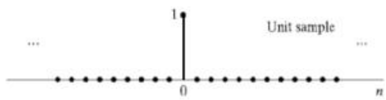
Fig 2 (13) (a) Segment of a continuous-time speech signal $x_a(t)$.
(b) Sequence of samples $x[n] = x_a(nT_s)$ obtained from the signal in part (a) with $T_s = 125\ \mu s$.

Basic sequences and operations

- y[n] is said to be a delayed (or shifted) version of the sequence x[n] if
  $y[n]=x[n-n_0]$, with n_0 integer

- The unit sample sequence is defined as = unit pulse

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases}$$



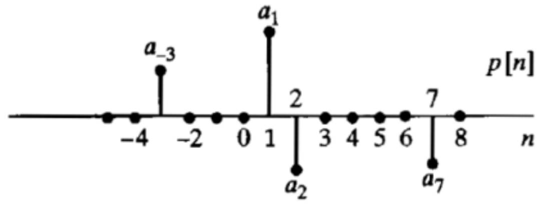
- An example of delayed unit sample sequence:

$$\delta[n-2] = \begin{cases} 0, & n \neq 2 \\ 1, & n = 2 \end{cases}$$

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} = \text{unit step!}$$

Basic sequences and operations

- An arbitrary sequence can be represented as a sum of scaled, delayed, impulses.



$$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_7\delta[n-7].$$

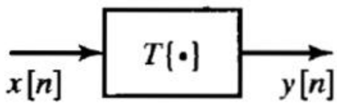
- More generally, any sequence can be expressed as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Discrete time systems

- A discrete-time system is an operator that maps an input sequence x[n] to an output sequence y[n]

$$y[n] = T\{x[n]\}$$



- Examples of operators:
  - Delay $y[n] = x[n - n_{delay}]$ $-\infty < n < \infty$
  - Moving average $y[n] = \frac{1}{M1 + M2 + 1} \sum_{k=-M1}^{M2} x[n-k]$
  - FIR filter $y[n] = \frac{1}{\sum_{k=0}^{M} b_k} \sum_{k=0}^{M} b_k \cdot x[n-k]$

Linear discrete time systems

- The class of linear system is defined by the principle of superposition.

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$
$$T\{a \cdot x[n]\} = a \cdot T\{x[n]\} = a \cdot y[n]$$

$$T\{a \cdot x_1[n] + b \cdot x_2[n]\} = a \cdot T\{x_1[n]\} + b \cdot T\{x_2[n]\} = a \cdot y_1[n] + b \cdot y_2[n]$$

Linear discrete time systems

- The accumulator system

$$y[n] = \sum_{k=-\infty}^{n} x[k] \quad \Rightarrow \quad \text{Linear system}$$

- Consider the following

$$w[n] = \log_{10}(|x[n]|). \quad \Rightarrow \quad \text{Non-Linear system}$$

Time invariant discrete systems

- A time invariant system is a system for which a time delay/shift of the input sequence causes a corresponding shift in the output sequence.

$$x_1[n] = x[n - n_0] \Rightarrow y_1[n] = y[n - n_0].$$

- The accumulator is a time invariant system.
- Compressor is a non-time invariant system

$$y[n] = x[Mn], \qquad -\infty < n < \infty,$$

Causal discrete time systems

- A system is causal if, for every choice of n_0, the output sequence at the index $n=n_0$ depends only on the input sequence values for $n<=n_0$.

- Forward difference system Afhængen af noget der kommer til at ske

$$y[n] = x[n + 1] - x[n]. \Rightarrow \text{Non- causal}$$

- Backward difference system

$$y[n] = x[n] - x[n - 1], \Rightarrow \text{Causal}$$

Back in time / noget sket

Stable discrete time systems

- A system is stable if and only if every bounded input sequence produces a bounded output sequence.

$$|x[n]| \le B_x < \infty, \quad \text{for all } n. \quad \Rightarrow \quad |y[n]| \le B_y < \infty, \quad \text{for all } n.$$

- Examples

$$y[n] = \sum_{k=-\infty}^{n} u[k] \quad \Rightarrow \quad \text{Not stable}$$

$$y[n] = x[n-n_d], \quad \Rightarrow \quad \text{Stable}$$

$$y[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x[n-k] \quad \Rightarrow \quad \text{Stable}$$

What about $y[n] = \log_{10}(|x[n]|)$?

Linear time invariant systems

- Linear systems → principle of superposition
- General sequence can be expressed as a linear combination of delayed and scaled unit pulses → a linear system can be completely characterized by its impulse response.

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n].$$

- If only linearity is imposed, $h_k[n]$ depends on both *k* and *n*.
- Time invariance: if h[n] is the response to delta[n], then the response to δ[n-k] is h[n-k].

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

convolution sum

$$y[n] = x[n] * h[n].$$

Linear time invariant systems

- Convolution operation is commutative

$$x[n] * h[n] = h[n] * x[n].$$

- Convolution operation distributes over addition

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

Linear time invariant systems

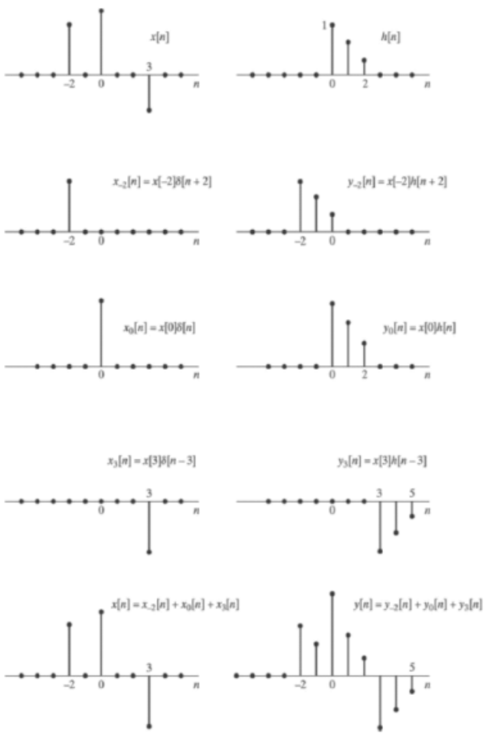


Fig 8 (27)

Representation of the output of an LTI system as the superposition of responses to individual samples of the input.

