Starting algorithm and modification for Newton – Raphson load-flow method

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A starting algorithm for the Newton-Raphson method of load-flow calculation is proposed. The algorithm provides good starting values for the system variables by defining two new functions, the coupling between which is weaker than the coupling between the normal active and reactive power mismatch functions. A modification to the Newton-Raphson method is also proposed that takes into account most of the 2nd-order terms that are normally neglected. Results of systems solved by the standard Newton-Raphson and other methods, and by the modified method with the new algorithm incorporated, demonstrate the advantages of the proposed algorithm and modification.

Keywords. network analysis, load flow, digital model

I. Notation

The superscript T denotes matrix transpose Over-barred letters indicate phasors or complex quantities

$P_i^{sp} + jQ_i^{sp}$	specified power at node i
$\Delta P_i + j\Delta Q_i$	complex power mismatch at node i
$\bar{Y}_{ij}$	element of the nodal admittance matrix
$G_{ij} + jB_{ij}$	$= \bar{Y}_{ij}$
θ_i	voltage angle at node i
V_i	voltage magnitude at node i
θ_{ij}	$= \theta_i - \theta_j$
γ_{ij}	angle of $\bar{Y}_{ij}$
β_i	$= \gamma_{ii} + \pi/2$
$\|\| A \|\|$	Euclidean norm of A, defined as the sum of the squares of all the elements of A
JK	Jacobian matrix of $(\Delta\mathbf{P}, \Delta\mathbf{Q})^T$
JK11, JK12, JK21, JK22	submatrices of **JK**
VR	exact solution voltage magnitude vector
θR	exact solution voltage angle vector
$\mathbf{S}(\mathbf{F}_x, \mathbf{y})$	coupling sensitivity matrix of **F**, a set of functions of **x** and **y**, the ijth element of S is $\partial x_i/\partial y_i$ calculated using F_i
$N + 1$	total number of nodes in the system
NL	number of load nodes in the system
k	set of nodes directly connected to node i including node i
axis $\angle\Phi$	axis, the angle of which is Φ in the complex plane
$i \in k$	i is an element of set k

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II. Introduction

Hundreds of papers have been written introducing methods to solve the load-flow problem[1] These methods differ in two aspects the form of the equations describing the system and the numerical algorithm used to solve them Essentially, the problem consists in finding **V** and $\boldsymbol{\theta}$ which satisfy the following set of nonlinear equations

$$\Delta P_i = P_i^{sp} - V_i \sum_{j \in k} V_j (G_{ij} \cos\theta_{ij} + B_{ij} \sin\theta_{ij}) \quad (1)$$

$$\Delta Q_i = Q_i^{sp} + V_i \sum_{j \in k} V_j (B_{ij} \cos\theta_{ij} - G_{ij} \sin\theta_{ij}) \quad (2)$$

The Newton-Raphson method of load-flow calculation has now gained widespread popularity It solves a set of linear equations of the following form

$$\begin{bmatrix} \Delta\mathbf{P} \\ \hline \Delta\mathbf{Q} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{JK11} & \mathbf{JK12} \\ \hline \mathbf{JK21} & \mathbf{JK22} \end{array}\right] \begin{bmatrix} \Delta\theta \\ \hline \Delta\mathbf{V}/\mathbf{V} \end{bmatrix} \quad (3)$$

during each iteration. The Jacobian and the power mismatches are to be calculated anew in each iteration of the formal Newton-Raphson method

To save computational time, it was suggested as early as 1967 to use the same Jacobian for more than one iteration[2] This method was found to have no advantage over the formal Newton-Raphson method, although favourable reports about it were published in 1978[3] The present author's experience is that the fixed Jacobian method could save computational time for lightly-loaded systems or when the estimated values of the variables are near the solution, usually after two Newton-Raphson iterations

To reduce the number of linear equations to be solved in each iteration, decoupling was suggested[4,5]. The decoupling method neglects **JK12** and **JK21** and solves two sets of linear equations in each iteration. $\Delta\mathbf{P} = \mathbf{JK11}\ \Delta\theta$ and $\Delta\mathbf{Q} = \mathbf{JK22} \cdot \Delta\mathbf{V}/\mathbf{V}$. Neglecting **JK21** and **JK12** sacrifices some accuracy in the neighbourhood of the solution, how-

ever, this disadvantage is more than offset when the improved values of the variables obtained in the first half of the iteration are used in the second half of the iteration, to calculate **JK22** and $\Delta\mathbf{Q}$

