Visualization
– Scalar & Vectorfields

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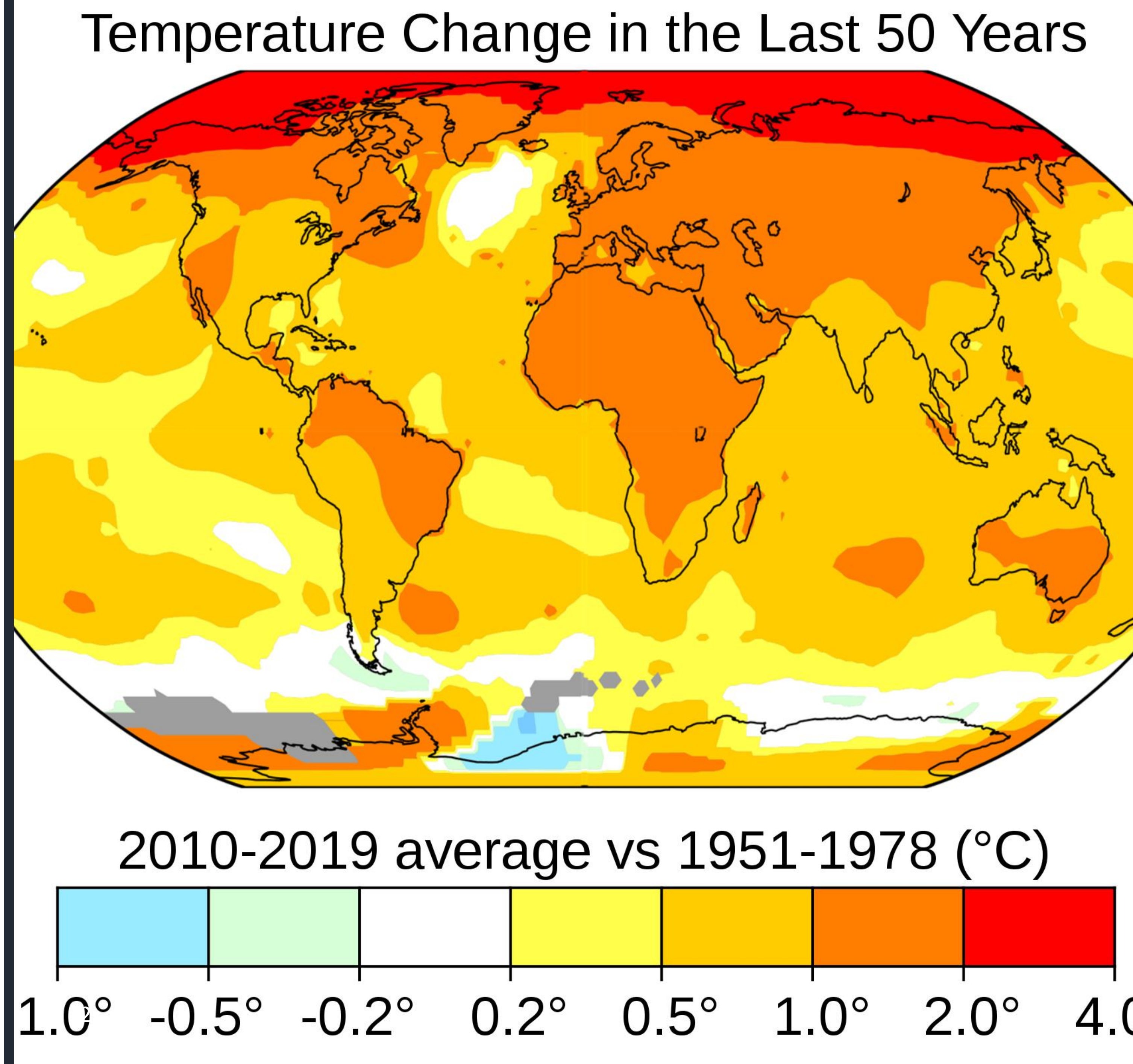
Motivation

Scalar fields can encode information on the surface



Vectorfields

Blood flow in an aneurysm

Martinez Esturo et al. Global Selection of Stream Surfaces, 2013.

Preliminaries

Preliminaries

- Derivative
- Gradient
- Divergence
- Laplace

Functions

- Function maps a real value to a real value

$$f : \mathbb{R} \to \mathbb{R}$$

- Multivariate function

$$f : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$$

Scalar Fields

- Relevant here: 2D and 3D scalar fields

- 2D:

$$f : \mathbb{R}^2 \to \mathbb{R}$$

- 3D:

$$f : \mathbb{R}^3 \to \mathbb{R}$$

- model for 2D: height field

Derivatives

Derivative

- Given is a 2D function: $f : \mathbb{R}^2 \rightarrow \mathbb{R}$
- Mathematically the derivative along the x direction at x_0 is defined as the limit from x to x_0:

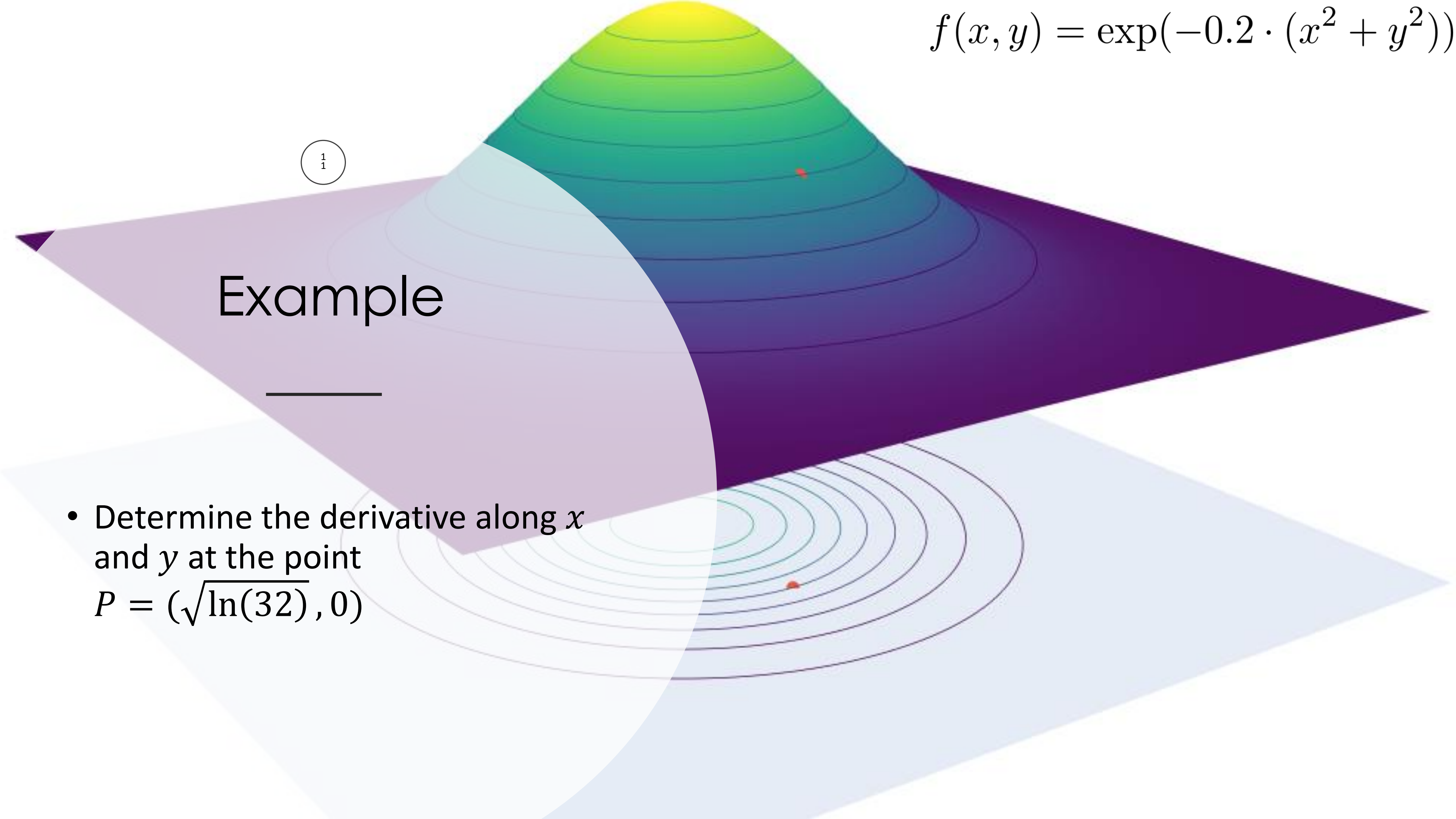
$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{f(x_0,y) - f(x,y)}{x_0 - x}$$

- The derivative of y and of 3D functions can be determined analogously

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

Example

- Determine the derivative along x and y at the point $P = (\sqrt{\ln(32)}, 0)$



$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

Example

- Derive:

$$\begin{aligned} \frac{\partial f(x, y)}{\partial x} &= -0.2 \cdot 2 \cdot x \cdot \exp(-0.2 \cdot (x^2 + y^2)) \\ &= -0.4 \cdot x \cdot f(x, y) \end{aligned}$$

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$
$$P = (\sqrt{\ln(32)}, 0)$$

Example

- Derive:

$$\frac{\partial f(x,y)}{\partial x} = -0.4 \cdot x \cdot f(x,y)$$

$$\frac{\partial f(x,y)}{\partial y} = -0.4 \cdot y \cdot f(x,y)$$

$$f(x,y) = \exp(-0.2 \cdot (x^2 + y^2))$$

$$P = (\sqrt{\ln(32)}, 0)$$

Example

- Set the point in the derivative:

$$\frac{\partial f(x,y)}{\partial x} = -0.4 \cdot x \cdot f(x,y) \xrightarrow{f(P)} -0.4 \cdot \sqrt{\ln 32} \cdot \frac{1}{2} = -0.2 \cdot \sqrt{\ln 32}$$

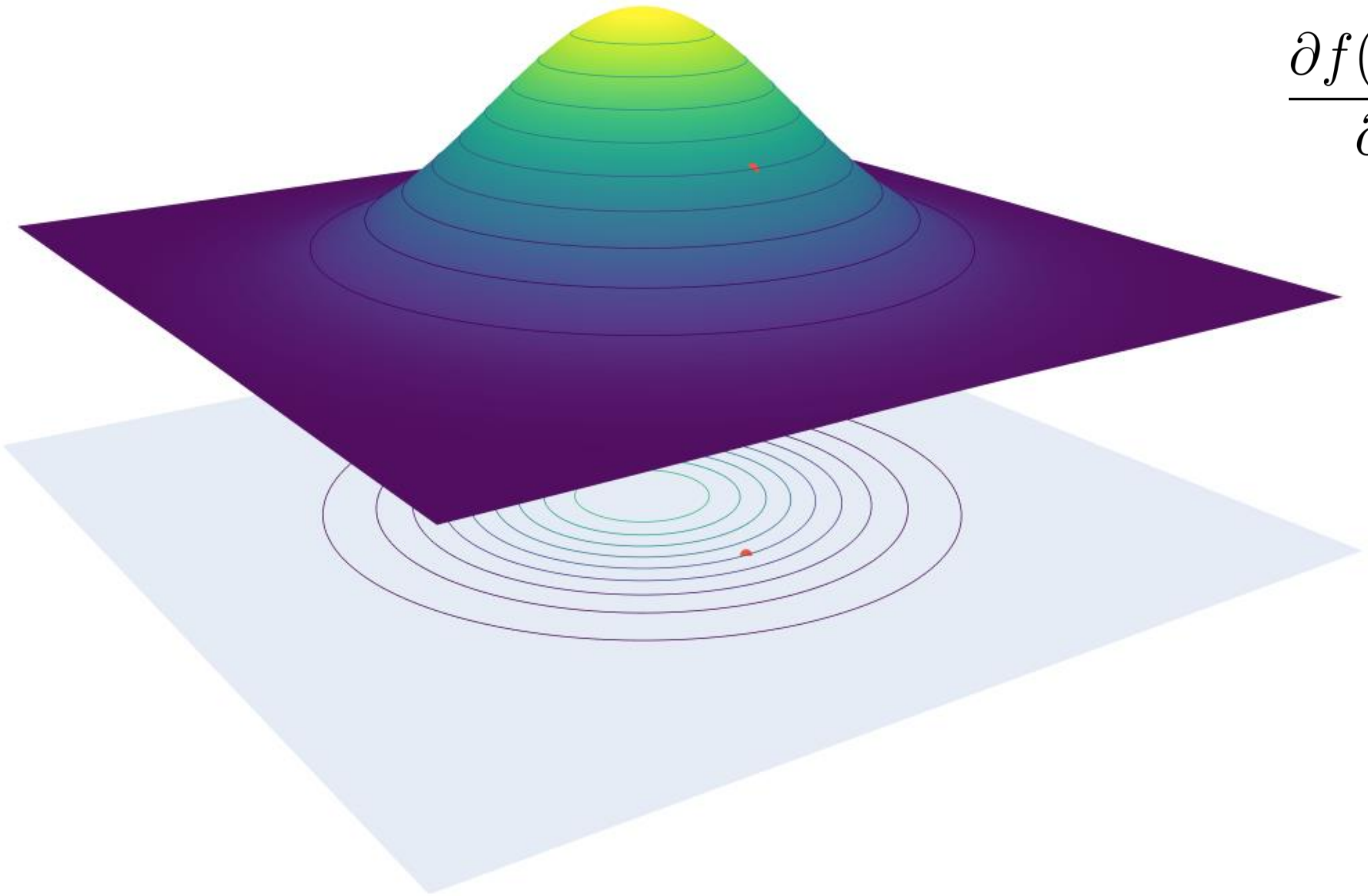
$$\frac{\partial f(x,y)}{\partial y} = -0.4 \cdot y \cdot f(x,y) \xrightarrow{f(P)} 0$$

Example

$$f(x,y) = \exp(-0.2 \cdot (x^2 + y^2))$$

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_P = -0.2 \cdot \sqrt{\ln 32}$$

$$\left.\frac{\partial f(x,y)}{\partial y}\right|_P = 0$$



Scalar Fields – Directional Derivatives

- Given a vector

$$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- How strong does f change in the direction of $\boldsymbol{u}$?

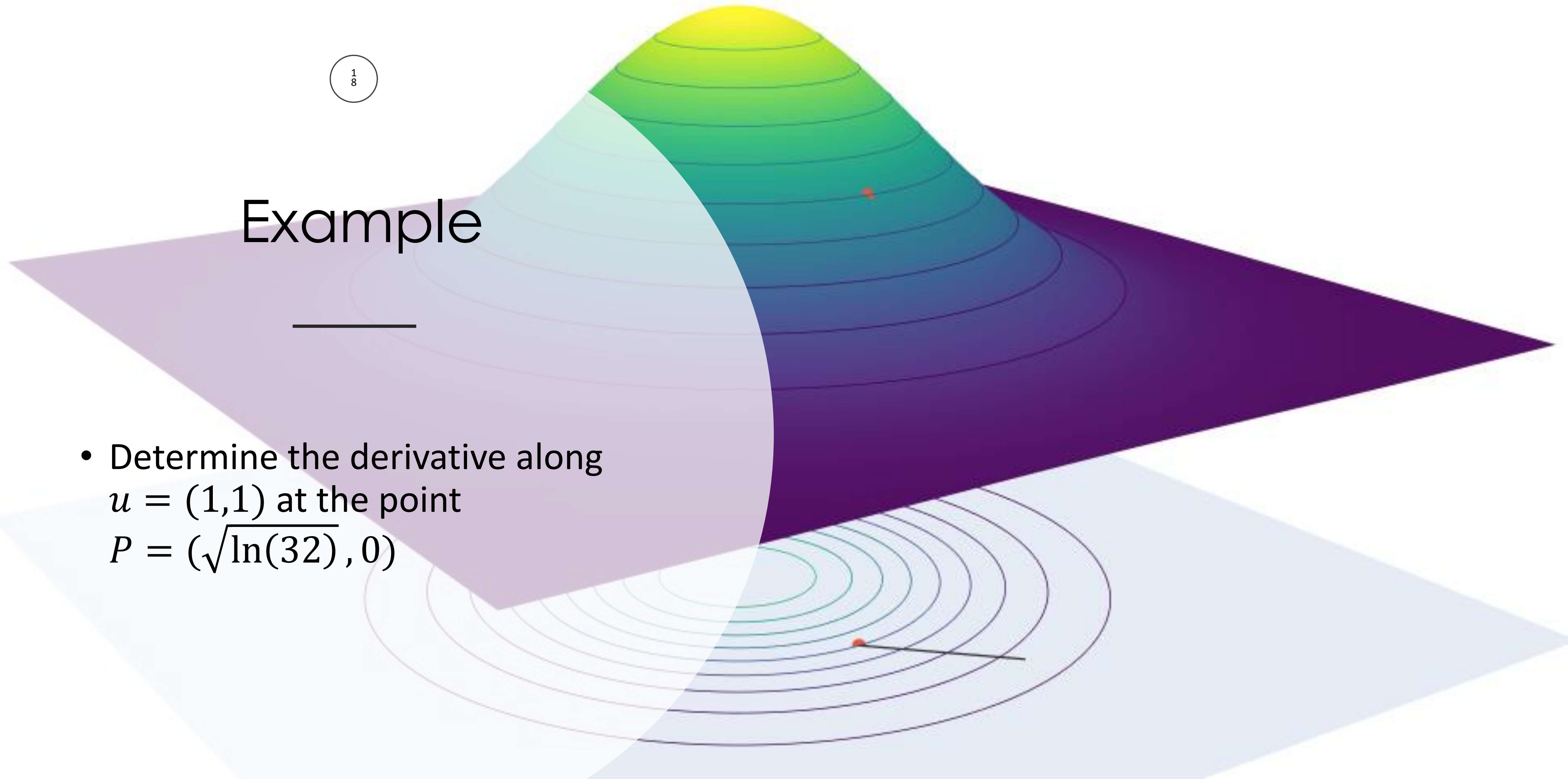
  Define line: $\mathbf{x}(t) = \mathbf{x}_0 + t\,\mathbf{u}$

- The directional derivative is defined as:

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{u}) - f(\mathbf{x}_0)}{t}$$

$$f(x,y) = \exp(-0.2 \cdot (x^2 + y^2))$$

Example



- Determine the derivative along $u = (1,1)$ at the point $P = (\sqrt{\ln(32)}, 0)$

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

Example

- Define:

$$f(\mathbf{x}_0 + t \cdot \mathbf{u}) = \exp(-0.2 \cdot ((x_0 + t \cdot u_x)^2 + (y_0 + t \cdot u_y)^2))$$

- Determine:

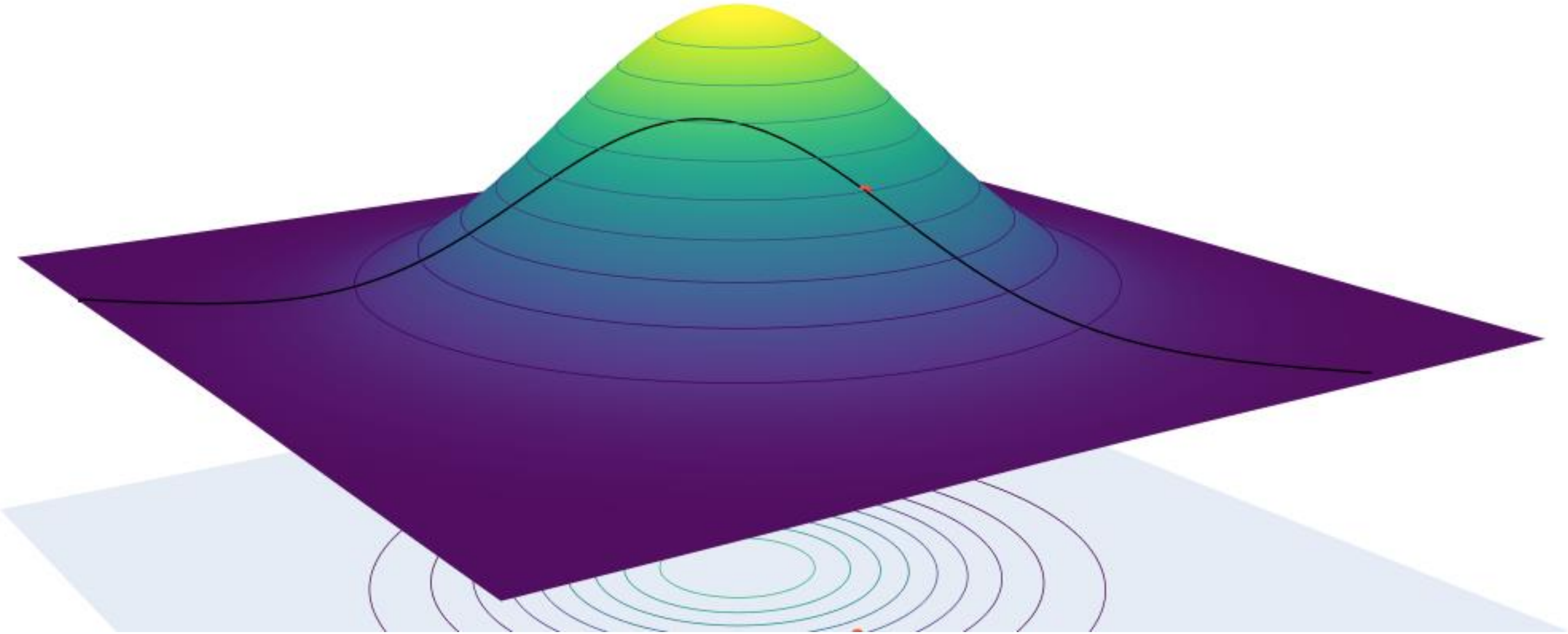
$$\frac{\partial f(\mathbf{x}_0 + t \cdot \mathbf{u})}{\partial t} = -0.2 \cdot (2u_x \cdot (x_0 + t \cdot u_x) + 2u_y \cdot (y_0 + t \cdot u_y)) \cdot f(\mathbf{x}_0 + t \cdot \mathbf{u})$$

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

Example

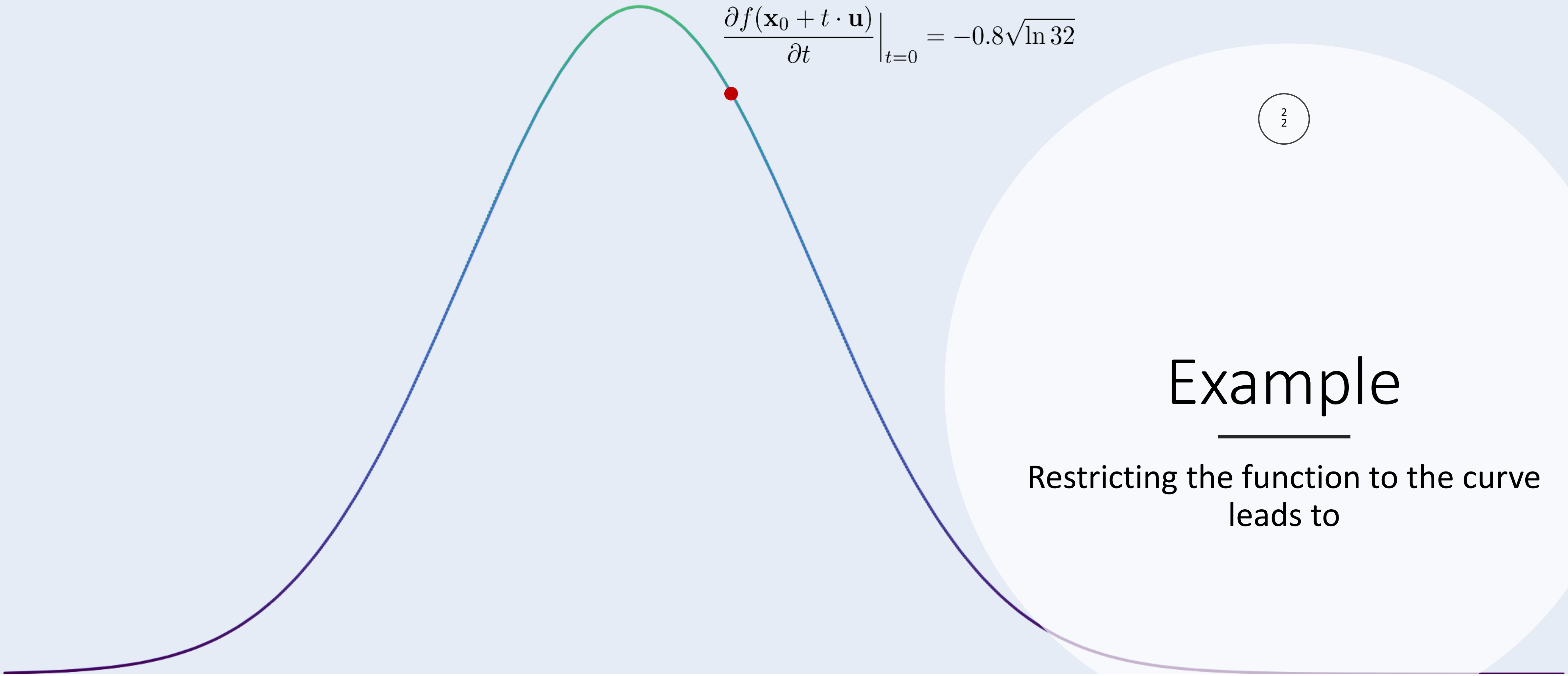
- Inserting the direction into the derivative:

$$\begin{aligned}
\left.\frac{\partial f(\mathbf{x}_0 + t \cdot \mathbf{u})}{\partial t}\right|_{t=0} &= -0.2 \cdot (2u_x \cdot (x_0 + t \cdot u_x) + 2u_y \cdot (y_0 + t \cdot u_y)) \cdot f(\mathbf{x}_0 + t \cdot \mathbf{u})\Big|_{t=0} \\
&= -0.2 \cdot (2u_x \cdot x_0 + 2u_y \cdot y_0) \cdot f(\mathbf{x}_0) \\
&= -0.2 \cdot (2\sqrt{\ln 32}) \cdot \frac{1}{2} \\
&= -0.2 \cdot \sqrt{\ln 32}
\end{aligned}$$



Example

Curve on the surface



$$\left.\frac{\partial f(\mathbf{x}_0 + t \cdot \mathbf{u})}{\partial t}\right|_{t=0} = -0.8\sqrt{\ln 32}$$

Example

Restricting the function to the curve leads to

Remark

- Some authors/texts restrict the directional vector $\boldsymbol{u}$ to be a unit vector

Derivative – Image

- On an image we cannot determine the limit due to the discrete samples
- But we can estimate the derivative by applying the formula to neighbor points:

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{x=x_0} = \frac{f(x_0,y) - f(x_0-1,y)}{1}$$

Derivative – Image

- … but in contrast to the mathematical definition the derivative is not unique for all sequences from x to x_0
- Thus, we define the forward and backward derivative:

$$f'_{\overrightarrow{x}}(x, y) = f(x, y) - f(x - 1, y)$$

$$f'_{\overleftarrow{x}}(x, y) = f(x, y) - f(x + 1, y)$$

Derivative – Image

- Remark: If nothing else is stated, we use the forward derivative

$$f'_{\vec{x}}(x, y) = f(x, y) - f(x - 1, y)$$

Derivative – Image

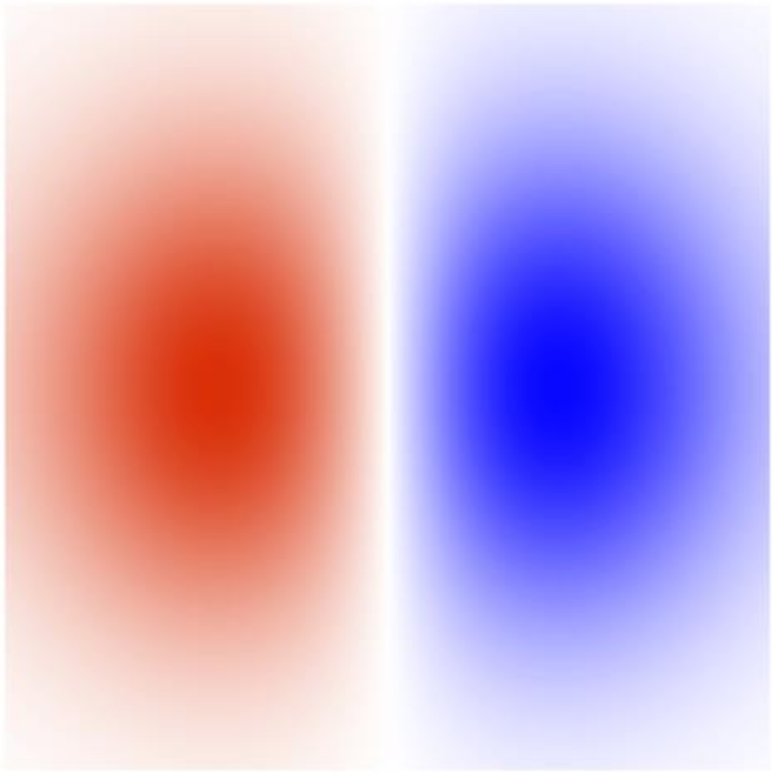
- In general:

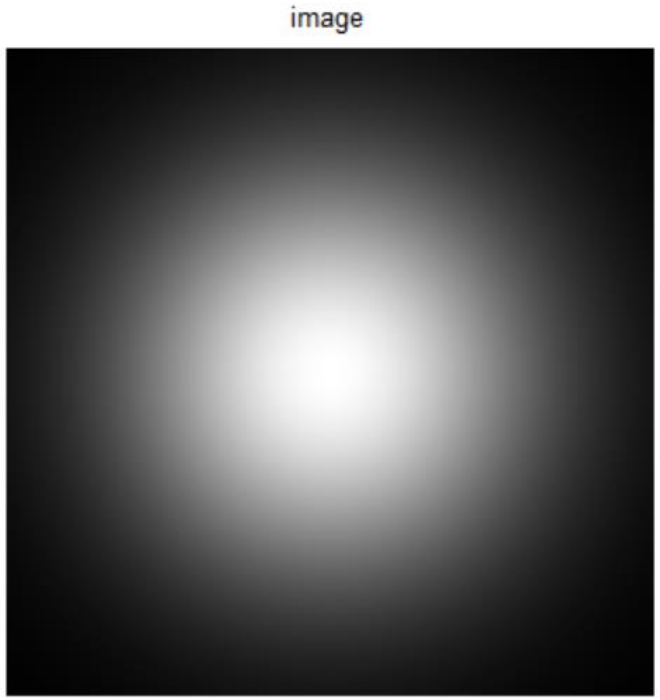
$$f'_{\overrightarrow{x}}(x,y) = \frac{f(x,y) - f(x-1,y)}{\Delta x}$$

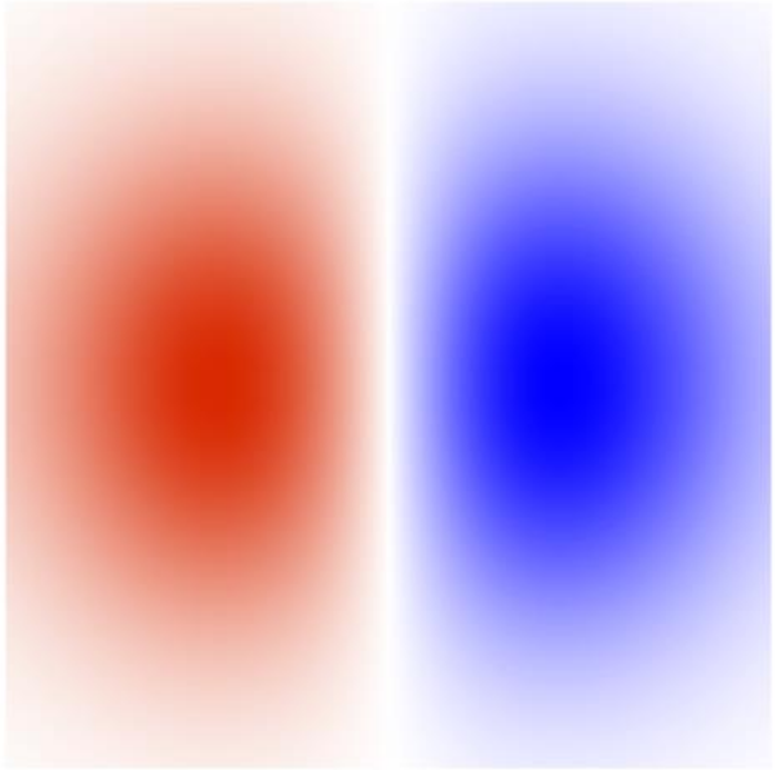
$$f'_{\overleftarrow{x}}(x,y) = \frac{f(x,y) - f(x+1,y)}{\Delta x}$$

Derivative – Image

- Left continuous derivative and right discrete derivative (of image middle)







Gradient

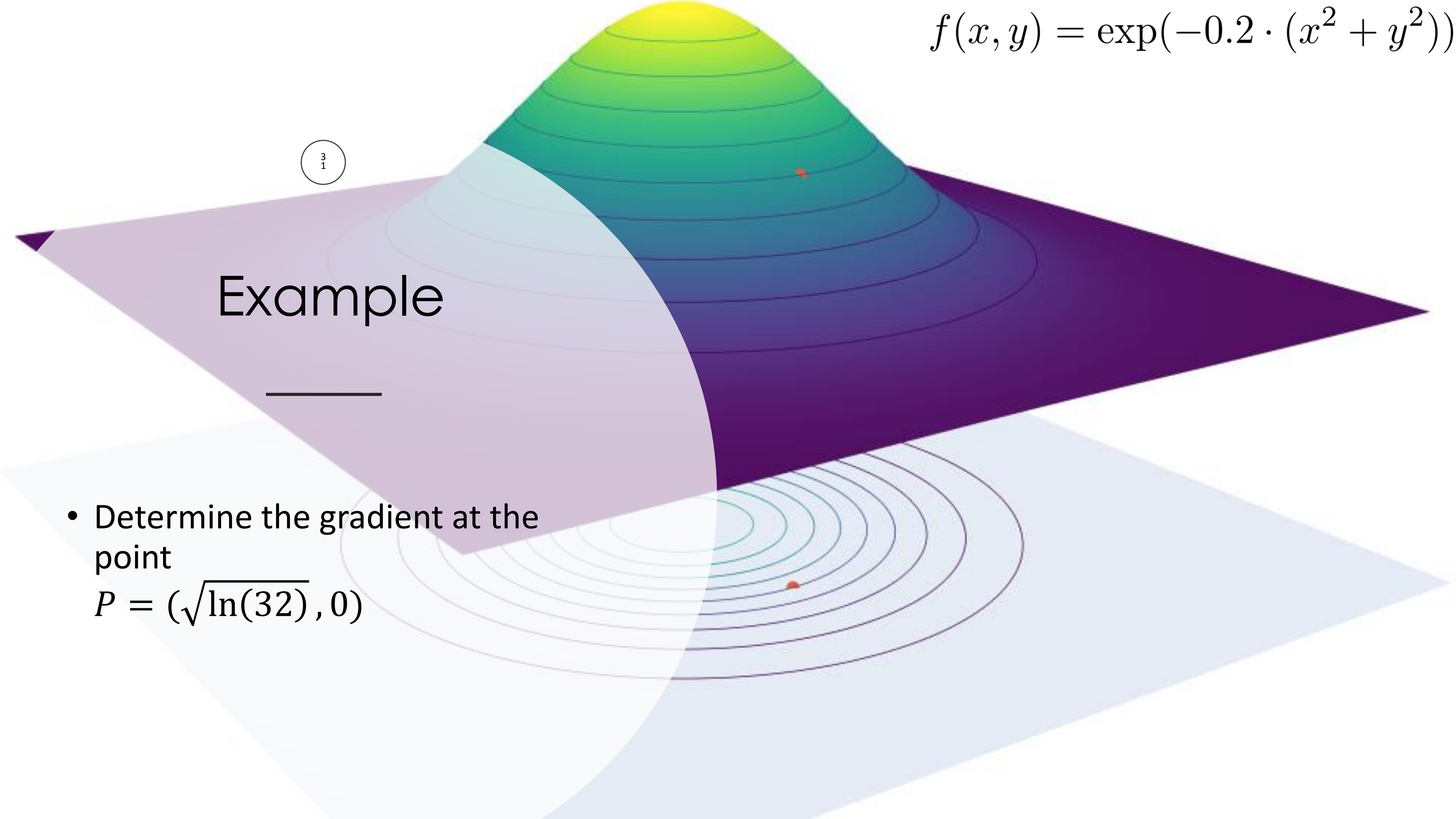
Gradient

- Given is a 2D function: $f : \mathbb{R}^2 \to \mathbb{R}$
- The gradient is a vector pointing in the direction of the highest slope

$$\nabla f(x, y) = \begin{pmatrix} \frac{\partial f(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} \end{pmatrix}$$

- The gradient of 3D (and higher) functions can be determined analogously

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$



Example

- Determine the gradient at the point $P = (\sqrt{\ln(32)}, 0)$

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

$$P = (\sqrt{\ln(32)}, 0)$$

Example

- Derive:

$$\frac{\partial f(x, y)}{\partial x} = -0.4 \cdot x \cdot f(x, y)$$

$$\frac{\partial f(x, y)}{\partial y} = -0.4 \cdot y \cdot f(x, y)$$

$$\Rightarrow \nabla f(x, y) = -0.4 \cdot f(x, y) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Example

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

$$P = (\sqrt{\ln(32)}, 0)$$

- Derive:

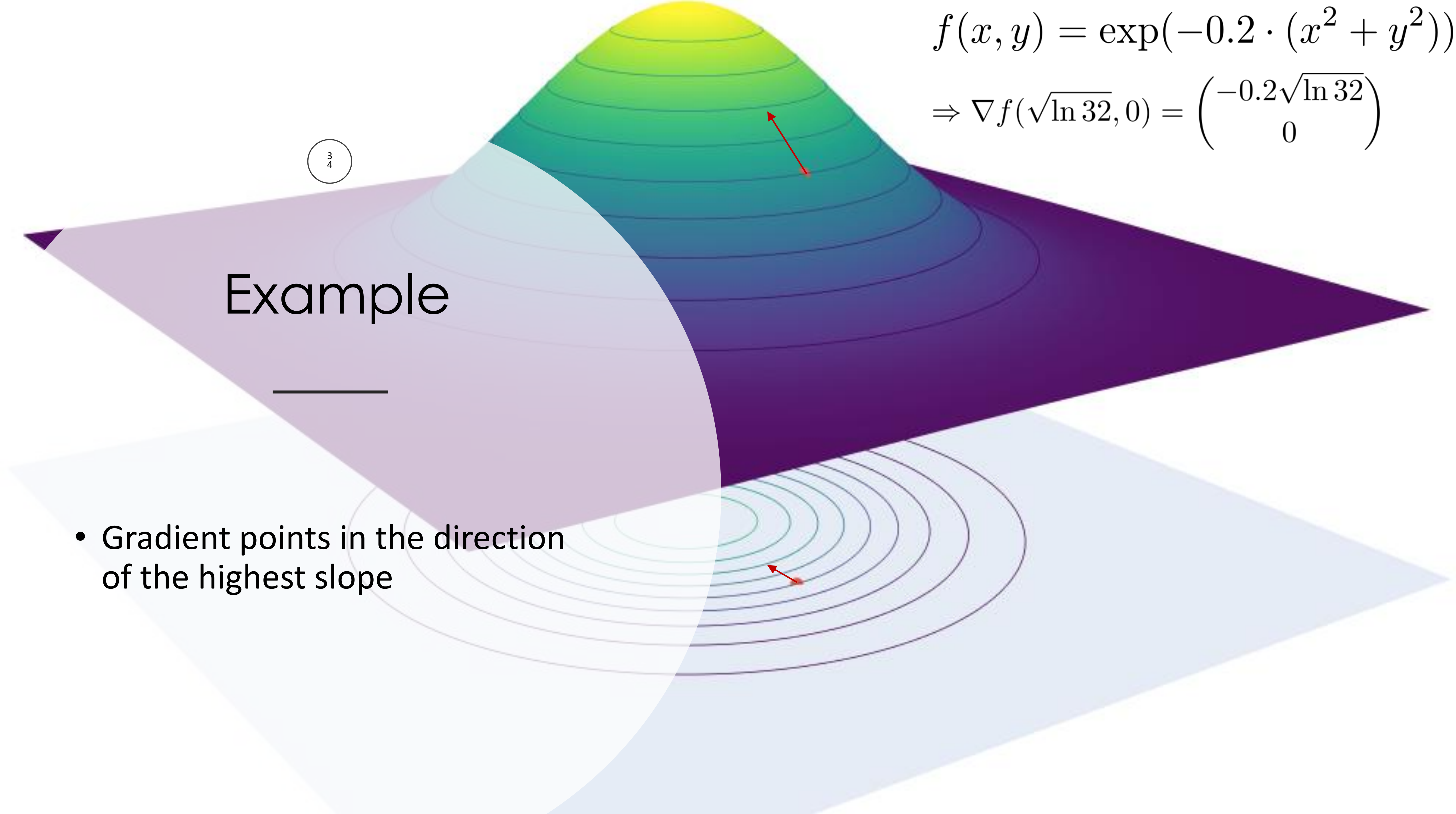
$$\frac{\partial f(x,y)}{\partial x} = -0.4 \cdot x \cdot f(x,y) \xrightarrow{f(P)} -0.4 \cdot \sqrt{\ln 32} \cdot \frac{1}{2} = -0.2 \cdot \sqrt{\ln 32}$$

$$\frac{\partial f(x,y)}{\partial y} = -0.4 \cdot y \cdot f(x,y) \xrightarrow{f(P)} 0$$

$$\Rightarrow \nabla f(\sqrt{\ln 32}, 0) = \begin{pmatrix} -0.2 \cdot \sqrt{\ln 32} \\ 0 \end{pmatrix}$$

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

$$\Rightarrow \nabla f(\sqrt{\ln 32}, 0) = \begin{pmatrix} -0.2\sqrt{\ln 32} \\ 0 \end{pmatrix}$$



Example

- Gradient points in the direction of the highest slope

Gradient – Directional Derivatives

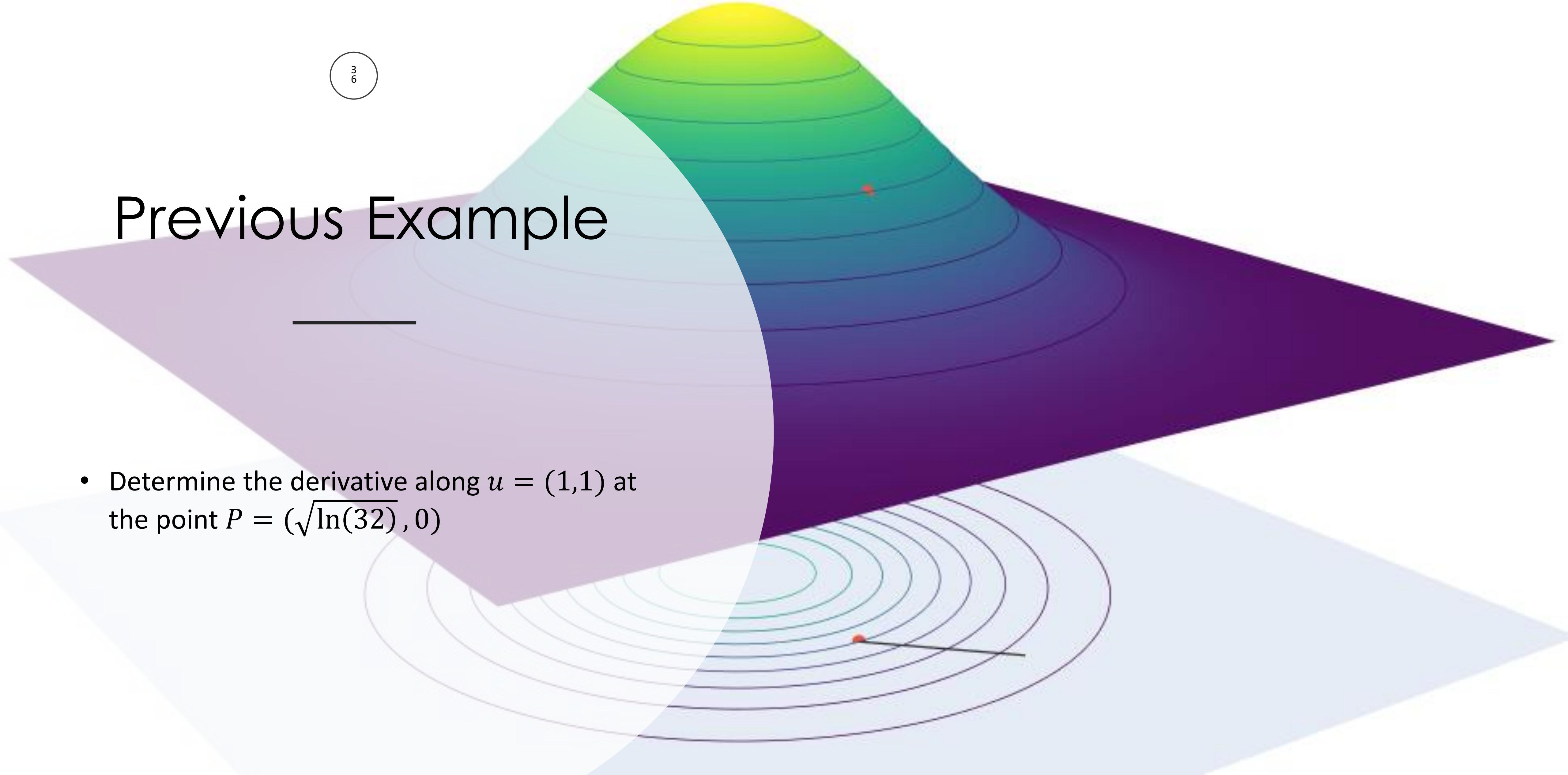
- If the function f is differentiable at P, then the directional derivative exists along any vector $\boldsymbol{u}$ and can be determined by:

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

- Where $\cdot$ denotes the Euclidean dot product

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

Previous Example



- Determine the derivative along $u = (1,1)$ at the point $P = (\sqrt{\ln(32)}, 0)$

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

$$P = (\sqrt{\ln(32)}, 0)$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

Example

- Calculate:

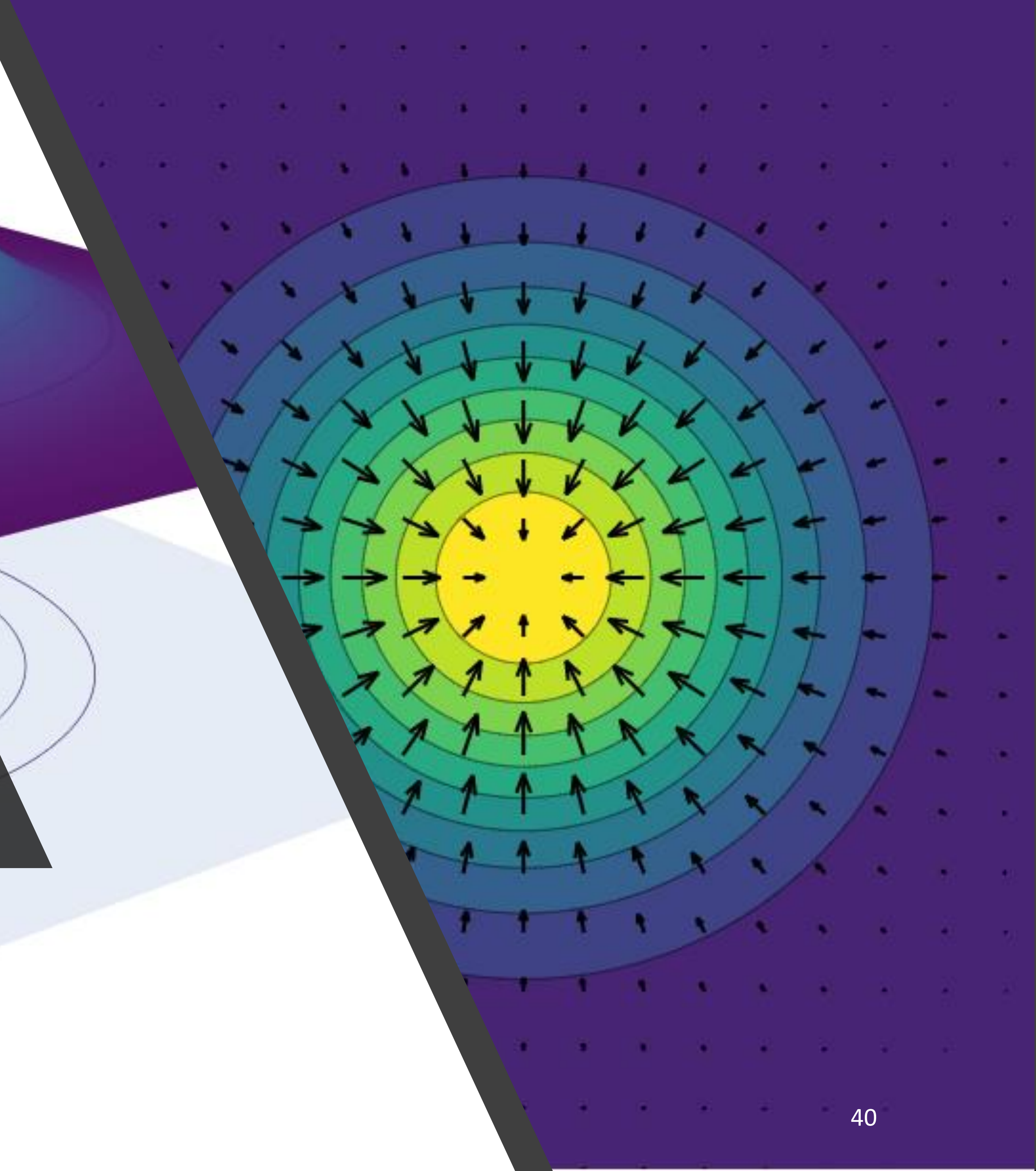
$$\begin{aligned} D_{\mathbf{u}} f &= \nabla f(\sqrt{\ln 32}, 0) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -0.2\sqrt{\ln 32} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= -0.2\sqrt{\ln 32} \end{aligned}$$

Scalar Fields – Gradient Properties

- $\mathbf{grad}\ f$ is a vector field
- $\mathbf{grad}\ f$ points in direction of maximal rate of increase of f
- $|\mathbf{grad}\ f|$ is the maximum rate of increase of f per unit distance
- Through every point $\mathbf{x}_0$ where $\mathbf{grad}\ f \neq 0$ there passes an isosurface $f(\mathbf{x}) = c$; $\mathbf{grad}\ f$ is normal (perpendicular) to this surface at $\mathbf{x}_0$.

Scalar Fields – Gradient Properties

- Gradients (right) point in the direction of the highest slope
- The length represents the rate of increase
- The gradient is perpendicular to the isolines



Gradient – Image

- On an image we cannot determine the gradient due to the discrete samples
- But we can estimate the gradient:

$$\nabla f(x,y) = \begin{pmatrix} f'_{\vec{x}} \\ f'_{\vec{y}} \end{pmatrix}$$

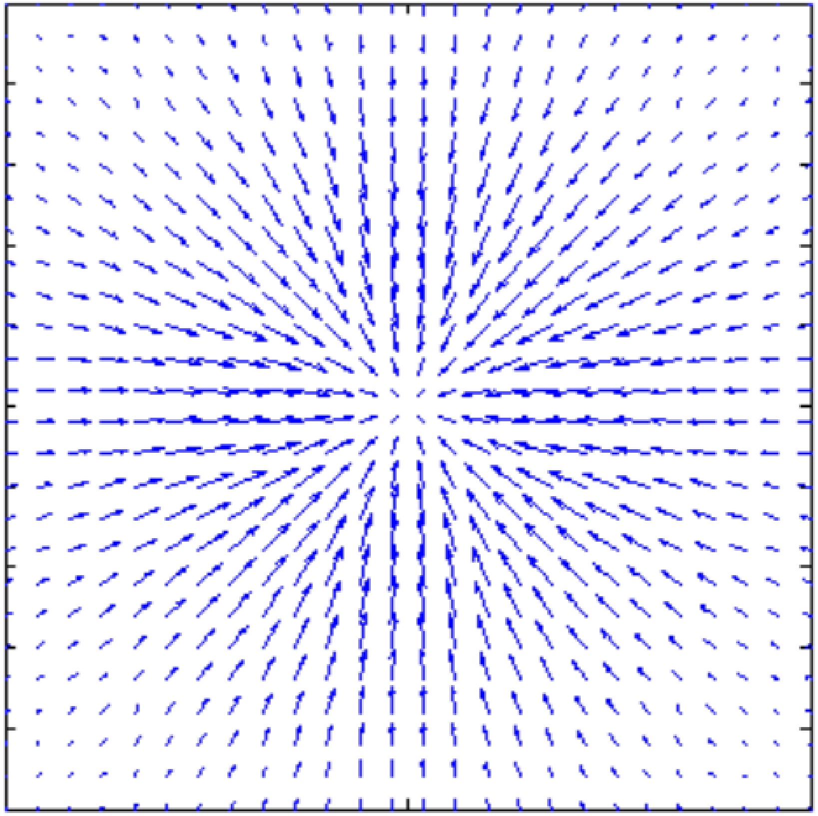
$$= \begin{pmatrix} f(x,y) - f(x-1,y) \\ f(x,y) - f(x,y-1) \end{pmatrix}$$

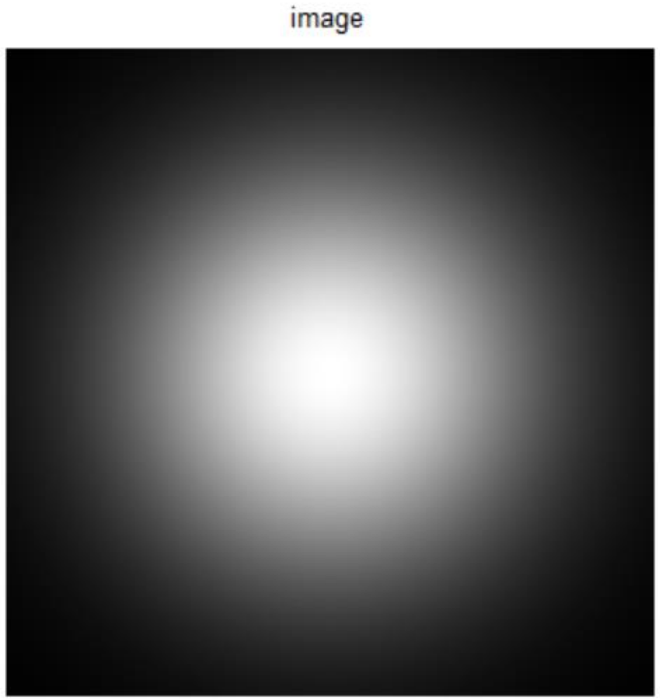
Gradient – Image

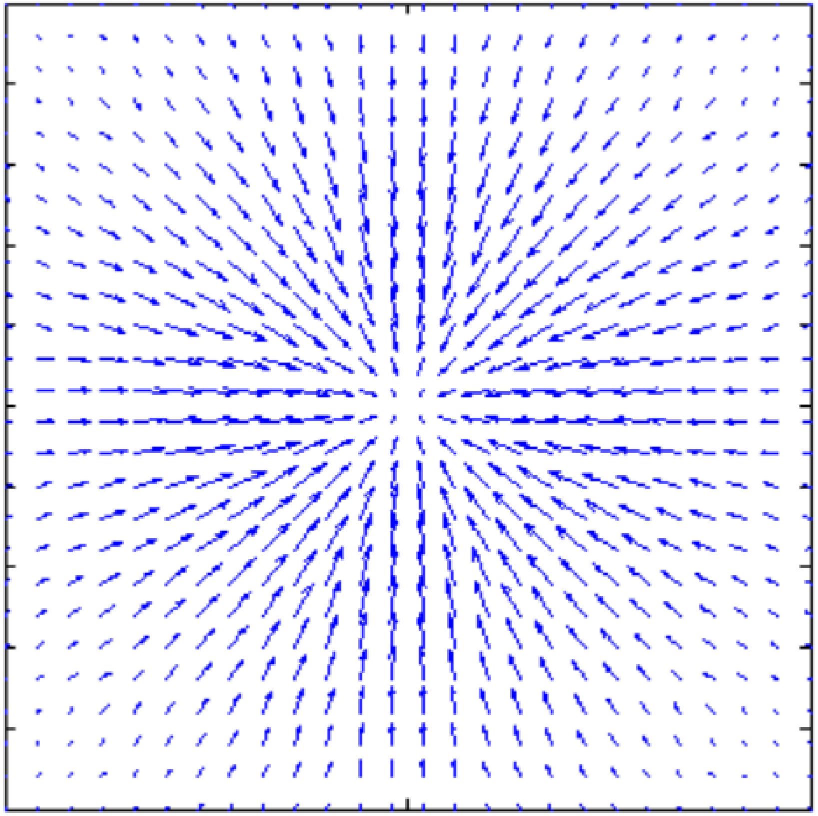
- Remark: We have a derivative in x and y direction, both have forward and backward derivatives -> 4 possibilities
- Again, we use forward