$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad \text{for all } n.$$

Linear time invariant systems

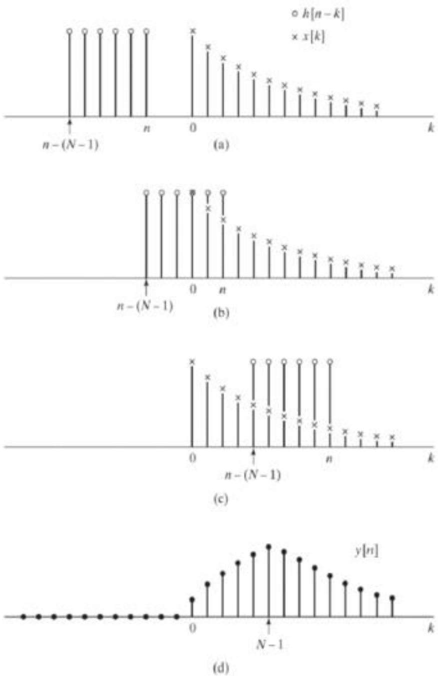


Fig 10 (30)

Sequence involved in computing a discrete convolution.

(a) – (c): The sequences $x[k]$ and $h[n-k]$ as a function of k for different values of n. (Only nonzero samples are shown.)

(d): Corresponding output sequence as a function of n.

h[n] = u[n] – u[n-N]

$$x[n] = \begin{cases} a^n, for\ n \geq 0 \\ 0, for\ n < 0 \end{cases}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad \text{for all } n.$$

Fourier transform

- The Fourier transform of the function *x(t)* is defined as

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

- The inverse Fourier transform is defined as

$$x(t) = \int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

Fourier transform

Physical interpretation: spectrum of a signal

Fourier transform

Time domain $\Rightarrow$ Frequency domain

Inverse Fourier transform

Time domain $\Leftarrow$ Frequency domain

- The Fourier transform returns the amplitude and phase of sinusoidal signals at the different frequencies that compose the time domain signal.
- The Inverse Fourier transform returns the time domain signal which is composed of the sinusoidal signals at different amplitude and phases of the frequency domain representation.

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Fourier transform

- What about the spectrum of a discrete signal, obtained by sampling x(t)?

time

frequency $X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \Rightarrow$??

- It can be shown that the spectrum of a discrete time signal can be calculated as

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Discrete time Fourier transform (DTFT)

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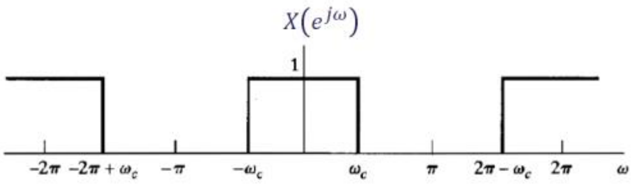
Fourier transform

- A common notation for $X(\omega)$ is $X\left(e^{j\omega}\right)$.
- $X\left(e^{j\omega}\right)$ is periodic of period 2π → periodic spectrum



- Since $X\left(e^{j\omega}\right)$ is 2π – periodic, the inverse Fourier transform can be calculated by integrating over a single period, e.g. [- π, π].

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

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Fourier transform

- The frequency response of a linear time invariant system is the Fourier transform of the impulse response,

$$\mathrm{H}(\omega) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}$$

$$\Downarrow$$

$$\mathrm{h}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

27

Fourier transform

- What is the condition of existence of the Fourier transform?

Determining the class of signals that can be represented by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega,$$

is equivalent to considering the convergence of the infinite sum in $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$

$$|X(e^{j\omega})| < \infty \qquad \text{for all } \omega, \quad \Rightarrow \quad |X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]|\,|e^{-j\omega n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

If x[n] is absolutely summable, than the Fourier transform exists.

Fourier transform

- Example: does the Fourier transform of the following sequence exist?

Let $x[n] = a^n u[n]$. The Fourier transform of this sequence is

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1-ae^{-j\omega}} \qquad \text{if } |ae^{-j\omega}| < 1 \quad \text{or} \quad |a| < 1.$$

Clearly, the condition $|a| < 1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty \qquad \text{if } |a| < 1. \tag{2.140}$$

Udregninger til LEC 8

Thursday, May 30, 2024 8:15 AM

$h[n] = 3n\mathcal{U}[n-1]$

$x[N] = 5\mathcal{U}[n]$

$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$

Rewrite $h[n-k]$

$h[n] = 3n\mathcal{U}[n-k]$

$h[n-k] = 3(n-k)\mathcal{U}[n-1-k]$

given that

$\mathcal{U}[n-k-1] = 0$, for $n-k-1 < 0$

And as it is an impulse respones meaning $k > n-1$ we need to sum for $k=0$ to $k=n-1$

Rewrite $x(n)$

$x(n) = 5\mathcal{U}[n]$

$x(k) = 5\mathcal{U}(k)$

for $k \geq 0$

So for $y(n)$

$y(n) = x(n) * h(n) = \sum_{k=0}^{n-1} x(k)\, h(n-k)$

$= \sum_{k=0}^{n-1} 5(k) \cdot 3n\mathcal{U}[n-1-k]$

$= 15 \sum_{k=0}^{n-1} (k) n\mathcal{U}[n-1-k]$

$$\boxed{\mathcal{U}[n] \begin{Bmatrix} 1, n \geq 0 \\ 0, n < 0 \end{Bmatrix}}$$

$n(n+1)$

$15 \dfrac{n(n+1)}{}$

$$15\,\frac{n(n+1)}{2}$$

$$\frac{15(n(n+1)}{2}$$

$$\frac{15}{2}n^2 + \frac{15}{n}n$$

$$\frac{15}{2}(3n) - \frac{15}{n}$$

$$-\frac{15}{n}u[n] + \frac{15}{2}3n\,u[n] \quad \text{for } n>0$$

$$h[n] = 3^n u(n-1) \rightarrow h[n-k] = 3^{n-k} u(n-k-1)$$

$$x[n] = 5u(n) \rightarrow 5u(k)$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x(k)\, h[n-k]$$