The idea behind the starting algorithm introduced in this paper is to define a set of functions $\mathbf{f}_{\theta i}, i = 1, \ldots, N$, for which the value of the Euclidean norm of their coupling sensitivity matrix $\| \mathbf{S}(\mathbf{f}_\theta, \mathbf{V}) \|$, is minimal. The resulting equations are linearized and solved for $\boldsymbol{\theta}$, they will be called the angle equations. Another set of functions $\mathbf{f}_{vi}, i = 1, \ldots, NL$ is defined, such that the Euclidean norm of their coupling sensitivity matrix $\| \mathbf{S}(\mathbf{f}_v, \boldsymbol{\theta}) \|$, is minimal, the resulting equations will be called the voltage equations The voltage equations are linearized and solved, the linearization, however, is performed in a special way so as to minimize the effect of neglecting the 2nd-order terms This method of generating starting values gives better results than any other reported method known to the authors.

Applying the proposed modification to the basic Newton-Raphson iteration implies taking into account most of the 2nd-order terms and some of the higher order terms of a Taylor's series expansion of the load-flow equations. This is achieved by finding relations between $f(x), f'(x_0)$ and $f''(x_0)$, which enables these 2nd-order terms to be generated and included implicitly in the linear approximating functions

III. The starting algorithm

The starting algorithm is a decoupling algorithm applied to a new set of equations One decoupling iteration as applied to equations (1) and (2) can be summarized as follows

(i) Use the best available estimate of $\boldsymbol{\theta}$ and $\mathbf{V}$ to calculate $\Delta\mathbf{P}$, **JK11** and solve $\Delta\mathbf{P} = \mathbf{JK11} \cdot \Delta\boldsymbol{\theta}$ for $\Delta\boldsymbol{\theta}$

(ii) Update $\boldsymbol{\theta}$, calculate $\Delta\mathbf{Q}$, **JK22** and solve

$$\Delta\mathbf{Q} = \mathbf{JK22} \cdot \Delta\mathbf{V}/\mathbf{V}$$

for $\Delta\mathbf{V}/\mathbf{V}$

(iii) Update $\mathbf{V}$ and test for convergence.

(iv) Go to 1 if not converged

From this it can be seen that in a $(\mathbf{P}-\boldsymbol{\theta}, \mathbf{Q}-\mathbf{V})$ decoupling, **JK12** and **JK21** are neglected in equation (3).

Decoupling can be applied to equations that have forms different from those of equations (1) and (2) but have the same solution[6,7] The question arises as to what is the best form of equations to decouple To answer that question, it is only logical to assume that the smaller the relative magnitude, in some sense, of the neglected coupling subJacobians, the better the updated values of $\boldsymbol{\theta}$ and $\mathbf{V}$, or the smaller the effect of the error in $\mathbf{V}$ upon the calculation of $\boldsymbol{\theta}$ in the first half of the iteration Also, the smaller the effect of the error in $\boldsymbol{\theta}$ upon the calculation of $\mathbf{V}$ in the second half of the iteration, the better. In other words, terms such as $\partial\theta/\partial V$ and $\partial V/\partial\theta$ should be minimized when calculating $\boldsymbol{\theta}$ and $\mathbf{V}$ respectively. If all these terms are zero, then the two sets of equations to be decoupled are reducible[8] and can be solved completely independently. With this in mind, the angle and the voltage equations are developed

III 1 *Angle equations*

The functions to be used for the calculation of $\boldsymbol{\theta}$ are assumed to be of the form.

$$f_{\theta i}(\theta) = P_i \cos\Phi_i + Q_i \sin\Phi_i \tag{4}$$

where Φ_i is an element of the constant n-dimensional vector $\boldsymbol{\Phi}$ to be determined. $\mathbf{S}(\mathbf{f}_\theta, \mathbf{V})$ is the $N \times NL$ coupling sensitivity matrix, the ijth element of which is $\partial\theta_i/\partial V_j$ calculated using equation (4) It is required to determine $\boldsymbol{\Phi}$ such that $\| \mathbf{S}(\mathbf{f}_\theta, \mathbf{V}) \|$ is minimal to calculate the ijth element of $\mathbf{S}(\mathbf{f}_\theta, \mathbf{V})$, the general formula

$$\frac{\partial\theta_i}{\partial V_j} = -\frac{\partial f_{\theta i}/\partial V_j}{\partial f_{\theta i}/\partial\theta_i} \tag{5}$$

will be used. It is evident from equations (3) and (4) that

$$\partial f_{\theta i}/\partial V_j = JK12_{ij} \cos\Phi_i + JK22_{ij} \sin\Phi_i \tag{6}$$

and

$$\partial f_{\theta i}/\partial\theta_i = JK11_{ii} \cos\Phi_i + JK21_{ii} \sin\Phi_i \tag{7}$$