Gradient – Image

- Left continuous gradient and right discrete gradient (of image middle)





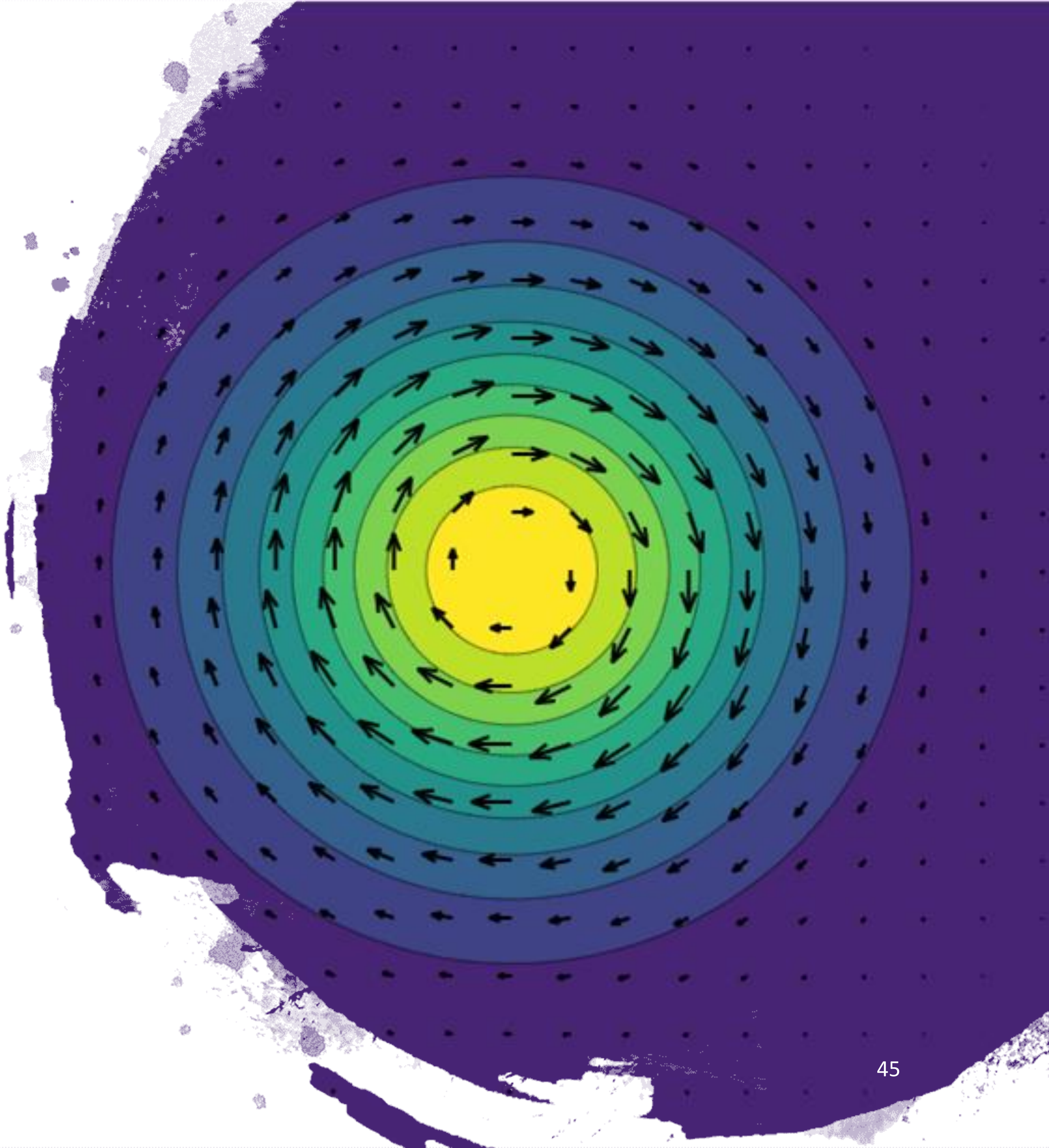


Scalar Fields – Co-gradient

- Exists only for 2D scalar fields, not for 3D fields!
- Is in every point perpendicular to the gradient

$$\mathbf{co-grad}\ f = \begin{pmatrix} -f_y \\ f_x \end{pmatrix}$$

- Points in the direction of isolines (height lines) of f



Hessian Matrix

Scalar Fields – Hessian Matrix

- For $\mathbf{grad}\ f \neq 0$, the gradient describes the local first-order properties of f.

- What happens at points with $\mathbf{grad}\ f = 0$?
  - Generally isolated points
  - Can be a local maximum, local minimum, or saddle
  - Characterized by Hessian matrix

Scalar Fields – Hessian Matrix

- Hessian Matrix:

$$\mathbf{H}(f) = \begin{pmatrix} \dfrac{\delta^2 f}{\delta x^2} & \dfrac{\delta^2 f}{\delta x \delta y} & \dfrac{\delta^2 f}{\delta x \delta z} \\ \dfrac{\delta^2 f}{\delta x \delta y} & \dfrac{\delta^2 f}{\delta y^2} & \dfrac{\delta^2 f}{\delta y \delta z} \\ \dfrac{\delta^2 f}{\delta x \delta z} & \dfrac{\delta^2 f}{\delta y \delta z} & \dfrac{\delta^2 f}{\delta z^2} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}$$

- $\mathbf{H}(f)$ is symmetric.

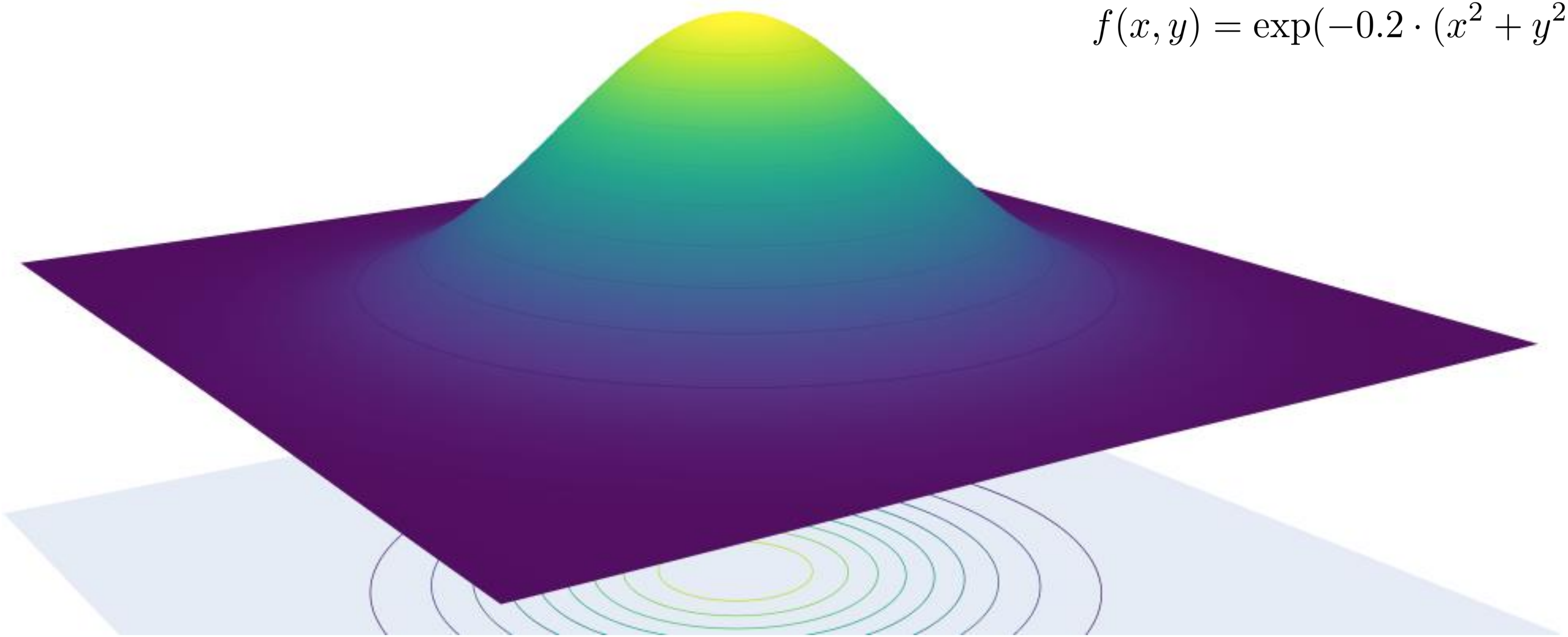
Scalar Fields – Extremal Points

- Isolated point $\mathbf{x}_0$ with $\mathbf{grad}\ f(\mathbf{x}_0) = 0$
  - Can be classified by Eigenanalysis of $\mathbf{H}(f(\mathbf{x}_0))$ if $\det(\mathbf{H}) \neq 0$
  - Also called critical points

- Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be eigenvalues of $\mathbf{H}$ and $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ the corresponding eigenvectors at $\mathbf{x}_0$ with $\mathbf{grad}\ f(\mathbf{x}_0) = 0$.
  - If $\mathbf{H}$ symmetric: $\lambda_1, \lambda_2, \lambda_3$ real
    $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ orthogonal

$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ → local minimum
$\lambda_1 < 0 < \lambda_3$ → saddle
$\lambda_1 \leq \lambda_2 \leq \lambda_3 < 0$ → local maximum

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$



Example

Determine isolated points

Example

- First, determine the gradient and check for $grad\ f\ = 0$:

$$\nabla f(x,y) = -0.4 \cdot f(x,y) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\nabla f(x,y) = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Example

$$H(x,y) = \begin{pmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial x \partial y & \partial^2 f/\partial y^2 \end{pmatrix}$$

- Determine the Hessian Matrix:

$$\frac{\partial^2 f}{\partial x^2} f(x,y) = (0.16x^2 - 0.4) \cdot f(x,y)$$

$$\frac{\partial^2 f}{\partial x \partial y} f(x,y) = 0.16xy \cdot f(x,y)$$

$$\frac{\partial^2 f}{\partial y^2} f(x,y) = (0.16y^2 - 0.4) \cdot f(x,y)$$

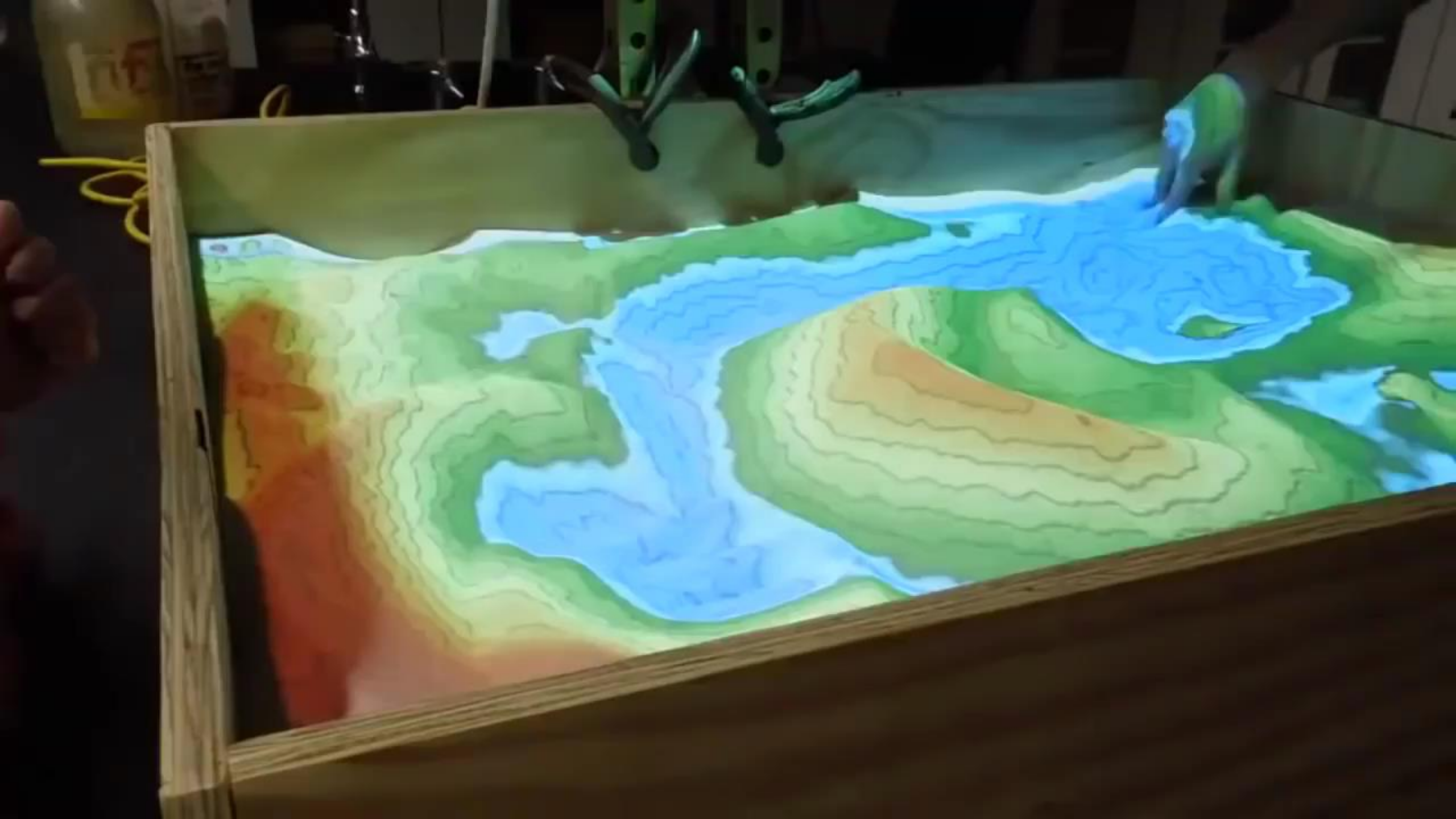
Example

$$H(x,y) = \begin{pmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial x \partial y & \partial^2 f/\partial y^2 \end{pmatrix}$$

- Get the eigenvalues at the isolated point:

$$H(0,0) = \begin{pmatrix} -0.4 & 0 \\ 0 & -0.4 \end{pmatrix}$$

$\lambda_{1,2} = -0.4$ → local maximum



Morse–Smale Complex

Scalar Fields – Morse–Smale Complex

- Here, for 2D scalar fields
- Imagine as height field: let water flow from every point
  - flows to local minimum

Scalar Fields – Morse–Smale Complex

- Here, for 2D scalar fields
- Imagine as height field: let water flow from every point
  - flows to local minimum

- ω-basin (unstable manifold):
  - collection of all points where the water flows to the same local minimum
  - Backward gradient (height field)

- α-basin (stable manifold):
  - collection of all points where the water flows backwards to the same local maximum
  - (Forward) gradient (height field)

Scalar Fields – Integral Lines

- Integral lines: maximal parametric curves $\mathbf{p}(t)$ whose tangent vector agrees with the gradient

$$\frac{d\mathbf{p}}{dt} = \mathbf{grad}\, f(\mathbf{p}(t)) \quad \text{for all} \quad t \in \Re$$

$$\mathbf{org}\,\mathbf{p} = \lim_{t\to-\infty} \mathbf{p}(t)$$ - origin of $\mathbf{p}$

$$\mathbf{dest}\,\mathbf{p} = \lim_{t\to\infty} \mathbf{p}(t)$$ - destination of $\mathbf{p}$

Scalar Fields – Integral Lines

Properties of integral lines:

- Two integral lines are either disjoint or the same
- The images of all integral lines cover all non-critical points of D
- The limits $\mathbf{org\ p}$ and $\mathbf{dest\ p}$ are critical points of f

Scalar Fields – Stable and Unstable Manifolds

Stable manifolds around critical point $\mathbf{x}_0$:

$$\mathbf{S}(\mathbf{x}_0) = \{\mathbf{x}_0\} \cup \{\mathbf{x} \in D : \exists \mathbf{p}(t), t_0 : \mathbf{p}(t_0) = \mathbf{x} \wedge \mathbf{dest}\,\mathbf{p}(t) = \mathbf{x}_0\}$$

- Pairwise disjoint
- Partition of D

Unstable manifolds around critical point $\mathbf{x}$_0:

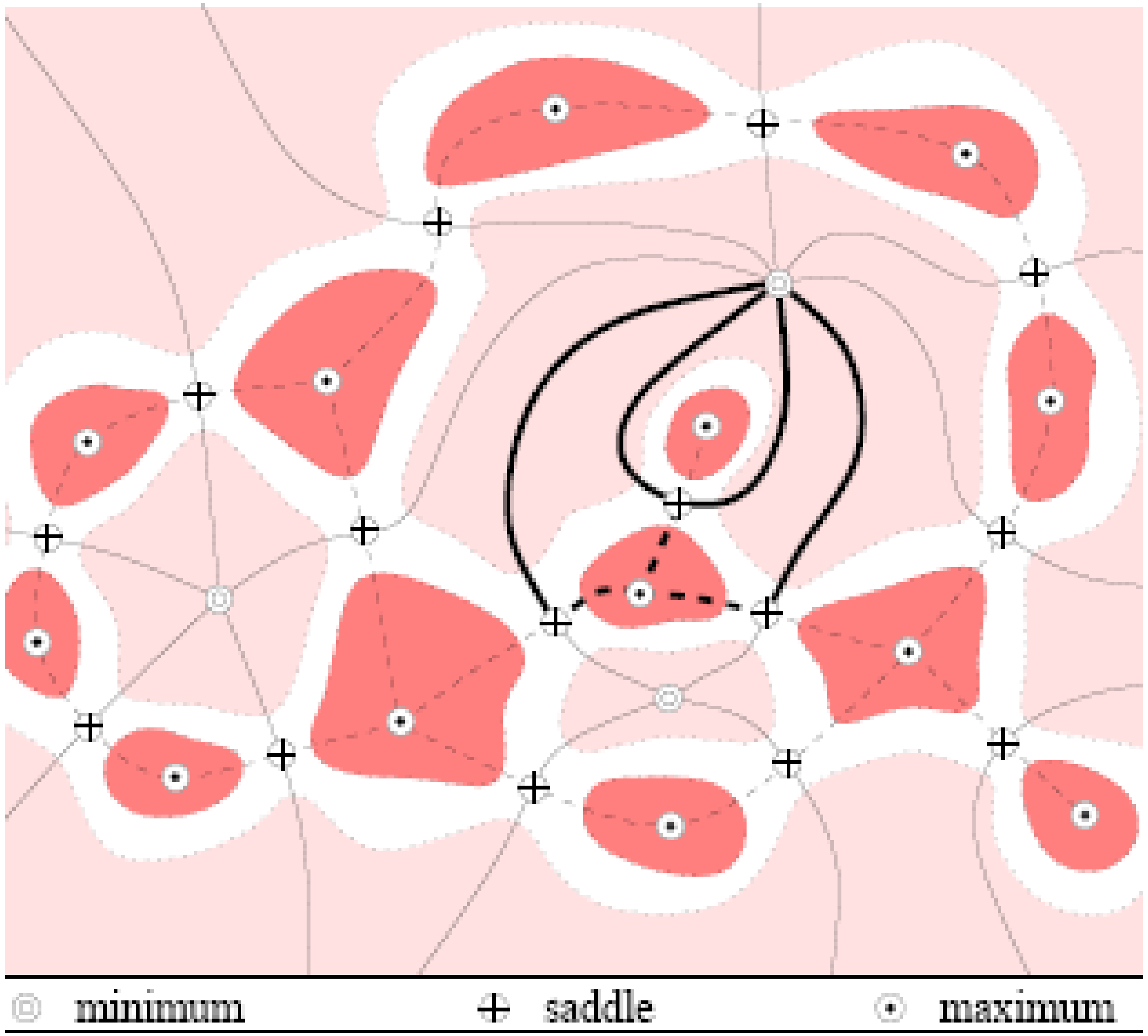
$$\mathbf{U}(\mathbf{x}_0) = \{\mathbf{x}_0\} \cup \{\mathbf{x} \in D : \exists \mathbf{p}(t), t_0 : \mathbf{p}(t_0) = \mathbf{x} \wedge \mathbf{org}\,\mathbf{p}(t) = \mathbf{x}_0\}$$

- Pairwise disjoint
- Partition of D

Scalar Fields – Morse–Smale Complex

Morse-Smale complex:

- Intersection of stable and unstable manifolds
- The lines that connect the critical points are called **separatrices**



H.Edelsbrunner, J.Harer, A. Zomorodian Hierarchical Morse - Smale Complexes for Piecewise Linear 2-Manifolds. Discrete & Computational Geometry, 30(3), pp. 87-107, 2003.