$$= \sum_{k-\infty}^{\infty} 5u(k)\, 3^{n-k} u(n-k-1)$$

$$h(n) = 3^n u(n-1)$$

$$h(n-k) = 3^{n-k} u(n-k-1)$$

$$3^n\left(\frac{1}{3}\right)^n$$

$$yn = 5\sum_{k=0}^{n-1} 3^{n-k}$$

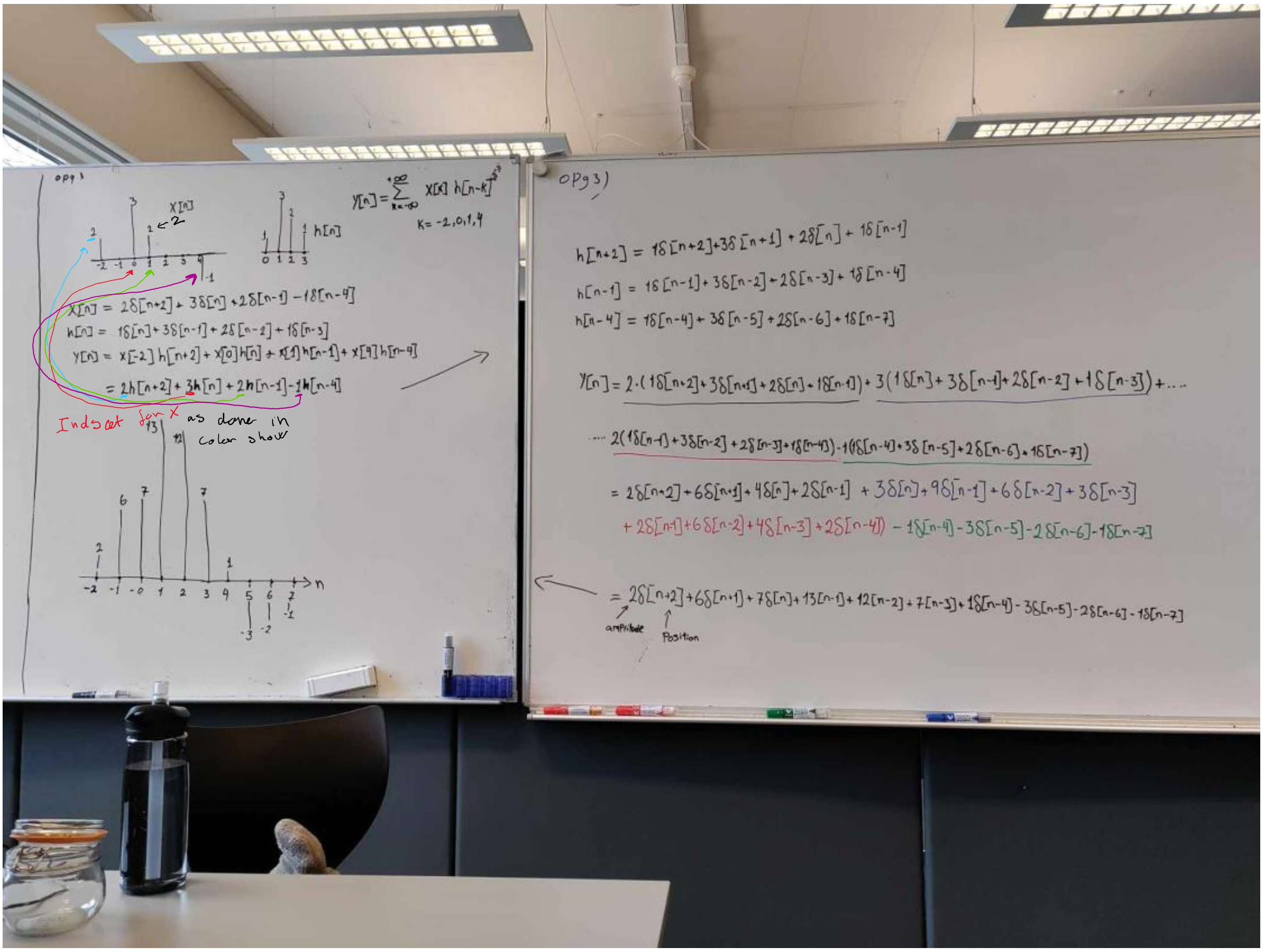
$$3^n \frac{1^n}{3^n} \Rightarrow 3n\frac{1}{3^n}$$

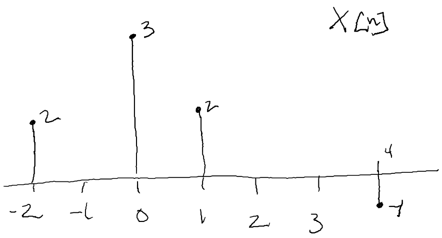
$$yn = 5\cdot 3^n \sum^{n-1}\left(\frac{1}{3}\right)^k$$

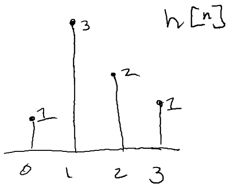
$$y_n = 5\cdot 3^n \sum^{n-1} \left(\frac{1}{3}\right)^k$$