Using flat-start values to calculate the subJacobian elements of equations (6) and (7), and then substituting equations (6) and (7) in equation (5), one gets

$$\frac{\partial\theta_i}{\partial V_j} = \frac{G_{ij} \cos\Phi_i - B_{ij} \sin\Phi_i}{B_{ii} \cos\Phi_i + G_{ii} \sin\Phi_i} \tag{8}$$

$i = 1, \ldots, N, j = 1, \ldots, NL$. It is shown in Appendix 1 that choosing $\Phi_i = -\beta_i = \tan^{-1}(G_{ii}/B_{ii})$ makes $\| \mathbf{S}(\mathbf{f}_{\theta i}, \mathbf{V}) \|$ minimal, doing so, equation (4) can be written as

$$f_{\theta i} = B_{ii} \Delta P_i + G_{ii} \Delta Q_i = 0 \qquad i = 1, \ldots, N \tag{9}$$

III 2 *Voltage equations*

Applying reasoning similar to that of the previous section and trying to minimize $\| \mathbf{S}(\mathbf{f}_v, \boldsymbol{\theta}) \|$, the following functions for $\mathbf{V}$ calculation are obtained

$$f_{vi} = G_{ii} \Delta P_i - B_{ii} \Delta Q_i = 0 \qquad i = 1, \ldots, NL \tag{10}$$

It can be seen that $f_{\theta i}$ is the complex power mismatch component at node i along axis $\angle -\beta_i$ and f_{vi} is the component along axis $\angle -\gamma_{ii}$

Use of the voltage and the angle equations is analogous to the elimination method for solving two linear equations. To illustrate this, consider the equations

$$F_1 \equiv A_1 X + B_1 Y = C_1 \tag{11a}$$

$$F_2 \equiv A_2 X + B_2 Y = C_2 \tag{11b}$$

The first step in the elimination method is to derive two new functions from equations (11) such that $\partial X/\partial Y$ vanishes for the first and $\partial Y/\partial X$ vanishes for the second. $\partial Y/\partial X$ and $\partial X/\partial Y$ resemble the coupling sensitivity matrices

A graphical illustration of the effectiveness of the angle and the voltage equations is given in Appendix 2.

IV. Modifying the Newton-Raphson equations

It can be shown that, in the various diagonal elements of the subJacobians, there are expressions of functions whose zeros are sought, e.g Q_i appears in $JK11_{ii}$. Evidently, the Q_i must be evaluated using the best estimate of the variables. The proposed modification is to use Q_i^m and P_i^m in place of Q_i and P_i, respectively, whenever either of them appears in a diagonal element of a subJacobian, where

$$Q_i^m = 0.5(Q_i + Q_i^{sp})$$

and

$$P_i^m = 0.5(P_i + P_i^{sp}) \tag{12}$$

The resulting modified diagonal elements will be distinguished from the unmodified ones by the superscript m.

To examine the effect of this modification, Q_i^{sp} and P_i^{sp} in equation (12) must be expanded and the resulting expressions of Q_i^m and P_i^m must be substituted in the expressions of the diagonal elements of the subJacobians. The result is that each modified diagonal element implicitly contains the original diagonal element plus 2nd-order terms. In this way, most of the 2nd-order terms that were neglected in the usual Newton-Raphson method are included in the modified version. Equation (13) shows the 2nd-order terms that are generated by applying the modification to one linearized ΔP_i equation

$$\begin{aligned} &JK11_{ii}^m \Delta\theta_i + JK12_{ii}^m \frac{\Delta V_i}{V_i} \\ &= JK11_{ii}\Delta\theta_i + JK12_{ii}\frac{\Delta V_i}{V_i} + \frac{1}{2}\sum_{j\in k}\frac{\partial^2 \Delta P_i}{\partial\theta_i \partial\theta_j}\Delta\theta_i \Delta\theta_j \\ &+ \frac{1}{2}\sum_{j\in k}\frac{\partial^2 \Delta P_i}{\partial V_i \partial V_j}\Delta V_i \Delta V_j + \frac{\partial^2 \Delta P_i}{\partial\theta_i \partial V_i}\Delta\theta_i \Delta V_i \\ &+ \frac{1}{2}\sum_{j\in L}\frac{\partial^2 \Delta P_i}{\partial\theta_i \partial V_j}\Delta\theta_i \Delta V_j + \frac{JK12_{ii}}{2}\left(\frac{\Delta V_i}{V_i}\right)^2 \\ &+ \frac{1}{2}\sum_{j\in L}\frac{\partial^2 \Delta P_i}{\partial\theta_j \partial V_i}\Delta\theta_j \Delta V_i + V_i^2 B_{ii}\Delta\theta_i \frac{\Delta V_i}{V_i} \end{aligned} \tag{13}$$

More detailed equations appear in Appendix 3.