Scalar Fields – Morse–Smale Complex

Computation of Separatrices:

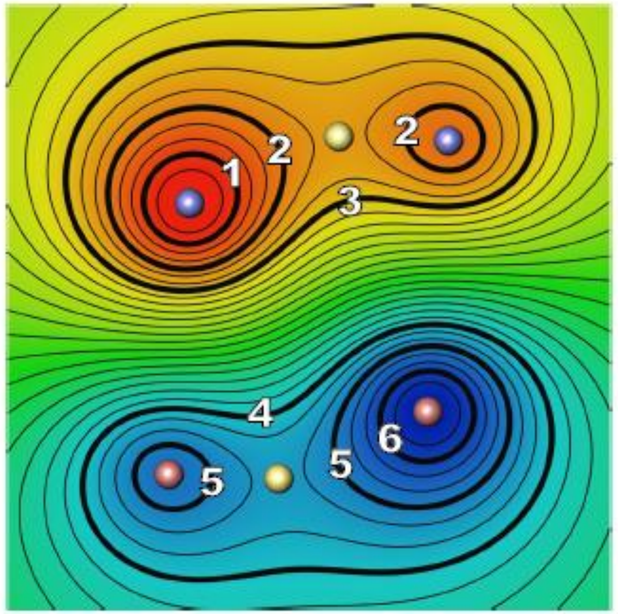
- Compute locations of critical points (sinks, sources, saddles)
- At each saddle position $\mathbf{x}_c$:
  - Compute eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{c}_1, \mathbf{c}_2$
  - Go an epsilon away from $\mathbf{x}_c$ in direction of $\mathbf{c}_1$ and $\mathbf{c}_2$ and trace an integral line of the gradient field until it ends in a sink, source or exits the domain. So, the seed points are:

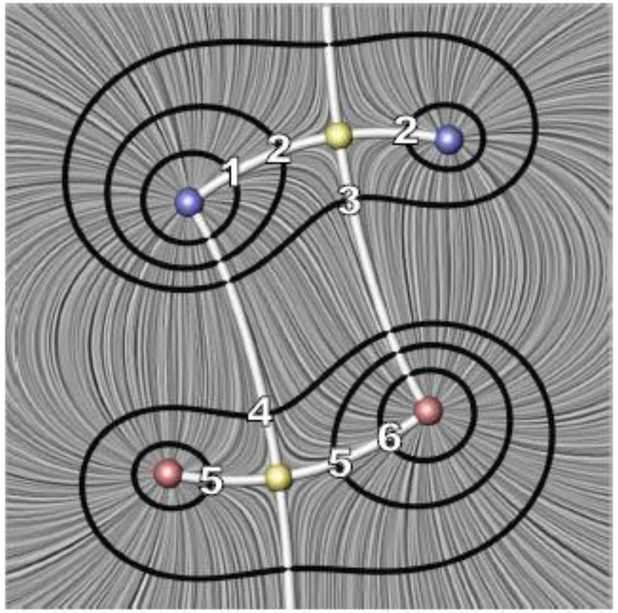
$$\begin{aligned}
&\mathbf{x}_c + \epsilon\ \mathbf{c}_1 \\
&\mathbf{x}_c - \epsilon\ \mathbf{c}_1 \\
&\mathbf{x}_c + \epsilon\ \mathbf{c}_2 \\
&\mathbf{x}_c - \epsilon\ \mathbf{c}_2
\end{aligned}$$

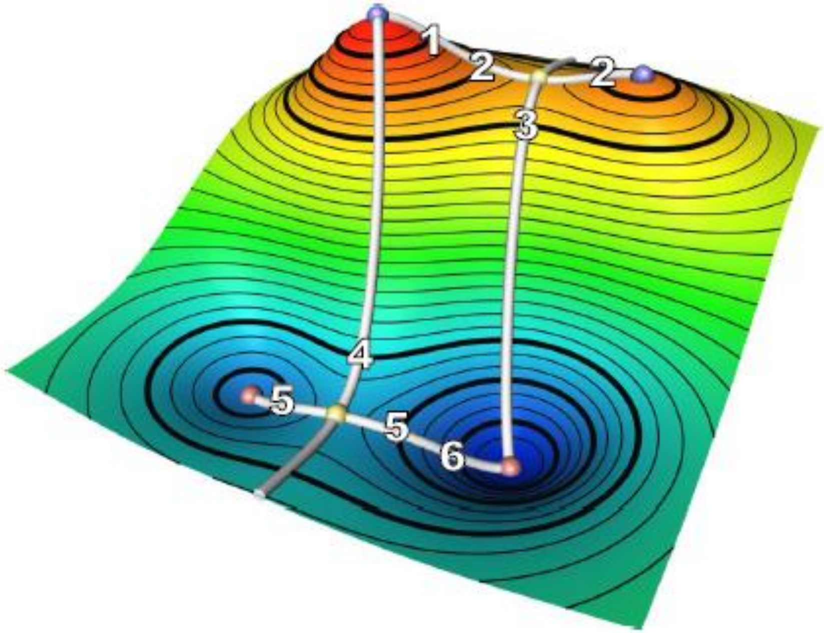
Scalar Fields – Morse–Smale Complex

Quadrangle Lemma:

Each region (without boundary) of the Morse-Smale complex is a quadrangle with vertices **minimum, saddle, maximum, saddle**, in this order around the region.





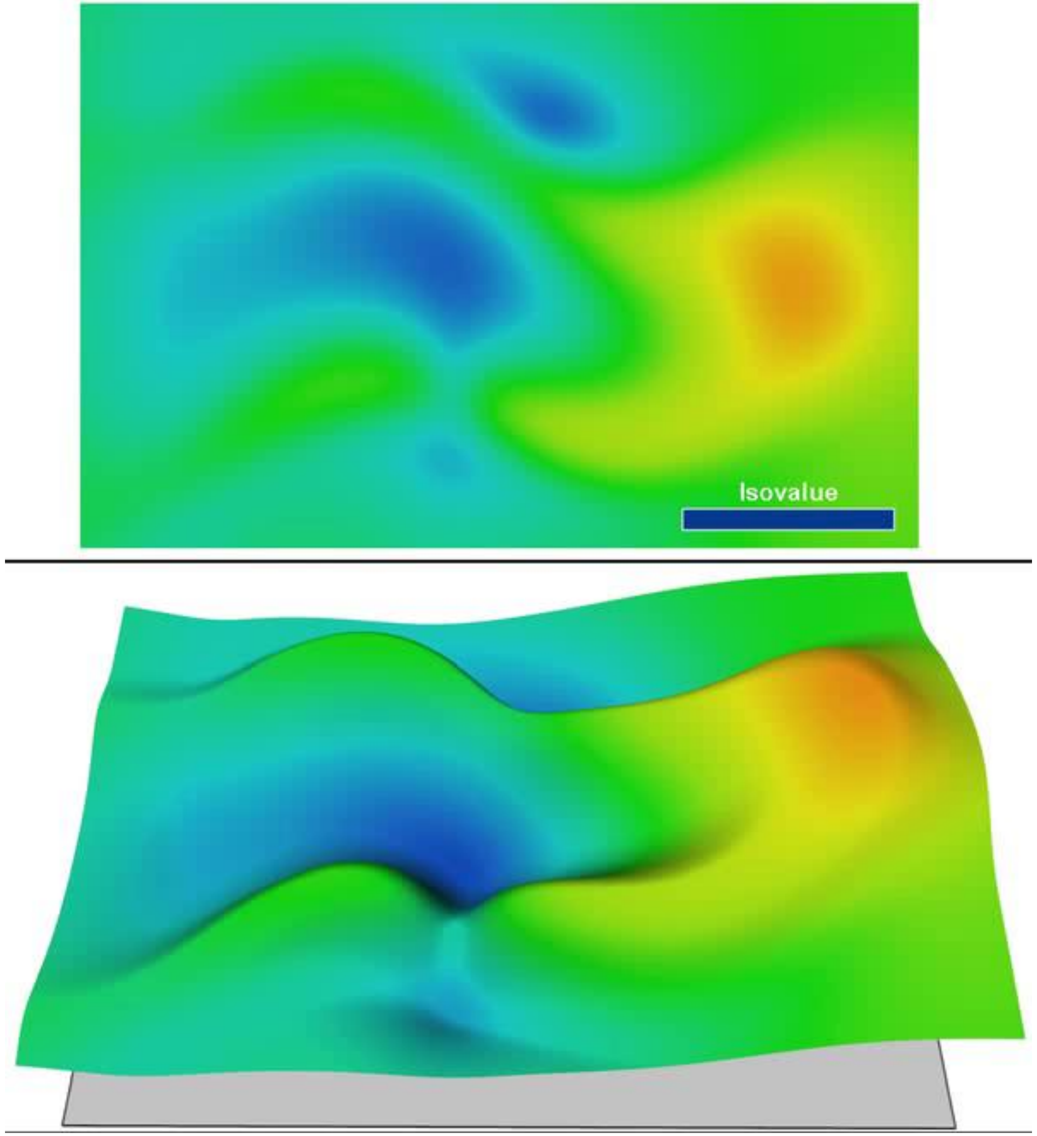


Scalar Fields – Morse–Smale Complex

Isolines are also called „height lines“.
(Example: Pressure lines in weather reports.)

Weinkauf: PhD thesis

Scalar Fields – Critical Points



Ridges & Valleys

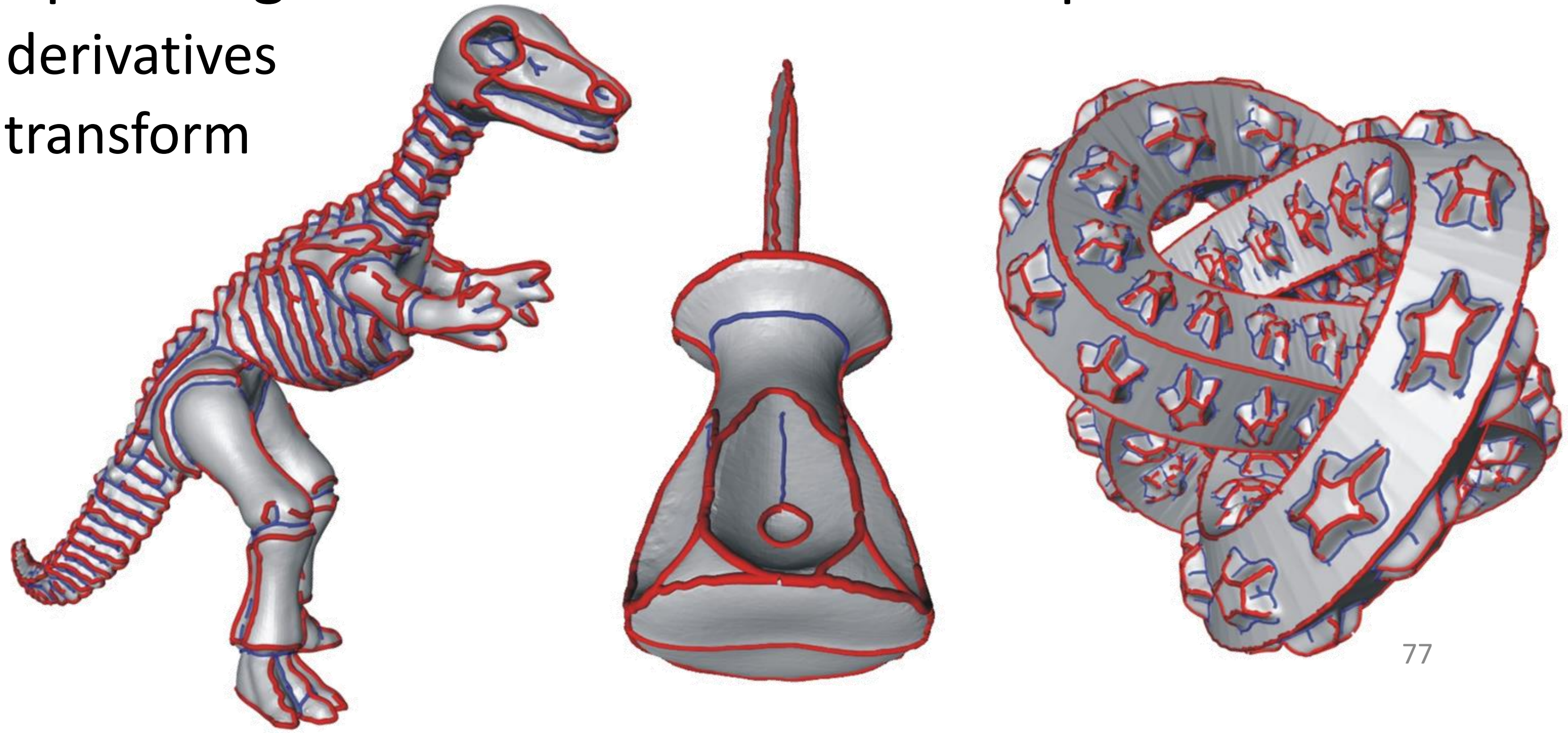
- Ridges along a path

Scalar Fields – Ridge and Valley Lines

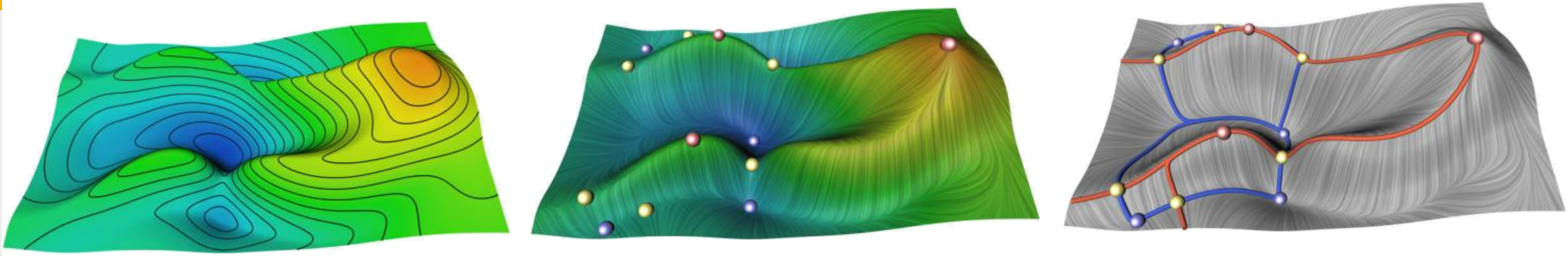
For 2D and 3D scalar fields:

Ridge/Valley lines are separating lines of Morse-Smale complex

- needs only 1st order derivatives
- similar to watershed transform



T. Weinkauf and D. Günther, 2009



Scalar Fields – Ridge and Valley Lines

- Ridge (red)
- Valley (blue)

T. Weinkauf and D. Günther, 2009

Vector Fields

Vector Fields

- 2D Vector field:

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

- 3D Vector field :

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$

Vector Fields – Partial Derivatives

- Vector field $\mathbf{v}(x,y,z) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$

$$\frac{\delta \mathbf{v}}{\delta x} = \mathbf{v}_x = \begin{pmatrix} \frac{\delta u}{\delta x} \\ \frac{\delta v}{\delta x} \\ \frac{\delta w}{\delta x} \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix}$$

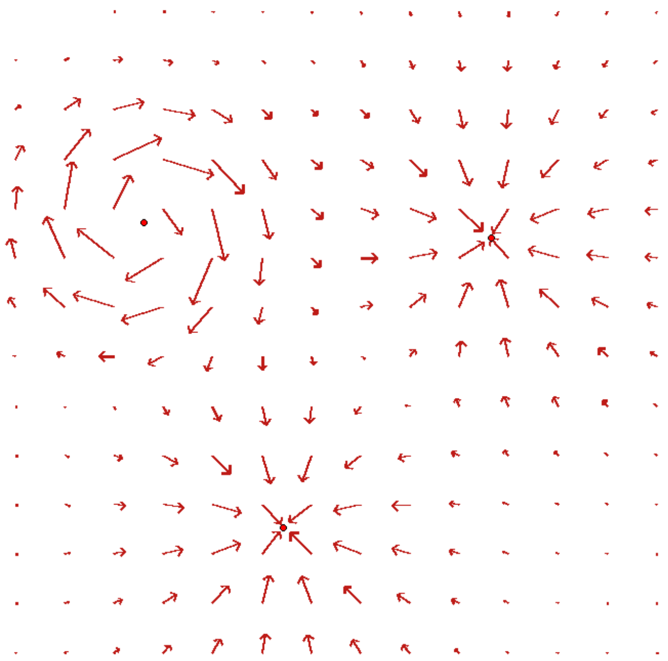
$$\frac{\delta \mathbf{v}}{\delta y} = \mathbf{v}_y = \begin{pmatrix} \frac{\delta u}{\delta y} \\ \frac{\delta y}{\delta y} \\ \frac{\delta w}{\delta y} \end{pmatrix} = \begin{pmatrix} u_y \\ v_y \\ w_y \end{pmatrix}$$

