+ Z-Transform and INVERSE Z-Transform

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The z-transform

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TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Pole finding:

$$\frac{1}{2-6z^{-1}}$$

$2-6z^{-1}=0 \Leftrightarrow 2=6z^{-1}$

$z^{-1}=\frac{2}{6} \Leftrightarrow \underline{z=3}$

$|z|<|a|$ and left side

$\frac{1}{2-\frac{1}{2}z^{-1}} \rightarrow 2=\frac{1}{2}z^{-1} \rightarrow$

$4=z^{-1} \Rightarrow \underline{z=\frac{1}{4}}$

$|z|>|a|$ and right sided

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Negativ / standart

$$\frac{Q z^{-M}}{1-Az^{-1}} \overset{\text{using 5}}{\Longrightarrow} Q A^{n-\boxed{M}} u[n-\boxed{M}]$$

Husk fortejn for M. Som standart er neg

$$\frac{Q z^{-M}}{1-Az^{-1}} \overset{\text{using 6}}{\Longrightarrow} Q-(A)^{n-\boxed{M}} u[-n-1--\boxed{M}]$$

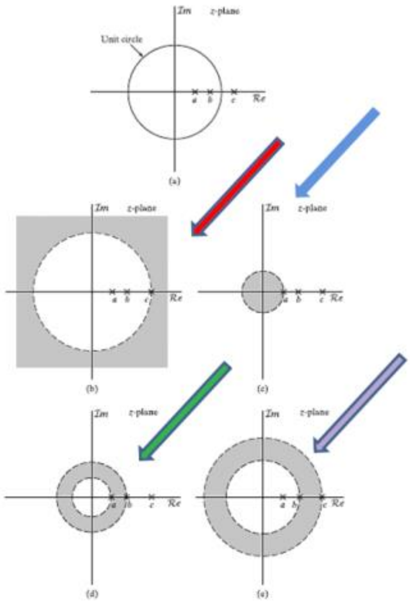
Positive / -- Standart

$$\frac{Q z^{+M}}{1-Az^{-1}} \Rightarrow A^{n+\boxed{M}} u[n+\boxed{M}]$$

Her er den positiv

$$\frac{Q z^{+M}}{1-Az^{-1}} \Rightarrow Q-(A)^{n+\boxed{M}} u[-n-1-\boxed{M}]$$

The z-transform



Examples of four z-transforms with the same pole–zero locations, illustrating the different possibilities for the ROC, each of which corresponds to a different sequence:
(b) to a right-sided sequence,
(c) to a left-sided sequence,
(d) to a two-sided sequence, and
(e) to a two-sided sequence.

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The z-transform

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

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$$H(z) = \frac{Y(z)}{X(z)}$$

; $Y(z)(a+b+c\ldots) = X(z)(d+e+f)\ldots$

$$\frac{Y(z)}{X(z)} = \frac{d+ef}{a+b+c} = H(z)$$

How to inverse z-transform!

$$X(z) = \frac{2z^{-1}}{1-z^{-1}} + \frac{z^{3}}{1-\frac{4}{7}z^{-1}}$$

given ROC $|z|>1$

Det betyder vi bruger denne basis

5) $a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}$

$$\frac{2z^{-1}}{1-z^{-1}} = 2\frac{z^{-1}}{1-1\cdot z^{-1}} \; ; \; \frac{z^{3}}{1-\frac{4}{7}z^{-1}}$$

$$2\frac{z^{-1}}{1-1z^{-1}} \qquad \frac{z^{3}}{1-\frac{4}{7}z^{-1}}$$

$$2 \cdot 1^{n-1}[n-1] \qquad \frac{4}{7}^{n+3} u[n+3]$$

$$2[n-1]$$

$$2[n-1] + \left(\frac{4}{7}\right)^{n+3} u[n+3]$$

Let's inverse some more

$$X(z) = \frac{3}{1-2z^{-1}} + \frac{z^{-1}}{1-z^{-1}}$$

ROC $|z|<1$

Which means we use this

6) $-a^n u[-n-1] \leftrightarrow \frac{1}{1-az^{-1}}$

$$\frac{3}{1-2z^{-1}} + \frac{z^{-1}}{1-z^{-1}}$$

de bliver

$-- = +$

da har det

der minus

$$3 \quad -(2)^n u[-n-1] + -1^{n-1} u[-n-1--1]$$

from formula

$-1^{n-1} = 1$

$-n-1+1 = -n$

$$\underline{\underline{-3(2)^n u[-n-1] - u[-n]}}$$

A stable LTI system is characterized by this difference equation:

$$y[n] - 3y[n-1] = 5(x)[n-2]$$

$$y[n-0] - 3y[n-1] = 5(x)[n-2]$$

$$\mathcal{Z}\{\Updownarrow\}$$

$$Y(z)\cdot z^{0} - 3Y(z)\cdot z^{-1} = 5X(z)\cdot z^{-2}$$

$$Y(z)\cdot(z^{0} - 3\cdot z^{-1}) = X(z)(5\cdot z^{-2})$$

And then we know that $H(z) = \frac{Y(z)}{X(z)}$

$X(z)$

$$\frac{Y(z)}{X(z)}(1-3z^{-1}) = 5z^{-2} \Rightarrow \frac{Y(z)}{X(z)} = \frac{5z^{-2}}{1-3z^{-1}} = H(z)$$

THIS IS IMPORTANT

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5 \cdot z^{-2}}{1-3z^{-1}}$$

og det kan vi bruge inverse Z transform på

ROC $= 3$ which means leftsided and therefore $|z| < |a|$

Som betyder vi bruger

$$-a^n u[-n-1] \Rightarrow \frac{1}{1-az^{-1}}$$

$$5 \cdot \frac{z^{-2}}{1-3z^{-1}} \rightarrow \mathcal{Z}^{-1}\{5 - 3^{n-2} u[-n-1--2]\}$$

$$-5(3)^{n-2} u[-n+1]$$

and is non causal as $[-n+1]$ is positive

Apply $\mathcal{Z}^{-1}\{\ \}$ to

$$H(z) = \frac{1-3z^{-1}}{2-4z^{-1}} = \frac{1}{2-4z^{-1}} - \frac{3z^{-1}}{2-4z^{-1}}$$

for ROC $z-4z^{-1} \to z-4\frac{1}{2}z^{-1} = 0$

Then ROC $|z|<|a|$

6) $-a^n u[-n-1] \Leftrightarrow \frac{1}{1-az^{-1}}$

$$\frac{1}{2-4z^{-1}} - 3\frac{z^{-1}}{2-4z^{-1}} = \frac{1}{2}\frac{1}{1-2z^{-1}} - \frac{3}{2}\frac{z^{-1}}{1-2z^{-1}}$$

$$\frac{1}{2}\cdot -(2)^n u[-n-1] - \frac{3}{2}\cdot -(2)^{n+1} u[-n-1--1]$$

$$-\frac{1}{2}2^n u[-n-1] + \frac{3}{2}(2)^{n+1} u[-n]$$

5 $a^n u[n] \Leftrightarrow \frac{1}{1-az^{-1}}$ ROC $|z|>|a|$

$$\frac{1}{2}\frac{1}{1-2z^{-1}} - \frac{3}{2}\frac{z^{-1}}{1-2z^{-1}}$$

$$\frac{1}{2}2^n[n] - \frac{3}{2}2^{n-1}[n-1]$$

Opgaveeks. 7

Given a causal linear time invariant system with the following transfer function:

$$H(z) = \frac{1-3z^{-1}}{2-4z^{-1}},$$

calculate the impulse response $h_i[n]$ of the inverse system.

Solution: $h_i[n] = 2(3)^n u[n] - 4(3)^{n-1}u[n-1]$, or $h_i[n] = -2(3)^n u[-n-1] + 4(3)^{n-1}u[-n]$.

Så opdag nu lige at det skal

være den inverse

$$H(z) = \mathcal{Z}^{-1}\left\{\frac{1}{H(z)}\right\}$$

$$H(z) = \mathcal{Z}^{-1}\left\{\frac{1}{H(z)}\right\}$$

$$H(z)^{-1} = \frac{2-4z^{-1}}{1-3z^{-1}} = 2\frac{1}{1-3z^{-1}} - \frac{4z^{-1}}{1-3z}$$

$$2(3)^{n}[n] - 4(3)^{n-1}[n-1]$$ formel 5)

OR

$$-2(3)^{n}[-n-1] + 4(3)^{n-1}[-n]$$ formel 6)