V. Implementation on existing Newton-Raphson program

The implementation of the starting algorithm is simple and requires no additional core The Jacobian of equation (9) is straightforward to construct since $\partial\Delta P_i/\partial\theta_j$ and $\partial\Delta Q_i/\partial\theta_j$ have already been calculated during the process of deriving the nodal active and reactive power mismatches. These terms are $JK11_{ij}$ and $JK21_{ij}$. Element JK_{ij} of the Jacobian of equation (9) is then

$$JK_{ij} = B_{ii}JK11_{ij} - G_{ii}JK21_{ij} \tag{14}$$

Since the sparsity of the Jacobian resulting from the angle equations is similar to the sparsity of the Jacobian resulting from the $(P-\theta)$ decoupling algorithm, the only extra computations necessary for each element of the Jacobian are the two multiplications and the one subtraction indicated by equation (14) Because of sparsity, the extra time needed is insignificant. Similar considerations apply to the voltage equations. In this case

$$JK_{ij} = G_{ii}JK12_{ij} + B_{ii}JK22_{ij} \tag{15}$$

The implementation of the proposed modification in the Newton-Raphson method requires no extra storage and no extra computation. It requires only the substitution of the specified active and reactive powers in the diagonal elements of the subJacobians. For the real power equations of the voltage-controlled nodes, the reactive power is not specified, therefore, only the specified real power is substituted for its calculated counterpart

VI. Tests and comparison of results

For testing the numerical performance of the method, the following procedure was adopted

(1) The solution voltage magnitude vector **VR** and the solution angle vector $\boldsymbol{\theta}$**R** are assumed.

(2) The nodal power components $\mathbf{P}^{sp}$ and $\mathbf{Q}^{sp}$ are calculated using **VR** and $\boldsymbol{\theta}$**R**

(3) Flat-start is assumed and a solution found

Four networks were used for testing the proposed algorithm the 5-node system of Reference 5, the 39-node IEEE system, and 23-node and 28-node systems taken from the literature. Each system was solved for ten different loading conditions The results of using Stott's algorithm[9], the proposed algorithm and the Newton-Raphson method are summarized in Tables 1-6

The solution of the angle equations $\boldsymbol{\theta}$ is used to form the voltage equations, these are linearized with a

Table 1. Algorithm test | average $\theta_{ij} - \theta R_{ij}$ |

No of nodes	No of lines	Angle equations solution, deg	$(P-\theta)$ decoupling, deg	1 Newton-Raphson iteration, deg
39	46	0 23	0.34	0 22
28	41	0.54	0.95	0.91
23	30	0.37	0 70	0 63
5	7	1 00	2.58	1 13

Table 2. Algorithm test average | $V_i - VR_i$ |

No. of nodes	No of load nodes	Proposed algorithm, p u.	Stott's algorithm, p.u	2 Newton-Raphson iterations, p u
39	29	0.001 2	0.002 4	0.002 0
28	29	0.002 6	0.007 7	0.002 0
23	19	0.002 9	0.003 9	0.012 8
5	4	0.020 5	0.028 3	0 054 9

Table 3. Algorithm test: average ΔP_i

No. of nodes	Proposed algorithm, p.u.	Stott's algorithm, p.u.	1 Newton-Raphson iteration, p.u.	2 Newton-Raphson iterations, p.u.
39	0.20	0.40	0.56	0.02
28	0.15	0 85	0.38	0.03
23	0.13	0.28	1.27	0.12
5	0.25	0.73	5.69	0.90

Table 4. Algorithm test. average ΔQ_i

No of nodes	Proposed algorithm, p.u	Stott's algorithm, p.u.	1 Newton-Raphson iteration, p.u.	2 Newton-Raphson iterations, p.u.
39	0.03	0 04	0.61	0.03
28	0.03	0 15	0.28	0.03
23	0 02	0.03	0 54	0.05
5	0.08	0 15	3.65	0.51

Table 5. Time ratios for starting algorithm

No. of nodes	39	28	23	5
Time ratio	0.54	0.67	0.68	1.00

Table 6. Results obtained using starting algorithm with proposed modification

No. of nodes	Average no. of Newton-Raphson iterations	Average no. of modified Newton-Raphson iterations
39	3.86	2.00
28	3.60	2.10
23	4.40	2 00
5	4.50	2.60

modification similar to the one described earlier and then solved.