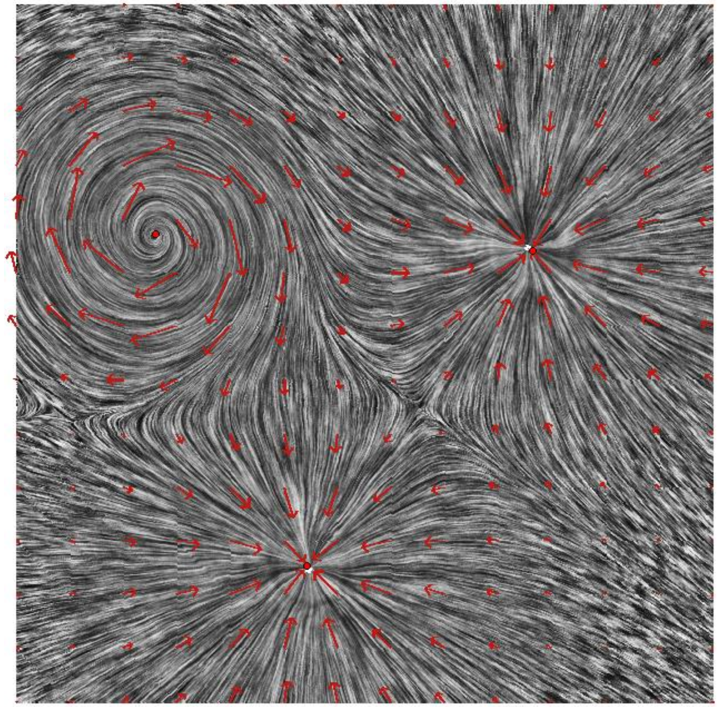
$$\frac{\delta \mathbf{v}}{\delta z} = \mathbf{v}_z = \begin{pmatrix} \frac{\delta u}{\delta z} \\ \frac{\delta v}{\delta z} \\ \frac{\delta w}{\delta z} \end{pmatrix} = \begin{pmatrix} u_z \\ v_z \\ w_z \end{pmatrix}$$

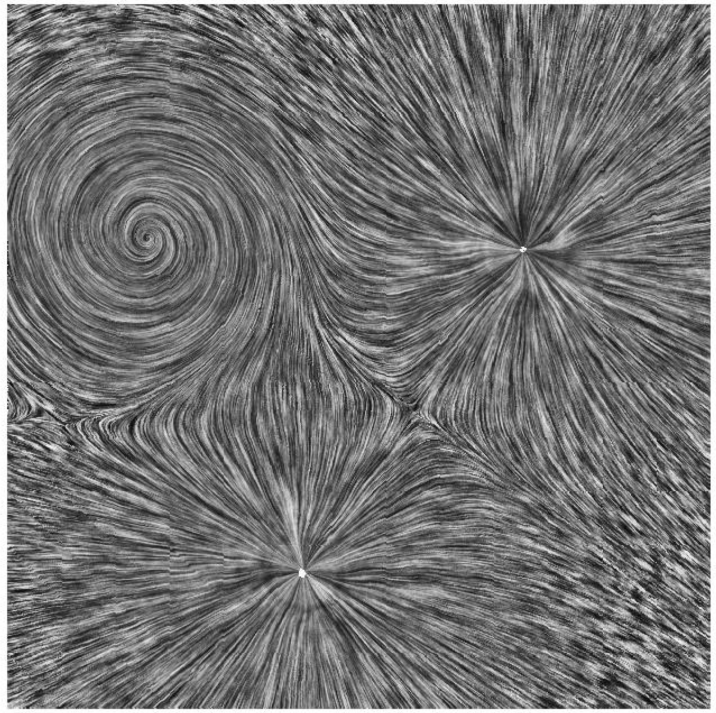
Vector Fields – Jacobian Matrix

- Contains all first-order information of $\mathbf{v}$:

$$\mathbf{J}_{\mathbf{v}}(\mathbf{x}) = (\mathbf{v}_{x,}\mathbf{v}_{y,}\mathbf{v}_{z}) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$





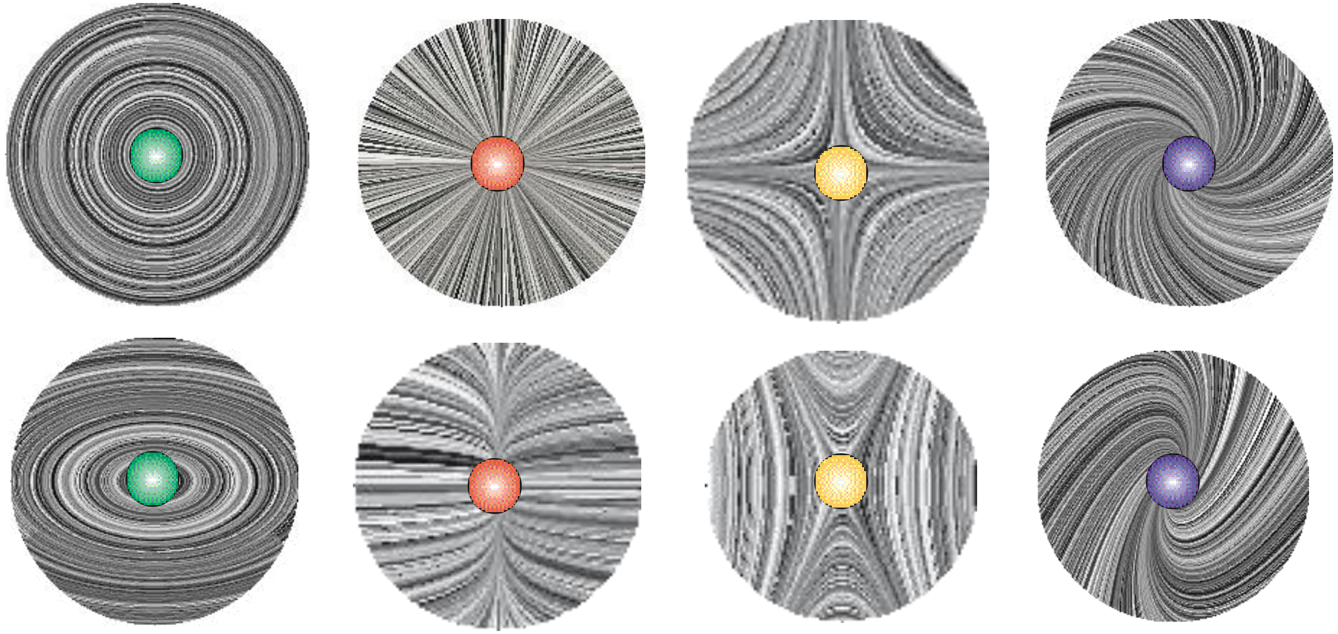


Vector Fields

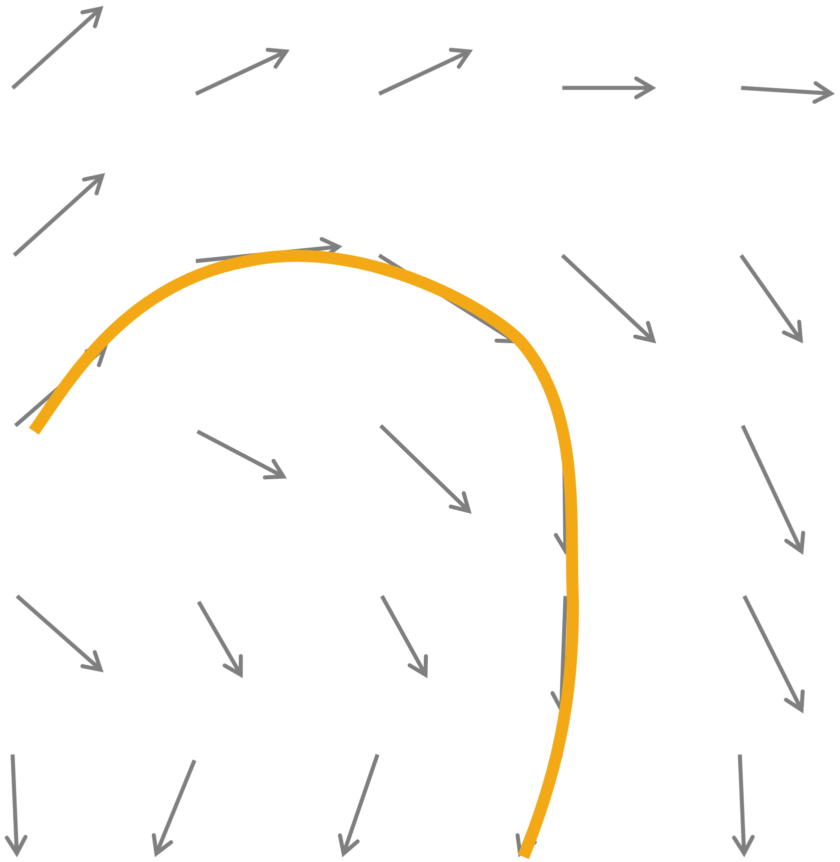
Visualization (more on this later)

Vector Fields – Critical Points

A point $\mathbf{x}_0 \in E_2$ is called a critical point iff $\mathbf{v}(\mathbf{x}_0) = 0$ and $\mathbf{v}(\mathbf{x}) \neq 0$ for any $\mathbf{x} \neq \mathbf{x}_0$ in a certain neighborhood of $\mathbf{x}_0$.



Vector Fields – Stream Lines



Stream line (tangent curve):

A stream line $\mathbf{s}(t)$ of the vector field $\mathbf{v}$ is a curve in E_2 with

$$\dot{\mathbf{s}}(t) = \mathbf{v}(\mathbf{s}(t))$$

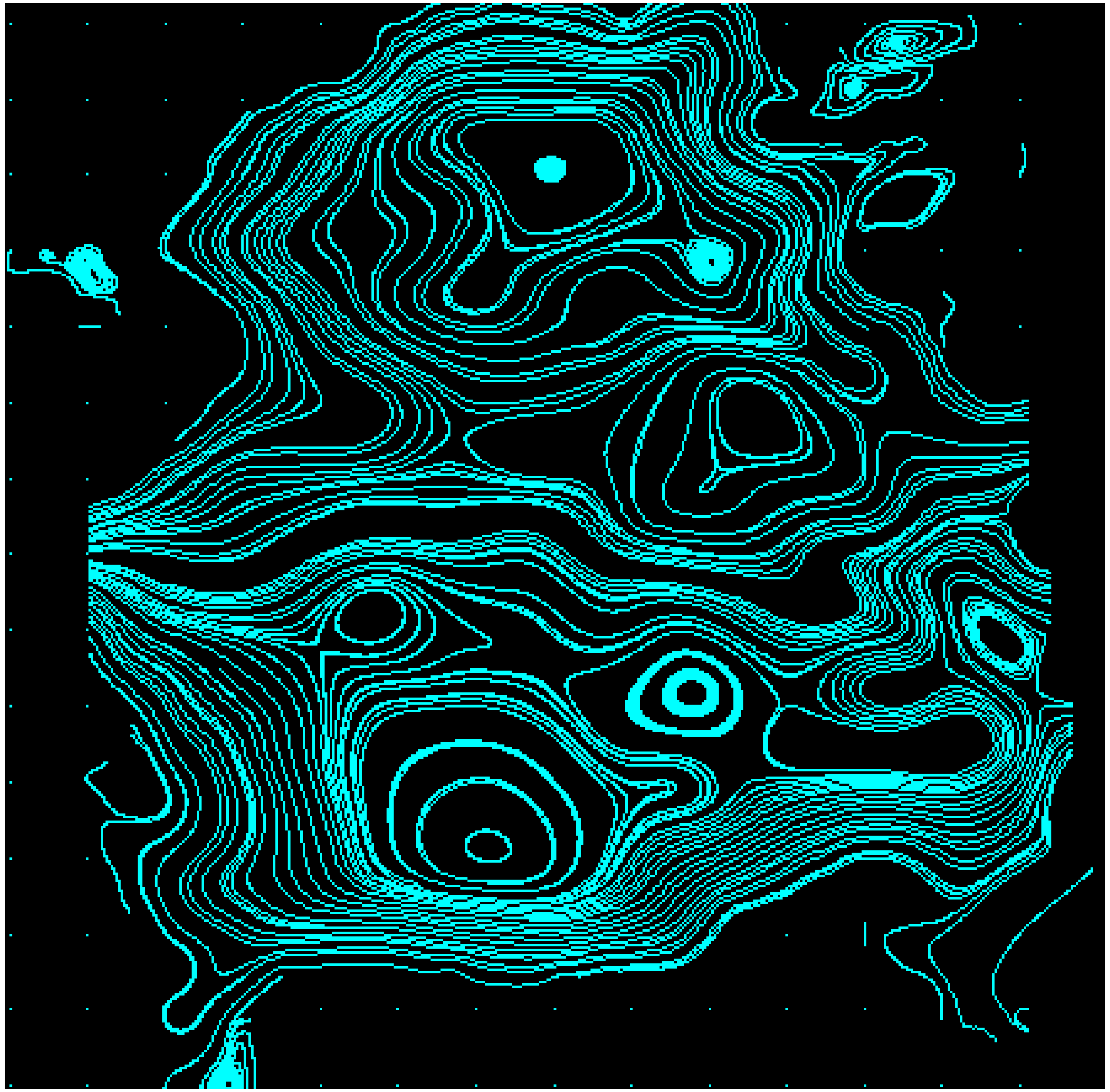
for any t of the domain of $\mathbf{s}$. ($\dot{\mathbf{s}}(t)$ denotes the tangent vectors of $\mathbf{s}(t)$).

Interpretation:

Stream line is the path of a massless particle in a flow described by $\mathbf{v}$.

Examples of stream lines

Vector Fields – Stream Lines



Vector Fields – Stream Lines

Properties of stream lines:

- Stream lines do not intersect each other (except for critical points of $\mathbf{v}$).

- Given a point in the vector field $\mathbf{v}$, there is one and only one stream line through it (except for critical points of $\mathbf{v}$).

- A parametric description of stream lines is usually not possible.

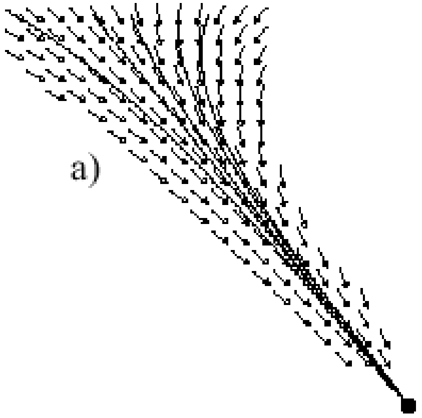
Vector Fields – Higher–Order Critical Points

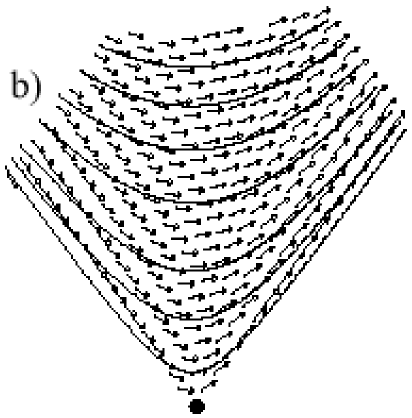
2D critical points are classified by distinguishing regions of different flow behavior around a critical point.

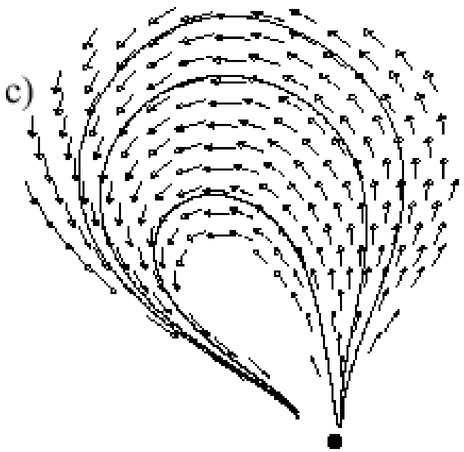
3 cases are possible:

- In a *parabolic* sector either all stream lines end, or all stream lines originate, in the critical point.
- In a *hyperbolic* sector all stream lines go by the critical point, except for two stream lines making the boundaries of the sector. One of these two stream lines ends in the critical point while the other one originates in it
- In an *elliptic* sector all stream lines originate and end in the critical point.

Vector Fields – Critical Points



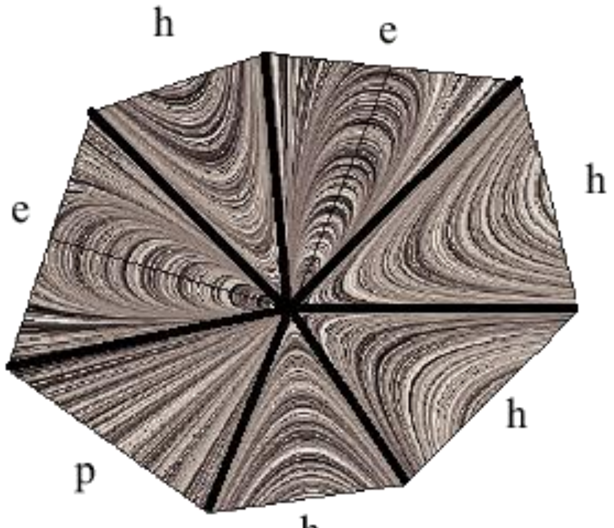




a) parabolic sector
sector

b) hyperbolic sector

c) elliptic



critical point consisting of 7 sectors

Vector Fields – First–Order 2D Critical Points

The point $\mathbf{x}_0$ is a **first-order** critical point of vector field $\mathbf{v}$ iff

1. $\mathbf{x}_0$ is a critical point of $\mathbf{v}$, and
2. $\mathbf{v}$ is differentiable at $\mathbf{x}_0$,
3. $\det\big(\mathbf{J}_\mathbf{v}(\mathbf{x}_0)\big) \neq 0.$

They can be analyzed by an eigenanalysis of $\mathbf{J}_\mathbf{v}(\mathbf{x}_0)$

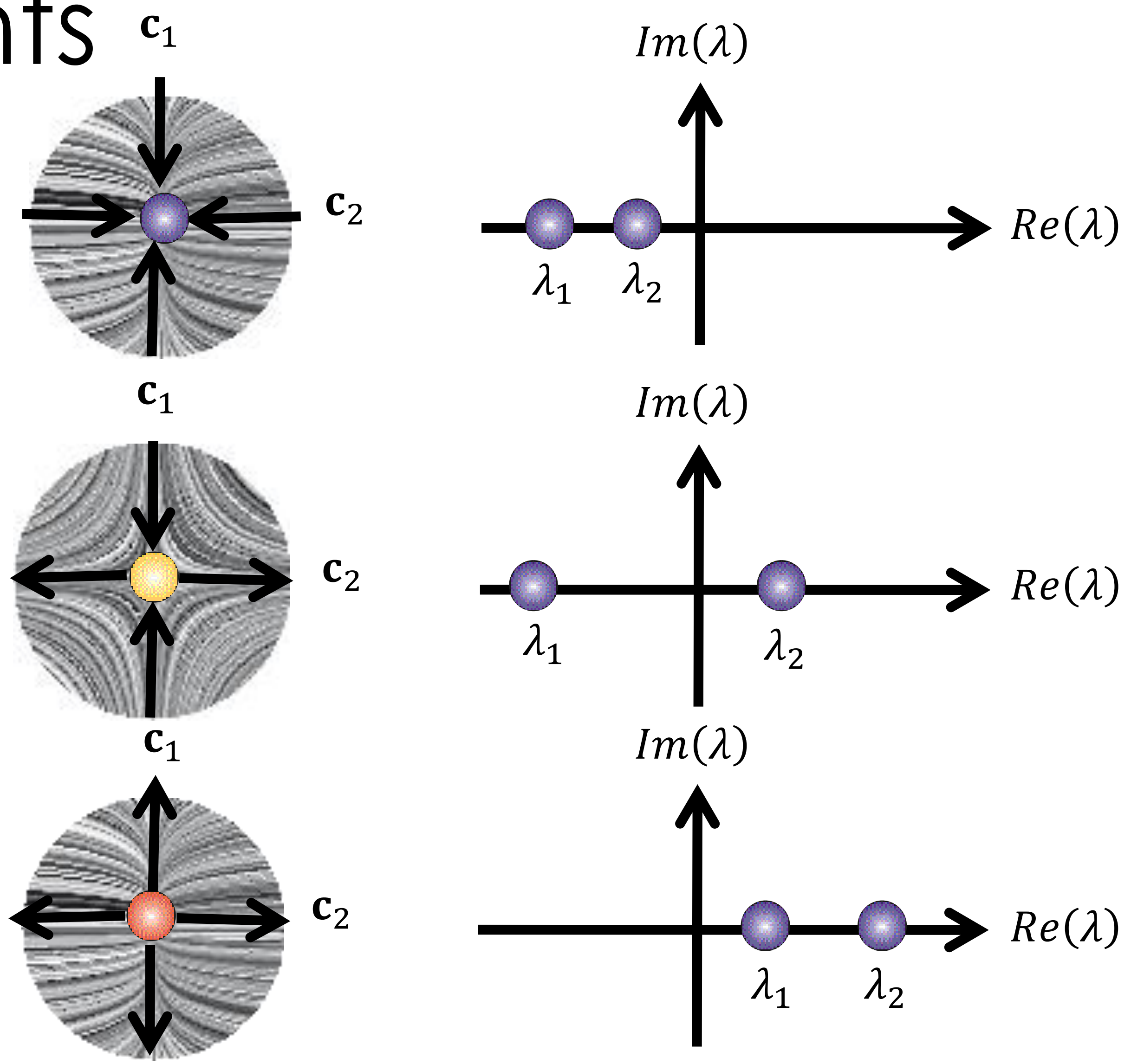
Vector Fields – First–Order 2D Critical Points

Let λ_1, λ_2 be the eigenvalues of $\mathbf{J_v}(\mathbf{x}_0)$ with $Re(\lambda_1) \leq Re(\lambda_2)$, and let $\mathbf{c}_1, \mathbf{c}_2$ be the corresponding eigenvectors.