$$3^n \frac{1}{3^n} \Rightarrow 3n \frac{1}{3^n}$$

$$\frac{3^n}{3^n} = 1$$

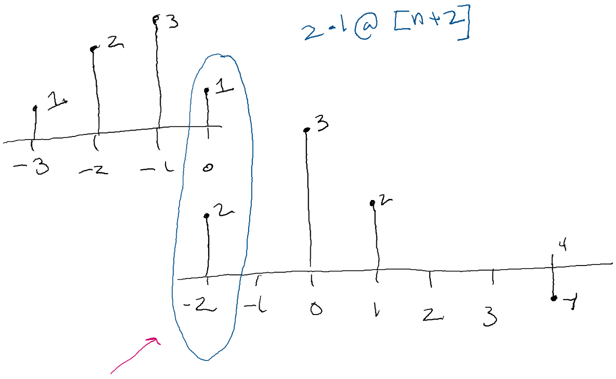






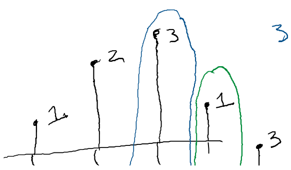
— Now for the convoluting —

$2 \cdot 1 @ [n+2]$



$= 2 \cdot 1 [n+2]$

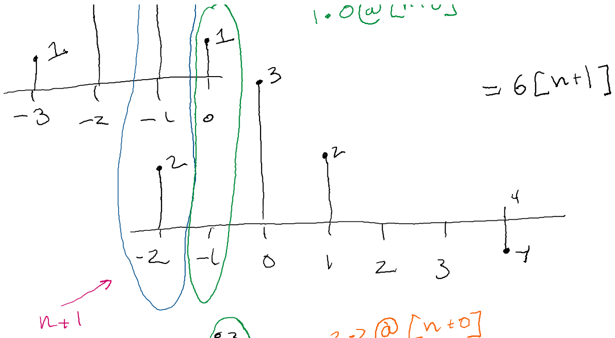
n+2



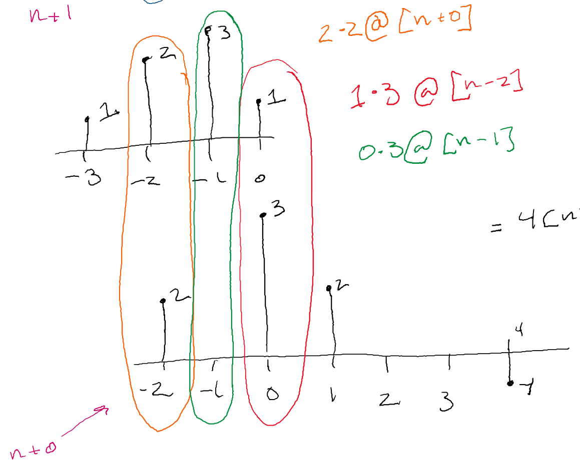
$3 \cdot 2 @ [n+1]$

$1 \cdot 0 @ [n+0]$

$= 6 [n+1]$



$= 6[n+1]$



$2 \cdot 2 @ [n+0]$

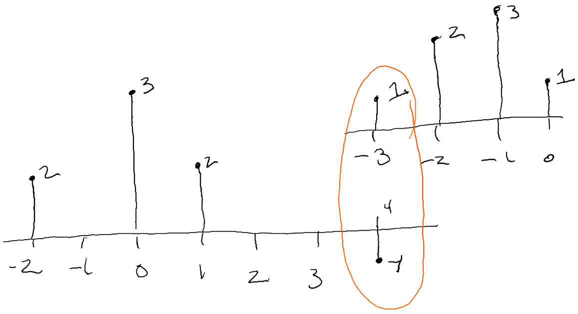
$1 \cdot 3 @ [n-2]$

$0 \cdot 3 @ [n-1]$

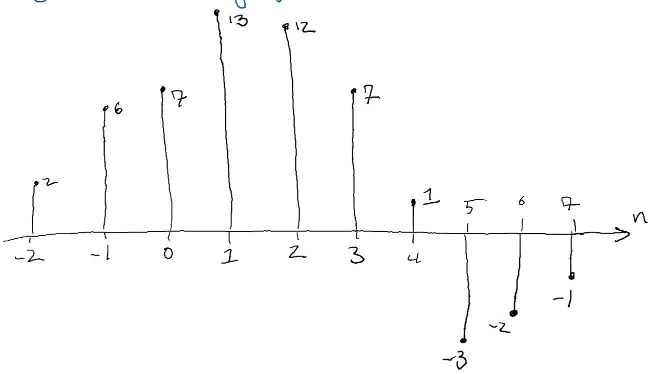
$= 4[n] + 3[n-2]$

OSV OSV OSV indtil man er <u>her!</u>

$1 \cdot 1 [n-7]$



Så ligger man alle ens $[n-?]$ sammen i størrelse og for!



$$\mathrm{Len}(y[n]) = \big(\mathrm{len}(h[n]) + \mathrm{len}(x[n])\big) - 1$$

$$H(z) = \frac{1-3z^{-1}}{2-4z^{-1}} = \frac{1}{2-4z^{-1}} - 3\frac{z^{-1}}{2-4z^{-1}}$$

$$y(n) - 3y[n-1] = 5(x)[n-2]$$

$$y(n) = 5(x)[n-2] + 3y[n-1]$$

$$y[n] - 3y[n-1] = 5x(n-2].h[x] \Rightarrow \frac{1}{h(x)} = \frac{y(n)}{x(n)}$$

$$z^0 y(z) - 3y(z)^{-1} = 5x(z)^{-2}$$

$$1 - 3y(z)^{-1} = 5x(z)z^{-1}$$

$$H(z) = \frac{5z^{-1}}{1-3z^{-1}}$$

$$y(n) - 3y(n-1) = 5x(n-2)$$

$$Y(z) - 3Y(z)z^{-1} = 5X(z)z^{-2}$$

$$Y(z)(1-3z^{-1}) = 5X(z)z^{-2}$$

$$Y(z) = \quad 5X(z)z^{-2}$$

What we have learned in the previous lecture

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- The Discrete Time Fourier transform of a sequence x[n] is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$ ω is the radian frequency

- The Fourier transform determines how much of each frequency component is required to synthetize x[n].

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$ Inverse Fourier transform

- If x[n] is absolutely summable, then the Fourier transform exists.

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

2

The z-transform

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- The z-transform of a sequence x[n] is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

- The z-transform operator transforms the sequence x[n] into the function X[z], where z is a continuous complex variable.
- The z-transform reduces to the Fourier transform if $z = e^{j\omega}$
- More generally, the complex variable z can be expressed as $z = re^{j\omega}$.