In Table 5, the time ratio is calculated as the time, on an ICL 1906S, needed to perform the starting algorithm, divided by the time required for one Newton-Raphson iteration.

The nearer the ratio $G_{ii}/B_{ij}, j \in L$, is to the ratio G_{ii}/B_{ii}, the weaker the coupling between the angle equations and the voltage equations, consequently, the method is especially powerful in weakly interconnected systems.

VII. Conclusion

A method for providing starting values for Newton-Raphson load-flow calculation has been described. It involves forming the angle equations and the voltage equations. The solution of the linearized angle equations gives values of θ_{ij} that on average, differ by 0.41° from the exact solution. The solution of the linearized voltage equations provides voltage magnitudes that differ, on average, by 0.003 2 from the exact solution, this is to be compared with 0.008 2, the average voltage magnitude error after two complete Newton-Raphson iterations The reactive power mismatches obtained from the proposed starting algorithm are better than those obtained using two Newton-Raphson iterations. Also, the real power mismatches obtained from the starting algorithm are comparable with those obtained using two iterations. The implementation of the starting algorithm in existing Newton-Raphson programs requires no additional storage.

A simple modification of the well-known Newton-Raphson iteration has been suggested, and it has been shown that this modification takes into account most of the 2nd-order terms of the Taylor's series expansion This reduces the number of iterations, while the computational effort per iteration remains the same and no extra storage is needed

Owing to the better starting values that the algorithm provides and to the fact that the modification takes into account the 2nd-order terms, it can be concluded that the reliability of the Newton-Raphson load-flow method is enhanced by using the techniques presented.

IX. References

1 **Stott, B** 'Review of load-flow calculation methods' *Proc. IEEE* Vol 62 No 7 (July 1974)

2 **Tinney, W F and Hart, C C** 'Power flow solutions by Newtons method' *IEEE Trans. Power Appar. & Syst.* Vol PAS-86 (November 1967) pp 1449-1460

3 **Iwamoto, S and Tamura, Y** 'A fast load flow solution retaining nonlinearity' *IEEE Trans. Power Appar. & Syst.* Vol PAS-97 (1978) pp 1586-1599

4 **Stott, B and Alsac, O** 'Fast decoupled load-flow' *IEEE Trans. Power Appar. & Syst.* Vol PAS-93 (1974) pp 859-869

5 **Krishnaparandhama, T et al.** 'Fast power-flow solution by the method of reduction and restoration' *IEE Proc.* Vol 127 Pt C No 2 (March 1980)

6 **Park, T and Sekine, Y** 'A method for analysing multi-solution in power flow analysis' *Electr. Eng. Jap.* Vol 99 No 2 (1979) pp 95-103

7 **Stagg, G W and El-Abiad, A H** *Computer methods in power system analysis* McGraw-Hill, USA (1968)

8 **Ortega, J M and Rheinboldt, W C** *Iterative solution of nonlinear equations in several variables* Academic Press, USA (1970)

9 **Stott, B** 'Effective starting process for Newton-Raphson load-flow' *Proc. IEE* Vol 118 (1971) p 983

Appendix 1: Minimizing the coupling sensitivity matrices norms

It is required to determine Φ such that $\| \mathbf{S}(\mathbf{f}_\theta, \mathbf{V}) \|$ is minimal. Equation (8) can be written as.

$$\frac{\partial \theta_i}{\partial V_j} = \frac{Y_{ij} \cos(\gamma_{ij} + \Phi_i)}{Y_{ii} \sin(\gamma_{ii} + \Phi_i)} \tag{16}$$

and the problem is to minimize

$$\frac{\sum_{j \in k} Y_{ij}^2 \cos^2(\gamma_{ij} + \Phi_i)}{Y_{ii}^2 \sin^2(\gamma_{ii} + \Phi_i)} \tag{17}$$

Expression (17) is the sum of the squares of the elements in row i of $\mathbf{S}(\mathbf{f}_\theta, \mathbf{V})$, this has a limit when

$$\Phi_i = \tan^{-1} \frac{\sum_{j \in k} Y_{ij} G_{ij} \cos(\gamma_{ij} - \gamma_{ii})}{\sum_{j \in k} Y_{ij} B_{ij} \cos(\gamma_{ij} - \gamma_{ii})} \tag{18}$$