- $Re(\lambda_i) < 0$ → inflow behavior
- $Re(\lambda_i) > 0$ → outflow behavior
- $Im(\lambda_1) = -Im(\lambda_2) \neq 0$ → swirling behavior

- $\mathbf{c}_i$ describes the direction of straight inflow/outflow.

Vector Fields – First–Order 2D Critical Points

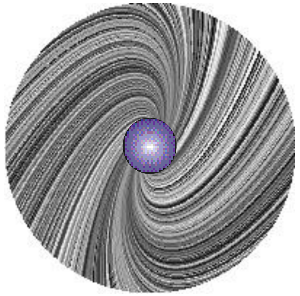


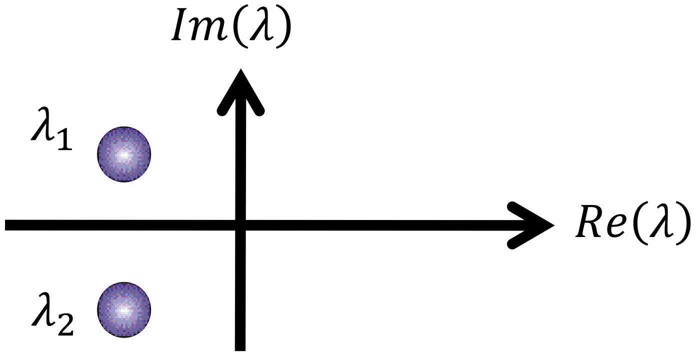
Attracting node (sink)

Saddle node

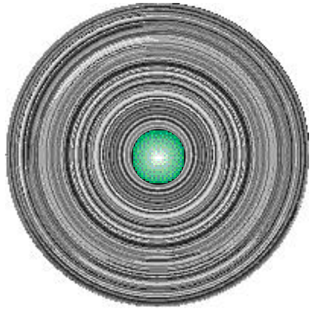
Repelling node (source)

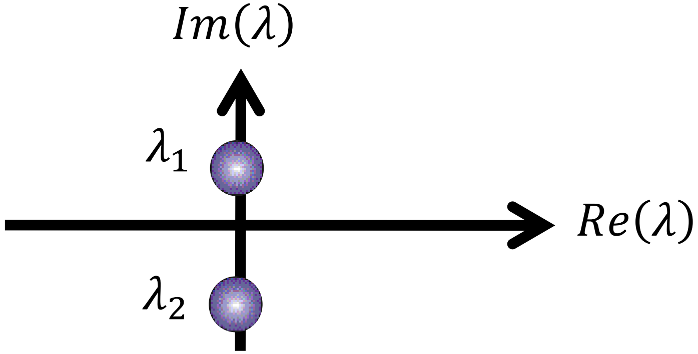
Vector Fields – First–Order 2D Critical Points



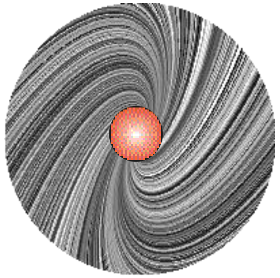


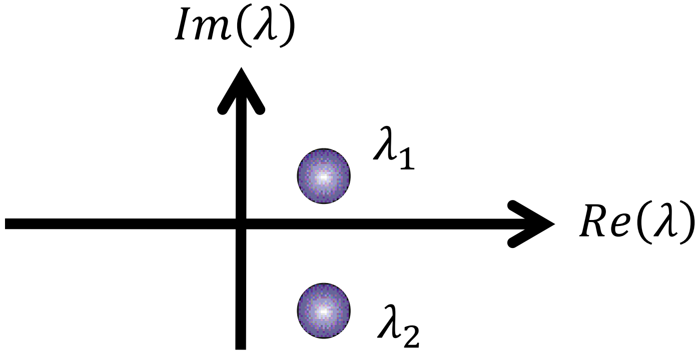
Attracting focus (sink)





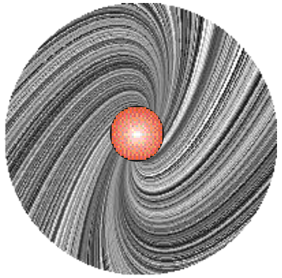
Center (structurally unstable)





Repelling focus (source)

Vector Fields – First–Order 2D Critical Points



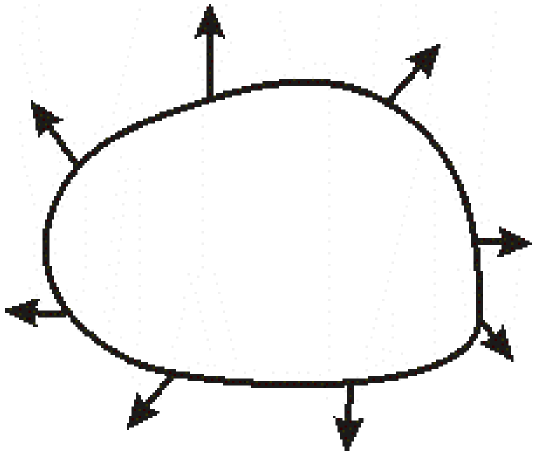
Rotation direction of a “swirling” critical point?

- Simple heuristic:
  Go an epsilon to the “right” and test if the velocity vector points up or down.

Poincaré Hopf Theorem

Vector Fields – Poincaré–index

- Consider a closed curve $\mathbf{c}$ in D such that no critical point of $\mathbf{v}$ in on $\mathbf{c}$.

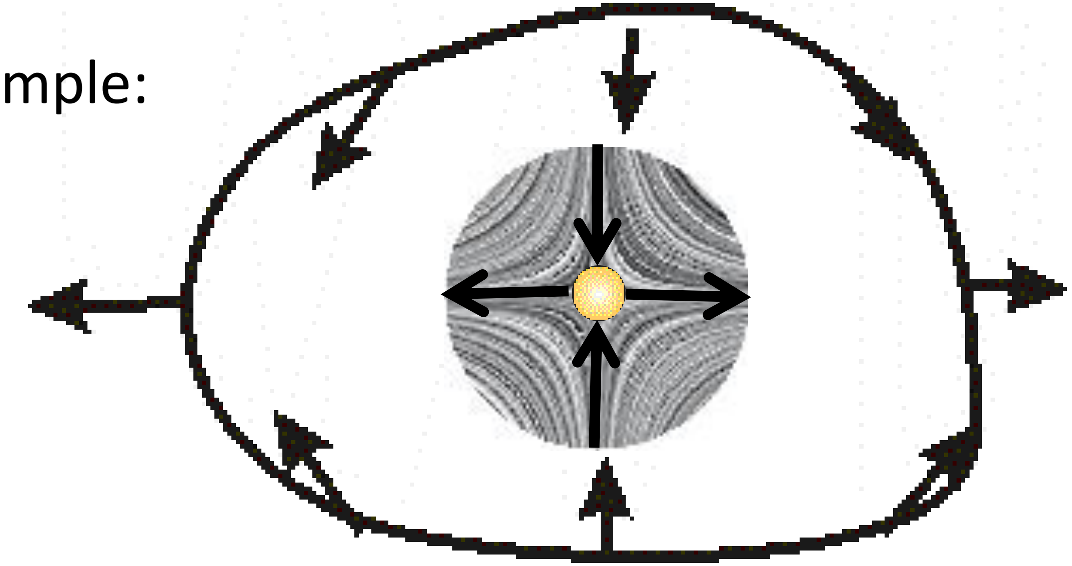
- Index: number of counterclockwise rotations of the vectors of $\mathbf{v}$ while travelling counterclockwise on the closed curve
  - index is (possibly negative) integer
  - Example:  index = 1

Vector Fields – Poincaré–index

Index of a critical point:

- Place **c** around critical point such that no other critical point lies inside **c**.

- Example:



index = -1

Vector Fields – Poincaré–index

First order critical points

- have index +1 or -1
- index(saddle) = -1
- index(source/sink/center) = +1

General critical point: index can be obtained by counting different sectors:

$$\text{index} = 1 + \frac{n_e - n_h}{2}$$

- n_e : number of elliptic sectors
- n_h : number of hyperbolic sectors

Vector Fields – Poincaré–index

Index Theorem:

- $$\text{index(area)} = \sum_{\substack{\text{all critical}\\ \text{points cp}\\ \text{in the area}}} \text{index(cp)}$$

Vector Fields – First–Order 3D Critical Points

Classification of first order 3D critical points by eigenanalysis of Jacobian matrix:

- $\lambda_1, \lambda_2, \lambda_3$: eigenvalues of $\mathbf{J_v}$
- $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$: eigenvectors of $\mathbf{J_v}$

Vector Fields – First–Order 3D Critical Points

- Positive eigenvalues: outflow
- Negative eigenvalues: inflow

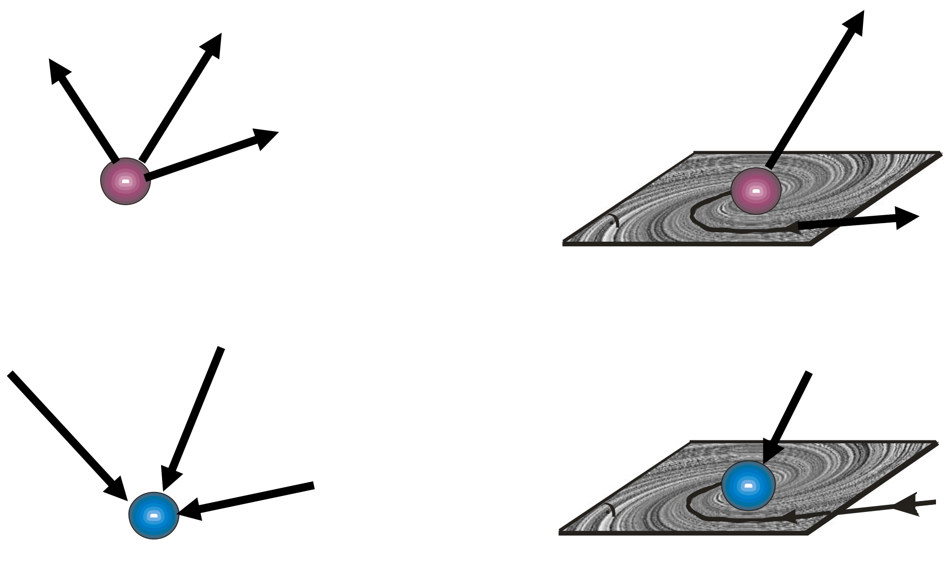
Sources:	$0 < Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3)$
Repelling saddles:	$Re(\lambda_1) < 0 < Re(\lambda_2) \leq Re(\lambda_3)$
Attracting saddles:	$Re(\lambda_1) \leq Re(\lambda_2) < 0 < Re(\lambda_3)$
Sinks:	$Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3) < 0$

- Each of these 4 cases can be further subdivided by considering the imaginary parts:

Foci:	$Im(\lambda_1) = 0$ and $Im(\lambda_2) = -Im(\lambda_3) \neq 0$
Nodes:	$Im(\lambda_1) = Im(\lambda_2) = Im(\lambda_3) = 0$

Vector Fields – First–Order 3D Critical Points

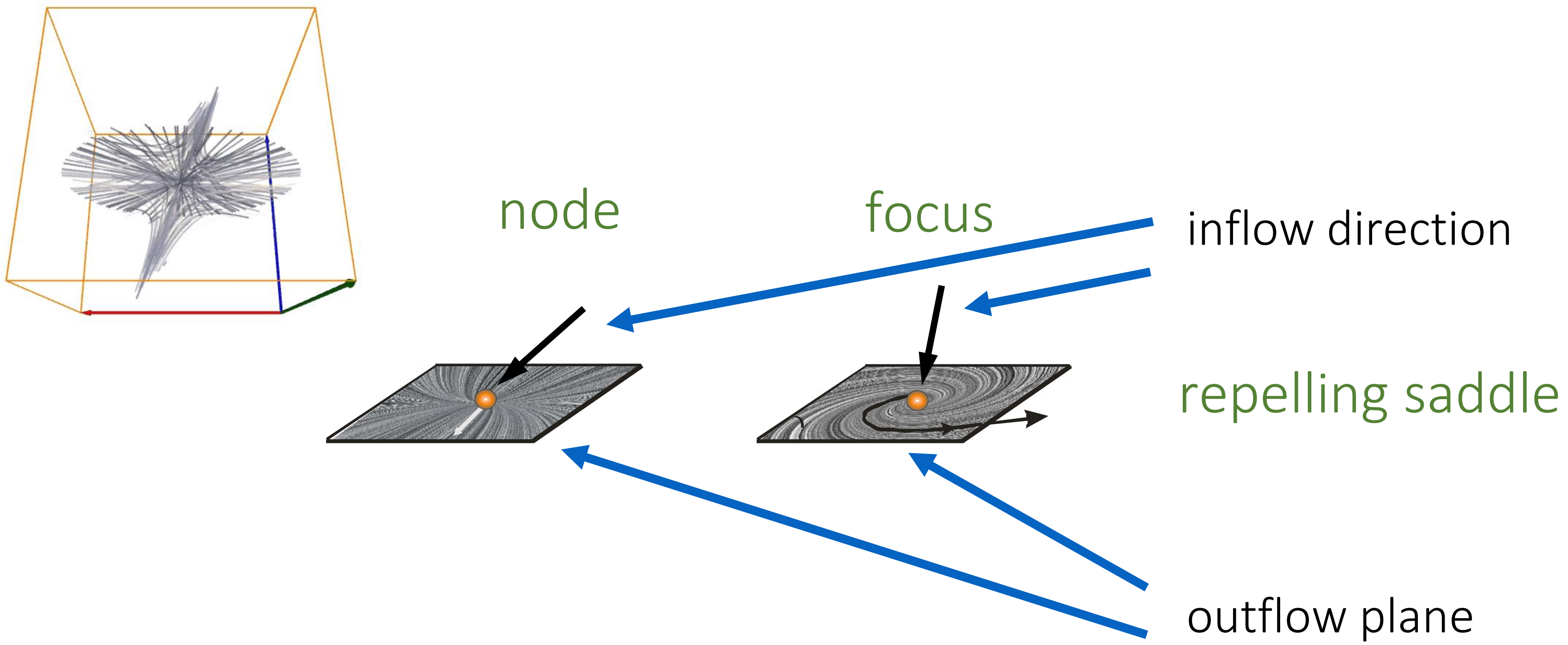
node focus



source

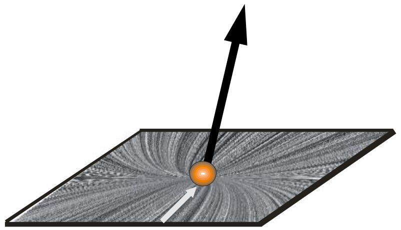
sink

Vector Fields – First-Order 3D Critical Points

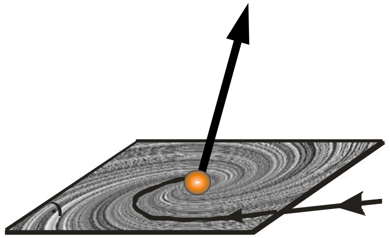


Vector Fields – First-Order 3D Critical Points

node



focus



attracting saddle

Vector Fields – First–Order 3D Critical Points

Index of first-order critical points:

		Index:
Sources:	$0 < Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3)$	1
Repelling saddles:	$Re(\lambda_1) < 0 < Re(\lambda_2) \leq Re(\lambda_3)$	-1
Attracting saddles:	$Re(\lambda_1) \leq Re(\lambda_2) < 0 < Re(\lambda_3)$	1
Sinks:	$Re(\lambda_1) \leq Re(\lambda_2) \leq Re(\lambda_3) < 0$	-1

Divergence

Vector Fields – Properties: Divergence

Divergence of $\mathbf{v}$:

- scalar field
- observe transport of a small ball around a point
  - expand volume → positive divergence
  - reduce volume → negative volume
  - constant volume → zero divergence

$$\boxed{\operatorname{div}\,\mathbf{v} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z} = u_x + v_y + w_z}$$

  - $\operatorname{div}\mathbf{v} \equiv 0 \Leftrightarrow \mathbf{v}$ is incompressible

Vector Fields – Properties: Divergence

- As an example, we want to determine the divergence of the gradient of

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

$$f(x,y) = \exp(-0.2 \cdot (x^2 + y^2))$$

Vector Fields – Properties: Divergence

- The gradient

$$\nabla f(x,y) = -0.4 \cdot f(x,y) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Vector Fields – Properties: Divergence