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n},$$

$$\Downarrow$$

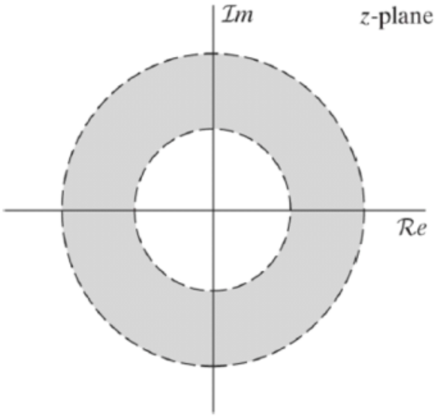
$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}.$$

4

The z-transform

- The z-transform does not converge for all sequences or all values of *z.*
- The set of values of z for which the z-tranform converges is called the region of convergence (ROC).

$$\sum_{n=-\infty}^{\infty} |x[n]| \cdot |z|^{-n} < \infty$$



The ROC is a ring in the *z*-plane

The z-transform

- There is possibility that the z-transform converges even if Fourier transform diverges

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot r^{-n} \cdot e^{-j\omega n}$$

$\sum_{n=-\infty}^{\infty} |x[n] \cdot r^{-n}|$ converges even if $\sum_{n=-\infty}^{\infty} |x[n]|$ diverges

The z-transform

- The z-trasform is most useful when the infinite sum can be expressed in closed form.
- Among the most important and useful z-transforms, are those for which X(z) is a rational function inside the ROC, i.e.

$$X(z) = \frac{P(z)}{Q(z)},$$

where P(z) and Q(z) are polynomials in z.

- The values of z for which X(z)=0 are called the zeros of X(z), while the values of z for which Q(z)=0 are the poles of X(z).

The z-transform

- Right-sided exponential sequence

What is the ROC for the z-transform of $x[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$, $-\infty < n < \infty$?

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

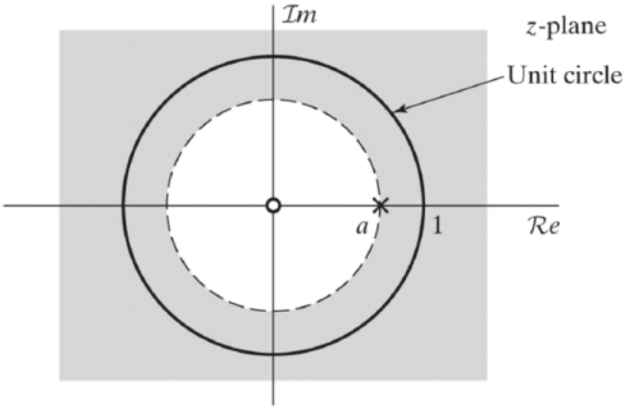
For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

Thus, the ROC is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Inside the ROC, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|. \tag{12}$$

The z-transform



$$x[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$$

$$a = |a| = \left(\frac{1}{2}\right)$$

- For $|a| > 1$, the ROC does not contain the unit circle → Fourier transform does not exist

The z-transform

- Left-sided exponential sequence

What is the ROC for the z-transform of $x[n] = -\left(\frac{3}{2}\right)^n \cdot u[-n-1]$? $-\infty < n < \infty$?

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n \le -1 \\ 0 & n > -1. \end{cases}$$

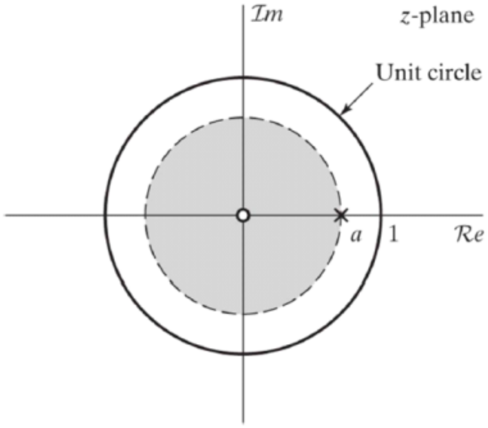
Since the sequence is nonzero only for $n \le -1$, this is a *left-sided* sequence. The z-transform in this case is

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n. \end{aligned} \quad (15)$$

If $|a^{-1}z| < 1$ or, equivalently, $|z| < |a|$, the last sum in Eq. (15) converges, and using again the formula for the sum of terms in a geometric series,

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|. \quad (16)$$

The z-transform



$$x[n] = -\left(\frac{3}{2}\right)^n \cdot u[-n-1]$$

$$a = \left(\frac{3}{2}\right)$$

- For $|a| < 1$, the sequence grows exponentially and therefore the Fourier transform does not exist.

The z-transform

- Sum of two exponential sequences

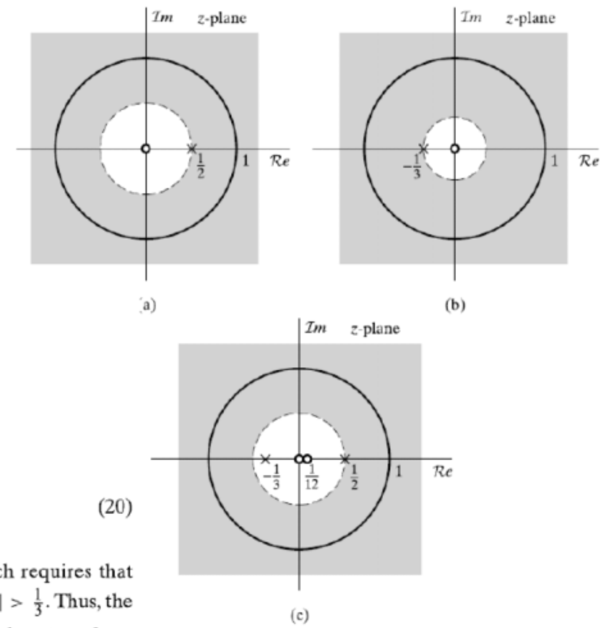
Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n].$$

The z-transform is

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n \\
&= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} = \frac{2\left(1 - \frac{1}{12} z^{-1}\right)}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{3} z^{-1}\right)} \\
&= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}. \qquad (20)
\end{aligned}$$

For convergence of $X(z)$, both sums in Eq. (19) must converge, which requires that both $\left|\frac{1}{2} z^{-1}\right| < 1$ and $\left|\left(-\frac{1}{3}\right) z^{-1}\right| < 1$ or, equivalently, $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3}$. Thus, the ROC is the region of overlap, $|z| > \frac{1}{2}$. The pole–zero plot and ROC for the z-transform of each of the individual terms and for the combined signal are shown in Figure 5.