But

$$\gamma_{ij} \simeq \gamma_{ii} - \pi$$

$$\sum_{i \neq j} G_{ij} \simeq -G_{ii}$$

and

$$\sum_{i \neq j} B_{ij} \simeq -B_{ii}$$

therefore

$$\Phi_i \simeq \tan^{-1} \frac{G_{ii}}{B_{ii}} = -\beta_i \tag{19}$$

Testing the constants of the 28-node system showed that the error introduced by the approximation in equation (19) is less than 1%

Similar steps can be applied to the voltage equations to minimize $\| \mathbf{S}(\mathbf{f}_v, \theta \|$ The result in this case is

$$\Phi_i \simeq -\gamma_{ii} \tag{20}$$

Appendix 2: Effectiveness of angle and voltage equations

To illustrate the effectiveness of $\mathbf{f}_\theta$ and $\mathbf{f}_v$, consider a simple two-bus system. The voltage of the slack bus is $1\angle 0$, and that of the load bus is $V\angle\theta$ The admittance of the line connecting the two nodes is $(0.05 - j0.10)$ p.u. The contours of constant P and constant Q are shown in Figure 1. The values of the load power, P, were varied between -0 006 and 0.048 in ten steps, and those of Q were varied between 0.00 and -0.09, also in ten steps

The contours of constant f_θ and constant f_v are shown in Figure 2. The values of f_θ were varied between -0 004 and 0.005 and those of f_v between -0.002 and 0.007, in ten steps each. The solution paths are shown as dotted lines ΔV and $\Delta\theta$ shown in the figures are the errors after one complete iteration.

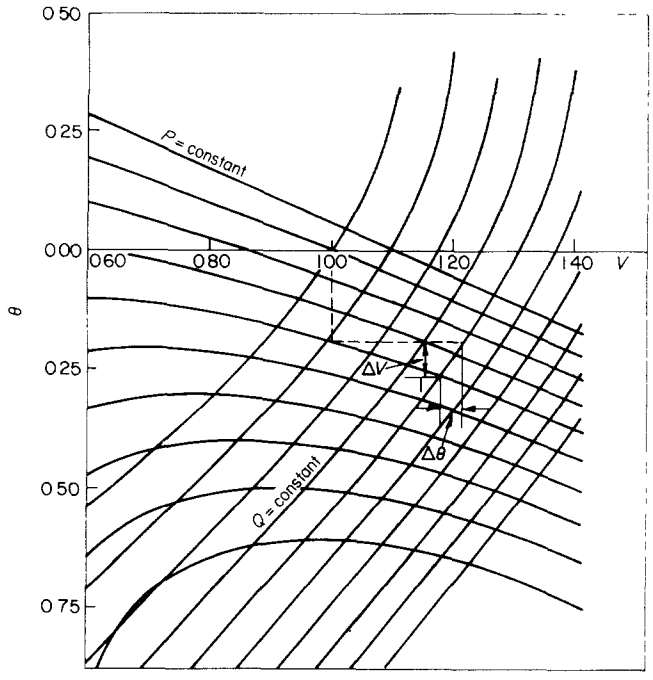

Figure 1 Contours of constant P and constant Q for a two-node network, point 1. desired solution, ---- solution path, ΔV and $\Delta\theta$ errors after one iteration

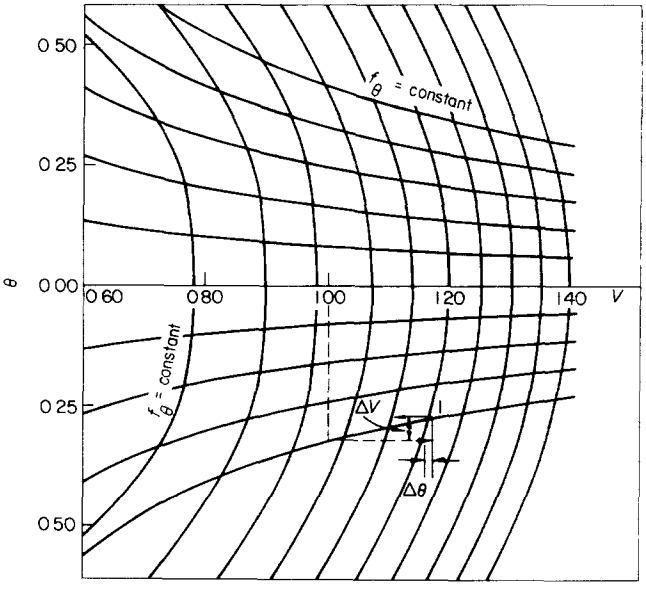

Figure 2 Contours of constant f_θ and constant f_V for a two-node network, point 1 desired solution, ---- solution path