- We need the x and y derivative of the first and the second component of the gradient, respectively (we already determined it with the Hessian matrix)

$$\frac{\partial^2 f}{\partial x^2} f(x,y) = (0.16x^2 - 0.4) \cdot f(x,y)$$
$$\frac{\partial^2 f}{\partial y^2} f(x,y) = (0.16y^2 - 0.4) \cdot f(x,y)$$

Vector Fields – Properties: Divergence

- This yields the divergence

$$\frac{\partial^2 f}{\partial x^2} f(x,y) = (0.16x^2 - 0.4) \cdot f(x,y)$$

$$\frac{\partial^2 f}{\partial y^2} f(x,y) = (0.16y^2 - 0.4) \cdot f(x,y)$$

$$\Rightarrow \quad \operatorname{div} \nabla f = 0.16 \cdot (x^2 + y^2 - 5) \cdot f(x,y)$$

Divergence – Image

- Estimated divergence:

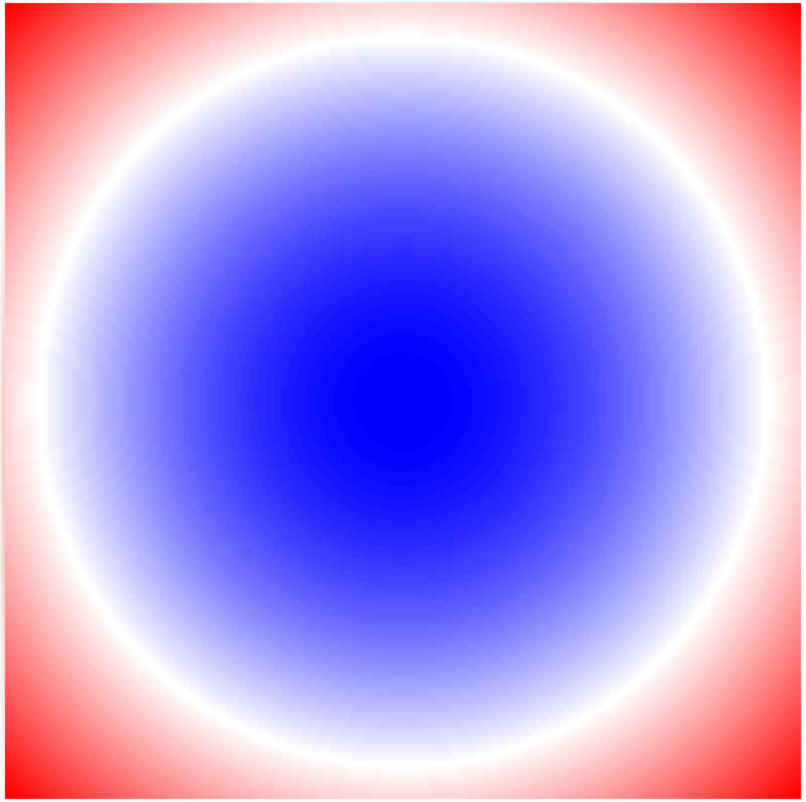
$$div f(x,y) = f_x(x,y) - f_x(x-1,y) + f_y(x,y) - f_y(x,y-1)$$

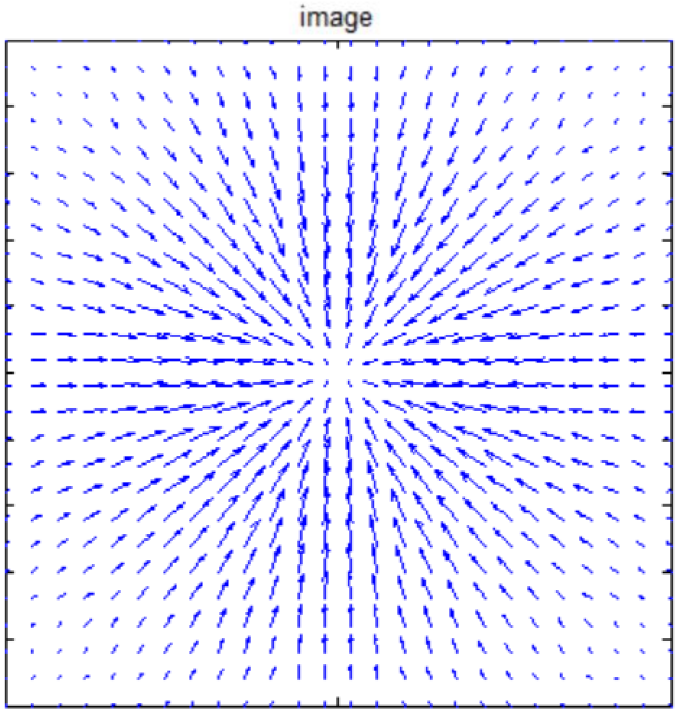
Divergence – Image

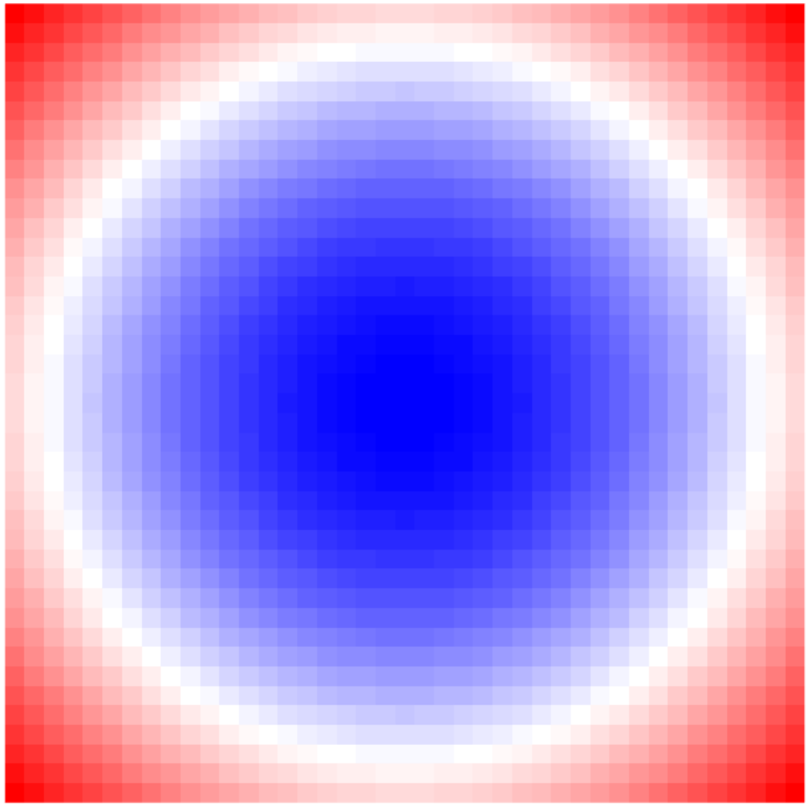
- Remark: We have a derivative in x and y direction, both have forward and backward derivatives -> 4 possibilities
- Again, we use forward

Divergence – Image

- Left continuous divergence and right discrete divergence (of image middle)







Curl

Vector Fields – Properties: Curl

Curl of $\mathbf{v}$:

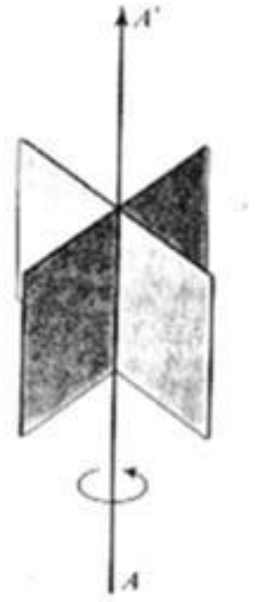
- vector field
- also called rotation (rot) or vorticity
- indication of how the field swirls at a point

$$\mathbf{curl\ v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ u & v & w \end{vmatrix} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

Vector Fields – Properties: Curl

Paddle wheel model

- insert paddle wheel in a flow
- orient such that its rate of rotation is maximal
- $\rightarrow$ $\mathbf{curl\ v}$ is parallel to main rotation axis
- $\rightarrow$ $|\mathbf{curl\ v}|$ is corresponds to rate of rotation



Golf ball model

- consider golf ball in v
- rotates
- $\rightarrow$ $\mathbf{curl\ v}$ is parallel to main rotation axis
- $\rightarrow$ $|\mathbf{curl\ v}|$ is corresponds to rate of rotation

Vector Fields – Properties

- $\mathbf{curl}\,\mathbf{v} \equiv 0 \Leftrightarrow \mathbf{v}$ is irrotational or curl-free

- $\mathbf{v} = \operatorname{grad} f \Leftrightarrow \mathbf{v}$ is conservative

- Conservative is subclass of curl-free, since $\mathbf{curl}\operatorname{grad} f \equiv 0$ for any scalar field f

Laplace Operator

Scalar Fields – Properties: Laplacian

Laplacian of a scalar field:

- scalar field

$$\boxed{\begin{aligned} Lf &= \operatorname{div} \mathbf{grad}\, f = \operatorname{div} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \\ &= \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} + \frac{\delta^2 f}{\delta z^2} = f_{xx} + f_{yy} + f_{zz} \end{aligned}}$$

Scalar Fields – Properties: Laplacian

- L invariant under rotation and translation of the underlying coordinate system
- $L\,f \equiv 0 \Leftrightarrow f$ is harmonic function

- Interpretation of Laplacian:
  - Measure of the difference between the average value of f in the immediate neighborhood of the point and the precise value of the field at the point.

Scalar Fields – Properties: Laplacian

- Laplace of our example?
- We already determined it

$$f(x, y) = \exp(-0.2 \cdot (x^2 + y^2))$$

$$f(x,y) = \exp(-0.2 \cdot (x^2 + y^2))$$

Scalar Fields – Properties: Laplacian

- This yields the Laplacian

$$\frac{\partial^2 f}{\partial x^2} f(x,y) = (0.16x^2 - 0.4) \cdot f(x,y)$$
$$\frac{\partial^2 f}{\partial y^2} f(x,y) = (0.16y^2 - 0.4) \cdot f(x,y)$$

$$\Longrightarrow \quad \operatorname{div} \nabla f = 0.16 \cdot (x^2 + y^2 - 5) \cdot f(x,y)$$

Laplace Operator – Image

- 2nd derivative is twice the 1st derivative
- 2nd derivative in x direction:

$$
\begin{aligned}
f'_{\vec{x}}(x,y) &= f(x,y) - f(x-1,y) \\
f''_{\vec{x}\,\vec{x}}(x,y) &= f'_{\vec{x}}(x,y) - f'_{\vec{x}}(x-1,y) \\
&= f(x,y) - f(x-1,y) - (f(x-1,y) - f(x-2,y)) \\
&= f(x,y) - 2 \cdot f(x-1,y) + f(x-2,y)
\end{aligned}
$$

Laplace Operator – Image

- 2nd derivative in x direction depends on neighbor and the neighbor's neighbor
- Not intuitive!

$$f''_{\vec{x}\vec{x}}(x, y) = f(x, y) - 2 \cdot f(x-1, y) + f(x-2, y)$$

Laplace Operator – Image

- Change the direction:

$$
\begin{aligned}
f''_{\overrightarrow{x}\overleftarrow{x}}(x,y) &= f'_{\overleftarrow{x}}(x,y) - f'_{\overleftarrow{x}}(x-1,y) \\
&= f(x,y) - f(x+1,y) - (f(x-1,y) - f(x,y)) \\
&= -f(x-1,y) + 2\cdot f(x,y) - f(x+1,y)
\end{aligned}
$$

Laplace Operator – Image

- Rewrite in matrix notation:

$$f''_{\overrightarrow{x}\overleftarrow{x}}(x,y) = \begin{pmatrix} -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} f(x-1,y) \\ f(x,y) \\ f(x+1,y) \end{pmatrix}$$

$$f''_{\overrightarrow{y}\overleftarrow{y}}(x,y) = \begin{pmatrix} -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} f(x,y-1) \\ f(x,y) \\ f(x,y+1) \end{pmatrix}$$

Laplace Operator

$$f''_{\overrightarrow{x}\overleftarrow{x}}(x,y) = \begin{pmatrix}-1 & 2 & -1\end{pmatrix}\begin{pmatrix}f(x-1,y)\\ f(x,y)\\ f(x+1,y)\end{pmatrix}$$

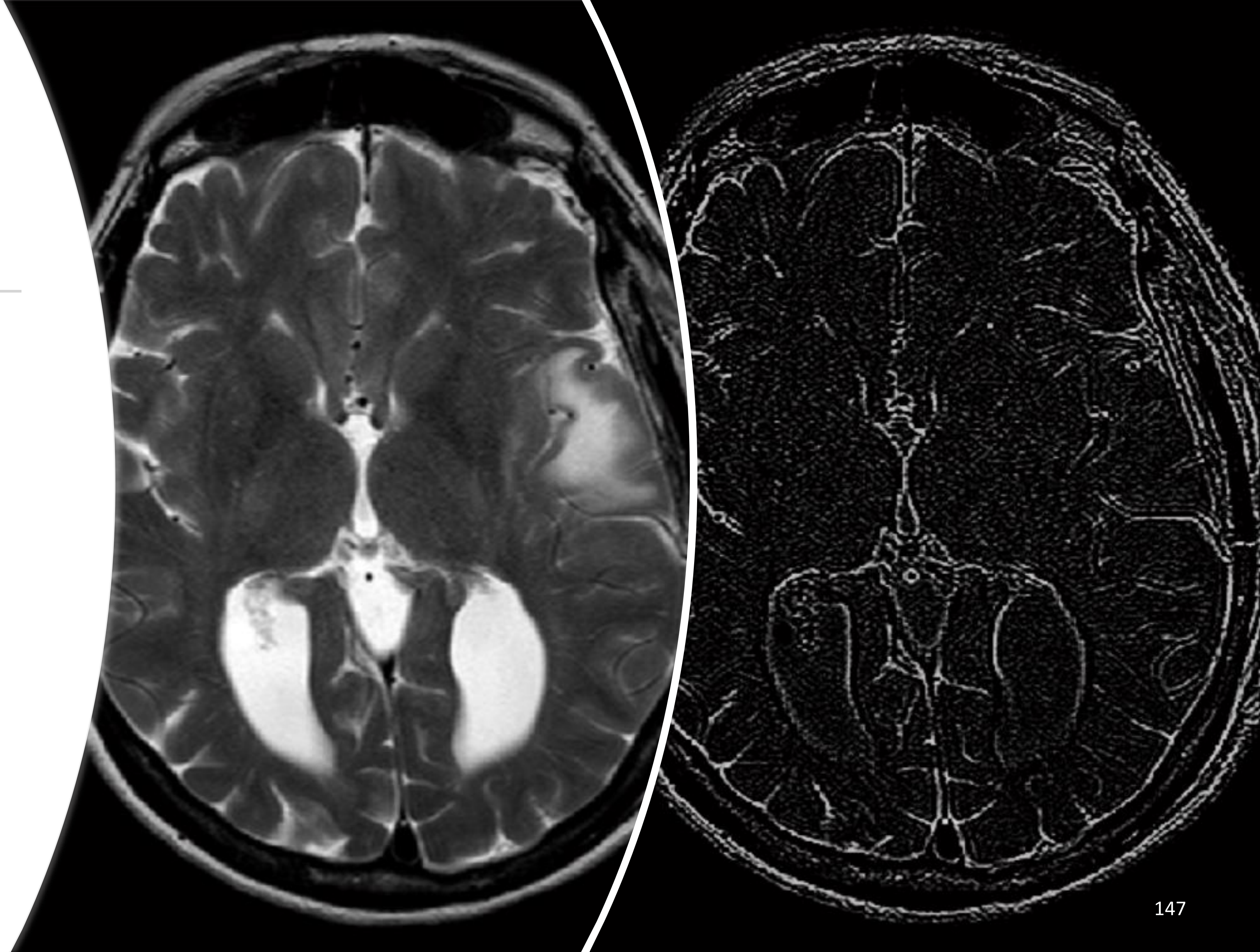
$$f''_{\overrightarrow{y}\overleftarrow{y}}(x,y) = \begin{pmatrix}-1 & 2 & -1\end{pmatrix}\begin{pmatrix}f(x,y-1)\\ f(x,y)\\ f(x,y+1)\end{pmatrix}$$

- Laplace kernel:

$$\begin{pmatrix}0 & -1 & 0\\ -1 & 4 & -1\\ 0 & -1 & 0\end{pmatrix}$$

Laplace – Image

- Laplace filter (right)



Nabla Operator

Nabla–Operator

The Nabla-operator:

- also called “Del”-operator
- abbreviation: ∇
- symbolically written as:

$$\nabla = \mathbf{i}\,\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\,\frac{\delta}{\delta z} = \begin{pmatrix} \dfrac{\delta}{\delta x} \\ \dfrac{\delta}{\delta y} \\ \dfrac{\delta}{\delta z} \end{pmatrix}$$

Nabla–Operator

This allows to write the other operators as

$$\mathbf{grad}\, f = \nabla f$$

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v}$$

$$\mathbf{J}_{\mathbf{v}} = \nabla \mathbf{v}$$

$$\mathbf{curl}\, \mathbf{v} = \nabla \times \mathbf{v}$$

$$L\, f = \operatorname{div}(\mathbf{grad}\, f) = \nabla \cdot (\nabla f) = \nabla^2 f$$

$$L\, \mathbf{v} = \nabla^2 \mathbf{v}$$

Identities

Scalar and Vector Identities

$$\nabla(f+g) = \nabla f + \nabla g$$

$$\nabla(cf) = c\nabla f$$ for a constant c

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$$ at points $\mathbf{x}$ where $g(\mathbf{x}) \neq 0$

$$\operatorname{div}(\mathbf{v}+\mathbf{w}) = \operatorname{div}\mathbf{v} + \operatorname{div}\mathbf{w}$$

$$\mathbf{curl}(\mathbf{v}+\mathbf{w}) = \mathbf{curl}\,\mathbf{v} + \mathbf{curl}\,\mathbf{w}$$

$$\operatorname{div}(f\,\mathbf{v}) = f\operatorname{div}\mathbf{v} + \mathbf{v}\cdot\nabla f$$

$$\operatorname{div}(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot\mathbf{curl}\,\mathbf{v} - \mathbf{v}\cdot\mathbf{curl}\,\mathbf{w}$$

Scalar and Vector Identities

$$\operatorname{div} \mathbf{curl}\, \mathbf{v} = 0$$

$$\mathbf{curl}\left(f\ \mathbf{v}\right) = f\ \mathbf{curl}\, \mathbf{v} + \nabla f \times \mathbf{v}$$

$$\mathbf{curl}\, \nabla f = \mathbf{0}$$

$$\nabla^2\left(f\ g\right) = f\nabla^2 g + g\nabla^2 f + 2\left(\nabla f \cdot \nabla g\right)$$

$$\operatorname{div}\left(\nabla f \times \nabla g\right) = 0$$

$$\operatorname{div}\left(f\ \nabla g - g\ \nabla f\right) = f\ \nabla^2 g - g\ \nabla^2 f$$

Questions???