The z-transform

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- Two-sided exponential sequences

Consider the sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]. \qquad (24)$$

Note that this sequence grows exponentially as $n \to -\infty$. Using the general result of Example 1 with $a = -\frac{1}{3}$, we obtain

$$\left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3},$$

and using the result of Example 2 with $a = \frac{1}{2}$ yields

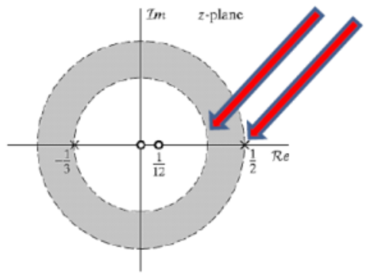
$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}.$$

Thus, by the linearity of the z-transform,

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad \frac{1}{3} < |z| \text{ and } |z| < \frac{1}{2},$$

$$= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}.$$

Since the ROC does not contain the unit circle, the sequence in Eq. (24) does not have a Fourier transform.



The z-transform

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- What is the z-transform of $x[n] = \begin{cases} 0{,}8^n, & for\ 0 \ll n \ll N-1 \\ 0, & ellers \end{cases}$??

$$\sum_{n=0}^{\infty}(az^{-1})^n - \sum_{n=N}^{\infty}(az^{-1})^n$$

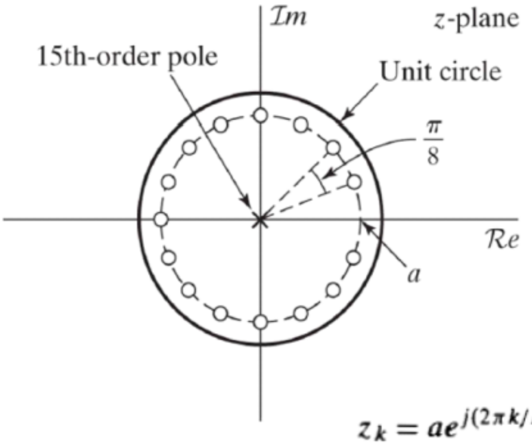
$$\sum_{n=0}^{\infty}(az^{-1})^n - (az^{-1})^N \cdot \sum_{n=0}^{\infty}(az^{-1})^n$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1}(az^{-1})^n$$
$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}}\frac{z^N - a^N}{z-a}, \qquad (3.23)$$

where we have used the general formula in Eq. (2.56) to sum the finite series. The ROC is determined by the set of values of z for which

$$\sum_{n=0}^{N-1}|az^{-1}|^n < \infty.$$

The z-transform



Pole–zero plot
N = 16 and a = 0,8

The ROC in this example consists of all values of z except z = 0.

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 1, \ldots, N-1.$$

The z-transform

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

The z-transform

PROPERTY 1: The ROC will either be of the form $0 \le r_R < |z|$, or $|z| < r_L \le \infty$, or, in general the annulus, i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

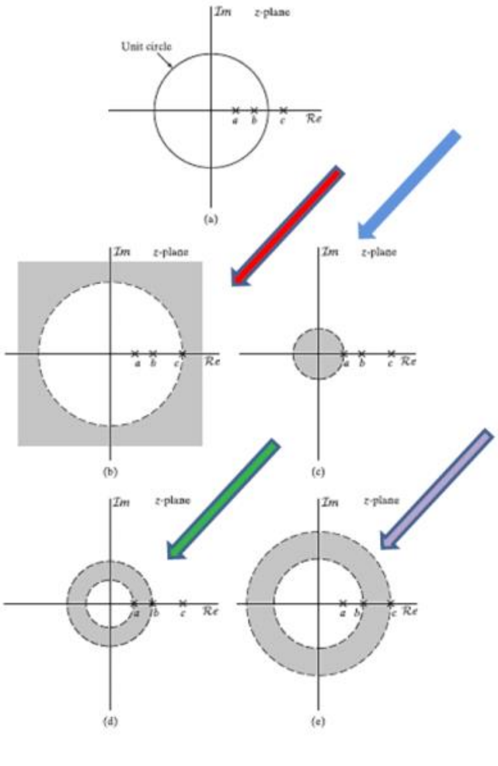
PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

The z-transform



Examples of four z-transforms with the same pole–zero locations, illustrating the different possibilities for the ROC, each of which corresponds to a different sequence:
(b) to a right-sided sequence,
(c) to a left-sided sequence,
(d) to a two-sided sequence, and
(e) to a two-sided sequence.

The z-transform

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

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The z-transform

Convolution in time ←→ multiplication in frequency domain

$$y[n]=\sum_{k=-\infty}^{\infty} x_1[k]\cdot x_2[n-k] \quad eller \quad y[n]=\sum_{k=-\infty}^{\infty} x_2[k]\cdot x_1[n-k] \quad for\ alle\ n$$

(Foldning er kommutativ)

$$y[n] = (x_1 * x_2)[n] = \sum_{k=-\infty}^{\infty} x_1[k]\cdot x_2[n-k]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]\cdot z^{-n} = \sum_{n=-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]\right\} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]\, z^{-n}$$

Ved substitution: $m = n-k$,

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k]\left\{\sum_{m=-\infty}^{\infty} x_2[m]\cdot z^{-m}\right\} z^{-k}$$

$$= X_1(z)\cdot X_2(z)$$

Altså:

$$x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1(z)\cdot X_2(z)$$

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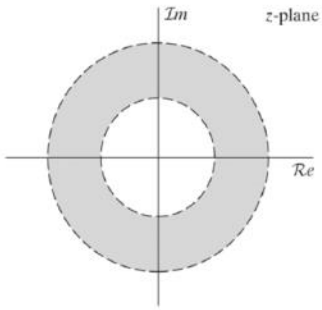
LEC 10 - Inverse Z-transform

Wednesday, 29 May 2024 13.35

What we have learned in the previous lecture

Z- transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$



The set of values of z for which the z-tranform converges is called the region of convergence (ROC).

We have calculated z-trasform and ROC of
- Right sided exponential sequence
- Left Sided exponential sequence
- Sum of exponential sequences

We have studied the properties of the ROC, and common z-transform pairs.