Appendix 3

The equations corresponding to P_i and Q_i that are included in equation (3) are

$$P_i^{sp} = P_i + JK11_{ii} \Delta\theta_i + \sum_{j \in L} JK11_{ij} \Delta\theta_j + JK12_{ii} \frac{\Delta V_i}{V_i}$$

$$+ \sum_{j \in L} JK12_{ij} \frac{\Delta V_j}{V_j} \tag{21}$$

$$Q_i^{sp} = Q_i + JK21_{ii}\Delta\theta_i + \sum_{j\in L} JK21_{ij}\Delta\theta_j + JK22_{ii}\frac{\Delta V_i}{V_i} + \sum_{j\in L} JK22_{ij}\frac{\Delta V_j}{V_j} \quad (22)$$

All the coefficients on the right-hand side of equation (21) are to be calculated using the $\mathbf{V}$ and $\boldsymbol{\theta}$ around which the expansion was taken It can be verified that

$$JK11_{ii} = -Q_i - V_i^2 B_{ii} \quad (23)$$

and

$$JK12_{ii} = P_i + V_i^2 G_{ii} \quad (24)$$

The suggested modification is to use the expression

$$JK11_{ii}^m = -0\,5(Q_i + Q_i^{sp}) - V_i^2 B_{ii} \quad (25)$$

for $JK11_{ii}$ and

$$JK12_{ii}^m = 0.5(P_i + P_i^{sp}) + V_i^2 G_{ii} \quad (26)$$

for $JK12_{ii}$

To see the effect of this modification, the value of Q_i^{sp} as given by equation (22) is substituted into equation (25) and that of P_i^{sp} as given by equation (21) is substituted into equation (26) and the resultant $JK11_{ii}^m$ and $JK12_{ii}^m$ are substituted back into equation (21) as follows

$$\begin{aligned} P_i^{sp} = P_i + \Bigg[-0\,5\Bigg(Q_i + \Bigg(Q_i + JK21_{ii}\Delta\theta_i \\ + \sum_{j\in L} JK21_{ij}\Delta\theta_j + JK22_{ii}\frac{\Delta V_i}{V_i} \\ + \sum_{j\in L} JK22_{ij}\frac{\Delta V_j}{V_j}\Bigg)\Bigg) - V_i^2 B_{ii}\Bigg]\Delta\theta_i \\ + \sum_{j\in L} JK11_{ij}\Delta\theta_j + \Bigg[0\,5\Bigg(P_i + \Bigg(P_i + JK11_{ii}\Delta\theta_i \\ + \sum_{j\in L} JK11_{ij}\Delta\theta_j + JK12_{ii}\frac{\Delta V_i}{V_i} \\ + \sum_{j\in L} JK12_{ij}\frac{\Delta V_j}{V_j}\Bigg)\Bigg) + V_i^2 G_{ii}\Bigg]\frac{\Delta V_i}{V_i} \\ + \sum_{j\in L} JK12_{ij}\frac{\Delta V_j}{V_j} \end{aligned} \quad (27)$$

Regrouping terms, it can be easily verified that the right-hand side of equation (27) contains all the terms of the right-hand side of equation (21) in addition to the following 2nd-order terms

$$-0.5\,JK21_{ii}\Delta\theta_i^2$$

$$-0\,5\sum_{j\in L} JK21_{ij}\Delta\theta_i\Delta\theta_j$$

$$-0\,5\sum_{j\in L} JK22_{ij}\Delta\theta_i\Delta V_j/V_j$$

$$0\,5\sum_{j\in L} JK11_{ij}\Delta\theta_j\Delta V_i/V_i$$

$$0.5\sum_{j\in L} JK12_{ij}(\Delta V_i/V_i)(\Delta V_j/V_j)$$

$$0\,5\,(-JK22_{ii} + JK11_{ii})\Delta\theta_i\Delta V_i/V_i$$

$$0\,5\,JK12_{ii}(\Delta V_i/V_i)^2$$

It can easily be shown that the first five terms of the above list can be written as

$$0\,5\frac{\delta^2 P_i}{\delta\theta_i^2}(\Delta\theta_i)^2$$

$$0.5\sum_{j\in L}\frac{\delta^2 P_i}{\delta\theta_i\delta\theta_j}\Delta\theta_i\Delta\theta_j$$

$$0.5\sum_{j\in L}\frac{\delta^2 P_i}{\delta\theta_i\delta V_j}\Delta\theta_i\Delta V_j$$

$$0.5\sum_{j\in L}\frac{\delta^2 P_i}{\delta\theta_i\delta V_i}\Delta\theta_j\Delta V_i$$

$$0.5\sum_{j\in L}\frac{\delta^2 P_i}{\delta V_i\delta V_j}\Delta V_i\Delta V_j$$

The coefficient of $(\Delta\theta_i)^2$ is the same as it is in the Taylor's series expansion of P_i. The coefficients of the other terms, apart from the 0.5 multiplier, are also the same as the corresponding coefficients in the Taylor's series. The coefficients of $\Delta\theta_i(\Delta V_i/V_i)$ and $(\Delta V_i/V_i)^2$ are the same as they are in the Taylor's series expansion assuming flat-start values of the variables.

To summarize, the matrix $\mathbf{H}_p$ is the Hessian matrix calculated using flat-start values of $\mathbf{V}$ and $\boldsymbol{\theta}$. The matrix $2\mathbf{H}_p$ is as follows

$$2\mathbf{H}_p = \begin{array}{c} \\ \theta_1 \\ \\ \theta_i \\ \theta_n \\ V_1 \\ \\ V_i \\ V_n \end{array} \begin{array}{c} \begin{array}{cc} \theta_1 & \theta_i\ \theta_n\ V_1 \end{array} \quad\quad \begin{array}{cc} V_i & V_n \end{array} \\ \left[\begin{array}{cc|cc} G_{ij} & -G_{ij} & B_{ij} & B_{ij} \\ -G_{ij} & -G_{ii} & -B_{ij} & B_{ii} \\ \hline B_{ij} & -B_{ij} & 0 & -G_{ij} \\ B_{ij} & B_{ii} & -G_{ij} & -2G_{ii} \end{array}\right] \end{array}$$

$\mathbf{H}'_p$ is the matrix representing the generated 2nd-order terms calculated using flat-start values also, and the matrix $4\mathbf{H}'_p$ is given as

$$
4\mathbf{H}'_p = \begin{array}{c} \\ \theta_1 \\ \\ \theta_i \\ \\ \theta_n \\ V_1 \\ \\ V_i \\ \\ V_n \end{array}
\begin{array}{c}
\begin{array}{cccc} \theta_1 & & \theta_i\ \theta_n\ V_1 & & V_i & V_n \end{array} \\
\left[\begin{array}{cc|cc}
0 & -G_{ij} & 0 & B_{ij} \\
-G_{ij} & -2G_{ii} & -B_{ij} & 0 \\
\hline
0 & -B_{ij} & 0 & -G_{ij} \\
B_{ij} & 0 & -G_{ij} & -2G_{ii}
\end{array}\right]
\end{array}
$$

Similar steps can be followed to see the effect of the proposed modifications on the Q equations. The following two matrices summarize the results.

$$
2\mathbf{H}_q = \begin{array}{c} \\ \theta_1 \\ \\ \theta_i \\ \\ \theta_n \\ V_1 \\ \\ V_i \\ \\ V_n \end{array}
\begin{array}{c}
\begin{array}{cccc} \theta_1 & & \theta_i\ \theta_n\ V_1 & & V_i & V_n \end{array} \\
\left[\begin{array}{cc|cc}
B_{ij} & B_{ij} & B_{ij} & B_{ij} \\
B_{ij} & B_{ii} & B_{ij} & B_{ii} \\
\hline
B_{ij} & B_{ij} & 0 & B_{ij} \\
B_{ij} & B_{ii} & B_{ij} & 2B_{ii}
\end{array}\right]
\end{array}
$$

where $\mathbf{H}_q$ is the Hessian matrix corresponding to the Q_i equations, calculated using a flat start

$$
4\mathbf{H}'_q = \begin{array}{c} \\ \theta_1 \\ \\ \theta_i \\ \\ \theta_n \\ V_1 \\ \\ V_i \\ \\ V_n \end{array}
\begin{array}{c}
\begin{array}{cccc} \theta_1 & & \theta_i\ \theta_n\ V_1 & & V_i & V_n \end{array} \\
\left[\begin{array}{cc|cc}
0 & B_{ij} & 0 & B_{ij} \\
B_{ij} & 2B_{ii} & B_{ij} & 0 \\
\hline
0 & B_{ij} & 0 & B_{ij} \\
B_{ij} & 0 & B_{ij} & 2B_{ii}
\end{array}\right]
\end{array}
$$

where $\mathbf{H}'_q$ is the matrix representing the generated 2nd-order terms calculated using a flat start