Inverse z-transform

- Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} \qquad \begin{array}{l} z \in C \\ ROC\colon \{z |\, X(z) < \infty\} \end{array}$$

- Inverse z-transform:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz,$$

where C is a closed circle around the origin which includes all the poles for X(z). In case C is the unit circle:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (X(e^{i\omega}) \cdot e^{i\omega n}) d\omega$$

Inverse z-transform

- Less formal procedure than the Cauchy-Schwartz integral are preferable.
- **Inspection method:** ”recognizing” certain transform pairs

$$x[n] = \underline{(a)^n \cdot u[n]} \quad -\infty < n < \infty \quad \xleftrightarrow{Z} \quad X(z)_h = \sum_{n=0}^{\infty} a^n \cdot z^{-n} = \frac{1}{1-a\cdot z^{-1}}; \quad \underline{|a| < |z|}$$

$$x[n] = \underline{-(a)^n \cdot u[-n-1]} \quad -\infty < n < \infty \xleftrightarrow{Z} X(z)_v = -\sum_{n=1}^{\infty} a^{-n} \cdot z^{n} = \frac{1}{1-a\cdot z^{-1}}; \quad \underline{|z| < |a|}$$

- Example

$$X(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right); \quad |z| > \frac{1}{2}$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Inverse z-transform

- Less formal procedure than the Cauchy-Schwartz integral are preferable.
- **Inspection method:** ”recognizing” certain transform pairs

$$x[n] = \underline{(a)^n \cdot u[n]} \quad -\infty < n < \infty \quad \xleftrightarrow{Z} \quad X(z)_h = \sum_{n=0}^{\infty} a^n \cdot z^{-n} = \frac{1}{1-a\cdot z^{-1}}; \quad \underline{|a| < |z|}$$

$$x[n] = \underline{-(a)^n \cdot u[-n-1]} \quad -\infty < n < \infty \xleftrightarrow{Z} X(z)_v = -\sum_{n=1}^{\infty} a^{-n} \cdot z^{n} = \frac{1}{1-a\cdot z^{-1}}; \quad \underline{|z| < |a|}$$

- Example

$$X(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right); \quad |z| > \frac{1}{2}$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

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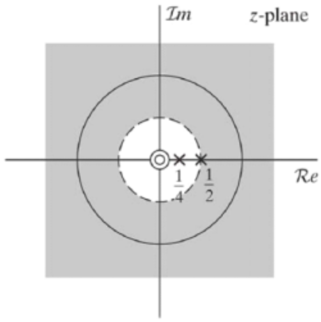
Inverse z-transform

- Example



$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}.$$

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

$$A_1 = \left(1 - \frac{1}{4}z^{-1}\right) X(z)\Big|_{z=1/4} = -1,$$

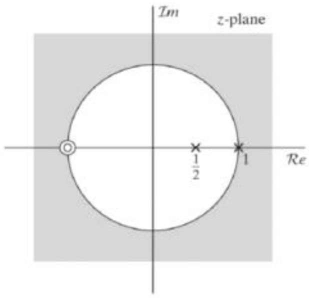
$$A_2 = \left(1 - \frac{1}{2}z^{-1}\right) X(z)\Big|_{z=1/2} = 2.$$

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n].$$

Inverse z-transform

- Example



$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}, \qquad |z| > 1.$$

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

$$\begin{array}{r|l} & 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & z^{-2} - 3z^{-1} + 2 \\ \hline & 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}.$$

$$A_1 = \left[\left(\frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)\left(1 - \frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(\frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)(1 - z^{-1})\right]_{z=1} = 8.$$

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

$$F(s) = \frac{b(s)}{a(s)} = \frac{-4s+8}{s^2+6s+8}.$$

```
b = [-4 8];
a = [1 6 8];
[r,p,k] = residue(b,a)
```

Get

```
r = 2×1

   -12
     8

p = 2×1

    -4
    -2

k =

     []
```

This represents the partial fraction expansion

$$\frac{-4s+8}{s^2+6s+8} = \frac{-12}{s+4} + \frac{8}{s+2}.$$

Partial defrac in matlab.

$$X(z) = 5 + \frac{1}{1-2z^{-1}} - \frac{3}{1-\frac{1}{2}z^{-1}}$$

$$X(z) = 5 + \frac{1}{1-2z^{-1}} - 3\frac{1}{1-\frac{1}{2}z^{-1}}$$

For ROC |z|

$\frac{1}{2} < |z| < 2$

$5\,\delta[n]$

$|z| < |a|$ as it is largest

$-2^n u[-n-1]$

$\frac{1}{2} > |z|$ as it is smallest

$3\,\frac{1}{2}^n u[n]$

$$x = 5\,\delta[n] - 2^n u[-n-1] - 3\left(\frac{1}{2}\right)^n u[n]$$

$$X = \frac{2z^{-1}}{1-z^{-1}} + \frac{z^3}{1-\frac{4}{7}z^{-1}} \qquad \text{ROC } |z| > 1$$

$$2\frac{z^{1}}{1-z^{-1}} \Rightarrow x = 2u[n-1] + \left(\frac{4}{7}\right)^{n+3} u[n+3]$$

$$X = \frac{3}{1-2z^{-1}} + \frac{z^{-1}}{1-z^{-1}} \qquad \text{ROC } |z| < 1$$

$$-na^n u[-n-1] \quad \& \quad -a^n u[-n-1]$$

$$-3(2)^n u[-n-1] - 1^n u[-n-1]$$

$$x = -3(2)^n u[-n-1] - u[-n-1]$$

+ SOLVE y(n)

Friday, 31 May 2024 15.29

$$\boxed{y(n) = h(n) * x(n) \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)}$$

Basis formel! Husk den!

$h(n) = 3^n u(n-1)$

$x(n) = 5u(n)$

Basis ligningen som man for givet

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

rewrite for K

$x(n) = 5u[n]$

$x(k) = 5u(k)$ ✓

$h(n) = 3^n u(n-1)$

$h(n-k) = 3^{n-k} u(n-k-1)$ ✓

Boundaries

$K = 0$, as step funktion

$n-1$, is upper limit

$$A) \quad u\begin{cases} 1, & n \geq 1 \\ 0, & n < 0 \end{cases}$$

$$\sum_{k=0}^{n-1} 5\,u(k)\cdot 3^{n-k}\,(n-k-1)$$

$$\frac{3^n}{3^k} = 3^n \frac{1}{3^k} = \frac{1^k}{3^k} = \left(\frac{1}{3}\right)^k$$

$$5 \sum_{k=0}^{n-1} u(k)\cdot 3^{n-k}\cdot u(n-k-1)$$

$$5\cdot 3^n \sum_{k=0}^{n-1} u(k)\left(\frac{1}{3}\right)^k u(n-k-1)$$

da: Thus $u \geq 1 = 0$ $u \leq 1 = 0$ og derfor så alt bliver nul og lige gyldig

$$5\cdot 3^n \sum_{k=0}^{n-1} \left(\frac{1}{3}\right)^k$$

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r} \quad \text{for } r \neq 1$$

$$5\cdot 3^n \cdot \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}} \rightarrow \frac{1-\left(\frac{1}{3}\right)^n}{\frac{2}{3}} \rightarrow \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$$

$$5\cdot 3^n\left(\frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)\right) = \frac{15\cdot 3^n}{2}\cdot\left(1-\left(\frac{1}{3}\right)^n\right)$$

$$\left(\frac{1}{3}\right)^n = \frac{1^n}{3^n} = \frac{1}{3^n}$$

$$\frac{15\cdot 3^n}{-} - \frac{\frac{15}{2}3^n \cdot 1}{-}$$

$$\frac{15\cdot 3^{\cdot}}{2} - \frac{z \cdot 1}{3n} \rightarrow \frac{15}{2}\cdot\frac{3^n}{3^n} = \frac{15}{2}$$

$$= \frac{15}{2} 3n\, u[n] - \frac{15}{2} u[n] \quad \text{for } \underline{\underline{n > 1}}$$

Ellers så plugg det ind i Maple

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k)$$ basisformel

De ligningen du for:

$x(n) = 5u(n)$

$h(n) = 3^n u(n-1)$

Omskrivning fra $n \to k$

$$5u(n) \longrightarrow 5u(k)$$

$$3^n u(n-1) \longrightarrow 3^{n-k}(n-1-k)$$

$$\boxed{u \begin{cases} 1, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases}}$$

$$\sum 5u(k)\cdot 3^{n-k} u(n-1-k)$$

$$5 \sum u(k) \cdot \frac{3^n}{3^k} \, u(n-1-k)$$

$$5 \cdot 3^n \sum u(k) \cdot \frac{1}{3^k} \, u(n-1-k)$$

Så fjerner vi step funktionerne $u(n)$
da vi endten ganger med 0 eller 1

$$5 \cdot 3^n \sum \frac{1}{3^k}$$

$$\boxed{\sum r^k = \frac{1-r^n}{1-r} \quad \text{for } r \neq 1 \qquad \frac{1}{3^k} = \frac{1^k}{3^k} = \left(\frac{1}{3}\right)^k}$$

$$5 \cdot 3^n \sum \left(\frac{1}{3}\right)^k$$

$$5 \cdot 3^n \cdot \frac{1-\frac{1}{3}^n}{1-\frac{1}{3}}$$

which you can maple plugg

$$\frac{15}{2} 3^n - \frac{15}{2}$$

Som en facit

+ Z-transform practice

Sunday, 2 June 2024 13.39

1) Find formula and ROC

$$X(z) = \frac{2z^{-1}}{1-z^{-1}} + \frac{z^3}{1-\frac{4}{7}z^{-1}}$$

with a given ROC $|z| > 1$

5) with ROC $|z|>1$

$$\frac{1}{1-az^{-1}} \Rightarrow a^n u[n]$$

$$2\frac{z^{-1}}{1-z^{-1}} + \frac{z^3}{1-\frac{4}{7}z^{-1}}$$

$$2 \cdot 1^{n-1} u[n-1] + \left(\frac{4}{7}\right)^{n+3} u[n+3]$$

$$X(z) \frac{3}{1-2z^{-1}} + \frac{z^{-1}}{1-z^{-1}}$$

with ROC $|z|<1$

We use 6) $\frac{1}{1-az^{-1}} \Rightarrow -a^n u[n-1]$

$$3\frac{1}{1-2z^{-1}} + \frac{z^{-1}}{\ldots}$$

$$3\frac{1}{1-2z^{-1}} + \frac{z^{-1}}{1-z^{-1}}$$

$$3\cdot-(2)u[n-1]+(-1)u[n-1+1]$$

$$-3(2)u[n-1]-u[n]$$

$$y[n]-3y[n-1]=5(x)[n-2]$$

$$y[n-0]-3y[n-1]=5(x)[n-2]$$

$\Updownarrow$ Z transform

$$Y(z)z^{0}-3Y(z)z^{-1}=5X(z)z^{-2}$$

$$Y(z)(z^{0}-3z^{-1})=X(z)(5z^{-2})$$

$$\boxed{H(z)=\frac{Y(z)}{X(z)} \quad ; \quad Y(z)(a+b+c\ldots)=X(z)(d+e+f)\ldots \quad \frac{Y(z)}{X(z)}=\frac{d+e+f}{a+b+c}=H(z)}$$

$$Y(z)\left(z^{0}-3z^{-1}\right)=X(z)\left(5z^{-2}\right)$$

$$\Updownarrow$$

$$\frac{Y(z)}{X(z)}=\frac{5z^{-2}}{z^{0}-3z^{-1}}=H(z)$$

$$5\,\frac{z^{-2}}{1-3z^{-1}}$$

Pole finding

$$1-3x^{-1}=0$$

$$-3x^{-1}=-1$$

$$3x^{-1}=1$$

$$x^{-1}=\frac{1}{3}$$

$$x=3$$

So $|z|>|a|$

and it is left sided

6) $\frac{1}{1-az^{-1}} \Rightarrow -a^{n}u[-n-1]$

$$5-(3)^{n-2}\quad u[-n-1--2]$$

$$-5(3)^{n-2}\quad u[-n+1]$$